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THÈSE DE DOCTORAT DE

L'UNIVERSITE DE BRETAGNE SUD

ECOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : Mathématiques et leurs Interactions

Par
Hui XIAO

Grandes déviations pour les produits de matrices aléatoires

Thèse présentée et soutenue à Vannes, le 11/06/2020
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Résumé

L'objet de cette thèse est d'étudier les asymptotiques précises des grandes déviations et des déviations modérées pour les produits de matrices aléatoires. La thèse se compose de six parties.

Dans la première partie, nous établissons des asymptotiques exactes de types Bahadur-Rao et Petrov pour les probabilités de grandes déviations supérieurs et inférieurs pour le cocycle de la norme, $\log|G_n x|$, où $G_n = g_n \dots g_1$ est le produit des matrices aléatoires g_i, de type $d \times d$, indépendantes et identiquement distribuées, x est un vecteur unitaire de $\mathbb{R}^d$. Nos résultats sont valables à la fois pour les matrices inversibles et les matrices positives. Plus généralement, nous prouvons des résultats de grandes déviation de types Bahadur-Rao et Petrov pour le couple $(X_n^x, \log|G_n x|)$ avec des fonctions cibles φ sur $X_n^x = G_n x/|G_n x|$ et ψ sur $\log|G_n x|)$. Comme applications, pour la norme d'opérateur $\|G_n\|$, nous déduisons un principe de grandes déviations pour les matrices inversibles et un principe de grandes déviations renforcé pour les matrices positives. Nous dérivons aussi des théorèmes de limites locales avec grandes déviations pour le cocycle de la norme.

La deuxième partie est consacrée à l'établissement des résultats de grandes déviations de types Bahadur-Rao et Petrov pour les entrées (i,j)-ème $G_n^{i,j}$ de G_n, pour les matrices inversibles et les matrices positives. En particulier, nous obtenons un principe de grandes déviations avec une fonction de taux explicite, qui améliore de manière significative les bornes de grandes déviations établies récemment dans la littérature. Pour les preuves, une étape importante, qui a un intérêt indépendant, est d'établir la propriété de régularité höldérienne pour la mesure stationnaire de la chaîne de Markov X_n^x sous la mesure changée sur l'espace projectif. Comme applications, nous obtenons des théorèmes limites locales avec grandes déviations pour les entrées $G_n^{i,j}$, et un principe renforcé de grandes déviations pour le rayon spectral $\rho(G_n)$, pour les matrices positives.

Dans la troisième partie, nous obtenons la borne de Berry-Esseen pour la vitesse de convergence dans le théorème central limite, et le développement de déviations modérées de type Cramér pour le cocycle de la norme des produits de matrices aléatoires. Nous établissons d'abord une borne de Berry-Esseen et un développement de type Edgeworth pour le couple $(X_n^x, \log|G_n x|)$ avec une fonction cible φ sur la chaîne de Markov X_n^x, pour les matrices inversibles et les matrices positives. Ces résultats sont prouvés en élaborant une nouvelle approche basée sur une inégalité de lissage dans le plan complexe et sur la méthode du point-selle. Le développement de déviations modérées de type Cramér pour le couple $(X_n^x, \log|G_n x|)$ est démontré en utilisant la borne de Berry-Esseen sous la mesure changée.

La quatrième partie est consacrée à l'étude des bornes de type Berry-Esseen et des développements de déviations modérées de type Cramér pour la norme d'opérateur $\|G_n\|$, les entrées $G_n^{i,j}$ et le rayon spectral $\rho(G_n)$ des produits de matrices aléatoires pos-

itives. Les résultats sur $\|G_n\|$ sont prouvés sous les conditions usuelles d'admissibilité et de positivité; les résultats sur $G_n^{i,j}$ et $\rho(G_n)$ sont établis sous une condition de bornetude plus faible que celle de Furstenberg-Kesten.

Dans la cinquième partie, nous étudions les bornes de type Berry-Esseen et les déviations modérées pour la norme d'opérateur $\|G_n\|$ et le rayon spectral $\rho(G_n)$, pour les matrices inversibles. Nous établissons d'abord, sous la condition de proximalité, les principes de déviations modérées pour les couples $(X_n^x, \log \|G_n\|)$ et $(X_n^x, \log \rho(G_n))$ avec des fonctions cibles sur la chaîne de Markov X_n^x, en utilisant les résultats de déviations modérées sur le couple $(X_n^x, \log |G_n x|)$ prouvés dans la troisième partie. Nous prouvons ensuite les principes de déviations modérées pour $\|G_n\|$ et $\rho(G_n)$ sans supposer la condition de proximalité. Nous prouvons également des développements de déviations modérées dans la zone $[0, o(n^{1/6})]$ pour les couples $(X_n^x, \log \|G_n\|)$ et $(X_n^x, \log \rho(G_n))$ avec des fonctions cibles.

La sixième partie est consacrée à la démonstration des développements de déviation modérée de type Cramér pour les entrées $G_n^{i,j}$ de produits de matrices inversibles dans le groupe linéaire spécial $SL_2(\mathbb{R})$. Notre résultat implique un principe de déviation modérée pour $\log |G_n^{i,j}|$ et des théorèmes de limites locales avec des déviations modérées. Dans notre preuve, nous utilisons la méthode du point selle, la régularité höldérienne de la mesure stationnaire de la chaîne de Markov X_n^x, et les progrès récents sur la propriété non-arithmétique forte de l'opérateur perturbé.

Abstract

The purpose of this Ph.D. thesis is to study precise large and moderate deviation asymptotics for products of random matrices. The thesis consists of six parts corresponding to Chapters 2-7.

In the first part (Chapter 2), we establish Bahadur-Rao type and Petrov type exact asymptotics of the upper and lower large deviation probabilities for the norm cocycle $\log|G_n x|$, where $G_n = g_n \dots g_1$ is the product of independent and identically distributed random $d \times d$ matrices g_i, x is a unit vector in $\mathbb{R}^d$. Our results are valid for both invertible matrices and positive matrices. More generally, we prove analogous Bahadur-Rao-Petrov type large deviation results for the couple $(X_n^x, \log|G_n x|)$ with target functions φ on $X_n^x = G_n x/|G_n x|$ and ψ on $\log|G_n x|$. As applications, for the operator norm $\|G_n\|$, we deduce large deviation principles for invertible matrices, and reinforced large deviation principles for positive matrices. We also derive precise local limit theorems with large deviations for the norm cocycle.

The second part (Chapter 3) is devoted to establishing Bahadur-Rao type and Petrov type large deviations for the (i,j)-th entries $G_n^{i,j}$ of G_n, for both invertible matrices and positive matrices. In particular, we obtain a large deviation principle with an explicit rate function, which improves significantly the large deviation bounds established recently in the literature. In our proof, an important issue is to prove the Hölder regularity property for the stationary measure of the Markov chain X_n^x under the changed measure on the projective space, which is of independent interest. As applications, we obtain precise local limit theorems with large deviations for the entries and reinforced large deviation principles for the spectral radius of products of positive random matrices.

In the third part (Chapter 4), we investigate the Berry-Esseen bound of the rate of convergence in the central limit theorem, and Cramér type moderate deviation expansion, for the norm cocycle of products of random matrices. We first establish the Berry-Esseen bound and the Edgeworth expansion for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on the Markov chain X_n^x, for both invertible matrices and positive matrices. This is proved by elaborating a new approach based on a smoothing inequality in the complex plane and on the saddle point method. Using the Berry-Esseen bound under the changed measure, we then establish the Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$.

The fourth part (Chapter 5) is devoted to studying Berry-Esseen bounds and Cramér type moderate deviation expansions for the operator norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$, for positive matrices. The approach is based on the results established in the third part. The results on $\|G_n\|$ are proved under the usual allowability and positivity conditions; the results on $G_n^{i,j}$ and $\rho(G_n)$ are established under a boundedness condition weaker than that of Furstenberg-Kesten.

In the fifth part (Chapter 6), we study the Berry-Esseen type bounds and moderate

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deviations for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$, for invertible matrices. First, under the proximality condition, we establish moderate deviation principles for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions on the Markov chain X_n^x, using the moderate deviation results on the couple $(X_n^x, \log |G_n x|)$ proved in the third part. Next, we prove the moderate deviation principles for $\|G_n\|$ and $\rho(G_n)$ without assuming the proximality condition. We then prove the moderate deviation expansions in the range $[0, o(n^{1/6})]$ for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions.

The sixth part (Chapter 7) is devoted to proving the Cramér type moderate deviation expansion for the entries $G_n^{i,j}$ of products of invertible matrices in the special linear group $SL_2(\mathbb{R})$. Our result implies the moderate deviation principle for $\log |G_n^{i,j}|$ and local limit theorems with moderate deviations, which are also new. In our proof, we use the saddle point method, the Hölder regularity of the stationary measure of the Markov chain X_n^x and the strong non-lattice property of the perturbed operator established recently.

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Chapter 1

Introduction

1.1 Context

The topic of this thesis is concerned with limit theorems, especially with large deviations for products of independent and identically distributed random matrices. The products of random matrices can be viewed as random walks on linear groups or semigroups. The topic is therefore closely related to probability theory, group theory and dynamical systems. The study of products of random matrices was formally initialed by Furstenberg and Kesten [37], and then greatly developed by Guivarc'h, Ledrappier, Le Page, Raugi and many others, cf. for example the book of Bougerol and Lacroix [13]. Important progress on this subject has been made recently, see for example Guivarc'h and Le Page [50], and Benoist and Quint [6, 7, 8, 10]. This topic has not only its own vitality, but also broad application prospects in many related areas.

The study of products of random matrices can be dated back to the middle of 20th century when Bellman [5] conjectured the central limit theorem in the non-commutative case. His conjecture was confirmed in the pioneering work of Furstenberg and Kesten [37] for the semigroup of positive matrices, where they also established the law of large numbers. Important developments can be found in Kesten [66]. In the setting of general linear groups, Furstenberg [36] established the law of large numbers; Tutubalin [81] proved the central limit theorem in the restrictive case where the common law of the random matrices has a density with respect to the Haar measure. Without assuming the density condition, Le Page [69] established the central limit theorem by proving the spectral gap property for the associated Markov chain on a special designed Banach space. Other laws of large numbers and central limit theorems were established by Guivarc'h and Raugi [51], Guivarc'h [48], and Benoist and Quint [9, 10].

The law of large numbers and the central limit theorem are the most fundamental results in probability theory and statistic sciences. For a number of applications, we need to know the rates of convergence in these limit theorems. It is therefore important to study large deviations and Berry-Esseen bounds, which describe respectively the rate of convergence in the law of large numbers and that in the central limit theorem. This motivates us to study such limit theorems for products of random matrices. Another motivation of this study lies in the fact that the theory of products of random matrices turns out to be very useful in a number of branches of mathematics. For instance, it plays crucial roles in [13] to investigate the spectral theory of random

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Schrödinger operators, in [14] to study the stationary measure and the quantitative distribution properties on the torus, in [50] to obtain the tail behavior of the multidimensional affine stochastic recursions, in [6, 7, 8] to understand the dynamics of group actions on finite volume homogeneous spaces, in [16] to study the multidimensional Mandelbrot cascades, and in [70] to investigate the survival probability of critical multi-type branching processes in random environment.

For sums of independent and identically distributed real-valued random variables, precise large and moderate deviation results were established respectively by Bahadur-Rao and Petrov [4, 73], and by Cramér and Petrov [26, 74]. The main goal of this thesis is to prove analogous results for products of random matrices.

1.2 Background and main objectives

1.2.1 Background

For any integer $d \geqslant 2$, let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) $d\times d$ real random matrices defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with common law μ. Consider the random matrix product G_n defined by

$$G_n = g_n \dots g_1.$$

It has been of great interest to investigate the asymptotic behaviors of the product G_n since the groundwork of Furstenberg and Kesten [37], where the strong law of large numbers (SLLN) for the *operator norm* $\|G_n\|$ was established: if $\mathbb{E}(\log^+ \|g_1\|) < \infty$ (with $\log^+ a = \max\{\log a, 0\}$ for $a > 0$), then

$$\lim_{n\to\infty} \frac{1}{n} \log \|G_n\| = \lambda, \quad \text{a.s.}, \tag{1.2.1}$$

with λ a constant called the upper Lyapunov exponent of the product sequence (G_n). Note that (1.2.1) remains valid when the operator norm is replaced by any matrix norm since all matrix norms are equivalent. The SLLN (1.2.1) can be considered as a direct consequence of Kingman's subaddtive ergodic theorem [67]. The central limit theorem (CLT) for $(\|G_n\|)$ was considered by Tutubalin [81] under the restrictive density assumption that the measure μ is absolute continuous with respect to the Haar measure. Further developments can be found in the work of Kaijser [64]. A cornerstone result in this direction is due to Le Page [69]. To state this result, we need some notation and conditions. Denote by Γ_μ the smallest subsemigroup generated by $\operatorname{supp}\mu$, the support of μ. We say that: (a) Γ_μ is *irreducible* (resp. *strongly irreducible*) if there is no proper subspace (resp. finite union of proper subspaces) of $\mathbb{R}^d$ which is Γ_μ-invariant; (b) Γ_μ is *proximal* if Γ_μ contains at least one matrix with a dominant eigenvalue; (c) μ has exponential moment (second moment) if $\mathbb{E}[N(g_1)^\eta] < \infty$ for some constant $\eta > 0$ ($\mathbb{E}[\log^2 N(g_1)] < \infty$), where $N(g) = \max\{\|g\|, \|g^{-1}\|\}$. Le Page [69] established the following CLT for $\|G_n\|$: under the strong irreducibility, proximality and the exponential moment conditions on μ, for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right) = \Phi(y), \tag{1.2.2}$$

where $\sigma^2 := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log\|G_n\| - n\lambda)^2\Big] > 0$ is the asymptotic variance, Φ is the standard normal distribution function on $\mathbb{R}$. Gol'dsheid and Margulis [41] showed that the strong irreducibility and proximality conditions are equivalent to the condition that the action of the Zariski closure of the Γ_μ is strogly irreducible and proximal. Gol'dsheid and Guivarc'h [40] extended the CLT to the Cartan projection under the condition that Γ_μ is Zariski dense in $GL(d,\mathbb{R})$. Later on, Bougerol and Lacroix [13] proved the CLT (1.2.2) when the proximality condition is replaced by the unboundedness of the semigroup Γ_μ. The latter condition holds if and only if the multiplicity of the dominating eigenvalue lies between 1 and $d-1$. Note that in the case when Γ_μ is not assumed to be proximal, the Markov chain $X_n^x := G_n x/|G_n x|$ may not have a unique stationary measure ν on the projective space in $\mathbb{R}^d$. The only remaining assumption to relax was the exponential moment condition. Jan [62] proved the CLT (1.2.2) under the weaker condition that all the p-th moments of μ are finite. Very recently, under the optimal second moment condition on μ, Benoist and Quint [9] have established the CLT (1.2.2) based on Gordin's martingale approximation method and the log-regularity of the stationary measure ν.

Equip the Euclidean space $\mathbb{R}^d$ with the canonical Euclidean norm $|\cdot|$. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$, and $\mathbb{P}^{d-1} = \mathbb{S}^{d-1}/\pm$ be the projective space obtained from $\mathbb{S}^{d-1}$ by identifying $-x$ with x. We consider the random walk $(G_n x)$ starting from a point $x \in \mathbb{P}^{d-1}$, governed by the products G_n of the random matrices (g_i). Now we recall some results for the *Euclidean norm* $|G_n x|$ of the position $G_n x$ at time n of the random walk. The SLLN for $(|G_n x|)$ was established by Furstenberg [36]: if $\mathbb{E}(\log^+\|g_1\|) < \infty$ and Γ_μ is irreducible, then for any $x \in \mathbb{P}^{d-1}$,

$$\lim_{n\to\infty} \frac{1}{n} \log|G_n x| = \lambda, \quad \text{a.s.}. \tag{1.2.3}$$

For related results on the Iwasawa cocycle on the flag variety we refer to the nice work of Gol'dsheid and Margulis [41]. Under the same conditions as those used for (1.2.2), Le Page [69] proved the following CLT for $(|G_n x|)$: for any $y \in \mathbb{R}$, it holds uniformly in $x \in \mathbb{P}^{d-1}$ that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log|G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right) = \Phi(y). \tag{1.2.4}$$

Applying Gordin's martingale approximation method, Benoist and Quint [9] have recently improved the CLT (1.2.4) by relaxing the exponential moment condition to the optimal second moment condition on μ.

We now turn to the *entries* of the product G_n. Denote by $G_n^{i,j}$ the (i,j)-th entry of G_n. It turns out that the study of the asymptotic behavior of $G_n^{i,j}$ is more delicate and difficult. Guivarc'h and Raugi [51] proved the following SLLN for $G_n^{i,j}$: under the strong irreducibility, proximality and the exponential moment conditions on μ, for any $1 \leqslant i, j \leqslant d$,

$$\lim_{n\to\infty} \frac{1}{n} \log|G_n^{i,j}| = \lambda, \quad \text{a.s.}. \tag{1.2.5}$$

Under the same conditions, in [51] a CLT for the entries has been established: for any $1 \leqslant i, j \leqslant d$ and $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log|G_n^{i,j}| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right) = \Phi(y). \tag{1.2.6}$$

Compared to the one dimensional case, both the exponential moment condition and the proximality condition for (1.2.5) and (1.2.6) are unnatural. However, it is still open how to relax these conditions to the optimal ones. In our work, we shall also use these conditions.

Denote by $\rho(G_n)$ the *spectral radius* of the product G_n. Compared with the operator norm $\|G_n\|$, the submultiplicativity is no longer valid for the spectral radius $\rho(G_n)$, but a SLLN still holds for $\rho(G_n)$, as established by Guivarc'h [48]: under the strong irreducibility, proximality and the exponential moment conditions on μ,

$$\lim_{n\to\infty} \frac{1}{n} \log \rho(G_n) = \lambda, \quad \text{a.s..} \tag{1.2.7}$$

Recently, Aoun and Sert [3] have proved (1.2.7) under the second moment condition on μ, without assuming the strong irreducibility and proximality assumptions. However, it still remains an open question to prove (1.2.7) under the first moment condition. As to the CLT for $\rho(G_n)$, Benoist and Quint [10] established the following: under the strong irreducibility, proximality and the exponential moment conditions on μ, for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leqslant y\right) = \Phi(y). \tag{1.2.8}$$

This result has been recently improved by Aoun [2] by relaxing the exponential moment condition to the second moment condition, and by relaxing the proximality condition to the assumption of the unboundedness of the semigroup Γ_μ.

1.2.2 Classical results on precise large and moderate deviations

In this section we briefly recall some classical results on precise large and moderate deviations for sums of i.i.d. real-valued random variables.

Let $(X_i)_{i\geqslant 1}$ be a sequence of i.i.d. real-valued random variables and $S_n = \sum_{i=1}^n X_i$. Denote by I_Λ the set of real numbers $s \geqslant 0$ such that

$$\Lambda(s) := \log \mathbb{E}[e^{sX_1}] < +\infty \tag{1.2.9}$$

and by I_Λ° the interior of the interval I_Λ. Let Λ^* be the Frenchel-Legendre transform of Λ. Assume that $s \in I_\Lambda^\circ$ and q are related by $q = \Lambda'(s)$. Set $\sigma_s^2 = \Lambda''(s)$. The following theorem is due to Bahadur and Rao [4]:

Theorem 1.2.1 ([4]). *Let $s \in I_\Lambda^\circ$ and $q = \Lambda'(s)$. Assume that the law of X_1 is non-lattice. Then, as $n \to \infty$,*

$$\mathbb{P}(S_n \geqslant nq) = \frac{\exp(-n\Lambda^*(q))}{s\sigma_s\sqrt{2\pi n}}(1 + o(1)). \tag{1.2.10}$$

When the law of X_1 is lattice, an analogous precise large deviation result has also been established in [4].

Later on, Petrov [73] improved Theorem 1.2.1 by considering a vanishing perturbation l on q:

Theorem 1.2.2 ([73]). *Let $s \in I_\Lambda^\circ$ and $q = \Lambda'(s)$. Assume that the law of X_1 is non-lattice. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $|l| \leqslant l_n$,*

$$\mathbb{P}(S_n \geqslant n(q+l)) = \frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}(1+o(1)). \tag{1.2.11}$$

Note that the exact asymptotics for the lower tail large deviation probabilities can be deduced easily from upper tail large deviation asymptotics (1.2.10) and (1.2.11) by considering $-X_1$ instead of X_1.

Denote $\gamma_k = \Lambda^{(k)}(0)$, $k \geqslant 1$, where Λ is the cumulant generating function of X_1 defined in (1.2.9). Let $\lambda := \gamma_1 = \mathbb{E}X_1$ and $\sigma^2 := \gamma_2 = \mathbb{E}(X_1-\lambda)^2$ be the mean and variance of X_1. Denote by ζ the Cramér series of Λ (see [26] and [74]):

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \cdots$$

which converges for $|t|$ small enough. Let Φ be the standard normal distribution function on $\mathbb{R}$. We recall the following Cramér type moderate deviation expansion for S_n.

Theorem 1.2.3 ([74]). *Assume that $\sigma^2 > 0$ and that $\mathbb{E}e^{\delta|X_1|} < \infty$ for some $\delta > 0$. Then, as $n \to \infty$, uniformly in $y \in [0, o(\sqrt{n}))$,*

$$\frac{\mathbb{P}\left(\frac{S_n - n\lambda}{\sigma\sqrt{n}} \geqslant y\right)}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\left[1 + O\left(\frac{y+1}{\sqrt{n}}\right)\right],$$

and

$$\frac{\mathbb{P}\left(\frac{S_n - n\lambda}{\sigma\sqrt{n}} \leqslant -y\right)}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\left[1 + O\left(\frac{y+1}{\sqrt{n}}\right)\right].$$

1.2.3 Main objectives and previous results

1. Our first objective is to establish the Bahadur-Rao type and Petrov type precise large deviation results for the norm cocycle $\log|G_n x|$. In other words, we are interested in the precise asymptotic behavior of the forms (1.2.10) and (1.2.11) for the large deviation probabilities

$$\mathbb{P}(\log|G_n x| \geqslant nq) \quad \text{and} \quad \mathbb{P}(\log|G_n x| \geqslant n(q+l))$$

with $q > \lambda$, $x \in \mathbb{P}^{d-1}$, and l a vanishing perturbation on q. The case when the norm cocycle $\log|G_n x|$ is replaced by the logarithm of the operator norm $\log\|G_n\|$ will also be considered.

Previously, for the norm cocycle $\log|G_n x|$, Le Page [69] first established a Bahadur-Rao type precise large deviation result (of form (1.2.10)) for invertible matrices in the case when q is sufficiently near to λ; Buraczewski and Mentemeier [17] recently proved the following precise large deviation bounds (also called reinforced large deviation principle) for $\log|G_n x|$: for some constants $c, C > 0$,

$$\frac{\exp(-n\Lambda^*(q))}{C\sqrt{n}} \leqslant \mathbb{P}(\log|G_n x| \geqslant nq) \leqslant \frac{\exp(-n\Lambda^*(q))}{c\sqrt{n}}, \quad q > \lambda \tag{1.2.12}$$

for both invertible matrices and positive matrices: see (1.3.7) for details. For the operator norm $\|G_n\|$, much less results are known in the literature. In Bougerol and Lacroix [13], an exponential upper bound for the large deviation probability $\mathbb{P}(\log \|G_n\| \geqslant nq)$ was proved, without giving an explicit form of the rate function. Benoist and Quint [10] have generalized this result to the setting of general framework of reductive groups. Very recently, Sert [78] has given the rate function of the large deviation principle for $\log \|G_n\|$ under a stronger exponential moment condition, i.e., the exponential moments of all orders are finite.

2. Our second objective is to establish the Bahadur-Rao type and Petrov type precise large deviation results of the forms (1.2.10) and (1.2.11), for the entries $G_n^{i,j}$: we want to study the precise asymptotic behavior of the large deviation probabilities

$$\mathbb{P}(\log |G_n^{i,j}| \geqslant nq) \quad \text{and} \quad \mathbb{P}(\log |G_n^{i,j}| \geqslant n(q+l)),$$

with $q > \lambda$ and l a vanishing perturbation on q. The case when the entry $G_n^{i,j}$ is replaced by the spectral radius $\rho(G_n)$ will also be considered.

Previously, Benoist and Quint [10] established upper exponential large deviation bounds for the entries $G_n^{i,j}$ and for the spectral radius $\rho(G_n)$; however, the rate functions were not known in the literature. Sert [78] conjectured that the large deviation principle for the spectral radius $\rho(G_n)$ holds with the same rate function as for the norm cocycle $\log |G_n x|$.

3. Our third objective is to quantify the error in the normal approximation (1.2.4), which can be achieved in two ways. The first way is to estimate the following absolute error: for any starting point $x \in \mathbb{P}^{d-1}$ and $n \geqslant 1$,

$$\sup_{y\in\mathbb{R}} \left| \mathbb{P}\Big(\frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y \Big) - \Phi(y) \right|, \tag{1.2.13}$$

and, more generally, with a target function φ on the Markov chain X_n^x,

$$\sup_{y\in\mathbb{R}} \left| \mathbb{E}\left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y \right\}} \right] - \nu(\varphi)\Phi(y) \right|. \tag{1.2.14}$$

The second way is to study the relative error in the normal zone $y \in [0, o(\sqrt{n})]$: for any starting point $x \in \mathbb{P}^{d-1}$ and $n \geqslant 1$,

$$\frac{\mathbb{P}\Big(\frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \geqslant y \Big)}{1 - \Phi(y)} \tag{1.2.15}$$

and

$$\frac{\mathbb{E}\Big[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \geqslant \sqrt{n}\sigma y\}} \Big]}{1 - \Phi(y)}. \tag{1.2.16}$$

Previously, Le Page [69] established the Berry-Esseen type rate of convergence in (1.2.13) for invertible matrices. For positive matrices, Hennion and Hervé

[55] proved an analog of the Berry-Esseen type bound in (1.2.13). However, the Berry-Esseen bound for the couple $(X_n^x, \log|G_n x|)$ with general target function φ ($\varphi \neq \mathbf{1}$) in (1.2.14) becomes more complicated and delicate and were not known in the literature, neither for invertible matrices nor for positive matrices.

Concerning (1.2.15) and (1.2.16), Benoist and Quint [10] have recently established the moderate deviation principle for $\log|G_n x|$ using the strategy of Kolmogorov for the law of iterated logarithm. Cuny, Dedecker and Merlevède [23] have proved the functional moderate deviation principle based on the martingale approximation method developed in [9]. However, the Cramér type moderate deviation expansion for $\log|G_n x|$ was not known in the literature, neither for invertible matrices nor for positive matrices. Moreover, it turns out to be delicate to establish the Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$ with a target function, even for the moderate deviation principle for the couple.

4. Our fourth objective is to quantify the absolute errors as well as the relative errors in the normal approximations (1.2.2), (1.2.6) and (1.2.8) for the operator norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$, respectively. In other words, we are interested in the asymptotic behaviors of (1.2.13), (1.2.14), (1.2.15) and (1.2.16) when the vector $G_n x$ is respectively replaced by the operator norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$.

   Previously, very few were solved for the above-mentioned problems. Guivarc'h and Raugi [51] announced the Berry-Esseen bound for the entries $G_n^{i,j}$ of products of invertible matrices. We shall prove the Berry-Esseen bounds and Cramér type moderate deviation expansions for the operator norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$ for products of positive matrices. The situation for invertible matrices is more complicated and delicate and will be considered in ongoing work [87, 90].

1.3 Presentation of main results of the thesis

As already mentioned, the main goal of the present thesis is to study precise large and moderate deviation asymptotics for products of random matrices. The remaining part of the thesis consists of six chapters.

Chapter 2 is mainly devoted to the study of Bahadur-Rao type and Petrov type large deviations for the norm cocycle $\log|G_n x|$, for both invertible matrices and positive matrices; see Section 1.3.2 for the presentation of the main results. Using the spectral gap theory for products of random matrices and the saddle point method, we establish Bahadur-Rao-Petrov type exact asymptotics of the upper and lower tail large deviation probabilities for the norm cocycle. More generally, we also prove analogous Bahadur-Rao-Petrov type large deviation results for the couple $(X_n^x, \log|G_n x|)$ with target functions. As applications, we deduce new results on large deviation principles for the operator norm $\|G_n\|$ for invertible matrices, and reinforced large deviation principles for $\|G_n\|$ for positive matrices. Moreover, we derive precise local limit theorems with large deviations for the norm cocycle.

In Chapter 3, our main goal is to establish Bahadur-Rao type and Petrov type large deviations for the entries $G_n^{i,j}$, for both invertible matrices and positive matrices; see Section 1.3.3 for the presentation of the main results. As in Chapter 2, we also prove Bahadur-Rao-Petrov type upper and lower tail large deviation results for the couple $(X_n^{e_i}, \log|G_n^{i,j}|)$ with target functions. In particular, we obtain the large deviation principle with an explicit rate function, thus improving significantly the large deviation bounds established earlier. In our proof, a very important issue is to prove the Hölder regularity property for the stationary measure corresponding to the Markov chain X_n^x under the changed measure on the projective space, which is one of the central points of the proof and is of independent interest. As applications, we obtain new results on precise local limit theorems with large deviations for the entries and on reinforced large deviation principles for the spectral radius of products of positive random matrices.

Chapter 4 is devoted to investigating the Berry-Esseen bound and Cramér type moderate deviation expansion for the norm cocycle of products of random matrices; see Section 1.3.4 for the presentation of the main results. We first establish the Berry-Esseen bound and the Edgeworth expansion for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on the Markov chain X_n^x, for both invertible matrices and positive matrices. This is proved by elaborating a new approach based on a smoothing inequality in the complex plane and on the saddle point method. Using the Berry-Esseen bound under the changed measure, we then establish Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$.

In Chapter 5, we study Berry-Esseen bounds and Cramér type moderate deviation expansions for the operator norm $\|G_n\|$, entries $G_n^{i,j}$ and spectral radius $\rho(G_n)$ of products of positive random matrices; see Section 1.3.5 for the presentation of the main results. The results for the operator norm $\|G_n\|$ are proved under general conditions; the results for the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$ are established under a boundedness condition weaker than that of Furstenberg-Kesten.

Chapter 6 is devoted to studying the Berry-Esseen type bounds and moderate deviations for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$ of products of random matrices in the general linear group $GL_d(\mathbb{R})$; see Section 1.3.6 for the presentation of the main results. Under the proximality condition, we first prove the moderate deviation principles for the couples $(X_n^x, \log\|G_n\|)$ and $(X_n^x, \log\rho(G_n))$ with target functions based on the moderate deviation results for the norm cocycle $\log|G_n x|$ established in Chapter 2. Then we prove the moderate deviation principles for $\|G_n\|$ and $\rho(G_n)$ without assuming the proximality condition. We also prove the moderate deviation expansions in the range $[0, o(n^{1/6})]$ for the couples $(X_n^x, \log\|G_n\|)$ and $(X_n^x, \log\rho(G_n))$ with target functions.

In Chapter 7, we establish the Cramér type moderate deviation expansions for the entries $G_n^{i,j}$ of products of invertible matrices in the special linear group $SL_2(\mathbb{R})$; see Section 1.3.7 for the presentation of the main results. Our result implies the moderate deviation principle for $\log|G_n^{i,j}|$ and local limit theorems with moderate deviations, which are also new. In our proof, we use the saddle point method, the Hölder regularity of the stationary measure corresponding to the Markov chain X_n^x, and the recent progress on the strong non-lattice result for the perturbed transfer operator.

1.3.1 Notation and conditions

For $d \geqslant 2$, let $M(d,\mathbb{R})$ be the set of $d \times d$ matrices with entries in $\mathbb{R}$. Denote by $\mathscr{G} = GL(d,\mathbb{R})$ the general linear group of invertible matrices of $M(d,\mathbb{R})$. A positive matrix $g \in M(d,\mathbb{R})$ (we use the terminology positive in the wide sense, i.e. each entry is non-negative) is said to be *allowable*, if every row and every column of g has a strictly positive entry. Denote by $\mathscr{G}_+$ the multiplicative semigroup of allowable positive matrices of $M(d,\mathbb{R})$, and by $\mathscr{G}_+^\circ$ the subsemigroup of $\mathscr{G}_+$ with strictly positive entries.

Consider the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. Denote by $(e_i)_{1 \leqslant i \leqslant d}$ the canonical orthonormal basis in $\mathbb{R}^d$. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$, and $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ be its intersection with the positive quadrant. Consider the projective space $\mathbb{P}^{d-1} := \mathbb{S}^{d-1}/\pm$ by identifying $-x$ with x. To unify the exposition, we use the symbol $\mathcal{S}$ to denote $\mathbb{S}^{d-1}$ in the case of invertible matrices, and $\mathbb{S}_+^{d-1}$ in the case of positive matrices. The space $\mathcal{S}$ is equipped with the metric $\mathbf{d}$ which we proceed to introduce. For invertible matrices, the distance $\mathbf{d}$ is defined as the angular distance (see [50]), i.e., for any $x, y \in \mathbb{P}^{d-1}$, $\mathbf{d}(x,y) = |\sin\theta(x,y)|$, where $\theta(x,y)$ is the angle between x and y. For positive matrices, the distance $\mathbf{d}$ is the Hilbert cross-ratio metric (see [53]) defined by $\mathbf{d}(x,y) = \frac{1-m(x,y)m(y,x)}{1+m(x,y)m(y,x)}$, where $m(x,y) = \sup\{\lambda > 0 : \ \lambda y_i \leqslant x_i, \ \forall i = 1, \dots, d\}$, for any two vectors $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in $\mathbb{S}_+^{d-1}$.

Let $\mathcal{C}(\mathcal{S})$ be the space of complex-valued continuous functions on $\mathcal{S}$. We write $\mathbf{1}$ for the identity function $1(x)$, $x \in \mathcal{S}$. In the sequel, let $\gamma > 0$ be a fixed small constant. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\infty := \sup_{x \in \mathcal{S}} |\varphi(x)| \quad \text{and} \quad \|\varphi\|_\gamma := \|\varphi\|_\infty + \sup_{x,y \in \mathcal{S}} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}(x,y)^\gamma},$$

and introduce the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma < +\infty\}$.

For any $g \in M(d,\mathbb{R})$, set $\|g\| := \sup_{x \in \mathcal{S}} |gx|$ and $\iota(g) := \inf_{x \in \mathcal{S}} |gx|$. For both invertible matrices and allowable positive matrices, it holds that $\iota(g) > 0$. Note that for any invertible matrix g, we have $\iota(g) = \|g^{-1}\|^{-1}$.

Let $(g_n)_{n \geqslant 1}$ be a sequence of i.i.d. random matrices of the same probability law μ on $M(d,\mathbb{R})$. Set $G_n = g_n \dots g_1$, for $n \geqslant 1$. We denote by $\Gamma_\mu := [\operatorname{supp} \mu]$ the smallest closed semigroup of $M(d,\mathbb{R})$ generated by $\operatorname{supp} \mu$ (the support of μ). Let

$$I_\mu = \{s \geqslant 0 : \mathbb{E}(\|g_1\|^s) < +\infty\},$$

and I_μ° be its interior. We always assume that I_μ° is non-empty. One can easily see that I_μ is an interval by applying Hölder's inequality to $\mathbb{E}(\|g_1\|^s)$. We make use of the following exponential moment condition.

A1. There exist $s \in I_\mu^\circ$ and $0 < \alpha < 1$ such that $\mathbb{E}\|g_1\|^{s+\alpha} \iota(g_1)^{-\alpha} < +\infty$.

For $g \in \Gamma_\mu$, set $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$, which reduces to $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for invertible matrices. The following is the two-sided exponential moment condition.

A2. There exists a constant $0 < \eta < 1$ such that $\mathbb{E}[N(g_1)^\eta] < +\infty$.

A matrix g is said to be *proximal* if it has an algebraic simple dominant eigenvalue. For invertible matrices, we introduce the following conditions:

A3. (i)(Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant.
(ii)(Proximality) Γ_μ contains at least one proximal matrix.

For positive matrices, introduce the following conditions:

A4. (i) (Allowability) Every $g \in \Gamma_\mu$ is allowable.
(ii) (Positivity) Γ_μ contains at least one matrix belonging to $\mathscr{G}_+^\circ$.

In the groundwork [37], Furstenberg and Kesten established the strong law of large numbers and the central limit theorem for the entries of positive matrices under the condition that there exists a constant $C > 1$ such that for any $g = (g^{i,j})_{1\leqslant i,j\leqslant d} \in \operatorname{supp}\mu$,

$$1 \leqslant \frac{\max_{1\leqslant i,j\leqslant d} g^{i,j}}{\min_{1\leqslant i,j\leqslant d} g^{i,j}} \leqslant C. \tag{1.3.1}$$

In this thesis we shall relax it to:

A5. There exists a constant $C > 1$ such that for any $g = (g^{i,j})_{1\leqslant i,j\leqslant d} \in \operatorname{supp}\mu$, and $1 \leqslant j \leqslant d$, we have

$$1 \leqslant \frac{\max_{1\leqslant i\leqslant d} g^{i,j}}{\min_{1\leqslant i\leqslant d} g^{i,j}} \leqslant C. \tag{1.3.2}$$

Condition **A5** is clearly weaker than (1.3.1); the latter one means that all the entries $g^{i,j}$ of the matrix $g \in \operatorname{supp}\mu$ are comparable, while **A5** requires only that all the entries in the same columns of the matrix $g \in \operatorname{supp}\mu$ are comparable.

The following condition ensures the harmonic moments of the entries of g_1:

A6. For any $1 \leqslant i, j \leqslant d$, there exists a constant $\delta > 0$ such that

$$\mathbb{E}\Big[(g_1^{i,j})^{-\delta}\Big] < \infty.$$

Clearly, condition **A6** implies condition **A2**. Note also that the conditions **A5** and **A6** do not imply each other. However, under the assumption **A2**, condition **A5** (and therefore also (1.3.1)) implies condition **A6**. The converse is not true.

For $g \in \Gamma_\mu$ and $x \in \mathcal{S}$, we write $g \cdot x = \frac{gx}{|gx|}$ for the projective action of g on $\mathcal{S}$. With the starting point $x \in \mathcal{S}$, define a Markov chain on the projective space $\mathcal{S}$ by setting

$$X_n^x := G_n \cdot x = \frac{G_n x}{|G_n x|}, \quad n \geqslant 1.$$

Under either condition **A3** for invertible matrices, or condition **A4** (**A5** or **A6**) for positive matrices, the Markov chain $(X_n^x)_{n\geqslant 1}$ has a unique stationary measure ν on $\mathcal{S}$ such that for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$\int_{\mathcal{S}}\int_{\Gamma_\mu} \varphi(g_1 \cdot x)\mu(dg_1)\nu(dx) = \int_{\mathcal{S}} \varphi(x)\nu(dx). \tag{1.3.3}$$

For positive matrices, it will be shown in Proposition 4.3.15 that under conditions **A2** and **A4**, for any $x \in \mathcal{S}$, the asymptotic variance

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log|G_n x| - n\lambda)^2\Big]$$

exists with value in $[0, \infty)$. To establish the Berry-Esseen theorem and the moderate deviation expansion, we need the following condition:

A7. The asymptotic variance σ^2 satisfies $\sigma^2 > 0$.

We need the following non-arithmeticity condition for positive matrices:

A8. (Non-arithmeticity) For $t > 0$, $\theta \in [0, 2\pi)$ and a function $\varphi : \mathcal{S} \to \mathbb{R}$, the equation

$$\varphi(g \cdot x)|gx|^{it} = e^{i\theta}\varphi(x), \quad \forall g \in \Gamma_\mu, \forall x \in \operatorname{supp} \nu,$$

has no trivial solution except that $t = 0$, $\theta = 0$ and φ is a constant.

It can be easily checked that in the unidimensional case $d = 1$, this condition coincides with the usual non-lattice condition that $\log |g|$ is not concentrated on a set of the form $a\mathbb{Z} + b$. For positive matrices, a simple sufficient condition established in [66] for the measure μ to be non-arithmetic is that the additive subgroup of $\mathbb{R}$ generated by the set $\{\log \lambda_g : g \in \Gamma_\mu,\ g \in \mathscr{G}_+^\circ\}$ is dense in $\mathbb{R}$ (see [17]).

For any $s \in (-s_0, 0) \cup I_\mu$ with small enough $s_0 > 0$, define the transfer operator P_s and the conjugate transfer operator P_s^* as follows: for any $\varphi \in \mathcal{C}(\mathcal{S})$ and $x \in \mathcal{S}$,

$$P_s\varphi(x) = \int_{\Gamma_\mu} |g_1 x|^s \varphi(g_1 \cdot x)\mu(dg_1), \ P_s^*\varphi(x) = \int_{\Gamma_\mu} |g_1^{\mathrm{T}} x|^s \varphi(g_1^{\mathrm{T}} \cdot x)\mu(dg_1), \tag{1.3.4}$$

where the matrix g_1^{T} denotes the transpose of g_1. Under suitable conditions, the operator P_s (resp. P_s^*) has a unique probability eigenmeasure ν_s (resp. ν_s^*) on $\mathcal{S}$ corresponding to the eigenvalue $\kappa(s)$:

$$P_s\nu_s = \kappa(s)\nu_s \quad \text{and} \quad P_s^*\nu_s^* = \kappa(s)\nu_s^*. \tag{1.3.5}$$

For $x \in \mathcal{S}$, let

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy).$$

Then, r_s is the unique, up to a scaling constant, strictly positive eigenfunction of P_s: $P_s r_s = \kappa(s) r_s$; similarly r_s^* is the unique, up to a scaling constant, strictly positive eigenfunction of P_s^*: $P_s^* r_s^* = \kappa(s) r_s^*$. It is easy to see that $\nu_s(r_s) = \nu_s^*(r_s^*) := \varrho_s$. The stationary measure π_s is defined by $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\varrho_s}$, for any $\varphi \in \mathcal{C}(\mathcal{S})$.

Define $\Lambda = \log \kappa : (-s_0, 0) \cup I_\mu \to \mathbb{R}$, then the function Λ is convex and analytic (it plays the same role as the log-Laplace transform of X_1 in the real i.i.d. case). Condition **A8** implies that $\sigma_s = \Lambda''(s)$ is strictly positive for any $s \in (-s_0, 0) \cup I_\mu$. For invertible matrices, condition **A3** implies **A8**, hence σ_s is also strictly positive under condition **A3** (see [52]). Denote by Λ^* the Fenchel-Legendre transform of Λ, then we have $\Lambda^*(q) = sq - \Lambda(s) > 0$ if $q = \Lambda'(s)$ for $s \in (-s_0, 0) \cup I_\mu^\circ$.

1.3.2 Precise large deviations for the norm cocycle of products of random matrices

Precise large deviations for the norm cocycle

The large deviation theory, which is an important and active research area in probability theory, allows us to describe the rate of convergence in the law of large numbers. For sums of i.i.d. real-valued random variables, the most remarkable large deviation

results in this direction are due to Bahadur-Rao [4] and Petrov [73]. These milestone results have numerous applications in various domains of probability and statistics; see, for example, Buraczewski, Collamore, Damek and Zienkiewicz [18] for a recent application to the asymptotic of the ruin time in some models of financial mathematics. Our main goal of this section is to present the analogous Bahadur-Rao type and Petrov type precise large deviation asymptotics for the norm cocycle $\log|G_n x|$. Our results are valid for both invertible matrices and positive matrices. As applications we improve previous results on large deviation principles for the operator norm $\|G_n\|$ and we obtain precise local limit theorems with large deviations.

The standard approach to establish Bahadur-Rao [4] and Petrov [73] types large deviation asymptotics for sums of i.i.d. real-valued random variables consists of making a change of measure and then proving an Edgeworth expansion under the changed measure, see also Dembo and Zeitouni [30]. In the case of products of random matrices, this approach has been recently employed by Buraczewski and Mentemeier [17], where the main result is the following:

Theorem 1.3.1 ([17]). *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A1**, **A3** for invertible matrices, or conditions **A1**, **A4**, **A8** for positive matrices. Then,*

$$\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \Big| s\sigma_s \sqrt{2\pi n}\, e^{n\Lambda^*(q)} \mathbb{E}\Big[r_s(X_n^x) \mathbb{1}_{\{\log|G_n x| \geqslant nq\}} \Big] - r_s(x) \Big| = 0. \tag{1.3.6}$$

In particular, there exist constants $c, C > 0$ such that for all $x \in \mathcal{S}$,

$$\begin{aligned} c < & \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q)} \mathbb{P}\Big(\log|G_n x| \geqslant nq \Big) \\ \leqslant & \limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q)} \mathbb{P}\Big(\log|G_n x| \geqslant nq \Big) < C. \end{aligned} \tag{1.3.7}$$

It is worth mentioning that Theorem 1.3.1 turns out to be useful in [17] to investigate the precise tail asymptotics for multidimensional affine stochastic recursion.

The appearance of the eigenfunction r_s inside the expectation in the statement (1.3.6) is somehow unpleasant, even though we know that r_s is strictly positive and uniformly bounded on the projective space $\mathcal{S}$. We would like to sharpen the inequality (1.3.7) by giving an exact limit instead of upper and lower bounds. To achieve this goal, our approach becomes different from the standard one employed in [4, 73, 30, 17], as mentioned above. Our proof is carried out by making use of the spectral gap theory developed in [50] for invertible matrices and in [16, 17] for positive matrices, by employing the saddle point method (see for instance [34]), and by using smoothing and approximation techniques; we refer to Chapter 2 for details.

The following result concerns the exact asymptotics of the upper tail large deviation probabilities for the norm cocycle.

Theorem 1.3.2. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A1**, **A3** for invertible matrices, or conditions **A1**, **A4**, **A8** for positive matrices. Then, as $n \to \infty$, uniformly in $x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|G_n x| \geqslant nq \Big) = \frac{r_s(x)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q)\right)}{s\sigma_s\sqrt{2\pi n}} (1 + o(1)). \tag{1.3.8}$$

More generally, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\mathbb{P}\Big(\log|G_n x| \geqslant n(q+l) \Big) = \frac{r_s(x)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}} (1 + o(1)). \tag{1.3.9}$$

Our first result (1.3.8) is parallel to Bahadur-Rao type [4] large deviation asymptotic in the case of sums of i.i.d. real-valued random variables, and the second one (1.3.9) is parallel to Petrov type [73] large deviation asymptotic. For invertible matrices, in the case when $s > 0$ is sufficiently small, under a stronger exponential moment condition, a non-uniform version of (1.3.8), without $\sup_{x\in\mathcal{S}}$, has been first established by Le Page [69]. For positive matrices, the asymptotic (1.3.8) is new and implies the large deviation bounds (1.3.7) established by Buraczewski and Mentemeier [17].

Petrov's proof of the extension (1.2.11) requires the analyticity of the function $s \mapsto \Lambda(s)$ over compact subsets of I_Λ°. However, for products of random matrices it is necessary that the constants appearing in the spectral gap property of the perturbed transfer operator $R_{s,it}^n$ (cf. Proposition 2.3.4) do not depend on s over compact subsets of I_μ°.

Denote $\gamma_{k,s} = \Lambda^{(k)}(s)$, $k \geqslant 1$, where $\Lambda = \log\kappa$ with the function κ defined in (1.3.5). In particular, we have $\gamma_{1,s} = q$ and $\gamma_{2,s} = \sigma_s^2$. For l in a small neighborhood of 0, the rate function $\Lambda^*(q+l)$ has the following expansion:

$$\Lambda^*(q+l) = \Lambda^*(q) + sl + \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s\Big(\frac{l}{\sigma_s}\Big),$$

where $\zeta_s(t)$ is the Cramér series (under the changed measure $\mathbb{Q}_s^x$) given by

$$\zeta_s(t) = \frac{\gamma_{3,s}}{6\gamma_{2,s}^{3/2}} + \frac{\gamma_{4,s}\gamma_{2,s} - 3\gamma_{3,s}^2}{24\gamma_{2,s}^3}t + \frac{\gamma_{5,s}\gamma_{2,s}^2 - 10\gamma_{4,s}\gamma_{3,s}\gamma_{2,s} + 15\gamma_{3,s}^3}{120\gamma_{2,s}^{9/2}}t^2 + \cdots$$

which converges for $|t|$ small enough.

Now we consider the precise large deviations for the couple $(X_n^x, \log|G_nx|)$ with target functions φ and ψ respectively on $X_n^x := G_n\cdot x$ and $\log|G_nx|$.

Theorem 1.3.3. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A1**, **A3** for invertible matrices, or conditions **A1**, **A4**, **A8** for positive matrices. Then, for any $\varphi \in \mathcal{B}_\gamma$ and any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for some $s' \in (s-\eta, s+\eta)$ with $\eta > 0$ sufficiently small, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\begin{aligned}\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_nx| - n(q+l))\Big] \\ = \frac{r_s(x)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}\Big[\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + o(1)\Big].\end{aligned} \tag{1.3.10}$$

Theorem 1.3.3 readily implies Theorem 1.3.2 by taking $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y\geqslant 0\}}$ for $y \in \mathbb{R}$. For invertible matrices, with $l = 0$, the asymptotic (1.3.10) strengthens the large deviation result stated in Theorem 3.3 of Guivarc'h [49], since we do not assume the function ψ to be compactly supported. By the way we would like to remark that in Theorem 3.3 of [49] $\kappa^n(s)$ should be replaced by $\kappa^{-n}(s)$, and $\nu_s(\varphi r_s^{-1})$ should be replaced by $\frac{\nu_s(\varphi)}{\nu_s(r_s)}$. For positive matrices, Theorem 1.3.3 is new. Since r_s is a strictly positive and Hölder continuous function on $\mathcal{S}$, taking $l = 0$, $\varphi = r_s$ and $\psi(y) = \mathbb{1}_{\{y\geqslant 0\}}$ in (1.3.10), we get Theorem 1.3.1.

As mentioned before, for sums of i.i.d. real-valued random variables, Bahadur-Rao-Petrov type upper tail large deviation asymptotics (1.2.10) and (1.2.11) imply

the lower tail large deviation asymptotics by considering $-X_1$ instead of X_1. However, the situation is more delicate for products of random matrices; Theorems 1.3.2 and 1.3.3 do not imply the similar asymptotic for lower tail large deviation probabilities $\mathbb{P}(\log|G_n x| \leqslant nq)$, where $q < \Lambda'(0)$. We state our results under the two-sided exponential moment condition **A2**.

Theorem 1.3.4. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A2**, **A4**, **A8** for positive matrices. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|G_n x| \leqslant nq\Big) = \frac{r_s(x)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)).$$

More generally, for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\mathbb{P}\Big(\log|G_n x| \leqslant n(q+l)\Big) = \frac{r_s(x)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)).$$

For invertible matrices, this result sharpens the large deviation principle established in [13]. For positive matrices, our result is new, even for the large deviation principle. It would be interesting and challenging to investigate the precise lower tail large deviation asymptotic when $s < 0$ is far from the origin.

More generally, we have the precise large deviation asymptotic for the couple $(X_n^x, \log|G_n x|)$ with target functions.

Theorem 1.3.5. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A2**, **A4**, **A8** for positive matrices. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, for any $\varphi \in \mathcal{B}_\gamma$ and any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for all $s' \in (s-\eta, s+\eta)$ with $\eta > 0$ sufficiently small, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\begin{aligned}\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] \\ = \frac{r_s(x)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}\Big[\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + o(1)\Big].\end{aligned}$$

In particular, taking $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y\leqslant 0\}}$ for $y \in \mathbb{R}$, we obtain Theorem 1.3.4.

Applications to large deviation principle for the matrix norm

Theorems 1.3.2 and 1.3.4 can be used to deduce large deviation principles for the operator norm $\|G_n\|$. Our first result concerns invertible matrices and the second one deals with positive matrices.

Theorem 1.3.6. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume conditions **A1**, **A3** for invertible matrices. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, uniformly in $|l| \leqslant l_n$,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log\|G_n\| \geqslant n(q+l)\Big) = -\Lambda^*(q).$$

With $l = 0$, Theorem 1.3.6 improves the large deviation bounds established by Benoist and Quint [10, Theorem 14.19], where the authors consider general groups, but without giving the rate function Λ^*.

Theorem 1.3.6 concerns the upper tail large deviation principle for the operator norm $\|G_n\|$. The situation for its lower tail large deviation principle is more delicate. In fact, we can deduce easily from Theorem 1.3.4 the following upper bound: there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, for any $\epsilon > 0$, we have, uniformly in $|l| \leqslant l_n$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log \|G_n\| \leqslant n(q+l)\Big) \leqslant -\Lambda^*(q).$$

However, it remains an open question to prove the following lower bound:

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log \|G_n\| \leqslant n(q+l)\Big) \geqslant -\Lambda^*(q). \tag{1.3.11}$$

We are able to give an affirmative answer to the question (1.3.11) for positive matrices, and moreover, we have the reinforced large deviation principles:

Theorem 1.3.7. *The following two assertions hold:*

(1). *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume conditions **A1**, **A4**, **A8** for positive matrices. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, there exist constants $0 < c < C < +\infty$ such that uniformly in $|l| \leqslant l_n$,*

$$\begin{aligned} c < & \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \geqslant n(q+l)\Big) \\ \leqslant & \limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \geqslant n(q+l)\Big) < C. \end{aligned}$$

(2). *Let $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, where $s_0 > 0$ is a small constant. Assume conditions **A2**, **A4**, **A8** for positive matrices. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, there exist constants $0 < c < C < +\infty$ such that uniformly in $|l| \leqslant l_n$,*

$$\begin{aligned} c < & \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \leqslant n(q+l)\Big) \\ \leqslant & \limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \leqslant n(q+l)\Big) < C. \end{aligned}$$

Theorem 1.3.7 clearly implies the upper and lower tail large deviation principles for $\log \|G_n\|$, which are also new to our knowledge. An interesting open question is to obtain the exact value of the constants in statements (1) and (2) in Theorem 1.3.7, i.e., to establish the Bahadur-Rao type and Petrov type large deviation asymptotics for the operator norm $\|G_n\|$.

Local limit theorems with large deviations

For sums of i.i.d. real-valued random variables, local limit theorems with large and moderate deviations have been studied by Gnedenko [39], Sheep [79], Stone [80], Breuillard [15], Borovkov and Borovkov [11]. For products of random matrices, local limit

theorems with moderate deviations have been recently established by Benoist and Quint [10, Theorems 17.9 and 17.10], and have important applications to study dynamics of group actions on finite volume homogeneous spaces, see [7, Proposition 4.7]. In Theorem 1.3.3, taking $\psi = \mathbb{1}_{[a_1,a_2]}$, where $a_1 < a_2$ are fixed real numbers, we get the following a local limit theorem with large deviations.

Theorem 1.3.8. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A1**, **A3** for invertible matrices, or conditions **A1**, **A4**, **A8** for positive matrices. Then, for any $\varphi \in \mathcal{B}_\gamma$, for any real numbers $-\infty < a_1 < a_2 < \infty$, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|\in n(q+l)+[a_1,a_2)\}}\Big] = \frac{r_s(x)}{\varrho_s}\Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}\Big[\nu_s(\varphi) + \|\varphi\|_\gamma o(1)\Big].$$

In particular, with $\varphi = \mathbf{1}$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\mathbb{P}\Big(\log|G_nx| \in n(q+l) + [a_1,a_2)\Big) = \frac{r_s(x)}{\varrho_s}\Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big].$$

Theorem 1.3.8 can be compared with Theorem 3.3 in [49], where the above equivalences are established for $l = 0$.

From Theorem 1.3.5, we can obtain that the assertions of Theorem 1.3.8 remain valid for $s < 0$ small enough under the additional assumption **A2**.

1.3.3 Precise large deviations for entries of products of random matrices

The goal of this section is to investigate exact large deviation asymptotics for the entries $G_n^{i,j}$, and more generally for the scalar product $\langle f, G_nx\rangle$, where $f, x \in \mathcal{S}$. To the best of our knowledge, the precise large deviations for the entries $G_n^{i,j}$ and, in particular, even the large deviation principles for the entries $G_n^{i,j}$, have not been studied by now in the literature.

*Throughout this section, for invertible matrices, all the statements are valid only for 2×2 matrices; for positive matrices, all the statements are valid for $d\times d$ ($d \geqslant 2$) matrices under condition **A5**, and only for 2×2 matrices otherwise.*

As mentioned before, the standard approach to establish precise large deviation asymptotics for sums of i.i.d. real-valued random variables consists in performing a change of measure and proving an Edgeworth expansion under the changed measure (see [4, 73, 30]). Applying this strategy to the entries $G_n^{i,j}$ of products of random matrices turns out to be way more difficult. We have to overcome three main difficulties: prove an Edgeworth expansion for the couple $(X_n^{e_j}, \log|G_n^{i,j}|)$ with a target function φ on the Markov chain $X_n^{e_j}$ under the changed measure; give a precise control of the difference between $\log|G_n^{i,j}|$ and $\log|G_ne_j|$; establish the Hölder regularity of the stationary measure π_s.

For the first point, it turn out that the techniques which work for $\log|G_n^{i,j}|$ alone cannot be applied for the couple. Dealing with the couple $(X_n^{e_j}, \log|G_n^{i,j}|)$ with a target function φ on X_n^x needs considering a new kind of smoothing inequality on a complex contour, instead of the usual Esseen one on the real line. We make use of the saddle point method to obtain precise asymptotics for the integrals of the corresponding Laplace transforms on the complex plane. For this method we refer to Chapter 4 where an Edgeworth expansion with a target function on X_n^x for the norm cocycle $\log|G_n x|$ has been established.

Secondly, from the previous works (see e.g. [51, 13, 53, 10]) on limit theorems such as the law of large numbers, the central limit theorem and the law of iterated logarithm for $G_n^{i,j}$, we know that the difference $|\log|G_n^{i,j}| - \log|G_n e_j||$ generally diverges to infinity as $n \to \infty$. It is controlled by the corresponding norming factors in the above-mentioned limit theorems. However, such a control is not enough to obtain precise large deviation expansions for $G_n^{i,j}$, nor even for a large deviation principle with explicit rate function. A precise account of the contribution of the error term is given by the following decomposition:

$$\log|G_n^{i,j}| = \log|G_n x| + \log|f(X_n^x)|, \quad n \geqslant 1, \tag{1.3.12}$$

where $x = e_j$, $f = e_i$, and $f(X_n^x)$ is seen as a linear functional f acting on the Markov chain X_n^x. The main difficulty here is that the Markov chain X_n^x may stay in or very close to the hyperplane $\ker f := \{x \in \mathcal{S} : \langle f, x\rangle = 0\}$, which makes $\log|f(X_n^x)|$ meaningless. Our idea to circumvent this difficulty is to discretize the values of the function $x \mapsto \log|\langle f, x\rangle|$. Consequently, this discretization together with the exact decomposition (1.3.12), allows to deduce the precise large deviation asymptotic from the results for the couple $(X_n^x, \log|G_n x|)$ with a target function on X_n^x established in [85]. The details will be presented in Chapter 3.

The third important difficulty is to prove the Hölder regularity property of the stationary measure π_s: there exist constants $c, C > 0$ such that for any $0 < t < 1$,

$$\sup_{f \in \mathcal{S}} \pi_s\Big(\{x \in \mathcal{S} : |\langle f, x\rangle| \leqslant t\}\Big) \leqslant C t^c. \tag{1.3.13}$$

This is one of the central points of the proof and is of independent interest. The inequality (1.3.13) for $s = 0$ has been proved in [51] and further studied in [13]. With $s = 0$ it was used to establish limit theorems for the entries $G_n^{i,j}$, see [51, 13, 9, 10]. For other applications see [7, 14]. Overall, to prove (1.3.13) when $s > 0$ for invertible matrices, we adapt the arguments from [51, 13] where (1.3.13) was established for $s = 0$. For $s > 0$ the arguments are much more delicate. One of the difficulties is that the sequence $(g_n)_{n \geqslant 1}$ becomes dependent under the changed measure $\mathbb{Q}_s^x$. We need to extend the results in [13] to this case. Of crucial importance are the simplicity of the dominant Lyapunov exponent for G_n under the changed measure recently established in [50], and the key proximality property which states that $M_n \cdot m$ (with $M_n = g_1 \dots g_n$) converges weakly to the Dirac measure δ_{Z_s}, where the law of the random variable Z_s is the stationary measure π_s and m is the unique rotation invariant measure on the projective space $\mathcal{S}$.

We also establish an analog of (1.3.13) for positive matrices, but under either assumption **A5** of Furstenberg-Kesten type, or assumption **A6** on the harmonic moments of the entries. The techniques of the proofs are quite different from those used in the

case of invertible matrices. Under condition **A5**, they rely on the fact that the Markov chain X_n^x is separated from the coordinates e_i and the support of the stationary measure π_s of X_n^x coincides with the support of the stationary measure $\pi_0 = \nu$. Under condition **A6**, the proofs are based on the large deviation bounds for the norm cocycle $\log|G_n x|$ under the changed measure, see Theorem 3.4.4.

We now present the Bahadur-Rao type upper tail large deviation asymptotic for the scalar product $\langle f, G_n x\rangle$ for both invertible matrices and positive matrices.

Theorem 1.3.9. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A1**, **A5**, **A8** (or conditions **A1**, **A6**, **A8**) for positive matrices. Then, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant nq\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q)\right)}{s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big]. \tag{1.3.14}$$

The result (1.3.14) particularly implies the large deviation asymptotic for the entries $G_n^{i,j}$ by taking $f = e_i$ and $x = e_j$. It is easy to see that (1.3.14) implies a large deviation principle for the scalar product $\langle f, G_n x\rangle$: under the assumptions of Theorem 1.3.9, we have, uniformly in $f, x \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant nq\Big) = -\Lambda^*(q). \tag{1.3.15}$$

The asymptotic (1.3.15) clearly improves the following large deviation bound established by Benoist and Quint [10] for invertible matrices: there exists a constant $c > 0$ such that uniformly in $f, x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}(\log|\langle f, G_n x\rangle| > nq) \leqslant e^{-cn}.$$

An important field of applications of large deviation asymptotics for the entries of type (1.3.14) is the study of the asymptotic behavior of the branching processes in random environment with several types of particles. For results in the case of single type branching processes we refer to [46, 47] and for the relation between the entries of products of random matrices and the multi-type branching processes we refer to [21].

It is worth some comments on the moment assumptions for Theorem 1.3.9. For positive matrices, if we assume the Furstenberg-Kesten type condition **A5**, the assertion of Theorem 1.3.9 holds without assuming the moment condition **A2**. However, it is not clear whether condition **A2** is necessary for invertible matrices. This question is open, the main difficulty being to establish the Hölder regularity of the stationary measure π_s for invertible matrices without assuming condition **A2** (see Proposition 3.3.4). In the same line, we note that a Bahadur-Rao-Petrov type large deviation result for the norm cocycle $\log|G_n x|$ has been recently shown in Theorem 1.3.2 for invertible matrices under conditions **A1** and **A3**.

Our next result is an improvement of Theorem 1.3.9 by allowing a vanishing perturbation l on $q = \Lambda'(s)$, in the spirit of the Petrov result [73], called the Bahadur-Rao-Petrov type large deviation asymptotics.

Theorem 1.3.10. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A1**, **A5**, **A8** (or conditions **A1**, **A6**, **A8**) for*

positive matrices. Then, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant n(q+l)\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big], \tag{1.3.16}$$

and, more generally, for any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for some $s' \in (0, s)$, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\gamma$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} &\mathbb{E}\Big[\varphi(X_n^x)\psi\Big(\log|\langle f, G_n x\rangle| - n(q+l)\Big)\Big] \qquad (1.3.17)\\ =&\frac{r_s(x)}{\varrho_s}\frac{\exp(-n\Lambda^*(q+l))}{\sigma_s\sqrt{2\pi n}}\left[\int_{\mathcal{S}}\varphi(u)|\langle f, u\rangle|^s \nu_s(du)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + \|\varphi\|_\gamma o(1)\right]. \end{aligned}$$

Now we are going to give asymptotics of the lower tail large deviation probabilities $\mathbb{P}(\log|\langle f, G_n x\rangle| \leqslant nq)$, where $q = \Lambda'(s) < \lambda = \Lambda'(0)$ for $s < 0$. These results cannot be deduced from Theorems 1.3.9 and 1.3.10; moreover the proofs are quite different and require to develop the corresponding spectral gap theory for the transfer operator P_s. In addition we need the Hölder regularity for the eigenmeasure ν_s for $s < 0$ sufficiently close to 0, which is of independent interest; this is established using a different approach compared to the case $s > 0$.

Theorem 1.3.11. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A6**, **A8** for positive matrices. Then, there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \leqslant nq\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big]. \tag{1.3.18}$$

In particular, with $f = e_i$ and $x = e_j$ in (1.3.18), we obtain the Bahadur-Rao type lower tail large deviation asymptotic for the entries $G_n^{i,j}$. From (1.3.18) one gets a lower tail large deviation principle for the scalar product $\langle f, G_n x\rangle$: under the assumptions of Theorem 1.3.11, we have, uniformly in $f, x \in \mathcal{S}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \leqslant nq\Big) = -\Lambda^*(q). \tag{1.3.19}$$

The result (1.3.19) sharpens the following lower tail large deviation bound established by Benoist and Quint [10, Theorem 14.21] for invertible matrices: there exists a constant $c > 0$ such that uniformly in $f, x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}(\log|\langle f, G_n x\rangle| \leqslant nq) \leqslant e^{-cn}.$$

Now we give a Bahadur-Rao-Petrov version of the above theorem.

Theorem 1.3.12. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A6**, **A8** for positive matrices. Then, there exists $s_0 > 0$ such that, for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \leqslant n(q+l)\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big], \tag{1.3.20}$$

and, more generally, for any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for some $s' \in (-s_0, s)$, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\gamma$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\mathbb{E}\Big[\varphi(X_n^x)\psi\Big(\log|\langle f, G_n x\rangle| - n(q+l)\Big)\Big] \tag{1.3.21}$$
$$= \frac{r_s(x)}{\varrho_s}\frac{\exp(-n\Lambda^*(q+l))}{\sigma_s\sqrt{2\pi n}}\left[\int_{\mathcal{S}}\varphi(u)|\langle f, u\rangle|^s\nu_s(du)\int_{\mathbb{R}}e^{-sy}\psi(y)dy + \|\varphi\|_\gamma o(1)\right].$$

Consider the reversed random walk M_n defined by $M_n = g_1 \dots g_n$. Since the two probabilities $\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant n(q+l)\Big)$ and $\mathbb{P}\Big(\log|\langle f, M_n x\rangle| \geqslant n(q+l)\Big)$ are equal (as G_n and M_n have the same law), for M_n we have the same large deviation expansions as for G_n. It is interesting to note that, this fact and the symmetry in the definition of the eigenfunctions r_s and r_s^*, imply that in condition **A5** one can replace the bound (1.3.2) on columns of g by a similar one on rows, namely, by the bound: there exists a constant $C > 1$ such that for any $g \in \operatorname{supp}\mu$, and $1 \leqslant i \leqslant d$, we have

$$1 \leqslant \frac{\max_{1\leqslant j\leqslant d} g^{i,j}}{\min_{1\leqslant j\leqslant d} g^{i,j}} \leqslant C. \tag{1.3.22}$$

Local limit theorems with large deviations

From Theorem 1.3.10, we easily get the following local limit theorem with large deviations for scalar products $\langle f, G_n x\rangle$.

Theorem 1.3.13. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A1**, **A5**, **A8** (or conditions **A1**, **A6**, **A8**) for positive matrices. Then, for any real numbers $-\infty < a_1 < a_2 < \infty$, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$.*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \in n(q+l) + [a_1, a_2)\Big) \tag{1.3.23}$$
$$= \Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{r_s(x)r_s^*(f)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big],$$

More generally, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\gamma$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| \in n(q+l)+[a_1,a_2)\}}\Big] \tag{1.3.24}$$
$$= \Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{r_s(x)}{\varrho_s}\frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}\left[\int_{\mathcal{S}}\varphi(u)|\langle f, u\rangle|^s\nu_s(du) + \|\varphi\|_\gamma o(1)\right].$$

The following local limit theorem with large deviations for $s < 0$ is deduced from Theorem 1.3.12.

Theorem 1.3.14. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A6**, **A8** for positive matrices. Then, for any real numbers $-\infty < a_1 < a_2 < \infty$, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \in n(q+l) + [a_1, a_2)\Big)$$
$$= \Big(e^{-sa_2} - e^{-sa_1}\Big)\frac{r_s(x)r_s^*(f)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big].$$

More generally, we have, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\gamma$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}&\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| \in n(q+l)+[a_1,a_2)\}}\Big]\\ &= \left(e^{-sa_2} - e^{-sa_1}\right)\frac{r_s(x)}{\varrho_s}\frac{\exp(-n\Lambda^*(q+l))}{-s\sigma_s\sqrt{2\pi n}}\left[\int_{\mathcal{S}}\varphi(u)|\langle f,u\rangle|^s\nu_s(du) + \|\varphi\|_\gamma o(1)\right].\end{aligned}$$

Large deviation principle for the spectral radius of positive matrices

We are able to derive from Theorems 1.3.10 and 1.3.12 the reinforced large deviation principles for the spectral radius of products of positive random matrices. According to the Perron-Frobenius theory, the spectral radius $\rho(g)$ of a positive matrix g actually coincides with its largest eigenvalue.

Theorem 1.3.15.
(1) *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume conditions **A1**, **A5**, **A8** (or conditions **A1**, **A6**, **A8**) for positive matrices. Then, there exist constants $0 < c < C < +\infty$ such that uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,*

$$\begin{aligned}c <& \liminf_{n\to\infty}\sqrt{n}\,e^{n\Lambda^*(q+l)}\mathbb{P}\Big(\log\rho(G_n) \geqslant n(q+l)\Big)\\ \leqslant& \limsup_{n\to\infty}\sqrt{n}\,e^{n\Lambda^*(q+l)}\mathbb{P}\Big(\log\rho(G_n) \geqslant n(q+l)\Big) < C.\end{aligned} \tag{1.3.25}$$

(2) *Assume conditions **A6**, **A8** for positive matrices. Then, there exist constants $s_0 > 0$ and $0 < c < C < +\infty$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,*

$$\begin{aligned}c <& \liminf_{n\to\infty}\sqrt{n}\,e^{n\Lambda^*(q+l)}\mathbb{P}\Big(\log\rho(G_n) \leqslant n(q+l)\Big)\\ \leqslant& \limsup_{n\to\infty}\sqrt{n}\,e^{n\Lambda^*(q+l)}\mathbb{P}\Big(\log\rho(G_n) \leqslant n(q+l)\Big) < C.\end{aligned}$$

A more general version of Theorem 1.3.15 with a target function ψ on $\log\rho(G_n)$ will be presented in Section 3.8: see Theorem 3.8.1. The statements (1) and (2) in Theorem 1.3.15 clearly imply the following large deviation principle for $\log\rho(G_n)$: under the assumptions of Theorem 1.3.15, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log\rho(G_n) \geqslant n(q+l)\Big) = -\Lambda^*(q); \tag{1.3.26}$$

a similar assertion also holds for the lower tail. Note also that statements (1) and (2) of Theorem 1.3.15 still hold when the product G_n is replaced by $M_n := g_1 \dots g_n$.

The upper bound of part (1) in Theorem 1.3.15 follows from the reinforced large deviation principle for the matrix norm $\|G_n\|$ recently established in [85]. The lower bound is deduced from Theorem 1.3.10 in conjunction with the Collatz-Wielandt formula for positive matrices. Note that, the Collatz-Wielandt formula does not hold in general for invertible matrices, hence the question of proving Theorem 1.3.15 for invertible matrices remains open, even for the large deviation principle for $\rho(G_n)$; the latter has been recently conjectured by Sert [78]. The corresponding upper bound in large deviation principle for invertible matrices can be easily deduced from the results in [85]: under conditions **A2**, **A3**, for any $s \in I_\mu^\circ$ and $q = \Lambda'(s)$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log\rho(G_n) \geqslant n(q+l)\Big) \leqslant -\Lambda^*(q).$$

1.3.4 Berry-Esseen bound and precise moderate deviations for the norm cocycle of products of random matrices

The goal of this section is to present the Berry-Esseen bound, the Edgeworth expansion and the Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on the Markov chain X_n^x. It is surprising that our proof of the Berry-Esseen bound and of the Edgeworth expansion with a non-trivial target function $\varphi \neq \mathbf{1}$ is way more difficult than the analogous results with $\varphi = \mathbf{1}$. On one hand, instead of using Esseen's smoothing inequality, we have to develop a new smoothing inequality on the complex plane $\mathbb{C}$; see Proposition 4.4.1. An important issue is to construct a Schwartz density function ρ such that its Fourier transform $\widehat{\rho}$ has a compact support on the real line $\mathbb{R}$ and can be extended analytically on a neighborhood of 0. On the other hand, we have to rework on the spectral gap theory for the transfer operators P_z and $R_{s,z}$, by considering the case when s can take values in the interval $(-\eta, \eta)$ with $\eta > 0$ small, and z belongs to a small complex ball centered at the origin, which allows to define the change of measure $\mathbb{Q}_s^x$ for $s < 0$. The new smoothing inequality, the extended spectral gap theory, and the saddle point method enable us to establish the Berry-Esseen bound for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on X_n^x. Note that the non-arithmeticity condition is not needed for the validity for the Berry-Esseen bound. Under the non-arithmeticity condition, we prove an Edgeworth expansion for the couple $(X_n^x, |G_n x|)$ with a target function φ on X_n^x. For the Cramér type moderate deviation expansions, our proof is different from those in [10] and [23]: in [10] the moderate deviation principle for $\log|G_n x|$ is obtained by following the strategy of Kolmogorov [68] suited to show the law of iterated logarithm (see also de Acosta [29] and Wittman [84]); in [23] the proof of the functional moderate deviation principle is based on the martingale approximation method developed in [9]. In our proof, the moderate deviation expansion is established by using the Berry-Esseen bound for the the couple $(X_n^x, \log|G_n x|)$ under the changed measure and by adapting the techniques from Petrov [74].

It is worth mentioning that the approach developed in this section opens a way to investigate the Berry-Esseen theorem, the Egdeworth expansion and the Cramér type moderate deviation expansion with target functions for general Markov chains.

The following result concerns the Berry-Esseen bound for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on X_n^x.

Theorem 1.3.16. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A2**, **A4**, **A7** for positive matrices. Then, there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left|\mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\right] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{1.3.27}$$

Theorem 1.3.16 extends the Berry-Esseen bounds from [69, 62] for invertible matrices, and [55] for positive matrices to versions with target functions on X_n^x. Note that the results in [62, 55] have been established under some polynomial moment conditions. However, proving (1.3.27) with the target function $\varphi \neq \mathbf{1}$ under the polynomial moments is still an open problem.

The following result gives an Edgeworth expansion for $\log|G_n x|$ with the target function φ on X_n^x. To formulate the result, we introduce the necessary notation. It is

shown in Lemma 4.3.11 that for any $\varphi \in \mathcal{B}_\gamma$, the function

$$b_\varphi(x) = \lim_{n\to\infty} \mathbb{E}\Big[(\log|G_n x| - n\lambda)\varphi(X_n^x)\Big], \quad x \in \mathcal{S} \tag{1.3.28}$$

is well defined, belongs to $\mathcal{B}_\gamma$ and has an equivalent expression (4.3.39) in terms of derivative of the projection operator $\Pi_{0,z}$, see Proposition 4.3.8.

Theorem 1.3.17. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A2**, **A4**, **A8** for positive matrices. Then, as $n \to \infty$, uniformly in $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\begin{aligned}\Big|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] \\ - \nu(\varphi)\Big[\Phi(y) + \frac{\Lambda'''(0)}{6\sigma^3\sqrt{n}}(1-y^2)\phi(y)\Big] + \frac{b_\varphi(x)}{\sigma\sqrt{n}}\phi(y)\Big| = \|\varphi\|_\gamma o\Big(\frac{1}{\sqrt{n}}\Big).\end{aligned} \tag{1.3.29}$$

The proof of this theorem is based on a new smoothing inequality on the complex plane $\mathbb{C}$ (see Proposition 4.4.1) and the saddle point method. Theorem 1.3.17 is new even for $\varphi = \mathbf{1}$.

Moderate deviation expansions

Denote $\gamma_k = \Lambda^{(k)}(0)$, $k \geqslant 1$, where $\Lambda = \log\kappa$ with the function κ defined in (1.3.5). In particular, we have $\gamma_1 = \lambda$ and $\gamma_2 = \sigma^2$. We write ζ for the Cramér series of Λ:

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \cdots \tag{1.3.30}$$

which converges for $|t|$ small enough.

We start by formulating a Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on X_n^x, for both invertible matrices and positive matrices.

Theorem 1.3.18. *Assume either conditions **A2**, **A3** for invertible matrices, or conditions **A2**, **A4**, **A7** for positive matrices. Then, uniformly in $x \in \mathcal{S}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \to \infty$,*

$$\begin{aligned}\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} &= e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \\ \frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} &= e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big].\end{aligned}$$

In particular, under conditions of Theorem 1.3.18, with $\varphi = \mathbf{1}$ we obtain:

$$\begin{aligned}\frac{\mathbb{P}\left(\frac{\log|G_n x|-n\lambda}{\sigma\sqrt{n}} \geqslant y\right)}{1-\Phi(y)} &= e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[1 + O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \\ \frac{\mathbb{P}\left(\frac{\log|G_n x|-n\lambda}{\sigma\sqrt{n}} \leqslant -y\right)}{\Phi(-y)} &= e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[1 + O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big].\end{aligned}$$

When $\varphi \in \mathcal{B}_\gamma$ is a real-valued function satisfying $\nu(\varphi) > 0$, Theorem 1.3.18 clearly implies the following moderate deviation principle for $\log|G_n x|$ with target function on X_n^x: for any Borel set $B \subseteq \mathbb{R}$, and positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, uniformly in $x \in \mathcal{S}$,

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log|G_n x|-n\lambda}{b_n}\in B\}}\Big] \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log|G_n x|-n\lambda}{b_n}\in B\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned} \tag{1.3.31}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B. With $\varphi = \mathbf{1}$, our result (1.3.31) implies the moderate deviation principle established in [10, Proposition 12.12] for invertible matrices. The moderate deviation principle (1.3.31) with a target function φ on X_n^x is new for both invertible matrices and positive matrices. Note that in (1.3.31) the function φ is not necessarily positive.

The asymptotic expansions in Theorem 1.3.18 remain valid even when $\nu(\varphi) = 0$. In this case, for example, the first expansion becomes

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big] = (1-\Phi(y))e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big).$$

It is an open question to extend the results of Theorem 1.3.18 to higher order expansions under the additional condition of non-arithmeticity. We refer to Saulis [77] and Rozovsky [76] for relevant results in the i.i.d. real-valued case. In the case of products of random matrices this problem seems to us challenging because of the presence of the derivatives in s of the eigenfunction r_s and of the eigenmeasure ν_s in the higher order terms.

1.3.5 Berry-Esseen bounds and moderate deviations for the norm, entries and spectral radius of products of positive random matrices

In the previous Section 1.3.4 we presented the Berry-Esseen bound and moderate deviation expansions for the norm cocycle $\log|G_n x|$ jointly with the Markov chain X_n^x. However, this type of results for other quantities like the matrix norm, the entries and the spectral radius of G_n are notably absent in the literature. The goal of this section is to fill the gap and to extend the results of Section 1.3.4 to the matrix norm $\|G_n\|$, to the entries $G_n^{i,j}$ and to the spectral radius $\rho(G_n)$ for the product G_n of positive random matrices, jointly with X_n^x.

Berry-Esseen bounds for the norm, entries and spectral radius of products of positive random matrices

In this section, we formulate the Berry-Esseen bounds for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$. Let us first state the result for the operator norm $\|G_n\|$. We denote $(\mathbb{S}_+^{d-1})^\circ = \{x > 0 : |x| = 1\}$, which is the interior of the projective space $\mathbb{S}_+^{d-1}$.

Theorem 1.3.19. *Assume conditions* ***A2***, ***A4*** *and* ***A7****. Then, for any compact set* $K \subset (\mathbb{S}_+^{d-1})^\circ$*, we have, uniformly in* $\varphi \in \mathcal{B}_\gamma$*,*

$$\sup_{y\in\mathbb{R}} \sup_{x\in K} \left| \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{1.3.32}$$

It can be easily checked that in Theorem 1.3.19, the operator norm $\|\cdot\|$ can be replaced by any matrix norm since all matrix norms are equivalent. It would be interesting to show that (1.3.32) holds uniformly in $x \in \mathbb{S}_+^{d-1}$ instead of $x \in K$. Note that Theorem 1.3.19 is proved under the exponential moment condition **A2**. It is not clear how to establish Theorem 1.3.19 under the polynomial moment condition on the matrix law μ.

If the stronger condition **A5** holds instead of condition **A4**, then we are able to prove the following Berry-Esseen bounds for the scalar product $\langle f, G_n x\rangle$ and for the spectral radius $\rho(G_n)$.

Theorem 1.3.20. *Under conditions* ***A2***, ***A5*** *and* ***A7****, we have:*

(1) *uniformly in* $\varphi \in \mathcal{B}_\gamma$*,*

$$\sup_{y\in\mathbb{R}} \sup_{f,x\in\mathbb{S}_+^{d-1}} \left| \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\langle f,G_n x\rangle-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma; \tag{1.3.33}$$

(2) *for any compact set* $K \subset (\mathbb{S}_+^{d-1})^\circ$*, uniformly in* $\varphi \in \mathcal{B}_\gamma$*,*

$$\sup_{y\in\mathbb{R}} \sup_{x\in K} \left| \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{1.3.34}$$

In particular, taking $\varphi = \mathbf{1}$, $f = e_i$ and $x = e_j$ in (1.3.33), we get the Berry-Esseen bound for the entries $G_n^{i,j}$. The Berry-Esseen bounds (1.3.33) and (1.3.34) are all new. It would be interesting to establish these bounds under some condition weaker than **A5**.

Precise moderate deviation expansions

The goal of this section is to formulate the Cramér type moderate deviation expansions for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$.

The following result concerns the Cramér type moderate deviations for the operator norm $\|G_n\|$. Recall that $(\mathbb{S}_+^{d-1})^\circ = \{x > 0 : |x| = 1\}$ and the Cramér series ζ is defined by (1.3.30).

Theorem 1.3.21. *Assume conditions* ***A2***, ***A4*** *and* ***A7****. Then, for any compact set* $K \subset (\mathbb{S}_+^{d-1})^\circ$*, we have, uniformly in* $x \in K$*,* $y \in [0, o(\sqrt{n})]$ *and* $\varphi \in \mathcal{B}_\gamma$*,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{1.3.35}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]. \tag{1.3.36}$$

Like in Theorem 1.3.19, it can also be checked that in Theorem 1.3.21 the operator norm $\|\cdot\|$ can be replaced by any matrix norm. Note that condition **A5** is not required in Theorem 1.3.21. Theorem 1.3.21 is new even for $\varphi = \mathbf{1}$ and the expansions (1.3.35) and (1.3.36) remain valid even when $\nu(\varphi) = 0$. As a particular case, Theorem 1.3.21 implies the following moderate deviation principle for $\log \|G_n\|$ with a target function φ on the Markov chain X_n^x.

Corollary 1.3.22. *Assume conditions **A2**, **A4** and **A7**. Then, for any real-valued function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, we have, uniformly in $x \in K$,*

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\}}\Big] \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned} \tag{1.3.37}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

The target function φ in (1.3.37) is not necessary positive and it can vanish on some part of the projective space $\mathbb{S}_+^{d-1}$. The moderate deviation principle (1.3.37) is new, even for $\varphi = \mathbf{1}$. As in Theorem 1.3.19, it would be interesting to prove that Theorem 1.3.21 holds uniformly in $x \in \mathbb{S}_+^{d-1}$ instead of $x \in K$.

Now we formulate Cramér type moderate deviation expansions for the scalar product $\langle f, G_n x\rangle$ as well as for the spectral radius $\rho(G_n)$.

Theorem 1.3.23. *Assume conditions **A2**, **A5** and **A7**. Then, we have:*
(1) uniformly in $f, x \in \mathbb{S}_+^{d-1}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\langle f,G_n x\rangle - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{1.3.38}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\langle f,G_n x\rangle - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]; \tag{1.3.39}$$

(2) for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, uniformly in $x \in K$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n) - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{1.3.40}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n) - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]. \tag{1.3.41}$$

As a particular case of (1.3.38) and (1.3.39) with $f = e_i$ and $x = e_j$, we get the Cramér type moderate deviation expansions for the entries $G_n^{i,j}$. The expansions (1.3.38)-(1.3.41) are all new even for $\varphi = \mathbf{1}$.

From Theorem 1.3.23 we can get the moderate deviation principles with target functions for the scalar product $\langle f, G_n x\rangle$ and the spectral radius $\rho(G_n)$, just as we obtained (1.3.37) from Theorem 1.3.21.

Corollary 1.3.24. *Assume conditions* ***A2***, ***A5*** *and* ***A7***. *Then, for any real-valued function* $\varphi \in \mathcal{B}_\gamma$ *satisfying* $\nu(\varphi) > 0$, *for any Borel set* $B \subseteq \mathbb{R}$ *and any positive sequence* $(b_n)_{n\geqslant 1}$ *satisfying* $\frac{b_n}{n} \to 0$ *and* $\frac{b_n}{\sqrt{n}} \to \infty$, *we have*

(1) *uniformly in* $f, x \in \mathbb{S}_+^{d-1}$,

$$-\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} \leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\langle f, G_n x\rangle - n\lambda}{b_n} \in B\right\}}\Big]$$
$$\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\langle f, G_n x\rangle - n\lambda}{b_n} \in B\right\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2};$$

(2) *for any compact set* $K \subset (\mathbb{S}_+^{d-1})^\circ$, *uniformly in* $x \in K$,

$$-\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} \leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log \rho(G_n) - n\lambda}{b_n} \in B\right\}}\Big]$$
$$\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log \rho(G_n) - n\lambda}{b_n} \in B\right\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}.$$

1.3.6 Berry-Esseen bounds and moderate deviation principles for the random walk on $GL_d(\mathbb{R})$

The goal of this section is to present the Berry-Esseen type bounds and moderate deviation principles for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$ of products of random matrices in $GL_d(\mathbb{R})$.

Berry-Esseen type bounds for $\|G_n\|$ and $\rho(G_n)$ of invertible matrices

We formulate the Berry-Esseen type bounds for the couples $(X_n^x, \log\|G_n\|)$ and $(X_n^x, \log\rho(G_n))$ with a target function φ on the Markov chain X_n^x.

Theorem 1.3.25. *Assume conditions* ***A2*** *and* ***A3*** *for invertible matrices. Then, we have, uniformly in* $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ *and* $\varphi \in \mathcal{B}_\gamma$,

$$\Big|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\Big| \leqslant \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma; \tag{1.3.42}$$

and

$$\Big|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\Big| \leqslant \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma. \tag{1.3.43}$$

Since all matrix norms are equivalent, the operator norm $\|\cdot\|$ in (1.3.42) can be replaced by any matrix norm. The proof of (1.3.42) is based on the Berry-Esseen bound for the norm cocycle $\log|G_n x|$ established in Theorem 1.3.16 and on the comparison between $\|G_n\|$ and $|G_n x|$ established in [10]: see Lemma 6.3.2. The proof of (1.3.43) relies on (1.3.42) and the comparison between $\rho(G_n)$ and $\|G_n\|$ established in [10]: see Lemma 6.3.3.

One can make conjectures that the optimal rates of convergence on the right hand sides of (1.3.42) and (1.3.43) should be of order $O(\frac{1}{\sqrt{n}})$ instead of $O(\frac{\log n}{\sqrt{n}})$. However, the proofs seem to be delicate and new ideas and techniques are required.

Moderate deviation principles for $\|G_n\|$ and $\rho(G_n)$ of invertible matrices

We first state moderate deviation principles for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$.

Theorem 1.3.26. *Assume conditions **A2** and **A3** for invertible matrices. Then, for any non-negative function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have, uniformly in $x \in \mathbb{P}^{d-1}$,*

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}; \end{aligned} \tag{1.3.44}$$

and

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}; \end{aligned} \tag{1.3.45}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

Note that the target function φ in (1.3.44) is not necessary strictly positive and it can vanish on some part of the projective space $\mathbb{P}^{d-1}$. The moderate deviation principles (1.3.44) and (1.3.45) are all new, even for $\varphi = \mathbf{1}$.

When the proximality condition **A3** (ii) fails, we are still able to establish moderate deviation principles for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$:

Theorem 1.3.27. *Assume conditions **A2** and **A3** (i) for invertible matrices. Assume also that $\{|det(g)|^{-1/d} g : g \in \Gamma_\mu\}$ is not contained in a compact subgroup of $\mathbb{G}$. Then, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have,*

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right) \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right) \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}; \end{aligned} \tag{1.3.46}$$

and

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right) \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right) \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}; \end{aligned} \tag{1.3.47}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

The proof of (1.3.46) is based on the moderate deviation principle (1.3.26), Chevalley's result (Lemma 6.4.2) on the irreducible representation of the subgroup of $GL(d,\mathbb{R})$, and techniques from Bougerol and Lacroix [13]. Using (1.3.46), the proof of (1.3.47) is carried out by a result established by Benoist and Quint [10] (see Lemma 6.3.3) on the comparison between $\rho(G_n)$ and $\|G_n\|$.

Moderate deviation expansions for $\|G_n\|$ and $\rho(G_n)$ of invertible matrices

We formulate the moderate deviation expansions for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. The following result concerns the operator norm $\|G_n\|$.

Theorem 1.3.28. *Assume conditions **A2** and **A3** for invertible matrices. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1),$$
$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1).$$

Theorem 1.3.28 is new even when $\varphi = \mathbf{1}$. Its proof is based on the Cramér type moderate deviation expansions for the couple $(X_n^x, \log|G_n x|)$ shown in Theorem 1.3.18, and on a delicate comparison between the operator norm $\|G_n\|$ and the vector norm $|G_n x|$ established in [10] (see Lemma 6.3.2). Note that Theorem 1.3.28 remains valid even in the case when $\nu(\varphi) = 0$.

The following result concerns the moderate deviation expansions for spectral radius $\rho(G_n)$.

Theorem 1.3.29. *Assume conditions **A2** and **A3** for invertible matrices. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n)-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1),$$
$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n)-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1).$$

Theorem 1.3.29 is new even when $\varphi = \mathbf{1}$. Its proof relies on Theorem 6.2.6 and on an estimate of the difference between spectral radius $\rho(G_n)$ and the operator norm $\|G_n\|$ established in [10] (see Lemma 6.3.3). Note that Theorem 1.3.29 remains valid even in the case when $\nu(\varphi) = 0$.

1.3.7 Moderate deviation expansions for the entries of the random walk on $SL_2(\mathbb{R})$

The purpose of this section is to formulate the Cramér type moderate deviation expansion for the entries $G_n^{i,j}$ of products of random matrices G_n in the special linear group $SL_2(\mathbb{R})$.

Theorem 1.3.30. *Assume condition **A2** and that Γ_μ is Zariski dense in $SL_2(\mathbb{R})$. Then, we have, uniformly in $f, x \in \mathbb{P}^1$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_n x\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma o(1)\Big],$$
$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_n x\rangle|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma o(1)\Big].$$

Theorem 1.3.30 clearly implies the following moderate deviation principle for the couple $(X_n^x, \log|\langle f, G_n x\rangle|)$ with a target function φ on the Markov chain X_n^x. Let $(b_n)_{n\geqslant 1}$ be a positive sequence satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$.

Corollary 1.3.31. *Assume condition **A2** and that Γ_μ is Zariski dense in $SL_2(\mathbb{R})$. Then, for any Borel set $B \subseteq \mathbb{R}$ and for any real-valued function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, we have, uniformly in $f, x \in \mathbb{P}^1$,*

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|\langle f, G_n x\rangle| - n\lambda}{b_n} \in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|\langle f, G_n x\rangle| - n\lambda}{b_n} \in B\right\}}\Big] \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned} \tag{1.3.48}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

The moderate deviation principle (1.3.48) is new even for $\varphi = \mathbf{1}$.

Chapter 2

Precise large deviation asymptotics for products of random matrices

Abstract. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent identically distributed $d \times d$ real random matrices with Lyapunov exponent λ. For any starting point x on the unit sphere in $\mathbb{R}^d$, we deal with the norm $|G_n x|$, where $G_n := g_n \dots g_1$. The goal of this paper is to establish precise asymptotics for large deviation probabilities $\mathbb{P}(\log |G_n x| \geqslant n(q+l))$, where $q > \lambda$ is fixed and l is vanishing as $n \to \infty$. We study both invertible matrices and positive matrices and give analogous results for the couple $(X_n^x, \log |G_n x|)$ with target functions, where $X_n^x = G_n x/|G_n x|$. As applications we improve previous results on the large deviation principle for the matrix norm $\|G_n\|$ and obtain a precise local limit theorem with large deviations.

2.1 Introduction

2.1.1 Background and main objectives

One of the fundamental results in the probability theory is the law of large numbers. The large deviation theory describes the rate of convergence in the law of large numbers. The most important results in this direction are the Bahadur-Rao and the Petrov precise large deviation asymptotics that we recall below for independent and identically distributed (i.i.d.) real-valued random variables $(X_i)_{i\geqslant 1}$. Let $S_n = \sum_{i=1}^n X_i$. Denote by I_Λ the set of real numbers $s \geqslant 0$ such that $\Lambda(s) := \log \mathbb{E}[e^{sX_1}] < +\infty$ and by I_Λ° the interior of I_Λ. Let Λ^* be the Frenchel-Legendre transform of Λ. Assume that $s \in I_\Lambda^\circ$ and q are related by $q = \Lambda'(s)$. Set $\sigma_s^2 = \Lambda''(s)$. From the results of Bahadur and Rao [4] and Petrov [73] it follows that if the law of X_1 is non-lattice, then the following large deviation asymptotic holds true:

$$\mathbb{P}(S_n \geqslant n(q+l)) \sim \frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}, \ n \to \infty, \tag{2.1.1}$$

where $\Lambda^*(q+l) = \Lambda^*(q) + sl + \frac{l^2}{2\sigma_s^2} + O(l^3)$ and l is a vanishing perturbation as $n \to \infty$. Bahadur and Rao [4] have established the equivalence (2.1.1) with $l = 0$. Petrov improved it by showing that (2.1.1) holds uniformly in $|l| \leqslant l_n \to 0$ as $n \to \infty$. Actually, Petrov's result is also uniform in q and is therefore stronger than Bahadur-Rao's theorem even with $l = 0$. The relation (2.1.1) with $l = 0$ and its extension to

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$|l| \leqslant l_n \to 0$ have multiple implications in various domains of probability and statistics. The main goal of the present paper is to establish an equivalence similar to (2.1.1) for products of i.i.d. random matrices.

Let $(g_n)_{n\geqslant 1}$ be a sequence of i.i.d. $d \times d$ real random matrices defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common law μ. Denote by $\|\cdot\|$ the operator norm of a matrix and by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$. Set for brevity $G_n := g_n \dots g_1$, $n \geqslant 1$. The study of asymptotic behavior of the product G_n attracted much attention, since the fundamental work of Furstenberg and Kesten [37], where the following strong law of large numbers for $\log \|G_n\|$ has been established:

$$\lim_{n\to\infty} \frac{1}{n} \log \|G_n\| = \lambda, \quad a.s.,$$

with λ a constant called the first Lyapunov exponent of the sequence $(g_n)_{n\geqslant 1}$. Under additional assumptions, Furstenberg [36] extended it to $\log |G_n x|$, for any starting point x on the projective space $\mathbb{P}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}/\pm$. A number of noteworthy results in this area can be found in Kesten [66], Kingman [67], Le Page [69], Guivarc'h and Raugi [51], Bougerol and Lacroix [13], Goldsheid and Guivarc'h [40], Hennion [53], Furman [35], Hennion and Hervé [55], Guivarc'h [49], Guivarc'h and Le Page [50], Benoist and Quint [9, 10] to name only a few.

In this paper we are interested in asymptotic behaviour of large deviation probabilities for $\log |G_n x|$ where $x \in \mathbb{P}^{d-1}$. Set $I_\mu = \{s \geqslant 0 : \mathbb{E}(\|g_1\|^s) < +\infty\}$. For $s \in I_\mu$, let $\kappa(s) = \lim_{n\to\infty} (\mathbb{E}\|G_n\|^s)^{\frac{1}{n}}$. Define the convex function $\Lambda(s) = \log \kappa(s)$, $s \in I_\mu$, and consider its Fenchel-Legendre transform $\Lambda^*(q) = \sup_{s\in I_\mu}\{sq - \Lambda(s)\}$, $q \in \Lambda'(I_\mu)$. Our first objective is to establish the following Bahadur-Rao type precise large deviation asymptotic: with $q = \Lambda'(s)$,

$$\mathbb{P}(\log |G_n x| \geqslant nq) \sim \bar{r}_s(x) \frac{\exp(-n\Lambda^*(q))}{s\sigma_s\sqrt{2\pi n}}, \ n \to \infty, \tag{2.1.2}$$

where $\sigma_s > 0$, $\bar{r}_s = \frac{r_s}{\nu_s(r_s)} > 0$, r_s and ν_s are, respectively, the unique up to a constant eigenfunction and unique probability eigenmeasure of the transfer operator P_s corresponding to the eigenvalue $\kappa(s)$ (see Section 2.2.2 for precise statements). In fact, to enlarge the area of applications in (2.1.2) it is useful to add a vanishing perturbation for q. In this line we obtain the following Petrov type large deviation expansion: under appropriate conditions, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, uniformly in $|l| \leqslant l_n \to 0$ as $n \to \infty$,

$$\mathbb{P}(\log |G_n x| \geqslant n(q+l)) \sim \bar{r}_s(x) \frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}, \quad n \to \infty. \tag{2.1.3}$$

As a consequence of (2.1.3) we are able to infer new results, such as large deviation principles for $\log \|G_n\|$, see Theorems 2.2.5 and 2.2.6.

Our results are established for both invertible matrices and positive matrices. For invertible matrices, Le Page [69] has obtained (2.1.2) for $s > 0$ small enough under more restrictive conditions, such as the existence of exponential moments of $\|g_1\|$ and $\|g_1^{-1}\|$. The asymptotic (2.1.2) clearly implies a large deviation result due to Buraczewski and Mentemeier [17] which holds for invertible matrices and positive matrices: for $q = \Lambda'(s)$

and $s \in I_\mu^\circ$, there exist two constants $0 < c_s < C_s < +\infty$ such that

$$c_s \leqslant \liminf_{n\to\infty} \frac{\mathbb{P}(\log|G_n x| \geqslant nq)}{\frac{1}{\sqrt{n}}\, e^{-n\Lambda^*(q)}} \leqslant \limsup_{n\to\infty} \frac{\mathbb{P}(\log|G_n x| \geqslant nq)}{\frac{1}{\sqrt{n}}\, e^{-n\Lambda^*(q)}} \leqslant C_s. \tag{2.1.4}$$

Consider the Markov chain $X_n^x := G_n x/|G_n x|$. Our second objective is to give precise large deviations for the couple $(X_n^x, \log|G_n x|)$ with target functions. We prove that for any Hölder continuous target function φ on X_n^x, and any target function ψ on $\log|G_n x|$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for all $s' \in (s-\eta, s+\eta)$ with $\eta > 0$ sufficiently small, it holds uniformly in $|l| \leqslant l_n \to 0$ that

$$\begin{aligned}&\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big]\\ &\qquad \sim \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy\, \frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}, \quad n\to\infty.\end{aligned} \tag{2.1.5}$$

From (2.1.5) we can deduce a local limit theorem with large deviations: for any real numbers $-\infty < a_1 < a_2 < \infty$, we have uniformly in $|l| \leqslant l_n \to 0$,

$$\mathbb{P}(\log|G_n x| \in [a_1, a_2] + n(q+l)) \sim (e^{-sa_1} - e^{-sa_2})\frac{\bar{r}_s(x)}{s\sigma_s\sqrt{2\pi n}} e^{-n\Lambda^*(q+l)}, \ n\to\infty. \tag{2.1.6}$$

As a special case of (2.1.6) with $l = 0$ we obtain Theorem 3.3 of Guivarc'h [49]. With $l = 0$, ψ the indicator function of the interval $[0,\infty)$ and $\varphi = r_s$, we get the main result in [17].

Our third objective is to establish asymptotics for lower large deviation probabilities: we prove that for $q = \Lambda'(s)$ with $s < 0$ sufficiently close to 0, it holds uniformly in $|l| \leqslant l_n$ that

$$\mathbb{P}\Big(\log|G_n x| \leqslant n(q+l)\Big) = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)). \tag{2.1.7}$$

This sharpens the large deviation principle established in [13, Theorem 6.1] for invertible matrices. Moreover, we extend the large deviation asymptotic (2.1.7) to the couple $(X_n^x, \log|G_n x|)$ with target functions.

It is expected that the results of this paper can be applied to study large deviations for multivariate perpetuity sequences arising in some models of financial mathematics; we refer, for example, to Buraczewski, Collamore, Damek and Zienkiewicz [18].

2.1.2 Proof outline

Our proof is different from the standard approach of Dembo and Zeitouni [30] based on the Edgeworth expansion, which has been employed for instance in [17]. In contrast to [17], we start with the identity

$$\frac{e^{n\Lambda^*(q)}}{r_s(x)}\mathbb{P}\Big(\log|G_n x| \geqslant nq\Big) = \mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{\psi_s(\log|G_n x| - nq)}{r_s(X_n^x)}\Big), \tag{2.1.8}$$

where $\mathbb{Q}_s^x$ is the change of measure defined in Section 2.3 for the norm cocycle $\log|G_n x|$, $\psi_s(y) = e^{-sy}\mathbb{1}_{\{y\geqslant 0\}}$. Usually the expectation in the right-hand side of (2.1.8) is handled via the Edgeworth expansion for the distribution function $\mathbb{Q}_s^x\big(\frac{\log|G_n x|-nq}{\sqrt{n}\sigma_s} \leqslant t\big)$; however, the presence of the multiplier $r_s(X_n^x)^{-1}$ makes this impossible. Our idea is to replace the function ψ_s with some upper and lower smoothed bounds using a technique from Grama, Lauvergnat and Le Page [42]. For simplicity we deal only with the upper bound $\psi_s \leqslant \psi_{s,\varepsilon}^+ * \rho_{\varepsilon^2}$, where $\psi_{s,\varepsilon}^+(y) = \sup_{y':|y'-y|\leqslant\varepsilon}\psi_s(y')$, for some $\varepsilon > 0$, and ρ_{ε^2} is a density function on the real line satisfying the following properties: the Fourier transform $\widehat{\rho}_{\varepsilon^2}$ is supported on $[-\varepsilon^{-2}, \varepsilon^{-2}]$. Let $R_{s,it}$ be the perturbed operator defined by $R_{s,it}(\varphi)(x) = \mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_1)e^{it(\log|g_1x|-q)}]$, for any Hölder continuous function φ on the projective space $\mathbb{P}^{d-1}$. Using the inversion formula we obtain the following upper bound:

$$\mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{\psi_s(\log|G_n x|-nq)}{r_s(X_n^x)}\Big) \leqslant \frac{1}{2\pi}\int_{\mathbb{R}} R_{s,it}^n(r_s^{-1})(x)\widehat{\psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \tag{2.1.9}$$

where $R_{s,it}^n$ is the n-th iteration of $R_{s,it}$. The integral in the right-hand side of (2.1.9) is decomposed into two parts:

$$\Big\{\int_{|t|<\delta} + \int_{|t|\geqslant\delta}\Big\} R_{s,it}^n(r_s^{-1})(x)\widehat{\psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt. \tag{2.1.10}$$

Since $\widehat{\rho}_{\varepsilon^2}$ is compactly supported on $\mathbb{R}$ and μ is non-arithmetic, for any fixed $s \in I_\mu^\circ$, the second integral in (2.1.10) decays exponentially fast to 0. The difficulty is to prove that the spectral radius $\varrho(R_{s,it})$ decays exponentially fast to 0 uniformly in $s \in K$ and $t \in T$, where K is a compact set of I_μ° and T is a compact set of $\mathbb{R}\setminus\{0\}$. This is overcome by employing the techniques from Hennion and Hervé [54]. To deal with the first integral in (2.1.10), we make use of spectral gap decomposition for the perturbed operator $R_{s,it}$: $R_{s,it}^n = \lambda_{s,it}^n\Pi_{s,it} + N_{s,it}^n$. Of great importance is to prove that the remainder term $N_{s,it}^n$ decays exponentially fast to 0, uniformly in $s \in K$.

The lower bound of the integral in (2.1.8) is a little more delicate, but can be treated in a similar way. The passage to the targeted version is done by using approximation techniques.

We end this section by fixing some notation, which will be used throughout the paper. We denote by c, C, eventually supplied with indices, absolute constants whose values may change from line to line. By c_α, C_α we mean constants depending only on the index α. The interior of a set A is denoted by A°. Let $\mathbb{N} = \{1, 2, \ldots\}$. For any integrable function $\psi : \mathbb{R} \to \mathbb{C}$, define its Fourier transform by $\widehat{\psi}(t) = \int_{\mathbb{R}} e^{-ity}\psi(y)dy$, $t \in \mathbb{R}$. For a matrix g, its transpose is denoted by g^{T}. For a measure ν and a function φ we write $\nu(\varphi) = \int \varphi d\nu$.

2.2 Main results

2.2.1 Notation and conditions

The space $\mathbb{R}^d$ is equipped with the standard scalar product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $|\cdot|$. For $d \geqslant 1$, let $M(d,\mathbb{R})$ be the set of $d\times d$ matrices with entries in $\mathbb{R}$ equipped with the operator norm $\|g\| = \sup_{x\in\mathbb{S}^{d-1}}|gx|$, for $g \in M(d,\mathbb{R})$, where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ is the unit sphere.

We shall work with products of invertible or positive matrices (all over the paper we use the term positive in the wide sense, i.e. each entry is non-negative). Denote by $\mathscr{G} = GL(d, \mathbb{R})$ the general linear group of invertible matrices of $M(d, \mathbb{R})$. A positive matrix $g \in M(d, \mathbb{R})$ is said to be *allowable*, if every row and every column of g has a strictly positive entry. Denote by $\mathscr{G}_+$ the multiplicative semigroup of allowable positive matrices of $M(d, \mathbb{R})$. We write $\mathscr{G}_+^\circ$ for the subsemigroup of $\mathscr{G}_+$ with strictly positive entries.

Consider the projective space $\mathbb{P}^{d-1} := \mathbb{S}^{d-1}/\pm$ by identifying $-x$ with x. Denote by $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ the intersection of the unit sphere with the positive quadrant. To unify the exposition, we use the symbol $\mathcal{S}$ to denote $\mathbb{P}^{d-1}$ in the case of invertible matrices, and $\mathbb{S}_+^{d-1}$ in the case of positive matrices. The space $\mathcal{S}$ is equipped with the metric $\mathbf{d}$ which we proceed to introduce. For invertible matrices, the distance $\mathbf{d}$ is defined as the angular distance (see [50]), i.e., for any $x, y \in \mathbb{P}^{d-1}$, $\mathbf{d}(x, y) = |\sin\theta(x, y)|$, where $\theta(x, y)$ is the angle between x and y. For positive matrices, the distance $\mathbf{d}$ is the Hilbert cross-ratio metric (see [53]) defined by $\mathbf{d}(x, y) = \frac{1-m(x,y)m(y,x)}{1+m(x,y)m(y,x)}$, where $m(x, y) = \sup\{\lambda > 0 : \ \lambda y_i \leqslant x_i, \ \forall i = 1, \ldots, d\}$, for any two vectors $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{S}_+^{d-1}$.

Let $\mathcal{C}(\mathcal{S})$ be the space of continuous functions on $\mathcal{S}$. We write $\mathbf{1}$ for the identity function $1(x)$, $x \in \mathcal{S}$. Throughout this paper, let $\gamma > 0$ be a fixed small constant. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\infty := \sup_{x\in\mathcal{S}} |\varphi(x)| \quad \text{and} \quad \|\varphi\|_\gamma := \|\varphi\|_\infty + \sup_{x,y\in\mathcal{S}} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}(x, y)^\gamma},$$

and introduce the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma < +\infty\}$.

For $g \in M(d, \mathbb{R})$ and $x \in \mathcal{S}$, write $g \cdot x = \frac{gx}{|gx|}$ for the projective action of g on $\mathcal{S}$. For any $g \in M(d, \mathbb{R})$, set $\iota(g) := \inf_{x\in\mathcal{S}} |gx|$. For both invertible matrices and allowable positive matrices, it holds that $\iota(g) > 0$. Note that for any invertible matrix g, we have $\iota(g) = \|g^{-1}\|^{-1}$.

Let $(g_n)_{n\geqslant 1}$ be a sequence of i.i.d. random matrices of the same probability law μ on $M(d, \mathbb{R})$. Set $G_n = g_n \ldots g_1$, for $n \geqslant 1$. Our aim is to establish, under suitable conditions, a large deviation equivalence of type (2.1.1) for the norm cocycle $\log |G_n x|$ for invertible matrices and positive matrices. In both cases, we denote by $\Gamma_\mu := [\operatorname{supp} \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by $\operatorname{supp} \mu$ (the support of μ), that is, $\Gamma_\mu = \overline{\cup_{n=1}^\infty \{\operatorname{supp} \mu\}^n}$.

Set

$$I_\mu = \{s \geqslant 0 : \mathbb{E}(\|g_1\|^s) < +\infty\}.$$

Applying Hölder's inequality to $\mathbb{E}(\|g_1\|^s)$, it is easily seen that I_μ is an interval. We make use of the following exponential moment condition, where we assume that $s \in I_\mu^\circ$ is given.

P1. There exist $s \in I_\mu^\circ$ and $\alpha \in (0, 1)$ such that $\mathbb{E}\|g_1\|^{s+\alpha} \iota(g_1)^{-\alpha} < +\infty$.

For invertible matrices, we introduce the following strong irreducibility and proximality conditions, where we recall that a matrix g is said to be *proximal* if it has an algebraic simple dominant eigenvalue.

P2. (i)(Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant.
(ii)(Proximality) Γ_μ contains at least one proximal matrix.

The conditions of strong irreducibility and proximality are always satisfied for $d = 1$. If g is proximal, denote by λ_g its dominant eigenvalue and by v_g the associated normalized eigenvector ($|v_g| = 1$). In fact, g is proximal iff the space $\mathbb{R}^d$ can be decomposed as $\mathbb{R}^d = \mathbb{R}\lambda_g \oplus V'$ such that $gV' \subset V'$ and the spectral radius of g on the invariant subspace V' is strictly less than $|\lambda_g|$. For invertible matrices, condition **P2** implies that the Markov chain X_n^x has a unique μ-stationary measure (see [50]), which is supported on

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{P}^{d-1} : g \in \Gamma_\mu,\ g \text{ is proximal}\}}.$$

For positive matrices, introduce the following condition:

P3. (i) (Allowability) Every $g \in \Gamma_\mu$ is allowable.
(ii) (Positivity) Γ_μ contains at least one matrix belonging to $\mathscr{G}_+^\circ$.

It can be shown (see [16, Lemma 4.3]) that for positive matrices, condition **P3** ensures the existence and uniqueness of the invariant measure for the Markov chain X_n^x supported on

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{S}_+^{d-1} : g \in \Gamma_\mu \cap \mathscr{G}_+^\circ\}}.$$

In addition, $V(\Gamma_\mu)$ is the unique minimal Γ_μ-invariant subset (see [16, Lemma 4.2]). According to the Perron-Frobenius theorem, a strictly positive matrix always has a unique dominant eigenvalue, so condition **P3**(ii) implies condition **P2**(ii) for $d > 1$.

For any $s \in I_\mu$, for invertible matrices and for positive matrices, the following limit exists (see [50] and [17]):

$$\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\|G_n\|^s\right)^{\frac{1}{n}}.$$

The function $\Lambda = \log \kappa : I_\mu \to \mathbb{R}$ is convex and analytic on I_μ° (it plays the same role as the log-Laplace transform of X_1 in the real i.i.d. case). Introduce the Fenchel-Legendre transform of Λ by $\Lambda^*(q) = \sup_{s\in I_\mu}\{sq - \Lambda(s)\}$, $q \in \Lambda'(I_\mu)$. We have that $\Lambda^*(q) = sq - \Lambda(s)$ if $q = \Lambda'(s)$ for some $s \in I_\mu$, which implies $\Lambda^*(q) \geqslant 0$ on $\Lambda'(I_\mu)$ since $\Lambda(0) = 0$ and $\Lambda(s)$ is convex on I_μ.

We say that the measure μ is *arithmetic*, if there exist $t > 0$, $\beta \in [0, 2\pi)$ and a function $\vartheta : \mathcal{S} \to \mathbb{R}$ such that for any $g \in \Gamma_\mu$ and any $x \in V(\Gamma_\mu)$, we have $\exp[it \log|gx| - i\beta + i\vartheta(g\cdot x) - i\vartheta(x)] = 1$. For positive matrices, we need the following condition:

P4. (Non-arithmeticity) The measure μ is non-arithmetic.

A simple sufficient condition established in [66] for the measure μ to be non-arithmetic is that the additive subgroup of $\mathbb{R}$ generated by the set $\{\log \lambda_g : g \in \Gamma_\mu \cap \mathscr{G}_+^\circ\}$ is dense in $\mathbb{R}$ (see [17, Lemma 2.7]). Note that condition **P4** for positive matrices ensures that $\sigma_s^2 = \Lambda''(s) > 0$.

For invertible matrices, condition **P4** is implied by condition **P2**, so that we also have $\sigma_s^2 > 0$; for a proof see Guivarc'h and Urban [52, Proposition 4.6].

For any $s \in I_\mu$, the transfer operator P_s and the conjugate transfer operator P_s^* are defined, for any $\varphi \in \mathcal{C}(\mathcal{S})$ and $x \in \mathcal{S}$, by

$$P_s\varphi(x) = \int_{\Gamma_\mu} |g_1 x|^s \varphi(g_1\cdot x)\mu(dg_1),\ \ P_s^*\varphi(x) = \int_{\Gamma_\mu} |g_1^{\mathrm{T}} x|^s \varphi(g_1^{\mathrm{T}}\cdot x)\mu(dg_1), \tag{2.2.1}$$

which are bounded linear on $\mathcal{C}(\mathcal{S})$. Under condition **P2** for invertible matrices, or condition **P3** for positive matrices, the operator P_s has a unique probability eigenmeasure ν_s on $\mathcal{S}$ corresponding to the eigenvalue $\kappa(s)$: $P_s\nu_s = \kappa(s)\nu_s$. Similarly, the operator P_s^* has a unique probability eigenmeasure ν_s^* corresponding to the eigenvalue $\kappa(s)$: $P_s\nu_s = \kappa(s)\nu_s$, where $P_s\nu_s$ stands for the measure on $\mathcal{S}$ such that $P_s\nu_s(\varphi) = \nu_s(P_s\varphi)$ for any $\varphi \in \mathcal{C}(\mathcal{S})$. Similarly, the operator P_s^* has a unique probability eigenmeasure ν_s^* corresponding to the eigenvalue $\kappa(s)$: $P_s^*\nu_s^* = \kappa(s)\nu_s^*$, where $P_s^*\nu_s^*$ is defined in a similar way as $P_s\nu_s$. Set, for $x \in \mathcal{S}$,

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy).$$

Then, r_s is the unique, up to a scaling constant, strictly positive eigenfunction of P_s: $P_s r_s = \kappa(s) r_s$; similarly r_s^* is the unique, up to a scaling constant, strictly positive eigenfunction of P_s^*: $P_s^* r_s^* = \kappa(s) r_s^*$. We refer for details to Section 2.3.

Below we shall also make use of normalized eigenfunction $\bar{r}_s$ defined by $\bar{r}_s(x) = \frac{r_s(x)}{\nu_s(r_s)}$, $x \in \mathcal{S}$, which is strictly positive and Hölder continuous on the projective space $\mathcal{S}$, see Proposition 2.3.1.

2.2.2 Large deviations for the norm cocycle

The following theorem gives the exact asymptotic behavior of the large deviation probabilities for the norm cocycle.

Theorem 2.2.1. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume that μ satisfies either conditions **P1**, **P2** for invertible matrices, or conditions **P1**, **P3**, **P4** for positive matrices. Then, as $n \to \infty$, uniformly in $x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|G_n x| \geqslant nq\Big) = \bar{r}_s(x) \frac{\exp\left(-n\Lambda^*(q)\right)}{s\sigma_s\sqrt{2\pi n}}(1 + o(1)). \tag{2.2.2}$$

More generally, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\mathbb{P}\Big(\log|G_n x| \geqslant n(q + l)\Big) = \bar{r}_s(x) \frac{\exp\left(-n\Lambda^*(q + l)\right)}{s\sigma_s\sqrt{2\pi n}}(1 + o(1)). \tag{2.2.3}$$

The rate function $\Lambda^*(q + l)$ admits the following expansion: for $q = \Lambda'(s)$ and l in a small neighborhood of 0, we have

$$\Lambda^*(q + l) = \Lambda^*(q) + sl + \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s\Big(\frac{l}{\sigma_s}\Big), \tag{2.2.4}$$

where $\zeta_s(t)$ is the Cramér series, $\zeta_s(t) = \sum_{k=3}^\infty c_{s,k} t^{k-3} = \frac{\Lambda'''(s)}{6\sigma_s^3} + O(t)$, with $\Lambda'''(s)$ and σ_s defined in Proposition 2.3.4. We refer for details to Lemma 2.4.1, where the coefficients $c_{s,k}$ are given in terms of the cumulant generating function $\Lambda = \log\kappa$.

For invertible matrices, a point-wise non-uniform version of (2.2.2), without $\sup_{x\in\mathcal{S}}$, namely the asymptotic (2.1.2), has been first established by Le Page [69, Theorem 8] for small enough $s > 0$ under a stronger exponential moment condition. For positive matrices, the asymptotic (2.2.2) is new and implies the large deviation bounds (2.1.4)

established in Buraczewski and Mentemeier [17, Corollary 3.2]. We note that there is a misprint in [17], where e^{sq} should be replaced by $e^{\Lambda^*(q)}$.

Now we consider the precise large deviations for the couple $(X_n^x, \log|G_n x|)$ with target functions φ and ψ on $X_n^x := G_n \cdot x$ and $\log|G_n x|$, respectively.

Theorem 2.2.2. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Assume the conditions of Theorem 2.2.1. Then, for any $\varphi \in \mathcal{B}_\gamma$, any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for all $s' \in (s-\eta, s+\eta)$ with $\eta > 0$ sufficiently small, and any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}\left[\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + o(1)\right]. \tag{2.2.5}$$

With $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y\geqslant 0\}}$ for $y \in \mathbb{R}$, we obtain Theorem 2.2.1. For invertible matrices and with $l = 0$, Theorem 2.2.2 strengthens the point-wise large deviation result stated in Theorem 3.3 of Guivarc'h [49], since we do not assume the function ψ to be compactly supported and our result is uniform in $x \in \mathcal{S}$. By the way we would like to remark that in Theorem 3.3 of [49] $\kappa^n(s)$ should be replaced by $\kappa^{-n}(s)$, and $\nu_s(\varphi r_s^{-1})$ should be replaced by $\frac{\nu_s(\varphi)}{\nu_s(r_s)}$. For positive matrices, Theorem 2.2.2 is new. Since r_s is a strictly positive and Hölder continuous function on $\mathcal{S}$ (see Proposition 2.3.1), taking $l = 0$, $\varphi = r_s$ and $\psi(y) = \mathbb{1}_{\{y\geqslant 0\}}$, $y \in \mathbb{R}$ in Theorem 2.2.2, we get the main result of [17] (Theorem 3.1).

Unlike the case of i.i.d. real-valued random variables, Theorems 2.2.1 and 2.2.2 do not imply the similar asymptotic for lower large deviation probabilities $\mathbb{P}(\log|G_n x| \leqslant n(q+l))$, where $q < \Lambda'(0)$. To formulate our results, we need an exponential moment condition, as in Le Page [69]. For $g \in \Gamma_\mu$, set $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$, which reduces to $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for invertible matrices.

P5. There exists a constant $\eta \in (0,1)$ such that $\mathbb{E}[N(g_1)^\eta] < +\infty$.

Under condition **P5**, the functions $s \mapsto \kappa(s)$ and $s \mapsto \Lambda(s) = \log\kappa(s)$ can be extended analytically in a small neighborhood of 0 of the complex plane; in this case the expansion (2.2.4) still holds and we have $\sigma_s = \Lambda''(s) > 0$ for $s < 0$ small enough. We also need to extend the function $\bar{r}_s$ for small $s < 0$, which is positive and Hölder continuous on the projective space $\mathcal{S}$, as in the case of $s > 0$: we refer to Proposition 2.3.2 for details.

Theorem 2.2.3. *Assume that μ satisfies either conditions **P2**, **P5** for invertible matrices or conditions **P3**, **P4**, **P5** for positive matrices. Then, there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|G_n x| \leqslant nq\Big) = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)).$$

More generally, for any $s \in (-\eta_0, 0)$ and $q = \Lambda'(s)$, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\mathbb{P}\Big(\log|G_n x| \leqslant n(q+l)\Big) = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)).$$

For invertible matrices, this result sharpens the large deviation principle established in [13]. For positive matrices, our result is new, even for the large deviation principle.

More generally, we also have the precise large deviations result for the couple $(X_n^x, \log|G_n x|)$ with target functions.

Theorem 2.2.4. *Assume the conditions of Theorem 2.2.3. Then, there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, for any $\varphi \in \mathcal{B}_\gamma$, for any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for all $s' \in (s-\eta, s+\eta)$ with $\eta > 0$ sufficiently small, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\begin{aligned}\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] \\ = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}\left[\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + \|\varphi\|_\gamma o(1)\right].\end{aligned}$$

With $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y\leqslant 0\}}$ for $y \in \mathbb{R}$, we obtain Theorem 2.2.3.

2.2.3 Applications to large deviation principle for the matrix norm

We use Theorems 2.2.1 and 2.2.3 to deduce large deviation principles for the matrix norm $\|G_n\|$. Our first result concerns invertible matrices and the second one deals with positive matrices.

Theorem 2.2.5. *Assume the conditions of Theorem 2.2.1 for invertible matrices. Let $q = \Lambda'(s)$, where $s \in I_\mu^\circ$. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, uniformly in $|l| \leqslant l_n$,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log\|G_n\| \geqslant n(q+l)\Big) = -\Lambda^*(q).$$

With $l = 0$, Theorem 2.2.5 improves the large deviation bounds in Benoist and Quint [10, Theorem 14.19], where the authors consider general groups, but without giving the rate function.

In the case of invertible matrices, under the conditions of Theorem 2.2.3, since $\log|G_n x| \leqslant \log\|G_n\|$, we deduce easily from Theorem 2.2.3 the following upper bound of the large deviation principle for $\log\|G_n\|$: there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, uniformly in $|l| \leqslant l_n$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log\|G_n\| \leqslant n(q+l)\Big) \leqslant -\Lambda^*(q).$$

However, it remains an open question to prove the following lower bound:

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\log\|G_n\| \leqslant n(q+l)\Big) \geqslant -\Lambda^*(q). \tag{2.2.6}$$

In the case of positive matrices, we are able to give an affirmative answer to the question (2.2.6), and moreover, we have the following reinforced large deviation principles.

Theorem 2.2.6. *The following two assertions hold:*

(1) *Let $q = \Lambda'(s)$, where $s \in I_\mu^\circ$. Assume conditions* ***P1***, ***P3***, ***P4*** *for positive matrices. Then, there exist two constants $0 < c < C < +\infty$ such that for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, uniformly in $|l| \leqslant l_n$,*

$$\begin{aligned} c < & \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \geqslant n(q+l)\Big) \\ \leqslant & \limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \geqslant n(q+l)\Big) < C. \end{aligned}$$

(2) *Let $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, where $s_0 > 0$ is a small constant. Assume conditions* ***P3***, ***P4***, ***P5*** *for positive matrices. Then, there exist two constants $0 < c < C < +\infty$ such that for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, uniformly in $|l| \leqslant l_n$,*

$$\begin{aligned} c < & \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \leqslant n(q+l)\Big) \\ \leqslant & \limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \leqslant n(q+l)\Big) < C. \end{aligned}$$

The statements (1) and (2) in Theorem 2.2.6 clearly implies the large deviation principles for $\log \|G_n\|$, which are also new to our knowledge.

2.2.4 Local limit theorems with large deviations

Local limit theorems and large and moderate deviations for sums of i.i.d. random variables have been studied by Gnedenko [39], Sheep [79], Stone [80], Breuillard [15], Borovkov and Borovkov [11]. Moderate deviation results in the local limit theorem for products of invertible random matrices have been obtained in [10, Theorems 17.9 and 17.10].

Taking $\varphi = \mathbf{1}$ and $\psi = \mathbb{1}_{[a,a+\Delta]}$ in Theorem 2.2.2, we can deduce the following local limit theorem with large deviations.

Theorem 2.2.7. *Assume conditions of Theorem 2.2.1 and let $q = \Lambda'(s)$. Then, for any $\varphi \in \mathcal{B}_\gamma$, any real numbers $-\infty < a_1 < a_2 < \infty$, and any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\begin{aligned} & \mathbb{E}\Big[\varphi(X_n^x) \mathbb{1}_{\left\{ \log |G_n x| \in [a_1,a_2] + n(q+l) \right\}}\Big] \\ & \qquad = \bar{r}_s(x)(e^{-sa_1} - e^{-sa_2}) \frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}} \Big[\nu_s(\varphi) + o(1)\Big]. \end{aligned}$$

Taking $\varphi = \mathbf{1}$, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\begin{aligned} & \mathbb{P}\Big(\log |G_n x| \in [a_1, a_2] + n(q+l)\Big) \\ & \qquad = \bar{r}_s(x)(e^{-sa_1} - e^{-sa_2}) \frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}} \Big[1 + o(1)\Big]. \end{aligned}$$

We can compare this result with Theorem 3.3 in [49], from which the above equivalence can be deduced for $l = 0$.

It is easy to see that, under additional assumption **P5**, the assertion of Theorem 2.2.7 remains true for $s < 0$ small enough. This can be deduced from Theorem 2.2.4: the details are left to the reader.

2.3 Spectral gap theory for the norm

2.3.1 Properties of the transfer operator

Recall that the transfer operator P_s and the conjugate operator P_s^* are defined by (2.2.1). Below $P_s\nu_s$ stands for the measure on $\mathcal{S}$ such that $P_s\nu_s(\varphi) = \nu_s(P_s\varphi)$, for continuous functions φ on $\mathcal{S}$, and $P_s^*\nu_s^*$ is defined similarly. The following result was proved in [16, 17] for positive matrices, and in [50] for invertible matrices.

Proposition 2.3.1. *Let $s \in I_\mu$. Assume either conditions **P1**, **P2** for invertible matrices, or conditions **P1**, **P3** for positive matrices. Then the spectral radii $\varrho(P_s)$ and $\varrho(P_s^*)$ are both equal to $\kappa(s)$, and there exist a unique, up to a scaling constant, strictly positive Hölder continuous function r_s and a unique probability measure ν_s on $\mathcal{S}$ such that*

$$P_s r_s = \kappa(s) r_s, \quad P_s\nu_s = \kappa(s)\nu_s.$$

Similarly, there exist a unique strictly positive Hölder continuous function r_s^* and a unique probability measure ν_s^* on $\mathcal{S}$ such that

$$P_s^* r_s^* = \kappa(s) r_s^*, \quad P_s^*\nu_s^* = \kappa(s)\nu_s^*.$$

Moreover, the functions r_s and r_s^* are given by

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy), \quad x \in \mathcal{S}.$$

It is easy to see that the family of kernels $q_n^s(x, g) = \frac{|gx|^s}{\kappa^n(s)} \frac{r_s(g\cdot x)}{r_s(x)}$, $n \geqslant 1$ satisfies the following cocycle property:

$$q_n^s(x, g_1) q_m^s(g_1\cdot x, g_2) = q_{n+m}^s(x, g_2 g_1). \tag{2.3.1}$$

The equation $P_s r_s = \kappa(s) r_s$ implies that, for any $x \in \mathcal{S}$ and $s \in I_\mu$, the probability measures $\mathbb{Q}_{s,n}^x(dg_1, \dots, dg_n) = q_n^s(x, g_n \dots g_1)\mu(dg_1)\dots\mu(dg_n)$, $n \geqslant 1$, form a projective system on $M(d, \mathbb{R})^{\mathbb{N}}$. By the Kolmogorov extension theorem, there is a unique probability measure $\mathbb{Q}_s^x$ on $M(d, \mathbb{R})^{\mathbb{N}}$, with marginals $\mathbb{Q}_{s,n}^x$; denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation.

If $(g_n)_{n\in\mathbb{N}}$ denotes the coordinate process on the space of trajectories $M(d, \mathbb{R})^{\mathbb{N}}$, then the sequence $(g_n)_{n\geqslant 1}$ is i.i.d. with the common law μ under $\mathbb{Q}_0^x$. However, for any $s \in I_\mu^\circ$ and $x \in \mathcal{S}$, the sequence $(g_n)_{n\geqslant 1}$ is Markov-dependent under the measure $\mathbb{Q}_s^x$. Let

$$X_0^x = x, \quad X_n^x = G_n\cdot x, \quad n \geqslant 1.$$

By the definition of $\mathbb{Q}_s^x$, for any bounded measurable function f on $(\mathcal{S}\times\mathbb{R})^n$, it holds that

$$\begin{aligned}\frac{1}{\kappa^n(s) r_s(x)} \mathbb{E}\Big[r_s(X_n^x) |G_n x|^s f\Big(X_1^x, \log|G_1 x|, \dots, X_n^x, \log|G_n x|\Big)\Big] \\ = \mathbb{E}_{\mathbb{Q}_s^x}\Big[f\Big(X_1^x, \log|G_1 x|, \dots, X_n^x, \log|G_n x|\Big)\Big].\end{aligned} \tag{2.3.2}$$

Under the measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n\in\mathbb{N}}$ is a Markov chain with the transition operator given by

$$Q_s\varphi(x) = \frac{1}{\kappa(s)r_s(x)}P_s(\varphi r_s)(x) = \frac{1}{\kappa(s)r_s(x)}\int_{\Gamma_\mu}|gx|^s\varphi(g\cdot x)r_s(g\cdot x)\mu(dg).$$

It has been proved in [16] for positive matrices, and in [50] for invertible matrices, that Q_s has a unique invariant probability measure π_s supported on $V(\Gamma_\mu)$ and that, for any $\varphi\in\mathcal{C}(\mathcal{S})$,

$$\lim_{n\to\infty}Q_s^n\varphi = \pi_s(\varphi), \quad \text{where } \pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}. \tag{2.3.3}$$

Moreover, letting $\mathbb{Q}_s = \int\mathbb{Q}_s^x\pi_s(dx)$, from the results of [16, 50], it follows that, under the assumptions of Theorem 2.2.1, for any $s\in I_\mu$, we have $\lim_{n\to\infty}\frac{\log|G_nx|}{n} = \Lambda'(s)$, $\mathbb{Q}_s$-a.s. and $\mathbb{Q}_s^x$-a.s., where $\Lambda'(s) = \frac{\kappa'(s)}{\kappa(s)}$.

When $s\in(-s_0,0)$ for small enough $s_0>0$, define the transfer operator P_s as follows: for any $\varphi\in\mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) = \int_{\Gamma_\mu}|g_1x|^s\varphi(g_1\cdot x)\mu(dg_1), \quad x\in\mathcal{S},$$

which is well-defined under condition **P5**. The following proposition is proved in [86].

Proposition 2.3.2. *Assume that μ satisfies either conditions **P2**, **P5** for invertible matrices, or conditions **P3**, **P5** for positive matrices. Then there exists a constant $s_0>0$ such that for any $s\in(-s_0,0)$, the spectral radius $\varrho(P_s)$ of the operator P_s is equal to $\kappa(s)$. Moreover there exist a unique, up to a scaling constant, strictly positive Hölder continuous function r_s and a unique probability measure ν_s on $\mathcal{S}$ such that*

$$P_sr_s = \kappa(s)r_s, \quad P_s\nu_s = \kappa(s)\nu_s.$$

Based on Proposition 2.3.2, in the same way as for $s>0$, one can define the measure $\mathbb{Q}_s^x$ for negative values $s<0$ sufficiently close to 0, and one can extend the change of measure formula (2.3.2) to $s<0$. Under the measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n\in\mathbb{N}}$ is a Markov chain with the transition operator Q_s and the assertion (2.3.3) holds true. We refer to [86] for details.

2.3.2 Spectral gap of the perturbed operator

Recall that the Banach space $\mathcal{B}_\gamma$ consists of all γ-Hölder continuous function on $\mathcal{S}$, where $\gamma>0$ is a fixed small constant. Denote by $\mathcal{L}(\mathcal{B}_\gamma,\mathcal{B}_\gamma)$ the set of all bounded linear operators from $\mathcal{B}_\gamma$ to $\mathcal{B}_\gamma$ equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma}$. For $s\in I_\mu^\circ$ and $z\in\mathbb{C}$ with $s+\Re z\in I_\mu$, define a family of perturbed operators $R_{s,z}$ as follows: for any $\varphi\in\mathcal{B}_\gamma$,

$$R_{s,z}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|g_1x|-q)}\varphi(X_1^x)\right], \quad x\in\mathcal{S}. \tag{2.3.4}$$

It follows from the cocycle property (2.3.1) that

$$R_{s,z}^n\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|G_nx|-nq)}\varphi(X_n^x)\right], \quad x\in\mathcal{S}.$$

We need the following lemma from [54, Lemma III.9].

Lemma 2.3.3. *Let $s \in \mathbb{R}$, $\delta > 0$ and $I_{s,\delta} = (s-\delta, s+\delta)$. Assume that $t \in I_{s,\delta} \mapsto P(t) \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is a continuous mapping. Let $r > \varrho(P(s))$. Then, there exist constants $\varepsilon = \varepsilon(s)$ and $c = c(s) > 0$ such that*

$$\sup_{t\in(s-\varepsilon,s+\varepsilon)} \|P^n(t)\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} < cr^n.$$

Moreover, it holds that

$$\limsup_{t\to s} \varrho(P(t)) \leqslant \varrho(P(s)).$$

It is easy to check that in the proof of [54, Lemma III.9], the constant c can be chosen to depend only on s.

The following proposition collects useful assertions that we will use in the proofs of our results. Denote $B_\delta(0) := \{z \in \mathbb{C} : |z| \leqslant \delta\}$.

Proposition 2.3.4. *Assume that μ satisfies either conditions **P1**, **P2** for invertible matrices, or conditions **P1**, **P3** for positive matrices. Then, for any $s \in I_\mu^\circ$, there exists $\delta = \delta(s) > 0$ such that for any $z \in B_\delta(0)$,*

$$R_{s,z}^n = \lambda_{s,z}^n \Pi_{s,z} + N_{s,z}^n, \ n \geqslant 1. \tag{2.3.5}$$

Moreover, for any $s \in I_\mu^\circ$, the following assertions hold:

(i) *$\Pi_{s,z}$ is a rank-one projection for $|z| \leqslant \delta$, with $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}_\gamma$ and $x \in \mathcal{S}$, $\Pi_{s,z}N_{s,z} = N_{s,z}\Pi_{s,z} = 0$ and*

$$\lambda_{s,z} = e^{-qz}\frac{\kappa(s+z)}{\kappa(s)}, \quad \text{for } z \in B_\delta(0), \tag{2.3.6}$$

where $q = \Lambda'(s)$.

For any compact set $K \subset I_\mu^\circ$ and fixed integer $k \geqslant 0$, there exist constants $\delta = \delta(k, K)$, $c = c(k, \delta, K)$ and $\varkappa = \varkappa(k, \delta, K) \in (0, 1)$ such that for all $n \geqslant 1$,

$$\sup_{s\in K}\sup_{|z|<\delta} \|\frac{d^k}{dz^k}\Pi_{s,z}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant c, \tag{2.3.7}$$

$$\sup_{s\in K}\sup_{|z|<\delta} \|\frac{d^k}{dz^k}N_{s,z}^n\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant c\varkappa^n. \tag{2.3.8}$$

In addition, the mappings $z \mapsto \Pi_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ and $z \mapsto N_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ are analytic in the strong operator sense.

(ii) *For any compact sets $K \subset I_\mu^\circ$ and $T \subseteq \mathbb{R}\backslash\{0\}$, there exists a constant $C = C(K, T) > 0$ such that for any $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$, we have*

$$\sup_{s\in K}\sup_{t\in T}\sup_{x\in\mathcal{S}} |R_{s,it}^n\varphi(x)| \leqslant e^{-Cn}\|\varphi\|_\gamma. \tag{2.3.9}$$

(iii) *For any* $s \in I_\mu^\circ$ *and* $q = \Lambda'(s)$*, the mapping* $z \mapsto \lambda_{s,z} : B_\delta(0) \to \mathbb{C}$ *is analytic, and*

$$\lambda_{s,z} = 1 + \frac{\sigma_s^2}{2} z^2 + \frac{\Lambda'''(s)}{6} z^3 + o(z^3) \quad as\ z \to 0,$$

where

$$\sigma_s^2 = \Lambda''(s) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s} (\log |G_n x| - nq)^2$$

and

$$\Lambda'''(s) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s} (\log |G_n x| - nq)^3.$$

In addition, if the measure μ *is non-arithmetic, then the asymptotic variance* σ_s^2 *is strictly positive.*

Proof. Let $s \in I_\mu^\circ$ be fixed. The spectral gap decomposition (2.3.5) and formulae (2.3.7), (2.3.8) and (2.3.9) are proved in [17] for imaginary-valued $z \in (-i\delta, i\delta)$. The extension for arbitrary complex valued $z \in B_\delta(0)$ is immediate.

We now prove (2.3.6). The operator P_s, its spectral radius $\kappa(s)$ and eigenvector r_s can be respectively extended analytically to P_{s+z}, the eigenvalue $\kappa(s+z)$ and the eigenvector r_{s+z}, with z in the small neighborhood of 0, see [50]. Specifically, since the transfer operator P_s has spectral gap properties and the mapping $z \mapsto P_{s+z}$ is analytic in a small neighborhood of 0 in the complex plane, by functional calculus and the perturbation theory (see [54]), the operator P_{s+z} has an isolated spectral value $\kappa(s+z)$, so that we have the following spectral gap decomposition of P_{s+z}:

$$P_{s+z}^n = \kappa^n(s+z) M_{s+z} + L_{s+z}^n, \tag{2.3.10}$$

where M_{s+z} is a rank-one projection for $|z| \leqslant \delta$, with

$$M_{s+z}(\varphi)(x) = \frac{\nu_{s+z}(\varphi)}{\nu_{s+z}(r_{s+z})} r_{s+z}(x)$$

for any $\varphi \in \mathcal{B}_\gamma$ and $x \in \mathcal{S}$, and $M_{s+z} L_{s+z} = L_{s+z} M_{s+z} = 0$. Moreover, for any fixed $s \in I_\mu^\circ$ and integer $k \geqslant 0$, there exist constants $\varepsilon(s) = \varepsilon(k,s) > 0$, $\delta(s) = \delta(k,s) > 0$, $\varkappa(s) = \varkappa(k,s) \in (0,1)$ and $c(s) = c(k,s) > 0$ such that

$$\sup_{s' \in (s-\varepsilon(s), s+\varepsilon(s))} \frac{1}{\kappa^n(s)} \sup_{|z| \leqslant \delta(s)} \| \frac{d^k}{dz^k} L_{s+z}^n \|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma} < c(s) \varkappa(s)^n. \tag{2.3.11}$$

By the definition of $R_{s,z}$ and P_z, using the change of measure (2.3.2), we obtain for any $\varphi \in \mathcal{B}_\gamma$, $n \geqslant 1$, $s \in I_\mu^\circ$ and $z \in B_\delta(0)$,

$$R_{s,z}^n(\varphi) = e^{-nz\Lambda'(s)} \frac{P_{s+z}^n(\varphi r_s)}{\kappa^n(s) r_s}. \tag{2.3.12}$$

Substituting the spectral gap decomposition (2.3.10) into the identity (2.3.12), we get

$$R_{s,z}^n = \lambda_{s,z}^n \Pi_{s,z} + N_{s,z}^n,$$

where $\lambda_{s,z}$ is given by (2.3.6), $\Pi_{s,z}$ and $N^n_{s,z}$ are given as follows: for any $\varphi \in \mathcal{B}_\gamma$, $n \geqslant 1$, $s \in I^\circ_\mu$ and $z \in B_\delta(0)$,

$$\Pi_{s,z}(\varphi) = \frac{\nu_{s+z}(\varphi r_s)}{\nu_{s+z}(r_{s+z})}\frac{r_{s+z}}{r_s}, \quad N^n_{s,z}(\varphi) = e^{-n\Lambda'(s)z}\frac{1}{\kappa^n(s)}\frac{L^n_{s+z}(\varphi r_s)}{r_s}. \tag{2.3.13}$$

We proceed to prove (2.3.7) and (2.3.8). From (2.3.11) and (2.3.13), it follows that for any $s \in I^\circ_\mu$ and integer $k \geqslant 0$, there exist constants $\varepsilon(s) = \varepsilon(k,s) > 0$, $\delta(s) = \delta(k,s) > 0$, $\varkappa(s) = \varkappa(k,s) \in (0,1)$ and $c(s) = c(k,s) > 0$ such that for all $n \geqslant 1$,

$$\sup_{s' \in (s-\varepsilon(s), s+\varepsilon(s))} \sup_{|z| \leqslant \delta(s)} \|\frac{d^k}{dz^k}\Pi_{s,z}\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma} \leqslant c(s), \tag{2.3.14}$$

and

$$\sup_{s' \in (s-\varepsilon(s), s+\varepsilon(s))} \sup_{|z| \leqslant \delta(s)} \|\frac{d^k}{dz^k}N^n_{s',z}\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma} < c(s)\varkappa(s)^n. \tag{2.3.15}$$

Let $K \subset I^\circ_\mu$ be a compact set. Since $\cup_{s\in K}(s-\varepsilon(s), s+\varepsilon(s)) \supset K$, applying Heine–Borel's theorem, there exists a sequence $\{s_m\}_{1\leqslant m\leqslant m_0}$ such that $\cup_{m=1}^{m_0}(s_m - \varepsilon_m, s_m + \varepsilon_m) \supset K$, where $\varepsilon_m = \varepsilon(s_m)$. Therefore, from (2.3.14), we take $\delta = \delta(k,K) = \min_{1\leqslant m\leqslant m_0}\delta(k,s_m)$ and $c = c(k,K) = \max_{1\leqslant m\leqslant m_0} c(k,s_m)$ to obtain (2.3.7). Similarly, from (2.3.15), taking $\delta = \delta(k,K) = \min_{1\leqslant m\leqslant m_0}\delta(k,s_m)$, $c = c(k,K) = \max_{1\leqslant m\leqslant m_0} c(k,s_m)$ and $\varkappa = \varkappa(k,K) = \max_{1\leqslant m\leqslant m_0}\varkappa(k,s_m)$, we conclude the proof of (2.3.8).

We finally prove (2.3.9). It was established in [17] that for any fixed $s \in I^\circ_\mu$ and $t \in \mathbb{R}\backslash\{0\}$, there exists a constant $C(s,t) > 0$ such that for any $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$, we have

$$\sup_{x\in\mathcal{S}} |R^n_{s,it}\varphi(x)| \leqslant e^{-nC(s,t)}\|\varphi\|_\gamma. \tag{2.3.16}$$

By the formula (2.3.12), we see that the operator $R_{s,it}$ is continuous in s and t. Using Lemma 2.3.3, we get that there exist constants $\varepsilon(s) > 0$ and $\delta(t) > 0$ such that

$$\sup_{s' \in (s-\varepsilon(s), s+\varepsilon(s))} \sup_{t' \in (t-\delta(t), t+\delta(t))} \sup_{x\in\mathcal{S}} |R^n_{s',it'}\varphi(x)| \leqslant e^{-nC(s,t)}\|\varphi\|_\gamma.$$

Let $K \subset I^\circ_\mu$ and $T \subseteq \mathbb{R}\backslash\{0\}$ be compact sets. Since

$$\cup_{(s,t)\in K\times T}\Big\{\Big(s-\varepsilon(s), s+\varepsilon(s)\Big) \times \Big(t-\delta(t), t+\delta(t)\Big)\Big\} \supset K \times T,$$

by Heine–Borel's theorem, there exists a sequence $\{s_m, t_m\}_{1\leqslant m\leqslant m_0}$ such that

$$\cup_{m=1}^{m_0}\Big\{(s_m - \varepsilon_m, s_m + \varepsilon_m) \times (t_m - \delta_m, t_m + \delta_m)\Big\} \supset K \times T,$$

where $\varepsilon_m = \varepsilon(s_m)$ and $\delta_m = \delta(s_m)$. This concludes the proof of (2.3.9) by taking $C = C(K,T) = \min_{1\leqslant m\leqslant m_0} C(s_m, t_m)$. □

For negative values $s < 0$ sufficiently close to 0, we can define the perturbed operator $R_{s,z}$ as in (2.3.4). The following spectral gap property of $R_{s,z}$ is established in [86].

Proposition 2.3.5. *Assume that μ satisfies conditions **P2**, **P5** for invertible matrices, or conditions **P3**, **P5** for positive matrices. Then, there exist constants $s_0 > 0$ and $\delta > 0$ such that for any $s \in (-s_0, 0)$ and $z \in B_\delta(0)$,*

$$R^n_{s,z} = \lambda^n_{s,z}\Pi_{s,z} + N^n_{s,z},\ n \geqslant 1.$$

Moreover, for any $s \in (-s_0, 0)$, the assertions (i), (ii), (iii) of Proposition 2.3.4 hold true.

2.4 Proof of precise large deviations for the norm cocycle

The goal of this section is to establish Theorems 2.2.1 and 2.2.3.

2.4.1 Auxiliary results

We need some preliminary statements. Following Petrov [74], under the changed measure $\mathbb{Q}^x_s$, define the Cramér series ζ_s by

$$\zeta_s(t) = \frac{\gamma_{s,3}}{6\gamma_{s,2}^{3/2}} + \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{24\gamma_{s,2}^3}t + \frac{\gamma_{s,5}\gamma_{s,2}^2 - 10\gamma_{s,4}\gamma_{s,3}\gamma_{s,2} + 15\gamma_{s,3}^3}{120\gamma_{s,2}^{9/2}}t^2 + \dots,$$

where $\gamma_{s,k} = \Lambda^{(k)}(s)$ and $\Lambda(s) = \log\kappa(s)$. The following lemma gives a full expansion of $\Lambda^*(q+l)$ in terms of power series in l in a neighborhood of 0, for $q = \Lambda'(s)$ and $s \in I^\circ_\mu \cup (-s_0, 0)$, where s_0 is from Proposition 2.3.5.

Lemma 2.4.1. *Assume conditions of Theorem 2.2.1 or Theorem 2.2.3. Let $q = \Lambda'(s)$. Then, there exists $\delta > 0$ such that, for any $|l| \leqslant \delta$,*

$$\Lambda^*(q+l) = \Lambda^*(q) + sl + h_s(l),$$

where h_s is linked to the Cramér series ζ_s by the identity

$$h_s(l) = \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s(\frac{l}{\sigma_s}). \tag{2.4.1}$$

Proof. Let $(\Lambda')^{-1}$ be the inverse function of Λ'. With the notation $l_s = (\Lambda')^{-1}(q+l) - s$, we have $\Lambda'(s+l_s) = q+l$. By the definition of Λ^*, it follows that $\Lambda^*(q+l) = (s+l_s)(q+l) - \Lambda(s+l_s)$. This, together with $\Lambda^*(q) = sq - \Lambda(s)$ and Taylor's formula, gives

$$h_s(l) := \Lambda^*(q+l) - \Lambda^*(q) - sl = l_s l - \sum_{k=2}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}l_s^k. \tag{2.4.2}$$

From $\Lambda'(s+l_s) = q+l$ and $\Lambda'(s) = q$, we deduce that $l = \Lambda'(s+l_s) - \Lambda'(s)$, so that, by Taylor's formula for $\Lambda'(s)$,

$$l = \sum_{k=1}^{\infty}\frac{\Lambda^{(k+1)}(s)}{k!}l_s^k. \tag{2.4.3}$$

The rest of the proof is similar to that in Petrov [74] (chapter VIII, section 2). For $|l|$ small enough, the equation (2.4.3) has a unique solution l_s given by

$$l_s = \frac{l}{\sigma_s^2} - \frac{\Lambda^{(3)}(s)}{2\sigma_s^6} l^2 - \frac{\Lambda^{(4)}(s)\sigma_s^2 - 3(\Lambda^{(3)}(s))^2}{6\sigma_s^{10}} l^3 + \cdots .$$

Together with (2.4.2) and (2.4.3), this implies

$$h_s(l) = \sum_{k=2}^{\infty} \Lambda^{(k)}(s) \frac{k-1}{k!} l_s^k = \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3} \zeta_s(\frac{l}{\sigma_s}).$$

□

We next provide an estimation for the eigenvalue $\lambda_{s,it}^n$ when $t = O(\sqrt{n})$, which will be used to deduce the asymptotic properties of the operator $R_{s,it}^n$ in Proposition 2.4.4. The proof is similar to that of Lemma 4.4 in [42] in the setting of Markov chains. Compared with [42], the novelty here consists in proving the uniformity with respect to s, which plays a crucial role in establishing Petrov type large deviation asymptotics; see Theorem 2.2.1.

Lemma 2.4.2. *Assume conditions of Theorem 2.2.1. Then, for any compact set $K \subset I_\mu^\circ$, there exist constants $\delta_1 = \delta_1(K) > 0$ and $c_1 = c_1(K) > 0$ such that for all $s \in K$, $t \in [-\delta_1\sqrt{n}, \delta_1\sqrt{n}]$ and $n \geqslant 1$,*

$$\left| \lambda_{s,\frac{it}{\sqrt{n}}}^n - e^{-\frac{\sigma_s^2 t^2}{2}} \right| \leqslant \frac{c_1}{\sqrt{n}} e^{-\frac{\sigma_s^2 t^2}{4}}.$$

Proof. For any $s \in K \subset I_\mu^\circ$, consider the complex-valued function

$$f_s(t) = \lambda_{s,it} - 1 + \frac{\sigma_s^2}{2} t^2, \quad t \in \mathbb{R}.$$

From Proposition 2.3.4 (iii), we infer that $f_s(0) = f_s'(0) = f_s''(0) = 0$ and $f_s'''(0) = -i\Lambda'''(s)$. In view of (2.3.6) we see that for any $s \in K \subset I_\mu^\circ$, there exist constants $\varepsilon(s) > 0$ and $\delta(s) > 0$ such that the mapping $(s,t) \mapsto f_s(t)$ is analytic on $(s - \varepsilon(s), s + \varepsilon(s)) \times (t - \delta(s), t + \delta(s))$. Hence, by Taylor's formula, we get that there exists a constant $c = c(s) > 0$ such that for all $t \in (t - \delta(s), t + \delta(s))$,

$$\sup_{s' \in (s-\varepsilon(s), s+\varepsilon(s))} |f_{s'}(t)| \leqslant c|t|^3.$$

Since $\cup_{s\in K}(s - \varepsilon(s), s + \varepsilon(s)) \supset K$, by Heine–Borel's theorem, there exists a sequence $\{s_m\}_{1\leqslant m\leqslant m_0}$ such that $\cup_{m=1}^{m_0}(s_m - \varepsilon_m, s_m + \varepsilon_m) \supset K$, where $\varepsilon_m = \varepsilon(s_m)$. Therefore, taking $\delta_1 = \delta(K) = \min_{1\leqslant m\leqslant m_0} \delta(s_m)$ and $c_1 = c_1(K) = \max_{1\leqslant m\leqslant m_0} c(s_m)$, we obtain that for all $t \in (t - \delta_1, t + \delta_1)$,

$$\sup_{s\in K} |f_s(t)| \leqslant c_1 |t|^3. \tag{2.4.4}$$

Without loss of generality, we assume that $\delta_1 \leqslant \inf_{s\in K} \frac{1}{\sigma_s}$. For $t \in [-\delta_1\sqrt{n}, \delta_1\sqrt{n}]$, we write

$$\begin{aligned} \left| \lambda_{s,\frac{it}{\sqrt{n}}}^n - e^{-\frac{\sigma_s^2 t^2}{2}} \right| &\leqslant \left| \left(1 - \frac{\sigma_s^2 t^2}{2n}\right)^n \left(\left(1 + \frac{f_s(\frac{t}{\sqrt{n}})}{1 - \frac{\sigma_s^2 t^2}{2n}}\right)^n - 1 \right) \right| + \left| \left(1 - \frac{\sigma_s^2 t^2}{2n}\right)^n - e^{-\frac{\sigma_s^2 t^2}{2}} \right| \\ &= I_1(s) + I_2(s). \end{aligned} \tag{2.4.5}$$

For $I_1(s)$, from $\delta_1 \leqslant \inf_{s\in K} \frac{1}{\sigma_s}$ and $|\frac{t}{\sqrt{n}}| \leqslant \delta_1$ we get $1 - \frac{\sigma_s^2 t^2}{2n} \geqslant 1/2$, uniformly in $s \in K$. Hence, using the basic inequality $|(1+z)^n - 1| \leqslant (1+|z|)^n - 1$ which is valid for any $z \in \mathbb{C}$, we have

$$I_1(s) \leqslant \left|1 - \frac{\sigma_s^2 t^2}{2n}\right|^n \left(\left(1 + \left|\frac{f_s(\frac{t}{\sqrt{n}})}{1 - \frac{\sigma_s^2 t^2}{2n}}\right|\right)^n - 1\right).$$

Noting that $1 + y \leqslant e^y$ holds for any $y \in \mathbb{R}$, we get $\left|1 - \frac{\sigma_s^2 t^2}{2n}\right|^n \leqslant e^{-\frac{\sigma_s^2 t^2}{2}}$. Recall that from (2.4.4) we have $\sup_{s\in K} |f_s(\frac{t}{\sqrt{n}})| \leqslant c_1(\frac{|t|}{\sqrt{n}})^3$ for any $t \in [-\delta_1\sqrt{n}, \delta_1\sqrt{n}]$. Using again the inequality $1 + y \leqslant e^y$ leads to

$$I_1(s) \leqslant e^{-\frac{\sigma_s^2 t^2}{2}} \left(e^{c_1 \frac{|t|^3}{\sqrt{n}}} - 1\right).$$

From the inequality $e^y - 1 \leqslant ye^y$, $y \in \mathbb{R}$, and the fact $|\frac{t}{\sqrt{n}}| \leqslant \delta_1$, it follows that

$$I_1(s) \leqslant e^{-\frac{\sigma_s^2 t^2}{2}} \frac{c_1}{\sqrt{n}} |t|^3 e^{c_1\delta_1 t^2} \leqslant e^{-\frac{\sigma_s^2 t^2}{2}} \frac{c_1 C_1}{\sqrt{n}} e^{2c_1\delta_1 t^2},$$

where in the last inequality we use the fact that $|t|^3 \leqslant C_1 e^{c_1\delta_1 t^2}$ for sufficiently large constant $C_1 > 0$. Choosing $0 < \delta_1 < \min\{\inf_{s\in K} \frac{1}{\sigma_s}, \inf_{s\in K} \frac{\sigma_s^2}{8c_1\delta_1}\}$, we obtain that for all $s \in K$ and $t \in [-\delta_1\sqrt{n}, \delta_1\sqrt{n}]$,

$$I_1(s) \leqslant \frac{c_1 C_1}{\sqrt{n}} e^{-\frac{\sigma_s^2 t^2}{4}}. \tag{2.4.6}$$

For $I_2(s)$, applying the inequalities $\log(1-y) \geqslant -y - y^2$, $y \in [0,1]$, and $1 - e^{-y} \leqslant y$, $y \in \mathbb{R}$, we get

$$I_2(s) = e^{-\frac{\sigma_s^2 t^2}{2}} - e^{n\log\left(1 - \frac{\sigma_s^2 t^2}{2n}\right)} \leqslant e^{-\frac{\sigma_s^2 t^2}{2}} - e^{-\frac{\sigma_s^2 t^2}{2} - \frac{\sigma_s^4 t^4}{4n}} \leqslant \frac{\sigma_s^4 t^4}{4n} e^{-\frac{\sigma_s^2 t^2}{2}} \leqslant \frac{4}{n} e^{-\frac{\sigma_s^2 t^2}{4}}, \tag{2.4.7}$$

where in the last inequality we use the inequality $\sigma_s^4 t^4 \leqslant 16 e^{\frac{\sigma_s^2 t^2}{4}}$. We conclude the proof of Lemma 2.4.2 by Combining (2.4.5), (2.4.6) and (2.4.7). □

Now let us fix a density function ρ satisfying the property that $\rho(y) \leqslant \frac{C}{y^4+1}$, $y \in \mathbb{R}$, for some constant $C > 0$. Moreover, the Fourier transform $\hat{\rho}(t) = \int_{\mathbb{R}} e^{-ity}\rho(y)dy$, $t \in \mathbb{R}$, is compactly supported in $[-1,1]$ and is differentiable at the point 0 on the real line. For the existence of such a function, see [61, 63]. For any $\varepsilon > 0$, we define a rescaled density function $\rho_\varepsilon(y) = \frac{1}{\varepsilon}\rho(\frac{y}{\varepsilon})$, $y \in \mathbb{R}$. It is easy to see that its Fourier transform $\hat{\rho}_\varepsilon(t) = \int_{\mathbb{R}} e^{-ity}\rho_\varepsilon(y)dy$, $t \in \mathbb{R}$, has a compact support in $[-\varepsilon^{-1}, \varepsilon^{-1}]$ and is differentiable in a small neighborhood of 0 on the real line.

For any non-negative integrable function ψ, following the paper [44], we introduce two modified functions related to ψ and establish some two-sided bounds. For any $\varepsilon > 0$ and $y \in \mathbb{R}$, set $\mathbb{B}_\varepsilon(y) = \{y' \in \mathbb{R} : |y' - y| \leqslant \varepsilon\}$ and

$$\psi_\varepsilon^+(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)} \psi(y') \quad \text{and} \quad \psi_\varepsilon^-(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)} \psi(y'). \tag{2.4.8}$$

Lemma 2.4.3. *Suppose that ψ is a non-negative integrable function and that ψ_ε^+ and ψ_ε^- are measurable for any $\varepsilon > 0$, then for sufficiently small ε, there exists a positive constant $C_\rho(\varepsilon)$ with $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that, for any $x \in \mathbb{R}$,*

$$\psi_\varepsilon^- * \rho_{\varepsilon^2}(x) - \int_{|y|\geqslant\varepsilon} \psi_\varepsilon^-(x-y)\rho_{\varepsilon^2}(y)dy \leqslant \psi(x) \leqslant (1+C_\rho(\varepsilon))\psi_\varepsilon^+ * \rho_{\varepsilon^2}(x).$$

The proof of the above lemma, being similar to that of Lemma 5.2 in [42], will not be detailed here.

The next assertion is the key point in establishing Theorem 2.2.1. Its proof is based on the spectral gap properties of the perturbed operator $R_{s,z}$ (see Proposition 2.3.4). Let us introduce the necessary notation. In the following, let φ be a γ-Hölder continuous function on $\mathcal{S}$. Assume that $\psi : \mathbb{R} \mapsto \mathbb{C}$ is a continuous function with compact support in $\mathbb{R}$, and moreover, ψ is differentiable in a small neighborhood of 0 on the real line. Recall that π_s is the invariant measure of the Markov chain X_n^x under the changed measure $\mathbb{Q}_s^x$, see (2.3.3).

Proposition 2.4.4. *Assume conditions of Theorem 2.2.1. Then, for any compact set $K \subset I_\mu^\circ$, there exist constants $\delta = \delta(K) > 0$, $c = c(K) > 0$, $C = C(K) > 0$ such that for all $x \in \mathcal{S}$, $s \in K$, $|l| \leqslant \frac{1}{\sqrt{n}}$, $\varphi \in \mathcal{B}_\gamma$ and $n \geqslant 1$,*

$$\begin{aligned}&\left|\sqrt{n}\,\sigma_s e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - \sqrt{2\pi}\psi(0)\pi_s(\varphi)\right| \\ &\leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma + \frac{C}{n}\|\varphi\|_\gamma \sup_{|t|\leqslant\delta}\Big(|\psi(t)| + |\psi'(t)|\Big) + Ce^{-cn}\|\varphi\|_\gamma \int_{\mathbb{R}} |\psi(t)|dt.\end{aligned}$$

Proof. We denote

$$c_s(\psi) = \frac{\sqrt{2\pi}}{\sigma_s}\psi(0)\pi_s(\varphi).$$

Taking sufficiently small $\delta > 0$, we write

$$\begin{aligned}&\left|\sqrt{n}\; e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - c_s(\psi)\right| \\ &\leqslant \left|\sqrt{n}\; e^{nh_s(l)} \int_{|t|\geqslant\delta} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt\right| \\ &\quad + \left|\sqrt{n}\; e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - c_s(\psi)\right| \\ &= I(n) + J(n). \end{aligned} \tag{2.4.9}$$

For $I(n)$, since ψ is bounded and compactly supported on the real line, taking into account Proposition 2.3.4 (ii), the fact $|e^{-itln}| = 1$ and equality (2.4.1), we get

$$\sup_{s\in K}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} |I(n)| \leqslant C_\delta e^{-c_\delta n}\|\varphi\|_\gamma \int_{|t|\geqslant\delta} |\psi(t)|dt. \tag{2.4.10}$$

For $J(n)$, by Proposition 2.3.4 (i), we have

$$R_{s,it}^n(\varphi)(x) = \lambda_{s,it}^n \Pi_{s,it}(\varphi)(x) + N_{s,it}^n(\varphi)(x).$$

Set for brevity

$$\psi_{s,x}(t) = \Pi_{s,it}(\varphi)(x)\psi(t). \tag{2.4.11}$$

It follows that

$$\begin{aligned} J(n) \leqslant & \left|\sqrt{n}\, e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} \lambda_{s,it}^n \psi_{s,x}(t)dt - c_s(\psi)\right| \\ & + \left|\sqrt{n}\, e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} N_{s,it}^n(\varphi)(x)\psi(t)dt\right| \\ = & \; J_1(n) + J_2(n). \end{aligned} \tag{2.4.12}$$

For the second term $J_2(n)$, applying Proposition 2.3.4 (i), we get that there exist constants $c_\delta = c_\delta(K) > 0$ and $\varkappa = \varkappa(K) \in (0,1)$ such that

$$\sup_{s\in K}\sup_{x\in\mathcal{S}}\sup_{|t|<\delta} |N_{s,it}^n(\varphi)(x)| \leqslant \sup_{s\in K}\sup_{|t|<\delta} \|N_{s,it}^n\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma}\|\varphi\|_\gamma \leqslant c_\delta \varkappa^n \|\varphi\|_\gamma.$$

Together with the fact $|e^{-itln}| = 1$, this implies that uniformly in $s \in K$, $|l| \leqslant \frac{1}{\sqrt{n}}$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$J_2(n) \leqslant C_\delta e^{-c_\delta n}\|\varphi\|_\gamma \int_{|t|<\delta} |\psi(t)|dt. \tag{2.4.13}$$

For the first term $J_1(n)$, we make a change of variable $t = t'/\sqrt{n}$ to get

$$J_1(n) = \left|e^{nh_s(l)} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-itl\sqrt{n}} \lambda_{s,\frac{it}{\sqrt{n}}}^n \psi_{s,x}(t/\sqrt{n})dt - c_s(\psi)\right|.$$

It follows that

$$\begin{aligned} J_1(n) \leqslant & \left|e^{nh_s(l)} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-itl\sqrt{n}} \lambda_{s,\frac{it}{\sqrt{n}}}^n \left[\psi_{s,x}(t/\sqrt{n}) - \psi_{s,x}(0)\right] dt\right| \\ & + \left|\psi_{s,x}(0) e^{nh_s(l)} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-itl\sqrt{n}} \lambda_{s,\frac{it}{\sqrt{n}}}^n dt - c_s(\psi)\right| \\ = & J_{11}(n) + J_{12}(n). \end{aligned} \tag{2.4.14}$$

By Lemma 2.4.2, for any compact set $K \subset I_\mu^\circ$, there exist constants $\delta = \delta(K) > 0$ and $c = c(K) > 0$ such that for all $s \in K$, $t \in [-\delta\sqrt{n}, \delta\sqrt{n}]$ and $n \geqslant 1$,

$$\left|\lambda_{s,\frac{it}{\sqrt{n}}}^n - e^{-\frac{\sigma_s^2 t^2}{2}}\right| \leqslant \frac{c}{\sqrt{n}} e^{-\frac{\sigma_s^2 t^2}{4}}. \tag{2.4.15}$$

We now give a bound for $J_{11}(n)$ in (2.4.14). From (2.3.7) we get that there exists a constant $c = c(K) > 0$ such that for all $s \in K$, $x \in \mathcal{S}$ and $t \in [-\delta\sqrt{n}, \delta\sqrt{n}]$,

$$\left|\Pi_{s,\frac{it}{\sqrt{n}}}(\varphi)(x) - \Pi_{s,0}(\varphi)(x)\right| \leqslant \|\Pi_{s,\frac{it}{\sqrt{n}}} - \Pi_{s,0}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma}\|\varphi\|_\gamma \leqslant c\frac{|t|}{\sqrt{n}}\|\varphi\|_\gamma.$$

Combining this with (2.4.11), and the fact that the function ψ is differentiable in a small neighborhood of 0, we obtain that there exists a constant $c = c(K) > 0$ such

that for all $s \in K$, $x \in \mathcal{S}$ and $t \in [-\delta\sqrt{n}, \delta\sqrt{n}]$,

$$\begin{aligned}
\Big|\psi_{s,x}\Big(\frac{t}{\sqrt{n}}\Big) - \psi_{s,x}(0)\Big| &= \Big|\Pi_{s,\frac{it}{\sqrt{n}}}(\varphi)(x)\psi\Big(\frac{t}{\sqrt{n}}\Big) - \Pi_{s,0}(\varphi)(x)\psi(0)\Big| \\
&\leqslant \Big|\Pi_{s,\frac{it}{\sqrt{n}}}(\varphi)(x)\psi\Big(\frac{t}{\sqrt{n}}\Big) - \Pi_{s,0}(\varphi)(x)\psi\Big(\frac{t}{\sqrt{n}}\Big)\Big| \\
&\quad + \Big|\Pi_{s,0}(\varphi)(x)\psi\Big(\frac{t}{\sqrt{n}}\Big) - \Pi_{s,0}(\varphi)(x)\psi(0)\Big| \\
&\leqslant c\frac{|t|}{\sqrt{n}}\|\varphi\|_\gamma \sup_{|t|\leqslant\delta}|\psi(t)| + c\frac{|t|}{\sqrt{n}}\|\varphi\|_\infty \sup_{|t|\leqslant\delta}|\psi'(t)| \\
&\leqslant c\frac{|t|}{\sqrt{n}}\|\varphi\|_\gamma \sup_{|t|\leqslant\delta}\Big(|\psi(t)| + |\psi'(t)|\Big).
\end{aligned}$$

Taking into account (2.4.15) and the fact that $e^{nh_s(l)} \leqslant c$ uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$, we get the desired bound for $J_{11}(n)$: there exists a constant $c = c(K) > 0$ such that for all $|l| \leqslant \frac{1}{\sqrt{n}}$, $s \in K$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned}
J_{11}(n) &\leqslant c\|\varphi\|_\gamma \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \frac{|t|}{\sqrt{n}} e^{-\frac{\sigma_s^2 t^2}{2}} dt + c\|\varphi\|_\gamma \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \frac{|t|}{n} e^{-\frac{\sigma_s^2 t^2}{4}} dt \sup_{|t|\leqslant\delta}\Big(|\psi(t)| + |\psi'(t)|\Big) \\
&= \|\varphi\|_\gamma O(\frac{1}{\sqrt{n}}) + \|\varphi\|_\gamma \sup_{|t|\leqslant\delta}\Big(|\psi(t)| + |\psi'(t)|\Big) O(\frac{1}{n}).
\end{aligned} \tag{2.4.16}$$

To estimate $J_{12}(n)$ in (2.4.14), we first write

$$\begin{aligned}
J_{12}(n) &= \pi_s(\varphi)\psi(0)\left|e^{nh_s(l)} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-itl\sqrt{n}} \lambda_{s,\frac{it}{\sqrt{n}}}^n dt - \frac{\sqrt{2\pi}}{\sigma_s}\right| \\
&\leqslant \pi_s(\varphi)\psi(0)\left|\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{\frac{nl^2}{2\sigma_s^2}} e^{-itl\sqrt{n}} \left(\lambda_{s,\frac{it}{\sqrt{n}}}^n - e^{-\frac{\sigma_s^2 t^2}{2}}\right) dt\right| \\
&\quad + \pi_s(\varphi)\psi(0)\left|\int_{\mathbb{R}} e^{\frac{nl^2}{2\sigma_s^2}} e^{-itl\sqrt{n}} e^{-\frac{\sigma_s^2 t^2}{2}} dt - \int_{\mathbb{R}} e^{-\frac{\sigma_s^2 t^2}{2}} dt\right| \\
&\quad + \pi_s(\varphi)\psi(0)\left|\int_{|t|\geqslant\delta\sqrt{n}} e^{\frac{nl^2}{2\sigma_s^2}} e^{-itl\sqrt{n}} e^{-\frac{\sigma_s^2 t^2}{2}} dt\right| \\
&= J_{121}(n) + J_{122}(n) + J_{123}(n).
\end{aligned} \tag{2.4.17}$$

For $J_{121}(n)$, using (2.4.15) we get that there exists a constant $c_1 = c_1(K) > 0$ such that $J_{121}(n) \leqslant \frac{c_1}{\sqrt{n}}\|\varphi\|_\infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$, $s \in K$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. It is easy to see that $J_{122}(n) = 0$. For $J_{123}(n)$, using the inequality $\int_y^\infty e^{-\frac{t^2}{2}} dt \leqslant \frac{1}{y} e^{-\frac{y^2}{2}}$ for $y > 0$, we get that $J_{123}(n) \leqslant e^{-cn}\|\varphi\|_\infty$. Combining these bounds yield that there exists a constant $c_1 = c_1(K)$ such that for all $|l| \leqslant \frac{1}{\sqrt{n}}$, $s \in K$ and $\varphi \in \mathcal{B}_\gamma$,

$$J_{12}(n) \leqslant \frac{c_1}{\sqrt{n}}\|\varphi\|_\infty$$

This, together with (2.4.16) and (2.4.14), implies the desired bound for $J_1(n)$: uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$, $s \in K$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$J_1(n) = \|\varphi\|_\gamma O(\frac{1}{\sqrt{n}}) + \|\varphi\|_\gamma \sup_{|t|\leqslant\delta}\Big(|\psi(t)| + |\psi'(t)|\Big) O(\frac{1}{n}).$$

Combining this with (2.4.13) and (2.4.10), we conclude the proof of Proposition 2.4.4. □

Assume that the functions φ and ψ satisfy the same properties as in Proposition 2.4.4. The following result, for $s < 0$ small enough, will be used to prove Theorem 2.2.3.

Proposition 2.4.5. *Assume conditions of Theorem 2.2.3. Then, there exists $s_0 > 0$ such that for any compact set $K \subset (-s_0, 0)$, the following assertion holds: there exist constants $\delta = \delta(K) > 0$, $c = c(K) > 0$, $C = C(K) > 0$ such that for all $x \in \mathcal{S}$, $s \in K$, $|l| \leqslant \frac{1}{\sqrt{n}}$, $\varphi \in \mathcal{B}_\gamma$ and $n \geqslant 1$,*

$$\begin{aligned}&\left|\sqrt{n}\,\sigma_s e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R^n_{s,it}(\varphi)(x)\psi(t)dt - \sqrt{2\pi}\psi(0)\pi_s(\varphi)\right| \\ &\leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma + \frac{C}{n}\|\varphi\|_\gamma \sup_{|t|\leqslant\delta}\Big(|\psi(t)| + |\psi'(t)|\Big) + Ce^{-cn}\|\varphi\|_\gamma \int_{\mathbb{R}} |\psi(t)|dt.\end{aligned}$$

Proof. Using Propositions 2.3.2 and 2.3.5, the proof of Proposition 2.4.5 can be carried out as the proof of Proposition 2.4.4. We omit the details. □

2.4.2 Proof of Theorem 2.2.1

Recall that $q = \Lambda'(s)$, $\Lambda^*(q+l) = \Lambda^*(q) + sl + h_s(l)$, $x \in \mathcal{S}$, and $|l| \leqslant \frac{1}{\sqrt{n}} \to 0$, as $n \to \infty$. Let $K \subset I^\circ_\mu$ be a compact set. Taking into account that $e^{n\Lambda^*(q)} = e^{sqn}/\kappa^n(s)$ and using the change of measure (2.3.2), we write

$$\begin{aligned}A_n(x,l) &:= \sqrt{2\pi n}\; s\sigma_s e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{P}(\log|G_n x| \geqslant n(q+l)) \\ &= \sqrt{2\pi n}\; s\sigma_s e^{nsl} e^{nh_s(l)} e^{sqn} \mathbb{E}_{\mathbb{Q}^x_s}\left(\frac{1}{r_s(X^x_n)} e^{-s\log|G_n x|} \mathbb{1}_{\{\log|G_n x| \geqslant n(q+l)\}}\right).\end{aligned} \tag{2.4.18}$$

Setting $T^x_n = \log|G_n x| - nq$ and $\psi_s(y) = e^{-sy}\mathbb{1}_{\{y\geqslant 0\}}$, from (2.4.18) we get

$$A_n(x,l) = \sqrt{2\pi n}\; s\sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}^x_s}\left(\frac{1}{r_s(X^x_n)}\psi_s(T^x_n - nl)\right). \tag{2.4.19}$$

Upper bound. Let $\varepsilon \in (0,1)$ and $\psi^+_{s,\varepsilon}(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)} \psi_s(y')$ be defined as in (2.4.8), i.e.,

$$\psi^+_{s,\varepsilon}(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)} \psi_s(y') = \mathbb{1}_{\{-\varepsilon\leqslant y<\varepsilon\}} + e^{-s(y-\varepsilon)}\mathbb{1}_{\{y\geqslant\varepsilon\}}. \tag{2.4.20}$$

Using Lemma 2.4.3, we get that there exists a constant $C_\rho(\varepsilon) > 0$ depending on the density function ρ and the constant $\varepsilon > 0$ such that for all $s \in K$,

$$\begin{aligned}A_n(x,l) &\leqslant (1 + C_\rho(\varepsilon))\sqrt{2\pi n}\; s\sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}^x_s}\left[\frac{1}{r_s(X^x_n)}(\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2})(T^x_n - nl)\right] \\ &=: B^+_n(x,l).\end{aligned} \tag{2.4.21}$$

Denote by $\widehat{\psi}^+_{s,\varepsilon}$ the Fourier transform of $\psi^+_{s,\varepsilon}$. From (2.4.20), elementary calculations yield that

$$\widehat{\psi}^+_{s,\varepsilon}(t) = \int_{\mathbb{R}} e^{-ity}\psi^+_{s,\varepsilon}(y)dy = 2\frac{\sin(\varepsilon t)}{t} + e^{-i\varepsilon t}\frac{1}{s+it}, \quad t \in \mathbb{R}, \tag{2.4.22}$$

and for all $s \in K$,

$$\sup_{t\in\mathbb{R}} |\widehat{\psi}^+_{s,\varepsilon}(t)| \leqslant \widehat{\psi}^+_{s,\varepsilon}(0) = \int_{-\varepsilon}^{\varepsilon} dy + \int_{\varepsilon}^{+\infty} e^{-s(y-\varepsilon)}dy = \frac{1+2s\varepsilon}{s}. \tag{2.4.23}$$

By the inversion formula, for any $y \in \mathbb{R}$,

$$\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2}(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ity}\widehat{\psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt.$$

Substituting $y = T^x_n - nl$, taking expectation with respect to $\mathbb{E}_{\mathbb{Q}^x_s}$, and using Fubini's theorem, we get

$$\mathbb{E}_{\mathbb{Q}^x_s}\left[\frac{1}{r_s(X^x_n)}(\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2})(T^x_n - nl)\right] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itln}R^n_{s,it}(r_s^{-1})(x)\widehat{\psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \tag{2.4.24}$$

where

$$R^n_{s,it}(r_s^{-1})(x) = \mathbb{E}_{\mathbb{Q}^x_s}\left[e^{itT^x_n}\frac{1}{r_s(X^x_n)}\right].$$

Note that $\widehat{\psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ is compactly supported in $\mathbb{R}$ since the function $\widehat{\rho}_{\varepsilon^2}$ has a compact support. Note that $\widehat{\rho}_{\varepsilon^2}$ is differentiable in a small neighborhood of 0 on the real line. From (2.4.22), one can verify that for any $s \in K$, the function $\widehat{\psi}^+_{s,\varepsilon}$ is also differentiable in a small neighborhood of 0 on the real line. Using Proposition 2.4.4 with $\varphi = r_s^{-1}$ and $\psi = \widehat{\psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$, it follows that

$$\lim_{n\to\infty}\sup_{s\in K}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}}\left|B^+_n(x,l) - (1+C_\rho(\varepsilon))\pi_s(r_s^{-1})s\widehat{\psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\right| = 0. \tag{2.4.25}$$

Since $\widehat{\rho}_{\varepsilon^2}(0) = 1$, from (2.4.19)-(2.4.25), we have that uniformly in $s \in K$,

$$\begin{aligned}\limsup_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} A_n(x,l) &\leqslant (1+C_\rho(\varepsilon))s\pi_s(r_s^{-1})\widehat{\psi}^+_{s,\varepsilon}(0)\widehat{\rho}_\varepsilon(0)\\ &\leqslant (1+C_\rho(\varepsilon))(1+2s\varepsilon)\pi_s(r_s^{-1}).\end{aligned}$$

Letting $\varepsilon \to 0$ and noting that $C_\rho(\varepsilon) \to 0$, we obtain the desired upper bound: uniformly in $s \in K$,

$$\limsup_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} A_n(x,l) \leqslant \pi_s(r_s^{-1}) = \frac{1}{\nu_s(r_s)}. \tag{2.4.26}$$

Lower bound. For $\varepsilon \in (0,1)$, let $\psi^-_{s,\varepsilon}(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)}\psi_s(y')$ be defined as in (2.4.8), i.e.,

$$\psi^-_{s,\varepsilon}(y) = e^{-s(y+\varepsilon)}\mathbb{1}_{\{y\geqslant\varepsilon\}}, \quad y \in \mathbb{R}.$$

Its Fourier transform is given by

$$\widehat{\psi}^-_{s,\varepsilon}(t) = \int_{\mathbb{R}} e^{-ity}\psi^-_{s,\varepsilon}(y)dy = e^{-2\varepsilon s}\frac{e^{-i\varepsilon t}}{s+it}, \quad t \in \mathbb{R}.$$

From (2.4.19) and Lemma 2.4.3, we get

$$\begin{aligned} A_n(x,l) &\geqslant \sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}^x_s}\left[\frac{1}{r_s(X^x_n)}(\psi^-_{s,\varepsilon}*\rho_{\varepsilon^2})(T^x_n - nl)\right] \\ &\quad - \sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)}\int_{|y|\geqslant\varepsilon}\mathbb{E}_{\mathbb{Q}^x_s}\left[\frac{1}{r_s(X^x_n)}\psi^-_{s,\varepsilon}(T^x_n - nl - y)\right]\rho_{\varepsilon^2}(y)dy \\ &:= B^-_n(x,l) - C^-_n(x,l). \end{aligned} \tag{2.4.27}$$

For the first term $B^-_n(x,l)$, applying (2.4.24) with $\psi^+_{s,\varepsilon}\rho_{\varepsilon^2}$ replaced by $\psi^-_{s,\varepsilon}\rho_{\varepsilon^2}$, we get

$$B^-_n(x,l) = \sqrt{\frac{n}{2\pi}}\ s\sigma_s e^{nh_s(l)}\int_{\mathbb{R}} e^{-itln}R^n_{s,it}(r_s^{-1})(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt.$$

In the same way as for the upper bound, using $\widehat{\psi}^-_{s,\varepsilon}(0) = \frac{e^{-2s\varepsilon}}{s}$ and Proposition 2.4.4 with $\varphi = r_s^{-1}$ and $\psi = \widehat{\psi}^-_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ (one can check that the functions φ and ψ satisfy the required conditions in Proposition 2.4.4), we obtain the desired lower bound: for all $s \in K$,

$$\liminf_{n\to\infty}\inf_{x\in\mathcal{S}}\inf_{|l|\leqslant\frac{1}{\sqrt{n}}} B^-_n(x,l) \geqslant \pi_s(r_s^{-1}) = \frac{1}{\nu_s(r_s)}. \tag{2.4.28}$$

For the second term $C^-_n(x,l)$, noting that $\psi^-_{s,\varepsilon} \leqslant \psi_s$ and applying Lemma 2.4.3 to ψ_s, we get $\psi^-_{s,\varepsilon} \leqslant \psi_s \leqslant (1+C_\rho(\varepsilon))\psi^+_{s,\varepsilon}*\rho_{\varepsilon^2}$. We use the same argument as in (2.4.24) to obtain

$$\begin{aligned} C^-_n(x,l) &\leqslant (1+C_\rho(\varepsilon))\sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)} \\ &\quad\times\int_{|y|\geqslant\varepsilon}\mathbb{E}_{\mathbb{Q}^x_s}\left[\frac{1}{r_s(X^x_n)}(\psi^+_{s,\varepsilon}*\rho_{\varepsilon^2})(T^x_n - nl - y)\right]\rho_{\varepsilon^2}(y)dy \\ &= (1+C_\rho(\varepsilon))\sqrt{\frac{n}{2\pi}}\ s\sigma_s e^{nh_s(l)} \\ &\quad\times\int_{|y|\geqslant\varepsilon}\left(\int_{\mathbb{R}} e^{-it(ln+y)}R^n_{s,it}(r_s^{-1})(x)\widehat{\psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt\right)\rho_{\varepsilon^2}(y)dy. \end{aligned} \tag{2.4.29}$$

To obtain a precise asymptotic for the above integral, we shall apply the Lebesgue dominated convergence theorem to pass the limit $n\to\infty$ through the integral $\int_{|y|\geqslant\varepsilon}$. The applicability of this theorem is justified below. We split the integral $\int_{|y|\geqslant\varepsilon}$ on the right hand side of (2.4.29) into two parts: $\int_{\sqrt{n}\geqslant|y|\geqslant\varepsilon}$ and $\int_{|y|>\sqrt{n}}$. For the first part, from Lemma 2.4.1, it holds uniformly in $|y|\leqslant\sqrt{n}$ and $|l|\leqslant\frac{1}{\sqrt{n}}$ that $e^{nh_s(l)-nh_s(l+\frac{y}{n})}\to 1$ as $n\to\infty$. Hence, by Proposition 2.4.4, the function on the right hand side of (2.4.29) under the integral $\int_{\sqrt{n}\geqslant|y|\geqslant\varepsilon}$ is dominated by $C\rho_{\varepsilon^2}$, which is integrable on $\mathbb{R}$. For the second part $\int_{|y|>\sqrt{n}}$, since the density function has polynomial decay, i.e. $\rho_{\varepsilon^2}(y)\leqslant\frac{C_\varepsilon}{1+y^4}$, $|y|>\sqrt{n}$, we get that $\sqrt{n}e^{nh_s(l)}\rho_{\varepsilon^2}(y)\leqslant\frac{C_\varepsilon}{1+|y|^3}$, which is integrable on $\mathbb{R}$. Consequently, we can interchange the limit as $n\to\infty$ and the integral $\int_{|y|\geqslant\varepsilon}$, and then use Proposition

2.4.4 again with $\varphi = r_s^{-1}$, $\psi = \widehat{\psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ to obtain that uniformly in $s \in K$, $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}\limsup_{n\to\infty} C_n^-(x,l) &\leqslant (1+C_\rho(\varepsilon))s\pi_s(r_s^{-1})\widehat{\psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\int_{|y|\geqslant\varepsilon}\rho_{\varepsilon^2}(y)dy\\ &= (1+C_\rho(\varepsilon))\pi_s(r_s^{-1})(1+2s\varepsilon)\int_{|y|\geqslant\frac{1}{\varepsilon}}\rho(y)dy \to 0, \quad \text{as } \varepsilon\to 0,\end{aligned}$$

since ρ is integrable on $\mathbb{R}$. This, together with (2.4.27)-(2.4.28), implies the lower bound: uniformly in $s \in K$,

$$\liminf_{n\to\infty}\inf_{x\in\mathcal{S}}\inf_{|l|\leqslant\frac{1}{\sqrt{n}}} A_n(x,l) \geqslant \pi_s(r_s^{-1}) = \frac{1}{\nu_s(r_s)}, \tag{2.4.30}$$

as required. We conclude the proof of Theorem 2.2.1 by combining (2.4.26) and (2.4.30).

2.4.3 Proof of Theorem 2.2.3

Since the change of measure formula can be extended for small $s < 0$, under the conditions of Theorem 2.2.3, we have, in the same way as in (2.4.18),

$$\begin{aligned}&- s\sigma_s\sqrt{2\pi n}\, e^{n\Lambda^*(q+l)}\frac{1}{r_s(x)}\mathbb{P}(\log|G_nx| \leqslant n(q+l))\\ =&- s\sigma_s\sqrt{2\pi n}\, e^{nsl}e^{nh_s(l)}e^{sqn}\mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{1}{r_s(X_n^x)}e^{-s\log|G_nx|}\mathbb{1}_{\{\log|G_nx|\leqslant n(q+l)\}}\Big).\end{aligned}$$

Applying Proposition 2.4.5, we can follow the proof of Theorem 2.2.1 to show Theorem 2.2.3. We omit the details.

2.5 Proof of precise large deviations with target functions

We first establish the following assertion which will be used to prove Theorem 2.2.2, but which is of independent interest. Let ψ be a measurable function on $\mathbb{R}$ and $\varepsilon > 0$. Denote, for brevity, $\psi_s(y) = e^{-sy}\psi(y)$ and

$$\psi^+_{s,\varepsilon}(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)}\psi_s(y'), \quad \psi^-_{s,\varepsilon}(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)}\psi_s(y').$$

Introduce the following condition: for any $s \in I_\mu^\circ$ and $\varepsilon > 0$, the functions $y \mapsto \psi^+_{s,\varepsilon}(y)$ and $y \mapsto \psi^-_{s,\varepsilon}(y)$ are measurable and

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\psi^+_{s,\varepsilon}(y)dy = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\psi^-_{s,\varepsilon}(y)dy = \int_{\mathbb{R}} e^{-sy}\psi(y)dy < +\infty. \tag{2.5.1}$$

Theorem 2.5.1. *Suppose the assumptions of Theorem 2.2.1 hold true. Let $K \subset I_\mu^\circ$ be a compact set. Assume that φ is a Hölder continuous function on $\mathcal{S}$ and that ψ is a measurable function on $\mathbb{R}$ satisfying condition (2.5.1). Assume also that $y \mapsto e^{-sy}\psi(y)$*

is directly Riemann integrable on $\mathbb{R}$ *for all* $s' \in (s-\eta, s+\eta)$ *with* $\eta > 0$ *sufficiently small, Then, we have, as* $n \to \infty$*, uniformly in* $s \in K$*,* $x \in \mathcal{S}$ *and* $|l| \leqslant \frac{1}{\sqrt{n}}$*,*

$$\sqrt{2\pi n}\,\sigma_s e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] \\ = \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + o(1). \tag{2.5.2}$$

Before proceeding with the proof of this theorem, let us give some examples of functions satisfying condition (2.5.1). It is easy to see that (2.5.1) holds for increasing non-negative functions ψ satisfying $\int_{\mathbb{R}} e^{-sy}\psi(y)dy < +\infty$, in particular, for the indicator function $\psi(y) = \mathbb{1}_{\{y \geqslant c\}}$, $y \in \mathbb{R}$, where $c \in \mathbb{R}$ is a fixed constant. Another example for which (2.5.1) holds true is when ψ is non-negative, continuous and there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} e^{-sy}\psi_\varepsilon^+(y)dy < +\infty, \tag{2.5.3}$$

where the function $\psi_\varepsilon^+(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \psi(y')$ is assumed to be measurable.

Proof of Theorem 2.5.1. Without loss of generality, we assume that both φ and ψ are non-negative (otherwise, we decompose the functions $\varphi = \varphi^+ - \varphi^-$ and $\psi = \psi^+ - \psi^-$). Let $T_n^x = \log|G_n x| - nq$. Since $e^{n\Lambda^*(q)} = e^{sqn}/\kappa^n(s)$, using the change of measure (2.3.2), we have

$$\begin{aligned} A_n(x,l) &:= \sqrt{2\pi n}\ \sigma_s e^{n\Lambda^*(q+l)}\frac{1}{r_s(x)}\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] \\ &= \sqrt{2\pi n}\ \sigma_s e^{nsl}e^{nh_s(l)}e^{sqn}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-s\log|G_n x|}\psi(T_n^x - nl)\right] \\ &= \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-s(T_n^x - nl)}\psi(T_n^x - nl)\right]. \end{aligned}$$

For brevity, set

$$\Phi_s(x) = \left(\varphi r_s^{-1}\right)(x), \quad x \in \mathcal{S}, \quad \text{and} \quad \Psi_s(y) = e^{-sy}\psi(y), \quad y \in \mathbb{R}.$$

Then,

$$A_n(x,l) = \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}_s^x}\left[\Phi_s(X_n^x)\Psi_s(T_n^x - nl)\right]. \tag{2.5.4}$$

Upper bound. We wish to write the expectation in (2.5.4) as an integral of the Fourier transform of Ψ_s, which, however, may not belong to the space $L^1(\mathbb{R})$. As in the proof of Theorem 2.2.1 (see Section 2.4.2), we make use of the convolution technique to overcome this difficulty. Applying Lemma 2.4.3 to Ψ_s, one has, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} A_n(x,l) &\leqslant (1 + C_\rho(\varepsilon))\sqrt{2\pi n}\ \sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}_s^x}\left[\Phi_s(X_n^x)(\Psi_{s,\varepsilon}^+ * \rho_{\varepsilon^2})(T_n^x - nl)\right] \\ &:= B_n(x,l), \end{aligned} \tag{2.5.5}$$

where $\Psi_{s,\varepsilon}^+(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \Psi_s(y')$, $y \in \mathbb{R}$. Using the same arguments as for deducing (2.4.24), we have

$$B_n(x,l) = (1 + C_\rho(\varepsilon))\frac{\sigma_s}{\sqrt{2\pi}}\sqrt{n}\ e^{nh_s(l)}\int_{\mathbb{R}} e^{-itln}R_{s,it}^n\Phi_s(x)\widehat{\Psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \tag{2.5.6}$$

where $R^n_{s,it}\Phi_s(x) = \mathbb{E}_{\mathbb{Q}^x_s}\left[e^{itT^x_n}\Phi_s(X^x_n)\right]$ and $\widehat{\Psi}^+_{s,\varepsilon}$ is the Fourier transform of $\Psi^+_{s,\varepsilon}$. Note that Φ_s is strictly positive and γ-Hölder continuous function on $\mathcal{S}$, and $\widehat{\Psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ has a compact support in $\mathbb{R}$. Applying Proposition 2.4.4 with $\varphi = \Phi_s$ and $\psi = \widehat{\Psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ (one can verify that the functions φ and ψ satisfy the required conditions in Proposition 2.4.4), we obtain, as $n \to \infty$, uniformly in $s \in K$, $x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\gamma$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} B_n(x,l) &\leqslant (1+C_\rho(\varepsilon))\pi_s(\Phi_s)\widehat{\Psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0) + \frac{C}{\sqrt{n}}\|\Phi_s\|_\gamma \\ &\quad + \frac{C}{n}\|\Phi_s\|_\gamma \sup_{|t|\leqslant\delta}\left(|\widehat{\Psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)| + |(\widehat{\Psi}^+_{s,\varepsilon})'(t)\widehat{\rho}_{\varepsilon^2}(t) + \widehat{\Psi}^+_{s,\varepsilon}(t)(\widehat{\rho}_{\varepsilon^2})'(t)|\right) \\ &\quad + Ce^{-cn}\|\Phi_s\|_\gamma \int_{\mathbb{R}} |\widehat{\Psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)|dt. \end{aligned} \tag{2.5.7}$$

Note that the function $\widehat{\rho}_{\varepsilon^2}$ is compactly supported in $\mathbb{R}$ and differentiable at the point 0 on the real line, hence $\widehat{\rho}_{\varepsilon^2}$ and $(\widehat{\rho}_{\varepsilon^2})'$ are uniformly bounded in the interval $(-\delta,\delta)$. Since

$$\widehat{\Psi}^+_{s,\varepsilon}(t) = \int_{\mathbb{R}} e^{-ity} \sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi(y')dy, \quad t \in \mathbb{R},$$

under the condition that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable on $\mathbb{R}$, for all $s' \in (s-\eta, s+\eta)$ with $\eta > 0$ sufficiently small, we can verify that the last two terms on the right-hand side of (2.5.7) converge to 0 as $n \to \infty$, uniformly in $s \in K$. Thus, noting that $\widehat{\Psi}^+_{s,\varepsilon}(0) = \int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi(y')dy$ and $\widehat{\rho}_{\varepsilon^2}(0) = 1$, we obtain that uniformly in $s \in K$,

$$\lim_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} B_n(x,l) \leqslant (1+C_\rho(\varepsilon))\pi_s(\Phi_s)\int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)}\left\{e^{-sy'}\psi(y')\right\}dy. \tag{2.5.8}$$

From (2.5.5) and (2.5.8), letting ε go to 0, using the condition (2.5.1) and the fact that $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, we get the desired upper bound: uniformly in $s \in K$,

$$\lim_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} A_n(x,l) \leqslant \pi_s(\Phi_s)\int_{\mathbb{R}} e^{-sy}\psi(y)dy. \tag{2.5.9}$$

Lower bound. Denote $\Psi^-_{s,\varepsilon}(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)}\Psi_s(y')$. From (2.5.4), using Lemma 2.4.3, we get

$$\begin{aligned} A_n(x,l) &\geqslant \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}^x_s}\left[\Phi_s(X^x_n)(\Psi^-_{s,\varepsilon}*\rho_{\varepsilon^2})(T^x_n - nl)\right] \\ &\quad - \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)}\int_{|y|\geqslant\varepsilon}\mathbb{E}_{\mathbb{Q}^x_s}\left[\Phi_s(X^x_n)\Psi^-_{s,\varepsilon}(T^x_n - nl - y)\right]\rho_{\varepsilon^2}(y)dy \\ &:= B^-_n(x,l) - C^-_n(x,l). \end{aligned} \tag{2.5.10}$$

For $B^-_n(x,l)$, we proceed as for (2.5.5) and (2.5.6), with $\Psi^+_{s,\varepsilon}$ replaced by $\Psi^-_{s,\varepsilon}$. Using Proposition 2.4.4, with $\varphi = \Phi_s$ and $\psi = \widehat{\Psi}^-_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$, and the fact that $\widehat{\rho}_{\varepsilon^2}(0) = 1$ and $\widehat{\Psi}^-_{s,\varepsilon}(0) = \int_{\mathbb{R}}\inf_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi(y')dy$, in an analogous way as in (2.5.9), we obtain that uniformly in $s \in K$,

$$\begin{aligned} &\lim_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} B^-_n(x,l) \\ &= \pi_s(\Phi_s)\int_{\mathbb{R}}\inf_{y\in\mathbb{B}_\varepsilon(z)} e^{-sy}\psi(y)dz \to \pi_s(\Phi_s)\int_{\mathbb{R}} e^{-sy}\psi(y)dy, \text{ as } \varepsilon \to 0, \end{aligned} \tag{2.5.11}$$

where the last convergence is due to the condition (2.5.1). For $C_n^-(x,l)$, noting that $\Psi_{s,\varepsilon}^- \leqslant \Psi_s$, applying Lemma 2.4.3 to Ψ_s we get $\Psi_{s,\varepsilon}^- \leqslant (1+C_\rho(\varepsilon))\widehat{\Psi}_{s,\varepsilon}^+\widehat{\rho}_{\varepsilon^2}$. Similarly to (2.5.6), we show that

$$C_n^-(x,l) \leqslant (1+C_\rho(\varepsilon))\sqrt{\frac{n}{2\pi}}\,\sigma_s e^{nh_s(l)} \\ \times \int_{|y|\geqslant\varepsilon}\left(\int_{\mathbb{R}} e^{-it(ln+y)}R_{s,it}^n(\Phi_s)(x)\widehat{\Psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt\right)\rho_{\varepsilon^2}(y)dy.$$

We want to use the Lebesgue dominated convergence theorem to pass the limit $n\to\infty$ through the integral $\int_{|y|\geqslant\varepsilon}$. One can justify the applicability of this theorem by following the strategy for estimating the integral in (2.4.29). Hence, applying Proposition 2.4.4 with $\varphi=\Phi_s$ and $\psi=\widehat{\Psi}_{s,\varepsilon}^+\widehat{\rho}_{\varepsilon^2}$, it follows from the Lebesgue dominated convergence theorem that uniformly in $s\in K$,

$$\limsup_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant\frac{1}{\sqrt{n}}} C_n^-(x,l) \leqslant (1+C_\rho(\varepsilon))\pi_s(\Phi_s)\widehat{\Psi}_{s,\varepsilon}^+(0)\widehat{\rho}_{\varepsilon^2}(0)\int_{|y|\geqslant\varepsilon}\rho_{\varepsilon^2}(y)dy \to 0$$

as $\varepsilon\to 0$. Combining this with (2.5.10)-(2.5.11), we get the desired lower bound: uniformly in $s\in K$,

$$\liminf_{n\to\infty}\inf_{x\in\mathcal{S}}\inf_{|l|\leqslant\frac{1}{\sqrt{n}}} A_n(x,l) \geqslant \pi_s(\Phi_s)\int_{\mathbb{R}} e^{-sy}\psi(y)dy. \tag{2.5.12}$$

Putting together (2.5.9) and (2.5.12), and noting that $\pi_s(\Phi_s)=\pi_s(\varphi r_s^{-1})=\frac{\nu_s(\varphi)}{\nu_s(r_s)}$, the result follows. □

In the sequel, we deduce Theorem 2.2.2 from Theorem 2.5.1 using approximation techniques.

Proof of Theorem 2.2.2. Without loss of generality, we assume that $\varphi\geqslant 0$ and $\psi\geqslant 0$. Let $\Psi_s(y)=e^{-sy}\psi(y)$, $y\in\mathbb{R}$. We construct two step functions as follows: for any $\eta\in(0,1)$, $m\in\mathbb{Z}$ and $y\in[m\eta,(m+1)\eta)$, set

$$\Psi_{s,\eta}^+(y)=\sup_{y\in[m\eta,(m+1)\eta)}\Psi_s(y) \quad\text{and}\quad \Psi_{s,\eta}^-(y)=\inf_{y\in[m\eta,(m+1)\eta)}\Psi_s(y).$$

By the definition of the direct Riemann integrability, the following two limits exist and are equal:

$$\lim_{\eta\to 0^+}\int_{\mathbb{R}}\Psi_{s,\eta}^+(y)dy=\lim_{\eta\to 0^+}\int_{\mathbb{R}}\Psi_{s,\eta}^-(y)dy. \tag{2.5.13}$$

Since Ψ_s is directly Riemann integrable, we have $M:=\sup_{y\in\mathbb{R}}\Psi_s(y)<+\infty$. Let $\varepsilon\in(0,M\eta)$ be fixed. Denote $I_m=[(m-1)\eta,m\eta)$, $I_m^-=\left(m\eta-\frac{\varepsilon}{M4^{|m|}},m\eta\right)$, and $I_m^+=\left[m\eta,m\eta+\frac{\varepsilon}{M4^{|m|}}\right)$, $m\in\mathbb{Z}$. Set $k_m^+:=M4^{|m|}\frac{\Psi_{s,\eta}^+(m\eta)-\Psi_{s,\eta}^+((m-1)\eta)}{\varepsilon}$, $m\in\mathbb{Z}$. For the step function $\Psi_{s,\eta}^+$, in the neighborhood of every possible discontinuous point $m\eta$, $m\in\mathbb{Z}$, if $\Psi_{s,\eta}^+(m\eta)\geqslant\Psi_{s,\eta}^+((m-1)\eta)$, then for any $y\in I_m\cup I_{m+1}$, $m\in\mathbb{Z}$, we define

$$\Psi_{s,\eta,\varepsilon}^+(y)=\begin{cases}\Psi_{s,\eta}^+((m-1)\eta), & y\in I_m\setminus I_m^-\\ \Psi_{s,\eta}^+((m-1)\eta)+k_m^+\left(y-m\eta+\frac{\varepsilon}{M4^{|m|}}\right), & y\in I_m^-\\ \Psi_{s,\eta}^+(m\eta), & y\in I_{m+1}.\end{cases}$$

If $\Psi^+_{s,\eta}(m\eta) < \Psi^+_{s,\eta}((m-1)\eta)$, then we define

$$\Psi^+_{s,\eta,\varepsilon}(y) = \begin{cases} \Psi^+_{s,\eta}((m-1)\eta), & y \in I_m \\ \Psi^+_{s,\eta}((m-1)\eta) + k^+_m(y - m\eta), & y \in I^+_m \\ \Psi^+_{s,\eta}(m\eta), & y \in I_{m+1} \setminus I^+_m. \end{cases}$$

From this construction, the non-negative continuous function $\Psi^+_{s,\eta,\varepsilon}$ satisfies $\Psi^+_{s,\eta} \leqslant \Psi^+_{s,\eta,\varepsilon}$ and $\int_{\mathbb{R}}[\Psi^+_{s,\eta,\varepsilon}(y) - \Psi^+_{s,\eta}(y)]dy < \varepsilon$. Similarly, for the step function $\Psi^-_{s,\eta}$, one can construct a non-negative continuous function $\Psi^-_{s,\eta,\varepsilon}$ which satisfies $\Psi^-_{s,\eta,\varepsilon} \leqslant \Psi^-_{s,\eta}$ and $\int_{\mathbb{R}}[\Psi^-_{s,\eta}(y) - \Psi^-_{s,\eta,\varepsilon}(y)]dy < \varepsilon$. Consequently, in view of (2.5.13), we obtain that, for η small enough,

$$\int_{\mathbb{R}} |\Psi^+_{s,\eta,\varepsilon}(y) - \Psi^-_{s,\eta,\varepsilon}(y)|dy < 3\varepsilon. \tag{2.5.14}$$

For brevity, set $c_{s,l,n} = \sqrt{2\pi n}\, \sigma_s e^{n\Lambda^*(q+l)}$ and $T^x_{n,l} = \log|G_n x| - n(q+l)$. Recalling that $\Psi_s(y) = e^{-sy}\psi(y)$, we write

$$\begin{aligned} &\left| c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)\psi(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi_s(y)dy \right| \\ \leqslant\ &\left| c_{s,l,n}\mathbb{E}\left\{\varphi(X^x_n)e^{sT^x_{n,l}}\left[\Psi_s(T^x_{n,l}) - \Psi^+_{s,\eta,\varepsilon}(T^x_{n,l})\right]\right\}\right| \\ &+ \left| c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)e^{sT^x_{n,l}}\Psi^+_{s,\eta,\varepsilon}(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy \right| \\ &+ \left| r_s(x)\pi_s(\varphi r_s^{-1})\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi_s(y)dy \right| \\ =\ &J_1 + J_2 + J_3. \end{aligned} \tag{2.5.15}$$

To control J_2, we shall verify the conditions of Theorem 2.5.1. Noting that the function $y \mapsto e^{sy}\Psi^+_{s,\eta,\varepsilon}(y)$ is non-negative and continuous, it remains to check the condition (2.5.3). By the construction of $\Psi^+_{s,\eta,\varepsilon}$ one can verify that there exists a constant $\varepsilon_1 \in (0, \min\{M\eta, \eta/3\})$ such that

$$\begin{aligned} \int_{\mathbb{R}} \sup_{y' \in \mathbb{B}_{\varepsilon_1}(y)} \Psi^+_{s,\eta,\varepsilon}(y')dy &\leqslant 2\eta \sum_{m\in\mathbb{Z}} \sup_{y\in[m\eta,(m+1)\eta)} \Psi^+_{s,\eta}(y) \\ &= 2\eta \sum_{m\in\mathbb{Z}} \sup_{y\in[m\eta,(m+1)\eta)} \Psi_s(y) < +\infty, \end{aligned} \tag{2.5.16}$$

where the series is finite since the function Ψ_s is directly Riemann integrable. Hence, applying Theorem 2.5.1 to $y \mapsto e^{sy}\Psi^+_{s,\eta,\varepsilon}(y)$, we get that uniformly in $s \in K$,

$$\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant\frac{1}{\sqrt{n}}} J_2 = 0. \tag{2.5.17}$$

For $J_3(x)$, recall that $\Psi^-_{s,\eta,\varepsilon} \leqslant \Psi_s \leqslant \Psi^+_{s,\eta,\varepsilon}$. Using (2.5.14) and the fact that r_s is uniformly bounded on $\mathcal{S}$, we get that there exists a constant $C_s > 0$ such that

$$\sup_{x\in\mathcal{S}} J_3 \leqslant C_s\varepsilon\|\varphi\|_\infty. \tag{2.5.18}$$

For J_1, note that $e^{sy}\Psi^-_{s,\eta,\varepsilon}(y) \leqslant e^{sy}\Psi_s(y) \leqslant e^{sy}\Psi^+_{s,\eta,\varepsilon}(y)$, $y \in \mathbb{R}$. Combining this with the positivity of φ, it holds that

$$\begin{aligned}
|J_1| &\leqslant \left|c_{s,l,n}\mathbb{E}\left\{\varphi(X_n^x)e^{sT^x_{n,l}}\left[\Psi^+_{s,\eta,\varepsilon}(T^x_{n,l}) - \Psi^-_{s,\eta,\varepsilon}(T^x_{n,l})\right]\right\}\right| \\
&\leqslant \left|c_{s,l,n}\mathbb{E}\left[\varphi(X_n^x)e^{sT^x_{n,l}}\Psi^+_{s,\eta,\varepsilon}(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy\right| \\
&\quad + \left|c_{s,l,n}\mathbb{E}\left[\varphi(X_n^x)e^{sT^x_{n,l}}\Psi^-_{s,\eta,\varepsilon}(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^-_{s,\eta,\varepsilon}(y)dy\right| \\
&\quad + \left|\bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^-_{s,\eta,\varepsilon}(y)dy\right| \\
&= J_{11} + J_{12} + J_{13}.
\end{aligned}$$

Using (2.5.17), as $n \to \infty$, it holds that $J_{11} \to 0$, uniformly in $s \in K$, $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$. For J_{12}, note that the function $y \mapsto e^{sy}\Psi^-_{s,\eta,\varepsilon}(y)$ is non-negative and continuous. By the construction of $\Psi^-_{s,\eta,\varepsilon}$, similarly to (2.5.16), one can verify that there exists $\varepsilon_2 > 0$ such that $\int_{\mathbb{R}}\sup_{y'\in\mathbb{B}_{\varepsilon_2}(y)}\Psi^-_{s,\eta,\varepsilon}(y')dy < +\infty$. We deduce from Theorem 2.5.1 that $J_{12} \to 0$ as $n \to \infty$, uniformly in $s \in K$, $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$. For J_{13}, we use (2.5.14) to get that $J_{13} \leqslant C_s\varepsilon$. Consequently, we obtain that, as $n \to \infty$, $J_1 \leqslant C_s\varepsilon$, uniformly in $s \in K$, $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$. This, together with (2.5.15), (2.5.17)-(2.5.18), implies that as $n \to \infty$, uniformly in $s \in K$, $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\left|c_{s,l,n}\mathbb{E}\left[\varphi(X_n^x)\psi(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi_s(y)dy\right| \leqslant C_s\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude the proof of Theorem 2.2.2. □

Proof of Theorem 2.2.4. Following the proof of Theorem 2.5.1, one can verify that the asymptotic (2.5.2) holds true for $s < 0$ small enough and for ψ satisfying condition (2.5.1). The passage to a directly Riemann integrable function ψ can be done by using the same approximation techniques as in the proof of Theorem 2.2.2. □

2.6 Proofs of LDP for $\log\|G_n\|$ and local limit theorems with large deviations

Proof of Theorem 2.2.5. Since $\log|G_nx| \leqslant \log\|G_n\|$ and the function $\bar{r}_s$ is strictly positive and uniformly bounded on $\mathbb{P}^{d-1}$, applying Theorem 2.2.1 we get the lower bound:

$$\liminf_{n\to\infty}\inf_{|l|\leqslant l_n}\frac{1}{n}\log\mathbb{P}(\log\|G_n\| \geqslant n(q+l)) \geqslant -\Lambda^*(q). \tag{2.6.1}$$

For the upper bound, since all matrix norms are equivalent, there exists a positive constant C which does not depend on the product G_n such that $\log\|G_n\| \leqslant \max_{1\leqslant i\leqslant d}\log|G_ne_i| + C$, where $(e_i)_{1\leqslant i\leqslant d}$ is the canonical orthonormal basis in $\mathbb{R}^d$. From this inequality, we deduce that

$$\mathbb{P}(\log\|G_n\| \geqslant n(q+l)) \leqslant \sum_{i=1}^{d}\mathbb{P}\Big(\log|G_ne_i| \geqslant n\big(q+l-C/n\big)\Big).$$

Using Lemma 2.4.1, there exists a constant $C_s > 0$ such that $e^{n[\Lambda^*(q+l-C/n)-\Lambda^*(q+l)]} \leqslant C_s$, uniformly in $|l| \leqslant l_n$ and $n \geqslant 1$. Again by Theorem 2.2.1, we obtain the upper bound:

$$\limsup_{n\to\infty} \sup_{|l|\leqslant l_n} \frac{1}{n} \log \mathbb{P}(\log \|G_n\| \geqslant n(q+l)) \leqslant -\Lambda^*(q).$$

This, together with (2.6.1), proves Theorem 2.2.5. □

Proof of Theorem 2.2.6. We first prove the assertion (1). The proof consists of lower and upper bounds. Since $\log \|G_n\| \geqslant \log |G_n x|$, using Theorem 2.2.1 and the fact that the function r_s is strictly positive on $\mathbb{S}_+^{d-1}$, we deduce that there exists a constant $c > 0$ such that uniformly in $|l| \leqslant l_n$,

$$c < \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \geqslant n(q+l)\Big).$$

It remains to prove the upper bound. Denote by $(\mathbb{S}_+^{d-1})^\circ$ the interior of the projective space $\mathbb{S}_+^{d-1}$. In other words, $x \in (\mathbb{S}_+^{d-1})^\circ$ if and only if each component of the vector x is strictly positive. Recalling that Γ_μ is the smallest closed semigroup generated by the support of the measure μ, it was shown in [16, Lemma 4.5] that for any fixed $x \in (\mathbb{S}_+^{d-1})^\circ$,

$$\inf_{g\in\Gamma_\mu} \frac{|gx|}{\|g\|} > 0.$$

This implies that for any fixed $x \in (\mathbb{S}_+^{d-1})^\circ$, there exists a constant $C > 1$ such that for all $n \geqslant 1$, we have $\log \|G_n\| \leqslant \log |G_n x| + C$ and hence

$$\mathbb{P}(\log \|G_n\| \geqslant n(q+l)) \leqslant \mathbb{P}\Big(\log |G_n x| \geqslant n(q+l-C/n)\Big).$$

As in the proof of Theorem 2.2.5, by Lemma 2.4.1, there exists a constant $C_s > 0$ such that $e^{n[\Lambda^*(q+l-C/n)-\Lambda^*(q+l)]} \leqslant C_s$, uniformly in $|l| \leqslant l_n$ and $n \geqslant 1$. Consequently, by Theorem 2.2.1, we obtain the following upper bound: uniformly in $|l| \leqslant l_n$,

$$\limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \|G_n\| \geqslant n(q+l)\Big) < C.$$

This ends the proof of the assertion (1). Using Theorem 2.2.3, the proof of the assertion (2) can be carried out in the same way. □

Chapter 3

Large deviation expansions for the entries of products of random matrices

Abstract. Assume that $(g_n)_{n\geqslant 1}$ is a sequence of independent and identically distributed $d \times d$ real random matrices. Consider the product $G_n = g_n \dots g_1$. For both invertible matrices and positive matrices, we establish precise large deviation expansions for (i,j)-th entry $G_n^{i,j}$ of G_n, jointly with $X_n^{e_j}$, where $X_n^x = G_n x/|G_n x|$ is a Markov chain on the projective space with x a starting point. In particular, for the entries $G_n^{i,j}$ we obtain the large deviation principle with an explicit rate function, thus improving significantly the large deviation bounds established earlier. Toward this end we prove the Hölder regularity of the stationary measure corresponding to X_n^x under the changed measure, which is of independent interest. As an application, we obtain the large deviation principle and its reinforced form for the spectral radius of products of positive matrices. We also derive local limit theorems with large deviations for the entries.

3.1 Introduction

3.1.1 Background and objectives

Let $d \geqslant 2$ be an integer. Assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a sequence $(g_n)_{n\geqslant 1}$ of $d \times d$ real random matrices which are independent and identically distributed (i.i.d.) with common law μ. A great deal of research has been devoted to studying the random matrix product $G_n := g_n \dots g_1$. Many fundamental results related to G_n, such that the strong law of large numbers (SLLN), the central limit theorem (CLT), the law of iterated logarithm (LIL) and large deviations (LD) have been established by Furstenberg and Kesten [37], Kingman [67], Le Page [69], Guivarc'h and Raugi [51], Bougerol and Lacroix [13], Hennion [53], Furman [35], Guivarc'h and Le Page [50], Benoist and Quint [9, 10], to name only a few. These limit theorems turn out to be very useful in various applications, such as in [13, 20] to study the spectral theory of random Schrödinger operators, in [27] to explore disordered systems and chaotic dynamics coming from statistical physics, in [66, 50] to investigate the multidimensional stochastic recursion, in [14, 7] to study the dynamics of group actions, and in [45, 70, 43] to investigate conditioned limit theorems and survival probabilities

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of branching processes in random environment.

Denote by $G_n^{i,j}$ the (i,j)-th entry of G_n, where $1 \leqslant i,j \leqslant d$. There has been growing interest in the study of the asymptotic behavior of $G_n^{i,j}$, since the pioneering work of Furstenberg and Kesten [37], where the following SLLN has been established for positive matrices:

$$\lim_{n\to\infty} \frac{1}{n} \log |G_n^{i,j}| = \lambda,$$

with λ a constant called the first Lyapunov exponent of the sequence $(g_n)_{n\geqslant 1}$. In [37] the CLT has also been proved, thus giving an affirmative answer to Bellman's conjecture in [5]. In the case of invertible matrices, Guivarc'h and Raugi [51] have established the SLLN and CLT for the entries $G_n^{i,j}$, where the proof turns out to be more involved than that in [37], and is based on the regularity of the stationary measure of the Markov chain $X_n^x = G_n x/|G_n x|$ with x a starting point on the projective space. Recently, Benoist and Quint [10] have proved the following large deviation bound: for $q > \lambda$ and some constant $c > 0$,

$$\mathbb{P}(\log |G_n^{i,j}| > nq) \leqslant e^{-cn}. \tag{3.1.1}$$

But the precise decay rate on the large deviation probability in (3.1.1) is not known, neither for invertible matrices nor for positive matrices. The goal of this paper is to establish an exact large deviation asymptotic for the entries $G_n^{i,j}$, which will be called Bahadur-Rao-Petrov type large deviations following the groundwork by Bahadur-Rao [4] and Petrov [73] for sums of i.i.d. real-valued random variables. Our results will imply the large deviation principle with an explicit rate function and the local limit theorem with large deviations, which clearly improves (3.1.1). Moreover, we establish the corresponding results for the spectral radius of products of positive matrices. Similar results for lower large deviations are also obtained.

3.1.2 Brief overview of the results

Let $I_\mu = \{s \geqslant 0 : \mathbb{E}(\|g_1\|^s) < \infty\}$, where $\|\cdot\|$ is the operator norm. For any $s \in I_\mu$, set $\kappa(s) = \lim_{n\to\infty} (\mathbb{E}\|G_n\|^s)^{\frac{1}{n}}$. Denote $\Lambda = \log \kappa$ and consider its Fenchel-Legendre transform Λ^*, which satisfies $\Lambda^*(q) = sq - \Lambda(s) > 0$ for $q = \Lambda'(s) > \lambda$ and $s \in I_\mu^\circ$ (the interior of I_μ). The projective sphere is $\mathbb{P}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}/\pm$. Consider the transfer operator P_s defined by $P_s\varphi(x) = \mathbb{E}[e^{s\log|g_1 x|}\varphi(\frac{g_1 x}{|g_1 x|})]$, $x \in \mathbb{P}^{d-1}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$ and φ is a continuous function on $\mathbb{P}^{d-1}$; the conjugate transfer operator P_s^* is defined similarly: see (3.2.5). The operators P_s and P_s^* have unique continuous strictly positive eigenfunctions r_s and r_s^* on $\mathbb{P}^{d-1}$ and unique probability eigenmeasures ν_s and ν_s^*, respectively, satisfying $P_s r_s = \kappa(s) r_s$, $P_s\nu_s = \kappa(s)\nu_s$, $P_s^* r_s^* = \kappa(s) r_s^*$ and $P_s^*\nu_s^* = \kappa(s)\nu_s^*$. Denote $\sigma_s := \sqrt{\Lambda''(s)} > 0$. For details see Section 3.3.1.

Our first objective is to establish a Bahadur-Rao type large deviation asymptotic for the entries $G_n^{i,j}$ for both invertible matrices and positive matrices; we refer to Bahadur and Rao [4] for the case of i.i.d. real-valued random variables. More precisely, we prove that, for $q = \Lambda'(s)$ with $s \in I_\mu^\circ$, as $n \to \infty$,

$$\mathbb{P}(\log |G_n^{i,j}| \geqslant nq) = \frac{r_s(e_j) r_s^*(e_i)}{\varrho_s} \frac{\exp\big(-n\Lambda^*(q)\big)}{s\sigma_s\sqrt{2\pi n}} \big[1 + o(1)\big], \tag{3.1.2}$$

where $(e_i)_{1\leqslant i\leqslant d}$ is the canonical orthonormal basis of $\mathbb{R}^d$ and $\varrho_s = \nu_s(r_s) = \nu_s^*(r_s^*) > 0$. For invertible matrices, the asymptotic (3.1.2) clearly implies the large deviation principle for $G_n^{i,j}$ with rate function Λ^*, which obviously improves the large deviation bound (3.1.1). In addition, we show that the asymptotic (3.1.2) also holds for positive matrices.

In fact, we shall extend (3.1.2) to the scalar product $\langle f, G_n x\rangle$, and more generally, to the couple $(X_n^x, \log|\langle f, G_n x\rangle|)$ with target functions, where $X_n^x = G_n x/|G_n x|$, $f, x \in \mathbb{P}^{d-1}$ and $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^d$. Precisely, we prove that for any Hölder continuous function φ on $\mathbb{P}^{d-1}$ and any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-sy}\psi(y)$ is directly Riemann integrable, as $n \to \infty$, uniformly in $f, x \in \mathbb{P}^{d-1}$,

$$\begin{aligned}&\mathbb{E}\Big[\varphi(X_n^x)\psi\Big(\log|\langle f, G_n x\rangle| - nq\Big)\Big]\\ &= \frac{r_s(x)}{\varrho_s}\int_{\mathbb{P}^{d-1}}\varphi(u)|\langle f, u\rangle|^s\nu_s(du)\int_{\mathbb{R}} e^{-sy}\psi(y)dy\frac{\exp(-n\Lambda^*(q))}{\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big]. \qquad (3.1.3)\end{aligned}$$

Our second objective is to establish a Bahadur-Rao type result on the lower large deviation probabilities $\mathbb{P}\Big(\log|G_n^{i,j}| \leqslant nq\Big)$, where $q = \Lambda'(s) < \lambda$ with $s < 0$ sufficiently close to 0. Specifically, we prove that, as $n \to \infty$,

$$\mathbb{P}\Big(\log|G_n^{i,j}| \leqslant nq\Big) = \frac{r_s(e_j)r_s^*(e_i)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big], \qquad (3.1.4)$$

where the quantities r_s, r_s^*, ν_s, ϱ_s, Λ^* and $\sigma_s > 0$, for small $s < 0$, are defined in Section 3.3.1 similarly to the case $s > 0$. The asymptotic (3.1.4) is of course much sharper than the corresponding lower large deviation principle for $G_n^{i,j}$. More generally, we extend the scope of the lower large asymptotic (3.1.4) to the couple $(X_n^x, \log|\langle f, G_n x\rangle|)$ with target functions, in the same line as the asymptotic (3.1.3). As an application of (3.1.2) and (3.1.4), for positive matrices we derive reinforced large deviation principles for the spectral radius $\rho(G_n)$ of G_n: see Theorem 3.2.7.

The assertions (3.1.2), (3.1.3) and (3.1.4) stated above concern Bahadur-Rao type large deviation asymptotics, but we shall actually establish an extended version of these results with an additional vanishing perturbation, which in the literature is known as Baghadur-Rao-Petrov type large deviation results. It is worth mentioning that such type of extensions has important and interesting implications, for instance, to local limit theorems with large deviations for the entries $G_n^{i,j}$: see Theorems 3.2.5 and 3.2.6.

3.1.3 Proof strategy

The standard approach to obtain precise large deviations for i.i.d. real-valued random variables consists in performing a change of measure and proving an Edgeworth expansion under the changed measure (see [30]). Applying this strategy to the entries $G_n^{i,j}$ of products of random matrices turns out to be way more difficult. We have to overcome three main difficulties: state an Edgeworth expansion for the couple $(X_n^{e_j}, \log|G_n^{i,j}|)$ with a target function φ on the Markov chain $X_n^{e_j}$ under the changed measure; give a precise control of the difference between $\log|G_n^{i,j}|$ and $\log|G_n e_j|$; establish the regularity of the eigenmeasure ν_s.

For the first point, it turn out that the techniques which work for $\log|G_n^{i,j}|$ alone cannot be applied for the couple. Dealing with a couple $(X_n^{e_j}, \log|G_n^{i,j}|)$ with a target

function on X_n^x needs considering a new kind of smoothing inequality on a complex contour, instead of the usual Esseen one on the real line. We make use of the saddle point method to obtain precise asymptotics for the integrals of the corresponding Laplace transforms on the complex plane. For this method we refer to a recent work of the authors [86] where the Edgeworth expansion with a target function on X_n^x for the norm cocycle $\log|G_n x|$ has been established.

Secondly, from the previous works on limit theorems such as SLLN, CLT and LIL for $G_n^{i,j}$, see e.g. [51, 13, 53, 10], we know that the difference $|\log|G_n^{i,j}| - \log|G_n e_j||$ generally diverges to infinity as $n \to \infty$. It is controlled by the corresponding norming factors in SLLN, CLT and LIL. However, such a control is not enough to obtain precise large deviation expansions for $G_n^{i,j}$, nor even for a large deviation principle with explicit rate function. A precise account of the contribution of the error term is given by the following decomposition:

$$\log|G_n^{i,j}| = \log|G_n x| + \log|f(X_n^x)|, \quad n \geqslant 1, \tag{3.1.5}$$

where $x = e_j$, $f = e_i$, and $f(X_n^x)$ is seen as a linear functional f acting on the Markov chain X_n^x. The exact decomposition (3.1.5) allows to deduce the precise large deviation asymptotic from the results for the couple $(X_n^x, \log|G_n x|)$ with a target function on X_n^x established in [85]. The idea is as follows: with $\mathbb{Q}_s^x$ the changed measure defined in Section 3.3.1, we have

$$\frac{e^{n\Lambda^*(q)}}{r_s(x)}\mathbb{P}\Big(\log|G_n^{i,j}| \geqslant nq\Big) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[\frac{e^{-s(\log|G_n x|-nq)}}{r_s(X_n^x)}\mathbb{1}_{\{\log|G_n^{i,j}|-nq\geqslant 0\}}\Big]. \tag{3.1.6}$$

We only sketch how to cope with the upper bound of the right-hand side of (3.1.6). Consider a partition $I_k := (-\delta k, -\delta(k-1)]$, $k \geqslant 1$, of the interval $(-\infty, 0]$, where $\delta > 0$. Using (3.1.5) we get the upper bound

$$\mathbb{1}_{\{\log|G_n^{i,j}|-nq\geqslant 0\}} \leqslant \sum_{k=1}^{\infty} \mathbb{1}_{\{\log|G_n x|-nq-\delta(k-1)\geqslant 0\}}\mathbb{1}_{\{\log|f(X_n^x)|\in I_k\}},$$

which we substitute into (3.1.6). Thus we are led to the estimation of the sum

$$\sum_{k=1}^{\infty} e^{-s\delta(k-1)}\mathbb{E}_{\mathbb{Q}_s^x}\left[\frac{\psi_s(\log|G_n x| - nq - \delta(k-1))}{r_s(X_n^x)}\mathbb{1}_{\{\log|f(X_n^x)|\in I_k\}}\right], \tag{3.1.7}$$

where $\psi_s(y) = e^{-sy}\mathbb{1}_{\{y\geqslant 0\}}$, $y \in \mathbb{R}$. Let $R_{s,it}(\varphi)(x) = \mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_1)e^{it(\log|g_1 x|-q)}]$ be the perturbed transfer operator defined for any Hölder continuous function φ on $\mathbb{P}^{d-1}$, and $R_{s,it}^n$ be its n-th iteration. Then, by the inversion formula, the sum in (3.1.7) is bounded from above by

$$\frac{1}{2\pi}\sum_{k=1}^{\infty} e^{-s\delta(k-1)}\int_{\mathbb{R}} e^{-it\delta(k-1)}R_{s,it}^n(r_s^{-1}\Phi_{s,k,\varepsilon_2})(x)\widehat{\Psi}_{s,\varepsilon_1}(t)dt, \tag{3.1.8}$$

where we choose some appropriate smooth functions Φ_{s,k,ε_2} and Ψ_{s,ε_1}, for $\varepsilon_1, \varepsilon_2 > 0$, which dominate $\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}$ and ψ_s, respectively. It has been established recently in [85] that, for any $k \geqslant 1$, the term under the sign of the infinite sum in (3.1.8), say $I_n(k)$, converges as $n \to \infty$ to a limit, say $I(k) = \frac{\sqrt{2\pi}}{s\sigma_s\nu_s(r_s)}e^{-s\delta(k-1)}\nu_s(\Phi_{s,k,\varepsilon_2})$. The

interchangeability of the limit as $n \to \infty$ and of the summation over k in (3.1.8) is justified by specifying the rate in the convergence of $I_n(k)$ to $I(k)$, as argued in [85]. This implies that as $n \to \infty$ and $\varepsilon_1 \to 0$, (3.1.8) converges to

$$\frac{1}{s\sigma_s\sqrt{2\pi}\nu_s(r_s)} \sum_{k=1}^{\infty} e^{-s\delta(k-1)}\nu_s(\Phi_{s,k,\varepsilon_2}).$$

It remains to show that the last sum converges to $r_s^*(f)$, as $\delta \to 0$ and $\varepsilon_2 \to 0$. For this we have to overcome the third important difficulty of the paper: prove the Hölder regularity property of the eigenmeasure ν_s, i.e. that there exist two constants $c, C > 0$ such that for any $0 < t < 1$,

$$\sup_{f \in \mathbb{P}^{d-1}} \nu_s\Big(\{x : |\langle f, x\rangle| \leqslant t\}\Big) \leqslant Ct^c. \tag{3.1.9}$$

This one of the the central points of the paper which is of independent interest. The inequality (3.1.9) for $s = 0$ has been proved in [51] and further studied in [13]. With $s = 0$ it was used to establish limit theorems for the entries $G_n^{i,j}$, see [51, 13, 9, 10]. For other applications see [7, 14].

To prove (3.1.9) when $s > 0$, for invertible matrices, we adapt the arguments from [51] and [13] where (3.1.9) was established for $s = 0$. For $s > 0$ the arguments are much more delicate. One of the difficulties is that the sequence $(g_n)_{n\geqslant 1}$ becomes dependent under the changed measure. We need to extend the results in [13] to this case. Of crucial importance are the simplicity of the dominant Lyapunov exponent for G_n under the changed measure $\mathbb{Q}_s^x$ recently established in [50] (see Lemma 3.5.6), and the key proximality property which states that $M_n \cdot m$ (here $M_n = g_1 \dots g_n$) converges weakly to the Dirac measure δ_{Z_s}, where Z_s is a random variable whose law is the stationary measure π_s of X_n^x, for $s > 0$ (see Lemma 3.5.2), and m is the unique rotation invariant measure on $\mathbb{P}^{d-1}$.

We also establish an analog of (3.1.9) for positive matrices, but under either assumption **M4** of Furstenberg-Kesten type, or assumption **M5** on the harmonic moments of the entries. The techniques of the proofs are quite different from those used in the case of invertible matrices. Under condition **M4**, they rely on the fact that the Markov chain X_n^x is separated from the coordinates e_i and the support of the stationary measure π_s of X_n^x coincides with the support of the stationary measure $\pi_0 = \nu$. Under condition **M5**, the proofs are based on the large deviation bounds under the changed measure, see Theorem 3.4.4.

The passage to the large deviation asymptotic (3.1.3) with target functions is achieved by using approximation techniques (see [85]).

The proof of the lower large deviation asymptotic (3.1.4) can be carried out in the same way as that of upper large deviation asymptotic (3.1.2). The novelty here consists in the use of the change of measure formula for $\mathbb{Q}_s^x$ when $s < 0$ and of the spectral gap theory under the changed measure as stated in [86] for $s < 0$. In addition we need the Hölder regularity for the eigenmeasure ν_s for $s < 0$ sufficiently close to 0, which is of independent interest; this is established using a different approach compared to the case $s > 0$.

3.2 Main results

3.2.1 Notation and conditions

Denote by c, C absolute constants whose values may change from line to line. By c_α, C_α we mean constants depending only on the parameter α. For any integrable function $\rho : \mathbb{R} \to \mathbb{C}$, denote its Fourier transform by $\widehat{\rho}(t) = \int_{\mathbb{R}} e^{-ity}\rho(y)dy$, $t \in \mathbb{R}$. For a measure ν and a function φ we write $\nu(\varphi) = \int \varphi d\nu$. Let $\mathbb{N} = \{1, 2, \ldots\}$. By convention $\log 0 = -\infty$.

The space $\mathbb{R}^d$ is equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. Let $(e_i)_{1\leqslant i\leqslant d}$ be the canonical orthonormal basis of $\mathbb{R}^d$. For $d \geqslant 2$, let $M(d, \mathbb{R})$ be the set of $d \times d$ matrices with entries in $\mathbb{R}$. Denote by $\mathscr{G} = GL(d, \mathbb{R})$ the general linear group of invertible matrices of $M(d, \mathbb{R})$, and by $\mathscr{G}_+^\circ$ the subsemigroup of $M(d, \mathbb{R})$ with strictly positive entries. We shall work with products of invertible matrices and positive matrices (all over the paper we use the term positive matrix in the strict sense that each entry is strictly positive).

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$, and $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ be its intersection with the positive quadrant. It will be convenient to consider the projective space $\mathbb{P}^{d-1} := \mathbb{S}^{d-1}/\pm$ by identifying $-x$ with x. To unify the exposition, we use the symbol $\mathcal{S}$ to denote $\mathbb{P}^{d-1}$ in the case of invertible matrices, and $\mathbb{S}_+^{d-1}$ in the case of positive matrices. The projective space $\mathcal{S}$ is equipped with the metric $\mathbf{d}$ defined as follows. For invertible matrices, $\mathbf{d}$ is the angular distance (see [50]), i.e., for any $x, y \in \mathbb{P}^{d-1}$, $\mathbf{d}(x, y) = |x \wedge y|$, where $x \wedge y$ denotes the exterior product of two vectors x and y. For positive matrices, $\mathbf{d}$ is the Hilbert cross-ratio distance (see [53]) defined by $\mathbf{d}(x, y) = \frac{1-m(x,y)m(y,x)}{1+m(x,y)m(y,x)}$, where $m(x, y) = \sup\{\beta > 0 : \beta y_i \leqslant x_i, \forall i = 1, \ldots, d\}$ for any two vectors $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{S}_+^{d-1}$.

Let $\mathcal{C}(\mathcal{S})$ be the space of complex-valued continuous functions on $\mathcal{S}$. We write $\mathbf{1}$ for the identity function $1(x)$, $x \in \mathcal{S}$. Throughout this paper, $\gamma > 0$ is a fixed sufficiently small constant. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\infty := \sup_{x\in\mathcal{S}} |\varphi(x)| \quad \text{and} \quad \|\varphi\|_\gamma := \|\varphi\|_\infty + \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}(x, y)^\gamma},$$

and consider the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma < +\infty\}$.

All over the paper $(g_n)_{n\geqslant 1}$ is a sequence of i.i.d. random matrices of the same probability law μ on $M(d, \mathbb{R})$. Denote by $\Gamma_\mu := [\operatorname{supp} \mu]$ the smallest closed subsemigroup of $M(d, \mathbb{R})$ generated by $\operatorname{supp} \mu$, the support of μ. For any $g \in \mathscr{G}$ (or $g \in \mathscr{G}_+^\circ$), denote $\|g\| = \sup_{x\in\mathcal{S}} |gx|$. Let

$$I_\mu = \{s \geqslant 0 : \mathbb{E}(\|g_1\|^s) < +\infty\}.$$

and denote by I_μ° its interior. In the sequel we always assume that there exists $s > 0$ such that

$$\mathbb{E}(\|g_1\|^s) < +\infty, \tag{3.2.1}$$

so that I_μ° is non-empty and is an interval of $\mathbb{R}$.

For any $g \in \mathscr{G}$ (or $g \in \mathscr{G}_+^\circ$), set $\iota(g) = \inf_{x\in\mathcal{S}} |gx|$. It is easy to see that $\iota(g) > 0$ for both invertible matrices and positive matrices. We will need the following exponential moment condition:

M1. There exist $s \in I_\mu^\circ$ and $\alpha \in (0,1)$ such that $\mathbb{E}\|g_1\|^{s+\alpha}\iota(g_1)^{-\alpha} < +\infty$.

Moreover, we shall use the following one-sided moment condition:

M2. There exists a small constant $\eta > 0$ such that $\mathbb{E}\Big(\iota(g_1)^{-\eta}\Big) < +\infty$.

For an invertible g we have $\iota(g) = \|g^{-1}\|^{-1}$, so **M2** reads as $\mathbb{E}\Big(\|g_1^{-1}\|^{\eta}\Big) < +\infty$. By Hölder's inequality, condition **M2** together with (3.2.1) implies condition **M1**.

A matrix g is said to be *proximal* if it has an algebraic simple dominant eigenvalue, that is, g has an eigenvalue λ_g satisfying $|\lambda_g| > |\lambda_g'|$ for all other eigenvalues λ_g' of g. It is easy to verify that $\lambda_g \in \mathbb{R}$. The eigenvector v_g with unit norm $|v_g| = 1$, corresponding to the eigenvalue λ_g, is called the dominant eigenvector. For invertible matrices, we need the following strong irreducibility and proximality conditions:

M3. (i)(Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant.
(ii)(Proximality) Γ_μ contains at least one proximal matrix.

For positive matrices, the condition **M3**(ii) is always satisfied, since by the Perron-Frobenius theorem, any positive matrix g has a dominant eigenvalue $\lambda_g > 0$, with the corresponding eigenvector $v_g \in \mathbb{S}_+^{d-1}$.

In the groundwork [37], Furstenberg and Kesten studied the SLLN and CLT for the entries of positive matrices under the condition that there exists a constant $C > 1$ such that for any $g = (g^{i,j})_{1\leqslant i,j\leqslant d} \in \operatorname{supp}\mu$,

$$1 \leqslant \frac{\max_{1\leqslant i,j\leqslant d} g^{i,j}}{\min_{1\leqslant i,j\leqslant d} g^{i,j}} \leqslant C. \tag{3.2.2}$$

In our paper we shall relax it to:

M4. There exists a constant $C > 1$ such that for any $g = (g^{i,j})_{1\leqslant i,j\leqslant d} \in \operatorname{supp}\mu$, and $1 \leqslant j \leqslant d$, we have

$$1 \leqslant \frac{\max_{1\leqslant i\leqslant d} g^{i,j}}{\min_{1\leqslant i\leqslant d} g^{i,j}} \leqslant C. \tag{3.2.3}$$

Condition **M4** is clearly weaker than (3.2.2); the latter one means that all the entries $g^{i,j}$ of the matrix $g \in \operatorname{supp}\mu$ are comparable, while **M4** requires only that all the entries in the same columns of the matrix $g \in \operatorname{supp}\mu$ are comparable. An equivalent formulation of **M4** will be given in Lemma 3.5.8, from which it follows that the set of matrices satisfying **M4** forms a subsemigroup of $M(d,\mathbb{R})$. We will see below that **M4** can be replaced by the corresponding condition that all the entries in the same rows are comparable. Finally, we can replace **M4**, by assuming the existence of the harmonic moments of the entries of g_1:

M5. For any $1 \leqslant i,j \leqslant d$, there exists a constant $\delta > 0$ such that

$$\mathbb{E}\Big[\Big(g_1^{i,j}\Big)^{-\delta}\Big] < \infty.$$

One can easily verify that condition **M5** implies condition **M2**. Note also that the conditions **M4** and **M5** do not imply each other. However, under the assumption **M2**, condition **M4** (and therefore also (3.2.2)) implies condition **M5**. The converse is not true.

For any $g \in \mathscr{G}$ (or $g \in \mathscr{G}_+^\circ$) and $x \in \mathcal{S}$, we write $g \cdot x = \frac{gx}{|gx|}$ for the projective action of g on $\mathcal{S}$. With the starting point $x \in \mathcal{S}$, define a Markov chain on the projective space $\mathcal{S}$ by setting

$$X_n^x := G_n \cdot x = \frac{G_n x}{|G_n x|}, \quad n \geqslant 1.$$

Under either condition **M3** for invertible matrices, or condition **M4** (or **M5**) for positive matrices, the Markov chain $(X_n^x)_{n\geqslant 1}$ has a unique stationary measure ν on $\mathcal{S}$ such that for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$\int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(g_1 \cdot x)\mu(dg_1)\nu(dx) = \int_{\mathcal{S}} \varphi(x)\nu(dx). \tag{3.2.4}$$

Moreover, the support of ν is given by $\operatorname{supp}\nu = \overline{\{v_g \in \mathbb{P}^{d-1} : g \in \Gamma_\mu,\ g \text{ is proximal}\}}$ for invertible matrices, and by $\operatorname{supp}\nu = \overline{\{v_g \in \mathbb{S}_+^{d-1} : g \in \Gamma_\mu\}}$ for positive matrices. In addition, for both cases, $\operatorname{supp}\nu$ is indeed the unique minimal Γ_μ-invariant subset: see [50] and [16] for the proof.

We need the following non-arithmeticity condition for positive matrices:

M6. (Non-arithmeticity) For $t > 0$, $\theta \in [0, 2\pi)$ and a function $\varphi : \mathcal{S} \to \mathbb{R}$, the equation

$$\varphi(g \cdot x)|gx|^{it} = e^{i\theta}\varphi(x), \quad \forall g \in \Gamma_\mu, \forall x \in \operatorname{supp}\nu,$$

has no trivial solution except that $t = 0$, $\theta = 0$ and φ is a constant.

For positive matrices, if the additive subgroup of $\mathbb{R}$ generated by the set $\{\log \lambda_g : g \in \Gamma_\mu\}$ is dense in $\mathbb{R}$, then condition **M6** is fulfilled (see [17]). This sufficient condition was introduced by Kesten [66] and is usually easier to verify in practice. For invertible matrices, it was proved in [52] that condition **M3** implies **M6**.

For any $s \in (-s_0, 0) \cup I_\mu$ with small enough $s_0 > 0$, define the transfer operator P_s and the conjugate transfer operator P_s^* as follows: for any $\varphi \in \mathcal{C}(\mathcal{S})$ and $x \in \mathcal{S}$,

$$P_s\varphi(x) = \int_{\Gamma_\mu} |g_1 x|^s \varphi(g_1 \cdot x)\mu(dg_1), \ P_s^*\varphi(x) = \int_{\Gamma_\mu} |g_1^{\mathrm{T}} x|^s \varphi(g_1^{\mathrm{T}} \cdot x)\mu(dg_1). \tag{3.2.5}$$

Under suitable conditions, the transfer operator P_s has a unique probability eigenmeasure ν_s on $\mathcal{S}$ corresponding to the eigenvalue $\kappa(s)$: $P_s\nu_s = \kappa(s)\nu_s$. Similarly, the conjugate transfer operator P_s^* has a unique probability eigenmeasure ν_s^* corresponding to the eigenvalue $\kappa(s)$: $P_s^*\nu_s^* = \kappa(s)\nu_s^*$. Set, for $x \in \mathcal{S}$,

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy).$$

Then, r_s is the unique, up to a scaling constant, strictly positive eigenfunction of P_s: $P_s r_s = \kappa(s) r_s$; similarly r_s^* is the unique, up to a scaling constant, strictly positive eigenfunction of P_s^*: $P_s^* r_s^* = \kappa(s) r_s^*$. It is easy to see that the eigenfunctions satisfy

$\nu_s(r_s) = \nu_s^*(r_s^*) := \varrho_s$. The stationary measure π_s is defined by $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\varrho_s}$, for any $\varphi \in \mathcal{C}(\mathcal{S})$. We refer to Section 3.3.1 for details.

Define $\Lambda = \log \kappa : (-s_0, 0) \cup I_\mu \to \mathbb{R}$, then the function Λ is convex and analytic. For any $s \in (-s_0, 0) \cup I_\mu$, condition **M6** implies that $\sigma_s = \Lambda''(s)$ is strictly positive. Denote by Λ^* the Fenchel-Legendre transform of Λ, then we have $\Lambda^*(q) = sq - \Lambda(s) > 0$ if $q = \Lambda'(s)$ for $s \in (-s_0, 0) \cup I_\mu^\circ$.

3.2.2 Precise large deviations for the scalar product

The goal of this section is to state exact large deviation asymptotics for the scalar product $\langle f, G_n x\rangle$, where $f, x \in \mathcal{S}$. To the best of our knowledge, the precise large deviations for the scalar product $\langle f, G_n x\rangle$ and, in particular, for the entries $G_n^{i,j}$, have not been studied by now in the literature. Our first result is a large deviation asymptotic of the Bahadur-Rao type (see [4]) for the upper tails of $\langle f, G_n x\rangle$.

*Note that throughout this section, for invertible matrices, all the statements are valid only for 2×2 matrices; for positive matrices, all the statements are valid for $d \times d$ ($d \geqslant 2$) matrices under condition **M4**, and only for 2×2 matrices otherwise.*

Theorem 3.2.1. *Assume either conditions **M2**, **M3** for invertible matrices, or conditions **M1**, **M4**, **M6** (or conditions **M1**, **M5**, **M6**) for positive matrices. Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Then, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant nq\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q)\right)}{s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big]. \tag{3.2.6}$$

The large deviation asymptotic (3.1.2) for the entries $G_n^{i,j}$ announced in the introduction, is obtained from (3.2.6) with $f = e_i$ and $x = e_j$. It is easy to verify that the asymptotic (3.2.6) implies the following large deviation principle: under the assumptions of Theorem 3.2.1, we have, uniformly in $f, x \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant nq\Big) = -\Lambda^*(q). \tag{3.2.7}$$

In its turn, the asymptotic (3.2.7) improves significantly the bound (3.1.1).

An important field of applications of large deviation asymptotics for the entries of type (3.2.6) is the study of the asymptotic behavior of the branching processes in random environment with several types of particles. For results in the case of single type branching processes we refer to [46, 47] and for the relation between the entries of products of random matrices and the multi-type branching processes we refer to [21].

It may be interesting to precise some details on the moment assumptions for Theorem 3.2.1 to hold true. For positive matrices, if we assume the Furstenberg-Kesten type condition **M4**, the assertion holds without assuming the moment condition **M2**. However, it is not clear whether condition **M2** is necessary for invertible matrices. This question is open, the main difficulty being to establish the Hölder regularity of the stationary measure π_s for invertible matrices without assuming condition **M2** (see Proposition 3.3.4). In the same line, we note that a Bahadur-Rao-Petrov type large deviation result for the norm cocycle $\log|G_n x|$ has been recently established in [85] for invertible matrices under conditions **M1** and **M3**.

Our next result is an improvement of Theorem 3.2.1 by allowing a vanishing perturbation l on $q = \Lambda'(s)$, in the spirit of the Petrov result [73], called the Bahadur-Rao-Petrov type large deviation. Large deviations with a perturbation l have been used for example in Buraczewski, Collamore, Damek and Zienkiewicz [18], for a recent application to the asymptotic of the ruin time in some models of financial mathematics. These results are also useful to deduce local limit theorems with large deviations, see Subsection 3.2.3.

Theorem 3.2.2. *Assume either conditions **M2**, **M3** for invertible matrices, or conditions **M1**, **M4**, **M6** (or conditions **M1**, **M5**, **M6**) for positive matrices. Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Then, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant n(q+l)\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big], \tag{3.2.8}$$

and, more generally, for any $\varphi \in \mathcal{B}_\gamma$ and any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for some $s' \in (0, s)$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,

$$\begin{aligned} &\mathbb{E}\Big[\varphi(X_n^x)\psi\Big(\log|\langle f, G_n x\rangle| - n(q+l)\Big)\Big] \qquad (3.2.9)\\ &= \frac{r_s(x)}{\varrho_s}\int_{\mathcal{S}} \varphi(u)|\langle f, u\rangle|^s \nu_s(du) \int_{\mathbb{R}} e^{-sy}\psi(y)dy \frac{\exp(-n\Lambda^*(q+l))}{\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big]. \end{aligned}$$

Now we are going to give asymptotics of the lower tail large deviation probabilities $\mathbb{P}(\log|\langle f, G_n x\rangle| \leqslant nq)$, where $q = \Lambda'(s) < \lambda = \Lambda'(0)$ for $s < 0$. These results cannot be deduced from Theorems 3.2.1 and 3.2.2; moreover the proofs are quite different and require to develop the corresponding spectral gap theory for the transfer operator P_s and to establish the Hölder regularity for the stationary measure π_s with $s < 0$. Recall that all over the paper we assume condition (3.2.1).

Theorem 3.2.3. *Assume either conditions **M2**, **M3** for invertible matrices, or conditions **M5**, **M6** for positive matrices. Then, there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \leqslant nq\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big]. \tag{3.2.10}$$

In particular, with $f = e_i$ and $x = e_j$ in (3.2.10), we obtain the Bahadur-Rao type lower tail large deviation asymptotic (3.1.4) for the entries $G_n^{i,j}$. From (3.2.10) one gets a lower tail large deviation principle: under the assumptions of Theorem 3.2.3, we have, uniformly in $f, x \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log|\langle f, G_n x\rangle| \leqslant nq\Big) = -\Lambda^*(q). \tag{3.2.11}$$

The result (3.2.11) sharpens the following lower tail large deviation bound established by Benoist and Quint [10, Theorem 14.21] for invertible matrices: for $q < \lambda$, there exists a constant $c > 0$ such that uniformly in $f, x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}(\log|\langle f, G_n x\rangle| < nq) \leqslant e^{-cn}.$$

Now we give a Bahadur-Rao-Petrov version of the above theorem.

Theorem 3.2.4. *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or conditions* ***M5***, ***M6*** *for positive matrices. Then, there exists $s_0 > 0$ such that, for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,*

$$\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \leqslant n(q+l)\Big) = \frac{r_s(x) r_s^*(f)}{\varrho_s} \frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big], \tag{3.2.12}$$

and, more generally, for any $\varphi \in \mathcal{B}_\gamma$ and any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable for some $s' \in (-s_0, s)$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,

$$\begin{aligned} &\mathbb{E}\Big[\varphi(X_n^x)\psi\Big(\log|\langle f, G_n x\rangle| - n(q+l)\Big)\Big] \qquad (3.2.13)\\ &= \frac{r_s(x)}{\varrho_s}\int_{\mathcal{S}} \varphi(u)|\langle f, u\rangle|^s \nu_s(du) \int_{\mathbb{R}} e^{-sy}\psi(y)dy \frac{\exp(-n\Lambda^*(q+l))}{\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big]. \end{aligned}$$

Consider the reversed random walk M_n defined by $M_n = g_1 \dots g_n$. Since the two probabilities $\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \geqslant n(q+l)\Big)$ and $\mathbb{P}\Big(\log|\langle f, M_n x\rangle| \geqslant n(q+l)\Big)$ are equal (as G_n and M_n have the same law), for M_n we have the same large deviation expansions as for G_n. It is interesting to note that, this fact and the symmetry in the definition of the eigenfunctions r_s and r_s^*, imply that in condition **M4** one can replace the bound (3.2.3) on columns of g by a similar one on rows, namely, by the bound: there exists a constant $C > 1$ such that for any $g \in \operatorname{supp}\mu$, and $1 \leqslant i \leqslant d$, we have

$$1 \leqslant \frac{\max_{1\leqslant j\leqslant d} g^{i,j}}{\min_{1\leqslant j\leqslant d} g^{i,j}} \leqslant C. \tag{3.2.14}$$

3.2.3 Local limit theorems with large deviations

As before, throughout this section, for invertible matrices, all the statements are valid only for 2×2 matrices; for positive matrices, all the statements are valid for $d \times d$ ($d \geqslant 2$) matrices under condition ***M4***, *and only for 2×2 matrices otherwise.*

In Theorem 3.2.2, taking $\psi = \mathbb{1}_{[a_1,a_2]}$ with fixed real numbers $a_1 < a_2$, we get the following local limit theorem with large deviations for the scalar products $\langle f, G_n x\rangle$.

Theorem 3.2.5. *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or conditions* ***M1***, ***M4***, ***M6*** *(or conditions* ***M1***, ***M5***, ***M6****) for positive matrices. Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Then, for any real numbers $-\infty < a_1 < a_2 < \infty$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,*

$$\begin{aligned} &\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \in n(q+l) + [a_1, a_2)\Big)\\ &\qquad = \Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{r_s(x) r_s^*(f)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big], \qquad (3.2.15) \end{aligned}$$

and, more generally, for any $\varphi \in \mathcal{B}_\gamma$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,

$$\begin{aligned} &\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| \in n(q+l)+[a_1,a_2)\}}\Big]\\ &= \Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{r_s(x)}{\varrho_s}\int_{\mathcal{S}} \varphi(u)|\langle f, u\rangle|^s \nu_s(du)\frac{\exp(-n\Lambda^*(q+l))}{\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big]. \qquad (3.2.16) \end{aligned}$$

It is clear that (3.2.15) follows from (3.2.16) if we set $\varphi = \mathbf{1}$.

For sums of independent real-valued random variables, local limit theorems with large and moderate deviations can be found for instance in Gnedenko [39], Sheep [79], Stone [80], Borovkov and Borovkov [11], Breuillard [15]. For products of random matrices, such types of local limit theorems for the norm cocycle $\log|G_n x|$ have been recently established in [10, 85, 86]. Our results (3.2.15) and (3.2.16) extend the results in [85, 86] to the case of the scalar product $\langle f, G_n x\rangle$.

The local limit theorem with large deviations for $s < 0$ can be deduced from Theorem 3.2.4 in the same way as the corresponding results of Theorem 3.2.5.

Theorem 3.2.6. *Assume either conditions **M2**, **M3** for invertible matrices, or conditions **M5**, **M6** for positive matrices. Then, there exist $s_0 > 0$ and a sequence $\Delta_n > 0$ converging to 0 as $n \to \infty$ such that, for any $s \in (-s_0, 0)$ and $q = \Lambda'(s)$, and any real numbers $-\infty < a_1 < a_2 < \infty$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,*

$$\begin{aligned}\mathbb{P}\Big(\log|\langle f, G_n x\rangle| \in n(q+l) + [a_1, a_2)\Big)\\ = \Big(e^{-sa_2} - e^{-sa_1}\Big)\frac{r_s(x) r_s^*(f)}{\varrho_s}\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big],\end{aligned}$$

and, more generally, for any $\varphi \in \mathcal{B}_\gamma$, we have, as $n \to \infty$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$,

$$\begin{aligned}&\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| \in n(q+l)+[a_1,a_2)\}}\Big]\\ &= \Big(e^{-sa_1} - e^{-sa_2}\Big)\frac{r_s(x)}{\varrho_s}\int_{\mathcal{S}}\varphi(u)|\langle f, u\rangle|^s \nu_s(du)\frac{\exp(-n\Lambda^*(q+l))}{-s\sigma_s\sqrt{2\pi n}}\Big[1 + o(1)\Big].\end{aligned}$$

3.2.4 Large deviation principle for the spectral radius of positive matrices

*As before, in this section, the statements are valid for $d \times d$ ($d \geqslant 2$) matrices under condition **M4**, and for 2×2 matrices otherwise.*

Using Theorems 3.2.2 and 3.2.4, we are able to derive reinforced large deviation principles for the spectral radius of products of positive random matrices. Recall that the spectral radius of a matrix $g \in M(d, \mathbb{R})$ is defined by $\rho(g) = \lim_{k\to\infty}\|g^k\|^{1/k}$. According to the Perron-Frobenius theory, the spectral radius $\rho(g)$ of a positive matrix g actually coincides with its largest eigenvalue. Below we state the results with a perturbation l on $q = \Lambda'(s)$, but, of course, they remain true for $l = 0$.

Theorem 3.2.7.
(1) *Assume conditions **M1**, **M4**, **M6** (or conditions **M1**, **M5**, **M6**) for positive matrices. Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Then, there exist constants $0 < c < C < +\infty$ such that uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,*

$$\begin{aligned}c < &\liminf_{n\to\infty}\sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{P}\Big(\log\rho(G_n) \geqslant n(q+l)\Big)\\ \leqslant &\limsup_{n\to\infty}\sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{P}\Big(\log\rho(G_n) \geqslant n(q+l)\Big) < C. \qquad (3.2.17)\end{aligned}$$

(2) *Assume conditions* ***M5***, ***M6*** *for positive matrices. Then, there exist constants* $s_0 > 0$ *and* $0 < c < C < +\infty$ *such that for any* $s \in (-s_0, 0)$ *and* $q = \Lambda'(s)$, *uniformly in* $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} c < & \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \rho(G_n) \leqslant n(q+l) \Big) \\ \leqslant & \limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{P}\Big(\log \rho(G_n) \leqslant n(q+l) \Big) < C. \end{aligned}$$

A more general version of Theorem 3.2.7 with a target function ψ on $\log \rho(G_n)$ will be presented in Section 3.8: see Theorem 3.8.1. The statements (1) and (2) in Theorem 3.2.7 clearly imply the following large deviation principle for $\log \rho(G_n)$: under the assumptions of Theorem 3.2.7, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log \rho(G_n) \geqslant n(q+l) \Big) = -\Lambda^*(q); \tag{3.2.18}$$

a similar assertion also holds for the lower tail. Note also that statements (1) and (2) of Theorem 3.2.7 still hold when the product G_n is replaced by M_n defined by $M_n = g_1 \dots g_n$.

The upper bound of part (1) in Theorem 3.2.7 follows from the reinforced large deviation principle for the matrix norm $\|G_n\|$ recently established in [85]. The lower bound will be deduced from our Theorem 3.2.2 in conjunction with the Collatz-Wielandt formula for positive matrices. Note that, the Collatz-Wielandt formula does not hold in general for invertible matrices, hence the question of proving Theorem 3.2.7 for invertible matrices remains open, even for the large deviation principle for $\rho(G_n)$; the latter has been recently conjectured by Sert [78]. The corresponding upper bound in large deviation principle for invertible matrices can be easily deduced from the results in [85]: under conditions **M2**, **M3**, for any $s \in I_\mu^\circ$ and $q = \Lambda'(s)$, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\log \rho(G_n) \geqslant n(q+l) \Big) \leqslant -\Lambda^*(q).$$

3.3 Hölder regularity of the stationary measure

3.3.1 Spectral gap properties and a change of measure

Recall that the transfer operator P_s and the conjugate transfer operator P_s^* are defined by (3.2.5). Below $P_s\nu_s$ stands for the measure on $\mathcal{S}$ such that $P_s\nu_s(\varphi) = \nu_s(P_s\varphi)$, for continuous functions φ on $\mathcal{S}$, and $P_s^*\nu_s^*$ is defined similarly. The spectral gap properties of P_s and P_s^* are summarized in the following proposition which was proved in [16, 17] for positive matrices, and in [50] for invertible matrices.

Proposition 3.3.1. *Assume either condition* ***M3*** *for invertible matrices. Then, for any* $s \in I_\mu^\circ$, *the following assertions hold:*
(1) *the spectral radii of the operators* P_s *and* P_s^* *are both equal to* $\kappa(s)$ *and there exist a unique, up to a scaling constant, strictly positive Hölder continuous function* r_s *and a unique probability measure* ν_s *on* $\mathcal{S}$ *such that*

$$P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s)\nu_s;$$

(2) *there exist a unique strictly positive Hölder continuous function* r_s^* *and a unique probability measure* ν_s^* *on* $\mathcal{S}$ *such that*

$$P_s^* r_s^* = \kappa(s) r_s^*, \quad P_s^* \nu_s^* = \kappa(s) \nu_s^*;$$

Moreover, the function $\kappa : I_\mu^\circ \mapsto \mathbb{R}$ *is analytic.*

The case of $s < 0$ is not covered by Proposition 3.3.1. We state below the corresponding result, which is proved in [85].

Proposition 3.3.2. *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or condition* ***M5*** *for positive matrices. Then there exists a constant* $s_0 > 0$ *such that for any* $s \in (-s_0, 0)$*, the assertions (1) and (2) of Proposition 3.3.1 remain valid. Moreover, the function* $\kappa : (-s_0, 0) \mapsto \mathbb{R}$ *is analytic.*

The following lemma gives explicit formulae for the eigenfunctions r_s and r_s^*.

Lemma 3.3.3.
(1) *Assume condition* ***M3*** *for invertible matrices. Then, for* $s \in I_\mu^\circ$*, the eigenfunctions* r_s *and* r_s^* *are given by*

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy), \quad x \in \mathcal{S}. \tag{3.3.1}$$

(2) *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or condition* ***M5*** *for positive matrices. Then there exists a constant* $s_0 > 0$ *such that for any* $s \in (-s_0, 0)$*, the expressions* (3.3.1) *hold.*

The first assertion of Lemma 3.3.3 for $s > 0$ was proved in [16, 17] for positive matrices, and in [50] for invertible matrices. The proof of the second one for $s < 0$ is quite different from that in the case $s > 0$ and is postponed to Section 3.6. It is based on the Hölder regularity of the eigenmeasures ν_s and ν_s^* which is the subject of the next section.

In Propositions 3.3.1 and 3.3.2, the eigenvalue $\kappa(s)$ and the eigenfunction r_s are both strictly positive. This fact allows us to perform a change of measure, as shown below. Under the corresponding assumptions of Propositions 3.3.1 and 3.3.2, for any $s \in (-s_0, 0) \cup I_\mu$, the family of probability kernels $q_n^s(x, g) = \frac{|gx|^s}{\kappa^n(s)} \frac{r_s(g\cdot x)}{r_s(x)}$, $n \geqslant 1$, satisfies the following cocycle property: for any $x \in \mathcal{S}$ and $g_1, g_2 \in \Gamma_\mu$,

$$q_n^s(x, g_1) q_m^s(g_1 \cdot x, g_2) = q_{n+m}^s(x, g_2 g_1). \tag{3.3.2}$$

Thus, the probability measures $\mathbb{Q}_{s,n}^x(dg_1, \dots, dg_n) = q_n^s(x, g_n \dots g_1)\mu(dg_1) \dots \mu(dg_n)$, $n \geqslant 1$, form a projective system on $M(d, \mathbb{R})^{\mathbb{N}}$. By the Kolmogorov extension theorem, there exists a unique probability measure $\mathbb{Q}_s^x$ on $M(d, \mathbb{R})^{\mathbb{N}}$, with marginals $\mathbb{Q}_{s,n}^x$. We denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation. By the definition of $\mathbb{Q}_s^x$, for any measurable function φ on $(\mathcal{S} \times \mathbb{R})^n$, it holds that

$$\begin{aligned} &\frac{1}{\kappa^n(s) r_s(x)} \mathbb{E}\Big[r_s(X_n^x) |G_n x|^s \varphi\Big(X_1^x, \log|G_1 x|, \dots, X_n^x, \log|G_n x|\Big)\Big] \\ &\qquad = \mathbb{E}_{\mathbb{Q}_s^x}\Big[\varphi\Big(X_1^x, \log|G_1 x|, \dots, X_n^x, \log|G_n x|\Big)\Big]. \end{aligned} \tag{3.3.3}$$

Under the changed measure $\mathbb{Q}_s^x$, the process

$$X_0^x = x, \quad X_n^x = G_n \cdot x, \quad n \geqslant 1.$$

forms a Markov chain on $\mathcal{S}$ with the transition operator given by

$$Q_s\varphi(x) = \frac{1}{\kappa(s) r_s(x)} P_s(\varphi r_s)(x) = \frac{1}{\kappa(s) r_s(x)} \int_{\Gamma_\mu} |g_1 x|^s \varphi(g_1 \cdot x) r_s(g_1 \cdot x) \mu(dg_1).$$

The Markov operator Q_s has a unique stationary probability measure π_s such that for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$\lim_{n\to\infty} Q_s^n \varphi = \pi_s(\varphi), \quad \text{where} \quad \pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}. \tag{3.3.4}$$

Let $\mathbb{Q}_s = \int_{\mathcal{S}} \mathbb{Q}_s^x \pi_s(dx)$. Then, the measure $\mathbb{Q}_s$ is shift-invariant and ergodic since π_s is the unique stationary measure of the Markov operator Q_s. When $s \in I_\mu$, the following SLLN was established in [50] for invertible matrices and in [16] for positive matrices, and in our setting can be read as follows: under either conditions **M2**, **M3** for invertible matrices, or condition **M4** (or condition **M5**) for positive matrices, for any $x \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \log |G_n x| = \lim_{n\to\infty} \frac{1}{n} \log \|G_n\| = \Lambda'(s), \quad \mathbb{Q}_s^x\text{-a.s. and } \mathbb{Q}_s\text{-a.s.}, \tag{3.3.5}$$

where $\Lambda'(s) = \frac{\kappa'(s)}{\kappa(s)}$ with the function κ defined in Proposition 3.3.1. Moreover, the CLT for $\log |G_n x|$ under the changed measures $\mathbb{Q}_s^x$ and $\mathbb{Q}_s$ was proved in [17]. When $s \in (-s_0, 0)$ with small enough $s_0 > 0$, the SLLN and the CLT for $\log |G_n x|$ under the measures $\mathbb{Q}_s^x$ and $\mathbb{Q}_s$ have been recently established in [86] for both invertible matrices and positive matrices.

3.3.2 Hölder regularity of the stationary measure

We shall present our results on the Hölder regularity of the stationary measure π_s and of the eigenmeasure ν_s for both invertible matrices and positive matrices. The regularity of π_s and ν_s is central to establishing the precise large deviation asymptotics for the entries $G_n^{i,j}$ and is also of independent interest.

Proposition 3.3.4. *Assume either conditions **M2**, **M3** for invertible matrices, or conditions **M4**, **M6** (or conditions **M1** **M5**, **M6**) for positive matrices. Then, for any $s \in I_\mu^\circ$, there exists a constant $c > 0$ such that*

$$\sup_{f\in\mathcal{S}} \int_{\mathcal{S}} \frac{1}{|\langle f, x\rangle|^c} \pi_s(dx) < +\infty \quad \textit{and} \quad \sup_{f\in\mathcal{S}} \int_{\mathcal{S}} \frac{1}{|\langle f, x\rangle|^c} \nu_s(dx) < +\infty. \tag{3.3.6}$$

In particular, for any $s \in I_\mu^\circ$, there exist constants $c, C > 0$ such that for any $0 < t < 1$,

$$\sup_{f\in\mathcal{S}} \pi_s\Big(\{x : |\langle f, x\rangle| \leqslant t\}\Big) \leqslant Ct^c \ \textit{and} \ \sup_{f\in\mathcal{S}} \nu_s\Big(\{x : |\langle f, x\rangle| \leqslant t\}\Big) \leqslant Ct^c. \tag{3.3.7}$$

*Moreover, for positive matrices, under conditions **M4**, **M6**, the constant $c > 0$ in (3.3.6) and (3.3.7) can be sufficiently large (independent of the dimension $d-1$ of the projective space $\mathcal{S} = \mathbb{S}_+^{d-1}$).*

For $s = 0$ such regularity was used to obtain limit theorems for the entries $G_n^{i,j}$ and for the spectral radius $\rho(G_n)$ by many authors, however, such a result has not been established before neither for invertible matrices, nor for positive matrices under the changed measure. For invertible matrices, the proof of the assertion (3.3.6) is based on the asymptotic properties of the components in the Cartan and Iwasawa decompositions of the reversed random matrix product $M_n = g_1 \dots g_n$ and on the simplicity of the dominant Lyapunov exponent of G_n under the changed measure $\mathbb{Q}_s^x$: see Section 3.5.1.

For positive matrices, under conditions **M4** and **M6**, the proof of the assertion (3.3.6) relies on the fact that $\operatorname{supp} \nu = \operatorname{supp} \nu_s$ $(s > 0)$ established in [16] and essentially on condition **M4** which ensures that the Markov chain $(X_n^x)_{n \geqslant 1}$ stays forever in the interior of the projective space $\mathbb{S}_+^{d-1}$: see Section 3.5.2. If condition **M4** is replaced by **M5**, the main difficulty to prove (3.3.6) is that the Markov chain $(X_n^x)_{n \geqslant 1}$ is no longer separated from the coordinates $(e_k)_{1 \leqslant k \leqslant d}$, hence the proof can not follow directly from the fact that $\operatorname{supp} \nu = \operatorname{supp} \nu_s$. Instead, the main ingredient in our proof consists in the large deviation asymptotic for the norm cocycle $\log |G_n x|$ under the changed measure $\mathbb{Q}_s^x$ established in Theorem 3.4.4.

When s is non-positive and sufficiently close to 0, the Hölder regularity of the stationary measure π_s is given by the following proposition.

Proposition 3.3.5. *Assume either conditions **M2**, **M3** for invertible matrices, or conditions **M5**, **M6** for positive matrices. Then, there exist constants $c > 0$ and $s_0 > 0$ such that uniformly in $s \in (-s_0, 0]$, the statements (3.3.6) and (3.3.7) are valid.*

For positive matrices, Proposition 3.3.5 is new even for $s = 0$: in this case $\pi_0 = \nu_0 = \nu$ with the stationary measure ν defined by (3.2.4). The corresponding results for invertible matrices with $s = 0$ (in this case also $\pi_0 = \nu_0 = \nu$) has been obtained in [51]; we also refer to [13] for the detailed description of the method used in [51] and to [14] and [10] for a different approach of the proof. When $s < 0$, Proposition 3.3.5 is deduced from the Hölder regularity of the measure ν and the analyticity of the eigenfunction κ.

The proofs of Propositions 3.3.4 and 3.3.5 are technically involved and are postponed to Sections 3.5 and 3.6.

3.4 Auxiliary statements

Under conditions of Theorem 3.2.1, the function $\Lambda = \log \kappa$ is convex and analytic on $(-s_0, 0) \cup I_\mu$. Set $\gamma_{s,k} = \Lambda^{(k)}(s)$, $k \geqslant 1$. Under the changed measure $\mathbb{Q}_s^x$, define the Cramér series ζ_s (see Petrov [74]) by

$$\zeta_s(t) = \frac{\gamma_{s,3}}{6\gamma_{s,2}^{3/2}} + \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{24\gamma_{s,2}^3} t + \frac{\gamma_{s,5}\gamma_{s,2}^2 - 10\gamma_{s,4}\gamma_{s,3}\gamma_{s,2} + 15\gamma_{s,3}^3}{120\gamma_{s,2}^{9/2}} t^2 + \dots,$$

which converges for small enough $|t|$. We need the following expansion of $\Lambda^*(q + l)$ with respect to the perturbation l, which is taken from [85]:

Lemma 3.4.1. *Assume either conditions of Proposition 3.3.4 when $s \in I_\mu^\circ$, or conditions of Proposition 3.3.5 when $s \in (-s_0, 0)$ with small enough $s_0 > 0$. Let $q = \Lambda'(s)$. Then, there exists $\delta > 0$ such that for any $|l| \leqslant \delta$,*

$$\Lambda^*(q+l) = \Lambda^*(q) + sl + h_s(l),$$

where h_s is linked to the Cramér series ζ_s by the identity

$$h_s(l) = \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s(\frac{l}{\sigma_s}).$$

Now let us fix a density function ρ on the real line. Its Fourier transform $\hat{\rho}$ is supported on a the interval $[-1, 1]$. For any $\varepsilon > 0$, define the scaled density function ρ_ε by $\rho_\varepsilon(y) = \frac{1}{\varepsilon}\rho(\frac{y}{\varepsilon})$, $y \in \mathbb{R}$, whose Fourier transform $\widehat{\rho}_\varepsilon$ is compactly support on $[-\varepsilon^{-1}, \varepsilon^{-1}]$. For any non-negative integrable function ψ on $\mathbb{R}$, we introduce two modified functions related to ψ as follows: for any $y \in \mathbb{R}$, set $\mathbb{B}_\varepsilon(y) = \{y' \in \mathbb{R} : |y' - y| \leqslant \varepsilon\}$ and

$$\psi_\varepsilon^+(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \psi(y') \quad \text{and} \quad \psi_\varepsilon^-(y) = \inf_{y' \in \mathbb{B}_\varepsilon(y)} \psi(y'). \tag{3.4.1}$$

The following smoothing inequality gives the two-sided bounds of the function ψ.

Lemma 3.4.2. *Suppose that ψ is a non-negative integrable function and that ψ_ε^+ and ψ_ε^- are measurable for any $\varepsilon > 0$, then for sufficiently small ε, there exists a positive constant $C_\rho(\varepsilon)$ with $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that, for any $x \in \mathbb{R}$,*

$$\psi_\varepsilon^- * \rho_{\varepsilon^2}(x) - \int_{|y| \geqslant \varepsilon} \psi_\varepsilon^-(x-y)\rho_{\varepsilon^2}(y)dy \leqslant \psi(x) \leqslant (1 + C_\rho(\varepsilon))\psi_\varepsilon^+ * \rho_{\varepsilon^2}(x).$$

The proof of the above lemma, being similar to that of Lemma 5.2 in [42], will not be detailed here.

For any $s \in (-s_0, 0) \cup I_\mu$ and $t \in \mathbb{R}$, define a family of perturbed operators $R_{s,it}$ as follows: for any $\varphi \in \mathcal{B}_\gamma$,

$$R_{s,it}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[e^{it(\log|g_1 x| - q)}\varphi(X_1^x)\Big], \quad x \in \mathcal{S}. \tag{3.4.2}$$

It follows from the cocycle property (3.3.2) that

$$R_{s,it}^n\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[e^{it(\log|G_n x| - nq)}\varphi(X_n^x)\Big], \quad x \in \mathcal{S}.$$

Under various restrictions on s, it was shown in [17, 85, 86] that the operator $R_{s,it}$ acts onto the Banach space $\mathcal{B}_\gamma$ and has a spectral gap.

The next proposition is taken from [85]. Its proof is based on the spectral gap properties of the perturbed operator $R_{s,it}$. In the following, let φ be a strictly positive and γ-Hölder continuous function on the projective space $\mathcal{S}$. Assume that the function $\psi : \mathbb{R} \mapsto \mathbb{C}$ is a bounded measurable function with compact support in $\mathbb{R}$, and moreover, ψ is differentiable in a small neighborhood of 0 on the real line.

Proposition 3.4.3.

(1) *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or conditions* ***M1***, ***M4***, ***M6*** *(or conditions* ***M1***, ***M5***, ***M6****) for positive matrices. Let* $s \in I_\mu^\circ$ *and* $q = \Lambda'(s)$. *Then, we have, as* $n \to \infty$, *uniformly in* $x \in \mathcal{S}$, $|l| \leqslant \frac{1}{\sqrt{n}}$ *and* $\varphi \in \mathcal{B}_\gamma$,

$$\Big|\sqrt{n}\ \sigma_s e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R^n_{s,it}(\varphi)(x)\psi(t)dt - \sqrt{2\pi}\psi(0)\pi_s(\varphi)\Big| \leqslant \Big(\frac{C}{\sqrt{n}} + |l|\Big)\|\varphi\|_\gamma. \quad (3.4.3)$$

(2) *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or conditions* ***M5***, ***M6*** *for positive matrices. Then, there exists* $s_0 > 0$ *such that for any* $s \in (-s_0, 0)$ *and* $q = \Lambda'(s)$, *the inequality* (3.4.3) *holds uniformly in* $x \in \mathcal{S}$, $|l| \leqslant \frac{1}{\sqrt{n}}$ *and* $\varphi \in \mathcal{B}_\gamma$.

The perturbation l as well as the explicit rate of convergence in (3.4.3) are important in the sequel. They play the crucial role to establish the Bahadur-Rao type large deviations for the entries $G_n^{i,j}$ in Theorems 3.2.1 and 3.2.3.

We end this section by giving a precise large deviation result for the norm cocycle $\log|G_n x|$ under the changed measure $\mathbb{Q}_s^x$. It is is deduced from [85, Theorem 2.2] and will be used in the proof of regularity of the stationary measure π_s for positive matrices under conditions **M5**, **M6** (see Proposition 3.3.4).

Theorem 3.4.4. *Assume either conditions* ***M2***, ***M3*** *for invertible matrices, or conditions* ***M1***, ***M4***, ***M6*** *(or conditions* ***M1***, ***M5***, ***M6****) for positive matrices. Let* $s \in I_\mu$, $t \in I_\mu^\circ$ *be such that* $s < t$ *and set* $q_s = \Lambda'(s)$ *and* $q_t = \Lambda'(t)$. *Then, we have, as* $n \to \infty$, *uniformly in* $x \in \mathcal{S}$ *and* $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}&\mathbb{Q}_s^x\Big(\log|G_n x| \geqslant n(q_t + l)\Big)\\ &\qquad = \frac{\nu_t(r_s)}{\nu_t(r_t)}\frac{r_t(x)}{r_s(x)}\frac{\exp\{-n(\Lambda^*(q_t+l) - \Lambda^*(q_s) - s(q_t - q_s + l))\}}{(t-s)\sigma_t\sqrt{2\pi n}}[1 + o(1)].\end{aligned}$$

Proof. By (3.3.3), we get

$$\begin{aligned}&\mathbb{Q}_s^x(\log|G_n x| \geqslant n(q_t + l))\\ &= \frac{1}{\kappa^n(s)r_s(x)}\mathbb{E}\left(r_s(X_n^x)|G_n x|^s \mathbb{1}_{\{\log|G_n x| \geqslant n(q_t+l)\}}\right)\\ &= \frac{1}{\kappa^n(s)r_s(x)}e^{sn(q_t+l)}\mathbb{E}\Big[r_s(X_n^x)\psi_s\Big(\log|G_n x| - n(q_t+l)\Big)\Big],\end{aligned}$$

where $\psi_s(y) = e^{sy}\mathbb{1}_{\{y \geqslant 0\}}$, $y \in \mathbb{R}$. From Theorem 2.2 in [85,] it follows that for any $t \in I_\mu^\circ$, $q_t = \Lambda'(t)$, $\varphi \in \mathcal{B}_\gamma$ and measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-t'y}\psi(y)$ is directly Riemann integrable for some $t' \in (0, s)$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}&\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q_t + l))\Big]\\ &\qquad = \frac{r_t(x)}{\varrho_t}\frac{\exp(-n\Lambda^*(q_t+l))}{\sigma_t\sqrt{2\pi n}}\Big[\nu_t(\varphi)\int_{\mathbb{R}} e^{-ty}\psi(y)dy + o(1)\Big]. \qquad (3.4.4)\end{aligned}$$

Using (3.4.4) with $\varphi = r_s$ and $\psi = \psi_s$, we obtain that, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\mathbb{E}\Big[r_s(X_n^x)\psi_s\Big(\log|G_n x| - n(q_t+l)\Big)\Big] = \frac{r_t(x)}{\varrho_t}\nu_t(r_s)\frac{e^{-n\Lambda^*(q_t+l)}}{(t-s)\sigma_t\sqrt{2\pi n}}\Big[1 + o(1)\Big].$$

We conclude the proof of Theorem 3.4.4 by using the fact that $\Lambda^*(q) = sq - \Lambda(s)$, $\Lambda(s) = \log\kappa(s)$ and $\varrho_t = \nu_t(r_t)$. □

3.5 Proof of the Hölder regularity of π_s for positive s

In this section we prove Proposition 3.3.4, i.e. the Hölder regularity of the stationary measure π_s for both invertible matrices and positive matrices for $s > 0$. These results are of independent interest and play a crucial role in establishing the precise large deviation asymptotics for scalar products and entries, see Theorem 3.2.2.

The study of the regularity of the stationary measure ν defined by (3.2.4), attracted a great deal of attention, see e.g. [1, 7, 9, 10, 13, 14, 27, 31, 48, 51]. As far as we know, there are three different approaches to establish the regularity of ν. The first one, consists in making use of the simplicity of the dominant Lyapunov exponent λ and investigating the asymptotic behaviors of the components in the Cartan and Iwasawa decompositions of the random matrix product M_n and is originally due to Guivarc'h and Raugi [51], see also [13, 48]. The second one is developed in [14] for the study of the regularity of the stationary measure on the torus, and has been applied to the setting of products of random matrices in [9, 10], where the large deviation bounds for the Iwasawa cocycle and for the Cartan projection play a crucial role. The third one, which is recently developed in [31] for the special linear group $SL(2, \mathbb{C})$ consisting of complex 2×2 matrices with determinant one, is based on the theory of super-potentials introduced in [32]. All of the results mentioned above are concerned with the regularity of the stationary measure ν. However, the regularity of the eigenmeasure ν_s or of the stationary measure π_s for s different from 0 was not known before in the literature, neither for invertible matrices nor for positive matrices.

As the proofs are rather long we split them into two parts. In Subsection 3.5.1 we establish Proposition 3.3.4 for invertible matrices. In Subsection 3.5.2 we prove Proposition 3.3.4 for positive matrices.

3.5.1 Regularity of the stationary measure for invertible matrices

In order to prove Proposition 3.3.4 for invertible matrices, we first extend the some convergence results concerning the Cartan and Iwasawa decompositions of the matrix product M_n established earlier in [13] under the measure $\mathbb{P}$, to the framework of the changed measure $\mathbb{Q}_s$.

Asymptotics for the Cartan decomposition

Recall that $G_n = g_n \dots g_1$. We are going to investigate the asymptotic behavior of the components of the Cartan decomposition of the reversed matrix product

$$M_n = g_1 g_2 \dots g_n, \quad n \geqslant 1.$$

Note that the distributions of the random walks M_n and G_n coincide. However, as it will be seen below, the asymptotics of the components of their Cartan decompositions are not exactly the same. Let $K = SO(d, \mathbb{R})$ be the orthogonal group, and A^+ be the set of diagonal matrices whose diagonal entries starting from the upper left corner are strictly positive and decreasing. With these notation, the well known Cartan decomposition states that $GL(d, \mathbb{R}) = KA^+K$. The Cartan decomposition of the

matrix product M_n will be denoted by $M_n = k_n a_n k'_n$, where $k_n, k'_n \in K$ and $a_n \in A^+$ with its diagonal elements (singular values) satisfying $a_n^{1,1} \geqslant a_n^{2,2} \geqslant \ldots \geqslant a_n^{d,d} > 0$. Note that the diagonal matrix a_n is uniquely determined, but the orthogonal matrices k_n and k'_n are not unique. We choose one such decomposition of M_n. The vector $k_n e_1 \in \mathbb{P}^{d-1}$ is called the *density point* of M_n. It plays an important role in the study of products of random matrices: see [14, 10]. The following result shows that the density point converges almost surely to the random variable Z_s of the law π_s under the changed measure $\mathbb{Q}_s = \int_{\mathbb{P}^{d-1}} \mathbb{Q}_s^x \pi_s(dx)$.

Lemma 3.5.1. *Let $s \in I_\mu^\circ$. Under condition **M3**, with the above notation, we have*

$$\lim_{n\to\infty} \frac{a_n^{2,2}}{a_n^{1,1}} = 0, \ \mathbb{Q}_s\text{-a.s.} \quad \text{and} \quad \lim_{n\to\infty} k_n e_1 = Z_s, \ \mathbb{Q}_s\text{-a.s.}, \tag{3.5.1}$$

and for any $x \in \mathbb{P}^{d-1}$,

$$\lim_{n\to\infty} \frac{|M_n^{\mathrm{T}} x|}{\|M_n^{\mathrm{T}}\|} = |\langle x, Z_s\rangle|, \ \mathbb{Q}_s\text{-a.s.}, \tag{3.5.2}$$

where the law of the random variable Z_s (on $\mathbb{P}^{d-1}$) is the stationary measure π_s. Moreover, the assertions (3.5.1) and (3.5.2) also hold true with the measure $\mathbb{Q}_s$ replaced by $\mathbb{Q}_s^x$, for any starting point $x \in \mathbb{P}^{d-1}$.

Before proceeding to proving Lemma 3.5.1, let us first recall the following two results which were established in [50]. In the sequel, let m be the unique rotation invariant probability measure on the projective space $\mathbb{P}^{d-1}$. For any matrix $g \in GL(d, \mathbb{R})$, denote by $g \cdot m$ the probability measure on $\mathbb{P}^{d-1}$ such that for any measurable function φ on $\mathbb{P}^{d-1}$,

$$\int_{\mathbb{P}^{d-1}} \varphi(x)(g \cdot m)(dx) = \int_{\mathbb{P}^{d-1}} \varphi(g \cdot x) m(dx).$$

Lemma 3.5.2. *Assume condition **M3**. Let $s \in I_\mu^\circ$. Then, the probability measure $M_n \cdot m$ converges weakly to the Dirac measure δ_{Z_s}, $\mathbb{Q}_s$-a.s., where the law of the random variable Z_s is given by π_s.*

Proof. This result is a direct consequence of [50, Theorem 3.2]. More specifically, according to [50], the probability measure $G_n^{\mathrm{T}} \cdot m$ converges weakly to a Dirac measure $\delta_{Z_s^*}$, $\mathbb{Q}_s$-a.s., where the law of the random variable Z_s^* is given by π_s^*. Here π_s^* is the stationary measure of the conjugate Markov operator Q_s^* defined by $Q_s^* \varphi(x) = \frac{1}{\kappa(s) r_s^*(x)} P_s^*(\varphi r_s^*)(x)$, $x \in \mathbb{P}^{d-1}$, for any $\varphi \in \mathcal{C}(\mathcal{S})$. Denote by μ^{T} the image of the measure μ by the map $g \mapsto g^{\mathrm{T}}$. Then, Lemma 3.5.2 follows since the measure μ^{T} also satisfies condition **M3**. □

The following result is proved in [50, Lemma 3.5].

Lemma 3.5.3. *Assume condition **M3**. Let $s \in I_\mu^\circ$. Then, there exists a constant $c_s > 0$ such that for any $x \in \mathbb{P}^{d-1}$, it holds that $\mathbb{Q}_s^x \leqslant c_s \mathbb{Q}_s$.*

The assertion of Lemma 3.5.3 implies that the measure $\mathbb{Q}_s^x$ is absolutely continuous with respect to $\mathbb{Q}_s$. Using Lemmas 3.5.2 and 3.5.3, we are now in a position to prove Lemma 3.5.1.

Proof of Lemma 3.5.1. By the Cartan decomposition of M_n, we have $M_n = k_n a_n k'_n$, where $k_n, k'_n \in K$ and $a_n \in A^+$. By Lemma 3.5.2, the probability measure $M_n \cdot m$ converges weakly to the Dirac measure δ_{Z_s}, $\mathbb{Q}_s$-a.s.. Since m is a rotation invariant measure on $\mathbb{P}^{d-1}$, we deduce that $k_n a_n \cdot m$ converges weakly to the random variable Z_s, $\mathbb{Q}_s$-a.s.. Taking into account that a_n is a diagonal random matrix with decreasing diagonal entries, this implies that, as $n \to \infty$, we have $a_n \cdot m \to \delta_{e_1}$, $a_n^{2,2}/a_n^{1,1} \to 0$ and $k_n e_1 \to Z_s$, $\mathbb{Q}_s$-a.s.. This concludes the proof of the assertion (3.5.1). To show (3.5.2), using again the decomposition $M_n = k_n a_n k'_n$, it follows that for any $x \in \mathbb{P}^{d-1}$,

$$|M_n^{\mathrm{T}} x|^2 = \langle M_n^{\mathrm{T}} x, M_n^{\mathrm{T}} x\rangle = \langle a_n k_n^{\mathrm{T}} x, a_n k_n^{\mathrm{T}} x\rangle = \sum_{j=1}^{d} (a_n^{j,j})^2 |\langle k_n^{\mathrm{T}} x, e_j\rangle|^2. \tag{3.5.3}$$

The statement (3.5.2) follows from the identity (3.5.3) and the fact that $\|M_n^{\mathrm{T}}\| = a_n^{1,1}$. Taking into account of Lemma 3.5.3, we see that the assertions (3.5.1) and (3.5.2) remain valid with the measure $\mathbb{Q}_s$ replaced by $\mathbb{Q}_s^x$. □

Asymptotics for the Iwasawa decomposition

In this subsection we study the asymptotic of the components in the Iwasawa decomposition of M_n under the changed measure $\mathbb{Q}_s^x$. Denote by L the group of lower triangular matrices with 1 in the diagonal elements, by A the group of diagonal matrices with strictly positive entries in the diagonal elements, and as before by K the group of orthogonal matrices. The Iwasawa decomposition states that $GL(d, \mathbb{R}) = LAK$ and such decomposition is unique. Therefore, for the product M_n, there exist unique $L(M_n) \in L$, $A(M_n) \in A$ and $K(M_n) \in K$ such that $M_n = L(M_n)A(M_n)K(M_n)$. With these notation, we have the following:

Lemma 3.5.4. *Let $s \in I_\mu^\circ$. Under condition **M3**, for any $x \in \mathbb{P}^{d-1}$,*

$$\lim_{n\to\infty} L(M_n)e_1 = \frac{Z_s}{\langle Z_s, e_1\rangle}, \quad \mathbb{Q}_s\text{-a.s. and } \mathbb{Q}_s^x\text{-a.s..}$$

where Z_s is a random variable given by Lemma 3.5.1.

Proof. In view of Lemma 3.5.3, it suffices to prove the assertion under the measure $\mathbb{Q}_s$. Using the Iwasawa decomposition $M_n = L(M_n)A(M_n)K(M_n)$ and noticing that $K(M_n)$ is an orthogonal matrix, it follows that

$$\frac{M_n M_n^{\mathrm{T}} e_1}{|M_n^{\mathrm{T}} e_1|^2} = \frac{L(M_n)A(M_n)^2 L(M_n)^{\mathrm{T}} e_1}{|A(M_n)L(M_n)^{\mathrm{T}} e_1|^2} = L(M_n)e_1, \tag{3.5.4}$$

where the second equality holds since $A(M_n)^2 L(M_n)^{\mathrm{T}} e_1 = |A(M_n)L(M_n)e_1|^2 e_1$. By the Cartan decomposition of M_n we have $M_n = k_n a_n k'_n$, where k_n, k'_n are two orthogonal matrices, hence we get that, for any $y \in \mathbb{R}^d$,

$$\begin{aligned}\langle M_n M_n^{\mathrm{T}} e_1, y\rangle &= \langle (a_n)^2 k_n^{\mathrm{T}} e_1, k_n^{\mathrm{T}} y\rangle \\ &= (a_n^{1,1})^2 \langle k_n^{\mathrm{T}} e_1, e_1\rangle \langle k_n^{\mathrm{T}} y, e_1\rangle + O(a_n^{1,1} a_n^{2,2}) \\ &= (a_n^{1,1})^2 \langle k_n e_1, e_1\rangle \langle k_n e_1, y\rangle + O(a_n^{1,1} a_n^{2,2}).\end{aligned} \tag{3.5.5}$$

Consequently, we obtain that $\mathbb{Q}_s$-a.s.,

$$\lim_{n\to\infty}\langle L(M_n)e_1, y\rangle = \lim_{n\to\infty}\frac{\langle M_n M_n^{\mathrm{T}} e_1, y\rangle}{\langle M_n M_n^{\mathrm{T}} e_1, e_1\rangle} = \lim_{n\to\infty}\frac{\langle k_n e_1, y\rangle}{\langle k_n e_1, e_1\rangle} = \frac{\langle Z_s, y\rangle}{\langle Z_s, e_1\rangle},$$

where in the first equality we use (3.5.4), in the second one we use (3.5.5) and Lemma 3.5.1, and in the last one we apply again Lemma 3.5.1. Since $y \in \mathbb{R}^d$ is arbitrary, the proof of Lemma 3.5.4 is complete. □

For any $1 \leqslant k \leqslant d$, we briefly recall the notion of exterior algebra $\wedge^k(\mathbb{R}^d)$ of the vector space $\mathbb{R}^d$. The space $\wedge^k(\mathbb{R}^d)$ is endowed with the scalar product $\langle\cdot,\cdot\rangle$ and the norm $|\cdot|$; we use the same notation as in $\mathbb{R}^d$ and the distinction should be clear from the context. The scalar product in $\wedge^k(\mathbb{R}^d)$ satisfies the following property: for any u_i, $v_j \in \mathbb{R}^d$, $1 \leqslant i, j \leqslant d$,

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k\rangle = det(\langle u_i, v_j\rangle)_{1\leqslant i,j\leqslant d},$$

where $det(\langle u_i, v_j\rangle)_{1\leqslant i,j\leqslant d}$ denotes the determinant of the associated matrix. It is well known that $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}, 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant d\}$ forms a basis of $\wedge^k(\mathbb{R}^d)$, $1 \leqslant k \leqslant d$, and that $v_1 \wedge \cdots \wedge v_k$ is nonzero if and only if $v_1, \ldots, v_k$ are linearly independent in $\mathbb{R}^d$. For any $g \in GL(d,\mathbb{R})$ and $1 \leqslant k \leqslant d$, the exterior product $\wedge^k g$ of the matrix g is defined as follows: for any $v_1, \ldots, v_k \in \mathbb{R}^d$,

$$\wedge^k g(v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k.$$

Set $\| \wedge^k g\| = \sup\{|(\wedge^k g)v| : v \in \wedge^k(\mathbb{R}^d), |v| = 1\}$. Since $\wedge^k(gg') = (\wedge^k g)(\wedge^k g')$, it holds that $\| \wedge^k (gg')\| \leqslant \| \wedge^k g\|\| \wedge^k g'\|$ for any $g, g' \in GL(d,\mathbb{R})$. Besides, if we denote by $a_{11}, \ldots, a_{dd}$ the singular values of the matrix g, then $\| \wedge^k g\| = a_{11} \ldots a_{kk}$. In particular, we have $\| \wedge^k g\| \leqslant \|g\|^k$.

The following lemma was proved in [13]. For any $g \in GL(d,\mathbb{R})$, by the Iwasawa decomposition we have $g = L(g)A(g)K(g)$, where $L(g) \in L$, $A(g) \in A$ and $K(g) \in K$. In the sequel, we denote $N(g) = \max\{\|g\|, \|g^{-1}\|\}$. Recall that $M_n = g_1 g_2 \ldots g_n$.

Lemma 3.5.5. *For any integers $n, m \geqslant 0$, we have*

$$|L(M_{n+m})e_1 - L(M_n)e_1| \leqslant \sum_{j=n}^{n+m-1} \frac{\| \wedge^2 M_j^{\mathrm{T}}\|}{|M_j^{\mathrm{T}} e_1|^2} e^{2\log N(g_{j+1})},$$

where we use the convention that $L(M_0) = 0$ and $\frac{\|\wedge^2 M_0^{\mathrm{T}}\|}{|M_0^{\mathrm{T}} e_1|^2} = 0$.

The following result shows the simplicity of the dominant Lyapunov exponent for M_n^{T} under the changed measure $\mathbb{Q}_s^x$.

Lemma 3.5.6. *Assume condition **M3**. Let $s \in I_\mu^\circ$. Then, uniformly in $x \in \mathbb{P}^{d-1}$,*

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathbb{Q}_s^x}(\log|M_n^{\mathrm{T}} x|) = \lambda_1(s), \tag{3.5.6}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathbb{Q}_s^x}(\log\| \wedge^2 M_n^{\mathrm{T}}\|) = \lambda_1(s) + \lambda_2(s), \tag{3.5.7}$$

where $\lambda_1(s) > \lambda_2(s)$ are called the first two Lyapunov exponents of M_n^{T} under the measure $\mathbb{Q}_s^x$.

The assertion (3.5.6) is proved in [50, Theorem 3.10]. The assertion (3.5.7) follows by combining Theorems 3.10 and 3.17 in [50]. The fact that $\lambda_1(s) > \lambda_2(s)$ will play an essential role in the proof of the Hölder regularity of the stationary measure π_s for invertible matrices, see Proposition 3.3.4.

Using the simplicity of the Lyapunov exponent (see Lemma 3.5.6) we can complement the convergence result in Lemma 3.5.4 by giving the rate of convergence. This result is not used in the proofs, but is of independent interest.

Proposition 3.5.7. *Assume condition **M3**. Let $s \in I_\mu^\circ$. Then, there exist constants $c, C > 0$ such that uniformly in $x \in \mathbb{P}^{d-1}$ and $n \geqslant 1$,*

$$\mathbb{E}_{\mathbb{Q}_s^x}\left|L(M_n)e_1 - \frac{Z_s}{\langle Z_s, e_1\rangle}\right|^c \leqslant e^{-Cn}. \tag{3.5.8}$$

Moreover, the assertion (3.5.8) remains valid when the measure $\mathbb{Q}_s^x$ is replaced by $\mathbb{Q}_s$.

The proof of Proposition 3.5.7 is postponed to Subsection 3.5.1.

By Jensen's inequality, the bound (3.5.8) implies that there exists a constant $C > 0$ such that uniformly in $x \in \mathbb{P}^{d-1}$,

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{Q}_s^x} \log\left|L(M_n)e_1 - \frac{Z_s}{\langle Z_s, e_1\rangle}\right| \leqslant -C.$$

When $s = 0$, it was proved in [13] that $C = \lambda_1(0) - \lambda_2(0)$. We conjecture that $C = \lambda_1(s) - \lambda_2(s)$ also for $s > 0$, but the proof eluded us.

Proof of Propositions 3.3.4 and 3.5.7

With the results established in Sections 3.5.1 and 3.5.1, we are well equipped to prove Propositions 3.3.4 and 3.5.7 for invertible matrices.

Proof of Proposition 3.3.4 for invertible matrices. Since r_s is bounded away from infinity and 0 uniformly on $\mathbb{P}^{d-1}$, it suffices to establish (3.3.6) and (3.3.7) for the stationary measure π_s.

Define the function $\rho : GL(d,\mathbb{R}) \times \mathbb{P}^{d-1} \to \mathbb{R}$ as follows: for $g \in GL(d,\mathbb{R})$ and $x \in \mathbb{P}^{d-1}$,

$$\rho(g,x) = \log\|\wedge^2 g\| - 2\log|gx|.$$

It is clear that

$$\mathbb{E}_{\mathbb{Q}_s^x}\rho(M_n^{\mathrm{T}}, x) = \mathbb{E}_{\mathbb{Q}_s^x}\Big(\log\|\wedge^2 M_n^{\mathrm{T}}\|\Big) - 2\mathbb{E}_{\mathbb{Q}_s^x}\Big(\log|M_n^{\mathrm{T}}x|\Big).$$

By Lemma 3.5.6, we see that

$$\lim_{n\to\infty}\frac{1}{n}\sup_{x\in\mathbb{P}^{d-1}}\mathbb{E}_{\mathbb{Q}_s^x}\rho(M_n^{\mathrm{T}}, x) < 0,$$

which clearly implies that, for large enough n,

$$\sup_{x\in\mathbb{P}^{d-1}}\mathbb{E}_{\mathbb{Q}_s^x}\rho(M_n^{\mathrm{T}}, x) < 0. \tag{3.5.9}$$

We claim that there exists a small constant $c > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in\mathbb{P}^{d-1}} \mathbb{E}_{\mathbb{Q}_s^x} \frac{\| \wedge^2 M_n^{\mathrm{T}}\|^c}{|M_n^{\mathrm{T}} x|^{2c}} < 0. \tag{3.5.10}$$

To prove (3.5.10), we denote $a_n = \log\Big(\sup_{x\in\mathbb{P}^{d-1}} \mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{c\rho(M_n^{\mathrm{T}},x)}\Big)\Big)$, for sufficiently small constant $c > 0$. Using the cocycle property (3.3.2) and the fact that ρ is subadditive, we get that for any $n, m \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{c\rho(M_{n+m}^{\mathrm{T}},x)}\Big)\\
&\leqslant \mathbb{E}\Big(q_m^s(x, G_m)e^{c\rho(M_m^{\mathrm{T}},x)}\Big)\mathbb{E}\Big(q_n^s(x, g_{m+1}\dots g_{m+n})e^{c\rho(g_{m+n}^{\mathrm{T}}\cdots g_{m+1}^{\mathrm{T}},x)}\Big)\\
&= \mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{c\rho(M_m^{\mathrm{T}},x)}\Big)\mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{c\rho(M_n^{\mathrm{T}},x)}\Big).
\end{aligned}$$

Taking supremum on both sides of the above inequality, this yields that the sequence $(a_n)_{n\geqslant 1}$ satisfies the subadditive property: $a_{n+m} \leqslant a_m + a_n$, hence we get $a = \lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geqslant 1} \frac{a_n}{n}$. To show that $a < 0$, it suffices to check that there exists some integer $p \geqslant 1$ such that

$$\sup_{x\in\mathbb{P}^{d-1}} \mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{c\rho(M_p^{\mathrm{T}},x)}\Big) < 1. \tag{3.5.11}$$

We proceed to verify (3.5.11). Using the fact that $\sup_x |\rho(g, x)| \leqslant 4 \log N(g)$ and the basic inequality $e^y \leqslant 1 + y + \frac{y^2}{2} e^{|y|}$, $y \in \mathbb{R}$, we obtain

$$\mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{c\rho(M_p^{\mathrm{T}},x)}\Big) \leqslant 1 + c\mathbb{E}_{\mathbb{Q}_s^x}\Big(\rho(M_p^{\mathrm{T}}, x)\Big) + \frac{c^2}{2}\mathbb{E}_{\mathbb{Q}_s^x}\Big(16 \log^2 N(M_p^{\mathrm{T}})e^{4c\log N(M_p^{\mathrm{T}})}\Big). \tag{3.5.12}$$

The second term on the right-hand side of (3.5.12) is strictly negative by using the bound (3.5.9) and taking large enough p. The third term is finite due to the moment condition (3.2.1). Consequently, taking $c > 0$ small enough, we obtain the inequality (3.5.11) and thus the desired assertion (3.5.10) follows.

Since the bound (3.5.10) holds uniformly over $x \in \mathbb{P}^{d-1}$, taking into account that $\mathbb{Q}_s = \int_{\mathbb{P}^{d-1}} \mathbb{Q}_s^x \pi_s(dx)$, it follows that there exist constants $C > 0$ and $0 < r < 1$ such that for any $x \in \mathbb{P}^{d-1}$,

$$\mathbb{E}_{\mathbb{Q}_s} \frac{\| \wedge^2 M_n^{\mathrm{T}}\|^c}{|M_n^{\mathrm{T}} x|^{2c}} \leqslant Cr^n. \tag{3.5.13}$$

Using Lemma 3.5.4, Fatou's lemma and the fact that $|Z_s| = 1$, we have that for sufficiently small constant $c > 0$,

$$\mathbb{E}_{\mathbb{Q}_s} \frac{1}{|\langle Z_s, e_1\rangle|^c} \leqslant \liminf_{n\to\infty} \mathbb{E}_{\mathbb{Q}_s}\Big(|L(M_n)e_1|^c\Big). \tag{3.5.14}$$

From Lemma 3.5.5 with $n = 0$, it follows that

$$|L(M_n)e_1|^c \leqslant \sum_{j=1}^{\infty} \frac{\| \wedge^2 M_j^{\mathrm{T}}\|^c}{|M_j^{\mathrm{T}} e_1|^{2c}} e^{2c\log N(g_{j+1})}.$$

Notice that M_j^{T} and g_{j+1} are not independent under the measure $\mathbb{Q}_s$. Using Fubini's theorem, Hölder's inequality and the bound (3.5.13), we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_s}\Big(|L(M_n)e_1|^c\Big) &\leqslant \sum_{j=1}^{\infty}\Big[\mathbb{E}_{\mathbb{Q}_s}\frac{\|\wedge^2 M_j^{\mathrm{T}}\|^{2c}}{|M_j^{\mathrm{T}}e_1|^{4c}}\Big]^{1/2}\Big[\mathbb{E}_{\mathbb{Q}_s}e^{4c\log N(g_{j+1})}\Big]^{1/2}\\ &\leqslant C\mathbb{E}_{\mathbb{Q}_s}e^{4c\log N(g_1)}\sum_{j=1}^{\infty}r^j<+\infty.\end{aligned}$$

Combining this with (3.5.14) leads to $\mathbb{E}_{\mathbb{Q}_s}\frac{1}{|\langle Z_s,e_1\rangle|^c}<+\infty$. Note that for any $f\in\mathbb{P}^{d-1}$, we can choose an orthogonal matrix k such that $ke_1=f$. If we replace g_i by $k^{-1}g_ik$, then it is easy to see that M_n is replaced by $k^{-1}M_nk$. Moreover, in view of Lemma 3.5.2, the random variable Z_s is replaced by $k^{-1}Z_s$. Since the bound (3.5.13) holds uniformly in $x\in\mathbb{P}^{d-1}$, it follows that

$$\mathbb{E}_{\mathbb{Q}_s}\frac{1}{|\langle k^{-1}Z_s,e_1\rangle|^c}\leqslant C\mathbb{E}_{\mathbb{Q}_s}e^{4c\log N(k^{-1}g_1k)}\sum_{j=1}^{\infty}r^j<+\infty.$$

Observe that $N(k^{-1}g_1k)=N(g_1)$ and $\langle k^{-1}Z_s,e_1\rangle=\langle Z_s,f\rangle$. Therefore, for any $s\in I_\mu^\circ$, there exists a constant $c>0$ such that

$$\sup_{f\in\mathbb{P}^{d-1}}\int_{\mathbb{P}^{d-1}}\frac{1}{|\langle f,x\rangle|^c}\pi_s(dx)=\sup_{f\in\mathbb{P}^{d-1}}\mathbb{E}_{\mathbb{Q}_s}\frac{1}{|\langle f,Z_s\rangle|^c}<+\infty.$$

This implies that there exists a constant $C>0$ such that for any $0<t<1$, uniformly in $f\in\mathbb{P}^{d-1}$,

$$\begin{aligned}\pi_s\Big(\{x:|\langle f,x\rangle|\leqslant t\}\Big) &= \int_{\{x:|\langle f,x\rangle|\leqslant t\}}\frac{|\langle f,x\rangle|^c}{|\langle f,x\rangle|^c}\pi_s(dx)\\ &\leqslant t^c\int_{\mathbb{P}^{d-1}}\frac{1}{|\langle f,x\rangle|^c}\pi_s(dx)\leqslant Ct^c.\end{aligned}$$

The proof of Proposition 3.3.4 for invertible matrices is complete. □

Proof of Proposition 3.5.7. In view of Lemma 3.5.3, it suffices to prove the assertion of the proposition with $\mathbb{Q}_s$ instead of $\mathbb{Q}_s^x$, i.e. we show that there exist constants $c,C>0$ such that for all $n\geqslant 1$,

$$\mathbb{E}_{\mathbb{Q}_s}\left|L(M_n)e_1-\frac{Z_s}{\langle Z_s,e_1\rangle}\right|^c<e^{-Cn}. \tag{3.5.15}$$

Using Lemma 3.5.5 and Hölder's inequality, for sufficiently small constant $c>0$ and for any $n,m\geqslant 1$, we get

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_s}\Big|L(M_{n+m})e_1-L(M_n)e_1\Big|^c\\ &\leqslant \sum_{j=n}^{n+m-1}\Big[\mathbb{E}_{\mathbb{Q}_s}\frac{\|\wedge^2 M_j^{\mathrm{T}}\|^{2c}}{|M_j^{\mathrm{T}}e_1|^{4c}}\Big]^{1/2}\Big[\mathbb{E}_{\mathbb{Q}_s}e^{4c\log N(g_{j+1})}\Big]^{1/2}\\ &\leqslant C\sum_{j=n}^{n+m-1}\Big[\mathbb{E}_{\mathbb{Q}_s}\frac{\|\wedge^2 M_j^{\mathrm{T}}\|^{2c}}{|M_j^{\mathrm{T}}e_1|^{4c}}\Big]^{1/2},\end{aligned} \tag{3.5.16}$$

where the last inequality holds due to the moment condition (3.2.1). By the Fatou lemma, taking the limit as $m \to \infty$, we see that

$$\mathbb{E}_{\mathbb{Q}_s}\left|L(M_n)e_1 - \frac{Z_s}{\langle Z_s, e_1\rangle}\right|^c \leqslant C\sum_{j=n}^{\infty}\left[\mathbb{E}_{\mathbb{Q}_s}\frac{\|\wedge^2 M_j^{\mathrm{T}}\|^{2c}}{|M_j^{\mathrm{T}}e_1|^{4c}}\right]^{1/2} \leqslant Ce^{-Cn},$$

where the last inequality holds due to the bound (3.5.13). □

3.5.2 The regularity of the stationary measure for positive matrices

The aim of this section is to prove Proposition 3.3.4 for positive matrices. For small constant $\epsilon > 0$, we denote

$$\mathbb{S}_{+,\epsilon}^{d-1} = \{x \in \mathbb{S}_+^{d-1} : \langle e_i, x\rangle \geqslant \epsilon \text{ for all } 1 \leqslant i \leqslant d\}.$$

For any $g \in \operatorname{supp}\mu$, set $g \cdot \mathbb{S}_+^{d-1} = \{g \cdot x : x \in \mathbb{S}_+^{d-1}\}$. The following result gives equivalent formulations of conditions **M4** and (3.2.14).

Lemma 3.5.8. *For positive matrices, the following assertions hold:*

(1) *condition* ***M4*** *is equivalent to the following statement: there exists* $\epsilon \in (0, \frac{\sqrt{2}}{2})$ *such that*

$$g \cdot \mathbb{S}_+^{d-1} \subset \mathbb{S}_{+,\epsilon}^{d-1}, \quad \textit{for any } g \in \operatorname{supp}\mu;$$

(2) *condition* (3.2.14) *is equivalent to the following statement: there exists* $\epsilon \in (0, \frac{\sqrt{2}}{2})$ *such that*

$$g^{\mathrm{T}} \cdot \mathbb{S}_+^{d-1} \subset \mathbb{S}_{+,\epsilon}^{d-1}, \quad \textit{for any } g \in \operatorname{supp}\mu.$$

Proof. Part (1) is proved in [88]. The proof of part (2) can be carried out in a similar way as in [88] and therefore the details are left to the reader. □

Proof of Proposition 3.3.4 for positive matrices. As mentioned before, we only need to establish (3.3.6) and (3.3.7) for the stationary measure π_s since r_s is bounded away from infinity and 0 uniformly on $\mathbb{S}_+^{d-1}$.

We first prove Proposition 3.3.4 for positive matrices under conditions **M4**, **M6**. By Lemma 3.5.8, the Markov chain $(X_n^x)_{n\geqslant 1}$ stays in the space $\mathbb{S}_{+,\epsilon}^{d-1}$, and therefore the support of its stationary measure ν is included in $\mathbb{S}_{+,\epsilon}^{d-1}$. Since $\operatorname{supp}\nu_s = \operatorname{supp}\nu$ for $s \in I_\mu$ (by [16, Proposition 3.1]), it holds that $\operatorname{supp}\nu_s \subset \mathbb{S}_{+,\epsilon}^{d-1}$. As a consequence we also have $\operatorname{supp}\pi_s \subset \mathbb{S}_{+,\epsilon}^{d-1}$. This implies that $\langle f, x\rangle \geqslant \epsilon$ for all $f \in \mathbb{S}_+^{d-1}$, $x \in \operatorname{supp}\pi_s$, and so the bounds (3.3.6) and (3.3.7) hold for positive matrices.

We next prove Proposition 3.3.4 for positive matrices under conditions **M5**, **M6**. The proof is split into two steps. It is worth mentioning that the assertions shown below remain valid when $s = 0$.

Step 1. We prove that there exist two constants $C_1, C_2 > 0$ and an integer $n_0 \geqslant 1$ satisfying $C_1 > \Lambda'(s)$ such that, for any $n \geqslant n_0$, it holds uniformly in $f, x \in \mathbb{S}_+^{d-1}$ that

$$I_n := \mathbb{Q}_s^x\Big(\langle f, X_n^x\rangle \leqslant e^{-C_1 n}\Big) \leqslant e^{-C_2 n}. \tag{3.5.17}$$

Let $s \in I_\mu$, $t \in I_\mu^\circ$ be such that $s < t$ and set $q_s = \Lambda'(s)$ and $q_t = \Lambda'(t)$ (we allow s to be 0). Substituting $X_n^x = \frac{G_n x}{|G_n x|}$ into (3.5.17) , we have

$$I_n \leqslant \mathbb{Q}_s^x\Big(\log|G_n x| > nq_t\Big) + \mathbb{Q}_s^x\Big(\log\langle f, G_n x\rangle \leqslant -(C_1 - q_t)n\Big). \tag{3.5.18}$$

Since $s < t$, using Theorem 3.4.4, we get that there exists a constant $c > 0$ such that the first term on the right-hand side of (3.5.18) is bounded by e^{-cn}, uniformly in $x \in \mathbb{S}_+^{d-1}$. For the second term on the right-hand side of (3.5.18), applying the Markov inequality and the change of measure formula (3.3.3), it follows that for a sufficiently small constant $c_1 > 0$, uniformly in $f, x \in \mathbb{S}_+^{d-1}$,

$$\begin{aligned}
\mathbb{Q}_s^x\Big(\log\langle f, G_n x\rangle \leqslant -(C_1 - q_t)n\Big) &\leqslant e^{-c_1(C_1-q_t)n}\mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{1}{\langle f, G_n x\rangle^{c_1}}\Big) \\
&= e^{-c_1(C_1-q_t)n}\mathbb{E}\left(\frac{|G_n x|^s}{\kappa^n(s)}\frac{r_s(X_n^x)}{r_s(x)}\frac{1}{\langle f, G_n x\rangle^{c_1}}\right) \\
&\leqslant e^{-c_1(C_1-q_t)n}\mathbb{E}\left(\frac{|G_n x|^s}{\kappa^n(s)}\frac{r_s(X_n^x)}{r_s(x)}\frac{1}{\min_{1\leqslant i,j\leqslant d}\langle e_i, G_n e_j\rangle^{c_1}}\right),
\end{aligned} \tag{3.5.19}$$

where for the last line we use the fact that $\min_{1\leqslant i,j\leqslant d}\langle e_i, ge_j\rangle = \inf_{f,x\in\mathbb{S}_+^{d-1}}\langle f, gx\rangle$ for any $g \in \Gamma_\mu$. Since $|G_n x| \leqslant \|G_n\|$ and the function r_s is uniformly bounded and strictly positive on $\mathbb{S}_+^{d-1}$, using the Hölder inequality leads to

$$\begin{aligned}
\mathbb{E}\Big(\frac{|G_n x|^s}{\kappa^n(s)}\frac{r_s(X_n^x)}{r_s(x)}\frac{1}{\langle e_i, G_n e_j\rangle^{c_1}}\Big) &\leqslant \kappa^{-n}(s)\mathbb{E}^{\frac{1}{p}}\Big(\|G_n\|^{sp}\Big)\mathbb{E}^{\frac{1}{p'}}\Big(\frac{1}{\langle e_i, G_n e_j\rangle^{c_1 p'}}\Big) \\
&\leqslant \kappa^{-n}(s)\mathbb{E}^{\frac{1}{p}}\Big(\|G_n\|^{sp}\Big)\mathbb{E}^{\frac{n}{p'}}\Big(\frac{1}{\min_{1\leqslant i,j\leqslant d}\langle e_i, g_1 e_j\rangle^{c_1 p'}}\Big),
\end{aligned} \tag{3.5.20}$$

where $1/p + 1/p' = 1$ with $p, p' > 1$. Recall that $c_1 > 0$ can be taken sufficiently small. Taking p sufficiently close to 1 (p' sufficiently large) and using conditions (3.2.1) and **M5**, we get that the right-hand side of (3.5.20) is dominated by e^{Cn} with some constant $C > 0$. Consequently, in view of (3.5.19), choosing the constant $C_1 > 0$ sufficiently large, we obtain that the right-hand side of (3.5.19) is bounded by $e^{-C_2 n}$ with some constant $C_2 > 0$, uniformly in $f, x \in \mathbb{S}_+^{d-1}$.

Step 2. From Proposition 3.3.1, the construction of $\mathbb{Q}_s^x$, and (3.3.4), one can verify that for any $x \in \mathbb{S}_+^{d-1}$ and $n \geqslant 1$, $\pi_s = (\mathbb{Q}_s^x)^{*n} * \pi_s$, where $*$ denotes the convolution between two measures. Combining this with (3.5.17), we get that for any $s \in I_\mu$, uniformly in $f \in \mathbb{S}_+^{d-1}$,

$$\pi_s\Big(\{x : \langle f, x\rangle \leqslant e^{-C_1 n}\}\Big) = \int_{\mathbb{S}_+^{d-1}} (\mathbb{Q}_s^x)^{*n}\Big(\langle f, X_n^x\rangle \leqslant e^{-C_1 n}\Big)\pi_s(dx) \leqslant e^{-C_2 n}, \tag{3.5.21}$$

where the constants C_1 and C_2 are given in step 1. This proves the assertion (3.3.7). It remains to show (3.3.6). For $n \geqslant 1$, denote $B_{f,n} := \{x \in \mathbb{S}_+^{d-1} : e^{-C_1(n+1)} \leqslant \langle f, x\rangle \leqslant e^{-C_1 n}\}$. Choosing $c \in (0, C_2/C_1)$, we deduce from (3.5.21) that, uniformly in $f \in \mathbb{S}_+^{d-1}$,

$$\begin{aligned}
\int_{\mathbb{S}_+^{d-1}} \frac{1}{\langle f, x\rangle^c}\pi_s(dx) &= \int_{\{x:\langle f,x\rangle > e^{-C_1 n_0}\}} \frac{1}{\langle f, x\rangle^c}\pi_s(dx) + \sum_{n=n_0}^{\infty}\int_{B_{f,n}} \frac{1}{\langle f, x\rangle^c}\pi_s(dx) \\
&\leqslant e^{cC_1 n_0} + \sum_{n=n_0}^{\infty} e^{cC_1}e^{-(C_2 - cC_1)n} < +\infty.
\end{aligned} \tag{3.5.22}$$

This concludes the proof of Proposition 3.3.4 for positive matrices. □

3.6 The Hölder regularity of π_s for negative s

In this section, we first establish Proposition 3.3.5. Then, we use Proposition 3.3.5 to prove the assertion (2) in Lemma 3.3.3.

Proof of Proposition 3.3.5. We split the proof into two steps.

Step 1. In this step we choose a small enough constant $s_0 > 0$ and we show that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $n_0 \geqslant 1$ such that, for $n \geqslant n_0$,

$$\sup_{s\in(-s_0,0]} \sup_{f,x\in\mathcal{S}} \mathbb{Q}_s^x\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant e^{-c_1 n}. \tag{3.6.1}$$

To prove this, according to the change of measure formula (3.3.3), we write

$$\mathbb{Q}_s^x\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) = \mathbb{E}\left(\frac{|G_n x|^s}{\kappa^n(s)} \frac{r_s(X_n^x)}{r_s(x)} \mathbb{1}_{\{|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\}}\right).$$

By Proposition 3.3.1, the eigenfunction r_s is strictly positive and bounded on $\mathcal{S}$, uniformly in $s \in (-s_0, 0]$. Using Hölder's inequality, it follows that, uniformly in $s \in (-s_0, 0]$,

$$\mathbb{Q}_s^x\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant \frac{c}{\kappa^n(s)} \Big(\mathbb{E}|G_n x|^{2s}\Big)^{1/2} \Big[\mathbb{P}\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big)\Big]^{1/2}. \tag{3.6.2}$$

Under conditions of Proposition 3.3.5, it has been proved in [10, Proposition 14.3] for invertible matrices and in (3.5.17) (when $s = 0$) for positive matrices that there exist $c_2 > 0$ and $n_0 \geqslant 1$, such that for $n \geqslant n_0$, uniformly in $f, x \in \mathcal{S}$,

$$\mathbb{P}\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant e^{-c_2 n}. \tag{3.6.3}$$

We now give a control of $\mathbb{E}|G_n x|^{2s}$ in (3.6.2). Since s is negative, by the definition of ι and the fact that $\iota(gg') \geqslant \iota(g)\iota(g')$, it is easy to see that

$$\mathbb{E}|G_n x|^{2s} \leqslant \mathbb{E}\iota(G_n)^{2s} \leqslant \Big[\mathbb{E}\iota(g_1)^{2s}\Big]^n.$$

Since $\kappa(0) = 1$ and the function κ is continuous in a small neighborhood of 0, we deduce that for sufficiently small $s_0 > 0$, there exists a constant $c_3 \in (0, c_2/2)$ such that

$$\sup_{s\in(-s_0,0]} \sup_{x\in\mathcal{S}} \frac{1}{\kappa^n(s)} \Big(\mathbb{E}|G_n x|^{2s}\Big)^{1/2} \leqslant e^{c_3 n}.$$

This, together with (3.6.2)-(3.6.3), proves the desired bound (3.6.1).

Step 2. By Proposition 3.3.2 and the construction of $\mathbb{Q}_s^x$, it is easy to check that for any $s \in (-s_0, 0]$, $x \in \mathcal{S}$ and $n \geqslant 1$, $\pi_s = (\mathbb{Q}_s^x)^{*n} * \pi_s$. Combining this with (3.6.1), we get that, uniformly in $s \in (-s_0, 0]$ and $f \in \mathcal{S}$,

$$\pi_s\Big(\{x : |\langle f, x\rangle| \leqslant e^{-\varepsilon n}\}\Big) = \int_{\mathcal{S}} (\mathbb{Q}_s^x)^{*n}\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big)\pi_s(dx) \leqslant e^{-c_1 n}. \tag{3.6.4}$$

For any $n \geqslant 1$ and $f \in \mathcal{S}$, set $B_{f,n} := \{x \in \mathbb{P}^{d-1} : e^{-\varepsilon(n+1)} \leqslant |\langle f, x\rangle| \leqslant e^{-\varepsilon n}\}$. In an analogous way as in (3.5.22), taking a constant $c \in (0, c_1/\varepsilon)$ with c_1 and ε given in (3.6.4), we deduce from (3.6.4) that, uniformly in $f \in \mathcal{S}$ and $s \in (-s_0, 0]$,

$$\int_{\mathcal{S}} \frac{1}{|\langle f, x\rangle|^c} \pi_s(dx) \leqslant e^{\varepsilon c n_0} + \sum_{n=n_0}^{\infty} e^{\varepsilon c(n+1)} e^{-c_1 n} < +\infty.$$

This ends the proof of Proposition 3.3.5. □

We now give a proof of the assertion (2) in Lemma 3.3.3 based on the Hölder regularity of the stationary measure π_s established in Proposition 3.3.5. The fact that the Hölder regularity of π_s holds uniformly in $s \in (-s_0, 0]$ plays an important role.

Proof of (2) in Lemma 3.3.3. For $s \in (-s_0, 0)$, set

$$r_s(x) = \int_{\mathcal{S}} \frac{1}{|\langle x, y\rangle|^{-s}} \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} \frac{1}{|\langle x, y\rangle|^{-s}} \nu_s(dy), \quad x \in \mathcal{S}. \tag{3.6.5}$$

We first prove that there exist two constants $s_0 > 0$ and $C > 0$ such that $r_s \leqslant C$ and $r_s^* \leqslant C$ for all $x \in \mathcal{S}$ and $s \in (-s_0, 0]$. The boundedness of r_s^* can be obtained from Proposition 3.3.5. For the function r_s, denote by $(\mathbb{Q}_s^x)^*$ and μ^* the images of the measures $\mathbb{Q}_s^x$ and μ under the map $g \mapsto g^{\mathrm{T}}$. Taking into account that π_s^* is $(\mathbb{Q}_s^x)^*$-invariant ($(\mathbb{Q}_s^x)^* * \pi_s^* = \pi_s^*$), and that the measure μ^* fulfills all the assumptions of Proposition 3.3.5, for both invertible matrices and positive matrices, we can follow the proof of Proposition 3.3.5 to show that there exist constants $c > 0$ and $s_0 > 0$ such that uniformly in $s \in (-s_0, 0]$ and $f \in \mathcal{S}$,

$$\int_{\mathcal{S}} \frac{1}{|\langle f, x\rangle|^c} \pi_s^*(dx) < +\infty, \qquad \int_{\mathcal{S}} \frac{1}{|\langle f, x\rangle|^c} \nu_s^*(dx) < +\infty.$$

This implies that the function r_s is uniformly bounded on $\mathcal{S}$.

By the uniqueness of the eigenfunctions of the operators P_s and P_s^*, to prove the lemma it is enough to show that r_s and r_s^* are the eigenfunctions of P_s and P_s^* corresponding to the eigenvalue $\kappa(s)$. We show that r_s satisfies the equation

$$P_s r_s(x) = \kappa(s) r_s(x), \quad x \in \mathcal{S}. \tag{3.6.6}$$

In fact, since r_s is uniformly bounded on $\mathcal{S}$, using Fubini's theorem, we get

$$\begin{aligned} P_s r_s(x) &= \int_{\Gamma_\mu} |g_1 x|^s \Big(\int_{\mathcal{S}} |\langle g_1 \cdot x, y\rangle|^s \nu_s^*(dy) \Big) \mu(dg_1) \\ &= \int_{\mathcal{S}} \int_{\Gamma_\mu} |g_1^{\mathrm{T}} y|^s |\langle x, g_1^{\mathrm{T}} \cdot y\rangle|^s \mu(dg_1) \nu_s^*(dy). \end{aligned} \tag{3.6.7}$$

Recall that for any bounded measurable function φ on $\mathcal{S}$, we have

$$(\nu_s^* P_s^*)\varphi = \int_{\mathcal{S}} \int_{\Gamma_\mu} |g_1^{\mathrm{T}} y|^s \varphi(g_1^{\mathrm{T}} \cdot y) \mu(dg_1) \nu_s^*(dy). \tag{3.6.8}$$

For any fixed $x \in \mathcal{S}$ and $n \geqslant 1$, set

$$\varphi_{n,x}(y) = \frac{1}{|\langle x, y\rangle|^{-s}} \mathbb{1}_{\{|\langle x,y\rangle| \in (\frac{1}{n+1}, \frac{1}{n}]\}}, \quad y \in \mathcal{S}.$$

From (3.6.7) and (3.6.8), using Fubini's theorem, it follows that

$$P_s r_s(x) = (\nu_s^* P_s^*) \sum_{n=1}^{\infty} \varphi_{n,x}(y) = \sum_{n=1}^{\infty} (\nu_s^* P_s^*) \varphi_{n,x}(y). \tag{3.6.9}$$

Since the function $\varphi_{n,x}(y)$ is bounded on $\mathcal{S}$ for any fixed $n \geqslant 1$, using $\nu_s^* P_s^* = \kappa(s)\nu_s^*$ gives that for any $n \geqslant 1$,

$$(\nu_s^* P_s^*)\varphi_{n,x}(y) = \kappa(s)\nu_s^*(\varphi_{n,x}) = \kappa(s) \int_{\{y \in \mathcal{S} : |\langle x,y\rangle| \in (\frac{1}{n+1}, \frac{1}{n}]\}} \frac{1}{|\langle x, y\rangle|^{-s}} \nu_s^*(dy).$$

Summing up the above equality with respect to n, using (3.6.9) and the definition of r_s, the identity (3.6.6) follows. The proof for r_s^* is similar. □

3.7 Proof of precise large deviations for scalar products

The aim of this section is to establish Theorems 3.2.1, 3.2.2, 3.2.3 and 3.2.4.

3.7.1 Proof of Theorems 3.2.2 and 3.2.4

Let ψ be a measurable function on $\mathbb{R}$ and $\varepsilon > 0$. Denote, for brevity, $\psi_s(y) = e^{-sy}\psi(y)$ for some $s \in (-s_0, 0) \cup I_\mu^\circ$, and

$$\psi_{s,\varepsilon}^+(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \psi_s(y'), \quad \psi_{s,\varepsilon}^-(y) = \inf_{y' \in \mathbb{B}_\varepsilon(y)} \psi_s(y').$$

Introduce the following condition: for any $\varepsilon > 0$, the functions $y \mapsto \psi_{s,\varepsilon}^+(y)$ and $y \mapsto \psi_{s,\varepsilon}^-(y)$ are measurable and

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \psi_{s,\varepsilon}^+(y)dy = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \psi_{s,\varepsilon}^-(y)dy = \int_{\mathbb{R}} e^{-sy}\psi(y)dy < +\infty. \tag{3.7.1}$$

Let ψ be a function on $\mathbb{R}$ such that $y \mapsto e^{-sy}\psi(y)$ is directly Riemann integrable for some $s \in I_\mu^\circ$. In particular, it follows that $\lim_{y \to -\infty} \psi(y) = 0$.

Proof of Theorem 3.2.2. We shall first establish Theorem 3.2.2 under conditions **M2** and **M3** for invertible 2×2 matrices, and under conditions **M5** and **M6** for positive 2×2 matrices. The proof of Theorem 3.2.2 under conditions **M4**, **M6** for positive $d \times d$ ($d \geqslant 2$) matrices will be considered separately.

We will only prove the second assertion in Theorem 3.2.2, since the first one follows from the second by choosing $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y \geqslant 0\}}$, $y \in \mathbb{R}$. Moreover, with no loss of generality, we assume that the functions φ and ψ are non-negative. We assume additionally that the function ψ satisfies the condition (3.7.1), which will be relaxed subsequenly to directly Riemann integrability condition.

Note that $\log|\langle f, G_n x\rangle| = \log|G_n x| + \log|f(X_n^x)|$, and that $\log|\langle f, G_n x\rangle| = -\infty$ if and only if $\log|f(X_n^x)| = -\infty$. Taking into account that $\psi(-\infty) = 0$, we can replace the logarithm of the scalar product $\log|\langle f, G_n x\rangle|$ by the sum $\log|G_n x| + \log|f(X_n^x)|$ as follows:

$$\begin{aligned} A_n &:= \sqrt{2\pi n}\,\sigma_s e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{E}\Big[\varphi(X_n^x)\psi(\log|\langle f, G_n x\rangle| - n(q+l))\Big] \\ &= \sqrt{2\pi n}\,\sigma_s e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| + \log|f(X_n^x)| - n(q+l))\Big]. \end{aligned}$$

Recall that $q = \Lambda'(s)$, $\Lambda^*(q+l) = \Lambda^*(q) + sl + h_s(l)$, $x \in \mathbb{P}^{d-1}$, and $|l| \leqslant \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. Taking into account that $e^{n\Lambda^*(q)} = e^{sqn}/\kappa^n(s)$ and using the change of measure (3.3.3), we get

$$\begin{aligned} A_n &= \sqrt{2\pi n}\,\sigma_s e^{nsl + nh_s(l) + nsq} \\ &\quad \times \mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x) e^{-s\log|G_n x|} \psi\Big(\log|G_n x| + \log|f(X_n^x)| - n(q+l)\Big)\Big]. \end{aligned}$$

With the notation $T_n^x = \log|G_n x| - nq$, we have

$$A_n = \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)e^{-s(T_n^x-nl)}\psi(T_n^x + \log|f(X_n^x)| - nl)\Big]. \tag{3.7.2}$$

For short, set $Y_n^{f,x} := \log|f(X_n^x)|$. For any fixed $0 < \delta < 1$ and $k \geqslant 1$ denote $I_k := (-\delta k, -\delta(k-1)]$. Let $M_n := [n^{1/4}]$, where $[a]$ is the integer part of a. Then, the expectation in (3.7.2) can be written as a sum

$$\begin{aligned} A_n &= \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)} \\ &\quad\times \sum_{k=1}^{\infty} \mathbb{1}_{\{k\leqslant M_n\}}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-s(T_n^x-nl)}\psi(T_n^x - nl + Y_n^{f,x})\mathbb{1}_{\{Y_n^{f,x}\in I_k\}}\right] \\ &\quad+\sqrt{2\pi n}\,\sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-s(T_n^x-nl)}\psi(T_n^x - nl + Y_n^{f,x})\mathbb{1}_{\{Y_n^{f,x}\leqslant -\delta M_n\}}\right] \\ &= A_{n,1} + A_{n,2}. \end{aligned} \tag{3.7.3}$$

We now give a bound for the second term $A_{n,2}$. Since the function $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable on $\mathbb{R}$ for some $s' \in (0, s)$, one can verify that the function $y \mapsto e^{-sy}\psi(y)$ is bounded on $\mathbb{R}$ and hence there exists a constant $c_s > 0$ such that

$$e^{-s(T_n^x-nl)}\psi(T_n^x - nl + Y_n^{f,x}) \leqslant c_s e^{sY_n^{f,x}}.$$

Using Lemma 3.4.1 and the fact that the function φr_s^{-1} is uniformly bounded on $\mathcal{S}$, we get that, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$ and $f, x \in \mathcal{S}$, as $n \to \infty$,

$$A_{n,2} \leqslant c_s\sqrt{2\pi n}\,\sigma_s e^{nh_s(l)-s\delta M_n} \to 0. \tag{3.7.4}$$

The remaining part of the proof is devoted to establishing upper and lower bounds for the first term $A_{n,1}$.

Upper bound for $A_{n,1}$. On the event $\{Y_n^{f,x} \in I_k\}$, we have $Y_n^{f,x} + \delta(k-1) \in (0, \delta]$. With the notation $\psi_\delta^+(y) = \sup_{y'\in\mathbb{B}_\delta(y)}\psi(y')$, we get

$$\psi(T_n^x - nl + Y_n^{f,x}) \leqslant \psi_\delta^+(T_n^x - nl - \delta(k-1)).$$

It follows that

$$\begin{aligned} A_{n,1} &\leqslant \sqrt{2\pi n}\;\sigma_s e^{nh_s(l)} \\ &\quad\times \sum_{k=1}^{\infty} \mathbb{1}_{\{k\leqslant M_n\}}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-s(T_n^x-nl)}\psi_\delta^+(T_n^x - nl - \delta(k-1))\mathbb{1}_{\{Y_n^{f,x}\in I_k\}}\right]. \end{aligned}$$

Set $\Psi_{s,\delta}(y) = e^{-sy}\psi_\delta^+(y)$, $y \in \mathbb{R}$. Recall that $\Psi_{s,\delta,\varepsilon}^+(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)}\Psi_{s,\delta}(y')$, $y \in \mathbb{R}$, is defined by (3.4.1), for $\varepsilon \in (0,1)$. From now on we choose $\delta \in (0, \varepsilon)$. Since the function $\Psi_{s,\delta,\varepsilon}^+$ is non-negative and integrable on the real line, using Lemma 3.4.2, we get

$$\begin{aligned} A_{n,1} &\leqslant (1 + C_\rho(\varepsilon))\sqrt{2\pi n}\,\sigma_s e^{nh_s(l)}\sum_{k=1}^{\infty}\mathbb{1}_{\{k\leqslant M_n\}}e^{-s\delta(k-1)} \\ &\qquad\times \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\mathbb{1}_{\{Y_n^{f,x}\in I_k\}}(\Psi_{s,\delta,\varepsilon}^+ * \rho_{\varepsilon^2})(T_n^x - nl - \delta(k-1))\right] \\ &=: B_n^+, \end{aligned} \tag{3.7.5}$$

where the constant $C_\rho(\varepsilon)$ converges to 0 as $\varepsilon \to 0$. For fixed small constant $\varepsilon_1 > 0$, introduce the density function $\bar{\rho}_{\varepsilon_1}$ defined as follows: $\bar{\rho}_{\varepsilon_1}(x) = \frac{1}{\varepsilon_1}(1 - \frac{|x|}{\varepsilon_1})$ for $x \in [-\varepsilon_1, \varepsilon_1]$, and $\bar{\rho}_{\varepsilon_1}(x) = 0$ otherwise. For any $k \geqslant 1$, setting $\chi_k(x) := \mathbb{1}_{\{x \in I_k\}}$ and $\chi^+_{k,\varepsilon_1}(x) = \sup_{x' \in \mathbb{B}_{\varepsilon_1}(x)} \chi_k(x')$, one can verify that the following smoothing inequality holds:

$$\chi_k(x) \leqslant (\chi^+_{k,\varepsilon_1} * \bar{\rho}_{\varepsilon_1})(x) \leqslant \chi^+_{k,2\varepsilon_1}(x), \quad x \in \mathbb{R}. \tag{3.7.6}$$

For short, we denote $\tilde{\chi}_k(x) := (\chi^+_{k,\varepsilon_1} * \bar{\rho}_{\varepsilon_1})(x)$, $x \in \mathbb{R}$, which is Hölder continuous on $\mathbb{R}$. In view of (3.7.5), using the smoothing inequality (3.7.6) leads to

$$\begin{aligned} B_n^+ &\leqslant (1 + C_\rho(\varepsilon))\sqrt{2\pi n}\, \sigma_s e^{nh_s(l)} \sum_{k=1}^{\infty} \mathbb{1}_{\{k \leqslant M_n\}} e^{-s\delta(k-1)} \\ &\times \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\tilde{\chi}_k(Y_n^{f,x})(\Psi^+_{s,\delta,\varepsilon} * \rho_{\varepsilon^2})(T_n^x - nl - \delta(k-1))\right] =: B_n^{++}. \end{aligned} \tag{3.7.7}$$

Let $\widehat{\Psi}^+_{s,\delta,\varepsilon}$ be the Fourier transform of $\Psi^+_{s,\delta,\varepsilon}$. By the inversion formula, we have

$$\Psi^+_{s,\delta,\varepsilon} * \rho_{\varepsilon^2}(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ity}\widehat{\Psi}^+_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \quad y \in \mathbb{R}.$$

Note that $\tilde{\chi}_k(Y_n^{f,x}) = \tilde{\chi}_k(\log|f(X_n^x)|) = (\tilde{\chi}_k \circ \log|f|)(X_n^x)$. Substituting $y = T_n^x - nl - \delta(k-1)$ and taking expectation with respect to $\mathbb{E}_{\mathbb{Q}_s^x}$, by Fubini's theorem, it holds that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\tilde{\chi}_k(Y_n^{f,x})(\Psi^+_{s,\delta,\varepsilon} * \rho_{\varepsilon^2})(T_n^x - nl - \delta(k-1))\right] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it[ln+\delta(k-1)]} R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\right)(x)\widehat{\Psi}^+_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \end{aligned} \tag{3.7.8}$$

where

$$R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\right)(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{itT_n^x}(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)(X_n^x)\right].$$

Substituting the equality (3.7.8) into (3.7.7), we get

$$\begin{aligned} B_n^{++} &= (1 + C_\rho(\varepsilon))\sqrt{\frac{n}{2\pi}}\, \sigma_s \sum_{k=1}^{\infty} \mathbb{1}_{\{k \leqslant M_n\}} e^{-s\delta(k-1)} e^{nh_s(l)} \\ &\times \int_{\mathbb{R}} e^{-it[ln+\delta(k-1)]} R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\right)(x)\widehat{\Psi}^+_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt. \end{aligned} \tag{3.7.9}$$

Since the function $\tilde{\chi}_k$ is Hölder continuous on the real line, one can check that $(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)$ is also Hölder continuous on the projective space $\mathcal{S}$. Using the fact that the function $y \mapsto e^{-s'y}\psi(y)$ is directly Riemann integrable on $\mathbb{R}$ for some $s' \in (0, s)$, one can also verify that the function $\widehat{\Psi}^+_{s,\delta,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ is compactly supported in $\mathbb{R}$, and moreover, $\widehat{\Psi}^+_{s,\delta,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ is differentiable in a small neighborhood of 0 on the real line. Using Proposition 3.4.3 with $\varphi = (\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)$, $\psi = \widehat{\Psi}^+_{s,\delta,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ and with $l_{n,k} = l + \frac{\delta(k-1)}{n}$ instead of l, we obtain that for any fixed $k \geqslant 1$ and for sufficiently large n, it holds uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$ that

$$\begin{aligned} &\left|\sqrt{n}\sigma_s e^{nh_s(l_{n,k})}\int_{\mathbb{R}} e^{-itl_{n,k}n} R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\right)(x)\widehat{\Psi}^+_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt - B^+(k)\right| \\ &\leqslant C\|(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\|_\gamma\left(\frac{1}{\sqrt{n}} + \frac{\delta(k-1)}{n}\right), \end{aligned} \tag{3.7.10}$$

where

$$B^+(k) := \sqrt{2\pi}\widehat{\Psi}^+_{s,\delta,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\pi_s\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\right).$$

Taking into account that $1 \leqslant k \leqslant M_n = [n^{1/4}]$, by Lemma 3.4.1, for any fixed $k \geqslant 1$ we have that $|e^{nh_s(l)-nh_s(l_{n,k})} - 1| \leqslant C\delta k l_n e^{c\delta k l_n}$ as $n \to \infty$ uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$. Using (3.7.10) and the fact that $B^+(k)$ is dominated by $\|(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\|_\gamma$, we can replace $e^{nh_s(l_{n,k})}$ by $e^{nh_s(l)}$, yielding that uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}
&\left|\sqrt{n}\,\sigma_s e^{nh_s(l)}\int_{\mathbb{R}} e^{-itl_{n,k}n}R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\right)(x)\widehat{\Psi}^+_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt - D(k)\right| \\
&\qquad \leqslant C\delta k l_n e^{c\delta k l_n}\|(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\|_\gamma\Big(1 + \frac{1}{\sqrt{n}} + \frac{\delta(k-1)}{n}\Big) \\
&\qquad\quad + C\|(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\|_\gamma\Big(\frac{1}{\sqrt{n}} + \frac{\delta(k-1)}{n}\Big) \\
&\qquad =: E_{n,k}. \qquad (3.7.11)
\end{aligned}$$

One can easily calculate that γ-Hölder norm $\|(\varphi r_s^{-1})(\tilde{\chi}_k \circ \log|f|)\|_\gamma$ is bounded by $\frac{e^{\delta\gamma k}}{(1-e^{-2\varepsilon_1})^\gamma}$. Taking sufficiently small $\gamma > 0$, we obtain that, for n large enough, one can verify that the series $\sum_{k=1}^\infty e^{-s\delta(k-1)}E_{n,k}$ is convergent, and moreover, its limit is 0 as $n \to \infty$. Consequently, we are allowed to interchange the limit as $n \to \infty$ and the infinite summation over k in (3.7.9). Therefore, from (3.7.9), (3.7.10) and (3.7.11) we deduce that, as $n \to \infty$, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$B_n^{++} \leqslant (1 + C_\rho(\varepsilon))\widehat{\Psi}^+_{s,\delta,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\sum_{k=1}^\infty e^{-s\delta(k-1)}\pi_s\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log f)\right). \qquad (3.7.12)$$

To calculate the sum in (3.7.12), we shall make use of the regularity of the stationary measure π_s. Note that $\widehat{\Psi}^+_{s,\delta,\varepsilon}(0) = \int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi^+_\delta(y')dy$ and $\widehat{\rho}_{\varepsilon^2}(0) = 1$. Using (3.7.6), we have $\tilde{\chi}_k \leqslant \chi^+_{k,2\varepsilon_1}$. Therefore, recalling that $B_n^+ \leqslant B_n^{++}$, we obtain that, uniformly in $f \in \mathcal{S}$,

$$\begin{aligned}
\limsup_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant l_n} B_n^+ &\leqslant (1 + C_\rho(\varepsilon))\int_{\mathbb{R}}\sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi^+_\delta(y')dy \\
&\quad \times \sum_{k=1}^\infty e^{-s\delta(k-1)}\pi_s\left((\varphi r_s^{-1})(\chi^+_{k,2\varepsilon_1} \circ \log|f|)\right). \qquad (3.7.13)
\end{aligned}$$

Notice that

$$\chi^+_{k,2\varepsilon_1}(x) \circ \log|f| = \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x) + \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_{k,\varepsilon_1}\}}(x), \qquad (3.7.14)$$

where $I_{k,\varepsilon_1} = \Big(-\delta k - 2\varepsilon_1, -\delta k\Big] \cup \Big(-\delta(k-1), -\delta(k-1) + 2\varepsilon_1\Big]$. For the first term on the right hand-side of (3.7.14), we have

$$\lim_{\delta\to 0}\sum_{k=1}^\infty e^{-s\delta(k-1)}\pi_s\Big((\varphi r_s^{-1})\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}\Big) = \int_{\mathcal{S}}|\langle f,x\rangle|^s\varphi(x)r_s^{-1}(x)\pi_s(dx). \qquad (3.7.15)$$

To deal with the second term on the right-hand side of (3.7.14), we shall apply the Hölder regularity of the stationary measure π_s established in Proposition 3.3.4. Set

for brevity $J_{k,\delta}(\varepsilon_1) = (e^{-\delta k-2\varepsilon_1}, e^{-\delta k}]$. Notice that for 2×2 matrices, the projective space $\mathcal{S}$ is one-dimensional ($\mathcal{S} = \mathbb{P}^1$ for invertible matrices and $\mathcal{S} = \mathbb{S}^1_+$ for positive matrices), so there exists $f_1 = f_1(\delta, k, \varepsilon) \in \mathcal{S}$ and $f_2 = f_2(\delta, k, \varepsilon) \in \mathcal{S}$ such that

$$\pi_s\Big(\Big\{x \in \mathcal{S} : |\langle f, x\rangle| \in J_{l,\delta}(\varepsilon_1)\Big\}\Big) \leqslant \pi_s\Big(\Big\{x \in \mathcal{S} : |\langle f_1, x\rangle| \in (0, e^{-\delta k} - e^{-\delta k-2\varepsilon_1}]\Big\}\Big) + \pi_s\Big(\Big\{x \in \mathcal{S} : |\langle f_2, x\rangle| \in (0, e^{-\delta k} - e^{-\delta k-2\varepsilon_1}]\Big\}\Big). \tag{3.7.16}$$

Applying Proposition 3.3.4, we get that there exists a constant $c > 0$ such that, uniformly in $f_1 \in \mathcal{S}$,

$$\pi_s\Big(\Big\{x \in \mathcal{S} : |\langle f_1, x\rangle| \in (0, e^{-\delta k} - e^{-\delta k-2\varepsilon_1}]\Big\}\Big) \leqslant C_s(e^{-\delta k} - e^{-\delta k-2\varepsilon_1})^c,$$

which converges to 0 as $\varepsilon_1 \to 0$. A similar inequality also holds for the second term in (3.7.16). Hence we get that uniformly in $f \in \mathcal{S}$,

$$\lim_{\varepsilon_1\to 0} \pi_s\Big(\Big\{x \in \mathcal{S} : |\langle f, x\rangle| \in J_{k,\delta}(\varepsilon_1)\Big\}\Big) = 0. \tag{3.7.17}$$

In the same way we show that (3.7.17) holds with $J'_{k,\delta}(\varepsilon_1) = (e^{-\delta(k-1)+2\varepsilon_1}, e^{-\delta(k-1)}]$ instead of $J_{k,\delta}(\varepsilon_1)$. Consequently, we have, uniformly in $f \in \mathcal{S}$,

$$\lim_{\delta\to 0}\lim_{\varepsilon_1\to 0} \pi_s(\{x \in \mathcal{S} : \log|\langle f, x\rangle| \in I_{k,\varepsilon_1}\}) = 0. \tag{3.7.18}$$

By the Lebesgue convergence theorem, from (3.7.18) we deduce that

$$\lim_{\delta\to 0}\lim_{\varepsilon_1\to 0}\sup_{f\in\mathcal{S}}\sum_{k=1}^{\infty} e^{-s\delta(k-1)}\pi_s\left((\varphi r_s^{-1})\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_{k,\varepsilon_1}\}}\right) = 0. \tag{3.7.19}$$

Combining (3.7.13)-(3.7.19), letting $n \to \infty$, $\varepsilon_1 \to 0$, $\delta \to 0$, $\varepsilon \to 0$ and noting that $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain that, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$A_{n,1} \leqslant B_n^+ \leqslant \int_{\mathcal{S}} |\langle f, x\rangle|^s \varphi(x) r_s^{-1}(x)\pi_s(dx) \limsup_{\varepsilon\to 0}\int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)} \left(e^{-sy'}\psi(y')\right)dy. \tag{3.7.20}$$

Lower bound for $A_{n,1}$. We are going to establish the lower bound for $A_{n,1}$ given by (3.7.3). Recall that $Y_n^{f,x} = \log|f(X_n^x)|$. On the event $\{Y_n^{f,x} \in I_k\}$ we have $Y_n^{f,x} + \delta(k-1) \in (0, \delta]$. With the notation $\psi_\delta^-(y) = \inf_{y'\in\mathbb{B}_\delta(y)} \psi(y)$, we get

$$\psi(T_n^x - nl + Y_n^{f,x}) \geqslant \psi_\delta^-(T_n^x - nl - \delta k).$$

In view of (3.7.3), using Fatou's lemma, it follows that

$$\liminf_{n\to\infty} A_{n,1} \geqslant \sum_{k=1}^{\infty} \liminf_{n\to\infty} \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)}\mathbb{1}_{\{k\leqslant M_n\}} \times \mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)e^{-s(T_n^x-nl)}\psi_\delta^-(T_n^x - nl - \delta k)\mathbb{1}_{\{Y_n^{f,x}\in I_k\}}\Big].$$

Set $\Psi_{s,\delta}(y) = e^{-sy}\psi_\delta^-(y)$, $y \in \mathbb{R}$. Recall that $\Psi_{s,\delta,\varepsilon}^-(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)} \Psi_{s,\delta}(y')$, $y \in \mathbb{R}$, is defined by (3.4.1), for $\varepsilon \in (0, 1)$. In the sequel we choose $\delta \in (0, \varepsilon)$. Noting that the

function $\Psi^-_{s,\delta,\varepsilon}$ is non-negative and integrable on the real line, by Lemma 3.4.2, we get the following lower bound:

$$\liminf_{n\to\infty} A_{n,1} \geqslant \sum_{k=1}^{\infty} \liminf_{n\to\infty} B_n^-(k) - \sum_{k=1}^{\infty} \limsup_{n\to\infty} D_n^-(k), \tag{3.7.21}$$

where

$$\begin{aligned} B_n^-(k) &= \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)} \mathbb{1}_{\{k\leqslant M_n\}} e^{-s\delta k} \\ &\quad \times \mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)\mathbb{1}_{\{Y_n^{f,x}\in I_k\}}(\Psi^-_{s,\delta,\varepsilon}*\rho_{\varepsilon^2})(T_n^x - nl - \delta k)\Big], \\ D_n^-(k) &= \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)} \mathbb{1}_{\{k\leqslant M_n\}} e^{-s\delta k} \\ &\quad \times \int_{|y|\geqslant\varepsilon} \mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)\mathbb{1}_{\{Y_n^{f,x}\in I_k\}}\Psi^-_{s,\delta,\varepsilon}(T_n^x - nl - \delta k - y)\Big]\rho_{\varepsilon^2}(y)dy. \end{aligned}$$

We are going to give a lower bound for $B_n^-(k)$. We denote $\chi_k(x) = \mathbb{1}_{\{x\in I_k\}}$ and $\chi^-_{k,\varepsilon_1}(x) = \inf_{x'\in\mathbb{B}_{\varepsilon_1}(x)} \chi_k(x')$, where $\varepsilon_1 > 0$ is a fixed small constant. Similarly the inequality (3.7.6), one can easily get the following smoothing inequality:

$$\chi^-_{k,2\varepsilon_1}(x) \leqslant (\chi^-_{k,\varepsilon_1} * \bar\rho_{\varepsilon_1})(x) \leqslant \chi_k(x), \quad x\in\mathbb{R}, \tag{3.7.22}$$

where the density function is the same as that in (3.7.6). For the first term $B_n^-(k)$ in (3.7.21), using the inequality (3.7.22) gives

$$\begin{aligned} B_n^-(k) &\geqslant \sqrt{2\pi n}\,\sigma_s e^{nh_s(l)} \mathbb{1}_{\{k\leqslant M_n\}} e^{-s\delta k} \\ &\quad \times \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\tilde\chi_k^-(Y_n^{f,x})(\Psi^-_{s,\delta,\varepsilon}*\rho_{\varepsilon^2})(T_n^x - nl - \delta k)\right], \end{aligned} \tag{3.7.23}$$

where, for short, $\tilde\chi_k^-(x) := (\chi^-_{k,\varepsilon_1} * \bar\rho_{\varepsilon_1})(x)$. Denote by $\widehat\Psi^-_{s,\delta,\varepsilon}$ the Fourier transform of $\Psi^-_{s,\delta,\varepsilon}$. Applying the inversion formula to $\Psi^-_{s,\delta,\varepsilon}*\rho_{\varepsilon^2}$, and using Fubini's theorem, we get

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\tilde\chi_k^-(Y_n^{f,x})(\Psi^-_{s,\delta,\varepsilon}*\rho_{\varepsilon^2})(T_n^x - nl - \delta k)\right] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itn[l+\delta\frac{k}{n}]} R^n_{s,it}\left((\varphi r_s^{-1})(\tilde\chi_k^- \circ \log|f|)\right)(x)\widehat\Psi^-_{s,\delta,\varepsilon}(t)\widehat\rho_{\varepsilon^2}(t)dt, \end{aligned} \tag{3.7.24}$$

where

$$R^n_{s,it}\left((\varphi r_s^{-1})(\tilde\chi_k^- \circ \log|f|)\right)(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{itT_n^x}(\varphi r_s^{-1})(\tilde\chi_k^- \circ \log|f|)(X_n^x)\right].$$

Substituting (3.7.24) into (3.7.23), we obtain

$$\begin{aligned} B_n^-(k) \geqslant &\sqrt{\frac{n}{2\pi}}\,\sigma_s \mathbb{1}_{\{k\leqslant M_n\}} e^{-s\delta k} \\ &\times e^{nh_s(l)}\int_{\mathbb{R}} e^{-itn[l+\delta\frac{k}{n}]} R^n_{s,it}\left((\varphi r_s^{-1})\tilde\chi_k^- \circ \log|f|\right)(x)\widehat\Psi^-_{s,\delta,\varepsilon}(t)\widehat\rho_{\varepsilon^2}(t)dt. \end{aligned} \tag{3.7.25}$$

Since the function $\tilde\chi_k^-$ is Hölder continuous for any fixed $k\geqslant 1$, one can also check that $(\varphi r_s^{-1})(\tilde\chi_k^- \circ \log|f|)$ is Hölder continuous on the projective space $\mathcal{S}$. One can also verify that the function $\widehat\Psi^-_{s,\delta,\varepsilon}\widehat\rho_{\varepsilon^2}$ is compactly supported in $\mathbb{R}$, and moreover, $\widehat\Psi^-_{s,\delta,\varepsilon}\widehat\rho_{\varepsilon^2}$ has a continuous extension in the complex plane and has an analytic extension in the domain

$\{z \in \mathbb{C} : |z| < \varepsilon^2, \Im z \neq 0\}$. Using Proposition 3.4.3 with $\varphi = (\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)$, $\psi = \widehat{\Psi}^-_{s,\delta,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ and $l'_{n,k} = l + \frac{\delta k}{n}$, we have that for any fixed $k \geqslant 1$, as $n \to \infty$, it holds uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$ that

$$\begin{aligned}\Big|\sqrt{n}\sigma_s e^{nh_s(l'_{n,k})}\int_{\mathbb{R}} e^{-itl'_{n,k}n} R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\right)(x)\widehat{\Psi}^-_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt - D^-(k)\Big| \\ \leqslant C\|(\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\|_\gamma\Big(\frac{1}{\sqrt{n}} + \frac{\delta k}{n}\Big), \end{aligned} \tag{3.7.26}$$

where

$$D^-(k) := \sqrt{2\pi}\widehat{\Psi}^-_{s,\delta,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\pi_s\left((\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\right).$$

Since $1 \leqslant k \leqslant M_n = [n^{1/4}]$, using Lemma 3.4.1, for any fixed $k \geqslant 1$ we get that $|e^{nh_s(l)-nh_s(l'_{n,k})} - 1| \leqslant C\delta\frac{k}{\sqrt{n}}e^{c\delta k/\sqrt{n}}$ as $n \to \infty$ uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$. In the same way as in (3.7.11), we can replace $e^{nh_s(l'_{n,k})}$ by $e^{nh_s(l)}$ to obtain that, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}&\Big|\sqrt{n}\,\sigma_s e^{nh_s(l)}\int_{\mathbb{R}} e^{-itl'_{n,k}n} R^n_{s,it}\left((\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\right)(x)\widehat{\Psi}^-_{s,\delta,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt - D^-(k)\Big| \\ &\quad\leqslant C\delta k l_n e^{c\delta k l_n}\|(\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\|_\gamma\Big(1 + \frac{1}{\sqrt{n}} + \frac{\delta k}{n}\Big) \\ &\qquad + C\|(\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\|_\gamma\Big(\frac{1}{\sqrt{n}} + \frac{\delta k}{n}\Big) \\ &\quad =: E^-_{n,k}.\end{aligned}$$

Since the γ-Hölder norm $\|(\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\|_\gamma$ is bounded by $\frac{e^{\delta\gamma k}}{(e^{2\varepsilon_1}-1)^\gamma}$, taking sufficiently small $\gamma > 0$, we obtain that the series $\sum_{k=1}^\infty e^{-s\delta(k-1)}E^-_{n,k}$ converges to 0 as $n \to \infty$. As a result, by virtue of (3.7.26), we obtain that, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\sum_{k=1}^\infty \liminf_{n\to\infty} B^-_n(k) \geqslant \widehat{\Psi}^-_{s,\delta,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\sum_{k=1}^\infty e^{-s\delta k}\pi_s\left((\varphi r_s^{-1})(\tilde{\chi}_k^- \circ \log|f|)\right).$$

Note that $\widehat{\Psi}^-_{s,\delta,\varepsilon}(0) = \int_{\mathbb{R}} \inf_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi^-_\delta(y')dy$ and $\widehat{\rho}_{\varepsilon^2}(0) = 1$. Using (3.7.22), we have that $\tilde{\chi}_k \geqslant \chi^-_{k,2\varepsilon_1}$. Consequently, we obtain the lower bound for the first term on the right hand side of (3.7.21): uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned}&\sum_{k=1}^\infty \liminf_{n\to\infty} B^-_n(k) \\ &\geqslant \int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi^-_\delta(y')dy\sum_{k=1}^\infty e^{-s\delta k}\pi_s\left((\varphi r_s^{-1})(\chi^-_{k,2\varepsilon_1} \circ \log|f|)\right).\end{aligned} \tag{3.7.27}$$

Notice that

$$\chi^-_{k,2\varepsilon_1}(x) \circ \log|f| = \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x) - \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I'_{k,\varepsilon_1}\}}(x), \tag{3.7.28}$$

where $I'_{k,\varepsilon_1} = \Big(-\delta k, -\delta k + 2\varepsilon_1\Big] \cup \Big(-\delta(k-1) - 2\varepsilon_1, -\delta(k-1)\Big]$. By the Lebesgue convergence theorem, we have that uniformly in $f \in \mathcal{S}$,

$$\lim_{\varepsilon_1 \to 0} \lim_{\delta \to 0} \sum_{k=1}^{\infty} e^{-s\delta k} \pi_s \Big((\varphi r_s^{-1}) \mathbb{1}_{\{\log|\langle f, \cdot\rangle| \in I'_{k,\varepsilon_1}\}} \Big) = 0. \tag{3.7.29}$$

Combining (3.7.27)-(3.7.29), taking $\liminf_{n\to\infty}$ and letting $\delta \to 0$, $\varepsilon_1 \to 0$, $\varepsilon \to 0$, we obtain that uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} &\sum_{k=1}^{\infty} \liminf_{n\to\infty} B_n^-(k) \\ &\geqslant \int_{\mathcal{S}} |\langle f, x\rangle|^s \varphi(x) r_s^{-1}(x) \pi_s(dx) \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \inf_{y' \in \mathbb{B}_\varepsilon(y)} e^{-sy'} \psi_\delta^-(y') dy. \end{aligned} \tag{3.7.30}$$

We now proceed to establish an upper bound for the second term $D_n^-(k)$ in (3.7.21). Note that $\Psi_{s,\delta,\varepsilon}^- \leqslant \Psi_{s,\delta}$, where $\Psi_{s,\delta}(y) = e^{-sy}\psi_\delta^+(y)$, $y \in \mathbb{R}$. Then it follows from Lemma 3.4.2 that $\Psi_{s,\delta,\varepsilon}^- \leqslant (1+C_\rho(\varepsilon))\widehat{\Psi}_{s,\delta,\varepsilon}^+ \widehat{\rho}_{\varepsilon^2}$, where $\Psi_{s,\delta,\varepsilon}^+(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \Psi_{s,\delta}(y')$, $y \in \mathbb{R}$. Moreover, using (3.7.6), we get $\mathbb{1}_{\{Y_n^{f,x} \in I_k\}} \leqslant \tilde{\chi}_k(x) = (\chi_{k,\varepsilon_1}^+ * \bar{\rho}_{\varepsilon_1})(x)$. Consequently, similarly to (3.7.9), we have the upper bound for $D_n^-(k)$:

$$\begin{aligned} D_n^-(k) &\leqslant (1 + C_\rho(\varepsilon)) \int_{|y| \geqslant \varepsilon} \sqrt{\frac{n}{2\pi}}\, \sigma_s \mathbb{1}_{\{k \leqslant M_n\}} e^{nh_s(l)} e^{-s\delta k} \\ &\times \left[\int_{\mathbb{R}} e^{-itn[l + \delta\frac{k}{n} + \frac{y}{n}]} R_{s,it}^n \Big((\varphi r_s^{-1}) \tilde{\chi}_k \circ \log|f| \Big)(x) \widehat{\Psi}_{s,\delta,\varepsilon}^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt \right] \rho_{\varepsilon^2}(y) dy. \end{aligned} \tag{3.7.31}$$

As in [85], we use the Lebesgue dominated convergence theorem to interchange the limit $n \to \infty$ and the integral $\int_{|y| \geqslant \varepsilon}$. Hence, using Proposition 3.4.3 we obtain that, uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} &\sum_{k=1}^{\infty} \limsup_{n\to\infty} D_n^-(k) \\ &\leqslant (1 + C_\rho(\varepsilon)) \sum_{k=1}^{\infty} \pi_s \Big[(\varphi r_s^{-1}) \tilde{\chi}_k \circ \log|f| \Big] \widehat{\Psi}_{s,\delta,\varepsilon}^+(0) \widehat{\rho}_{\varepsilon^2}(0) \int_{|y| \geqslant \varepsilon} \rho_{\varepsilon^2}(y) dy, \end{aligned}$$

where $\int_{|y| \geqslant \varepsilon} \rho_{\varepsilon^2}(y) dy \to 0$ as $\varepsilon \to 0$. Combining this with (3.7.21) and (3.7.30), we get the lower bound for $A_{n,1}$: uniformly in $f, x \in \mathcal{S}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\liminf_{n\to\infty} A_{n,1} \geqslant \int_{\mathcal{S}} |\langle f, x\rangle|^s \varphi(x) r_s^{-1}(x) \pi_s(dx) \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \inf_{y' \in \mathbb{B}_\varepsilon(y)} e^{-sy'} \psi(y') dy. \tag{3.7.32}$$

This, together with (3.7.3), (3.7.4) and (3.7.20), proves the desired asymptotic (3.2.9) under the condition (3.7.1) on the function ψ. Using the approximation techniques similar to that in [85], we extend it to the case of a function which is directly Riemann integrable. This part of the proof is left to the reader. This concludes the proof of Theorem 3.2.2 under either conditions **M2**, **M3** for invertible matrices, or conditions **M5**, **M6** for positive matrices. □

Next we proceed to prove Theorem 3.2.2 under conditions **M4**, **M6** for positive $d \times d$ matrices.

*Proof of Theorem 3.2.2 for positive matrices under conditions **M4**, **M6**.* The proof is similar to that of Theorem 3.2.2 in the case of invertible matrices, therefore we will only sketch the differences. Recall that $\mathcal{S} = \mathbb{S}_+^{d-1}$ for positive matrices.

Proceeding as in the proof of Theorem 3.2.2 in the case of invertible matrices, we have to deal with the quantity

$$A_n := \sqrt{2\pi n}\ \sigma_s e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{E}\left[\varphi(X_n^x)\psi(\log\langle f, G_n x\rangle - n(q+l))\right].$$

By the assertion (1) in Lemma 3.5.8, there exists a constant $0 < \epsilon < \frac{\sqrt{2}}{2}$ such that $f(X_n^x) \in [\epsilon, 1]$ for all $n \geqslant 1$ and $f, x \in \mathbb{S}_+^{d-1}$. For simplicity, set $Y_n^{f,x} := \log|f(X_n^x)|$ and so we can write $\log\langle f, G_n x\rangle = \log|G_n x| + Y_n^{f,x}$. Therefore, in the same way as in (3.7.3), we get

$$\begin{aligned} A_n &= \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)} \\ &\times \sum_{k=1}^{M} \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x) e^{-s(T_n^x - nl)} \psi(T_n^x + Y_n^{f,x} - nl) \mathbb{1}_{\{Y_n^{f,x} \in I_k\}}\right], \end{aligned} \tag{3.7.33}$$

where, in contrast to the proof of Theorem 3.2.2 for invertible matrices, M is a fixed positive integer. Similarly to the proof of (3.7.9), taking into account that $\tilde{\chi}_k(Y_n^{f,x}) = \tilde{\chi}_k(\log f(X_n^x)) = (\tilde{\chi}_k \circ \log f)(X_n^x)$, we obtain

$$\begin{aligned} A_n &\leqslant B_n^+ := (1 + C_\rho(\varepsilon))\sqrt{\frac{n}{2\pi}}\, \sigma_s \sum_{k=1}^{M} e^{-s\delta(k-1)} e^{nh_s(l)} \\ &\times \int_{\mathbb{R}} e^{-it[ln+\delta(k-1)]} R_{s,it}^n\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log f)\right)(x) \widehat{\Psi}_{s,\delta,\varepsilon}^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt. \end{aligned} \tag{3.7.34}$$

Denote $l'_{n,k} = l + \frac{\delta(k-1)}{n}$. For each fixed k, uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$, it holds that $e^{nh_s(l) - nh_s(l'_{n,k})} \to 1$, as $n \to \infty$. Hence using Proposition 3.4.3, the term under the sign of the sum in (3.7.34) converges to

$$e^{-s\delta(k-1)} \sqrt{2\pi} \widehat{\Psi}_{s,\delta,\varepsilon}^+(0) \widehat{\rho}_{\varepsilon^2}(0) \pi_s\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log f)\right),$$

in the same way as in (3.7.10). Moreover, this convergence is uniform in $k \geqslant 1$, since M is finite. This allows us to interchange the limit as $n \to \infty$ and the finite summation over k. By the Lebesgue dominated convergence theorem, similarly to (3.7.12), we deduce that, as $n \to \infty$, uniformly in $f, x \in \mathbb{S}_+^{d-1}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$B_n^+ \leqslant (1 + C_\rho(\varepsilon)) \widehat{\Psi}_{s,\delta,\varepsilon}^+(0) \widehat{\rho}_{\varepsilon^2}(0) \sum_{k=1}^{M} e^{-s\delta(k-1)} \pi_s\left((\varphi r_s^{-1})(\tilde{\chi}_k \circ \log f)\right). \tag{3.7.35}$$

Now we are going to calculate the finite sum in (3.7.35). Following the path of the proof of Theorem 3.2.2 (see (3.7.13), (3.7.14), (3.7.15)), we obtain that

$$\lim_{\delta \to 0} \sum_{k=1}^{M} e^{-s\delta(k-1)} \pi_s\left((\varphi r_s^{-1}) \mathbb{1}_{\{\log\langle f, \cdot\rangle \in I_k\}}\right) = \int_{\mathbb{S}_+^{d-1}} \langle f, x\rangle^s (\varphi r_s^{-1})(x) \pi_s(dx). \tag{3.7.36}$$

In order to show a relation similar to (3.7.18), we have to make use of the Hölder regularity of the stationary measure π_s established in Proposition 3.3.4 under conditions **M4** and **M6** for positive $d \times d$ matrices: the main distinct feature in this

case is that the constant C_1 in Proposition 3.3.4 can be sufficiently large. Set for brevity $J_{l,\delta}(\varepsilon_1) = (e^{-\delta k-2\varepsilon_1}, e^{-\delta k}]$. One can verify that for the set $\{x \in \mathbb{S}_+^{d-1} : \langle f, x\rangle \in (e^{-\delta k-2\varepsilon_1}, e^{-\delta k}]\}$ on the $d-1$ dimensional unit sphere, we can use N ball to cover this set, where N is proportional to $\frac{1}{e^{-\delta k}-e^{-\delta k-2\varepsilon_1}}$. Specifically, we have, there exists $f_1 = f_1(\delta, k, \varepsilon), f_2 = f_2(\delta, k, \varepsilon), \ldots, f_N = f_1(\delta, k, \varepsilon) \in \mathbb{S}_+^{d-1}$ such that

$$\pi_s\Big(\Big\{x : \langle f, x\rangle \in J_{l,\delta}(\varepsilon_1)\Big\}\Big) \leqslant \sum_{j=1}^{N} \pi_s\Big(\Big\{x : \langle f_j, x\rangle \in (0, e^{-\delta k} - e^{-\delta k-2\varepsilon_1}]\Big\}\Big).$$

According to (3.3.7), we get that there exist constants $c_s, C_s > 0$ such that, uniformly in $f_j \in \mathbb{S}_+^{d-1}$,

$$\pi_s\Big(\Big\{x : \langle f_j, x\rangle \in (0, e^{-\delta k} - e^{-\delta k-2\varepsilon_1}]\Big\}\Big) \leqslant C_s(e^{-\delta k} - e^{-\delta k-2\varepsilon_1})^{C_s},$$

where $C_s > 0$ can be sufficiently large (independent of the dimension d). Therefore, we obtain that, uniformly in $f \in \mathbb{S}_+^{d-1}$,

$$\lim_{\varepsilon_1\to 0} \pi_s\Big(\Big\{x : \langle f, x\rangle \in J_{l,\delta}(\varepsilon_1)\Big\}\Big) = 0. \tag{3.7.37}$$

One can also show that (3.7.37) holds with $J'_{l,\delta}(\varepsilon_1) = (e^{-\delta(k-1)+2\varepsilon_1}, e^{-\delta(k-1)}]$ instead of $J_{l,\delta}(\varepsilon_1)$. Therefore, using the Lebesgue dominated convergence theorem, similarly to (3.7.19), we deduce that

$$\lim_{\varepsilon_1\to 0} \sup_{f\in\mathbb{S}_+^{d-1}} \sum_{k=1}^{M} e^{-s\delta(k-1)} \pi_s\left((\varphi r_s^{-1})\mathbb{1}_{\{\log\langle f,\cdot\rangle\in I_{k,\varepsilon_1}\}}\right) = 0. \tag{3.7.38}$$

Combining (3.7.33)-(3.7.36) and (3.7.38), letting $n \to \infty$, $\varepsilon_1 \to 0$, $\delta \to 0$, $\varepsilon \to 0$, and noting that $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain the upper bound: uniformly in $f, x \in \mathbb{S}_+^{d-1}$ and $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$B_n^+ \leqslant \int_{\mathbb{S}_+^{d-1}} \langle f, x\rangle^s \varphi(x) r_s^{-1}(x) \pi_s(dx) \limsup_{\varepsilon\to 0} \int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi(y')dy.$$

The lower bound can be deduced using the same techniques. The proof can now be completed as in [85] and therefore the details are omitted. This concludes the proof. □

Proof of Theorem 3.2.4. Using Propositions 3.3.5, 3.3.2 and 3.4.3 (2) instead of Propositions 3.3.4, 3.3.1 and 3.4.3 (1), the proof of Theorem 3.2.4 can be carried out by following the proof of Theorem 3.2.2. The details are left to the reader. □

3.8 Proofs of large deviation principles for spectral radius

We next give a proof of Theorem 3.2.7, which is based on the reinforced large deviation principles for the matrix norm $\|G_n\|$ established in [85], the Collatz-Wielandt formula and the precise large deviation results for the entries $G_n^{i,j}$ established in Theorems 3.2.2 and 3.2.4.

In fact, we are able to establish the following more general version of large deviation results for the spectral radius $\rho(G_n)$ with a target function ψ on $\log\rho(G_n)$.

Theorem 3.8.1.
(1) *Assume conditions* ***M1***, ***M4***, ***M6*** *(or conditions* ***M1***, ***M5***, ***M6****) for positive matrices. Let* $s \in I_\mu^\circ$ *and* $q = \Lambda'(s)$. *Let* ψ *be a non-decreasing function on* $\mathbb{R}$ *such that* $y \mapsto e^{-s'y}\psi(y)$ *is directly Riemann integrable for some* $s' \in (0, s)$ *and* $\int_{\mathbb{R}} e^{-sy}\psi(y)dy > 0$. *Then, there exist two constants* $0 < c < C < +\infty$ *such that uniformly in* $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} c < &\liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\rho(G_n) - n(q+l)\Big)\Big] \\ \leqslant &\limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\rho(G_n) - n(q+l)\Big)\Big] < C. \end{aligned} \tag{3.8.1}$$

(2) *Assume conditions* ***M5***, ***M6*** *for positive matrices. Let* $q = \Lambda'(s)$ *and* $s \in (-s_0, 0)$, *where* $s_0 > 0$ *is small enough. Let* ψ *be a non-increasing function on* $\mathbb{R}$ *such that* $y \mapsto e^{-s'y}\psi(y)$ *is directly Riemann integrable for some* $s' \in (-s_0, s)$, *and* $\int_{\mathbb{R}} e^{-sy}\psi(y)dy > 0$. *Then, there exist two constants* $0 < c < C < +\infty$ *such that uniformly in* $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\begin{aligned} c < &\liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\rho(G_n) - n(q+l)\Big)\Big] \\ \leqslant &\limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\rho(G_n) - n(q+l)\Big)\Big] < C. \end{aligned} \tag{3.8.2}$$

Proof of Theorems 3.2.7 and 3.8.1. We only give a proof of Theorem 3.8.1 since Theorem 3.2.7 is a particular case of Theorem 3.8.1.

(1) Recall that the following large deviation bound for the matrix norm $\|G_n\|$ is proved in [85]: uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\|G_n\| - n(q+l)\Big)\Big] < C.$$

Since $\rho(G_n) \leqslant \|G_n\|$ and the function ψ is non-decreasing on $\mathbb{R}$, the upper bound in (3.8.1) easily follows: uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\limsup_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\rho(G_n) - n(q+l)\Big)\Big] < C.$$

We now proceed to prove the lower bound in (3.8.1) using the Collatz-Wielandt formula in conjugation with Theorem 3.2.2 for positive matrices. By the Collatz-Wielandt formula, we have for any $g \in \mathscr{G}_+^\circ$,

$$\rho(g) = \sup_{x\in\mathbb{S}_+^{d-1}} \min_{1\leqslant i\leqslant d,\, \langle e_i, x\rangle > 0} \frac{\langle e_i, gx\rangle}{\langle e_i, x\rangle}. \tag{3.8.3}$$

Taking $x = e_1$, we get $\rho(G_n) \geqslant G_n^{1,1}$. Using Theorem 3.2.2 for $f = x = e_1$ and taking into account that ψ is non-decreasing on $\mathbb{R}$, we obtain that there exists a constant $c > 0$ such that uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$c < \liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)}\mathbb{E}\Big[\psi\Big(\log\rho(G_n) - n(q+l)\Big)\Big], \tag{3.8.4}$$

which concludes the proof of (3.8.1).

(2) Recall that the following large deviation bound for the matrix norm $\|G_n\|$ is proved in [85]: there exists a constant $c > 0$ such that uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{E}\Big[\psi\Big(\log \|G_n\| - n(q+l)\Big)\Big] > c.$$

Since $\rho(G_n) \leqslant \|G_n\|$ and the function ψ is non-increasing on $\mathbb{R}$, we get the following lower bound: uniformly in $|l| \leqslant \frac{1}{\sqrt{n}}$,

$$\liminf_{n\to\infty} \sqrt{n}\, e^{n\Lambda^*(q+l)} \mathbb{E}\Big[\psi\Big(\log \rho(G_n) - n(q+l)\Big)\Big] > c.$$

In a similar way as in the proof of (3.8.4), using the Collatz-Wielandt formula (3.8.3) and Theorem 3.2.4, one can verify the upper bound of (3.8.2). The proof of (3.8.2) is complete. □

Chapter 4

Berry-Esseen bound and precise moderate deviations for products of random matrices

Abstract. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) $d\times d$ real random matrices. Set $G_n = g_n g_{n-1}\dots g_1$ and $X_n^x = G_n x/|G_n x|$, $n\geqslant 1$, where $|\cdot|$ is an arbitrary norm in $\mathbb{R}^d$ and $x\in\mathbb{R}^d$ is a starting point with $|x|=1$. For both invertible matrices and positive matrices, under suitable conditions we prove a Berry-Esseen type theorem and an Edgeworth expansion for the couple $(X_n^x, \log|G_n x|)$. These results are established using a brand new smoothing inequality on complex plane, the saddle point method and additional spectral gap properties of the transfer operator related to the Markov chain X_n^x. Cramér type moderate deviation expansions are derived for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on the Markov chain X_n^x.

4.1 Introduction

4.1.1 Background and objectives

For any integer $d\geqslant 2$, denote by $GL(d,\mathbb{R})$ the general linear group of $d\times d$ invertible matrices. Equip $\mathbb{R}^d$ with any norm $|\cdot|$, denote by $\mathbb{P}^{d-1}=\{x\in\mathbb{R}^d, |x|=1\}/\pm$ the projective space in $\mathbb{R}^d$, and let $\|g\|=\sup_{x\in\mathbb{P}^{d-1}}|gx|$ be the operator norm for $g\in GL(d,\mathbb{R})$. Let $(g_n)_{n\geqslant 1}$ be a sequence of i.i.d. $d\times d$ real random matrices of the same law μ on $GL(d,\mathbb{R})$, and consider the product $G_n = g_n g_{n-1}\dots g_1$ and the process $X_n^x = G_n x/|G_n x|$, $n\geqslant 1$, with starting point $x\in\mathbb{P}^{d-1}$.

The study of the asymptotic properties of the Markov chain $(X_n^x)_{n\geqslant 1}$ and of the product $(G_n)_{n\geqslant 1}$ has attracted a good deal of attention since the groundwork of Furstenberg and Kesten [37], where the strong law of large numbers for $\log\|G_n\|$ has been established, which is a fundamental result for the products of random matrices. Furstenberg [36] proved the following version of the law of large numbers: for any $x\in\mathbb{R}^d$,

$$\lim_{n\to\infty}\frac{1}{n}\log|G_n x| = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log|G_n x| = \lambda \quad \mathbb{P}\text{-a.s.},$$

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where the real number λ is called upper Lyapunov exponent associated with the product G_n. Another cornerstone result is the central limit theorem (CLT) for the couple $(X_n^x, \log|G_n x|)$, established under contracting type assumptions by Le Page [69]: for any fixed $y \in \mathbb{R}$ and any Hölder continuous function $\varphi : \mathbb{P}^{d-1} \mapsto \mathbb{R}$, it holds uniformly in $x \in \mathbb{P}^{d-1}$ that

$$\lim_{n\to\infty} \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] = \nu(\varphi)\Phi(y),$$

where ν is the unique stationary probability measure of the Markov chain X_n^x on $\mathbb{P}^{d-1}$, $\sigma^2 = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\Big[(\log|G_n x| - n\lambda)^2\Big]$ is the asymptotic variance independent of x, and Φ is the standard normal distribution function. The optimal conditions for the CLT to hold true have been established recently by Benoist and Quint [9].

A very important topic is the study of large and moderate deviation probabilities, which describe the rate of convergence in the law of large numbers. For an account to the theory of large deviations for sums of independent random variables we refer to Cramér [26], Petrov [74], Stroock [82], Varadhan [83] and Dembo and Zeitouni [30]. For products of random matrices, precise large deviations asymptotics have been considered e.g. by Le Page [69], Buraczewski and Mentemeier [17], Guivarc'h [49], Benoist and Quint [10], Sert [78], Xiao, Grama and Liu [85]. For moderate deviations, very little results are known. Benoist and Quint [10] have recently established the asymptotic for the logarithm of probabilities of moderate deviations for reductive groups, which in our setting reads as follows: for any interval $B \subseteq \mathbb{R}$, and positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, it holds, uniformly in $x \in \mathbb{P}^{d-1}$, that

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log|G_n x| - n\lambda}{b_n} \in B\Big) = -\inf_{y\in B} \frac{y^2}{2\sigma^2}. \tag{4.1.1}$$

A functional moderate deviation principle has been established by Cuny, Dedecker and Jan [23].

The first objective of our paper is to improve on the result (4.1.1) by establishing a Cramér type moderate deviation expansion for $\log|G_n x|$: we prove that uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, o(\sqrt{n})]$,

$$\frac{\mathbb{P}\Big(\log|G_n x| - n\lambda \geqslant \sqrt{n}\sigma y\Big)}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\left[1 + O\left(\frac{y+1}{\sqrt{n}}\right)\right], \tag{4.1.2}$$

where $t \mapsto \zeta(t)$ is the Cramér series of the logarithm of the eigenfunction related to the transfer operator of the Markov walk associated to the product of random matrices (see Section 4.2.3).

It is useful to extend the moderate deviation expantion (4.1.2) for the couple $(X_n^x, \log|G_n x|)$ which describes completely the random walk $(G_n x)_{n\geqslant 1}$. We prove that, for any Hölder continuous function φ on $\mathbb{P}^{d-1}$, uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, o(\sqrt{n})]$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\left[\nu(\varphi) + O\left(\frac{y+1}{\sqrt{n}}\right)\right], \tag{4.1.3}$$

see Theorem 4.2.3. Our second objective, which is also the key point in proving (4.1.3), is a Berry-Esseen bound for the couple $(X_n^x, \log|G_n x|)$: for any Hölder continuous

function φ on $\mathbb{P}^{d-1}$,

$$\sup_{x\in\mathbb{P}^{d-1},\, y\in\mathbb{R}} \left|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| = O\Big(\frac{1}{\sqrt{n}}\Big), \tag{4.1.4}$$

see Theorem 4.2.1. This extends the result of Le Page [69] established for the particular target function $\varphi = \mathbf{1}$ (see also Jan [62]). We further upgrade (4.1.4) to an Edgeworth expansion under a non-arithmeticity condition, see Theorem 4.2.2, which is new even for $\varphi = \mathbf{1}$.

All the results stated above concern invertible matrices, but we also establish analogous theorems for positive matrices. Some limit theorems for $\log|G_n x|$ in case of positive matrices such as central limit theorem and Berry-Esseen theorem have been established earlier by Furstenberg and Kesten [37], Hennion [53], and Hennion and Hervé [55]. Here, we extend the Berry-Esseen theorem of [55] to the couple $(X_n^x, \log|G_n x|)$ with a target function φ on the Markov chain X_n^x. We also complement the results in [37, 53, 55] by giving a Cramér type moderate deviation expansion.

4.1.2 Key ideas of the approach

For the moderate deviation expansions (4.1.2) and (4.1.3), our proof is different from those in [10] and [23]: in [10] the moderate deviation principle (4.1.1) is obtained by following the strategy of Kolmogorov [68] suited to show the law of iterated logarithm (see also de Acosta [29] and Wittman [84]); in [23] the proof of the functional moderate deviation principle is based on the martingale approximation method developed in [9].

In order to prove (4.1.3) we have to rework the spectral gap theory for the transfer operators P_z and $R_{s,z}$, by considering the case when s can take values in the interval $(-\eta, \eta)$ with $\eta > 0$ small, and z belongs to a small complex ball centered at the origin, see Section 4.3. This allows to define the change of measure $\mathbb{Q}_s^x$ and to extend the Berry-Esseen bound (4.1.4) for the changed measure $\mathbb{Q}_s^x$, see Theorem 4.5.1. The moderate deviation expansion (4.1.3) is established by adapting the techniques from Petrov [74].

It is surprising that the proof of the Berry-Esseen bound and of the Edgeworth expansion with a non-trivial target function $\varphi \neq \mathbf{1}$ is way more difficult than the analogous results with $\varphi = \mathbf{1}$. This can be seen from the following sketch of the proof.

For simplicity, we assume that $\sigma = 1$. Introduce the transfer operator P_z: for any Hölder continuous function φ on $\mathbb{P}^{d-1}$ and $z \in \mathbb{C}$,

$$P_z\varphi(x) = \mathbb{E}\Big[e^{z\log|g_1 x|}\varphi(X_1^x)\Big], \quad x \in \mathbb{P}^{d-1}. \tag{4.1.5}$$

Let F be the distribution function of $\frac{\log|G_n x|-n\lambda}{\sqrt{n}}$ and f be its Fourier transform: $f(t) = e^{it\sqrt{n}\lambda}(P^n_{-it/\sqrt{n}}\mathbf{1})(x)$, $t \in \mathbb{R}$. The Berry-Esseen bound (4.1.4) with target function $\varphi = \mathbf{1}$ is usually proved using Esseen's smoothing inequality: for all $T > 0$,

$$\sup_{y\in\mathbb{R}} |F(y) - \Phi(y)| \leqslant \frac{1}{\pi}\int_{-T}^{T}\left|\frac{f(t) - e^{-t^2/2}}{t}\right| dt + \frac{C}{T}. \tag{4.1.6}$$

Inserting the spectral gap decomposition

$$P_z^n = \kappa^n(z)M_z + L_z^n \quad (n \geqslant 1) \tag{4.1.7}$$

into (4.1.6) allows us to obtain the Berry-Esseen bound (4.1.4) with $\varphi = \mathbf{1}$: after some straightforward calculations, it reduces to showing that, with $T = c\sqrt{n}$,

$$\int_{-T}^{T} \left|(L^n_{-it/\sqrt{n}}\mathbf{1})(x)\right|/|t|dt < \infty. \tag{4.1.8}$$

The bound (4.1.8) is proved using Taylor's expansion $L^n_z\mathbf{1} = L^n_0\mathbf{1} + z\frac{d}{dz}(L^n_z\mathbf{1}) + o(z)$ with $z = -it/\sqrt{n}$, and the fact that $L^n_0\mathbf{1} = 0$. However, when we replace the unit function $\mathbf{1}$ by a target function φ for which in general $L^n_0\varphi \neq 0$, instead of (4.1.8), we have

$$\int_{-T}^{T} |L^n_{-it/\sqrt{n}}\varphi(x)|/|t|dt = \infty, \tag{4.1.9}$$

even though $|L^n_0\varphi(x)|$ decays exponentially fast to 0 as $n \to \infty$. To overcome it, we have elaborated a new approach based on smoothing inequality on complex contours and on the saddle point method, see Daniels [28] and Fedoryuk [34].

Specifically, our smoothing inequality is formulated as follows: for any $T > r > 0$,

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} |F(y) - H(y)| \\
&\leqslant \frac{1}{\pi}\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-} \frac{f(z)-h(z)}{z} e^{izy}e^{-ib\frac{z}{T}}dz\right| + \frac{1}{\pi}\sup_{y>0}\left|\int_{\mathcal{C}_r^+} \frac{f(z)-h(z)}{z} e^{izy}e^{-ib\frac{z}{T}}dz\right| \\
&+ \frac{1}{\pi}\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-} \frac{f(z)-h(z)}{z} e^{izy}e^{ib\frac{z}{T}}dz\right| + \frac{1}{\pi}\sup_{y>0}\left|\int_{\mathcal{C}_r^+} \frac{f(z)-h(z)}{z} e^{izy}e^{ib\frac{z}{T}}dz\right| \\
&+ \frac{1}{\pi}\int_{r\leqslant|t|\leqslant T}\left|\frac{f(t)-h(t)}{t}\right|dt + \frac{2}{\pi T}\int_{-T}^{T}|f(t)-h(t)|dt + \frac{3b}{T}\sup_{y\in\mathbb{R}}|H'(y)|,
\end{aligned} \tag{4.1.10}$$

where $b > 0$ is a fixed sufficiently large constant, $\mathcal{C}_r^-$ and $\mathcal{C}_r^+$ are semicircles given by $\mathcal{C}_r^- = \{z \in \mathbb{C} : |z| = r, \Im z < 0\}$ and $\mathcal{C}_r^+ = \{z \in \mathbb{C} : |z| = r, \Im z > 0\}$; see Section 4.4 for details. Use Cauchy's integral theorem enables us to establish (4.1.10) and also to give the estimation of the integrals therein.

The smoothing inequality (4.1.10) together with the spectral gap property (4.1.7) leads to the estimation of the following integrals:

$$\int_{\mathcal{C}_r^-} \frac{\kappa^n(z)M_z\varphi(x) - e^{-z^2/2}}{z} e^{izy}e^{-ib\frac{z}{T}}dz, \tag{4.1.11}$$

$$\int_{\mathcal{C}_r^-} \frac{L^n_z\varphi(x)}{z} e^{izy}e^{ib\frac{z}{T}}dz. \tag{4.1.12}$$

The integral (4.1.11) is handled by using the saddle point method choosing a suitable path for the integration in Section 4.5.2, which is one of the challenging parts of the proof. For the integral (4.1.12) we use the facts that $|L^n_z\varphi(x)|$ decays exponentially fast as $n \to \infty$ and that $|\frac{e^{izy}}{z}| \leqslant \frac{1}{T}$ on the contour $\mathcal{C}_r^-$ for $y \leqslant 0$, where $T = c\sqrt{n}$. In contrast to (4.1.9), this shows that (4.1.12) is bounded by Ce^{-cn} uniformly in y.

The integrals on the semicircle $\mathcal{C}_r^+$ is treated similarly, which allows us to establish (4.1.4). Note that the non-arithmeticity condition is not needed for the validity of (4.1.4). Under the non-arithmeticity condition, in Theorem 4.2.2 we obtain an Edgeworth expansion for $(X_n^x, |G_n x|)$ with the target function φ on X_n^x, which is of independent interest.

4.2 Main results

4.2.1 Notation and conditions

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. The real part, imaginary part and the conjugate of a complex number z are denoted by $\Re z$, $\Im z$ and $\overline{z}$ respectively. For $y \in \mathbb{R}$, we write $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ and $\Phi(y) = \int_{-\infty}^{y} \phi(t)dt$. For any $\eta > 0$, set $B_\eta(0) = \{z \in \mathbb{C} : |z| < \eta\}$ for the ball with center 0 and radius η in the complex plane $\mathbb{C}$. We denote by c, C, positive absolute constants whose values may change from line to line. By c_α, C_α we mean positive constants depending only on the index α. We write $\mathbb{1}_A$ for the indicator function of an event A. For a measure ν and a function φ we denote $\nu(\varphi) = \int \varphi d\nu$.

For $d \geqslant 2$, let $M(d, \mathbb{R}) := \mathscr{M}$ be the set of $d \times d$ matrices with entries in $\mathbb{R}$. We shall work with products of invertible or non-negative matrices. Denote by $\mathscr{G} = GL(d, \mathbb{R})$ the group of invertible matrices of $\mathscr{M}$. A non-negative matrix $g \in \mathscr{M}$ is said to be *allowable*, if every row and every column of g has a strictly positive entry. Denote by $\mathscr{G}_+$ the multiplicative semigroup of allowable non-negative matrices of $\mathscr{M}$, which will be called simply positive. We write $\mathscr{G}_+^\circ$ for the subsemigroup of $\mathscr{G}_+$ with strictly positive entries.

The space $\mathbb{R}^d$ is equipped with any given norm $|\cdot|$. Denote by $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ the unit sphere, and by $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ the intersection of the unit sphere with the positive quadrant. It will be convenient to consider the projective space $\mathbb{P}^{d-1} = \mathbb{S}^{d-1}/\pm$ by identifying x with $-x$. To unify the exposition, we use the symbol $\mathcal{S}$ to denote $\mathbb{P}^{d-1}$ in case of invertible matrices and $\mathbb{S}_+^{d-1}$ in case of positive matrices. For $x \in \mathcal{S}$ and $g \in \mathscr{G}$ or $g \in \mathscr{G}_+$, we write $g \cdot x = \frac{gx}{|gx|}$ for the projective action of g on $\mathcal{S}$. The space $\mathcal{S}$ is endowed with the metric $\mathbf{d}$: for invertible matrices, $\mathbf{d}$ is the angular distance, i.e., for any $x, y \in \mathbb{P}^{d-1}$, $\mathbf{d}(x, y) = |\sin\theta(x, y)|$, where $\theta(x, y)$ is the angle between x and y; for positive matrices, $\mathbf{d}$ is the Hilbert cross-ratio metric, i.e., for any $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{S}_+^{d-1}$, $\mathbf{d}(x, y) = \frac{1-m(x,y)m(y,x)}{1+m(x,y)m(y,x)}$, where $m(x, y) = \sup\{\lambda > 0 : \ \lambda y_i \leqslant x_i, \ \forall i = 1, \ldots, d\}$. In both cases, there exists a constant $C > 0$ such that

$$|x - y| \leqslant C\mathbf{d}(x, y), \quad \text{for any} \quad x, y \in \mathcal{S}. \tag{4.2.1}$$

We refer to [50] and [53] for more details.

Let $\mathcal{C}(\mathcal{S})$ be the space of continuous complex-valued functions on $\mathcal{S}$ and $\mathbf{1}$ be the constant function with value 1. Let $\gamma > 0$. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_\gamma, \quad \|\varphi\|_\infty := \sup_{x \in \mathcal{S}} |\varphi(x)|, \quad [\varphi]_\gamma := \sup_{x,y \in \mathcal{S}} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}^\gamma(x, y)}.$$

Introduce the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma < +\infty\}$.

Let $(g_n)_{n \geqslant 1}$ be a sequence of i.i.d. random matrices with the same law μ, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set $G_n = g_n \ldots g_1$, $n \geqslant 1$, then for any starting point $x \in \mathcal{S}$, the process

$$X_0^x = x, \quad X_n^x = G_n \cdot x, \quad n \geqslant 1$$

forms a Markov chain on $\mathcal{S}$. The goal of the present paper is to establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$

with a target function φ on the Markov chain (X_n^x), for both invertible matrices and positive matrices.

For any $g \in \mathscr{M}$, set $\|g\| = \sup_{x\in\mathcal{S}} |gx|$ and $\iota(g) = \inf_{x\in\mathcal{S}} |gx| > 0$, where $\iota(g) > 0$ for both $g \in \mathscr{G}$ and $g \in \mathscr{G}_+$. In the following we use the notation $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$. From the Cartan decomposition it follows that the norm $\|g\|$ coincides with the largest singular value of g, i.e. $\|g\|$ is the square root of the largest eigenvalue of $g^{\mathrm{T}}g$, where g^{T} denotes the transpose of g. For an invertible matrix $g \in \mathscr{G}$, $\iota(g) = \|g^{-1}\|^{-1}$, hence $\iota(g)$ is the smallest singular value of g and $N(g) = \max\{\|g\|, \|g^{-1}\|\}$. We need the two-sided exponential moment condition:

C1. There exists a constant $\eta_0 \in (0,1)$ such that $\mathbb{E}[N(g_1)^{\eta_0}] < +\infty$.

We denote by $\Gamma_\mu := [\operatorname{supp}\mu]$ the smallest closed semigroup of $\mathscr{M}$ generated by $\operatorname{supp}\mu$, the support of μ.

For invertible matrices, we will need the strong irreducibility and proximality conditions. Recall that a matrix g is said to be *proximal* if g has an eigenvalue λ_g satisfying $|\lambda_g| > |\lambda_g'|$ for all other eigenvalues λ_g' of g. The normalized eigenvector v_g ($|v_g| = 1$) corresponding to the eigenvalue λ_g is called the dominant eigenvector. It is easy to verify that $\lambda_g \in \mathbb{R}$.

C2. (i)(Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant.
(ii)(Proximality) Γ_μ contains at least one proximal matrix.

For positive matrices, we will use the allowability and positivity conditions:

C3. (i) (Allowability) Every $g \in \Gamma_\mu$ is allowable.
(ii) (Positivity) Γ_μ contains at least one matrix belonging to $\mathscr{G}_+^\circ$.

It follows from the Perron-Frobenius theorem that every $g \in \mathscr{G}_+^\circ$ has a dominant eigenvalue $\lambda_g > 0$, with the corresponding eigenvector $v_g \in \mathbb{S}_+^{d-1}$.

Under conditions **C1** and **C2** for invertible matrices, or conditions **C1** and **C3** for positive matrices, there exists a unique μ-stationary probability measure ν on $\mathcal{S}$ ([50, 16]): for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$(\mu * \nu)(\varphi) = \int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(g_1 \cdot x)\mu(dg_1)\nu(dx) = \int_{\mathcal{S}} \varphi(x)\nu(dx) = \nu(\varphi). \tag{4.2.2}$$

Moreover, for invertible matrices, $\operatorname{supp}\nu$ (the support of ν) is given by

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{P}^{d-1} : g \in \Gamma_\mu,\ g \text{ is proximal}\}}; \tag{4.2.3}$$

for positive matrices, $\operatorname{supp}\nu$ is given by

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{S}_+^{d-1} : g \in \Gamma_\mu,\ g \in \mathscr{G}_+^\circ\}}. \tag{4.2.4}$$

In addition, for both cases, $V(\Gamma_\mu)$ is the unique minimal Γ_μ-invariant subset (see [50] and [16]).

For positive matrices, it will be shown in Proposition 4.3.15 that under conditions **C1** and **C3**, the asymptotic variance

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log|G_n x| - n\lambda)^2\Big]$$

exists with value in $[0,\infty)$. To establish the Berry-Esseen theorem and the moderate deviation expansion, we need the following condition:

C4. The asymptotic variance σ^2 satisfies $\sigma^2 > 0$.

We say that the measure μ is *arithmetic*, if there exist $t > 0$, $\beta \in [0, 2\pi)$ and a function $\vartheta : \mathcal{S} \to \mathbb{R}$ such that $\exp[it \log|gx| - i\beta + i\vartheta(g \cdot x) - i\vartheta(x)] = 1$ for any $g \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$. To establish the Edgeworth expansion for positive matrices, we impose the following condition:

C5. (Non-arithmeticity) The measure μ is non-arithmetic.

A simple sufficient condition introduced in [66] for the measure μ to be non-arithmetic is that the additive subgroup of $\mathbb{R}$ generated by the set $\{\log \lambda_g : g \in \Gamma_\mu,\ g \in \mathscr{G}_+^\circ\}$ is dense in $\mathbb{R}$, see [17, Lemma 2.7].

We end this subsection by giving some implications among the above conditions. For invertible matrices, it was proved in [52] that condition **C2** implies condition **C5**. For positive matrices, conditions **C1**, **C3** and **C5** imply condition **C4**, see Proposition 4.3.15.

4.2.2 Berry-Esseen bound and Edgeworth expansion

In this subsection we formulate the Berry-Esseen theorem and the Edgeworth expansion for $(X_n^x, \log|G_n x|)$. We first state the Berry-Esseen theorem with a target function on X_n^x. Through the rest of the paper we assume that $\gamma > 0$ is a fixed small enough constant so that the spectral properties stated in Proposition 4.3.1 hold true.

Theorem 4.2.1. *Assume either conditions **C1** and **C2** for invertible matrices, or conditions **C1**, **C3** and **C4** for positive matrices. Then, there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant C\frac{\|\varphi\|_\gamma}{\sqrt{n}}. \tag{4.2.5}$$

The proof of this theorem follows the same line as the proof of the Edgeworth expansion in Theorem 4.2.2 formulated below, and will be sketched at the end of Section 4.5. The presence of the target function in Theorem 4.2.1 turns out to be crucial in the study of the asymptotic of moderate deviations of the scalar product $\log|\langle f, G_n x\rangle|$, which will be done in a forthcoming paper.

Theorem 4.2.1 extends the Berry-Esseen bounds from [69, 62] for invertible matrices, and [55] for positive matrices to versions with target functions on X_n^x. Note that the results in [62, 55] have been established under some polynomial moment conditions. However, proving (4.2.5) with the target function $\varphi \neq \mathbf{1}$ under the polynomial moments is still an open problem.

The following result gives an Edgeworth expansion for $\log|G_n x|$ with the target function φ on X_n^x. To formulate the result, we introduce the necessary notation. Consider the following transfer operator: for any $s \in (-\eta, \eta)$ with $\eta > 0$ small, and $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) = \mathbb{E}\Big[e^{s\log|g_1 x|}\varphi(g_1 \cdot x)\Big] \quad x \in \mathcal{S}.$$

It will be shown in Proposition 4.3.1 that there exist a measure ν_s and a Hölder continuous function r_s on $\mathcal{S}$ such that

$$\nu_s P_s = \kappa(s)\nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \tag{4.2.6}$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s. Set $\Lambda(s) = \log \kappa(s)$. It is shown in Lemma 4.3.11 that for any $\varphi \in \mathcal{B}_\gamma$, the function

$$b_\varphi(x) = \lim_{n\to\infty} \mathbb{E}\Big[(\log|G_n x| - n\lambda)\varphi(X_n^x)\Big], \quad x \in \mathcal{S} \tag{4.2.7}$$

is well defined, belongs to $\mathcal{B}_\gamma$ and has an equivalent expression (4.3.39) in terms of derivative of the projection operator $\Pi_{0,z}$, see Proposition 4.3.8.

Theorem 4.2.2. *Assume either conditions **C1** and **C2** for invertible matrices, or conditions **C1**, **C3** and **C5** for positive matrices. Then, as $n \to \infty$, uniformly in $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\begin{aligned}\Bigg|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] \\ - \nu(\varphi)\Big[\Phi(y) + \frac{\Lambda'''(0)}{6\sigma^3\sqrt{n}}(1-y^2)\phi(y)\Big] + \frac{b_\varphi(x)}{\sigma\sqrt{n}}\phi(y)\Bigg| = \|\varphi\|_\gamma o\Big(\frac{1}{\sqrt{n}}\Big).\end{aligned} \tag{4.2.8}$$

The proof of this theorem is postponed to Section 4.5 and is based on a new smoothing inequality (Proposition 4.4.1) and the saddle point method. Even for $\varphi = \mathbf{1}$, Theorem 4.2.2 is new.

4.2.3 Moderate deviation expansions

Denote $\gamma_k = \Lambda^{(k)}(0)$, $k \geqslant 1$, where $\Lambda = \log\kappa$ with the function κ defined in (4.2.6). In particular, $\gamma_1 = \lambda$ and $\gamma_2 = \sigma^2$, see Propositions 4.3.13 and 4.3.15, where we give also an expression for γ_3. Throughout the paper, we write ζ for the Cramér series of Λ (see [26] and [74]):

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \cdots \tag{4.2.9}$$

which converges for $|t|$ small enough.

We start by formulating a Cramér type moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$ with target function on X_n^x, for both invertible matrices and positive matrices.

Theorem 4.2.3. *Assume either conditions **C1** and **C2** for invertible matrices, or conditions **C1**, **C3** and **C4** for positive matrices. Then, uniformly in $x \in \mathcal{S}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \to \infty$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big],$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big].$$

Note that the above asymptotic expansions remain valid even when $\nu(\varphi) = 0$. In this case, for example, the first expansion becomes

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big] = (1-\Phi(y))e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big).$$

It is an open question to extend the results of Theorem 4.2.3 to higher order expansions under the additional condition of non-arithmeticity. We refer to Saulis [77] and Rozovsky [76] for relevant results in the i.i.d. real-valued case. In the case of products of random matrices this problem seems to us challenging because of the presence of the derivatives in s of the eigenfunction r_s and of the eigenmeasure ν_s in the higher order terms.

In particular, under conditions of Theorem 4.2.3, with $\varphi = \mathbf{1}$ we obtain:

$$\frac{\mathbb{P}\left(\frac{\log|G_nx|-n\lambda}{\sigma\sqrt{n}}\geqslant y\right)}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[1+O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big],$$
$$\frac{\mathbb{P}\left(\frac{\log|G_nx|-n\lambda}{\sigma\sqrt{n}}\leqslant -y\right)}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[1+O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big].$$

When $\varphi \in \mathcal{B}_\gamma$ is a real-valued function satisfying $\nu(\varphi) > 0$, Theorem 4.2.3 clearly implies the following moderate deviation principle for $\log|G_nx|$ with target function on X_n^x: for any Borel set $B \subseteq \mathbb{R}$, and positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned}-\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log|G_nx|-n\lambda}{b_n}\in B\}}\Big]\\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log|G_nx|-n\lambda}{b_n}\in B\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2},\end{aligned} \tag{4.2.10}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B. In fact it is enough to show (4.2.10) only for the case where B is an interval, the result for general B can be established using Lemma 4.4 of Huang and Liu [58]. With $\varphi = \mathbf{1}$, (4.2.10) implies the moderate deviation principle (4.1.1) established in [10, Proposition 12.12] for invertible matrices. The moderate deviation principle (4.2.10) with target function on X_n^x is new for both invertible matrices and positive matrices; (4.1.1) is new for positive matrices. Note that in (4.2.10) the function φ is not necessarily positive.

4.3 Spectral gap theory

This section is devoted to investigating the spectral gap properties of some operators to be introduced below: the transfer operator P_z, its normalization Q_s which is a Markov operator, and the perturbed operator $R_{s,z}$, for real-valued s and complex-valued z. The properties for these operators have been intensively studied in recent years, for instance in [69, 16, 50, 17, 10], where various results have been established under different restrictions on s and z, which are not enough for obtaining the results of the paper. We shall complete these results by investigating the case when $s \in (-\eta, \eta)$ with $\eta > 0$ small, and z belongs to a small ball of the complex plane centered at the origin.

The case of $s < 0$ turns out to be more difficult than the case $s \geqslant 0$ and requires a deeper analysis. We also complement the previous results with some new properties to be used in the proofs of the main results of the paper.

4.3.1 Properties of the transfer operator P_z

Recall that the Banach space $\mathcal{B}_\gamma$ consists of all the γ-Hölder continuous complex-valued functions on $\mathcal{S}$. We write $\mathcal{B}'_\gamma$ for the topological dual of $\mathcal{B}_\gamma$ endowed with the norm $\|\nu\|_{\mathcal{B}'_\gamma} = \sup_{\|\varphi\|_\gamma=1} |\nu(\varphi)|$, for any linear functional $\nu \in \mathcal{B}'_\gamma$. Let $\mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ be the set of all bounded linear operators from $\mathcal{B}_\gamma$ to $\mathcal{B}_\gamma$ equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma}$. Denote by $\varrho(Q)$ the spectral radius of an operator $Q \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and by $Q|_E$ its restriction to the subspace $E \subseteq \mathcal{B}_\gamma$.

For any $z \in \mathbb{C}$ with $|z| < \eta_0$, where η_0 is given in condition **C1**, define the transfer operator P_z as follows: for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_z\varphi(x) = \mathbb{E}\Big[e^{z \log |g_1 x|}\varphi(g_1 \cdot x)\Big] \quad x \in \mathcal{S}. \tag{4.3.1}$$

The transfer operator P_z acts from $\mathcal{C}(\mathcal{S})$ to the space of bounded functions on $\mathcal{S}$. The following proposition gives the spectral gap properties of the operator P_z for z in a small enough neighborhood of 0 in the complex plane.

Proposition 4.3.1. *Assume that μ satisfies either conditions **C1** and **C2** for invertible matrices, or conditions **C1** and **C3** for positive matrices. Then, $P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ for any $z \in B_{\frac{\eta_0}{2}}(0)$, and the mapping $z \mapsto P_z : B_{\frac{\eta_0}{2}}(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is analytic for $\gamma > 0$ small enough, where η_0 is given in condition **C1**. Moreover, there exists a small $\eta > 0$ such that for any $z \in B_\eta(0)$ and $n \geqslant 1$, we have the decomposition*

$$P_z^n = \kappa^n(z) M_z + L_z^n, \tag{4.3.2}$$

where the operator $M_z := \nu_z \otimes r_z$ is a rank one projection on $\mathcal{B}_\gamma$ defined by $M_z\varphi = \frac{\nu_z(\varphi)}{\nu_z(r_z)} r_z$ for any $\varphi \in \mathcal{B}_\gamma$, and the mappings on $B_\eta(0)$

$$z \mapsto \kappa(z) \in \mathbb{C}, \quad z \mapsto r_z \in \mathcal{B}_\gamma, \quad z \mapsto \nu_z \in \mathcal{B}'_\gamma, \quad z \mapsto L_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$$

are unique under the normalizing conditions $\nu(r_z) = 1$ and $\nu_z(\mathbf{1}) = 1$, where ν is defined in (4.2.2); all these mappings are analytic in $B_\eta(0)$, and possess the following properties:

(a) *for any $z \in B_\eta(0)$, it holds that $M_z L_z = L_z M_z = 0$;*

(b) *for any $z \in B_\eta(0)$, $P_z r_z = \kappa(z) r_z$ and $\nu_z P_z = \kappa(z)\nu_z$;*

(c) *$\kappa(0) = 1$, $r_0 = \mathbf{1}$, $\nu_0 = \nu$, and $\kappa(s)$ and r_s are real-valued and satisfy $\kappa(s) > 0$ and $r_s(x) > 0$ for any $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$;*

(d) *for any $k \in \mathbb{N}$, there exist $0 < a_1 < a_2 < 1$ such that $|\kappa(z)| > 1 - a_1$ and $\|\frac{d^k}{dz^k} L_z^n\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma} \leqslant C_k(1 - a_2)^n$ for all $z \in B_\eta(0)$.*

Let us point out the differences between Proposition 4.3.1 and the previous results in [69, 16, 10]. Firstly, we complement the results in [69, 10] by giving the explicit formula $M_z\varphi = \frac{\nu_z(\varphi)}{\nu_z(r_z)} r_z$ in (4.3.2), for $z \in B_\eta(0)$, which is one of the crucial points in the proofs of the results of the paper. Basically, it permits us to deduce the spectral gap properties of the operators Q_s and $R_{s,z}$ from those of P_z. In particular this will enable us to obtain an explicit formula for the operators N_s and $N_{s,z}$ in Propositions 4.3.4 and 4.3.8, and the uniformity of the bounds (4.3.36) and (4.3.37). Secondly, for positive matrices, some points of Proposition 4.3.1 have been obtained in [16] only for real $z \geqslant 0$. The difficulty here is the case when $z \in \mathbb{R}$ is negative and when z is not real, so Proposition 4.3.1 is new for positive matrices when $|z| \leqslant \eta$. Thirdly, we show that $\kappa(z)$ and r_z take real positive values when z is real, which allows to define the change of measure $\mathbb{Q}_s^x$ for real s, for both invertible matrices and positive matrices. Previously it was shown in [10] that $\kappa(z)$ is real-valued for real $z \in (-\eta, \eta)$ for invertible matrices.

In the sequel, without explicitly stated, we always assume that $\gamma > 0$ is a sufficiently small constant.

Remark 4.3.2. Define the conjugate transfer operator P_z^* by

$$P_z^*\varphi(x) = \mathbb{E}\Big[e^{z\log|g_1^{\mathrm{T}}x|}\varphi(g_1^{\mathrm{T}}\cdot x)\Big] \quad x \in \mathcal{S},$$

where $z \in \mathbb{C}$ with $\Re z \in (-\eta_0, \eta_0)$, and g_1^{T} denotes the transpose of the matrix g_1. One can verify that P_z^* satisfies all the properties of Proposition 4.3.1: under conditions of Proposition 4.3.1, we have the decomposition

$$P_z^{*n} = \kappa^{*n}(z)\nu_z^* \otimes r_z^* + L_z^{*n}, \quad z \in B_\eta(0),\ n \geqslant 1, \tag{4.3.3}$$

and all the assertions in Proposition 4.3.1 hold for P_z^*, $\kappa^*(z)$, ν_z^*, r_z^*, L_z^* instead of P_z, $\kappa(z)$, ν_z, r_z, L_z.

Proof of Proposition 4.3.1. We split the proof into three steps. In steps 1 and 2 we concentrate on the case of positive matrices, since for invertible matrices the results of these steps have been proved in [69, 10]. In step 1 we follow the same lines as in [69, 10]. In step 2 we follow [56] to prove the spectral gap property of the operator P_0 and we use the perturbation theory to extend it to P_z. In step 3 the proof is new and is provided for both invertible and positive matrices by complementing the results in [69, 16, 10].

Step 1. We only need to consider the case of positive matrices. We will show that there exists $\gamma \in (0, \frac{\eta_0}{6})$ such that $P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and that the mapping $z \mapsto P_z$ is analytic on $B_{\frac{\eta_0}{2}}(0)$. For any $m \geqslant 0$, $z \in B_{\frac{\eta_0}{2}}(0)$ and $\varphi \in \mathcal{B}_\gamma$, let

$$P_z^{(m)}\varphi(x) = \mathbb{E}\Big[(\log|g_1x|)^m|g_1x|^z\varphi(g_1\cdot x)\Big], \quad x \in \mathbb{S}_+^{d-1}.$$

It suffices to show that for $z \in B_{\frac{\eta_0}{2}}(0)$ and $\theta \in B_{\frac{\eta_0}{6}}(0)$,

$$P_{z+\theta} = \sum_{m=0}^{\infty} \frac{\theta^m}{m!} P_z^{(m)}, \tag{4.3.4}$$

and that there exists a constant $C > 0$ not depending on θ and z such that

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} \|P_z^{(m)}\varphi\|_\gamma \leqslant C\|\varphi\|_\gamma. \tag{4.3.5}$$

From (4.3.5) we deduce that $P_z^{(0)} = P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$. Moreover, the bound (4.3.5) ensures the validity of (4.3.4) which implies the analyticity of the mapping $z \mapsto P_z$ on $B_{\frac{\eta_0}{2}}(0)$.

It remains to prove (4.3.5). We first give a control of $\|P_z^{(m)}\varphi\|_\infty$. Since $|\log|gx|| \leqslant \log N(g)$ for $g \in \Gamma_\mu$ and $x \in \mathbb{S}_+^{d-1}$, we get

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} \|P_z^{(m)}\varphi\|_\infty \leqslant \|\varphi\|_\infty \mathbb{E}\Big[e^{(|\theta|+|\Re z|)\log N(g_1)}\Big] \leqslant C\|\varphi\|_\infty. \tag{4.3.6}$$

To control $[P_z^{(m)}\varphi]_\gamma$, note that for any $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned}
[P_z^{(m)}\varphi]_\gamma \leqslant & \sup_{x,y\in\mathbb{S}_+^{d-1}, x\neq y} \Big|\mathbb{E}\Big[\frac{(\log|g_1x|)^m - (\log|g_1y|)^m}{\mathbf{d}^\gamma(x,y)}|g_1x|^z\varphi(g_1\cdot x)\Big]\Big| \\
& + \sup_{x,y\in\mathbb{S}_+^{d-1}, x\neq y} \Big|\mathbb{E}\Big[(\log|g_1y|)^m\frac{|g_1x|^z - |g_1y|^z}{\mathbf{d}^\gamma(x,y)}\varphi(g_1\cdot x)\Big]\Big| \\
& + \sup_{x,y\in\mathbb{S}_+^{d-1}, x\neq y} \Big|\mathbb{E}\Big[(\log|g_1y|)^m|g_1y|^z\frac{\varphi(g_1\cdot x) - \varphi(g_1\cdot y)}{\mathbf{d}^\gamma(x,y)}\Big]\Big| \\
= & \, I_{1,m} + I_{2,m} + I_{3,m}.
\end{aligned} \tag{4.3.7}$$

We then control each of the three terms $I_{1,m}, I_{2,m}, I_{3,m}$.

Control of $I_{1,m}$. Since for any $a, b \in \mathbb{C}$, $m \in \mathbb{N}$ and $0 < \gamma < 1$,

$$|a^m - b^m| \leqslant 2m\max\{|a|^{m-\gamma}, |b|^{m-\gamma}\}|a-b|^\gamma, \tag{4.3.8}$$

we get

$$I_{1,m} \leqslant 2m\|\varphi\|_\infty \sup_{x,y\in\mathbb{S}_+^{d-1}, x\neq y} \mathbb{E}\Big[\frac{(\log N(g_1))^{m-\gamma}N(g_1)^{|\Re z|}}{\mathbf{d}^\gamma(x,y)}\Big|\log\frac{|g_1x|}{|g_1y|}\Big|^\gamma\Big].$$

Using (4.2.1), we deduce that for any $g \in \Gamma_\mu$,

$$\Big|\log\frac{|gx|}{|gy|}\Big| \leqslant \frac{|g(x-y)|}{|gy|} \leqslant \|g\|\iota(g)^{-1}|x-y| \leqslant C\|g\|\iota(g)^{-1}\mathbf{d}(x,y), \tag{4.3.9}$$

and hence

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} I_{1,m} \leqslant 2\|\varphi\|_\infty \mathbb{E}\Big[(\log N(g_1))^{1-\gamma}e^{(|\theta|+|\Re z|+2\gamma)\log N(g_1)}\Big]. \tag{4.3.10}$$

Control of $I_{2,m}$. Using (4.3.8), we deduce that for any $z_1, z_2 \in \mathbb{C}$,

$$|e^{z_1} - e^{z_2}| \leqslant 2\max\{|z_1|^{1-\gamma}, |z_2|^{1-\gamma}\}\max\{e^{\Re z_1}, e^{\Re z_2}\}|z_1 - z_2|^\gamma. \tag{4.3.11}$$

By this inequality, we find that for any $g \in \Gamma_\mu$,

$$\left|e^{z\log|gx|} - e^{z\log|gy|}\right| \leqslant 2\log N(g))^{1-\gamma} e^{|\Re z|\log N(g)} |\log|gx| - \log|gy||^\gamma.$$

Combining this with (4.3.9) implies that

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} I_{2,m} \leqslant 2\|\varphi\|_\infty \mathbb{E}\Big[(\log N(g_1))^{1-\gamma} e^{(|\theta|+|\Re z|+2\gamma)\log N(g_1)}\Big]. \tag{4.3.12}$$

Control of $I_{3,m}$. Since $\varphi \in \mathcal{B}_\gamma$ and $\mathbf{d}(g\cdot x, g\cdot y) \leqslant \mathbf{d}(x,y)$ for any $g \in \Gamma_\mu$, we get

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} I_{3,m} \leqslant \|\varphi\|_\gamma \mathbb{E}\Big[e^{(|\theta|+|\Re z|+2\gamma)\log N(g_1)}\Big].$$

Combining this with (4.3.6), (4.3.7), (4.3.10) and (4.3.12), we obtain (4.3.5).

Step 2. Again we need only to consider the case of positive matrices. We will prove the decomposition formula (4.3.2) together with parts (a), (b) and (d). Our proof follows closely [56]. Define the operator M on $\mathcal{B}_\gamma$ by $M\varphi = \nu(\varphi)\mathbf{1}$, $\varphi \in \mathcal{B}_\gamma$. Set $E = \ker M \cap \mathcal{B}_\gamma$. We first show that $\|\varphi\|_\infty \leqslant [\varphi]_\gamma$ for any $\varphi \in E$. Since $\nu(\varphi) = 0$ for any $\varphi \in E$, there exist $x_1, x_2 \in \mathbb{S}_+^{d-1}$ such that $\Re\varphi(x_1) = \Im\varphi(x_2) = 0$. Since $\mathbf{d}(x,y) \in [0,1]$, it follows that

$$\|\varphi\|_\infty \leqslant \sup_{x\in\mathbb{S}_+^{d-1}} |\Re\varphi(x) - \Re\varphi(x_1)| + \sup_{x\in\mathbb{S}_+^{d-1}} |\Im\varphi(x) - \Im\varphi(x_2)| \leqslant 2[\varphi]_\gamma. \tag{4.3.13}$$

We next show that $\varrho(P|_E) < 1$, where $P = P_0$ (see (4.3.1)). For any $x, y \in \mathbb{S}_+^{d-1}, x \neq y$, and $\varphi \in \mathcal{B}_\gamma$, there exists $a \in (0,1)$ such that for large $n \geqslant 1$,

$$\frac{|P^n\varphi(x) - P^n\varphi(y)|}{\mathbf{d}^\gamma(x,y)} \leqslant \|\varphi\|_\gamma \mathbb{E}\Big[\frac{\mathbf{d}^\gamma(G_n\cdot x, G_n\cdot y)}{\mathbf{d}^\gamma(x,y)}\Big] \leqslant \|\varphi\|_\gamma a^n,$$

where for the last inequality we use [53, Lemma 3.2]. Observe that for any $\varphi \in \mathcal{B}_\gamma$, we have $\varphi - M\varphi \in E$, thus $P^n(\varphi - M\varphi) \in E$ for any $n \geqslant 1$ since $\nu P = \nu$. Combining this with (4.3.13) and the above inequality, we get

$$\|P^n(\varphi - M\varphi)\|_\gamma \leqslant 2[P^n(\varphi - M\varphi)]_\gamma \leqslant 2a^n[\varphi]_\gamma \leqslant 2a^n\|\varphi\|_\gamma,$$

which implies $\varrho(P|_E) < 1$. This, together with the definition of E and the fact that $P\mathbf{1} = \mathbf{1}$, shows that 1 is the isolated dominant eigenvalue of the operator P. Using this and the analyticity of $P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and applying the perturbation theorem (see [54, Theorem III.8]), we obtain the decomposition formula (4.3.2) with $M_z(\varphi) = c_1\nu_z(\varphi)r_z$ for some constant $c_1 \neq 0$, as well as parts (a), (b) and (d). Using $P_z r_z = \kappa(z) r_z$, we get $c_1 = 1/\nu_z(r_z)$ and thus $M_z\varphi = \frac{\nu_z(\varphi)}{\nu_z(r_z)} r_z$ for any $\varphi \in \mathcal{B}_\gamma$.

Step 3. We prove part (c) for invertible matrices and positive matrices. From $P\mathbf{1} = \mathbf{1}$, we see that $\kappa(0) = 1$ and $r_0 = \mathbf{1}$. Letting $z = 0$ in $\nu_z P_z = \kappa(z)\nu_z$, we get $\nu_0 P = \nu_0$ and thus $\nu_0 = \nu$ since ν is the unique μ-stationary probability measure. Now we fix $z \in (-\eta, \eta)$ and we show that $\kappa(z)$ and r_z are real-valued. Taking the conjugate in the equality $P_z r_z = \kappa(z) r_z$, we get $P_z \overline{r_z} = \overline{\kappa(z)}\,\overline{r_z}$, so that $\overline{\kappa(z)}$ is an eigenvalue of the operator P_z. By the uniqueness of the dominant eigenvalue of P_z, it follows that $\overline{\kappa(z)} = \kappa(z)$, showing that $\kappa(z)$ is real-valued for $z \in (-\eta, \eta)$. We now prove that r_z

is real-valued. Write r_z in the form $r_z = u_z + iv_z$, where u_z and v_z are real-valued functions on $\mathcal{S}$. From the normalization condition $\nu(r_z) = 1$, we get $\nu(u_z) = 1$ and $\nu(v_z) = 0$. From the equation $P_z r_z = \kappa(z) r_z$ and the fact that $\kappa(z)$ is real-valued, we get that $P_z u_z = \kappa(z) u_z$ and $P_z v_z = \kappa(z) v_z$. By part (a), the space of eigenvectors corresponding to the eigenvalue $\kappa(z)$ is one dimensional. Therefore, we have either $u_z = c v_z$ for some constant $c \in \mathbb{R}$, or $v_z = 0$. However, the equality $u_z = c v_z$ is impossible because we have seen that $\nu(u_z) = 1$ and $\nu(v_z) = 0$. Hence $v_z = 0$ and r_z is real-valued for $z \in (-\eta, \eta)$. The positivity of $\kappa(z)$ and r_z then follows from $\kappa(0) = 1$, $r_0 = \mathbf{1}$ and the analyticity of the mappings $z \mapsto \kappa(z)$ and $z \mapsto r_z$. This ends the proof of part (c), as well as the proof of Proposition 4.3.1. □

4.3.2 Definition of the change of measure $\mathbb{Q}_s^x$

Proposition 4.3.1 allows us to perform a change of measure. Note that this change of measure for positive s has been extensively studied in [16, 17, 50]; however, for negative s it is new. For any $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $g \in \Gamma_\mu$, denote

$$q_n^s(x, g) = \frac{|gx|^s}{\kappa^n(s)} \frac{r_s(g \cdot x)}{r_s(x)}, \quad n \geqslant 1. \tag{4.3.14}$$

Note that (q_n^s) verifies the cocycle property: for any $n, m \geqslant 1$ and $g_1, g_2 \in \Gamma_\mu$,

$$q_n^s(x, g_1) q_m^s(g_1 \cdot x, g_2) = q_{n+m}^s(x, g_2 g_1). \tag{4.3.15}$$

Since $\kappa(s)$ and r_s are strictly positive, $q_n^s(x, G_n)\mu(dg_1)\dots\mu(dg_n)$, $n \geqslant 1$, is a sequence of probability measures, and forms a projective system on $\mathscr{M}^{\mathbb{N}^*}$. By the Kolmogorov extension theorem, there is a unique probability measure $\mathbb{Q}_s^x$ on $\mathscr{M}^{\mathbb{N}^*}$ with marginals $q_n^s(x, G_n)\mu(dg_1)\dots\mu(dg_n)$. Denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation. For any $n \in \mathbb{N}$ and any bounded measurable function h on $(\mathcal{S} \times \mathbb{R})^{n+1}$, it holds that

$$\begin{aligned} &\mathbb{E}\Big[\frac{r_s(X_n^x)|G_n x|^s}{\kappa^n(s) r_s(x)} h\Big(X_0^x, \log|x|, \dots, X_n^x, \log|G_n x|\Big)\Big] \\ &= \mathbb{E}_{\mathbb{Q}_s^x}\Big[h\Big(X_0^x, \log|x|, \dots, X_n^x, \log|G_n x|\Big)\Big]. \end{aligned} \tag{4.3.16}$$

4.3.3 Properties of the Markov operator Q_s

For any $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$, define the Markov operator Q_s by

$$Q_s \varphi(x) = \frac{1}{\kappa(s) r_s(x)} P_s(\varphi r_s)(x), \quad x \in \mathcal{S}.$$

Under the changed measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n \in \mathbb{N}}$ is a Markov chain with the transition operator given by Q_s.

The following assertion will be useful to prove that the function κ is strictly convex (see Lemma 4.3.16). Recall that $V(\Gamma_\mu)$ is the support of the measure ν (cf. (4.2.3), (4.2.4)).

Lemma 4.3.3. *Assume the conditions of Proposition 4.3.1. Let $s \in (-\eta, \eta)$ where η is small. If $\varphi \leqslant Q_s \varphi$ for some real-valued $\varphi \in \mathcal{C}(\mathcal{S})$, then $\varphi(x) = \sup_{y \in \mathcal{S}} \varphi(y)$ for any $x \in V(\Gamma_\mu)$.*

Proof. We use the approach developed in [50]. Set $\mathcal{M} = \sup_{y\in\mathcal{S}} \varphi(y)$ and $\mathcal{S}^+ = \{x \in \mathcal{S} : \varphi(x) = \mathcal{M}\}$. From the condition $\varphi \leqslant Q_s\varphi$ and the fact that $\int q_1^s(x, g_1)\mu(dg_1) = 1$, we get that if $x \in \mathcal{S}^+$, then $g{\cdot}x \in \mathcal{S}^+$ for any $g \in \Gamma_\mu$, so that $\Gamma_\mu\mathcal{S}^+ \subseteq \mathcal{S}^+$. Since $V(\Gamma_\mu)$ is the unique minimal Γ_μ-invariant set (see [50] and [16]), we obtain $V(\Gamma_\mu) \subseteq \mathcal{S}^+$ and the claim follows. □

We state the spectral gap property of the Markov operator Q_s, whose proof is postponed to Section 4.3.5.

Proposition 4.3.4. *Assume the conditions of Proposition 4.3.1. Then there exists $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $n \geqslant 1$, we have*

$$Q_s^n = \Pi_s + N_s^n,$$

where the mappings $s \mapsto \Pi_s$, $s \mapsto N_s \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ are analytic and satisfy the following properties:

(a) *with $\pi_s(\varphi) := \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$, we have for any $\varphi \in \mathcal{B}_\gamma$,*

$$\Pi_s(\varphi)(x) = \pi_s(\varphi)\mathbf{1}, \quad N_s^n(\varphi)(x) = \frac{1}{\kappa^n(s)}\frac{L_s^n(\varphi r_s)(x)}{r_s(x)}, \quad x \in \mathcal{S}$$

where ν_s, r_s, L_s are given in Proposition 4.3.1;

(b) *$\Pi_s N_s = N_s\Pi_s = 0$, and for each $k \in \mathbb{N}$, there exist constants $C_k > 0$ and $a \in (0, 1)$ such that*

$$\sup_{s\in(-\eta,\eta)} \|\frac{d^k}{ds^k} N_s^n\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant C_k a^n. \tag{4.3.17}$$

4.3.4 Quasi-compactness of the operator Q_{s+it}

For $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, define the operator Q_{s+it} as follows: for any $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} Q_{s+it}\varphi(x) &= \frac{1}{\kappa(s)r_s(x)} P_{s+it}(\varphi r_s)(x) \\ &= \frac{1}{\kappa(s)r_s(x)} \mathbb{E}\Big[|g_1x|^{s+it}\varphi(g_1{\cdot}x)r_s(g_1{\cdot}x)\Big], \quad x \in \mathcal{S}. \end{aligned}$$

The spectral gap properties of the operator Q_{s+it} for $|t|$ small enough can be deduced from Proposition 4.3.1. However, this approach does not work for large $|t|$. In order to investigate the spectral gap properties of the operator Q_{s+it} for $t \in \mathbb{R}$, we first prove the Doeblin-Fortet inequality and then we apply the theorem of Ionescu-Tulcea and Marinescu [60] to establish the quasi-compactness of the operator Q_{s+it}. Based on this property, we shall use the non-arithmeticicty condition **C5** to prove that the spectral radius of Q_{s+it} is strictly less than 1 when t is different from 0.

The following is the Doeblin-Fortet inequality for the operator Q_{s+it}.

Lemma 4.3.5. *Assume the conditions of Proposition 4.3.1. Then, there exist constants $0 < a < 1$, and $\eta > 0$ small enough, such that for any $s \in (-\eta, \eta)$, $t \in \mathbb{R}$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$, we have*

$$[Q_{s+it}^n\varphi]_\gamma \leqslant C_{s,t,n}\|\varphi\|_\infty + C_s a^n [\varphi]_\gamma. \tag{4.3.18}$$

For positive-valued s, analogous results can be found in [50] for invertible matrices and in [17] for positive matrices. The proofs in [50, 17] rely essentially on the Hölder continuity of the mapping $x \mapsto q_n^s(x,g)$ defined in (4.3.14). However, this property doesn't hold any more in the case when s is negative. Our proof of Lemma 4.3.5 is carried out using the Hölder inequality and the spectral gap properties of the operator P_s established in Proposition 4.3.1.

Proof of Lemma 4.3.5. Using the definition of Q_{s+it} and (4.3.15), we get that for any $n \geqslant 1$,

$$Q_{s+it}^n \varphi(x) = \frac{1}{\kappa^n(s) r_s(x)} P_{s+it}^n(\varphi r_s)(x), \quad x \in \mathcal{S}.$$

It follows that

$$\sup_{x,y\in\mathcal{S}, x\neq y} \frac{|Q_{s+it}^n \varphi(x) - Q_{s+it}^n \varphi(y)|}{\mathbf{d}^\gamma(x,y)} \leqslant J_1(n) + J_2(n), \tag{4.3.19}$$

where

$$\begin{aligned} J_1(n) &= \sup_{x,y\in\mathcal{S}, x\neq y} \frac{1}{\mathbf{d}^\gamma(x,y)\kappa^n(s)} \left| \frac{1}{r_s(x)} - \frac{1}{r_s(y)} \right| \left| P_{s+it}^n(\varphi r_s)(x) \right|, \\ J_2(n) &= \sup_{x,y\in\mathcal{S}, x\neq y} \frac{1}{r_s(y)\mathbf{d}^\gamma(x,y)\kappa^n(s)} \left| P_{s+it}^n(\varphi r_s)(x) - P_{s+it}^n(\varphi r_s)(y) \right|. \end{aligned}$$

Note that by Proposition 4.3.1, for any $s \in (-\eta, \eta)$, we have $\min_{x\in\mathcal{S}} r_s(x) > 0$, $\max_{x\in\mathcal{S}} r_s(x) < \infty$ and $\kappa(s) > 0$.

Control of $J_1(n)$. Observe that uniformly in $x \in \mathcal{S}$,

$$|P_{s+it}^n(\varphi r_s)(x)| \leqslant P_s^n(|\varphi| r_s)(x) \leqslant \|\varphi\|_\infty \kappa^n(s) \|r_s\|_\infty \leqslant C_s \|\varphi\|_\infty \kappa^n(s).$$

Since $r_s \in \mathcal{B}_\gamma$, this implies that for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$,

$$J_1(n) \leqslant C_s \|\varphi\|_\infty. \tag{4.3.20}$$

Control of $J_2(n)$. Using the definition of P_{s+it} and taking into account that r_s is strictly positive and bounded on $\mathcal{S}$, we have

$$J_2(n) \leqslant C_s (J_{21}(n) + J_{22}(n) + J_{23}(n)), \tag{4.3.21}$$

where

$$\begin{aligned} J_{21}(n) &= \sup_{x,y\in\mathcal{S}, x\neq y} \frac{1}{\mathbf{d}^\gamma(x,y)\kappa^n(s)} \left| \mathbb{E}\Big[(|G_n x|^{s+it} - |G_n y|^{s+it}) \varphi(X_n^x) \Big] \right|, \\ J_{22}(n) &= \sup_{x,y\in\mathcal{S}, x\neq y} \frac{1}{\mathbf{d}^\gamma(x,y)\kappa^n(s)} \left| \mathbb{E}\Big[|G_n y|^{s+it} (\varphi(X_n^x) - \varphi(X_n^y)) \Big] \right|, \\ J_{23}(n) &= \sup_{x,y\in\mathcal{S}, x\neq y} \frac{1}{\mathbf{d}^\gamma(x,y)\kappa^n(s)} \left| \mathbb{E}\Big\{ |G_n y|^{s+it} \varphi(X_n^y) [r_s(X_n^x) - r_s(X_n^y)] \Big\} \right|. \end{aligned}$$

Control of $J_{21}(n)$. Using (4.3.11) and the inequality $\log u \leqslant u^\varepsilon$, $u > 1$, for $\varepsilon > 0$ small enough, we obtain

$$\Big| |G_n x|^{s+it} - |G_n y|^{s+it} \Big| \leqslant 2(N(G_n))^{|s|+\varepsilon} \Big| \log|G_n x| - \log|G_n y| \Big|^\gamma. \tag{4.3.22}$$

From the inequality (4.2.1), by arguing as in the estimate of (4.3.9), we get

$$\Big|\log|G_n x| - \log|G_n y|\Big|^\gamma \leqslant C\|G_n\|^\gamma \iota(G_n)^{-\gamma}\mathbf{d}^\gamma(x,y).$$

Using first (4.3.22) and then the last bound, we deduce that

$$J_{21}(n) \leqslant \frac{C\|\varphi\|_\infty}{\kappa^n(s)}\Big\{\mathbb{E}\Big[(N(g_1))^{|s|+\varepsilon}\|g_1\|^\gamma \iota(g_1)^{-\gamma}\Big]\Big\}^n \leqslant C_{s,t,n}\|\varphi\|_\infty, \tag{4.3.23}$$

where the last inequality holds by condition **C1**.

Control of $J_{22}(n)$. Since $\varphi \in \mathcal{B}_\gamma$, applying the Hölder inequality leads to

$$\begin{aligned} J_{22}(n) &\leqslant \frac{C_s[\varphi]_\gamma}{\kappa^n(s)} \sup_{x,y\in\mathcal{S},x\neq y} \mathbb{E}\Big[|G_n y|^s \frac{\mathbf{d}^\gamma(X_n^x, X_n^y)}{\mathbf{d}^\gamma(x,y)}\Big] \\ &\leqslant C_s[\varphi]_\gamma \sup_{x,y\in\mathcal{S},x\neq y} \frac{\{\mathbb{E}\,[|G_n y|^{2s}]\}^{1/2}}{\kappa^n(s)}\Big\{\mathbb{E}\frac{\mathbf{d}^{2\gamma}(X_n^x, X_n^y)}{\mathbf{d}^{2\gamma}(x,y)}\Big\}^{1/2}. \end{aligned} \tag{4.3.24}$$

Since $\gamma > 0$ is small enough, by [69, Theorem 1] for invertible matrices and [53, Lemma 3.2] for positive matrices, there exists a constant $a_0 \in (0,1)$ such that for sufficiently large n,

$$\sup_{x,y\in\mathcal{S},x\neq y}\Big\{\mathbb{E}\frac{\mathbf{d}^{2\gamma}(X_n^x, X_n^y)}{\mathbf{d}^{2\gamma}(x,y)}\Big\}^{1/2} \leqslant a_0^n. \tag{4.3.25}$$

In view of Proposition 4.3.1, we have

$$\mathbb{E}[|G_n y|^{2s}] = \kappa^n(2s)(M_{2s}\mathbf{1})(y) + L_{2s}^n\mathbf{1}(y), \quad y \in \mathcal{S}.$$

Since, by Proposition 4.3.1(d), $\|M_{2s}\mathbf{1}\|_\infty$ is bounded by some constant C_s, and $\|L_{2s}^n\mathbf{1}\|_\infty$ is bounded by $C_s\kappa^n(2s)$ uniformly in $n \geqslant 1$, it follows that

$$\sup_{n\geqslant 1}\sup_{y\in\mathcal{S}} \frac{\mathbb{E}[|G_n y|^{2s}]}{\kappa^n(2s)} \leqslant C_s. \tag{4.3.26}$$

As κ is continuous in the neighborhood of 0 and $\kappa(0) = 1$, one can choose $\eta > 0$ small enough and a constant $\alpha \in (0, 1/a_0)$ such that $\kappa^{n/2}(2s)/\kappa^n(s) \leqslant \alpha^n$, uniformly in $s \in (-\eta, \eta)$. Substituting this inequality together with (4.3.25) and (4.3.26) into (4.3.24), we obtain that for any $s \in (-\eta, \eta)$ with $\eta > 0$ small, there exists $0 < a < 1$ such that uniformly in $n \geqslant 1$,

$$J_{22}(n) \leqslant C_s a^n [\varphi]_\gamma, \tag{4.3.27}$$

Control of $J_{23}(n)$. Using (4.3.26) and the fact that $r_s \in \mathcal{B}_\gamma$, and applying similar techniques as in the control of $J_{22}(n)$, one can verify that there exists a constant $0 < a < 1$ such that uniformly in $n \geqslant 1$,

$$J_{23}(n) \leqslant C_s a^n \|\varphi\|_\infty [r_s]_\gamma \leqslant C_s a^n \|\varphi\|_\infty. \tag{4.3.28}$$

Inserting (4.3.23), (4.3.27) and (4.3.28) into (4.3.21), we conclude that

$$J_2(n) \leqslant C_{s,t,n}\|\varphi\|_\infty + C_s a^n [\varphi]_\gamma.$$

Combining this with (4.3.20) and (4.3.19), we obtain the inequality (4.3.18). □

From Lemma 4.3.5 and the fact that $\|Q_{s+it}\varphi\|_\infty \leqslant C_s\|\varphi\|_\infty$ for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, we can deduce that $Q_{s+it} \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$. We next prove that the operator Q_{s+it} is quasi-compact. Recall that an operator $Q \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is *quasi-compact* if $\mathcal{B}_\gamma$ can be decomposed into two Q invariant closed subspaces $\mathcal{B}_\gamma = E \oplus F$, such that $\dim E < \infty$, each eigenvalue of $Q|_E$ has modulus $\varrho(Q)$, and $\varrho(Q|_F) < \varrho(Q)$ (see [54] for more details).

Proposition 4.3.6. *Assume the conditions of Proposition 4.3.1. Then, there exists a small $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, the operator Q_{s+it} is quasi-compact.*

Proof. The proof consists of verifying the conditions of the theorem of Ionescu-Tulcea and Marinescu [60]. We follow the formulation in [54, Theorem II.5].

Firstly, by the definition of Q_{s+it}, there exists a constant $C_s > 0$ such that $\|Q_{s+it}\varphi\|_\infty \leqslant C_s\|\varphi\|_\infty$ for any $s \in (-\eta, \eta)$, $t \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$.

Secondly, by Lemma 4.3.5, the Doeblin-Fortet inequality (4.3.18) holds for the operator Q_{s+it}.

Thirdly, denoting $K = Q_{s+it}\{\varphi : \|\varphi\|_\gamma \leqslant 1\}$, we claim that for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, the set K is conditionally compact in $(\mathcal{B}_\gamma, \|\cdot\|_\infty)$. Since $\|Q_{s+it}\varphi\|_\infty \leqslant C_s\|\varphi\|_\infty$ for any $\varphi \in \mathcal{B}_\gamma$, we conclude that K is uniformly bounded in $(\mathcal{B}_\gamma, \|\cdot\|_\infty)$. Moreover, by taking $n = 1$ in (4.3.18), we get that uniformly in $\varphi \in \mathcal{B}_\gamma$ with $\|\varphi\|_\gamma \leqslant 1$,

$$|Q_{s+it}\varphi(x) - Q_{s+it}\varphi(y)| \leqslant C_{s,t}\mathbf{d}^\gamma(x, y).$$

This shows that K is equicontinuous in $(\mathcal{B}_\gamma, \|\cdot\|_\infty)$. Therefore, we obtain the claim by the Arzelà-Ascoli theorem.

The assertion of the proposition now follows from the theorem of Ionescu-Tulcea and Marinescu. □

The following proposition shows that the spectral radius of the operator Q_{s+it} is strictly less than 1 when t is different from 0. The proof which relies on the non-arithmeticity condition **C5**, follows the standard pattern in [50, 17]; it is included for the commodity of the reader.

Proposition 4.3.7. *Assume either conditions **C1** and **C2** for invertible matrices, or conditions **C1**, **C3** and **C5** for positive matrices. Then, for any $s \in (-\eta, \eta)$ with small $\eta > 0$, and any $t \in \mathbb{R}\backslash\{0\}$, we have $\varrho(Q_{s+it}) < 1$.*

Proof. By the definition of Q_{s+it}, we have $\varrho(Q_{s+it}) \leqslant \varrho(Q_s) = 1$. Suppose that $\varrho(Q_{s+it}) = 1$ for some $t \neq 0$. Then, applying Proposition 4.3.6, there exist $\varphi \in \mathcal{B}_\gamma$ and $\beta \in \mathbb{R}$ such that $Q_{s+it}\varphi = e^{i\beta}\varphi$. From this equation, we deduce that $|\varphi| \leqslant Q_s|\varphi|$. Using Lemma 4.3.3, this implies that $|\varphi(x)| = \sup_{y\in\mathcal{S}} |\varphi(y)|$ for any $x \in V(\Gamma_\mu)$, so that $\varphi(x) = ce^{i\vartheta(x)}$, where $c \neq 0$ is a constant and ϑ is a real-valued continuous function on $\mathcal{S}$. Substituting this into the equation $Q_{s+it}\varphi = e^{i\beta}\varphi$ gives that for any $x \in V(\Gamma_\mu)$,

$$\mathbb{E}_{\mathbb{Q}_s^x} \exp[it \log |g_1 x| - i\beta + i\vartheta(g_1 \cdot x) - i\vartheta(x)] = 1.$$

Since ϑ is real-valued, this implies $\exp[it \log |gx| - i\beta + i\vartheta(g \cdot x) - i\vartheta(x)] = 1$ for any $x \in V(\Gamma_\mu)$ and μ-a.e. $g \in \Gamma_\mu$, which contradicts to condition **C5**. Therefore $\varrho(Q_{s+it}) < 1$ for any $t \neq 0$. Recalling that condition **C2** implies condition **C5**, the proof of Proposition 4.3.7 is complete. □

4.3.5 Spectral gap properties of the perturbed operator $R_{s,z}$

For any $s \in (-\eta, \eta)$ and $z \in \mathbb{C}$ such that $s + \Re z \in (-\eta_0, \eta_0)$, define the perturbed operator $R_{s,z}$ as follows: for any $\varphi \in \mathcal{B}_\gamma$,

$$R_{s,z}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|g_1 x| - \Lambda'(s))}\varphi(X_1^x)\right], \quad x \in \mathcal{S}. \tag{4.3.29}$$

With some calculations using (4.3.15), it follows that for any $n \geqslant 1$,

$$R_{s,z}^n\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|G_n x| - n\Lambda'(s))}\varphi(X_n^x)\right], \quad x \in \mathcal{S}. \tag{4.3.30}$$

The following formula relates the operator $R_{s,z}^n$ to the operator P_{s+z}^n and is of independent interest: for any $\varphi \in \mathcal{B}_\gamma$, $n \geqslant 1$, $s \in (-\eta, \eta)$ and $z \in B_\eta(0)$,

$$R_{s,z}^n(\varphi) = e^{-nz\Lambda'(s)}\frac{P_{s+z}^n(\varphi r_s)}{\kappa^n(s) r_s}. \tag{4.3.31}$$

The identity (4.3.31) is obtained by the definitions of $R_{s,z}$ and P_z using the change of measure (4.3.16).

There are two ways to establish spectral gap properties of the operator $R_{s,z}$: one is to use the perturbation theory of operators [54, Theorem III.8], another is based on the Ionescu-Tulcea and Marinescu theorem [60] about the quasi-compactness of operators. The representation (4.3.31) allows us to deduce the spectral gap properties of $R_{s,z}$ directly from the properties of the operator P_z. This has some advantages: it ensures the uniformity in $s \in (-\eta, \eta)$, allows to deal with negative-vaued s and provides an explicit formula for the projection operator $\Pi_{s,z}$ and the remainder operator $N_{s,z}^n$ defined below.

Recall that $\Lambda = \log \kappa$, where κ is defined in (4.2.6).

Proposition 4.3.8. *Assume the conditions of Proposition 4.3.1. Then, there exist $\eta > 0$ and $\delta \in (0, \eta)$ such that for any $s \in (-\eta, \eta)$ and $z \in B_\delta(0)$,*

$$R_{s,z}^n = \lambda_{s,z}^n \Pi_{s,z} + N_{s,z}^n, \quad n \geqslant 1, \tag{4.3.32}$$

$$\lambda_{s,z} = e^{\Lambda(s+z) - \Lambda(s) - \Lambda'(s)z} \tag{4.3.33}$$

and for $\varphi \in \mathcal{B}_\gamma$,

$$\Pi_{s,z}(\varphi) = \frac{\nu_{s+z}(\varphi r_s)}{\nu_{s+z}(r_{s+z})}\frac{r_{s+z}}{r_s}, \tag{4.3.34}$$

$$N_{s,z}^n(\varphi) = e^{-n[\Lambda(s) + \Lambda'(s)z]}\frac{L_{s+z}^n(\varphi r_s)}{r_s}, \tag{4.3.35}$$

where r_z, ν_z and L_z are given in Proposition 4.3.1. In addition, we have:

(a) *for fixed s, the mappings $z \mapsto \Pi_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, $z \mapsto N_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ and $z \mapsto \lambda_{s,z} : B_\delta(0) \to \mathbb{C}$ are analytic,*

(b) *for fixed s and z, $\Pi_{s,z}$ is a rank-one projection with $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}_\gamma$ and $x \in \mathcal{S}$, and $\Pi_{s,z}N_{s,z} = N_{s,z}\Pi_{s,z} = 0$,*

(c) *for* $k \in \mathbb{N}$, *there exist* $0 < a < 1$ *and* $C_k > 0$ *such that*

$$\sup_{s\in(-\eta,\eta)} \sup_{z\in B_\delta(0)} \|\frac{d^k}{dz^k}\Pi_{s,z}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant C_k, \tag{4.3.36}$$

$$\sup_{s\in(-\eta,\eta)} \sup_{z\in B_\delta(0)} \|\frac{d^k}{dz^k}N^n_{s,z}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant C_k a^n. \tag{4.3.37}$$

Note that, for $s > 0$, similar results have been obtained in [17]. The novelty here is that s can account for negative values $s \in (-\eta, 0]$ and that the bounds (4.3.36) and (4.3.37) hold uniformity in $s \in (-\eta, \eta)$. This plays a crucial role in establishing Theorem 4.2.3.

Proof of Proposition 4.3.8. The proof is divided into three steps.

Step 1. By Proposition 4.3.1,

$$P^n_{s+z}(\varphi r_s) = \kappa^n(s+z)\frac{\nu_{s+z}(\varphi r_s)}{\nu_{s+z}(r_{s+z})}r_{s+z} + L^n_{s+z}(\varphi r_s).$$

Substituting this into (4.3.31) shows (4.3.32), (4.3.33), (4.3.34) and (4.3.35).

Step 2. We prove parts (a) and (b). The assertion in part (a) follows from the expressions (4.3.33), (4.3.34) and (4.3.35), and the analyticity of the mappings $z \mapsto \kappa(z)$, $z \mapsto r_z$, $z \mapsto \nu_z$ and $z \mapsto L_z$ defined in Proposition 4.3.1. To show part (b), by (4.3.34), we have that $\Pi_{s,z}$ is a rank-one projection on the subspace $\left\{w\frac{r_{s+z}}{r_s} : w \in \mathbb{C}\right\}$. The identity $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ follows from (4.3.34) and the fact that $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$. Using Proposition 4.3.1, we get that $L_z r_z = 0$ and $\nu_z(L_z\varphi) = 0$ for any $\varphi \in \mathcal{B}_\gamma$. This, together with (4.3.34) and (4.3.35), shows $\Pi_{s,z}N_{s,z} = N_{s,z}\Pi_{s,z} = 0$.

Step 3. We prove part (c). By Proposition 4.3.1, there exists $\eta > 0$ such that the mappings $z \mapsto \kappa(z)$, $z \mapsto r_z$, $z \mapsto \nu_z$ are analytic and uniformly bounded on $B_{2\eta}(0)$. Combining this with (4.3.34), we obtain (4.3.36). We now prove (4.3.37). Since the function r_s is strictly positive on the compact set $\mathcal{S}$, by Proposition 4.3.1(d), we deduce that there exists $0 < a_0 < 1$ such that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)} \sup_{z\in B_\eta(0)} \left\|\frac{L^n_{s+z}(\varphi r_s)}{r_s}\right\|_\gamma \leqslant C\|\varphi\|_\gamma a_0^n. \tag{4.3.38}$$

Using the fact that the function Λ is continuous and $\Lambda(0) = 0$, there exist a small $\eta > 0$, $\delta \in (0, \eta)$ and a constant $a_1 < \frac{1}{a_0}$ such that

$$\sup_{s\in(-\eta,\eta)} \sup_{z\in B_\delta(0)} \left|e^{-n[\Lambda(s)+\Lambda'(s)z]}\right| \leqslant Ca_1^n.$$

Combining this with (4.3.38) proves (4.3.37) with $k = 0$. The proof of (4.3.37) when $k \geqslant 1$ can be carried out in the same way as in the case of $k = 0$. □

Proof of Proposition 4.3.4. The assertion is obtained from Proposition 4.3.8 taking $z = 0$. □

We need the following lemma from [54, Lemma III.9].

Lemma 4.3.9. *Let $s \in \mathbb{R}$, $\delta > 0$ and $I_{s,\delta} = (s-\delta, s+\delta)$. Assume that $t \in I_{s,\delta} \mapsto P(t) \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is a continuous mapping. Let $r > \varrho(P(s))$. Then, there exist constants $\varepsilon = \varepsilon(s)$ and $c = c(s) > 0$ such that*

$$\sup_{t\in(s-\varepsilon,s+\varepsilon)} \|P^n(t)\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} < cr^n.$$

Moreover, it holds that

$$\limsup_{t\to s} \varrho(P(t)) \leqslant \varrho(P(s)).$$

Proposition 4.3.10. *Assume the conditions of Proposition 4.3.7. For any compact set $K \subseteq \mathbb{R}\backslash\{0\}$, there exist a constant $C_K > 0$ and small $\eta > 0$ such that for any $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\sup_{s\in(-\eta,\eta)} \sup_{t\in K} \sup_{x\in\mathcal{S}} |R^n_{s,it}\varphi(x)| \leqslant e^{-nC_K}\|\varphi\|_\gamma.$$

Proof. By Proposition 4.3.7, we have $\rho(s_0, t_0) \leqslant \varrho(R_{s_0+it_0}) = \varrho(Q_{s_0+it_0}) < 1$. Hence it follows that for any fixed $s \in (-\eta, \eta)$ and $t \in \mathbb{R}\backslash\{0\}$, there exists a constant $C(s,t) > 0$ such that for any $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$, we have

$$\sup_{x\in\mathcal{S}} |R^n_{s,it}\varphi(x)| \leqslant e^{-nC(s,t)}\|\varphi\|_\gamma.$$

From (4.3.31) we see that the operator $R_{s,it}$ is continuous in s and t. By Lemma 4.3.9, it follows that there exist constants $\varepsilon(s) > 0$ and $\delta(t) > 0$ such that

$$\sup_{s'\in(s-\varepsilon(s),s+\varepsilon(s))} \sup_{t'\in(t-\delta(t),t+\delta(t))} \sup_{x\in\mathcal{S}} |R^n_{s',it'}\varphi(x)| \leqslant e^{-nC(s,t)}\|\varphi\|_\gamma.$$

Let $K \subset (-\eta, \eta)$ and $T \subseteq \mathbb{R}\backslash\{0\}$ be compact sets. Since

$$\cup_{(s,t)\in K\times T}\Big\{\Big(s-\varepsilon(s), s+\varepsilon(s)\Big) \times \Big(t-\delta(t), t+\delta(t)\Big)\Big\} \supset K \times T,$$

by Heine–Borel's theorem, there exists a sequence $\{s_m, t_m\}_{1\leqslant m\leqslant m_0}$ such that

$$\cup_{m=1}^{m_0}\Big\{(s_m-\varepsilon_m, s_m+\varepsilon_m) \times (t_m-\delta_m, t_m+\delta_m)\Big\} \supset K \times T,$$

where $\varepsilon_m = \varepsilon(s_m)$ and $\delta_m = \delta(s_m)$. This concludes the proof of Proposition 4.3.10 by taking $C = C(K,T) = \min_{1\leqslant m\leqslant m_0} C(s_m, t_m)$. □

We now give some properties of the function $b_{s,\varphi}$ defined as follows: for any $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,

$$b_{s,\varphi}(x) = \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}^x_s}\Big[(\log|G_n x| - n\Lambda'(s))\varphi(X^x_n)\Big], \quad x \in \mathcal{S}.$$

In particular, with $s = 0$, $b_{0,\varphi} = b_\varphi$, which is defined in (4.2.7).

Lemma 4.3.11. *Assume the conditions of Proposition 4.3.1. Then the function $b_{s,\varphi}$ is well-defined, $b_{s,\varphi} \in \mathcal{B}_\gamma$ and*

$$b_{s,\varphi}(x) = \frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi(x), \quad x \in \mathcal{S}. \tag{4.3.39}$$

Proof. In view of Proposition 4.3.8, we have that for any $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|G_n x|-n\Lambda'(s))}\varphi(X_n^x)\right] = \lambda_{s,z}^n \Pi_{s,z}\varphi(x) + N_{s,z}^n\varphi(x), \quad x \in \mathcal{S}.$$

From (4.3.33), we have $\lambda_{s,0} = 1$ and $\frac{d\lambda_{s,z}}{dz}|_{z=0} = 0$. Differentiating both sides of the above equation w.r.t. z at the point 0 gives that for any $x \in \mathcal{S}$,

$$\mathbb{E}_{\mathbb{Q}_s^x}\Big[(\log|G_n x| - n\Lambda'(s))\varphi(X_n^x)\Big] = \frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi(x) + \frac{dN_{s,z}^n}{dz}\Big|_{z=0}\varphi(x). \tag{4.3.40}$$

Using the bounds (4.3.36) and (4.3.37), we find that the first term on the right-hand side of (4.3.40) belongs to $\mathcal{B}_\gamma$, and the second term converges to 0 exponentially fast as $n \to \infty$. Hence, letting $n \to \infty$ in (4.3.40), we obtain (4.3.39). This shows that the function $b_{s,\varphi}$ is well-defined and $b_{s,\varphi} \in \mathcal{B}_\gamma$. □

For any $s \in (-\eta, \eta)$ with $\eta > 0$ small, define $\mathbb{Q}_s = \int_{\mathcal{S}} \mathbb{Q}_s^x \pi_s(dx)$. The following result will be used to prove the strong law of large numbers for $\log|G_n x|$ under the changed measure $\mathbb{Q}_s$.

Lemma 4.3.12. *Assume the conditions of Proposition 4.3.1. There exist $\eta > 0$ and $c, C > 0$ such that uniformly in $s \in (-\eta, \eta)$, $\varphi \in \mathcal{B}_\gamma$ and $n \geqslant 1$,*

$$\Big|\mathbb{E}_{\mathbb{Q}_s}\Big[(\log|G_n x| - n\Lambda'(s))\varphi(X_n^x)\Big]\Big| \leqslant C\|\varphi\|_\gamma e^{-cn}. \tag{4.3.41}$$

Proof. We follow the proof of the previous lemma. Integrating both sides of the identity (4.3.40) w.r.t. π_s, we get, for any $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_s}\Big[(\log|G_n x| - n\Lambda'(s))\varphi(X_n^x)\Big]\\ &\quad= \pi_s\Big(\frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi\Big) + \pi_s\Big(\frac{dN_{s,z}^n}{dz}\Big|_{z=0}\varphi\Big).\end{aligned} \tag{4.3.42}$$

Since $\Pi_{s,z}^2\varphi = \Pi_{s,z}\varphi$, it follows that $2\Pi_{s,0}(\frac{d\Pi_{s,z}}{dz}|_{z=0}\varphi) = \frac{d\Pi_{s,z}}{dz}|_{z=0}\varphi$. Integrating both sides of this equation w.r.t. π_s and using the fact that $\Pi_{s,0} = \pi_s$, we find that

$$\pi_s\Big(\frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi\Big) = 0. \tag{4.3.43}$$

It follows from (4.3.37) that uniformly in $\varphi \in \mathcal{B}_\gamma$ and $s \in (-\eta, \eta)$, the second term on the right-hand side of (4.3.42) is bounded by $C\|\varphi\|_\gamma e^{-cn}$. Therefore, from (4.3.42) and (4.3.43) we obtain (4.3.41). □

We now establish the strong laws of large numbers for $\log|G_n x|$ under the measures $\mathbb{Q}_s^x$ and $\mathbb{Q}_s$, which are of independent interest.

Proposition 4.3.13. *Assume the conditions of Proposition 4.3.1. Then, there exists $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$,*

$$\lim_{n\to\infty}\frac{1}{n}\log|G_n x| = \Lambda'(s), \quad \mathbb{Q}_s^x\text{-a.s.}.$$

Proof. By the Borel-Cantelli lemma, it suffices to show that for any $\varepsilon > 0$, $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$, we have

$$\sum_{n=1}^{\infty} \mathbb{Q}_s^x\Big(\Big|\log|G_n x| - n\Lambda'(s)\Big| \geqslant n\varepsilon\Big) < \infty. \tag{4.3.44}$$

Now let us prove (4.3.44). By Markov's inequality, we have for small $\delta > 0$,

$$\begin{aligned}&\mathbb{Q}_s^x\Big(\Big|\log|G_n x| - n\Lambda'(s)\Big| \geqslant n\varepsilon\Big)\\ \leqslant\ & e^{-n\delta\varepsilon}\mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{\delta(\log|G_n x| - n\Lambda'(s))}\Big) + e^{-n\delta\varepsilon}\mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{-\delta(\log|G_n x| - n\Lambda'(s))}\Big).\end{aligned}$$

From (4.3.30) and Proposition 4.3.8, we deduce that there exist positive constants c, C independent of s, x, δ such that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{\delta(\log|G_n x| - n\Lambda'(s))}\Big) + \mathbb{E}_{\mathbb{Q}_s^x}\Big(e^{-\delta(\log|G_n x| - n\Lambda'(s))}\Big)\\ \leqslant\ & Ce^{n[\Lambda(s+\delta) - \Lambda(s) - \Lambda'(s)\delta]} + Ce^{n[\Lambda(s-\delta) - \Lambda(s) + \Lambda'(s)\delta]} + Ce^{-cn}.\end{aligned}$$

Using Taylor's formula and taking $\delta > 0$ small enough, we conclude that

$$\mathbb{Q}_s^x\Big(\Big|\log|G_n x| - n\Lambda'(s)\Big| \geqslant n\varepsilon\Big) \leqslant Ce^{-n\frac{\delta}{2}\varepsilon},$$

which implies the assertion (4.3.44). □

Proposition 4.3.14. *Assume the conditions of Proposition 4.3.1. Then, there exists $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$,*

$$\lim_{n\to\infty} \frac{1}{n}\log|G_n x| = \Lambda'(s), \quad \mathbb{Q}_s\text{-a.s.}.$$

Proof. Taking $\varphi = \mathbf{1}$ in (4.3.41) leads to

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{Q}_s}\log|G_n x| = \Lambda'(s). \tag{4.3.45}$$

Let $\Omega = M(d, \mathbb{R})^{\mathbb{N}^*}$ and $\widehat{\Omega} = \mathcal{S} \times \Omega$. Following [50, Theorem 3.10], define the shift operator $\widehat{\theta}$ on $\widehat{\Omega}$ by $\widehat{\theta}(x, \omega) = (g_1 \cdot x, \theta\omega)$, where $\omega \in \Omega$ and θ is the shift operator on Ω. For any $x \in \mathcal{S}$ and $\omega \in \Omega$, set $h(x, \omega) = \log|g_1(\omega)x|$. Then h is $\mathbb{Q}_s$-integrable. Since $\log|G_n x| = \sum_{k=0}^{n-1}(h \circ \widehat{\theta}^k)(x, \omega)$ and $\mathbb{Q}_s$ is $\widehat{\theta}$-ergodic, it follows from Birkhoff's ergodic theorem that $\frac{1}{n}\log|G_n x|$ converges $\mathbb{Q}_s$-a.s. to some constant c_s as $n \to \infty$. If we suppose that c_s is different from $\Lambda'(s)$, then this contradicts to (4.3.45). Thus $c_s = \Lambda'(s)$ and the assertion of the lemma follows. □

Now we give the third-order Taylor expansion of $\lambda_{s,z}$ defined by (4.3.33), w.r.t. z at the origin in the complex plane.

Proposition 4.3.15. *Assume the conditions of Proposition 4.3.1. Then, there exist $\eta > 0$ and $\delta > 0$ such that for any $s \in (-\eta, \eta)$ and $z \in B_\delta(0)$,*

$$\lambda_{s,z} = 1 + \frac{\sigma_s^2}{2}z^2 + \frac{\Lambda'''(s)}{6}z^3 + o(|z|^3) \quad \text{as } |z| \to 0, \tag{4.3.46}$$

where

(a) $\sigma_s^2 = \Lambda''(s) \geqslant 0$ *and* $\Lambda'''(s) \in \mathbb{R}$*;*

(b) *for invertible matrices,* $\sigma_s > 0$ *under the stated conditions; for positive matrices,* $\sigma_s > 0$ *if additionally* $\sigma = \sigma_0 > 0$ *or if the measure* μ *is non-arithmetic;*

(c) *uniformly in* $s \in (-\eta, \eta)$ *and* $x \in \mathcal{S}$*,*

$$\begin{aligned}\sigma_s^2 &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{Q}_s^x}\Big[\log|G_n x| - n\Lambda'(s)\Big]^2 \\ &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{Q}_s}\Big[\log|G_n x| - n\Lambda'(s)\Big]^2;\end{aligned}$$

(d) *uniformly in* $s \in (-\eta, \eta)$*,*

$$\Lambda'''(s) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathbb{Q}_s}\Big[\log|G_n x| - n\Lambda'(s)\Big]^3.$$

The proof of Proposition 4.3.15 is based on the following lemma.

Lemma 4.3.16. *Assume the conditions of Proposition 4.3.1. Then the functions* Λ *and* κ *are convex on* $(-\eta, \eta)$ *for* $\eta > 0$ *small enough. Moreover,* Λ *and* κ *are strictly convex for invertible matrices under the given conditions, and for positive matrices under the additional condition* ***C5****.*

Proof. The proof follows [50]. Since $\Lambda = \log\kappa$, it suffices to prove Lemma 4.3.16 for the function Λ. For any $t \in (0,1)$, $s_1, s_2 \in (-\eta, \eta)$, set $s' = ts_1 + (1-t)s_2$. Using Hölder's inequality and the fact that $P_s r_s = \kappa(s) r_s$,

$$P_{s'}(r_{s_1}^t r_{s_2}^{1-t}) \leqslant \kappa^t(s_1)\kappa^{1-t}(s_2) r_{s_1}^t r_{s_2}^{1-t}. \tag{4.3.47}$$

Since $\kappa(s')$ is the dominant eigenvalue of the operator $P_{s'}$, we obtain $\kappa(s') \leqslant \kappa^t(s_1)\kappa^{1-t}(s_2)$ and thus Λ is convex.

To show that the function Λ is strictly convex, we suppose, by absurd, that there exist $s_1 \neq s_2$ and some $t \in (0,1)$ such that $\kappa(s') = \kappa^t(s_1)\kappa^{1-t}(s_2)$. Using this equality, the definition of Q_s and (4.3.47), we get $Q_{s'}(r_{s_1}^t r_{s_2}^{1-t}/r_{s'}) \leqslant r_{s_1}^t r_{s_2}^{1-t}/r_{s'}$. By Lemma 4.3.3, this implies that $r_{s_1}^t r_{s_2}^{1-t} = c r_{s'}$ on $V(\Gamma_\mu)$ for some constant $c > 0$. Substituting this equality and the identity $\kappa(s') = \kappa^t(s_1)\kappa^{1-t}(s_2)$ into (4.3.47), we see that the Hölder inequality in (4.3.47) is actually an equality. This yields that there exists a function $c(x) > 0$ such that for any $g \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$, we have

$$|gx|^{s_1} r_{s_1}(g\cdot x) = c(x)|gx|^{s_2} r_{s_2}(g\cdot x). \tag{4.3.48}$$

Integrating both sides of the equation (4.3.48) w.r.t. μ gives $c(x) = \frac{\kappa(s_1) r_{s_1}(x)}{\kappa(s_2) r_{s_2}(x)}$. Substituting this into (4.3.48) and noting that $s_1 \neq s_2$, we find that there exist a constant $c_1 > 0$ and a real-valued function φ on $\mathcal{S}$ such that $|gx| = c_1 \frac{\varphi(g\cdot x)}{\varphi(x)}$ for any $g \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$. This contradicts to condition **C5**. Recall that condition **C2** implies condition **C5** for invertible matrices. Hence Λ is strictly convex for invertible matrices under stated conditions. □

Proof of Proposition 4.3.15. The expansion (4.3.46) follows from the identity (4.3.33) and Taylor's formula.

For part (a), by Lemma 4.3.16, we have $\Lambda''(s) \geqslant 0$. Since $\Lambda = \log \kappa$ and it is shown in Proposition 4.3.1 that the function κ is real-valued and strictly positive on $(-\eta, \eta)$, we get $\Lambda'''(s) \in \mathbb{R}$.

For part (b), recall that it was shown in [17] that $\sigma_0 > 0$ for invertible matrices under the stated conditions, and for positive matrices under the additional condition of non-arithmeticity. Hence, using the continuity of the function Λ'', we obtain that $\sigma_s > 0$.

For part (c), by Proposition 4.3.8, we get that for $|z|$ small,

$$\mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|G_n x| - n\Lambda'(s))}\right] = \lambda_{s,z}^n(\Pi_{s,z}\mathbf{1})(x) + (N_{s,z}^n\mathbf{1})(x). \tag{4.3.49}$$

It follows from (4.3.46) that for $|z| = o(n^{-1/3})$,

$$\lambda_{s,z}^n = 1 + n\sigma_s^2\frac{z^2}{2} + n\Lambda'''(s)\frac{z^3}{6} + o(n|z|^3). \tag{4.3.50}$$

Using Taylor's formula, the bound (4.3.36) and the fact $\Pi_{s,0}\mathbf{1} = \mathbf{1}$, we obtain

$$(\Pi_{s,z}\mathbf{1})(x) = 1 + c_{s,x,1}z + c_{s,x,2}z^2 + c_{s,x,3}z^3 + o(|z|^3), \tag{4.3.51}$$

where the constants $c_{s,x,1}, c_{s,x,2}, c_{s,x,3} \in \mathbb{C}$ are bounded as functions of s and x. Similarly, using the fact $N_{s,0}\mathbf{1} = 0$ and the bound (4.3.37), there exist constants $C_{s,x,n,1}$, $C_{s,x,n,2}$, $C_{s,x,n,3} \in \mathbb{C}$ which are bounded as functions of s, x and n such that

$$(N_{s,z}^n\mathbf{1})(x) = C_{s,x,n,1}z + C_{s,x,n,2}z^2 + C_{s,x,n,3}z^3 + o(|z|^3). \tag{4.3.52}$$

Taking the second derivative on both sides of the equation (4.3.49) with respect to z at 0, and using the expansions (4.3.50)-(4.3.52), we deduce that

$$\mathbb{E}_{\mathbb{Q}_s^x}\left[\log|G_n x| - n\Lambda'(s)\right]^2 = n\sigma_s^2 + 2c_{s,x,2} + 2C_{s,x,n,2}. \tag{4.3.53}$$

This, together with the definition of $\mathbb{Q}_s$ and the fact that the constants $c_{s,x,2}$, $C_{s,x,n,2}$ are bounded as functions of s, x, n, concludes the proof of part (c).

For part (d), integrating both sides of the equations (4.3.49), (4.3.51) and (4.3.52) with respect to the invariant measure π_s, and using the property (4.3.43) with $\varphi = \mathbf{1}$ (thus the second term on the right-hand side of (4.3.51) vanishes), in the same way as in the proof of (4.3.53), we get

$$\mathbb{E}_{\mathbb{Q}_s}\left[\log|G_n x| - n\Lambda'(s)\right]^3 = n\Lambda'''(s) + 6c_{s,3} + 6C_{s,n,3}.$$

This implies the assertion in part (d). □

Remark 4.3.17. Inspecting the proof of Proposition 4.3.15, it is easy to see that the results in parts (c) and (d) can be reinforced to the following bounds:

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}\left|\frac{1}{n}\mathbb{E}_{\mathbb{Q}_s^x}\Big[\log|G_n x| - n\Lambda'(s)\Big]^2 - \sigma_s^2\right| \leqslant \frac{C}{n},$$

$$\sup_{s\in(-\eta,\eta)}\left|\frac{1}{n}\mathbb{E}_{\mathbb{Q}_s}\Big[\log|G_n x| - n\Lambda'(s)\Big]^3 - \Lambda'''(s)\right| \leqslant \frac{C}{n}.$$

The first bound above also holds with the measure $\mathbb{Q}_s^x$ replaced by $\mathbb{Q}_s$.

4.4 Smoothing inequality on the complex plane

In this section we aim to establish a new smoothing inequality, which plays a crucial role in proving the Berry-Esseen theorem and Edgeworth expansion with a target function φ on X_n^x; see Theorems 4.2.1, 4.2.2, 4.5.1 and 4.5.3.

From now on, for any integrable function $h : \mathbb{R} \to \mathbb{C}$, denote its Fourier transform by $\widehat{h}(t) = \int_{\mathbb{R}} e^{-ity} h(y)dy$, $t \in \mathbb{R}$. If $\widehat{h}$ is integrable on $\mathbb{R}$, then using the inverse Fourier transform gives $h(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ity} \widehat{h}(t)dt$, for almost all $y \in \mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$. Denote by $h_1 * h_2$ the convolution of the functions h_1, h_2 on the real line. For any $r > 0$, we denote

$$D_r = \{z \in \mathbb{C} : |z| < r\},$$

and

$$D_r^+ = \{z \in \mathbb{C} : |z| < r, \Im z > 0\} \quad \text{and} \quad D_r^- = \{z \in \mathbb{C} : |z| < r, \Im z < 0\}.$$

Now we construct a density function ρ_T which plays an important role in establishing a new smoothing inequality. As in [74], we define a density function ρ on the real line $\mathbb{R}$ by setting $\rho(0) = 1/2\pi$ and

$$\rho(y) = \frac{1}{2\pi} \Big(\frac{\sin \frac{y}{2}}{\frac{y}{2}} \Big)^2, \quad y \in \mathbb{R} \setminus \{0\}.$$

Then ρ is a non-negative function bounded by $\frac{1}{2\pi}$ and $\int_{\mathbb{R}} \rho(y)dy = 1$. Its Fourier transform $\widehat{\rho}$ is given by

$$\widehat{\rho}(t) = 1 - |t|, \quad t \in [-1, 1],$$

and $\widehat{\rho}(t) = 0$ otherwise.

For any $T > 0$ and the fixed constant $b > 0$ satisfying $\int_{-b}^{b} \rho(y)dy = 3/4$, define the density function

$$\rho_T(y) = T\rho(Ty - b), \quad y \in \mathbb{R},$$

whose Fourier transform $\widehat{\rho}_T$ is given by

$$\widehat{\rho}_T(t) = e^{-ib\frac{t}{T}} \Big(1 - \frac{|t|}{T} \Big), \quad t \in [-T, T], \tag{4.4.1}$$

and $\widehat{\rho}_T(t) = 0$ otherwise. Note that the function $\widehat{\rho}_T$ is not smooth at the point 0, so that it can not have an analytic extension in a small neighborhood of 0 in the complex plane $\mathbb{C}$.

Now we are ready to establish a new smoothing inequality. Its proof is based on the properties of the density function ρ_T, Cauchy's integral theorem and some techniques from [33, 74].

Proposition 4.4.1. *Assume that F is non-decreasing on $\mathbb{R}$, and that H is differentiable of bounded variation on $\mathbb{R}$ such that $\sup_{y\in\mathbb{R}} |H'(y)| < \infty$. Suppose that $F(-\infty) = H(-\infty)$ and $F(\infty) = H(\infty)$. Let*

$$f(t) = \int_{\mathbb{R}} e^{-ity} dF(y) \quad \text{and} \quad h(t) = \int_{\mathbb{R}} e^{-ity} dH(y), \quad t \in \mathbb{R}.$$

Suppose that $r > 0$ and that f and h have analytic extensions on D_r. Then, for any $T \geqslant r$,

$$\begin{aligned}\sup_{y\in\mathbb{R}}|F(y)-H(y)| \leqslant\ & \frac{1}{\pi}\sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big| \\ & +\frac{1}{\pi}\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big| \\ & +\frac{1}{\pi}\sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\Big| \\ & +\frac{1}{\pi}\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\Big| \\ & +\frac{1}{\pi}\int_{r\leqslant|t|\leqslant T}\Big|\frac{f(t)-h(t)}{t}\Big|dt \\ & +\frac{2}{\pi T}\int_{-T}^{T}|f(t)-h(t)|dt+\frac{3b}{T}\sup_{y\in\mathbb{R}}|H'(y)|,\end{aligned}$$

where $b > 0$ is a fixed constant satisfying $\int_{-b}^{b}\rho(y)dy = 3/4$, and the semicircles $\mathcal{C}_r^-$ and $\mathcal{C}_r^+$ are given by

$$\mathcal{C}_r^- = \{z\in\mathbb{C} : |z| = r, \Im z < 0\},\quad \mathcal{C}_r^+ = \{z\in\mathbb{C} : |z| = r, \Im z > 0\}. \tag{4.4.2}$$

Proof. Let $T \geqslant r$. From the definition of ρ_T and the choice of the constant b, we have $\int_0^{2b/T}\rho_T(y)dy = 3/4$. Since $\rho \leqslant \frac{1}{2\pi}$, the function ρ_T is bounded by $T/2\pi$. The proof of Proposition 4.4.1 consists in establishing first an upper bound and then a lower bound.

Upper bound. Since the function F is non-decreasing on $\mathbb{R}$ and ρ_T is a density function on $\mathbb{R}$, we find that for any $y\in\mathbb{R}$,

$$\begin{aligned}F(y) \leqslant\ & \frac{4}{3}\int_y^{y+\frac{2b}{T}}F(u)\rho_T(u-y)du \\ =& H(y)+\frac{4}{3}\int_y^{y+\frac{2b}{T}}\Big[(F(u)-H(u))\rho_T(u-y)+(H(u)-H(y))\rho_T(u-y)\Big]du \\ \leqslant& H(y)+\frac{4}{3}\int_y^{y+\frac{2b}{T}}(F(u)-H(u))\rho_T(u-y)du+\frac{2b}{T}\sup_{y\in\mathbb{R}}|H'(y)|.\end{aligned} \tag{4.4.3}$$

Let $F_1(y) = \int_{\mathbb{R}}F(u)\rho_T(u-y)du$, and $H_1(y) = \int_{\mathbb{R}}H(u)\rho_T(u-y)du$, $y\in\mathbb{R}$. Elementary calculations lead to

$$\int_{\mathbb{R}}e^{-ity}dF_1(y) = f(t)\widehat{\rho}_T(-t),\quad \int_{\mathbb{R}}e^{-ity}dH_1(y) = h(t)\widehat{\rho}_T(-t),\quad t\in\mathbb{R}.$$

Restricted on the real line, the function $\widehat{\rho}_T$ is supported on $[-T,T]$. By the inversion formula we get

$$\begin{aligned}F_1(y)-F_1(v) &= \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{ity}-e^{itv}}{it}f(t)\widehat{\rho}_T(-t)dt, \qquad y,v\in\mathbb{R}, \\ H_1(y)-H_1(v) &= \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{ity}-e^{itv}}{it}h(t)\widehat{\rho}_T(-t)dt \qquad y,v\in\mathbb{R}.\end{aligned}$$

By the definition of $\hat{\rho}_T$ (cf. (4.4.1)), we get

$$\begin{aligned}
&F_1(y) - H_1(y) - (F_1(v) - H_1(v)) \\
&= \frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{ity} e^{-ib\frac{t}{T}} dt - \frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{itv} e^{-ib\frac{t}{T}} dt \\
&\quad - \frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{ity} \frac{|t|}{T} dt + \frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{itv} \frac{|t|}{T} dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Big|F_1(y) - H_1(y) - (F_1(v) - H_1(v))\Big| \\
&\leqslant \left|\frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{ity} e^{-ib\frac{t}{T}} dt - \frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{itv} e^{-ib\frac{t}{T}} dt\right| \\
&\quad + \frac{1}{\pi T}\int_{-T}^{T} |f(t)-h(t)| dt. \qquad (4.4.4)
\end{aligned}$$

We shall use Cauchy's integral theorem to change the integration path $[-T, T]$ to a contour in the complex plane. In order to estimate the difference $|F_1(y) - H_1(y)|$, we are led to consider two cases: $y \leqslant 0$ and $y > 0$.

Control of $|F_1(y) - H_1(y)|$ *when* $y \leqslant 0$. Let $\mathcal{C}_- = \mathcal{C}_{r,T} \cup \mathcal{C}_r^-$, where $\mathcal{C}_{r,T} = [-T, -r] \cup [r, T]$ and the lower semicircle $\mathcal{C}_r^-$ is given in (4.4.2). Since the functions f, h and $t \mapsto e^{-ib\frac{t}{T}}$ are analytic on the domain D_r, applying Cauchy's integral theorem gives

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{ity} e^{-ib\frac{t}{T}} dt - \frac{1}{2\pi}\int_{-T}^{T} \frac{f(t)-h(t)}{it} e^{itv} e^{-ib\frac{t}{T}} dt \\
&= \frac{1}{2\pi}\Big[\int_{\mathcal{C}_-} \frac{f(z)-h(z)}{iz} e^{izy} e^{-ib\frac{z}{T}} dz - \int_{\mathcal{C}_-} \frac{f(z)-h(z)}{iz} e^{izv} e^{-ib\frac{z}{T}} dz\Big], \qquad (4.4.5)
\end{aligned}$$

where the integration is over the complex curve $\mathcal{C}_-$ oriented from $-T$ to T. The second integral in (4.4.5) converges to 0 as $v \to -\infty$, by using the Riemann-Lebesgue lemma on the real segment $\mathcal{C}_{r,T}$ and by applying the Lebesgue convergence theorem on the semicircle $\mathcal{C}_r^-$. Note that $F_1(-\infty) = H_1(-\infty)$ since $F(-\infty) = H(-\infty)$. Consequently, letting $v \to -\infty$ in (4.4.5) and substituting it into (4.4.4) we get

$$\Big|F_1(y) - H_1(y)\Big| \leqslant \left|\frac{1}{2\pi}\int_{\mathcal{C}_-} \frac{f(z)-h(z)}{iz} e^{izy} e^{-ib\frac{z}{T}} dz\right| + \frac{1}{\pi T}\int_{-T}^{T} |f(t)-h(t)| dt.$$

Therefore,

$$\begin{aligned}
\sup_{y\leqslant 0} |F_1(y) - H_1(y)| &\leqslant \frac{1}{2\pi}\int_{\mathcal{C}_{r,T}} \left|\frac{f(t)-h(t)}{t}\right| dt \\
&\quad + \frac{1}{2\pi}\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-} \frac{f(z)-h(z)}{z} e^{izy} e^{-ib\frac{z}{T}} dz\right| \\
&\quad + \frac{1}{\pi T}\int_{-T}^{T} |f(t)-h(t)| dt. \qquad (4.4.6)
\end{aligned}$$

Control of $|F_1(y) - H_1(y)|$ *when* $y > 0$. Let $\mathcal{C}_+ = \mathcal{C}_{r,T} \cup \mathcal{C}_r^+$, where $\mathcal{C}_{r,T} = [-T, -r] \cup [r, T]$ and the upper semicircle $\mathcal{C}_r^+$ is given in (4.4.2). In an analogous way as in (4.4.5),

we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{-T}^{T}\frac{f(t)-h(t)}{it}e^{ity}e^{-ib\frac{t}{T}}dt-\frac{1}{2\pi}\int_{-T}^{T}\frac{f(t)-h(t)}{it}e^{itv}e^{-ib\frac{t}{T}}dt\\&=\frac{1}{2\pi}\Big[\int_{\mathcal{C}_+}\frac{f(z)-h(z)}{iz}e^{izy}e^{-ib\frac{z}{T}}dz-\int_{\mathcal{C}_+}\frac{f(z)-h(z)}{iz}e^{izv}e^{-ib\frac{z}{T}}dz\Big],\end{aligned}\tag{4.4.7}$$

where the integration is over the complex curve $\mathcal{C}_+$ also oriented from $-T$ to T. The second integral in (4.4.7) converges to 0 as $v\to+\infty$, by using again the Riemann-Lebesgue lemma on the real segment $\mathcal{C}_{r,T}$ and by applying the Lebesgue convergence theorem on the upper semicircle $\mathcal{C}_r^+$. Note that $F_1(\infty)=H_1(\infty)$ since $F(\infty)=H(\infty)$. Hence, letting $v\to+\infty$ in (4.4.7), similarly to (4.4.6), we obtain

$$\begin{aligned}\sup_{y>0}|F_1(y)-H_1(y)|\leqslant&\frac{1}{2\pi}\int_{\mathcal{C}_{r,T}}\Big|\frac{f(t)-h(t)}{t}\Big|dt\\&+\frac{1}{2\pi}\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|\\&+\frac{1}{\pi T}\int_{-T}^{T}|f(t)-h(t)|dt.\end{aligned}\tag{4.4.8}$$

As a result, putting together (4.4.6) and (4.4.8) leads to

$$\begin{aligned}\sup_{y\in\mathbb{R}}|F_1(y)-H_1(y)|\leqslant&\frac{1}{2\pi}\int_{\mathcal{C}_{r,T}}\Big|\frac{f(t)-h(t)}{t}\Big|dt\\&+\frac{1}{2\pi}\sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|\\&+\frac{1}{2\pi}\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|\\&+\frac{1}{\pi T}\int_{-T}^{T}|f(t)-h(t)|dt.\end{aligned}\tag{4.4.9}$$

Denote $\Delta=\sup_{y\in\mathbb{R}}|F(y)-H(y)|$. Then, taking into account that ρ_T is a density function on $\mathbb{R}$, using (4.4.9) and the fact that $\int_0^{2b/T}\rho_T(y)dy=3/4$, we get

$$\begin{aligned}&\Big|\int_y^{y+\frac{2b}{T}}(F(u)-H(u))\rho_T(u-y)du\Big|\\&\leqslant|F_1(y)-H_1(y)|+\Delta\Big(1-\int_0^{2b/T}\rho_T(u)du\Big)\\&\leqslant\frac{1}{2\pi}\int_{\mathcal{C}_{r,T}}\Big|\frac{f(t)-h(t)}{t}\Big|dt+\frac{1}{2\pi}\sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|\\&\quad+\frac{1}{2\pi}\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|+\frac{1}{\pi T}\int_{-T}^{T}|f(t)-h(t)|dt+\frac{\Delta}{4}.\end{aligned}$$

Substituting this inequality into (4.4.3), we obtain the following upper bound:

$$\begin{aligned}F(y)-H(y) \leqslant & \frac{2}{3\pi}\int_{\mathcal{C}_{r,T}}\left|\frac{f(t)-h(t)}{t}\right|dt \\ & +\frac{2}{3\pi}\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\right| \\ & +\frac{2}{3\pi}\sup_{y>0}\left|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\right| \\ & +\frac{4}{3\pi T}\int_{-T}^{T}|f(t)-h(t)|dt+\frac{\Delta}{3}+\frac{2b}{T}\sup_{y\in\mathbb{R}}|H'(y)|. \end{aligned} \tag{4.4.10}$$

Lower bound. Similarly to (4.4.3), we have for any $y\in\mathbb{R}$,

$$\begin{aligned}F(y) &\geqslant \frac{4}{3}\int_{y-\frac{2b}{T}}^{y}F(u)\rho_T(y-u)du \\ &\geqslant H(y)+\frac{4}{3}\int_{y-\frac{2b}{T}}^{y}(F(u)-H(u))\rho_T(y-u)du-\frac{2b}{T}\sup_{y\in\mathbb{R}}|H'(y)|.\end{aligned}$$

Let $F_2(y)=(F*\rho_T)(y)$ and $H_2(y)=(H*\rho_T)(y)$, $y\in\mathbb{R}$, then

$$\int_{\mathbb{R}}e^{-ity}dF_2(y)=f(t)\widehat{\rho}_T(t),\quad \int_{\mathbb{R}}e^{-ity}dH_2(y)=h(t)\widehat{\rho}_T(t),\quad t\in\mathbb{R}.$$

Proceeding in the same way as in the proof of (4.4.9), we get

$$\begin{aligned}\sup_{y\in\mathbb{R}}|F_2(y)-H_2(y)| \leqslant & \frac{1}{2\pi}\int_{\mathcal{C}_{r,T}}\left|\frac{f(t)-h(t)}{t}\right|dt \\ & +\frac{1}{2\pi}\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\right| \\ & +\frac{1}{2\pi}\sup_{y>0}\left|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\right| \\ & +\frac{1}{\pi T}\int_{-T}^{T}|f(t)-h(t)|dt.\end{aligned}$$

Following the proof of (4.4.10), we obtain the lower bound:

$$\begin{aligned}F(y)-H(y) \geqslant & -\frac{2}{3\pi}\int_{\mathcal{C}_{r,T}}\left|\frac{f(t)-h(t)}{t}\right|dt \\ & -\frac{2}{3\pi}\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\right| \\ & -\frac{2}{3\pi}\sup_{y>0}\left|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\right| \\ & -\frac{4}{3\pi T}\int_{-T}^{T}|f(t)-h(t)|dt-\frac{\Delta}{3}-\frac{2b}{T}\sup_{y\in\mathbb{R}}|H'(y)|.\end{aligned} \tag{4.4.11}$$

Combining (4.4.10) and (4.4.11), we conclude the proof of Proposition 4.4.1. □

4.5 Proofs of Berry-Esseen bound and Edgeworth expansion

4.5.1 Berry-Esseen bound and Edgeworth expansion under the changed measure

We first formulate a Berry-Esseen bound under the changed measure $\mathbb{Q}_s^x$.

Theorem 4.5.1. *Assume either conditions* ***C1*** *and* ***C2*** *for invertible matrices, or conditions* ***C1***, ***C3*** *and* ***C4*** *for positive matrices. Then there exist constants $\eta > 0$ and $C > 0$ such that for all $n \geqslant 1$, $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left|\mathbb{E}_{\mathbb{Q}_s^x}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant y\right\}}\right] - \pi_s(\varphi)\Phi(y)\right| \leqslant C\frac{\|\varphi\|_\gamma}{\sqrt{n}}. \tag{4.5.1}$$

The following result gives an Edgeworth expansion for $\log|G_n x|$ with the target function φ on X_n^x under $\mathbb{Q}_s^x$. The function $b_{s,\varphi}(x)$, $x \in \mathcal{S}$, which will be used in the formulation of this result, is defined in Lemma 4.3.11 and has an equivalent expression (4.3.39) in terms of derivative of the projection operator $\Pi_{s,z}$, see Proposition 4.3.8.

Theorem 4.5.2. *Assume either conditions* ***C1*** *and* ***C2*** *for invertible matrices, or conditions* ***C1***, ***C3*** *and* ***C5*** *for positive matrices. Then there exists $\eta > 0$ such that as $n \to \infty$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\begin{aligned}&\Bigg|\mathbb{E}_{\mathbb{Q}_s^x}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant y\right\}}\right]\\ &- \mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)]\left[\Phi(y) + \frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}(1-y^2)\phi(y)\right] + \frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\phi(y)\Bigg| = \|\varphi\|_\gamma o\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

The following asymptotic expansion is slightly different from that in Theorem 4.5.2, with the term $\mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)]$ replaced by $\pi_s(\varphi)$.

Theorem 4.5.3. *Under the conditions of Theorem 4.5.2, there exists $\eta > 0$ such that, as $n \to \infty$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\begin{aligned}&\Bigg|\mathbb{E}_{\mathbb{Q}_s^x}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant y\right\}}\right]\\ &- \pi_s(\varphi)\left[\Phi(y) + \frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}(1-y^2)\phi(y)\right] + \frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\phi(y)\Bigg| = \|\varphi\|_\gamma o\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{4.5.2}$$

With fixed $s > 0$ and $\varphi = \mathbf{1}$, the expansion (4.5.2) has been established earlier in [17].

The assertion of Theorem 4.5.3 follows from Theorem 4.5.2, since the bound (4.3.17) implies that there exist constants $c, C > 0$ such that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}|\mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)] - \pi_s(\varphi)| \leqslant Ce^{-cn}\|\varphi\|_\gamma. \tag{4.5.3}$$

Theorems 4.2.1 and 4.2.2 follow from the above theorems taking $s = 0$ and recalling the fact that $\Lambda'(0) = \lambda$, $\sigma_0 = \sigma$ and $b_{0,\varphi} = b_\varphi$.

4.5.2 Proof of Theorem 4.5.2

Without loss of generality, we assume that φ is non-negative. Denote

$$F(y) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\frac{\log|G_n x|-n\Lambda'(s)}{\sigma_s\sqrt{n}}\leqslant y\Big\}}\Big], \quad y\in\mathbb{R},$$
$$H(y) = \mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)]\Big[\Phi(y)+\frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}(1-y^2)\phi(y)\Big]-\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\phi(y), \quad y\in\mathbb{R}.$$

Define

$$f(t)=\int_{\mathbb{R}} e^{-ity}dF(y), \qquad h(t)=\int_{\mathbb{R}} e^{-ity}dH(y), \quad t\in\mathbb{R}.$$

By straightforward calculations we have

$$f(t) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[\varphi(X_n^x)e^{-it\frac{\log|G_n x|-n\Lambda'(s)}{\sigma_s\sqrt{n}}}\Big] = R^n_{s,\frac{-it}{\sigma_s\sqrt{n}}}\varphi(x), \quad t\in\mathbb{R}, \tag{4.5.4}$$
$$h(t) = e^{-\frac{t^2}{2}}\Big\{\Big[1-(it)^3\frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}\Big]R_{s,0}\varphi(x)-it\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\Big\}, \quad t\in\mathbb{R}. \tag{4.5.5}$$

It is clear that $F(-\infty)=H(-\infty)=0$ and $F(\infty)=H(\infty)$. Moreover, one can verify that the functions F,H and their corresponding Fourier-Stieltjes transforms f,h satisfy the conditions of Proposition 4.4.1 for $r=\delta_1\sqrt{n}$, with some $\delta_1>0$ sufficiently small. Then, for any real $T\geqslant r$,

$$\sup_{y\in\mathbb{R}}|F(y)-H(y)|\leqslant\frac{1}{\pi}(I_1+I_2+I_3+I_4), \tag{4.5.6}$$

where

$$\begin{aligned}
I_1 &= \frac{3\pi b}{T}\sup_{y\in\mathbb{R}}|H'(y)|, \qquad I_2=\int_{r\leqslant|t|\leqslant T}\Big|\frac{f(t)-h(t)}{t}\Big|dt,\\
I_3 &= \sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|\\
&\quad+\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\Big|\\
&\quad+\sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\Big|\\
&\quad+\sup_{y>0}\Big|\int_{\mathcal{C}_r^+}\frac{f(z)-h(z)}{z}e^{izy}e^{ib\frac{z}{T}}dz\Big|\\
&=: I_{31}+I_{32}+I_{33}+I_{34},\\
I_4 &= \frac{2}{T}\int_{-T}^{T}|f(t)-h(t)|dt,
\end{aligned} \tag{4.5.7}$$

with the constant $b>0$ and the complex contours $\mathcal{C}_r^-,\mathcal{C}_r^+$ defined in (4.4.2).

By virtue of (4.5.6), in order to prove Theorem 4.5.2 it suffices to show that uniformly in $s\in(-\eta,\eta)$, $x\in\mathcal{S}$ and $\varphi\in\mathcal{B}_\gamma$,

$$I_1+I_2+I_3+I_4=\|\varphi\|_\gamma o\Big(\frac{1}{\sqrt{n}}\Big). \tag{4.5.8}$$

Control of I_1. From (4.5.3) we deduce that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)]| \leqslant C\|\varphi\|_\gamma. \tag{4.5.9}$$

By the formula (4.3.39) and the bound (4.3.36), we get that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} |b_{s,\varphi}(x)| \leqslant C\|\varphi\|_\gamma. \tag{4.5.10}$$

Using the bounds (4.5.9) and (4.5.10), and taking into account that $\sigma_s^2 > 0$ and $\Lambda'''(s) \in \mathbb{R}$ are bounded by a constant independent of $s \in (-\eta,\eta)$, we obtain that $|H'(y)|$ is bounded by $c_1\|\varphi\|_\gamma$, uniformly in $s \in (-\eta,\eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$. Hence, for any $\varepsilon > 0$, we can choose $a > 0$ large enough, such that, for $T = a\sqrt{n}$, uniformly in $\varphi \in \mathcal{B}_\gamma$, we have

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} I_1 \leqslant \frac{6\pi b c_1}{T}\|\varphi\|_\gamma < \frac{\varepsilon}{\sqrt{n}}\|\varphi\|_\gamma. \tag{4.5.11}$$

Control of I_2. Since $\sigma_m := \inf_{s\in(-\eta,\eta)} \sigma_s > 0$, we can pick δ_1 small enough, such that $0 < \delta_1 < \min\{a, \delta\sigma_m/2\}$, where the constant $\delta > 0$ is given in Proposition 4.3.8. Then with $r = \delta_1\sqrt{n}$ we bound I_2 as follows:

$$I_2 \leqslant \int_{\delta_1\sqrt{n}<|t|\leqslant a\sqrt{n}} \left|\frac{f(t)}{t}\right| dt + \int_{\delta_1\sqrt{n}<|t|\leqslant a\sqrt{n}} \left|\frac{h(t)}{t}\right| dt. \tag{4.5.12}$$

Let $\sigma_M := \sup_{s\in(-\eta,\eta)} \sigma_s < \infty$. On the right-hand side of (4.5.12), using Proposition 4.3.10 with $K = \{t \in \mathbb{R} : \delta_1/\sigma_M \leqslant |t| \leqslant a/\sigma_m\}$, the first integral is bounded by $Ce^{-cn}\|\varphi\|_\gamma$, uniformly in $s \in (-\eta,\eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$; the second integral, by the bounds (4.5.9) and (4.5.10) and direct calculations, is bounded by $Ce^{-c\sqrt{n}}\|\varphi\|_\gamma$, also uniformly in $s \in (-\eta,\eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. Consequently, we conclude that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} I_2 \leqslant Ce^{-c\sqrt{n}}\|\varphi\|_\gamma. \tag{4.5.13}$$

Control of I_3. Recall that the term I_3 is decomposed into four terms in (4.5.7). We will only deal with I_{31}, since I_{32}, I_{33}, I_{34} can be treated in a similar way. In view of (4.5.4) and (4.5.5), by the spectral gap decomposition (4.3.32), we get

$$f(z) - h(z) = J_1(z) + J_2(z) + J_3(z) + J_4(z), \tag{4.5.14}$$

where

$$J_1(z) = \pi_s(\varphi)\Big\{\lambda^n_{s,\frac{-iz}{\sigma_s\sqrt{n}}} - e^{-\frac{z^2}{2}}\Big[1 - (iz)^3\frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}\Big]\Big\}, \tag{4.5.15}$$

$$J_2(z) = \lambda^n_{s,\frac{-iz}{\sigma_s\sqrt{n}}}\Big[\Pi_{s,\frac{-iz}{\sigma_s\sqrt{n}}}\varphi(x) - \pi_s(\varphi) - iz\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\Big], \tag{4.5.16}$$

$$J_3(z) = iz\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\Big(\lambda^n_{s,\frac{-iz}{\sigma_s\sqrt{n}}} - e^{-\frac{z^2}{2}}\Big), \tag{4.5.17}$$

$$J_4(z) = N^n_{s,\frac{-iz}{\sigma_s\sqrt{n}}}\varphi(x) + N^n_{s,0}\varphi(x)e^{-\frac{z^2}{2}}\Big[1 - (iz)^3\frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}\Big]. \tag{4.5.18}$$

With the above notation, we use the decomposition (4.5.14) to bound I_{31} in (4.5.7) as follows:

$$I_{31} \leqslant \sum_{k=1}^{4} A_k, \quad \text{where} \quad A_k = \sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-} \frac{J_k(z)}{z} e^{izy} e^{-ib\frac{z}{T}} dz\right|. \tag{4.5.19}$$

We now give bounds of A_k, $1 \leqslant k \leqslant 4$, in a series of lemmata. Let us start by giving an elementary inequality, which will be used repeatedly in the sequel. Let $[z_1, z_2] = \{z_1 + \theta(z_2 - z_1)) : 0 \leqslant \theta \leqslant 1\}$ be the complex segment with the endpoints z_1 and z_2.

Lemma 4.5.4. *Let f be an analytic function on the open convex domain $D \subseteq \mathbb{C}$. Then for any $z_1, z_2 \in D$, and $n \geqslant 1$,*

$$\left|f(z_2) - \sum_{k=0}^{n-1} \frac{f^{(k)}(z_1)}{k!}(z_2 - z_1)^k\right| \leqslant \frac{\sup_{z\in[z_1,z_2]} |f^{(n)}(z)|}{n!}|z_2 - z_1|^n.$$

Proof. The proof of this inequality can be carried out by induction. The inequality clearly holds for $n = 1$ since for any $z_1, z_2 \in D$,

$$|f(z_2) - f(z_1)| = \left|\int_{[z_1,z_2]} f'(z)dz\right| \leqslant \sup_{z\in[z_1,z_2]} |f'(z)||z_2 - z_1|. \tag{4.5.20}$$

For $n \geqslant 2$, applying (4.5.20) to $F(z) = f(z) - \sum_{k=1}^{n-1} \frac{f^{(k)}(z_1)}{k!}(z - z_1)^k$, $z \in D$, leads to the desired assertion. □

Now we are ready to establish a bound of each term A_k. The proof is based on the saddle point method. To be more precise, we deform the integration path, which passes through a suitable point related to the saddle point, to minimise the integral in A_k (see (4.5.19)).

Lemma 4.5.5. *Let $\mathcal{C}_r^-$ be defined by (4.4.2) with $r = \delta_1\sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_1 = \sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-} \frac{J_1(z)}{z} e^{izy} e^{-ib\frac{z}{T}} dz\right| \leqslant \frac{c}{n}\|\varphi\|_\infty.$$

Proof. In view of (4.3.33), using $\Lambda = \log\kappa$ and Taylor's formula, we have

$$\lambda^n_{s,\frac{-iz}{\sigma_s\sqrt{n}}} = e^{-\frac{z^2}{2}} e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(-\frac{iz}{\sigma_s\sqrt{n}})^k}. \tag{4.5.21}$$

For brevity, for any $z \in \mathcal{C}_r^-$, denote

$$h_1(z) = \frac{1}{z}\left[e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(-\frac{iz}{\sigma_s\sqrt{n}})^k} - 1 - (-iz)^3\frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}\right]e^{-ib\frac{z}{T}}. \tag{4.5.22}$$

Then, in view of (4.5.15), the term A_1 can be rewritten as

$$A_1 = \pi_s(\varphi)\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-} e^{-\frac{z^2}{2}+izy} h_1(z)dz\right|. \tag{4.5.23}$$

The main contribution to the integral in (4.5.23) is given by the saddle point $z = iy$ which is the solution of the equation $\frac{d}{dz}(-\frac{z^2}{2} + izy) = 0$. Denote by $D_{2r}^- = \{z \in \mathbb{C} : |z| < 2r, \Im z < 0\}$ the domain on analyticity of h_1, where $r = \delta_1\sqrt{n}$. Set

$$y_n = \min\{-y, \delta_1\sqrt{n}\}. \tag{4.5.24}$$

When $-\delta_1\sqrt{n} \leqslant y \leqslant 0$, the saddle point iy belongs to D_{2r}^-. By Cauchy's integral theorem, we change the integration in (4.5.23) to a rectangular path inside the domain on analyticity D_{2r}^- which passes through the saddle point. When $y < -\delta_1\sqrt{n}$ is large, the saddle point iy is outside the domain D_{2r}^-. In this case we choose a rectangular path inside D_{2r}^- which passes through the point $-iy_n = -i\delta_1\sqrt{n}$. Note that $\pi_s(\varphi)$ is bounded by $c_1\|\varphi\|_\infty$ uniformly in $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$. Since the function h_1 has an analytic extension on the domain D_{2r}^- with $r = \delta_1\sqrt{n}$, applying Cauchy's integral theorem, we deduce that

$$\begin{aligned} A_1 &\leqslant c_1\|\varphi\|_\infty \sup_{y\leqslant 0}\Big|\Big\{\int_{-\delta_1\sqrt{n}}^{-\delta_1\sqrt{n}-iy_n} + \int_{\delta_1\sqrt{n}-iy_n}^{\delta_1\sqrt{n}}\Big\}e^{-\frac{z^2}{2}+izy}h_1(z)dz\Big| \\ &\quad + c_1\|\varphi\|_\infty \sup_{y\leqslant 0}\Big|\int_{-\delta_1\sqrt{n}-iy_n}^{\delta_1\sqrt{n}-iy_n} e^{-\frac{z^2}{2}+izy}h_1(z)dz\Big| \\ &= c_1\|\varphi\|_\infty(A_{11} + A_{12}). \end{aligned} \tag{4.5.25}$$

Control of A_{11}. Using a change of variable, we get

$$\begin{aligned} A_{11} &= e^{-\frac{\delta_1^2}{2}n}\sup_{y\leqslant 0}\Big|\int_0^{y_n} e^{\frac{t^2}{2}+ty-i\delta_1\sqrt{n}(t+y)}h_1(-\delta_1\sqrt{n}-it)dt \\ &\qquad - \int_0^{y_n} e^{\frac{t^2}{2}+ty+i\delta_1\sqrt{n}(t+y)}h_1(\delta_1\sqrt{n}-it)dt\Big| \\ &\leqslant e^{-\frac{\delta_1^2}{2}n}\sup_{y\leqslant 0}\Big|\int_0^{y_n} e^{\frac{t^2}{2}+ty}\Big\{|h_1(-\delta_1\sqrt{n}-it)| + |h_1(\delta_1\sqrt{n}-it)|\Big\}dt\Big|. \end{aligned} \tag{4.5.26}$$

We first bound $|h_1(\pm\delta_1\sqrt{n}-it)|$. Since $t \in [0, y_n]$ and $y_n \leqslant \delta_1\sqrt{n}$, direct calculations give

$$\Re\Big[(-i)^3(\pm\delta_1\sqrt{n}-it)^3\Big] = 3\delta_1^2nt - t^3 \leqslant 2\delta_1^3n^{3/2},$$

which implies that for $\delta_1 > 0$ sufficiently small,

$$\Re\Big\{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}\frac{(-i)^k(\pm\delta_1\sqrt{n}-it)^k}{(\sigma_s\sqrt{n})^k}\Big\} \leqslant \frac{1}{4}\delta_1^2n. \tag{4.5.27}$$

Observe that there exists a constant $c > 0$ such that uniformly in $t \in [0, y_n]$ and $s \in (-\eta, \eta)$,

$$\Big|\frac{1}{z}\Big| = \Big|\frac{1}{\pm\delta_1\sqrt{n}-it}\Big| \leqslant \frac{c}{\delta_1\sqrt{n}}, \quad \Big|i^3(\pm\delta_1\sqrt{n}-it)^3\frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}}\Big| \leqslant cn. \tag{4.5.28}$$

Since the function $t \mapsto e^{-ib\frac{z}{T}}$ is analytic on the domain D_T, we infer that $|\exp\{\frac{ib}{T}(\pm\delta_1\sqrt{n}+it)\}|$ is bounded uniformly in $t \in [0, y_n]$ and $n \geqslant 1$. Combining this with the bounds (4.5.27) and (4.5.28), uniformly in $s \in (-\eta, \eta)$,

$$|h_1(-\delta_1\sqrt{n}-it)| + |h_1(\delta_1\sqrt{n}-it)| \leqslant \frac{c}{\delta_1\sqrt{n}}\Big(e^{\frac{\delta_1^2}{4}n} + cn\Big) \leqslant \frac{c_{\delta_1}}{\sqrt{n}}e^{\frac{\delta_1^2}{4}n}.$$

In view of (4.5.24), we have $t \leqslant y_n \leqslant -y$ and thus $e^{\frac{t^2}{2}+ty} \leqslant 1$ for any $t \in [0, y_n]$. Note that $y_n \leqslant \delta_1\sqrt{n}$ by (4.5.24). Consequently, we obtain the bound:

$$\sup_{s\in(-\eta,\eta)} A_{11} \leqslant c_{\delta_1}\frac{y_n}{\sqrt{n}}e^{-\frac{\delta_1^2}{2}n}e^{\frac{\delta_1^2}{4}n} \leqslant c_{\delta_1}e^{-\frac{\delta_1^2}{4}n}. \tag{4.5.29}$$

Control of A_{12}. Using a change of variable $z = t - iy_n$ leads to

$$\begin{aligned} A_{12} &= \sup_{y\leqslant 0}\left|e^{\frac{1}{2}y_n^2+y_ny}\int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2}+it(y_n+y)}h_1(t-iy_n)dt\right| \\ &\leqslant \sup_{y\leqslant 0}\left|e^{\frac{1}{2}y_n^2+y_ny}\int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2}}|h_1(t-iy_n)|dt\right|, \end{aligned} \tag{4.5.30}$$

where the function h_1 is defined by (4.5.22). To estimate the term A_{12}, the main task is to give a control of $|h_1(t-iy_n)|$. It follows from Lemma 4.5.4 that $|e^{z_1}-e^{z_2}| \leqslant e^{\max\{\Re z_1,\Re z_2\}}|z_1-z_2|$ and $|e^{z_2}-1-z_2| \leqslant \frac{1}{2}|z_2|^2e^{|z_2|}$ for any $z_1, z_2 \in \mathbb{C}$, and hence

$$|e^{z_1}-1-z_2| \leqslant e^{\max\{\Re z_1,\Re z_2\}}|z_1-z_2| + \frac{1}{2}|z_2|^2e^{|z_2|}. \tag{4.5.31}$$

We shall make use of the inequality (4.5.31) to derive a bound of $|h_1(t-iy_n)|$. Since $\frac{y_n}{\sqrt{n}} \leqslant \delta_1$ where $\delta_1 > 0$ can be sufficiently small, for any $|t| \leqslant \delta_1\sqrt{n}$, we get that uniformly in $s \in (-\eta,\eta)$,

$$\Re\Big\{[-i(t-iy_n)]^3\frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}}\Big\} = \frac{y_n}{\sqrt{n}}\frac{(3t^2-y_n^2)\Lambda^{(3)}(s)}{6\sigma_s^3} \leqslant \frac{1}{4}t^2, \tag{4.5.32}$$

$$\Re\Big[n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}\Big(-\frac{i(t-iy_n)}{\sigma_s\sqrt{n}}\Big)^k\Big] \leqslant \frac{y_n}{\sqrt{n}}\frac{(6t^2-\frac{1}{2}y_n^2)\Lambda^{(3)}(s)}{6\sigma_s^3} \leqslant \frac{1}{4}t^2. \tag{4.5.33}$$

Moreover, elementary calculations yield that there exists a constant $c > 0$ such that uniformly in $s \in (-\eta,\eta)$,

$$\begin{aligned} &\Big|n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}\Big(-\frac{i(t-iy_n)}{\sigma_s\sqrt{n}}\Big)^k - [-i(t-iy_n)]^3\frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}}\Big| \\ &= \Big|n\sum_{k=4}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}\Big(-\frac{i(t-iy_n)}{\sigma_s\sqrt{n}}\Big)^k\Big| \leqslant c\frac{t^4+y_n^4}{n}. \end{aligned} \tag{4.5.34}$$

It is clear that

$$\sup_{s\in(-\eta,\eta)}\Big|[-i(t-iy_n)]^3\frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}}\Big|^2 \leqslant c\frac{t^6+y_n^6}{n}. \tag{4.5.35}$$

Taking into account that both $|t|$ and y_n are less than $\delta_1\sqrt{n}$, and the fact $\delta_1 > 0$ can be small enough, it follows that

$$\sup_{s\in(-\eta,\eta)}\exp\Big\{|[-i(t-iy_n)]^3\frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}}|\Big\} \leqslant e^{\frac{1}{4}(t^2+y_n^2)}.$$

Combining this with the bounds (4.5.32), (4.5.33), (4.5.34) and (4.5.35), and using the inequality (4.5.31), we conclude that

$$\begin{aligned}&\sup_{s\in(-\eta,\eta)}\left|e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(-\frac{iz}{\sigma_s\sqrt{n}})^k}-1-(-iz)^3\frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}}\right|\\&\leqslant c\frac{t^4+y_n^4}{n}e^{\frac14 t^2}+c\frac{t^6+y_n^6}{n}e^{\frac14(t^2+y_n^2)}\leqslant c\frac{t^4+y_n^4+t^6+y_n^6}{n}e^{\frac14(t^2+y_n^2)}.\end{aligned}\tag{4.5.36}$$

Since the function $\widehat{\rho}_T$ has a continuous extension on the domain $\overline{D}_T$, we get that $|\widehat{\rho}_T(-t+iy_n)|$ is bounded uniformly in $|t|\leqslant\delta_1\sqrt{n}$ and $n\geqslant 1$. Combining this with (4.5.36) and the fact $|\frac{1}{t-iy_n}|=1/\sqrt{t^2+y_n^2}$ leads to

$$\sup_{s\in(-\eta,\eta)}|h_1(t-iy_n)|\leqslant c\frac{|t|^3+y_n^3+|t|^5+y_n^5}{n}e^{\frac14(t^2+y_n^2)}.$$

Therefore, noting that $y\leqslant -y_n$ and $0\leqslant y_n\leqslant\delta_1\sqrt{n}$, we obtain

$$\begin{aligned}\sup_{s\in(-\eta,\eta)}A_{12}&\leqslant\frac{c}{n}\sup_{y\leqslant 0}\left|e^{\frac34 y_n^2+y_ny}\int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}}e^{-\frac{t^2}{4}}(|t|^3+y_n^3+|t|^5+y_n^5)dt\right|\\&\leqslant\frac{c}{n}\sup_{y_n\in[0,\delta_1\sqrt{n}]}e^{-\frac14 y_n^2}(1+y_n^3+y_n^5)\leqslant\frac{c}{n}.\end{aligned}$$

Substituting this and (4.5.29) into (4.5.25), we conclude the proof of Lemma 4.5.5. □

Lemma 4.5.6. *Let $J_2(z)$ be defined by (4.5.16), and $\mathcal{C}_r^-$ be defined by (4.4.2) with $r=\delta_1\sqrt{n}$ and $\delta_1>0$ small enough. Then, for $T=a\sqrt{n}$ with $a>0$ large enough, uniformly in $x\in\mathcal{S}$, $s\in(-\eta,\eta)$ and $\varphi\in\mathcal{B}_\gamma$,*

$$A_2=\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-}\frac{J_2(z)}{z}e^{izy}e^{-ib\frac{z}{T}}dz\right|\leqslant\frac{c}{n}\|\varphi\|_\gamma.$$

Proof. Denote

$$h_2(z)=e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(-\frac{iz}{\sigma_s\sqrt{n}})^k}\left[\Pi_{s,\frac{-iz}{\sigma_s\sqrt{n}}}\varphi(x)-\pi_s(\varphi)-iz\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}}\right]\frac{\widehat{\rho}_T(-z)}{z}.$$

Using (4.5.21), we rewrite A_2 as

$$A_2=\sup_{y\leqslant 0}\left|\int_{\mathcal{C}_r^-}e^{-\frac{z^2}{2}+izy}h_2(z)dz\right|.$$

As in the estimation of Lemma 4.5.5, the solution of the saddle point equation $\frac{d}{dz}(-\frac{z^2}{2}+izy)=0$ is $z=iy$. Set $y_n=\min\{-y,\delta_1\sqrt{n}\}$. Since $y_n\in D_{2r}^-$, where $r=\delta_1\sqrt{n}$, and the function h_2 is analytic on the domain D_{2r}^-, by Cauchy's integral theorem we obtain

$$\begin{aligned}A_2\leqslant&\sup_{y\leqslant 0}\left|\left\{\int_{-\delta_1\sqrt{n}}^{-\delta_1\sqrt{n}-iy_n}+\int_{\delta_1\sqrt{n}-iy_n}^{\delta_1\sqrt{n}}\right\}e^{-\frac{z^2}{2}+izy}h_2(z)dz\right|\\&+\sup_{y\leqslant 0}\left|\int_{-\delta_1\sqrt{n}-iy_n}^{\delta_1\sqrt{n}-iy_n}e^{-\frac{z^2}{2}+izy}h_2(z)dz\right|=:\ A_{21}+A_{22}.\end{aligned}$$

Control of A_{21}. Similarly to (4.5.26), we use a change of variable to get

$$A_{21} \leqslant e^{-\frac{\delta_1^2}{2}n} \sup_{y\leqslant 0} \Big| \int_0^{y_n} e^{\frac{t^2}{2}+ty} \Big[|h_2(-\delta_1\sqrt{n}-it)| + |h_2(\delta_1\sqrt{n}-it)| \Big] dt \Big|.$$

Using Lemma 4.5.4, the formula (4.3.39) and the bound (4.3.36), for any $z = \pm\delta_1\sqrt{n} - it$ with $t \in [0, y_n]$, we get that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\Big|\frac{1}{z}\Big| \Big| \Pi_{s,\frac{-iz}{\sigma_s\sqrt{n}}}\varphi(x) - \pi_s(\varphi) - iz\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}} \Big| \leqslant c\frac{|z|}{n}\|\varphi\|_\gamma \leqslant \frac{c}{\sqrt{n}}\|\varphi\|_\gamma. \tag{4.5.37}$$

Recall that the function $t \mapsto e^{-ib\frac{z}{T}}$ is continuous on the domain D_T, so that $|e^{-ib\frac{z}{T}}|$ is bounded uniformly in $z = \pm\delta_1\sqrt{n} - it$, where $t \in [0, y_n]$. Therefore, taking into account the bound (4.5.27), we get that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$|h_2(-\delta_1\sqrt{n}-it)| + |h_2(\delta_1\sqrt{n}-it)| \leqslant \frac{c}{\sqrt{n}} e^{\frac{\delta_1^2}{4}n}\|\varphi\|_\gamma.$$

Since $y \leqslant 0$, for any $t \in [0, y_n]$, it follows that $\frac{t^2}{2} + ty \leqslant 0$ and thus $e^{\frac{t^2}{2}+ty} \leqslant 1$. Combining this with the above inequality yields that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} A_{21} \leqslant ce^{-\frac{\delta_1^2}{2}n}\frac{y_n}{\sqrt{n}} e^{\frac{\delta_1^2}{4}n}\|\varphi\|_\gamma \leqslant ce^{-\frac{\delta_1^2}{4}n}\|\varphi\|_\gamma. \tag{4.5.38}$$

Control of A_{22}. Similarly to (4.5.30), we use a change of variable to get

$$A_{22} \leqslant \sup_{y\leqslant 0} \Big| e^{\frac{1}{2}y_n^2+y_n y} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2}} |h_2(t-iy_n)| dt \Big|.$$

We first estimate $|h_2(t - iy_n)|$. In the same way as in (4.5.37), with $z = t - iy_n$, we obtain that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\Big|\frac{1}{z}\Big| \Big| \Pi_{s,\frac{-iz}{\sigma_s\sqrt{n}}}\varphi(x) - \pi_s(\varphi) - iz\frac{b_{s,\varphi}(x)}{\sigma_s\sqrt{n}} \Big| \leqslant c\frac{|z|}{n}\|\varphi\|_\gamma \leqslant c\frac{|t|+y_n}{n}\|\varphi\|_\gamma.$$

Combining this with the bound (4.5.33), we get that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} A_{22} &\leqslant \frac{c}{n}\|\varphi\|_\gamma \sup_{y\leqslant 0} \Big| e^{\frac{1}{2}y_n^2+y_n y} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{4}} (|t|+y_n) dt \Big| \\ &\leqslant \frac{c}{n}\|\varphi\|_\gamma \sup_{y_n\in[0,\delta_1\sqrt{n}]} e^{-\frac{1}{2}y_n^2}(1+y_n) \leqslant \frac{c}{n}\|\varphi\|_\gamma. \end{aligned} \tag{4.5.39}$$

Putting together (4.5.38) and (4.5.39) completes the proof. □

Lemma 4.5.7. *Let $J_3(z)$ be defined by (4.5.17), and $\mathcal{C}_r^-$ be defined by (4.4.2) with $r = \delta_1\sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_3 = \sup_{y\leqslant 0} \Big| \int_{\mathcal{C}_r^-} \frac{J_3(z)}{z} e^{izy} e^{-ib\frac{z}{T}} dz \Big| \leqslant \frac{c}{n}\|\varphi\|_\gamma.$$

Proof. Denote

$$h_3(z) = \frac{1}{\sigma_s\sqrt{n}}\Big[e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(-\frac{iz}{\sigma_s\sqrt{n}})^k} - 1\Big]e^{-ib\frac{z}{T}}. \tag{4.5.40}$$

Using the expansion (4.5.21) and the bound (4.5.10), we have that uniformly in $s \in (-\eta,\eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$A_3 \leqslant c\|\varphi\|_\gamma \sup_{y\leqslant 0}\Big|\int_{\mathcal{C}_r^-} e^{-\frac{z^2}{2}+izy}h_3(z)dz\Big|.$$

As in Lemma 4.5.5, the saddle point equation $\frac{d}{dz}(-\frac{z^2}{2}+izy) = 0$ has the solution $z = iy$. Set $y_n = \min\{-y, \delta_1\sqrt{n}\}$. It follows from Cauchy's integral theorem that

$$\begin{aligned} A_3 \leqslant c\|\varphi\|_\gamma \sup_{y\leqslant 0}&\Big|\Big\{\int_{-\delta_1\sqrt{n}}^{-\delta_1\sqrt{n}-iy_n} + \int_{\delta_1\sqrt{n}-iy_n}^{\delta_1\sqrt{n}}\Big\}e^{-\frac{z^2}{2}+izy}h_3(z)dz\Big| \\ &+ c\|\varphi\|_\gamma \sup_{y\leqslant 0}\Big|\int_{-\delta_1\sqrt{n}-iy_n}^{\delta_1\sqrt{n}-iy_n} e^{-\frac{z^2}{2}+izy}h_3(z)dz\Big| =: A_{31} + A_{32}. \end{aligned}$$

Control of A_{31}. Similarly to (4.5.26), we use a change of variable to get

$$A_{31} \leqslant c\|\varphi\|_\gamma e^{-\frac{\delta_1^2}{2}n}\sup_{y\leqslant 0}\Big|\int_0^{y_n} e^{\frac{t^2}{2}+ty}\Big[|h_3(-\delta_1\sqrt{n}-it)| + |h_3(\delta_1\sqrt{n}-it)|\Big]dt\Big|.$$

Using the bounds (4.5.10) and (4.5.27), we deduce that uniformly in $s \in (-\eta,\eta)$ and $x \in \mathcal{S}$,

$$|h_3(-\delta_1\sqrt{n}-it)| + |h_3(\delta_1\sqrt{n}-it)| \leqslant \frac{c}{\sqrt{n}}\Big(e^{\frac{\delta_1^2}{4}n}+1\Big) \leqslant \frac{c}{\sqrt{n}}e^{\frac{\delta_1^2}{4}n}.$$

Since $\frac{t^2}{2}+ty \leqslant 0$ for any $t \in [0, y_n]$ and $y \leqslant 0$, it follows that $e^{\frac{t^2}{2}+ty} \leqslant 1$. This, together with the above inequality, implies that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}} A_{31} \leqslant c\frac{y_n}{\sqrt{n}}e^{-\frac{\delta_1^2}{4}n}\|\varphi\|_\gamma \leqslant ce^{-\frac{\delta_1^2}{4}n}\|\varphi\|_\gamma. \tag{4.5.41}$$

Control of A_{32}. Similarly to (4.5.30), we use a change of variable to get

$$A_{32} \leqslant c\|\varphi\|_\gamma \sup_{y\leqslant 0}\Big|e^{\frac{1}{2}y_n^2+y_ny}\int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2}}|h_3(t-iy_n)|dt\Big|.$$

We first give a control of $|h_3(t-iy_n)|$. It follows from Lemma 4.5.4 that $|e^z-1| \leqslant e^{\max\{\Re z,0\}}|z|$ for any $z \in \mathbb{C}$. Using this inequality and taking into account of the bound (4.5.33), we obtain

$$\sup_{s\in(-\eta,\eta)}\Big|e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(-\frac{iz}{\sigma_s\sqrt{n}})^k} - 1\Big| \leqslant ce^{\frac{1}{4}t^2}\frac{|t|^3+y_n^3}{\sqrt{n}},$$

and hence

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}|h_3(t-iy_n)| \leqslant ce^{\frac{1}{4}t^2}\frac{|t|^3+y_n^3}{n}.$$

It follows that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$A_{32} \leqslant \frac{c}{n}\|\varphi\|_\gamma \sup_{y \leqslant 0}\left|e^{-\frac{1}{2}y_n^2}\int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{4}}(|t|^3 + y_n^3)dt\right| \leqslant \frac{c}{n}\|\varphi\|_\gamma. \tag{4.5.42}$$

Putting together (4.5.41) and (4.5.42), we conclude the proof. □

Lemma 4.5.8. *Let $J_4(z)$ be defined by (4.5.18), and $\mathcal{C}_r^-$ be defined by (4.4.2) with $r = \delta_1\sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_4 = \sup_{y \leqslant 0}\left|\int_{\mathcal{C}_r^-} \frac{J_4(z)}{z} e^{izy} e^{-ib\frac{z}{T}} dz\right| \leqslant ce^{-cn}\|\varphi\|_\gamma.$$

Proof. Since $\Im z \leqslant 0$ on $\mathcal{C}_r^-$ and $y \leqslant 0$, we have $|e^{izy}| \leqslant 1$. Since the function $z \mapsto e^{-ib\frac{z}{T}}$ is analytic on the domain $\overline{D}_T$, we have that $|e^{-ib\frac{z}{T}}|$ is uniformly bounded on $\mathcal{C}_r^-$. Using the bound (4.3.37) and the fact that $\delta_1 > 0$ can be sufficiently small, we deduce that $|J_4(z)| \leqslant ce^{-cn}\|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. Therefore, noting that $|\frac{1}{z}| = (\delta_1\sqrt{n})^{-1}$ and that the length of $\mathcal{C}_r^-$ is $\pi\delta_1\sqrt{n}$, the desired result follows. □

End of the proof of Theorem 4.5.2. Combining Lemmata 4.5.5-4.5.8, we obtain that $I_{31} \leqslant \frac{c}{n}\|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$.

Now we give a control of the term I_{32} defined in (4.5.7). Note that $y > 0$ in I_{32} and the integral in I_{32} is taken over the semicircle $\mathcal{C}_r^+$, which lies in the upper part of the complex plane. In this case we have the saddle point equation $\frac{d}{dz}(-\frac{z^2}{2} + izy) = 0$ whose solution $z = iy$ also lies in the upper part of the complex plane. Similarly to (4.5.24), we choose a suitable point $y_n = \min\{y, \delta_1\sqrt{n}\}$. Proceeding in the same way as for bounding I_{31} we obtain that $I_{32} \leqslant \frac{c}{n}\|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$.

Let us now bound the terms I_{33} and I_{34} defined in (4.5.7). Since the mapping $z \mapsto \widehat{\rho}_T(z)$ is analytic on $\mathcal{C}_r^-$ and $\mathcal{C}_r^+$, the estimates of I_{33} and I_{34} are similar to those of I_{31} and I_{32}, respectively. From these bounds, we conclude that there exists a constant $c > 0$ such that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_3 \leqslant \frac{c}{n}\|\varphi\|_\gamma. \tag{4.5.43}$$

It remains to estimate I_4 defined in (4.5.7). We can decompose the difference $|f(t) - h(t)|$ in the same way as we did in (4.5.14) (with real-valued $t = z$). Then proceeding in a similar way as in the estimation of I_{31}, I_{32}, I_{33} and I_{34}, one can verify that there exists a constant $c > 0$ such that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_4 \leqslant \frac{c}{n}\|\varphi\|_\gamma. \tag{4.5.44}$$

Combining (4.5.43), (4.5.43) and the bounds for I_1 and I_2 in (4.5.11) and (4.5.13), and using the fact that ε can be arbitrary small, we obtain (4.5.8), which finishes the proof of Theorem 4.5.2. □

4.5.3 Proof of Theorem 4.5.1

Since the proof of Theorem 4.5.1 is quite similar to that of Theorem 4.5.3, we only sketch the main differences. Denote

$$F(y) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\frac{\log|G_n x| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant y\Big\}}\Big], \quad y \in \mathbb{R},$$
$$H(y) = \mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)]\ \Phi(y), \quad y \in \mathbb{R}.$$

By the definition of the operator $R_{s,z}$ in (4.3.29), direct calculations lead to

$$f(t) = \int_{\mathbb{R}} e^{-ity} dF(y) = R^n_{s,\frac{-it}{\sigma_s\sqrt{n}}}\varphi(x), \quad t \in \mathbb{R},$$
$$h(t) = \int_{\mathbb{R}} e^{-ity} dH(y) = e^{-\frac{t^2}{2}} R_{s,0}\varphi(x), \quad t \in \mathbb{R}.$$

One can verify that the functions F, H and their corresponding Fourier-Stieljes transforms f, h satisfy all the conditions stated in Proposition 4.4.1. Instead of using Proposition 4.4.1 with $r < T$ in the proof of Theorem 4.5.3, we apply Proposition 4.4.1 with $r = T = \delta_1\sqrt{n}$, where $\delta_1 > 0$ is a sufficiently small constant. Then we obtain a similar inequality as (4.5.6) but with the term $I_2 = 0$. Since the non-arithmeticity condition **C5** is only used in the bound of the term I_2, following the proof of Theorem 4.5.3 we show that under the conditions of Theorem 4.5.1, the terms I_1 and I_3 defined in (4.5.7) are bounded by $c\|\varphi\|_\gamma/\sqrt{n}$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. We omit the details of the rest of the proof.

4.6 Proof of moderate deviation expansions

In this section we prove Theorem 4.2.3. The proof is based on the Berry-Esseen bound in Theorem 4.5.1 and follows the standard techniques in Petrov [74], and therefore some details will be left to the reader.

We start with the following lemma whose proof uses the analyticity of the eigenfunction r_s and the eigenmeasure ν_s, see Proposition 4.3.1.

Lemma 4.6.1. *Assume either conditions **C1** and **C2** for invertible matrices, or conditions **C1** and **C3** for positive matrices. Then, there exists $\eta > 0$ such that uniformly in $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\|r_s - \mathbf{1}\|_\infty \leqslant C|s| \quad \textit{and} \quad |\nu_s(\varphi) - \nu(\varphi)| \leqslant C|s|\|\varphi\|_\gamma.$$

Proof. According to Proposition 4.3.1, we have $r_0 = \mathbf{1}$, $\nu_0 = \nu$. In addition, the mappings $s \mapsto r_s$ and $s \mapsto \nu_s$ are analytic on $(-\eta, \eta)$. The assertions follow using Taylor's formula. □

Now we prove Theorem 4.2.3. When $y \in [0, 1]$, Theorem 4.2.3 is a direct consequence of Theorem 4.5.1, so it remains to prove Theorem 4.2.3 in the case when $y > 1$ with $y = o(\sqrt{n})$. We proceed to prove the first assertion Theorem 4.2.3. Applying the change of measure formula (4.3.16), we have

$$\begin{aligned} I &:= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x| \geqslant n\Lambda'(0) + \sqrt{n}\sigma_0 y\}}\Big] \\ &= r_s(x)\kappa^n(s)\mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)e^{-s\log|G_n x|}\mathbb{1}_{\{\log|G_n x| \geqslant n\Lambda'(0) + \sqrt{n}\sigma_0 y\}}\Big]. \end{aligned} \tag{4.6.1}$$

Under the assumptions of Theorem 4.2.3, by Proposition 4.3.15, $\sigma_s^2 = \Lambda''(s) > 0$, for any $s \in (-\eta, \eta)$ and $\eta > 0$ small enough. Denote $W_n^x = \frac{\log|G_n x| - n\Lambda'(s)}{\sigma_s\sqrt{n}}$. Recalling that $\Lambda = \log\kappa$, we rewrite (4.6.1) as follows:

$$\begin{aligned} I &= r_s(x)e^{-n[s\Lambda'(s)-\Lambda(s)]} \\ &\quad \times \mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)e^{-s\sigma_s\sqrt{n}W_n^x}\mathbb{1}_{\Big\{W_n^x \geqslant \frac{\sqrt{n}[\Lambda'(0)-\Lambda'(s)]}{\sigma_s}+\frac{\sigma_0 y}{\sigma_s}\Big\}}\Big]. \end{aligned} \tag{4.6.2}$$

By Proposition 4.3.1, the function Λ is analytic and hence for $s \in (-\eta, \eta)$, $\Lambda(s) = \sum_{k=1}^{\infty}\frac{\gamma_k}{k!}s^k$, where $\gamma_k = \Lambda^{(k)}(0)$. For any $y > 1$ with $y = o(\sqrt{n})$, consider the equation

$$\sqrt{n}[\Lambda'(s) - \Lambda'(0)] = \sigma_0 y. \tag{4.6.3}$$

Choosing the unique real root s of (4.6.3), it follows from Petrov [74] that

$$s\Lambda'(s) - \Lambda(s) = \frac{y^2}{2n} - \frac{y^3}{n^{3/2}}\zeta(\frac{y}{\sqrt{n}}), \tag{4.6.4}$$

where ζ is the Cramér series defined by (4.2.9). Substituting (4.6.3) into (4.6.2), and using (4.6.4), we get

$$I = r_s(x)e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)e^{-s\sigma_s\sqrt{n}W_n^x}\mathbb{1}_{\{W_n^x \geqslant 0\}}\Big]. \tag{4.6.5}$$

For brevity, denote $F(u) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)\mathbb{1}_{\{W_n^x \leqslant u\}}\Big]$, $u \in \mathbb{R}$. In view of (4.6.5), using Fubini's theorem and the integration by parts, we deduce that

$$\begin{aligned} I &= r_s(x)e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)\int_0^\infty \mathbb{1}_{\{0 \leqslant W_n^x \leqslant u\}}s\sigma_s\sqrt{n}\, e^{-s\sigma_s\sqrt{n}u}du\Big] \\ &= r_s(x)e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\int_0^\infty e^{-s\sqrt{n}\sigma_s u}dF(u). \end{aligned} \tag{4.6.6}$$

Letting $l(u) = F(u) - \pi_s(\varphi r_s^{-1})\Phi(u)$, $u \in \mathbb{R}$, we have

$$\int_0^\infty e^{-s\sqrt{n}\sigma_s u}dF(u) = I_1 + \frac{\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}}I_2, \tag{4.6.7}$$

$$I_1 = \int_0^\infty e^{-s\sqrt{n}\sigma_s u}dl(u), \quad I_2 = \int_0^\infty e^{-s\sqrt{n}\sigma_s u - \frac{u^2}{2}}du. \tag{4.6.8}$$

Estimate of I_1. Integrating by parts, using the fact that $r_s \in \mathcal{B}_\gamma$ and the Berry-Esseen bound in Theorem 4.5.1 implies that uniformly in $s \in [0, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$|I_1| \leqslant |l(0)| + s\sqrt{n}\sigma_s\int_0^\infty e^{-s\sqrt{n}\sigma_s u}|l(u)|du \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{4.6.9}$$

Estimate of I_2. The function Λ is analytic on $(-\eta, \eta)$ and $\sigma_s^2 = \Lambda''(s) > 0$. By Taylor's formula, $\Lambda'(s) - \Lambda'(0) = s\sigma_0^2\Big[1 + O(s)\Big]$ and $\sigma_s^2 = \sigma_0^2\Big[1 + O(s)\Big]$. Then, using standard techniques from Petrov [74], we obtain

$$I_2 = I_3 + O\Big(\frac{1}{\sqrt{n}}\Big), \quad \text{where } I_3 = \int_0^\infty e^{-\frac{\sqrt{n}[\Lambda'(s)-\Lambda'(0)]}{\sigma_0}u - \frac{u^2}{2}}du. \tag{4.6.10}$$

Since σ_s is strictly positive and bounded uniformly in $s \in (0, \eta)$, using (4.6.3) and the fact that $y > 1$, for sufficiently large n, we get that $s\sqrt{n}\,\sigma_s \geqslant \frac{y}{2\sigma_0}\sigma_s \geqslant c_1 > 0$. This implies that $C_1 \leqslant s\sqrt{n}I_2 \leqslant C_2$ holds for large enough n, where $C_1 < C_2$ are two positive constants independent of n and s. Combining this two-sided bound with (4.6.7), (4.6.9) and (4.6.10),

$$\int_0^\infty e^{-s\sqrt{n}\sigma_s u} dF(u) = I_3\Big[\frac{\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} + \|\varphi\|_\gamma O(s)\Big]. \tag{4.6.11}$$

Substituting (4.6.3) into (4.6.11), it follows that

$$\int_0^\infty e^{-s\sqrt{n}\sigma_s u} dF(u) = e^{\frac{y^2}{2}} \int_y^\infty e^{-\frac{1}{2}u^2} du\Big[\frac{\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} + \|\varphi\|_\gamma O(s)\Big].$$

Together with (4.6.6), this implies

$$I = r_s(x) e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[1 - \Phi(y)\Big]\Big[\pi_s(\varphi r_s^{-1}) + \|\varphi\|_\gamma O(s)\Big], \tag{4.6.12}$$

where $\pi_s(\varphi r_s^{-1}) = \frac{\nu_s(\varphi)}{\nu_s(r_s)}$. By Lemma 4.6.1, we have $\|r_s - \mathbf{1}\|_\infty \leqslant Cs$ and $|\pi_s(\varphi r_s^{-1}) - \nu(\varphi)| \leqslant Cs\|\varphi\|_\gamma$, uniformly in $s \in [0, \eta)$ and $\varphi \in \mathcal{B}_\gamma$. Since $s = O(\frac{y}{\sqrt{n}})$, this concludes the proof of the first assertion of Theorem 4.2.3.

The proof of the second assertion of Theorem 4.2.3 can be carried out in a similar way. Instead of (4.6.3), we consider the equation $\sqrt{n}[\Lambda'(s) - \Lambda'(0)] = -\sigma_0 y$, where $y > 1$ and $s \in (-\eta, 0]$. We then apply the spectral gap properties of operators $P_s, Q_s, R_{s,z}$ (see Section 4.3) for negative valued s to deduce the second assertion by following the proof of the first one. We omit the details.

Chapter 5

Berry-Esseen bounds and moderate deviations for the norm, entries and spectral radius of products of positive random matrices

Abstract. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) $d\times d$ positive random matrices and consider the matrix product $G_n = g_n \dots g_1$. Denote by $\|G_n\|$ any matrix norm of G_n, by $G_n^{i,j}$ its (i,j)-th entry, and by $\rho(G_n)$ its spectral radius. Under suitable conditions, we establish Berry-Esseen bounds and precise moderate deviation expansions for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$, and the spectral radius $\rho(G_n)$. As corollaries, moderate deviation principles are derived.

5.1 Introduction

Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) $d \times d$ $(d \geqslant 2)$ positive random matrices of the same probability law μ. Set $G_n = g_n \dots g_1$ and denote by $\|G_n\|$ any matrix norm of the product G_n. It has been of great interest in recent years to investigate the asymptotic behaviors of the random matrix product G_n since the fundamental work of Furstenberg and Kesten [37], in which the following strong law of large numbers (SLLN) for the matrix norm $\|G_n\|$ was established: if $\mathbb{E}(\max\{0, \log \|g_1\|\}) < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \|G_n\| = \lambda, \quad a.s., \tag{5.1.1}$$

where λ is a constant called the upper Lyapunov exponent of the product G_n. This result can be seen as a direct consequence of Kingman's subaddtive ergodic theorem [67]. The central limit theorem (CLT) for $\|G_n\|$ was also considered in [37]; it was established by Hennion [53] under the second moment condition together with the allowability and positivity condition (that we will present later): for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} dt =: \Phi(y), \tag{5.1.2}$$

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where $\sigma^2 > 0$ is the asymptotic variance corresponding to the product G_n. In the case of products of invertible random matrices, the CLT (5.1.2) was established by Le Page [69], and has been extended by Goldsheid and Guivarc'h [40] to a multidimensional version, and by Benoist-Quint [9] to the general framework of reductive groups.

In [86] the authors proved a Berry-Esseen bound and a moderate deviation expansion for the norm cocycle $\log|G_n x|$ jointly with the Markov chain $X_n^x = G_n x/|G_n x|$, where x is any starting point on the unit sphere and $|\cdot|$ is the euclidean norm. For related results for $\log|G_n x|$ we refer to [69, 13, 1, 49, 17, 10, 22, 23, 78, 85]. However, this type of results for other quantities like the matrix norm, the entries and the spectral radius of G_n are notably absent in the literature. The goal of the paper is to fill the gap and to extend the results of [86] to the matrix norm $\|G_n\|$, to the entries $G_n^{i,j}$ and to the spectral radius $\rho(G_n)$ for the product G_n of positive random matrices, jointly with X_n^x.

Let us explain briefly the main results that we obtain for the matrix norm. We would like to quantify the error in the normal approximation (5.1.2). We do this in two ways. The first way is to estimate the absolute error. In this spirit, under suitable conditions we prove the following Berry-Esseen bound:

$$\sup_{y\in\mathbb{R}}\Big|\mathbb{P}\Big(\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\leqslant y\Big)-\Phi(y)\Big|\leqslant\frac{C}{\sqrt{n}}. \tag{5.1.3}$$

Our result (5.1.3) is clearly a refinement of (5.1.2) by giving the rate of convergence. In fact a more general version of the Berry-Esseen bound with a target function is given in Theorem 5.2.1.

The second way is to study the relative error. Along this line we prove the following Cramér type moderate deviation expansion: uniformly in $y\in[0,o(\sqrt{n})]$,

$$\frac{\mathbb{P}\Big(\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\geqslant y\Big)}{1-\Phi(y)}=e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[1+O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{5.1.4}$$

where ζ is the Cramér series (see (5.2.6)). Note that the expansion (5.1.4) clearly implies the moderate deviation principle for the matrix norm $\|G_n\|$, see Corollary 5.2.4, which to the best of our knowledge was not known before.

The above results concern the matrix norm $\|G_n\|$, but we also prove that all the assertions (5.1.3) and (5.1.4) remain valid (under stronger conditions) when the matrix norm $\|G_n\|$ is replaced by the entries $G_n^{i,j}$ or the spectral radius $\rho(G_n)$: see Theorems 5.2.2 and 5.2.5. The corresponding strong law of large numbers and the central limit theorem were established in [37, 25, 53] for the entries $G_n^{i,j}$ and in [53] for the spectral radius $\rho(G_n)$. However, our Theorems 5.2.2 and 5.2.5 on Berry-Esseen bounds and Cramér type moderate deviation expansions for the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$ are new.

The proofs of (5.1.3) and (5.1.4) are based on the recent results established in [86] about the Berry-Esseen bound and the Cramér type moderate deviation expansion for the norm cocyle $\log|G_n x|$ and on a comparison between $\|G_n\|$ and $|G_n x|$ (Lemma 5.3.1), where x is a vector in $\mathbb{R}^d$ with strictly positive components.

To prove (5.1.3) and (5.1.4) when the matrix norm $\|G_n\|$ is replaced by the entries $G_n^{i,j}$, in addition to the use of the aforementioned results established in [86], we do a careful quantitative analysis of the comparison between $\log G_n^{i,j} := \log\langle e_i, G_n e_j\rangle$ and

$\log|G_n e_j|$, where $(e_i)_{1\leqslant k\leqslant d}$ is the canonical orthonormal basis in $\mathbb{R}^d$. This comparison is possible due to a regularity condition which ensures that all the entries in the same column of the matrix $g \in \operatorname{supp}\mu$ (the support of μ) are comparable: see condition **H3**. Note that this condition is nevertheless weaker than the Furstenberg-Kesten condition (5.2.1) used in [37], which says that all the entries of the matrix $g \in \operatorname{supp}\mu$ are comparable.

Using the results mentioned above for the matrix norm $\|G_n\|$ and for the norm cocycle $\log|G_n x|$ established in [86], we then prove the corresponding results for the spectral radius $\rho(G_n)$ based on the Collatz-Wielandt formula: see Theorems 5.2.2 and 5.2.5.

5.2 Main results

5.2.1 Notation and conditions

For any integer $d \geqslant 2$, denote by $\mathscr{G}_+$ the multiplicative semigroup of $d\times d$ matrices with non-negative entries in $\mathbb{R}$. A non-negative matrix $g \in \mathscr{G}_+$ is said to be *allowable*, if every row and every column of g contains a strictly positive entry. Any allowable matrix g will be simply called positive matrix. We write $\mathscr{G}_+^\circ$ for the subsemigroup of $\mathscr{G}_+$ with strictly positive entries. Equip the space $\mathbb{R}^d$ with the standard scalar product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $|\cdot|$. For a vector x, we write $x \geqslant 0$ (resp. $x > 0$) if all its components are non-negative (resp. strictly positive). Denote by $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ the intersection of the unit sphere with the positive quadrant. The space $\mathbb{S}_+^{d-1}$ is endowed with the Hilbert cross-ratio metric $\mathbf{d}$, i.e., for any $x = (x_1,\dots,x_d)$ and $y = (y_1,\dots,y_d)$ in $\mathbb{S}_+^{d-1}$, $\mathbf{d}(x,y) = \frac{1-m(x,y)m(y,x)}{1+m(x,y)m(y,x)}$, where $m(x,y) = \sup\{\alpha > 0 : \alpha y_i \leqslant x_i,\ \forall i = 1,\dots,d\}$. It is shown in [53] that there exists a constant $C > 0$ such that $|x-y| \leqslant C\mathbf{d}(x,y)$ for any $x,y \in \mathbb{S}_+^{d-1}$. We refer to [53] for more properties of the metric $\mathbf{d}$.

Let $\mathcal{C}(\mathbb{S}_+^{d-1})$ be the space of continuous complex-valued functions on $\mathbb{S}_+^{d-1}$ and $\mathbf{1}$ be the constant function with value 1. Throughout the paper we always assume that $\gamma > 0$ is a fixed small enough constant. For any $\varphi \in \mathcal{C}(\mathbb{S}_+^{d-1})$, set

$$\|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_\gamma, \quad \|\varphi\|_\infty := \sup_{x\in\mathbb{S}_+^{d-1}} |\varphi(x)|, \quad [\varphi]_\gamma := \sup_{x,y\in\mathbb{S}_+^{d-1}} \frac{|\varphi(x)-\varphi(y)|}{\mathbf{d}^\gamma(x,y)}.$$

Introduce the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathbb{S}_+^{d-1}) : \|\varphi\|_\gamma < +\infty\}$.

Let $(g_n)_{n\geqslant 1}$ be a sequence of i.i.d. positive random matrices of the same probability law μ on $\mathscr{G}_+$. Consider the matrix product $G_n = g_n \dots g_1$ and denote by $G_n^{i,j}$ the (i,j)-th entry of G_n, where $1 \leqslant i,j \leqslant d$. It holds that $G_n^{i,j} = \langle e_i, G_n e_j\rangle$, where $(e_k)_{1\leqslant k\leqslant d}$ is the canonical orthonormal basis in $\mathbb{R}^d$. For any $g \in \mathscr{G}_+$, denote by $\rho(g)$ the spectral radius of g, and by $\|g\|$ its operator norm as follows: $\|g\| = \sup_{x\in\mathbb{S}_+^{d-1}} |gx|$. By Gelfand's formula, it holds that $\rho(g) = \lim_{k\to\infty} \|g^k\|^{1/k}$. In this paper, we are interested in Berry-Esseen bounds and Cramér type moderate deviation expansions for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$.

Let $\iota(g) = \inf_{x\in\mathbb{S}_+^{d-1}} |gx|$ and $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$. We need the following exponential moment condition:

H1. There exists a constant $\eta \in (0,1)$ such that $\mathbb{E}[N(g_1)^\eta] < +\infty$.

Let $\Gamma_\mu := [\operatorname{supp}\mu]$ be the smallest closed semigroup of $\mathscr{G}_+$ generated by $\operatorname{supp}\mu$ (the support of μ). We will use the allowability and positivity conditions:

H2. (i) (Allowability) Every $g \in \Gamma_\mu$ is allowable.
(ii) (Positivity) Γ_μ contains at least one matrix belonging to $\mathscr{G}_+^\circ$.

It follows from the Perron-Frobenius theorem that every $g \in \mathscr{G}_+^\circ$ has a dominant eigenvalue which coincides with its spectral radius $\rho(g)$.

The following condition ensures that all the entries in each column of the matrix $g \in \operatorname{supp}\mu$ are comparable.

H3. For any $1 \leqslant j \leqslant d$, there exists a constant $C > 1$ such that for any $g \in \operatorname{supp}\mu$,

$$1 \leqslant \frac{\max_{1\leqslant i\leqslant d} g^{i,j}}{\min_{1\leqslant i\leqslant d} g^{i,j}} \leqslant C.$$

Note that the set of such type of matrices forms a subsemigroup of $\mathscr{G}_+$, because if two positive matrices g_1 and g_2 satisfy condition **H3**, then so does the product $g_2 g_1$, as will be seen from Lemma 5.3.2 where an equivalent description of condition **H3** will be provided.

It is easy to see that condition **H3** implies condition **H2**. However, our condition **H3** is clearly weaker than the Furstenberg-Kesten condition used in [37]: there exists a constant $C > 1$ such that for any $g \in \operatorname{supp}\mu$,

$$1 \leqslant \frac{\max_{1\leqslant i,j\leqslant d} g^{i,j}}{\min_{1\leqslant i,j\leqslant d} g^{i,j}} \leqslant C. \tag{5.2.1}$$

This condition plays an essential role in [37] for the proofs of the strong law of large numbers and the central limit theorem for entries $G_n^{i,j}$.

Under conditions **H1** and **H2**, it is shown in [86] that uniformly in $x \in \mathbb{S}_+^{d-1}$,

$$\sigma^2 := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log|G_n x| - n\lambda)^2\Big] \in [0,\infty), \tag{5.2.2}$$

where the constant λ is the upper Lyapunov exponent defined by (5.1.1). Equivalent formulations of σ^2 will be given in Proposition 5.2.7. We shall need the following condition:

H4. The asymptotic variance σ^2 satisfies $\sigma^2 > 0$.

Condition **H4** holds if the additive subgroup of $\mathbb{R}$ generated by the set $\{\log\rho(g) : g \in \Gamma_\mu,\ g \in \mathscr{G}_+^\circ\}$ is dense in $\mathbb{R}$, see [66, 17, 86] for details.

For any $x \in \mathbb{S}_+^{d-1}$ and allowable matrix $g \in \mathscr{G}_+$, we write $g \cdot x := \frac{gx}{|gx|}$ for the projective action of g on the projective space $\mathbb{S}_+^{d-1}$. For any starting point $x \in \mathbb{S}_+^{d-1}$, set $X_0^x = x$ and

$$X_n^x = G_n \cdot x, \quad n \geqslant 1,$$

which forms a Markov chain on the projective space $\mathbb{S}_+^{d-1}$. Under conditions **H1** and **H2**, the Markov chain $(X_n^x)_{n\geqslant 0}$ possesses a unique stationary measure ν on $\mathbb{S}_+^{d-1}$ such that for any continuous function φ on $\mathbb{S}_+^{d-1}$,

$$\int_{\mathbb{S}_+^{d-1}} \int_{\Gamma_\mu} \varphi(g_1 \cdot x)\mu(dg_1)\nu(dx) = \int_{\mathbb{S}_+^{d-1}} \varphi(x)\nu(dx).$$

We refer to [66, 53, 16, 86] for more details.

5.2.2 Berry-Esseen bounds

The goal of this section is to present our results on the Berry-Esseen bounds for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$. Let us first state the result for the operator norm $\|G_n\|$. Denote $(\mathbb{S}_+^{d-1})^\circ = \{x > 0 : |x| = 1\}$, which is the interior of the projective space $\mathbb{S}_+^{d-1}$.

Theorem 5.2.1. *Assume conditions **H1**, **H2** and **H4**. Then, for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, we have, uniformly in $\varphi \in \mathcal{B}_\gamma$,*

$$\sup_{y\in\mathbb{R}} \sup_{x\in K} \left| \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{5.2.3}$$

Since all matrix norms are equivalent, it can be easily checked that in Theorem 5.2.1, the operator norm $\|\cdot\|$ can be replaced by any matrix norm.

It would be interesting to show that (5.2.3) holds uniformly in $x \in \mathbb{S}_+^{d-1}$ instead of $x \in K$. Note that Theorem 5.2.1 is proved under the exponential moment condition **H1**. It is not clear how to establish Theorem 5.2.1 under the polynomial moment condition on the matrix law μ.

If the stronger condition **H3** holds instead of condition **H2**, then we are able to prove the following Berry-Esseen bounds for the scalar product $\langle f, G_n x\rangle$ and for the spectral radius $\rho(G_n)$.

Theorem 5.2.2. *Under conditions **H1**, **H3** and **H4**, we have:*

(1) *uniformly in $\varphi \in \mathcal{B}_\gamma$,*

$$\sup_{y\in\mathbb{R}} \sup_{f,x\in\mathbb{S}_+^{d-1}} \left| \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\langle f,G_n x\rangle-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma; \tag{5.2.4}$$

(2) *for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, uniformly in $\varphi \in \mathcal{B}_\gamma$,*

$$\sup_{y\in\mathbb{R}} \sup_{x\in K} \left| \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{5.2.5}$$

In particular, taking $\varphi = \mathbf{1}$, $f = e_i$ and $x = e_j$ in (5.2.4), we get the Berry-Esseen bound for the entries $G_n^{i,j}$. The Berry-Esseen bounds (5.2.4) and (5.2.5) are all new. It would be interesting to establish these bounds under some condition weaker than **H3**.

5.2.3 Precise moderate deviation expansions

In this section we formulate the Cramér type moderate deviation expansions for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$.

To present our results, we need some notation. For any $s \in (-\eta, \eta)$, define the transfer operator P_s as follows: $P_s\varphi(x) = \mathbb{E}[e^{s\log|g_1x|}\varphi(g_1\cdot x)]$, where $\varphi \in \mathcal{C}(\mathbb{S}_+^{d-1})$ and $x \in \mathbb{S}_+^{d-1}$. Based on the perturbation theorem, it is shown in [86] that under conditions **H1** and **H2**, the transfer operator P_s has spectral gap properties on the Banach space $\mathcal{B}_\gamma$ and possesses a dominating eigenvalue $\kappa(s)$. Moreover, the function

κ is analytic, real-valued and strictly convex in a small neighborhood of 0 under the additional condition **H4**. Denote $\Lambda = \log \kappa$ and $\gamma_k = \Lambda^{(k)}(0)$, $k \geqslant 1$, then it holds that $\gamma_1 = \lambda$ and $\gamma_2 = \sigma^2$. Throughout this paper, we write ζ for the Cramér series of Λ:

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \cdots, \tag{5.2.6}$$

which converges for $|t|$ small enough. We refer to [86] for more details.

The following result concerns the Cramér type moderate deviations for the operator norm $\|G_n\|$. Recall that $(\mathbb{S}_+^{d-1})^\circ = \{x > 0 : |x| = 1\}$.

Theorem 5.2.3. *Assume conditions **H1**, **H2** and **H4**. Then, for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, we have, uniformly in $x \in K$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\| - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{5.2.7}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\| - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]. \tag{5.2.8}$$

Like in Theorem 5.2.1, it can also be checked that in Theorem 5.2.3 the operator norm $\|\cdot\|$ can be replaced by any matrix norm.

Note that condition **H3** is not required in Theorem 5.2.3. Theorem 5.2.3 is new even for $\varphi = \mathbf{1}$ and the expansions (5.2.7) and (5.2.8) remain valid even when $\nu(\varphi) = 0$. As a particular case, Theorem 5.2.3 implies the following moderate deviation principle for $\log\|G_n\|$ with a target function φ on the Markov chain X_n^x, where the operator norm $\|\cdot\|$ can be replaced by any matrix norm.

Corollary 5.2.4. *Assume conditions **H1**, **H2** and **H4**. Then, for any real-valued function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, we have, uniformly in $x \in K$,*

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}, \end{aligned} \tag{5.2.9}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

Note that the target function φ in (5.2.9) is not necessarily positive and it can vanish on some part of the projective space $\mathbb{S}_+^{d-1}$. The moderate deviation principle (5.2.9) is new, even for $\varphi = \mathbf{1}$.

As in Theorem 5.2.1, it would be interesting to prove that Theorem 5.2.3 holds uniformly in $x \in \mathbb{S}_+^{d-1}$ instead of $x \in K$.

Now we formulate Cramér type moderate deviation expansions for the scalar product $\langle f, G_n x\rangle$ as well as for the spectral radius $\rho(G_n)$.

Theorem 5.2.5. *Assume conditions* ***H1***, ***H3*** *and* ***H4***. *Then, we have:*
(1) uniformly in $f, x \in \mathbb{S}_+^{d-1}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\langle f, G_n x\rangle - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{5.2.10}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\langle f, G_n x\rangle - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]; \tag{5.2.11}$$

(2) for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, uniformly in $x \in K$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n) - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{5.2.12}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n) - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]. \tag{5.2.13}$$

As a particular case of (5.2.10) and (5.2.11) with $f = e_i$ and $x = e_j$, we get the Cramér type moderate deviation expansions for the entries $G_n^{i,j}$. The expansions (5.2.10)-(5.2.13) are all new even for $\varphi = \mathbf{1}$.

From Theorem 5.2.5 we can get the moderate deviation principles with target functions for the scalar product $\langle f, G_n x\rangle$ and the spectral radius $\rho(G_n)$, just as we obtained (5.2.9) from Theorem 5.2.3. Let us state them below. Recall that for a set B, we write respectively B° and $\bar{B}$ for its interior and closure.

Corollary 5.2.6. *Assume conditions* ***H1***, ***H3*** *and* ***H4***. *Then, for any real-valued function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, we have*

(1) *uniformly in $f, x \in \mathbb{S}_+^{d-1}$,*

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\langle f, G_n x\rangle - n\lambda}{b_n}\in B\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\langle f, G_n x\rangle - n\lambda}{b_n}\in B\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}; \end{aligned}$$

(2) *for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, uniformly in $x \in K$,*

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\rho(G_n) - n\lambda}{b_n}\in B\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\frac{\log\rho(G_n) - n\lambda}{b_n}\in B\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}. \end{aligned}$$

5.2.4 Formulas for the asymptotic variance

In this section, we give alternative expresssions for the asymptotic variance σ^2 defined by (5.2.2). These expressions can be useful while applying the theorems and the corollaries stated before, where σ appears.

Proposition 5.2.7. *(1) Under conditions* ***H1*** *and* ***H2****, we have*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log\|G_n\| - n\lambda)^2\Big]. \tag{5.2.14}$$

(2) Under conditions ***H1*** *and* ***H3****, we have*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log\langle f, G_n x\rangle - n\lambda)^2\Big] = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[(\log\rho(G_n) - n\lambda)^2\Big], \tag{5.2.15}$$

where the convergence in the first equality holds uniformly in $f, x \in \mathbb{S}_+^{d-1}$.

For invertible matrices, the expression (5.2.14) has been established in [10, Proposition 14.7]. For positive matrices, both (5.2.14) and (5.2.15) are new.

5.3 Proofs of Berry-Esseen bounds

The goal of this section is to prove Theorems 5.2.1 and 5.2.2. In order to prove Theorem 5.2.1, we shall use the following result which is shown in [16, Lemma 4.5].

Lemma 5.3.1. *Under condition* ***H2*** *(i), for any* $x \in (\mathbb{S}_+^{d-1})^\circ$*, we have*

$$\tau(x) := \inf_{g\in\Gamma_\mu} \frac{|gx|}{\|g\|} > 0.$$

Moreover, for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$*, it holds that* $\inf_{x\in K}\tau(x) > 0$.

We now proceed to prove Theorem 5.2.1 based on Lemma 5.3.1 and the Berry-Esseen bound for the norm cocycle $\log|G_n x|$ established in [86].

Proof of Theorem 5.2.1. Without loss of generality, we assume that the function φ is non-negative. Under conditions of Theorem 5.2.1, the following Berry-Esseen bound for the norm cocycle $\log|G_n x|$ with a target function φ on the Markov chain X_n^x has been recently established in [86]: there exists a constant $C > 0$ such that uniformly in $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{y\in\mathbb{R}} \sup_{x\in\mathbb{S}_+^{d-1}} \left|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{5.3.1}$$

On the one hand, using the fact that $\log|G_n x| \leqslant \log\|G_n\|$, we deduce from (5.3.1) that there exists a constant $C > 0$ such that uniformly in $y \in \mathbb{R}$, $x \in \mathbb{S}_+^{d-1}$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] \leqslant \nu(\varphi)\Phi(y) + \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

On the other hand, by Lemma 5.3.1, we see that for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, there exists a constant $C_1 > 0$ such that uniformly in $n \geqslant 1$ and $x \in K$,

$$\log\|G_n\| \leqslant \log|G_n x| + C_1. \tag{5.3.2}$$

Combining this inequality with (5.3.1), we obtain that, with $y_1 = y - \frac{C_1}{\sigma\sqrt{n}}$, uniformly in $y \in \mathbb{R}$, $x \in K$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] \geqslant \nu(\varphi)\Phi(y_1) - \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

Elementary calculation gives that $\Phi(y_1) - \Phi(y) \geqslant -\frac{C}{\sqrt{n}}$, uniformly in $y \in \mathbb{R}$ and $n \geqslant 1$. This, together with the above inequality, yields the desired lower bound. The proof of Theorem 5.2.1 is complete. □

Now we turn to prove Theorem 5.2.2. For any $0 < \epsilon < 1$, set

$$\mathbb{S}_{+,\epsilon}^{d-1} = \{x \in \mathbb{S}_+^{d-1} : \langle x, e_j\rangle \geqslant \epsilon \text{ for all } 1 \leqslant j \leqslant d\}.$$

The following result provides an equivalent formulation of condition **H3**, which will be used to prove Theorems 5.2.2 and 5.2.5. For any matrix $g \in \operatorname{supp}\mu$, we denote $g \cdot \mathbb{S}_+^{d-1} = \{g \cdot x : x \in \mathbb{S}_+^{d-1}\}$.

Lemma 5.3.2. *Condition **H3** is equivalent to the following statement: there exists a constant $\epsilon \in (0, \frac{\sqrt{2}}{2})$ such that*

$$g \cdot \mathbb{S}_+^{d-1} \subset \mathbb{S}_{+,\epsilon}^{d-1}, \quad \textit{for any } g \in \operatorname{supp}\mu. \tag{5.3.3}$$

Proof. We first show that (5.3.3) implies condition **H3**. For any matrix $g = (g^{i,j})_{1\leqslant i,j\leqslant d} \in \operatorname{supp}\mu$, we have that for any $1 \leqslant i, j \leqslant d$,

$$\langle e_i, g \cdot e_j\rangle = \frac{g^{i,j}}{\sqrt{\sum_{i=1}^d (g^{i,j})^2}}. \tag{5.3.4}$$

Using (5.3.3) and the definition of $\mathbb{S}_{+,\epsilon}^{d-1}$, we get that there exists $\epsilon \in (0, \frac{\sqrt{2}}{2})$ such that $\langle e_i, g\cdot e_j\rangle \geqslant \epsilon$ for all $1 \leqslant i, j \leqslant d$. This implies condition **H3** with $C = \sqrt{\frac{1}{d-1}(\frac{1}{\epsilon^2} - 1)}$ by taking maxima and minima by rows in (5.3.4).

We next prove that condition **H3** implies (5.3.3). For any $x \in \mathbb{S}_+^{d-1}$, we write $x = \sum_{j=1}^d x_j e_j$, where $x_j \geqslant 0$ satisfies $\sum_{j=1}^d x_j^2 = 1$. It is easy to see that $\sum_{j=1}^d x_j \geqslant 1$. For any $1 \leqslant i \leqslant d$, it holds that

$$\langle e_i, g\cdot x\rangle = \frac{1}{|gx|}\sum_{j=1}^d x_j\langle e_i, ge_j\rangle = \frac{\sum_{j=1}^d x_j g^{i,j}}{\sqrt{\sum_{i=1}^d(\sum_{j=1}^d g^{i,j}x_j)^2}}.$$

Since $\sum_{j=1}^d x_j^2 = 1$, using the Cauchy-Schwarz inequality gives $(\sum_{j=1}^d g^{i,j}x_j)^2 \leqslant \sum_{j=1}^d (g^{i,j})^2$. Combining this inequality with condition **H3** and the fact that $\sum_{j=1}^d x_j \geqslant 1$, we obtain $\langle e_i, g\cdot x\rangle \geqslant \sum_{j=1}^d \frac{x_j}{\sqrt{C^2d^2}} \geqslant \frac{1}{Cd}$, so that (5.3.3) holds with $\epsilon = \frac{1}{Cd}$. □

Using Lemma 5.3.2, Theorem 5.2.1 and the Berry-Esseen bound (5.3.1), we can prove Theorem 5.2.2.

Proof of Theorem 5.2.2. Without loss of generality, we assume that φ is non-negative.

We first prove the Berry-Esseen bound (5.2.4) for the scalar product $\langle f, G_n x\rangle$. On the one hand, using the fact that $\log\langle f, G_n x\rangle \leqslant \log|G_n x|$, we deduce from the Berry-Esseen bound (5.3.1) that there exists a constant $C > 0$ such that uniformly in $y \in \mathbb{R}$, $f, x \in \mathbb{S}_+^{d-1}$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\langle f, G_n x\rangle - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] \geqslant \nu(\varphi)\Phi(y) - \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

On the other hand, note that $\log\langle f, G_n x\rangle = \log|G_n x| + \log\langle f, X_n^x\rangle$. By Lemma 5.3.2, we see that there exists a constant $C_1 > 0$ such that uniformly in $f, x \in \mathbb{S}_+^{d-1}$ and $n \geqslant 1$,

$$\log|G_n x| \leqslant \log\langle f, G_n x\rangle + C_1. \tag{5.3.5}$$

Using this inequality and again the Berry-Esseen bound (5.3.1), we obtain that, with $y_1 = y + \frac{C_1}{\sigma\sqrt{n}}$, uniformly in $y \in \mathbb{R}$, $f, x \in \mathbb{S}_+^{d-1}$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\langle f, G_n x\rangle - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] \leqslant \nu(\varphi)\Phi(y_1) + \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

It is easy to show that $\Phi(y_1) - \Phi(y) \leqslant \frac{C}{\sqrt{n}}$, uniformly in $y \in \mathbb{R}$. Together with the above inequality, this leads to the desired upper bound and ends the proof of the Berry-Esseen bound (5.2.4).

We next prove the bound (5.2.5) for the spectral radius $\rho(G_n)$. Since $\rho(G_n) \leqslant \|G_n\|$, using Theorem 5.2.1, we get the following lower bound: there exists a constant $C > 0$ such that uniformly in $y \in \mathbb{R}$, $x \in K$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] \geqslant \nu(\varphi)\Phi(y) - \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

The upper bound is carried out by using the Collatz-Wielandt formula in conjugation with the Berry-Esseen bound (5.2.4) for the entries $G_n^{i,i}$. Denote by $\mathcal{C}_+ = \{x \in \mathbb{R}^d : x \geqslant 0\} \setminus \{0\}$ the positive quadrant in $\mathbb{R}^d$ except the origin. According to the Collatz-Wielandt formula, the spectral radius of the positive matrix G_n can be represented as follows:

$$\rho(G_n) = \sup_{x \in \mathcal{C}_+} \min_{1 \leqslant i \leqslant d, \langle e_i, x\rangle > 0} \frac{\langle e_i, G_n x\rangle}{\langle e_i, x\rangle}. \tag{5.3.6}$$

It follows that there exists a constant $\epsilon \in (0, \frac{\sqrt{2}}{2})$ such that uniformly in $x \in \mathbb{S}_+^{d-1}$,

$$\rho(G_n) \geqslant \min_{1 \leqslant i \leqslant d} \langle e_i, G_n x\rangle \geqslant \min_{1 \leqslant i \leqslant d} \langle e_i, X_n^x\rangle |G_n x| \geqslant \epsilon |G_n x|, \tag{5.3.7}$$

where in the last inequality we use Lemma 5.3.2. Using the bound (5.3.1) and the inequality (5.3.7), we deduce that there exists a constant $C > 0$ such that uniformly in $x \in \mathbb{S}_+^{d-1}$, $y \in \mathbb{R}$, $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] \leqslant \nu(\varphi)\Phi(y) + \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

This ends the proof of the bound (5.2.5) for the spectral radius $\rho(G_n)$. □

5.4 Proofs of moderate deviation expansions

The aim of this section is to establish Theorems 5.2.3 and 5.2.5 on moderate deviation expansions, and Proposition 5.2.7 about the expressions of the asymptotic variance σ^2.

For the proof of Theorem 5.2.3 we need the following Cramér type moderate deviation expansion for the norm cocycle $\log|G_n x|$.

Lemma 5.4.1. *Assume conditions* ***H1***, ***H2*** *and* ***H4***. *Then, as $n \to \infty$, we have, uniformly in $x \in \mathbb{S}_+^{d-1}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{5.4.1}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]. \tag{5.4.2}$$

Lemma 5.4.1 has been recently established in [86] by developing a new smoothing inequality, applying a saddle point method and spectral gap properties of the transfer operator corresponding to the Markov chain $(X_n^x)_{n\geqslant 0}$. Note that condition **H3** is not assumed in Lemma 5.4.1 and the expansions (5.4.1) and (5.4.2) hold uniformly with respect to the starting point x on the whole projective space $\mathbb{S}_+^{d-1}$.

We now prove Theorem 5.2.3 using Lemmas 5.3.1 and 5.4.1.

Proof of Theorem 5.2.3. Without loss of generality, we assume that φ is non-negative on $\mathbb{S}_+^{d-1}$. We first prove (5.2.7). The proof consists of lower and upper bounds.

Lower bound. Since $\log\|G_n\| \geqslant \log|G_n x|$, applying Lemma 5.4.1, there exists a constant $C > 0$ such that, as $n \to \infty$, uniformly in $x \in \mathbb{S}_+^{d-1}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} \geqslant e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)-C\frac{y+1}{\sqrt{n}}\|\varphi\|_\gamma\Big]. \tag{5.4.3}$$

Upper bound. Using Lemmas 5.3.1 and 5.4.1, for any compact set $K \subset (\mathbb{S}_+^{d-1})^\circ$, we obtain that with $y_1 = y - \frac{C_1}{\sigma\sqrt{n}}$, there exists a constant $C > 0$ such that, as $n \to \infty$, uniformly in $x \in K$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y_1)} \leqslant e^{\frac{y_1^3}{\sqrt{n}}\zeta(\frac{y_1}{\sqrt{n}})}\Big[\nu(\varphi)+C\frac{y_1+1}{\sqrt{n}}\|\varphi\|_\gamma\Big]. \tag{5.4.4}$$

Since the Cramér series ζ is uniformly continuous in a small neighborhood of 0, we have $|\zeta(\frac{y_1}{\sqrt{n}})-\zeta(\frac{y}{\sqrt{n}})| \leqslant \frac{C}{n}$, uniformly in $y \in [0, o(\sqrt{n})]$. Combining this with some simple calculations yield that uniformly in $y \in [0, o(\sqrt{n})]$,

$$e^{\frac{y_1^3}{\sqrt{n}}\zeta(\frac{y_1}{\sqrt{n}})-\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} = 1+O(\frac{y^2+1}{n}), \quad \frac{1-\Phi(y_1)}{1-\Phi(y)} = 1+O(\frac{y+1}{\sqrt{n}}). \tag{5.4.5}$$

Note that $\frac{y_1+1}{\sqrt{n}} = O(\frac{y+1}{\sqrt{n}})$. Combining this with (5.4.4)-(5.4.5), we obtain

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} \leqslant e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)+C\frac{y+1}{\sqrt{n}}\|\varphi\|_\gamma\Big].$$

Together with (5.4.3), this concludes the proof of (5.2.7). The proof of the expansion (5.2.8) is similar to that of (5.2.7) by using the expansion (5.4.2) and Lemma 5.3.1. □

We next prove Theorem 5.2.5 based on Lemmas 5.3.2 and 5.4.1.

Proof of Theorem 5.2.5. Without loss of generality, we assume that φ is non-negative on $\mathbb{S}_+^{d-1}$. We first prove (5.2.10). The proof consists of upper and lower bounds.

Upper bound. Since $\log\langle f, G_n x\rangle \leqslant \log|G_n x|$, applying Lemma 5.4.1, this implies that there exists a constant $C>0$ such that, as $n\to\infty$, uniformly in $f,x\in\mathbb{S}_+^{d-1}$, $y\in[0,o(\sqrt{n})]$ and $\varphi\in\mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\langle f,G_n x\rangle-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} \leqslant e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)+C\frac{y+1}{\sqrt{n}}\|\varphi\|_\gamma\Big]. \tag{5.4.6}$$

Lower bound. Using (5.3.5) and applying (5.4.1) in Lemma 5.4.1, with $y_1 = y+\frac{C_1}{\sigma\sqrt{n}}$, we obtain that there exists a constant $c>0$ such that

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_n x\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y_1)} \geqslant e^{\frac{y_1^3}{\sqrt{n}}\zeta(\frac{y_1}{\sqrt{n}})}\Big[\nu(\varphi)-c\frac{y_1+1}{\sqrt{n}}\|\varphi\|_\gamma\Big]. \tag{5.4.7}$$

In an analogous way as in the proof of the upper bound in Theorem 5.2.3, one can verify that $|\zeta(\frac{y_1}{\sqrt{n}})-\zeta(\frac{y}{\sqrt{n}})|\leqslant\frac{C}{n}$, uniformly in $y\in[0,o(\sqrt{n})]$. Moreover, elementary calculations yield that uniformly in $y\in[0,o(\sqrt{n})]$, it holds that $e^{\frac{y_1^3}{\sqrt{n}}\zeta(\frac{y_1}{\sqrt{n}})-\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} = 1+O(\frac{y^2+1}{n})$, $\frac{1-\Phi(y_1)}{1-\Phi(y)} = 1+O(\frac{y+1}{\sqrt{n}})$ and $\frac{y_1+1}{\sqrt{n}} = O(\frac{y+1}{\sqrt{n}})$. Combining this with (5.4.7), we obtain

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_n x\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} \geqslant e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)-c\frac{y+1}{\sqrt{n}}\|\varphi\|_\gamma\Big].$$

Together with the upper bound (5.4.6), this concludes the proof of (5.2.10). The proof of (5.2.11) is similar to that of (5.2.10) by using (5.4.2) and Lemma 5.3.2.

The proof of the expansions (5.2.12) and (5.2.13) for the spectral radius $\rho(G_n)$ can be carried out in an analogous way using Theorem 5.2.3, Lemma 5.4.1 and inequality (5.3.7). We omit the details. □

We finally prove Proposition 5.2.7 based on Lemmas 5.3.1, 5.3.2 and the Collatz-Wielandt formula (5.3.6).

Proof of Proposition 5.2.7. We first prove part (1). For fixed $x\in K\subset(\mathbb{S}_+^{d-1})^\circ$, we denote

$$A_n = \mathbb{E}\Big(\log|G_n x|-n\lambda\Big)^2, \quad B_n = \mathbb{E}\Big(\log\|G_n\|-n\lambda\Big)^2.$$

Since $\frac{1}{n}A_n \to \sigma^2$ as $n \to \infty$ (see (5.2.2)), it suffices to show that $\frac{1}{n}(B_n - A_n) \to 0$ as $n \to \infty$. Using Minkowski's inequality, we see that there exists a constant $C > 0$ independent of $x \in K$ such that

$$\left|\sqrt{B_n} - \sqrt{A_n}\right| \leqslant \left[\mathbb{E}\Big(\log \frac{\|G_n\|}{|G_n x|}\Big)^2\right]^{1/2} \leqslant C,$$

where the last inequality holds by Lemma 5.3.1. Consequently, it follows that

$$|B_n - A_n| \leqslant |\sqrt{B_n} - \sqrt{A_n}|(|\sqrt{B_n} - \sqrt{A_n}| + 2\sqrt{A_n}) \leqslant C(C + O(\sqrt{n})), \tag{5.4.8}$$

which leads to the desired assertion in part (1).

Now we proceed to prove part (2). Denote

$$D_n = \mathbb{E}\Big[(\log\langle f, G_n x\rangle - n\lambda)^2\Big], \quad E_n = \mathbb{E}\Big[(\log \rho(G_n) - n\lambda)^2\Big].$$

As in the proof of part (1), by Minkowski's inequality, we have, uniformly in $f, x \in \mathbb{S}_+^{d-1}$,

$$\left|\sqrt{D_n} - \sqrt{A_n}\right| \leqslant \left[\mathbb{E}\Big(\log\langle f, X_n^x\rangle\Big)^2\right]^{1/2} \leqslant C,$$

where the last inequality holds by Lemma 5.3.2. In the same way as in the proof of (5.4.8), one can verify that $\frac{1}{n}(D_n - A_n) \to 0$, as $n \to \infty$, uniformly in $f, x \in \mathbb{S}_+^{d-1}$. This ends the proof of the first equality in part (2). To prove the second one in part (2), using again the Minkowski inequality, we have

$$\left|\sqrt{E_n} - \sqrt{B_n}\right| \leqslant \left[\mathbb{E}\Big(\log \frac{\|G_n\|}{\rho(G_n)}\Big)^2\right]^{1/2}.$$

Taking into account of the Collatz-Wielandt formula (5.3.6) with $i = 1$ and $x_0 = (1, 1, \ldots, 1)^{\mathrm{T}}$, we get that $\rho(G_n) \geqslant \langle e_1, G_n x_0\rangle$. Since $\rho(G_n) \leqslant \|G_n\|$ and $\|G_n\| \leqslant C|G_n x_0|$ (see Lemma 5.3.1), it follows from Lemma 5.3.2 that

$$\left|\sqrt{E_n} - \sqrt{B_n}\right| \leqslant C + \left[\mathbb{E}\Big(\log\langle e_1, X_n^{x_0}\rangle\Big)^2\right]^{1/2} \leqslant C.$$

Together with part (1), this proves the second equality in part (2). □

Chapter 6

Berry-Esseen bounds and moderate deviations for the random walk on $GL_d(\mathbb{R})$

Abstract. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed random $d \times d$ matrices with law μ. Consider the random walk $G_n := g_n \dots g_1$ on the general linear group $GL_d(\mathbb{R})$ and the Markov chain $X_n^x := G_n x/|G_n x|$, where x is a starting point on the unit sphere $\mathbb{S}^{d-1}$. Denote respectively by $\|G_n\|$ and $\rho(G_n)$ the operator norm and the spectral radius of G_n. For $\log \|G_n\|$ and $\log \rho(G_n)$, we prove moderate deviation principles under exponential moment and strong irreducibility conditions on μ; we also establish moderate deviation expansions in the normal range $[0, o(n^{1/6})]$ and Berry-Esseen bounds under the additional proximality condition on μ. Similar results are found for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions.

6.1 Introduction

6.1.1 Background and previous results

For any integer $d \geqslant 2$, denote by $\mathbb{G} = GL_d(\mathbb{R})$ the general linear group of real invertible $d \times d$ matrices. We equip $\mathbb{R}^d$ with the canonical Euclidean norm $|\cdot|$. Let $\mathbb{P}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}/\pm$ be the projective space in $\mathbb{R}^d$. For any $g \in \mathbb{G}$, denote by $\|g\| = \sup_{x\in\mathbb{P}^{d-1}} |gx|$ its operator norm, and by $\rho(g)$ its spectral radius. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) random matrices with law μ on the group $\mathbb{G}$. Consider the random walk $G_n = g_n \dots g_1$ on $\mathbb{G}$, and the Markov chain $X_n^x := G_n x/|G_n x|$ on $\mathbb{P}^{d-1}$ with any starting point $x \in \mathbb{P}^{d-1}$. The goal of this paper is to investigate Berry-Esseen type bounds and moderate deviation results for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$, and more generally, for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions on X_n^x.

Let Γ_μ be the smallest closed subsemigroup of $\mathbb{G}$ generated by the support of μ. Denote $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for any $g \in \mathbb{G}$. Consider the following conditions.

L1. (Exponential moments) There exists a constant $\delta > 0$ such that $\mathbb{E}[N(g_1)^\delta] < \infty$.

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L2. (Strong irreducibility) No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant, that is, there do not exist a finite number of proper subspaces $V_1, \cdots, V_m$ of $\mathbb{R}^d$ such that $g(V_1 \cup \ldots \cup V_m) = V_1 \cup \ldots \cup V_m$ for all $g \in \Gamma_\mu$.

L3. (Proximality) Γ_μ contains at least one proximal matrix with a unique eigenvalue of maximal modulus.

Notice that in condition **L2**, Γ_μ can be replaced by G_μ, the smallest closed subgroup of $\mathbb{G}$ generated by the support of μ. In fact, the set $I = \{g \in \mathbb{G} : g(V_1 \cup \ldots \cup V_m) = V_1 \cup \ldots \cup V_m\}$ is a subgroup of $\mathbb{G}$, so that $\Gamma_\mu \subset I$ if and only if $G_\mu \subset I$, meaning that $V_1 \cup \ldots \cup V_m$ is Γ_μ-invariant if and only if $V_1 \cup \ldots \cup V_m$ is G_μ-invariant.

The topic of products of random matrices has a very rich history and has been studied by many authors. The main distinct feature compared with the case of a sum of i.i.d. real-valued random variables lies in the fact that the matrix product is no longer commutative. Furstenberg and Kesten [37] first established the strong law of large numbers for the operator norm $\|G_n\|$: if $\mathbb{E}[\max\{0, \log \|g_1\|\}] < \infty$, then as $n \to \infty$, $\frac{1}{n} \log \|G_n\| \to \lambda$ almost surely, where λ is a constant called top Lyapunov exponent of the product G_n. This result can be considered as an immediate consequence of Kingman's subadditive ergodic theorem [67]. The central limit theorem for $\|G_n\|$ is due to Le Page [69] (see also Bougerol and Lacroix [13]): if conditions **L1**, **L2** and **L3** hold, then for any $y \in \mathbb{R}$ and any continuous function φ on $\mathbb{P}^{d-1}$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\lim_{n\to\infty} \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] = \nu(\varphi)\Phi(y), \tag{6.1.1}$$

where $\sigma^2 > 0$ is the asymptotic variance of the random walk $(G_n)_{n\geqslant 1}$, Φ is the standard normal distribution function, and ν is the unique stationary probability measure of the Markov chain $(X_n^x)_{n\geqslant 1}$. Recently, using Gordin's martingale approximation method, Benoist and Quint [9] have relaxed the exponential moment condition **L1** to the optimal second moment condition that $\mathbb{E}[\log^2 N(g_1)] < \infty$.

Similar law of large numbers and central limit theorem have been known for the spectral radius $\rho(G_n)$. Using the Hölder regularity of the stationary measure ν (see [51, 48]), Guivarc'h [48] has established the strong law of large numbers for $\rho(G_n)$: under conditions **L1**, **L2** and **L3**, $\frac{1}{n} \log \rho(G_n) \to \lambda$ almost surely. Recently, under the same conditions, Benoist and Quint [10] established the central limit theorem for $\rho(G_n)$: for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leqslant y\Big) = \Phi(y),$$

Further improvements have been done very recently: Aoun and Sert [3] proved the strong law of large numbers for $\rho(G_n)$ assuming only the second moment condition $\mathbb{E}[\log^2 N(g_1)] < \infty$, while Aoun [2] proved the central limit theorem for $\rho(G_n)$ under the second moment condition, the strong irreducibility condition **L2** and the unboundedness assumption of the semigroup Γ_μ.

Very little has been known about the Berry-Esseen bounds and moderate and large deviations, for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. For Berry-Esseen type bounds, Cuny, Dedecker and Jan [22] (see also Cuny, Dedecker

and Merlevède [24] in a more general setting) have recently established the following result about the rate of convergence in the central limit theorem for $\|G_n\|$: assuming $\mathbb{E}[\log^3 N(g_1)] < \infty$, **L2** and **L3**, we have

$$\sup_{y\in\mathbb{R}}\left|\mathbb{P}\Big(\frac{\log\|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\Big) - \Phi(y)\right| \leqslant \frac{C\sqrt{\log n}}{n^{1/4}}. \tag{6.1.2}$$

However, the rate of convergence in the central limit theorem for the couple $(X_n^x, \log\|G_n\|)$ (cf. (6.1.1)) has not been known in the literature. For the spectral radius $\rho(G_n)$ and the couple $(X_n^x, \log\rho(G_n))$, such type of result has not yet been considered.

Moderate deviations have not yet been studied neither for $\|G_n\|$ nor for $\rho(G_n)$, to the best of our knowledge. For large deviations, the upper tail large deviation principle for $\|G_n\|$ has been established in [78] and [85] under different conditions; it is conjectured in [78] that the usual large deviation principle holds for $\rho(G_n)$.

6.1.2 Objectives

In this paper, we shall establish Berry-Esseen type bounds and moderate deviation results for both the operation norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. Such kinds of results are important in applications because they give the rate of convergence in the central limit theorem and in the law of large numbers.

Our first objective is to establish the following Berry-Esseen type bound concerning the rate of convergence in the central limit theorem (6.1.1): under conditions **L1**, **L2** and **L3**, for any continuous function φ on $\mathbb{P}^{d-1}$, we have,

$$\sup_{x\in\mathbb{P}^{d-1}}\sup_{y\in\mathbb{R}}\left|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C\log n}{\sqrt{n}}. \tag{6.1.3}$$

In particular, with $\varphi = \mathbf{1}$, the bound (6.1.3) clearly improves (6.1.2).

Our second objective is to prove the moderate deviation principle for the couple $(X_n^x, \log\|G_n\|)$: under conditions **L1**, **L2** and **L3**, for any non-negative Hölder continuous function φ on $\mathbb{P}^{d-1}$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}, \end{aligned} \tag{6.1.4}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B. Note that the moderate deviation principle (6.1.4) is proved under the proximality condition **L3**. This condition ensures that the Markov chain (X_n^x) has a unique stationary measure ν on the projective space $\mathbb{P}^{d-1}$. When condition **L3** is replaced by the weaker one that the set $\{|det(g)|^{-1/d}g : g \in \Gamma_\mu\}$ is not contained in a compact subgroup of $\mathbb{G}$, we are still able to prove the following moderate deviation principle for $\|G_n\|$: for any Borel

set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have,

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log\|G_n\| - n\lambda}{b_n} \in B\right) \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log\|G_n\| - n\lambda}{b_n} \in B\right) \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}. \end{aligned} \tag{6.1.5}$$

Our third objective is to establish the moderate deviation expansion in the normal range $[0, o(n^{1/6})]$ for the couples $(X_n^x, \log\|G_n\|)$ with a target function: under conditions **L1**, **L2** and **L3**, for any Hölder continuous function φ on $\mathbb{P}^{d-1}$, we have, uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, o(n^{1/6})]$,

$$\frac{\mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y\}}\right]}{1-\Phi(y)} = \nu(\varphi) + o(1). \tag{6.1.6}$$

The expansion (6.1.6) has not been considered in the literature even when $\varphi = \mathbf{1}$.

All the above mentioned results (6.1.3), (6.1.4), (6.1.5), and (6.1.6) are concerned with the operator norm $\|G_n\|$, but we shall also establish the analog of these results for the spectral radius $\rho(G_n)$.

6.1.3 Proof outline

In the work [22, 24], the proof of (6.1.2) consists of establishing the central limit theorem with rate of convergence in Wasserstein's distance utilizing the martingale approximation method developed in [9]. However, using this approach it is difficult to obtain a better convergence rate than that in (6.1.2). Instead, our proof of (6.1.3) is based on the Berry-Esseen bound for the couple $(X_n^x, \log|G_n x|)$ recently established in [86] and on the following precise comparison between $\|G_n\|$ and $|G_n x|$ established in [10]: for any $a > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geqslant k \geqslant k_0$ and $x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}\left(\left|\log\frac{\|G_n\|}{\|G_k\|} - \log\frac{|G_n x|}{|G_k x|}\right| \leqslant e^{-ak}\right) > 1 - e^{-ck}. \tag{6.1.7}$$

The basic idea to utilize this powerful inequality consists in carefully choosing certain integer k, taking the conditional expectation with respect to the filtration $\sigma(g_1, \dots, g_k)$ and using the large deviation bounds for $\log\|G_k\|$. This technique, in conjugation with limit theorems for the norm cocycle $\log|G_{n-k}x|$, makes it possible to prove corresponding results for $\log\|G_n\|$; see [10] where a local limit theorem for $\log\|G_n\|$ has been established by taking $k = [\log^2 n]$, where $[a]$ denotes the integral part of a. In this paper, the proof of (6.1.3) is carried out by choosing $k = [C_1 \log n]$ with a sufficiently large constant $C_1 > 0$ and by using the Berry-Esseen bound for the couple $(X_n^x, \log|G_n x|)$ with a target function φ on X_n^x. In the same spirit, the moderate deviation principle (6.1.4) for the couple $(X_n^x, \log\|G_n\|)$ is established using the moderate deviation principle for the couple $(X_n^x, \log|G_n x|)$ proved in [86], together with the inequality (6.1.7) with $k = \left[C_1 \frac{b_n^2}{n}\right]$, where the constant C_1 is sufficiently large and the sequence $(b_n)_{n\geqslant 1}$ is given in (6.1.4).

As to the moderate deviation principle (6.1.5) for $\log \|G_n\|$ without assuming the proximality condition **L3**, its proof is more technical and delicate than that of (6.1.4). Indeed, when condition **L3** fails, the transfer operator of the Markov chain $(X_n^x)_{n\geqslant 0}$ has no spectral gap in general and it may happen that $(X_n^x)_{n\geqslant 0}$ possesses several stationary measures on the projective space $\mathbb{P}^{d-1}$. In this case, it becomes hopeless to prove a general form of (6.1.5) when a target function φ on X_n^x is taken into account. Nevertheless, the proof of (6.1.5) can be carried out by following the approach of Bougerol and Lacroix [13] (first announced in [12]), where central limit theorems and exponential large deviation bounds for $\|G_n\|$ and $|G_n x|$ were established without giving the rate function. Specifically, employing this approach consists in finding the proximal dimension p of the semigroup Γ_μ generated by the matrix law μ and then applying Chevaley's algebraic irreducible representation [19] of the exterior powers $\wedge^p\mathbb{R}^d$, to show that the action of the semigroup Γ_μ is strongly irreducible and proximal on $\wedge^p\mathbb{R}^d$. Using this strategy together with (6.1.4) for $\varphi = \mathbf{1}$, we are able to establish (6.1.5) .

For the proof of the Cramér type moderate deviation expansion (6.1.6), when $y \in [0, \frac{1}{2}\sqrt{\log n}]$, we deduce the desired result from the Berry-Esseen type bound (6.1.3); when $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we make use of the moderate deviation expansion for the couple $(X_n^x, \log|G_n x|)$ recently established in [86] and the aforementioned inequality (6.1.7) with $k = [C_1 y^2]$, where $C_1 > 0$ is a sufficiently large constant.

All of the aforementioned results (6.1.3), (6.1.4), (6.1.5) and (6.1.6) for the operator norm $\|G_n\|$ turn out to be essential to establish analogous Berry-Esseen type bounds and moderate deviation results for the spectral radius $\rho(G_n)$. Another important ingredient in our proof is the precise comparison between $\rho(G_n)$ and $\|G_n\|$ established in [10]; see Lemma 6.3.3 below.

6.2 Main results

To formulate our main results, we introduce some notation below. Let $\mathcal{C}(\mathbb{P}^{d-1})$ be the space of continuous complex-valued functions on $\mathbb{P}^{d-1}$ and $\mathbf{1}$ be the constant function with value 1. All over the paper we assume that $\gamma > 0$ is a fixed small enough constant. We equip the projective space $\mathbb{P}^{d-1}$ with the angular distance $\mathbf{d}$ defined by $\mathbf{d}(x, y) = |x \wedge y|$ for $x, y \in \mathbb{P}^{d-1}$, where $x \wedge y$ denotes the exterior product of x and y. Consider the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathbb{P}^{d-1}) : \|\varphi\|_\gamma < +\infty\}$, where

$$\|\varphi\|_\gamma := \|\varphi\|_\infty + \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}^\gamma(x, y)} \quad \text{with} \quad \|\varphi\|_\infty := \sup_{x\in\mathbb{P}^{d-1}} |\varphi(x)|.$$

For any $g \in \mathbb{G}$ and $x \in \mathbb{P}^{d-1}$, we write $g \cdot x := \frac{gx}{|gx|}$ for the projective action of g on the projective space $\mathbb{P}^{d-1}$. Consider the Markov chain

$$X_0^x = x, \qquad X_n^x = G_n \cdot x, \quad n \geqslant 1.$$

Under conditions **L1**, **L2** and **L3**, the chain $(X_n^x)_{n\geqslant 0}$ possesses a unique stationary measure ν on $\mathbb{P}^{d-1}$ such that $\mu * \nu = \nu$ (see [50]), where $\mu * \nu$ denotes the convolution of μ and ν. It is worth mentioning that if the proximality condition **L3** fails, then the stationary measure ν may not be unique (see [13, 10]). It was shown in [10, Proposition

14.17] that under conditions **L1** and **L2**, the asymptotic variance σ^2 of the random walk $(G_n)_{n\geqslant 1}$ can be given by

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[\Big(\log\|G_n\| - n\lambda\Big)^2\Big].$$

Throughout the paper, we denote by Φ the standard normal distribution function on $\mathbb{R}$. We write c, C for positive constants whose values may change from line to line.

6.2.1 Berry-Esseen type bounds

In this section, we present Berry-Esseen type bounds for the couples $(X_n^x, \log\|G_n\|)$ and $(X_n^x, \log\rho(G_n))$ with target functions on the Markov chain $(X_n^x)_{n\geqslant 0}$. Recall that by Gelfand's formula, it holds that $\rho(g) = \lim_{k\to\infty}\|g^k\|^{1/k}$ for any $g \in \mathbb{G}$.

Theorem 6.2.1. *Assume conditions **L1**, **L2** and **L3**. Then there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\Big|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\Big| \leqslant \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma \tag{6.2.1}$$

and

$$\Big|\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] - \nu(\varphi)\Phi(y)\Big| \leqslant \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma. \tag{6.2.2}$$

Using the fact that all matrix norms are equivalent, one can verify that in (6.2.1), the operator norm $\|\cdot\|$ can be replaced by any matrix norm.

In particular, taking $\varphi = \mathbf{1}$ in (6.2.1) and (6.2.2), we have: under conditions **L1**, **L2** and **L3**, for all $n \geqslant 1$ and $y \in \mathbb{R}$,

$$\Big|\mathbb{P}\Big(\frac{\log\|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\Big) - \Phi(y)\Big| \leqslant \frac{C\log n}{\sqrt{n}}, \tag{6.2.3}$$

$$\Big|\mathbb{P}\Big(\frac{\log\rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leqslant y\Big) - \Phi(y)\Big| \leqslant \frac{C\log n}{\sqrt{n}}. \tag{6.2.4}$$

As mentioned before, the Berry-Esseen type bound (6.2.3) improves (6.1.2) established recently by Cuny, Dedecker and Jan [22].

It is natural to make the conjecture that the optimal rate of convergence on the right hand sides of (6.2.1), (6.2.2), (6.2.3) and (6.2.4) should be $\frac{C}{\sqrt{n}}$ instead of $\frac{C\log n}{\sqrt{n}}$. For positive matrices, these optimal bounds have been proved in [89]. However, the proofs of the conjecture for invertible matrices seem to be rather delicate, for which new ideas and techniques are required. Nevertheless, we can prove the optimal bound $\frac{C}{\sqrt{n}}$ for large values of $|y|$, as indicated in the following remark which will be seen in the proof of Theorem 6.2.1.

Remark 6.2.2. Under the same conditions as in Theorem 6.2.1, if we consider $|y| > \sqrt{2\log\log n}$ instead of $y \in \mathbb{R}$, then the bound $\frac{C\log n}{\sqrt{n}}$ in (6.2.1), (6.2.2), (6.2.3) and (6.2.4) can be improved to be $\frac{C}{\sqrt{n}}$.

6.2.2 Moderate deviation principles

We first state moderate deviation principles for the couples $(X_n^x, \log\|G_n\|)$ and $(X_n^x, \log\rho(G_n))$ with target functions on the Markov chain $(X_n^x)_{n\geqslant 0}$.

Theorem 6.2.3. *Assume conditions* ***L1***, ***L2*** *and* ***L3***. *Then, for any non-negative function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have, uniformly in $x \in \mathbb{P}^{d-1}$,*

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}, \end{aligned} \tag{6.2.5}$$

and

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}, \end{aligned} \tag{6.2.6}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

Note that the target function φ in (6.2.5) and (6.2.6) is not necessarily strictly positive, and it can vanish somewhere on the projective space $\mathbb{P}^{d-1}$. The moderate deviation principles (6.2.5) and (6.2.6) are all new, even for $\varphi = \mathbf{1}$.

If we only consider the operator norm $\|G_n\|$ or the spectral radius $\rho(G_n)$, instead of the couples $(X_n^x, \log\|G_n\|)$ and $(X_n^x, \log\rho(G_n))$, we are still able to establish moderate deviation principles without assuming the proximality condition **L3**:

Theorem 6.2.4. *Assume conditions* ***L1***, ***L2*** *and $\sigma^2 > 0$. Then, there exists a constant $\sigma_0 > 0$ such that for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have,*

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma_0^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\left(\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right) \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\left(\frac{\log\|G_n\|-n\lambda}{b_n}\in B\right) \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma_0^2}, \end{aligned} \tag{6.2.7}$$

and

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma_0^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\left(\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right) \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\left(\frac{\log\rho(G_n)-n\lambda}{b_n}\in B\right) \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma_0^2}, \end{aligned} \tag{6.2.8}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

Remark 6.2.5. Assume conditions **L1** and **L2**. Let

$$\Gamma_{\mu,1} = \{|det(g)|^{-1/d}g : g \in \Gamma_\mu\}$$

be the set of elements of Γ_μ normalized to have determinant 1.

1. If $\Gamma_{\mu,1}$ is not contained in a compact subgroup of $\mathbb{G}$, then $\sigma > 0$, as will be seen in the proof of Theorem 6.2.4 .

2. If $\Gamma_{\mu,1}$ is contained in a compact subgroup of $\mathbb{G}$, then $c_1 = \inf\{\|g\| : g \in \Gamma_{\mu,1}\} > 0$ and $c_2 = \{\|g\| : g \in \Gamma_{\mu,1}\} < \infty$, so that

$$c_1^d \, |det(g)| \leqslant \|g\|^d \leqslant c_2^d \, |det(g)| \quad \forall g \in \Gamma_\mu. \tag{6.2.9}$$

   Since $\log |det(G_n)| = \sum_{i=1}^n \log |det(g_i)|$ is a sum of i.i.d. real-valued random variables, from (6.2.9) (applied to $g = G_n$) it follows directly that the moderate deviation principle (6.2.7) holds with $\lambda = \frac{1}{d}\mathbb{E} \log |det(g_1)|$ and $\sigma^2 = \mathbb{E}[(\frac{1}{d} \log |det(g_1)| - \lambda)^2]$ (which coincide with their original definitions), provided that $|det(g_1)|$ is not a.s. a constant (which is equivalent to $\sigma^2 > 0$).

In fact, in the second case of the remark above, $\log \|G_n\|$ can be expressed exactly as a sum of of i.i.d. real-valued random variables when the norm $\|\cdot\|$ is suitably chosen, as observed by Bougerol and Lacroix [13]. See also Lemma 6.4.5 in subsection 6.2.3.

6.2.3 Moderate deviation expansions

In this subsection we formulate the Cramér type moderate deviation expansions in the normal range for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. Our first result concerns the operator norm $\|G_n\|$.

Theorem 6.2.6. *Assume conditions* ***L1***, ***L2*** *and* ***L3***. *Then, we have, uniformly in* $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ *and* $\varphi \in \mathcal{B}_\gamma$, *as* $n \to \infty$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log \|G_n\| - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1 - \Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1), \tag{6.2.10}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log \|G_n\| - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1). \tag{6.2.11}$$

Theorem 6.2.6 is new even when $\varphi = \mathbf{1}$. Its proof is based on the Cramér type moderate deviation expansion for the couple $(X_n^x, \log |G_n x|)$ proved recently in [86], and on a fine comparison between the operator norm $\|G_n\|$ and the vector norm $|G_n x|$ established in [10] (see Lemma 6.3.2 below). Note that Theorem 6.2.6 covers the special case where $\nu(\varphi) = 0$; in this case the exact comparison with the normal tail remains open.

Our second result concerns the moderate deviation expansions for the spectral radius $\rho(G_n)$, also in the normal range.

Theorem 6.2.7. *Assume conditions* ***L1***, ***L2*** *and* ***L3***. *Then, we have, uniformly in* $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ *and* $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log \rho(G_n) - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1 - \Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1), \tag{6.2.12}$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log \rho(G_n) - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1). \tag{6.2.13}$$

Theorem 6.2.7 is new even when $\varphi = \mathbf{1}$. Its proof relies on Theorem 6.2.6 and on an estimate of the difference between spectral radius $\rho(G_n)$ and the operator norm $\|G_n\|$ established in [10] (see Lemma 6.3.3). Like in Theorem 6.2.6, Theorem 6.2.7 also covers the case where $\nu(\varphi) = 0$, for which the exact comparaison with the normal tail is not known.

6.3 Berry-Esseen type bounds

The goal of this section is to prove Theorem 6.2.1 about Berry-Esseen type bounds for the operator norm $\|G_n\|$ and for the spectral radius $\rho(G_n)$. We will need the following Berry-Esseen bound for the norm cocycle $\log|G_n x|$ established in [86].

Lemma 6.3.1. *Assume conditions **L1**, **L2** and **L3**. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left|\mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\right] - \nu(\varphi)\Phi(y)\right| \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma.$$

The following result is an interesting comparison theorem for $\log\|G_n\|$ and $\log|G_n x|$. It shows that the difference between $\log\frac{\|G_n\|}{\|G_k\|}$ and $\log\frac{|G_n x|}{|G_k x|}$ is smaller than any exponential rate e^{-ak} for any $a > 0$ and $k \leqslant n$, with large probability.

Lemma 6.3.2. *Assume conditions **L1**, **L2** and **L3**. Then, for any $a > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$ and $x \in \mathbb{P}^{d-1}$,*

$$\mathbb{P}\left(\left|\log\frac{\|G_n\|}{\|G_k\|} - \log\frac{|G_n x|}{|G_k x|}\right| \leqslant e^{-ak}\right) > 1 - e^{-ck}.$$

Lemma 6.3.2 was established in [10, Lemma 17.8] and has been used to prove the local limit theorem for the operator norm $\|G_n\|$: see [10, Theorem 17.9].

Proof of (6.2.1) *of Theorem 6.2.1.* Without loss of generality, we assume that the target function φ is non-negative. On the one hand, using Lemma 6.3.1 and the fact that $\log\|G_n\| \geqslant \log|G_n x|$, we easily get the following upper bound: there exists a constant $C > 0$ such that for all $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_n := \mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\right] \leqslant \nu(\varphi)\Phi(y) + \frac{C}{\sqrt{n}}\|\varphi\|_\gamma. \tag{6.3.1}$$

On the other hand, applying Lemma 6.3.2, we deduce that for any $a > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$, it holds uniformly in $x \in \mathbb{P}^{d-1}$ and $\varphi \in \mathcal{B}_\gamma$ that

$$\begin{aligned} I_n &\geqslant \mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\| - n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\mathbb{1}_{\left\{\left|\log\|G_n\| - \log\frac{|G_n x|}{|G_k x|} - \log\|G_k\|\right| \leqslant e^{-ak}\right\}}\right] \\ &\geqslant \mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - \log|G_k x| + \log\|G_k\| - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leqslant y\right\}}\right] - e^{-ck}\|\varphi\|_\infty. \end{aligned} \tag{6.3.2}$$

For simplicity, for any $n > k \geqslant 1$, we write $G_n = G_{n,k} G_k$ with

$$G_{n,k} = g_n \dots g_{k+1}, \qquad G_k = g_k \dots g_1.$$

From the large deviation bounds for $\log \|G_k\|$ (see [10] or [85]), we have that for any $q > \lambda$, there exists a constant $c > 0$ such that for sufficiently large $k \geqslant 1$,

$$\mathbb{P}(\log \|G_k\| > kq) \leqslant e^{-ck}. \tag{6.3.3}$$

Denote the σ-algebra $\mathcal{F}_k = \sigma(g_1, \dots, g_k)$. From (6.3.2), taking the conditional expectation with respect to the filtration $\mathcal{F}_k$, we derive that for any $q > \lambda$,

$$\begin{aligned}
I_n &\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - \log|G_k x| + \log\|G_k\| - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leqslant y\right\}}\Big|\mathcal{F}_k\Big]\right\} - e^{-ck}\|\varphi\|_\infty \\
&\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - \log|G_k x| + \log\|G_k\| - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leqslant y\right\}}\mathbb{1}_{\{\log\|G_k\| \leqslant kq\}}\Big|\mathcal{F}_k\Big]\right\} \\
&\quad - e^{-ck}\|\varphi\|_\infty \\
&\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x| - \log|G_k x| + kq - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leqslant y\right\}}\Big|\mathcal{F}_k\Big]\right\} - 2e^{-c_1 k}\|\varphi\|_\infty,
\end{aligned}$$

where in the last step we use the large deviation bound (6.3.3) and the constant $c_1 > 0$ is taken to be small enough. Note that $X_n^x = G_n \cdot x = G_{n,k} \cdot X_k^x$ and $\log|G_n x| - \log|G_k x| = \log|G_{n,k} X_k^x|$. It follows that

$$I_n \geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(G_{n,k} \cdot X_k^x)\mathbb{1}_{\left\{\frac{\log|G_{n,k} X_k^x| + kq - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leqslant y\right\}}\Big|\mathcal{F}_k\Big]\right\} - 2e^{-c_1 k}\|\varphi\|_\infty.$$

Since it is shown in Lemma 6.3.1 that the Berry-Esseen bound for the norm cocycle $\log|G_n x|$ holds uniformly in $x \in \mathbb{P}^{d-1}$, we obtain

$$I_n \geqslant \nu(\varphi)\Phi(y_1) - \frac{C}{\sqrt{n-k}}\|\varphi\|_\gamma - 2e^{-c_1 k}\|\varphi\|_\infty,$$

where

$$y_1 = \frac{\sqrt{n}}{\sqrt{n-k}} y - \frac{k(q-\lambda) + e^{-ak}}{\sigma\sqrt{n-k}}.$$

Taking $k = [C_1 \log n]$ with $C_1 = \frac{1}{2c_1}$, we get that there exists a constant $C > 0$ such that $\frac{1}{\sqrt{n-k}} \leqslant \frac{C}{\sqrt{n}}$ and $e^{-c_1 k} \leqslant \frac{C}{\sqrt{n}}$. Since Φ is the standard normal distribution function, we write

$$\Phi(y_1) = \Phi(y) - \frac{1}{\sqrt{2\pi}} \int_{y_1}^{y} e^{-\frac{t^2}{2}} dt.$$

To estimate the above integral, by elementary calculations, there exists a constant $C_2 > 0$ such that

$$|y - y_1| \leqslant C_2\Big(\frac{\log n}{n}|y| + \frac{\log n}{\sqrt{n}}\Big),$$

and for $n > k_0$ large enough,

$$\begin{aligned} e^{-\frac{y_1^2}{2}} &\leqslant \exp\Big\{ -\frac{1}{2}\frac{n}{n-k}y^2 + \sqrt{n}\frac{k(q-\lambda)+e^{-ak}}{n-k}y\Big\} \\ &\leqslant \exp\Big\{ -\frac{1}{2}y^2 + C_2\frac{\log n}{\sqrt{n}}|y|\Big\}. \end{aligned}$$

Thus, it follows that there exists a constant $C > 0$ such that

$$\begin{aligned} \Big|\int_{y_1}^{y} e^{-\frac{t^2}{2}}dt\Big| &\leqslant |y-y_1|\max\Big(e^{-\frac{y^2}{2}}, e^{-\frac{y_1^2}{2}}\Big) \\ &\leqslant C_2\Big(\frac{\log n}{n}|y| + \frac{\log n}{\sqrt{n}}\Big)\exp\Big\{ -\frac{1}{2}y^2 + C_2\frac{\log n}{\sqrt{n}}|y|\Big\} \\ &\leqslant \begin{cases} C\frac{\log n}{\sqrt{n}} & \forall y \in \mathbb{R}, \\ C\frac{1}{\sqrt{n}} & \text{if } |y| > \sqrt{2\log\log n}. \end{cases} \end{aligned}$$

Consequently, we get the following lower bound for I_n: there exists a constant $C > 0$ such that for all $x \in \mathbb{P}^{d-1}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_n \geqslant \begin{cases} \nu(\varphi)\Phi(y) - \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma & \forall y \in \mathbb{R}, \\ \nu(\varphi)\Phi(y) - \frac{C}{\sqrt{n}}\|\varphi\|_\gamma & \text{if } |y| > \sqrt{2\log\log n}. \end{cases}$$

Together with the upper bound (6.3.1), this concludes the proof of (6.2.1) of Theorem 6.2.1 and the corresponding results in Remark 6.2.2. □

We now proceed to prove the Berry-Esseen type bound (6.2.2) of Theorem 6.2.1 for the spectral radius $\rho(G_n)$. The proof relies on the following comparison lemma between the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$, established in [10, Lemma 14.13] using the Hölder regularity properties of the stationary measure ν. It shows that $\rho(G_n)$ behaves like $\|G_n\|$ up to a factor lying between 1 and $e^{-\varepsilon k}$ for any $\varepsilon > 0$ and $k \leqslant n$, with high probability. Notice that $\rho(G_n) \leqslant \|G_n\|$ by Gelfand's formula and the submultiplicity of the operator norm.

Lemma 6.3.3. *Assume conditions **L1** and **L2**. Then, for any $\varepsilon > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$,*

$$\mathbb{P}\Big(1 \geqslant \frac{\rho(G_n)}{\|G_n\|} > e^{-\varepsilon k}\Big) \geqslant 1 - e^{-ck}.$$

Proof of (6.2.2) of Theorem 6.2.1. Without loss of generality, we assume that the target function φ is non-negative.

The lower bound is a direct consequence of (6.2.1) together with Remark 6.2.2 on it, and the inequality $\log\rho(G_n) \leqslant \log\|G_n\|$, from which we get that, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} I_n &:= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{\sigma\sqrt{n}} \leqslant y\right\}}\Big] \\ &\geqslant \begin{cases} \nu(\varphi)\Phi(y) - \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma & \forall y \in \mathbb{R}, \\ \nu(\varphi)\Phi(y) - \frac{C}{\sqrt{n}}\|\varphi\|_\gamma & \text{if } |y| > \sqrt{2\log\log n}. \end{cases} \end{aligned}$$

The upper bound is a consequence of (6.2.1) together with Remark 6.2.2 on it and Lemma 6.3.3. In fact, applying Lemma 6.3.3, we deduce that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$,

$$\begin{aligned} I_n &\leqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\mathbb{1}_{\left\{\log\rho(G_n)-\log\|G_n\|>-\varepsilon k\right\}}\Big] + e^{-c_1 k}\|\varphi\|_\infty \\ &\leqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-\varepsilon k-n\lambda}{\sigma\sqrt{n}}\leqslant y\right\}}\Big] + e^{-c_1 k}\|\varphi\|_\infty. \end{aligned}$$

Taking $k = [C_1 \log n]$ with $C_1 = \frac{1}{2c_1}$, we have $e^{-c_1 k} \leqslant \frac{C}{\sqrt{n}}$ for some constant $C > 0$. Using the bound (6.2.1) with y replaced by $y_1 := y + \frac{\varepsilon k}{\sigma\sqrt{n}}$, we obtain the following upper bound for I_n: there exists a constant $C > 0$ such that for all $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$, $\varphi \in \mathcal{B}_\gamma$, and $n \geqslant k_0$ with k_0 large enough,

$$I_n \leqslant \begin{cases} \nu(\varphi)\Phi(y_1) + \frac{C\log n}{\sqrt{n}}\|\varphi\|_\gamma & \forall y \in \mathbb{R}, \\ \nu(\varphi)\Phi(y_1) + \frac{C}{\sqrt{n}}\|\varphi\|_\gamma & \text{if } |y| > \sqrt{2\log\log n}. \end{cases}$$

(Notice that $|y| > \sqrt{2\log\log n}$ implies $|y_1| > \sqrt{2\log\log n}$ for n large enough.) By an argument similar to that used in the proof of (6.2.1), it can be seen that

$$\Phi(y_1) \leqslant \begin{cases} \Phi(y) + \frac{C\log n}{\sqrt{n}} & \forall y \in \mathbb{R}, \\ \Phi(y) + \frac{C}{\sqrt{n}} & \text{if } |y| > \sqrt{2\log\log n}. \end{cases}$$

This concludes the proof of (6.2.2) and Remark 6.2.2 on it. □

6.4 moderate deviation principles

The goal of this section is to establish Theorems 6.2.3 and 6.2.4 about moderate deviation principles for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. Notice that in the first theorem, we need the proximality condition, while in the second, we do not need it.

6.4.1 Proof of Theorem 6.2.3

We shall make use of the following moderate deviation principle for the couple $(X_n^x, \log|G_n x|)$, which is a direct consequence of Cramér type moderate deviation expansion for $(X_n^x, \log|G_n x|)$ recently established in [86].

Lemma 6.4.1. *Assume conditions **L1**, **L2** and **L3**. Then, for any non-negative function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{\sqrt{n}} \to \infty$ and $\frac{b_n}{n} \to 0$, we have, uniformly in $x \in \mathbb{P}^{d-1}$,*

$$\begin{aligned} -\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{b_n}\in B\right\}}\Big] \\ &\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{b_n}\in B\right\}}\Big] \leqslant -\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2}, \end{aligned}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

Proof of (6.2.5) *of Theorem 6.2.3.* Since the rate function $I(y) := \frac{y^2}{2\sigma^2}$, $y \in \mathbb{R}$, is strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty,0]$ with $I(0)=0$, by Lemma 4.4 of [58], it suffices to prove the following moderate deviation asymptotics: for any $y>0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\geqslant y\right\}}\Big] = -\frac{y^2}{2\sigma^2}, \tag{6.4.1}$$

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\leqslant -y\right\}}\Big] = -\frac{y^2}{2\sigma^2}. \tag{6.4.2}$$

We first prove (6.4.1) using the moderate deviation principle for the norm cocycle $\log|G_n x|$ stated in Lemma 6.4.1.

For the lower bound, by Lemma 6.4.1 and the fact that $\log|G_n x| \leqslant \log\|G_n\|$, we easily get: for any $y>0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\geqslant y\right\}}\Big] \geqslant -\frac{y^2}{2\sigma^2}.$$

We now prove the upper bound. Denote by $(e_i)_{1\leqslant i\leqslant d}$ the standard orthonormal basis of $\mathbb{R}^d$. Since all matrix norms in $\mathbb{R}^d$ are equivalent, and both $g \mapsto \|g\|$ and $g \mapsto \max_{1\leqslant i\leqslant d}|ge_i|$ are matrix norms, there exists a positive constant c_1 such that $\log\|G_n\| \leqslant \max_{1\leqslant i\leqslant d}\log|G_n e_i| + c_1$. From this inequality, we derive that

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\geqslant y\right\}}\Big] \leqslant \sum_{i=1}^d \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n e_i|-n\lambda+c_1}{b_n}\geqslant y\right\}}\Big].$$

Since $b_n \to \infty$ as $n\to\infty$, we have that for any $\varepsilon>0$, it holds that $\frac{c_1}{b_n} < \varepsilon$ for large enough n. Thus using Lemma 6.4.1, we obtain for any $y>0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n e_i|-n\lambda+c_1}{b_n}\geqslant y\right\}}\Big] \\
&\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n e_i|-n\lambda}{b_n}\geqslant y-\varepsilon\right\}}\Big] = -\frac{(y-\varepsilon)^2}{2\sigma^2}.
\end{aligned}$$

Since $\varepsilon>0$ can be arbitrary small, we get the desired upper bound. This concludes the proof of (6.4.1).

We next prove (6.4.2). The upper bound is easy: using Lemma 6.4.1 and the fact that $\log|G_n x| \leqslant \log\|G_n\|$, we get that for any $y>0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\leqslant -y\right\}}\Big] \leqslant -\frac{y^2}{2\sigma^2}. \tag{6.4.3}$$

We now come to the proof of the lower bound using Lemma 6.3.2. For any $n \geqslant k$, consider the event

$$A_{n,k} = \left\{\left|\log\|G_n\| - \log\frac{|G_n x|}{|G_k x|} - \log\|G_k\|\right| \leqslant e^{-ak}\right\},$$

and denote by $A_{n,k}^c$ its complement. From Lemma 6.3.2, we know that for any $a > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$ and $x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}(A_{n,k}^c) \leqslant e^{-c_1 k}. \tag{6.4.4}$$

Using (6.4.4), we see that

$$\begin{aligned}
I_n &:= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\leqslant -y\right\}}\Big] \\
&\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\leqslant -y\right\}}\mathbb{1}_{\{A_{n,k}\}}\Big] \\
&\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-\log|G_k x|+\log\|G_k\|-n\lambda+e^{-ak}}{b_n}\leqslant -y\right\}}\mathbb{1}_{\{A_{n,k}\}}\Big] \\
&\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-\log|G_k x|+\log\|G_k\|-n\lambda+e^{-ak}}{b_n}\leqslant -y\right\}}\Big] - e^{-c_1 k}\|\varphi\|_\infty.
\end{aligned} \tag{6.4.5}$$

As in the proof of (6.2.1), for any $n \geqslant k \geqslant k_0$, we write $G_n = G_{n,k}G_k$ with

$$G_{n,k} = g_n \dots g_{k+1}, \qquad G_k = g_k \dots g_1.$$

Taking the conditional expectation with respect to the filtration $\mathcal{F}_k = \sigma(g_1, \dots, g_k)$, and using the large deviation bound (6.3.3) for the operator norm $\|G_k\|$, we derive that for any $q > \lambda$, there exists a constant $c_2 > 0$ such that

$$\begin{aligned}
I_n &\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-\log|G_k x|+\log\|G_k\|-n\lambda+e^{-ak}}{b_n}\leqslant -y\right\}}\Big|\mathcal{F}_k\Big]\right\} - e^{-c_1 k}\|\varphi\|_\infty \\
&\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-\log|G_k x|+\log\|G_k\|-n\lambda+e^{-ak}}{b_n}\leqslant -y\right\}}\mathbb{1}_{\{\log\|G_k\|\leqslant kq\}}\Big|\mathcal{F}_k\Big]\right\} \\
&\qquad - e^{-c_1 k}\|\varphi\|_\infty \\
&\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-\log|G_k x|+kq-n\lambda+e^{-ak}}{b_n}\leqslant -y\right\}}\Big|\mathcal{F}_k\Big]\right\} - 2e^{-c_2 k}\|\varphi\|_\infty \\
&=: J_n - 2e^{-c_2 k}\|\varphi\|_\infty.
\end{aligned} \tag{6.4.6}$$

According to Lemma 6.4.1, for any $y > 0$ and $\epsilon > 0$, we have that for sufficiently large n, uniformly in $x \in \mathbb{P}^{d-1}$,

$$e^{-\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)} \leqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|G_n x|-n\lambda}{b_n}\leqslant -y\right\}}\Big] \leqslant e^{-\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma^2}-\epsilon\right)}. \tag{6.4.7}$$

Note that $X_n^x = G_{n,k} \cdot X_k^x$ and $\log|G_n x| - \log|G_k x| = \log|G_{n,k}X_k^x|$. In the sequel, we take

$$k = \left[C_1 \frac{b_n^2}{n}\right], \tag{6.4.8}$$

where $C_1 > 0$ is a constant whose value will be chosen sufficiently large. If we denote

$$b_n' = b_n + \frac{k(q-\lambda) + e^{-ak}}{y},$$

then the term J_n defined in (6.4.6) can be rewritten as

$$J_n = \mathbb{E}\left\{\mathbb{E}\Big[\varphi(G_{n,k}\cdot X_k^x)\mathbb{1}_{\left\{\frac{\log|G_{n,k}X_k^x|-(n-k)\lambda}{b'_n}\leqslant -y\right\}}\Big|\mathcal{F}_k\Big]\right\}.$$

Note that the sequence $(b'_n)_{n\geqslant 1}$ satisfies $\frac{b'_n}{\sqrt{n}}\to\infty$ and $\frac{b'_n}{n}\to 0$, as $n\to\infty$. Applying the moderate deviation bound (6.4.7) with n replaced by $n-k$, and with b_n replaced by b'_n, we obtain the following upper and lower bounds for J_n: with fixed $y>0$ and $\epsilon>0$, for n large enough, uniformly in $x\in\mathbb{P}^{d-1}$,

$$e^{-\frac{(b'_n)^2}{n-k}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)}\leqslant J_n\leqslant e^{-\frac{(b'_n)^2}{n-k}\left(\frac{y^2}{2\sigma^2}-\epsilon\right)}. \tag{6.4.9}$$

From (6.4.6) and (6.4.9), it follows that, there exists a constant $c_3>0$ such that

$$\begin{aligned}\liminf_{n\to\infty}\frac{n}{b_n^2}\log I_n &\geqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\left[e^{-\frac{(b'_n)^2}{n-k}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)}-2e^{-c_2k}\right]\\ &\geqslant \liminf_{n\to\infty}\frac{n}{b_n^2}\log\left[e^{-\frac{(b'_n)^2}{n-k}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)}(1-2e^{-c_3k})\right],\end{aligned}$$

where the last inequality holds due to the fact that as $n\to\infty$,

$$\frac{(b'_n)^2}{k(n-k)}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)\to\frac{1}{C_1}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)<c_2$$

by choosing $C_1>\left(\frac{y^2}{2\sigma^2}+\epsilon\right)/c_2$. Recalling that $k=\left[C_1\frac{b_n^2}{n}\right]\to\infty$, by elementary calculations, we get $\lim_{n\to\infty}\frac{n}{b_n^2}\log(1-2e^{-c_3k})=0$ and

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{b_n^2}\log e^{-\frac{(b'_n)^2}{n-k}\left(\frac{y^2}{2\sigma^2}+\epsilon\right)} &= -\lim_{n\to\infty}\frac{n}{n-k}\left[1+\frac{k(q-\lambda)+e^{-ak}}{yb_n}\right]^2\left(\frac{y^2}{2\sigma^2}+\epsilon\right)\\ &= -\left(\frac{y^2}{2\sigma^2}+\epsilon\right).\end{aligned}$$

Taking $\epsilon\to 0$, we get the desired lower bound: for any $y>0$, uniformly in $x\in\mathbb{P}^{d-1}$,

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\leqslant -y\right\}}\Big]\geqslant -\frac{y^2}{2\sigma^2}.$$

Combining this with the upper bound (6.4.3), we obtain (6.4.2) and thus conclude the proof of (6.2.5). □

Using the moderate deviation principle (6.2.5) for the couple $(X_n^x,\log\|G_n\|)$ and Lemma 6.3.3, we are now in a position to establish the moderate deviation principle (6.2.6) for the couple $(X_n^x,\log\rho(G_n))$.

Proof of (6.2.6) *of Theorem 6.2.3.* As explained in the proof of (6.2.5), according to Lemma 4.4 of [58], it is sufficient to prove the following moderate deviation asymptotics: for any $y>0$, uniformly in $x\in\mathbb{P}^{d-1}$,

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\geqslant y\right\}}\Big]=-\frac{y^2}{2\sigma^2}, \tag{6.4.10}$$

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\leqslant -y\right\}}\Big]=-\frac{y^2}{2\sigma^2}. \tag{6.4.11}$$

We first prove (6.4.10). On the one hand, since the function φ is non-negative, using the moderate deviation principle (6.2.5) for the operator norm $\|G_n\|$ and the fact that $\rho(G_n) \leqslant \|G_n\|$, we immediately get the upper bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\geqslant y\right\}}\Big] \leqslant -\frac{y^2}{2\sigma^2}. \tag{6.4.12}$$

On the other hand, using Lemma 6.3.3, we obtain that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$,

$$\begin{aligned} I_n :&= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\geqslant y\right\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\geqslant y\right\}}\mathbb{1}_{\{\log\rho(G_n)-\log\|G_n\|\geqslant-\varepsilon k\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant yb_n+\varepsilon k\}}\mathbb{1}_{\{\log\rho(G_n)-\log\|G_n\|\geqslant-\varepsilon k\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant yb_n+\varepsilon k\}}\Big] - e^{-c_1 k}\|\varphi\|_\infty. \end{aligned} \tag{6.4.13}$$

As in the proof of (6.2.5) of Theorem 6.2.3, we take

$$k = \Big[C_1\frac{b_n^2}{n}\Big], \tag{6.4.14}$$

where $C_1 > 0$ is a constant whose value will be chosen sufficiently large. By the moderate deviation principle (6.2.5) for the couple $(X_n^x, \log\|G_n\|)$, it follows that for any $y > 0$ and $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\geqslant y\right\}}\Big] \geqslant e^{-\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma^2}+\eta\right)}. \tag{6.4.15}$$

Set

$$b_n' = b_n + \frac{\varepsilon k}{y}.$$

It is easy to see that the sequence $(b_n')_{n\geqslant1}$ satisfies $\frac{b_n'}{\sqrt{n}} \to \infty$ and $\frac{b_n'}{n} \to 0$, as $n \to \infty$. Using the bound (6.4.15), we get that uniformly in $x \in \mathbb{P}^{d-1}$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\geqslant yb_n+\varepsilon k\}}\Big] \geqslant e^{-\frac{(b_n')^2}{n}\left(\frac{y^2}{2\sigma^2}+\eta\right)}.$$

Substituting this bound into (6.4.13), we obtain

$$I_n \geqslant e^{-\frac{(b_n')^2}{n}\left(\frac{y^2}{2\sigma^2}+\eta\right)}\Big[1 - e^{-c_1 k+\frac{(b_n')^2}{n}\left(\frac{y^2}{2\sigma^2}+\eta\right)}\|\varphi\|_\infty\Big].$$

Taking into account of (6.4.14), by elementary calculations, choosing $C_1 > \frac{1}{c_1}\left(\frac{y^2}{2\sigma^2}+\eta\right)$, we have, as $n \to \infty$,

$$\frac{(b_n')^2}{kn}\Big(\frac{y^2}{2\sigma^2}+\eta\Big) \to \frac{1}{C_1}\Big(\frac{y^2}{2\sigma^2}+\eta\Big) < c_1.$$

Hence we get for some constant $c_2 > 0$,

$$I_n \geqslant e^{-\frac{(b_n')^2}{n}\left(\frac{y^2}{2\sigma^2}+\eta\right)}\Big[1 - e^{-c_2 k}\|\varphi\|_\infty\Big].$$

Therefore, using $k = \left[C_1 \frac{b_n^2}{n}\right] \to \infty$, we obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{n}{b_n^2}\log I_n &\geqslant \lim_{n\to\infty}\frac{n}{b_n^2}\Big[-\frac{(b_n')^2}{n}\Big(\frac{y^2}{2\sigma^2}+\eta\Big)\Big] + \lim_{n\to\infty}\frac{n}{b_n^2}\log(1-e^{-c_2k}\|\varphi\|_\infty)\\ &= \lim_{n\to\infty}\Big[-\Big(1+\frac{\varepsilon k}{yb_n}\Big)^2\Big(\frac{y^2}{2\sigma^2}+\eta\Big)\Big] + 0\\ &= -\Big(\frac{y^2}{2\sigma^2}+\eta\Big).\end{aligned}$$

Taking $\eta \to 0$, we obtain the desired lower bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\geqslant y\right\}}\Big] \geqslant -\frac{y^2}{2\sigma^2}.$$

Together with the upper bound (6.4.12), this concludes the proof of (6.4.10).

We next prove (6.4.11). Using (6.4.2) and the fact that $\rho(G_n) \leqslant \|G_n\|$, we easily get the desired lower bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\log \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\leqslant -y\right\}}\Big] \geqslant -\frac{y^2}{2\sigma^2}. \tag{6.4.16}$$

For the upper bound, we still choose k as before:

$$k = \Big[C_2\frac{b_n^2}{n}\Big], \tag{6.4.17}$$

where $C_2 > 0$ is a constant whose value will be chosen sufficiently large. By Lemma 6.3.3, we see that for any $\varepsilon > 0$, there exist $c_3 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geqslant k \geqslant k_0$,

$$\begin{aligned}J_n :&= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\leqslant -y\right\}}\Big]\\ &= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\leqslant -y\right\}}\mathbb{1}_{\left\{\log\rho(G_n)-\log\|G_n\|\geqslant -\varepsilon k\right\}}\Big]\\ &\quad + \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\rho(G_n)-n\lambda}{b_n}\leqslant -y\right\}}\mathbb{1}_{\left\{\log\rho(G_n)-\log\|G_n\|< -\varepsilon k\right\}}\Big]\\ &\leqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log\|G_n\|-n\lambda\leqslant -yb_n+\varepsilon k\right\}}\Big] + e^{-c_3k}\|\varphi\|_\infty.\end{aligned}$$

By the moderate deviation principle (6.4.2) for $\|G_n\|$, we have that for any $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log\|G_n\|-n\lambda}{b_n}\leqslant -y\right\}}\Big] \leqslant e^{-\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma^2}-\eta\right)}. \tag{6.4.18}$$

Let $b'_n = b_n - \frac{\varepsilon k}{y}$. In view of (6.4.17), we see that $\frac{b'_n}{\sqrt{n}} \to \infty$ and $\frac{b'_n}{n} \to 0$, as $n \to \infty$. From (6.4.18), it follows that uniformly in $x \in \mathbb{P}^{d-1}$,

$$J_n \leqslant e^{-\frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2}-\eta\right)} + e^{-c_3 k}\|\varphi\|_\infty. \tag{6.4.19}$$

Since $b'_n = b_n - \frac{\varepsilon k}{y}$ and $k = \left[C_1 \frac{b_n^2}{n}\right]$, it holds that as $n \to \infty$, $\frac{b'_n}{b_n} \to 1$ and, choosing $C_1 > \frac{1}{c_3}\left(\frac{y^2}{2\sigma^2} - \eta\right)$, we have,

$$\frac{(b'_n)^2}{kn}\left(\frac{y^2}{2\sigma^2} - \eta\right) \to \frac{1}{C_1}\left(\frac{y^2}{2\sigma^2} - \eta\right) < c_3.$$

Thus,

$$\begin{aligned}\limsup_{n\to\infty} \frac{n}{b_n^2} \log J_n &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log e^{-\frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2}-\eta\right)} \\ &= -\lim_{n\to\infty} \left(\frac{b'_n}{b_n}\right)^2 \left(\frac{y^2}{2\sigma^2} - \eta\right)\Big] = -\left(\frac{y^2}{2\sigma^2} - \eta\right).\end{aligned}$$

Since $\eta > 0$ is arbitrary, we get the desired upper bound for J_n:

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log J_n \leqslant -\frac{y^2}{2\sigma^2}.$$

Combining this with the lower bound (6.4.16), we conclude the proof of (6.4.11).

Combining (6.4.10) and (6.4.11), we obtain the moderate deviation principle (6.2.6), as mentioned in the beginning of the proof. □

6.4.2 Proof of Theorem 6.2.4

We now come to the proof of Theorem 6.2.4 without assuming the proximality condition **L3**. The proof is based on Theorem 6.2.3 applied to $\| \wedge^p G_n\|$ which will be introduced below and which satisfies the proximality condition **L3**, using the p-th exterior power representation approach developed in [13]. In [13, Theorem V. 6.2], this approach is used to establish large deviation bounds for the vector norm $|G_n x|$ and the operator norm $\|G_n\|$; it allows to relax the proximality condition **L3**, but fails to give the rate function. For moderate deviations, the situation is different: with this approach we are able to get the rate function explicitely.

In order to prove Theorem 6.2.4, we need to introduce some additional notation. For any integer $1 \leqslant p \leqslant d$, the p-th exterior power $\wedge^p(\mathbb{R}^d)$ is the $\binom{d}{p}$-dimensional vector space with basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}, 1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant d\},$$

where $(e_i)_{1\leqslant i\leqslant d}$ is the standard orthonormal basis of $\mathbb{R}^d$; it is endowed with the standard norm still denoted by $|\cdot|$ as in the case of $\mathbb{R}^d$ (there should be no confusion in the context). For any $v_1, \ldots, v_p \in \mathbb{R}^d$, the vector $v_1 \wedge \cdots \wedge v_p$ is nonzero if and only if $v_1, \ldots, v_p$ are linearly independent in $\mathbb{R}^d$. We write $\wedge^p g$ for the image of $g \in GL_d(\mathbb{R})$

under the representation $\wedge^p(\mathbb{R}^d)$; for any $v_1, \ldots, v_p \in \mathbb{R}^d$, the action of the matrix $\wedge^p g$ on the vector $v_1 \wedge \cdots \wedge v_p$ is given by

$$\wedge^p g(v_1 \wedge \cdots \wedge v_p) = gv_1 \wedge \cdots \wedge gv_p.$$

The associated operator norm of $\wedge^p g$ is defined by

$$\| \wedge^p g\| = \sup\{|(\wedge^p g)v| : v \in \wedge^p(\mathbb{R}^d), |v| = 1\}.$$

Since $\wedge^p(gg') = (\wedge^p g)(\wedge^p g')$ for any $g, g' \in GL_d(\mathbb{R})$, the submultiplicative property holds: $\| \wedge^p (gg')\| \leqslant \| \wedge^p g\|\| \wedge^p g'\|$. If the singular values of a matrix $g \in GL_d(\mathbb{R})$ is given by $a_{11}, \ldots, a_{dd}$ (arranged in decreasing order), then it holds that

$$\| \wedge^p g\| = a_{11} \ldots a_{pp}. \tag{6.4.20}$$

As a consequence, we have $\| \wedge^p g\| \leqslant \|g\|^p$ and $\| \wedge^p g\|\| \wedge^{p+2} g\| \leqslant \| \wedge^{p+1} g\|^2$.

Let V be a subspace of $\wedge^p(\mathbb{R}^d)$. A set $S \subset \wedge^p(\mathbb{G}) := \{\wedge^p g : g \in \mathbb{G}\}$ is said to be *irreducible* on V if there is no proper subspace V_1 of V such that $gV_1 = V_1$ for all $g \in S$. A set $S \subset \wedge^p(\mathbb{G})$ is said to be *strongly irreducible* on V if there are no finite number of subspaces $V_1, \ldots, V_m$ of V such that $g(V_1 \cup \ldots \cup V_m) = V_1 \cup \ldots \cup V_m$ for all $g \in S$. The strong irreducibility condition **L2** means that Γ_μ is strongly irreducible on $\mathbb{R}^d$, or, equivalently, G_μ is strongly irreducible on $\mathbb{R}^d$. We refer to [13] for more details.

The following purely algebraic result is due to Chevalley [19]; see also Bougerol and Lacroix [13].

Lemma 6.4.2. *Let G be an irreducible subgroup of $GL_d(\mathbb{R})$. Then, for any integer $1 \leqslant p \leqslant d$, there exists a direct-sum decomposition of the p-th exterior power: $\wedge^p(\mathbb{R}^d) = V_1 \oplus \ldots \oplus V_k$ such that $(\wedge^p g)V_j = V_j$ for any $g \in G$ and $1 \leqslant j \leqslant k$. Moreover, $\wedge^p(G) := \{\wedge^p g : g \in G\}$ is irreducible on each subspace $V_j, j = 1, \cdots, k$.*

We say that an integer $1 \leqslant p \leqslant d$ is the *proximal dimension* of the semigroup Γ_μ, if p is the smallest integer with the following property: there exists a sequence of matrices $\{M_n\}_{n\geqslant 1} \subset \Gamma_\mu$ such that $\frac{M_n}{\|M_n\|}$ converges to a matrix with rank p. By definition, it is easy to verify that the proximality condition **L3** implies that the proximal dimension of Γ_μ is 1. The converse is also true if we assume that Γ_μ is irreducible, see [10] for the proof. Under the first moment condition $\mathbb{E}(\log N(g_1)) < \infty$, according to Kingman's subadditive ergodic theorem [67], the Lyapunov exponents $(\lambda_p)_{1\leqslant p\leqslant d}$ of μ are defined recursively by

$$\lambda_1 + \ldots + \lambda_p = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log \| \wedge^p G_n\|) = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^p G_n\|, \quad a.s..$$

This formula, together with the fact that $\| \wedge^{p-1} G_n\|\| \wedge^{p+1} G_n\| \leqslant \| \wedge^p G_n\|^2$, yields that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d$. The following fundamental result is due to Guivarc'h and Raugi [51] and gives a sufficient condition for ensuring that two successive Lyapunov exponents are distinct. It can also be found in [13, Proposition III. 6.2].

Lemma 6.4.3. *Assume condition **L2**. If $\mathbb{E} \log N(g_1) < \infty$ and the proximal dimension of the semigroup Γ_μ is p, then we have*

$$\lambda_1 = \lambda_2 = \ldots = \lambda_p > \lambda_{p+1}.$$

From (6.4.20) we have seen that $\| \wedge^p g\| \leqslant \|g\|^p$ for $1 \leqslant p \leqslant d$. The following result (see [13, Lemma III. 1.4]) provides a two-sided comparison between $\| \wedge^p g\|$ and $\|g\|^p$, where p is the proximal dimension of the semigroup Γ_μ.

Lemma 6.4.4. *Assume condition **L2**. If $\mathbb{E} \log N(g_1) < \infty$ and the proximal dimension of the semigroup Γ_μ is p, then there exists a constant $c > 0$ such that for all $g \in \Gamma_\mu$,*

$$c\|g\|^p \leqslant \| \wedge^p g\| \leqslant \|g\|^p.$$

The following lemma was proved in [13, Proposition III. 1.7 and Remark III. 1.8]. Recall that $\Gamma_{\mu,1} = \{|det(g)|^{-1/d} g : g \in \Gamma_\mu\}$.

Lemma 6.4.5. *(a) If the set $\Gamma_{\mu,1}$ is not contained in a compact subgroup of $\mathbb{G}$, then the proximal dimension p of the semigroup Γ_μ satisfies $1 \leqslant p \leqslant d-1$.*

(b) If the set $\Gamma_{\mu,1}$ is contained in a compact subgroup of $\mathbb{G}$, then there exists a scalar product on $\mathbb{R}^d$ for which all the matrices in $\Gamma_{\mu,1}$ are orthogonal. In this case, $\log \|G_n\|$ can be written as a sum of i.i.d. real-valued random variables.

Now we are equipped to prove the moderate deviation principle (6.2.7) for the operator norm $\|G_n\|$ without assuming the proximality condition **L3**.

Proof of (6.2.7) of Theorem 6.2.4. We assume that $\Gamma_{\mu,1}$ is not contained in a compact subgroup of $\mathbb{G}$; the opposite case was already proved in Remark 6.2.5(2). Note that $\lambda = \lambda_1$. Without loss of generality, we assume that $\lambda_1 = 0$ since otherwise we can replace each matrix $g \in \Gamma_\mu$ by $e^{-\lambda_1} g$. As mentioned before, by Lemma 4.4 of [58], in order to prove (6.2.7), it is sufficient to establish the following moderate deviation asymptotics: for any $y > 0$,

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \|G_n\|}{b_n} \geqslant y\Big) = -\frac{y^2}{2\sigma_0^2}, \tag{6.4.21}$$

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \|G_n\|}{b_n} \leqslant -y\Big) = -\frac{y^2}{2\sigma_0^2}. \tag{6.4.22}$$

We first give a proof of (6.4.21). Let p be the proximal dimension of the semigroup Γ_μ. Since the set $\Gamma_{\mu,1}$ is not contained in a compact subgroup of $\mathbb{G}$, by Lemma 6.4.5 (a), we have $1 \leqslant p \leqslant d-1$. Using Lemma 6.4.3, under condition **L2**, this implies that the Lyapunov exponents $(\lambda_p)_{1\leqslant p\leqslant d}$ of μ satisfy

$$\lambda_1 = \ldots = \lambda_p = 0 > \lambda_{p+1}.$$

It follows that the two largest Lyapunov exponents of $\wedge^p G_n$ are given by $\lambda_1 + \cdots + \lambda_p = 0$ and $\lambda_2 + \cdots + \lambda_{p+1} = \lambda_{p+1} < 0$ (see [13, Proposition III. 1,2]). Applying Lemma 6.4.2 to $G = G_\mu$ (the smallest closed subgroup of $\mathbb{G}$ generated by the support of μ), we get the following direct-sum decomposition of the p-th exterior power $\wedge^p(\mathbb{R}^d)$:

$$\wedge^p(\mathbb{R}^d) = V_1 \oplus V_2 \oplus \ldots \oplus V_k,$$

where V_j are subspaces of $\wedge^p(\mathbb{R}^d)$ such that $(\wedge^p g)V_j = V_j$ for any $g \in G_\mu$ and $1 \leqslant j \leqslant k$, i.e. each V_j is invariant under $\wedge^p(G_\mu) := \{\wedge^p g : g \in G_\mu\}$. Moreover, $\wedge^p(G_\mu)$ is irreducible on each subspace V_j. Note that the set of all Lyapunov exponents of $\wedge^p G_n$

on the space $\wedge^p(\mathbb{R}^d)$ coincides with the union of all the Lyapunov exponents of $(\wedge^p G_n)$ restricted to each subspace V_j, $1 \leqslant j \leqslant k$. Hence we can choose V_1 in such a way that the restrictions of $\wedge^p G_n$ to V_1 and $V_2 \oplus \ldots \oplus V_k$, denoted respectively by G'_n and G''_n (as usual we identify the linear transform with the corresponding matrice), satisfy:

$$\lim_{n\to\infty} \frac{1}{n} \log \|G'_n\| = \lambda_1 + \cdots + \lambda_p = 0, \quad \text{a.s.}, \tag{6.4.23}$$

$$\lim_{n\to\infty} \frac{1}{n} \log \|G''_n\| = \lambda_2 + \cdots + \lambda_{p+1} = \lambda_{p+1} < 0, \quad \text{a.s.}, \tag{6.4.24}$$

$$\| \wedge^p G_n\| = \max\{\|G'_n\|, \|G''_n\|\}. \tag{6.4.25}$$

As in the case of G_n, G'_n and G''_n are products of i.i.d. invertible matrices of the form $G'_n = g'_n \cdots g'_1$ and $G''_n = g''_n \cdots g''_1$. We denote by μ_1 the law of the random matrix g_1, by d_1 the dimension of the vector space V_1, and by Γ_{μ_1} the smallest closed subsemigroup of $GL_{d_1}(\mathbb{R})$ generated by the support of μ_1. Then, following the analogous argument used in the proof of the central limit theorem for $\|G_n\|$ (see [13, Theorem V.5.4]), one can verify, under Condition **L2** on μ, that the semigroup Γ_{μ_1} is strongly irreducible and proximal on $\mathbb{R}^{d_1}$. Therefore, μ_1 satisfies Conditions **L2** and **L3**, so that we are allowed to apply the moderate deviation principle (6.2.5) with $\varphi = \mathbf{1}$ and G_n replaced by G'_n, to get the following moderate deviation asymptotics: for any $y > 0$,

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \|G'_n\|}{b_n} \geqslant y\Big) = -\frac{y^2}{2\sigma_1^2}, \tag{6.4.26}$$

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \|G'_n\|}{b_n} \leqslant -y\Big) = -\frac{y^2}{2\sigma_1^2}, \tag{6.4.27}$$

where $\sigma_1^2 > 0$ is the asymptotic variance of the sequence $(G'_n)_{n\geqslant 1}$ given by

$$\sigma_1^2 = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\Big[(\log \|G'_n\|)^2\Big]. \tag{6.4.28}$$

From (6.4.25), it follows that $\log \| \wedge^p G_n\| \geqslant \log \|G'_n\|$. By (6.4.26), we get the lower bound for $\| \wedge^p G_n\|$: for any $y > 0$,

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \| \wedge^p G_n\|}{b_n} \geqslant y\Big) \geqslant -\frac{y^2}{2\sigma_1^2}. \tag{6.4.29}$$

On the other hand, since the upper Lyapunov exponent of the sequence $(G''_n)_{n\geqslant 1}$ is strictly less than 0 (see (6.4.24)), we have $\mathbb{E}(\log \|G''_m\|) < 0$ for sufficiently large integer $m \geqslant 1$. Write $n = km + r$ with $k \geqslant 1$ and $0 \leqslant r < m$. By the identity $G''_n = [G''_n (G''_{km})^{-1}]\, [G''_{km}(G''_{(k-1)m})^{-1}] \cdots [G''_{2m}(G''_m)^{-1}]\, G''_m$, it follows that

$$\log \|G''_n\| \leqslant \log \|G''_n (G''_{km})^{-1}\| + \log \|G''_{km}(G''_{(k-1)m})^{-1}\| + \cdots + \log \|G''_m\|. \tag{6.4.30}$$

For fixed integer $m \geqslant 1$, we denote $u_m := -\mathbb{E}(\log \|G''_m\|) > 0$. Notice that

$$\begin{aligned}\mathbb{P}(\log \|G''_n\| \geqslant 0) &\leqslant \mathbb{P}\Big(\log \|G''_n (G''_{km})^{-1}\| \geqslant k\frac{u_m}{2}\Big) \\ &\quad + \mathbb{P}\Big(\log \|G''_{km}(G''_{(k-1)m})^{-1}\| + \cdots + \log \|G''_m\| + ku_m \geqslant k\frac{u_m}{2}\Big).\end{aligned} \tag{6.4.31}$$

Using (6.4.25) and the fact that $\|\wedge^p g\| \leqslant \|g\|^p$ for any $g \in \Gamma_\mu$, we get that for constant $c > 0$ small enough,

$$\mathbb{E}(\|G''_n(G''_{km})^{-1}\|^c) = \mathbb{E}(\|G''_r\|^c) \leqslant \mathbb{E}(\|\wedge^p G_r\|^c) \leqslant \mathbb{E}(\|G_r\|^{cp}) \leqslant \Big[\mathbb{E}(\|g_1\|^{cp})\Big]^r,$$

which is finite by Condition **L1**. By Markov's inequality and the fact that $u_m > 0$ is a constant, it follows that there exist constants $c, C > 0$ such that

$$\mathbb{P}\Big(\log\|G''_n(G''_{km})^{-1}\| \geqslant k\frac{u_m}{2}\Big) \leqslant \mathbb{E}(\|G''_n(G''_{km})^{-1}\|^c)e^{-ck\frac{u_m}{2}} \leqslant Ce^{-ck}. \tag{6.4.32}$$

Using the large deviation bounds for sums of i.i.d. real-valued random variables, the second term in the right hand side of (6.4.31) is dominated by Ce^{-ck}. Substituting this bound and (6.4.32) into (6.4.31), and taking into account $k \geqslant n/(m+1)$, we obtain

$$\mathbb{P}(\log\|G''_n\| \geqslant 0) \leqslant Ce^{-cn}. \tag{6.4.33}$$

From (6.4.26) we derive that for any $y > 0$ and $\epsilon > 0$, there exists $n_0 \geqslant 1$ such that for any $n \geqslant n_0$,

$$\mathbb{P}\Big(\log\|G'_n\| \geqslant yb_n\Big) \leqslant \exp\Big\{-\frac{b_n^2}{n}\Big(\frac{y^2}{2\sigma_1^2} - \epsilon\Big)\Big\}.$$

This, together with (6.4.33) and (6.4.25), yields the upper bound for $\|\wedge^p G_n\|$: for any $y > 0$ and $\epsilon > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\Big(\frac{\log\|\wedge^p G_n\|}{b_n} \geqslant y\Big)\\
&\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\Big[\mathbb{P}\Big(\frac{\log\|G'_n\|}{b_n} \geqslant y\Big) + \mathbb{P}\Big(\frac{\log\|G''_n\|}{b_n} \geqslant y\Big)\Big]\\
&\leqslant \limsup_{n\to\infty}\frac{n}{b_n^2}\log\Big[e^{-\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma_1^2}-\epsilon\right)} + Ce^{-cn}\Big]\\
&= \limsup_{n\to\infty}\frac{n}{b_n^2}\log\Big[e^{-\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma_1^2}-\epsilon\right)}\Big(1 + Ce^{-cn+\frac{b_n^2}{n}\left(\frac{y^2}{2\sigma_1^2}-\epsilon\right)}\Big)\Big]\\
&= -\Big(\frac{y^2}{2\sigma_1^2} - \epsilon\Big).
\end{aligned}$$

Since $\epsilon > 0$ can be arbitrary small, it follows that

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\Big(\frac{\log\|\wedge^p G_n\|}{b_n} \geqslant y\Big) \leqslant -\frac{y^2}{2\sigma_1^2}. \tag{6.4.34}$$

The lower and upper bounds (6.4.29) and (6.4.34) lead to

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\Big(\frac{\log\|\wedge^p G_n\|}{b_n} \geqslant y\Big) = -\frac{y^2}{2\sigma_1^2}. \tag{6.4.35}$$

Since the proximal dimension of the semigroup Γ_μ is p, by Lemma 6.4.4, the sequence $\{\log\|\wedge^p G_n\| - p\log\|G_n\|\}_{n\geqslant 1}$ is bounded by a constant from above and below. Combining this with (6.4.35), we deduce the following moderate deviation asymptotic for the operator norm $\|G_n\|$: for any $y > 0$,

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{P}\Big(\frac{\log\|G_n\|}{b_n} \geqslant y\Big) = -\frac{y^2}{2\sigma_0^2}, \tag{6.4.36}$$

where $\sigma_0^2 = (\sigma_1^2)/p^2 > 0$.

We next give a proof of (6.4.22). From (6.4.25) and (6.4.27), the upper bound easily follows: for any $y > 0$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \| \wedge^p G_n\|}{b_n} \leqslant -y\Big) \leqslant -\frac{y^2}{2\sigma_1^2}. \tag{6.4.37}$$

To prove the lower bound, observe that from (6.4.25) we have

$$\begin{aligned}\mathbb{P}\Big(\log \| \wedge^p G_n\| \leqslant -yb_n\Big) &= \mathbb{P}\Big(\log \|G_n'\| \leqslant -yb_n, \log \|G_n''\| \leqslant -yb_n\Big)\\ &\geqslant \mathbb{P}\Big(\log \|G_n'\| \leqslant -yb_n\Big) - \mathbb{P}(\log \|G_n''\| > -yb_n).\end{aligned}$$

Similarly to (6.4.33), with fixed integer $m \geqslant 1$ and $u_m = -\mathbb{E}(\log \|G_m''\|) > 0$, taking into account (6.4.30), we write

$$\begin{aligned}\mathbb{P}(\log \|G_n''\| > -yb_n) &\leqslant \mathbb{P}\Big(\log \|G_n''(G_{km}'')^{-1}\| > k\frac{u_m}{2} - yb_n\Big)\\ &\quad + \mathbb{P}\Big(\log \|G_{km}''(G_{(k-1)m}'')^{-1}\| + \cdots + \log \|G_m''\| + ku_m > k\frac{u_m}{2}\Big).\end{aligned}$$

In an analogous way as in the proof of (6.4.32), by Markov's inequality and the fact that $k = O(n)$ and $b_n = o(n)$, the first term on the right hand side of the above inequality is bounded by Ce^{-ck}, where $c, C > 0$ are constants. It has been shown in the proof of (6.4.33) that the second term is also bounded by Ce^{-ck}. Therefore, taking into account $k \geqslant n/(m+1)$, we get that there exist constants $c, C > 0$ such that

$$\mathbb{P}(\log \|G_n''\| > -yb_n) \leqslant Ce^{-cn}.$$

Combining this bound with (6.4.27), we obtain

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \| \wedge^p G_n\|}{b_n} \leqslant -y\Big) \geqslant -\frac{y^2}{2\sigma_1^2}.$$

By Lemma 6.4.4, this implies

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log \|G_n\|}{b_n} \leqslant -y\Big) \geqslant -\frac{y^2}{2\sigma_0^2}, \tag{6.4.38}$$

where $\sigma_0^2 = (\sigma_1^2)/p^2 > 0$. Putting together (6.4.37) and (6.4.38), we conclude the proof of (6.4.22). Combining (6.4.21) and (6.4.22), we get the desired moderate deviation principle (6.2.7) for the operator norm $\|G_n\|$. □

Proof of (6.2.8) *of Theorem 6.2.4.* Using Lemma 6.3.3, we can obtain (6.2.8) from (6.2.7) just as we obtained (6.2.6) from (6.2.5). The details are omitted. □

6.5 Moderate deviation expansions

This section is devoted to proving Theorems 6.2.6 and 6.2.7 about Cramér type moderate deviation expansions in the normal range, for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. We will use the following moderate deviation expansion for the norm cocycle $\log |G_n x|$ recently established in [86] (where the usual range $y \in [0, o(n^{1/2})]$ is considered):

Lemma 6.5.1. *Assume conditions* ***L1***, ***L2*** *and* ***L3***. *Then, we have, uniformly in* $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ *and* $\varphi \in \mathcal{B}_\gamma$, *as* $n \to \infty$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big),$$
$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big).$$

Proof of Theorem 6.2.6. Using the Berry-Esseen type bound (6.2.1) for the couple $(X_n^x, \log\|G_n\|)$, we get that there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{P}^{d-1}$, $y > 0$ and $\varphi \in \mathcal{B}_\gamma$,

$$\left|\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log\|G_n\|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} - \nu(\varphi)\right| \leqslant C\frac{\log n}{\sqrt{n}\Phi(-y)}\|\varphi\|_\gamma. \tag{6.5.1}$$

Using the basic inequality

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{y}-\frac{1}{y^3}\Big)e^{-\frac{y^2}{2}} \leqslant \Phi(-y) \leqslant \frac{1}{\sqrt{2\pi}y}e^{-\frac{y^2}{2}} \quad \text{for } y > 0, \tag{6.5.2}$$

it is easy to see that $\frac{\log n}{\sqrt{n}\Phi(-y)} = O(n^{-3/8}(\log n)^{3/2}) \to 0$, as $n \to \infty$, uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$. Therefore, from (6.5.1) we see that the expansion (6.2.11) holds uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$. In the same way, using again the Berry-Esseen type bound (6.2.1) but together with the fact that $|\mathbb{E}\varphi(X_n^x) - \nu(\varphi)| \leqslant Ce^{-cn}\|\varphi\|_\gamma$, one can also verify that the expansion (6.2.10) also holds uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$.

It remains to prove that expansions (6.2.10) and (6.2.11) hold uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$. Without loss of generality we assume that the target function φ is non-negative (otherwise we can consider the positive and negative parts of φ). We only give a proof of (6.2.11), since (6.2.10) can be established in a similar way. For simplicity, we denote for any $x \in \mathbb{P}^{d-1}$ and $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$I_n := \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\|G_n\|-n\lambda\leqslant-\sqrt{n}\sigma y\Big\}}\Big].$$

The proof consists of establishing the upper and lower bounds. From Lemma 6.5.1 on $\log|G_nx|$ and the fact that $|G_nx| \leqslant \|G_n\|$, the upper bound of I_n immediately follows: there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{I_n}{\Phi(-y)} \leqslant \nu(\varphi) + C\|\varphi\|_\gamma\frac{y+1}{\sqrt{n}}. \tag{6.5.3}$$

For the lower bound of I_n, we shall use Lemma 6.3.2. For any $a > 0$ and $n > k \geqslant 1$, consider the event

$$A_{n,k} = \Big\{\Big|\log\|G_n\| - \log\frac{|G_nx|}{|G_kx|} - \log\|G_k\|\Big| \leqslant e^{-ak}\Big\},$$

and we write $A_{n,k}^c$ for its complement. Since the function φ is assumed to be non-negative, using Lemma 6.3.2 we get that for any $a > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geqslant k \geqslant k_0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned} I_n &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log\|G_n\|-n\lambda\leqslant-\sqrt{n}\sigma y\right\}}\mathbb{1}_{\{A_{n,k}\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log|G_nx|-\log|G_kx|+\log\|G_k\|-n\lambda+e^{-ak}\leqslant-\sqrt{n}\sigma y\right\}}\mathbb{1}_{\{A_{n,k}\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log|G_nx|-\log|G_kx|+\log\|G_k\|-n\lambda+e^{-ak}\leqslant-\sqrt{n}\sigma y\right\}}\Big] - e^{-c_1k}\|\varphi\|_\infty \\ &=: J_n - e^{-c_1k}\|\varphi\|_\infty. \end{aligned} \tag{6.5.4}$$

We now give a lower bound for J_n. As before, for any $n > k \geqslant 1$, we write $G_n = G_{n,k}G_k$ with $G_{n,k} = g_n \dots g_{k+1}$ and $G_k = g_k \dots g_1$. We take the conditional expectation with respect to the filtration $\mathcal{F}_k = \sigma(g_1, \dots, g_k)$ and use the large deviation bound (6.3.3) for the operator norm $\|G_k\|$, to obtain that, for any $q > \lambda$, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} J_n &= \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log|G_nx|-\log|G_kx|+\log\|G_k\|-n\lambda+e^{-ak}\leqslant-\sqrt{n}\sigma y\right\}}\Big|\mathcal{F}_k\Big]\right\} \\ &\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log|G_nx|-\log|G_kx|+\log\|G_k\|-n\lambda+e^{-ak}\leqslant-\sqrt{n}\sigma y\right\}}\mathbb{1}_{\{\log\|G_k\|\leqslant kq\}}\Big|\mathcal{F}_k\Big]\right\} \\ &\geqslant \mathbb{E}\left\{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\log|G_nx|-\log|G_kx|+kq-n\lambda+e^{-ak}\leqslant-\sqrt{n}\sigma y\right\}}\Big|\mathcal{F}_k\Big]\right\} - e^{-c_2k}\|\varphi\|_\infty \\ &=: J_n' - e^{-c_2k}\|\varphi\|_\infty. \end{aligned} \tag{6.5.5}$$

For brevity, we set

$$y_1 = y\sqrt{\frac{n}{n-k}} - \frac{k(q-\lambda)}{\sigma\sqrt{n-k}} - \frac{e^{-ak}}{\sigma\sqrt{n-k}}, \tag{6.5.6}$$

then J_n' can be rewritten as

$$J_n' = \mathbb{E}\left\{\mathbb{E}\Big[\varphi(G_{n,k}\cdot X_k^x)\mathbb{1}_{\left\{\log|G_{n,k}X_k^x|-(n-k)\lambda\leqslant-\sqrt{n-k}\sigma y_1\right\}}\Big|\mathcal{F}_k\Big]\right\}.$$

For any $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we take

$$k = \Big[C_1y^2\Big], \tag{6.5.7}$$

where $C_1 > 0$ is a constant whose value will be chosen large enough. From (6.5.6) and (6.5.7), we see that $y \sim y_1 = o(n^{1/6})$ as $n \to \infty$. Hence we are allowed to apply Lemma 6.5.1 to obtain the following moderate deviation expansion for J_n': as $n \to \infty$, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{J_n'}{\Phi(-y_1)} = \nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y_1+1}{\sqrt{n}}\Big). \tag{6.5.8}$$

Using the asymptotic expansion $\sqrt{2\pi}\Phi(-y) = \frac{1}{y}e^{-\frac{y^2}{2}}[1+O(\frac{1}{y^2})]$ as $y \to \infty$ (cf. (6.5.2)), we get that as $n \to \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$\frac{\Phi(-y)}{\Phi(-y_1)} = \frac{\frac{1}{y}e^{-\frac{y^2}{2}}[1+O(\frac{1}{y^2})]}{\frac{1}{y_1}e^{-\frac{y_1^2}{2}}[1+O(\frac{1}{y_1^2})]} = \frac{y_1}{y}e^{-\frac{y^2}{2}+\frac{y_1^2}{2}}(1+o(1)).$$

Taking into account (6.5.6) and (6.5.7), one can calculate that, as $n \to \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we have $\frac{y_1}{y} = 1 + o(1)$ and

$$\begin{aligned} e^{-\frac{y^2}{2}+\frac{y_1^2}{2}} &= \exp\left\{\frac{ky^2}{2(n-k)} - \frac{y\sqrt{n}(e^{-ak}+k(q-\lambda))}{\sigma(n-k)} + \frac{(e^{-ak}+k(q-\lambda))^2}{2\sigma^2(n-k)}\right\} \\ &= 1 + o(1). \end{aligned}$$

Consequently, substituting the above estimates into (6.5.8), we get that, as $n \to \infty$, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{J_n'}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1).$$

This, together with (6.5.5), implies the following lower bound for I_n: there exists a constant $c_3 > 0$ such that uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \frac{I_n}{\Phi(-y)} &\geqslant \nu(\varphi) + \|\varphi\|_\gamma o(1) - \|\varphi\|_\infty \frac{2e^{-c_3 k}}{\Phi(-y)} \\ &= \nu(\varphi) + \|\varphi\|_\gamma o(1) - \|\varphi\|_\infty \frac{2e^{-c_3[C_1 y^2]}}{\Phi(-y)}. \end{aligned}$$

Using the inequality (6.5.2) and taking $C_1 > \frac{1}{c_3}$, it follows that, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \frac{I_n}{\Phi(-y)} &\geqslant \nu(\varphi) + \|\varphi\|_\gamma o(1) - 4\|\varphi\|_\infty y e^{-c_3[C_1 y^2]+\frac{y^2}{2}} \\ &= \nu(\varphi) + \|\varphi\|_\gamma o(1). \end{aligned} \tag{6.5.9}$$

Combining this with the upper bound (6.5.3), we conclude the proof of the expansion (6.2.11). □

We proceed to establish Theorem 6.2.7 based on Theorem 6.2.6, Lemma 6.3.3 and the Berry-Esseen type bound (6.2.2).

Proof of Theorem 6.2.7. We only prove the first expansion (6.2.12) since the proof of the second one (6.2.13) can be carried out in an analogous way. Using the Berry-Esseen type bound (6.2.2) and the fact that $|\mathbb{E}\varphi(X_n^x) - \nu(\varphi)| \leqslant Ce^{-cn}\|\varphi\|_\gamma$, we derive that uniformly in $x \in \mathbb{P}^{d-1}$, $y > 0$ and $\varphi \in \mathcal{B}_\gamma$,

$$\left|\frac{\mathbb{E}\left[\varphi(X_n^x)\mathbb{1}_{\{\log\rho(G_n)-n\lambda \geqslant \sqrt{n}\sigma y\}}\right]}{1-\Phi(y)} - \nu(\varphi)\right| \leqslant C\frac{\log n}{\sqrt{n}(1-\Phi(y))}\|\varphi\|_\gamma.$$

Using the inequality (6.5.2), Using the inequality (6.5.2), one can verify that $\frac{\log n}{\sqrt{n}(1-\Phi(y))} \to 0$, as $n \to \infty$, uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$. Hence the expansion (6.2.12) holds uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$.

In the sequel we prove that (6.2.12) holds uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$. Without loss of generality, we assume that the target function φ is non-negative. For brevity, we denote for any $x \in \mathbb{P}^{d-1}$ and $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$I_n := \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\rho(G_n)-n\lambda\geqslant\sqrt{n}\sigma y\Big\}}\Big].$$

The remaining part of the proof consists of establishing upper and lower bounds of I_n. Since $\rho(G_n) \leqslant \|G_n\|$, using the moderate deviation expansion (6.2.10) for the operator norm $\|G_n\|$, we get the upper bound for I_n: as $n \to \infty$, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{I_n}{1-\Phi(y)} \leqslant \nu(\varphi) + \|\varphi\|_\gamma o(1). \tag{6.5.10}$$

We shall apply Lemma 6.3.3 to derive a lower bound for I_n. For any $\varepsilon > 0$ and $n > k \geqslant 1$, we denote

$$A_{n,k} = \Big\{\log\rho(G_n) - \log\|G_n\| > -\varepsilon k\Big\}.$$

From Lemma 6.3.3 we know that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geqslant k \geqslant k_0$, we have $\mathbb{P}(A_{n,k}) > 1 - e^{-c_1 k}$. Thus,

$$\begin{aligned} I_n &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\rho(G_n)-n\lambda\geqslant\sqrt{n}\sigma y\Big\}}\mathbb{1}_{\{A_{n,k}\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\Big\}}\mathbb{1}_{\{A_{n,k}\}}\Big] \\ &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\Big\}}\Big] - e^{-c_1 k}\|\varphi\|_\infty. \end{aligned} \tag{6.5.11}$$

By Theorem 6.2.6, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\Big\}}\Big]}{1-\Phi(y+\frac{\varepsilon k}{\sqrt{n}\sigma})} = \nu(\varphi) + \|\varphi\|_\gamma o(1). \tag{6.5.12}$$

For $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we take

$$k = \Big[C_1 y^2\Big], \tag{6.5.13}$$

where $C_1 > 0$ is a constant whose value will be chosen large enough. From the asymptotic expansion $\int_y^\infty e^{-\frac{t^2}{2}}dt = \frac{1}{y}e^{-\frac{y^2}{2}}[1 + O(\frac{1}{y^2})]$ as $y \to \infty$, we infer that as $n \to \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$\begin{aligned} \frac{1-\Phi(y+\frac{\varepsilon k}{\sqrt{n}\sigma})}{1-\Phi(y)} &= \frac{y}{y+\frac{\varepsilon k}{\sqrt{n}\sigma}}\exp\Big\{\frac{1}{2}y^2 - \frac{1}{2}(y+\frac{\varepsilon k}{\sqrt{n}\sigma})^2\Big\}(1+o(1)) \\ &= \frac{y}{y+\frac{\varepsilon k}{\sqrt{n}\sigma}}\exp\Big\{-y\frac{\varepsilon k}{\sqrt{n}\sigma} - \frac{\varepsilon^2 k^2}{2n\sigma^2}\Big\}(1+o(1)). \end{aligned}$$

Since $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, taking into account (6.5.13), we get

$$\frac{y}{y+\frac{\varepsilon k}{\sqrt{n}\sigma}} = 1 - \frac{\frac{\varepsilon k}{\sqrt{n}\sigma}}{y+\frac{\varepsilon k}{\sqrt{n}\sigma}} \geqslant 1 - \frac{\varepsilon k}{y\sqrt{n}\sigma} = 1 - \frac{\varepsilon[C_1 y^2]}{y\sqrt{n}\sigma} = 1 + o(1),$$

and

$$\exp\Big\{-y\frac{\varepsilon k}{\sqrt{n}\sigma} - \frac{\varepsilon^2 k^2}{2n\sigma^2}\Big\} = \exp\Big\{-y\frac{\varepsilon[C_1 y^2]}{\sqrt{n}\sigma} - \frac{\varepsilon^2[C_1 y^2]^2}{2n\sigma^2}\Big\} = 1 + o(1).$$

Hence, substituting the above estimates into (6.5.12), we get, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\Big\{\log\|G_n\|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\Big\}}\Big]}{1-\Phi(y)} \geqslant \nu(\varphi) + \|\varphi\|_\gamma o(1).$$

This, together with (6.5.11), implies the lower bound for I_n: uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned}\frac{I_n}{1-\Phi(y)} &\geqslant \nu(\varphi) + \|\varphi\|_\gamma o(1) - \frac{e^{-c_1 k}}{1-\Phi(y)}\|\varphi\|_\infty \\ &\geqslant \nu(\varphi) + \|\varphi\|_\gamma o(1) - 2y\exp\Big\{-c_1[C_1 y^2] + \frac{y^2}{2}\Big\}\|\varphi\|_\infty \\ &\geqslant \nu(\varphi) + \|\varphi\|_\gamma o(1),\end{aligned}$$

where in the second inequality we use (6.5.2) and in the last inequality we take $C_1 > \frac{1}{c_1}$. Consequently, combining this with the upper bound (6.5.10), we conclude the proof of Theorem 6.2.7. □

Chapter 7

Cramér type moderate deviation expansions for entries of products of random invertible matrices

Abstract. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed random invertible 2×2 matrices. Consider the product matrix $G_n := g_n \dots g_1$ and its (i,j)-th entry $G_n^{i,j}$, where $1 \leqslant i,j \leqslant 2$. Under suitable conditions, we establish Cramér type moderate deviation expansions for $G_n^{i,j}$. Our result implies a moderate deviation principle for $G_n^{i,j}$, which is also new. The proof is based on the saddle point method and the Hölder regularity of the stationary measure of a Markov chain on the unit sphere.

7.1 Introduction

7.1.1 Background and main objective

Equip $\mathbb{R}^2$ with the standard inner product $\langle\cdot,\cdot\rangle$ and the canonical Euclidean norm $|\cdot|$. Denote by $(e_k)_{1\leqslant k\leqslant 2}$ the standard orthonormal basis, and by $\mathbb{P}^1 = \{x\in\mathbb{R}^2, |x|=1\}/\pm$ the projective space obtained from the unit sphere of $\mathbb{R}^2$ by identifying x and $-x$. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) real random matrices with law μ on the special linear group $\mathbb{G} := SL_2(\mathbb{R})$. Consider the random walk $G_n := g_n \dots g_1$ on the group $\mathbb{G}$. Denote by $G_n^{i,j} := \langle e_i, G_n e_j\rangle$ the (i,j)-th entry of the product matrix G_n, where $1\leqslant i,j\leqslant 2$. The study of asymptotic properties of the entries $G_n^{i,j}$ has attracted a lot of attention since the pioneering work of Furstenberg and Kesten [37]. The goal of the present paper is to investigate the Cramér type moderate deviation expansion for the entries $G_n^{i,j}$, and more generally, for the scalar product $\langle f, G_n x\rangle$, where $f,x\in\mathbb{P}^1$.

For positive matrices, Furstenberg and Kesten [37] established the strong law of large numbers and the central limit theorem for $G_n^{i,j}$. Under weaker assumptions, these results were subsequently generalized by Kingman [67], Cohn, Nerman and Peligrad [25], Hennion [53]. Precise large and moderate deviation expansions for $G_n^{i,j}$ have been recently established in [88, 89].

For invertible matrices, the situation is more complicated and delicate. For $g\in\mathbb{G}$, set $\|g\| = \sup_{x\in\mathbb{P}^1}|gx|$. Let Γ_μ be the smallest closed subgroup of $\mathbb{G}$ generated by the

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support of μ. Consider the following conditions.

B1. There exists a constant $\eta > 0$ such that $\int_{\mathbb{G}} \|g_1\|^\eta \mu(dg_1) < \infty$.

B2. The subgroup Γ_μ is Zariski dense in $SL(2, \mathbb{R})$.

Guivarc'h and Raugi [51] first established the strong law of large numbers for $G_n^{i,j}$: under conditions **B1** and **B2**, for any $1 \leqslant i, j \leqslant 2$,

$$\lim_{n\to\infty} \frac{1}{n} \log |G_n^{i,j}| = \lambda \quad \text{a.s.}, \tag{7.1.1}$$

where the constant λ is called the upper Lyapunov exponent of the product sequence (G_n). Under the same conditions, the central limit theorem for $G_n^{i,j}$ was also established in [51]: for any $1 \leqslant i, j \leqslant 2$ and $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\log |G_n^{i,j}| - n\lambda}{\sigma\sqrt{n}} \leqslant y\Big) = \Phi(y), \tag{7.1.2}$$

where Φ is the standard normal distribution function and $\sigma^2 > 0$ is the asymptotic variance of $\frac{1}{\sqrt{n}} \log |G_n^{i,j}|$. The proof of (7.1.1) and (7.1.2) is based on the Hölder regularity of the stationary measure of the Markov chain $X_n^{e_j} := G_n e_j / |G_n e_j|$ on the projective space $\mathbb{P}^1$. Recently, Benoist and Quint [10] have extended (7.1.1) and (7.1.2) to the framework of the general linear group $GL(d, \mathbb{K})$, where $\mathbb{K}$ is a local field. Due to a large number of applications in probability theory and statistics, it is important to investigate the rate of convergence in (7.1.1) and (7.1.2). Bahadur-Rao type and Petrov type large deviation asymptotics for $G_n^{i,j}$ have been recently established in [89]. However, moderate deviations have not yet been considered in the literature.

The objective of this paper is to establish the following Cramér type moderate deviation expansion for the entries $G_n^{i,j}$: under conditions **B1** and **B2**, for any $1 \leqslant i, j \leqslant 2$, we have, uniformly in $y \in [0, o(\sqrt{n})]$,

$$\frac{\mathbb{P}\Big(\frac{\log |G_n^{i,j}| - n\lambda}{\sigma\sqrt{n}} \geqslant y\Big)}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[1 + o(1)\Big], \tag{7.1.3}$$

$$\frac{\mathbb{P}\Big(\frac{\log |G_n^{i,j}| - n\lambda}{\sigma\sqrt{n}} \leqslant -y\Big)}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[1 + o(1)\Big], \tag{7.1.4}$$

where ζ is the Cramér series (see (7.2.1)). More generally, we shall prove the Cramér type moderate deviation expansion for the scalar product $\langle f, G_n x\rangle$ and for the couple $(X_n^x, \log |\langle f, G_n x\rangle|)$ with a target function φ on the Markov chain $X_n^x := G_n x / |G_n x|$ on the projective space $\mathbb{P}^1$ (see Theorem 7.2.2). In particular, the expansions (7.1.3) and (7.1.4) clearly imply the following moderate deviation principle which is also new: for any Borel set $B \subseteq \mathbb{R}$, any positive sequence $(b_n)_{n\geqslant 1}$ satisfying $\frac{b_n}{n} \to 0$ and $\frac{b_n}{\sqrt{n}} \to \infty$, we have,

$$\begin{aligned} -\inf_{y\in B^\circ} \frac{y^2}{2\sigma^2} &\leqslant \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log |G_n^{i,j}| - n\lambda}{b_n} \in B\Big) \\ &\leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big(\frac{\log |G_n^{i,j}| - n\lambda}{b_n} \in B\Big) \leqslant -\inf_{y\in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned}$$

where B° and $\bar{B}$ are respectively the interior and the closure of B.

7.1.2 Proof strategy

The standard approach to prove the Cramér type moderate deviation expansion for sums of i.i.d. real-valued random variables consists in performing a change of measure and proving a Berry-Esseen bound under the changed measure; see for example Cramér [26] and Petrov [74]. For random walks on groups or semigroups, this approach has been recently employed in [86] to establish the Cramér type moderate deviation expansion for the norm cocycle $\log|G_n x|$ with $x \in \mathbb{P}^1$; in this case performing a change of measure is essentially reduced to proving the spectral gap properties of a transfer operator of the underlying Markov chain X_n^x. However, the change of measure formula for the log entry $\log|G_n^{i,j}|$ has not yet been established and needs some new ideas. Our basic idea is to decompose the log entry $\log|G_n^{i,j}|$ as the sum of the norm cocycle $\log|G_n e_j|$ and the log scalar product of e_i and $X_n^{e_j} := G_n e_j/|G_n e_j|$:

$$\log|G_n^{i,j}| = \log|G_n e_j| + \log|f(X_n^{e_j})|, \text{ with } f(u) = \langle e_i, u\rangle \ \forall u \in \mathbb{P}^1. \tag{7.1.5}$$

In order to prove the expansions (7.1.3) and (7.1.4), we can not use the method employed in [88] where a precise moderate deviation expansion for $G_n^{i,j}$ has been proved in the case of positive matrices under a boundedness assumption of type Furstenberg-Kesten. The later assumption plays a key role to ensure that the Markov chain $(X_n^{e_j})$ stays separated from the coordinates $(e_i)_{1\leqslant i\leqslant 2}$, so that $\langle f, X_n^{e_j}\rangle$ is strictly positive. However, such kind of analysis breaks down for invertible matrices. The reason is that the random walk $(G_n e_j)_{n\geqslant 1}$ may stay in the hyperplane $\ker f$, making $\log|f(X_n^{e_j})|$ meaningless. To circumvent this, according to the values of $\log|f(X_n^{e_j})|$, we split the interval $(-\infty, 0]$ into equal pieces $I_k := (-\delta k, -\delta(k-1)]$, $k \in \mathbb{N}$, for some small constant $\delta > 0$. To deal with each piece I_k, we first define the transfer operator of the Markov chain $(X_n^{e_j})_{n\geqslant 1}$ as follows: for any $z \in \mathbb{C}$ with $|z|$ small enough, and any continuous function φ on the projective space $\mathbb{P}^1$,

$$P_z\varphi(e_j) = \int e^{z\log|g_1 e_j|}\varphi(X_1^{e_j})\mu(dg_1).$$

Since spectral gap properties of P_z holds on the Banach space of Hölder continuous functions, an important issue is to make the indicator function $\mathbb{1}_{\{\log|f(X_n^{e_j})|\in I_k\}}$ to be smooth. This smoothing techniques, together with the discretization and the decomposition (7.1.5), permits to investigate the precise moderate deviation asymptotics for the couple $(X_n^{e_j}, \log|G_n e_j|)$ with a target function on $X_n^{e_j}$. To patch up all the pieces I_k, $k \in \mathbb{N}$, of central importance is to establish the Hölder regularity of the stationary measure π_s of the Markov chain $X_n^{e_j}$: there exist constants $c, C > 0$ such that for any $0 < t < 1$,

$$\pi_s(\{x \in \mathbb{P}^1 : |\langle f, x\rangle| \leqslant t\}) \leqslant Ct^c.$$

We refer to Theorem 7.4.1 for details.

7.2 Main results

To formulate our results, we need some notation. Denote $\gamma_k = \Lambda^{(k)}(0)$, $k \geqslant 1$, where $\Lambda = \log\kappa$ with the function κ defined by the dominant eigenvalue of the operator P_s

(see Proposition 7.3.1). In particular, under conditions **B1** and **B2**, we have $\gamma_1 = \lambda > 0$ and $\gamma_2 = \sigma^2 > 0$. Throughout the paper, we write ζ for the Cramér series:

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \cdots, \tag{7.2.1}$$

which converges for $|t|$ small enough. For any $g \in \mathbb{G}$ and $x \in \mathbb{P}^1$, we write $g \cdot x := \frac{gx}{|gx|}$ for the projective action of g on the projective space $\mathbb{P}^1$. Consider the Markov chain

$$X_0^x = x, \qquad X_n^x = G_n \cdot x, \quad n \geqslant 1.$$

Under conditions **B1** and **B2**, the chain $(X_n^x)_{n\geqslant 0}$ has a unique stationary probability measure ν on $\mathbb{P}^1$ such that for any continuous function φ on $\mathbb{P}^1$,

$$\int_{\mathbb{P}^1}\int_{\Gamma_\mu} \varphi(g_1 \cdot x)\mu(dg_1)\nu(dx) = \int_{\mathbb{P}^1} \varphi(x)\nu(dx) := \nu(\varphi). \tag{7.2.2}$$

Now we state the Cramér type moderate deviation expansion for the entry $G_n^{i,j}$, and more generally for the scalar product $\langle f, G_n x\rangle$, where $f, x \in \mathbb{P}^1$.

Theorem 7.2.1. *Assume conditions **B1** and **B2**. Then, we have, uniformly in $f, x \in \mathbb{P}^1$ and $y \in [0, o(\sqrt{n})]$,*

$$\frac{\mathbb{P}\left(\frac{\log|\langle f, G_n x\rangle| - n\lambda}{\sigma\sqrt{n}} \geqslant y\right)}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[1 + o(1)\Big],$$
$$\frac{\mathbb{P}\left(\frac{\log|\langle f, G_n x\rangle| - n\lambda}{\sigma\sqrt{n}} \leqslant -y\right)}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[1 + o(1)\Big].$$

In particular, taking $f = e_i$ and $x = e_j$ in Theorem 7.2.1, we obtain the moderate deviation expansions (7.1.3) and (7.1.4) for entries $G_n^{i,j}$.

More generally, we shall establish the Cramér type moderate deviation expansion for the couple $(X_n^x, \log|\langle f, G_n x\rangle|)$ with a target function φ on the Markov chain $(X_n^x)_{n\geqslant 0}$. Let us first introduce some notation. Denote by $\mathcal{B}_\gamma$ the Banach space of complex-valued γ-Hölder continuous functions on the projective space $\mathbb{P}^1$, see Section 7.3.1 for details.

As a generalization of Theorem 7.2.1, the following result concerns the moderate deviation expansion for the couple $(X_n^x, \log|\langle f, G_n x\rangle|)$.

Theorem 7.2.2. *Assume conditions **B1** and **B2**. Then, we have, uniformly in $f, x \in \mathbb{P}^1$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma o(1)\Big],$$
$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma o(1)\Big].$$

In particular, with $\varphi = \mathbf{1}$, Theorem 7.2.2 implies Theorem 7.2.1. Theorem 7.2.2 clearly implies the following moderate deviation principle for the couple $(X_n^x, \log|\langle f, G_n x\rangle|)$ with a target function φ on the Markov chain X_n^x.

Corollary 7.2.3. *Assume conditions* ***B1*** *and* ***B2****. Let* $(b_n)_{n\geqslant 1}$ *be a positive sequence satisfying* $\frac{b_n}{n}\to 0$ *and* $\frac{b_n}{\sqrt{n}}\to\infty$*. Then, for any Borel set* $B\subseteq\mathbb{R}$ *and for any real-valued function* $\varphi\in\mathcal{B}_\gamma$ *satisfying* $\nu(\varphi)>0$*, we have, uniformly in* $f,x\in\mathbb{P}^1$*,*

$$-\inf_{y\in B^\circ}\frac{y^2}{2\sigma^2}\leqslant\liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|\langle f,G_nx\rangle|-n\lambda}{b_n}\in B\right\}}\Big]$$
$$\leqslant\limsup_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\left\{\frac{\log|\langle f,G_nx\rangle|-n\lambda}{b_n}\in B\right\}}\Big]\leqslant-\inf_{y\in\bar{B}}\frac{y^2}{2\sigma^2},\tag{7.2.3}$$

where B° *and* $\bar{B}$ *are respectively the interior and the closure of* B*.*

The moderate deviation principle (7.2.3) is new even for $\varphi=\mathbf{1}$.

7.3 Spectral gap theory

7.3.1 A change of measure

We equip the projective space $\mathbb{P}^1$ with the angular distance $\mathbf{d}$, i.e. $\mathbf{d}(x,y)=\sqrt{1-|\langle x,y\rangle|^2}$ for any $x,y\in\mathbb{P}^1$. Denote by $\mathcal{C}(\mathbb{P}^1)$ the space of all continuous complex-valued functions on $\mathbb{P}^1$. In particular, we write $\mathbf{1}$ for the constant function with value 1 on the space $\mathbb{P}^1$. Throughout the present paper, we assume that $\gamma>0$ is a fixed small enough constant. For any $\varphi\in\mathcal{C}(\mathbb{P}^1)$, set

$$\|\varphi\|_\infty:=\sup_{x\in\mathbb{P}^1}|\varphi(x)|,\quad \|\varphi\|_\gamma:=\|\varphi\|_\infty+\sup_{x,y\in\mathbb{P}^1}\frac{|\varphi(x)-\varphi(y)|}{\mathbf{d}^\gamma(x,y)},$$

and consider the Banach space $\mathcal{B}_\gamma:=\{\varphi\in\mathcal{C}(\mathbb{P}^1):\|\varphi\|_\gamma<\infty\}$. We write $\mathcal{B}'_\gamma$ for the dual space of $\mathcal{B}_\gamma$. Denote by $\mathcal{L}(\mathcal{B}_\gamma,\mathcal{B}_\gamma)$ the set of all bounded linear operators from $\mathcal{B}_\gamma$ to $\mathcal{B}_\gamma$ equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma}$. For $z\in\mathbb{C}$ with the absolute value of the real part $|\Re z|$ small enough, we define the transfer operator P_z as follows: for any $\varphi\in\mathcal{C}(\mathbb{P}^1)$,

$$P_z\varphi(x)=\int_{\Gamma_\mu}e^{z\log|g_1x|}\varphi(g_1\cdot x)\mu(dg_1),\quad x\in\mathbb{P}^1.\tag{7.3.1}$$

The following result, which is obtained utilizing the perturbation theory of linear operators [65, 54], shows that the operator P_z has spectral gap properties when z lies in a small neighborhood of 0 in the complex plane; we refer to [69, 50, 10, 86] for the proof. Throughout this paper let $B_\eta(0):=\{z\in\mathbb{C}:|z|<\eta\}$ be the open disc with center 0 and radius $\eta>0$ in the complex plane $\mathbb{C}$.

Proposition 7.3.1. *Assume conditions* ***B1*** *and* ***B2****. Then, there exists a constant* $\eta>0$ *such that for any* $z\in B_\eta(0)$*, we have*

$$P_z^n=\kappa^n(z)\nu_z\otimes r_z+L_z^n,\quad n\geqslant 1,\tag{7.3.2}$$

where

$$z\mapsto\kappa(z)\in\mathbb{C},\quad z\mapsto r_z\in\mathcal{B}_\gamma,\quad z\mapsto\nu_z\in\mathcal{B}'_\gamma,\quad z\mapsto L_z\in\mathcal{L}(\mathcal{B}_\gamma,\mathcal{B}_\gamma)$$

are analytic mappings which satisfy, for any $z\in B_\eta(0)$*,*

(a) *the operator* $M_z := \nu_z \otimes r_z$ *is the rank one projection on* $\mathcal{B}_\gamma$*, i.e.* $M_z\varphi = \nu_z(\varphi)r_z$ *for any* $\varphi \in \mathcal{B}_\gamma$*;*

(b) $M_zL_z = L_zM_z = 0$*,* $P_zr_z = \kappa(z)r_z$ *with* $\nu(r_z) = 1$*, and* $\nu_zP_z = \kappa(z)\nu_z$*;*

(c) $\kappa(0) = 1$*,* $r_0 = \mathbf{1}$*,* $\nu_0 = \nu$ *with* ν *defined by* (7.2.2)*, and* $\kappa(z)$ *and* r_z *are strictly positive for* $z \in (-\eta, \eta)$*;*

(d) *for any* $k \in \mathbb{N}$*, there exist* $0 < a_1 < a_2 < 1$ *such that* $|\kappa(z)| > 1 - a_1$ *and* $\|\frac{d^k}{dz^k}L_z^n\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant c(1-a_2)^n$*, uniformly in* $z \in B_\eta(0)$*.*

A change of measure can be performed utilizing the fact that the eigenvalue $\kappa(s)$ and the eigenfunction r_s of the operator P_s are strictly positive for $s \in (-\eta, \eta)$. For any $x \in \mathbb{P}^1$, $s \in (-\eta, \eta)$ and $n \geqslant 1$, denote $q_n^s(x, G_n) = \frac{|G_nx|^s}{\kappa^n(s)}\frac{r_s(X_n^x)}{r_s(x)}$. Then the probability measures $q_n^s(x, G_n)\mu(dg_1)\dots\mu(dg_n)$ form a projective system on $\mathbb{G}^{\mathbb{N}}$, so that there is a unique probability measure $\mathbb{Q}_s^x$ on $\mathbb{G}^{\mathbb{N}}$ by the Kolmogorov extension theorem. Denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation. With these notation, the following change of measure formula holds: for any $n \geqslant 1$ and bounded measurable function h on $(\mathbb{P}^1\times\mathbb{R})^n$,

$$\begin{aligned}&\mathbb{E}\Big[\frac{r_s(X_n^x)|G_nx|^s}{\kappa^n(s)r_s(x)}h\Big(X_1^x, \log|g_1x|, \dots, X_n^x, \log|G_nx|\Big)\Big]\\ &= \mathbb{E}_{\mathbb{Q}_s^x}\Big[h\Big(X_1^x, \log|g_1x|, \dots, X_n^x, \log|G_nx|\Big)\Big]. \qquad (7.3.3)\end{aligned}$$

Under the changed measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n\in\mathbb{N}}$ is a Markov chain with the transition operator Q_s defined as follows: for any $\varphi \in \mathcal{C}(\mathbb{P}^1)$,

$$Q_s\varphi(x) = \frac{1}{\kappa(s)r_s(x)}P_s(\varphi r_s)(x), \quad x \in \mathbb{P}^1.$$

It was shown in [86] that for any $s \in (-\eta, \eta)$, the Markov operator Q_s has a unique stationary measure π_s given by

$$\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}. \qquad (7.3.4)$$

7.3.2 Spectral gap and strong non-lattice

From Proposition 7.3.1, it follows that the function $\Lambda := \log\kappa$ is convex in a small neighborhood of 0. Indeed, the function Λ plays the same role as the log-Laplace transform in the case of i.i.d. real-valued random variables. For any $s \in (-\eta, \eta)$ and $z \in \mathbb{C}$ with $|\Re z|$ sufficiently small, define the perturbed operator $R_{s,z}$ as follows: for $\varphi \in \mathcal{B}_\gamma$,

$$R_{s,z}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|g_1x| - \Lambda'(s))}\varphi(X_1^x)\right], \quad x \in \mathbb{P}^1. \qquad (7.3.5)$$

It follows that for $n \geqslant 1$,

$$R_{s,z}^n\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|G_nx| - n\Lambda'(s))}\varphi(X_n^x)\right], \quad x \in \mathbb{P}^1.$$

The following proposition provides spectral gap properties of the perturbed operator $R_{s,z}$; see [86] for the proof.

Proposition 7.3.2. *Assume conditions* ***B1*** *and* ***B2****. Then, there exist constants* $\eta, \delta > 0$ *such that for any* $s \in (-\eta, \eta)$ *and* $z \in B_\delta(0)$,

$$R^n_{s,z} = \lambda^n_{s,z}\Pi_{s,z} + N^n_{s,z}, \quad \text{with } \lambda_{s,z} = e^{\Lambda(s+z)-\Lambda(s)-\Lambda'(s)z}, \tag{7.3.6}$$

where, for fixed s*, the mappings* $z \mapsto \Pi_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, $z \mapsto N_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ *and* $z \mapsto \lambda_{s,z} : B_\delta(0) \to \mathbb{R}$ *are analytic. In addition, for fixed* s *and* z*, the operator* $\Pi_{s,z}$ *is a rank-one projection with* $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ *for any* $\varphi \in \mathcal{B}_\gamma$ *and* $x \in \mathbb{P}^1$*, and* $\Pi_{s,z}N_{s,z} = N_{s,z}\Pi_{s,z} = 0$*. Moreover, for any* $k \in \mathbb{N}$*, there exist constants* $c > 0$ *and* $0 < a < 1$ *such that*

$$\sup_{|s|<\eta}\sup_{|z|<\delta} \|\frac{d^k}{dz^k}\Pi^n_{s,z}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant c, \quad \sup_{|s|<\eta}\sup_{|z|<\delta} \|\frac{d^k}{dz^k}N^n_{s,z}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \leqslant ca^n. \tag{7.3.7}$$

Recall that the transfer operator P_{s+it} for $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$ is defined by (7.3.1). Due to a lot of applications to the study of limit theorems for products of random matrices (see [69, 51, 50, 17, 10]) and to related topics such as the decay of Fourier coefficients of stationary measure ν (see [71]), it is of interest to investigate the non-arithmeticity property of P_{s+it}. In the context of general linear group $GL_d(\mathbb{R})$, Le Page [69] (see also Guivarc'h and Le Page [50]) proved that the spectral radius of P_{s+it} is strictly less than 1 when t lies in a compact set of $\mathbb{R} \setminus \{0\}$. In the context of special linear group $SL_d(\mathbb{R})$, Li [72] recently established the following result based on the polynomial decay of Fourier coefficients of stationary measure ν.

Lemma 7.3.3. *Assume conditions* ***B1*** *and* ***B2****. Then for any fixed* $\delta > 0$*, there exist constants* $0 < \eta, \alpha < 1$ *such that*

$$\sup_{s\in(-\eta,\eta)}\sup_{|t|>\delta} \varrho(P_{s+it}) < 1 - \alpha.$$

This result, together with Proposition 7.3.2, allows to deduce the following:

Lemma 7.3.4. *Assume conditions* ***B1*** *and* ***B2****. Then for any fixed* $\delta > 0$*, there exist constants* $0 < \eta, \alpha < 1$ *such that*

$$\sup_{s\in(-\eta,\eta)}\sup_{|t|>\delta} \varrho(R_{s,it}) < 1 - \alpha.$$

Proof. For $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, using the definition of operators P_{s+it} and $R_{s,it}$ (see (7.3.1) and (7.3.5)), together with the change of measure formula (7.3.3), we have that for any $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$,

$$R^n_{s,it}(\varphi) = e^{-n\Lambda(s)-itn\Lambda'(s)}\frac{P^n_{s+it}(\varphi r_s)}{r_s}.$$

By Proposition 7.3.1, we have that, uniformly in $s \in (-\eta, \eta)$, the eigenfunction r_s is strictly positive and bounded on $\mathbb{P}^1$. This yields that

$$\varrho(R_{s,it}) \leqslant e^{-\Lambda(s)}\varrho(P_{s+it}).$$

According to Lemma 7.3.3, the spectral radius $\varrho(P_{s+it})$ is strictly less than 1, uniformly in $s \in (-\eta, \eta)$ and $|t| \geqslant \delta$. Noting that $\Lambda(0) = 0$ and the function Λ is continuous in a neighborhood of 0, the desired result follows by taking $\eta > 0$ small enough. □

7.4 Regularity of the stationary measure

This section is devoted to establishing the Hölder regularity of the stationary measure π_s defined in (7.3.4). Recall that when $s = 0$, we have $\pi_0 = \nu$, where the stationary measure ν is defined in (7.2.2). The Hölder regularity of the stationary measure ν has been established in [51] (see also [13, 48, 10]): under conditions **B1** and **B2**, there exists a constant $\alpha > 0$ such that

$$\sup_{f\in\mathbb{P}^1}\int_{\mathbb{P}^1}\frac{1}{|\langle f,x\rangle|^\alpha}\nu(dx) < +\infty. \tag{7.4.1}$$

Using (7.4.1) and the spectral gap properties of the transfer operator P_z shown in Proposition 7.3.1, the following result gives the Hölder regularity of the stationary measure π_s uniformly with respect to s in a small neighborhood of 0.

Theorem 7.4.1. *Assume conditions **B1** and **B2**. Then, there exist constants $\eta, \alpha > 0$ such that*

$$\sup_{s\in(-\eta,\eta)}\sup_{f\in\mathbb{P}^1}\int_{\mathbb{P}^1}\frac{1}{|\langle f,x\rangle|^\alpha}\pi_s(dx) < +\infty. \tag{7.4.2}$$

In particular, there exist constants $c, C > 0$ such that for any $0 < t < 1$, we have

$$\sup_{s\in(-\eta,\eta)}\sup_{f\in\mathbb{P}^1}\pi_s\Big(\{x\in\mathbb{P}^1 : |\langle f,x\rangle| \leqslant t\}\Big) \leqslant Ct^c. \tag{7.4.3}$$

Proof. Step 1. We choose a small enough constant $\eta > 0$ and we show that for any $\varepsilon > 0$, there exist $c_0 > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$,

$$\sup_{s\in(-\eta,\eta)}\sup_{f,x\in\mathbb{P}^1}\mathbb{Q}_s^x\Big(|\langle f,X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant e^{-c_0 n}. \tag{7.4.4}$$

To prove this, using (7.3.3), we write

$$\mathbb{Q}_s^x\Big(|\langle f,X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) = \mathbb{E}\left(\frac{|G_nx|^s}{\kappa^n(s)}\frac{r_s(X_n^x)}{r_s(x)}\mathbb{1}_{\{|\langle f,X_n^x\rangle|\leqslant e^{-\varepsilon n}\}}\right).$$

By Proposition 7.3.1, the eigenfunction $x \mapsto r_s(x)$ is strictly positive and bounded on $\mathbb{P}^1$, uniformly with respect to $s \in (-\eta,\eta)$. It follows that

$$\mathbb{Q}_s^x\Big(|\langle f,X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant c\mathbb{E}\left(\frac{|G_nx|^s}{\kappa^n(s)}\mathbb{1}_{\{|\langle f,X_n^x\rangle|\leqslant e^{-\varepsilon n}\}}\right).$$

Using Hölder's inequality, we get

$$\mathbb{Q}_s^x\Big(|\langle f,X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant \frac{c}{\kappa^n(s)}\Big(\mathbb{E}|G_nx|^{2s}\Big)^{1/2}\Big[\mathbb{P}\Big(|\langle f,X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big)\Big]^{1/2}. \tag{7.4.5}$$

It was shown in [10, Proposition 14.3] that there exist $c_1 > 0$ and $n_0 \in \mathbb{N}$, such that for $n \geqslant n_0$, uniformly in $f, x \in \mathbb{P}^1$,

$$\mathbb{P}\Big(|\langle f,X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big) \leqslant e^{-c_1 n}. \tag{7.4.6}$$

Let us now give a control of $\mathbb{E}|G_n x|^{2s}$. By Proposition 7.3.1, we have

$$\mathbb{E}|G_n x|^{2s} = P_{2s}^n \mathbf{1}(x) = \kappa^n(2s)(M_{2s}\mathbf{1})(x) + (L_{2s}^n \mathbf{1})(x).$$

Using Proposition 7.3.1 again, we get that uniformly in $s \in (-\eta, \eta)$ and $x \in \mathbb{P}^1$, the first term on the right-hand side of the above equality is bounded by $c\kappa^n(2s)$, and the second term is bounded by ce^{-cn}. Hence we have $\mathbb{E}|G_n x|^{2s} \leqslant c\kappa^n(2s) + ce^{-cn}$, uniformly in $s \in (-\eta, \eta)$ and $x \in \mathbb{P}^1$. Since $\kappa(0) = 1$ and the function κ is continuous in a small neighborhood of 0, we deduce that there exists $\eta > 0$ such that

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathbb{P}^1} \frac{1}{\kappa^n(s)} \left(\mathbb{E}|G_n x|^{2s}\right)^{1/2} \leqslant e^{c_2 n},$$

where the constant $c_2 \in (0, c_1/2)$. This, together with (7.4.5)-(7.4.6), concludes the proof of the desired bound (7.4.4).

Step 2. From Proposition 7.3.1 and the construction of $\mathbb{Q}_s^x$, one can verify that for any $x \in \mathbb{P}^1$ and $n \geqslant 1$, $\pi_s = (\mathbb{Q}_s^x)^{*n} * \pi_s$. Combining this with (7.4.4), we get that, uniformly in $s \in (-\eta, \eta)$ and $f \in \mathbb{P}^1$,

$$\pi_s\Big(\{x : |\langle f, x\rangle| \leqslant e^{-\varepsilon n}\}\Big) = \int_{\mathbb{P}^1} (\mathbb{Q}_s^x)^{*n}\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon n}\Big)\pi_s(dx) \leqslant e^{-c_0 n}. \tag{7.4.7}$$

For any $n \in \mathbb{N}$ and $f \in \mathbb{P}^1$, denote $B_{f,n} := \{x \in \mathbb{P}^1 : e^{-\varepsilon(n+1)} \leqslant |\langle f, x\rangle| \leqslant e^{-\varepsilon n}\}$. Choosing $\alpha \in (0, c_0/\varepsilon)$, we deduce from (7.4.7) that, uniformly in $s \in (-\eta, \eta)$ and $f \in \mathbb{P}^1$,

$$\begin{aligned}
&\int_{\mathbb{P}^1} \frac{1}{|\langle f, x\rangle|^\alpha} \pi_s(dx) \\
&= \int_{\{x\in\mathbb{P}^1 : |\langle f,x\rangle| > e^{-\varepsilon n_0}\}} \frac{1}{|\langle f, x\rangle|^\alpha} \pi_s(dx) + \sum_{n=n_0}^{\infty} \int_{B_{f,n}} \frac{1}{|\langle f, x\rangle|^\alpha} \pi_s(dx) \\
&\leqslant e^{\varepsilon n_0 \alpha} + \sum_{n=n_0}^{\infty} e^{\varepsilon\alpha(n+1)} e^{-c_0 n} < +\infty.
\end{aligned}$$

This ends the proof of Theorem 7.4.1. □

7.5 The saddle point approximation

7.5.1 Preliminaries

For any integrable function $h : \mathbb{R} \to \mathbb{C}$, denote its Fourier transform by $\widehat{h}(t) = \int_{\mathbb{R}} e^{-ity} h(y)dy$, $t \in \mathbb{R}$. If $\widehat{h}$ is integrable on $\mathbb{R}$, then using the inverse Fourier transform gives $h(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ity} \widehat{h}(t)dt$, for almost all $y \in \mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$. Denote by $h_1 * h_2$ the convolution of the functions h_1 and h_2 on the real line.

For $s > 0$, let

$$\psi_s(u) = e^{-su} \mathbb{1}_{\{u \geqslant 0\}}, \quad u \in \mathbb{R}.$$

For $0 < \varepsilon < 1$ and $u \in \mathbb{R}$, set $\mathbb{B}_\varepsilon(u) = \{u' \in \mathbb{R} : |u' - u| \leqslant \varepsilon\}$. With this notation, define

$$\psi_{s,\varepsilon}^+(u) = \sup_{u'\in\mathbb{B}_\varepsilon(u)} \psi_s(u'), \quad \psi_{s,\varepsilon}^-(u) = \inf_{u'\in\mathbb{B}_\varepsilon(u)} \psi_s(u'), \quad u \in \mathbb{R}. \tag{7.5.1}$$

That is, $\psi^+_{s,\varepsilon}(u) = 0$ when $u < -\varepsilon$; $\psi^+_{s,\varepsilon}(u) = 1$ when $u \in [-\varepsilon, \varepsilon]$; $\psi^+_{s,\varepsilon}(u) = e^{-s(u-\varepsilon)}$ when $u > \varepsilon$. Similarly, we have $\psi^-_{s,\varepsilon}(u) = e^{-s(u+\varepsilon)}\mathbb{1}_{\{u \geqslant \varepsilon\}}$ for $u \in \mathbb{R}$. By elementary calculations, one can give the explicit expressions for Fourier transforms of $\psi^+_{s,\varepsilon}$ and $\psi^-_{s,\varepsilon}$:

$$\widehat{\psi}^+_{s,\varepsilon}(t) = 2\frac{\sin(\varepsilon t)}{t} + e^{-i\varepsilon t}\frac{1}{s+it}, \quad \widehat{\psi}^-_{s,\varepsilon}(t) = e^{-2\varepsilon s}\frac{e^{-i\varepsilon t}}{s+it}, \quad t \in \mathbb{R}. \tag{7.5.2}$$

These two functions can be extended analytically to a small neighborhood of 0 in the complex plane: for any $z \in \mathbb{C}$ with $|z| < s$,

$$\widehat{\psi}^+_{s,\varepsilon}(z) = 2\frac{\sin(\varepsilon z)}{z} + e^{-i\varepsilon z}\frac{1}{s+iz}, \quad \widehat{\psi}^-_{s,\varepsilon}(z) = e^{-2\varepsilon s}\frac{e^{-i\varepsilon z}}{s+iz}. \tag{7.5.3}$$

Note that $z = 0$ is a removable singular point of the function $z \mapsto \frac{\sin(\varepsilon z)}{z}$. From (7.5.2) we see that the functions $\widehat{\psi}^+_{s,\varepsilon}$ and $\widehat{\psi}^-_{s,\varepsilon}$ are not integrable on the real line. We will see in the proof of Theorem 7.6.3 that smoothing techniques are required. From now on let us fix a non-negative density function ρ on $\mathbb{R}$ with compact support $[-1, 1]$, whose Fourier transform $\widehat{\rho}$ is integrable on $\mathbb{R}$ and has an analytic extension in a neighborhood of 0 in the complex plane $\mathbb{C}$. For any $0 < \varepsilon < 1$, define the scaled density function $\rho_\varepsilon(y) = \frac{1}{\varepsilon}\rho(\frac{y}{\varepsilon})$, $y \in \mathbb{R}$, which has a compact support on $[-\varepsilon^{-1}, \varepsilon^{-1}]$.

7.5.2 The saddle point approximation

In the sequel, for any fixed $y > 1$, we shall choose $s > 0$ satisfying the following equation:

$$\Lambda'(s) - \Lambda'(0) = \frac{\sigma y}{\sqrt{n}}. \tag{7.5.4}$$

For brevity we denote $\sigma_s = \sqrt{\Lambda''(s)}$. Then $\sigma_s > 0$ uniformly in $s \in (0, \eta)$ since the function Λ is strictly convex in a small neighborhood of 0.

Proposition 7.5.1. *Assume conditions **B1** and **B2**. Let $\psi^-_{s,\varepsilon}(t)$ be defined in (7.5.1). Suppose that $s > 0$ satisfies the equation (7.5.4). Then, for any $0 < \varepsilon < 1$ and sufficiently small $\eta > 0$, we have, as $n \to \infty$, uniformly in $s \in (0, \eta)$, $x \in \mathbb{P}^1$, $y \in [1, o(\sqrt{n})]$, $\varphi \in \mathcal{B}_\gamma$ and $l \in \mathbb{R}$ with $|l| = o(s)$,*

$$\begin{aligned}&\left| s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R^n_{s,it}(\varphi)(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt - \sqrt{2\pi}\pi_s(\varphi)\right| \\ &\leqslant c\left(\frac{1}{\sqrt{n}} + |l|\right)\|\varphi\|_\gamma + c\left(\frac{|l|}{s} + \frac{1}{s^2 n}\right)\|\varphi\|_\infty.\end{aligned}$$

Proof. It is sufficient to prove the result in the case when φ is nonnegative. Taking $\delta > 0$ small enough, we use Proposition 7.3.2 to decompose the integral into three parts:

$$s\sigma_s\sqrt{n}\, e^{-nh_s(l)} \int_{\mathbb{R}} e^{-itln} R^n_{s,it}(\varphi)(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt = I_1(n) + I_2(n) + I_3(n), \tag{7.5.5}$$

where

$$I_1(n) = s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{|t|\geqslant\delta} e^{-itln} R^n_{s,it}(\varphi)(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt,$$
$$I_2(n) = s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} N^n_{s,it}(\varphi)(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt,$$
$$I_3(n) = s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} \lambda^n_{s,it}\Pi^n_{s,it}(\varphi)(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt.$$

For simplicity we denote $K_s(z) = \log \lambda_{s,z}$ and we choose the branch where $K_s(0) = 0$. It follows from (7.3.6) and Taylor's formula that for any $z \in \mathbb{C}$ with $|z| < \delta$,

$$K_s(z) = \sum_{k=2}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} z^k, \quad \text{where } \Lambda(s) = \log \kappa(s). \tag{7.5.6}$$

Since $\gamma_2 = \sigma^2 > 0$, we deduce that for any $y > 1$ and sufficiently large n, the equation (7.5.4) has a unique solution given by

$$s = \frac{1}{\gamma_2^{1/2}} \frac{y}{\sqrt{n}} - \frac{\gamma_3}{2\gamma_2^2}\left(\frac{y}{\sqrt{n}}\right)^2 - \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{6\gamma_2^{7/2}}\left(\frac{y}{\sqrt{n}}\right)^3 + \cdots. \tag{7.5.7}$$

For sufficiently large n, the series on the right-hand side of (7.5.7) is absolutely convergent according to the theorem on the inversion of analytic functions. In addition, from (7.5.4) and $y = o(\sqrt{n})$ we see that $s \to 0^+$ as $n \to \infty$, so that we can assume $s \in (0, \eta)$ for sufficiently small constant $\eta > 0$.

Estimate of $I_1(n)$. By Lemma 7.3.4, there exist constants $c, C > 0$ such that

$$\sup_{s\in(0,\eta)} \sup_{|t|\geqslant\delta} \sup_{x\in\mathbb{P}^1} |R^n_{s,it}\varphi(x)| \leqslant Ce^{-cn}\|\varphi\|_\gamma. \tag{7.5.8}$$

From (7.5.2) and the fact that ρ_{ε^2} is a density function on $\mathbb{R}$, we see that

$$\sup_{t\in\mathbb{R}} |\widehat{\psi}^-_{s,\varepsilon}(t)| \leqslant \widehat{\psi}^-_{s,\varepsilon}(0) = \frac{1}{s}e^{-2\varepsilon s}, \quad \sup_{t\in\mathbb{R}} |\widehat{\rho}_{\varepsilon^2}(t)| \leqslant \widehat{\rho}_{\varepsilon^2}(0) = 1. \tag{7.5.9}$$

Combining (7.5.8) and the first inequality in (7.5.9), and taking into account that the function $\widehat{\rho}_{\varepsilon^2}$ is integrable on $\mathbb{R}$, we obtain

$$|I_1(n)| \leqslant Ce^{-cn}\sqrt{n}\, e^{nh_s(l)}e^{-2\varepsilon s}\|\varphi\|_\gamma \leqslant Ce^{-cn}\|\varphi\|_\gamma. \tag{7.5.10}$$

Estimate of $I_2(n)$. Using the bound (7.3.7), we have that uniformly in $s \in (0, \eta)$, $t \in [-\delta, \delta]$ and $x \in \mathbb{P}^1$,

$$|N^n_{s,it}(\varphi)(x)| \leqslant \|N^n_{s,it}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma}\|\varphi\|_\gamma \leqslant Ce^{-cn}\|\varphi\|_\gamma.$$

Together with (7.5.9), this implies

$$|I_2(n)| \leqslant Ce^{-cn}\sqrt{n}\, e^{nh_s(l)}e^{-2\varepsilon s}\|\varphi\|_\gamma \leqslant Ce^{-cn}\|\varphi\|_\gamma. \tag{7.5.11}$$

Estimate of $I_3(n)$. The saddle point method plays a crucial role in deriving a precise asymptotic expansion for the integral $I_3(n)$. For brevity, we define the function $\Psi_{s,x}$ as follows: for $z \in \mathbb{C}$ with $|z| < 2\delta$,

$$\Psi_{s,x}(z) := \Pi_{s,iz}(\varphi)(x)\widehat{\psi}^-_{s,\varepsilon}(z)\widehat{\rho}_{\varepsilon^2}(z), \tag{7.5.12}$$

which is analytic in the open disc $B_{2\delta}(0) = \{z \in \mathbb{C} : |z| < 2\delta\}$ due to the analyticity of the mappings $z \mapsto \Pi_{s,iz}$, $z \mapsto \widehat{\psi}^{-}_{s,\varepsilon}(z)$ and $z \mapsto \widehat{\rho}_{\varepsilon^2}(z)$. Recalling that $K_s(z) = \log \lambda_{s,z}$, making a change of variable $z = it$, we rewrite $I_3(n)$ as

$$I_3(n) = -is\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{-i\delta}^{i\delta} e^{n(K_s(z)-zl)}\Psi_{s,x}(-iz)dz. \tag{7.5.13}$$

Consider the saddle point equation: $K_s'(z) - l = 0$, that is,

$$\sum_{k=2}^{\infty} \gamma_{s,k} \frac{z^{k-1}}{(k-1)!} = l, \quad \text{where } \gamma_{s,k} = \Lambda^{(k)}(s). \tag{7.5.14}$$

This equation has a unique solution z_s which is called the saddle point given by

$$z_s = z_s(l) := \frac{1}{\gamma_{s,2}} l - \frac{\gamma_{s,3}}{2\gamma_{s,2}^3} l^2 - \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{6\gamma_{s,2}^5} l^3 + \cdots . \tag{7.5.15}$$

Similarly as in (7.5.7), for sufficiently small l, the series on the right-hand side of (7.5.15) is absolutely convergent by the theorem of the inversion of analytic function. It follows from (7.5.15) that z_s is real-valued and $z_s \to 0$ as $l \to 0$. Moreover, $z_s > 0$ for sufficiently small $l > 0$, and $z_s < 0$ for sufficiently small $l < 0$. Since the function $\Psi_{s,x}$ defined in (7.5.12) is analytic in the open disc $B_{2\delta}(0)$, applying Cauchy's integral theorem, we are allowed to choose a integration path which passes through the saddle point z_s to rewrite $I_3(n)$ as

$$I_3(n) = -is\sigma_s\sqrt{n}\, e^{nh_s(l)} \Big\{ \int_{L_1} + \int_{L_2} + \int_{L_3} \Big\} e^{n(K_s(z)-zl)}\Psi_{s,x}(-iz)dz. \tag{7.5.16}$$

where $L_1 = (-i\delta, z_s - i\delta)$, $L_2 = (z_s - i\delta, z_s + i\delta)$, $L_3 = (z_s + i\delta, i\delta)$.

The remaining part of the proof consists of giving precise estimates of these integrals over L_1, L_2, L_3 in (7.5.16).

We first deal with the integrals over L_1 and L_3. From (7.3.7), we get that $|\Pi_{s,z}(\varphi)(x)| \leqslant c\|\varphi\|_\infty$ uniformly in $s \in (0,\eta)$, $|z| \leqslant 2\delta$ and $x \in \mathbb{P}^1$. In view of (7.5.3), by simple calculations we see that $|\widehat{\psi}^{-}_{s,\varepsilon}(-iz)| \leqslant \frac{c}{\delta}$, uniformly in $s \in (0,\eta)$ and $z \in L_1 \cup L_3$. Since $\widehat{\rho}_{\varepsilon^2}$ is analytic in a small neighbourhood of 0, we have $|\widehat{\rho}_{\varepsilon^2}(-iz)| \leqslant c$ for all $z \in L_1 \cup L_3$. Hence, we get

$$\sup_{s\in(0,\eta)} \sup_{z\in L_1\cup L_3} \sup_{x\in\mathbb{P}^1} |\Psi_{s,x}(-iz)| \leqslant \frac{c}{\delta}\|\varphi\|_\infty \leqslant C\|\varphi\|_\infty. \tag{7.5.17}$$

Recall that from (7.5.15) we have that $z_s \to 0$ as $l \to 0$. From (7.5.6), we get $K_s(it) = -\frac{1}{2}\sigma_s^2 t^2 + O(t^3)$, which implies that $|e^{nK_s(it)}| \leqslant e^{-\frac{n}{3}\sigma_s^2 t^2}$, for sufficiently small t. Combining this with the continuity of the mapping $z \mapsto K_s(z)$ in the neighborhood of 0 yields that $|e^{nK_s(z)}| \leqslant e^{-\frac{n}{4}\sigma_s^2\delta^2}$, for any $z \in L_1 \cup L_3$. Since $lz_s > 0$ for sufficiently small l, it holds that $|e^{-nzl}| \leqslant 1$ for $z \in L_1 \cup L_3$. Consequently, using (7.5.17), for sufficiently large n, we obtain that uniformly in $s \in (0,\eta)$, $x \in \mathbb{P}^1$ and $|l| = o(1)$,

$$\left| -i \left\{ \int_{L_1} + \int_{L_3} \right\} e^{n(K_s(z)-zl)}\Psi_{s,x}(-iz)dz \right| \leqslant Cse^{-cn}\|\varphi\|_\infty. \tag{7.5.18}$$

In the sequel, the proof is devoted to dealing with the integral over L_2 in (7.5.16). We make a change of variable $z = z_s + it$ to get

$$\begin{aligned}
&- is\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{L_2} e^{n(K_s(z)-zl)}\Psi_{s,x}(-iz)dz \\
&= s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{-\delta}^{\delta} e^{n[K_s(z_s+it)-(z_s+it)l]}\Psi_{s,x}(t-iz_s)dt = I_{31}(n) + I_{32}(n),
\end{aligned}$$

where we assume $n \geqslant 3$ and

$$\begin{aligned}
I_{31}(n) &= s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{n^{-\frac{1}{2}}\log n\leqslant|t|\leqslant\delta} e^{n[K_s(z_s+it)-(z_s+it)l]}\Psi_{s,x}(t-iz_s)dt, \\
I_{32}(n) &= s\sigma_s\sqrt{n}\, e^{nh_s(l)} \int_{|t|<n^{-\frac{1}{2}}\log n} e^{n[K_s(z_s+it)-(z_s+it)l]}\Psi_{s,x}(t-iz_s)dt.
\end{aligned}$$

To control $I_{31}(n)$ and $I_{32}(n)$, we first derive an expansion for $K_s(z_s+it)-(z_s+it)l$. Using Taylor's formula for K_s, we get that for $|t|<\delta$,

$$\begin{aligned}
K_s(z_s+it)-(z_s+it)l &= K_s(z_s) - z_s l + \sum_{k=2}^{\infty}\frac{K_s^{(k)}(z_s)(it)^k}{k!} \\
&=: -h_s(l) + \sum_{k=2}^{\infty}\frac{K_s^{(k)}(z_s)(it)^k}{k!},
\end{aligned} \tag{7.5.19}$$

where using (7.5.6) and (7.5.15) we have

$$h_s(l) = -\frac{l^2}{2\sigma_s^2} + O(l^3). \tag{7.5.20}$$

Hence $I_{31}(n)$ and $I_{32}(n)$ can be rewritten as follows:

$$\begin{aligned}
I_{31}(n) &= s\sigma_s\sqrt{n}\int_{n^{-\frac{1}{2}}\log n\leqslant|t|\leqslant\delta} e^{n\sum_{k=2}^{\infty}\frac{K_s^{(k)}(z_s)(it)^k}{k!}}\Psi_{s,x}(t-iz_s)dt, \\
I_{32}(n) &= s\sigma_s\sqrt{n}\int_{|t|<n^{-\frac{1}{2}}\log n} e^{n\sum_{k=2}^{\infty}\frac{K_s^{(k)}(z_s)(it)^k}{k!}}\Psi_{s,x}(t-iz_s)dt.
\end{aligned}$$

Estimate of $I_{31}(n)$. Since $\Lambda''(s)>0$, using (7.5.6) we see that $K_s^{(k)}(z_s)>0$ and thus $\Re(\sum_{k=2}^{\infty}\frac{K_s^{(k)}(z_s)(it)^k}{k!}) < -\frac{1}{8}\sigma_s^2t^2 < -ct^2$, uniformly in $s\in(0,\eta)$. By (7.5.3), it holds uniformly in $n^{-\frac{1}{2}}\log n \leqslant |t| \leqslant \delta$ that

$$|\widehat{\psi}_{s,\varepsilon}^{-}(t-iz_s)| = \left|\frac{e^{-2\varepsilon s-i\varepsilon t+\varepsilon z_s}}{s+z_s+it}\right| \leqslant \frac{c}{\sqrt{(s+z_s)^2+t^2}} \leqslant \frac{c}{|s+z_s|} \leqslant \frac{c}{s}.$$

This, together with the fact that the functions $z\mapsto\Pi_{s,z}$ and $z\mapsto\widehat{\rho}_{\varepsilon^2}(z)$ are uniformly bounded in a small neighborhood of 0 in the complex plane, yields that $|\Psi_{s,x}(t-iz_s)|\leqslant \frac{c}{s}\|\varphi\|_\infty$, uniformly in $s\in(0,\eta)$, $x\in\mathbb{P}^1$ and $n^{-\frac{1}{2}}\log n\leqslant|t|\leqslant\delta$. Consequently, we get

$$|I_{31}(n)| \leqslant c\|\varphi\|_\infty\sqrt{n}\int_{n^{-\frac{1}{2}}\log n\leqslant|t|\leqslant\delta} e^{-cnt^2}dt \leqslant Ce^{-c(\log n)^2}\|\varphi\|_\infty. \tag{7.5.21}$$

Estimate of $I_{32}(n)$. Since $\sigma_s = \sqrt{\Lambda''(s)} > 0$, by a change of variable $t' = t\sigma_s\sqrt{n}$, we get

$$I_{32}(n) = s\int_{-\sigma_s \log n}^{\sigma_s \log n} e^{-\frac{t^2}{2}} e^{\sum_{k=3}^{\infty} \frac{\alpha_k(s)(it)^k}{k! n^{k/2-1}}} \Psi_{s,x}\Big(\frac{t}{\sqrt{n\Lambda''(s)}} - iz_s\Big)dt,$$

where $\alpha_k(s) = \frac{K_s^{(k)}(z_s)}{\sigma_s^k}$. Note that as $n \to \infty$, uniformly in $|t| \leqslant \sigma_s \log n$ and $s \in (0, \eta)$, it holds that

$$e^{\sum_{k=3}^{\infty} \frac{\alpha_k(s)(it)^k}{k! n^{k/2-1}}} = 1 - \frac{i\alpha_3(s)t^3}{6\sqrt{n}} + O\Big(\frac{\log^6 n}{n}\Big). \tag{7.5.22}$$

For brevity, we denote

$$t_n = \frac{t}{\sigma_s\sqrt{n}} - iz_s. \tag{7.5.23}$$

In view of (7.5.12), we decompose the function $\Psi_{s,x}$ into four terms:

$$\Psi_{s,x}(t_n) = h_1(t_n) + h_2(t_n) + h_3(t_n) + h_4(t_n),$$

where

$$\begin{aligned}
h_1(t_n) &= [\Pi_{s,it_n}(\varphi)(x) - \pi_s(\varphi)]\widehat{\psi}^-_{s,\varepsilon}(t_n)\widehat{\rho}_{\varepsilon^2}(t_n),\\
h_2(t_n) &= \pi_s(\varphi)\widehat{\psi}^-_{s,\varepsilon}(t_n)[\widehat{\rho}_{\varepsilon^2}(t_n) - \widehat{\rho}_{\varepsilon^2}(0)],\\
h_3(t_n) &= \pi_s(\varphi)[\widehat{\psi}^-_{s,\varepsilon}(t_n) - \widehat{\psi}^-_{s,\varepsilon}(0)]\widehat{\rho}_{\varepsilon^2}(0),\\
h_4(t_n) &= \pi_s(\varphi)\widehat{\psi}^-_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0).
\end{aligned}$$

Then we have

$$I_{32}(n) = J_1(n) + J_2(n) + J_3(n) + J_4(n), \tag{7.5.24}$$

where for $j = 1, \ldots, 4$,

$$J_j(n) = s\int_{-\sigma_s \log n}^{\sigma_s \log n} e^{-\frac{t^2}{2}} \left[1 - \frac{i\alpha_3(s)t^3}{6\sqrt{n}} + O\Big(\frac{\log^6 n}{n}\Big)\right] h_j(t_n)dt.$$

Estimate of $J_1(n)$. Using Proposition 7.3.2 and (7.5.15), we have, uniformly in $x \in \mathbb{P}^1$, $s \in (0, \eta)$ and $|t| \leqslant \sigma_s \log n$,

$$|\Pi_{s,it_n}(\varphi)(x) - \pi_s(\varphi)| \leqslant c|t_n|\|\varphi\|_\gamma \leqslant c\left(\frac{|t|}{\sqrt{n}} + |l|\right)\|\varphi\|_\gamma.$$

From (7.5.3), we get

$$|\widehat{\psi}^-_{s,\varepsilon}(t_n)| = \left|\frac{e^{-2\varepsilon s - i\varepsilon t_n}}{s + it_n}\right| \leqslant \left|\frac{c}{s + it_n}\right| = \frac{c}{\sqrt{(s + z_s)^2 + \frac{t^2}{\sigma_s^2 n}}} \leqslant \frac{c}{|s + z_s|} \leqslant \frac{c}{s}, \tag{7.5.25}$$

where in the last inequality we use (7.5.7) and (7.5.15). Combining the above two inequalities with (7.5.9) gives $|h_1(t_n)| \leqslant \frac{c}{s}(\frac{|t|}{\sqrt{n}} + |l|)\|\varphi\|_\gamma$. Hence, we obtain that uniformly in $s \in (0,\eta)$, $x \in \mathbb{P}^1$ and $\varphi \in \mathcal{B}_\gamma$,

$$|J_1(n)| \leqslant c\left(\frac{1}{\sqrt{n}} + |l|\right)\|\varphi\|_\gamma. \tag{7.5.26}$$

Estimate of $J_2(n)$. It is easy to verify that

$$|\widehat{\rho}_{\varepsilon^2}(t_n) - \widehat{\rho}_{\varepsilon^2}(0)| \leqslant \frac{c}{\varepsilon^4}|t_n| \leqslant \frac{c}{\varepsilon^4}\left(\frac{|t|}{\sqrt{n}} + |l|\right).$$

Combining this with (7.5.25) gives $|h_2(t_n)| \leqslant \frac{c}{\varepsilon^4}\frac{1}{s}(\frac{|t|}{\sqrt{n}} + |l|)\|\varphi\|_\infty$. Hence, uniformly in $s \in (0,\eta)$, $x \in \mathbb{P}^1$ and $\varphi \in \mathcal{B}_\gamma$,

$$|J_2(n)| \leqslant c\left(\frac{1}{\sqrt{n}} + |l|\right)\|\varphi\|_\infty. \tag{7.5.27}$$

Estimate of $J_3(n)$. A careful quantitative analysis is required to provide a precise estimate of $J_3(n)$. In view of (7.5.23), we have

$$\widehat{\psi}^-_{s,\varepsilon}(t_n) - \widehat{\psi}^-_{s,\varepsilon}(0) = e^{-2\varepsilon s}\frac{e^{-i\varepsilon t_n} - 1}{s + it_n} + e^{-2\varepsilon s}\left(\frac{1}{s + it_n} - \frac{1}{s}\right).$$

Using the inequality $|e^z - 1| \leqslant e^{\Re z}|z|$ for $z \in \mathbb{C}$, and taking into account of (7.5.23) and(7.5.25), we have

$$e^{-2\varepsilon s}\left|\frac{e^{-i\varepsilon t_n} - 1}{s + it_n}\right| = e^{-2\varepsilon s}\frac{|e^{-i\varepsilon t_n} - 1|}{|s + it_n|} \leqslant \frac{c}{s}\left(\frac{|t|}{\sqrt{n}} + |l|\right).$$

Thus, using the fact that $\widehat{\rho}_{\varepsilon^2}(0) = 1$ and $|\pi_s(\varphi)| \leqslant c\|\varphi\|_\infty$,

$$\begin{aligned}&\left|s\int_{-\sigma_s\log n}^{\sigma_s\log n} e^{-\frac{t^2}{2}}\left[1 - \frac{i\alpha_3(s)t^3}{6\sqrt{n}} + O(\frac{\log^6 n}{n})\right]e^{-2\varepsilon s}\frac{e^{-i\varepsilon t_n} - 1}{s + it_n}dt\widehat{\rho}_{\varepsilon^2}(0)\pi_s(\varphi)\right| \\ &\leqslant c\left(\frac{1}{\sqrt{n}} + |l|\right)\|\varphi\|_\infty.\end{aligned} \tag{7.5.28}$$

Note that

$$\frac{1}{s + it_n} - \frac{1}{s} = \frac{1}{s}\frac{-z_s(s + z_s) - is\frac{t}{\sigma_s\sqrt{n}} - \frac{t^2}{\sigma_s^2 n}}{(s + z_s)^2 + \frac{t^2}{\sigma_s^2 n}}.$$

Since the integral of an odd function over a symmetric interval is identically zero, by simple calculations, we obtain

$$\begin{aligned}&\left|s\int_{-\sigma_s\log n}^{\sigma_s\log n} e^{-\frac{t^2}{2}}\left[1 - \frac{i\alpha_3(s)t^3}{6\sqrt{n}} + O(\frac{\log^6 n}{n})\right]\left(\frac{1}{s + it_n} - \frac{1}{s}\right)dt\widehat{\rho}_{\varepsilon^2}(0)\pi_s(\varphi)\right| \\ &\leqslant c\left(\frac{|l|}{|s + l|} + \frac{1}{(s + l)^2 n}\right)\|\varphi\|_\infty \leqslant c\left(\frac{|l|}{s} + \frac{1}{s^2 n}\right)\|\varphi\|_\infty.\end{aligned} \tag{7.5.29}$$

Consequently we conclude from (7.5.28) and (7.5.29) that

$$|J_3(n)| \leqslant c\left(\frac{|l|}{s}+\frac{1}{s^2n}+\frac{1}{\sqrt{n}}\right)\|\varphi\|_\infty. \tag{7.5.30}$$

Estimate of $J_4(n)$. From (7.5.2) and $\widehat{\rho}_{\varepsilon^2}(0)=1$, we see that

$$J_4(n)=\pi_s(\varphi)e^{-2\varepsilon s}\int_{-\sigma_s\log n}^{\sigma_s\log n}e^{-\frac{t^2}{2}}\Big[1+O(\frac{\log^6 n}{n})\Big]dt. \tag{7.5.31}$$

Since $\sqrt{2\pi}>\int_{-\sigma_s\log n}^{\sigma_s\log n}e^{-\frac{t^2}{2}}dt>\sqrt{2\pi}-\frac{1}{n}$ and $e^{-2\varepsilon s}=1+O(s)$, substituting these estimates into (7.5.31) gives

$$J_4(n)=\sqrt{2\pi}\pi_s(\varphi)e^{-2\varepsilon s}\Big[1+O(\frac{\log^6 n}{n})\Big]=\sqrt{2\pi}\pi_s(\varphi)(1+O(s)).$$

It follows that

$$\Big|J_4(n)-\sqrt{2\pi}\pi_s(\varphi)\Big|\leqslant Cs\|\varphi\|_\infty. \tag{7.5.32}$$

In view of (7.5.24), putting the bounds (7.5.26), (7.5.27), (7.5.30) and (7.5.32) together, we obtain that uniformly in $s\in(0,\eta)$, $x\in\mathbb{P}^1$, $\varphi\in\mathcal{B}_\gamma$ and $|l|=o(1)$,

$$\Big|I_{32}(n)-\sqrt{2\pi}\pi_s(\varphi)\Big|\leqslant c\left(\frac{1}{\sqrt{n}}+|l|\right)\|\varphi\|_\gamma+c\left(\frac{|l|}{s}+\frac{1}{s^2n}\right)\|\varphi\|_\infty.$$

Substituting this and (7.5.21), (7.5.18) into (7.5.16), we conclude that

$$\Big|I_3(n)-\sqrt{2\pi}\pi_s(\varphi)\Big|\leqslant c\left(\frac{1}{\sqrt{n}}+|l|\right)\|\varphi\|_\gamma+c\left(\frac{|l|}{s}+\frac{1}{s^2n}\right)\|\varphi\|_\infty.$$

Together with (7.5.10) and (7.5.11), this completes the proof of Proposition 7.5.1. □

For $s<0$, let

$$\phi_s(u)=e^{-su}\mathbb{1}_{\{u\leqslant 0\}},\quad u\in\mathbb{R}.$$

With the notation in (7.5.1), for $0<\varepsilon<1$ the function $\phi^+_{s,\varepsilon}$ is defined as follows: $\phi^+_{s,\varepsilon}(u)=0$ when $u>\varepsilon$; $\phi^+_{s,\varepsilon}(u)=1$ when $u\in[-\varepsilon,\varepsilon]$; $\phi^+_{s,\varepsilon}(u)=e^{-s(u+\varepsilon)}$ when $u<-\varepsilon$. From basic calculations, one can give the explicit expression for the Fourier transform of $\phi^+_{s,\varepsilon}$:

$$\widehat{\phi}^+_{s,\varepsilon}(t)=\int_{\mathbb{R}}e^{-itu}\phi^+_{s,\varepsilon}(u)du=2i\frac{\cos(\varepsilon t)}{t}+e^{i\varepsilon t}\frac{1}{-s-it},\quad t\in\mathbb{R}. \tag{7.5.33}$$

This function can be extended analytically to a small neighborhood of 0 in the complex plane: for any $z\in\mathbb{C}$ with $|z|<s$,

$$\widehat{\phi}^+_{s,\varepsilon}(z)=2i\frac{\cos(\varepsilon z)}{z}+e^{i\varepsilon z}\frac{1}{-s-iz}. \tag{7.5.34}$$

In the sequel, for any fixed $y>1$, we choose $s<0$ satisfying the equation:

$$\Lambda'(s)-\Lambda'(0)=-\frac{\sigma y}{\sqrt{n}}. \tag{7.5.35}$$

Proposition 7.5.2. *Assume conditions* ***B1*** *and* ***B2****. Let* $\phi^+_{s,\varepsilon}(t)$ *be defined in* (7.5.33)*. Suppose that* $s<0$ *satisfies the equation* (7.5.35)*. Then, for any* $0<\varepsilon<1$ *and sufficiently small* $\eta>0$*, we have, uniformly in* $s\in(-\eta,0)$*,* $x\in\mathbb{P}^1$*,* $y\in[1,o(\sqrt{n})]$*,* $\varphi\in\mathcal{B}_\gamma$ *and* $l\in\mathbb{R}$ *with* $|l|=o(|s|)$*,*

$$\begin{aligned}&\Big|-s\sigma_s\sqrt{n}\,e^{nh_s(l)}\int_{\mathbb{R}}e^{-itln}R^n_{s,it}(\varphi)(x)\widehat{\phi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt-\sqrt{2\pi}\pi_s(\varphi)\Big|\\&\leqslant c\left(\frac{1}{\sqrt{n}}+|l|\right)\|\varphi\|_\gamma+c\left(\frac{|l|}{-s}+\frac{1}{s^2n}\right)\|\varphi\|_\infty.\end{aligned}$$

The proof of Proposition 7.5.2 can be carried out in an analogous way as that Proposition 7.5.1. We omit the details.

7.6 Proof of Cramér type moderate deviation expansions

We now formulate the moderate deviation expansion in the normal range $y\in[0,o(n^{1/6})]$ for the couple $(X^x_n,\log|\langle f,G_nx\rangle|)$.

Theorem 7.6.1. *Assume conditions* ***B1*** *and* ***B2****. Then, we have:*

(1) uniformly in $f,x\in\mathbb{P}^1$*,* $y\in[0,\sqrt{\log n}]$ *and* $\varphi\in\mathcal{B}_\gamma$*,*

$$\begin{aligned}\frac{\mathbb{E}\Big[\varphi(X^x_n)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)}&=\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\log n\Big),\\\frac{\mathbb{E}\Big[\varphi(X^x_n)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)}&=\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\log n\Big);\end{aligned}$$

(2) uniformly in $f,x\in\mathbb{P}^1$*,* $y\in[\sqrt{\log n},n^{1/10}]$ *and* $\varphi\in\mathcal{B}_\gamma$*,*

$$\begin{aligned}\frac{\mathbb{E}\Big[\varphi(X^x_n)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)}&=\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{1}{y^2}\Big),\\\frac{\mathbb{E}\Big[\varphi(X^x_n)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)}&=\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{1}{y^2}\Big);\end{aligned}$$

(3) uniformly in $f,x\in\mathbb{P}^1$*,* $y\in[n^{1/10},o(n^{1/6})]$ *and* $\varphi\in\mathcal{B}_\gamma$*,*

$$\begin{aligned}\frac{\mathbb{E}\Big[\varphi(X^x_n)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)}&=\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{y^3}{\sqrt{n}}\Big),\\\frac{\mathbb{E}\Big[\varphi(X^x_n)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\leqslant-\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)}&=\nu(\varphi)+\|\varphi\|_\gamma O\Big(\frac{y^3}{\sqrt{n}}\Big).\end{aligned}$$

The proof of Theorem 7.6.1 is based on the moderate deviation expansion for the norm cocycle $\log|G_nx|$ proved in [86] and on the following lemma established in [10, Lemma 14.11].

Lemma 7.6.2. *Under conditions* ***B1*** *and* ***B2****, for all* $\varepsilon > 0$*, there exist* $c > 0$ *and* $k_0 \in \mathbb{N}$*, such that for all* $n \geqslant k \geqslant k_0$ *and* $f, x \in \mathbb{P}^1$*,*

$$\mathbb{P}\Big(|\langle f, X_n^x\rangle| \leqslant e^{-\varepsilon k}\Big) \leqslant ce^{-ck}.$$

Proof of Theorem 7.6.1. Without loss of generality, we assume that φ is non-negative. The proof of Theorem 7.6.1 consists of establishing the upper and lower bounds. We only establish the first assertion in Theorem 7.6.1 since the proof of the second one can be carried out in a similar way.

Upper bound. Recall that it is proved in [86] that under conditions **B1** and **B2**, it holds uniformly in $x \in \mathbb{P}^1$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$ that

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big]. \tag{7.6.1}$$

Since $\log|\langle f, G_nx\rangle| \leqslant \log|G_nx|$, this implies that there exists a constant $C > 0$ such that uniformly in $f, x \in \mathbb{P}^1$ and $y \in [0, o(\sqrt{n})]$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} \leqslant e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big(\nu(\varphi) + C\|\varphi\|_\gamma\frac{y+1}{\sqrt{n}}\Big). \tag{7.6.2}$$

Lower bounds. By Lemma 7.6.2, we get that for all $\varepsilon > 0$, there exist $c > 0$ and $k_0 \geqslant 1$, such that for all $n \geqslant k \geqslant k_0$, and for all $f, x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}\Big(\log|\langle f, G_nx\rangle| - \log|G_nx| \leqslant -\varepsilon k\Big) \leqslant ce^{-ck}.$$

Using this inequality, we get

$$\begin{aligned}
I_n &\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-\log|G_nx|>-\varepsilon k\}}\Big]\\
&\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\}}\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-\log|G_nx|>-\varepsilon k\}}\Big]\\
&\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\}}\Big] - \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-\log|G_nx|\leqslant-\varepsilon k\}}\Big]\\
&\geqslant \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\}}\Big] - ce^{-ck}\|\varphi\|_\infty.
\end{aligned} \tag{7.6.3}$$

To establish the first expansion in assertion (1), it suffices to prove that uniformly in $f, x \in \mathbb{P}^1$, $y \in [0, \sqrt{\log n}]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f,G_nx\rangle|-n\lambda\geqslant\sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} \geqslant \nu(\varphi) - c\|\varphi\|_\gamma\Big(\frac{y+1}{\sqrt{n}}\log n\Big). \tag{7.6.4}$$

Take $k = [C_1 \log n]$ in (7.6.3) and denote $y_1 = y + \frac{C_1\log n}{\sigma\sqrt{n}}$, where $C_1 > 0$ is a fixed sufficiently large constant. Applying (7.6.1), there exists a constant $c > 0$ such that uniformly in $f, x \in \mathbb{P}^{d-1}$, $y \in [0, \sqrt{\log n}]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|-n\lambda\geqslant\sqrt{n}\sigma y+\varepsilon k\}}\Big]}{1-\Phi(y_1)} = \nu(\varphi) - c\|\varphi\|_\gamma\frac{y_1^3+1}{\sqrt{n}}. \tag{7.6.5}$$

We claim that uniformly in $y \in [0, \sqrt{\log n}]$,

$$1 > \frac{1-\Phi(y_1)}{1-\Phi(y)} = 1 - \frac{\int_y^{y_1} e^{-\frac{t^2}{2}} dt}{\int_y^{\infty} e^{-\frac{t^2}{2}} dt} > 1 - c\frac{y+1}{\sqrt{n}} \log n.$$

Indeed, when $y \in [0,2]$, one can easily see that the inequality holds; when $y \in [2, \sqrt{\log n}]$, one uses the basic inequality $e^{\frac{y^2}{2}} \int_y^{\infty} e^{-\frac{t^2}{2}} dt \geqslant \frac{1}{y} - \frac{1}{y^3} > \frac{1}{2y}$. It is easy to check that $\frac{y_1^3+1}{\sqrt{n}} = O(\frac{y^3+1}{\sqrt{n}})$, uniformly in $y \in [0, \sqrt{\log n}]$. Consequently, we get, uniformly in $f, x \in \mathbb{P}^{d-1}$, $y \in [0, \sqrt{\log n}]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x| - n\lambda \geqslant \sqrt{n}\sigma y + \varepsilon k\}}\Big]}{1-\Phi(y)} \geqslant \nu(\varphi) - c\|\varphi\|_\gamma \frac{y+1}{\sqrt{n}} \log n.$$

This, together with the fact that $e^{-ck}/[1-\Phi(y)]$ decays to 0 faster than $\frac{1}{n}$, concludes the proof of (7.6.4).

To establish the first expansions in assertions (2) and (3), we take $k = C_1 y^2$ in (7.6.3), where $C_1 > 0$ is a fixed sufficiently large constant. In the same way as in (7.6.5), we get that, with $y_1 = y + \frac{C_1 y^2}{\sigma\sqrt{n}}$, uniformly in $f, x \in \mathbb{P}^{d-1}$, $y \in [\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x| - n\lambda \geqslant \sqrt{n}\sigma y + \varepsilon k\}}\Big]}{1-\Phi(y_1)} = \nu(\varphi) - c\|\varphi\|_\gamma \frac{y_1^3+1}{\sqrt{n}}. \tag{7.6.6}$$

Using the inequality $\frac{1}{y} \geqslant e^{\frac{y^2}{2}} \int_y^{\infty} e^{-\frac{t^2}{2}} dt \geqslant \frac{1}{y} - \frac{1}{y^3}$ for $y > 0$, by elementary calculations, we get that uniformly in $y \in [\sqrt{\log n}, o(n^{1/6})]$,

$$1 > \frac{1-\Phi(y_1)}{1-\Phi(y)} \geqslant \frac{y}{y_1}\Big(1 - \frac{1}{y_1^2}\Big) e^{\frac{y^2}{2} - \frac{y_1^2}{2}} > \Big(1 - \frac{c}{y^2}\Big)\Big(1 - \frac{cy^3}{\sqrt{n}}\Big).$$

Taking into account that $\frac{y_1^3+1}{\sqrt{n}} = O(\frac{y^3+1}{\sqrt{n}})$ and that $e^{-cC_1 y^2}/[1-\Phi(y)]$ decays to 0 faster than $\frac{1}{n}$, we obtain the first expansions in (2) and (3). □

Theorem 7.6.3. *Assume conditions **B1** and **B2**. Then, we have, uniformly in $f, x \in \mathbb{P}^1$, $y \in [n^{1/6}, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| - n\lambda \geqslant \sqrt{n}\sigma y\}}\Big]}{1-\Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y\log n}{\sqrt{n}}\Big)\Big],$$

$$\frac{\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| - n\lambda \leqslant -\sqrt{n}\sigma y\}}\Big]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\Big[\nu(\varphi) + \|\varphi\|_\gamma O\Big(\frac{y\log n}{\sqrt{n}}\Big)\Big].$$

Proof. The proof of Theorem 7.6.3 is based on the Cramér type moderate deviation expansion for the norm cocycle $\log|G_n x|$, Proposition 7.5.1, and the Hölder regularity of the stationary measure π_s on the projective space $\mathbb{P}^1$, see Theorem 7.4.1.

Without loss of generality, we assume that φ is non-negative. We first establish the first assertion in Theorem 7.6.3. Using the Cramér type moderate deviation expansion

for the norm cocycle $\log|G_n x|$, the upper bound has been shown in (7.6.2), hence it remains to establish the lower bound.

From the change of measure formula (7.3.3) and the fact $\lambda = \Lambda'(0)$, we get

$$\begin{aligned} A_n &:= \mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|\langle f, G_n x\rangle| - n\Lambda'(0) \geqslant \sqrt{n}\sigma y\}}\Big] \\ &= r_s(x)\kappa^n(s)\mathbb{E}_{\mathbb{Q}_s^x}\Big[(\varphi r_s^{-1})(X_n^x)e^{-s\log|G_n x|}\mathbb{1}_{\{\log|\langle f, G_n x\rangle| \geqslant n\Lambda'(0)+\sqrt{n}\sigma y\}}\Big]. \end{aligned} \tag{7.6.7}$$

Denote $T_n^x = \log|G_n x| - n\Lambda'(s)$ and observing that $\log|\langle f, G_n x\rangle| = \log|G_n x| + \log|f(X_n^x)|$. Choosing $s > 0$ satisfying the equation (7.5.4), it follows from (7.6.7) that

$$A_n = r_s(x)e^{-n[s\Lambda'(s)-\Lambda(s)]}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-sT_n^x}\mathbb{1}_{\{T_n^x + \log|f(X_n^x)| \geqslant 0\}}\right].$$

Using (7.5.4), one can verify that

$$s\Lambda'(s) - \Lambda(s) = \frac{y^2}{2n} - \frac{y^3}{n^{3/2}}\zeta(\frac{y}{\sqrt{n}}), \tag{7.6.8}$$

where ζ is the Cramér series given by (7.2.1). Therefore, A_n can be rewritten as

$$A_n = r_s(x)e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-sT_n^x}\mathbb{1}_{\{T_n^x + \log|f(X_n^x)| \geqslant 0\}}\right].$$

For brevity, set $Y_n^{f,x} := \log|f(X_n^x)|$ and $M_n := [\log n]$, where $[a]$ denotes the integer part of a real number a. For fixed $0 < \delta < 1$, denote

$$I_k := (-\delta k, -\delta(k-1)], \quad k \in \mathbb{N}.$$

Since the functions φ and r_s are positive, we have

$$\begin{aligned} A_n &\geqslant r_s(x)e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\sum_{k=1}^{M_n}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)e^{-sT_n^x}\mathbb{1}_{\{T_n^x + Y_n^{f,x} \geqslant 0\}}\mathbb{1}_{\{Y_n^{f,x} \in I_k\}}\right] \\ &\geqslant r_s(x)e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\sum_{k=1}^{M_n}\mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\mathbb{1}_{\{Y_n^{f,x} \in I_k\}}e^{-sT_n^x}\mathbb{1}_{\{T_n^x - \delta k \geqslant 0\}}\right]. \end{aligned}$$

Introduce $\bar{\rho}_{\varepsilon_1}(x) := \frac{1}{\varepsilon_1}(1 - \frac{|x|}{\varepsilon_1})$ for $x \in [-\varepsilon_1, \varepsilon_1]$, and $\bar{\rho}_{\varepsilon_1}(x) = 0$ otherwise. For any $k \in \mathbb{N}$, denoting $\chi_k(x) := \mathbb{1}_{\{x \in I_k\}}$ and $\chi_{k,\varepsilon_1}^-(x) = \inf_{x' \in \mathbb{B}_{\varepsilon_1}(x)}\chi_k(x')$, one can verify that

$$\chi_{k,2\varepsilon_1}^-(x) \leqslant (\chi_{k,\varepsilon_1}^- * \bar{\rho}_{\varepsilon_1})(x) \leqslant \chi_k(x), \quad x \in \mathbb{R}. \tag{7.6.9}$$

For short, denote $\tilde{\chi}_k(x) := (\chi_{k,\varepsilon_1}^- * \bar{\rho}_{\varepsilon_1})(x)$, $x \in \mathbb{P}^1$, which is Hölder continuous. Let

$$\psi_s(u) = e^{-su}\mathbb{1}_{\{u \geqslant 0\}}, \quad u \in \mathbb{R},$$

and

$$\varphi_{s,k,\varepsilon_1}^f(x) = (\varphi r_s^{-1})(x)\tilde{\chi}_k(\log|\langle f, x\rangle|), \quad x \in \mathbb{P}^1.$$

With the above notation, it follows from the inequalities (7.6.9) and $\psi_s \geqslant \psi_{s,\varepsilon}^- * \rho_{\varepsilon^2}$ that

$$\begin{aligned}
A_n &\geqslant r_s(x) e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \sum_{k=1}^{M_n} \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x)\tilde{\chi}_k(Y_n^x) e^{-sT_n^x} \mathbb{1}_{\{T_n^x-\delta k\geqslant 0\}}\right] \\
&= r_s(x) e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \sum_{k=1}^{M_n} e^{-s\delta k} \mathbb{E}_{\mathbb{Q}_s^x}\left[\varphi_{s,k,\varepsilon_1}^f(X_n^x)\psi_s(T_n^x-\delta k)\right] \\
&\geqslant r_s(x) e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \sum_{k=1}^{M_n} e^{-s\delta k} \mathbb{E}_{\mathbb{Q}_s^x}\left[\varphi_{s,k,\varepsilon_1}^f(X_n^x)(\psi_{s,\varepsilon}^- * \rho_{\varepsilon^2})(T_n^x-\delta k)\right] \\
&=: B_n. \qquad (7.6.10)
\end{aligned}$$

Since the function $\widehat{\rho}_{\varepsilon^2}$ is integrable on $\mathbb{R}$, by the inversion formula, we have

$$\psi_{s,\varepsilon}^- * \rho_{\varepsilon^2}(u) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itu}\widehat{\psi}_{s,\varepsilon}^-(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \quad u \in \mathbb{R}.$$

Substituting $u = T_n^x - \delta k$, taking the expectation with respect to $\mathbb{E}_{\mathbb{Q}_s^x}$, and using the Fubini theorem, we get

$$B_n = \frac{r_s(x)}{2\pi} e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \sum_{k=1}^{M_n} e^{-s\delta k} \int_{\mathbb{R}} e^{-it\delta k} R_{s,it}^n(\varphi_{s,k,\varepsilon_1}^f)(x)\widehat{\psi}_{s,\varepsilon}^-(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \qquad (7.6.11)$$

where

$$R_{s,it}^n(\varphi_{s,k,\varepsilon_1}^f)(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi_{s,k,\varepsilon_1}^f)(X_n^x)e^{itT_n^x}\right].$$

Since $|l| = O(\frac{\log n}{n})$, using (7.5.20), we have that uniformly in $s \in (0,\eta)$,

$$e^{-nh_s(l)} = 1 + O(nl^2) = 1 + O(\frac{\log^2 n}{n}). \qquad (7.6.12)$$

Applying Proposition 7.5.1 with $l = \frac{\delta k}{n}$ and $\varphi = \varphi_{s,k,\varepsilon_1}^f$, and taking into account of (7.6.12), by elementary calculations, we conclude that uniformly in $x \in \mathbb{P}^1$, $k \in [1, M_n]$, and $y \in [n^{1/6}, o(\sqrt{n})]$,

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-it\delta k} R_{s,it}^n(\varphi_{s,k,\varepsilon_1}^f)(x)\widehat{\psi}_{s,\varepsilon}^-(t)\widehat{\rho}_{\varepsilon^2}(t)dt - \pi_s(\varphi_{s,k,\varepsilon_1}^f)\frac{\sqrt{2\pi}}{s\sqrt{n\Lambda''(s)}} \\
&\geqslant -\frac{c}{s\sqrt{n}}\left(\frac{1}{\sqrt{n}} + |l|\right)\|\varphi_{s,k,\varepsilon_1}^f\|_\gamma - \frac{c}{s\sqrt{n}}\left(\frac{|l|}{s} + \frac{1}{s^2 n} + s\right)\|\varphi_{s,k,\varepsilon_1}^f\|_\infty \\
&\geqslant -\frac{c}{y\sqrt{n}}\|\varphi_{s,k,\varepsilon_1}^f\|_\gamma - \frac{c}{\sqrt{n}}\|\varphi_{s,k,\varepsilon_1}^f\|_\infty.
\end{aligned}$$

Using the inequality $\sqrt{2\pi}ye^{\frac{y^2}{2}}[1-\Phi(y)] \geqslant 1$ for any $y > 0$, it follows that

$$\begin{aligned}
&\frac{\int_{\mathbb{R}} e^{-it\delta k} R_{s,it}^n(\varphi_{s,k,\varepsilon_1}^f)(x)\widehat{\psi}_{s,\varepsilon}^-(t)\widehat{\rho}_{\varepsilon^2}(t)dt}{e^{\frac{y^2}{2}}[1-\Phi(y)]} - \pi_s(\varphi_{s,k,\varepsilon_1}^f)\frac{2\pi y}{s\sqrt{n\Lambda''(s)}} \\
&\geqslant -\frac{c}{\sqrt{n}}\|\varphi_{s,k,\varepsilon_1}^f\|_\gamma - \frac{cy}{\sqrt{n}}\|\varphi_{s,k,\varepsilon_1}^f\|_\infty. \qquad (7.6.13)
\end{aligned}$$

Note that $\sqrt{\Lambda''(s)} = \sigma[1+O(s)]$. From (7.5.7), we have $\frac{y}{s\sigma\sqrt{n}} = 1+O(s)$, and thus $\frac{y}{s\sqrt{n\Lambda''(s)}} = 1+O(s)$. Note also that $|r_s(x)-\mathbf{1}| = O(s)$, uniformly in $x \in \mathbb{P}^1$. Therefore, taking the summation with respect to $k \in [1, M_n]$ in (7.6.13), we get

$$\begin{aligned}\frac{B_n}{e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}[1-\Phi(y)]} &\geqslant (1-Cs)\sum_{k=1}^{M_n} e^{-s\delta k}\pi_s(\varphi^f_{s,k,\varepsilon_1}) \\ &\quad - C\sum_{k=1}^{M_n} e^{-s\delta k}\left(\frac{c}{\sqrt{n}}\|\varphi^f_{s,k,\varepsilon_1}\|_\gamma + \frac{cy}{\sqrt{n}}\|\varphi^f_{s,k,\varepsilon_1}\|_\infty\right). \end{aligned} \tag{7.6.14}$$

We now provide an estimate of the first term in (7.6.14). Using (7.6.9), we have

$$\begin{aligned}\sum_{k=1}^{M_n} e^{-s\delta k}\pi_s(\varphi^f_{s,k,\varepsilon_1}) &= \sum_{k=1}^{M_n} e^{-s\delta k}\int_{\mathbb{P}^1}\tilde{\chi}_k(\log|\langle f,x\rangle|)\varphi(x)\pi_s(dx) \\ &\geqslant \sum_{k=1}^{M_n} e^{-s\delta k}\int_{\mathbb{P}^1}\chi^-_{k,2\varepsilon_1}(\log|\langle f,x\rangle|)\varphi(x)\pi_s(dx).\end{aligned}$$

Notice that

$$\chi^-_{k,2\varepsilon_1}(x)\circ\log|f| = \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x) - \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_{k,\varepsilon_1}\}}(x),$$

where

$$I_{k,\varepsilon_1} = \Big(-\delta k, -\delta k+2\varepsilon_1\Big] \cup \Big(-\delta(k-1)-2\varepsilon_1, -\delta(k-1)\Big].$$

It follows that

$$\sum_{k=1}^{M_n} e^{-s\delta k}\pi_s(\varphi^f_{s,k,\varepsilon_1}) \geqslant B_{n,1} - B_{n,2}, \tag{7.6.15}$$

where

$$\begin{aligned}B_{n,1} &= \sum_{k=1}^{M_n} e^{-s\delta k}\int_{\mathbb{P}^1}\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x)\varphi(x)\pi_s(dx), \\ B_{n,2} &= \sum_{k=1}^{M_n} e^{-s\delta k}\int_{\mathbb{P}^1}\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_{k,\varepsilon_1}\}}(x)\varphi(x)\pi_s(dx).\end{aligned}$$

Control of $B_{n,1}$. It is easy to see that

$$\begin{aligned}B_{n,1} &= \sum_{k=1}^{\infty} e^{-s\delta k}\int_{\mathbb{P}^1}\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x)\varphi(x)\pi_s(dx) \\ &\quad - \sum_{k=M_n+1}^{\infty} e^{-s\delta k}\int_{\mathbb{P}^1}\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x)\varphi(x)\pi_s(dx) \\ &= \pi_s(\varphi|\langle f,\cdot\rangle|^s) - \sum_{k=M_n+1}^{\infty} e^{-s\delta k}\int_{\mathbb{P}^1}\mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x)\varphi(x)\pi_s(dx).\end{aligned}$$

To give a bound for the second term in the above equality, we need to apply the regularity of the stationary measure π_s. Specifically, by Theorem 7.4.1, we get that uniformly in $s \in (0, \eta)$,

$$\begin{aligned}&\left|\sum_{k=M_n+1}^{\infty} e^{-s\delta k} \int_{\mathbb{P}^1} \mathbb{1}_{\{\log|\langle f,\cdot\rangle|\in I_k\}}(x)\varphi(x)\pi_s(dx)\right| \\ &\leqslant \|\varphi\|_\infty \pi_s\Big(\{x : |\langle f, x\rangle| \in (0, e^{-\delta M_n}]\Big) \leqslant Ce^{-\alpha\delta M_n}\|\varphi\|_\infty \leqslant \frac{1}{\sqrt{n}}\|\varphi\|_\infty.\end{aligned}$$

Consequently, we obtain

$$B_{n,1} \geqslant \pi_s(\varphi|\langle f, \cdot\rangle|^s) - \frac{1}{\sqrt{n}}\|\varphi\|_\infty. \tag{7.6.16}$$

Control of $B_{n,2}$. Again, the Hölder regularity of the stationary measure π_s is required to handle $B_{n,2}$. Note that

$$B_{n,2} \leqslant \|\varphi\|_\infty \sum_{k=1}^{M_n} \pi_s\left(\{x : \log|\langle f, x\rangle| \in I_{k,\varepsilon_1}\}\right).$$

Observe that

$$\begin{aligned}\pi_s\left(\{x : \log|\langle f, x\rangle| \in I_{k,\varepsilon_1}\}\right) &= \pi_s\Big(\Big\{x : |\langle f, x\rangle| \in (e^{-\delta k}, e^{-\delta k+2\varepsilon_1}]\Big\}\Big) \\ &\quad + \pi_s\Big(\Big\{x : |\langle f, x\rangle| \in (e^{-\delta(k-1)-2\varepsilon_1}, e^{-\delta(k-1)}]\Big\}\Big).\end{aligned} \tag{7.6.17}$$

Notice that there exist $f_1 = f_1(\delta, k, \varepsilon) \in \mathbb{P}^1$ and $f_2 = f_2(\delta, k, \varepsilon) \in \mathbb{P}^1$ such that

$$\begin{aligned}&\pi_s\Big(\Big\{x : |\langle f, x\rangle| \in (e^{-\delta k}, e^{-\delta k+2\varepsilon_1}]\Big\}\Big) \\ &\leqslant \pi_s\Big(\Big\{x : |\langle f_1, x\rangle| \in (0, e^{-\delta k+2\varepsilon_1} - e^{-\delta k}]\Big\}\Big) \\ &\quad + \pi_s\Big(\Big\{x : |\langle f_2, x\rangle| \in (0, e^{-\delta k+2\varepsilon_1} - e^{-\delta k}]\Big\}\Big).\end{aligned} \tag{7.6.18}$$

From Theorem 7.4.1, we get that there exists a constant $c > 0$ such that, uniformly in $f_1 \in \mathbb{P}^1$,

$$\begin{aligned}\pi_s\Big(\Big\{x : |\langle f_1, x\rangle| \in (0, e^{-\delta k+2\varepsilon_1} - e^{-\delta k}]\Big\}\Big) &\leqslant C(e^{-\delta k+2\varepsilon_1} - e^{-\delta k})^c \\ &= Ce^{-c\delta k}(e^{2\varepsilon_1} - 1)^c.\end{aligned}$$

A similar assertion also holds for the second term in (7.6.18). Hence

$$\sum_{k=1}^{M_n} \pi_s\Big(\Big\{x : |\langle f, x\rangle| \in (e^{-\delta k}, e^{-\delta k+2\varepsilon_1}]\Big\}\Big) \leqslant C(e^{2\varepsilon_1} - 1)^c. \tag{7.6.19}$$

In the same way as in the proof of (7.6.19), one can check that

$$\sum_{k=1}^{M_n} \pi_s\left(\Big\{x : |\langle f, x\rangle| \in (e^{-\delta(k-1)-2\varepsilon_1}, e^{-\delta(k-1)}]\Big\}\right) \leqslant C(1 - e^{-2\varepsilon_1})^c. \tag{7.6.20}$$

Combining (7.6.19) and (7.6.20), taking $\varepsilon_1 = n^{-\beta}$ with $\beta > 0$ sufficiently large, we obtain

$$B_{n,2} \leqslant C\|\varphi\|_\infty (e^{2\varepsilon_1} - 1)^c + C\|\varphi\|_\infty (1 - e^{-2\varepsilon_1})^c \leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\infty. \tag{7.6.21}$$

Substituting (7.6.16) and (7.6.21) into (7.6.15), we conclude that

$$\sum_{k=1}^{M_n} e^{-s\delta k}\pi_s(\varphi^f_{s,k,\varepsilon_1}) \geqslant \pi_s(\varphi|\langle f,\cdot\rangle|^s) - \frac{C}{\sqrt{n}}\|\varphi\|_\infty. \tag{7.6.22}$$

We now give an estimate of the second term in (7.6.14). Note that uniformly in $s \in (0,\eta)$, $f \in \mathbb{P}^1$, $\varphi \in \mathcal{B}_\gamma$ and $k \in [1, M_n]$,

$$\|\varphi^f_{s,k,\varepsilon_1}\|_\infty \leqslant C\|\varphi\|_\infty$$

and

$$\|\varphi^f_{s,k,\varepsilon_1}\|_\gamma \leqslant C\|\varphi\|_\gamma + \frac{C\|\varphi\|_\infty}{[e^{-\delta k+2\varepsilon_1} - e^{-\delta k}]\gamma} + \frac{C\|\varphi\|_\infty}{[e^{-\delta(k-1)} - e^{-\delta(k-1)-2\varepsilon_1}]\gamma}.$$

Recalling that $M_n = [\log n]$ and $\varepsilon_1 = n^{-\beta}$ with $\beta > 0$ large enough, taking $\gamma > 0$ sufficiently small, we get an upper bound for the second term in (7.6.14):

$$\begin{aligned}
&\sum_{k=1}^{M_n} e^{-s\delta k}\left(\frac{1}{\sqrt{n}}\|\varphi^f_{s,k,\varepsilon_1}\|_\gamma + \frac{y}{\sqrt{n}}\|\varphi^f_{s,k,\varepsilon_1}\|_\infty\right)\\
&\leqslant \sum_{k=1}^{M_n}\left(\frac{1}{\sqrt{n}}\|\varphi^f_{s,k,\varepsilon_1}\|_\gamma + \frac{y}{\sqrt{n}}\|\varphi^f_{s,k,\varepsilon_1}\|_\infty\right)\\
&\leqslant C\frac{\log n}{\sqrt{n}}\|\varphi\|_\gamma + C\frac{n^\epsilon}{\sqrt{n}}\|\varphi\|_\gamma + C\frac{n^\epsilon}{\sqrt{n}}\|\varphi\|_\gamma + C\frac{y\log n}{\sqrt{n}}\|\varphi\|_\infty\\
&\leqslant C\frac{n^\epsilon}{\sqrt{n}}\|\varphi\|_\gamma + C\frac{y\log n}{\sqrt{n}}\|\varphi\|_\infty.
\end{aligned}$$

Combining this with (7.6.22), we conclude that

$$\begin{aligned}
\frac{B_n}{e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}[1-\Phi(y)]} &\geqslant [1+O(s)]\left[\pi_s(\varphi|\langle f,\cdot\rangle|^s) - \frac{C}{\sqrt{n}}\|\varphi\|_\infty\right]\\
&\quad - C\frac{n^\epsilon}{\sqrt{n}}\|\varphi\|_\gamma - C\frac{y\log n}{\sqrt{n}}\|\varphi\|_\infty.
\end{aligned}$$

Note that uniformly in $s \in (0,\eta)$, we have

$$\pi_s(\varphi|\langle f,\cdot\rangle|^s) = \nu(\varphi) + O(s).$$

This implies the desired lower bound: there exists a constant $c > 0$ such that uniformly in $f, x \in \mathbb{P}^1$ and $y \in [n^{1/6}, o(\sqrt{n})]$,

$$\frac{A_n}{1-\Phi(y)} \geqslant e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\left[\nu(\varphi) - c\frac{y}{\sqrt{n}}\|\varphi\|_\gamma - c\frac{y\log n}{\sqrt{n}}\|\varphi\|_\infty\right].$$

The proof of the first assertion in Theorem 7.6.3 is completed. Using Proposition 7.5.2 instead of Proposition 7.5.1, the proof of the second assertion in Theorem 7.6.3 can be carried out in a similar way. □

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