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L'UNIVERSITÉ DE BRETAGNE SUD

ÉCOLE DOCTORALE N° 601
*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*
Spécialité : *Mathématiques et leurs Interactions*

Par

Thi Thuy BUI

Limit theorems for branching random walks and products of random matrices

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Théorèmes limites pour les marches aléatoires avec branchement et produits de matrices aléatoires

Résumé.

L'objectif principal de ma thèse est d'établir des théorèmes limites pour une marche aléatoire avec branchement gouvernée par des produits de matrices aléatoires. La thèse est composée de quatre chapitres.

Le chapitre 2 est consacré au modèle classique d'une marche aléatoire avec branchement, dans lequel chaque particule donne naissance à un nombre aléatoire de particules de la génération suivante, qui se déplacent sur la ligne réelle; le nombre d'enfants et les déplacements des enfants suivent une loi fixe. Nous établissons une borne Berry-Esseen et une expansion modérée de type Cramér pour la mesure de comptage qui compte le nombre de particules de n -ème génération située dans une région donnée.

Le chapitre 3 est consacré à l'étude des produits de matrices aléatoires qui sera utilisé dans les chapitres suivants pour l'étude la marche aléatoire avec branchement gouvernée par des produits de matrices aléatoires. Soit $(A_n)_{n \geq 1}$ une suite de matrices aléatoires réelles de type $d \times d$, indépendantes et identiquement distribuées. Considérons le produit $G_n = A_n \dots A_1$ et la marche aléatoire $(G_n x)$, où x est un point de départ avec la norme unitaire $|x| = 1$. Le vecteur $G_n x$ est uniquement déterminé par sa direction $X_n^x = \frac{G_n x}{|G_n x|}$ et sa norme $S_n^x = \log |G_n x|$. Nous nous sommes intéressés par les propriétés asymptotiques de la chaîne de Markov (X_n^x, S_n^x) . Pour matrices inversibles, Le Page [63] a établi un théorème central limite et un théorème limite local pour (X_n^x, S_n^x) . Avec une motivation pour des applications aux marches aléatoires avec branchement gouvernées par des produits de matrices aléatoires, ses résultats sont améliorés et étendus dans deux aspects : 1) le théorème central limite est établi uniformément en x et un développement asymptotique est donné dans le théorème local limite avec une fonction continue f agissant sur X_n^x et une fonction h directement Riemann intégrable agissant sur S_n^x ; 2) les résultats sont aussi établis au cas des matrices non-négatives.

Au chapitre 4, nous considérons un modèle de marches aléatoires avec branchement, où les mouvements des individus sont gouvernés par des produits de matrices aléatoires, où les particules donnent naissance à un nombre aléatoire d'enfants selon un processus de Galton-Watson, qui se déplacent dans \mathbb{R}^d dont les positions sont déterminées par l'action de matrices aléatoires indépendantes et identiquement distribuées sur la position du parent. Nous nous intéressons aux propriétés asymptotiques de la mesure de comptage

Z_n^x sur \mathbb{R}^d qui compte le nombre de particules de génération n située dans une région donnée, lorsque le processus démarre avec une particule initiale située à $x \in \mathbb{R}^d \setminus \{0\}$. A l'aide des résultats établis au chapitre 3 pour les produits de matrices aléatoires, nous établissons un théorème central limite et une expansion asymptotique à grande déviation de type Bahadur-Rao pour Z_n^x avec une normalisation appropriée. En tant que sous-produit, nous obtenons une condition nécessaire et suffisante pour la non-dégénérescence de la limite de la martingale fondamentale, qui étend le théorème de type Kesten-Stigum de Biggins.

Dans le chapitre 5, nous considérons toujours la marche aléatoire avec branchement gouvernée par des produits de matrices aléatoires. Dans ce chapitre nous établissons une borne Berry-Esseen et une expansion modérée de type Cramér pour la mesure de comptage Z_n^x définie comme ci-dessus. Dans la preuve, nous construisons une nouvelle martingale, et établissons son uniforme convergence ainsi que celle de la martingale fondamentale.

Limit theorems for branching random walks and products of random matrices

Abstract.

The main objective of my thesis is to establish limit theorems for a branching random walk with products of random matrices. The thesis is composed of four chapters.

Chapter 2 is devoted to the classical branching random walk, in which each particle gives birth to a random number of particles of the next generation, which move on the real line; the number of children and the displacements of the children are governed by a fixed law. We establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for the counting measure which counts the number of particles of n th generation situated in a given region.

Chapter 3 is devoted to establishing limit theorems for products of random matrices which will be used in the following chapters for the study of a branching random walk with products of random matrices. Let $(A_n)_{n \geq 1}$ be a sequence of independent and identically distributed random $d \times d$ real matrices. Consider the product $G_n = A_n \dots A_1$ and the random walk $(G_n x)$, where x is a starting point with unit norm $|x| = 1$. The vector $G_n x$ is uniquely determined by its direction $X_n^x = \frac{G_n x}{|G_n x|}$ and its log norm $S_n^x = \log |G_n x|$. We consider asymptotic properties of the Markov chain (X_n^x, S_n^x) . For invertible matrices, Le Page [63] established a central limit theorem and a local limit theorem on (X_n^x, S_n^x) . Motivated by some applications in branching random walks, we improve and extend his theorems in the sense that: 1) we prove that the central limit theorem holds uniformly in x , and give an asymptotic expansion in the local limit theorem with a continuous function f acting on X_n^x and a directly Riemann integrable function h acting on S_n^x ; 2) we extend the results to the case of nonnegative matrices.

In Chapter 4, we consider a branching random walk with products of random matrices, where particles give birth to a random number of children as a Galton-Watson process, which move in \mathbb{R}^d whose positions are determined by the action of independent and identically distributed random matrices on the position of the parent. We are interested in asymptotic properties of the counting measure Z_n^x on \mathbb{R}^d which counts the number of particles of generation n situated in a given region, when the process starts with one initial particle located at $x \in \mathbb{R}^d \setminus \{0\}$. With the help of the results established in Chapter 3 for products of random matrices, we establish a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for Z_n^x with suitable norming.

As a by-product, we obtain a sufficient and necessary condition for the non-degeneracy of the limit of the fundamental martingale, which extends the Kesten-Stigum type theorem of Biggins.

In Chapter 5 we still consider a branching random walk with products of random matrices. In this chapter we establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for the counting measure Z_n^x defined as above. In the proof, we construct a new martingale, and establish its uniform convergence as well as that of the fundamental martingale.

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Chapitre 1

Introduction

1.1 Context

My thesis focuses on limit theorems for a supercritical branching random walk. As the name suggests, a branching random walk is a system of particles performing random walks while branching. An introduction to branching random walks and an overview of classical results can e.g. be found in the books [77, 84, 81]. A striking feature of the model is the large number of interactions that it possesses, not only with other fields in probability and mathematics, but also in other sciences such as statistical mechanics [79, 74] and biology [37, 61, 58, 42]. In recent years, this topic has attracted the attention of many authors, see for example, [1, 52, 35, 5, 30, 55, 65]. The model is closely related to various applied probability settings, such as Mandelbrot's cascades (cf. e.g. [56, 67, 6, 25, 72]), perpetuities (see e.g. [77, 26, 54]) and branching Brownian motion (cf. e.g. [59, 29, 12, 71]). For extensions to random environments in space and time, see e.g. [44, 33] and [19, 62, 69, 39, 40]. For other related works and many references, see e.g. the recent books [77, 26, 54].

In the classical branching random walk, the moving is a simple random translation, that is, a particle, whose parent is at position y , moves to position $y + l$, with independent and identically distributed (i.i.d.) displacements l . Although this model can be applied to many fields, it does not cover the interesting cases occurring in many situations where the movements are determined by linear transformations such as rotations, dilations, shears, reflections, projections etc. Motivated by this observation, we consider a *branching random walk with products of random matrices*, in which the position of a particle in \mathbb{R}^d ($d \geq 1$) is obtained by the action of a matrix on the position of its parent. In other words, the positions of particles are obtained by the action of products of random matrices on the position of one initial particle. This permits us to extend significantly the domains of applications of the theory of branching random walks, but the study of the model becomes much more involved. For such a model, we consider the counting measure Z_n^x , which counts the number of particles of generation n situated in a given region, when the

process begins with one initial particle situated at x . The main goal of this thesis is to give precise asymptotics of the counting measure Z_n^x as $n \rightarrow \infty$, by establishing central limit theorems, large and moderate deviation results. The study is interesting because it gives a good description of the configuration of the process at time n . In fact, finding the asymptotic properties of the counting measure is one of the fundamental problems in the theory of branching random walks.

In recent years, important progress has been made in the study of products of random matrices, see for example Guivarc'h and Le Page [46], and Benoist and Quint [7, 8, 9, 10, 11]. In our approach, we benefit much from the results and methods recently developed on this subject (e.g. [46, 28, 82, 83]).

1.2 Background and main objectives

1.2.1 The classical branching random walk

The classical branching random walk on the real line can be defined as follows. At time 0, there is one initial particle \emptyset generation 0, located at $S_\emptyset = 0$. At time 1, it is replaced by $N = N_\emptyset$ new particles $\emptyset i$ ($1 \leq i \leq N$) of generation 1, located at $L_i = L_{\emptyset i}$, $1 \leq i \leq N$, where N is of distribution $p = \{p_k : k \in \mathbb{N}\}$, each random variable L_i is of distribution G . Both N and L_i are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In general, each particle $u = u_1 \dots u_n$ of generation n is replaced at time $n + 1$ by N_u new particles of generation $n + 1$, with displacements $L_{u1}, L_{u2}, \dots, L_{uN_u}$, so that the i -th child is located at

$$S_{ui} = S_u + L_{ui},$$

where N_u is of distribution p and each L_{ui} is of distribution G . All the random variables N_u and L_{ui} , indexed by all finite sequences $u \in \mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$ (by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$), are independent of each other.

Denote by \mathbb{T} the genealogical tree associated to the elements $\{N_u : u \in \mathbb{U}\}$. It is defined by the following properties : 1) $\emptyset \in \mathbb{T}$; 2) when $u \in \mathbb{T}$, then for $i \in \mathbb{N}$, $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$; 3) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n , where $|u|$ denotes the length of the sequence u and

represents the number of generation to which u belongs; by convention $|\emptyset| = 0$. Consider the counting measure

$$Z_n(A) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{S_u \in A\}}, \quad A \subset \mathbb{R}, \quad (1.1)$$

which counts the number of particles of n -th generation situated in A , where for a set D , $\mathbb{1}_D$ denotes its indicator function.

We assume that $m := \mathbb{E}N = \mathbb{E}[Z_1(\mathbb{R})] \in (1, \infty)$ and that $N > 0$ a.s., so that the Galton-Watson process formed by the generation sizes survives with positive probability. Denote

$$m_0 = \int xG(dx) \quad \text{and} \quad \sigma_0^2 = \int (x - m_0)^2 G(dx).$$

Harris [48, Chapter III. §16] conjectured a central limit theorem for Z_n , which states that if $0 < \sigma_0 < \infty$, then for any $x \in \mathbb{R}$,

$$\frac{1}{m^n} Z_n \left((-\infty, x\sigma_0\sqrt{n} + nm_0] \right) \xrightarrow{n \rightarrow \infty} W\Phi(x) \quad (1.2)$$

in probability, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the normal distribution function and W is the a.s. limit of the fundamental martingale $\left(\frac{Z_n(\mathbb{R})}{m^n}\right)$ of the Galton-Watson process $(Z_n(\mathbb{R}))$. This conjecture has first been solved by Stam [78], then improved by Asmussen and Kaplan [2] to L^2 -convergence and almost sure (a.s.) convergence. A more general process where the two families $(N_u)_{u \in \mathbb{U}}$ and $(L_u)_{u \in \mathbb{U}}$ are not necessarily independent, and the family $(L_u)_{u \in \mathbb{U}}$ is not necessarily i.i.d., was introduced by Biggins [13]: instead, he just assumed that the random vectors $(N_u, L_{u1}, L_{u2}, \dots)$ indexed by all the finite sequences u are i.i.d. This model is called the *general branching random walk*. For this model, results like (1.2) were established by Klebaner [60] and Biggins [16].

The rate of convergence in (1.2) has been studied in several papers. Révész [76] considered the special case where the displacements follow the same Gaussian law and conjectured the exact convergence rate; his conjecture was solved by Chen [29]. Gao and Liu [39] improved and extended Chen's result to the general non-lattice case, while the lattice case has been considered by Grübel and Kabluchko [45]. All the above mentioned results are about the point-wise convergence without uniformity in x .

Objective 1 : establish a uniform bound for the rate of convergence in (1.2) of type

Berry-Esseen. We will prove that, under suitable conditions, a.s. for $n \geq 1$,

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{m^n} Z_n \left((-\infty, x\sigma_0\sqrt{n} + nm_0] \right) - W\Phi(x) \right| \leq \frac{M}{\sqrt{n}}, \quad (1.3)$$

where M is a positive and finite random variable. In fact, a similar result will be established for the general branching random walk.

The problem of large deviations for the counting measure $Z_n(\cdot)$ has been considered by Biggins : he established in [14] a large deviation principle, which was subsequently improved in [15] to a Bahadur-Rao large deviation asymptotic. Here we consider the moderate deviations :

Objective 2 : establish a Cramér type moderate deviation expansion for Z_n . We will prove that, under suitable conditions, for $x \in [0, o(\sqrt{n})]$, as $n \rightarrow \infty$, a.s.,

$$\frac{Z_n \left((x\sigma_0\sqrt{n} + nm_0, +\infty) \right)}{m^n W[1 - \Phi(x)]} = e^{\frac{x^3}{\sqrt{n}} \mathcal{L}(\frac{x}{\sqrt{n}})} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right) \right], \quad (1.4)$$

where $t \mapsto \mathcal{L}(t)$ is the Cramér series (see (1.13) for details). Actually, as in the case of Berry-Esseen bound, a similar result will be established for the general branching random walk.

An important role in the study of large deviations for Z_n is played by the martingale of Biggins with complex parameter :

$$\begin{aligned} W_n(\lambda) &= \frac{1}{\left(\mathbb{E} \left[\sum_{i=1}^N e^{\lambda L_i} \right] \right)^n} \int_{\mathbb{R}} e^{\lambda t} Z_n(dt) \\ &= \frac{1}{\left(\mathbb{E} \left[\sum_{i=1}^N e^{\lambda L_i} \right] \right)^n} \sum_{u \in \mathbb{T}_n} e^{\lambda S_u}, \quad n \geq 0, \lambda \in \mathbb{C}. \end{aligned}$$

Set $m(\lambda) = \mathbb{E} \left[\sum_{i=1}^N e^{\lambda L_i} \right]$, $\lambda \in \mathbb{C}$ and $\mathcal{D} = \text{int}\{\theta \in \mathbb{R} : m(\theta) < \infty\} \neq \emptyset$, where $\text{int}(A)$ denotes the interior of the set A . When $\lambda = 0$, $W_n := W_n(0) = \frac{Z_n(\mathbb{R})}{m^n}$ is the fundamental martingale of the Galton -Watson process $(Z_n(\mathbb{R}))$, whose limit is denoted by W . The famous Kesten-Stigum theorem states that W is non degenerate if and only if $\mathbb{E} N \log_+ N < \infty$ (see [4]), where $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$. By the martingale convergence theorem for non-negative martingales, we have for

all $\theta \in \mathcal{D}$,

$$W_n(\theta) \xrightarrow{n \rightarrow \infty} W(\theta), \quad \text{a.s.}$$

Biggins [13, Theorem A] gave a necessary and sufficient condition for the non-degeneracy of $W(\theta) : \mathbb{E}W(\theta) > 0$ if and only if

$$\mathbb{E}[W_1(\theta) \log_+ W_1(\theta)] < \infty \quad \text{and} \quad \theta \in (\theta_-, \theta_+), \quad (1.5)$$

where $(\theta_-, \theta_+) \subset \mathcal{D}$ denotes by the open interval on which $\frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta)$, i.e.

$$\begin{aligned} \theta_- &= \inf \left\{ \theta \in \mathcal{D} : \frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta) \right\}, \\ \theta_+ &= \sup \left\{ \theta \in \mathcal{D} : \frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta) \right\}. \end{aligned}$$

Moreover, when $N > 0$ a.s. and (1.5) hold,

$$W(\theta) > 0 \text{ a.s. and } \mathbb{E}W(\theta) = 1.$$

It has been shown in [18, Theorem 2] that if

$$\mathbb{E}[W_1^\gamma(\theta)] < \infty, \quad \text{for some } \gamma > 1 \text{ and for all } \theta \in (\theta_-, \theta_+), \quad (1.6)$$

then for every compact subset C in the trip $V := \{\lambda = \theta + i\eta : \theta \in (\theta_-, \theta_+), \eta \in \mathbb{R}\}$, a.s.

$$\sup_{\lambda \in C} |W_n(\lambda) - W(\lambda)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad W(\lambda) \text{ is analytic in } V.$$

Our next objectives are to establish analogous results for a branching random walk with products of random matrices.

1.2.2 Branching random walks with products of random matrices

To introduce the model we need some notation. Assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a set of independent identically distributed random variables $(N_u)_{u \in \mathbb{U}}$ of the same law $p = \{p_k : k \in \mathbb{N}\}$, and a set of independent identically dis-

tributed $d \times d$ random matrices $(A_u)_{u \in \mathbb{U}}$ of the same law μ on the set of $d \times d$ matrices $M(d, \mathbb{R})$, where $d \geq 1$. The two families $(N_u)_{u \in \mathbb{U}}$ and $(A_u)_{u \in \mathbb{U}}$ are also assumed to be independent.

A branching random walk with products of random matrices is defined as follows. At time 0, there is one initial particle \emptyset of generation 0, with initial position $Y_\emptyset := x \in \mathbb{R}^d \setminus \{0\}$. At time 1, the initial particle \emptyset is replaced by $N = N_\emptyset$ new particles $i = \emptyset i$ of generation 1, located at $Y_i = A_i Y_\emptyset$, $1 \leq i \leq N$. In general, at time $n + 1$, each particle $u = u_1 \dots u_n$ of generation n , located at $Y_u \in \mathbb{R}^d$, is replaced by N_u new particles ui of generation $n + 1$, located at $Y_{ui} = A_{ui} Y_u$, $1 \leq i \leq N_u$. Namely, the position of the particle ui is obtained from the position Y_u of u by the action of the matrix A_{ui} , so that the position Y_u of a particle u in generation $n \geq 1$ is given by the action of products of random matrices on the initial position x :

$$Y_u = G_u x, \quad \text{where} \quad G_u = A_{u_1 \dots u_n} \dots A_{u_1}.$$

The space \mathbb{R}^d is equipped with the Euclidean norm $|\cdot|$. The position $G_u x$ of the particle u is completely described by two components : its norm $|G_u x|$ and its projection on the unit sphere $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d, |y| = 1\}$ denoted by

$$X_u^x := \frac{G_u x}{|G_u x|}.$$

Accordingly, we consider the following counting measure of particles of generation n which describes the configuration of the branching random walk at time n : for measurable sets $B_1 \subset \mathbb{S}^{d-1}$ and $B_2 \subset \mathbb{R}$,

$$Z_n^x(B_1, B_2) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B_1, \log |G_u x| \in B_2\}}. \quad (1.7)$$

In particular when $B = \mathbb{S}^{d-1}$ the measure (1.7) reduces to

$$Z_n^x(\mathbb{S}^{d-1}, B_2) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{\log |G_u x| \in B_2\}}. \quad (1.8)$$

When $d = 1$, $x = 1$ and $A_u \neq 0$ for all $u \in \mathbb{T}$, the measure defined by (1.8) is exactly the counting measure considered in the classical branching random walk on \mathbb{R} starting from the origin $0 \in \mathbb{R}$, where the position S_u of a particle $u = u_1 \dots u_n$ is given by $S_u = L_{u_1} + \dots + L_{u_1 \dots u_n}$, with $L_u = \log |A_u|$. So our model in the one dimensional case

$d = 1$ reduces essentially to the classical (additive) branching random walk. For this reason, in the following we will focus on the case $d \geq 2$.

We will prove limit theorems for the counting measure Z_n^x both in the case when the matrices A_u are nonnegative, and in the case when the matrices A_u are invertible.

Objective 3 : establish a central limit theorem for the counting measure Z_n^x with suitable norming. In particular, we will prove that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $y \in \mathbb{R}$, as $n \rightarrow \infty$, a.s.,

$$\frac{1}{m^n} Z_n^x \left(\mathbb{S}^{d-1}, (-\infty, n\gamma + y\sigma\sqrt{n}] \right) - W\Phi(y) \rightarrow 0, \quad (1.9)$$

where the constants γ and σ are explicitly defined (see (C6) and (1.15) in the following section). In fact, the result is established in a more general setting : a result similar to (1.9) is proved when \mathbb{S}^{d-1} is replaced by a measurable set $B_1 \subseteq \mathbb{S}^{d-1}$.

Objective 4 : strengthen the central limit theorem (1.9) and its integral version (with a target function on X_u^x) to a Berry-Esseen bound for the counting measure Z_n^x . We prove that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $n \geq 1$, a.s.,

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{m^n} Z_n^x \left(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}(-\infty, y] \right) - W\Phi(y) \right| \leq \frac{M}{\sqrt{n}}, \quad (1.10)$$

where M is a finite and positive random variable. In fact, an integral version of (1.10) is established with a target function φ on X_u^x , which reduces to (1.10) when $\varphi = 1$.

Objective 5 : establish the Cramér type moderate deviation expansion for Z_n^x . We prove that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $0 \leq y = o(\sqrt{n})$, as $n \rightarrow \infty$, a.s.,

$$\frac{Z_n^x \left(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}(y, +\infty) \right)}{m^n W[1 - \Phi(y)]} = e^{\frac{y^3}{\sqrt{n}} \zeta\left(\frac{y}{\sqrt{n}}\right)} \left[1 + O\left(\frac{y+1}{\sqrt{n}}\right) \right],$$

where $t \mapsto \zeta(t)$ is the Cramér series (see (1.34)). In fact, a similar result will be established with a target function φ on X_u^x .

Objective 6 : establish a large deviation asymptotic expansion of Bahadur-Rao type for Z_n^x . We prove that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$, with $q_s > \gamma$, we have

a.s.,

$$\frac{1}{m^n} Z_n^x \left(\mathbb{S}^{d-1}, [nq_s, +\infty) \right) = \frac{W_s^x r_s(x) e^{-n\Lambda^*(q_s)}}{s\sigma_s \sqrt{2\pi n}} \left(1 + o(1) \right), \quad (1.11)$$

where r_s is a function bounded from below and from above by two positive constants, $s, \sigma_s, \Lambda^*(q_s)$ are positive constants and W_s^x is the limit of a martingale associated to branching random walks with products of random matrices. As in the case of central limit theorem, a result similar to (1.11) is proved when \mathbb{S}^{d-1} is replaced by a measurable subset B_1 of \mathbb{S}^{d-1} . Moreover, an integral version for the large deviation expansion with target functions on the two components X_u^x and $\log |G_u x|$ is also established.

To achieve our objectives, as important ingredients in the approach, we mention in particular the following :

- asymptotic expansions in central and local limit theorems for products of random matrices ;
- the non-degeneracy of the limit of the fundamental martingale associated to branching random walk with products of random matrices ;
- the uniform convergence of the fundamental martingale and the construction of a new martingale.

1.3 Main results

The main results of the thesis, together with some key ideas of the proofs, are presented below in four subsections. For the detailed proofs, we refer to Chapters 1-4. Each of the four subsections corresponds to the content of one of the four chapters.

1.3.1 Berry-Esseen bound and Cramér moderate deviation expansion for a supercritical branching random walk

In this subsection we present our main results on the Berry-Esseen bound and Cramér moderate deviation expansion for the counting measure Z_n defined in (1.1), for a general branching random walk on the real line, generated by $(N_u, L_{u1}, L_{u2}, \dots)$, which are independent copies of (N, L_1, L_2, \dots) .

Conditions and statement of main results

We will use the following standard assumptions.

C1. $N > 0$ a.s. with $m = \mathbb{E}N \in (1, \infty)$, and $\mathbb{E}\left[\sum_{i=1}^N L_i^2\right] < \infty$.

The first condition in **C1** implies that the underlying Galton -Watson process is super-critical and

$$F(A) = \mathbb{E}[Z_1(A)], \quad A \subset \mathbb{R},$$

is a finite measure on \mathbb{R} with mass m . Let \bar{F} be the probability measure on \mathbb{R} defined by

$$\bar{F}(A) = \frac{F(A)}{m}, \quad A \subset \mathbb{R}.$$

Denote its mean and variance by

$$m_0 = \int x \bar{F}(dx) \quad \text{and} \quad \sigma_0^2 = \int (x - m_0)^2 \bar{F}(dx). \quad (1.12)$$

We will assume that

C2. F is non-degenerate, i.e. it is not concentrated on a single point.

The last condition in **C1**, together with condition **C2**, implies that the mean m_0 and the variance σ_0^2 defined by (1.12) are finite with $\sigma_0 > 0$.

C3. \mathcal{D} is non-empty.

We will need the following moment condition which is weaker than (1.6).

C4. There are $\gamma > 1$ and $K_0 > 0$ with $(-K_0, K_0) \subset (\theta_-, \theta_+)$ such that

$$\mathbb{E}W_1^\gamma(\theta) < \infty \quad \forall \theta \in (-K_0, K_0).$$

By the argument of the proof of [18, Theorem 2], we know that under hypothesis **C4**, for every compact subset C of $V := \{\lambda = \theta + i\eta : \theta \in (-K_0, K_0), \eta \in \mathbb{R}\}$, a.s.

$$\sup_{\lambda \in C} |W_n(\lambda) - W(\lambda)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad W(\lambda) \text{ is analytic in } C.$$

Our first result gives the Berry-Esseen bound for Z_n :

Theorem 1.3.1. *Assume conditions **C1** -**C4**. Then, a.s. for all $n \geq 1$,*

$$\sup_{x \in \mathbb{R}} \left| \frac{Z_n \left((-\infty, x\sigma_0\sqrt{n} + nm_0] \right)}{m^n} - W\Phi(x) \right| \leq \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable.

To state the result corresponding to the Cramér type moderate deviation expansion for Z_n , we need more notation. Consider the measure

$$F_\theta(dx) = \frac{e^{\theta x}}{m(\theta)} F(dx), \quad \theta \in \mathcal{D}.$$

We see that F_θ is a distribution function with finite mean m_θ and variance σ_θ^2 , given by

$$m_\theta = \frac{m'(\theta)}{m(\theta)}, \quad \sigma_\theta^2 = \frac{m''(\theta)}{m(\theta)} - \left(\frac{m'(\theta)}{m(\theta)} \right)^2;$$

moreover, $\sigma_\theta > 0$ when F is non-degenerate. Consider the change of measure of type Cramér for Z_n : for $\theta \in \mathcal{D}$,

$$Z_n^\theta(dx) = e^{\theta x} Z_n(dx),$$

namely,

$$Z_n^\theta(A) = \sum_{u \in \mathbb{T}_n} e^{\theta S_u} \mathbb{1}_{\{S_u \in A\}}, \quad A \subset \mathbb{R}.$$

Let X be a random variable with distribution $\bar{F} := \frac{F}{m}$, and

$$\Lambda(\theta) := \log \mathbb{E} e^{\theta X} = \log m(\theta) - \log m$$

be its cumulant generating function. Then $\Lambda(\theta)$ is analytic on \mathcal{D} , with $\Lambda'(\theta) = m_\theta$ and $\Lambda''(\theta) = \sigma_\theta^2$. Denote by $\gamma_k := \Lambda^{(k)}(0)$ the cumulant of order k of the random variable X . We shall use the Cramér series (see [75, Theorem VIII.2.2]) :

$$\mathcal{L}(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \dots \quad (1.13)$$

which converges for $|t|$ small enough.

Theorem 1.3.2. *Assume conditions **C1** -**C4**. Then we have, for $0 \leq x = o(\sqrt{n})$, as*

$n \rightarrow \infty$, a.s.

$$\frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)]} = e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right],$$

and

$$\frac{Z_n\left((-\infty, -x\sigma_0\sqrt{n} + nm_0)\right)}{m^n W\Phi(-x)} = e^{-\frac{x^3}{\sqrt{n}} \mathcal{L}\left(-\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right].$$

As a by-product in the proof of Theorem 1.3.2, we obtain a Berry - Esseen bound for the changed measure Z_n^θ with uniformity in θ .

Theorem 1.3.3. *Assume conditions C1 - C4. Then, there exists a constant $0 < K < K_0$ such that a.s. for all $n \geq 1$,*

$$\sup_{\theta \in [-K, K]} \sup_{x \in \mathbb{R}} \left| \frac{Z_n^\theta\left((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta)\right)}{m(\theta)^n} - W(\theta)\Phi(x) \right| \leq \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable.

Key ideas of the proofs

Let us explain briefly the key ideas in the proofs. To prove the Berry-Esseen bound (1.3), we use Esseen's smoothing inequality ([75, Theorem V.2.2.]). The key point in this proof is the formula of the characteristic function of $\frac{1}{m^n} Z_n\left((-\infty, x\sigma_0\sqrt{n} + nm_0)\right)$, which can be interpreted as $W_n\left(\frac{it}{\sigma_0\sqrt{n}}\right) f_n(t)$, $t \in \mathbb{R}$, where $(W_n(\lambda))$ is Biggins' martingale with complex valued parameter λ for the branching random walk (see [17, 18]), and $f_n(t)$ is the characteristic function of the n -fold convolution of \bar{F} . Using the results of Biggins [17, 18], Grübel and Kabluchko [45] about the uniform convergence of $W_n(\lambda)$, together with the approach of Petrov [75] for the proof of the Berry-Esseen bound for sums of i.i.d. random variables, we are able to establish (1.3). The Berry-Esseen bound (1.3) is then extended to the changed measure of type Cramér, $Z_n^\theta(A) = \int_A e^{\theta t} Z_n(dt)$, $A \subset \mathbb{R}$, $\theta \in \mathbb{R}$. This is an important step in establishing the moderate deviation expansion (1.4). Our approach in proving (1.4) is very different to the method of Biggins [15] on the Bahadur-Rao large deviation asymptotic, but is inspired by the ideas in the proof of Cramér's moderate deviation expansion on sums of i.i.d. random variables (see [75]), and the arguments in [18] that Biggins used to prove the local limit theorem with large deviations for Z_n .

1.3.2 Asymptotic expansions in central and local limit theorems for products of random matrices

This subsection is devoted to the presentation of our main results about a central limit theorem and a local limit theorem for products of random matrices, which will be used to establish limit theorems for the branching random walk with products of random matrices.

Conditions and statement of main results

Let $M(d, \mathbb{R})$ be equipped with the operator norm : for any $\mathbf{a} \in M(d, \mathbb{R})$ we set $\|\mathbf{a}\| = \sup_{x \in \mathbb{S}^{d-1}} |\mathbf{a}x|$, where $|\cdot|$ is a given vectorial norm on \mathbb{R}^d , and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere in \mathbb{R}^d . Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by the support of μ . Let us recall some definitions in matrix theory. A matrix $\mathbf{a} \in M(d, \mathbb{R})$ is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by \mathcal{M}_+ the set of matrices with nonnegative entries. A nonnegative matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column has a positive entry.

We say that the measure μ is *arithmetic* if there is $t > 0$ together with $\theta \in [0, 2\pi)$ and a function $\vartheta : \mathbb{S}_+^{d-1} \rightarrow \mathbb{R}$ such that

$$\forall \mathbf{a} \in \Gamma, \forall x \in V(\Gamma) : \exp[it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a} \cdot x) - \vartheta(x))] = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Notice when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form : $\log a$ is almost surely concentrated on an arithmetic progression $a_0 + a_1\mathbb{N}$ for some $a_0, a_1 > 0$.

We will need the following assumptions on the law μ .

C5.

1. For invertible matrices :

- (a) (Strong irreducibility) There is no finite union $\mathcal{W} = \bigcup_{i=1}^n W_i$ of proper subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ -invariant (in the sense that $a\mathcal{W} = \mathcal{W}$ for each $a \in \Gamma_\mu$)
- (b) (Proximality) Γ_μ contains at least one proximal matrix.

2. For nonnegative matrices :

- (a) (Allowability) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.

(b) (Positivity) Γ_μ contains at least one matrix belonging to $\text{int}(\mathcal{M}_+)$.

(c) (Non-arithmeticity) The measure μ is non-arithmetic.

Notice that when $d = 1$, the strong irreducibility and proximality conditions are always satisfied. It is known that when $d \geq 2$, condition **C5.1** implies **C5.2.c** (see [47, Proposition 4.6]).

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathcal{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a} \cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|} \quad \text{when } \mathbf{a}x \neq 0,$$

where $\mathbf{a} \cdot x$ is called the projective action of the matrix \mathbf{a} on the vector $x \in \mathbb{S}^{d-1}$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices. Set, for an invertible or nonnegative matrix \mathbf{a} ,

$$N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \iota(\mathbf{a})^{-1}\}.$$

For invertible matrices we have $\iota(\mathbf{a}) = \|\mathbf{a}^{-1}\|^{-1}$ and $N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \|\mathbf{a}^{-1}\|\}$.

C6. (Moment condition) There exists $\eta_0 \in (0, 1)$ such that

$$\mathbb{E}[N(A_1)^{\eta_0}] < \infty.$$

We will consider the action of invertible matrices on the projective space \mathbb{P}^{d-1} which is obtained from \mathbb{S}^{d-1} by identifying x and $-x$, and the action of nonnegative matrices on \mathbb{S}_+^{d-1} . For convenience, we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} for invertible matrices, and \mathbb{S}_+^{d-1} for nonnegative matrices. The space \mathcal{S} will be equipped with the metric \mathbf{d} , which is the angular distance (see [20]) for invertible matrices, and the Hilbert cross-ratio metric (see [49]) for nonnegative matrices. Moreover, \mathcal{S} is a separable metric space equipped with Borel σ -field.

Let $G_n = A_n \dots A_2 A_1$ be the product of i.i.d. $d \times d$ real random matrices A_i , defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with common law μ . Let $x \in \mathcal{S}$ be a starting point. As mentioned in the introduction, the random walk $G_n x$ is completely determined by its log

norm and its projection on \mathcal{S} , denoted respectively by

$$S_n^x := \log |G_n x|, \quad X_n^x := G_n \cdot x = \frac{G_n x}{|G_n x|}, \quad n \geq 0,$$

with the convention that $G_0 x = x$. Since $S_n^x = \log |A_n X_{n-1}^x| + S_{n-1}^x$ and $X_n^x = A_n \cdot X_{n-1}^x$, the sequence $(S_n^x, X_n^x)_{n \geq 0}$ is a Markov chain.

Denote by \mathbb{E} the expectation corresponding to \mathbb{P} . By the law of large numbers of Furstenberg [38], under conditions **C5** and **C6**, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n^x = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[S_n^x] = \gamma \quad \mathbb{P}\text{-a.s.}, \quad (1.14)$$

where $\gamma = \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E} \log \|G_n\|$ is the upper Lyapunov exponent associated with the product sequence (G_n) . Le Page [63] and Henion [49] showed that

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} (S_n^x - n\gamma)^2 \quad (1.15)$$

exists and is independent of x for invertible matrices and nonnegative matrices, respectively. Moreover, there exists a unique μ -stationary probability measure ν on \mathcal{S} (see [46, 25]); the μ -stationarity of ν means that $\mu * \nu = \nu$, that is, for any continuous function φ on \mathcal{S} ,

$$(\mu * \nu)(\varphi) := \int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(\mathbf{a} \cdot x) \mu(d\mathbf{a}) \nu(dx) = \nu(\varphi).$$

where $\nu(\varphi) = \int_{\mathcal{S}} \varphi(x) \nu(dx)$. This notation for the integral will be used for any function and any measure.

We state first a central limit theorem for the couple (X_n^x, S_n^x) with uniform convergence in $x \in \mathcal{S}$.

Theorem 1.3.4. *For invertible matrices, assume **C5.1** if $d > 1$, and **C5.2.c** if $d = 1$. For nonnegative matrices, assume **C5.2**. For both cases, assume additionally **C6**.*

1. *For any continuous function f on \mathcal{S} , we have*

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[f(X_n^x) \mathbf{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(f) \Phi(t) \right| = 0. \quad (1.16)$$

2. For any measurable set $B \subset \mathcal{S}$ with $\nu(\partial B) = 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_n^x \in B, \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right) - \nu(B)\Phi(t) \right| = 0. \quad (1.17)$$

For invertible matrices, a point-wise version (by considering a fixed $x \in \mathcal{S}$ instead of $\sup_{x \in \mathcal{S}}$) has been established by Le Page in [63, Theorem 4]. For nonnegative matrices, the asymptotic for the Markov chain (X_n^x, S_n^x) is new even for a fixed x . The uniformity in $x \in \mathcal{S}$ is new for both invertible matrices and nonnegative matrices. Theorem 1.3.4 will be deduced from a result on the convergence rate in (1.16) which has been established in [83] for the case when f is Hölder continuous.

The following theorem gives the asymptotic expansion in the local limit theorem for products of random matrices.

Theorem 1.3.5. *Assume the conditions of Theorem 1.3.4.*

1. For any continuous function f on \mathcal{S} and any directly Riemann integrable function h on \mathbb{R} , we have as $n \rightarrow \infty$,

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma\sqrt{n} \mathbb{E} [f(X_n^x)h(y + S_n^x - n\gamma)] - \nu(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz \right| \rightarrow 0, \quad (1.18)$$

where

$$H_x(u) = 1 - \frac{b(x)}{\sigma^2\sqrt{n}}u + \frac{m_3}{6\sigma^6\sqrt{n}}(3\sigma^2u - u^3),$$

with m_3 and $b(x)$ defined in Proposition 3.3.3.

2. For any measurable set $B \subset \mathcal{S}$ with $\nu(\partial B) = 0$ and any directly Riemann integrable function h on \mathbb{R} , we have as $n \rightarrow \infty$

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma\sqrt{n} \mathbb{E} [\mathbf{1}_B(X_n^x)h(y + S_n^x - n\gamma)] - \nu(B) \int_{\mathbb{R}} h(z) \phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz \right| \rightarrow 0. \quad (1.19)$$

When $y = 0$, $f = 1$ and $h = \mathbf{1}_{[a,b]}$, the integral $\mathbb{E} [f(X_n^x)h(y + S_n^x - n\gamma)]$ reduces to the local probability $\mathbb{P}(S_n^x \in n\gamma + [a, b])$, which is the usual object studied in local limit theorems.

The expansions (1.18) and (1.19) are new for both invertible matrices and nonnegative matrices. The first expansion implies the local limit theorem established in [63, Theorem 6] for invertible matrices, which states that (1.18) holds when the polynomial $H_x(\cdot)$ is replaced by 1 and when f, h are continuous functions with compact supports.

The case $d = 1$ is worth some comments. In this case, Theorem 1.3.4 follows from Theorem VII.2.7 of Petrov [75], while expansion (1.18) in Theorem 1.3.5 was proved by Feller (see [36, Theorem XVI.4.1]) under the same non-arithmetic condition on μ and when $h = \mathbb{1}_{[a,b]}$ is the indicator function of an interval. Breuillard (see [21, Theorem 3.2]) proved an expansion like (1.18) but for any finite order, when μ is strongly non-arithmetic (in the sense that its characteristic function $\hat{\mu}(t) = \int e^{itx} \mu(dx)$ satisfies Cramér's condition $\limsup_{|t| \rightarrow \infty} |\hat{\mu}(t)| < 1$) with finite moments of order high enough and when h is integrable and regular enough (he assumed in particular that h has continuous and integrable derivatives $h^{(k)}$ for $0 \leq k \leq K$ with $K \geq 2$ large enough). Compared with the result of Breuillard, the novelty in Theorem 1.3.5 is that we assume the non-arithmetic condition instead of the strongly arithmetic condition, and we use the direct Riemann integrability of h instead of the smoothness condition on h .

Key ideas of the proofs

Our approach is mainly based on the spectral gap theory recently developed for the norm cocycle by Guivarc'h and Le Page [46] for invertible matrices, and by Buraczewski, Damek, Guivarc'h and Mentemeier (see [25, 28]) for nonnegative matrices. Smoothing techniques are also used for the approximation of functions : in the proof of Theorem 1.3.4, we use a smooth approximation of the indicator function of a Borel set (see Lemma 3.4.1), while in the proof of Theorem 1.3.5, we use a suitable approximation of a directly Riemann integrable function with the techniques developed in [82].

1.3.3 Central limit theorem and precise large deviations for branching random walks with products of random matrices

The goal of this section is to present our main results about a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type on Z_n^x for a branching random walk with products of random matrices defined in (1.7). Note that in our model, along each branch we encounter a product of random matrices. We introduce some notation

and the necessary assumptions on products of random matrices in order to formulate our main results. We shall consider two cases, the case when the matrices are nonnegative and the case when the matrices are invertible.

Notation on products of random matrices

In addition to the notation introduced in the precedent, we need some others. Let $\mathcal{C}(\mathcal{S})$ be the space of continuous complex-valued functions on \mathcal{S} . Set

$$I_\mu = \{s \geq 0 : \mathbb{E}\|A_1\|^s < \infty\}.$$

Note that I_μ is an interval of \mathbb{R}_+ . Let $s_\infty = \sup I_\mu$. Define the transfer operator on the set $\mathcal{C}(\mathcal{S})$ of continuous functions on \mathcal{S} as follows : for any $s \in (-\eta_0, s_\infty)$, and $f \in \mathcal{C}(\mathcal{S})$,

$$P_s f(x) = \mathbb{E}[|A_1 x|^s f(A_1 \cdot x)], \quad \text{for all } x \in \mathcal{S}. \quad (1.20)$$

It is known that under conditions **C5**, and **C6**, there exists a small constant $0 < \eta_1 < \eta_0$ such that for any $s \in (-\eta_1, s_\infty)$, there are a unique probability measure ν_s and a unique Hölder continuous normalized function r_s (under the normalizing condition $\nu_s(r_s) = 1$) on \mathcal{S} satisfying

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \quad (1.21)$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s , $\nu_s P_s$ is the measure on \mathcal{S} such that $(\nu_s P_s)(f) = \nu_s(P_s f)$ for all $f \in \mathcal{C}(\mathcal{S})$. For $s \in [0, s_\infty)$, the property (1.21) is proved in [25, Proposition 3.1] and [28, Corollary 7.3] for positive matrices, and in [46, Theorem 2.6 and Corollary 3.20] for invertible matrices. For both positive matrices and invertible matrices, the existence of $\eta_1 > 0$ and the property (1.21) for $s \in (-\eta_1, \eta_1)$ are proved in [83, Proposition 3.1], where the following properties are also established : the functions $s \mapsto \kappa(s)$ and $s \mapsto r_s(x)$ are strictly positive and analytic in $(-\eta_1, s_\infty)$, for $x \in \mathcal{S}$. Below we shall make use of normalized function r_s , i.e. $r_s(x) \leq 1$ for all $x \in \mathcal{S}, s \in (-\eta_1, s_\infty)$. Moreover, it is proved (see [46, Lemma 3.5], [25, Lemma 6.2], [83, Propositions 3.12 and 3.14]) that, under conditions **C5** and **C6**, the function

$$\Lambda(s) = \log \kappa(s)$$

is finite and analytic on $(-\eta_1, s_\infty)$, and satisfies

$$\Lambda(0) = 0, \quad \Lambda'(0) = \gamma, \quad \Lambda''(0) = \sigma^2 > 0, \quad \text{and} \quad \Lambda''(s) > 0 \quad \forall s \in (-\eta_1, s_\infty).$$

Statement of main results

Note that the population size at time n is $Z_n = Z_n^x(S, \mathbb{R})$, which does not depend on the starting point x and forms a Galton-Watson process with $Z_0 = 1$ and $Z_1 = N$. Recall that $m = \mathbb{E}N$ which is supposed that $1 < m < \infty$. We will need the following condition.

C7. *There exists a constant $\eta > 1$ such that*

$$\mathbb{E}N \log_+^{\eta+1} N < \infty.$$

We start with a central limit theorem for the normalized counting measure (1.7). For $t \in \mathbb{R}$, let

$$Z_n^x(B, t) = Z_n^x\left(B, (-\infty, n\gamma + t\sigma\sqrt{n}]\right) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B, \frac{\log |G_{ux}| - n\gamma}{\sigma\sqrt{n}} \leq t\}}.$$

Theorem 1.3.6. *Assume that the law μ of the random matrices satisfies conditions **C5** and **C6**. Assume also that the offspring distribution satisfies condition **C7**. Then, for any $x \in \mathcal{S}$, any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$ and any $t \in \mathbb{R}$, we have, as $n \rightarrow \infty$,*

$$\frac{Z_n^x(B, t)}{m^n} \rightarrow \nu(B)\Phi(t)W \quad \mathbb{P}\text{-a.s.} \quad (1.22)$$

For the one dimensional case (where $d = 1$), the result is due to Asmussen and Kaplan [3, Theorem 1]. Theorem 1.22 opens ways for extending some results on central limit theorem in [2, 29, 41, 39, 40] to the multi-dimensional case where the moving of particles is determined by products of random matrices.

Our second main result is on the large deviation for the counting measure Z_n^x . To study the large deviation of the measure Z_n^x , a natural way would be to consider its Laplace transform defined by, for $(s_1, s_2) \in \mathbb{R}^d \times \mathbb{R}$,

$$\tilde{Z}_n^x(s_1, s_2) = \int_{\mathbb{R}^d \times \mathbb{R}} e^{s_1 y_1 + s_2 y_2} Z_n^x(dy_1, dy_2) = \sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}, \quad (1.23)$$

where $s_1 y_1$ is the inner product of vectors s_1 and y_1 in \mathbb{R}^d .

In the one dimensional case, when $x = 1$ and $A_n > 0$, we have $X_u^x = 1$, so that $\tilde{Z}_n^x(s_1, s_2)/\mathbb{E}\tilde{Z}_n^x(s_1, s_2)$ reduces to Biggins' fundamental martingale of the branching random walk :

$$\frac{\sum_{u \in \mathbb{T}_n} e^{s_2 S_u^x}}{\mathbb{E} [\sum_{u \in \mathbb{T}_n} e^{s_2 S_u^x}]}, \quad n \geq 0, \quad (1.24)$$

which has been well studied (see [13], for example), and which plays an essential role in many problems. However, in the multidimensional case, in general the sequence (1.24) is no longer a martingale, nor the sequence

$$\frac{\tilde{Z}_n^x(s_1, s_2)}{\mathbb{E}\tilde{Z}_n^x(s_1, s_2)} = \frac{\sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}}{\mathbb{E} [\sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}]}, \quad n \geq 0,$$

for $(s_1, s_2) \in \mathbb{R}^d \times \mathbb{R}$. So an important difficulty arises when we mimic Cramér's change of measure for random walks by use of the Laplace transform of Z_n^x .

However, there is still a natural martingale in the present setting. By the spectral gap property (1.21), it is easy to verify that (see Section 4.4 for more details), for $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$, the sequence

$$W_{s,n}^x := \frac{\sum_{u \in \mathbb{T}_n} e^{s S_u^x} r_s(X_u^x)}{m^n \kappa(s)^n r_s(x)}, \quad n \geq 0,$$

constitutes a positive martingale with respect to the natural filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{F}_n = \sigma(N_u, A_{ui} : i \geq 1, |u| < n) \text{ for } n \geq 1,$$

as observed by Mentemeier [72] in the study of the multivariate smoothing transform. By the martingale convergence theorem, the limit

$$W_s^x := \lim_{n \rightarrow \infty} W_{s,n}^x \quad \text{exists in } \mathbb{R} \quad \mathbb{P}\text{-a.s.}$$

It turns out that the martingale $(W_{s,n}^x)$ in the multidimensional case plays the same role as Biggins' fundamental martingale for one dimensional case, for large deviations.

Just as in the case of Biggins' martingale, it is crucial to know when the limit variable W_s^x of the fundamental martingale $W_{s,n}^x$ is non-degenerate. When the matrices A_u are nonnegative and $s > 0$, Mentemeier [72, Proposition 4.4] gave a sufficient condition for W_s^x to be non-degenerate. In the following we complete his result by considering the

necessary and sufficient conditions, and by treating meanwhile the case $s < 0$ and the case of invertible matrices. To state the result, we need some notation. For $s \in (-\eta_1, s_\infty)$, set $\Lambda^*(q_s) = sq_s - \Lambda(s)$ with $q_s = \Lambda'(s)$. Since $\Lambda''(s) > 0$ and $\frac{\partial}{\partial s}\Lambda^*(q_s) = s\Lambda''(s)$, $\Lambda^*(q_s)$ attains its minimum at $s = 0$, so that $\Lambda^*(q_s) \geq \Lambda^*(q_0) = -\Lambda(0) = 0$ for all $s \in (-\eta_1, s_\infty)$.

Theorem 1.3.7. *Assume conditions [C5](#), [C6](#). If*

$$\Lambda^*(q_s) - \log m < 0 \tag{1.25}$$

and

$$\mathbb{E}[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x] < \infty, \tag{1.26}$$

then for all $x \in \mathcal{S}$,

$$\mathbb{E}[W_s^x] = 1. \tag{1.27}$$

Conversely, if

$$\mathbb{E}[W_s^x] > 0 \tag{1.28}$$

for some $x \in \mathcal{S}$, then [\(1.25\)](#) holds, and

$$\mathbb{E}[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x] < \infty. \tag{1.29}$$

Corollary 1.3.8. *Suppose the conditions [C5](#), [C6](#).*

1. *Assume [\(1.25\)](#) together with*

$$\mathbb{E}N \log_+ N < \infty \quad \text{and} \quad \mathbb{E}\|A_1\|^s \log_+ \|A_1\| < \infty. \tag{1.30}$$

Then [\(1.27\)](#) holds for all $x \in \mathcal{S}$.

2. *Assume that the random matrix $A_1 = (A_1(i, j))$ satisfies the Furstenberg- Kesten condition : there exists a constant $C > 1$ such that*

$$\frac{\max_{1 \leq i, j \leq d} |A_1(i, j)|}{\min_{1 \leq i, j \leq d} |A_1(i, j)|} \leq C \quad \text{a.s.}$$

Then the three conditions [\(1.26\)](#), [\(1.29\)](#) and [\(1.30\)](#) are equivalent, and [\(1.28\)](#) holds for all $x \in \mathcal{S}$ if and only if [\(1.25\)](#) and [\(1.30\)](#) hold. Moreover, if [\(1.28\)](#) holds for some $x \in \mathcal{S}$, then [\(1.27\)](#) holds for all $x \in \mathcal{S}$.

Notice that by Sheffé's theorem, for each $x \in \mathcal{S}$, if (1.28) holds, then $W_{s,n}^x \rightarrow W_s^x$ in L^1 . So the martingale $(W_{s,n}^x)$ converges in L^1 for some $x \in \mathcal{S}$ if and only if (1.25) and (1.30) hold; moreover, when the martingale converges in L^1 for some $x \in \mathcal{S}$, then it converges in L^1 for all $x \in \mathcal{S}$.

When the matrices A_u are nonnegative and $s > 0$, Part (1) has been established by Mentemeier [72, Proposition 4.4]. When $d = 1$, Part (2) is essentially the well-known Kesten-Stigum type theorem for the classical branching random walk on the real line, due to Biggins [13]; see also [56] for Mandelbrot's cascades and [70, 66] for versions which are slightly different to the initial result of Biggins [13].

Now we consider the precise large deviations for Z_n^x with target functions f and g on the components $X_u^x = G_u \cdot x$ and $S_u^x = \log |G_u x|$. More precisely, we shall study the asymptotic of the large deviations of the following integral :

$$\int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) = \sum_{u \in \mathbb{T}_n} f(X_u^x)g(S_u^x - nq_s). \quad (1.31)$$

Our result will be stated under the very general assumption that $e^{-sz}g(z)$, $z \in \mathbb{R}$ is directly Riemann integrable, see Feller [36], Chapter XI.

Theorem 1.3.9. *Assume conditions C5 and C6, and let $s \in (-\eta_1, s_\infty)$ be fixed such that $\Lambda^*(q_s) - \log m < 0$ and that*

$$\mathbb{E} \left[\max_{x \in \mathcal{S}} W_1^x(s) \log_+^{\delta+1} \max_{x \in \mathcal{S}} W_1^x(s) \right] < \infty \quad \text{for some } \delta > 3/2. \quad (1.32)$$

Then for any continuous function f on \mathcal{S} and any measurable function g on \mathbb{R} such that $z \mapsto e^{-sz}g(z)$ is directly Riemann integrable, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)}}{m^n} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) \\ = W_s^x r_s(x) \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} e^{-sz} g(z) dz, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where $\pi_s \left(\frac{f}{r_s} \right) = \frac{\nu_s(f)}{\nu_s(r_s)}$, and $\sigma_s^2 = \Lambda''(s)$.

When $s = 0$ this result reduces to the following local limit theorem for the counting measure Z_n^x :

Corollary 1.3.10. *Assume conditions C5 and C6. Assume also that (1.32) holds with*

$s = 0$. Then

$$\lim_{n \rightarrow \infty} \frac{\sigma \sqrt{2\pi n}}{m^n} \int_{\mathcal{S} \times \mathbb{R}} f(y) g(z - n\gamma) Z_n^x(dy, dz) = W\nu(f) \int_{\mathbb{R}} g(z) dz.$$

When $f = 1$ and $g = \mathbb{1}_{[a,b]}$ with $-\infty < a < b < \infty$, it gives the precise asymptotic of $Z_n^x(\mathcal{S}, n\gamma + [a, b])$ as $n \rightarrow \infty$.

The following theorem describes the asymptotic size of the number of particles in n -th generation situated in the regions $(B, [e^{nq_s}, +\infty))$ for $s > 0$, and $(B, (0, e^{nq_s}))$ for $s < 0$, where $B \subseteq \mathcal{S}$.

Theorem 1.3.11. *Assume the conditions of Theorem 1.3.9. Then, for any $x \in \mathcal{S}$, any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$, we have, \mathbb{P} -a.s., for $s > 0$,*

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)} \frac{Z_n^x(B, [nq_s, +\infty))}{m^n} = \frac{1}{s} W_s^x r_s(x) \int_B \frac{1}{r_s(y)} \pi_s(dy),$$

and for $s < 0$,

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)} \frac{Z_n^x(B, (-\infty, nq_s])}{m^n} = \frac{1}{s} W_s^x r_s(x) \int_B \frac{1}{r_s(y)} \pi_s(dy).$$

This theorem is obtained from Theorem 1.3.9 by taking $g = \mathbb{1}_{[0,+\infty)}$ when $s > 0$, and $g = \mathbb{1}_{(-\infty,0]}$ when $s < 0$, and by using a smooth approximation of indicator function (see [22, Lemma 4.1]).

In the one dimensional case (where $d = 1$), Theorems 1.3.9 and 1.3.11 reduce to the Bahadur-Rao type results of Biggins [15]. The large deviation principle was established earlier by Biggins in [14].

Key ideas of the proofs

The starting point in the proofs of our results is a decomposition formula which permits to express the counting measure as the sum of conditionally independent random variables, using the branching property like in the one dimensional case for which we may refer to [3, 15]. However, there is much to do to arrive to the conclusions in the multidimensional case, due to the appearance of products of random matrices. In particular, for the proof of Theorem 1.3.6 about the central limit theorem and Theorem 1.3.9 about the precise large deviation with target functions, we use respectively the central limit theorem and the recent progress on the spectral gap theory and precise large deviations for products

of random matrices. Another step forward in the proof of Theorem 1.3.9 concerns the extension of Biggins' martingale to the case of branching products of random matrices, for which we prove a criterion for the non-degeneracy of the limit of the fundamental martingale (see Theorem 1.3.7) which completes a result of Mentemeier [72] obtained in the context of the multivariate smoothing transform, and extends the Kesten-Stigum type theorem of Biggins [13] on the classical branching random walk.

1.3.4 Berry-Esseen bound and Cramér moderate deviation expansion for a branching random walk with products of random matrices

This subsection is to present our main results about the Berry-Esseen bound and Cramér moderate deviation expansion on Z_n^x defined in (1.7), for a branching random walk with products of random matrices. We will use the assumptions on products of random matrices introduced in the subsection 1.3.3.

Statement of main results

Set

$$J = \{s \in (-\eta_1, \eta_1) : \Lambda^*(q_s) - \log m < 0\},$$

which is an open interval containing 0. We assume the following moment condition slightly stronger than (1.26) :

C8. *There are constants $\gamma_0 > 1$ and $0 < \eta_2 < \frac{\eta_1}{2}$ with $[-\eta_2, \eta_2] \subset J$ such that*

$$\mathbb{E} \left[\max_{x \in \mathcal{S}} \left(W_1^x(s) \right)^{\gamma_0} \right] < \infty \quad \forall s \in [-\eta_2, \eta_2].$$

It is clear that conditions **C5-C8** (together with the hypothesis $\mathbb{P}(N = 0) = 0$ that we assume always), imply that for all $x \in \mathcal{S}$, $W^x(s) > 0$ a.s. and $\mathbb{E}[W^x(s)] = 1$; in particular (when $s = 0$), $W > 0$ a.s. and $\mathbb{E}[W] = 1$.

For $\beta > 0$ sufficiently small, we introduce the Banach space $\mathcal{B}_\beta = \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\beta < +\infty\}$, where

$$\|f\|_\beta := \|f\|_\infty + |f|_\beta,$$

with

$$\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|, \quad |f|_\beta := \sup_{x, y \in \mathcal{S}, x \neq y} \frac{|f(x) - f(y)|}{\mathbf{d}^\beta(x, y)}.$$

Our first result is the Berry-Esseen bound for the counting measure Z_n^x :

Theorem 1.3.12. *Assume conditions C5-C8. Then, for any $x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\beta$ and $n \geq 1$, we have, a.s.,*

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\left\{ \frac{S_u^x - n\gamma}{\sigma\sqrt{n}} \leq y \right\}} - W\nu(\varphi)\Phi(y) \right| \leq \frac{M}{\sqrt{n}}, \quad (1.33)$$

where M is a finite and positive random variable.

This is a Berry-Esseen type bound for the counting measure Z_n^x with suitable norming because the sum in (1.33) is an integral with respect to Z_n^x :

$$\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\left\{ \frac{S_u^x - n\gamma}{\sigma\sqrt{n}} \leq y \right\}} = \int_{\mathcal{S} \times \mathbb{R}} \varphi(z_1) \mathbb{1}_{\left\{ \frac{z_2 - n\gamma}{\sigma\sqrt{n}} \leq y \right\}} Z_n(dz_1, dz_2).$$

Denote $\gamma_k = \Lambda^k(0)$, $k \geq 1$, where $\Lambda = \log \kappa$ defined in (1.21). We write for the Cramér series associated to Λ (see [75]) :

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \dots \quad (1.34)$$

which converges for $|t|$ small enough. Our second result is the Cramér's moderate deviation expansion for Z_n^x .

Theorem 1.3.13. *Assume conditions C5-C8. Then, we have for any $x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\beta$, $0 \leq y = o(\sqrt{n})$, as $n \rightarrow \infty$, a.s.,*

$$\frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \geq \sqrt{n}\sigma y\}}}{m^n W[1 - \Phi(y)]} = e^{\frac{y^3}{\sqrt{n}} \zeta\left(\frac{y}{\sqrt{n}}\right)} \left[\nu(\varphi) + O\left(\frac{y+1}{\sqrt{n}}\right) \right],$$

and

$$\frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \leq -\sqrt{n}\sigma y\}}}{m^n W\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}} \zeta\left(-\frac{y}{\sqrt{n}}\right)} \left[\nu(\varphi) + O\left(\frac{y+1}{\sqrt{n}}\right) \right].$$

An important step in the proof of the moderate deviation expansion is to establish a

Berry-Esseen bound for the changed measure $Z_{s,n}^x$ defined by for measurable sets $B_1 \subset \mathbb{S}^{d-1}$ and $B_2 \subset \mathbb{R}$,

$$\begin{aligned} Z_{s,n}^x(B_1, B_2) &= \int_{B_1 \times B_2} \frac{e^{sz_2} r_s(z_1)}{[m\kappa(s)]^n r_s(x)} Z_n^x(dz_1, dz_2) \\ &= \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\{X_u^x \in B_1, S_u^x \in B_2\}}. \end{aligned}$$

Our third result is a Berry-Esseen bound for the changed measure $Z_{s,n}^x$:

Theorem 1.3.14. *Assume conditions **C5-C8**. Then, for any $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\beta$ there exists a constant $0 < \eta < \eta_2$ such that a.s., for $n \geq 1$,*

$$\sup_{s \in (-\eta, \eta)} \sup_{y \in \mathbb{R}} \left| \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} - W^x(s) \pi_s(\varphi) \Phi(y) \right| \leq \frac{M}{\sqrt{n}}, \quad (1.35)$$

where M is a positive and finite random variable.

This is a Berry-Esseen type bound for $Z_{s,n}^x$ because, similar to the case of Theorem 1.3.12, the sum in (1.35) is an integral with respect to $Z_{s,n}^x$:

$$\begin{aligned} & \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \\ &= \int_{\mathcal{S} \times \mathbb{R}} \frac{e^{sz_2} r_s(z_1) \varphi(z_1)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{z_2 - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} Z_{s,n}^x(dz_1, dz_2). \end{aligned}$$

Key ideas of the proofs

An important step in the proof of Theorems 1.3.12 and 1.3.13 is to establish a Berry-Esseen bound for the Cramér type changed measure $Z_{s,n}^x$. This will be done in Theorem 1.3.14. Theorem 1.3.12 will be obtained from Theorem 1.3.14 by taking $s = 0$, and Theorem 1.3.13 will be established by using Theorem 1.3.14 and by adapting the techniques from Petrov [75]. We would like to give some ideas on in the proof of Theorem 1.3.14. As in [24] where the one dimensional case is considered, we need to study the asymptotic of the characteristic function of the changed measure $Z_{s,n}^x$. Inspired by the approach in [24], we would like to express the characteristic function of $Z_{s,n}^x$ in terms of a martingale and a quantity that can be controlled by the theory of products of random matrices. However, in contrast to the one dimensional case, we cannot obtain directly an expression of the

characteristic function in terms of a martingale. Fortunately, using the spectral gap theory for products of random matrices established in [46, 25, 28] and recently developed in [83], we have been able to define a new martingale which is similar to the fundamental martingale and which can be used for a suitable approximation of the characteristic function of $Z_{s,n}^x$. We conclude by proving the uniform convergence and analyticity with respect to a complex parameter of the new martingale, and by using the asymptotic properties of the eigenvalue of the perturbed transfer operator related to the products of random matrices.

Chapter 2

Berry - Esseen bound and Cramér moderate deviation expansion for a supercritical branching random walk

We consider a supercritical branching random walk where each particle gives birth to a random number of particles of the next generation, which move on the real line, according to a fixed law. Let Z_n be the counting measure which counts the number of particles of n th generation situated in a given region. Under suitable conditions, we establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for Z_n with suitable norming.

2.1 Introduction

A branching random walk is a system of particles, in which each particle gives birth to new particles of the next generation, whose children move on \mathbb{R} . The particles behave independently; the number of children and their displacements are governed by the same probability law for all particles. Important research topics on the model include the study of the asymptotic properties of the counting measure Z_n which counts the number of particles of generation n situated in a Borel set (see e.g. [2, 3, 14, 15, 16, 29, 39, 40, 30]), the study of the fundamental martingale, the norming problem, and the properties of the limit variable (see e.g. [13, 19, 66, 70, 67, 1, 55, 65]), and the positions of the extreme particles (which constitute the boundary of the support of the counting measure Z_n (see e.g. [53, 52, 5, 27]), etc. The study of this model is very interesting especially due to a large number of applications and its close relation with other important models in applied probability settings, such as multiplicative cascades, fractals, perpetuities, branching Brownian motion, the quick sort algorithm and infinite particle systems. For close relations to Mandelbrot's cascades, see e.g. [56, 67, 6, 25, 72]; for relations to other important models, see e.g. the recent books [77, 26, 54] and many references therein. In this paper, we consider the asymptotic properties of the counting measure Z_n as $n \rightarrow \infty$,

by establishing the Berry - Esseen bound and Cramér’s moderate deviation expansion for a suitable norming of Z_n . The study of asymptotic properties of Z_n is interesting because it gives a good description of the configuration of the system at time n .

The branching random walk on the real line can be defined precisely as follows. The process begins with one initial particle denoted by the null sequence \emptyset , situated at the origin $S_\emptyset = 0$. It gives birth to N children denoted by $\emptyset i = i$, with displacements L_i , $i = 1, \dots, N$. In general, each particle of generation n , denoted by a sequence $u = u_1 \cdots u_n$ of length n , situated at $S_u \in \mathbb{R}$, gives birth to N_u particles of the next generation, denoted by ui , which move on the real line with displacements L_{ui} so that their positions are $S_{ui} = S_u + L_{ui}$, $i = 1, \dots, N_u$. All the random variables $(N_u, L_{u1}, L_{u2}, \dots)$, indexed by all finite sequences $u \in \mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$ (by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$), are independent and identically distributed, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{N} \times \mathbb{R} \times \mathbb{R} \times \dots$.

For $n \geq 0$, let \mathbb{T}_n be the set of particles of n -th generation. Consider the counting measure

$$Z_n(A) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{S_u \in A\}}, \quad A \subset \mathbb{R},$$

which counts the number of particles of n -th generation situated in A .

Throughout this paper we assume that

$$m := \mathbb{E}N = \mathbb{E}[Z_1(\mathbb{R})] \in (1, \infty),$$

so that the Galton-Watson process formed by the generation sizes survives with positive probability, and

$$F(A) = \mathbb{E}[Z_1(A)], \quad A \subset \mathbb{R},$$

is a finite measure on \mathbb{R} with mass m . Let \bar{F} be the probability measure on \mathbb{R} defined by

$$\bar{F}(A) = \frac{F(A)}{m}, \quad A \subset \mathbb{R}.$$

Denote its mean and variance by

$$m_0 = \int x \bar{F}(dx) \quad \text{and} \quad \sigma_0^2 = \int (x - m_0)^2 \bar{F}(dx). \tag{2.1}$$

We will assume that $\mathbb{E}(\sum_{i=1}^N L_i^2) < \infty$, so that m_0 and σ_0^2 are finite, with

$$m_0 = \frac{1}{m} \mathbb{E} \left[\sum_{i=1}^N L_i \right] \quad \text{and} \quad \sigma_0^2 = \frac{1}{m} \mathbb{E} \left[\sum_{i=1}^N L_i^2 \right] - m_0^2.$$

A central limit theorem for the special case where $(N_u)_{u \in \mathbb{U}}$ and $(L_u)_{u \in \mathbb{U}}$ are two independent families of independent and identically distributed (i.i.d.) random variables was conjectured by Harris [48]. His conjecture states that under suitable conditions we have, for any $x \in \mathbb{R}$,

$$\frac{1}{m^n} Z_n \left((-\infty, x\sigma_0\sqrt{n} + nm_0] \right) \xrightarrow{n \rightarrow \infty} W\Phi(x) \quad (2.2)$$

in probability, where $\Phi(x)$ is the normal distribution function and W is the a.s. limit of the fundamental martingale $\left(\frac{Z_n(\mathbb{R})}{m^n} \right)$ of the Galton-Watson process $(Z_n(\mathbb{R}))$. This conjecture has first been solved by Stam [78], then improved by Asmussen and Kaplan [2, 3] to L^2 -convergence and almost sure (a.s.) convergence. The general case has been considered by Klebaner [60] and Biggins [16].

In this paper we will study the Berry-Esseen bound about the rate of convergence in (2.2), and the associated Cramér's moderate deviation expansion.

The rate of convergence in (2.2) has been studied in several papers. Révész [76] considered the special case where the displacements follow the same Gaussian law and conjectured the exact convergence rate; his conjecture was solved by Chen [29]. Gao and Liu [39] improved and extended Chen's result to the general non-lattice case while the lattice case has been considered by Grübel and Kabluchko [45]. All the above mentioned results are about the point-wise convergence without uniformity in x . In this paper, our first objective is to find a uniform bound for the rate of convergence in (2.2) of type Berry-Esseen: we will prove that, under suitable conditions, a.s. for $n \geq 1$,

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{m^n} Z_n \left((-\infty, x\sigma_0\sqrt{n} + nm_0] \right) - W\Phi(x) \right| \leq \frac{M}{\sqrt{n}}, \quad (2.3)$$

where M is a positive and finite random variable (see Theorem 2.2.1).

The problem of large deviations for the counting measure $Z_n(\cdot)$ has been considered by Biggins: he established in [14] a large deviation principle, which was subsequently improved in [15] to a Bahadur-Rao large deviation asymptotic. Our second objective in this paper is to establish a Cramér type moderate deviation expansion for Z_n (see

Theorem 2.2.2): we will prove that a.s. for $n \rightarrow \infty$ and $x \in [0, o(\sqrt{n})]$,

$$\frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)]} = e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right], \quad (2.4)$$

where $t \mapsto \mathcal{L}(t)$ is the Cramér series (see (2.11)). Here we use the usual notation $b_n = O(a_n)$ to mean that the sequence (b_n/a_n) is bounded. (We mention that as (2.4) holds a.s., the bound in $O\left(\frac{x+1}{\sqrt{n}}\right)$ may be random.)

Let us explain briefly the key ideas in the proofs. To prove the Berry-Esseen bound (2.3), we use Esseen’s smoothing inequality ([75, Theorem V.2.2.]). The key point in this proof is the formula of the characteristic function of $\frac{1}{m^n} Z_n\left((-\infty, x\sigma_0\sqrt{n} + nm_0]\right)$, which can be interpreted as $W_n\left(\frac{it}{\sigma_0\sqrt{n}}\right) f_n(t)$, $t \in \mathbb{R}$, where $(W_n(\lambda))$ is Biggins’ martingale with complex valued parameter λ for the branching random walk (see [17, 18]), and $f_n(t)$ is the characteristic function of the n -fold convolution of \overline{F} . Using the results of Biggins [17, 18], Grübel and Kabluchko [45] about the uniform convergence of $W_n(\lambda)$, together with the approach of Petrov [75] for the proof of the Berry-Esseen bound for sums of i.i.d. random variables, we are able to establish (2.3). The Berry-Esseen bound (2.3) is then extended to the changed measure of type Cramér, $Z_n^\theta(A) = \int_A e^{\theta t} Z_n(dt)$, $A \subset \mathbb{R}$, $\theta \in \mathbb{R}$. This is an important step in establishing the moderate deviation expansion (2.4). Our approach in proving (2.4) is very different from the method of Biggins [15] on the Bahadur-Rao large deviation asymptotic; instead, it is inspired by the ideas of the proof of Cramér’s moderate deviation expansion on sums of i.i.d. random variables (see [75]), and the arguments in [18] for the proof of the local limit theorem with large deviations for Z_n .

The main results, Theorems 2.2.1-2.2.3, are presented in Section 2.2. Theorems 2.2.1 and 2.2.3 about the Berry-Esseen bound are proved in Section 2.3, while Theorem 2.2.2 about the moderate deviation is established in Section 2.4.

2.2 Notation and results

We will use the following standard assumptions.

H1. $N > 0$ a.s. with $m = \mathbb{E}N \in (1, \infty)$, and $\mathbb{E}\left[\sum_{i=1}^N L_i^2\right] < \infty$.

H2. F is non-degenerate, i.e. it is not concentrated on a single point.

The first condition in **H1** implies that the underlying Galton -Watson process is supercritical; the second condition in **H1**, together with condition **H2**, implies that the mean

m_0 and the variance σ_0^2 defined by (2.1) are finite with $\sigma_0 > 0$.

The Laplace transform of F will be denoted by

$$m(\lambda) = \int_{\mathbb{R}} e^{\lambda t} F(dt) = \mathbb{E} \left[\sum_{i=1}^N e^{\lambda L_i} \right], \quad \lambda \in \mathbb{C}. \quad (2.5)$$

Denote by $\text{int}(A)$ the interior of the set A . Set

$$\mathcal{D} = \text{int}\{\theta \in \mathbb{R} : m(\theta) < \infty\}. \quad (2.6)$$

Throughout, we assume that

H3. \mathcal{D} is non-empty.

Denote by $\text{Re}(\lambda)$ the real part of $\lambda \in \mathbb{C}$. An important role in the proof of Berry-Esseen bound and moderate deviation expansion is played by the martingale of Biggins with complex parameter:

$$W_n(\lambda) = \frac{1}{m(\lambda)^n} \int_{\mathbb{R}} e^{\lambda t} Z_n(dt) = \sum_{u \in \mathbb{T}_n} \frac{e^{\lambda S_u}}{m(\lambda)^n}, \quad n \geq 0, \text{Re}(\lambda) \in \mathcal{D}.$$

When $\lambda = 0$, $W_n := W_n(0) = \frac{Z_n(\mathbb{R})}{m^n}$ is the fundamental martingale of the Galton - Watson process $(Z_n(\mathbb{R}))$, whose a.s. limit is denoted by W . The famous Kesten-Stigum theorem states that W is non degenerate if and only if $\mathbb{E}N \log_+ N < \infty$ (see [4]), where $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$. By the martingale convergence theorem for non-negative martingales, we have for all $\theta \in \mathcal{D}$,

$$W_n(\theta) \xrightarrow{n \rightarrow \infty} W(\theta), \quad \text{a.s.}$$

Notice that when $N > 0$ a.s. we have $W_n(\theta) > 0$ a.s. for all $n \geq 0$ and $\theta \in \mathcal{D}$. Biggins [13, Theorem A] gave a necessary and sufficient condition for the non-degeneracy of $W(\theta)$: $\mathbb{E}W(\theta) > 0$ if and only if

$$\mathbb{E}[W_1(\theta) \log_+ W_1(\theta)] < \infty \quad \text{and} \quad \theta \in (\theta_-, \theta_+), \quad (2.7)$$

where $(\theta_-, \theta_+) \subset \mathcal{D}$ denotes by the open interval on which $\frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta)$, i.e.

$$\begin{aligned}\theta_- &= \inf \left\{ \theta \in \mathcal{D} : \frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta) \right\}, \\ \theta_+ &= \sup \left\{ \theta \in \mathcal{D} : \frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta) \right\}.\end{aligned}$$

Moreover, when **H1** and (2.7) hold,

$$W(\theta) > 0 \text{ a.s. and } \mathbb{E}W(\theta) = 1. \tag{2.8}$$

We see that $0 \in (\theta_-, \theta_+)$, so that this interval is non-empty. The endpoints of the interval \mathcal{D} and the quantities θ_-, θ_+ are allowed to be infinite. We will need the following moment condition which is slightly stronger than (2.7).

H4. *There are $\gamma > 1$ and $K_0 > 0$ with $(-K_0, K_0) \subset (\theta_-, \theta_+)$ such that*

$$\mathbb{E}W_1^\gamma(\theta) < \infty \quad \forall \theta \in (-K_0, K_0).$$

By the argument of the proof of [18, Theorem 2], we know that under hypothesis **H4**, for every compact subset C of $V := \{\lambda = \theta + i\eta : \theta \in (-K_0, K_0), \eta \in \mathbb{R}\}$, a.s.

$$\sup_{\lambda \in C} |W_n(\lambda) - W(\lambda)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad W(\lambda) \text{ is analytic in } C. \tag{2.9}$$

Our first result gives the Berry-Esseen bound for Z_n :

Theorem 2.2.1. *Assume conditions **H1** -**H4**. Then, a.s. for all $n \geq 1$,*

$$\sup_{x \in \mathbb{R}} \left| \frac{Z_n((-\infty, x\sigma_0\sqrt{n} + nm_0])}{m^n} - W\Phi(x) \right| \leq \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable.

To state the result corresponding to the Cramér type moderate deviation expansion for Z_n , we need more notation. Consider the measure

$$F_\theta(dx) = \frac{e^{\theta x}}{m(\theta)} F(dx), \quad \theta \in \mathcal{D}. \tag{2.10}$$

We see that F_θ is a distribution function with finite mean m_θ and variance σ_θ^2 , given by

$$m_\theta = \frac{m'(\theta)}{m(\theta)}, \quad \sigma_\theta^2 = \frac{m''(\theta)}{m(\theta)} - \left(\frac{m'(\theta)}{m(\theta)}\right)^2;$$

moreover, $\sigma_\theta > 0$ when F is non-degenerate. Consider the change of measure of type Cramér for Z_n : for $\theta \in \mathcal{D}$,

$$Z_n^\theta(dx) = e^{\theta x} Z_n(dx),$$

namely,

$$Z_n^\theta(A) = \sum_{u \in \mathbb{T}_n} e^{\theta S_u} \mathbf{1}_{\{S_u \in A\}}, \quad A \subset \mathbb{R}.$$

Let X be a random variable with distribution $\bar{F} := \frac{F}{m}$, and

$$\Lambda(\theta) := \log \mathbb{E} e^{\theta X} = \log m(\theta) - \log m$$

be its cumulant generating function. Then $\Lambda(\theta)$ is analytic on \mathcal{D} , with $\Lambda'(\theta) = m_\theta$ and $\Lambda''(\theta) = \sigma_\theta^2$. Denote by $\gamma_k := \Lambda^{(k)}(0)$ the cumulant of order k of the random variable X . We shall use the Cramér series (see [75, Theorem VIII.2.2]):

$$\mathcal{L}(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \dots \quad (2.11)$$

which converges for $|t|$ small enough.

Theorem 2.2.2. *Assume conditions **H1** -**H4**. Then we have, for $0 \leq x = o(\sqrt{n})$, as $n \rightarrow \infty$, a.s.*

$$\frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)]} = e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right], \quad (2.12)$$

and

$$\frac{Z_n\left((-\infty, -x\sigma_0\sqrt{n} + nm_0)\right)}{m^n W\Phi(-x)} = e^{-\frac{x^3}{\sqrt{n}} \mathcal{L}\left(-\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right]. \quad (2.13)$$

As a by-product in the proof of Theorem 2.2.2, we obtain a Berry - Esseen bound for the changed measure Z_n^θ with uniformity in θ .

Theorem 2.2.3. *Assume conditions **H1** - **H4**. Then, there exists a constant $0 < K < K_0$ such that a.s. for all $n \geq 1$,*

$$\sup_{\theta \in [-K, K]} \sup_{x \in \mathbb{R}} \left| \frac{Z_n^\theta((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta])}{m(\theta)^n} - W(\theta)\Phi(x) \right| \leq \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable.

2.3 Proof of Theorems 2.2.1 and 2.2.3

We first recall some known results in the form of two lemmas which will be used for the proof of Theorems 2.2.1 and 2.2.3.

The first lemma concerns the Cramér change of measure (2.10), see [75, Theorem VIII.2.2, inequalities (2.31) and (2.32)].

Lemma 2.3.1. *Let X be a real random variable with distribution G . Suppose that $\text{Var}(X) > 0$ and that there exist strictly positive constants H, c such that*

$$|\log \mathbb{E}e^{\theta X}| \leq c \text{ for all } \theta \in (-H, H).$$

Let X_θ be a real random variable with distribution G_θ defined by

$$G_\theta(dx) = \frac{e^{\theta x} G(dx)}{\mathbb{E}e^{\theta X}}, \quad \theta \in (-H, H).$$

Then there exist strictly positive constants H_1, c_1, c_2 with $H_1 < H$, such that for all $\theta \in (-H_1, H_1)$,

$$\text{Var}(X_\theta) \geq c_1 \text{ and } \mathbb{E}|X_\theta - \mathbb{E}X_\theta|^3 \leq c_2.$$

We see that under **H2** and **H3**, the distribution $G = \bar{F}$ satisfies the conditions of this lemma. Indeed, if X is a random variable with distribution \bar{F} , then by condition **H2** about the non-degeneracy of F , we have $\text{Var}(X) > 0$. By condition **H3**, the set \mathcal{D} defined by (2.6) is an open interval containing 0. Notice that $\log \mathbb{E}e^{\theta X} = \log \frac{m(\theta)}{m} < \infty$ for all $\theta \in \mathcal{D}$. Hence there exist constants $H, c > 0$ such that $|\log \mathbb{E}e^{\theta X}| \leq c$ for all $\theta \in (-H, H)$.

The second lemma is about the exponential convergence rate of $W_n(\theta)$, see [45, Lemma 3.3]. In fact in [45, Lemma 3.3] the result is only given for the lattice case, but the proof

therein remains valid for the non-lattice case.

Lemma 2.3.2. *Assume conditions **H1-H4**. There exist two constants $0 < K < K_0$ and $c \in (0, 1)$ such that a.s. for all $n \geq 0$,*

$$\sup_{\theta \in [-K, K]} |W_n(\theta) - W(\theta)| \leq M_1 c^n,$$

where M_1 is a positive and finite random variable.

Notice that Theorem 2.2.1 follows from Theorem 2.2.3 with $\theta = 0$, by the fact that $m(0) = m$ and $W(0) = W$. So we only proceed to prove Theorem 2.2.3.

Proof of Theorem 2.2.3. From Lemma 2.3.2, to prove Theorem 2.2.3, it is enough to show that there is a constant $0 < K < K_0$ such that

$$\sup_{\theta \in [-K, K]} \sup_{x \in \mathbb{R}} \left| \frac{Z_n^\theta((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta])}{m(\theta)^n} - W_n(\theta)\Phi(x) \right| \leq \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable. Consider the random measure

$$\nu_n^\theta(A) = \frac{Z_n^\theta(\sigma_\theta\sqrt{n}A + nm_\theta)}{m(\theta)^n}, \quad A \subset \mathbb{R},$$

with the usual notation $aA + b = \{ax + b : x \in A\}$. Its distribution function is

$$\nu_n^\theta(x) = \frac{Z_n^\theta((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta])}{m(\theta)^n}, \quad x \in \mathbb{R}.$$

The characteristic function of the random measure ν_n^θ is

$$\begin{aligned} \psi_n^\theta(t) &= \int_{\mathbb{R}} e^{itx} \nu_n^\theta(dx) = \frac{1}{m(\theta)^n} \sum_{u \in \mathbb{T}_n} \exp \left\{ \left(\theta + \frac{it}{\sigma_\theta\sqrt{n}} \right) S_u - \frac{it nm_\theta}{\sigma_\theta\sqrt{n}} \right\} \\ &= W_n \left(\theta + \frac{it}{\sigma_\theta\sqrt{n}} \right) f_n^\theta(t), \quad t \in \mathbb{R}, \end{aligned} \quad (2.14)$$

where $f_n^\theta(t) = \frac{1}{m(\theta)^n} m \left(\theta + \frac{it}{\sigma_\theta\sqrt{n}} \right)^n e^{-\frac{it nm_\theta}{\sigma_\theta\sqrt{n}}}$. Denote by F_θ^{*n} the n -fold convolution of F_θ . It is not difficult to see that

$$f_n^\theta(t) = \int_{\mathbb{R}} e^{\frac{it(x - nm_\theta)}{\sigma_\theta\sqrt{n}}} F_\theta^{*n}(dx),$$

which is the characteristic function of $\frac{\bar{S}_n - nm_\theta}{\sigma_\theta \sqrt{n}}$, where \bar{S}_n is the sum of independent random variables $\{\bar{X}_i\}_{i=1}^n$ with the same law F_θ .

By Esseen's smoothing inequality (see [75, Theorem V.2.2]), we get for all $T > 0$, a.s.

$$\begin{aligned} & \sup_{x \in \mathbb{R}} \left| \nu_n^\theta(x) - W_n(\theta) \Phi(x) \right| \\ & \leq \frac{1}{\pi} \int_{-T}^T \left| \frac{W_n\left(\theta + \frac{it}{\sigma_\theta \sqrt{n}}\right) f_n^\theta(t) - W_n(\theta) e^{-t^2/2}}{t} \right| dt + W_n(\theta) \frac{c}{T}, \end{aligned} \quad (2.15)$$

where c is a deterministic positive constant. From Lemma 2.3.1, there exist strictly positive constants K, c_1, c_2 with $K < \min\{H_1, K_0\}$ such that for all $|\theta| \leq K$

$$\sigma_\theta^2 \geq c_1 \quad \text{and} \quad \mathbb{E}|\bar{X} - m_\theta|^3 \leq c_2. \quad (2.16)$$

Take $T = a\sigma_\theta \sqrt{n}$ with $a = \inf_{\theta \in [-K, K]} \frac{\sigma_\theta^2}{4\mathbb{E}|\bar{X} - m_\theta|^3} \geq \frac{c_1}{c_2} > 0$. For $0 < \varepsilon < a$, we split the integral on the right-hand side of (2.15) into two parts $|t| < \varepsilon\sigma_\theta \sqrt{n}$ and $\varepsilon\sigma_\theta \sqrt{n} \leq |t| \leq a\sigma_\theta \sqrt{n}$ to get

$$\sup_{\theta \in [-K, K]} \sup_{x \in \mathbb{R}} \left| \nu_n^\theta(x) - W_n(\theta) \Phi(x) \right| \leq \frac{1}{\pi} (I_1 + I_2) + \frac{c}{a\sqrt{n}} \sup_{\theta \in [-K, K]} \frac{W_n(\theta)}{\sigma_\theta},$$

where

$$\begin{aligned} I_1 &= \sup_{\theta \in [-K, K]} \int_{|t| < \varepsilon\sigma_\theta \sqrt{n}} \left| \frac{W_n\left(\theta + \frac{it}{\sigma_\theta \sqrt{n}}\right) f_n^\theta(t) - W_n(\theta) e^{-t^2/2}}{t} \right| dt, \\ I_2 &= \sup_{\theta \in [-K, K]} \int_{\varepsilon\sigma_\theta \sqrt{n} \leq |t| \leq a\sigma_\theta \sqrt{n}} \left| \frac{W_n\left(\theta + \frac{it}{\sigma_\theta \sqrt{n}}\right) f_n^\theta(t) - W_n(\theta) e^{-t^2/2}}{t} \right| dt. \end{aligned}$$

In the following, M_i denotes a positive and finite random variable. By Lemma 2.3.2 and the lower bound (2.16) of σ_θ , $\sup_{\theta \in [-K, K]} \frac{W_n(\theta)}{\sigma_\theta} \leq M_2$ a.s. Hence, it remains to show that a.s., $I_1 \leq \frac{M_3}{\sqrt{n}}$ and $I_2 \leq \frac{M_4}{\sqrt{n}}$.

For I_1 , we see that

$$\begin{aligned} I_1 \leq & \sup_{\theta \in [-K, K]} \sup_{\frac{|t|}{\sigma_\theta \sqrt{n}} \leq \varepsilon} \left| W_n \left(\theta + \frac{it}{\sigma_\theta \sqrt{n}} \right) \right| \int_{|t| < \varepsilon \sigma_\theta \sqrt{n}} \frac{|f_n^\theta(t) - e^{-t^2/2}|}{|t|} dt \\ & + \sup_{\theta \in [-K, K]} \int_{|t| < \varepsilon \sigma_\theta \sqrt{n}} \frac{|W_n \left(\theta + \frac{it}{\sigma_\theta \sqrt{n}} \right) - W_n(\theta)|}{|t|} e^{-t^2/2} dt. \end{aligned} \quad (2.17)$$

By the uniform convergence (2.9) of $W_n(\cdot)$, we have

$$\sup_{\theta \in [-K, K]} \sup_{\frac{|t|}{\sigma_\theta \sqrt{n}} \leq \varepsilon} \left| W_n \left(\theta + \frac{it}{\sigma_\theta \sqrt{n}} \right) \right| \leq M_5. \quad (2.18)$$

Recall that $t \mapsto f_n^\theta(t)$ is the characteristic function of $\frac{\bar{S}_n - nm_\theta}{\sigma_\theta \sqrt{n}}$. Then by [75, Lemma V.2.1], for $|t| \leq \frac{\sigma_\theta^3 \sqrt{n}}{4\mathbb{E}|\bar{X} - m_\theta|^3}$, we have

$$\frac{|f_n^\theta(t) - e^{-t^2/2}|}{|t|} \leq \frac{\mathbb{E}|\bar{X} - m_\theta|^3}{\sigma_\theta^3 \sqrt{n}} t^2 e^{-t^2/3} \leq \frac{c_2}{c_1 \sqrt{n}} t^2 e^{-t^2/3}. \quad (2.19)$$

Therefore (2.19) holds for $|t| \leq \varepsilon \sigma_\theta \sqrt{n}$ since $\varepsilon \sigma_\theta \sqrt{n} \leq \frac{\sigma_\theta^3 \sqrt{n}}{4\mathbb{E}|\bar{X} - m_\theta|^3}$. From (2.18), (2.19) and the fact that $\int_{\mathbb{R}} |t|^2 e^{-t^2/3} dt < \infty$, we see that the first term in (2.17) is bounded by $\frac{M_6}{\sqrt{n}}$.

Now we consider the second term in (2.17). Since $\int_{\mathbb{R}} e^{-t^2/2} dt = \sqrt{2\pi}$, we need only to show that

$$\sup_{\theta \in [-K, K]} \sup_{\frac{|t|}{\sigma_\theta \sqrt{n}} \leq \varepsilon} \frac{1}{|t|} \left| W_n \left(\theta + \frac{it}{\sigma_\theta \sqrt{n}} \right) - W_n(\theta) \right| \leq \frac{M_7}{\sqrt{n}}. \quad (2.20)$$

Notice that $W_n(\lambda)$ is a.s. analytic in the strip $Re(\lambda) \in (-K_0, K_0)$. Let $0 < K_1 < K_0$. By the mean value theorem, when $\theta \in [-K_1, K_1]$ and $\frac{|t|}{\sigma_\theta \sqrt{n}} \leq \varepsilon$, we have

$$\left| W_n \left(\theta + \frac{it}{\sigma_\theta \sqrt{n}} \right) - W_n(\theta) \right| \leq \frac{|t|}{\sigma_\theta \sqrt{n}} \max_{\eta \in [-\varepsilon, \varepsilon]} |W_n'(\theta + i\eta)| \quad (2.21)$$

By Cauchy's formula, when $|\lambda| < K_1$,

$$W'_n(\lambda) = \frac{1}{2\pi i} \int_{|z|=K_1} \frac{W_n(z)}{(z-\lambda)^2} dz.$$

By (2.9), a.s. for all $n \geq 1$ and all $z \in \mathbb{C}$ with $|z| \leq K_1$, $|W_n(z)| \leq M_8$. When $|\lambda| \leq K_1/2$ and $|z| = K_1$, $|z-\lambda| \geq K_1 - K_1/2 = K_1/2$, so that $|\frac{W_n(z)}{(z-\lambda)^2}| \leq \frac{4M_8}{K_1^2}$. Therefore for all $n \geq 1$, a.s.

$$\max_{|\lambda| \leq K_1/2} |W'_n(\lambda)| \leq \frac{4M_8}{K_1}.$$

Therefore from (2.21) and (2.16), we see that (2.20) holds when $K < K_1/4$ and $\varepsilon < K_1/4$. This concludes that the second term in (2.17) is bounded by $\frac{M_9}{\sqrt{n}}$. Therefore from (2.17) we get $I_1 \leq \frac{M_{10}}{\sqrt{n}}$.

For I_2 , using the constraint in the integral of I_2 , we have $\frac{1}{|t|} \leq \frac{1}{\varepsilon\sigma_\theta\sqrt{n}}$, so that

$$\begin{aligned} I_2 &\leq \sup_{\theta \in [-K, K]} \frac{1}{\varepsilon\sigma_\theta\sqrt{n}} \int_{\varepsilon \leq \frac{|t|}{\sigma_\theta\sqrt{n}} \leq a} \left| W_n\left(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\right) f_n^\theta(t) \right| dt \\ &\quad + \sup_{\theta \in [-K, K]} \frac{W_n(\theta)}{\varepsilon\sigma_\theta\sqrt{n}} \int_{\varepsilon \leq \frac{|t|}{\sigma_\theta\sqrt{n}} \leq a} e^{-t^2/2} dt. \end{aligned}$$

It is shown in the proof of [18, Lemma 5] that as $n \rightarrow \infty$,

$$\sup_{\theta \in [-K, K]} \sqrt{n} \int_{\varepsilon \leq \eta \leq a} \left| W_n(\theta + i\eta) f_n^\theta(\sigma_\theta\sqrt{n}\eta) \right| d\eta \rightarrow 0 \quad \text{a.s.},$$

which can be rewritten as

$$\sup_{\theta \in [-K, K]} \frac{1}{\sigma_\theta} \int_{\varepsilon \leq \frac{|t|}{\sigma_\theta\sqrt{n}} \leq a} \left| W_n\left(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\right) f_n^\theta(t) \right| dt \rightarrow 0 \quad \text{a.s.}$$

Therefore,

$$\sup_{\theta \in [-K, K]} \frac{1}{\varepsilon\sigma_\theta\sqrt{n}} \int_{\varepsilon \leq \frac{|t|}{\sigma_\theta\sqrt{n}} \leq a} \left| W_n\left(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\right) f_n^\theta(t) \right| dt \leq \frac{M_{11}}{\sqrt{n}} \quad \text{a.s.}$$

This, together with $\sup_{\theta \in [-K, K]} \frac{W_n(\theta)}{\sigma_\theta} \leq M_{12}$, implies that $I_2 \leq \frac{M_{13}}{\sqrt{n}}$. Thus the proof of

Theorem 2.2.3 is completed. \square

2.4 Proof of Theorem 2.2.2

In this section we prove Theorem 2.2.2, the Cramér type moderate deviation expansion for Z_n .

Proof of Theorem 2.2.2. We will only prove (2.12), as the proof of (2.13) is similar.

For $x \in [0, 1]$, Theorem 2.12 is a direct consequence of Theorem 2.2.1, as we will see in the following. For $n \geq 1$,

$$\begin{aligned} & \left| \frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)]e^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)}} - 1 \right| \\ &= \frac{1}{W[1 - \Phi(x)]e^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)}} \left| \frac{Z_n(\mathbb{R})}{m^n} - \frac{Z_n\left((-\infty, x\sigma_0\sqrt{n} + nm_0)\right)}{m^n} \right. \\ & \quad \left. - W(1 - \Phi(x))e^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \right|. \end{aligned} \quad (2.22)$$

Since $\sup_{x \in [0, 1]} \left| \frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right) \right| \rightarrow 0$, there exists n_0 large enough such that for all $x \in [0, 1]$ and $n \geq n_0$, $e^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \geq 1/2$. Using this and the fact that $1 - \Phi(x) \geq c := 1 - \Phi(1)$ for all $x \in [0, 1]$, from (2.22) we get for all $n \geq n_0$,

$$\begin{aligned} & \left| \frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)]e^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)}} - 1 \right| \\ & \leq \frac{2}{cW} \left| \frac{Z_n(\mathbb{R})}{m^n} - W \right| + \frac{2}{cW} \left| - \frac{Z_n\left((-\infty, x\sigma_0\sqrt{n} + nm_0)\right)}{m^n} + W\Phi(x) \right| \\ & \quad + \frac{2}{cW} \left| W(1 - \Phi(x)) \left(1 - e^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \right) \right|. \end{aligned} \quad (2.23)$$

In the last display, by Theorem 2.2.1, when $n \rightarrow \infty$, the two first terms are $O\left(\frac{1}{\sqrt{n}}\right)$. We will show below that the third term is also $O\left(\frac{1}{\sqrt{n}}\right)$. In fact, using the inequality $|1 - e^t| \leq |t|e^t$ for $t \in \mathbb{R}$ and the fact that $\sup_{x \in [0, 1]} \left| \mathcal{L}\left(\frac{x}{\sqrt{n}}\right) \right|$ is bounded for $n \geq n_0$, we

obtain for $x \in [0, 1]$, as $n \rightarrow \infty$,

$$\left| 1 - e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \right| \leq \left| \frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right) \right| e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} = O\left(\frac{1}{\sqrt{n}}\right).$$

This implies that the third term in (2.23) is $O\left(\frac{1}{\sqrt{n}}\right)$. From (2.23) and the above estimations, we see that for $x \in [0, 1]$, as $n \rightarrow \infty$,

$$\left| \frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)] e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)}} - 1 \right| = O\left(\frac{1}{\sqrt{n}}\right),$$

which implies

$$\frac{Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right)}{m^n W[1 - \Phi(x)]} = e^{\frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{1}{\sqrt{n}}\right) \right].$$

This ends the proof of (2.12) in the case where $x \in [0, 1]$.

We now deal with the case $1 < x = o(\sqrt{n})$. For $u \in (\mathbb{N}^*)^n$, set

$$V_u = \frac{S_u - nm_\theta}{\sigma_\theta \sqrt{n}}.$$

Recalling that $\Lambda(\theta) = \log \mathbb{E}e^{\theta X} = \log \frac{m(\theta)}{m}$ and $\Lambda'(\theta) = m_\theta$, we have

$$\begin{aligned} I &:= \frac{1}{m^n} Z_n\left((x\sigma_0\sqrt{n} + nm_0, +\infty)\right) = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \mathbf{1}_{\{S_u > x\sigma_0\sqrt{n} + nm_0\}} \\ &= e^{-n[\theta\Lambda'(\theta) - \Lambda(\theta)]} \sum_{u \in \mathbb{T}_n} e^{-\theta\sigma_\theta\sqrt{n}V_u} \cdot \frac{e^{\theta S_u}}{m(\theta)^n} \mathbf{1}_{\left\{V_u > \frac{\sigma_0 x}{\sigma_\theta} + \frac{(m_0 - m_\theta)\sqrt{n}}{\sigma_\theta}\right\}}. \end{aligned} \quad (2.24)$$

Because $\Lambda(\theta)$ is analytic on \mathcal{D} with $\Lambda(0) = 0$, it has the Taylor expansion

$$\Lambda(\theta) = \sum_{k=1}^{\infty} \frac{\gamma_k}{k!} \theta^k, \quad \text{where } \gamma_k = \Lambda^{(k)}(0), \quad \theta \in \mathcal{D}, \quad (2.25)$$

which implies that

$$\Lambda'(\theta) - \Lambda'(0) = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!} \theta^{k-1}. \quad (2.26)$$

Consider the equation

$$\sqrt{n}(m_\theta - m_0) = \sigma_0 x, \quad \text{namely} \quad \Lambda'(\theta) - \Lambda'(0) = \frac{\sigma_0 x}{\sqrt{n}}. \quad (2.27)$$

Set $t = \frac{x}{\sqrt{n}}$, from (2.26) and (2.27), we get

$$\sigma_0 t = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!} \theta^{k-1}. \quad (2.28)$$

Since $\gamma_2 = \sigma_0^2 > 0$, the equation (2.28) has the unique solution given by

$$\theta = \frac{t}{\gamma_2^{1/2}} - \frac{\gamma_3}{2\gamma_2^2} t^2 - \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{6\gamma_2^{7/2}} t^3 + \dots \quad (2.29)$$

Observe that from (2.25) and (2.26), for any $\theta \in \mathcal{D}$,

$$\theta\Lambda'(\theta) - \Lambda(\theta) = \sum_{k=1}^{\infty} \frac{\gamma_k}{(k-1)!} \theta^k - \sum_{k=1}^{\infty} \frac{\gamma_k}{k!} \theta^k = \sum_{k=2}^{\infty} \frac{k-1}{k!} \gamma_k \theta^k.$$

Choosing θ to be the unique real root of the equation (2.28), which is given by (2.29), we obtain (see [75, Theorem VIII.2.2] for details)

$$\theta\Lambda'(\theta) - \Lambda(\theta) = \frac{t^2}{2} - t^3 \mathcal{L}(t) = \frac{x^2}{2n} - \frac{x^3}{n^{3/2}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right), \quad (2.30)$$

where $\mathcal{L}(t)$ is the Cramér series defined in (2.11), which converges for $|t|$ small enough. Substituting (2.27) into (2.24) and using (2.30), we get

$$\begin{aligned} I &= e^{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \sum_{u \in \mathbb{T}_n} e^{-\theta \sigma_\theta \sqrt{n} V_u} \frac{e^{\theta S_u}}{m(\theta)^n} \mathbb{1}_{\{V_u > 0\}} \\ &= e^{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \mathcal{L}\left(\frac{x}{\sqrt{n}}\right)} \int_0^\infty e^{-\theta \sigma_\theta \sqrt{n} y} \bar{Z}_n^\theta(dy), \end{aligned} \quad (2.31)$$

where \bar{Z}_n^θ is the finite measure on \mathbb{R} defined by

$$\bar{Z}_n^\theta(A) = \sum_{u \in \mathbb{T}_n} \frac{e^{\theta S_u}}{m(\theta)^n} \mathbb{1}_{\{V_u \in A\}}, \quad A \subset \mathbb{R},$$

whose mass satisfies $\mathbb{E} \bar{Z}_n^\theta(\mathbb{R}) = 1$. From $t = \frac{x}{\sqrt{n}}$ and $x = o(\sqrt{n})$, it follows that $t \rightarrow 0$

as $n \rightarrow \infty$. By the inverse function theorem for analytic functions, the series on the right-hand side of (2.29) is absolutely convergent for $|t|$ small enough. Moreover, from (2.29), we have $\theta \rightarrow 0^+$ as $n \rightarrow \infty$. Hence, for sufficiently large n_0 and all $n \geq n_0$, we have $|\theta| \leq K$, where K is defined as in Theorem 2.2.3. Therefore, denoting

$$l_{n,\theta}(y) = \bar{Z}_n^\theta((-\infty, y]) - W(\theta)\Phi(y), \quad y \in \mathbb{R},$$

from Theorem 2.2.3 we get for all $n \geq n_0$,

$$\sup_{y \in \mathbb{R}} |l_{n,\theta}(y)| \leq \frac{M}{\sqrt{n}}, \tag{2.32}$$

where M is a positive and finite random variable independent of n and θ . Notice that

$$\begin{aligned} \int_0^\infty e^{-\theta\sigma_\theta\sqrt{ny}} \bar{Z}_n^\theta(dy) &= \int_0^\infty e^{-\theta\sigma_\theta\sqrt{ny}} dl_{n,\theta}(y) + \frac{W(\theta)}{\sqrt{2\pi}} \int_0^\infty e^{-\theta\sigma_\theta\sqrt{ny} - \frac{y^2}{2}} dy \\ &=: I_1 + W(\theta)I_2. \end{aligned} \tag{2.33}$$

Estimate of I_1 . Using the integration by parts and the bound (2.32), we get that for $n \geq n_0$,

$$|I_1| \leq |l_{n,\theta}(0)| + \theta\sigma_\theta\sqrt{n} \int_0^\infty e^{-\theta\sigma_\theta\sqrt{ny}} |l_{n,\theta}(y)| dy \leq \frac{2M}{\sqrt{n}}. \tag{2.34}$$

Estimate of I_2 . The integral I_2 appears in the proof of Cramér’s moderate deviation expansion theorem for sums of i.i.d. random variables (see [75, Theorem VIII.2.2]), where the following results have been proved:

- (i) there exist some positive constants C_1, C_2 such that for all $\theta \in [-K, K]$ and all n large enough,

$$C_1 \leq \theta\sigma_\theta\sqrt{n}I_2 \leq C_2;$$

- (ii) the integral I_2 admits the following asymptotic expansion :

$$I_2 = e^{\frac{x^2}{2}} [1 - \Phi(x)] \left[1 + O\left(\frac{x}{\sqrt{n}}\right) \right]. \tag{2.35}$$

By the definition of σ_θ , the mapping $\theta \mapsto \sigma_\theta$ is strictly positive and continuous on $[-K, K]$.

Hence, there exist positive constants C_3, C_4 such that for all $\theta \in [-K, K]$,

$$C_3 \leq \theta\sqrt{n}I_2 \leq C_4. \quad (2.36)$$

Notice that by (2.8), for all $\theta \in [-K, K]$, $W(\theta) > 0$ a.s. Moreover, $W(\theta)$ is a.s. continuous in θ by the continuity and uniform convergence of $W_n(\theta)$ on $[-K, K]$. Combining this with (2.36), we get

$$M_3 \leq \theta\sqrt{n}W(\theta)I_2 \leq M_4. \quad (2.37)$$

We now come back to (2.33), and let θ be defined by (2.29). Recall that for $n \geq n_0$, $|\theta| \leq K$. From (2.33), (2.37) and (2.34), we have, as $n \rightarrow \infty$,

$$\int_0^\infty e^{-\theta\sigma_\theta\sqrt{ny}}\bar{Z}_n^\theta(dy) = W(\theta)I_2\left(1 + \frac{\sqrt{n}I_1}{\sqrt{n}W(\theta)I_2}\right) = W(\theta)I_2(1 + O(\theta)). \quad (2.38)$$

According to the analyticity of $W(\theta)$ on $[-K, K]$ and using the mean value theorem one see that $|W(\theta) - W| = |W(\theta) - W(0)| \leq M_5\theta$. Since $\theta = O\left(\frac{x}{\sqrt{n}}\right)$ by (2.29), it follows from (2.38) and (2.35) that

$$\begin{aligned} \int_0^\infty e^{-\theta\sigma_\theta\sqrt{ny}}\bar{Z}_n^\theta(dy) &= (W + O(\theta))I_2(1 + O(\theta)) \\ &= We^{\frac{x^2}{2}}[1 - \Phi(x)]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right]. \end{aligned} \quad (2.39)$$

Combining this with (2.31) yields

$$I = We^{\frac{x^3}{\sqrt{n}}\mathcal{L}\left(\frac{x}{\sqrt{n}}\right)}[1 - \Phi(x)]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

which concludes the proof of (2.12). □

Chapter 3

Asymptotic Expansions in central and local limit theorems for products of random matrices

Let $(A_n)_{n \geq 1}$ be a sequence of independent and identically distributed random $d \times d$ real matrices. Set $G_n = A_n \dots A_1$, $X_n^x = \frac{G_n x}{|G_n x|}$ and $S_n^x := \log |G_n x|$. We consider asymptotic properties of the Markov chain (X_n^x, S_n^x) . For invertible matrices, Le Page [63] established a central limit theorem and a local limit theorem on (X_n^x, S_n^x) with x a starting point on the unit sphere in \mathbb{R}^d . In this paper, motivated by some applications in branching random walks, we improve and extend his theorems in the sense that: 1) we prove that the central limit theorem holds uniformly in x , and give an asymptotic expansion in the local limit theorem with a continuous function f acting on X_n^x and a directly Riemann integrable function h acting on S_n^x ; 2) we extend the results to the case of nonnegative matrices. Our approach is mainly based on the spectral gap theory recently developed for products of random matrices, and smoothing techniques for the approximation of functions.

3.1 Introduction

Let μ be a probability measure on the set of $d \times d$ matrices $M(d, \mathbb{R})$ ($d \geq 1$), and let $(A_n)_{n \geq 1}$ be a sequence of independent and identically distributed random matrices with law μ , defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are interested in the asymptotic behavior of the random walk $G_n x$, where

$$G_n = A_n \dots A_1$$

is the product of the random matrices A_i , x is a starting point on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, with $|\cdot|$ an arbitrary norm on \mathbb{R}^d . Notice that $G_n x$ is completely

determined by its log norm and its projection on the unit sphere, denoted respectively by

$$S_n^x := \log |G_n x| \quad \text{and} \quad X_n^x = G_n \cdot x := \frac{G_n x}{|G_n x|}.$$

We will use the convention that $G_0 x = x$, and introduce conditions such that $G_n x \neq 0$. Many authors have contributed to the study of asymptotic properties of S_n^x . For example, central limit theorems have been established by Benoist and Quint [10] for invertible matrices, and by Hennion [49] for nonnegative matrices.

While studying branching random walks in \mathbb{R}^d governed by products of random matrices, we need some asymptotic properties as those given in a central limit theorem and a local limit theorem on the couple (X_n^x, S_n^x) , but we find that the known results on this topic are not sharp enough for our purposes. We thus focus our study on the Markov chain (X_n^x, S_n^x) for establishing finer results. The applications in branching random walks will be considered in a forth coming paper [22].

For invertible matrices, Le Page [63] established a central limit theorem for (X_n^x, S_n^x) with x a given point in \mathbb{S}^{d-1} , and a local limit theorem for (X_n^x, S_n^x) with target functions f and h acting on X_n^x and S_n^x respectively, which are supposed to be continuous and of compact support. Such kind of limit theorems have also been established by Hennion and Hervé [50] in a more general setting by considering (X_n, S_n) instead of (X_n^x, S_n^x) , where (X_n) is a general Markov chain, $S_n = \sum_{i=1}^n \xi(X_i)$ with ξ a measurable and real valued function. Very recently, in parallel to the present work, a Berry- Essen type theorem on the rate of convergence in the central limit theorem has been established for (X_n^x, S_n^x) in [83] for both invertible and nonnegative matrices.

In this paper, our first objective is to improve the central limit theorem of Le Page [63] for invertible matrices with a uniform convergence in x (see Theorem 3.2.1), and deepen his local limit theorem by giving an asymptotic expansion under the weaker condition that the target functions f and h are respectively continuous and directly Riemann integrable (see Theorem 3.2.2). Our second objective is to prove that the results also hold for nonnegative matrices.

Our approach is mainly based on the spectral gap theory recently developed for the norm cocycle by Guivarc'h and Le Page [46] for invertible matrices, and by Buraczewski, Damek, Guivarc'h and Mentemeier (see [25, 28]) for nonnegative matrices. Smoothing techniques are also used for the approximation of functions: in the proof of Theorem 3.2.1, we use a smooth approximation of the indicator function of a Borel set (see Lemma

3.4.1), while in the proof of Theorem 3.2.2, we use a suitable approximation of a directly Riemann integrable function with the techniques developed in [82].

The paper is organized as follows. In Section 3.2, we fix some notation, introduce our assumptions on the branching products of random matrices and state the main results. In Section 3.3, we recall some results on spectral theory for products of random matrices which will be used in proofs of main results. These proofs are given in Sections 3.4 and 3.5.

3.2 Main results

3.2.1 Notation and preliminaries

We first fix some notation. We denote by c or C a constant whose value may change from line to line. For a set B , we use the symbols $\mathbb{1}_B$, \overline{B} , B° and $\partial B = \overline{B} \setminus B^\circ$ to denote respectively its indicator function, its closure, interior and boundary. For $t \in \mathbb{R}$, we write $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, $\Phi(t) = \int_{-\infty}^t \phi(u)du$, and $\phi_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/(2\sigma^2)}$. For a measure ν and a measurable function f we denote $\nu(f) = \int f d\nu$. For two functions f and g , we write $f(t) = o(g(t))$ or $f(t) = O(g(t))$ ($t \rightarrow 0$) when $\lim_{t \rightarrow 0} f(t)/g(t) = 0$ or $f(t)/g(t)$ is bounded for $|t|$ small enough, respectively. Denote by L^1 the class of complex valued measurable and Lebesgue integrable functions on \mathbb{R} ; for $f \in L^1$, denote its L^1 norm by $\|f\|_{L^1} = \int_{\mathbb{R}} |f(x)| dx$.

Let $M(d, \mathbb{R})$ be equipped with the operator norm $\|\mathbf{a}\| = \sup_{x \in \mathbb{S}^{d-1}} |\mathbf{a}x|$ for $\mathbf{a} \in M(d, \mathbb{R})$. Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by the support of μ . Let us recall some definitions in matrix theory. A matrix \mathbf{a} is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by \mathcal{M}_+ the set of matrices with nonnegative entries. A matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column has a positive entry.

For invertible matrices, we will use the strong irreducibility and proximality conditions.

M1. (i) (Strong irreducibility) There is no finite union $\mathcal{W} = \bigcup_{i=1}^n W_i$ of subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ -invariant (in the sense that $\Gamma_\mu \mathcal{W} = \mathcal{W}$).

(ii) (Proximality) Γ_μ contains at least one proximal matrix.

Notice that when $d = 1$, the strong irreducibility and proximality conditions are always satisfied.

For nonnegative matrices, we will need the allowability, positivity and non-arithmeticity conditions.

M2. (i) (Allowability) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.

(ii) (Positivity) Γ_μ contains at least one matrix belonging to \mathcal{M}_+^o .

We say that the measure μ is *arithmetic* if there are $t > 0$, $\theta \in [0, 2\pi)$ and a function $\vartheta : \mathbb{S}_+^{d-1} \rightarrow \mathbb{R}$ such that for all $\mathbf{a} \in \Gamma_\mu$ and all $x \in V(\Gamma_\mu)$,

$$\exp\{it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a} \cdot x) - \vartheta(x))\} = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Notice when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form: $\log a$ is almost surely concentrated on an arithmetic progression $a_0 + a_2\mathbb{N}$ for some $a_0, a_2 \geq 0$.

M3. (Non-arithmeticity) The measure μ is non-arithmetic.

It is known that when $d \geq 2$, condition **M1** implies **M3** (see [47, Proposition 4.6]).

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathcal{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a}.x := \frac{\mathbf{a}x}{|\mathbf{a}x|}$$

for the projective action of a matrix \mathbf{a} on $x \in \mathbb{S}^{d-1}$ when $\mathbf{a}x \neq 0$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices.

M4. There is $\alpha \in (0, 1)$ such that

$$\mathbb{E}N(A_1)^\alpha < \infty.$$

For invertible matrices, this condition is equivalent to the following two-sided exponential moment condition which is usually used in the literature: there is $\alpha_1 > 0$ such that $\mathbb{E}N(A_1)^{\alpha_1} < \infty$, where $N(A_1) = \max\{\|A_1\|, \|A_1^{-1}\|\}$.

We will consider the action of invertible matrices on the projective space \mathbb{P}^{d-1} which is obtained from \mathbb{S}^{d-1} by identifying x and $-x$, and the action of nonnegative matrices on \mathbb{S}_+^{d-1} . When convenient we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} for invertible matrices, and

\mathbb{S}_+^{d-1} for nonnegative matrices. The space \mathcal{S} will be equipped with the metric \mathbf{d} , which is the angular distance (see [20]) for invertible matrices, and the Hilbert cross-ratio metric (see [49]) for nonnegative matrices. Moreover, \mathcal{S} is a separable metric space with Borel- σ algebra. For any starting point $x \in \mathcal{S}$, as mentioned in the introduction, $G_n x$ is completely described by (X_n^x, S_n^x) . With the above conditions, X_n^x is well defined and the sequence (X_n^x, S_n^x) is a Markov chain because $X_n^x = A_n \cdot X_{n-1}^x$ and $S_n^x = \log |A_n X_{n-1}^x| + S_{n-1}^x$.

For invertible matrices, it was proved in [46, Theorem 2.6] that if condition **M1** holds, then the Markov chain X_n^x has a unique μ -stationary measure, which is supported on

$$V(\Gamma_\mu) := \overline{\{v_{\mathbf{a}} \in \mathbb{P}^{d-1} : \mathbf{a} \in \Gamma_\mu, \mathbf{a} \text{ is proximal}\}},$$

where $v_{\mathbf{a}}$ denotes the eigenvector with norm $|v_{\mathbf{a}}| = 1$ associated to the dominant eigenvalue of the proximal matrix \mathbf{a} .

For nonnegative matrices, it was shown in [25, Lemma 4.3] that condition **M2** ensures the existence and uniqueness of the invariant measure for the Markov chain (X_n^x) supported on

$$V(\Gamma_\mu) := \overline{\{v_{\mathbf{a}} \in \mathbb{S}_+^{d-1} : \mathbf{a} \in \Gamma_\mu, \mathbf{a} \in \mathcal{M}_+^o\}}.$$

In both cases, we write ν for the unique invariant measure of (X_n^x) .

Central limit theorems have been established in the literature. For invertible matrices, under condition **M1** and the two-sided exponential moment condition, Le Page [63] proved that

$$\frac{1}{\sqrt{n}}(S_n^x - n\gamma) \rightarrow N(0, \sigma^2) \quad \text{in law,} \tag{3.1}$$

where $\gamma = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \log \|A_n \cdots A_1\|$ is the Lyapunov exponent, and $\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} (S_n^x - n\gamma)^2$ is the asymptotic variance which is positive and independent of x . For nonnegative matrices, under condition **M2** and a second moment condition, Hennion [49] proved that (3.1) holds for some $\sigma \geq 0$; he also gave a condition of tightness of the sequence $(G_n)_{n \geq 0}$ to ensure that $\sigma > 0$. As a by-product of our approach, we will show that $\sigma > 0$ under the non-arithmeticity condition **M3** (see Proposition 3.3.3).

3.2.2 Main results

We state first a central limit theorem for the couple (X_n^x, S_n^x) with uniform convergence in $x \in \mathcal{S}$.

Theorem 3.2.1. *For invertible matrices, assume **M1** if $d > 1$, and **M3** if $d = 1$. For nonnegative matrices, assume **M2** and **M3**. For both cases, assume additionally **M4**.*

1. For any continuous function f on \mathcal{S} , we have

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[f(X_n^x) \mathbf{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(f) \Phi(t) \right| = 0. \quad (3.2)$$

2. For any measurable set $B \subset \mathcal{S}$ with $\nu(\partial B) = 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_n^x \in B, \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right) - \nu(B) \Phi(t) \right| = 0. \quad (3.3)$$

For invertible matrices, a point-wise version (by considering a fixed $x \in \mathcal{S}$ instead of $\sup_{x \in \mathcal{S}}$) has been established by Le Page in [63, Theorem 4]. For nonnegative matrices, the asymptotic for the Markov chain (X_n^x, S_n^x) is new even for a fixed x . The uniformity in $x \in \mathcal{S}$ is new for both invertible matrices and nonnegative matrices. Theorem 3.2.1 will be deduced from a result on the convergence rate in (3.2) which has been established in [83] for the case when f is Hölder continuous.

The following theorem gives the asymptotic expansion in the local limit theorem for products of random matrices.

Theorem 3.2.2. *Assume the conditions of Theorem 3.2.1.*

1. For any continuous function f on \mathcal{S} and any directly Riemann integrable function h on \mathbb{R} , we have as $n \rightarrow \infty$,

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma\sqrt{n} \mathbb{E} [f(X_n^x) h(y + S_n^x - n\gamma)] - \nu(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz \right| \rightarrow 0, \quad (3.4)$$

where

$$H_x(u) = 1 - \frac{b(x)}{\sigma^2\sqrt{n}}u + \frac{m_3}{6\sigma^6\sqrt{n}}(3\sigma^2u - u^3),$$

with m_3 and $b(x)$ defined in Proposition 3.3.3.

2. For any measurable set $B \subset \mathcal{S}$ with $\nu(\partial B) = 0$ and any directly Riemann integrable function h on \mathbb{R} , we have as $n \rightarrow \infty$

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |\sigma\sqrt{n}\mathbb{E} [\mathbf{1}_B(X_n^x)h(y + S_n^x - n\gamma)] - \nu(B) \int_{\mathbb{R}} h(z)\phi\left(\frac{y-z}{\sigma\sqrt{n}}\right)H_x\left(\frac{y-z}{\sqrt{n}}\right)dz| \rightarrow 0. \quad (3.5)$$

When $y = 0$, $f = 1$ and $h = \mathbf{1}_{[a,b]}$, the integral $\mathbb{E} [f(X_n^x)h(y + S_n^x - n\gamma)]$ reduces to the local probability $\mathbb{P}(S_n^x \in n\gamma + [a, b])$, which is the usual object studied in local limit theorems.

The expansions (3.4) and (3.5) are new for both invertible matrices and nonnegative matrices. The first expansion implies the local limit theorem established in [63, Theorem 6] for invertible matrices, which states that (3.4) holds when the polynomial $H_x(\cdot)$ is replaced by 1 and when f, h are continuous functions with compact supports.

The case $d = 1$ is worth some comments. In this case, Theorem 3.2.1 follows from Theorem VII.2.7 of Petrov [75], while expansion (3.4) in Theorem 3.2.2 was proved by Feller (see [36, Theorem XVI.4.1]) under the same non-arithmetic condition on μ and when $h = \mathbf{1}_{[a,b]}$ is the indicator function of an interval. Breuillard (see [21, Theorem 3.2]) proved an expansion like (3.4) but for any finite order, when μ is strongly non-arithmetic (in the sense that its characteristic function $\hat{\mu}(t) = \int e^{itx}\mu(dx)$ satisfies Cramér’s condition $\limsup_{|t| \rightarrow \infty} |\hat{\mu}(t)| < 1$) with finite moments of order high enough and when h is integrable and regular enough (he assumed in particular that h has continuous and integrable derivatives $h^{(k)}$ for $0 \leq k \leq K$ with $K \geq 2$ large enough). Compared with the result of Breuillard, the novelty in Theorem 3.2.2 is that we assume the non-arithmetic condition instead of the strongly arithmetic condition, and we use the direct Riemann integrability of h instead of the smoothness condition on h .

3.3 Spectral gap property

In this section we recall some spectral gap properties studied in [63, 28, 83] which will be used for the proofs of main results.

For $z \in \mathbb{C}$, define the operator P_z on the set $\mathcal{C}(\mathcal{S})$ of continuous functions on \mathcal{S} by

$$P_z f(x) = \int_{\mathcal{M}} |\mathbf{a}x|^z f(\mathbf{a} \cdot x)\mu(d\mathbf{a}), \quad \text{for all } x \in \mathcal{S}.$$

For $f \in \mathcal{C}(\mathcal{S})$ and $\beta > 0$, we introduce the norms

$$\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|; \quad |f|_\beta := \sup_{x, y \in \mathcal{S}} \frac{|f(x) - f(y)|}{\mathbf{d}^\beta(x, y)}; \quad \|f\|_\beta := \|f\|_\infty + |f|_\beta.$$

Consider the Banach space $\mathcal{B}_\beta := \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\beta < +\infty\}$. Denote by $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ the set of all bounded linear operators from \mathcal{B}_β to \mathcal{B}_β equipped with the operator norm

$$\|P\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta} := \sup_{f \neq 0} \frac{\|Pf\|_\beta}{\|f\|_\beta}, \quad \forall P \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta).$$

The following result describes the quasi-compactness of P_0 . It can be found in [63, Proposition 4] for invertible matrices and in [28, Proposition 4.2] for nonnegative matrices. For $P \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ and $n \in \mathbb{N}$, denote by P^n the n -fold iteration of P ; by convention P^0 is the identity operator.

Proposition 3.3.1. *Assume the conditions of Theorem 3.2.1. Let $\beta > 0$ be small enough. Then $P_0 \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$, and there is an operator $L \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ whose spectral radius is strictly less than 1, such that for all $n \in \mathbb{N}$,*

$$P_0^n = \Pi_0 + L^n,$$

where Π_0 is a rank-one projection satisfying $\Pi_0 \mathcal{B}_\beta = \{f \in \mathcal{B}_\beta : P_0 f = f\}$ and $\Pi_0 f(x) = \nu(f)$ for all $f \in \mathcal{B}_\beta$ and $x \in \mathcal{S}$.

For simplicity, in our proofs we will use a slightly different family of operators $(R_t)_{t \in \mathbb{R}}$ defined by

$$R_t f(x) := e^{-it\gamma} P_{it} f(x) = \mathbb{E}[e^{it(S_1^x - \gamma)} f(X_1^x)], \quad \text{for } f \in \mathcal{C}(\mathcal{S}), x \in \mathcal{S}.$$

By the cocycle property $\log |\mathbf{a}_2 \mathbf{a}_1 x| = \log |\mathbf{a}_2(\mathbf{a}_1 \cdot x)| + \log |\mathbf{a}_1 x|$ and an induction, we have

$$R_t^n f(x) = \mathbb{E}[e^{it(S_n^x - n\gamma)} f(X_n^x)], \quad n \geq 1.$$

We collect in the following two propositions some results from [83] that we will use. Although these results are stated in [83] only for $d \geq 2$, they remain valid for $d = 1$. The first proposition concerns the perturbation theory.

Proposition 3.3.2. *Assume the conditions of Theorem 3.2.1.*

1. There exists a real number $\delta > 0$ such that for $t \in [-\delta, \delta]$ we have:

(a) For all $f \in \mathcal{B}_\beta$ and $n \geq 1$

$$R_t^n f = \lambda^n(t) \Pi_t f + N_t^n f,$$

with $\Pi_t N_t = N_t \Pi_t = 0$.

(b) The mappings

$$\lambda : [-\delta, \delta] \rightarrow \mathbb{R}, \quad \Pi : [-\delta, \delta] \rightarrow \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta), \quad N : [-\delta, \delta] \rightarrow \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$$

are C^∞ .

(c) For $k_0 \in \mathbb{N}$, there exist $\rho = \rho(k_0) \in (0, 1)$ and $c = c(k_0) > 0$ such that for all $n \geq 1$,

$$\max_{0 \leq k \leq k_0} \sup_{|t| < \delta} \left\| \frac{d^k}{dt^k} N_t^n \right\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta} \leq c \rho^n.$$

2. Let $K \subset \mathbb{R} \setminus \{0\}$ be compact. Then for each $f \in \mathcal{B}_\beta$, there is $\rho_1 \in (0, 1)$ such that for all $n \geq 1$,

$$\sup_{t \in K} \|R_t^n f\|_\infty \leq \rho_1^n \|f\|_\infty.$$

The second proposition concerns the Taylor expansion of λ and the positivity of the asymptotic variance.

Proposition 3.3.3. *Assume the conditions of Theorem 3.2.1.*

1. The Taylor expansion of λ at 0 of order 3 is given by

$$\lambda(t) = 1 - \frac{\sigma^2}{2} t^2 - i \frac{m_3}{6} t^3 + o(t^3),$$

where $m_3 = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{V(\Gamma_\mu)} \mathbb{E}(S_n^x - n\gamma)^3 d\nu(x)$.

2. For each $x \in \mathcal{S}$, the limit $b(x) = \lim_{n \rightarrow \infty} \mathbb{E}(S_n^x - n\gamma)$ exists in \mathbb{R} , the function

$x \mapsto b(x)$ is in \mathcal{B}_β , and the derivative Π'_0 of Π_t at 0, satisfies

$$\Pi'_0 f(x) = i \nu(f) b(x), \quad \text{for } f \in \mathcal{B}_\beta, x \in \mathcal{S}.$$

3. If μ is non-arithmetic, then $\sigma > 0$.

3.4 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1, part (1). From [83, Theorem 2.1], for $f \in \mathcal{B}_\beta$, we have

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[f(X_n^x) \mathbf{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(f) \Phi(t) \right| = 0. \quad (3.6)$$

Combining this with the fact that the Banach space \mathcal{B}_β is dense in $\mathcal{C}(\mathcal{S})$ with respect to the norm $\|\cdot\|_\infty$ gives the conclusion of part (1). \square

To prove part (2), we first introduce a smooth approximation of an indicator function:

Lemma 3.4.1. (Smooth approximation of an indicator function)

1. Let $A, B \subset \mathcal{S}$ be non-empty closed sets with $A \cap B = \emptyset$. Then there is a continuous function $\varphi : \mathcal{S} \rightarrow [0, 1]$ such that $\varphi(x) = 1$ for all $x \in A$ and $\varphi(x) = 0$ for all $x \in B$.
2. Let $A \subset \mathcal{S}$ be a non-empty measurable set with $\nu(\partial A) = 0$, and let $\varepsilon > 0$. Then there exist two continuous functions $\varphi^-, \varphi^+ : \mathcal{S} \rightarrow [0, 1]$ such that

$$\varphi^- \leq \mathbf{1}_A \leq \varphi^+ \quad \text{and} \quad \nu\{x \in \mathcal{S} : \varphi^+ \neq \varphi^-\} < \varepsilon.$$

Proof. For a non-empty set $D \subset \mathcal{S}$ and $x \in \mathcal{S}$, define

$$\text{dist}(x, D) = \inf\{\mathbf{d}(x, z) : z \in D\}.$$

(1) Since A, B are closed and disjoint we have $\text{dist}(x, B) + \text{dist}(x, A) \neq 0$ for all $x \in \mathcal{S}$. The function defined below satisfies the desired properties:

$$\varphi(x) := \frac{\text{dist}(x, B)}{\text{dist}(x, B) + \text{dist}(x, A)}, \quad x \in \mathcal{S}.$$

(2) Since ν is a Borel measure on \mathcal{S} , we have

$$\begin{aligned}\nu(\bar{A}) &= \inf\{\nu(U) : \bar{A} \subseteq U, U \text{ open}\}, \\ \nu(A^\circ) &= \sup\{\nu(K) : K \subseteq A^\circ, K \text{ compact}\}.\end{aligned}$$

Hence for each $\varepsilon > 0$, there exists a compact $K \subset A^\circ$ such that $\nu(K) > \nu(A^\circ) - \frac{\varepsilon}{2}$. Since K and $(A^\circ)^c$ are disjoint closed sets, by part (1), there exists a continuous function $\varphi^- : \mathcal{S} \rightarrow [0, 1]$ such that $\varphi^-(x) = 1$ for $x \in K$ and $\varphi^-(x) = 0$ for $x \in (A^\circ)^c$.

Similarly, there exists an open set $U \supset \bar{A}$ such that $\nu(U) < \nu(\bar{A}) + \frac{\varepsilon}{2}$. Again by part (1) applied to the disjoint closed sets \bar{A} and U^c , we see that there is a continuous function $\varphi^+ : \mathcal{S} \rightarrow [0, 1]$ such that $\varphi^+(x) = 1$ for $x \in \bar{A}$ and $\varphi^+(x) = 0$ for $x \in U^c$. Therefore,

$$K \cup U^c \subset \{x \in \mathcal{S} : \varphi^+(x) = \varphi^-(x)\}.$$

Consequently,

$$\{x \in \mathcal{S} : \varphi^+(x) \neq \varphi^-(x)\} \subset K^c \cap U = U \setminus K.$$

Since $U \setminus K = (U \setminus \bar{A}) \cup (\bar{A} \setminus A^\circ) \cup (A^\circ \setminus K)$, it follows that

$$\nu\{x \in \mathcal{S} : \varphi^+(x) \neq \varphi^-(x)\} \leq \nu(U \setminus \bar{A}) + \nu(\bar{A} \setminus A^\circ) + \nu(A^\circ \setminus K) < \varepsilon,$$

where we have used the hypothesis that $\nu(\bar{A} \setminus A^\circ) = \nu(\partial A) = 0$. From the construction of φ^- and φ^+ , it is obvious that $\varphi^- \leq \mathbb{1}_B \leq \varphi^+$. □

Proof of Theorem 3.2.1, part (2). Let $\varepsilon > 0$ be given. By Lemma 3.4.1, there exists two continuous functions φ^+ and $\varphi^- : \mathcal{S} \rightarrow [0, 1]$ such that

$$\varphi^- \leq \mathbb{1}_B \leq \varphi^+ \quad \text{and} \quad \nu\{x \in \mathcal{S} : \varphi^+ \neq \varphi^-\} < \varepsilon. \tag{3.7}$$

By the triangular inequality in \mathbb{R} , we have

$$\begin{aligned}
& \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\mathbb{1}_B(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(B)\Phi(t) \right| \\
& \leq \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[(\mathbb{1}_B - \varphi^+)(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] \right| \\
& \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\varphi^+(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(\varphi^+)\Phi(t) \right| \\
& \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \nu(\varphi^+ - \mathbb{1}_B)\Phi(t) \right|. \tag{3.8}
\end{aligned}$$

By part (1), the second term in (3.8) is less than ε for n enough large. The third term is also less than ε by the property (3.7) since

$$\nu(\varphi^+ - \mathbb{1}_B) \leq \nu(\varphi^+ - \varphi^-) \leq \nu\{x \in \mathcal{S} : \varphi^+ \neq \varphi^-\} < \varepsilon. \tag{3.9}$$

The first term can be estimated as follows:

$$\begin{aligned}
& \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[(\mathbb{1}_B - \varphi^+)(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] \right| \\
& \leq \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[(\varphi^+ - \varphi^-)(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] \right| \\
& \leq \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\varphi^+(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(\varphi^+)\Phi(t) \right| \\
& \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\varphi^-(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(\varphi^-)\Phi(t) \right| \\
& \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \nu(\varphi^+ - \varphi^-)\Phi(t) \right|.
\end{aligned}$$

In the last display, the first two terms are less than ε for n large enough, again by part (1); the third one is also less than ε by (3.9). \square

3.5 Proof of Theorem 3.2.2

Proof of Theorem 3.2.2, part (1). We assume that both f and h are nonnegative; we can do this by considering the positive and negative parts. We will proceed the proof in 4 steps.

Step 1. We first establish (3.4) for $f \in \mathcal{B}_\beta$ and $h \in L^1$ whose Fourier transform

$$\widehat{h}(u) = \int_{\mathbb{R}} e^{-itu} h(t) dt, \quad \forall u \in \mathbb{R},$$

has a compact support $\text{supp}(\widehat{h}) \subset [-k, k]$. By the inversion formula of Fourier transform and Fubini's theorem, for any $x \in \mathcal{S}$ and $y \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} [f(X_n^x) h(y + S_n^x - n\gamma)] &= \mathbb{E} \left[\frac{f(X_n^x)}{2\pi} \int_{\mathbb{R}} e^{iu(y+S_n^x-n\gamma)} \widehat{h}(u) du \right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iuy} \widehat{h}(u) \mathbb{E} \left[e^{iu(S_n^x-n\gamma)} f(X_n^x) \right] du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iuy} \widehat{h}(u) R_u^n f(x) du. \end{aligned}$$

By the change of variables $u = \frac{t}{\sqrt{n}}$ and using Proposition 3.3.2, we have:

$$\begin{aligned} I &:= \sqrt{n} \mathbb{E} [f(X_n^x) h(y + S_n^x - n\gamma)] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) R_{\frac{t}{\sqrt{n}}}^n f(x) dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{|t| \leq \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) \lambda^n\left(\frac{t}{\sqrt{n}}\right) \Pi_{\frac{t}{\sqrt{n}}} f(x) dt, \\ I_2 &= \frac{1}{2\pi} \int_{|t| \leq \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) N_{\frac{t}{\sqrt{n}}}^n f(x) dt, \\ I_3 &= \frac{1}{2\pi} \int_{|t| > \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) R_{\frac{t}{\sqrt{n}}}^n f(x) dt, \end{aligned}$$

with $\delta_1 \in (0, \delta]$ a parameter which will be fixed later. We will prove that I_1 gives the main term of the desired expansion, while I_2 and I_3 tend to 0.

Estimation of I_1 . By Proposition 3.3.3 and an elementary calculation, we obtain, as $\frac{t}{\sqrt{n}} \rightarrow 0$,

$$\lambda^n\left(\frac{t}{\sqrt{n}}\right) = e^{-\frac{\sigma^2 t^2}{2}} \exp\left(-\frac{im_3 t^3}{6\sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right)\right). \quad (3.10)$$

By Taylor's expansion of the operator Π_t on a neighborhood of 0, we have

$$\Pi_{\frac{t}{\sqrt{n}}} f(x) = \Pi_0 f(x) + \frac{t}{\sqrt{n}} \Pi'_0 f(x) + O\left(\frac{t^2}{n}\right), \quad (3.11)$$

where Π_0 and Π'_0 are bounded operators on \mathcal{B}_β defined in Propositions 3.3.1 and 3.3.3(2). Notice that $\Pi_0 f(x) = \nu(f)$ and $\Pi'_0 f(x) = i\nu(f)b(x)$. With (3.10) and (3.11) in mind, we do the composition

$$\begin{aligned} \lambda^n\left(\frac{t}{\sqrt{n}}\right) \Pi_{\frac{t}{\sqrt{n}}} f(x) &= \lambda^n\left(\frac{t}{\sqrt{n}}\right) \left[\Pi_{\frac{t}{\sqrt{n}}} f(x) - \nu(f) - \frac{it\nu(f)b(x)}{\sqrt{n}} \right] \\ &\quad + \left[\lambda^n\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{\sigma^2 t^2}{2}} \left(1 - \frac{im_3 t^3}{6\sqrt{n}}\right) \right] \nu(f) \\ &\quad + \left[\lambda^n\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{\sigma^2 t^2}{2}} \right] \frac{it\nu(f)b(x)}{\sqrt{n}} \\ &\quad + \nu(f) e^{-\frac{\sigma^2 t^2}{2}} \left(1 + \frac{itb(x)}{\sqrt{n}} - \frac{im_3 t^3}{6\sqrt{n}}\right). \end{aligned} \quad (3.12)$$

Choosing $\delta_1 \leq \delta$ small enough such that when $|t|/\sqrt{n} \leq \delta_1$, we have

$$\left| -\frac{im_3 t^3}{6\sigma^3 \sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right) \right| \leq \frac{\sigma^2 t^2}{4} \quad \text{and} \quad \lambda^n\left(\frac{t}{\sqrt{n}}\right) \leq e^{-\frac{\sigma^2 t^2}{4}}. \quad (3.13)$$

In the definition of I_1 , we substitute $\lambda^n(\frac{t}{\sqrt{n}})\Pi_{\frac{t}{\sqrt{n}}} f(x)$ by the decomposition (3.12); this leads to a decomposition of I_1 which we write accordingly as

$$I_1 = J_1 + J_2 + J_3 + J_4.$$

We first estimate the integral J_1 . Using (3.11), (3.13) and the fact that \widehat{h} is bounded, we obtain

$$\begin{aligned} |J_1| &\leq \frac{1}{2\pi} \sup_{t \in \mathbb{R}} |\widehat{h}(t)| \int_{|t| \leq \delta_1 \sqrt{n}} e^{-\frac{\sigma^2 t^2}{4}} O\left(\frac{t^2}{n}\right) dt \\ &\leq \int_{\mathbb{R}} e^{-\frac{\sigma^2 t^2}{4}} |t| dt \cdot O\left(\frac{1}{\sqrt{n}}\right) \leq \frac{C}{\sqrt{n}}. \end{aligned}$$

To estimate J_2 , we use the following inequality (see inequality XVI(2.8) in [36]): for all

$u, v \in \mathbb{C}$,

$$|e^u - 1 - v| \leq \left(|u - v| + \frac{1}{2}|v|^2 \right) e^{\max(|u|, |v|)}. \quad (3.14)$$

Using (3.14) with $u = -\frac{im_3 t^3}{6\sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right)$ and $v = -\frac{im_3 t^3}{6\sqrt{n}}$, we have

$$\begin{aligned} |J_2| &\leq \frac{\nu(f)}{2\pi} \sup_{t \in \mathbb{R}} |\widehat{h}(t)| \int_{|t| \leq \delta_1 \sqrt{n}} e^{-\frac{\sigma^2 t^2}{2}} \left[|t|^3 o\left(\frac{1}{\sqrt{n}}\right) + t^6 O\left(\frac{1}{n}\right) \right] e^{\frac{\sigma^2 t^2}{4}} dt \\ &\leq \int_{\mathbb{R}} (|t|^3 + |t^5|) e^{-\frac{\sigma^2 t^2}{4}} dt \cdot O\left(\frac{1}{\sqrt{n}}\right) \leq \frac{C}{\sqrt{n}}. \end{aligned}$$

For J_3 , we use again inequality (3.14) and the fact that the mapping b belongs to \mathcal{B}_β , to conclude that

$$\begin{aligned} |J_3| &\leq \int_{|t| \leq \delta_1 \sqrt{n}} \frac{|t\nu(f)b(x)|}{\sqrt{n}} e^{-\frac{\sigma^2 t^2}{2}} \left| 1 - \lambda_{\frac{it}{\sqrt{n}}}^n \right| \\ &\leq \nu(f) \|b\|_\infty \delta_1 \int_{\mathbb{R}} e^{-\frac{\sigma^2 t^2}{4}} \left| -\frac{im_3 t^3}{6\sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right) \right| dt \\ &\leq \int_{\mathbb{R}} e^{-\frac{\sigma^2 t^2}{4}} |t|^3 dt \cdot O\left(\frac{1}{\sqrt{n}}\right) \leq \frac{C}{\sqrt{n}}. \end{aligned}$$

Hence, we obtain the following estimate of I_1 :

$$|I_1 - J_4| \leq \frac{C}{\sqrt{n}}. \quad (3.15)$$

Estimation of I_2 . From Proposition 3.3.2, we know that for $|t| \leq \delta_1 \sqrt{n}$ there exists a constant $c > 0$ and $\rho \in (0, 1)$ such that $\left\| N_{\frac{it}{\sqrt{n}}}^n f \right\|_\beta \leq c\rho^n$. Hence,

$$|I_2| \leq C\rho^n \int_{\mathbb{R}} |\widehat{h}(t)| dt. \quad (3.16)$$

Estimation of I_3 . From Proposition 3.3.2(2), we have

$$\sup_{\delta_1 \sqrt{n} \leq |t| \leq c\sqrt{n}} \left\| R_{\frac{it}{\sqrt{n}}}^n f \right\|_\infty < \rho_1^n \|f\|_\infty.$$

Using this together with the condition that $\text{supp}(\widehat{h}) \subset [-k, k]$, we get

$$|I_3| \leq C\rho^n \int_{\mathbb{R}} |\widehat{h}(t)| dt. \quad (3.17)$$

Collecting the bounds (3.15), (3.16) and (3.17), we have

$$|I - J_4| \leq C \left(\frac{1}{\sqrt{n}} + \rho_1^n + \rho^n \right). \quad (3.18)$$

Set

$$Q(t) = 1 + \frac{it}{\sqrt{n}} b(x) - \frac{im_3 t^3}{6\sqrt{n}}.$$

It is not difficult to see that

$$\begin{aligned} & \left| \frac{\nu(f)}{2\pi} \int_{|t| > \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) e^{-\frac{\sigma^2 t^2}{2}} Q(t) dt \right| \\ & \leq C \left(e^{-\frac{\sigma^2 \delta_1^2 n}{2}} \int_{\mathbb{R}} |\widehat{h}(t)| dt + \frac{1}{\sqrt{n}} \sup_{t \in \mathbb{R}} |\widehat{h}(t)| \right). \end{aligned}$$

Hence we replace the integral on $|t| \leq \delta_1 \sqrt{n}$ of J_4 in (3.18) by an integral on \mathbb{R} . We get

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| I - \frac{\nu(f)}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) e^{-\frac{\sigma^2 t^2}{2}} Q(t) dt \right| \xrightarrow{n \rightarrow \infty} 0.$$

Note that $\widehat{\phi}(\sigma t) = e^{-\frac{\sigma^2 t^2}{2}}$ is the Fourier transform of $\phi_\sigma(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$. Then

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| I - \frac{\nu(f)}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) \widehat{\phi}(t\sigma) Q(t) dt \right| = 0.$$

But one has for all $p \geq 0$, $(\sigma t)^p \widehat{\phi}(\sigma t) = (-i)^p \widehat{\phi_\sigma^{(p)}}(t)$ where the notation $f^{(p)}$ is the derivative of order p of f , it implies

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) \widehat{\phi}(t\sigma) Q(t) dt = \sqrt{n} \int_{\mathbb{R}} h\left(y + u\sqrt{n}\right) \phi_\sigma(u) H_x(u) du,$$

where $H_x(u)$ is a polynomial such that $Q\left(-i\frac{d}{dx}\right) \phi_\sigma(u) = H_x(u) \phi_\sigma(u)$. With an elemen-

tary calculation, we get

$$H_x(u) = 1 - \frac{b(x)}{\sigma^2\sqrt{n}}u + \frac{m_3}{6\sigma^6\sqrt{n}}(3\sigma^2u - u^3).$$

Using the change of variables $z = y + u\sqrt{n}$, we get

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| I - \nu(f) \int_{\mathbb{R}} h(z) \phi_{\sigma} \left(\frac{y-z}{\sqrt{n}} \right) H_x \left(\frac{y-z}{\sqrt{n}} \right) dz \right| = 0,$$

or, equivalently,

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma I - \nu(f) \int_{\mathbb{R}} h(z) \phi \left(\frac{y-z}{\sigma\sqrt{n}} \right) H_x \left(\frac{y-z}{\sqrt{n}} \right) dz \right| = 0.$$

So we have established (3.4) for $f \in \mathcal{B}_{\beta}$ and Lebesgue integrable function h whose Fourier transform \hat{h} has a compact support.

Step 2. We establish (3.4) for $f \in \mathcal{B}_{\beta}$ and $h \in L^1$ satisfying $h \in \mathcal{H}_{\varepsilon}$ for all $\varepsilon \in (0, \frac{1}{4})$ and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \bar{h}_{\varepsilon}(u) du = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \underline{h}_{\varepsilon}(u) du = \int_{\mathbb{R}} h(u) du, \quad (3.19)$$

where $\mathcal{H}_{\varepsilon}$, \bar{h}_{ε} and $\underline{h}_{\varepsilon}$ are defined below. For any nonnegative Lebesgue integrable function h defined on \mathbb{R} , and for any $\varepsilon > 0$ and $u \in \mathbb{R}$, set $\mathbb{B}_{\varepsilon}(u) = \{u' \in \mathbb{R} : |u' - u| \leq \varepsilon\}$,

$$\bar{h}_{\varepsilon}(u) = \sup_{v \in \mathbb{B}_{\varepsilon}(u)} h(v) \quad \text{and} \quad \underline{h}_{\varepsilon}(u) = \inf_{v \in \mathbb{B}_{\varepsilon}(u)} h(v).$$

For any $\varepsilon > 0$, denote by $\mathcal{H}_{\varepsilon}$ the set of nonnegative Lebesgue integrable functions h such that \bar{h}_{ε} and $\underline{h}_{\varepsilon}$ are measurable and Lebesgue integrable:

$$\mathcal{H}_{\varepsilon} = \{h \in L^1 : h \geq 0, \bar{h}_{\varepsilon} \text{ and } \underline{h}_{\varepsilon} \text{ are measurable and integrable}\}.$$

We shall use the following result proved in [43, Lemma 5.2].

Lemma 3.5.1. *Let $h \in L^1$ be such that $h \in \mathcal{H}_{\varepsilon}$ for all $\varepsilon \in (0, 1/4)$. Then we have, for all $\varepsilon \in (0, 1/4)$ and $u \in \mathbb{R}$,*

$$\underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}(u) - \int_{|v| \geq \varepsilon} \underline{h}_{\varepsilon}(u-v) \kappa_{\varepsilon^2}(v) dv \leq h(u) \leq (1 + 4\varepsilon) \bar{h}_{\varepsilon} * \kappa_{\varepsilon^2}(u),$$

where κ is defined on \mathbb{R} by

$$\kappa(u) = \frac{1}{2\pi} \left(\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}} \right)^2 \text{ for } u \in \mathbb{R}^*, \text{ and } \kappa(0) = \frac{1}{2\pi}.$$

Moreover, we need some properties of the kernel κ that we state in the following. The function κ is integrable and its Fourier transform is given by

$$\widehat{\kappa}(t) = 1 - |t| \text{ for all } t \in [-1, 1], \text{ and } \widehat{\kappa}(t) = 0 \text{ otherwise.}$$

Note that

$$\int_{\mathbb{R}} \kappa(u) du = \widehat{\kappa}(0) = 1 = \int_{\mathbb{R}} \widehat{\kappa}(t) dt.$$

For any $\varepsilon > 0$, we define the function κ_ε on \mathbb{R} by

$$\kappa_\varepsilon(u) = \frac{1}{\varepsilon} \kappa\left(\frac{u}{\varepsilon}\right), \quad u \in \mathbb{R}. \quad (3.20)$$

Its Fourier transform is given by $\widehat{\kappa}_\varepsilon(t) = \widehat{\kappa}(\varepsilon t)$. Note also that, for any $\varepsilon > 0$, we have

$$\int_{|u| \geq \frac{1}{\varepsilon}} \kappa(u) du \leq \frac{1}{\pi} \int_{\frac{1}{\varepsilon}}^{+\infty} \frac{4}{u^2} du = \frac{4\varepsilon}{\pi}. \quad (3.21)$$

For simplicity, we denote, for any $f \in \mathcal{C}(\mathcal{S})$ and $h \in L^1$,

$$I(n, f, h) = \sigma\sqrt{n}\mathbb{E}[f(X_n^x)h(y + S_n^x - n\gamma)], \quad (3.22)$$

$$K(n, f, h) = \nu(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz. \quad (3.23)$$

Notice that $\sup_{x \in \mathcal{S}, u \in \mathbb{R}} |\phi\left(\frac{u}{\sigma}\right) H_x(u)| < \infty$. This implies the following uniform bound in $x \in \mathcal{S}, y, z \in \mathbb{R}, n \geq 1$:

$$\phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) \leq C. \quad (3.24)$$

From this we see that for $f \in \mathcal{C}(\mathcal{S})$ and $h \in L^1$,

$$K(n, f, h) \leq C\nu(f)\|h\|_{L^1}. \quad (3.25)$$

Remark that, for f and h fixed as in the beginning of Step 2, with the notation (3.22) and (3.23), to prove the desired conclusion (3.4), it suffices to establish the following result: for all $\varepsilon > 0$ small enough, when n is large enough,

$$\sup_{(x,y) \in (\mathcal{S}, \mathbb{R})} |I(n, f, h) - K(n, f, h)| \leq \nu(f)\delta_h(\varepsilon) + \varepsilon, \quad (3.26)$$

where

$$\begin{aligned} \delta_h(\varepsilon) &= C \left(\|h - \underline{h}_{2\varepsilon}\|_{L^1} + \|\bar{h}_{2\varepsilon} - h\|_{L^1} \right) + C \|\bar{h}_{2\varepsilon}\|_{L^1} (\varepsilon + \varepsilon^2) \\ &\rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ (due to (3.19)).} \end{aligned}$$

Below we will prove (3.26) by giving upper and lower bounds of $I(n, f, h) - K(n, f, h)$.

Upper bound of $I(n, f, h) - K(n, f, h)$. By Lemma 3.5.1, we have, for any $x \in \mathcal{S}, n \geq 1, y \in \mathbb{R}$ and $\varepsilon \in (0, 1/4)$,

$$I(n, f, h) \leq (1 + 4\varepsilon)I(n, f, \bar{h}_\varepsilon * \kappa_{\varepsilon^2}).$$

Since \bar{h}_ε and κ_{ε^2} are integrable, the function $\bar{h}_\varepsilon * \kappa_{\varepsilon^2}$ is also integrable; its Fourier transform $\widehat{\bar{h}_\varepsilon * \kappa_{\varepsilon^2}}$ has a compact support included in $[-1/\varepsilon^2, 1/\varepsilon^2]$. Consequently, we can use the result proved in Step 1, applied to f and $\bar{h}_\varepsilon * \kappa_{\varepsilon^2}$, to conclude that for n large enough,

$$I(n, f, h) \leq (1 + 4\varepsilon)K(n, f, \bar{h}_\varepsilon * \kappa_{\varepsilon^2}) + \varepsilon. \quad (3.27)$$

Notice that for $|v| \leq \varepsilon$ and $u \in \mathbb{R}$, we have $[u - v - \varepsilon, u - v + \varepsilon] \subset [u - 2\varepsilon, u + 2\varepsilon]$. Therefore, by definition,

$$\underline{h}_\varepsilon(u - v) \geq \underline{h}_{2\varepsilon}(u) \quad \text{and} \quad \bar{h}_\varepsilon(u - v) \leq \bar{h}_{2\varepsilon}(u). \quad (3.28)$$

Consequently, for any $u \in \mathbb{R}$,

$$\begin{aligned} \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u) &\leq \bar{h}_{2\varepsilon}(u) \int_{|v| \leq \varepsilon} \kappa_{\varepsilon^2}(v) dv + \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv \\ &\leq \bar{h}_{2\varepsilon}(u) + \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv. \end{aligned}$$

From this together with the bound (3.24), inequality (3.27) implies

$$\begin{aligned} I(n, f, h) &\leq (1 + 4\varepsilon)K(n, f, \bar{h}_{2\varepsilon}) \\ &\quad + (1 + 4\varepsilon)C\nu(f) \int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_{\varepsilon}(z - v) \kappa_{\varepsilon^2}(v) dv dz + \varepsilon. \end{aligned}$$

For a bound of the first term in the right hand side, we use the decomposition $K(n, f, \bar{h}_{2\varepsilon}) = K(n, f, h) + K(n, f, \bar{h}_{2\varepsilon} - h)$ and the inequality (3.25) for $K(n, f, \bar{h}_{2\varepsilon} - h)$ to get $K(n, f, \bar{h}_{2\varepsilon}) \leq K(n, f, h) + C\nu(f) \|\bar{h}_{2\varepsilon} - h\|_{L^1}$. Therefore

$$\begin{aligned} I(n, f, h) &\leq (1 + 4\varepsilon) \left[K(n, f, h) + C\nu(f) \|\bar{h}_{2\varepsilon} - h\|_{L^1} \right] \\ &\quad + (1 + 4\varepsilon)C\nu(f) \int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_{\varepsilon}(z - v) \kappa_{\varepsilon^2}(v) dv dz + \varepsilon. \end{aligned}$$

For a bound of the last integral, we use (3.20) and (3.21), to obtain

$$\int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_{\varepsilon}(z - v) \kappa_{\varepsilon^2}(v) dv dz \leq \|\bar{h}_{\varepsilon}\|_{L^1} \frac{4\varepsilon}{\pi}.$$

Using this and the bound of $K(n, f, h)$ in (3.25), from the preceding bound of $I(n, f, h)$ we get for $f \in \mathcal{B}_{\beta}$ and $h \in \mathcal{H}_{\varepsilon}$,

$$\begin{aligned} I(n, f, h) - K(n, f, h) &\leq 4\nu(f)\varepsilon \|h\|_{L^1} + C\nu(f)(1 + 4\varepsilon) \|\bar{h}_{2\varepsilon} - h\|_{L^1} \\ &\quad + \frac{4C\nu(f)(1 + 4\varepsilon)\varepsilon}{\pi} \|\bar{h}_{\varepsilon}\|_{L^1} + \varepsilon \\ &\leq C\nu(f) \|\bar{h}_{2\varepsilon} - h\|_{L^1} + C\nu(f)(\varepsilon + \varepsilon^2) \|\bar{h}_{2\varepsilon}\|_{L^1} + \varepsilon. \end{aligned} \quad (3.29)$$

Lower bound of $I(n, f, h) - K(n, f, h)$. With the notation $g_{v, \varepsilon}(u) = \underline{h}_{\varepsilon}(u - v)$ and by Lemma 3.5.1, we have:

$$I(n, f, h) \geq I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) - \int_{|v| \geq \varepsilon} I(n, f, g_{v, \varepsilon}) \kappa_{\varepsilon^2}(v) dv. \quad (3.30)$$

*Bound of $I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2})$.* The Fourier transform of $\underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}$ has a compact support included in $[-1/\varepsilon^2, 1/\varepsilon^2]$. So by the the result proved in Step 1, for n large enough,

$$I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) \geq K(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) - \varepsilon. \quad (3.31)$$

By (3.28) and (3.21),

$$\underline{h}_\varepsilon * \kappa_{\varepsilon^2}(z) \geq \underline{h}_{2\varepsilon}(z) \int_{|v| \leq \varepsilon} \kappa_{\varepsilon^2}(v) dv \geq \left(1 - \frac{4\varepsilon}{\pi}\right) \underline{h}_{2\varepsilon}(z).$$

From this and the bound (3.25) for $K(n, f, \underline{h}_{2\varepsilon})$, the inequality (3.31) implies

$$I(n, f, \underline{h}_\varepsilon * \kappa_{\varepsilon^2}) \geq K(n, f, \underline{h}_{2\varepsilon}) - \frac{4\varepsilon}{\pi} C\nu(f) \|\underline{h}_{2\varepsilon}\|_{L^1} - \varepsilon.$$

Using $K(n, f, \underline{h}_{2\varepsilon}) = K(n, f, h) + K(n, f, \underline{h}_{2\varepsilon} - h)$ and the bound (3.25) for $K(n, f, \underline{h}_{2\varepsilon} - h)$, we have

$$I(n, f, \underline{h}_\varepsilon * \kappa_{\varepsilon^2}) - K(n, f, h) \geq -C\nu(f) \|h - \underline{h}_{2\varepsilon}\|_{L^1} - \frac{4C\nu(f)\varepsilon}{\pi} \|\underline{h}_{2\varepsilon}\|_{L^1} - \varepsilon. \quad (3.32)$$

Bound of $\int_{|v| \geq \varepsilon} I(n, f, g_{v,\varepsilon}) \kappa_{\varepsilon^2}(v) dv$ in (3.30). Using (3.29) with h replaced by $g_{v,\varepsilon}$ (which lies in \mathcal{H}_ε) and (3.25) for $K(n, f, g_{v,\varepsilon})$, we get

$$\begin{aligned} I(n, f, g_{v,\varepsilon}) &\leq C\nu(f) \|g_{v,\varepsilon}\|_{L^1} + C\nu(f) \|\overline{(g_{v,\varepsilon})_{2\varepsilon}} - g_{v,\varepsilon}\|_{L^1} \\ &\quad + C\nu(f) (\varepsilon + \varepsilon^2) \|\overline{(g_{v,\varepsilon})_{2\varepsilon}}\|_{L^1} + \varepsilon. \end{aligned}$$

This implies that

$$I(n, f, g_{v,\varepsilon}) \leq C\nu(f) (1 + \varepsilon + \varepsilon^2) \|\overline{(g_{v,\varepsilon})_{2\varepsilon}}\|_{L^1} + \varepsilon. \quad (3.33)$$

Note that, for any $v \in \mathbb{R}$,

$$\overline{(g_{\varepsilon,v})_{2\varepsilon}}(u) = \sup_{w \in [u-2\varepsilon, u+2\varepsilon]} \underline{h}_\varepsilon(w-v) \leq \sup_{w \in [u-2\varepsilon, u+2\varepsilon]} h(w-v) = \overline{h}_{2\varepsilon}(u-v).$$

So, $\|\overline{(g_{\varepsilon,v})_{2\varepsilon}}\|_{L^1} \leq \|\overline{h}_{2\varepsilon}\|_{L^1}$. This together with (3.33) and (3.21) implies that

$$\begin{aligned} \int_{|v| \geq \varepsilon} I(n, f, g_{v,\varepsilon}) \kappa_{\varepsilon^2}(v) dv &\leq \left(C\nu(f) (1 + \varepsilon + \varepsilon^2) \|\overline{h}_{2\varepsilon}\|_{L^1} + \varepsilon \right) \int_{|v| \geq \varepsilon} \kappa_{\varepsilon^2}(v) dv \\ &\leq \left(C\nu(f) (1 + \varepsilon + \varepsilon^2) \|\overline{h}_{2\varepsilon}\|_{L^1} + \varepsilon \right) \frac{4\varepsilon}{\pi}. \end{aligned} \quad (3.34)$$

Putting together (3.30), (3.32) and (3.34), we obtain

$$I(n, f, h) - K(n, f, h) \geq -C\nu(f) \|h - \underline{h}_{2\varepsilon}\|_{L^1} + C\nu(f) \|\bar{h}_{2\varepsilon}\|_{L^1} (\varepsilon + \varepsilon^2) - \varepsilon. \quad (3.35)$$

Combining the upper bound (3.29), the lower bound (3.35) and the condition (3.19), the desired result (3.26) or (3.4) follows for $f \in \mathcal{B}_\beta$ and $h \in \mathcal{H}_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{4})$ and h satisfies (3.19).

Step 3. We prove (3.4) for $f \in \mathcal{B}_\beta$ and h which is nonnegative and directly Riemann integrable. Since h is directly Riemann integrable, $M := \sup_{y \in \mathbb{R}} h(y) < +\infty$. Let $\eta \in (0, 1)$ and $\varepsilon \in (0, M\eta)$.

By a result of approximation in the proof of Theorem 2.2 in [82], there exist two functions $h_{\eta, \varepsilon}^-$ and $h_{\eta, \varepsilon}^+$ which belong to $\mathcal{H}_{\varepsilon_1}$ for all $\varepsilon_1 \in (0, \min\{1/4, M\eta, \eta/3\})$ small enough, and which satisfy (3.19), together with

$$h_{\eta, \varepsilon}^- \leq h \leq h_{\eta, \varepsilon}^+ \text{ and } \int_{\mathbb{R}} [h_{\eta, \varepsilon}^+(t) - h_{\eta, \varepsilon}^-(t)] dt < 3\varepsilon. \quad (3.36)$$

The first inequality in (3.36) gives $K(n, f, h_{\eta, \varepsilon}^+ - h) \leq K(n, f, h_{\eta, \varepsilon}^+ - h_{\eta, \varepsilon}^-)$, so that

$$\begin{aligned} |I(n, f, h) - K(n, f, h)| &\leq |I(n, f, h) - I(n, f, h_{\eta, \varepsilon}^+)| \\ &\quad + |I(n, f, h_{\eta, \varepsilon}^+) - K(n, f, h_{\eta, \varepsilon}^+)| + |K(n, f, h_{\eta, \varepsilon}^+ - h_{\eta, \varepsilon}^-)|. \end{aligned} \quad (3.37)$$

In the right hand side, as $n \rightarrow \infty$, the second term tends to 0 uniformly in $x \in \mathcal{S}$ and $y \in \mathbb{R}$ by the result proved in Step 2. The third one is bounded by $C\nu(f)3\varepsilon$ from the bound (3.25) for $K(n, f, h_{\eta, \varepsilon}^+ - h_{\eta, \varepsilon}^-)$ and the property (3.36). Therefore, using (3.37) and passing to the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \sup_{(x, y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| \\ &\leq \limsup_{n \rightarrow \infty} \sup_{(x, y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - I(n, f, h_{\eta, \varepsilon}^+)| + C\nu(f)3\varepsilon. \end{aligned} \quad (3.38)$$

Now

$$\begin{aligned} I(n, f, h) - I(n, f, h_{\eta, \varepsilon}^+) &\leq I(n, f, h_{\eta, \varepsilon}^+) - I(n, f, h_{\eta, \varepsilon}^-) \\ &= [I(n, f, h_{\eta, \varepsilon}^+) - K(n, f, h_{\eta, \varepsilon}^+)] + K(n, f, h_{\eta, \varepsilon}^+ - h_{\eta, \varepsilon}^-) \\ &\quad + [K(n, f, h_{\eta, \varepsilon}^-) - I(n, f, h_{\eta, \varepsilon}^-)]. \end{aligned} \quad (3.39)$$

As in the preceding, in the last display, as $n \rightarrow \infty$, the first and third terms tend to 0 uniformly in $x \in \mathcal{S}$ and $y \in \mathbb{R}$ by the result proved in Step 2; the second one is bounded by $C\nu(f)3\varepsilon$ (by (3.25) and (3.36)). Therefore,

$$\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - I(n, f, h_{n,\varepsilon}^+)| \leq C\nu(f)3\varepsilon. \quad (3.40)$$

Combining (3.38) and (3.40), we obtain

$$\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| \leq C\nu(f)6\varepsilon. \quad (3.41)$$

Since $\varepsilon > 0$ est arbitrary, this gives (3.4).

Step 4. We establish (3.4) for $f \in \mathcal{C}(\mathcal{S})$ and h which is directly Riemann integrable. Let $\varepsilon > 0$. From the fact that \mathcal{B}_β is dense in $\mathcal{C}(\mathcal{S})$ with respect to the norm $\|\cdot\|_\infty$, there is a function $\tilde{f} \in \mathcal{B}_\beta$ such that $\|\tilde{f} - f\|_\infty < \varepsilon$. Hence we have

$$\begin{aligned} I(n, f, h) - K(n, f, h) &= I(n, f - \tilde{f}, h) + [I(n, \tilde{f}, h) - K(n, \tilde{f}, h)] \\ &\quad - K(n, f - \tilde{f}, h). \end{aligned} \quad (3.42)$$

It follows from the result proved in Step 3 that

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, \tilde{f}, h) - K(n, \tilde{f}, h)| = 0.$$

Consequently, by (3.42)

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| \\ &\leq \limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f - \tilde{f}, h) - K(n, f - \tilde{f}, h)| \\ &\leq \|f - \tilde{f}\|_\infty \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} [I(n, 1, h) + K(n, 1, h)], \end{aligned} \quad (3.43)$$

where the last inequality follows directly from the definition of I and K . By the result proved in Step 3 (applied to $f = 1$) and the bound (3.25) applied for $K(n, 1, h)$, we see that $I(n, 1, h) - K(n, 1, h)$ and $I(n, 1, h)$ are bounded uniformly in x, y and $n \geq 1$. Hence

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} [I(n, 1, h) + K(n, 1, h)] < \infty.$$

Since $\|\tilde{f} - f\|_\infty < \varepsilon$ and $\varepsilon > 0$ is arbitrary, this together with (3.43) implies

$$\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| = 0,$$

which completes the proof of part (1) of Theorem 3.2.2. □

Proof of Theorem 3.2.2, part (2). For the proof of part (2), we use the conclusion of part (1) and the approximation of the indicator function by a continuous function (see Lemma 3.4.1). Because the argument is quite similar to the proof of part (2) of Theorem 3.2.1, we omit the details. □

Chapter 4

Central limit theorem and precise large deviations for branching random walks with products of random matrices

We consider a branching random walk where particles give birth to children as a Galton-Watson process, which move in \mathbb{R}^d with positions determined by the action of independent and identically distributed random matrices on the position of the parent. We are interested in asymptotic properties of the counting measure Z_n^x which counts the number of particles of generation n situated in a given region, when the process starts with one initial particle located at x . We establish a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for Z_n^x with suitable norming. An integral version of the large deviation result is also established. One of the key points in the proofs is the study of the fundamental martingale related to the spectral gap theory for products of random matrices. As a by-product, we obtain a sufficient and necessary condition for the non-degeneracy of the limit of the fundamental martingale, which extends the Kesten-Stigum type theorem of Biggins.

4.1 Introduction

A branching random walk is a system of particles, in which each particle gives birth to new particles of the next generation, whose children move on \mathbb{R} or \mathbb{R}^d , according to some probability law. For early fundamental results on this model, see for example [2, 3, 13, 15, 16]. In recent years, this topic has attracted the attention of many authors, see for example, [1, 52, 35, 5, 34, 30, 55, 65]. The model is closely related to various applied probability settings, such as Mandelbrot's cascades (cf. e.g. [56, 67, 6, 25, 72]), perpetuities (see e.g. [77, 26, 54]) and branching Brownian motion (cf. e.g. [59, 29, 12, 71]). For extensions to random environments in space and time, see e.g. [44, 33] and [19, 62, 69, 39, 40]. For other related works and many references, see e.g. the recent books [77, 26, 54]. In the classical branching random walk, a particle whose parent is at position

y , moves to position $y + l$ with independent and identically distributed (i.i.d.) increments l for different particles, so that the moving is a simple random translation. The classical model does not cover the interesting cases occurring in many applications where the movements are determined by linear transformations such as rotations, dilations, shears, reflections, projections etc. In this paper, we deal with the case where the position of a particle is obtained by the action of a matrix A on the position of its parent, where the matrices A 's corresponding to different particles are i.i.d. In other words, the positions of particles are obtained by the action of products of random matrices on the position of one initial particle. This permits us to extend significantly the domains of applications of the theory of branching random walks. However, the study of this model becomes much more involved. One of the fundamental problems in the theory of branching random walks is to give a precise description of the configuration of the process at time n . We will consider this problem by giving precise asymptotics of the counting measure Z_n^x which counts the number of particles of generation n situated in a given region, when the process begins with one initial particle situated at x . More precisely, for the model that we introduce here, we will establish a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for Z_n^x with suitable norming.

To introduce the model we need some notation. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. Set $\mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$, where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. A particle of generation n will be denoted by a sequence $u = u_1 \cdots u_n = (u_1, \dots, u_n) \in (\mathbb{N}^*)^n$ of length n ; the initial particle will be denoted by the null sequence \emptyset . Assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a set of independent identically distributed random variables $(N_u)_{u \in \mathbb{U}}$ of the same law $p = \{p_k : k \in \mathbb{N}\}$, and a set of independent identically distributed $d \times d$ random matrices $(A_u)_{u \in \mathbb{U}}$ of the same law μ on the set of $d \times d$ matrices $M(d, \mathbb{R})$, where $d \geq 1$. The two families $(N_u)_{u \in \mathbb{U}}$ and $(A_u)_{u \in \mathbb{U}}$ are also assumed to be independent.

A branching random walk with products of random matrices is defined as follows. At time 0, there is one initial particle \emptyset of generation 0, with initial position $Y_\emptyset := x \in \mathbb{R}^d \setminus \{0\}$. At time 1, the initial particle \emptyset is replaced by $N = N_\emptyset$ new particles $i = \emptyset i$ of generation 1, located at $Y_i = A_i Y_\emptyset$, $1 \leq i \leq N$. In general, at time $n + 1$, each particle $u = u_1 \dots u_n$ of generation n , located at $Y_u \in \mathbb{R}^d$, is replaced by N_u new particles ui of generation $n + 1$, located at $Y_{ui} = A_{ui} Y_u$, $1 \leq i \leq N_u$. Namely, the position of the particle ui is obtained from the position Y_u of u by the action of the matrix A_{ui} , so that the position Y_u of a particle u in generation $n \geq 1$ is given by the action of products of random matrices on

the initial position x :

$$Y_u = G_u x, \quad \text{where} \quad G_u = A_{u_1 \dots u_n} \dots A_{u_1}. \quad (4.1)$$

Denote by \mathbb{T} the genealogical tree associated to the elements $\{N_u : u \in \mathbb{U}\}$. It is defined by the following properties: 1) $\emptyset \in \mathbb{T}$; 2) when $u \in \mathbb{T}$, then for $i \in \mathbb{N}$, $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$; 3) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n , where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs; by convention $|\emptyset| = 0$.

The space \mathbb{R}^d is equipped with the Euclidean norm $|\cdot|$. The position $G_u x$ of the particle u is completely described by two components: its norm $|G_u x|$ and its projection on the unit sphere $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d, |y| = 1\}$ denoted by

$$X_u^x := \frac{G_u x}{|G_u x|}.$$

Accordingly, we consider the following counting measure of particles of generation n which describes the configuration of the branching random walk at time n : for measurable sets $B \subset \mathbb{S}^{d-1}$ and $C \subset \mathbb{R}$,

$$Z_n^x(B, C) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B, \log |G_u x| \in C\}}, \quad (4.2)$$

where for a set D , $\mathbb{1}_D$ denotes its indicator function. In particular when $B = \mathbb{S}^{d-1}$ the measure (4.2) reduces to

$$Z_n^x(\mathbb{S}^{d-1}, C) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{\log |G_u x| \in C\}}. \quad (4.3)$$

The measure $C \mapsto Z_n^x(\mathbb{S}^{d-1}, C)$ counts the number of particles of generation n with a given distance to the origin; the distributional function $Z_n^x(\mathbb{S}^{d-1}, (-\infty, y])$ counts the number of particles of generation n situated in the ball centered at 0 with radius e^y . This information may be important for example when we consider a model describing the infection by a certain transmittable disease (an infected individual at time n leads to a random number of infected individuals at time $n+1$ who move according to random linear transformations

in \mathbb{R}^2 or \mathbb{R}^3), where we would like to know at time n how many infected individuals there are in a region with a given distance from the origin. The measure $(B, C) \mapsto Z_n^x(B, C)$ gives more information. For example, when $d = 2$ and $B = \{e^{i\theta} : \theta \in [\theta_1, \theta_2]\}$ is an arc, $Z_n^x(B, (-\infty, y])$ counts the number of particles of generation n situated in the region $\{re^{i\theta} : \theta \in [\theta_1, \theta_2], r \in [0, e^y]\}$.

When $d = 1$, $x = 1$ and $A_u \neq 0$ for all $u \in \mathbb{T}$, the measure defined by (4.3) is exactly the counting measure considered in the classical model of branching random walk on \mathbb{R} starting from the origin $0 \in \mathbb{R}$, where the position S_u of a particle $u = u_1 \cdots u_n$ is given by $S_u = L_{u_1} + \cdots + L_{u_1 \cdots u_n}$, with $L_u = \log |A_u|$. So our model in the one dimensional case $d = 1$ reduces essentially to the classical (additive) branching random walk. For this reason, in the following we will focus on the case $d \geq 2$.

The present work aims to establish asymptotic properties of the counting measure Z_n^x when it is suitably normalized, with $|x| = 1$ and $d \geq 2$. We will consider two cases: when the matrices A_u are nonnegative, and when the matrices A_u are invertible. Our first result is a central limit theorem for the counting measure Z_n^x (see Theorem 4.2.1). It states that for any fixed $B \subset \mathbb{S}^{d-1}$ and some constants γ, σ defined explicitly, the counting measure $C \mapsto Z_n^x(B, n\gamma + \sigma\sqrt{n}C)$ on \mathbb{R} with a suitable norming converges to the standard normal law. This result extends the corresponding one of Asmussen and Kaplan [3, Theorem 1] on the one dimensional case, which was first conjectured by Harris [48]. Our second result is a precise large deviation result of Kesten-Stigum type (see Theorem 4.2.6), namely we give an exact asymptotic for $Z_n^x(B, [na, +\infty))$ for fixed $B \subset \mathbb{S}^{d-1}$ and a in a natural range of \mathbb{R} . An extension to an integral version of the large deviation result with target functions on the two components X_u^x and $\log |G_u x|$ is also established (see Theorem 4.2.4). These results extend the corresponding ones of Biggins [15] on the one dimensional case to the multi-dimensional case.

The starting point in the proofs of our results is a decomposition formula which permits to express the counting measure as the sum of conditionally independent random variables, using the branching property like in the one dimensional case for which we may refer to [3, 15]. However, there is much to do to arrive to the conclusions in the multidimensional case, due to the appearance of products of random matrices. In particular, for the proof of Theorem 4.2.1 about the central limit theorem and Theorem 4.2.4 about the precise large deviation with target functions, we use respectively the central limit theorem and the recent progress on the spectral gap theory and precise large deviations for products of random matrices. Another step forward in the proof of Theorem 4.2.4 concerns the

extension of Biggins’ martingale to the case of branching products of random matrices, for which we prove a criterion for the non-degeneracy of the limit of the fundamental martingale (see Theorem 4.2.2) which completes a result of Mentemeier [72] obtained in the context of the multivariate smoothing transform, and extends the Kesten-Stigum type theorem of Biggins [13] on the classical branching random walk.

The outline of the paper is as follows. The main results will be stated in Section 4.2. Theorem 4.2.1 on the asymptotic normality of the counting measure is proved in Section 4.3. The necessary and sufficient condition for the non-degeneracy of the limit of the fundamental martingale is given in Theorem 4.2.2 and Corollary 4.2.3, which are proved in Section 4.4. Theorem 4.2.4 on the precise asymptotic of large deviations, which implies Theorem 4.2.6, is established in Section 4.5.

4.2 Main results

In this section, we introduce necessary notation and assumptions, and present the main results.

4.2.1 Notation and assumptions on products of random matrices

Note that in our model, along each branch we encounter a product of random matrices. In this section, we introduce some notation and the necessary assumptions on products of random matrices in order to formulate our main results. We shall consider two cases, the case when the matrices are nonnegative and the case when the matrices are invertible.

Let $M(d, \mathbb{R})$ be equipped with the operator norm: for any $\mathbf{a} \in M(d, \mathbb{R})$ we set $\|\mathbf{a}\| = \sup_{x \in \mathbb{S}^{d-1}} |\mathbf{a}x|$, where $|\cdot|$ is a given vectorial norm on \mathbb{R}^d , and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere in \mathbb{R}^d . Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by the support of μ . A matrix $\mathbf{a} \in M(d, \mathbb{R})$ is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by \mathcal{M}_+ the set of matrices with nonnegative entries. A nonnegative matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column has a strictly positive entry.

We say that the measure μ is *arithmetic* if there are $t > 0$, $\theta \in [0, 2\pi)$ and a function

$\vartheta : \mathbb{S}_+^{d-1} \rightarrow \mathbb{R}$ such that

$$\forall \mathbf{a} \in \Gamma_\mu, \forall x \in \text{supp } \nu : \exp[it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a} \cdot x) - \vartheta(x))] = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant, and ν is the μ -invariant measure (cf. (4.5)). Notice when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form: $\log a$ is a.s. concentrated on an arithmetic progression $a_0 + a_1\mathbb{N}$ for some $a_0, a_1 > 0$.

We will need the following assumptions on the law μ .

B1.

1. For invertible matrices:

- (a) (Strong irreducibility) There is no finite union $\mathcal{W} = \bigcup_{i=1}^n W_i$ of proper subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ -invariant (in the sense that $a\mathcal{W} = \mathcal{W}$ for each $a \in \Gamma_\mu$)
- (b) (Proximality) Γ_μ contains at least one proximal matrix.

2. For nonnegative matrices:

- (a) (Allowability) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.
- (b) (Positivity) Γ_μ contains at least one matrix belonging to $\text{int}(\mathcal{M}_+)$.
- (c) (Non-arithmeticity) The measure μ is non-arithmetic.

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathbb{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a} \cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|} \quad \text{when } \mathbf{a}x \neq 0,$$

where $\mathbf{a} \cdot x$ is called the projective action of the matrix \mathbf{a} on the vector $x \in \mathbb{S}^{d-1}$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices. Set, for an invertible or nonnegative matrix \mathbf{a} ,

$$N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \iota(\mathbf{a})^{-1}\}.$$

For invertible matrices we have $\iota(\mathbf{a}) = \|\mathbf{a}^{-1}\|^{-1}$ and $N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \|\mathbf{a}^{-1}\|\}$.

B2. (Moment condition) There exists $\eta_0 \in (0, 1)$ such that

$$\mathbb{E}[N(A_1)^{\eta_0}] < \infty.$$

We will consider the action of invertible matrices on the projective space \mathbb{P}^{d-1} which is obtained from \mathbb{S}^{d-1} by identifying x and $-x$, and the action of nonnegative matrices on \mathbb{S}_+^{d-1} . For convenience, we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} for invertible matrices, and \mathbb{S}_+^{d-1} for nonnegative matrices. The space \mathcal{S} will be equipped with the metric \mathbf{d} , which is the angular distance (see [20]) for invertible matrices, and the Hilbert cross-ratio metric (see [49]) for nonnegative matrices. Moreover, \mathcal{S} is a separable metric space equipped with Borel σ -field.

Let $G_n = A_n \dots A_2 A_1$ be the product of i.i.d. $d \times d$ real random matrices A_i , defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with common law μ . Let $x \in \mathcal{S}$ be a starting point. As mentioned in the introduction, the random walk $G_n x$ is completely determined by its log norm and its projection on \mathcal{S} , denoted respectively by

$$S_n^x := \log |G_n x|, \quad X_n^x := G_n \cdot x = \frac{G_n x}{|G_n x|}, \quad n \geq 0,$$

with the convention that $G_0 x = x$. Since $S_n^x = \log |A_n X_{n-1}^x| + S_{n-1}^x$ and $X_n^x = A_n \cdot X_{n-1}^x$, the sequence $(S_n^x, X_n^x)_{n \geq 0}$ is a Markov chain.

Denote by \mathbb{E} the expectation corresponding to \mathbb{P} . By the law of large numbers of Furstenberg [38], under conditions **B1** and **B2**, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n^x = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[S_n^x] = \gamma \quad \mathbb{P}\text{-a.s.},$$

where $\gamma = \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E} \log \|G_n\|$ is the upper Lyapunov exponent associated with the product sequence (G_n) . Le Page [63] and Henion [49] showed that

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} (S_n^x - n\gamma)^2 \tag{4.4}$$

exists and is independent of x for invertible matrices and nonnegative matrices, respectively. Moreover, there exists a unique μ -stationary probability measure ν on \mathcal{S} (see [46, 25]); the μ -stationarity of ν means that $\mu * \nu = \nu$, that is, for any continuous function φ

on \mathcal{S} ,

$$(\mu * \nu)(\varphi) := \int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(\mathbf{a} \cdot x) \mu(d\mathbf{a}) \nu(dx) = \nu(\varphi). \quad (4.5)$$

where $\nu(\varphi) = \int_{\mathcal{S}} \varphi(x) \nu(dx)$. This notation for the integral will be used for any function and any measure. Set

$$I_\mu = \{s \geq 0 : \mathbb{E}\|A_1\|^s < \infty\}.$$

Note that I_μ is an interval of \mathbb{R}_+ . Let $s_\infty = \sup I_\mu$. Define the transfer operator on the set $\mathcal{C}(\mathcal{S})$ of continuous functions on \mathcal{S} as follows: for any $s \in (-\eta_0, s_\infty)$, and $f \in \mathcal{C}(\mathcal{S})$,

$$P_s f(x) = \mathbb{E}[|A_1 x|^s f(A_1 \cdot x)], \quad \text{for all } x \in \mathcal{S}. \quad (4.6)$$

It is known that under conditions **B1**, and **B2**, there exists a small constant $0 < \eta_1 < \eta_0$ such that for any $s \in (-\eta_1, s_\infty)$, there are a unique probability measure ν_s and a unique Hölder continuous normalized function r_s (under the normalizing condition $\nu(r_s) = 1$) on \mathcal{S} satisfying

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \quad (4.7)$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s , $\nu_s P_s$ is the measure on \mathcal{S} such that $(\nu_s P_s)(f) = \nu_s(P_s f)$ for all $f \in \mathcal{C}(\mathcal{S})$. For $s \in [0, s_\infty)$, the property (4.7) is proved in [25, Proposition 3.1] and [28, Corollary 7.3] for positive matrices, and in [46, Theorem 2.6 and Corollary 3.20] for invertible matrices. For both positive matrices and invertible matrices, the existence of $\eta_1 > 0$ and the property (4.7) for $s \in (-\eta_1, \eta_1)$ are proved in [83, Proposition 3.1], where the following properties are also established: the functions $s \mapsto \kappa(s)$ and $s \mapsto r_s(x)$ are strictly positive and analytic in $(-\eta_1, s_\infty)$, for $x \in \mathcal{S}$. Moreover, it is proved (see [46, Lemma 3.5], [25, Lemma 6.2], [83, Propositions 3.12 and 3.14]) that, under conditions **B1** and **B2**, the function

$$\Lambda(s) = \log \kappa(s)$$

is finite and analytic on $(-\eta_1, s_\infty)$, and satisfies

$$\Lambda(0) = 0, \quad \Lambda'(0) = \gamma, \quad \Lambda''(0) = \sigma^2 > 0, \quad \text{and} \quad \Lambda''(s) > 0 \quad \forall s \in (-\eta_1, s_\infty).$$

4.2.2 Main results

Note that the population size at time n is $Z_n = Z_n^x(S, \mathbb{R})$, which does not depend on the starting point x and forms a Galton-Watson process with $Z_0 = 1$ and $Z_1 = N$. Denote by $m = \mathbb{E}N$ the expected value of the offspring distribution. Throughout the paper, we shall assume that $1 < m < \infty$, which ensures that the branching process (Z_n) is supercritical, so that $Z_n \rightarrow \infty$ as $n \rightarrow \infty$ with positive probability. It is well known that $\mathbb{E}Z_n = m^n$. Let

$$W_n = \frac{Z_n}{m^n} \quad \text{for } n \geq 0, \quad \text{and} \quad W = \lim_{n \rightarrow \infty} W_n.$$

The sequence $\{W_n\}$ is the fundamental martingale for the Galton-Watson process (Z_n) ; the limit above exists almost surely (a.s.) by the martingale convergence theorem. The famous Kesten-Stigum theorem states that W is non-degenerate if and only if $\mathbb{E}N \log_+ N < \infty$ (see [4]), where and through this paper $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$. We will need the following slightly stronger condition.

B3. *There exists a constant $\eta > 1$ such that*

$$\mathbb{E}N \log_+^{\eta+1} N < \infty. \tag{4.8}$$

We start with a central limit theorem for the normalized counting measure (4.2). For $t \in \mathbb{R}$, let

$$Z_n^x(B, t) = Z_n^x\left(B, (-\infty, n\gamma + t\sigma\sqrt{n}]\right) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B, \frac{\log |G_{ux}| - n\gamma}{\sigma\sqrt{n}} \leq t\}}.$$

Theorem 4.2.1. *Assume that the law μ of the random matrices satisfies conditions **B1** and **B2**. Assume also that the offspring distribution satisfies condition **B3**. Then, for any $x \in \mathcal{S}$, any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$ and any $t \in \mathbb{R}$, we have, as $n \rightarrow \infty$,*

$$\frac{Z_n^x(B, t)}{m^n} \rightarrow \nu(B)\Phi(t)W \quad \mathbb{P}\text{-a.s.}, \tag{4.9}$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx$ is the distribution function of the standard normal law.

For the one dimensional case (where $d = 1$), the result is due to Asmussen and Kaplan [3, Theorem 1], which was first conjectured by Harris [48, p.75] but with convergence in probability instead of the a.s. convergence in (4.9). Harris' conjecture was first solved

by Stam [78], then improved by Asmussen and Kaplan [2, 3] to L^2 -convergence and a.s. convergence. More general cases have been considered by Klebaner [60] and Biggins [16], who studied respectively the varying environment case and the general branching random walk where the displacements of particles with the same parent may have different laws. The random environment case has been considered by Gao, Liu and Wang [41]. The exact convergence rate in (4.9) has been considered by Chen [29] and Gao and Liu [39]. Asymptotic expansions have been obtained in [40]. Theorem 4.9 opens ways for extending some results in [2, 29, 41, 39, 40] to the multi-dimensional case where the moving of particles is determined by products of random matrices.

Our second main result is on the large deviation for the counting measure Z_n^x . To study the large deviation of the measure Z_n^x , a natural way would be to consider its Laplace transform defined by, for $(s_1, s_2) \in \mathbb{R}^d \times \mathbb{R}$,

$$\tilde{Z}_n^x(s_1, s_2) = \int_{\mathbb{R}^d \times \mathbb{R}} e^{s_1 y_1 + s_2 y_2} Z_n^x(dy_1, dy_2) = \sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}, \quad (4.10)$$

where $s_1 y_1$ is the inner product of vectors s_1 and y_1 in \mathbb{R}^d .

In the one dimensional case, when $x = 1$ and $A_n > 0$, we have $X_u^x = 1$, so that $\tilde{Z}_n^x(s_1, s_2)/\mathbb{E}\tilde{Z}_n^x(s_1, s_2)$ reduces to Biggins' fundamental martingale of the branching random walk:

$$\frac{\sum_{u \in \mathbb{T}_n} e^{s_2 S_u^x}}{\mathbb{E}[\sum_{u \in \mathbb{T}_n} e^{s_2 S_u^x}]}, \quad n \geq 0, \quad (4.11)$$

which has been well studied (see [13], for example), and which plays an essential role in many problems. However, in the multidimensional case, in general the sequence (4.11) is no longer a martingale, nor the sequence

$$\frac{\tilde{Z}_n^x(s_1, s_2)}{\mathbb{E}\tilde{Z}_n^x(s_1, s_2)} = \frac{\sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}}{\mathbb{E}[\sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}]}, \quad n \geq 0, \quad (4.12)$$

for $(s_1, s_2) \in \mathbb{R}^d \times \mathbb{R}$. So an important difficulty arises when we mimic Cramér's change of measure for random walks by use of the Laplace transform of Z_n^x .

However, there is still a natural martingale in the present setting. By the spectral gap property (4.7), it is easy to verify that (see Section 4.4 for more details), for $s \in (-\eta_1, s_\infty)$

and $x \in \mathcal{S}$, the sequence

$$W_{s,n}^x := \frac{\sum_{u \in \mathbb{T}_n} e^{sS_u^x} r_s(X_u^x)}{m^n \kappa(s)^n r_s(x)}, \quad n \geq 0, \quad (4.13)$$

constitutes a positive martingale with respect to the natural filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{F}_n = \sigma(N_u, A_{ui} : i \geq 1, |u| < n) \text{ for } n \geq 1,$$

as observed by Mentemeier [72] in the study of the multivariate smoothing transform. By the martingale convergence theorem, the limit

$$W_s^x := \lim_{n \rightarrow \infty} W_{s,n}^x \quad \text{exists in } \mathbb{R} \quad \mathbb{P}\text{-a.s.}$$

It turns out that the martingale $(W_{s,n}^x)$ in the multidimensional case plays the same rule as Biggins' fundamental martingale for one dimensional case, for large deviations.

Just as in the case of Biggins' martingale, it is crucial to know when the limit variable W_s^x of the fundamental martingale $W_{s,n}^x$ is non-degenerate. When the matrices A_u are nonnegative and $s > 0$, Mentemeier [72, Proposition 4.4] gave a sufficient condition for W_s^x to be non-degenerate. In the following we complete his result by considering the necessary and sufficient conditions, and by treating meanwhile the case $s < 0$ and the case of invertible matrices.

We first establish the following theorem, whose proof is deferred to Section 4.4. To state the result, we need some notation. For $s \in (-\eta_1, s_\infty)$, set $\Lambda^*(q_s) = sq_s - \Lambda(s)$ with $q_s = \Lambda'(s)$. Since $\Lambda''(s) > 0$ and $\frac{\partial}{\partial s} \Lambda^*(q_s) = s\Lambda''(s)$, $\Lambda^*(q_s)$ attains its minimum at $s = 0$, so that $\Lambda^*(q_s) \geq \Lambda^*(q_0) = -\Lambda(0) = 0$ for all $s \in (-\eta_1, s_\infty)$.

Theorem 4.2.2. *Assume conditions B1, B2. If*

$$\Lambda^*(q_s) - \log m < 0 \quad (4.14)$$

and

$$\mathbb{E}[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x] < \infty, \quad (4.15)$$

then for all $x \in \mathcal{S}$,

$$\mathbb{E}[W_s^x] = 1. \quad (4.16)$$

Conversely, if

$$\mathbb{E}[W_s^x] > 0 \tag{4.17}$$

for some $x \in \mathcal{S}$, then (4.14) holds, and

$$\mathbb{E}[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x] < \infty. \tag{4.18}$$

Corollary 4.2.3. *Suppose the conditions B1, B2.*

1. Assume (4.14) together with

$$\mathbb{E}N \log_+ N < \infty \quad \text{and} \quad \mathbb{E}\|A_1\|^s \log_+ \|A_1\| < \infty. \tag{4.19}$$

Then (4.16) holds for all $x \in \mathcal{S}$.

2. Assume that the random matrix $A_1 = (A_1(i, j))$ satisfies the Furstenberg- Kesten condition: there exists a constant $C > 1$ such that

$$\frac{\max_{1 \leq i, j \leq d} |A_1(i, j)|}{\min_{1 \leq i, j \leq d} |A_1(i, j)|} \leq C \quad \text{a.s.} \tag{4.20}$$

Then the three conditions (4.15), (4.18) and (4.19) are equivalent, and (4.17) holds for all $x \in \mathcal{S}$ if and only if (4.14) and (4.19) hold. Moreover, if (4.17) holds for some $x \in \mathcal{S}$, then (4.16) holds for all $x \in \mathcal{S}$.

Notice that by Sheffé's theorem, for each $x \in \mathcal{S}$, if (4.17) holds, then $W_{s,n}^x \rightarrow W_s^x$ in L^1 . So the martingale $(W_{s,n}^x)$ converges in L^1 for some $x \in \mathcal{S}$ if and only if (4.14) and (4.19) hold; moreover, when the martingale converges in L^1 for some $x \in \mathcal{S}$, then it converges in L^1 for all $x \in \mathcal{S}$.

When the matrices A_u are nonnegative and $s > 0$, Part (1) has been established by Mentemeier [72, Proposition 4.4]. When $d = 1$, Part (2) is essentially the well-known Kesten-Stigum type theorem for the classical branching random walk on the real line, due to Biggins [13]; see also [56] for Mandelbrot's cascades and [70, 66] for versions which are slightly different to the initial result of Biggins [13].

Now we consider the precise large deviations for Z_n^x with target functions f and g on the components $X_u^x = G_u \cdot x$ and $S_u^x = \log |G_u x|$. More precisely, we shall study the

asymptotic of the large deviations of the following integral:

$$\int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) = \sum_{u \in \mathbb{T}_n} f(X_u^x)g(S_u^x - nq_s). \quad (4.21)$$

Our result will be stated under the very general assumption that $e^{-sz}g(z)$, $z \in \mathbb{R}$ is directly Riemann integrable, see Feller [36], Chapter XI.

Theorem 4.2.4. *Assume conditions **B1** and **B2**, and let $s \in (-\eta_1, s_\infty)$ be fixed such that $\Lambda^*(q_s) - \log m < 0$ and that*

$$\mathbb{E} \left[\max_{x \in \mathcal{S}} W_1^x(s) \log_+^{\delta+1} \max_{x \in \mathcal{S}} W_1^x(s) \right] < \infty \quad \text{for some } \delta > 3/2. \quad (4.22)$$

Then for any continuous function f on \mathcal{S} and any measurable function g on \mathbb{R} such that $z \mapsto e^{-sz}g(z)$ is directly Riemann integrable, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)}}{m^n} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) \\ = W_s^x r_s(x) \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} e^{-sz}g(z)dz, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (4.23)$$

where $\pi_s \left(\frac{f}{r_s} \right) = \frac{\nu_s(f)}{\nu_s(r_s)}$, and $\sigma_s^2 = \Lambda''(s)$.

When $s = 0$ this result reduces to the following local limit theorem for the counting measure Z_n^x :

Corollary 4.2.5. *Assume conditions **B1** and **B2**. Assume also that (4.22) holds with $s = 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{\sigma \sqrt{2\pi n}}{m^n} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - n\gamma)Z_n^x(dy, dz) = W\nu(f) \int_{\mathbb{R}} g(z)dz.$$

When $f = 1$ and $g = \mathbb{1}_{[a,b]}$ with $-\infty < a < b < \infty$, it gives the precise asymptotic of $Z_n^x(\mathcal{S}, n\gamma + [a, b])$ as $n \rightarrow \infty$.

The following theorem describes the asymptotic size of the number of particles in n -th generation situated in the regions $(B, [e^{nq_s}, +\infty))$ for $s > 0$, and $(B, (0, e^{nq_s}])$ for $s < 0$, where $B \subseteq \mathcal{S}$.

Theorem 4.2.6. *Assume the conditions of Theorem 4.2.4. Then, for any $x \in \mathcal{S}$, any*

measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$, we have, \mathbb{P} -a.s., for $s > 0$,

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)} \frac{Z_n^x(B, [nq_s, +\infty))}{m^n} = \frac{1}{s} W_s^x r_s(x) \int_B \frac{1}{r_s(y)} \pi_s(dy),$$

and for $s < 0$,

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)} \frac{Z_n^x(B, (-\infty, nq_s])}{m^n} = \frac{1}{s} W_s^x r_s(x) \int_B \frac{1}{r_s(y)} \pi_s(dy).$$

This theorem is obtained from Theorem 4.2.4 by taking $g = \mathbb{1}_{[0, +\infty)}$ when $s > 0$, and $g = \mathbb{1}_{(-\infty, 0]}$ when $s < 0$, and by using a smooth approximation of indicator function (see [22, Lemma 4.1]).

In the one dimensional case (where $d = 1$), Theorems 4.2.4 and 4.2.6 reduce to the Bahadur-Rao type results of Biggins [15]. The large deviation principle was established earlier by Biggins in [14].

4.3 Proof of Theorem 4.2.1

This section is devoted to prove Theorem 4.2.1, the central limit theorem on the counting measure Z_n^x .

4.3.1 Basic decomposition

For all $u \in \mathbb{U}$, let $\mathbb{T}(u)$ be the shifted tree of \mathbb{T} at u associated to the elements $\{N_{uv}\}$. It is defined by the following properties: 1) $\emptyset \in \mathbb{T}(u)$, 2) $vi \in T(u)$ implies $v \in \mathbb{T}(u)$ and 3) if $v \in T(u)$, then $vi \in \mathbb{T}(u)$ if and only if $1 \leq i \leq N_{uv}$. Define $T_n(u) = \{v \in \mathbb{T}(u) : |v| = n\}$. Then $\mathbb{T} = \mathbb{T}(\emptyset)$ and $\mathbb{T}_n = \mathbb{T}_n(\emptyset)$.

It follows from the additive property of the branching process that, for $k \leq n$, any measurable set $B \subseteq \mathcal{S}$ and any Borel set C in \mathbb{R} ,

$$\begin{aligned} Z_n^x(B, C) &= \sum_{u \in \mathbb{T}_k} \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_{uv}^x \in B, S_{uv}^x \in C\}} \\ &= \sum_{u \in \mathbb{T}_k} Z_{n-k}^{X_u^x}(B, C - S_u^x), \end{aligned} \quad (4.24)$$

where

$$Z_{n-k}^{X_u^x}(B, C - S_u^x) = \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_v^{X_u^x} \in B, S_v^{X_u^x} \in C - S_u^x\}}$$

represents the number of descendants of u at time n in the region characterized by $(B, C - S_u^x)$, and $C - S_u^x = \{y - S_u^x : y \in C\}$. In this section, we consider $C = (-\infty, n\gamma + t\sigma\sqrt{n}]$, $t \in \mathbb{R}$. For simplicity, we will use the following notation:

$$\begin{aligned} Z_n^x(B, t) &= \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B, \frac{\log |G_{u,x}| - n\gamma}{\sigma\sqrt{n}} \leq t\}}, \\ Z_{n-k}^{X_u^x}(B, t) &= \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_v^{X_u^x} \in B, \frac{S_v^{X_u^x} - (n-k)\gamma}{\sigma\sqrt{n-k}} \leq t\}}, \\ Z_{n-k}^{X_u^x}(B, \mathbb{R}) &= \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_v^{X_u^x} \in B\}}, \\ W_{n-k}^{X_u^x}(B, t) &= \frac{Z_{n-k}^{X_u^x}(B, t)}{m^{n-k}}, \\ W_{n-k}^{X_u^x} &= W_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R}) = \frac{Z_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R})}{m^{n-k}}. \end{aligned}$$

Notice that $Z_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R})$ is the population size of generation $n - k$ of the Galton-Watson process beginning from the particle u (whose genealogical tree is the shifted tree of \mathbb{T} at u). So $Z_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R})$ and $W_{n-k}^{X_u^x}$ do not depend on the position of u .

For conditional probabilities and expectations, we write

$$\mathbb{P}_n(\cdot) = \mathbb{P}(\cdot | \mathcal{F}_n), \quad \mathbb{E}_n(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_n).$$

We obtain the following decomposition from (4.24), which will play a key role in our approach:

$$\begin{aligned} & \frac{1}{m^n} Z_n^x(B, t) - \nu(B)\Phi(t)W \\ &= \frac{1}{m^n} \sum_{u \in \mathbb{T}_{kn}} Z_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) - \nu(B)\Phi(t)W \\ &= A_n + B_n + C_n, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
A_n &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right. \\
&\quad \left. - \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right] \right\}, \\
B_n &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right] - \nu(B)\Phi(t) \right\}, \\
C_n &= (W_{k_n} - W) \nu(B)\Phi(t).
\end{aligned}$$

The idea is now to choose suitable k_n with $k_n \rightarrow \infty$ such that $A_n, B_n, C_n \rightarrow 0$ a.s.

4.3.2 Proof of Theorem 4.2.1

We choose β with $\frac{1}{\eta} < \beta < 1$ and $\alpha > \frac{2}{\beta-1}$. For each n , let $j = j(n) \in \mathbb{N}$ be such that $j^{\alpha/\beta} \leq n < (j+1)^{\alpha/\beta}$; set $k_n = a_j = \lfloor j^\alpha \rfloor$, the integer part of j^α . We will prove that with this choice of (k_n) , we have $A_n, B_n, C_n \rightarrow 0$ a.s. By the decomposition (4.25), this will imply Theorem 4.2.1.

By the convergence of the martingale W_n to W , we have clearly $C_n \rightarrow 0$, \mathbb{P} -a.s. It remains to show the following two lemmas.

Lemma 4.3.1. *Under the hypothesis of Theorem 4.2.1, we have*

$$A_n \xrightarrow{n \rightarrow \infty} 0 \quad \mathbb{P}\text{-a.s.} \quad (4.26)$$

Lemma 4.3.2. *Under the hypothesis of Theorem 4.2.1, then*

$$B_n \xrightarrow{n \rightarrow \infty} 0 \quad \mathbb{P}\text{-a.s.} \quad (4.27)$$

For the proof of Lemma 4.3.1, We shall use the following result on the weighted moments of $W^* := \sup_n \{W_n\}$, which is proved in Liang and Liu [64, Theorem 1.2].

Lemma 4.3.3. *Under the hypothesis of Theorem 4.2.1, then*

$$\mathbb{E}(W^* + 1) \log^\eta(W^* + 1) < \infty. \quad (4.28)$$

Proof of Lemma 4.3.1. By definition, $A_n = \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} Y_{n,u}$, where

$$Y_{n,u} = W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) - \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right].$$

We see that for any $u \in \mathbb{T}_{k_n}$,

$$\begin{aligned} |Y_{n,u}| &\leq W_{n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R}) + \mathbb{E}_{k_n} W_{n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R}) \\ &= W_{n-k_n}^{X_u^x} + 1, \end{aligned} \tag{4.29}$$

where the last equality holds because $W_{n-k_n}^{X_u^x} = W_{n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R})$ represents the fundamental martingale of the Galton-Watson process beginning with the particle $u \in \mathbb{T}_{k_n}$. Let

$$\bar{Y}_{n,u} = Y_{n,u} \mathbf{1}_{\{|Y_{n,u}| < m^{k_n}\}} \quad \text{and} \quad \bar{A}_n = \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \bar{Y}_{n,u}.$$

We will use the decomposition

$$A_n = (A_n - \bar{A}_n) + (\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]) + \mathbb{E}_{k_n}[\bar{A}_n],$$

and prove that each of the three terms on the right side of this identity tends to zero as $n \rightarrow \infty$. We divide the proof into 3 steps.

Step 1. We first prove that $A_n - \bar{A}_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$, as a consequence of

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n \neq \bar{A}_n) < \infty. \tag{4.30}$$

In fact, by the Lemma of Borel-Cantelli, (4.30) implies that a.s. $A_n - \bar{A}_n = 0$ when n is large enough. By the definition of $\bar{Y}_{n,u}$ and the inequality (4.29), we have:

$$\begin{aligned} \mathbb{P}_{k_n}(A_n \neq \bar{A}_n) &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(Y_{n,u} \neq \bar{Y}_{n,u}) = \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(|Y_{n,u}| \geq m^{k_n}) \\ &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(W_{n-k_n}^{X_u^x} + 1 \geq m^{k_n}). \end{aligned}$$

Since the law of $W_{n-k_n}^{X_u^x}$ conditioned upon \mathcal{F}_{k_n} is that of W_{n-k_n} , it follows that

$$\begin{aligned} \mathbb{P}_{k_n}(A_n \neq \bar{A}_n) &\leq Z_{k_n} \mathbb{P}(W_{n-k_n} + 1 \geq m^{k_n}) \leq W_{k_n} m^{k_n} \mathbb{P}(W^* + 1 \geq m^{k_n}) \\ &\leq W_{k_n} \mathbb{E} \left[(W^* + 1) \mathbf{1}_{\{W^* + 1 \geq m^{k_n}\}} \right] \\ &\leq \frac{W_{k_n}}{(\log m^{k_n})^\eta} \mathbb{E} [(W^* + 1) \log^\eta(W^* + 1)]. \end{aligned}$$

Taking expectation and denoting $C = \mathbb{E}[(W^* + 1) \log^\eta(W^* + 1)]$ (which is finite by Lemma 4.3.3), we get $\mathbb{P}(A_n \neq \bar{A}_n) \leq \frac{C}{(\log m)^\eta k_n^\eta}$. Since $k_n^\eta \sim j^{\alpha\eta} \sim n^{\beta\eta}$ and $\beta\eta > 1$, (4.30) is proved.

Step 2. We next prove that $\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n] \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$, as a consequence of

$$\sum_{n=1}^{\infty} \mathbb{P} \left(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon \right) < \infty \quad \forall \varepsilon > 0 \quad (4.31)$$

(by the Lemma of Borel-Cantelli). By Chebyshev's inequality we have

$$\mathbb{P}_{k_n}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \leq \frac{1}{\varepsilon^2} \mathbb{E}_{k_n}(\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n])^2. \quad (4.32)$$

By the definition of \bar{A}_n and \mathcal{F}_{k_n} , and the fact that $(\bar{Y}_{n,u})_{u \in \mathbb{T}_{k_n}}$ is a sequence of independent random variables under \mathbb{E}_{k_n} , we have

$$\begin{aligned} \mathbb{E}_{k_n}(\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n])^2 &= \frac{1}{m^{2k_n}} \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} (\bar{Y}_{n,u} - \mathbb{E}_{k_n}[\bar{Y}_{n,u}])^2 \right] \\ &= \frac{1}{m^{2k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left(\mathbb{E}_{k_n}[\bar{Y}_{n,u}^2] - [\mathbb{E}_{k_n} \bar{Y}_{n,u}]^2 \right). \end{aligned} \quad (4.33)$$

By the definition of $\bar{Y}_{n,u}$ and Fubini's theorem,

$$\begin{aligned} \mathbb{E}_{k_n}[\bar{Y}_{n,u}^2] &= \int_0^\infty 2x \mathbb{P}_{k_n}(|\bar{Y}_{n,u}| > x) dx \\ &= 2 \int_0^\infty x \mathbb{P}_{k_n}(|Y_{n,u}| \mathbf{1}_{\{|Y_{n,u}| < m^{k_n}\}} > x) dx \\ &= 2 \mathbb{E}_{k_n} \int_0^\infty x \mathbf{1}_{\{|Y_{n,u}| \mathbf{1}_{\{|Y_{n,u}| < m^{k_n}\}} > x\}} dx \\ &\leq 2 \mathbb{E}_{k_n} \int_0^{m^{k_n}} x \mathbf{1}_{\{|Y_{n,u}| > x\}} dx = \int_0^{m^{k_n}} x \mathbb{P}_{k_n}(|Y_{n,u}| > x) dx. \end{aligned}$$

Using (4.29), we obtain

$$\begin{aligned} \mathbb{P}_{k_n}(|Y_{n,u}| > x) &\leq \mathbb{P}_{k_n}(W_{n-k_n}^{X_u^x} + 1 > x) = \mathbb{P}(W_{n-k_n} + 1 > x) \\ &\leq \mathbb{P}(W^* + 1 > x). \end{aligned}$$

Therefore,

$$\mathbb{E}_{k_n}[\bar{Y}_{n,u}^2] \leq 2 \int_0^{m^{k_n}} x \mathbb{P}(W^* + 1 > x) dx. \quad (4.34)$$

Using (4.32), (4.33) and (4.34) and then taking expectation, we obtain

$$\mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \leq \frac{2}{\varepsilon^2 m^{k_n}} \int_0^{m^{k_n}} x \mathbb{P}(W^* + 1 > x) dx. \quad (4.35)$$

We split the above integral according to $x \in [0, e]$ and $x \in (e, m^{k_n}]$. Using $\mathbb{P}(W^* + 1 > x) \leq 1$, we see that

$$\int_0^e x \mathbb{P}(W^* + 1 > x) dx \leq \frac{e^2}{2}. \quad (4.36)$$

For the integral over $(e, m^{k_n}]$, using $x \mathbb{1}_{\{W^*+1>x\}} \leq (W^* + 1) \frac{\log^\eta(W^* + 1)}{\log^\eta x}$, we have

$$\int_e^{m^{k_n}} x \mathbb{P}(W^* + 1 > x) dx \leq \mathbb{E} \left[(W^* + 1) \log^\eta(W^* + 1) \right] \int_e^{m^{k_n}} \frac{1}{\log^\eta x} dx. \quad (4.37)$$

Taking a constant $b \in]1, m[$, we get

$$\begin{aligned} \int_e^{m^{k_n}} \frac{1}{\log^\eta x} dx &= \int_e^{b^{k_n}} \frac{1}{\log^\eta x} dx + \int_{b^{k_n}}^{m^{k_n}} \frac{1}{\log^\eta x} dx \\ &\leq b^{k_n} + \frac{(m^{k_n} - b^{k_n})}{(k_n \log b)^\eta}. \end{aligned}$$

From (4.35), (4.36) and (4.37), we obtain

$$\mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \leq \frac{1}{\varepsilon^2 m^{k_n}} \left(e^2 + 2C \left(b^{k_n} + \frac{m^{k_n} - b^{k_n}}{(k_n \log b)^\eta} \right) \right),$$

where $C = \mathbb{E} \left[(W^* + 1) \log^\eta(W^* + 1) \right] < \infty$ by Lemma 4.3.3. Therefore,

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \\ & \leq \frac{e^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{m^{k_n}} + \frac{2C}{\varepsilon^2} \left(\sum_{n=1}^{\infty} \frac{b^{k_n}}{m^{k_n}} + \sum_{n=1}^{\infty} \frac{m^{k_n} - b^{k_n}}{m^{k_n} (k_n \log b)^\eta} \right). \end{aligned} \quad (4.38)$$

Since $k_n \sim j^\alpha \sim n^\beta$, $1 < b < m$ and $\beta\eta > 1$, the three series $\sum_{n=1}^{\infty} \frac{1}{m^{k_n}}$, $\sum_{n=1}^{\infty} (\frac{b}{m})^{k_n}$ and $\sum_{n=1}^{\infty} \frac{1}{(k_n \log b)^\eta}$ converge. Therefore from (4.38), we get (4.31).

Step 3. We finally prove that $\mathbb{E}_{k_n}[\bar{A}_n] \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$, as a consequence of

$$\sum_{n=1}^{\infty} \mathbb{P} \left(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon \right) < \infty, \quad \forall \varepsilon > 0 \quad (4.39)$$

(again by the Lemma of Borel-Cantelli). By Markov's inequality, the fact that $0 = \mathbb{E}_{k_n}[A_n] = \mathbb{E}_{k_n}[\bar{A}_n] + \frac{1}{m^{k_n}} \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} Y_{n,u} \mathbf{1}_{\{|Y_{n,u}| \geq m^{k_n}\}} \right]$, and the inequality (4.29), we obtain:

$$\begin{aligned} \mathbb{P}(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) & \leq \frac{1}{\varepsilon} \mathbb{E} \left[|\mathbb{E}_{k_n}[\bar{A}_n]| \right] \\ & = \frac{1}{\varepsilon m^{k_n}} \mathbb{E} \left\{ \left| \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} (-Y_{n,u}) \mathbf{1}_{\{|Y_{n,u}| \geq m^{k_n}\}} \right] \right| \right\} \\ & \leq \frac{1}{\varepsilon m^{k_n}} \mathbb{E} \left\{ \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} (W_{n-k_n}^{X_u^x} + 1) \mathbf{1}_{\{W_{n-k_n}^{X_u^x} + 1 \geq m^{k_n}\}} \right] \right\} \\ & = \frac{1}{\varepsilon} \mathbb{E} \left[(W_{n-k_n} + 1) \mathbf{1}_{\{W_{n-k_n} + 1 \geq m^{k_n}\}} \right]. \end{aligned}$$

It follows that

$$\mathbb{P} \left(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon \right) \leq \frac{1}{\varepsilon (\log m^{k_n})^\eta} \mathbb{E} \left[(W^* + 1) \log^\eta(W^* + 1) \right].$$

Therefore, with $C = \mathbb{E} \left[(W^* + 1) \log^\eta(W^* + 1) \right] < \infty$ (by Lemma 4.3.3),

$$\sum_{n=1}^{\infty} \mathbb{P} \left(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon \right) \leq \frac{C}{\varepsilon \log^\eta m} \sum_{n=1}^{\infty} \frac{1}{k_n^\eta} < \infty$$

since $k_n^\eta \sim j^{\alpha\eta} \sim n^{\beta\eta}$ and $\beta\eta > 1$. So (4.39) is proved.

So the proof of Lemma 4.3.1 is finished. \square

The proof of Lemma 4.3.2 will be based on the following central limit theorem on the couple (X_n^x, S_n^x) (see Theorem 2.1, part (2) in [22]).

Lemma 4.3.4. *Under the hypothesis of Theorem 4.2.1, for any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$, we have*

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_n^x \in B, \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right) - \nu(B)\Phi(t) \right| = 0.$$

Proof of Lemma 4.3.2. We first calculate the conditional expectation in the definition of B_n . Denoting $t_n := \frac{t\sigma\sqrt{n} - S_u^x + k_n\gamma}{\sigma\sqrt{n - k_n}}$ and using the branching property, we have

$$\begin{aligned} \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x}(B, t_n) \right] &= \frac{1}{m^{k_n}} \mathbb{E}_{k_n} \left[\sum_{v \in \mathbb{T}_{n-k_n}(u)} \mathbb{1} \left\{ X_v^{X_u^x} \in B, \frac{S_v^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right\} \right] \\ &= \mathbb{P}_{k_n} \left(X_{n-k_n}^{X_u^x} \in B, \frac{S_{n-k_n}^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right). \end{aligned}$$

Therefore, by the definition of B_n ,

$$\begin{aligned} B_n &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ \mathbb{P}_{k_n} \left(X_{n-k_n}^{X_u^x} \in B, \frac{S_{n-k_n}^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right) - \nu(B)\Phi(t) \right\} \\ &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ \mathbb{P}_{k_n} \left(X_{n-k_n}^{X_u^x} \in B, \frac{S_{n-k_n}^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right) - \nu(B)\Phi(t_n) \right\} \\ &\quad + \frac{\nu(B)}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} [\Phi(t_n) - \Phi(t)]. \end{aligned}$$

Hence

$$|B_n| \leq W_{k_n} D_n + \frac{\nu(B)}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |\Phi(t_n) - \Phi(t)| \tag{4.40}$$

where

$$D_n = \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_{n-k_n}^x \in B, \frac{S_{n-k_n}^x - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t \right) - \nu(B)\Phi(t) \right|.$$

The first term in the right hand side of (4.40) tends to 0 a.s. because, by Lemma 4.3.4,

we have

$$D_n \xrightarrow{n \rightarrow \infty} 0 \quad \mathbb{P}\text{-a.s.}$$

We now prove that the second term in the right hand side of (4.40) also tends to 0 a.s. Remarking that

$$\begin{aligned} |\Phi(t_n) - \Phi(t)| &\leq \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t) \right| + \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t_n) \right| \\ &\leq \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t) \right| + \left| \frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}} - t_n \right| \end{aligned}$$

(since $|\Phi(x+h) - \Phi(x)| \leq |h|$ for any $x, h \in \mathbb{R}$), we obtain

$$\begin{aligned} &\frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |\Phi(t_n) - \Phi(t)| \\ &\leq W_{k_n} \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t) \right| + \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \frac{|S_u^x - k_n\gamma|}{\sigma\sqrt{n-k_n}}. \end{aligned} \quad (4.41)$$

It is clear that the first term in the above display tends to 0 a.s. as $n \rightarrow \infty$. So we need only to prove that the second term also tends to 0 a.s. Recall that $a_j = k_n$ and notice that $n - k_n \sim n \sim k_n^{1/\beta} = j^{\alpha/\beta}$. So it suffices to show that

$$M_j := \frac{1}{m^{a_j}} \sum_{u \in \mathbb{T}_{a_j}} j^{-\frac{\alpha}{2\beta}} |S_u^x - a_j\gamma| \rightarrow 0 \quad \mathbb{P}\text{-a.s. as } j \rightarrow +\infty. \quad (4.42)$$

Notice that

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbb{E}[M_j] &= \sum_{j=1}^{\infty} j^{-\frac{\alpha}{2\beta}} \mathbb{E}[|S_{a_j}^x - a_j\gamma|] \\ &\leq \sum_{j=1}^{\infty} j^{-\frac{\alpha}{2\beta}} \sqrt{\mathbb{E}[(S_{a_j}^x - a_j\gamma)^2]} \\ &= \sum_{j=1}^{\infty} j^{-\frac{\alpha}{2\beta}} a_j^{1/2} \sqrt{\frac{\mathbb{E}[(S_{a_j}^x - a_j\gamma)^2]}{a_j}} < \infty, \end{aligned}$$

where the last series converges by the expression of σ^2 (cf. (4.4)) and the fact that $j^{-\frac{\alpha}{2\beta}} a_j^{\frac{1}{2}} \sim j^{-\frac{\alpha}{2}} \left(\frac{1}{\beta} - 1\right)$ with $\frac{\alpha}{2} \left(\frac{1}{\beta} - 1\right) > 1$. Thus $\sum_{j=1}^{\infty} M_j < \infty$ a.s., which implies (4.42). So, by (4.41), the second term in the right hand side of (4.40) tends to 0 a.s. This ends

the proof of Lemma 4.3.2. □

4.4 Proof of Theorem 4.2.2 and Corollary 4.2.3

In this section we establish Theorem 4.2.2 and Corollary 4.2.3 about the non-degeneracy of the limit variable W_s^x of the fundamental martingale $(W_{s,n}^x)$. Let $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$ be fixed. Consider the positive function

$$H(n, y) = \frac{e^{s \log |y|} r_s\left(\frac{y}{|y|}\right)}{[m \kappa(s)]^n r_s(x)}, \quad n \geq 0, y \in \mathbb{R}^*.$$

Since r_s is the eigenfunction of the operator P_s with respect to the eigenvalue $\kappa(s)$ (see (4.7)), we see that H is a mean-harmonic function (see [19]) in the sense that for each $n \geq 0$ and $u \in \mathbb{T}_n$,

$$\mathbb{E} \left[\sum_{i=1}^{N_u} H(n+1, A_{ui} G_u x) \mid \mathcal{F}_n \right] = H(n, G_u x).$$

Indeed, we have:

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^{N_u} H(n+1, A_{ui} G_u x) \mid \mathcal{F}_n \right] &= \mathbb{E} \left[\sum_{i=1}^{N_u} \frac{e^{s \log |A_{ui} G_u x|} r_s(A_{ui} \cdot X_u^x)}{[m \kappa(s)]^{n+1} r_s(x)} \mid \mathcal{F}_n \right] \\ &= \frac{e^{s \log |G_u x|}}{[m \kappa(s)]^{n+1} r_s(x)} \mathbb{E} \left[\sum_{i=1}^{N_u} e^{s \log (|A_{ui} X_u^x|)} r_s(A_{ui} \cdot X_u^x) \mid \mathcal{F}_n \right] \\ &= \frac{e^{s \log |G_u x|} m P_s r_s(X_u^x)}{[m \kappa(s)]^{n+1} r_s(x)} \\ &= \frac{e^{s \log |G_u x|} r_s(X_u^x)}{[m \kappa(s)]^n r_s(x)}. \end{aligned}$$

Therefore

$$W_{s,n}^x = \sum_{u \in \mathbb{T}_n} H(n, G_u x) = \sum_{u \in \mathbb{T}_n} \frac{e^{s \log |G_u x|} r_s(X_u^x)}{[m \kappa(s)]^n r_s(x)}, \quad n \geq 0$$

is a positive martingale, so that the limit

$$W_s^x = \lim_{n \rightarrow \infty} W_{s,n}^x$$

exists a.s. with values in \mathbb{R}_+ . For $u \in \mathbb{U}$, denote

$$W_{s,1}^{X_u^x} = \sum_{i=1}^{N_u} \frac{e^{s \log |A_{ui} X_u^x|} r_s(A_{ui} \cdot X_u^x)}{m \kappa(s) r_s(X_u^x)},$$

which represents the first term of the fundamental martingale corresponding to the branching process starting from the particle u ; in particular for $u = \emptyset$, $W_{s,1}^{X_\emptyset^x}(\emptyset) = W_{s,1}^x$ with the usual convention that $X_\emptyset^x = x$.

For fixed $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$, the spectral gap property (4.7) allows to define a probability measure \mathbb{Q}_s^x on (Ω, \mathcal{F}) such that for any $n \in \mathbb{N}$ and any bounded and measurable function h on $(\mathcal{S} \times \mathbb{R})^{n+1}$,

$$\begin{aligned} & \mathbb{E} \left[\frac{e^{s S_n^x} r_s(X_n^x)}{\kappa^n(s) r_s(x)} h(X_0^x, S_0^x, \dots, X_n^x, S_n^x) \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} [h(X_0^x, S_0^x, \dots, X_n^x, S_n^x)], \end{aligned} \quad (4.43)$$

where $\mathbb{E}_{\mathbb{Q}_s^x}$ denotes the expectation with respect to \mathbb{Q}_s^x . See [25, 28, 46] for $s \geq 0$, and [83] for $s < 0$.

With the mean-harmonic function H specified above and the probability measure \mathbb{Q}_s^x introduced here, from [19, Theorem 2.1] we obtain the following result for the non-degeneracy of the limit W_s^x . We use the usual notation that $x \wedge y = \min\{x, y\}$, and we denote by $1_n = (1, \dots, 1) \in \mathbb{N}^{*n}$ the sequence of length n whose components are all equal to 1.

Lemma 4.4.1. *For $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$, we have:*

(i) $\mathbb{E}[W_s^x] = 1$ if

$$\sum_{n=1}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \left(H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} \wedge 1 \right) \middle| \mathcal{F}_n \right] < \infty \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.44)$$

(ii) $\mathbb{E}[W_s^x] = 0$ if either

$$\limsup_{n \rightarrow \infty} H(n, G_{1_n} x) = \infty \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.45)$$

or for all $y > 0$,

$$\sum_{n=1}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \mathbb{1}_{\{H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y\}} \middle| \mathcal{F}_n \right] = \infty \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.46)$$

4.4.1 Proof of Theorem 4.2.2

Sufficient condition

We assume conditions (4.14) and (4.15), together with **B1** and **B2**, and we will prove (4.44) which, by Lemma 4.4.1, will imply that $\mathbb{E}[W_s^x] = 1$. By the definition of \mathbb{Q}_s^x (cf. (4.43)), we have

$$\begin{aligned} & \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \left((H(n, G_{1_n}x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \right) \middle| \mathcal{F}_n \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{N_{1_n}} \frac{e^{s \log |A_{1_n i} X_{1_n}^x|} r_s(A_{1_n i} \cdot X_{1_n}^x)}{m \kappa(s) r_s(X_{1_n}^x)} \left((H(n, G_{1_n}x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \right) \middle| \mathcal{F}_n \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} \left[(H(n, G_{1_n}x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \middle| \mathcal{F}_n \right]. \end{aligned}$$

By the extended Borel-Cantelli Lemma (see [73, p. 151]), we have

$$\begin{aligned} & \left\{ \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[(H(n, G_{1_n}x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \middle| \mathcal{F}_n \right] < \infty \right\} \\ & \stackrel{\mathbb{Q}_s^x\text{-a.s.}}{=} \left\{ \sum_{n=1}^{\infty} \left((H(n, G_{1_n}x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \right) < \infty \right\}. \end{aligned} \quad (4.47)$$

We shall prove that \mathbb{Q}_s^x -a.s, the term $H(n, G_{1_n}x) \rightarrow 0$ exponentially and $W_{s,1}^{X_{1_n}^x} \rightarrow \infty$ subexponentially. This will imply that the two series in (4.47) converge \mathbb{Q}_s^x -a.s., and thus conclude the proof of (4.44).

We first prove that $H(n, G_{1_n}x) \rightarrow 0$ \mathbb{Q}_s^x -a.s. with an exponential rate. We start by rewriting $H(n, G_{1_n}x)$ in form

$$H(n, G_{1_n}x) = \frac{r_s(X_{1_n}^x)}{r_s(x)} \exp \left\{ n \left[s \left(\frac{S_{1_n}^x}{n} - q_s \right) + \left(s q_s - \log[m \kappa(s)] \right) \right] \right\}. \quad (4.48)$$

Recall that the function $r_s(\cdot)$ is strictly positive and Hölder continuous on the compact set \mathcal{S} . It is therefore bounded from above and from below by two positive constants. By the strong law of large numbers for $S_{1_n}^x$ under \mathbb{Q}_s^x (see [83, Proposition 3.12], [25, Theorem 6.1], [46, Theorem 3.10]),

$$\lim_{n \rightarrow \infty} \frac{S_{1_n}^x}{n} = q_s \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.49)$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log H(n, G_{1_n} x) = sq_s - \log[m \kappa(s)] \quad \mathbb{Q}_s^x \text{-a.s.} \quad (4.50)$$

By hypothesis $sq_s - \log[m \kappa(s)] < 0$, so $H(n, G_{1_n} x) \rightarrow 0$ exponentially \mathbb{Q}_s^x -a.s.

We next prove that $W_{s,1}^{X_{1_n}^x}$ grows to infinity subexponentially \mathbb{Q}_s^x -a.s., in the sense that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log_+ W_{s,1}^{X_{1_n}^x} = 0 \quad \mathbb{Q}_s^x \text{-a.s.}$$

By the lemma of Borel-Cantelli, it is enough to prove that

$$\sum_{n=0}^{\infty} \mathbb{Q}_s^x \left(\log_+ W_{s,1}^{X_{1_n}^x} > \varepsilon n \right) < \infty \quad \forall \varepsilon > 0. \quad (4.51)$$

By the definition of \mathbb{Q}_s^x and Fubini's Theorem, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{Q}_s^x \left(\log_+ W_{s,1}^{X_{1_n}^x} > \varepsilon n \right) &= \sum_{n=0}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \mathbb{1}_{\{\log_+ W_{s,1}^{X_{1_n}^x} > \varepsilon n\}} \right] \\ &\leq \mathbb{E} \left[\sum_{n=0}^{\infty} \max_{x \in \mathcal{S}} W_{s,1}^x \mathbb{1}_{\{\log_+ \max_{x \in \mathcal{S}} W_{s,1}^x > \varepsilon n\}} \right] \\ &\leq \frac{1}{\varepsilon} \mathbb{E} \left[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x + 1 \right], \end{aligned}$$

which is finite by hypothesis (4.15). Therefore, the property (4.51) is proved.

Necessary condition

It suffices to prove that if either $\Lambda^*(q_s) - \log m \geq 0$ or $\mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x \right] = \infty$, then $\mathbb{E} W_s^x = 0$ for all $x \in \mathcal{S}$. In the following we consider three cases.

Case 1. Suppose that $\Lambda^*(q_s) - \log m > 0$. Then by (4.48) and (4.49) we see that

$$\lim_{n \rightarrow \infty} H(n, G_{1_n} x) = \infty, \quad \mathbb{Q}_s^x \text{-a.s.} \quad (4.52)$$

This implies $\mathbb{E}[W_s^x] = 0$ by Lemma 4.4.1.

Case 2. Suppose that $\Lambda^*(q_s) - \log m = 0$. Then by (4.48),

$$H(n, G_{1_n} x) = \frac{r_s(X_{1_n}^x)}{r_s(x)} e^{s(S_{1_n}^x - nq_s)}.$$

Since r_s is bounded from below and from above by two positive constants, using Lemma 4.4.2 below, we see that (4.52) still holds, which implies $\mathbb{E}[W_s^x] = 0$ by Lemma 4.4.1, just as in the first case.

Here we have used the following law of iterated logarithm for products of random matrices. For $s = 0$, it was established in [63, Theorem 5].

Lemma 4.4.2. *Let $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$. Under conditions **B1** and **B2**,*

$$\limsup_{n \rightarrow \infty} \frac{S_n^x - nq_s}{\sigma_s \sqrt{2n \log \log n}} = 1 \quad \mathbb{Q}_s^x\text{-a.s.}$$

This lemma can be proved in the same way as in the proof of Theorem 5 of [63], using Berry-Esseen's bound for S_n^x under the changed measure \mathbb{Q}_s^x established in [83, Theorem 2.1] for $s \in (-\eta_1, 0]$ and in [28, Theorem 8.1] for $s \in (0, s_\infty)$. Since the proof is very similar, we omit the details.

Case 3. Assume that $\mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x \right] = \infty$ and $\Lambda^*(q_s) - \log m < 0$. We shall prove that (4.46) holds for all $y > 0$. By the definition of \mathbb{Q}_s^x , we have

$$\mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \mathbb{1}_{\{H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y\}} \mid \mathcal{F}_n \right] = \mathbb{E}_{\mathbb{Q}_s^x} \left[\mathbb{1}_{\{H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y\}} \mid \mathcal{F}_n \right].$$

By the extended Borel- Cantelli lemma, we get, for $y > 0$,

$$\left\{ \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\mathbb{1}_{\{H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y\}} \mid \mathcal{F}_n \right] = \infty \right\} \\ \stackrel{\mathbb{Q}_s^x\text{-a.s.}}{=} \limsup_{n \rightarrow \infty} \left\{ H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y \right\}.$$

Therefore, for $y > 0$, (4.46) holds if

$$\mathbb{Q}_s^x \left(\limsup_{n \rightarrow \infty} \left\{ H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y \right\} \right) = 1. \tag{4.53}$$

By (4.50), we see that (4.53) achieves if

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log_+ W_{s,1}^{X_{1_n}^x} = +\infty, \quad \mathbb{Q}_s^x\text{-a.s.},$$

which is equivalent to

$$\mathbb{Q}_s^x \left(\limsup_{n \rightarrow \infty} \underbrace{\left\{ \log_+ W_{s,1}^{X_{1n}^x} > Mn \right\}}_{=: B_{n+1}} \right) = 1 \quad \forall M > 0. \quad (4.54)$$

We see that (4.54) follows from

$$\mathbb{Q}_s^x \left(\sum_{n=0}^{\infty} \mathbb{1}_{B_{n+1}} = \infty \right) = 1. \quad (4.55)$$

To prove (4.55), notice that by the extended Borel-Cantelli lemma we have

$$\left\{ \sum_{n=0}^{\infty} \mathbb{1}_{B_{n+1}} < \infty \right\} \stackrel{\mathbb{Q}_s^x\text{-a.s.}}{=} \left\{ \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} [\mathbb{1}_{B_{n+1}} | \mathcal{F}_n] < \infty \right\}. \quad (4.56)$$

By the definition of \mathbb{Q}_s^x and Fubini's theorem, we have, \mathbb{Q}_s^x -a.s.,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} [\mathbb{1}_{B_{n+1}} | \mathcal{F}_n] &= \sum_{n=0}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1n}^x} \mathbb{1}_{\{\log_+ W_{s,1}^{X_{1n}^x} > Mn\}} | \mathcal{F}_n \right] \\ &\geq \sum_{n=0}^{\infty} \mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \mathbb{1}_{\{\log_+ \min_{x \in \mathcal{S}} W_{s,1}^x > Mn\}} | \mathcal{F}_n \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{\infty} \min_{x \in \mathcal{S}} W_{s,1}^x \mathbb{1}_{\{\log_+ \min_{x \in \mathcal{S}} W_{s,1}^x > Mn\}} \right] \\ &\geq \frac{1}{M} \mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x - 1 \right] \\ &= +\infty, \end{aligned}$$

where the second equality holds since $\min_{x \in \mathcal{S}} W_{s,1}^x$ is independent of \mathcal{F}_n , and the last equality holds by hypothesis. Hence (4.55) follows from (4.56).

4.4.2 Proof of Corollary 4.2.3

We will need the following result which was established in [68] in a slightly weaker form. We use the convention that the empty sum is taken to be 0.

Lemma 4.4.3. *Let N, X_1, X_2, \dots be independent random variables with $N \in \mathbb{N}$, $X_i \in \mathbb{R}_+$,*

$\mathbb{P}(N = 0) < 1$ and $\mathbb{P}(X = 0) < 1$. Assume that all the X_i have the same law. Then

$$\mathbb{E}\left[\left(\sum_{i=1}^N X_i\right) \log_+ \left(\sum_{i=1}^N X_i\right)\right] < \infty$$

if and only if

$$\mathbb{E}(X_1 \log_+ X_1) < \infty \quad \text{and} \quad \mathbb{E}(N \log_+ N) < \infty.$$

Proof. The "if" part has been proved in [68]. The "only if" part is slightly stronger than that in [68]. Let us give a short proof which is different to that used in [68]. Since the function $f(x) = x \log_+ x$ (with $f(0) = 0$) is increasing, we have

$$\mathbb{E}\left[f\left(\sum_{i=1}^N X_i\right)\right] \geq \mathbb{E}\left(X_1 \log_+ X_1 \mathbf{1}_{\{N \geq 1\}}\right) = \mathbb{E}(X_1 \log_+ X_1) \mathbb{P}(N \geq 1).$$

Therefore $\mathbb{E}(X_1 \log_+ X_1) < \infty$. Together with $\mathbb{P}(X_1 = 0) < 1$, this implies that $c := \mathbb{E}X_1 \in (0, \infty)$. Since f is convex on \mathbb{R}_+ , by Jensen's inequality, we have

$$\mathbb{E}\left[f\left(\sum_{i=1}^N X_i\right) \middle| N\right] \geq f\left(\mathbb{E}\left[\sum_{i=1}^N X_i \middle| N\right]\right) = f(cN) = (cN) \log_+(cN).$$

Taking expectation, we get $\mathbb{E}\left[\sum_{i=1}^N X_i \log_+ \left(\sum_{i=1}^N X_i\right)\right] \geq \mathbb{E}[(cN) \log_+(cN)]$. Hence we obtain $\mathbb{E}[(cN) \log_+(cN)] < \infty$, which is equivalent to $\mathbb{E}N \log_+ N < \infty$. \square

Then the three conditions (4.15), (4.18) and (4.19) are equivalent, and (4.16) holds for all $x \in \mathcal{S}$ if and only if (4.14) and (4.19) hold. Moreover, if (4.16) holds for some $x \in \mathcal{S}$, then it holds for all $x \in \mathcal{S}$.

Proof of Corollary 4.2.3. (1) Note that for $s \in (-\eta_1, s_\infty)$, the function $r_s(\cdot)$ is strictly positive and continuous on the compact set \mathcal{S} . It is therefore bounded from above and from below by two positive constants. From the definition of $W_{s,1}^x$ and $\|\cdot\|$, we observe that

$$\mathbb{E}\left[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x\right] \leq \mathbb{E}\left[\sum_{i=1}^N c \|A_i\|^s \log_+ \left(\sum_{i=1}^N c \|A_i\|^s\right)\right].$$

Therefore by Lemma 4.4.3, (4.19) implies (4.15). This ends the proof of the first part of Corollary 4.2.3.

(2) By Theorem 4.2.2, to prove the second part of Corollary 4.2.3, it is enough to show that (4.15), (4.18) and (4.19) are all equivalent.

Now we prove the equivalence according to the scheme: (4.15) \Rightarrow (4.18) \Rightarrow (4.19) \Rightarrow (4.15). The implication (4.15) \Rightarrow (4.18) is obvious; the implication (4.19) \Rightarrow (4.15) is just proved above in part (1). So we only need to show that (4.18) \Rightarrow (4.19). Set for $k \geq 1$,

$$\bar{A}_k = \max_{1 \leq i, j \leq d} |A_k(i, j)| \quad \text{and} \quad \underline{A}_k = \min_{1 \leq i, j \leq d} |A_k(i, j)|.$$

Since all norms on \mathbb{R}^d are equivalent, we can take the norm $|x| = |x_1| + |x_2| + \dots + |x_d|$. Then for $k \geq 1$ and $x \in \mathcal{S}$,

$$d\underline{A}_k \leq |A_k x| = \sum_{1 \leq i, j \leq d} |A_k(i, j)x_j| \leq d\bar{A}_k.$$

Since the function r_s on \mathcal{S} is bounded from above and from below by two positive constants, this implies that for some constant $c_1 > 0$,

$$\mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x \right] \geq \mathbb{E} \left[\sum_{k=1}^N c_1 \underline{A}_k^s \log_+ \left(\sum_{k=1}^N c_1 \underline{A}_k^s \right) \right].$$

Remark that under the Furstenberg-Kesten condition (4.20), $\mathbb{E} [\underline{A}_1^s \log_+ \underline{A}_1^s] < \infty$ if and only if $\mathbb{E} [\|A_1\|^s \log_+ \|A_1\|^s] < \infty$. Therefore, by Lemma 4.4.3 and the above inequality, (4.18) implies (4.19). \square

4.5 Proof of Theorem 4.2.4

In this section we will prove Theorem 4.2.4, the precise large deviation asymptotic of Bahadur-Rao type on the counting measure Z_n^x , using a uniform local limit theorem for products of random matrices that we recently established in [22].

4.5.1 Auxiliary results

In the proof of Theorem 4.2.4 we make use of the following three assertions. The first one is a local limit theorem for products of random matrices under the changed measure \mathbb{Q}_s^x (see Proposition 4.5.1). The second is an exponential bound of the large deviation probability of the products of random matrices under \mathbb{Q}_s^x (see Proposition 4.5.2). The

third gives a relationship between moment conditions on $W_{s,1}^x$ and on $W_{s,*}^x := \sup_n W_{s,n}^x$ (see Proposition 4.5.3).

We start with a uniform local limit theorem for products of random matrices under the changed measure \mathbb{Q}_s^x . Under the initial measure (when $s = 0$), it has been established in [22].

Proposition 4.5.1. *Under the conditions of Theorem 4.2.4, we have, for any continuous function f on \mathcal{S} and any directly Riemann integrable function h on \mathbb{R} ,*

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma_s \sqrt{n} \mathbb{E}_{\mathbb{Q}_s^x} [f(X_n^x) h(y + S_n^x - n\Lambda'(s))] - \pi_s(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{z-y}{\sigma_s \sqrt{n}}\right) dz \right| = 0, \quad (4.57)$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density function of the standard normal law.

Proof. For $\lambda > 0$ sufficiently small, we introduce the Banach space $\mathcal{B}_\lambda = \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\lambda < +\infty\}$, where

$$\|f\|_\lambda := \|f\|_\infty + |f|_\lambda,$$

with

$$\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|, \quad |f|_\lambda := \sup_{x,y \in \mathcal{S}, x \neq y} \frac{|f(x) - f(y)|}{\mathbf{d}^\lambda(x,y)}.$$

For $s \in (-\eta_1, s_\infty)$ and $t \in \mathbb{R}$, define the perturbed operator $R_{s,it}$ on \mathcal{B}_λ as follows: for any $\varphi \in \mathcal{B}_\lambda$,

$$R_{s,it}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it[S_1^x - \Lambda'(s)]} \varphi(X_1^x) \right], \quad x \in \mathcal{S}.$$

By induction, it follows that for any $n \geq 1$,

$$R_{s,it}^n \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it[S_n^x - n\Lambda'(s)]} \varphi(X_n^x) \right], \quad x \in \mathcal{S}.$$

For properties of this operator, we refer the reader to [82] for $s \in (0, s_\infty)$ and [83] for $s \in (-\eta_1, 0]$. Since the proof of Theorem 4.2.4 is quite similar to that of Theorem 2.2(1) in [22], we will not give the details here. The only difference is that, instead of the properties of the operator $R_{0,it}$ used in the proof in [22], here we use the properties of the

operator $R_{s,it}$ proved in [82, 83].

□

We next present an exponential bound of the large deviation probability of the products of random matrices under \mathbb{Q}_s^x . For $s = 0$, it has been established in [82].

Proposition 4.5.2. *Assume the conditions of Theorem 4.2.4. Let $\varepsilon > 0$. There are $C > 0$ and $0 < c < 1$ such that for all $n \geq 1$,*

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(\frac{|S_n^x - n\Lambda'(s)|}{n} > \varepsilon \right) \leq Cc^n. \quad (4.58)$$

Proof. By the monotonicity in ε of the large deviation probability, it is clear that it suffices to prove the inequality for $\varepsilon > 0$ small enough. By the formula of the changed measure (4.43), for any nonnegative and Borel function φ and any point $t \in (-\eta_1, s_\infty)$, we have

$$\mathbb{E}_{\mathbb{Q}_s^x} \varphi(S_n^x) = \frac{\kappa(t)^n r_t(x)}{\kappa(s)^n r_s(x)} \mathbb{E}_{\mathbb{Q}_t^x} \left[e^{-(t-s)S_n^x} (r_t^{-1} r_s)(X_n^x) \varphi(S_n^x) \right].$$

Take $\varphi(x) = \mathbb{1}_{(n\Lambda'(s)+n\varepsilon, +\infty)}(x)$. Because $\Lambda''(s) > 0$ for all $s \in (-\eta_1, s_\infty)$ and $\Lambda'(s)$ is continuous in s , for $\varepsilon > 0$ small enough, there is $t \in (-\eta_1, s_\infty)$ with $t > s$ such that $\Lambda'(t) = \Lambda'(s) + \varepsilon$. Hence

$$\begin{aligned} & \mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) > n\varepsilon \right) \\ &= \frac{\kappa(t)^n r_t(x)}{\kappa(s)^n r_s(x)} \mathbb{E}_{\mathbb{Q}_t^x} \left[\frac{e^{-(t-s)S_n^x} r_s(X_n^x)}{r_t(X_n^x)} \mathbb{1}_{\{S_n^x - n\Lambda'(s) > n\varepsilon\}} \right] \\ &= e^{n[-(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s)]} \frac{r_t(x)}{r_s(x)} \mathbb{E}_{\mathbb{Q}_t^x} \left[f(X_n^x) h(S_n^x - n\Lambda'(t)) \right], \end{aligned} \quad (4.59)$$

where $f(x) = \frac{r_s(x)}{r_t(x)}$ and $h(x) = e^{-(t-s)x} \mathbb{1}_{\{x > 0\}}$. Notice that $h(x) \leq 1$ and that $f(x)$ is bounded from below and above by two positive constants because r_t and r_s are continuous and strictly positive on the compact set \mathcal{S} . Therefore from (4.59), we see that there exists a constant $C_1 > 0$ such that for all $n \geq 1$,

$$\mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) > n\varepsilon \right) \leq C_1 e^{n[-(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s)]}. \quad (4.60)$$

We now prove that

$$-(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s) < 0. \tag{4.61}$$

To do this, we consider the function

$$\psi(y) = -y\Lambda'(s+y) + \Lambda(s+y) - \Lambda(s), \quad y \in [0, s_\infty - s],$$

which is continuous on the interval $[0, s_\infty - s]$. For $y \in (0, s_\infty - s)$, $\psi'(y) = -y\Lambda''(s+y) < 0$, so that $\psi(y) < \psi(0) = 0$. With $y = t - s$, this implies (4.61). From (4.60) and (4.61), we see that for all $n \geq 1$,

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) > n\varepsilon \right) \leq C_1 c_1^n, \tag{4.62}$$

where $c_1 = \exp \left\{ -(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s) \right\} < 1$. In the same way, if we take $\varphi(x) = \mathbb{1}_{(-\infty, n\Lambda'(s) - n\varepsilon)}(x)$ and $t \in (-\eta_1, s)$ such that $\Lambda'(t) = \Lambda'(s) - \varepsilon$, then there are constants $C_2 > 0$ and $0 < c_2 < 1$ such that

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) < -n\varepsilon \right) \leq C_2 c_2^n. \tag{4.63}$$

The conclusion of the proposition follows from (4.62) and (4.63). □

We finally establish a relationship between moment conditions on $W_{s,1}^x$ and on $W_{s,*}^x$.

Proposition 4.5.3. *Assume the conditions of Theorem 4.2.4. Then*

$$\sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_{s,*}^x + 1) \log^\delta (W_{s,*}^x + 1) \right] < \infty.$$

For the proof, we will adapt the approach of Biggins [15] on the classical branching random walk. The following recursive relations on $W_{s,n}^x$ and W_s^x will be used. First, it can be easily seen that for $1 \leq k \leq n$,

$$W_{s,n}^x = \sum_{u \in \mathbb{T}_k} H(k, G_u x) W_{s,n-k}^{X_u^x}, \quad \text{where } H(k, G_u x) = \frac{e^{sS_u^x} r_s(X_u^x)}{[m \kappa(s)]^k r_s(x)}. \tag{4.64}$$

From this recursive relation on $W_{s,n}^x$, taking $n \rightarrow \infty$ we obtain the following recursive

relation on W_s^x : for $k \geq 1$,

$$W_s^x = \sum_{u \in \mathbb{T}_k} H(k, G_u x) W_s^{X_u^x}, \quad (4.65)$$

by our notation. The proof of Proposition 4.5.3 will be done with the help of three lemmas.

Lemma 4.5.4. *Assume the conditions of Theorem 4.2.4. Then*

$$\sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_s^x + 1) \log^\delta (W_s^x + 1) \right] < \infty. \quad (4.66)$$

Proof of Lemma 4.5.4. Let $\frac{3}{2} < \delta < 2$ and

$$h(u) := \begin{cases} c_0 u & \text{for } 0 < u \leq x_0, \\ c_1 + c_2 \log^\delta u & \text{for } u \geq x_0 > 1, \end{cases}$$

where x_0, c_0, c_1 and c_2 are constants with $x_0 > 1, c_0, c_2 > 0$, which make h concave (and hence subadditive) and increasing. Then

$$h(uv) \leq C(1 + \log_+^\delta u + \log_+^\delta v), \quad \forall u, v > 0, \quad (4.67)$$

when $C > 0$ is a large enough constant. Notice that to prove (4.66) we only need show that

$$\sup_{x \in \mathcal{S}} \mathbb{E}[W_s^x h(W_s^x)] < \infty. \quad (4.68)$$

Using (4.64) and the subadditivity of h , we have

$$\begin{aligned} & \mathbb{E}[W_{s,n+1}^x h(W_{s,n+1}^x) \mid \mathcal{F}_n] \\ &= \mathbb{E} \left[\left(\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} \right) h \left(\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} \right) \mid \mathcal{F}_n \right] \\ &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x} \right) \mid \mathcal{F}_n \right] \\ &\quad + \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \mid \mathcal{F}_n \right]. \end{aligned} \quad (4.69)$$

For the first term, we see that $H(n, G_t x)$ is \mathcal{F}_n -measurable, $W_{s,1}^{X_u^x}$ and $\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x}$

are conditionally independent given \mathcal{F}_n . Hence

$$\begin{aligned} & \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x} \right) \middle| \mathcal{F}_n \right] \\ &= \sum_{u \in \mathbb{T}_n} H(n, G_u x) \mathbb{E} \left[W_{s,1}^{X_u^x} \middle| \mathcal{F}_n \right] \mathbb{E} \left[h \left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x} \right) \middle| \mathcal{F}_n \right] \\ &\leq \sum_{u \in \mathbb{T}_n} H(n, G_u x) h \left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) \mathbb{E} \left[W_{s,1}^{X_t^x} \middle| \mathcal{F}_n \right] \right) \\ &\leq W_{s,n}^x h(W_{s,n}^x), \end{aligned}$$

where the last two inequalities hold by Jensen's inequality and the fact that $\mathbb{E} \left[W_{s,1}^{X_u^x} \middle| \mathcal{F}_n \right] = 1$ and h is a concave and increasing function. Therefore, from (4.69),

$$\begin{aligned} \mathbb{E}[W_{s,n+1}^x h(W_{s,n+1}^x)] &\leq \mathbb{E}[W_{s,n}^x h(W_{s,n}^x)] \\ &\quad + \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \right]. \end{aligned}$$

So by recurrence on n and Fatou's lemma, we obtain

$$\begin{aligned} \mathbb{E}[W_s^x h(W_s^x)] &\leq \liminf_{n \rightarrow \infty} \mathbb{E}[W_n^x(s) h(W_n^x(s))] \\ &\leq \mathbb{E}[W_{s,1}^x h(W_{s,1}^x)] + \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \right]. \end{aligned}$$

Hence to prove (4.68), it is enough to prove that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \right] < \infty. \quad (4.70)$$

Note that the hypothesis $\Lambda^*(q_s) - \log m < 0$ implies that there exists $b > 0$ such that

$$\Lambda^*(q_s) - \log m < -b < 0. \quad (4.71)$$

Since r_s is strictly positive and continuous on \mathcal{S} ,

$$d_1 := \frac{\max_{x \in \mathcal{S}} r_s(x)}{\min_{x \in \mathcal{S}} r_s(x)} < \infty. \quad (4.72)$$

We write

$$\mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \right] = I_{n,1}(x) + I_{n,2}(x),$$

where

$$I_{n,1}(x) = \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \mathbb{1}_{\{H(n, G_u x) \leq d_1 e^{-bn}\}} \right],$$

$$I_{n,2}(x) = \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \mathbb{1}_{\{H(n, G_u x) > d_1 e^{-bn}\}} \right].$$

Control of $I_{n,1}(x)$. Using the facts that h is an increase function and $\mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \right] = \mathbb{E}[W_{s,n}^x] = 1$, we have

$$\begin{aligned} I_{n,1}(x) &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(d_1 e^{-bn} W_{s,1}^{X_u^x} \right) \right] \\ &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \mathbb{E} \left[W_{s,1}^{X_u^x} h \left(d_1 e^{-bn} W_{s,1}^{X_u^x} \right) \middle| \mathcal{F}_n \right] \right] \\ &\leq \mathbb{E} \left[\check{W}_{s,1} h \left(d_1 e^{-bn} \check{W}_{s,1} \right) \right], \end{aligned}$$

where $\check{W}_{s,1} = \max_{x \in \mathcal{S}} W_{s,1}^x$. Set $U = \left\{ d_1 e^{-bn} \check{W}_{s,1} \geq x_0 \right\}$ and its complement $U^c = \left\{ d_1 e^{-bn} \check{W}_{s,1} < x_0 \right\}$. From the definition and the property (4.67) of h , we have

$$\begin{aligned} I_{n,1}(x) &\leq C \mathbb{E} \left[\check{W}_{s,1} \left(1 + \log_+^\delta \left(d_1 e^{-bn} \right) + \log_+^\delta \check{W}_{s,1} \right) \mathbb{1}_U \right] \\ &\quad + c_0 \mathbb{E} \left[\check{W}_{s,1} \left(d_1 e^{-bn} \check{W}_{s,1} \right) \mathbb{1}_{U^c} \right] \\ &\leq C \mathbb{E} \left[\check{W}_{s,1} \left(1 + \log_+^\delta \check{W}_{s,1} \right) \mathbb{1}_U \right] + d_1 c_0 e^{-bn} \mathbb{E} \left[\left(\check{W}_{s,1} \right)^2 \mathbb{1}_{U^c} \right] \\ &= I_{n,1,1}(x) + I_{n,1,2}(x). \end{aligned} \tag{4.73}$$

We observe that $d_1 e^{-bn} \check{W}_{s,1} \geq x_0$ leads to $n \leq \frac{1}{b} \log \frac{d_1 \check{W}_{s,1}}{x_0}$. Let $J = \left\lfloor \frac{1}{b} \log_+ \frac{d_1 \check{W}_{s,1}}{x_0} \right\rfloor$. By

Fubini's theorem and hypothesis (4.22), we have

$$\begin{aligned} \sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,1,1}(x) &\leq C \mathbb{E} \left[\sum_{n=1}^{\infty} \check{W}_{s,1} \left(1 + \log_+^{\delta} \check{W}_{s,1} \right) \mathbf{1}_{\{n \leq J\}} \right] \\ &\leq C_1 \mathbb{E} \left[\check{W}_{s,1} \left(1 + \log_+^{\delta} \check{W}_{s,1} \right) \log_+ \check{W}_{s,1} \right] < \infty, \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,1,2}(x) &\leq d_1 c_0 \mathbb{E} \left[\left(\check{W}_{s,1} \right)^2 \sum_{n=1}^{\infty} e^{-bn} \mathbf{1}_{\{n \geq J+1\}} \right] \\ &\leq C_2 \mathbb{E} \left[\left(\check{W}_{s,1} \right)^2 e^{-b(J+1)} \right] \\ &\leq C_3 \mathbb{E} \left[\check{W}_{s,1} \right] < \infty. \end{aligned}$$

Hence we conclude that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,1}(x) < \infty. \quad (4.74)$$

Control of $I_{n,2}(x)$. Using the property (4.67) of the function h , we obtain

$$\begin{aligned} I_{n,2}(x) &\leq C \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} \left(1 + \log_+^{\delta} W_{s,1}^{X_u^x} \right) \mathbf{1}_{\{H(n, G_u x) > d_1 e^{-bn}\}} \right] \\ &\quad + C \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} \log_+^{\delta} \left(H(n, G_u x) \right) \mathbf{1}_{\{H(n, G_u x) > d_1 e^{-bn}\}} \right]. \end{aligned}$$

From hypothesis (4.22), we get for each $u \in \mathbb{T}_n$,

$$\mathbb{E} \left[W_{s,1}^{X_u^x} \left(1 + \log_+^{\delta} W_{s,1}^{X_u^x} \right) \middle| \mathcal{F}_n \right] < \infty.$$

Taking $C_1 > 0$ sufficiently large, we have

$$\begin{aligned} I_{n,2}(x) &\leq C_1 \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \left\{ 1 + \log_+^{\delta} \left(H(n, G_u x) \right) \right\} \mathbf{1}_{\{H(n, G_u x) > d_1 e^{-bn}\}} \right] \\ &\leq C_1 \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \left\{ 1 + \log_+^{\delta} \left(H(n, G_u x) \right) \right\} \mathbf{1}_{\{s S_u^x > n(\log[m\kappa(s)] - b)\}} \right]. \end{aligned}$$

By the definition of \mathbb{Q}_s^x , we have

$$I_{n,2}(x) \leq C_1 \mathbb{E}_{\mathbb{Q}_s^x} \left[\left\{ 1 + \log_+^\delta \left(H(n, G_{1_n} x) \right) \right\} \mathbb{1}_{\{sS_{1_n}^x > n(\log[m\kappa(s)]-b)\}} \right], \quad (4.75)$$

where $1_n = (1, \dots, 1) \in \mathbb{N}^{*n}$ denotes the sequence of length n whose components are all equal to 1, and $H(n, G_{1_n} x) = \frac{e^{sS_{1_n}^x r_s(X_{1_n}^x)}}{[m\kappa(s)]^n r_s(x)}$ by our notation. It is easy to see that if $s = 0$ then $I_{n,2}(x) = 0$ by the choice of b . Hence we only consider the case where $s \neq 0$. We will prove that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\left(1 + \log_+^\delta H(n, G_{1_n} x) \right) \mathbb{1}_{\{sS_{1_n}^x > n(\log[m\kappa(s)]-b)\}} \right] < \infty. \quad (4.76)$$

Set $w = \frac{\log[m\kappa(s)]-b}{s}$, which is equal to $\frac{\log m + \Lambda(s) - b}{s}$. By (4.71), we see that $w > \Lambda'(s)$ if and only if $s > 0$. From Proposition 4.5.2, we have for some constants $0 < c < 1$, $C > 0$, and all $k \geq 1$,

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(sS_{1_n}^x > nsw \right) \leq \sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(\frac{|S_{1_n}^x - n\Lambda'(s)|}{n} > |w - \Lambda'(s)| \right) \leq Cc^n. \quad (4.77)$$

Hence, to prove (4.76), it suffices to show that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\log_+^\delta H(n, G_{1_n} x) \mathbb{1}_{\left\{ \frac{|S_{1_n}^x - n\Lambda'(s)|}{n} > |w - \Lambda'(s)| \right\}} \right] < \infty. \quad (4.78)$$

Using Hölder's inequality and (4.77), for $1 < p < \frac{2}{\delta}$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} \left[\log_+^\delta H(n, G_{1_n} x) \mathbb{1}_{\left\{ \frac{|S_{1_n}^x - n\Lambda'(s)|}{n} > |w - \Lambda'(s)| \right\}} \right] \\ & \leq \left(\mathbb{E}_{\mathbb{Q}_s^x} \log_+^{\delta p} H(n, G_{1_n} x) \right)^{\frac{1}{p}} \left[\mathbb{Q}_s^x \left(\frac{|S_{1_n}^x - n\Lambda'(s)|}{n} > |w - \Lambda'(s)| \right) \right]^{\frac{1}{q}} \\ & \leq \left(\mathbb{E}_{\mathbb{Q}_s^x} \log_+^{\delta p} H(n, G_{1_n} x) \right)^{\frac{1}{p}} (Cc^n)^{\frac{1}{q}}. \end{aligned} \quad (4.79)$$

By (4.48) and (4.72), we have

$$\mathbb{E}_{\mathbb{Q}_s^x} \log_+^{\delta p} H(n, G_{1_n} x) \leq C_1 (n^{\delta p} + \mathbb{E}_{\mathbb{Q}_s^x} |S_{1_n}^x - nq_s|^{\delta p}),$$

where C_1 is independent of x . Since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s^x} \left(S_{1_n}^x - nq_s \right)^2 = \sigma_s^2,$$

(see [28, Lemma 7.1] for $s > 0$ and [83, Proposition 3.14] for $s \leq 0$), we get

$$\mathbb{E}_{\mathbb{Q}_s^x} \log_+^{\delta p} H(n, G_{1_n} x) \leq C_1 n^2.$$

Combining this with (4.79) gives

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\log_+^{\delta} H(n, G_{1_n} x) \mathbf{1}_{\left\{ \frac{|S_{1_n}^x - nq_s|}{n} > |w - \Lambda'(s)| \right\}} \right] < C^{\frac{1}{q}} C_1^{\frac{1}{p}} \sum_{n=1}^{\infty} n^{\frac{2}{p}} c^{\frac{n}{q}}.$$

This gives (4.78), which implies (4.76).

By (4.75) and (4.76),

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,2}(x) < \infty.$$

Hence Lemma 4.5.4 is proved. \square

Lemma 4.5.5. *Assume the conditions of Theorem 4.2.4. For any $\varepsilon > 0$, there exists a constant $B > 0$ such that for any $x \in \mathcal{S}$ and any $n \geq 0$,*

$$\mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > -\varepsilon \mid \mathcal{F}_n \right) \geq B.$$

Proof of Lemma 4.5.5. Let $\varepsilon > 0$, $x \in \mathcal{S}$ and $n \geq 0$. Let $T > 0$. For $u \in \mathbb{T}_n$, set $Y_u = W_s^{X_u} - 1$ and

$$Y_u^T := \begin{cases} Y_u & \text{if } Y_u < T \\ T & \text{if } Y_u \geq T. \end{cases}$$

Then $Y_u^T \leq Y_u$, and

$$\begin{aligned} \mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > -\varepsilon \mid \mathcal{F}_n \right) &= \mathbb{P} \left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u > -\varepsilon \mid \mathcal{F}_n \right) \\ &\geq \mathbb{P} \left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \mid \mathcal{F}_n \right). \end{aligned} \quad (4.80)$$

Using the facts that $Y_u^T \leq T$ and $\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) = 1$, we have

$$\begin{aligned}
& \mathbb{E} \left[\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \middle| \mathcal{F}_n \right] \\
&= \mathbb{E} \left[\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \right) \mathbf{1}_{\left\{ \frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \leq -\varepsilon \right\}} \middle| \mathcal{F}_n \right] \\
&\quad + \mathbb{E} \left[\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \right) \mathbf{1}_{\left\{ \frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \right\}} \middle| \mathcal{F}_n \right] \\
&\leq (-\varepsilon) + T \mathbb{P} \left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \middle| \mathcal{F}_n \right). \tag{4.81}
\end{aligned}$$

We now prove that the expectation in the above display is uniformly bounded from below by $-\varepsilon/2$ when T is large enough. By Theorem 4.2.2, for each $u \in \mathbb{T}_n$, Y_u satisfies

$$\mathbb{E}(Y_u | \mathcal{F}_n) = \mathbb{E} \left[W_s^{X_u^x} | \mathcal{F}_n \right] - 1 = 0.$$

Using this and the definition of Y_u^T , we have

$$\begin{aligned}
\mathbb{E} \left[Y_u^T \middle| \mathcal{F}_n \right] &= \mathbb{E} \left[Y_u \mathbf{1}_{\{Y_u < T\}} \middle| \mathcal{F}_n \right] + \mathbb{E} \left[T \mathbf{1}_{\{Y_u \geq T\}} \middle| \mathcal{F}_n \right] \\
&= \mathbb{E} \left[Y_u (1 - \mathbf{1}_{\{Y_u \geq T\}}) \middle| \mathcal{F}_n \right] + \mathbb{E} \left[(T \mathbf{1}_{\{Y_u \geq T\}}) \middle| \mathcal{F}_n \right] \\
&= -\mathbb{E} \left[(Y_u - T)_+ \middle| \mathcal{F}_n \right],
\end{aligned}$$

where $(Y_u - T)_+ = \max(Y_u - T, 0)$. Therefore

$$\mathbb{E} \left[\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \middle| \mathcal{F}_n \right] = -\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) \mathbb{E} \left[(Y_u - T)_+ \middle| \mathcal{F}_n \right].$$

Now

$$\begin{aligned}
\mathbb{E} \left[(Y_u - T)_+ \middle| \mathcal{F}_n \right] &\leq \sup_{y \in S} \mathbb{E} \left[(W_s^y - 1 - T)_+ \middle| \mathcal{F}_n \right] = \sup_{y \in S} \mathbb{E} \left[(W_s^y - 1 - T)_+ \right] \\
&\leq \sup_{y \in S} \mathbb{E} \left[W_s^y \mathbf{1}_{\{W_s^y > T\}} \right] \xrightarrow{T \rightarrow +\infty} 0,
\end{aligned}$$

where the last step holds because by Lemma 4.66, the family of random variables $W_s^y, y \in$

S is uniformly integrable. Let $T > 0$ be sufficiently large such that

$$\sup_{y \in S} \mathbb{E} \left[W_s^y 1_{\{W_s^y > T\}} \right] < \varepsilon/2.$$

Then for all $x \in \mathcal{S}$ and $n \geq 0$,

$$\mathbb{E} \left[\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \middle| \mathcal{F}_n \right] > -\frac{1}{2} \varepsilon. \quad (4.82)$$

Therefore, from (4.81), we obtain

$$\mathbb{P} \left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \middle| \mathcal{F}_n \right) \geq \frac{\varepsilon}{2T}.$$

Hence, it follows from (4.80) that the inequality in the lemma holds with $B = \frac{\varepsilon}{2T}$. \square

Lemma 4.5.6. *Assume the conditions of Theorem 4.2.4. For any $0 < a < 1$, there exists a constant $B > 0$ such that for any $x \in \mathcal{S}$ and any $t > 0$,*

$$\mathbb{P}(W_s^x \geq at) \geq B \mathbb{P}(W_{s,*}^x \geq t) \geq B \mathbb{P}(W_s^x \geq t).$$

Proof of Lemma 4.5.6. The second inequality is evident. We now prove the first one. For $t > 0$, let

$$E_n = \{W_{s,n}^x \geq t, W_{s,k}^x < t \text{ for } 0 \leq k < n\}, \quad n \geq 1.$$

As E_n are pairwise disjoint sets, for each $a \in (0, 1)$ and each $t > 0$,

$$\mathbb{P}(W_s^x > at) \geq \sum_{n=1}^{\infty} \mathbb{P}(W_s^x > at \mid E_n) \mathbb{P}(E_n). \quad (4.83)$$

By (4.65), we have for each $a \in (0, 1)$ and each $t > 0$,

$$\begin{aligned} \mathbb{P}(W_s^x > at \mid E_n) &= \mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > \frac{at}{W_{s,n}^x} - 1 \mid E_n \right) \\ &\geq \mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > a - 1 \mid E_n \right), \end{aligned}$$

where the last step holds because $W_{s,n}^x \geq t$ on E_n . By using the fact that $E_n \in \mathcal{F}_n$ and

applying Lemma 4.5.5, we have

$$\mathbb{P}\left(\frac{W_s^x}{W_{s,n}^x} - 1 > a - 1 \mid E_n\right) = \mathbb{E}\left[\mathbb{P}\left(\frac{W_s^x}{W_{s,n}^x} - 1 > a - 1 \mid \mathcal{F}_n\right) \mid E_n\right] \geq B > 0,$$

where B is a constant independent of n . It follows from (4.83) that

$$\mathbb{P}(W_s^x > at) \geq B \sum_{n=1}^{\infty} \mathbb{P}(E_n) = B\mathbb{P}(W_{s,*}^x \geq t),$$

which proves the first inequality of the lemma. □

4.5.2 Proof of Theorem 4.2.4

By the definition of $\Lambda^*(q_s)$ and $Z_n^x(dy, dz)$, we have

$$\begin{aligned} I &:= \frac{e^{n\Lambda^*(q_s)}}{m^n r_s(x)} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) \\ &= \frac{e^{snq_s}}{[m \kappa(s)]^n r_s(x)} \sum_{u \in \mathbb{T}_n} f(X_u^x)g(S_u^x - nq_s). \end{aligned}$$

Set $h(z) = e^{-sz}g(z)$, $z \in \mathbb{R}$. For $n \geq 1$ and $0 \leq k \leq n$, we have the decomposition

$$\begin{aligned} I &= \frac{1}{[m \kappa(s)]^n r_s(x)} \sum_{u \in \mathbb{T}_n} f(X_u^x)h(S_u^x - nq_s)e^{sS_u^x} \\ &= \frac{1}{[m \kappa(s)]^n r_s(x)} \sum_{u \in \mathbb{T}_{k_n}} e^{sS_u^x} \sum_{v \in \mathbb{T}_{n-k_n}(u)} e^{sS_v^{X_u^x}} f(X_v^{X_u^x})h(S_v^{X_u^x} + S_u^x - nq_s) \\ &= \frac{1}{[m \kappa(s)]^k r_s(x)} \sum_{u \in \mathbb{T}_{k_n}} e^{sS_u^x} r_s(X_u^x) \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)h(z + S_u^x - nq_s)}{r_s(y)} W_{s,n-k_n}^{X_u^x}(dy, dz), \end{aligned}$$

where $W_{s,n-k_n}^{X_u^x}(dy, dz)$ is the probability measure defined as follows: for measurable sets $B \subset \mathcal{S}$ and $C \subset \mathbb{R}$,

$$W_{s,n-k}^{X_u^x}(B, C) = \sum_{v \in \mathbb{T}_{n-k}(u)} \frac{e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x})}{[m \kappa(s)]^{n-k} r_s(X_u^x)} \mathbf{1}_{\{X_v^{X_u^x} \in B, S_v^{X_u^x} \in C\}}.$$

Recall that for $u \in \mathbb{T}_{k_n}$, $0 \leq k_n \leq n$, $H(k_n, G_u x) = \frac{e^{s S_u^x r_s(X_u^x)}}{[m \kappa(s)]^{k_n} r_s(x)}$. From the preceding decomposition of I , we obtain

$$\sigma_s \sqrt{2\pi n} I - W_s^x \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz = A_n + B_n + C_n, \quad (4.84)$$

where

$$\begin{aligned} A_n &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \int_{S \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \\ &\quad \left(W_{s, n-k_n}^{X_u^x}(dy, dz) - \mathbb{E}_{k_n} [W_{s, n-k_n}^{X_u^x}(dy, dz)] \right), \\ B_n &= \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left\{ \sigma_s \sqrt{2\pi n} \int_{S \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \right. \\ &\quad \left. \mathbb{E}_{k_n} [W_{s, n-k_n}^{X_u^x}(dy, dz)] - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz \right\}, \\ C_n &= (W_{s, k_n}^x - W_s^x) \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz. \end{aligned}$$

We choose k_n as follows. Let β be such that $\frac{3}{2\delta} < \beta < 1$ and $\alpha > \frac{2}{\beta-1-1}$. For each n , let $j = j(n) \in \mathbb{N}$ be such that $j^{\alpha/\beta} \leq n < (j+1)^{\alpha/\beta}$; set $k_n = a_j = \lfloor j^\alpha \rfloor$. Then $k_n \sim n^\beta$.

We will prove that with the above choice of (k_n) , $A_n, B_n, C_n \rightarrow 0$ a.s. By the decomposition (4.84), this will imply Theorem 4.2.4. By the convergence of the martingale $W_{s, n}^x$ to W_s^x , we have clearly $C_n \rightarrow 0$, \mathbb{P} -a.s. It remains to show that $A_n \rightarrow 0$ and $B_n \rightarrow 0$ \mathbb{P} -a.s.

A) We first prove that $A_n \rightarrow 0$ \mathbb{P} -a.s. For $u \in \mathbb{T}_{k_n}$, write

$$\begin{aligned} Y_u &= \int_{S \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \left(W_{s, n-k_n}^{X_u^x}(dy, dz) - \mathbb{E}_{k_n} [W_{s, n-k_n}^{X_u^x}(dy, dz)] \right), \\ \tilde{Y}_u &= Y_u \mathbf{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} \end{aligned}$$

Then

$$\begin{aligned}
A_n &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) Y_u \\
&= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{E}_{k_n}[\tilde{Y}_u] + \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (Y_u - \tilde{Y}_u) \\
&\quad + \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (\tilde{Y}_u - \mathbb{E}_{k_n}[\tilde{Y}_u]) \\
&= A_{n,1} + A_{n,2} + A_{n,3},
\end{aligned}$$

with $A_{n,i}$ denoting the corresponding sum. We will show that each of these three terms tends to zero a.s. as $n \rightarrow \infty$. We divide the proof into 3 steps.

Step 1. We prove that $A_{n,1} \xrightarrow{n \rightarrow \infty} 0$ a.s. From the fact that $0 = \mathbb{E}_{k_n}[Y_u] = \mathbb{E}_{k_n}[\tilde{Y}_u] + \mathbb{E}_{k_n}[Y_u \mathbf{1}_{\{|Y_u| > \frac{1}{H(k_n, G_u x)}\}}]$, we have

$$|A_{n,1}| \leq \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{E}_{k_n} \left[|Y_u| \mathbf{1}_{\{|Y_u| > \frac{1}{H(k_n, G_u x)}\}} \right]. \quad (4.85)$$

Notice that for $C > 0$ large enough, $\sup_{(y,z) \in \mathcal{S} \times \mathbb{R}} \left| \frac{f(y)h(z)}{r_s(y)} \right| \leq C$. Using this and the fact that $\mathbb{E}_{k_n} \left[W_{s,n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R}) \right] = \mathbb{E}_{k_n} \left[W_{s,n-k_n}^{X_u^x} \right] = 1$, we obtain

$$|Y_u| \leq C \left(W_{s,n-k_n}^{X_u^x} + 1 \right). \quad (4.86)$$

This implies that

$$\begin{aligned}
\mathbb{E}_{k_n} \left[|Y_u| \mathbf{1}_{\{|Y_u| > \frac{1}{H(k_n, G_u x)}\}} \right] &\leq C \mathbb{E}_{k_n} \left[\left(W_{s,n-k_n}^{X_u^x} + 1 \right) \mathbf{1}_{\{W_{s,n-k_n}^{X_u^x} + 1 > \frac{1}{CH(k_n, G_u x)}\}} \right] \\
&=: \mathcal{E}(W_{s,n-k_n}^{X_u^x}).
\end{aligned}$$

Let

$$U = \{H(k_n, G_u x) > d_1 e^{-bn}\}, \quad (4.87)$$

where b is chosen as in the proof of Lemma 4.5.4. Denote its complement by $U^c =$

$\{H(k_n, G_u x) \leq d_1 e^{-bn}\}$. We have

$$|A_{n,1}| \leq I_{n,1} + I_{n,2}, \tag{4.88}$$

where

$$\begin{aligned} I_{n,1} &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathcal{E}(W_{s,n-k_n}^{X_u^x}) \mathbf{1}_U, \\ I_{n,2} &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathcal{E}(W_{s,n-k_n}^{X_u^x}) \mathbf{1}_{U^c}. \end{aligned}$$

For the first term $I_{n,1}$, by using the facts that $\mathcal{E}(W_{s,n-k_n}^{X_u^x}) \leq 2C$ and $U \subseteq \{sS_u^x > k_n(\log[m\kappa(s)] - b)\}$, and the definition of \mathbb{Q}_s^x , we have

$$\begin{aligned} \mathbb{E}I_{n,1} &\leq \sigma_s 2C \sqrt{2\pi n} \mathbb{E} \left[\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbf{1}_{\{sS_u^x > k_n(\log[m\kappa(s)] - b)\}} \right] \\ &= \sigma_s 2C \sqrt{2\pi n} \mathbb{Q}_s^x \left(sS_{1_{k_n}}^x > k_n(\log[m\kappa(s)] - b) \right) \end{aligned} \tag{4.89}$$

(recall that $1_{k_n} = (1, \dots, 1)$ is the sequence of length k_n whose components are all equal to 1). If $s = 0$, then $\mathbb{E}[I_{n,1}] = 0$ by the choice of b . Hence we only need to consider the case where $s \neq 0$, which we assume below. As in the proof of Lemma 4.5.4, setting $w = \frac{\log[m\kappa(s)] - b}{s}$, we have $w > \Lambda'(s)$ if and only if $s > 0$ by the choice of b . From Proposition 4.5.2, there are constants $0 < c < 1$ and $C_1 > 0$ such that

$$\begin{aligned} \mathbb{Q}_s^x \left(sS_{1_{k_n}}^x > k_n(\log[m\kappa(s)] - b) \right) &\leq \mathbb{Q}_s^x \left(\frac{|S_{1_{k_n}}^x - k_n \Lambda'(s)|}{k_n} > |w - \Lambda'(s)| \right) \\ &\leq C_1 c^{k_n}. \end{aligned}$$

Hence, by (4.89), we get

$$\sum_{n=1}^{\infty} \mathbb{E}[I_{n,1}] \leq C_2 \sum_{n=1}^{\infty} c^{n\beta} n^{\frac{1}{2}} < \infty. \tag{4.90}$$

For the second term $I_{n,2}$, we see that

$$\begin{aligned} \mathcal{E}(W_{s,n-k_n}^{X_u^x}) \mathbf{1}_{U^c} &\leq C \mathbf{1}_{U^c} \mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \mathbf{1}_{\{W_{s,n-k_n}^{X_u^x} + 1 \geq \frac{e^{bk_n}}{Cd_1}\}} \right] \\ &\leq \frac{C \mathbf{1}_{U^c}}{\log_+^\delta \left(\frac{e^{bk_n}}{Cd_1} \right)} \mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \log^\delta (W_{s,n-k_n}^{X_u^x} + 1) \right] \\ &\leq \frac{CC_3 \mathbf{1}_{U^c}}{(bk_n)^\delta}, \end{aligned}$$

where $C_3 = \sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_{s,*}^x + 1) \log^\delta (W_{s,*}^x + 1) \right] < \infty$ by Proposition 4.5.3. Therefore, since $k_n^\delta \sim j^{\alpha\delta} \sim n^{\beta\delta}$ and $\beta\delta > 3/2$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}[I_{n,2}] &\leq \sum_{n=1}^{\infty} \frac{CC_3 \sigma_s \sqrt{2\pi n}}{(bk_n)^\delta} \mathbb{E}[W_{s,k_n}^x] \\ &\leq C_4 \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{\beta\delta}} < \infty. \end{aligned} \tag{4.91}$$

Putting together (4.88), (4.90) and (4.91), we get

$$\sum_{n=1}^{\infty} \mathbb{E}[|A_{n,1}|] < \infty.$$

Thus $\sum_{n=0}^{\infty} |A_{n,1}| < \infty$ a.s., which implies that $A_{n,1} \xrightarrow{n \rightarrow \infty} 0$ a.s.

Step 2. We prove that $A_{n,2} \xrightarrow{n \rightarrow \infty} 0$ a.s. By the definition of \tilde{Y}_u and inequality (4.86), for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}_{k_n}(|A_{n,2}| > \varepsilon) &\leq \mathbb{P}_{k_n} \left(\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (Y_u - \tilde{Y}_u) \neq 0 \right) \\ &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n} \left(Y_u \neq \tilde{Y}_u \right) = \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n} \left(|Y_u| \geq \frac{1}{H(k_n, G_u x)} \right) \\ &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{k_n} \left[\mathbf{1}_{\{W_{s,n-k_n}^{X_u^x} + 1 > \frac{1}{CH(k_n, G_u x)}\}} \right] \\ &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{k_n} \left[CH(k_n, G_u x) (W_{s,n-k_n}^{X_u^x} + 1) \mathbf{1}_{\{W_{s,n-k_n}^{X_u^x} + 1 > \frac{1}{CH(k_n, G_u x)}\}} \right] \\ &= \frac{C}{\sigma_s \sqrt{2\pi n}} (I_{n,1} + I_{n,2}), \end{aligned}$$

where $I_{n,1}$ and $I_{n,2}$ are defined in Step 1. Therefore, from (4.90) and (4.91), we get

$$\sum_{n=1}^{\infty} \mathbb{P}(|A_{n,2}| > \varepsilon) \leq \sum_{n=1}^{\infty} \frac{C}{\sigma_s \sqrt{2\pi n}} \mathbb{E}(I_{n,1} + I_{n,2}) < \infty.$$

So by the lemma of Borel-Cantelli, we conclude that $A_{n,2} \rightarrow 0$ \mathbb{P} -a.s.

Step 3. We prove that $A_{n,3} \xrightarrow{n \rightarrow \infty} 0$ a.s. By Markov's inequality and von Bahr-Esseen's inequality [80, Theorem 2] or Marcinkiewicz-Zygmund's inequality [32, Theorem 1.5],

we have for any $\varepsilon > 0$ and $1 < \theta < 2$

$$\begin{aligned} \mathbb{P}_{k_n}(|A_{n,3}| > \varepsilon) &\leq \frac{(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \mathbb{E}_{k_n} \left[\left| \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (\tilde{Y}_u - \mathbb{E}_{k_n} \tilde{Y}_u) \right|^\theta \right] \\ &\leq \frac{2(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{E}_{k_n} [|\tilde{Y}_u - \mathbb{E}_{k_n} \tilde{Y}_u|^\theta] \\ &\leq \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta] \\ &\leq K_{n,1} + K_{n,2}, \end{aligned} \tag{4.92}$$

where

$$\begin{aligned} K_{n,1} &= \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{1}_U \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta], \\ K_{n,2} &= \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{1}_{U^c} \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta]. \end{aligned}$$

For the first term $K_{n,1}$, by the definition of \tilde{Y}_u , it is easy to see that

$$\begin{aligned} K_{n,1} &= \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{1}_U \mathbb{E}_{k_n} \left[|Y_u|^\theta \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} \right] \\ &\leq \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{1}_U \mathbb{E}_{k_n} \left[|Y_u| \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} \right], \end{aligned}$$

where the last inequality holds because $|Y_u| < \frac{1}{H(k_n, G_u x)}$, and hence we have $[H(k_n, G_u x)|Y_u|]^{\theta-1} <$

1. Using the facts that $\mathbb{E}_{k_n}[|Y_u|] \leq 2$ and $U \subset \{sS_u^x > k_n(\log[m\kappa(s)] - b)\}$, we get that

$$\mathbb{E}[K_{n,1}] \leq \frac{8(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \mathbb{Q}_s^x \left(sS_{1_{k_n}}^x > k_n(\log[m\kappa(s)] - b) \right).$$

Similar to (4.90), with the same reason we get, for some constants $0 < c < 1$ and $C > 0$,

$$\sum_{n=1}^{\infty} \mathbb{E}[K_{n,1}] < C \sum_{n=1}^{\infty} c^{n^\beta} n^{\frac{\theta}{2}} < \infty. \quad (4.93)$$

For the second term $K_{n,2}$, using the definition of \tilde{Y}_u , Fubini's theorem and inequality (4.86), we have

$$\begin{aligned} \mathbb{E}_{k_n}[|\tilde{Y}_u|^\theta] &= \int_0^\infty \theta y^{\theta-1} \mathbb{P}_{k_n}(|\tilde{Y}_u| > y) dy \\ &= \theta \int_0^\infty y^{\theta-1} \mathbb{P}_{k_n}(|Y_u| \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_{ux})}\}} > y) dy \\ &= \theta \mathbb{E}_{k_n} \left[\int_0^{\frac{1}{H(k_n, G_{ux})}} y^{\theta-1} \mathbb{1}_{\{|Y_u| \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_{ux})}\}} > y\}} dy \right] \\ &\leq \theta \int_0^{\frac{1}{H(k_n, G_{ux})}} y^{\theta-1} \mathbb{P}_{k_n}(W_{s, n-k_n}^{X_u^x} + 1 > \frac{y}{C}) dy. \end{aligned}$$

By the change of variables $z = \left(\frac{y}{C}\right)^{\theta-1}$, we obtain

$$\begin{aligned} \mathbb{E}_{k_n}[|\tilde{Y}_u|^\theta] &\leq \theta C^\theta \int_0^{[CH(k_n, G_{ux})]^{1-\theta}} z^{\frac{1}{\theta-1}} \mathbb{P}_{k_n} \left((W_{s, n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right) dz \\ &\leq \theta C^\theta \int_0^{[CH(k_n, G_{ux})]^{1-\theta}} \mathbb{E}_{k_n} \left[(W_{s, n-k_n}^{X_u^x} + 1) \mathbb{1}_{\left\{ (W_{s, n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right\}} \right] dz. \quad (4.94) \end{aligned}$$

We split the above integral according to $z \in [0, e]$ and $z \in (e, [CH(k_n, G_{ux})]^{1-\theta})$. For the integral over $z \in [0, e]$, we use

$$\mathbb{E}_{k_n} \left[(W_{s, n-k_n}^{X_u^x} + 1) \mathbb{1}_{\left\{ (W_{s, n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right\}} \right] \leq \mathbb{E}_{k_n}(W_{s, n-k_n}^{X_u^x} + 1) = 2.$$

For the integral over $z \in (e, [CH(k_n, G_u x)]^{1-\theta}]$, we use

$$\begin{aligned} & \mathbb{E}_{k_n} \left[(W_{s, n-k_n}^{X_u^x} + 1) \mathbb{1}_{\left\{ (W_{s, n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right\}} \right] \\ & \leq \frac{\theta-1}{\log^\delta z} \mathbb{E}_{k_n} \left[(W_{s, n-k_n}^{X_{k_n}^x} + 1) \log^\delta (W_{s, n-k_n}^{X_{k_n}^x} + 1) \right] \leq \frac{C_3(\theta-1)}{\log^\delta z}, \end{aligned}$$

where $C_3 = \sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_{s,*}^x + 1) \log^\delta (W_{s,*}^x + 1) \right] < \infty$ by Proposition 4.5.3. Hence, by (4.94),

$$\begin{aligned} \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta] & \leq 2\theta C^2 \theta e \\ & + \theta(\theta-1) C^\theta C_3 \mathbb{1}_{\{[CH(k_n, G_u x)]^{1-\theta} > e\}} \int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz. \end{aligned} \quad (4.95)$$

By the definition of $K_{n,2}$ and inequality (4.95), we get

$$\begin{aligned} K_{n,2} & \leq C_4 n^{\frac{\theta}{2}} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{1}_{U^c} \\ & \cdot \left(1 + \mathbb{1}_{\{[CH(k_n, G_u x)]^{1-\theta} > e\}} \int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \right). \end{aligned} \quad (4.96)$$

Now consider the integral in the last expression. Take a constant $1 < d_2 < e^b$, we see that on U^c , we have $\frac{1}{CH(k_n, G_u x)} \geq \frac{1}{C} r_s(x) e^{bk_n} \geq C_5 d_2^{k_n}$. Let $n_0 \in \mathbb{N}^*$ be large enough such that $[C_5 d_2^{k_{n_0}}]^{\theta-1} > e$. Using $\log z \geq 1$ for $z \in [e, [C_5 d_2^{k_n}]^{\theta-1}]$, and $\log z \geq (\theta-1) \log(C_5 d_2^{k_n})$ for $z > [C_5 d_2^{k_n}]^{\theta-1}$, we see that when $[CH(k_n, G_u x)]^{1-\theta} > e$ and $n > n_0$, we have

$$\begin{aligned} & \int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \\ & = \int_e^{[C_5 d_2^{k_n}]^{\theta-1}} \frac{1}{\log^\delta z} dz + \int_{[C_5 d_2^{k_n}]^{\theta-1}}^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \\ & \leq [C_5 d_2^{k_n}]^{\theta-1} + \frac{[CH(k_n, G_u x)]^{1-\theta}}{\left((\theta-1) \log(C_5 d_2^{k_n}) \right)^\delta} \\ & \leq C_6 \left(d_2^{(\theta-1)k_n} + \frac{[CH(k_n, G_u x)]^{1-\theta}}{k_n^\delta} \right). \end{aligned} \quad (4.97)$$

When $[CH(k_n, G_u x)]^{1-\theta} > e$ and $n \leq n_0$, the above inequality (4.97) remains valid by

choosing C_6 large enough such that $C_6 > k_{n_0}^\delta / C^{\theta-1}$, since for all $1 \leq n \leq n_0$,

$$\int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \leq [CH(k_n, G_u x)]^{1-\theta} \leq \frac{C_6 [H(k_n, G_u x)]^{1-\theta}}{k_n^\delta}.$$

From (4.96) and (4.97) we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}[K_{n,2}] &\leq C_6 \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \mathbb{E} \left[\sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{1}_{U^c} \right. \\ &\quad \left. \cdot \left(1 + d_2^{(\theta-1)k_n} + \frac{[H(k_n, G_u x)]^{1-\theta}}{k_n^\delta} \right) \right] \\ &\leq \frac{C_6}{r_s(x)^{\theta-1}} \sum_{n=1}^{\infty} \mathbb{E}[W_{s,k_n}^x] \frac{n^{\frac{\theta}{2}} (1 + d_2^{(\theta-1)k_n})}{e^{b(\theta-1)k_n}} + C_6 \sum_{n=1}^{\infty} \mathbb{E}[W_{s,k_n}^x] \frac{n^{\frac{\theta}{2}}}{k_n^\delta}, \end{aligned} \quad (4.98)$$

where the last inequality holds because on U^c (see Eq. (4.87)), $H(k_n, G_u x) \leq \frac{1}{r_s(x)e^{bk_n}}$, so that $H^\theta(k_n, G_u x) = H(k_n, G_u x)H^{\theta-1}(k_n, G_u x) \leq \frac{H(k_n, G_u x)}{[r_s(x)e^{bk_n}]^{\theta-1}}$ (for the second term we just use the identity $H^\theta(k_n, G_u x)[H(k_n, G_u x)]^{1-\theta} = H(k_n, G_u x)$). We choose θ sufficiently close to 1. Since $\mathbb{E}[W_{s,k_n}^x] = 1$, $k_n \sim j^\alpha \sim n^\beta$, $1 < d_2 < e^b$ and $\beta\delta > \frac{3}{2}$, the two series $\sum_{n=1}^{\infty} \frac{n^{\frac{\theta}{2}}(1+d^{(\theta-1)k_n})}{e^{b(\theta-1)k_n}}$ and $\sum_{n=1}^{\infty} \frac{n^{\frac{\theta}{2}}}{1+k_n^\delta}$ converge. Therefore from (4.98), we get

$$\sum_{n=1}^{\infty} \mathbb{E}K_{n,2} < \infty. \quad (4.99)$$

Combining (4.92), (4.93) and (4.99), we conclude that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}(|A_{n,3}| > \varepsilon) < \infty.$$

By the Lemma of Borel-Cantelli, it follows that $A_{n,3} \xrightarrow{n \rightarrow \infty} 0$ a.s.

B) We then prove that $B_n \rightarrow 0$ \mathbb{P} -a.s. By the definition of $W_{s,n-k_n}^{X_u^x}(dy, dz)$, the

branching property, and the definition of \mathbb{Q}_s^x (cf.(4.43)), we obtain successively,

$$\begin{aligned}
 & \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \mathbb{E}_{k_n} \left[W_{s, n-k_n}^{X_u^x}(dy, dz) \right] \\
 &= \frac{\mathbb{E}_{k_n} \left[\sum_{v \in \mathbb{T}_{n-k_n}(u)} e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x}) \frac{f(X_v^{X_u^x})}{r_s(X_v^{X_u^x})} h(S_v^{X_u^x} + S_u^x - nq_s) \right]}{[m \kappa(s)]^{n-k_n} r_s(X_u^x)} \\
 &= \frac{\mathbb{E}_{k_n} \left[e^{sS_{n-k_n}^{X_u^x}} r_s(X_{n-k_n}^{X_u^x}) \frac{f(X_{n-k_n}^{X_u^x})}{r_s(X_{n-k_n}^{X_u^x})} h(S_{n-k_n}^{X_u^x} + S_u^x - nq_s) \right]}{[\kappa(s)]^{n-k_n} r_s(X_u^x)} \\
 &= \mathbb{E}_{\mathbb{Q}_s^x} \left[\frac{f(X_{n-k_n}^{X_u^x})}{r_s(X_{n-k_n}^{X_u^x})} h\left(S_{n-k_n}^{X_u^x} - (n-k_n)q_s + S_u^x - k_n q_s\right) \middle| \mathcal{F}_{k_n} \right] \tag{4.100} \\
 &=: \text{RHS}_{(4.100)}.
 \end{aligned}$$

Hence, by the definition of B_n ,

$$\begin{aligned}
 B_n &= \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left[\sigma_s \sqrt{2\pi n} \text{RHS}_{(4.100)} - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz \right] \\
 &= \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \sqrt{\frac{2\pi n}{n-k_n}} \left[\sigma_s \sqrt{(n-k_n)} \text{RHS}_{(4.100)} \right. \\
 &\quad \left. - \sqrt{\frac{n-k_n}{2\pi n}} \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz \right].
 \end{aligned}$$

By Proposition 4.5.1,

$$\begin{aligned}
 & \left| \sigma_s \sqrt{n-k_n} \text{RHS}_{(4.100)} - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) \phi \left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n-k_n}} \right) dz \right| \\
 & \leq \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma_s \sqrt{n-k_n} \mathbb{E}_{\mathbb{Q}_s^x} \left[\frac{f(X_{n-k_n}^x)}{r_s(X_{n-k_n}^x)} h(y + S_{n-k_n}^x - (n-k_n)q_s) \right] \right. \\
 & \quad \left. - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) \phi \left(\frac{z - y}{\sigma_s \sqrt{n-k_n}} \right) dz \right| \\
 & \xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

Since $\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \sqrt{\frac{2\pi n}{n-k_n}} \sim W_{s, k_n}^x \sqrt{2\pi} \rightarrow W_s^x \sqrt{2\pi}$ a.s., it follows that $B_n \xrightarrow{n \rightarrow \infty} 0$

a.s. if and only if

$$\left| \int_{\mathbb{R}} h(z) \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left[\sqrt{\frac{2\pi n}{n - k_n}} \phi\left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n - k_n}}\right) - 1 \right] dz \right| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

We shall prove this convergence by the dominated convergence theorem. Notice that the function in the above integral is bounded by $Ch(z)W_{s,k_n}^x \leq CW_{s,*}^x h(z)$ which is integrable on \mathbb{R} . So it suffices to prove that for $z \in \mathbb{R}$,

$$D_n(z) := \left| \sqrt{\frac{2\pi n}{n - k_n}} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \phi\left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n - k_n}}\right) - W_{s,k_n}^x \right| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.} \quad (4.101)$$

Using the fact that $|\phi(x) - \phi(y)| \leq C|x - y|$, we see that for all $z \in \mathbb{R}$

$$\begin{aligned} D_n(z) &\leq \sqrt{\frac{2\pi n}{n - k_n}} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left| \phi\left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n - k_n}}\right) - \phi\left(\frac{z}{\sigma_s \sqrt{n - k_n}}\right) \right| \\ &\quad + W_{s,k_n}^x \left| \sqrt{\frac{2\pi n}{n - k_n}} \phi\left(\frac{z}{\sigma_s \sqrt{n - k_n}}\right) - 1 \right| \\ &\leq C \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \frac{|S_u^x - k_n q_s|}{\sigma_s \sqrt{n - k_n}} + W_{s,k_n}^x \left| \sqrt{\frac{2\pi n}{n - k_n}} \phi\left(\frac{z}{\sigma_s \sqrt{n - k_n}}\right) - 1 \right|. \end{aligned}$$

It is clear that the second term converges to 0 a.s. as $n \rightarrow \infty$. For the first term, we use the same argument as the proof of (4.42), noting that

$$\mathbb{E} \left[\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \frac{|S_u^x - k_n q_s|}{\sigma_s \sqrt{n - k_n}} \right] = \frac{\mathbb{E}_{\mathbb{Q}_s^x} [|S_{k_n}^x - k_n q_s|]}{\sigma_s \sqrt{n - k_n}},$$

and (see [28, Lemma 7.1] for $s > 0$ and [83, Proposition 3.14] for $s \leq 0$)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s^x} (S_n^x - nq_s)^2 = \sigma_s^2.$$

Therefore, (4.101) holds. This shows that $B_n \xrightarrow{n \rightarrow \infty} 0$ a.s. The proof of Theorem 4.2.4 is therefore completed.

Chapter 5

Berry-Esseen bound and precise moderate deviations for branching random walks with products of random matrices

We consider a branching random walk where particles give birth to children as a Galton-Watson process, which move in \mathbb{R}^d according to products of independent and identically distributed random matrices. We establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for the counting measure which counts the number of particles in generation n situated in a region, as $n \rightarrow \infty$. In the proof, we construct a new martingale, and establish its uniform convergence as well as that of the fundamental martingale.

5.1 Introduction

A branching random walk in \mathbb{R}^d is a system of particles, where particles behave independently, and each particle gives birth to a random number of children which move in \mathbb{R}^d with independent and identically distributed (i.i.d.) displacements. One of the fundamental problems on this model is the study of the counting measure which counts the number of particles of generation n situated in a Borel set of \mathbb{R}^d . This problem has been studied by many authors, see e.g. [48, 78, 2, 3, 15, 16, 29, 39, 40, 30], where central limit theorems and large deviations have been considered. For other important topics and closely related models, see for example the recent papers [6, 25, 52, 72, 5], the recent books [77, 26, 54] and many references therein.

In the classical branching random walk, a particle whose parent is at position y , moves to position $y + l$ with i.i.d. increments l 's for different particles, so that the moving is a simple random translation. Recently, in [23] the authors consider a branching random walk in \mathbb{R}^d with products of random matrices, in which the position of a particle is obtained by the action of a matrix A on the position of its parent, where the matrices A 's corresponding to different particles are i.i.d. In other words, the positions of particles

are obtained by the action of products of random matrices on the position of the initial particle. This permits us to extend significantly the domains of applications of the theory of branching random walks, but the study of the model becomes much more involved. In [23], a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for the counting measure have been proved. In this paper, we will establish the Berry-Esseen bound about the rate of convergence in the central limit theorem, and a moderate deviation expansion of Cramér type.

For a precise description of the model we need some notation. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. Set $\mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$, where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. A particle of generation n will be denoted by a sequence $u = u_1 \cdots u_n = (u_1, \dots, u_n) \in (\mathbb{N}^*)^n$ of length n ; the initial particle will be denoted by the null sequence \emptyset . Assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a set of independent identically distributed random variables $(N_u)_{u \in \mathbb{U}}$ of the same law $p = \{p_k : k \in \mathbb{N}\}$, and a set of independent identically distributed $d \times d$ random matrices $(A_u)_{u \in \mathbb{U}}$ of the same law μ on the set of $d \times d$ matrices $M(d, \mathbb{R})$, where $d \geq 2$. The two families $(N_u)_{u \in \mathbb{U}}$ and $(A_u)_{u \in \mathbb{U}}$ are also assumed to be independent.

A branching random walk in \mathbb{R}^d with products of random matrices is defined as follows. At time 0, there is one initial particle \emptyset of generation 0, with initial position $Y_\emptyset := x \in \mathbb{R}^d \setminus \{0\}$. At time 1, the initial particle \emptyset is replaced by $N = N_\emptyset$ new particles $i = \emptyset i$ of generation 1, located at $Y_i = A_i Y_\emptyset$, $1 \leq i \leq N$. In general, at time $n + 1$, each particle $u = u_1 \dots u_n$ of generation n , located at $Y_u \in \mathbb{R}^d$, is replaced by N_u new particles ui of generation $n + 1$, located at $Y_{ui} = A_{ui} Y_u$, $1 \leq i \leq N_u$. Namely, the position of the particle ui is obtained from the position of u by the action of the matrix A_{ui} on the vector Y_u . Consequently the position Y_u of a particle u in generation $n \geq 1$ is given by the action of products of random matrices on the position x of the initial particle \emptyset :

$$Y_u = G_u x, \quad \text{where} \quad G_u = A_{u_1 \dots u_n} \cdots A_{u_1}. \quad (5.1)$$

Denote by \mathbb{T} the genealogical tree associated to the elements $\{N_u : u \in \mathbb{U}\}$, defined by the following properties: 1) $\emptyset \in \mathbb{T}$; 2) when $u \in \mathbb{T}$, then for $i \in \mathbb{N}$, $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$; 3) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n , where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs; by convention $|\emptyset| = 0$.

The space \mathbb{R}^d is equipped with the Euclidean norm $|\cdot|$. The position $G_u x$ of the particle u is completely described by two components: its norm $|G_u x|$ and its projection on the unit sphere $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d, |y| = 1\}$ denoted by

$$X_u^x := \frac{G_u x}{|G_u x|}.$$

Accordingly, we consider the following counting measure of particles of generation n which describes the configuration of the branching random walk at time n : for measurable sets $B_1 \subset \mathbb{S}^{d-1}$ and $B_2 \subset \mathbb{R}$,

$$Z_n^x(B_1, B_2) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_{\{X_u^x \in B_1, \log |G_u x| \in B_2\}}, \tag{5.2}$$

where for a set D , $\mathbf{1}_D$ denotes its indicator function.

In [23], a central limit theorem for the counting measure Z_n^x (with the starting point $x \in \mathbb{S}^{d-1}$) was established for both the case where the matrices A_u are nonnegative, and the case where the matrices A_u are invertible. It implies that, under suitable conditions, for some constants γ, σ explicitly defined (see (5.5) and (5.6)), the counting measure $B_2 \mapsto Z_n^x(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}B_2)$ on \mathbb{R} with a suitable norming converges to the standard normal law. In [23], a precise large deviation result of Bahadur-Rao type was also established, which gives in particular the exact asymptotic of $Z_n^x(\mathbb{S}^{d-1}, [na, +\infty))$ for $a > \gamma$.

In this paper, our first objective is to strengthen the central limit theorem in [23] to a Berry-Esseen bound for the counting measure Z_n^x with a target function φ on X_u^x : see Theorem 5.2.1. With $\varphi = 1$, it implies that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $n \geq 1$, we have, a.s.

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{m^n} Z_n^x(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}(-\infty, y]) - W\Phi(y) \right| \leq \frac{M}{\sqrt{n}}, \tag{5.3}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$ is the distribution function of the standard normal law and M is a finite and positive random variable.

Our second objective is to establish Cramér type moderate deviation expansion for Z_n^x with a target function φ on X_u^x : see Theorem 5.2.2. From this theorem with $\varphi = 1$, we know that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $0 \leq y = o(\sqrt{n})$, as $n \rightarrow \infty$,

a.s.,

$$\frac{Z_n^x\left(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}(-\infty, y]\right)}{m^n W[1 - \Phi(y)]} = e^{\frac{y^3}{\sqrt{n}}\zeta\left(\frac{y}{\sqrt{n}}\right)} \left[1 + O\left(\frac{y+1}{\sqrt{n}}\right)\right], \quad (5.4)$$

where $t \mapsto \zeta(t)$ is the Cramér series (see (5.9)).

An important step in attaining these two objectives is to establish a Berry-Esseen bound for the Cramér type changed measure $Z_{s,n}^x$ (see (5.18)). This will be done in Theorem 5.2.3. Theorem 5.2.1 will be obtained from Theorem 5.2.3 by taking $s = 0$, and Theorem 5.2.2 will be established by using Theorem 5.2.3 and by adapting the techniques from Petrov [75].

To facilitate the comprehension, let us present some ideas in the proof of Theorem 5.2.3. As in [24] where the one dimensional case is considered, we need to study the asymptotic of the characteristic function of the changed measure $Z_{s,n}^x$. Inspired by the approach in [24], we would like to express the characteristic function of $Z_{s,n}^x$ in terms of a martingale and a quantity that can be controlled by the theory of products of random matrices. However, in contrast to the one dimensional case, we cannot obtain directly an expression of the characteristic function in terms of a martingale. Fortunately, using the spectral gap theory for products of random matrices established in [46, 25, 28] and recently developed in [83], we have been able to define a new martingale which is similar to the fundamental martingale and which can be used for a suitable approximation of the characteristic function of $Z_{s,n}^x$. We conclude by proving the uniform convergence and analyticity with respect to a complex parameter of the new martingale, and by using the asymptotic properties of the eigenvalue of the perturbed transfer operator related to the products of random matrices. See Theorem 5.4.3 and Lemma 5.5.6 for details.

The rest of the paper is organized as follows. In Section 5.2, we fix some notation, introduce our assumptions on the branching products of random matrices, and state the main results. In Section 5.3, we recall some spectral gap properties on products of random matrices stated in [83]. In Section 5.4, the uniform convergence and analyticity of the constructed martingale are established. Sections 5.5 and 5.6 are devoted to the proofs of the main results.

5.2 Main results

5.2.1 Notation and assumptions on products of random matrices

Note that in our model, along each branch we encounter a product of random matrices. In this section, we introduce some notation and the necessary assumptions on products of random matrices in order to formulate our main results. We shall consider two cases, the case when the matrices are nonnegative and the case when the matrices are invertible.

The set $M(d, \mathbb{R})$ of $d \times d$ real matrices is equipped with the operator norm: $\|\mathbf{a}\| = \sup_{x \in \mathbb{S}^{d-1}} |\mathbf{a}x|$ for $\mathbf{a} \in M(d, \mathbb{R})$, where $|\cdot|$ is a given vectorial norm on \mathbb{R}^d , and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere in \mathbb{R}^d . A matrix $\mathbf{a} \in M(d, \mathbb{R})$ is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by \mathcal{M}_+ the set of matrices with nonnegative entries. A nonnegative matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column of \mathbf{a} has a strictly positive entry.

Let μ be a probability measure on $M(d, \mathbb{R})$. Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by the support of μ . We say that the measure μ is *arithmetic* if there are $t > 0$, $\theta \in [0, 2\pi)$ and a function $\vartheta : \mathbb{S}_+^{d-1} \rightarrow \mathbb{R}$ such that

$$\forall \mathbf{a} \in \Gamma, \forall x \in V(\Gamma) : \exp[it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a} \cdot x) - \vartheta(x))] = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Notice that when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form: $\log a$ is almost surely (a.s.) concentrated on an arithmetic progression $a_0 + a_1\mathbb{N}$ for some $a_0, a_1 > 0$.

We will need the following assumptions on the law μ .

L1.

1. *For invertible matrices:*

(a) *(Strong irreducibility)* There is no finite union $\mathcal{W} = \bigcup_{i=1}^m W_i$ of proper subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ -invariant (in the sense that $a\mathcal{W} = \mathcal{W}$ for each $a \in \Gamma_\mu$).

(b) *(Proximality)* Γ_μ contains at least one proximal matrix.

2. *For nonnegative matrices:*

- (a) (Allowability) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.
- (b) (Positivity) Γ_μ contains at least one matrix belonging to $\text{int}(\mathcal{M}_+)$.
- (c) (Non-arithmeticity) The measure μ is non-arithmetic.

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathcal{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a} \cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|} \quad \text{when } \mathbf{a}x \neq 0,$$

where $\mathbf{a} \cdot x$ is called the projective action of the matrix \mathbf{a} on the vector $x \in \mathbb{S}^{d-1}$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices. Set, for an invertible or nonnegative matrix \mathbf{a} ,

$$N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \iota(\mathbf{a})^{-1}\}.$$

For invertible matrices we have $\iota(\mathbf{a}) = \|\mathbf{a}^{-1}\|^{-1}$ and $N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \|\mathbf{a}^{-1}\|\}$.

L2. (Moment condition) There exists $\eta_0 \in (0, 1)$ such that

$$\mathbb{E}[N(A_1)^{\eta_0}] < \infty.$$

We will consider the action of invertible matrices on the projective space \mathbb{P}^{d-1} which is obtained from \mathbb{S}^{d-1} by identifying x and $-x$, and the action of nonnegative matrices on \mathbb{S}_+^{d-1} . For convenience we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} for invertible matrices, and \mathbb{S}_+^{d-1} for nonnegative matrices. The space \mathcal{S} will be equipped with the metric \mathbf{d} , which is the angular distance (see [20]) for invertible matrices, and the Hilbert cross-ratio metric (see [49]) for nonnegative matrices. Moreover, \mathcal{S} is a separable metric space equipped with Borel σ -field.

Let $\mathcal{C}(\mathcal{S})$ be the space of continuous complex-valued functions on \mathcal{S} . For $\beta > 0$ sufficiently small, we introduce the Banach space

$$\mathcal{B}_\beta = \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\beta < +\infty\},$$

equipped with the norm

$$\|f\|_\beta := \|f\|_\infty + |f|_\beta,$$

where

$$\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|, \quad |f|_\beta := \sup_{x, y \in \mathcal{S}, x \neq y} \frac{|f(x) - f(y)|}{\mathbf{d}^\beta(x, y)}.$$

Let $G_n = A_n \dots A_2 A_1$ be the product of i.i.d. $d \times d$ real random matrices A_i , defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with common law μ . Let $x \in \mathcal{S}$ be a starting point. As mentioned in the introduction, the random walk $G_n x$ is completely determined by its log norm and its projection on \mathcal{S} , denoted respectively by

$$S_n^x := \log |G_n x|, \quad X_n^x := G_n \cdot x = \frac{G_n x}{|G_n x|}, \quad n \geq 0,$$

with the convention that $G_0 x = x$. Since $S_{n+1}^x = \log |A_{n+1} X_n^x| + S_n^x$ and $X_{n+1}^x = A_{n+1} \cdot X_n^x$, the sequence $(S_n^x, X_n^x)_{n \geq 0}$ is a Markov chain.

Denote by \mathbb{E} the expectation with respect to \mathbb{P} . By the law of large numbers of Furstenberg [38], under conditions **L1** and **L2**, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n^x = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[S_n^x] = \gamma \quad \mathbb{P}\text{-a.s.}, \quad (5.5)$$

where $\gamma = \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E} \log \|G_n\|$ is the upper Lyapunov exponent associated with the product sequence (G_n) . Le Page [63] and Henion [49] showed that

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} (S_n^x - n\gamma)^2 \quad (5.6)$$

exists and is independent of x for invertible matrices and nonnegative matrices, respectively. Moreover, there exists a unique μ -stationary probability measure ν on \mathcal{S} (see [46, 25]): $\mu * \nu = \nu$, that is, for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$(\mu * \nu)(\varphi) := \int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(\mathbf{a} \cdot x) \mu(d\mathbf{a}) \nu(dx) = \nu(\varphi),$$

where $\nu(\varphi) = \int_{\mathcal{S}} \varphi(x) \nu(dx)$, and this notation for the integral will be used for any function and any measure. Define the transfer operator on $\mathcal{C}(\mathcal{S})$ as follows: for any $s \in (-\eta_0, \eta_0)$,

and $f \in \mathcal{C}(\mathcal{S})$,

$$P_s f(x) = \mathbb{E}[|A_1 x|^s f(A_1 \cdot x)], \quad \text{for all } x \in \mathcal{S}. \quad (5.7)$$

It is known that under conditions **L1** and **L2**, there exists a small constant $0 < \eta_1 < \eta_0$ such that for any $s \in (-\eta_1, \eta_1)$, there are a unique probability measure ν_s and a unique Hölder continuous function r_s on \mathcal{S} satisfying $\nu(r_s) = 1$ and

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \quad (5.8)$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s , $\nu_s P_s$ is the measure on \mathcal{S} such that $(\nu_s P_s)(f) = \nu_s(P_s f)$ for all $f \in \mathcal{C}(\mathcal{S})$. In particular, $r_0 = 1$ and $\kappa(0) = 1$. For $s \in [0, \eta_1)$, the property (5.8) is proved in [25, Proposition 3.1] and [28, Corollary 7.3] for positive matrices, and in [46, Theorem 2.6 and Corollary 3.20] for invertible matrices. For both positive matrices and invertible matrices, the existence of $\eta_1 > 0$ and the property (5.8) for $s \in (-\eta_1, \eta_1)$ are proved in [83, Proposition 3.1], where the following properties are also established: the functions $s \mapsto \kappa(s)$ and $s \mapsto r_s(x)$ are strictly positive and analytic in $(-\eta_1, \eta_1)$, for $x \in \mathcal{S}$. Moreover, it is proved (see [46, Lemma 3.5], [25, Lemma 6.2], [83, Propositions 3.12 and 3.14]) that, under conditions **L1** and **L2**, the function $\Lambda(s) = \log \kappa(s)$ is finite and analytic on $(-\eta_1, \eta_1)$, and satisfies

$$\Lambda'(0) = \gamma, \quad \Lambda''(0) = \sigma^2 > 0, \quad \text{and} \quad \Lambda''(s) > 0 \quad \forall s \in (-\eta_1, \eta_1).$$

Denote $\gamma_k = \Lambda^k(0)$, $k \geq 1$. Throughout the paper, we write ζ for the Cramér series associated to Λ (see [75, Theorem VIII.2.2] for details):

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \dots \quad (5.9)$$

which converges for $|t|$ small enough.

5.2.2 Main results

Let $Z_n = Z_n^x(S, \mathbb{R})$ be the population size at time n , which does not depend on the starting point x , and which forms a Galton-Watson process with $Z_0 = 1$ and $Z_1 = N$. Denote by $m = \mathbb{E}N$ the expected value of the offspring distribution. Throughout the paper, we

assume that

$$m \in (1, \infty) \quad \text{and} \quad \mathbb{P}(N = 0) = 0.$$

Therefore the branching process (Z_n) is supercritical, and $Z_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. It is well known that $\mathbb{E}Z_n = m^n$. Let

$$W = \lim_{n \rightarrow \infty} W_n, \quad \text{where} \quad W_n = \frac{Z_n}{m^n}, \quad n \geq 0,$$

is the fundamental martingale for the Galton-Watson process (Z_n) , and the limit exists a.s. by the martingale convergence theorem. An important ingredient in studying Berry-Esseen bound and moderate deviation expansion is the fundamental martingale associated to branching random walks with products of random matrices, defined for $s \in (-\eta_1, \eta_1)$ and $x \in \mathcal{S}$

$$W_n^x(s) := \frac{\sum_{u \in \mathbb{T}_n} e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^n r_s(x)}, \quad n \geq 0. \tag{5.10}$$

This is a positive martingale with respect to the natural filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_n = \sigma(N_u, A_{ui} : i \geq 1, |u| < n) \quad \text{for } n \geq 1.$$

By the martingale convergence theorem, the limit

$$W^x(s) := \lim_{n \rightarrow \infty} W_n^x(s) \quad \text{exists in } \mathbb{R} \quad \mathbb{P}\text{-a.s.}$$

Set $\Lambda^*(q_s) = sq_s - \Lambda(s)$ with $q_s = \Lambda'(s)$. It is proved in [23] that under conditions **L1** and **L2**, if

$$\Lambda^*(q_s) - \log m < 0 \tag{5.11}$$

and

$$\mathbb{E}[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x] < \infty, \tag{5.12}$$

where $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$, then for all $x \in \mathcal{S}$, $W^x(s)$ is non-degenerate with

$$\mathbb{E}[W^x(s)] = 1.$$

Set

$$J = \{s \in (-\eta_1, \eta_1) : \Lambda^*(q_s) - \log m < 0\}, \quad (5.13)$$

which is an open interval containing 0. We assume the following moment condition slightly stronger than (5.12):

L3. *There are constants $\gamma_0 > 1$ and $0 < \eta_2 < \frac{\eta_1}{2}$ with $[-\eta_2, \eta_2] \subset J$ such that*

$$\mathbb{E} \left[\max_{x \in \mathcal{S}} \left(W_1^x(s) \right)^{\gamma_0} \right] < \infty \quad \forall s \in [-\eta_2, \eta_2].$$

It is clear that conditions **L1-L3** (together with the hypothesis $\mathbb{P}(N = 0) = 0$ that we assume always), imply that for all $x \in \mathcal{S}$, $W^x(s) > 0$ a.s. and $\mathbb{E}[W^x(s)] = 1$; in particular (when $s = 0$), $W > 0$ a.s. and $\mathbb{E}[W] = 1$.

Our first result is the Berry-Esseen bound for the counting measure Z_n^x :

Theorem 5.2.1. *Assume conditions **L1-L3**. Then, for any $x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\beta$ and $n \geq 1$, we have, a.s.,*

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\left\{ \frac{S_u^x - n\gamma}{\sigma\sqrt{n}} \leq y \right\}} - W\nu(\varphi)\Phi(y) \right| \leq \frac{M}{\sqrt{n}}, \quad (5.14)$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$ is the distribution function of the standard normal law and M is a finite and positive random variable.

This is a Berry-Esseen type bound for the counting measure Z_n^x with suitable norming because the sum in (5.14) is an integral with respect to Z_n^x :

$$\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\left\{ \frac{S_u^x - n\gamma}{\sigma\sqrt{n}} \leq y \right\}} = \int_{\mathcal{S} \times \mathbb{R}} \varphi(z_1) \mathbb{1}_{\left\{ \frac{z_2 - n\gamma}{\sigma\sqrt{n}} \leq y \right\}} Z_n^x(dz_1, dz_2). \quad (5.15)$$

Our second result is the Cramér's moderate deviation expansion for Z_n^x .

Theorem 5.2.2. *Assume conditions **L1-L3**. Then, we have for any $x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\beta$, $0 \leq y = o(\sqrt{n})$, as $n \rightarrow \infty$, a.s.,*

$$\frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \geq \sqrt{n}\sigma y\}}}{m^n W[1 - \Phi(y)]} = e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \left[\nu(\varphi) + O\left(\frac{y+1}{\sqrt{n}}\right) \right], \quad (5.16)$$

and

$$\frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \leq -\sqrt{n}\sigma y\}}}{m^n W\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}} \zeta(-\frac{y}{\sqrt{n}})} \left[\nu(\varphi) + O\left(\frac{y+1}{\sqrt{n}}\right) \right]. \quad (5.17)$$

An important step in the proof of the moderate deviation expansion is to establish a Berry-Esseen bound for the changed measure $Z_{s,n}^x$ defined by for measurable sets $B_1 \subset \mathbb{S}^{d-1}$ and $B_2 \subset \mathbb{R}$,

$$\begin{aligned} Z_{s,n}^x(B_1, B_2) &= \int_{B_1 \times B_2} \frac{e^{sz_2} r_s(z_1)}{[m\kappa(s)]^n r_s(x)} Z_n^x(dz_1, dz_2) \\ &= \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\{X_u^x \in B_1, S_u^x \in B_2\}}. \end{aligned} \quad (5.18)$$

Our third result is a Berry-Esseen bound for the changed measure $Z_{s,n}^x$:

Theorem 5.2.3. *Assume conditions **L1-L3**. Then, for any $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\beta$ there exists a constant $0 < \eta < \eta_2$ such that a.s., for $n \geq 1$,*

$$\sup_{s \in (-\eta, \eta)} \sup_{y \in \mathbb{R}} \left| \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} - W^x(s) \pi_s(\varphi) \Phi(y) \right| \leq \frac{M}{\sqrt{n}}, \quad (5.19)$$

where M is a positive and finite random variable.

This is a Berry-Esseen type bound for $Z_{s,n}^x$ because, similar to the case of Theorem 5.2.1, the sum in (5.19) is an integral with respect to $Z_{s,n}^x$:

$$\begin{aligned} &\sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \\ &= \int_{\mathcal{S} \times \mathbb{R}} \varphi(z_1) \mathbb{1}_{\left\{ \frac{z_2 - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} Z_{s,n}^x(dz_1, dz_2). \end{aligned} \quad (5.20)$$

5.3 Preliminary results on products of random matrices

In this section we recall some spectral gap properties stated in [83] which will be used for the proofs of main results.

Define the operator P_z on $\mathcal{C}(\mathcal{S})$ by

$$P_z f(x) = \mathbb{E}[|A_1 x|^z f(A_1 \cdot x)], \quad \text{for all } x \in \mathcal{S}, z \in \mathbb{C}. \quad (5.21)$$

Denote by $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ the set of all bounded linear operators from \mathcal{B}_β to \mathcal{B}_β equipped with the operator norm

$$\|P\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta} := \sup_{f \neq 0} \frac{\|Pf\|_\beta}{\|f\|_\beta}, \quad \forall P \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta).$$

We write \mathcal{B}'_β for the topological dual of \mathcal{B}_β endowed with the norm $\|\nu\|_{\mathcal{B}'_\beta} = \sup_{\|\varphi\|_\beta=1} |\nu(\varphi)|$ for any linear functional $\nu \in \mathcal{B}'_\beta$. For any $\eta > 0$, set $B_\eta(0) = \{z \in \mathbb{C} : |z| < \eta\}$ for the ball with center 0 and radius η in the complex plane \mathbb{C} .

Lemma 5.3.1. *Assume conditions **L1** and **L2**. There exists a small $\eta_1 \in (0, \eta_0)$ such that for any $z \in B_{\eta_1}(0)$ and $n \geq 1$, we have the decomposition*

$$P_z^n = \kappa^n(z) M_z + L_z, \quad (5.22)$$

where the operator M_z is a rank one projection on \mathcal{B}_β , the mappings on $B_{\eta_1}(0)$,

$$z \mapsto \kappa(z) \in \mathbb{C}, \quad z \mapsto r_z \in \mathcal{B}_\beta, \quad z \mapsto \nu_z \in \mathcal{B}'_\beta, \quad z \mapsto L_z \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$$

are well-defined under the normalizing conditions $\nu_z(1) = \nu(r_z) = 1$. All these mappings are analytic in $B_{\eta_1}(0)$, and possess the following properties:

1. for any $z \in B_{\eta_1}(0)$, it holds that $M_z L_z = L_z M_z = 0$;
2. for any $z \in B_{\eta_1}(0)$, $P_z r_z = \kappa(z) r_z$ and $\nu_z P_z = \kappa(z) \nu_z$;
3. $\kappa(s)$ and r_s are real-valued and satisfy $\kappa(s) > 0$ and $r_s(x) > 0$ for any $s \in (-\eta_1, \eta)$ and $x \in \mathcal{S}$;

4. there exist two constants $0 < a_1 < a_2 < 1$ such that for all $z \in B_{\eta_1}(0)$ and all $n \in \mathbb{N}^*$, $|\kappa(z)| > 1 - a_1$ and $\|L_z^n\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta} \leq c(1 - a_2)^n$.

For fixed $s \in (-\eta_1, \eta_1)$ and $x \in \mathcal{S}$, the spectral gap property (5.8) allows to define a probability measure \mathbb{Q}_s^x on (Ω, \mathcal{F}) such that for any $n \in \mathbb{N}$ and any bounded and measurable function h on $(\mathcal{S} \times \mathbb{R})^{n+1}$,

$$\begin{aligned} & \mathbb{E} \left[\frac{e^{sS_n^x} r_s(X_n^x)}{\kappa^n(s) r_s(x)} h(X_0^x, S_0^x, \dots, X_n^x, S_n^x) \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} [h(X_0^x, S_0^x, \dots, X_n^x, S_n^x)], \end{aligned} \quad (5.23)$$

where $\mathbb{E}_{\mathbb{Q}_s^x}$ denotes the expectation with respect to \mathbb{Q}_s^x . See [25, 28, 46] for $s \geq 0$, and [83] for $s < 0$.

Under the changed measure \mathbb{Q}_s^x , the process $(X_n^x)_{n \in \mathbb{N}}$ is a Markov chain with the transition operator Q_s defined by, for any $s \in (-\eta_1, \eta_1)$ and $\varphi \in \mathcal{B}_\beta$,

$$Q_s \varphi(x) = \frac{1}{\kappa(s) r_s(x)} P_s(\varphi r_s)(x), \quad x \in \mathcal{S}.$$

It has been proved in [83, Proposition 3.4] that Q_s has a unique stationary probability measure defined by $\pi_s(\varphi) := \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$, $\varphi \in \mathcal{B}_\beta$, and there exist two constants $0 < a < 1$, $c_1 > 0$ such that

$$\sup_{s \in (-\eta_1, \eta_1)} \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)] - \pi_s(\varphi)| \leq c_1 a^n. \quad (5.24)$$

Moreover, the perturbed operator $R_{s,it}$ defined by

$$R_{s,it} \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it[S_1^x - \Lambda'(s)]} \varphi(X_1^x) \right], \quad s \in (-\eta_1, \eta_1) \text{ and } t \in \mathbb{R}, \quad (5.25)$$

satisfies for any compact $K \subset \mathbb{R} \setminus \{0\}$, $n \geq 1$ and $\varphi \in \mathcal{B}_\beta$,

$$\sup_{s \in (-\eta_1, \eta_1)} \sup_{t \in K} \sup_{x \in \mathcal{S}} |R_{s,it}^n \varphi(x)| \leq \|\varphi\|_\beta a_K^n, \quad 0 < a_K < 1. \quad (5.26)$$

The operator $R_{s,it}$ has eigenvalue $\lambda_{s,it}$ satisfying for $s \in (-\eta_1, \eta_1)$ and $t \in (-\delta, \delta) \subset (-\eta_1, \eta_1)$,

$$\lambda_{s,it} = e^{\Lambda(s+it) - \Lambda(s) - \Lambda'(s)it}. \quad (5.27)$$

5.4 Associated martingales

In this section, for the fundamental martingale $(W_n^x(s))$ we first reveal a relationship between the moments of $W_1^x(s)$ and $W_*^x(s) := \sup_{n \geq 0} W_n^x(s)$. We next prove the uniform convergence of $W_n^x(z)$ for $z \in B_{\eta_2}(0)$. We finally introduce a new martingale and establish its similar properties; this martingale will play a key role in the proof of the main results.

Theorem 5.4.1. *Assume conditions [L1-L3](#). Then there is a constant $\eta \in (0, \eta_2)$ such that*

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \mathbb{E}[W_*^x(s)]^{\gamma_0} < \infty. \quad (5.28)$$

Proof. In [[23](#), Lemma 5.6], it is proved that if $\mathbb{E}[W^x(s)] = 1$, then $W_*^x(s)$ and $W^x(s)$ have similar tail behaviour for $s \in (-\eta_2, \eta_2)$ and for all $x \in \mathcal{S}$, i.e. for $s \in (-\eta_2, \eta_2)$ and for any $a \in (0, 1)$, for $s \in (-\eta_2, \eta_2)$, there is a constant $b > 0$ such that for all $t > 0$, for all $x \in \mathcal{S}$

$$\mathbb{P}(W_s^x \geq at) \geq b\mathbb{P}(W_{s,*}^x \geq t) \geq b\mathbb{P}(W_s^x \geq t).$$

A slight modification in the proof of [[23](#), Lemmas 5.5 and 5.6] shows that we can choose b independent of $s \in (-\eta_2, \eta_2)$. (To see this, we just need to check the proof therein, and replace W_s^y by $\sup_{s \in (-\eta_2, \eta_2)} W_s^y$ in the formula $\sup_{y \in \mathcal{S}} \mathbb{E} \left[W_s^y 1_{\{W_s^y > T\}} \right] \xrightarrow{T \rightarrow +\infty} 0$ of the proof of Lemma 5.5, at the last line of page 34.) Recall that $\mathbb{E}[W^x(s)] = 1$ under the hypothesis of Theorem [5.4.1](#). Thus, in order to prove (5.28), it suffices to show that there is a constant $\eta \in (0, \eta_2)$ such that

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \mathbb{E}[W^x(s)]^{\gamma_0} < \infty. \quad (5.29)$$

Set $h(x) = x^\delta$ where $\delta = \gamma_0 - 1 \in (0, 1]$. Observe that

$$W_{n+1}^x(s) = \sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s), \text{ where } H_{n,u}^x = \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^n r_s(x)}. \quad (5.30)$$

Using (5.30) and the subadditivity of h , we have

$$\begin{aligned} \mathbb{E}_n \left[W_{n+1}^x(s) h \left(W_{n+1}^x(s) \right) \right] &\leq \mathbb{E}_n \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(H_{n,u}^x W_1^{X_u^x}(s) \right) \right] \\ &\quad + \mathbb{E}_n \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(\sum_{\substack{v \in \mathbb{T}_n \\ v \neq u}} H_{n,v}^x W_1^{X_v^x}(s) \right) \right]. \end{aligned}$$

Using Jensen's inequality for the conditional expectation and the facts that $\mathbb{E}_n \left[W_1^{X_u^x}(s) \right] = 1$ and h is an increasing function, the second term in the inequality above is less than $W_n^x(s) h(W_n^x(s))$. Then taking expectations in the two sides of the inequality above, we get

$$\begin{aligned} \mathbb{E} \left[W_{n+1}^x(s) h \left(W_{n+1}^x(s) \right) \right] &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(H_{n,u}^x W_1^{X_u^x}(s) \right) \right] \\ &\quad + \mathbb{E} \left[W_n^x(s) h \left(W_n^x(s) \right) \right]. \end{aligned}$$

So by recurrence on n and Fatou's lemma, we obtain

$$\begin{aligned} \mathbb{E} \left[W^x(s) h \left(W^x(s) \right) \right] &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[W_n^x(s) h \left(W_n^x(s) \right) \right] \\ &\leq \mathbb{E} \left[W_1^x(s) h \left(W_1^x(s) \right) \right] + \sum_{n=1}^{\infty} \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(H_{n,u}^x W_1^{X_u^x}(s) \right) \right]. \end{aligned}$$

To prove (5.29), it suffices to show that there is a constant $\eta \in (0, \eta_2)$ such that

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \mathbb{E} \left[W_1^x(s) h \left(W_1^x(s) \right) \right] \leq \sup_{s \in (-\eta_2, \eta_2)} \sup_{x \in \mathcal{S}} \mathbb{E} [\sup W_1^x(s)]^{\gamma_0} < \infty, \quad (5.31)$$

and

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(H_{n,u}^x W_1^{X_u^x}(s) \right) \right] < \infty. \quad (5.32)$$

For (5.31), we see that for all $s \in (-\eta_1, \eta_1)$,

$$\begin{aligned} W_1^x(s) &= \frac{1}{m\kappa(s)} \sum_{i=1}^N e^{s \log |A_i x|} r_s(A_i \cdot x) \\ &\leq \frac{\max_{x \in \mathcal{S}} |r_s(x)|}{m\kappa(s)} \left(\sum_{i=1}^N e^{-\eta_2 \log |A_i x|} + \sum_{i=1}^N e^{\eta_2 \log |A_i x|} \right). \end{aligned} \quad (5.33)$$

Since the functions $s \mapsto r_s$ and $s \mapsto \kappa(s)$ are strictly positive and analytic on $(-\eta_1, \eta_1)$ and $r_0 = 1, \kappa(0) = 1$, there are two constants $d_1, d_2 > 0$ such that

$$\frac{\max_{x \in \mathcal{S}} r_s(x)}{\min_{x \in \mathcal{S}} r_s(x)} \leq d_1 \quad \text{for all } s \in (-\eta_1, \eta_1) \quad (5.34)$$

and

$$\frac{\sup_{s \in (-\eta_1, \eta_1)} \kappa(s)}{\inf_{s \in (-\eta_1, \eta_1)} \kappa(s)} \leq d_2. \quad (5.35)$$

Hence, from (5.33), (5.34) and (5.35), for all $s \in (-\eta_1, \eta_1)$,

$$W_1^x(s) \leq d_1 d_2 \left(W_1^x(-\eta_2) + W_1^x(\eta_2) \right).$$

Therefore, by the inequality

$$(a + b)^{\gamma_0} \leq 2^{\gamma_0 - 1} (a^{\gamma_0} + b^{\gamma_0}), \quad a, b \in \mathbb{R}, \quad (5.36)$$

and condition **L3**,

$$\begin{aligned} &\sup_{s \in [-\eta_2, \eta_2]} \mathbb{E} \left[\sup_{x \in \mathcal{S}} W_1^x(s) \right]^{\gamma_0} \\ &\leq (d_1 d_2)^{\gamma_0} 2^{\gamma_0 - 1} \left(\mathbb{E} \sup_{x \in \mathcal{S}} [W_1^x(-\eta_2)]^{\gamma_0} + \mathbb{E} \sup_{x \in \mathcal{S}} [W_1^x(\eta_2)]^{\gamma_0} \right) < \infty. \end{aligned} \quad (5.37)$$

For (5.32), we consider the general term in its series. Since $h(x) = x^\delta$, we have, by (5.34),

for all $s \in (-\eta_2, \eta_2)$

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(H_{n,u}^x W_1^{X_u^x}(s) \right) \right] \\
 &= \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} \left(H_{n,u}^x W_1^{X_u^x}(s) \right)^{\gamma_0} \right] \\
 &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} \left(H_{n,u}^x \right)^{\gamma_0} \right] \mathbb{E} \sup_{x \in \mathcal{S}} W_1^x(s)^{\gamma_0} \\
 &\leq d_1^{2\gamma_0} \left(\frac{m\kappa(s\gamma_0)}{[m\kappa(s)]^{\gamma_0}} \right)^n \mathbb{E}[W_n^x(s\gamma_0)] \mathbb{E} \sup_{x \in \mathcal{S}} W_1^x(s)^{\gamma_0}. \tag{5.38}
 \end{aligned}$$

Set $f(s) = \frac{m\kappa(s\gamma_0)}{[m\kappa(s)]^{\gamma_0}}$, $s \in (-\eta_2, \eta_2)$. We see that $f(0) = m^{1-\gamma_0} < 1$ and f is continuous on $(-\eta_2, \eta_2)$ by the continuity of κ . Hence there is a small constant $\eta > 0$ with $(-\eta, \eta) \subset (-\eta_2, \eta_2)$ such that

$$c_1 := \sup_{s \in (-\eta, \eta)} \frac{m\kappa(s\gamma_0)}{[m\kappa(s)]^{\gamma_0}} < 1. \tag{5.39}$$

We can choose $\eta > 0$ sufficiently small so that $s\gamma_0 \in (-\eta_1, \eta_1)$. Then $W_n^x(s\gamma_0)$ is well-defined and a martingale, so $\mathbb{E}[W_n^x(s\gamma_0)] = 1$. Therefore, from (5.38), (5.39) and (5.37), we obtain

$$\begin{aligned}
 & \sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) h \left(H_{n,u}^x W_1^{X_u^x}(s) \right) \right] \\
 &\leq d_1^{2\gamma_0} \sup_{s \in [-\eta_2, \eta_2]} \mathbb{E} \sup_{x \in \mathcal{S}} W_1^x(s)^{\gamma_0} \sum_{n=1}^{\infty} c_1^n < \infty.
 \end{aligned}$$

This completes the proof of (5.32). Thus (5.29) is proved. □

Now we consider the martingale with complex parameter:

$$W_n^x(z) := \frac{\sum_{u \in \mathbb{T}_n} e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^n r_z(x)}, \quad n \geq 0, \quad z \in B_{\eta_1}(0). \tag{5.40}$$

For each fixed $z \in B_{\eta_1}(0)$, it can be easily checked that $(W_n^x(z))$ remains a martingale with respect to (\mathcal{F}_n) . Throughout, the real part of $z \in \mathbb{C}$ will be denoted by s , so that $z = s + iIm(z)$.

The next theorem gives the uniform convergence of $W_n^x(z)$. Let

$$\Omega_\alpha^1 = \text{int} \left\{ z \in B_{\eta_2}(0) : \frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} < 1 \right\} \quad \text{and} \quad \Omega_{\gamma_0} = \bigcup_{1 < \alpha \leq \gamma_0} \Omega_\alpha^1. \quad (5.41)$$

Since the derivative at 1 of the function $\alpha \mapsto \frac{m\kappa(\alpha s)}{[m\kappa(s)]^\alpha}$ is equal to $\Lambda^*(s) - \log m$ which is negative for $s \in (-\eta_2, \eta_2)$, we have, for these values of s , $\frac{m\kappa(\alpha s)}{[m\kappa(s)]^\alpha} < 1$ when $\alpha > 1$ is close to 1. This shows that the open set Ω_{γ_0} contains the segment $(-\eta_2, \eta_2)$, so that $(-\eta_2, \eta_2)$ is the intersection of Ω_{γ_0} with the real axis.

Theorem 5.4.2. *Assume conditions [L1-L3](#). Then the sequence $(W_n^x(z))_{n \geq 0}$ converges a.s. to some complex valued random variable $W^x(z)$, uniformly in z on any compact subset $K \subset \Omega_{\gamma_0}$. Moreover, we have a.s., for all $n \geq 0$,*

$$\sup_{z \in K} |W_n^x(z) - W^x(z)| \leq M\delta^n, \quad (5.42)$$

where M is a positive and finite random variable and $\delta \in (0, 1)$, and $W^x(z)$ is analytic on Ω_{γ_0} .

Proof. The basic ideas here are the same as those used in the proof of Theorem 2 in Biggins [18]. To prove the uniform convergence on a compact subset $K \subset \Omega_{\gamma_0}$, it suffices to show that for each $z_0 \in \Omega_{\gamma_0}$, the uniform convergence holds in a disc centred at z_0 . Given any $z_0 \in \Omega_{\gamma_0}$, we can find $1 < \alpha \leq \min\{2, \gamma_0\}$ and a small η such that $B_{2\eta}(z_0) \subset \Omega_\alpha^1$ and

$$c_1 = \sup_{z \in B_{2\eta}(z_0)} \frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} < 1. \quad (5.43)$$

For any $N \geq n$, $W_{N+1}^x(z) - W_n^x(z)$ is analytic in z on $B_{2\eta}(z_0)$, so by [18, Lemme 3], we deduce that for all $n \geq 0$,

$$\begin{aligned} \sup_{N \geq n} \sup_{z \in B_\eta(z_0)} |W_{N+1}^x(z) - W_n^x(z)| &\leq \sum_{k=n}^{\infty} \sup_{z \in B_\eta(z_0)} |W_{k+1}^x(z) - W_k^x(z)| \\ &\leq \frac{1}{\pi} \int_0^{2\pi} \sum_{k=n}^{\infty} |W_{k+1}^x(z(t)) - W_k^x(z(t))| dt, \end{aligned} \quad (5.44)$$

where $z(t) = z_0 + 2\eta e^{it}$, $0 \leq t \leq 2\pi$. (This can be easily proved by Cauchy's formula.)

Note that, by Fubini's theorem, for $n \geq 0$,

$$\mathbb{E} \int_0^{2\pi} \sum_{k=n}^{\infty} |W_{k+1}^x(z(t)) - W_k^x(z(t))| dt \leq 2\pi \sup_{z \in \partial B_{2\eta}(z_0)} \sum_{k=n}^{\infty} \mathbb{E} |W_{k+1}^x(z) - W_k^x(z)|, \quad (5.45)$$

where $\partial B_{2\eta}(z_0) = \{z \in \mathbb{C} : |z - z_0| = 2\eta\}$. Therefore, if the right hand side of (5.45) is finite for all $n \geq 0$, then the right-hand side of (5.44) goes to 0 a.s. as $n \rightarrow \infty$, so that a.s. the sequence $(W_n^x(z))$ converges uniformly on $B_\eta(z_0)$.

Now we prove that the right hand side of (5.45) is finite. Notice that

$$W_{k+1}^x(z) - W_k^x(z) = \sum_{u \in \mathbb{T}_k} \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^k r_z(x)} \left(W_1^{X_u^x}(z) - 1 \right). \quad (5.46)$$

Taking the α -th absolute moment at both sides of (5.46) conditional on \mathcal{F}_k and applying Lemma 1 of Biggins [18], we obtain

$$\mathbb{E}_k |W_{k+1}^x(z) - W_k^x(z)|^\alpha \leq 2^\alpha \sum_{u \in \mathbb{T}_k} \left| \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^k r_z(x)} \right|^\alpha \mathbb{E}_k |W_1^{X_u^x}(z) - 1|^\alpha. \quad (5.47)$$

Since the function $z \mapsto r_z$ is analytic on $B_{\eta_1}(0)$ and $r_0 = 1$, there is a constant $d_3 > 0$ such that

$$\frac{\max_{x \in \mathcal{S}} |r_z(x)|}{\min_{x \in \mathcal{S}} |r_z(x)|} \leq d_3 \quad \text{for all } z \in B_{\eta_1}(0). \quad (5.48)$$

Recall that s is the real part of z . Because $B_{3\eta}(z_0) \subset \Omega_\alpha^1 \subset B_{\eta_2}(0) \subset B_{\frac{\eta_1}{2}}(0)$, we have $z, \frac{\alpha s}{2} \in B_{\frac{\eta_1}{2}}(0)$ for $z \in \partial B_{2\eta}(z_0)$. It follows from (5.48) that for all $z \in \partial B_{2\eta}(z_0)$,

$$\begin{aligned} \left| \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^k r_z(x)} \right|^\alpha &\leq \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^k \frac{e^{\alpha s S_u^x} r_{\alpha s}(X_u^x)}{[m\kappa(\alpha s)]^k r_{\alpha s}(x)} \frac{|r_z(X_u^x)|^\alpha r_{\alpha s}(x)}{|r_z(x)|^\alpha r_{\alpha s}(X_u^x)} \\ &\leq d_3^{\alpha+1} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^k \frac{e^{\alpha s S_u^x} r_{\alpha s}(X_u^x)}{[m\kappa(\alpha s)]^k r_{\alpha s}(x)}. \end{aligned} \quad (5.49)$$

On the other hand, from (5.36) and (5.48), we obtain the following estimation, for all

$z \in \partial B_{2\eta}(z_0)$,

$$\begin{aligned}
& \mathbb{E}_k |W_1^{X_u^x}(z) - 1|^\alpha \\
& \leq 2^{\alpha-1} \left(\mathbb{E}_k |W_1^{X_u^x}(z)|^\alpha + 1 \right) = 2^{\alpha-1} \mathbb{E}_k \left| \sum_{v \in \mathbb{T}_1(u)} \frac{e^{zS_v^{X_u^x}} r_z(X_v^{X_u^x})}{m\kappa(z) r_z(X_u^x)} \right|^\alpha + 2^{\alpha-1} \\
& \leq 2^{\alpha-1} \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^\alpha \mathbb{E}_k \left[\sum_{v \in \mathbb{T}_1(u)} \frac{|r_z(X_v^{X_u^x})| r_s(X_u^x) e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x})}{|r_z(X_u^x)| r_s(X_v^{X_u^x}) m\kappa(s) r_s(X_u^x)} \right]^\alpha + 2^{\alpha-1} \\
& \leq d_3^{2\alpha} 2^{\alpha-1} \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^\alpha \mathbb{E} \sup_{x \in \mathcal{S}} (W_1^x(s))^\alpha + 2^{\alpha-1}.
\end{aligned}$$

Combining this with (5.47) and (5.49) gives, for all $z \in \partial B_{2\eta}(z_0)$,

$$\begin{aligned}
& \mathbb{E}_k |W_{k+1}^x(z) - W_k^x(z)|^\alpha \\
& \leq c \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^k W_n^x(\alpha s) \left[\left(\frac{\kappa(s)}{|\kappa(z)|} \right)^\alpha \mathbb{E} \sup_{x \in \mathcal{S}} (W_1^x(s))^\alpha + 1 \right].
\end{aligned}$$

Taking expectation at both sides of this inequality and using Jensen's inequality, we obtain for all $z \in \partial B_{2\eta}(z_0)$,

$$\mathbb{E} |W_{k+1}^x(z) - W_k^x(z)| \leq c^{\frac{1}{\alpha}} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{\frac{k}{\alpha}} \left[\left(\frac{\kappa(s)}{|\kappa(z)|} \right)^\alpha \mathbb{E} \sup_{x \in \mathcal{S}} (W_1^x(s))^\alpha + 1 \right]^{\frac{1}{\alpha}}.$$

From (5.43), (5.37), the analyticity of $\kappa(z)$ on $\partial B_{2\eta}(z_0) \subset B_{\eta_1}(0)$ and the fact that $|\kappa(z)| > 0$ for all $z \in B_{\eta_1}(0)$, we obtain

$$\sup_{z \in \partial B_{2\eta}(z_0)} \mathbb{E} |W_{k+1}^x(z) - W_k^x(z)| \leq C c_1^{\frac{k}{\alpha}}, \quad (5.50)$$

This concludes that (5.45) is finite for all $n \geq 0$. We have therefore proved that it is a.s. that the sequence $(W_n^x(z))$ converges uniformly on $B_\eta(z_0)$ for each $z_0 \in \Omega_{\gamma_0}$, which implies the uniform convergence on each compact subset $K \subset \Omega_{\gamma_0}$.

We now come to the speed of convergence (5.42). Clearly, it is enough to prove that there is a $\delta \in (0, 1)$ such that on each compact subset $K \subset \Omega_{\gamma_0}$,

$$\delta^{-n} \sup_{z \in K} |W_{n+1}^x(z) - W_n^x(z)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} \quad (5.51)$$

From (5.44), (5.45) and (5.50), we have for each $z_0 \in \Omega_{\gamma_0}$, there is $\eta > 0$ small enough

such that for all $n \geq 0$,

$$\mathbb{E} \sup_{z \in B_{\eta}(z_0)} |W_{n+1}^x(z) - W_n^x(z)| \leq 2 \sum_{k=n}^{\infty} C c_1^{\frac{k}{\alpha}},$$

where C and c_1 are constants which may depend on z_0 . Since K is compact, by Borel's theorem, K can be covered by a finite number of open balls $B_{\eta_i}(z_i)$, $i = 1, \dots, n_0$, so that there exist two constants $C_1 > 0$ and $c_2 \in (0, 1)$ which may depend on K , such that for $n \geq 0$,

$$\mathbb{E} \sup_{z \in K} |W_{n+1}^x(z) - W_n^x(z)| \leq 2 \sum_{k=n}^{\infty} C_1 c_2^k \leq C_2 c_2^n. \quad (5.52)$$

Taking $\delta \in (c_2, 1)$ and using Fubini's theorem we see that

$$\mathbb{E} \sum_{n=0}^{\infty} \delta^{-n} \sup_{z \in K} |W_{n+1}^x(z) - W_n^x(z)| \leq C_2 \sum_{n=0}^{\infty} \left(\frac{c_2}{\delta}\right)^n < \infty,$$

so that

$$\sum_{n=0}^{\infty} \delta^{-n} \sup_{z \in K} |W_{n+1}^x(z) - W_n^x(z)| < \infty \quad \text{a.s.}$$

Therefore, (5.51) is proved. This ends the proof of (5.42).

Finally, since a.s. each $W_n^x(z)$ is analytic on Ω_{γ_0} and the sequence $(W_n^x(z))$ converges uniformly on each compact set of Ω_{γ_0} , a standard result of complex analysis (see e.g. Corollary 2.2.4 in Hörmander [51]) gives the analyticity of $W^x(z)$ on Ω_{γ_0} . \square

In the following we introduce a new martingale and prove its uniform convergence and the analyticity of its limit. This is an important ingredient in the proof of Theorem 5.2.3 about the Berry-Esseen bound for the changed measure $Z_{s,n}^x$, which is crucial in establishing the main results of this paper. For $z \in B_{\eta_1}(0)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_{\beta}$, set

$$\widetilde{W}_n^x(z) = \sum_{u \in \mathbb{T}_n} \frac{e^{z S_u^x} M_z(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)}, \quad n \geq 0,$$

where M_z is defined in (5.22) and $(r_s \varphi)(X_u^x) := r_s(X_u^x) \varphi(X_u^x)$.

Theorem 5.4.3. *Assume conditions L1-L3. Then the sequence $(\widetilde{W}_n^x(z))_{n \geq 0}$ is a martingale with respect to the filtration (\mathcal{F}_n) and converges a.s. to some complex valued random*

variable $\widetilde{W}^x(z)$, uniformly in z on any compact subset $K \subset \Omega_{\gamma_0}$, and the limit $\widetilde{W}^x(z)$ is analytic on Ω_{γ_0} .

Proof. The fact that $(\widetilde{W}_n^x(z), \mathcal{F}_n)_{n \geq 0}$ is a martingale can be easily shown: it suffices to notice that

$$\mathbb{E}_n \widetilde{W}_{n+1}^x(z) = \sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x} M_z(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} \mathbb{E}_n \left[\sum_{v \in \mathbb{T}_1(u)} \frac{e^{zS_v^{X_u^x}} M_z(r_s \varphi)(X_v^{X_u^x})}{m\kappa(z) M_z(r_s \varphi)(X_u^x)} \right],$$

where $\mathbb{T}_1(u)$ represents the descendants of $u \in \mathbb{T}_n$ at time $n+1$. Moreover, by the branching property, the definition of P_z (5.21) and Lemma 5.3.1(1), we have for $u \in \mathbb{T}_n$,

$$\begin{aligned} \mathbb{E}_n \left[\sum_{v \in \mathbb{T}_1(u)} \frac{e^{zS_v^{X_u^x}} M_z(r_s \varphi)(X_v^{X_u^x})}{m\kappa(z) M_z(r_s \varphi)(X_u^x)} \right] &= \frac{\mathbb{E}_n \left[|A_{u1} X_u^x|^z M_z(r_s \varphi)(X_{u1}^{X_u^x}) \right]}{\kappa(z) M_z(r_s \varphi)(X_u^x)} \\ &= \frac{P_z(M_z(r_s \varphi))(X_u^x)}{\kappa(z) M_z(r_s \varphi)(X_u^x)} = 1. \end{aligned}$$

The proof of the uniform convergence and the analyticity of the limit is the same as in the proof of Theorem 5.4.2, whose details are omitted. \square

5.5 Proof of Theorems 5.2.1 and 5.2.3

Theorem 5.2.1 is a particular case of Theorem 5.2.3 with $s = 0$. Thus we only prove Theorem 5.2.3. Our proof is based on Petrov's method [75] for the proof of the Cramér's moderate deviation asymptotic on sums of i.i.d. real random variables. We split the proof of Theorem 5.2.3 into two theorems: Theorems 5.5.1 and 5.5.2, whose combination gives Theorem 5.2.3.

Theorem 5.5.1. *Under the conditions of Theorem 5.2.3. Then, for any $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\beta$ there exists a constant $\eta \in (0, \eta_2)$ such that a.s., for $n \geq 1$,*

$$\sup_{s \in (-\eta, \eta)} \left| \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} - W^x(s) \pi_s(\varphi) \right| \leq M \delta^n,$$

where M is a positive and finite random variable and $\delta \in (0, 1)$.

Theorem 5.5.2. *Under the conditions of Theorem 5.2.3. Then, for any $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\beta$ there exists a constant $\eta \in (0, \eta_2)$ such that uniformly in $s \in (-\eta, \eta)$ and $y \in \mathbb{R}$,*

a.s., for $n \geq 1$,

$$\left| \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} - \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x) \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \Phi(y) \right| \leq \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable (independent of s).

5.5.1 Proof of Theorem 5.5.1

The following decomposition which follows from the branching property will play a key role in our approach with a delicate choice of k for $0 < k \leq n$,

$$\sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} (r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} = \sum_{u \in \mathbb{T}_k} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^k r_s(x)} \sum_{v \in \mathbb{T}_{n-k}(u)} \frac{e^{sS_v^{X_u^x}} (r_s \varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k} r_s(X_u^x)}. \quad (5.53)$$

Recall that by our definition, for $u \in \mathbb{T}_k$, $\mathbb{T}_{n-k}(u)$ represents the descendants of u at time n .

For each n , we choose an integer $k_n = \lceil \frac{n}{2} \rceil$, which is the least integer greater than or equal to $\frac{n}{2}$. For brevity, we denote for $u \in \mathbb{T}_{k_n}$,

$$Y_{n-k_n}^u(s) = \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}} (r_s \varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)}.$$

Then by (5.53), the following decomposition holds:

$$\sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} (r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} - W^x(s) \pi_s(\varphi) = A_n(s) + B_n(s) + C_n(s), \quad (5.54)$$

where

$$\begin{aligned} A_n(s) &= \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \left[Y_{n-k_n}^u(s) - \mathbb{E}_{k_n} Y_{n-k_n}^u(s) \right], \\ B_n(s) &= \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \left[\mathbb{E}_{k_n} Y_{n-k_n}^u(s) - \pi_s(\varphi) \right], \\ C_n(s) &= [W_{k_n}^x(s) - W^x(s)] \pi_s(\varphi). \end{aligned}$$

By virtue of the decomposition (5.54), we shall divide the proof of Theorem 5.5.1 into

three lemmas.

Lemma 5.5.3. *Under the conditions of Theorem 5.2.3, there exist two constants $\eta \in (0, \eta_2)$ and $\delta \in (0, 1)$ such that*

$$\delta^{-n} \sup_{s \in (-\eta, \eta)} |A_n(s)| \xrightarrow{n \rightarrow \infty} 0, \quad a.s.$$

Proof. To prove Lemma 5.5.3, we will use the Borel-Cantelli Lemma. We can obtain the required result once we prove that there exist a small $\eta > 0$ and a constant $\delta \in (0, 1)$ such that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta^{-n} \sup_{s \in (-\eta, \eta)} |A_n(s)| > \varepsilon) < \infty. \tag{5.55}$$

By Markov’s inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta^{-n} \sup_{s \in (-\eta, \eta)} |A_n(s)| > \varepsilon) \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \delta^{-n} \mathbb{E} \sup_{s \in (-\eta, \eta)} |A_n(s)|. \tag{5.56}$$

Because Ω_{γ_0} is an open set containing 0, we can find a small $\rho > 0$ such that $B_\rho(0) \subset \Omega_\alpha^1$ for some $1 < \alpha \leq \min\{2, \gamma_0\}$. Let $\eta \in (0, \frac{\rho}{3})$ whose value will be fixed later. Then $B_{3\eta}(0) \subset B_\rho(0)$. We see that for every $n \in \mathbb{N}$, the function

$$z \mapsto A_n(z) = \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^{k_n} r_z(x)} \left[Y_{n-k_n}^u(z) - \mathbb{E}_{k_n} Y_{n-k_n}^u(z) \right]$$

is well-defined as an analytic function on $B_{\eta_1}(0)$. Recall that s is the real part of z . By Lemma 3 of Biggins [18], we have

$$\sup_{s \in (-\eta, \eta)} |A_n(s)| \leq \sup_{z \in B_\eta(0)} |A_n(z)| \leq \frac{1}{\pi} \int_0^{2\pi} |A_n(z(t))| dt,$$

where $z(t) = 2\eta e^{it}$, $0 \leq t \leq 2\pi$. Note that, by Fubini’s theorem,

$$\mathbb{E} \sup_{s \in (-\eta, \eta)} |A_n(s)| \leq \int_0^{2\pi} \mathbb{E} |A_n(z(t))| dt \leq 2\pi \sup_{|z|=2\eta} \mathbb{E} |A_n(z)|. \tag{5.57}$$

Consider now $\mathbb{E} |A_n(z)|$ for $|z| = 2\eta$. Taking the α -th absolute moment of $A_n(z)$

conditional on \mathcal{F}_k and applying Lemma 1 of Biggins [18], we obtain

$$\mathbb{E}_{k_n} |A_n(z)|^\alpha \leq 2^\alpha \sum_{u \in \mathbb{T}_{k_n}} \left| \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^{k_n} r_z(x)} \right|^\alpha \mathbb{E}_{k_n} |Y_{n-k_n}^u(z) - \mathbb{E}_{k_n} Y_{n-k_n}^u(z)|^\alpha. \quad (5.58)$$

Because $B_{3\eta}(0) \subset B_\rho(0) \subset B_{\eta_2}(0) \subset B_{\frac{\eta_1}{2}}(0)$, we see that if $|z| = 2\eta$, then $z, \frac{\alpha s}{2} \in B_{\frac{\eta_1}{2}}(0)$. Hence, by (5.48), we get for $|z| = 2\eta$,

$$\begin{aligned} \left| \frac{e^{zS_u^x} (r_z \varphi)(X_u^x)}{[m\kappa(z)]^{k_n} r_z(x)} \right|^\alpha &\leq \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{k_n} \frac{e^{\alpha s S_u^x} r_{\alpha s}(X_u^x)}{[m\kappa(\alpha s)]^{k_n} r_{\alpha s}(x)} \frac{|r_z(X_u^x)|^\alpha r_{\alpha s}(x)}{|r_z(x)|^\alpha r_{\alpha s}(X_u^x)} \\ &\leq d_3^{1+\alpha} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{k_n} \frac{e^{\alpha s S_u^x} r_{\alpha s}(X_u^x)}{[m\kappa(\alpha s)]^{k_n} r_{\alpha s}(x)}. \end{aligned} \quad (5.59)$$

We now estimate the expectation in (5.58). Using $|a+b|^\alpha \leq 2^{\alpha-1}(|a|^\alpha + |b|^\alpha) \leq 2(|a|^\alpha + |b|^\alpha)$ and (5.48), we have for $|z| = 2\eta$,

$$\begin{aligned} &\mathbb{E}_{k_n} |Y_{n-k_n}^u(z) - \mathbb{E}_{k_n} Y_{n-k_n}^u(z)|^\alpha \leq 2 \mathbb{E}_{k_n} |Y_{n-k_n}^u(z)|^\alpha \\ &\leq 2 \mathbb{E}_{k_n} \left[\sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} \frac{|r_z \varphi(X_v^{X_u^x})| r_s(X_u^x)}{|r_z(X_u^x)| r_s(X_v^{X_u^x})} \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^{n-k_n} \right]^\alpha \\ &\leq 2(d_3^2 \|\varphi\|_\beta)^\alpha \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^{\alpha(n-k_n)} \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x}(s) \right]^\alpha \\ &\leq 2(d_3^2 \|\varphi\|_\beta)^\alpha \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^{\alpha(n-k_n)} \sup_{x \in \mathcal{S}} \mathbb{E}(W_*^x(s))^\alpha. \end{aligned} \quad (5.60)$$

From (5.58), (5.59) and (5.60), we have for all $\eta > 0$ small enough and $|z| = 2\eta$,

$$\mathbb{E}_{k_n} |A_n(z)|^\alpha \leq c \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{k_n} \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^{\alpha(n-k_n)} W_{k_n}^x(\alpha s) \sup_{x \in \mathcal{S}} \mathbb{E}(W_*^x(s))^\alpha. \quad (5.61)$$

Since $\alpha s \in (-\eta_1, \eta_1)$, $(W_n^x(\alpha s))$ is a martingale, so $\mathbb{E}[W_n^x(\alpha s)] = 1$. Taking expectations at both sides of (5.61), we obtain for $|z| = 2\eta$,

$$\mathbb{E}|A_n(z)|^\alpha \leq c \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{k_n} \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^{\alpha(n-k_n)} \sup_{x \in \mathcal{S}} \mathbb{E}(W_*^x(s))^\alpha. \quad (5.62)$$

From (5.57), Jensen's inequality and (5.62), we get that

$$\begin{aligned} & \mathbb{E} \sup_{s \in (-\eta, \eta)} |A_n(s)| \\ & \leq c^{\frac{1}{\alpha}} \sup_{|z|=2\eta} \left\{ \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{\frac{k_n}{\alpha}} \left| e^{(n-k_n)[\Lambda(s)-\Lambda(z)]} \right| \left[\sup_{x \in \mathcal{S}} \mathbb{E}(W_*^x(s))^\alpha \right]^{\frac{1}{\alpha}} \right\}. \end{aligned} \quad (5.63)$$

From the facts that $B_{3\eta}(0) \subset B_\rho(0) \subset \Omega_\alpha^1$ and the definition of Ω_α^1 , we obtain

$$\sup_{|z|=2\eta} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{\frac{1}{\alpha}} \leq \sup_{z \in B_\rho(0)} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{\frac{1}{\alpha}} =: c_1 < 1. \quad (5.64)$$

From (5.63), (5.64) and the choice of k_n which implies that $k_n \geq n - k_n$, we get

$$\mathbb{E} \sup_{s \in (-\eta, \eta)} |A_n(s)| \leq c c_1^{n-k_n} \sup_{|z|=3\eta} \left\{ \left| e^{(n-k_n)[\Lambda(s)-\Lambda(z)]} \right| \left[\sup_{x \in \mathcal{S}} \mathbb{E}(W_*^x(s))^\alpha \right]^{\frac{1}{\alpha}} \right\}. \quad (5.65)$$

By Theorem 5.4.1, for $\eta > 0$ small enough,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \mathbb{E}(W_*^x(s))^\alpha < \infty. \quad (5.66)$$

Note that $c_1 < 1$ is independent of η . Let $c_2 \in (1, \frac{1}{c_1})$. Since Λ is continuous on $B_{\eta_1}(0)$ and $\Lambda(0) = 0$, there exists a small $\eta_3 > 0$ such that

$$\sup_{z \in B_{\eta_3}(0)} \left| e^{[\Lambda(s)-\Lambda(z)]} \right| \leq c_2. \quad (5.67)$$

Take η small enough such that $\eta < \eta_3$. Since $k_n = \lceil \frac{n}{2} \rceil$, we have $n - k_n \geq \frac{n}{2} - 1$. So combining (5.65), (5.66), (5.67) we obtain for all $\eta > 0$ small enough,

$$\mathbb{E} \sup_{s \in (-\eta, \eta)} |A_n(s)| \leq c(c_1 c_2)^{n-k_n} \leq c(c_1 c_2)^{\frac{n}{2}-1}.$$

Therefore, using (5.56) and taking $\delta \in \left((c_1 c_2)^{\frac{1}{2}}, 1 \right)$, we get that

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta^{-n} \sup_{s \in (-\eta, \eta)} |A_n(s)| > \varepsilon) \leq \frac{c}{\varepsilon c_1 c_2} \sum_{n=1}^{\infty} \left(\frac{(c_1 c_2)^{\frac{1}{2}}}{\delta} \right)^n < \infty.$$

This completes the proof of Lemma 5.5.3. □

Lemma 5.5.4. *Under the conditions of Theorem 5.2.3, there exist two constants $\eta \in (0, \eta_2)$ and $\delta \in (0, 1)$ such that*

$$\delta^{-n} \sup_{s \in (-\eta, \eta)} |B_n(s)| \xrightarrow{n \rightarrow \infty} 0 \quad a.s.$$

Proof. Using the branching property and the definition of \mathbb{Q}_s^x (5.23), we have for $u \in \mathbb{T}_{k_n}$,

$$\begin{aligned} \mathbb{E}_{k_n} Y_{n-k_n}^u(s) &= \mathbb{E}_{k_n} \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}}(r_s \varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} \\ &= \frac{\mathbb{E}_{k_n} \left[e^{sS_{n-k_n}^{X_u^x}}(r_s \varphi)(X_{n-k_n}^{X_u^x}) \right]}{\kappa^{n-k_n}(s) r_s(X_u^x)} \\ &= \mathbb{E}_{\mathbb{Q}_s^{X_u^x}} [\varphi(X_{n-k_n}^{X_u^x})]. \end{aligned}$$

Hence

$$\begin{aligned} |B_n(s)| &\leq \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{sS_u^{X_u^x}} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_{n-k_n}^x)] - \pi_s(\varphi)| \\ &\leq W_{k_n}^x(s) \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_{n-k_n}^x)] - \pi_s(\varphi)|. \end{aligned}$$

By Theorem 5.4.2 and the bound (5.24), for $\eta \in (0, \eta_2)$, there exist a constant $c \in (0, 1)$ and a positive finite random variable M such that for all $n \geq 0$,

$$\sup_{s \in (-\eta, \eta)} |B_n(s)| \leq M c^{n-k_n} \leq M c^{\frac{n}{2}-1}.$$

Therefore the conclusion of Lemma 5.5.4 holds for each $\delta \in (c^{\frac{1}{2}}, 1)$. □

Lemma 5.5.5. *Under the conditions of Theorem 5.2.3, there exist two constants $\eta \in (0, \eta_2)$ and $\delta \in (0, 1)$ such that*

$$\delta^{-n} \sup_{s \in (-\eta, \eta)} |C_n(s)| \xrightarrow{n \rightarrow \infty} 0 \quad a.s.$$

Proof. This is an immediate consequence of Theorem 5.4.2 and the fact that $|\pi_s(\varphi)| \leq \|\varphi\|_\infty$. □

5.5.2 Proof of Theorem 5.5.2

To prove Theorem 5.5.2, we need the following result.

Lemma 5.5.6. *Under the conditions of Theorem 5.5.2, there is a constant $\eta \in (0, \eta_2)$ such that*

$$\sup_{z \in B_\eta(0)} \left| \sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} - \widetilde{W}^x(z) \right| \xrightarrow{n \rightarrow \infty} 0 \quad a.s. \quad (5.68)$$

Moreover, $\sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)}$ is a.s. bounded by a positive and finite random variable uniformly in $z \in B_\eta(0)$ and $n \geq 0$.

Proof. By the branching property, for $k \leq n$,

$$\sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} = \sum_{u \in \mathbb{T}_k} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^k r_s(x)} \sum_{v \in \mathbb{T}_{n-k}(u)} \frac{e^{zS_v^{X_u^x}}(r_s \varphi)(X_v^{X_u^x})}{[m\kappa(z)]^{n-k} r_s(X_u^x)}. \quad (5.69)$$

As before, for each n , we take $k_n = \lfloor \frac{n}{2} \rfloor$. For brevity, we denote for $u \in \mathbb{T}_{k_n}$,

$$\widetilde{Y}_{n-k_n}^u(z) = \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{zS_v^{X_u^x}}(r_s \varphi)(X_v^{X_u^x})}{[m\kappa(z)]^{n-k_n} r_s(X_u^x)}.$$

Then by (5.69), the following decomposition holds:

$$\sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} - \widetilde{W}^x(z) = A_n(z) + B_n(z) + C_n(z), \quad (5.70)$$

where

$$\begin{aligned} A_n(z) &= \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \left[\widetilde{Y}_{n-k_n}^u(z) - \mathbb{E}_{k_n} \widetilde{Y}_{n-k_n}^u(z) \right], \\ B_n(z) &= \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \left[\mathbb{E}_{k_n} \widetilde{Y}_{n-k_n}^u(z) - \frac{M_z(r_s \varphi)(X_u^x)}{r_s(X_u^x)} \right], \\ C_n(z) &= \widetilde{W}_{k_n}^x(z) - \widetilde{W}^x(z). \end{aligned}$$

By virtue of the decomposition (5.70), in order to prove (5.68), it suffices to show that

there is a constant $\eta \in (0, \eta_2)$ such that

$$\sup_{z \in B_\eta(0)} |A_n(z)| \xrightarrow{n \rightarrow \infty} 0, \quad \text{a.s.}, \quad (5.71)$$

$$\sup_{z \in B_\eta(0)} |B_n(z)| \xrightarrow{n \rightarrow \infty} 0, \quad \text{a.s.}, \quad (5.72)$$

$$\sup_{z \in B_\eta(0)} |C_n(z)| \xrightarrow{n \rightarrow \infty} 0, \quad \text{a.s.} \quad (5.73)$$

The proof of (5.71) is similar to that of Lemma 5.5.3, and is omitted here. It is clear that (5.73) is an immediate consequence of Theorem 5.4.3. It remains to prove (5.72). By the branching property and the definition of the operator P_z (see (5.21)), we have

$$\begin{aligned} \mathbb{E}_{k_n} \tilde{Y}_{n-k_n}^u(z) &= \mathbb{E}_{k_n} \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{zS_v^{X_u^x}}(r_s \varphi)(X_v^{X_u^x})}{[m\kappa(z)]^{n-k_n} r_s(X_u^x)} \\ &= \frac{\mathbb{E}_{k_n} \left[e^{zS_{n-k_n}^{X_u^x}}(r_s \varphi)(X_{n-k_n}^{X_u^x}) \right]}{\kappa^{n-k_n}(z) r_s(X_u^x)} \\ &= \frac{P_z^{n-k_n}(r_s \varphi)(X_u^x)}{\kappa^{n-k_n}(z) r_s(X_u^x)}. \end{aligned}$$

Hence, by the decomposition (5.22) and Lemma 5.3.1(4), for any $z \in B_{\eta_1}(0)$, we have

$$\begin{aligned} |B_n(z)| &\leq \left| \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \left[\frac{P_z^{n-k_n}(r_s \varphi)(X_u^x)}{\kappa^{n-k_n}(z) r_s(X_u^x)} - \frac{M_z(r_s \varphi)(X_u^x)}{r_s(X_u^x)} \right] \right| \\ &\leq \sum_{u \in \mathbb{T}_{k_n}} \left| \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \frac{L_z^{n-k_n}(r_s \varphi)(X_u^x)}{\kappa^{n-k_n}(z) r_s(X_u^x)} \right| \\ &\leq \frac{\|L_z^{n-k_n}\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta}}{|k(z)|^{n-k_n}} \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \left(\frac{\kappa(s)}{|\kappa(z)|} \right)^{k_n} \frac{\|r_s \varphi\|_\beta}{\min_{y \in \mathcal{S}} r_s(y)} \\ &\leq c \left(\frac{1-a_2}{1-a_1} \right)^{n-k_n} \left| e^{k_n[\Lambda(s)-\Lambda(z)]} \right| W_{k_n}^x(s), \end{aligned} \quad (5.74)$$

where $0 < a_1 < a_2 < 1$ is defined in Lemma 5.3.1(4). In the last step we use the fact that $\|r_s \varphi\|_\beta \leq 3\|r_s\|_\beta \|\varphi\|_\beta \leq c$ and that the map $s \mapsto r_s$ is analytic with $r_0 = 1$. Since $k_n = \lceil \frac{n}{2} \rceil$, we have $n - k_n \geq \frac{n}{2} - 1 \geq k_n - 2$, so $\left(\frac{1-a_2}{1-a_1} \right)^{n-k_n} \leq \left(\frac{1-a_2}{1-a_1} \right)^{\frac{n}{2}-1} \leq \left(\frac{1-a_2}{1-a_1} \right)^{k_n-2}$. Let $c_1 \in \left(1, \frac{1-a_1}{1-a_2} \right)$. Using the facts that the function Λ is continuous on $B_{\eta_1}(0)$ and $\Lambda(0) = 0$,

there exist a small $\eta \in (0, \eta_1)$ such that

$$\sup_{z \in B_\eta(0)} \left| e^{[\Lambda(s) - \Lambda(z)]} \right| \leq c_1. \quad (5.75)$$

By Theorem 5.4.2, for $\eta \in (0, \eta_2)$ small enough, $\sup_{s \in (-\eta, \eta)} W_{k_n}^x(s) \leq M$, where M is a positive and finite random variable. This together with (5.74) and (5.75) implies that for $\eta \in (0, \eta_2)$ small enough,

$$\sup_{z \in B_\eta(0)} |B_n(z)| \leq c_2 M \left[\frac{c_1(1-a_2)}{1-a_1} \right]^{k_n} \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

This completes the proof of (5.72). So the proof of (5.68) is finished.

The uniform bound of $\sum_{u \in \mathbb{T}_n} \frac{e^{z S_u^x(r_s \varphi)(X_u^x)}}{[m\kappa(z)]^n r_s(x)}$ is an immediate consequence of (5.68) and the fact that $\widetilde{W}^x(z)$ is analytic in z (by Theorem 5.4.3). \square

Proof of Theorem 5.5.2. For simplicity, we suppose that $\varphi \geq 0$; otherwise we can consider the positive and negative parts of φ to conclude. Consider the distribution functions of finite measures:

$$F_{s,n}(y) = \sum_{u \in \mathbb{T}_n} \frac{e^{s S_u^x(r_s \varphi)(X_u^x)}}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}}, \quad y \in \mathbb{R},$$

$$H_{s,n}(y) = \sum_{u \in \mathbb{T}_n} \frac{e^{s S_u^x(r_s \varphi)(X_u^x)}}{[m\kappa(s)]^n r_s(x)} \Phi(y), \quad y \in \mathbb{R},$$

and their characteristic functions at $-t$:

$$f_{s,n}(t) = \int_{\mathbb{R}} e^{-ity} dF_{s,n}(y), \quad h_{s,n}(t) = \int_{\mathbb{R}} e^{-ity} dH_{s,n}(y), \quad t \in \mathbb{R}.$$

By straightforward calculations we have

$$h_{s,n}(t) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} e^{-\frac{t^2}{2}} \quad (5.76)$$

$$\begin{aligned} f_{s,n}(t) &= \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} e^{-it \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}}} \\ &= \sum_{u \in \mathbb{T}_n} \frac{e^{(s - \frac{it}{\sigma_s \sqrt{n}})S_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s - \frac{it}{\sigma_s \sqrt{n}})]^n r_s(x)} \left(\frac{\kappa(s - \frac{it}{\sigma_s \sqrt{n}})}{\kappa(s)} \right)^n e^{\frac{itn\Lambda'(s)}{\sigma_s \sqrt{n}}} \\ &= \sum_{u \in \mathbb{T}_n} \frac{e^{(s - \frac{it}{\sigma_s \sqrt{n}})S_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s - \frac{it}{\sigma_s \sqrt{n}})]^n r_s(x)} \lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n, \end{aligned} \quad (5.77)$$

where the last equality holds by the definition of $\lambda_{s,it}$ (see (5.27)).

Notice that $F_{s,n}(-\infty) = H_{s,n}(-\infty) = 0$, $F_{s,n}(+\infty) = H_{s,n}(+\infty) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)}$, $F_{s,n}$ and $H_{s,n}$ are non-decreasing on \mathbb{R} , and $H_{s,n}$ is differentiable on \mathbb{R} . So by Esseen's smoothing inequality (see [75, Theorem V.2.2.]), for all $T > 0$ and $s \in (-\eta_1, \eta_1)$,

$$\begin{aligned} \sup_{y \in \mathbb{R}} |F_{s,n}(y) - H_{s,n}(y)| &\leq \frac{1}{\pi} \int_{-T}^T \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt \\ &\quad + \frac{c_0}{T} \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)}, \end{aligned}$$

where c_0 is a positive constant. Therefore, to prove Theorem 5.5.2, it suffices to show that there exists a small $\eta \in (0, \eta_2)$ such that as $n \rightarrow \infty$, a.s.,

$$\sup_{s \in (-\eta, \eta)} \int_{-T}^T \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt = O\left(\frac{1}{\sqrt{n}}\right), \quad (5.78)$$

and

$$\sup_{s \in (-\eta, \eta)} \frac{c_0}{T} \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} = O\left(\frac{1}{\sqrt{n}}\right). \quad (5.79)$$

In the following, we denote by M_i a positive and finite random variable. Let $T := \eta \sigma \sqrt{n}$ with $\eta > 0$ small enough such that the conclusion in Lemma 5.5.6 holds, where $\underline{\sigma} := \inf_{s \in (-\eta, \eta)} \sigma_s > 0$. By Lemma 5.5.6, we have

$$\sup_{s \in (-\eta, \eta)} \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} \leq M_1.$$

Hence (5.79) is proved since

$$\sup_{s \in (-\eta, \eta)} \frac{c_0}{T} \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} \leq \frac{c_0 M_1}{\eta \underline{\sigma} \sqrt{n}}.$$

It remains to prove (5.78). We will prove this by showing that there exists a small $\eta \in (0, \eta_2)$ such that as $n \rightarrow \infty$, a.s.,

$$I_1(n) + I_2(n) = O\left(\frac{1}{\sqrt{n}}\right), \quad (5.80)$$

where

$$I_1(n) = \sup_{s \in (-\eta, \eta)} \int_{|t| < \delta_1 \underline{\sigma} \sqrt{n}} \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt,$$

$$I_2(n) = \sup_{s \in (-\eta, \eta)} \int_{\delta_1 \underline{\sigma} \sqrt{n} \leq |t| \leq \eta \underline{\sigma} \sqrt{n}} \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt,$$

with $\delta_1 \in (0, \eta)$ whose value will be fixed later.

Control of $I_1(n)$. Denote for $z = s + it$ with $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$,

$$U_n(z) = \sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x}(r_s \varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)}.$$

With this notation and using (5.76) and (5.77), we have

$$I_1(n) \leq I_{11}(n) + I_{12}(n),$$

where

$$I_{11}(n) = \sup_{s \in (-\eta, \eta)} \int_{|t| < \delta_1 \underline{\sigma} \sqrt{n}} \left| \frac{\lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n}{t} \left(U_n\left(s - \frac{it}{\sigma_s \sqrt{n}}\right) - U_n(s) \right) \right| dt$$

$$I_{12}(n) = \sup_{s \in (-\eta, \eta)} \int_{|t| < \delta_1 \underline{\sigma} \sqrt{n}} \left| \frac{\left(\lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n - e^{-\frac{t^2}{2}} \right) U_n(s)}{t} \right| dt.$$

For $I_{11}(n)$, by Taylor's formula and the fact that $\Lambda''(s) = \sigma_s^2$, we have

$$\begin{aligned} \lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n &= e^{n[\Lambda(s - \frac{it}{\sigma_s \sqrt{n}}) - \Lambda(s) + \Lambda'(s) \frac{it}{\sigma_s \sqrt{n}}]} \\ &= e^n \sum_{k=2}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k \\ &= e^{-\frac{t^2}{2}} e^n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k. \end{aligned} \tag{5.81}$$

By choosing δ_1 small enough, we have for all $s \in (-\eta, \eta)$ and $|t| < \delta_1 \sigma \sqrt{n}$,

$$\left| \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k \right| \leq \frac{t^2}{4n}, \tag{5.82}$$

and so, from (5.81),

$$\left| \lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n \right| \leq e^{-\frac{t^2}{4}}. \tag{5.83}$$

Therefore, for η and δ_1 small enough,

$$I_{11}(n) \leq \sup_{s \in (-\eta, \eta)} \int_{|t| < \delta_1 \sigma \sqrt{n}} \frac{e^{-\frac{t^2}{4}}}{|t|} \left| U_n\left(s - \frac{it}{\sigma_s \sqrt{n}}\right) - U_n(s) \right| dt. \tag{5.84}$$

By Lemma 5.5.6, there is a constant η_4 small enough such that for all $n \geq 0$,

$$\sup_{|z| = \frac{\eta_4}{2}} |U_n(z)| \leq M_2. \tag{5.85}$$

Notice that U_n is a.s. analytic on $B_{\eta_1}(0)$. Let $\eta, \delta_1 > 0$ be small enough such that $\eta + i\delta_1 \in B_{\frac{\eta_4}{3}}(0)$. By the mean value theorem, for $s \in (-\eta, \eta)$ and $\frac{t}{\sigma_s \sqrt{n}} \in (-\delta_1, \delta_1)$, we have

$$\begin{aligned} \left| U_n\left(s - \frac{it}{\sigma_s \sqrt{n}}\right) - U_n(s) \right| &\leq \frac{|t|}{\sigma_s \sqrt{n}} \sup_{t \in (-\delta_1, \delta_1)} \left| U_n'\left(s - \frac{it}{\sigma_s \sqrt{n}}\right) \right| \\ &\leq \frac{|t|}{\sigma_s \sqrt{n}} \sup_{z \in B_{\frac{\eta_4}{3}}(0)} |U_n'(z)|. \end{aligned} \tag{5.86}$$

By the Cauchy's formula, when $z \in B_{\frac{\eta_4}{2}}(0)$,

$$U'_n(z) = \frac{1}{2\pi i} \int_{|w|=\frac{\eta_4}{2}} \frac{U_n(w)}{(w-z)^2} dw.$$

Hence, by (5.85) and the fact that $|w-z| \geq \frac{\eta_4}{6}$ for $z \in B_{\frac{\eta_4}{3}}(0)$ and $|w| = \frac{\eta_4}{2}$, we have

$$\sup_{z \in B_{\frac{\eta_4}{3}}(0)} |U'_n(z)| \leq \frac{18M_2}{\eta_4}.$$

Combining this with (5.84), (5.86) and the fact that $\sigma_s > \underline{\sigma}$ for all $s \in (-\eta, \eta)$, we obtain

$$I_{11}(n) \leq \sup_{s \in (-\eta, \eta)} \frac{18M_2}{\sigma_s \sqrt{n} \eta_4} \int_{|t| < \delta_1 \underline{\sigma} \sqrt{n}} e^{-\frac{t^2}{4}} dt \leq \frac{M_3}{\sqrt{n}}. \quad (5.87)$$

For $I_{12}(n)$, using (5.81), the inequality $|e^z - 1| \leq |z|e^{|z|}$ for all $z \in \mathbb{C}$ and (5.82), we obtain

$$\begin{aligned} & \left| \lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n - e^{-\frac{t^2}{2}} \right| \leq e^{-\frac{t^2}{2}} \left| e^{n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k} - 1 \right| \\ & \leq e^{-\frac{t^2}{2}} \left| n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k \right| \left| n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k \right| \\ & \leq e^{-\frac{t^2}{4}} \left| n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k \right|. \end{aligned} \quad (5.88)$$

By choosing δ_1 small enough, we have for all $s \in (-\eta, \eta)$ and $|t| < \delta_1 \underline{\sigma} \sqrt{n}$,

$$\left| n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(\frac{-it}{\sigma_s \sqrt{n}}\right)^k \right| \leq C \left(\frac{|t|^3}{\sqrt{n}}\right). \quad (5.89)$$

From (5.88) and (5.89), we have for all $s \in (-\eta, \eta)$ and $|t| < \delta_1 \underline{\sigma} \sqrt{n}$,

$$\left| \frac{\lambda_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n - e^{-\frac{t^2}{2}}}{t} \right| \leq \frac{C}{\sqrt{n}} t^2 e^{-\frac{t^2}{4}}. \quad (5.90)$$

By Lemma 5.5.6, $U_n(s)$ is a.s. bounded uniformly in $s \in (-\eta, \eta)$. This together with

(5.90) and the fact that $\int_0^\infty t^2 e^{-\frac{t^2}{4}} dt < \infty$ implies that

$$I_{12}(n) \leq \frac{C \sup_{s \in (-\eta, \eta)} |U_n(s)|}{\sqrt{n}} \int_0^\infty t^2 e^{-\frac{t^2}{4}} dt \leq \frac{M_4}{\sqrt{n}}. \quad (5.91)$$

Putting together (5.87) and (5.91), we get $I_1(n) = O\left(\frac{1}{\sqrt{n}}\right)$.

Control of $I_2(n)$. Using the constraint $|t| \geq \delta_1 \sigma \sqrt{n}$, we have

$$\begin{aligned} I_2(n) &\leq \frac{1}{\delta_1 \sigma \sqrt{n}} \sup_{s \in (-\eta, \eta)} \int_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |f_{s,n}(t) - h_{s,n}(t)| dt \\ &\leq \frac{1}{\delta_1 \sigma \sqrt{n}} \sup_{s \in (-\eta, \eta)} \int_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} (|f_{s,n}(t)| + |h_{s,n}(t)|) dt \end{aligned} \quad (5.92)$$

By (5.76) and Lemma 5.5.6, for $\eta > 0$ small enough and $\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}$,

$$\sup_{s \in (-\eta, \eta)} |h_{s,n}(t)| \leq e^{-\frac{t^2}{2}} \sup_{s \in (-\eta, \eta)} |U_n(s)| \leq M_5 e^{-\frac{t^2}{2}}.$$

This implies that

$$\frac{1}{\delta_1 \sigma \sqrt{n}} \sup_{s \in (-\eta, \eta)} \int_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |h_{s,n}(t)| dt \leq \frac{M_6}{\sqrt{n}}.$$

Hence, from (5.92), to prove that $I_2(n) = O\left(\frac{1}{\sqrt{n}}\right)$, it remains to show that there exist a small $\eta \in (0, \eta_2)$ such that as $n \rightarrow \infty$, a.s.,

$$\frac{1}{\delta_1 \sigma \sqrt{n}} \sup_{s \in (-\eta, \eta)} \int_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |f_{s,n}(t)| dt = O\left(\frac{1}{\sqrt{n}}\right). \quad (5.93)$$

By the branching property, we have the following decomposition: for $n \geq 0$ and $k_n = \lfloor \frac{n}{2} \rfloor$,

$$f_{s,n}(t) = A_{s,n}(t) + B_{s,n}(t), \quad (5.94)$$

where

$$A_{s,n}(t) = \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{(s - \frac{it}{\sigma_s \sqrt{n}}) S_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} e^{\frac{itk_n \Lambda'(s)}{\sigma_s \sqrt{n}}} \left[\hat{Y}_{s,n-k_n}^u(t) - \mathbb{E}_{k_n} \hat{Y}_{s,n-k_n}^u(t) \right],$$

$$B_{s,n}(t) = \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{(s - \frac{it}{\sigma_s \sqrt{n}}) S_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} e^{\frac{itk_n \Lambda'(s)}{\sigma_s \sqrt{n}}} \mathbb{E}_{k_n} \hat{Y}_{s,n-k_n}^u(t),$$

with

$$\hat{Y}_{s,n-k_n}^u(t) = \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{s S_v^{X_u^x}}(r_s \varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} e^{\frac{-it}{\sigma_s \sqrt{n}} [S_v^{X_u^x} - (n-k_n) \Lambda'(s)]}.$$

For $A_{s,n}(t)$, using the same argument as in the proof of Lemma 5.5.3, we can prove that for $\eta > 0$ small enough, there exists $\delta \in (0, 1)$ such that

$$\sup_{s \in (-\eta, \eta)} \sup_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} \delta^{-n} |A_{s,n}(t)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

Therefore,

$$\sup_{s \in (-\eta, \eta)} \sup_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |A_{s,n}(t)| \leq M_{11} \delta^n. \quad (5.95)$$

For $B_{s,n}(t)$, using the branching property and the definitions of \mathbb{Q}_s^x (see (5.23)) and $R_{s,it}$ (see (5.25)), we have for $u \in \mathbb{T}_{k_n}$,

$$\begin{aligned} \mathbb{E}_{k_n} \hat{Y}_{s,n-k_n}^u(t) &= \mathbb{E}_{k_n} \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{s S_v^{X_u^x}}(r_s \varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} e^{\frac{-it}{\sigma_s \sqrt{n}} [S_v^{X_u^x} - (n-k_n) \Lambda'(s)]} \\ &= \frac{\mathbb{E}_{k_n} \left[e^{s S_{n-k_n}^{X_u^x}}(r_s \varphi)(X_{n-k_n}^{X_u^x}) e^{\frac{-it}{\sigma_s \sqrt{n}} [S_{n-k_n}^{X_u^x} - (n-k_n) \Lambda'(s)]} \right]}{[\kappa(s)]^{n-k_n} r_s(X_u^x)} \\ &= \mathbb{E}_{\mathbb{Q}_s^{X_u^x}} \left[\varphi(X_{n-k_n}^{X_u^x}) e^{\frac{-it}{\sigma_s \sqrt{n}} [S_{n-k_n}^{X_u^x} - (n-k_n) \Lambda'(s)]} \right] \\ &= R_{s, \frac{-it}{\sigma_s \sqrt{n}}}^{n-k_n} \varphi(X_u^x). \end{aligned}$$

Therefore, by (5.26) and Theorem 5.4.2, there is a constant $a \in (0, 1)$ such that for

$$k_n = \lceil \frac{n}{2} \rceil,$$

$$\sup_{s \in (-\eta, \eta)} \sup_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |B_{s,n}(t)| \leq \|\varphi\|_\beta a^{n-k_n} \sup_{s \in (-\eta, \eta)} W_{k_n}^x(s) \leq M_7 a^{\frac{n}{2}-1}. \quad (5.96)$$

From (5.94), (5.95) and (5.96), we obtain for $c_1 = \max\{\delta, a^{\frac{1}{2}}\}$,

$$\sup_{s \in (-\eta, \eta)} \sup_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |f_{s,n}(t)| \leq M_8 c_1^n.$$

Thus

$$\frac{1}{\delta_1 \sigma \sqrt{n}} \sup_{s \in (-\eta, \eta)} \int_{\delta_1 \sigma \sqrt{n} \leq |t| \leq \eta \sigma \sqrt{n}} |f_{s,n}(t)| dt \leq \frac{2(\eta - \delta_1) M_8 c_1^n}{\delta_1},$$

which implies (5.93). This concludes that $I_2(n) = O\left(\frac{1}{\sqrt{n}}\right)$, which ends the proof of (5.80) and (5.78). So Theorem 5.5.2 is proved. □

5.6 Proof of Theorem 5.2.2

For $y \in [0, 1]$, Theorem 5.2.2 is a direct consequence of Theorem 5.2.1, as we will see in the following. For $n \geq 1$,

$$\begin{aligned} & \left| \frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbf{1}_{\{S_u^x - n\gamma \geq \sqrt{n}\sigma y\}}}{m^n W[1 - \Phi(y)] e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})}} - \nu(\varphi) \right| \\ &= \frac{1}{W[1 - \Phi(y)] e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})}} \left| \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) - \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbf{1}_{\{\frac{S_u^x - n\gamma}{\sigma \sqrt{n}} \leq y\}} \right. \\ & \quad \left. - W \nu(\varphi) (1 - \Phi(y)) e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \right|. \end{aligned} \quad (5.97)$$

Since $\sup_{y \in [0, 1]} \left| \frac{y^3}{\sqrt{n}} \zeta\left(\frac{y}{\sqrt{n}}\right) \right| \rightarrow 0$, there exists n_0 large enough such that for all $y \in [0, 1]$ and $n \geq n_0$, $e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \geq 1/2$. Using this and the fact that $1 - \Phi(y) \geq c := 1 - \Phi(1)$ for all

$y \in [0, 1]$, from (5.97) we get for all $n \geq n_0$,

$$\begin{aligned}
& \left| \frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \geq \sqrt{n}\sigma y\}}}{m^n W [1 - \Phi(y)] e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})}} - \nu(\varphi) \right| \\
& \leq \frac{2}{cW} \left| \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) - W\nu(\varphi) \right| \\
& \quad + \frac{2}{cW} \left| -\frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{\frac{S_u^x - n\gamma}{\sigma\sqrt{n}} \leq y\}} + W\nu(\varphi)\Phi(y) \right| \\
& \quad + \frac{2}{cW} \left| W\nu(\varphi)(1 - \Phi(y)) \left(1 - e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})}\right) \right|. \tag{5.98}
\end{aligned}$$

In the last display, by Theorem 5.2.1, when $n \rightarrow \infty$, the two first terms are $O\left(\frac{1}{\sqrt{n}}\right)$. We will show below that the third term is also $O\left(\frac{1}{\sqrt{n}}\right)$. In fact, using the inequality $|1 - e^t| \leq |t|e^t$ for $t \in \mathbb{R}$ and the fact that $\sup_{y \in [0, 1]} \left|\zeta\left(\frac{y}{\sqrt{n}}\right)\right|$ is bounded for $n \geq n_0$, we obtain for $y \in [0, 1]$, as $n \rightarrow \infty$,

$$\left| 1 - e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \right| \leq \left| \frac{y^3}{\sqrt{n}} \zeta\left(\frac{y}{\sqrt{n}}\right) \right| e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} = O\left(\frac{1}{\sqrt{n}}\right).$$

Since $|\nu(\varphi)| \leq \|\varphi\|_\infty$, this implies that the third term in (5.98) is $O\left(\frac{1}{\sqrt{n}}\right)$.

From (5.98) and the above estimations, we see that for $y \in [0, 1]$, as $n \rightarrow \infty$,

$$\left| \frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \geq \sqrt{n}\sigma y\}}}{m^n W [1 - \Phi(y)] e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})}} - \nu(\varphi) \right| = O\left(\frac{1}{\sqrt{n}}\right),$$

which implies

$$\frac{\sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma \geq \sqrt{n}\sigma y\}}}{m^n W [1 - \Phi(y)]} = e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \left[\nu(\varphi) + O\left(\frac{1}{\sqrt{n}}\right) \right].$$

We now deal with the case $1 < y = o(\sqrt{n})$. We can suppose that $\varphi \geq 0$ by considering the positive and negative parts of φ . We will focus on the proof of (5.16), as the proof of (5.17) is similar. For $u \in (\mathbb{N}^*)^n$, set

$$V_u^x = \frac{S_u^x - n\Lambda'(s)}{\sigma_s \sqrt{n}}.$$

Then we have

$$\begin{aligned}
 I &:= \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\{S_u^x - n\gamma > y\sigma\sqrt{n}\}} \\
 &= r_s(x) \frac{e^{-ns\Lambda'(s)}}{\kappa^{-n}(s)} \sum_{u \in \mathbb{T}_n} e^{-s\sigma_s \sqrt{n} V_u^x} \frac{e^{sS_u^x} \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\{V_u^x > \frac{\sigma y}{\sigma_s} + \frac{\sqrt{n}[\gamma - \Lambda'(s)]}{\sigma_s}\}}. \tag{5.99}
 \end{aligned}$$

Because $\Lambda(s)$ is analytic on $(-\eta_1, \eta_1)$ with $\Lambda(0) = 0$, it has the Taylor expansion

$$\Lambda(s) = \sum_{k=1}^{\infty} \frac{\gamma_k}{k!} s^k, \quad \text{where } \gamma_k = \Lambda^{(k)}(0), \quad s \in (-\eta_1, \eta_1), \tag{5.100}$$

which implies that

$$\Lambda'(s) - \gamma = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!} s^{k-1}. \tag{5.101}$$

Consider the equation

$$\sqrt{n}[\Lambda'(s) - \gamma] = \sigma y. \tag{5.102}$$

Set $t = \frac{y}{\sqrt{n}}$. Using (5.101), we get

$$\sigma t = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!} s^{k-1}. \tag{5.103}$$

Since $\gamma_2 = \sigma^2 > 0$, the equation (5.103) has the unique solution given by

$$s = \frac{t}{\gamma_2^{1/2}} - \frac{\gamma_3}{2\gamma_2^2} t^2 - \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{6\gamma_2^{7/2}} t^3 + \dots, \tag{5.104}$$

which converges for $|t|$ small enough (see [75, Theorem VIII.2.2] for details). From (5.100) and (5.101), we see that

$$s\Lambda'(s) - \Lambda(s) = \sum_{k=2}^{\infty} \frac{k-1}{k!} \gamma_k s^k.$$

Choosing s given by (5.104), we obtain

$$s\Lambda'(s) - \Lambda(s) = \frac{t^2}{2} - t^3\zeta(t) = \frac{y^2}{2n} - \frac{y^3}{n^{3/2}}\zeta\left(\frac{x}{\sqrt{n}}\right), \quad (5.105)$$

where ζ is the Cramér series defined in (5.9), which converges for $|t|$ small enough (see [75, Theorem VIII.2.2] for details). Coming back to the expression of I (cf. (5.99)), using (5.105) together with (5.102) and the fact that $\frac{e^{-ns\Lambda'(s)}}{\kappa^{-n}(s)} = e^{-n[s\Lambda'(s) - \Lambda(s)]}$, we have

$$\begin{aligned} I &= r_s(x) e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \sum_{u \in \mathbb{T}_n} e^{-s\sigma_s \sqrt{n} V_u^x} \frac{e^{sS_u^x} \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbf{1}_{\{V_u^x > 0\}} \\ &= r_s(x) e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \int_0^\infty e^{-s\sigma_s \sqrt{ny}} \bar{Z}_{s,n}^x(dy), \end{aligned} \quad (5.106)$$

where $\bar{Z}_{s,n}^x$ is the finite measure on \mathbb{R} defined by:

$$\bar{Z}_{s,n}^x(B_2) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS_u^x} \varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbf{1}_{\{V_u^x \in B_2\}}, \quad B_2 \subset \mathbb{R}.$$

Its mass satisfies $\mathbb{E}[\bar{Z}_{s,n}^x(\mathbb{R})] \leq \left\| \frac{\varphi}{r_s} \right\|_\infty$.

Since $t = \frac{y}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$, by (5.104) we have $s \rightarrow 0^+$ as $n \rightarrow \infty$. Hence, for sufficiently large n_0 and all $n \geq n_0$, we have $s \in (0, \eta)$ where η is defined in Theorem 5.2.3. Therefore, denoting

$$l_{n,s}(y) = \bar{Z}_{s,n}^x((-\infty, y]) - W^x(s) \pi_s\left(\frac{\varphi}{r_s}\right) \Phi(y), \quad y \in \mathbb{R},$$

we get from Theorem 5.2.3 that for all $n \geq n_0$,

$$\sup_{y \in \mathbb{R}} |l_{n,s}(y)| \leq \frac{M}{\sqrt{n}}, \quad (5.107)$$

where M is a positive and finite random variable independent of n and s . In the following,

we write M_i for a positive and finite random variable. Notice that

$$\begin{aligned} & \int_0^\infty e^{-s\sigma_s\sqrt{ny}} \overline{Z}_{s,n}^x(dy) \\ &= \int_0^\infty e^{-s\sigma_s\sqrt{ny}} dl_{n,s}(y) + \frac{W^x(s)\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} \int_0^\infty e^{-s\sigma_s\sqrt{ny} - \frac{y^2}{2}} dy \\ &=: I_1 + W^x(s)\pi_s(\varphi r_s^{-1})I_2. \end{aligned} \tag{5.108}$$

Estimate of I_1 . Using the integration by parts and (5.107), we get for $n \geq n_0$,

$$|I_1| \leq |l_{n,s}(0)| + s\sigma_s\sqrt{n} \int_0^\infty e^{-s\sigma_s\sqrt{ny}} |l_{n,s}(y)| dy \leq \frac{2M}{\sqrt{n}}. \tag{5.109}$$

Estimate of I_2 . The integral I_2 appears in the proof of Cramér’s large deviation expansion theorem for sums of i.i.d random variables (see [75, Theorem VIII.2.2]), where the following results have been proved:

- (i) there exist some positive constants c_1, c_2 such that for all $s \in (-\eta, \eta)$ and n large enough,

$$c_1 \leq s\sigma_s\sqrt{n}I_2 \leq c_2;$$

- (ii) the integral I_2 admits the following asymptotic expansion:

$$I_2 = e^{\frac{y^2}{2}} [1 - \Phi(y)] \left[1 + O\left(\frac{y}{\sqrt{n}}\right) \right]. \tag{5.110}$$

By the definition of σ_s , the mapping $s \mapsto \sigma_s$ is strictly positive and continuous on $(-\eta, \eta)$. Hence, there exist constants $c_3, c_4 > 0$ such that

$$c_3 \leq s\sqrt{n}I_2 \leq c_4. \tag{5.111}$$

Notice that for all $s \in (-\eta, \eta)$, $W^x(s) > 0$ a.s. Moreover, $W^x(s)$ is a.s. continuous in $(-\eta, \eta)$ by the continuity and uniform convergence of $W_n^x(s)$ on $(-\eta, \eta)$. Combining this with (5.111), we get

$$M_3 \leq s\sqrt{n}W^x(s)I_2 \leq M_4. \tag{5.112}$$

We now come back to (5.108), and let s be defined by (5.104). Recall that for $n \geq n_0$, $s \in$

$(0, \eta)$. From (5.108), (5.109) and (5.112), we have, as $n \rightarrow \infty$,

$$\begin{aligned} \int_0^\infty e^{-s\sigma_s\sqrt{ny}} \bar{Z}_{s,n}^x(dy) &= W^x(s)I_2 \left[\pi_s(\varphi r_s^{-1}) + \frac{I_1}{W^x(s)I_2} \right] \\ &= W^x(s)I_2 \left[\pi_s(\varphi r_s^{-1}) + \frac{s\sqrt{n}I_1}{s\sqrt{n}W^x(s)I_2} \right] \\ &= W^x(s)I_2 \left[\pi_s(\varphi r_s^{-1}) + O(s) \right]. \end{aligned}$$

Substituting this into (5.106) and using (5.110), we obtain

$$I = r_s(x)W^x(s)e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}[1 - \Phi(y)] \left[1 + O\left(\frac{y}{\sqrt{n}}\right) \right] \left[\pi_s(\varphi r_s^{-1}) + O(s) \right]. \quad (5.113)$$

According to Theorem 5.4.2, $W^x(s)$ is analytic on $(-\eta, \eta)$ and using the mean theorem we see that $|W^x(s) - W^x| = |W^x(s) - W^x(0)| \leq M_5s$. On the other hand, by [83, Lemma 6.1], we have $\|r_s - 1\|_\infty \leq cs$ and $|\pi_s(\varphi r_s^{-1}) - \nu(\varphi)| = \left| \frac{\nu_s(\varphi)}{\nu_s(r_s)} - \nu(\varphi) \right| \leq cs\|\varphi\|_\beta$. Since $s = O\left(\frac{y}{\sqrt{n}}\right)$ by (5.104), it follows from (5.113) that

$$\begin{aligned} I &= \left[1 + O(s) \right] \left[W^x + O(s) \right] e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} [1 - \Phi(y)] \left[1 + O\left(\frac{y}{\sqrt{n}}\right) \right] \left[\nu(\varphi) + O(s) \right] \\ &= W^x e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} [1 - \Phi(y)] \left[\nu(\varphi) + O\left(\frac{y}{\sqrt{n}}\right) \right], \end{aligned}$$

which concludes the proof of (5.16).

The proof of (5.17) can be carried out in a similar way as that of (5.16). The only difference is that, instead of using (5.102), we consider the equation

$$\sqrt{n}[\Lambda'(s) - \Lambda'(0)] = -\sigma y,$$

where $1 < y = o(\sqrt{n})$ and $s \in (-\eta, 0)$. Since the rest of the argument is the same as that in the proof of (5.16), we omit the details.

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