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Wille Liu

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Université de Paris

École Doctorale de Sciences Mathématiques de Paris Centre (ED 386)
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Algèbres de Hecke doublement affines dégénérées de paramètres généraux : faisceaux pervers et foncteurs de Knizhnik–Zamolodchikov

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Thèse de doctorat de mathématiques

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Résumé

Le présent travail de thèse porte sur l'étude de la catégorie $\mathcal{O}$ des algèbres de Hecke doublement affines dégénérées (dDAHA) au point de vue de la théorie de Springer et celle des faisceaux pervers. Dans les premiers deux chapitres nous étudions de manière algébrique les dDAHA et leurs généralisations, algèbres de Hecke doubles carquois (QDHA). Nous introduisons le foncteur de Knizhnik–Zamolodchikov (KZ) pour les QDHA et démontrons qu'ils vérifient la propriété bicommutante dans chapitre 2. Les chapitres 3 et 4 sont consacrés à l'étude des faisceaux pervers sur les algèbres de Lie munies de graduations cycliques et la théorie de Springer pour les dDAHA avec certaines familles de paramètres. Dans le chapitre 5, nous expliquons comment le foncteur KZ se réalise en termes de faisceaux pervers et nous montrons comment des structures plus fines sur la catégorie $\mathcal{O}$ se déduisent de l'analyse faisceautique sur les algèbres de Lie cycliquement graduées.

Mots-clés : algèbres de Hecke doublement affines, algèbres de Cherednik, algèbres de Hecke carquois, foncteur de Knizhnik–Zamolodchikov, faisceux pervers, correspondance de Springer

Abstract

The present thesis work focuses on the study of the category $\mathcal{O}$ of degenerate double affine Hecke algebras (dDAHA) with the point of view of Springer theory and perverse sheaves. In the first two chapiters we study algebraically the dDAHAs and their generalisations, quiver double Hecke algebras (QDHA). We in introduce the Knizhnik–Zamolodchikov (KZ) functor for the QDHA and prove that it satisfies the double centraliser property in chapter 2. Chapters 3 and 4 are devoted to the study of perverse sheaves on a Lie algebra equipped with a cyclic grading and the Springer theory for the dDAHAs with certain families of parameters. In chapter 5, we explain how the KZ functor can be realised in terms of perverse sheaves and we show how finer structures on the category $\mathcal{O}$ can be deduced from the sheaf-theoretic analysis on cyclically graded Lie algebras.

Keywords : double affine Hecke algebras, Cherednik algebras, quiver Hecke algebras, Knizhnik–Zamolodchikov functors, perverse sheaves, Springer correspondence

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Introduction

Le présent travail trouve son origine dans un effort pour étudier la théorie des représentation des algèbres de Hecke doublement affines dégénérées (ci-dessous notée dDAHA) avec des méthodes algébro-géométriques.

Contexte

Algèbres de Hecke affines

Étant donné un système de Coxeter (W, S) avec la présentation $W = \langle s \in S \ ; \ (st)^{m_{s,t}} = 1 \rangle$ et un paramètre $q \in \mathbf{C}^\times$, on définit son algèbre d'Iwahori–Hecke comme l'algèbre associative unitaire H_W sur $\mathbf{C}$ engendrée par la famille $\{T_s\}_{s \in S}$ modulo les relations suivantes :

$$(T_s - q)(T_s + 1) = 0; \quad \underbrace{T_s T_t T_s \cdots}_{m_{s,t}} = \underbrace{T_t T_s T_t \cdots}_{m_{s,t}}, \quad \text{pour } s \neq t \in S, \ m_{s,t} \neq \infty.$$

Ces algèbres sont omniprésentes dans la théorie des représentations des groupes. Lorsque (W, S) est le groupe de Weyl affine d'un système de racines, on appelle H_W une algèbre de Hecke affine (AHA). Contentons-nous de mentionner seulement que les algèbres de Hecke affines font objet central dans la théorie des représentations des groupes p-adiques.

Équations de Knizhnik–Zamolodchikov affines et dDAHA

Les dDAHA furent introduites par I. Cherednik [11] dans ses études des équations de Knizhnik–Zamolodchikov (KZ) et leurs variants comme les affinisées des AHA. Au fil du temps, les dDAHA et ses variants apparaissent dans de nombreux contextes différents, comme celui des invariants des nœuds ou celui des intégrales orbitales sur les groupes p-adiques.

L'une des versions des équations KZ est connue sous le nom des équations KZ affines (AKZ) ou les équations KZ trigonométriques. C'est une famille de systèmes d'équations aux dérivées partielles linéaires et holomorphes sur l'espace des configurations du tore complexe $\mathbf{C}^\times$.

D'après Matsuo et Cherednik, le problème d'intégration des équations AKZ équivaut à celui de la détermination des fonctions propres des opérateurs de Dunkl sur $(\mathbf{C}^\times)^n$:

$$D_j = z_j \frac{\partial}{\partial z_j} - 2h \sum_{i<j} \frac{1 - s_{i,j}}{1 - z_j/z_i} + 2h \sum_{i>j} \frac{1 - s_{i,j}}{1 - z_i/z_j},$$

où $h \in \mathbf{C}$ est un paramètre et $s_{i,j} \in \mathfrak{S}_n$ est la permutation qui échange les coordonnées z_i et z_j et fixe les autres. La dDAHA $\mathbb{H}_n$ est l'algèbre d'opérateurs sur les fonctions holomorphes sur $(\mathbf{C}^\times)^n$ engendrée par les fonctions holomorphes, les opérateurs de Dunkl et le groupe symmétrique $\mathfrak{S}_n$ qui opère sur $(\mathbf{C}^\times)^n$ par permutation des coordonnées. Ainsi les systèmes AKZ peuvent être vus comme des modules à gauche de $\mathbb{H}_n$. La représentation monodromique d'un système AKZ, d'après un résultat fondamental de Cherednik, est un module sur l'algèbre de Hecke affine pour GL_n. Cette opération qui associe la représentation monodromique à un $\mathbb{H}$-module est fonctorielle. Le foncteur ainsi construit (appelé foncteur KZ) est fondamental dans l'étude des représentations des dDAHA.

Toutes ces constructions se généralisent à tous les systèmes de racines[1].

Étude géométrique des algèbres de Hecke affines

Soient R un système de racines fini, G un groupe algébrique complexe semisimple de type $R^\vee$, $B \subset G$ un sous-groupe de Borel et $T \subset B$ un tore maximal. Notons $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{b} = \operatorname{Lie} B$ et $\mathfrak{t} = \operatorname{Lie} T$ leurs algèbres de Lie. Soit $\mathfrak{n} \subset \mathfrak{b}$ l'algèbre de Lie du radical unipotent de B. La résolution de Springer $\pi : \mathcal{T} = G \times^B \mathfrak{n} \longrightarrow \mathfrak{g}^{\mathrm{nil}}$ est une désingularisation du cône nilpotent $\mathfrak{g}^{\mathrm{nil}}$ de $\mathfrak{g}$. Ginzburg et Kazhdan–Lusztig dans leurs travaux embématiques [13] [23], étudièrent la théorie des représentations de l'AHA étendue $\mathbb{K}$ associée au système de racines R via la K-théorie équivariante de la variété de Steinberg $\mathcal{Z} = \mathcal{T} \times_{\mathfrak{g}^{\mathrm{nil}}} \mathcal{T}$. Étant donnés $a \in T$ et $r \in \mathbf{C}^\times$, soient G_a le centralisateur de a dans G et

$$\mathfrak{g}^{\mathrm{nil}}_{(a,r)} = \left\{ x \in \mathfrak{g}^{\mathrm{nil}} \; ; \; \mathrm{Ad}(a)x = r^2 x \right\}.$$

Ginzburg et Kazhdan–Lusztig montrèrent que, lorsque $r \in \mathbf{C}^\times$ n'est pas une racine d'unité, les modules simples de $\mathbb{K}$ dans le bloc associé à (a, r) sont paramétrés par certains systèmes locaux irréductibles sur les G_a-orbites dans $\mathfrak{g}^{\mathrm{nil}}_{(a,r)}$. Ce résultat a connu un nombre important de généralisations.

Pourtant, la condition que $r \in \mathbf{C}^\times$ ne soit pas une racine d'unité est essentielle pour le paramétrage de Ginzburg–Kazhdan–Lusztig et par conséquent le cas où r est une racine d'unité nécessite d'autres méthodes. L'un des objectifs du présent travail est d'étudier le foncteur KZ qui relie la dDAHA et l'AHA étendue dans l'espoir de pouvoir mieux comprendre l'AHA à travers la dDAHA, dans l'esprit de [17] où les algèbres de Hecke associées aux groupes de réflexions complexes ont été étudiées à travers les algèbres de Cherednik rationnelles.

Algèbres de Hecke carquois

Les algèbres de Hecke carquois (QHA) furent introduites par M. Khovanov, A. Lauda [24] et R. Rouquier [44] dans le but de catégorifier des groupes quantiques de Drinfel'd–Jimbo. D'après un résultat de J. Brundan, A. Kleshchev [9] et Rouquier [44], ces algèbres peuvent être vues comme une vaste généralisation des algèbres de Hecke affines pour GL_n.

[1] En général, il y a un paramètre $h_a \in \mathbf{C}$ pour chaque racine affine $a \in S$ sous la condition que la famille $\{h_a\}_{a \in S}$ soit invariante sous l'action du groupe de Weyl affine étendu. Lorsque $h_a = h_b$ pour tous $a, b \in S$, on dit que la dDAHA est de *paramètres égaux*. Si le système de racines est simplement lacé, la dDAHA associée est toujours de paramètres égaux.

Le résultat de Brundan–Kleshchev–Rouquier se généralise aux AHA de type quelconque, à savoir on peut définir les algèbres de Hecke carquois sur les systèmes de racines finies au lieu de GL_n. Cette approche a déjà été prise dans plusieurs travaux précédents pour étudier les algèbres de Hecke affines de type classique, notamment [50] et [46]. Étant donné le fait que les dDAHA sont les affinisées des AHA, il est naturel d'emprunter cette méthode dans l'étude des dDAHA.

Présentation des résultats

Les résultats obtenus dans cette thèse pourraient mettre en lumière une idée selon laquelle les dDAHA avec leurs foncteurs KZ seraient le paradigme d'un phénomène qui se présente chez une classe plus vaste d'algèbres.

Le foncteur KZ des dDAHA

Dans §1, on rappelle la définition des systèmes de racines affines, les dDAHA, la catégorie $\mathcal{O}$ des dDAHA, les AHA et le foncteur de monodromie des équations KZ pour les dDAHA, noté $\mathbb{V}$. L'exposition insiste sur les thèmes en rapport avec $\mathbb{V}$. On démontre dans §1.4.5 que $\mathbb{V}$ est un foncteur de quotient qui fait de la catégorie des modules de dimension finie de l'AHA une catégorie quotient de la catégorie $\mathcal{O}$ de la dDAHA.

En envisageant §2, on montre dans §1.5 que le foncteur $\mathbf{V}$ construit dans §2.5 et $\mathbb{V}$ sont des foncteurs de quotient de même noyau. On conjecture que les deux foncteurs sont isomorphes.

QDHA comme dDAHA généralisées

Dans §2, on introduit les algèbres de Hecke double carquois $\mathbf{A}^\omega$ (QDHA) associées à un système de racines affine. Les QDHA peuvent être vues comme des QHA affinisées ou comme des dDAHA généralisées dans le sens de Brundan–Kleshchev–Rouquier. On fournit dans §2.1–§2.3 un socle théorique pour cette classe d'algèbres. Dans §2.4, on définit les algèbres de Hecke carquois $\mathbf{B}^\omega$ (QHA) associée à un système de racines fini, homologues des AHA dans le cas des dDAHA.

La section §2.5 est le cœur de ce chapitre. On établit dans §2.5.1 un isomorphisme qui identifie $\mathbf{B}^\omega$ à une sous-algèbre idempotente $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ de $\mathbf{A}^\omega$, ce qui nous permet de construire dans §2.5.6 le foncteur de Knizhnik–Zamolodchikov $\mathbf{V} : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{B}^\omega\text{-gmod}$ comme la troncation par l'idempotent $\mathbf{e}_\gamma$. Dans §2.5.7, on démontre une caractérisation asymptotique pour les $\mathbf{A}^\omega$ modules annulés par $\mathbf{V}$. Dans §2.5.8, on démontre la propriété bicommutante pour $\mathbf{V}$, ce qui rend $(\mathbf{A}^\omega, \mathbf{V})$ une « désingularisation crépante partielle » de $\mathbf{B}^\omega$. Dans §2.5.9, on établit une caractérisation catégorique pour $\mathbf{V}$, ce qui implique que son noyau $\ker \mathbf{V}$ est un invariant catégorique de la catégorie $\mathbf{A}^\omega$-gmod.

Faisceaux pervers et dDAHA

Dans §3 et §4, en s'appuyant sur la théorie de Lusztig–Yun [38] [39] [40], on étudie les dDAHA $\mathbb{H}$ à l'aide des faisceaux pervers sur le cône nilpotent des l'algèbres de Lie $\mathbf{Z}/m\mathbf{Z}$-graduées. Dans §3, on distingue deux types d'inductions géométriques, l'induction spirale et l'induction parabolique, et on démontre dans Theorem 3.20 que celle-ci est légèrement plus faible que celle-là dans un sens à préciser dans l'énoncé du théorème. Ce résultat servira de base géométrique pour une certaine propriété de fidèlité du foncteur $\mathbf{V}$.

Dans §4, en suivant la voie classique d'algèbres de convolution [13], on établit une construction géométrique de $\mathbb{H}$ en termes de la cohomologie équivariante à coefficients. Ce résultat repose sur celui de Lusztig dans le cas des AHA graduées. La construction géométrique, comme dans le cas des AHA graduées, nous permet d'établir un paramétrage géométrique à la Deligne–Langlands–Lusztig des modules simples de $\mathbb{H}$ (ce qui confirme une conjecture de Lusztig–Yun [40]) et une formule de multiplicité de Jordan–Hölder pour une classe de $\mathbb{H}$-modules dits standards propres en termes de cohomologie de certaines variétés de Hessenberg.

Dans §5.1, on discute la relation entre la construction géométrique de §4 et la construction algébrique du foncteur $\mathbf{V}$ de §2. On obtient une réalisation des AHA via la cohomologie équivariante au lieu de la K-théorie équivariante.

Dans §5.2, on applique les résultats généraux de S. Kato [21] sur les algèbres d'extensions aux dDAHA. La conséquence est que la catégorie des modules gradués de la dDAHA $\mathbb{H}$ que nous avons construites avec la géométrie dans §4 admet une structure de stratification (par les modules standards) propres dans le sens de Kleshchev [25].

Dans §5.3, on se tourne vers deux cas particulièrement simples de dDAHA qui viennent des carquois cycliques. Le thème de [48] sur l'équivalence entre les algèbres q-Schur affines et les dDAHA en type A y est repris.

Quelques précisions

Le sujet central du présent travail est les dDAHA. Pourtant, des hypothèses différentes sont mises sur les dDAHA selons les chapitres. Toutes les dDAHA concernées sont de niveau non-critique, ce qui implique que leurs centres sont triviaux. Dans §1, les dDAHA concernées sont de type quelconque et de paramètres quelconque. Dans §2, les algèbres $\mathbf{A}^\omega$ sont des généralisations des dDAHA considérées dans §1 et par conséquent, tous les résultats sur $\mathbf{A}^\omega$ s'appliquent également à toutes les dDAHA de paramètres quelconque. Dans §4, les dDAHA qui interviennent peuvent avoir des paramètres rationnels inégaux. Les rapports de ces paramètres sont ceux qui apparaissent dans la liste de Lusztig [35, §7]. Les résultats obtenus dans §4 généralisent ceux de [53], là où seules les DAHA de paramètres égaux sont prises en considération.

Chapter 1

Degenerate double affine Hecke algebras

Introduction

The degenerate double affine Hecke algebras (dDAHA), also known as trigonometric Cherednik algebras, were introduced by Cherednik in his study of integration of the trigonometric form of the Knizhnik–Zamolodchikov equations (KZ) [11].

The degenerate double affine Hecke algebras, different from their non-degenerate version and its rational degeneration, are not "symmetric": it contains a polynomial subalgebra and a Laurent polynomial subalgebra. Due to this asymmetry, one can adopt two different points of view to study the dDAHA: either viewing it as an algebra of differential operators on a torus attached to a root system, or as an algebra of difference operators on a root system. The former approach allows one to apply various techniques of $\mathcal{D}$-modules, symplectic geometry and is closer to the theory of rational Cherednik algebras [14]; the latter approach allows one to apply cohomological, K-theoretic or sheaf-theoretic methods [13] [53], and is closer to the non-degenerate DAHA *cf.* §4, §5.

In the present work, we will adopt the point of view of difference operators most of the time. We will see in §2 that with this point of view, the dDAHAs can be easily generalised and are quite flexible in the choice of parameters. We will also see that some of the features on the differential side can be recovered in this approach, namely the integration of the KZ equations.

This chapter serves mainly as preliminary materials for later chapters. The proof of most of the statements will not be presented since they can be found in the literature [32] [12] [42] [48] [49].

We define the affine root systems in §1.1, the dDAHAs in §1.2 and the affine Hecke algebras (AHA) in §1.3. We introduce the idempotent form of these algebras, which control blocks of the category $\mathcal{O}$ of both algebras. The definition of idempotent forms is a straightforward generalisation of the result of Brundan–Kleshchev [9] and Rouquier [43] on the equivalence between affine Hecke algebras for GL_n and quiver Hecke algebras for linear and cyclic quivers.

We recall the monodromy functor in §1.4 introduced in [48] as trigonometric analogue

of the KZ functor of [17]. We prove that it is a quotient functor in the sense of Gabriel.

In envisaging the next chapter, we discuss in §1.5 the relations between the monodromy functor of [48] and the functor which will be defined in algebraic terms in §2.5.6.

1.1 Affine root systems

We recall the notion of affine root systems. The reference is [41].

1.1.1 Affine reflections on euclidean spaces

Let E be an affine euclidean space of dimension $n > 0$ and let V be its vector space of translations. In particular, V is equipped with a positive definite scalar product $\langle -, - \rangle : V \times V \longrightarrow \mathbf{R}$. The dual space V^* is identified with V via the scalar product $\langle -, - \rangle$. Let $\mathbf{C}[E]^{\leq 1}$ be the space of affine functions on E. We have a map of differential $\partial : \mathbf{C}[E]^{\leq 1} \longrightarrow V$ whose kernel is the set of constant functions. The space $\mathbf{C}[E]^{\leq 1}$ is equipped with a symmetric bilinear form $\langle f, g \rangle = \langle \partial f, \partial g \rangle$. For any non-constant function $f \in \mathbf{C}[E]^{\leq 1}$, let $f^\vee = 2f/|f|^2$ and define the reflection with respect to the zero hyperplane of f:

$$s_f : E \longrightarrow E, \quad s_f(x) = x - f^\vee(x)\partial f$$

and

$$s_f : \mathbf{C}[E]^{\leq 1} \longrightarrow \mathbf{C}[E]^{\leq 1}, \quad s_f(g) = g - \langle f^\vee, g \rangle f.$$

It extends to an automorphism of the ring of regular functions $s_f : \mathbf{C}[E] \longrightarrow \mathbf{C}[E]$.

1.1.2 Affine root systems

An affine root system on E is a subset $S \subset \mathbf{C}[E]^{\leq 1}$ satisfying the following conditions:

(i) S spans $\mathbf{C}[E]^{\leq 1}$ and the elements of S are non-constant functions on E.

(ii) $s_a(b) \in S$ for all $a, b \in S$

(iii) $\langle a^\vee, b \rangle \in \mathbf{Z}$ for all $a, b \in S$.

(iv) the group W_S of auto-isometries on E generated by $\{s_a \ ; \ a \in S\}$ acts properly on E.

The group W_S is called the affine Weyl group (or simply the Weyl group of S). An affine root system (E, S) is irreducible if there is no partition $S = S_1 \sqcup S_2$ with $\langle -, - \rangle |_{S_1 \times S_2} = 0$ and $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$; it is reduced if $a \in S$ implies $2a \notin S$.

Let (E, S) be an affine root system. The set $R = \partial(S) \subset V^*$ is a finite root system on V. Let $P_R = \mathrm{P}(R) \subset V$ denote the weight lattice, $Q_R = \mathbf{Z}R$ the root lattice, $P_R^\vee = P(R^\vee)$ the coweight lattice and $Q_R^\vee = \mathbf{Z}R^\vee$ the coroot lattice.

Conversely, let (V, R) be an irreducible finite root system, reduced or not. Let P_R be the weight lattice, Q_R the root lattice.

We define the affinisation of (V, R) to be the affine root system (E, S) with $E = V$ and

$$S = \langle \alpha + n \ ; \ n \in \mathbf{Z}, \alpha \in R, \alpha \notin 2Q_R \rangle \sqcup \langle \alpha + 2m + 1 \ ; \ m \in \mathbf{Z}, \alpha \in R, \alpha \in 2Q_R \rangle.$$

Given a base $\Delta_0 \subset R$, we form $\Delta = \Delta_0 \cup \{a_0\}$, where $a_0 = 1 - \theta$ with $\theta \in R$ being the highest root with respect to the base Δ_0. It follows that $\delta = 1$. From now on, we will suppose that (E, S) comes from a finite root system (V, R) in this sense.

1.1.3 Affine Weyl group

Let (E, S) be an irreducible affine root system. A base of S is an $\mathbf{R}$-linearly independent subset $\Delta \subset S$ such that

(i) $S \subset \mathbf{N}\Delta \cup -\mathbf{N}\Delta$.

(ii) The set $\bigcap_{a \in \Delta} \{x \in E \ ; \ a(x) > 0\}$ is non-empty.

The W_S-action on S induces a simple transitive W_S-action on the set of bases of S. Upon fixing a base Δ of S, let $S^+ = S \cap \mathbf{N}\Delta$ and $S^- = S \cap -\mathbf{N}\Delta$ denote the sets of positive and negative roots.

We choose $a_0 \in \Delta$ such that the image of the set $\Delta_0 = \Delta \setminus \{a_0\}$ under the total differential $\partial : S \longrightarrow R$ forms a base for the finite root system R in the usual sense. This choice yields a base point $O \in E$ by $a(O) = 0$ for $a \in \Delta_0$. We will identify Δ_0 with its image $\partial\Delta_0 \in R$. It follows that the parabolic Coxeter subgroup $\langle s_a \ ; \ a \in \Delta_0 \rangle$ of W_S is isomorphic to the finite Weyl group $W_R = W(R, \Delta_0) = \langle s_\alpha \ ; \ \alpha \in \Delta_0 \rangle$ and there is an isomorphism

$$\begin{aligned} Q_R^\vee \rtimes W_R &\cong W_S \\ (\mu, w) &\mapsto X^\mu w, \end{aligned}$$

where the element X^μ acts on S by $a \mapsto a - \langle \partial a, \mu \rangle$. The extended affine Weyl group is defined to be $\tilde{W}_S = P_R^\vee \rtimes W_R$. It acts on S by extending the W_S action by the same formula $X^\mu a = a - \langle \partial a, \mu \rangle$ for $\mu \in P_R^\vee$.

The length function $\ell : \tilde{W}_S \longrightarrow \mathbf{N}$ is by $\ell(w) = \#(S^+ \cap w^{-1}S^-)$. This extends the usual length function on the Coxeter group W_S with respect to the set of generators $\{s_a\}_{a \in \Delta}$.

We will need the following formula for the length function.

Proposition 1.1. *For $\mu \in P_R^\vee$ and $w \in W_R$, we have*

$$\begin{aligned} \ell(wX^\mu) &= \ell(w) + \sum_{\alpha \in R^+ \cap w^{-1}R^-} |\langle \alpha, \mu \rangle + 1| + \sum_{\alpha \in R^+ \cap w^{-1}R^+} |\langle \alpha, \mu \rangle|. \\ \ell(X^\mu w) &= \ell(w) + \sum_{\alpha \in R^+ \cap w^{-1}R^-} |\langle \alpha, \mu \rangle| + \sum_{\alpha \in R^+ \cap w^{-1}R^+} |\langle \alpha, \mu \rangle + 1|. \end{aligned}$$

See [32] for a proof. □

1.1.4 Alcoves

For every affine root $a \in S$, let $H_a = \{\lambda \in E \; ; \; a(\lambda) = 0\}$ be the vanishing locus of a. The affine hyperplanes $\{H_a\}_{a \in S}$ yield a simplicial cellular decomposition of E. The open cells are called **alcoves**. Thus the set of connected componenents

$$\pi_0 \left(E \setminus \bigcup_{a \in S} H_a \right)$$

is the set of alcoves. The affine Weyl group W_S acts simply transitively on it. When a base $\Delta \subset S$ is fixed, the fundamental alcove is defined to be $\nu_0 = \bigcap_{a \in \Delta} \{x \in E \; ; \; a(x) > 0\}$.

1.2 Degenerate double affine Hecke algebra

Let (E, S, Δ) be an irreducible reduced affine root system with a base. We define in this section the degenerate double affine Hecke algebra $\mathbb{H}$ attached to (E, S, Δ) and its idempotent form $\mathbf{H}_{\lambda_0}$, which is a block algebra for the category $\mathcal{O}$ of $\mathbb{H}$.

1.2.1 Degenerate double affine Hecke algebra $\mathbb{H}$

Let $h = \{h_a\}_{a \in S}$ be a $\tilde{W}_S$-invariant family of complex numbers. The degenerate double affine Hecke algebra with parameters h attached to the affine root system S is the associative unital $\mathbf{C}$-algebra on the vector space $\mathbb{H} = \mathbf{C}W_S \otimes \mathbf{C}[E]$ whose multiplication satisfies following properties:

- Each of the subspaces $\mathbf{C}W_S$ and $\mathbf{C}[E]$ is given the usual ring structure, so that they are subalgebras of $\mathbb{H}$.
- $w \in \mathbf{C}W_S$ and $f \in \mathbf{C}[E]$ multiply by juxtaposition: $(w \otimes 1)(1 \otimes f) = w \otimes f$.
- $a \in \Delta$ and $f \in \mathbf{C}[E]$ satisfy the following:

$$(s_a \otimes 1)\,(1 \otimes f) - (1 \otimes s_a(f))\,(s_a \otimes 1) = 1 \otimes h_a \frac{f - s_a(f)}{a}.$$

1.2.2 Global dimension of $\mathbb{H}$

Put a filtration F on $\mathbb{H}$ as follows:

$$F_{\leq -1}\mathbb{H} = 0, \quad F_{\leq 0}\mathbb{H} = \mathbf{C}W_S, \quad F_{\leq 1}\mathbb{H} = (F_{\leq 0}\mathbb{H})\,\mathbf{C}[E]^{\leq 1}, \quad F_{\leq n}\mathbb{H} = (F_{\leq 1}\mathbb{H})^n, \quad n \geq 2.$$

Namely, $\mathbb{H}$ is filtered by its polynomial part $\mathbf{C}[E]$. The filtration F is compatible with the multiplication and its associated graded ring is given by the skew tensor product $\mathrm{gr}^F\,\mathbb{H} \cong \mathbf{C}W_R \ltimes (\mathbf{C}Q_R^\vee \otimes \mathbf{C}[V])$. Since $\mathrm{dim.gl}\;\mathbb{H} \leq \mathrm{dim.gl}\;\mathrm{gr}^F\,\mathbb{H}$ ([18, D.2.6]) and since $\mathrm{dim.gl}\;\mathbf{C}W_R \ltimes (\mathbf{C}Q_R^\vee \otimes \mathbf{C}[V]) = 2r$, where $r = \mathrm{rk}\,S = \dim E$, we have the following:

Proposition 1.2. *The global dimension of $\mathbb{H}$ is at most $2r$.*

□

1.2.3 Category $\mathcal{O}$

For each $\lambda \in E_{\mathbf{C}}$, let $\mathfrak{m}_\lambda \subset \mathbf{C}[E]$ be the maximal ideal generated by $f(\lambda) - \lambda$ for all $f \in \mathbf{C}[E]$. Given any module $M \in \mathbb{H}$-Mod, for each $\lambda \in E_{\mathbf{C}}$ consider the generalised λ-weight space in M:

$$M_\lambda = \bigcup_{N \geq 0} \left\{ a \in M \; ; \; \mathfrak{m}_\lambda^N a = 0 \right\}.$$

For any $\lambda_0 \in E_{\mathbf{C}}$, we define $\mathcal{O}_{\lambda_0}(\mathbb{H})$ to be the full subcategory of $\mathbb{H}$-mod consisting of those $M \in \mathbb{H}$-mod such that

$$M = \bigoplus_{\lambda \in W_S \lambda_0} M_\lambda.$$

In other words, the polynomial subalgebra $\mathbf{C}[E]$ acts locally finitely on M with eigenvalues in the W_S-orbit of $\lambda_0 \in E_{\mathbf{C}}$.

From the triangular decomposition $\mathbb{H} = \mathbf{C}Q_R^\vee \otimes \mathbf{C}W_R \otimes \mathbf{C}[E]$, we deduce the following:

Proposition 1.3. *For any $\lambda_0 \in E$, every object of $\mathcal{O}_{\lambda_0}(\mathbb{H})$ is a coherent $\mathbf{C}Q_R^\vee$-module.*

□

1.2.4 Block algebra $\mathbb{H}_{\lambda_0}^\wedge$

Fix once and for all $\lambda_0 \in E_{\mathbf{C}}$. Define for each $\lambda \in W_S\lambda_0$ a polynomial ring $\mathrm{Pol}_\lambda = \mathbf{C}[V]$ and let $\mathrm{Pol} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Pol}_\lambda$. Define the completion

$$\widehat{\mathrm{Pol}}_\lambda = \varprojlim_{N \longrightarrow \infty} \mathrm{Pol}_\lambda / \mathfrak{m}_0^N \mathrm{Pol}_\lambda = \mathbf{C}[\![V]\!], \quad \widehat{\mathrm{Pol}} = \bigoplus_{\lambda \in W_S\lambda_0} \widehat{\mathrm{Pol}}_\lambda,$$

where $\mathfrak{m}_0 \subset \mathrm{Pol}_\lambda$ is the maximal ideal given by $0 \in V$. The completion $\widehat{\mathrm{Pol}}_\lambda$ is equipped with the $\mathfrak{m}_0$-adic topology and $\widehat{\mathrm{Pol}}$ is equipped with the colimit topology of finite products.

For $\lambda \in W_S\lambda_0$, the translation $\lambda_* : V_{\mathbf{C}} \xrightarrow{\lambda+} E_{\mathbf{C}}$ yields an isomorphism

$$\lambda^* : \mathbf{C}[E] \cong \mathbf{C}[V] = \mathrm{Pol}_\lambda.$$

We define an action of $\mathbb{H}$ on Pol:

$$\psi = (\psi_\lambda)_\lambda : \mathbb{H} \longrightarrow \mathrm{End}(\widehat{\mathrm{Pol}}), \quad \psi_\lambda : \mathbb{H} \longrightarrow \mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}).$$

and where for $f \in \mathbf{C}[E]$ and $a \in \Delta$, set

$$\begin{aligned} \psi_\lambda(f) &= \lambda^* f \\ \psi_\lambda(s_a - 1) &= \begin{cases} -\frac{\lambda^*(a-h_a)}{\lambda^* a}(s_a - 1) \in \mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}_\lambda) & a(\lambda) = 0 \\ \frac{\lambda^*(h_a - a)}{\lambda^* a} - \frac{(s_a\lambda)^*(a-h_a)}{(s_a\lambda)^* a} s_a \in \mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}_\lambda \times \widehat{\mathrm{Pol}}_{s_a\lambda}) & a(\lambda) \neq 0 \end{cases}. \end{aligned} \tag{1.4}$$

Lemma 1.5. *The map ψ define a faithful continuous action of $\mathbb{H}$ on $\widehat{\mathrm{Pol}}$.*

□

Let $\mathbb{H}_{\lambda_0}^\wedge \subset \mathrm{End}^{\mathrm{cont}}(\widehat{\mathrm{Pol}})$ be the closure of the image of ψ. Let $\mathbb{H}_{\lambda_0}^\wedge$-mod^{sm} be the category of finitely generated $\mathbb{H}_{\lambda_0}^\wedge$-modules M such that for each element $m \in M$, the annihilator $\mathrm{ann}_{\mathbb{H}_{\lambda_0}^\wedge}(m)$ is an open left ideal of $\mathbb{H}_{\lambda_0}^\wedge$.

Lemma 1.6. *The restriction ψ^* yields an equivalence of categories*

$$\mathbb{H}^\wedge_{\lambda_0}\text{-mod} \cong \mathcal{O}_{\lambda_0}(\mathbb{H}).$$

□

The block algebra $\mathbb{H}^\wedge_{\lambda_0}$ is a topological algebra. It has a set of topological generators which reflects better than $\mathbb{H}$ the weight-space decomposition of objects of $\mathcal{O}_{\lambda_0}(\mathbb{H})$.

Lemma 1.7. *The topological algebra $\mathbb{H}^\wedge_{\lambda_0}$ is topologically generated by the following elements:*

(i) for each $\lambda \in W_S\lambda_0$, the projector $\mathbf{e}(\lambda) : \widehat{\mathrm{Pol}} \longrightarrow \widehat{\mathrm{Pol}}_\lambda$

(ii) the ring of formal power series $\mathbf{C}[\![V]\!]$

(iii) for each $a \in \Delta$ an operator $\tau_a : \widehat{\mathrm{Pol}} \longrightarrow \widehat{\mathrm{Pol}}$ such that

$$\tau_a\mathbf{e}(\lambda) = \mathbf{e}(s_a\lambda)\tau_a : \widehat{\mathrm{Pol}}_\lambda \longrightarrow \widehat{\mathrm{Pol}}_{s_a\lambda}, \quad \tau_a\mathbf{e}(\lambda) = \begin{cases} (\partial a)^{-1}(s_{\partial a} - 1) & a(\lambda) = 0 \ \wedge \ h_a \neq 0 \\ (\partial a)s_{\partial a} & a(\lambda) = h_a \ \wedge \ h_a \neq 0 \\ s_{\partial a} & a \notin \{0, h_a\} \ \vee \ h_a = 0 \end{cases}. \tag{1.8}$$

Proof. The idempotents $\mathbf{e}(\lambda)$ belong to $\mathbb{H}^\wedge_{\lambda_0}$ by the chinese remainder theorem. For the operators $s_a \in \mathbb{H}$, once we pass to the completion $\mathbb{H}^\wedge_{\lambda_0}$, only the pole / zero orders of the functions a^{-1} and $a - h_a$ appearing in the formula (1.4) at different points $\lambda \in W_S\lambda_0$ matter, which are given by (1.8). □

1.2.5 Idempotent form $\mathbf{H}_{\lambda_0}$

In view of Lemma 1.6 and Lemma 1.7, in order to study the block $\mathcal{O}_{\lambda_0}(\mathbb{H})$, it is convenient to consider the subalgebra generated by the generators given in Lemma 1.7.

For each λ, define $\mathbf{e}(\lambda) : \mathrm{Pol} \longrightarrow \mathrm{Pol}$ to be the idempotent linear endomorphism of projection onto the factor Pol_λ.

For $\lambda \in W_S\lambda_0$, we define a function $\mathrm{ord}_\lambda : S^+ \longrightarrow \mathbf{Z}_{\geq -1}$ by

$$\mathrm{ord}_\lambda(a) = \mathrm{ord}_{z=a(\lambda)}(z - h_a)z^{-1}. \tag{1.9}$$

For each $a \in \Delta$ and $\lambda \in W_S\lambda_0$, define an operator $\tau_a\mathbf{e}(\lambda) : \mathrm{Pol}_\lambda \longrightarrow \mathrm{Pol}_{s_a\lambda}$ by

$$\tau_a\mathbf{e}(\lambda) = \begin{cases} (\partial a)^{-1}(s_{\partial a} - 1) & \mathrm{ord}_\lambda(a) = -1 \\ (\partial a)^{\mathrm{ord}_\lambda(a)}s_{\partial a} & \mathrm{ord}_\lambda(a) \geq 0 \end{cases}. \tag{1.10}$$

Here $s_{\partial a} : \mathbf{C}[V] \longrightarrow \mathbf{C}[V]$ is the reflection with respect to the finite root $\partial a \in R$.

Let $\mathbf{H}_{\lambda_0}$ be the associative (non-unital) subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{Pol})$ generated by $f\mathbf{e}(\lambda)$ and $\tau_a\mathbf{e}(\lambda)$ for $f \in \mathbf{C}[V]$, $a \in \Delta$ and $\lambda \in W_S\lambda_0$.

Let $\mathbf{H}_{\lambda_0}\text{-mod}_0$ be the category of finitely generated $\mathbf{H}_{\lambda_0}\text{-mod}_0$-modules M such that $M = \bigoplus_{\lambda \in W_S\lambda_0} \mathbf{e}(\lambda)M$ and such that the subspace $V^* \subset \mathbf{C}[V]$ acts locally nilpotently on M.

Lemma 1.11. *The inclusion* $\mathbf{H}_{\lambda_0} \hookrightarrow \mathbb{H}^\wedge_{\lambda_0}$ *has dense image and induces an equivalence of categories*

$$\mathbb{H}^\wedge_{\lambda_0}\text{-mod}^{\text{sm}} \cong \mathbf{H}_{\lambda_0}\text{-mod}_0$$

□

Remark 1.12. *In §2, we will attach to each family of functions* $\{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ *an algebra* $\mathbf{A}^\omega$*. We will study them in a larger generality. The algebra* $\mathbf{H}_{\lambda_0}$ *is the special case where* $\omega_\lambda = \text{ord}_\lambda$ *for* $\lambda \in W_S\lambda_0$*.*

1.2.6 Centre $\mathcal{Z}^\wedge$

For $\lambda \in W_S\lambda_0$, let W_λ denote the stabiliser of λ in W_S. The stabiliser W_λ is a finite parabolic subgroup of the Coxeter group W_S. The affine Weyl group W_S acts on the vector space $V_{\mathbf{C}}$ via the finite quotient $\partial : W_S \longrightarrow W_S/Q^\vee_R \cong W_R$. Let $\mathcal{Z}^\wedge = \mathbf{C}[\![V]\!]^{W_{\lambda_0}}$ be the ring of W_{λ_0}-invariant formal power series. Since W_{λ_0} acts by reflections on V, the ring $\mathcal{Z}^\wedge$ a complete regular local ring. Let $\mathfrak{m}_{\mathcal{Z}} \subset \mathcal{Z}^\wedge$ be the maximal ideal.

For each $\lambda \in W_S\lambda_0$, we define a homomorphism $\mathcal{Z}^\wedge \longrightarrow \widehat{\text{Pol}}_\lambda$: choosing a $w \in W_S$ such that $w\lambda_0 = \lambda$, we let $f \mapsto w(f) \in \mathbf{C}[\![V]\!]^{W_{w\lambda}} \subset \widehat{\text{Pol}}_\lambda$. This map is clearly independent of the choice of w and it identify $\mathcal{Z}^\wedge$ with the invariant subspace $\mathbf{C}[\![V]\!]^{W_{w\lambda}}$. By the invariants theory, $\widehat{\text{Pol}}_\lambda$ is a free $\mathcal{Z}^\wedge$-module of rank $\#W_\lambda = \#W_{\lambda_0}$. The space $\widehat{\text{Pol}}$ is regarded as a $\mathcal{Z}^\wedge$-module via the diagonal action. It is easy to observe that $\mathcal{Z}^\wedge$ lies in the centre of $\mathbb{H}^\wedge_{\lambda_0}$.

Proposition 1.13. *The algebra of invariant formal power series* $\mathcal{Z}^\wedge$ *coincides with the centre of the block algebra* $\mathbb{H}^\wedge_{\lambda_0}$*.*

□

1.3 Affine Hecke algebra

We keep the notations (E, S, Δ), and $h = \{h_a\}_{a \in S}$ as above.

1.3.1 Extended affine Hecke algebras

We denote $R_0 = R \setminus 2R$. We put

$$v = \{v_\alpha\}_{\alpha \in R}, \quad v_\alpha = \begin{cases} \exp(\pi i h_\alpha) & \alpha \in R_0 \\ \exp(\pi i h_{\alpha+\delta}) & \alpha \in R \cap 2R \end{cases}.$$

Recall that $\tilde{W}^\vee_S = P_R \rtimes W_R$ is the dual extended Weyl group (we identify W_R with $W_{R^\vee}$ via the bijection $s_\alpha \leftrightarrow s_{\alpha^\vee}$).

Define the extended affine braid group $\mathfrak{B}_S$ for the dual root system $(V^*, R^\vee)$ to be the group generated by T_w for $w \in \tilde{W}^\vee_S$ with relations for $y, w \in \tilde{W}^\vee_S$:

$$T_y T_w = T_{yw}, \quad \text{if } l(yw) = l(y) + l(w).$$

The extended affine Hecke algebra of parameters v is an unital associative $\mathbf{C}$-algebra $\mathbb{K}$ is the group algebra $\mathbf{C}\mathfrak{B}_S$ quotient by the following relations for $\alpha \in \Delta_0$, in the case where R is reduced:

$$(T_{s_\alpha} - v_\alpha^2)(T_{s_{\theta^\vee}} + 1) = 0, \quad (T_{s_0} - v_\theta^2)(T_{s_0} + 1) = 0$$

where $s_0 \in \tilde{W}_S^\vee$ is the reflection with repect to the affine simple root and $\theta \in R^+$ is the highest root. In the case where R is non-reduced, let $\beta \in \Delta_0$ be the simple root such that $2\beta \in R$. Let $\mathbb{K}$ be the quotient of $\mathbf{C}\mathfrak{B}_S$ by the following relations for $\alpha \in \Delta_0 \setminus \{\beta\}$:

$$\begin{aligned}
&(T_{s_\alpha} + v_\alpha^2)(T_{s_\alpha} - 1) = 0\\
&(T_{s_\beta} + v_\beta^2 v_\theta)(T_{s_\beta} - 1) = 0\\
&(T_{s_0} + v_\beta^2 v_\theta^{-1})(T_{s_0} - 1) = 0.
\end{aligned}$$

1.3.2 Bernstein–Lusztig presentation

There is a subalgebra $\mathbf{C}P_R \subset \mathbb{K}$ given by $\beta \mapsto T_\beta$ for $\beta \in P_R$ dominant with respect to the base Δ_0. For $\beta \in P_R$ in general, decomposing it into $\beta = \beta' - \beta''$ with β' and β'' dominant, we let $Y^\beta = T_{\beta'} T_{\beta''}^{-1}$. Then there is a decomposition

$$\mathbb{K} = H_R \otimes \mathbf{C}P_R,$$

where H_R is the subalgebra generated by $\{T_{s_\alpha}\}_{a \in \Delta_0}$ with the following commutation relations: for $\alpha \in \Delta_0$ and $f \in \mathbf{C}P_R$:

$$\begin{aligned}
T_{s_\alpha} f - s_\alpha(f) T_{s_\alpha} &= (v_\alpha^2 - 1)\frac{f - s_\alpha(f)}{1 - X^{-\alpha}}, && 2\alpha \notin R\\
T_{s_\beta} f - s_\beta(f) T_{s_\beta} &= \left((v_\beta^2 v_\theta + 1) - \left(v_\beta^2 + v_\theta\right) X^{-\beta}\right) \frac{f - s_\beta(f)}{1 - X^{-2\beta}}, && 2\beta \in R,
\end{aligned} \tag{1.14}$$

1.3.3 Finite dimensional modules

Let T be the complex torus defined by $T = Q_R^\vee \otimes \mathbf{C}^\times$ so that we have $Q_R^\vee = \mathbf{X}_*(T)$ and $P_R = \mathbf{X}^*(T)$. For any $\beta \in P_R$, we denote by $Y^\beta \in \mathbf{C}[T]$ the corresponding function on T.

For each $\ell \in T$, let $\mathfrak{m}_\ell \subset \mathbf{C}P_R$ denote the maximal ideal corresponding to ℓ, which is generated by $Y^\beta - Y^\beta(\ell) \in \mathbf{C}P_R$ for all $\beta \in P_R$. Given any module $M \in \mathbb{K}$-Mod, for each $\ell \in T$ consider the generalised λ-weight space in M of the action of the subalgebra $\mathbf{C}P_R \subset \mathbb{K}$:

$$M_\ell = \bigcup_{N \geq 0} \left\{a \in M \ ; \ \mathfrak{m}_\ell^N a = 0\right\}$$

For any $\ell_0 \in T$. We define $\mathcal{O}_{\ell_0}(\mathbb{K})$ to be the full subcategory of $\mathbb{K}$-mod consisting of those $M \in \mathbb{K}$-mod such that

$$M = \bigoplus_{\ell \in W\,\ell_0} M_\ell.$$

1.3.4 Idempotent form $\mathbf{K}_{\ell_0}$

Fix $\ell_0 \in T$. As in the case of $\mathbb{H}$, we define an algebra which is more adapted to study the block $\mathcal{O}_{\ell_0}(\mathbb{K})$. Define for each $\ell \in W_R\ell_0$ a polynomial ring $\mathrm{Pol}_\ell = \mathbf{C}[V]$ and let $\mathrm{Pol} = \bigoplus_{\ell \in W_R\ell_0} \mathrm{Pol}_\ell$. For each ℓ, define $\mathbf{e}(\ell) : \mathrm{Pol} \longrightarrow \mathrm{Pol}$ to be the idempotent linear endomorphism of projection onto the factor Pol_ℓ. Let $R_0^+ = R^+ \setminus 2R^+$ denote the set of indivisible positive roots. In view of (1.14), for $\ell \in W_R\ell_0$, we define a function $\mathrm{ord}_\ell : R_0^+ \longrightarrow \mathbf{Z}$:

$$\mathrm{ord}_\ell(\alpha) = \begin{cases} \mathrm{ord}_{z=Y^\alpha(\ell)}(z - v_\alpha^2)(z-1)^{-1} & 2\alpha \notin R \\ \mathrm{ord}_{z=Y^\alpha(\ell)}(z - v_\alpha^2)(z - v_\theta)(z^2-1)^{-1} & 2\alpha \in R. \end{cases}$$

For each $\alpha \in \Delta_0$ and $\ell \in W_R\ell_0$, we define an operator $\tau_\alpha \mathbf{e}(\ell) : \mathrm{Pol}_\ell \longrightarrow \mathrm{Pol}_{s_\alpha \ell}$ by

$$\tau_\alpha \mathbf{e}(\ell) = \begin{cases} \alpha^{-1}(s_\alpha - 1) & \mathrm{ord}_\ell(\alpha) = -1 \\ \alpha^{\mathrm{ord}_\ell(\alpha)} s_\alpha & \mathrm{ord}_\ell(\alpha) \geq 0 \end{cases}.$$

Here $s_\alpha : \mathbf{C}[V] \longrightarrow \mathbf{C}[V]$ is the reflection with respect to α.

Let $\mathbf{K}_{\ell_0}$ be the associative subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{Pol})$ generated by $f\mathbf{e}(\ell)$ and $\tau_\alpha \mathbf{e}(\ell)$ for $f \in \mathbf{C}[V]$, $\alpha \in \Delta_0$ and $\ell \in W_R\ell_0$. Let $\mathbf{K}_{\ell_0}$-mod_0 be the category of finitely generated $\mathbf{K}_{\lambda_0}$-mod_0-modules M such that the subspace $V^* \subset \mathbf{C}[V]$ acts locally nilpotently on M. Same arguements as Lemma 1.6 and Lemma 1.11 shows that:

Lemma 1.15. *There is an equivalence of categories*

$$\mathcal{O}_{\ell_0}(\mathbb{K}) \cong \mathbf{K}_{\ell_0}\text{-}\mathrm{mod}_0$$

□

Remark 1.16. *In §2, we will attach to each family of functions $\{\omega_\ell\}_{\ell \in W_R\ell_0}$ an algebra $\mathbf{B}^\omega$. The algebra $\mathbf{K}_{\lambda_0}$ is the special case where $\omega_\ell = \mathrm{ord}_\ell$ for $\ell \in W_R\ell_0$.*

1.4 The monodromy functor $\mathbb{V}$

In this section, we recall the construction of the monodromy functor of [48], which is a trigonometric analogue of the Knizhnik–Zamolodchikov functor introduced in [17] for rational Cherednik algebras. We prove that this functor is a quotient functor.

Keep the notations (E, S, Δ), $a_0 \in \Delta$ as above. In addition, we fix $\lambda_0 \in E_{\mathbf{C}}$. Consider the following exponential map

$$E_{\mathbf{C}} \cong V_{\mathbf{C}} = Q^\vee \otimes \mathbf{C} \xrightarrow{\exp} Q^\vee \otimes \mathbf{C}^\times = T$$
$$\mu \otimes r \mapsto \mu \otimes e^{2\pi i r}.$$

Put $\ell_0 = \exp(\lambda_0)$. For simplying the notations, denote $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$ and $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$.

1.4.1 Dunkl operators

Consider the dual torus $T^\vee = P \otimes \mathbf{C}^\times$. The ring of regular functions $\mathbf{C}[T^\vee]$ is isomorphic to the group algebra of the coroot lattice $\mathbf{C}Q^\vee$:

$$\begin{aligned} \mathbf{C}Q^\vee &\xrightarrow{\cong} \mathbf{C}[T^\vee] \\ Q^\vee \ni \mu &\mapsto X^\mu \end{aligned}$$

For each $\xi \in V^*$, let $\partial_\xi \in \Gamma\left(T^\vee, \mathcal{T}_{T^\vee}\right)^{T^\vee}$ be the translation-invariant vector field on $T^\vee$ such that $\partial_\xi \mid_e = \xi$ under the isomorphism $\mathcal{T}_{T^\vee} \mid_e \cong V$. We view ∂_ξ as a linear differential operator on $T^\vee$, so that $\partial_\xi(X^\mu) = \langle \xi, \mu \rangle X^\mu$ for each $\mu \in Q^\vee$.

The regular part of $T^\vee$ is defined as $T^\vee_\circ = \bigcap_{\alpha \in R^+} \left\{ X^{\alpha^\vee} \neq 0 \right\} \subset T^\vee$. Let $\mathcal{D}(T^\vee_\circ)$ denotes the ring of differential operators on $T_\circ$.

For $\xi \in V^*$, the trigonometric Dunkl operator $D_\xi : \mathbf{C}[T^\vee] \longrightarrow \mathbf{C}[T^\vee]$ is a $\mathbf{C}$-linear operator defined as follows:

$$D_\xi(f) = \partial_\xi(f) - \sum_{\alpha \in R^+} h_\alpha \frac{f - s_\alpha(f)}{1 - e^{-\alpha^\vee}} + \langle \xi, \rho_h^\vee \rangle f, \quad \rho_h^\vee = \frac{1}{2} \sum_{\alpha \in \Delta^+} h_\alpha \alpha^\vee \in V_{\mathbf{C}}.$$

We consider D_ξ as an element in $\mathcal{D}(T^\vee_\circ) \rtimes W_R$.

The homomorphism of $\mathbf{C}$-algebras

$$\begin{aligned} \mathbf{C}[T^\vee_\circ] \otimes \mathbf{C}W_R \otimes \mathbf{C}[V] = \mathbb{H} &\longrightarrow \mathcal{D}(T^\vee_\circ) \rtimes W_R \\ e^\mu \otimes w \otimes 1 &\mapsto e^\mu \otimes w \\ 1 \otimes 1 \otimes \xi &\mapsto D_\xi \end{aligned}$$

induces an isomorphism $\mathbf{C}[T^\vee_\circ] \otimes_{\mathbf{C}[T^\vee]} \mathbb{H} \cong \mathcal{D}(T^\vee_\circ) \rtimes W_R$.

Let $[T^\vee_\circ / W]$ be the quotient stack. The orbifold fundamental group $\pi_1([T^\vee_\circ / W])$ is isomorphic to the extended affine braid group $\mathfrak{B}_S$.

If $M \in \mathcal{O}_{\lambda_0}(\mathbb{H})$, then

$$M_\circ = \mathbb{H}_\circ \otimes_{\mathbb{H}} M$$

is a W-equivariant $\mathcal{D}(T^\vee_\circ)$-module, which is in fact an integrable connection with regular singularities. Therefore the flat sections of M on (the universal covering of) the orbifold $[T^\vee_\circ / W]$ defines a $\mathfrak{B}_S$-module, which is denoted by $\mathbb{V}(M)$. It is shown in [48] that in fact the $\mathfrak{B}_S$-action on M factorises through the surjective homomorphism $\mathbf{C}\mathfrak{B}_S \longrightarrow \mathbb{K}$ and yields an exact functor

$$\mathbb{V} : \mathcal{O}_{\lambda_0}(\mathbb{H}) \longrightarrow \mathcal{O}_{\ell_0}(\mathbb{K}).$$

1.4.2 Central actions of $\mathcal{Z}^\wedge$ intertwined by $\mathbb{V}$

For convenient, we will denote $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$ and $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$. Recall the central algebra $\mathcal{Z}^\wedge = \mathbf{C}[\![V]\!]^{W_{\lambda_0}}$ defined in §1.2.6.

Let W_{λ_0} be the stabiliser of $\lambda_0 \in E_{\mathbf{C}}$ in W_S and let W_{ℓ_0} be the stabiliser of $\ell_0 \in T$ in W_R. Let $\bar{\lambda}_0$ be the image of λ_0 in $E_{\mathbf{C}}/W_{\lambda_0}$ and let $\bar{\ell}_0$ be the image of ℓ_0 in T/W_{ℓ_0}. The exponential map (1.4) induces an analytic map

$$\exp^{\lambda_0} : E_{\mathbf{C}}/W_{\lambda_0} \longrightarrow T/W_{\ell_0},$$

which is locally biholomorphic near $\bar{\lambda}_0$. The push-forward along $\exp^{\lambda_0}$ at $\bar{\lambda}_0$ yields an isomorphism of complete local rings

$$\exp^{\lambda_0}_* : \mathbf{C}\left[E_{\mathbf{C}}/W_{\lambda_0}\right]^{\wedge}_{\bar{\lambda}_0} \cong \mathbf{C}\left[T/W_{\ell_0}\right]^{\wedge}_{\bar{\ell}_0}.$$

Note that

$$\mathcal{Z}^{\wedge} \cong \mathbf{C}\left[E_{\mathbf{C}}/W_{\lambda_0}\right]^{\wedge}_{\bar{\lambda}_0}.$$

For any $w \in W_S$, the action of w on $E_{\mathbf{C}}$ and on T induces

$$w_* : \mathcal{Z}^{\wedge} \cong \mathbf{C}\left[E_{\mathbf{C}}/W_{w\lambda_0}\right]^{\wedge}_{w\bar{\lambda}_0}, \quad w_* : \mathbf{C}\left[T/W_{\ell_0}\right]^{\wedge}_{\bar{\ell}_0} \cong \mathbf{C}\left[T/W_{w\ell_0}\right]^{\wedge}_{w\bar{\ell}_0}.$$

We define homomorphisms $\mathcal{Z}^{\wedge} \longrightarrow Z(\mathcal{C}_0)$ and $\mathcal{Z}^{\wedge} \longrightarrow Z(\mathcal{B}_0)$ as follows: for any $M \in \mathcal{C}_0$, we decompose $M = \bigoplus_{\lambda \in W_S\lambda_0} M_\lambda$ and for each $\lambda = w\lambda_0$, $f \in \mathcal{Z}^{\wedge}$ acts by $w_* f$ on M_λ. This depends only on the weight λ but not on the choice of w. Simlarly, for any $N \in \mathcal{B}_0$, we decompose $N = \bigoplus_{\ell \in W_R\ell_0} M_\ell$. For each $\ell = w\ell_0$, $f \in \mathcal{Z}^{\wedge}$ acts by multiplication by $w_* \exp^{\lambda_0}_* f$ on N_ℓ.

Lemma 1.17. *The functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ intertwines the $\mathcal{Z}^{\wedge}$-actions on $\mathcal{C}_0$ and $\mathcal{B}_0$.*

Proof. Recall that the graded affine Hecke algebra is the subalgebra

$$\underline{\mathbb{H}} = \mathbf{C}W_R \otimes \operatorname{Sym} V^*_{\mathbf{C}} \subset \mathbb{H}.$$

For any weight $\lambda \in V_{\mathbf{C}}$, let $\mathcal{O}_\lambda(\underline{\mathbb{H}})$ be the category of finite dimensional $\underline{\mathbb{H}}$-modules on which the action of the polynomial part $\operatorname{Sym} V_{\mathbf{C}}$ has weights included in the orbit $W_S\lambda \subset V_{\mathbf{C}}$.

There is a functor of induction

$$\operatorname{Ind}^{\mathbb{H}}_{\underline{\mathbb{H}}} : \underline{\mathbb{H}}\text{-mod} \longrightarrow \mathbb{H}\text{-mod}, \quad \operatorname{Ind}^{\mathbb{H}}_{\underline{\mathbb{H}}} M = \mathbb{H} \otimes_{\underline{\mathbb{H}}} M$$

and for each weight $\lambda \in E_{\mathbf{C}}$, it restricts to

$$\operatorname{Ind}^{\mathbb{H}}_{\underline{\mathbb{H}}} : \mathcal{O}_\lambda(\underline{\mathbb{H}}) \longrightarrow \mathcal{O}_\lambda(\mathbb{H})$$

Let $\mathcal{I} \subset \mathcal{C}_0$ denote the essential image of $\operatorname{Ind}^{\mathbb{H}}_{\underline{\mathbb{H}}}$. It is known that $\mathcal{I}$ generates $\mathcal{C}_0$. Therefore, it suffices to show that the restriction $\mathbb{V}|_{\mathcal{I}}$ intertwines the actions of $\mathcal{Z}^{\wedge}$.

We shall apply the deformation argument to check this statement. The arguments are similar to [48, 5.1]. Let $\mathcal{O} = \mathbf{C}[\![\varpi]\!]$ and let $\mathcal{K} = \mathbf{C}(\!(\varpi)\!)$. Let $\varepsilon \in V^*_{\mathbf{C}}$ be any regular coweight and put $\lambda_{0,\mathcal{O}} = \lambda_0 + \varpi\varepsilon \in V^*_{\mathcal{O}}$. Put $\underline{\mathbb{H}}_{\mathcal{O}} = \underline{\mathbb{H}} \otimes \mathcal{O}$ and $\mathbb{K}_{\mathcal{O}} = \mathbb{K} \otimes \mathcal{O}$.

For each $\lambda_{\mathcal{O}} \in W_S\lambda_{0,\mathcal{O}}$ and for $n \in \mathbf{Z}_{\geq 1}$, let

$$\begin{aligned}
\mathfrak{m}_{\lambda_{\mathcal{O}}} &= \langle \beta_{\mathcal{O}} - \langle \beta_{\mathcal{O}}, \lambda_{\mathcal{O}} \rangle \,;\, \beta \in V_{\mathcal{O}} \rangle \subset \operatorname{Sym}_{\mathcal{O}} V^*_{\mathcal{O}}, \quad \mathfrak{m}_{\lambda_{\mathcal{K}}} = \mathfrak{m}_{\lambda_{\mathcal{O}}}[\varpi^{-1}] \\
\mathbb{S}_{\lambda^n_{\mathcal{O}}} &= \operatorname{Sym}_{\mathcal{O}} V^*_{\mathcal{O}}/\mathfrak{m}^n_{\lambda_{\mathcal{O}}}, \quad \mathbb{S}_{\lambda^n_{\mathcal{K}}} = \mathbb{S}_{\lambda^n_{\mathcal{O}}}[\varpi^{-1}] \\
\underline{P}(\lambda_{\mathcal{O}})_n &= \underline{\mathbb{H}}_{\mathcal{O}} \otimes_{\operatorname{Sym}_{\mathcal{O}} V^*_{\mathcal{O}}} \mathbb{S}_{\lambda^n_{\mathcal{O}}}, \quad \underline{P}(\lambda_{\mathcal{K}})_n = \underline{\mathbb{H}}_{\mathcal{K}}[\varpi^{-1}].
\end{aligned}$$

Note that all these objects are flat over $\mathcal{O}$. Let $\underline{P}(\lambda_{\mathcal{O}})_n^{\nabla}$ be the space of flat sections of the affine Knizhnik–Zamolodchikov equation (AKZ) on the constant vector bundle on $T_\circ^\vee$ of fibre $\underline{P}(\lambda_{\mathcal{O}})_n$. The monodromy representation yields a $\mathbb{K}_{\mathcal{O}} = \mathbb{K} \otimes \mathcal{O}$ action on $\underline{P}(\lambda_{\mathcal{O}})_n^{\nabla}$.

Since the stabiliser of $\lambda_{\mathcal{O}}$ in W_S is trivial, there is an eigenspace decomposition

$$P(\lambda_{\mathcal{K}})_n = \bigoplus_{w \in W} (P(\lambda_{\mathcal{K}})_n)_{w\lambda_{\mathcal{K}}}, \quad (P(\lambda_{\mathcal{K}})_n)_{w\lambda_{\mathcal{K}}} = b_w \mathbb{S}_{\lambda_{\mathcal{K}}^n}$$

where each $b_w \mathbb{S}_{\lambda_{\mathcal{K}}^n}$ is a free $\mathbb{S}_{\lambda_{\mathcal{K}}^n}$-module of rank 1. This means that there is a fundamental solution $\left\{b_w^{\nabla}\right\}_{w \in W_R}$ of the AKZ equation on $T_\circ^\vee$ which satisfies

$$\begin{aligned} b_w^{\nabla}(\exp(\mu)) &= e^{\sum_{j=1}^r \left(\langle \rho_h, \omega_j^\vee \rangle - \omega_j^\vee\right)\langle \alpha_j, \mu \rangle} \cdot (b_w + G(\mu)) \\ &\qquad \text{for } \mu \in V_{\mathbf{C}}^* \text{ such that } \mathfrak{Im}\left(\langle \alpha, \mu \rangle\right) \longrightarrow +\infty, \ \forall \alpha \in \Delta^+, \end{aligned}$$

where

$$\rho_h = \frac{1}{2} \sum_{\alpha \in \Delta^+} h_\alpha \alpha \in V_{\mathbf{C}},$$

$\omega_j^\vee \in V_{\mathbf{C}}$'s are the fundamental coweights, $\alpha_j \in V_{\mathbf{C}}^*$'s are the simple roots so that $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{i,j}$ and $G(\mu)$ is a $\underline{P}(\lambda_{\mathcal{K}})_n$-valued analytic function in μ with such that

$$G(\mu) \longrightarrow 0, \quad \text{when } \mathfrak{Im}\,\langle \alpha, \mu \rangle \longrightarrow +\infty, \forall \alpha \in \Delta^+.$$

The fundamental solution induces an $\mathbb{S}_{\lambda_{\mathcal{K}}^n}$-linear isomorphism

$$\underline{P}(\lambda_{\mathcal{K}})_n \longrightarrow \underline{P}(\lambda_{\mathcal{K}})_n^{\nabla}, \quad b_w \mapsto b_w^{\nabla}.$$

Under this isomorphism, the monodromy operator on the right-hand side corresponding to $\beta \in X$ is identified with $e^{2\pi i \beta}$ on the left-hand side. Put

$$\mathcal{Z}_{\mathcal{O}}^{\wedge} = \left((\operatorname{Sym} V_{\mathcal{O}}^*)^{W_{\lambda_0}}\right)_{\bar{\lambda}_{0,\mathcal{O}}}^{\wedge}, \quad \mathcal{Z}_{\mathcal{K}}^{\wedge} = \mathcal{Z}_{\mathcal{O}}^{\wedge}[\varpi^{-1}] \cong (\operatorname{Sym} V_{\mathcal{K}}^*)_{\bar{\lambda}_{0,\mathcal{O}}}^{\wedge}.$$

We define the action of $\mathcal{Z}_{\mathcal{O}}^{\wedge}$ and $\mathcal{Z}_{\mathcal{K}}^{\wedge}$ on $\underline{\mathbb{H}}_{\mathcal{O}}$-modules and $\underline{\mathbb{H}}_{\mathcal{K}}$ in a similar way.

Since the action of $\mathcal{Z}_{\mathcal{K}}^{\wedge}$ on $\underline{P}(\lambda_{\mathcal{K}})_n$ coincides with the action of the polynomial part $\operatorname{Sym} V_{\mathcal{K}} \subset \underline{\mathbb{H}}$ up to twists by elements of W_R, the induced action of $\mathcal{Z}_{\mathcal{K}}^{\wedge}$ on $\mathbb{K}_{\mathcal{K}}$-module $\underline{P}(\lambda_{\mathcal{K}})_n^{\nabla}$ is identified with the exponentiation of the action of $\mathcal{Z}_{\mathcal{K}}^{\wedge}$ on the $P(\lambda_{\mathcal{K}})_n$ under (1.4.2).

Since $P(\lambda_{\mathcal{O}})_n \subset P(\lambda_{\mathcal{K}})_n$ and $P(\lambda_{\mathcal{O}})_n^{\nabla} \subset P(\lambda_{\mathcal{K}})_n^{\nabla}$ are stable under the action of $\mathcal{Z}_{\mathcal{O}}^{\wedge} \subset \mathcal{Z}_{\mathcal{K}}^{\wedge}$, the functor $M \mapsto M^{\nabla}$ also intertwines the two $\mathcal{Z}_{\mathcal{O}}^{\wedge}$-actions. Put $\underline{P}(\lambda)_n = \underline{P}(\lambda_{\mathcal{O}})_n \otimes_{\mathcal{O}} \mathbf{C}$. Then $\underline{P}(\lambda)_n \mapsto \underline{P}(\lambda)_n^{\nabla} = \mathbb{V}(\operatorname{Ind}_{\underline{\mathbb{H}}}^{\mathbb{H}} P(\lambda)_n^{\nabla})$ also intertwines the two $\mathcal{Z}^{\wedge}$-actions. Finally, since the family of modules $\underline{P}(\lambda)_n$ for $\lambda \in W_S\lambda_0$ and $n \geq 1$ generates the category $\mathcal{O}_{\lambda_0}(\underline{\mathbb{H}})$, the functor $\mathbb{V}$ restricted to $\mathcal{I}$ intertwine the $\mathcal{Z}^{\wedge}$-actions as asserted. $\square$

1.4.3 Completion of categories

Since the affine Hecke algebra $\mathbb{K}$ is finite dimensional over its centre, $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$ is equivalent to the category of modules of finite length of some semi-perfect algebra. It

is also the case for $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$. In particular, they are both noetherian and artinian. Consider the category of pro-objects[1] $\mathrm{Pro}(\mathcal{C}_0)$ and $\mathrm{Pro}(\mathcal{B}_0)$. We have two central actions introduced in §1.4.2

$$\mathcal{Z}^\wedge \longrightarrow \mathrm{End}(\mathrm{id}_{\mathcal{C}_0}) \cong \mathrm{End}(\mathrm{id}_{\mathrm{Pro}(\mathcal{C}_0)})$$
$$\mathcal{Z}^\wedge \longrightarrow \mathrm{End}\left(\mathrm{id}_{\mathcal{B}_0}\right) \cong \mathrm{End}(\mathrm{id}_{\mathrm{Pro}(\mathcal{B}_0)}).$$

By Lemma 1.17, the functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ intertwines the $\mathcal{Z}^\wedge$-actions. The extension $\mathbb{V} : \mathrm{Pro}(\mathcal{C}_0) \longrightarrow \mathrm{Pro}(\mathcal{B}_0)$ still intertwine the $\mathcal{Z}^\wedge$-actions.

Define $\mathcal{C} \subset \mathrm{Pro}(\mathcal{C}_0)$ to be the subcategory consisting of objects $M \in \mathrm{Pro}(\mathcal{C}_0)$ such that $M/\mathfrak{m}_{\mathcal{Z}}^k M \in \mathcal{C}_0$ for all $k \geq 0$. Similarly we define $\mathcal{B} \subset \mathrm{Pro}(\mathcal{B}_0)$ to be the subcategory consisting of objects $N \in \mathrm{Pro}(\mathcal{B}_0)$ such that $M/\mathfrak{m}_{\mathcal{Z}}^k M \in \mathcal{C}_0$ for all $k \geq 0$.

Lemma 1.18. *For any simple object $L \in \mathcal{C}_0$ (resp. $L \in \mathcal{B}_0$), its projective cover $\mathcal{P}(L) \in \mathrm{Pro}(\mathcal{C}_0)$ (resp. $\mathcal{P}(L) \in \mathrm{Pro}(\mathcal{B}_0)$) lies in $\mathcal{C}$ (resp. $\mathcal{B}$).*

Proof. Notice that by a general result Lemma A.4, the objects of $\mathcal{C}_0$ (resp. $\mathcal{B}_0$) admit projective covers in $\mathrm{Pro}(\mathcal{C}_0)$ (resp. $\mathrm{Pro}(\mathcal{B}_0)$). The statement is obvious for $\mathcal{B}_0$ because $\mathbb{K}$ is of finite rank over its centre. For $\mathcal{C}_0$, by Lemma 1.6 and Lemma 1.11, there is an equivalence $\mathcal{C}_0 \cong \mathbf{H}_{\lambda_0}$ and the algebra $\mathbf{H}_{\lambda_0}$ is Morita-equivalent to an algebra of finite rank over its centre, *cf.* §2.2.4. □

Lemma 1.19. *The functor $\mathbb{V} : \mathrm{Pro}(\mathcal{C}_0) \longrightarrow \mathrm{Pro}(\mathcal{B}_0)$ restricts to $\mathbb{V} : \mathcal{C} \longrightarrow \mathcal{B}$.*

Proof. If $M \in \mathcal{C}$, then $M/\mathfrak{m}_{\mathcal{Z}}^k M \in \mathcal{C}_0$ and by Lemma 1.17, $\mathbb{V}(M)/\mathfrak{m}_{\mathcal{Z}}^k \mathbb{V}(M) \cong \mathbb{V}(M/\mathfrak{m}_{\mathcal{Z}}^k M) \in \mathcal{B}_0$. It follows that $\mathbb{V}(M) \in \mathcal{B}$. □

1.4.4 Right adjoint of $\mathbb{V}$

Recall that $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$ and $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$.

Lemma 1.20. *The functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ admits a right adjoint functor $\mathbb{V}^\top : \mathcal{B}_0 \longrightarrow \mathcal{C}_0$.*

Proof. We first define a functor $\mathbb{V}^\top : \mathcal{B}_0 \longrightarrow \mathrm{Ind}(\mathcal{C}_0)$ with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{B}_0}\left(\mathbb{V}(M), N\right) \cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}\left(M, \mathbb{V}^\top(N)\right) \tag{1.21}$$

for $M \in \mathcal{C}_0$ and $N \in \mathcal{B}_0$. For any $N \in \mathcal{B}_0$, let

$$F_N : \mathcal{C}_0^{\mathrm{op}} \longrightarrow \mathbf{C}\text{-Mod}, \quad F_N : M \mapsto \mathrm{Hom}_{\mathcal{B}_0}\left(\mathbb{V}(M), N\right)$$

and let

$$F_N(M)^{\min} = F_N(M) \setminus \bigcup_{0 \neq M' \subset M} F_N(M/M').$$

Here, we regard $F_N(M/M')$ as a subspace of $F_N(M)$ by the right exactness of F_N. Let $\mathcal{I}_N$ be the category whose objects are pairs (M, a), where $M \in \mathcal{C}_0$ and $a \in F_N(M)^{\min}$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{I}_N}\left((M, a), (M', a')\right) = \left\{f \in \mathrm{Hom}_{\mathcal{C}_0}(M, M') \; ; \; F_N(f)(a') = a\right\}.$$

[1] The generality of categories of pro-objects is recalled in §A.1.

We set

$$\mathbb{V}^{\top}(N) = \text{“}\varinjlim_{(M,a)\in\mathcal{I}_N}\text{”}\ M \in \mathrm{Ind}(\mathcal{C}_0).$$

According to [45, 3.5, Lemma 6], $\mathbb{V}^{\top}(N)$ represents the functor F_N, so $\mathbb{V}^{\top}$ satisfies the desired adjoint property (1.21).

Now we show that in fact the object $\mathbb{V}^{\top}(N)$ in $\mathrm{Ind}(\mathcal{C}_0)$ lies in the subcategory $\mathcal{C}_0$. Let $\mathcal{P}_{\mathcal{C}} \in \mathcal{C}$ be the sum of all projective indecomposable objects (up to isomorphism) of $\mathcal{C}$ so that for any $M \in \mathcal{C}_0$, the dimension of $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_{\mathcal{C}}, M)$ is equal to the length of M. Since $\mathbb{V}(\mathcal{P}_{\mathcal{C}}) \in \mathcal{B}$ is finitely generated $\mathbb{K}$-modules, the vector space $\mathrm{Hom}_{\mathcal{B}}(\mathbb{V}(\mathcal{P}_{\mathcal{C}}), N)$ is finite dimensional. Then there are isomorphisms

$$\begin{aligned}
\varinjlim_{\substack{M\subset\mathbb{V}^{\top}(N)\\ M\in\mathcal{C}_0}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_{\mathcal{C}}, M) &\cong \varinjlim_{\substack{M\subset\mathbb{V}^{\top}(N)\\ M\in\mathcal{C}_0}} \varinjlim_{\substack{\mathcal{Q}\subset\mathcal{P}_{\mathcal{C}}\\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q}\in\mathcal{C}_0}} \mathrm{Hom}_{\mathcal{C}_0}(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}, M)\\
&\cong \varinjlim_{\substack{\mathcal{Q}\subset\mathcal{P}_{\mathcal{C}}\\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q}\in\mathcal{C}_0}} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}\left(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}, \mathbb{V}^{\top}(N)\right)\\
&\cong \varinjlim_{\substack{\mathcal{Q}\subset\mathcal{P}_{\mathcal{C}}\\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q}\in\mathcal{C}_0}} \mathrm{Hom}_{\mathcal{B}_0}(\mathbb{V}(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}), N)\\
&\cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{B}_0)}\left(\varprojlim_{\substack{\mathcal{Q}\subset\mathcal{P}_{\mathcal{C}}\\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q}\in\mathcal{C}_0}} \mathbb{V}(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}), N\right) \cong \mathrm{Hom}_{\mathcal{B}}(\mathbb{V}(\mathcal{P}_{\mathcal{C}}), N).
\end{aligned} \tag{1.22}$$

The first and the fourth isomorphisms are due to (A.1) of § A.1; the second one is exchanging the order of the two colimits and the definition of morphisms between ind-objects; the third one is due to (1.21); the last one is by Lemma 1.19.

Since $N \in \mathcal{B}_0$, there is some integer n such that $\mathfrak{m}_{\mathcal{Z}}^n N = 0$. Since $\mathbb{V}(\mathcal{P}_{\mathcal{C}}) \in \mathcal{B}$, the quotient $\mathbb{V}(P_{\mathcal{C}})/\mathfrak{m}_{\mathcal{Z}}^n\mathbb{V}(P_{\mathcal{C}})$ lies in $\mathcal{B}_0$. Thus

$$\mathrm{Hom}_{\mathcal{B}}(\mathbb{V}(P_{\mathcal{C}}), N) \cong \mathrm{Hom}_{\mathcal{B}_0}(\mathbb{V}(P_{\mathcal{C}})/\mathfrak{m}_{\mathcal{Z}}^n\mathbb{V}(P_{\mathcal{C}}), N)$$

is finite-dimensional. The above isomorphisms (1.22) imply that the length of the subobjects $M \subset \mathbb{V}^{\top}(N)$ such that $M \in \mathcal{C}_0$ is bounded. It follows that $\mathbb{V}^{\top}(N)$ is in fact in $\mathcal{C}_0$ by Lemma A.2 (iii). Thus $\mathbb{V}^{\top} : \mathcal{B}_0 \longrightarrow \mathcal{C}_0$ is a right adjoint to $\mathbb{V}$.

□

1.4.5 Quotient functor $\mathbb{V}$

Proposition 1.23. *The monodromy functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ is a quotient functor.*

Proof. Recall that $\mathcal{D}(T_\circ^\vee)$ is the ring of algebraic linear differential operators on the regular part $T_\circ^\vee$ of the dual torus $T^\vee = P \otimes \mathbf{C}^\times$. By construction, the functor $\mathbb{V}$ factorises into

the following

$$\begin{tikzcd}
\mathbb{H}\text{-Mod} \arrow[r, "\mathrm{loc}"] & \mathcal{D}\left(T_\circ^\vee\right) \rtimes W_R\text{-Mod} & \\
\mathcal{C}_0 \arrow[u, hook] \arrow[r] \arrow[rrd, hook, "\mathbb{V}"] & \mathrm{conn}_{W_R}^{\mathrm{rs}}\left(T_\circ^\vee\right) \arrow[u, hook] \arrow[r, "\mathrm{RH}"] & \mathbf{C}\mathfrak{B}_S\text{-mod}^{\mathrm{fini}} \\
& & \mathcal{B}_0 \arrow[u, hook]
\end{tikzcd}$$

Here, $\mathrm{conn}_{W_R}^{\mathrm{rs}}\left(T_\circ^\vee\right)$ is the subcategory of $\mathcal{D}\left(T_\circ^\vee\right) \rtimes W_R$-mod consisting of W_R-equivariant connections on $T_\circ^\vee$ which have regular singularities along the boundary. The arrow in the first line is the localisation functor $\mathrm{loc} = \left(\mathcal{D}\left(T_\circ^\vee\right) \rtimes W_R\right) \otimes_{\mathbb{H}} -$, whose right adjoint $\mathrm{loc}^\top$ is the pull-back via $\mathbb{H} \subset \mathcal{D}\left(T_\circ^\vee\right) \rtimes W_R$. The restriction of loc to $\mathcal{C}_0$ factorises through the subcategory

$$\mathrm{conn}_{W_R}^{\mathrm{rs}}\left(T_\circ^\vee\right) \subset \mathcal{D}(T_\circ^\vee) \rtimes W_R\text{-Mod}$$

and gives the first arrow of the second line. The functor RH is the Riemann–Hilbert correspondence (the Knizhnik–Zamolodchikov equations have regular singularities [42]), due to Deligne, between algebraic connections with regular singularities and finite dimensional representations of the fundamental group $\pi_1\left([T_\circ^\vee/W_R]\right) \cong \mathfrak{B}_S$.

We show that $\mathbb{V}$ admits a section functor in the sense of Gabriel. Indeed, we have already seen that $\mathbb{V}$ admits a right adjoint functor $\mathbb{V}^\top$. The functor $\mathbb{V}^\top$ can be described as follows:

$$\mathcal{B}_0 \hookrightarrow \mathbf{C}\mathfrak{B}_S\text{-mod}^{\mathrm{fini}} \cong \mathrm{conn}_{W_R}^{\mathrm{rs}}\left(T_\circ^\vee\right) \longrightarrow \mathcal{C}_0,$$

where the last arrow is the functor which sends an object $N \in \mathrm{conn}_{W_R}^{\mathrm{rs}}(T_\circ^\vee)$ to the biggest $\mathbb{H}$-submodule of N which lies in $\mathcal{C}_0$. We show that the adjunction counit $\mathbb{V} \circ \mathbb{V}^\top \longrightarrow \mathrm{id}_{\mathcal{B}_1}$ is an isomorphism. We first show that it is a monomorphism. For any $M \in \mathcal{D}(T_\circ^\vee) \rtimes W_R$-Mod, we have $\mathbf{C}[T_\circ^\vee] \otimes_{\mathbf{C}[T^\vee]} M \cong M$. By the flatness of $\mathbf{C}[T_\circ^\vee]$ over $\mathbf{C}[T^\vee]$, the inclusion $M\,|_{\mathcal{C}_0} \hookrightarrow M$ gives rise to a monomorphism $\mathbf{C}[T_\circ^\vee] \otimes_{\mathbf{C}[T^\vee]} \left(M\,|_{\mathcal{C}_0}\right) \longrightarrow \mathbf{C}[T_\circ^\vee] \otimes_{\mathbf{C}[T^\vee]} M \cong M$. Composing it with the Riemann–Hilbert correspondence, we see that $\mathbb{V} \circ \mathbb{V}^\top \longrightarrow \mathrm{id}_{\mathcal{B}_0}$ is a monomorphism.

Let $N \in \mathcal{B}_0$. By the exactness of $\mathbb{V}$, to show that the adjunction counit $\mathbb{V}\mathbb{V}^\top N \hookrightarrow N$ is an isomorphism, it remains to find an $\mathbb{H}$-submodule of $\mathrm{RH}^{-1}(N)$ whose localisation to $T_\circ^\vee$ is equal to $\mathrm{RH}^{-1}(N)$. There exists a surjection

$$\bigoplus_{i \in \mathcal{I}} P\left(\ell_i\right)_{n_i} \longrightarrow N$$

where $\mathcal{I}$ is an index set and $P\left(\ell_i\right)_{n_i} = \mathbb{K}/\mathbb{K} \cdot \mathfrak{m}_{\ell_i}^{n_i}$. By [48, 5.1 (i)], for each $i \in \mathcal{I}$ there is an induced module $P\left(\lambda_i\right)_{n_i} = \mathbb{H}/\mathbb{H} \cdot \mathfrak{m}_{\lambda_i}^{n_i} \in \mathcal{C}_0$ such that $\exp(\lambda_i) = \ell_i$ and $\mathbb{V}\left(P\left(\lambda_i\right)_{n_i}\right) \cong P\left(\ell_i\right)_{n_i}$. Hence the image of $P(\lambda_i)_{n_i}$ in $\mathrm{RH}^{-1}(N)$ is an $\mathbb{H}$-submodule which satisfied the requirement. We conclude that $\mathbb{V} \circ \mathbb{V}^\top \cong \mathrm{id}_{\mathcal{B}_0}$.

By the well known result of Gabriel [16, 3.2, Prop 5], the functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ is a quotient functor. $\square$

1.5 Comparison of $\mathbb{V}$ and $\mathbf{V}$

1.5.1 The functors $\mathbb{V}$ and $\mathbf{V}$

In § 2, we will study the idempotent forms $\mathbf{H}_{\lambda_0}$ and $\mathbf{K}_{\ell_0}$ in a broader context, *cf.* Remark 1.12 and Remark 1.16. Specifically, in § 2.5.6, we will introduce a quotient functor for graded modules $\mathbf{V} : \mathbf{H}_{\lambda_0}\text{-gmod} \longrightarrow \mathbf{K}_{\ell_0}\text{-gmod}$. It has an ungraded version $\mathbf{V} : \mathbf{H}_{\lambda_0}\text{-mod}_0 \longrightarrow \mathbf{K}_{\lambda_0}\text{-mod}_0$. On the other hand, by Lemma 1.6, Lemma 1.11 and Lemma 1.15, we have equivalence of categories $\mathcal{O}_{\lambda_0}(\mathbb{H}) \cong \mathbf{H}_{\lambda_0}\text{-mod}_0$ and $\mathcal{O}_{\ell_0}(\mathbb{K}) \cong \mathbf{K}_{\lambda_0}\text{-mod}_0$. The situation can be depicted in a diagram:

$$\begin{array}{ccc} \mathcal{O}_{\lambda_0}(\mathbb{H}) & \xrightarrow{\mathbb{V}} & \mathcal{O}_{\ell_0}(\mathbb{K}) \\ \downarrow\cong & & \downarrow\cong \\ \mathbf{H}_{\lambda_0}\text{-mod}_0 & \xrightarrow{\mathbf{V}} & \mathbf{K}_{\ell_0}\text{-mod}_0 \end{array}$$

Conjecture 1.24. *There is an isomorphism of functors* $\mathbb{V} \cong \mathbf{V}$.

In the rest of this chapter, we prove a weaker version of this statement.

1.5.2 Comparison of the kernels

By Proposition 1.23 and §2.5.6, the functors $\mathbb{V}$ and $\mathbf{V}$ are already known to be quotient functors. We prove that $\mathbb{V}$ and $\mathbf{V}$ are quotient functors with the same kernel.

Proposition 1.25. *The kernels* $\ker \mathbb{V}$ *and* $\ker \mathbf{V}$ *are identified via the equivalence* $\mathcal{O}_{\lambda_0}(\mathbb{H}) \cong \mathbf{H}_{\lambda_0}\text{-mod}_0$.

Proof. Let $F : \mathcal{O}_{\lambda_0}(\mathbb{H}) \longrightarrow \mathbf{H}_{\lambda_0}\text{-mod}_0$ denote the equivalence obtained from Lemma 1.6 and Lemma 1.11. We show that for every object $M \in \mathcal{O}_{\lambda_0}(\mathbb{H})$, the condition Theorem 2.58 (iii) for FM implies $\mathbb{V}M = 0$. Let $M = \sum_{\lambda \in W_S\lambda_0} M_\lambda$ be the decomposition by generalised weight spaces of $\mathbf{C}[V]$ and let

$$M_{\leq t} = \sum_{\substack{\lambda \in W_S\lambda_0 \\ \|\lambda\| \leq t}} M_\lambda, \quad t \in \mathbf{R}_{\geq 0}$$

Note that under the equivalence F, the generalised weight space M_λ is identified with $\mathbf{e}(\lambda)F(M_\lambda)$. Following the same arguements as in the proof (iii)⇒(iv) of Theorem 2.58, we have $s_a M_t \leq M_{t+\delta}$ for every $t \in \mathbf{R}$ and $a \in \Delta$. Let $U = \mathbf{C}[E]^{\leq 1} + \sum_{a \in \Delta} \mathbf{C} \cdot s_a \subset \mathbb{H}$ so that U generates $\mathbb{H}$ as $\mathbf{C}$-algebra. Then, by the assumption (iii), we have that any finite dimensional subspace $L \subset M$,

$$\lim_{n \longrightarrow \infty} \dim\left(U^n L\right)/n^{r-1+\epsilon} = 0, \quad r = \operatorname{rk} R.$$

Hence we obtain $\dim_{\mathrm{GK},\mathbb{H}} M \leq r-1$, and in particular $\dim_{\mathrm{GK},\mathbf{C}[T^\vee]} M \leq r-1$ for the subalgebra $\mathbf{C}[T^\vee] = \mathbf{C}Q^\vee \subset \mathbb{H}$. As the algebra $\mathbf{C}[T^\vee]$ is commutative and by Proposition 1.3, M is coherent over $\mathbf{C}[T^\vee]$, the Gelfand–Kirillov dimension of M coincides with the Krull dimension of the subvariety $\operatorname{Supp} M \subset T^\vee$. As the localisation of M on the regular part

$T_\circ^\vee$ must be locally free, we see that it must be zero since $\dim T^\vee = r > \dim \operatorname{Supp} M$. Hence $\mathbb{V}M = 0$ by the definition of $\mathbb{V}$. We see that $\ker \mathbf{V} \subset F(\ker \mathbb{V})$.

Since $\mathbf{V}$ and $\mathbb{V}$ are both quotient functors of categories of finite length, by comparison of the rank of the Grothendieck groups

$$\begin{aligned}\operatorname{rk} K_0(\ker \mathbb{V}) &= \operatorname{rk} K_0(\mathcal{O}_{\lambda_0}(\mathbb{H})) - \operatorname{rk} K_0(\mathcal{O}_{\ell_0}(\mathbb{K})) \\ &= \operatorname{rk} K_0(\mathbf{H}_{\lambda_0}\text{-mod}_0) - \operatorname{rk} K_0(\mathbf{K}_{\ell_0}\text{-mod}_0) = \operatorname{rk} K_0(\ker \mathbf{V}),\end{aligned}$$

we see that $\ker \mathbf{V} = F(\ker \mathbb{V})$. $\square$

Chapter 2

Quiver double Hecke algebras

Introduction

The objective of this chapter is to introduce and study a class of algebras, called quiver double Hecke algebras (QDHA). They can be viewed as generalisation of degenerate double affine Hecke algebras (dDAHA) or as a double affine version of quiver Hecke algebras (QHA).

The quiver Hecke algebras, also known as Khovanov–Lauda–Rouquier algebras, were introduced in [24] and [43]. They were introduced in the purpose of categorifying the Drinfel'd–Jimbo quantum groups for Kac–Moody algebras as well as their integrable representations.

It was proven by Brundan–Kleshchev–McNamara [10] and Kato [20] that quiver Hecke algebras for Dynkin quivers of finite ADE types have pretty nice homological properties. Anachronically speaking, they proved that the categories of graded modules over these algebras carry an *affine highest weight structure* in the sense of [25]. As a consequence, they have finite global dimension. However, once one goes beyond the family of finite type, the quiver Hecke algebras often have infinite global dimension. The simpliest example would be the cyclic quivers of length ≥ 2. According to the result of Brundan–Kleshchev [9] and Rouquier [43], the quiver Hecke algebras of cyclic quivers are equivalent to affine Hecke algebras for GL_n with parameter at roots of unity. The representation theory of affine Hecke algebras at roots of unity are known to share several features of the modular representation theory finite groups. Notably, there are fewer simple modules in the modular case than there are in the ordinary case.

One approach to the modular representation theory is to resolve this lack of simple objects by finding a larger, but better behaved category, of which the modular category is a quotient. In the case of modular representation theory of symmetric groups, one uses the Schur algebras as resolution via the Schur–Weyl duality. In the same spirit, for Hecke algebras of complex reflection groups, the rational Cherednik algebras provides resolutions, as it was first established in [17]. For affine Hecke algebras, the resolution would be the degenerate double affine Hecke algebras. This perspective appeared in [48], where degenerate DAHAs are viewed as replacement for affine q-Schur algebras in relation with affine Hecke algebras *cf.* §1.2.2 and §1.4. In this chapter, we introduce quiver double Hecke algebras, which we believe to play the rôle of "resolution" for quiver Hecke algebras.

In §2.1, we introduce the quiver double Hecke algebras $\mathbf{A}^\omega$ attached to an affine root system (E, S) with spectrum being a W_S-orbit in E and with parameter ω. We define the filtration by length on $\mathbf{A}^\omega$ in §2.1.4 and prove the basis theorem in §2.1.5 with this filtration. We study the associated graded $\mathrm{gr}^F \mathbf{A}^\omega$ of the filtration by length in §2.1.6.

In §2.2, we study the categories of graded and ungraded $\mathbf{A}^\omega$-modules. We introduce in §2.2.6 a functor of induction from the quiver Hecke algebras attached to the finite root system (V, R) underlying (E, S).

In §2.3, we study good filtrations on $\mathbf{A}^\omega$-modules and the relation between induction and filtration.

In §2.4, we introduce the quiver Hecke algebra $\mathbf{B}^\omega$ attached to a finite root system (V, R) and with parameter ω. We prove a basis theorem for $\mathbf{B}^\omega$ and we introduce a Frobenius form on $\mathbf{B}^\omega$.

In §2.5, we prove that the algebra $\mathbf{B}^\omega$ is isomorphic to an idempotent subalgebra of $\mathbf{A}^\omega$. We use this isomorphism to define the Knizhnik–Zamolodchikov functor $\mathbf{V}$, which is a quotient functor. We give characterisations for the kernel of $\mathbf{V}$ in §2.5.7 and §2.5.9. The double centraliser property for $\mathbf{V}$ is proven in §2.5.8.

2.1 Quiver double Hecke algebra

Fix an irreducible based finite root system (V, R, Δ_0) and let (E, S, Δ) be its affinisation. In this section we will abbreviate $P = P_R$, $Q = Q_R$, $P^\vee = P_R^\vee$ and $Q^\vee = Q_R^\vee$.

2.1.1 The polynomial matrix algebra $\mathbf{A}^o$

Fix once and for all $\lambda_0 \in E$. Define for each $\lambda \in W_S\lambda_0$ a polynomial ring $\mathrm{Pol}_\lambda = \mathbf{C}[V]$ and let $\mathrm{Pol}_{W_S\lambda_0} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Pol}_\lambda$. For each λ, define $\mathbf{e}(\lambda) : \mathrm{Pol}_{W_S\lambda_0} \longrightarrow \mathrm{Pol}_\lambda$ to be the idempotent linear endomorphism of projection onto the factor Pol_λ.

For each $a \in \Delta$ and $\lambda \in W_S\lambda_0$, define an operator $\tau_a^o \mathbf{e}(\lambda) : \mathrm{Pol}_\lambda \longrightarrow \mathrm{Pol}_{s_a\lambda}$ by

$$\tau_a^o \mathbf{e}(\lambda) = \begin{cases} (\partial a)^{-1}(s_{\partial a} - 1) & a(\lambda) = 0 \\ s_{\partial a} & a(\lambda) \neq 0 \end{cases}.$$

Here $\partial a \in R$ is the differential of $a \in S$, *cf.* §1.1.

Let $\mathbf{A}^o$ be the associative (non-unital) subalgebra of $\mathrm{gEnd}_{\mathbf{C}}(\mathrm{Pol}_{W_S\lambda_0})$ generated by $f\mathbf{e}(\lambda)$ and $\tau_a^o \mathbf{e}(\lambda)$ for $f \in \mathbf{C}[V]$, $a \in \Delta$ and $\lambda \in W_S\lambda_0$.

2.1.2 Centre $\mathcal{Z}$

For $\lambda \in W_S\lambda_0$, let W_λ be the stabiliser of λ in W_S. The stabiliser W_λ is a finite parabolic subgroup of the Coxeter group W_S. The affine Weyl group W_S acts on the vector space V via the finite quotient $\partial : W_S \longrightarrow W_S/Q_R^\vee \cong W_R$. Let $\mathcal{Z} = \mathbf{C}[V]^{W_{\lambda_0}}$ be the ring of W_{λ_0}-invariant polynomials, graded by the degree of monomials. Since W_{λ_0} acts by reflections on V, the ring $\mathcal{Z}$ is a graded polynomial ring. Let $\mathfrak{m}_{\mathcal{Z}} \subset \mathcal{Z}$ be the unique homogeneous maximal ideal.

For each $\lambda \in W_S\lambda_0$, we define a homomorphism $\mathcal{Z} \longrightarrow \mathrm{Pol}_\lambda$: choosing a $w \in W_S$ such that $w\lambda_0 = \lambda$, we let $f \mapsto w(f) \in \mathbf{C}[V]^{W_{w\lambda}} \subset \mathrm{Pol}_\lambda$. This map is clearly independent of the choice of w and it identify $\mathcal{Z}$ with the invariant subspace $\mathbf{C}[V]^{W_{w\lambda}}$. The infinite sum $\mathrm{Pol}_{W_S\lambda_0}$ is regarded as a $\mathcal{Z}$-module via the diagonal action.

The following are standard results from the invariant theory for reflection groups:

Proposition 2.1. *The following statements hold:*

(i) For each $\lambda \in W_S\lambda_0$, the $\mathcal{Z}$-module Pol_λ is free of rank $\#W_\lambda = \#W_{\lambda_0}$.

(ii) For any $w \in W_S$, choose a reduced expression $w = s_{a_l} \cdots s_{a_1}$ and put $\tau_w^o \mathbf{e}(\lambda) = \tau_{a_l}^o \cdots \tau_{a_1}^o \mathbf{e}(\lambda)$ for each $\lambda \in W_S\lambda_0$. Then the element $\tau_w^o \mathbf{e}(\lambda)$ is independent of the choice of the reduced expression for w and moreover, there is a decomposition

$$\mathrm{Hom}_{\mathcal{Z}}\left(\mathrm{Pol}_\lambda, \mathrm{Pol}_{W_S\lambda_0}\right) = \bigoplus_{w \in W_S} \tau_w^o \mathbf{C}[V]\mathbf{e}(\lambda).$$

(iii) The $\mathbf{A}^o$-action on $\mathrm{Pol}_{W_S\lambda_0}$ commutes with $\mathcal{Z}$ and moreover, the map

$$\mathbf{A}^o \hookrightarrow \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Hom}_{\mathcal{Z}}(\mathrm{Pol}_\lambda, \mathrm{Pol}_{W_S\lambda_0}).$$

it is an isomorphism.

□

2.1.3 Subalgebras $\mathbf{A}^\omega$ of $\mathbf{A}^o$

Let $\omega = \{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ be a family of functions $\omega_\lambda : S^+ \longrightarrow \mathbf{Z}_{\geq -1}$ satisfying the properties:

(i) $\omega_\lambda(a) = -1$ implies $a(\lambda) = 0$.

(ii) For $w \in W_S$ and $b \in S^+ \cap w^{-1}S^+$ we have $\omega_\lambda(b) = \omega_{w\lambda}(wb)$.

One may extend ω_λ to a function $\tilde{\omega}_\lambda : S \longrightarrow \mathbf{Z}_{\geq -1}$ by choosing $w \in W_S$ such that $wa \in S^+$ and setting $\tilde{\omega}_\lambda(a) = \omega_{w\lambda}(wa)$. We will also require ω to satisfy the following property:

(iii) For any (thus every) $\lambda \in W_S\lambda_0$, the extended function $\tilde{\omega}_\lambda : S \longrightarrow \mathbf{Z}_{\geq -1}$ has finite support.

We call the family $\{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ a family of order functions.

Define an operator $\tau_a^\omega \mathbf{e}(\lambda) : \mathrm{Pol}_\lambda \longrightarrow \mathrm{Pol}_{s_a\lambda}$ by

$$\tau_a^\omega \mathbf{e}(\lambda) = \begin{cases} (\partial\alpha)^{-1}(s_{\partial a} - 1) & \omega_\lambda(a) = -1 \\ (\partial\alpha)^{\omega_\lambda(a)} s_{\partial a} & \omega_\lambda(a) \geq 0 \end{cases}$$

so that $\tau_a^\omega \mathbf{e}(\lambda) \in \mathbf{A}^o$.

Definition 2.2. *The quiver double Hecke algebra*[1,2] $\mathbf{A}^\omega$ *is defined to be the subalgebra of* $\mathbf{A}^o$ *generated by* $\mathbf{C}[V]\mathbf{e}(\lambda)$ *and* $\tau_a^\omega \mathbf{e}(\lambda)$ *for* $\lambda \in W_S$ *and* $a \in \Delta$.

We also introduce the homogeneous rational function rings and its matrix algebra:

$$\mathrm{Rat}_\lambda = \mathfrak{m}_{\mathcal{Z}}^{-1}\,\mathrm{Pol}_\lambda, \quad \mathrm{Rat} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Rat}_\lambda$$
$$\mathbf{A}^{-\infty} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Hom}_{\mathfrak{m}_{\mathcal{Z}}^{-1}\mathcal{Z}}(\mathrm{Rat}_\lambda, \mathrm{Rat}) = \mathfrak{m}_{\mathcal{Z}}^{-1}\mathbf{A}^o, \quad \tau_a^{-\infty} = s_a.$$

Example 2.3.

(i) *Let* $o = \{a \mapsto -\delta_{a(\lambda)=0}\}_{\lambda \in W_S\lambda_0}$ *denote the smallest of such families. We recover the matrix algebra* $\mathbf{A}^o$.

(ii) *Let* $\delta = \{0\}_{\lambda \in W_S\lambda_0}$ *be the zero constant function. Then* $\mathbf{A}^\delta = \mathrm{Pol}_{W_S\lambda_0} \rtimes W_S$ *is the skew tensor product. If* $W_{\lambda_0} = 1$, *then* $\mathbf{A}^\delta = \mathbf{C}[V] \wr W_S$ *is the wreath product.*

(iii) *Let* $E = \mathbf{R}$, *let* ϵ *be the coordinate function on* $\mathbf{R}$ *and let* $S = \{\pm 2\epsilon\} + \mathbf{Z}$, *so that* (E, S) *is the affine root system of type* $A_1^{(1)}$. *Choose the base* $\Delta = \{a_1 = 2\epsilon, a_0 = 1 - 2\epsilon\}$. *The affine Weyl group* W_S *is generated by* s_0 *and* s_1, *where* s_1 *(resp.* s_0*) is the orthogonal reflection with respect to* $0 \in E$ *(resp.* $1/2 \in E$*). Set* $\lambda_0 = 1/4 \in E$, *so that* $W_S\lambda_0 = 1/4 + (1/2)\mathbf{Z}$ *and* $W_{\lambda_0} = 1$. *It follows that* $\mathrm{Pol}_\lambda = \mathbf{C}[\epsilon]$ *for all* $\lambda \in W_S\lambda_0$ *and* $\mathbf{A}^o$ *is the matrix algebra over* $\mathbf{C}[\epsilon]$ *of rank* $W_S\lambda_0$.

Set

$$\tilde{\omega}_{\lambda_0}(a) = \begin{cases} 1 & a \in \Delta \\ 0 & a \in S \setminus \Delta \end{cases}$$

and define the family of order functions $\omega = \{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ *by* $\omega_{w\lambda_0}(a) = \tilde{\omega}_{\omega_0}(w^{-1}a)$. *It follows that* $\mathbf{A}^\omega$ *is equal to the idempotent form of the dDAHA* $\mathbf{H}_{\lambda_0}$ *introduced in §1.2.5 with parameter* $h_a = 1/2$ *for all* $a \in S$. *We can depict the algebra* $\mathbf{A}^\omega$ *with the following diagram:*

$$\begin{tikzcd}
\cdots & \mathrm{Pol}_{-3/4} \arrow[r, bend left, "s"] & \mathrm{Pol}_{3/4} \arrow[l, bend left, "\tau_1", "s"'] \arrow[r, bend left, "s"] & \mathrm{Pol}_{1/4} \arrow[l, bend left, "\tau_0", "-\epsilon s"'] \arrow[r, bend left, "\epsilon s"] & \mathrm{Pol}_{-1/4} \arrow[l, bend left, "\tau_1", "s"'] \arrow[r, bend left, "s"] & \mathrm{Pol}_{5/4} \arrow[l, bend left, "\tau_0", "s"'] & \cdots,
\end{tikzcd}$$

where $s : \mathbf{C}[\epsilon] \longrightarrow \mathbf{C}[\epsilon]$ *is given by the substitution* $\epsilon \mapsto -\epsilon$.

[1] The parameter ω is an analogue of the polynomials $Q_{i,j}(u, v)$ in Rouquier's definition of quiver Hecke algebras.

[2] In this definition, the requirement that $\lambda_0 \in E$ plays no essential role. We could have asked λ_0 to belong to some set on which W_S acts transitively with finite parabolic stabliser subgroups. However, the euclidean geometry of E will facilitate some arguements.

2.1.4 Filtration by length

Definition 2.4. *We define a filtration*

$$F_{\leq n}\mathbf{A}^{\omega} = \sum_{\lambda\in W_S\lambda_0}\sum_{k=0}^{n}\sum_{(a_1,\dots,a_k)\in\Delta^k}\mathbf{C}[V]\tau^{\omega}_{a_1}\cdots\tau^{\omega}_{a_k}\mathbf{e}(\lambda).$$

In general, it is hard to express the operators $\tau^{\omega}_{a_1}\cdots\tau^{\omega}_{a_k}$. However, the leading term is easy to describe.

Proposition 2.5. *Let $w = s_{a_l}\cdots s_{a_1}$ be a reduced expression and let $\lambda\in W_S\lambda_0$. Then*

(i) For any $f\in\mathbf{C}[V]$, and any family ω there is a commutation relation:

$$f\tau^{\omega}_{a_l}\cdots\tau^{\omega}_{a_1}\mathbf{e}(\lambda)\equiv\tau^{\omega}_{a_l}\cdots\tau^{\omega}_{a_1}\,w(f)\mathbf{e}(\lambda)\mod F_{\leq l-1}\mathbf{A}^{\omega}.$$

(ii) For any pair of families ω and ω' such that $\omega\leq\omega'$ (pointwise), there is a congruence relation:

$$\tau^{\omega'}_{a_l}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)\equiv\tau^{\omega}_{a_l}\cdots\tau^{\omega}_{a_1}\left(\prod_{b\in S^+\cap w^{-1}S^-}(-\partial b)^{\omega'_\lambda(b)-\omega_\lambda(b)}\right)\mathbf{e}(\lambda)\mod F_{\leq l-1}\mathbf{A}^{\omega}$$

Proof. We prove the statement (i) by induction on the length $l=\ell(w)$. It is trivial for $l=0$. For $l=1$:

$$(f\tau^{\omega}_a-\tau^{\omega}_a\,s_{\partial a}(f))\mathbf{e}(\lambda)=\begin{cases}(\partial a)^{\omega_\lambda(a)}\,\vartheta_{\partial a}(f))\mathbf{e}(\lambda) & \omega_\lambda(a)=-1\\ 0 & \omega_\lambda(a)\geq 0\end{cases}\tag{2.6}$$

It belongs to $F_{\leq 0}\mathbf{A}^{\omega}\mathbf{e}(\lambda)=\mathbf{C}[V]\mathbf{e}(\lambda)$ in both cases.

Let $l>1$, by the induction hypothesis, we get

$$\begin{aligned}&(f\tau^{\omega}_{a_l}\cdots\tau^{\omega}_{a_1}-\tau^{\omega}_{a_l}\cdots\tau^{\omega}_{a_1}\,w(f))\mathbf{e}(\lambda)\\&=(f\tau^{\omega}_{a_l}-\tau^{\omega}_{a_l}\,s_{a_l}(f))\tau^{\omega}_{a_{l-1}}\cdots\tau^{\omega}_{a_1}\mathbf{e}(\lambda)\\&+\tau^{\omega}_{a_l}(s_{a_l}(f)\tau^{\omega}_{a_{l-1}}\cdots\tau^{\omega}_{a_1}-\tau^{\omega}_{a_{l-1}}\cdots\tau^{\omega}_{a_1}\,w(f))\mathbf{e}(\lambda)\in F_{\leq l-1}\mathbf{A}^{\omega},\end{aligned}$$

whence (i).

Using this we prove (ii) by induction on $l=\ell(w)$. Denote $w'=s_{a_{l-1}}\cdots s_{a_1}$ and $\lambda'=w'\lambda$. Then

$$\begin{aligned}&\tau^{\omega'}_{a_l}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)=(\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^{\omega}_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)\\&=\left((\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^{\omega}_{a_l}-\tau^{\omega}_{a_l}(-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\right)\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)\\&+\tau^{\omega}_{a_l}(-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)\end{aligned}$$

Using (2.6), the first term belongs to $F_{\leq l-1}\mathbf{A}^{\omega}$. The second term, by the statement (i) for $w'=a_{i_{l-1}}\cdots a_{i_1}$, satisfies

$$\begin{aligned}\tau^{\omega}_{a_l}(-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)&\equiv\tau^{\omega}_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\,w'\left((-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\right)\mathbf{e}(\lambda)\\&=\tau^{\omega}_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\left(-\partial(w'^{-1}a_l)\right)^{\omega'_\lambda(w'^{-1}a_l)-\omega_\lambda(w'^{-1}a_l)}\mathbf{e}(\lambda).\end{aligned}$$

Here we have used the hypothesis that $\omega_\lambda(w'^{-1}a_l) = \omega_{\lambda'}(a_l)$. Using the induction hypothesis,

$$\begin{aligned}
&\tau_{a_l}^{\omega'} \cdots \tau_{a_1}^{\omega'} \mathbf{e}(\lambda) \equiv \tau_{a_l}^{\omega} \tau_{a_{l-1}}^{\omega'} \cdots \tau_{a_1}^{\omega'} (s_{a_l} w) \left((-\partial a_l)^{\omega'_{\lambda'} - \omega_{\lambda'}} \right) \mathbf{e}(\lambda) \\
&\equiv \tau_{a_l}^{\omega} \cdots \tau_{a_1}^{\omega} \left(\left((-\partial (w'^{-1} a_l))^{\omega'_{\lambda'} - \omega_{\lambda'}} ((w')^{-1} a_l) \right) \prod_{b \in S^+ \cap w'^{-1} S^-} (-\partial b)^{\omega'_\lambda(b) - \omega_\lambda(b)} \right) \mathbf{e}(\lambda) \mod F_{\leq l-1} \mathbf{A}^\omega \\
&= \tau_{a_l}^{\omega} \cdots \tau_{a_1}^{\omega} \left(\prod_{b \in S^+ \cap w^{-1} S^-} (-\partial b)^{\omega'_\lambda(b) - \omega_\lambda(b)} \right) \mathbf{e}(\lambda).
\end{aligned}$$

The last equation is due to the relation $S^+ \cap w^{-1} S^- = S^+ \cap w'^{-1} S^- \cup \{w'^{-1} a_l\}$. This proves (ii). □

2.1.5 Basis theorem

We aim to prove an analogue of Proposition 2.1 for the subalgebra $\mathbf{A}^\omega \subset \mathbf{A}^o$.

Lemma 2.7. *For any family $\{\omega_\lambda\}_{\lambda \in W_S \lambda_0}$ as above, the images of the operators $\tau_a^\omega \mathbf{e}(\lambda)$ in $\mathrm{gr}^F \mathbf{A}^\omega$ satisfies the braid relations: for $a, b \in \Delta$ with $a \neq b$, let $m_{a,b}$ be the order of $s_a s_b$ in W_S. If $m_{a,b} \neq \infty$, then*

$$\underbrace{\tau_a^\omega \tau_b^\omega \tau_a^\omega \cdots}_{m_{a,b}} \mathbf{e}(\lambda) \equiv \underbrace{\tau_b^\omega \tau_a^\omega \tau_b^\omega \cdots}_{m_{a,b}} \mathbf{e}(\lambda) \qquad \mod F_{\leq m_{a,b}-1} \mathbf{A}^\omega.$$

Proof. The statement is empty for $m_{a,b} = \infty$, so we assume $m_{a,b} \neq \infty$. Let $W_{a,b} \subset W_S$ be the parabolic subgroup generated by s_a and s_b, let $w_0 \in W_{a,b}$ be the longest element and let $S_{a,b} \subset S$ be the sub-root system spanned by a and b. Let $\mathbf{A}^\omega_{a,b}$ be the subalgebra of $\mathbf{A}^\omega$ generated by $f\mathbf{e}(\lambda)$, $\tau_a^\omega \mathbf{e}(\lambda)$ and $\tau_b^\omega \mathbf{e}(\lambda)$ for $\lambda \in W_S \lambda_0$ and let $F_{\leq n} \mathbf{A}^\omega_{a,b}$ be the filtration by length defined as in Definition 2.4. It suffices to show the following

$$\underbrace{\tau_a^\omega \tau_b^\omega \tau_a^\omega \cdots}_{m_{a,b}} \mathbf{e}(\lambda) \equiv \underbrace{\tau_b^\omega \tau_a^\omega \tau_b^\omega \cdots}_{m_{a,b}} \mathbf{e}(\lambda) \qquad \mod F_{\leq m_{a,b}-1} \mathbf{A}^\omega_{a,b}.$$

because there is an inclusion $F_{\leq m_{a,b}-1} \mathbf{A}^\omega_{a,b} \subset F_{\leq m_{a,b}-1} \mathbf{A}^\omega$. An analogue of Proposition 2.5 is valid for this subalgebra with the filtration $F_{\leq n} \mathbf{A}^\omega_{a,b}$.

We first prove the braid relation for the family $\omega' = \{\omega'_\lambda\}_{\lambda \in W_S \lambda_0}$, where $\omega'_\lambda(c) = \max\{\omega_\lambda(c), 0\}$. Since $\omega'_\lambda(c) \geq 0$ for all $c \in S^+_{a,b}$, the braid relation follows from the formula (with similar proof as Proposition 2.5 (ii))

$$\underbrace{\tau_a^{\omega'} \tau_b^{\omega'} \tau_a^{\omega'} \cdots}_{m_{a,b}} \mathbf{e}(\lambda) = \underbrace{s_{\partial a} s_{\partial b} s_{\partial b} \cdots}_{m_{a,b}} \prod_{c \in S^+_{a,b}} (-\partial c)^{\omega'_\lambda(c)} \mathbf{e}(\lambda).$$

Let $\mathfrak{d} = \prod_{\substack{c \in S^+_{a,b} \\ \omega_c(\lambda) = -1}} (\partial c)$. By Proposition 2.5 (ii), we have

$$\underbrace{\tau_a^{\omega'} \tau_b^{\omega'} \tau_a^{\omega'} \cdots}_{m_{a,b}} \mathbf{e}(\lambda) \equiv \underbrace{\tau_a^{\omega} \tau_b^{\omega} \tau_a^{\omega} \cdots}_{m_{a,b}} \mathfrak{d}\, \mathbf{e}(\lambda) \mod F_{\leq m_{a,b}-1} \mathbf{A}^\omega_{a,b}.$$

Write $X = (\underbrace{\tau_a^\omega\tau_b^\omega\tau_a^\omega\cdots}_{m_{a,b}} - \underbrace{\tau_b^\omega\tau_a^\omega\tau_b^\omega\cdots}_{m_{a,b}})\mathbf{e}(\lambda)$, so that $X\cdot\mathfrak{d} \in \mathbf{e}(w_0\lambda)\left(F_{\leq m_{a,b}-1}\mathbf{A}^\omega_{a,b}\right)\mathbf{e}(\lambda)$. Moreover, by Proposition 2.5 (ii), we have

$$X \equiv (\underbrace{\tau_a^o\tau_b^o\tau_a^o\cdots}_{m_{a,b}} - \underbrace{\tau_b^o\tau_a^o\tau_b^o\cdots}_{m_{a,b}}) \prod_{c\in S^+_{a,b}} (-\partial c)^{\omega_\lambda(c)-o_\lambda(c)}\mathbf{e}(\lambda) \mod F_{\leq m_{a,b}-1}\mathbf{A}^o_{a,b}$$

However, the elements $\tau_a^o\mathbf{e}(\lambda)$ satisfy the braid relations in $\mathbf{A}^o_{a,b}$ by Proposition 2.1 (ii). It follows that $X \in F_{\leq m_{a,b}-1}\mathbf{A}^o_{a,b}$ (notice that $\mathbf{A}^\omega \subseteq \mathbf{A}^o$). We claim that for $0 \leq j \leq m_{a,b}-1$, the quotient $F_{\leq j}\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)/F_{\leq j}\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda)$ is right $\mathfrak{d}$-torsion-free. This will imply that $X \in F_{\leq m_{a,b}-1}\mathbf{A}^\omega_{a,b}$ and complete the proof.

We prove the claim by induction on j. For $j = 0$, this is obvious since $F_{\leq 0}\mathbf{A}^o_{a,b} = F_{\leq 0}\mathbf{A}^\omega_{a,b}$. Assume $j \in [1, m_{a,b}-1]$. The quotient $\mathrm{gr}^F_j\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda)$ is spanned over $\mathbf{C}[V]$ by $\underbrace{\tau_a^\omega\tau_b^\omega\tau_a^\omega\cdots}_{j}\mathbf{e}(\lambda)$ and $\underbrace{\tau_b^\omega\tau_a^\omega\tau_b^\omega\cdots}_{j}\mathbf{e}(\lambda)$ since any non-reduced word in a, b of length $\leq j$ contains consecutive aa or bb and since $(\tau_a^\omega)^2, (\tau_b^\omega)^2 \in F_{\leq 1}\mathbf{A}^\omega_{a,b}$. Similarly, $\mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ is spanned over $\mathbf{C}[V]$ by $\underbrace{\tau_a^o\tau_b^o\tau_a^o\cdots}_{j}\mathbf{e}(\lambda)$ and $\underbrace{\tau_b^o\tau_a^o\tau_b^o\cdots}_{j}\mathbf{e}(\lambda)$. Moreover, by Proposition 2.1, $\mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ is free of rank 2 over $\mathbf{C}[V]$. Denote $w = \underbrace{s_as_bs_a\cdots}_{j}$. Since $\omega \geq o$, by Proposition 2.5 (ii), we have

$$\underbrace{\tau_a^\omega\tau_b^\omega\tau_a^\omega\cdots}_{j} \equiv \underbrace{\tau_a^o\tau_b^o\tau_a^o\cdots}_{j}\left(\prod_{c\in S^+_{a,b}\cap w^{-1}S^-_{a,b}} (-\partial c)^{\omega_\lambda(c)-o_\lambda(c)}\right) \mod F_{\leq j-1}\mathbf{A}^o_{a,b}.$$

The prime factors of $\mathfrak{d}$ are ∂c for $c \in S^+_{a,b}$ such that $\omega_\lambda(c) = -1$. Therefore $\mathfrak{d}$ and the product

$$\prod_{c\in S^+_{a,b}\cap w^{-1}S^-_{a,b}} (-\partial c)^{\omega_\lambda(c)-o_\lambda(c)}$$

are relatively prime. The same arguement applies to the other product $\tau_b^\omega\tau_a^\omega\tau_b^\omega\cdots$.

It follows that $\mathrm{gr}^F_j\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda)$ and $\mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ are both free over $\mathbf{C}[V]$ of rank 2, and the matrix representing the injective $\mathbf{C}[V]$-linear map $\varphi : \mathrm{gr}^F_j\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda) \longrightarrow \mathrm{gr}^F_j\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda)$ is diagonal with entries prime to $\mathfrak{d}$. Hence $\operatorname{coker}\varphi$ is $\mathfrak{d}$-torsion free. The snake lemma yields a short exact sequence

$$0 \longrightarrow \frac{F_{j-1}\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)}{F_{j-1}\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda)} \longrightarrow \frac{F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)}{F_j\mathbf{A}^\omega_{a,b}\mathbf{e}(\lambda)} \longrightarrow \operatorname{coker}\varphi \longrightarrow 0,$$

in which the first term is also $\mathfrak{d}$-torsion-free by induction hypothesis, and so is the middle term $\mathfrak{d}$-torsion-free, whence the claim is proven. □

Theorem 2.8. *For any $w \in W_S$, choose a reduced expression $w = s_{a_l}\cdots s_{a_1}$ and put $\tau_w^\omega = \tau_{a_l}^\omega\cdots\tau_{a_1}^\omega$. Then there is a decomposition*

$$\mathbf{A}^\omega = \bigoplus_{\lambda\in W_S\lambda_0}\bigoplus_{w\in W_S} \mathbf{C}[V]\tau_w^\omega\mathbf{e}(\lambda).$$

Proof. By dévissage, it suffices to show that for each $n \in \mathbf{N}$,

$$\mathrm{gr}_n^F \mathbf{A}^\omega = \bigoplus_{\lambda \in W_S\lambda_0} \bigoplus_{\substack{w \in W_S \\ \ell(w)=n}} \mathbf{C}[V]\tau_w^\omega \mathbf{e}(\lambda).$$

It follows from the braid relations for τ_a^ω in $\mathrm{gr}^F\mathbf{A}^\omega$ and the fact that $(\tau_a^\omega)^2\mathbf{e}(\lambda) \in F_{\leq 1}\mathbf{A}^\omega$, that these elements τ_w^ω span $\mathrm{gr}_n^F\mathbf{A}^\omega$. By the invariant theory of reflection groups, the family $(\tau_w^o\mathbf{e}(\lambda))_w$ is free over $\mathbf{C}[V]$ and forms a basis for $\mathrm{End}_{\mathcal{Z}}(\mathrm{Pol}_\lambda)$. In view of Proposition 2.5 (ii), the matrix of transition between between the families $(\tau_w^o\mathbf{e}(\lambda))_w$ and $(\tau_w^\omega\mathbf{e}(\lambda))_w$ is diagonal with non-zero entries, the latter is also free over $\mathbf{C}[V]$. □

Define the filtration $F_{\leq n}\mathbf{A}^{-\infty} = \mathfrak{m}_{\mathcal{Z}}^{-1}F_{\leq n}\mathbf{A}^{-o}$.

Corollary 2.9. *For each n, we have*

$$F_{\leq n}\mathbf{A}^\omega = F_{\leq n}\mathbf{A}^{-\infty} \cap \mathbf{A}^\omega.$$

Proof. Let $F'_{\leq n}\mathbf{A}^\omega = F_{\leq n}\mathbf{A}^{-\infty} \cap \mathbf{A}^\omega$. We have $F_{\leq n}\mathbf{A}^\omega \subset F'_{\leq n}\mathbf{A}^\omega$. Fix $\lambda, \lambda' \in W_S\lambda_0$. Put $N = \#\{w \in W_S\ ;\ w\lambda = \lambda'\}$, then we have $F_{\leq N}\mathbf{A}^\omega = \mathbf{A}^\omega$ by Theorem 2.8. We prove by induction on $k \in [0, N]$ that $F_{N-k}\mathbf{A} = F'_{N-k}\mathbf{A}$. It is already clear for $k = 0$. Suppose $k \geq 1$. Then we have the obvious diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_{N-k}\mathbf{A}^\omega \arrow[r] \arrow[d, "\varphi"] & F_{N-k+1}\mathbf{A}^\omega \arrow[r] \arrow[d, "\psi"] & \mathrm{gr}^F_{N-k+1}\mathbf{A}^\omega \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & F'_{N-k}\mathbf{A}^\omega \arrow[r] & F'_{N-k+1}\mathbf{A}^\omega \arrow[r] & \mathrm{gr}^{F'}_{N-k+1}\mathbf{A}^\omega \arrow[r] & 0.
\end{tikzcd}$$

The morphism ψ is an isomorphism by the induction hypothesis and φ is injective. By the snake lemma, we have $\ker\eta \cong \mathrm{coker}\,\varphi$. Theorem 2.8 implies that $\mathrm{gr}^F_{N-k+1}\mathbf{A}^\omega$ is $\mathbf{C}[V]$-torsion-free while $\mathrm{coker}\,\varphi$ is a $\mathbf{C}[V]$-torsion module. Therefore $\mathrm{coker}\,\varphi = 0$ and φ is an isomorphism. □

2.1.6 The associated graded $\mathrm{gr}^F\mathbf{A}^\omega$

We describe in greater details the structure of the associated graded $\mathrm{gr}^F\mathbf{A}^\omega$. In this subsection, we will omit the notation of congruence $\equiv$ and view the generators $\tau_a^\omega\mathbf{e}(\lambda)$ as in $\mathrm{gr}^F\mathbf{A}^\omega$.

Let $\mathrm{C}_0 \subset V^*$ denote the fundamental Weyl chamber and $\overline{\mathrm{C}_0}$ its closure in V^*. Let $P_+^\vee = P^\vee \cap \overline{\mathrm{C}_0}$ (resp. $Q_+^\vee = Q^\vee \cap \overline{\mathrm{C}_0}$) be the submonoid of $P^\vee$ consisting of dominant coweights (resp. dominant coroots). For any $\mu \in P^\vee$, we denote by $X^\mu \in \mathbf{C}P^\vee$ the corresponding element. Let $\mathbf{C}P_+^\vee$ (resp. $\mathbf{C}Q_+^\vee$) denote the monoid algebra of $P_+^\vee$ (resp. $P_+^\vee$). Recall the length formula in the extended affine Weyl group Proposition 1.1.

Endow the group algebras $\mathbf{C}P^\vee$ and $\mathbf{C}Q^\vee$ with a filtration:

$$F_{\leq n}\mathbf{C}P^\vee = \bigoplus_{\substack{\mu \in P^\vee \\ \ell(X^\mu) \leq n}} \mathbf{C} \cdot X^\mu, \quad F_{\leq n}\mathbf{C}Q^\vee = F_{\leq n}\mathbf{C}P^\vee \cap \mathbf{C}Q^\vee.$$

We define a map

$$\begin{aligned} \mathbf{C}P_+^\vee \times \mathrm{gr}^F \mathbf{C}P^\vee &\longrightarrow \mathrm{gr}^F \mathbf{C}P^\vee \\ (X^\mu, X^\nu) &\mapsto X^{w_0 w_\nu \mu + \nu}, \end{aligned} \tag{2.10}$$

where $w_\nu \in W_R$ is the unique shortest element such that $w_\nu^{-1}\nu$ is anti-dominant and w_0 is the longest element of W_R.

Lemma 2.11. *The following statements hold:*

(i) The ring $\mathbf{C}P_+^\vee$ is regular. Moreover, the map (2.10) defines a $\mathbf{C}P_+^\vee$-module structure on $\mathrm{gr}^F\mathbf{C}P^\vee$, which makes $\mathrm{gr}^F\mathbf{C}P^\vee$ a free module of rank $\#W$, and a basis of which is given by $\{b_w\}_{w\in W_R}$ with

$$b_w = \prod_{\substack{\alpha\in\Delta_0 \\ s_\alpha w < w}} X^{w_0 w \omega_\alpha^\vee}.$$

(ii) The ring $\mathbf{C}Q_+^\vee$ is Cohen–Macaulay and $\mathbf{Q}$-Gorenstein. Moreover, the $\mathbf{C}P_+^\vee$-module structure on $\mathrm{gr}^F\mathbf{C}P^\vee$ restricts to a $\mathbf{C}Q_+^\vee$-module structure on $\mathrm{gr}^F\mathbf{C}Q^\vee$, which makes $\mathrm{gr}^F\mathbf{C}Q^\vee$ a Cohen–Maucaulay module of maximal dimension and of rank $\#W$.

Proof. We observe that $\operatorname{Spec}\mathbf{C}Q_+^\vee$ and $\operatorname{Spec}\mathbf{C}P_+^\vee$ are toric varieties. The cone $\overline{\mathrm{C}}_0 \subset V^*$ is simplicial and are generated by the fundamental coweights $\omega_\alpha^\vee \in P_+^\vee$, whence $\mathbf{C}P_+^\vee$ a polynomial ring, hence regular, and $\mathbf{C}Q_+^\vee$ is a $\mathbf{Q}$-Gorenstein normal toric variety. Every normal toric variety is Cohen–Macaulay.

Consider the following injective $\mathbf{C}$-linear map

$$\zeta : \mathbf{C}P_+^\vee \longrightarrow \mathbf{C}P^\vee, \quad X^\mu \longrightarrow \sum_{\mu'\in W_R\mu} X^{\mu'}.$$

It maps $\mathbf{C}P_+^\vee$ onto the subring of W_R-invariants $(\mathbf{C}P^\vee)^{W_R}$. If $\ell(X^\mu) = k$, then $\tilde{\zeta}(X^\mu) \in F_{\le k}(\mathbf{C}W_P^\vee)^{W_R}$ since $\ell(X^\mu) = \ell(X^{w\mu})$ for any $\mu \in P_R^\vee$ and $w \in W_R$. The map $\tilde{\zeta}$ induces an injective linear map $\zeta : \mathbf{C}P_+^\vee \longrightarrow \mathrm{gr}^F P^\vee$. For any monomials $X^\mu \in \mathbf{C}P_+^\vee$ and $X^\nu \in \mathrm{gr}^F\mathbf{C}P^\vee$, we have

$$\zeta(X^\mu)X^\nu = \sum_{\substack{\mu'\in W_R\mu \\ \ell(X^{\mu'+\nu}) = \ell(X^{\mu'}) + \ell(X^\nu)}} X^{\mu'+\nu}.$$

By Proposition 1.1, the condition $\ell(X^{\mu'+\nu}) = \ell(X^{\mu'}) + \ell(X^\nu)$ is equivalent to that μ' and ν lie in the closure of a same Weyl chamber. The map ζ is a ring homomorphism. Indeed, it follows from the fact that ζ is injective and that for every pair $\mu, \mu' \in P_+^\vee$, the only dominant monomial which appears in $\zeta(X^\mu)\zeta(X^{\mu'})$ is equal to $X^{\mu+\mu'}$. This is not yet the desired $\mathbf{C}P_+^\vee$-module structure.

We can filter the $\mathbf{C}P_+^\vee$-module ζ by

$$(\mathrm{gr}^F\mathbf{C}P^\vee)^{\ge w} = \sum_{\substack{y\in W_R \\ w\le_{\mathrm{L}} y}} \left(\mathbf{C}P_+^\vee\right) b_y,$$

where $\le_{\mathrm{L}}$ is the left order on the Weyl group: $w \le_{\mathrm{L}} y \Leftrightarrow \ell(w) + \ell(yw^{-1}) = \ell(y)$. For any $w \in W_R$, we have

$$\zeta(X^\mu) b_w \equiv X^{w_0 w \mu} b_w \mod (\mathrm{gr}^F\mathbf{C}P^\vee)^{>w}, \tag{2.12}$$

so that the quotient $\left(\mathrm{gr}^F \mathbf{C}P^\vee\right)^{\geq w} / \left(\mathrm{gr}^F \mathbf{C}P^\vee\right)^{>w}$ is, endowed with the $\mathbf{C}P_+^\vee$-module structure induced from ζ, is a free of rank 1 generated by the image of b_w, denoted by b_w. Clearly, $w \in W_R$ is the shortest element among those $y \in W_R$ with the property

$$\sum_{\substack{\alpha \in \Delta_0 \\ s_\alpha w < w}} y^{-1} w_0 w \omega_\alpha^\vee \in -\overline{\mathrm{C}}_0.$$

If we identify the monomials $X^\nu \in \mathrm{gr}^F \mathbf{C}P^\vee$ with their images in these quotient, then (2.12) implies that formula (2.10) defines a $\mathbf{C}P_+^\vee$-module and the family $(b_w)_{w \in W_R}$ forms a basis, whence (i).

Since $\mathbf{C}Q_+^\vee$ is integrally closed and $\mathbf{C}P_+^\vee$ is an integral ring extension of it, by [7, X.2.6,coro 2], $\mathbf{C}P_+^\vee$ is a Cohen–Macaulay $\mathbf{C}Q_+^\vee$-module and so is $\mathrm{gr}^F P_R^\vee$ Cohen–Macaulay of maximal dimension over $\mathbf{C}Q_+^\vee$ and of rank $(\#W_R)(\#\Omega)$.

Let $\Omega = P/Q$. There is an action of Ω on group ring $\mathbf{C}P_R^\vee$ described by

$$\begin{aligned} \Omega \times \mathbf{C}P_R^\vee &\longrightarrow \mathbf{C}P_R^\vee \\ (\beta, X^\mu) &\mapsto e^{2\pi i \langle \beta, \mu \rangle} X^\mu. \end{aligned}$$

Since the formula (2.10) is Ω-equivariant, the $\mathbf{C}Q_+^\vee = \left(\mathbf{C}P_+^\vee\right)^\Omega$ action on $\mathrm{gr}^F \mathbf{C}P^\vee$ commutes with the Ω-action on the latter. As $\mathrm{gr}^F Q^\vee = (\mathrm{gr}^F P^\vee)^\Omega$ is a direct factor of the Cohen–Macaulay module $\mathrm{gr}^F P^\vee$, so is itself a Cohen–Macaulay $\mathbf{C}Q_+^\vee$-module of maximal dimension, which is of rank $\#W_R$. □

Let $W^R \subset W_S$ be the set of shortest representatives of the elements of W_S/W_R. The following maps

$$Q^\vee \hookrightarrow W_S \twoheadrightarrow W_S/W_R \leftarrow W^R$$

yield a bijection $\theta_R : Q^\vee \cong W^R$. The map θ_R induces a $\mathbf{C}$-linear map

$$\Theta_R : \mathrm{gr}^F \mathbf{C}Q^\vee \longrightarrow \mathrm{gr}^F W_S, \quad \Theta_R(X^\mu) = \theta_R(\mu).$$

where $\mathrm{gr}^F \mathbf{C}W_S$ is, as before, the associated graded of the filtration by length on the group algebra $\mathbf{C}W_S$.

By the braid relation Lemma 2.7, for each $\lambda \in W_S\lambda_0$, there is well-defined $\mathbf{C}$-linear map $\mathrm{gr}^F \mathbf{C}W_S \hookrightarrow \mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda)$ sending any simple expression $w = s_{a_l} \cdots s_{a_1}$ to $\tau_{a_l}^\omega \cdots \tau_{a_1}^\omega \mathbf{e}(\lambda)$, which is injective by the basis theorem Theorem 2.8. We denote by $\Theta_R^{\omega,\lambda} : \mathrm{gr}^F \mathbf{C}Q^\vee \longrightarrow \mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda)$ the composite of Θ_R with this injective map.

Define a $\mathbf{C}$-linear map

$$\mathbf{C}Q_+^\vee \longrightarrow \prod_{\lambda \in W_S\lambda_0} \mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda), \quad X^\mu \mapsto \left(\sum_{\mu' \in W_R\mu} \tau_{X^{\mu'}}^\omega \mathbf{e}(\lambda) \right)_\lambda, \tag{2.13}$$

where $\tau_{\mu'}^\omega \mathbf{e}(\lambda) = \tau_{a_l}^\omega \cdots \tau_{a_1}^\omega \mathbf{e}(\lambda)$ for any reduced expression $X^{\mu'} = s_{a_l} \cdots s_{a_1} \in W_S$.

Theorem 2.14. *The map (2.13) is a ring homomorphism and defines a structure of $\mathbf{C}Q_+^\vee$-module on $\mathrm{gr}^F \mathbf{A}^\omega$ which commutes with the right multiplication of $\mathbf{C}[V]$, such that the maps $\Theta_R^{\omega,\lambda}$ become $\mathbf{C}Q_+^\vee$-linear for all $\lambda \in W_S\lambda_0$.*

Moreover, there is a decomposition of $(\mathbf{C}Q_+^\vee, \mathbf{C}[V])$*-bimodule*

$$\mathrm{gr}^F\mathbf{A}^\omega = \bigoplus_{\lambda\in W_S\lambda_0} \Theta_R^{\omega,\lambda}(\mathrm{gr}^F\mathbf{C}Q^\vee)\otimes_{\mathbf{C}} \left(\bigoplus_{w\in W_R}\mathbf{C}[V]\tau_w^\omega\right)\mathbf{e}(\lambda),$$

where $\tau_w^\omega\mathbf{e}(\lambda) = \tau_{a_l}^\omega\cdots\tau_{a_1}^\omega\mathbf{e}(\lambda)$ *for any reduced expression* $w = s_{a_l}\cdots s_{a_1}$.

Proof. The map (2.13) is independent of the choice of the reduced expression $X^\mu = s_{a_l}\cdots s_{a_1}\in W_S$ by the braid relation Lemma 2.7. The assertion that (2.13) is a ring homomorphism follows with the same arguments as in the proof of Lemma 2.11. It is injective by Theorem 2.8. It yields a $\mathbf{C}Q_+^\vee$-module structure by left multiplication on each idempotent component $\mathrm{gr}^F\mathbf{A}^\omega\mathbf{e}(\lambda)$, which clearly commutes with the right multiplication of $\mathrm{gr}^F\mathbf{A}^\omega$.

We show that the maps $\Theta_R^{\omega,\lambda}$ are $\mathbf{C}Q_+^\vee$-linear with respect to the $\mathbf{C}Q_+^\vee$-actions (2.10) and (2.13). Let $\nu\in Q^\vee$ and let w_ν be the shortest element such that w_ν^{-1} is antidominant. By the length formula Proposition 1.1, we have $\theta_R(X^\nu) = X^\nu w$ with $w\in W_R$ such that $w^{-1}\nu$ is antidominant and w shortest among the elements of $wW_\nu\in W_R/W_{w^{-1}\nu}$, where $W_{w^{-1}\nu}$ is the stabiliser of $w^{-1}\nu$, which is a standard parabolic subgroup of W_R. Thus for each λ and $\mu\in Q_+^\vee$, the action (2.13) is given by

$$X^\mu\Theta_R^{\omega,\lambda}(X^\nu) = \left(\sum_{\mu'\in W_R\mu}\tau_{X^{\mu'}}^\omega\mathbf{e}(\lambda')\right)_{\lambda'=X^\nu w_\nu\lambda}\tau_{X^\nu w_\nu}^\omega\mathbf{e}(\lambda) = \sum_{\mu'\in W_R\mu}\tau_{X^{\mu'}}^\omega\tau_{X^\nu w_\nu}^\omega\mathbf{e}(\lambda).$$

For $\mu'\in W_R\mu$ with $y\in W_R$, the product $\tau_{X^{\mu'}}^\omega\tau_{X^\nu w_\nu}^\omega\mathbf{e}(\lambda)$ must be 0 in the associated graded $\mathrm{gr}^F\mathbf{A}^\omega$ unless $\ell(X^{\mu'})+\ell(X^\nu w_\nu) = \ell(X^{\mu'+\nu}w_\nu)$, which, according to Proposition 1.1, happens if and only if $w_{\mu'+\nu} = w_\nu$ and $\langle\alpha,\mu\rangle\geq 0$ for $\alpha\in R^+\cap w_\nu^{-1}R^-$ and $\langle\alpha,\mu\rangle\leq 0$ for $\alpha\in R^+\cap w_\nu^{-1}R^+$. In this case, $w_\nu^{-1}\mu'$ is antidominant. Such μ' is unique among in the orbit $W_R\mu$ and $\mu' = w_\nu w_0\mu$, where w_0 is the longest element of W_R. Hence

$$X^\mu\Theta_R^{\omega,\lambda}(X^\nu) = \tau_{X^{w_\nu w_0\mu+\nu}w_{w_\nu w_0\mu+\nu}}^\omega\mathbf{e}(\lambda) = \Theta_R^{\omega,\lambda}(X^{w_\nu w_0\mu+\nu}),$$

which proves the $\mathbf{C}Q_+^\vee$-linearity of $\Theta_R^{\omega,\lambda}$. The assertion about the decomposition follows immediately from Theorem 2.8. □

2.2 Module categories of $\mathbf{A}^\omega$

We keep the notations of §2.1. We put a $\mathbf{Z}$-grading on $\mathbf{A}^\omega$ as follows: the generators are homogeneous: $\deg\alpha\,\mathbf{e}(\lambda) = 2$ for $\alpha\in V^*$ and $\deg\tau_a^\omega\mathbf{e}(\lambda) = \omega_\lambda(a)+\omega_{s_a\lambda}(a)$. If $M = \bigoplus_n M_n$ is a graded vector space, denote by $M\langle m\rangle$ the grading shift given by $M\langle m\rangle_n = M_{m+n}$. For two graded vector spaces M and N, we denote by $\mathrm{Hom}(M,N)$ the space of $\mathbf{C}$-linear maps of degree 0 and $\mathrm{gHom}(M,N) = \bigoplus_{k\in\mathbf{Z}}\mathrm{Hom}(M,N\langle k\rangle)$.

Below, by "modules" we mean left modules. All statements can be turned into those for right modules by means of the anti-involution $\mathbf{A}^\omega\cong(\mathbf{A}^\omega)^{\mathrm{op}}$ defined by $\tau_a^\omega\mathbf{e}(\lambda)\mapsto\tau_a^\omega\mathbf{e}(s_a\lambda)$.

2.2.1 Graded $\mathbf{A}^\omega$-modules

An $\mathbf{A}^\omega$-module M is called a weight module if there is a decomposition

$$M = \bigoplus_{\lambda \in W_S\lambda_0} \mathbf{e}(\lambda)M.$$

Let $\mathbf{A}^\omega$-gMod denote the category of graded weight modules of $\mathbf{A}^\omega$. Let $\mathbf{A}^\omega$-gmod $\subset$ $\mathbf{A}^\omega$-gMod be the subcategory of compact objects and let $\mathbf{A}^\omega$-gmod$_0$ $\subset$ $\mathbf{A}^\omega$-gmod be the subcategory of $\mathfrak{m}_{\mathcal{Z}}$-nilpotent objects. It is clear that $\mathbf{A}^\omega$-gmod coincides with the category of graded finitely generated weight modules of $\mathbf{A}^\omega$. The following proposition is obvious.

Proposition 2.15. *A graded $\mathbf{A}^\omega$-module M is in $\mathbf{A}^\omega$-gmod if and only if there exists a finite sequences $(\lambda_1, \cdots, \lambda_r) \in (W_S\lambda_0)^r$, $(a_1, \cdots, a_r) \in \mathbf{Z}^r$ and a surjective homomorphism of graded $\mathbf{A}^\omega$-modules*

$$\bigoplus_{j=1}^{r} \mathbf{A}^\omega \mathbf{e}(\lambda_j)\langle a_j \rangle \twoheadrightarrow M.$$

□

We define a homomorphism of graded rings

$$\mathcal{Z} \longrightarrow \mathrm{gEnd}\,(\mathrm{id}_{\mathbf{A}^\omega\text{-gmod}}) \tag{2.16}$$

as follows: For every $f \in \mathbf{C}[V]^{W_{\lambda_0}}$ and $w \in W_S$, let f acts on $\mathbf{e}(w\lambda_0)M$ by multiplication with $(\partial w)(f) \in \mathbf{C}[V]^{W_{w\lambda_0}}$.

2.2.2 Intertwiners

For each $\lambda \in W_S\lambda_0$ and $a \in \Delta$, introduce the following element in $\mathbf{A}^\omega$:

$$\varphi_a \mathbf{e}(\lambda) = \begin{cases} ((\partial a)\tau_a^\omega + 1)\mathbf{e}(\lambda) & \omega_\lambda(a) = -1 \\ \tau_a^\omega \mathbf{e}(\lambda) & \omega_\lambda(a) \geq 0 \end{cases}.$$

It satisfies the following relations:

$$\varphi_a^2 \mathbf{e}(\lambda) = \begin{cases} \mathbf{e}(\lambda) & \omega_\lambda(a) = -1 \\ \pm(\partial a)^{n_{\lambda,a}} \mathbf{e}(\lambda) & \omega_\lambda(a) \geq 0 \end{cases}$$

$$\varphi_a f \mathbf{e}(\lambda) = s_a(f)\varphi_a \mathbf{e}(\lambda) \quad f \in \mathbf{C}[V],$$

where $n_{\lambda,a} = \max(\omega_\lambda(a) + \omega_{s_a\lambda}(-a), 0)$. They satisfy the usual braid relations. Thus, we may write $\varphi_w \mathbf{e}(\lambda) = \varphi_{a_l} \cdots \varphi_{a_1} \mathbf{e}(\lambda)$ by choosing any reduced expression $w = s_{a_l} \cdots s_{a_1}$.

Lemma 2.17. *Let $w \in W_S$ and $a \in \Delta$. Then the right multiplication by the intertwiner φ_a induces an isomorphism of $\mathbf{A}^\omega$-modules*

$$\mathbf{A}^\omega \mathbf{e}(\lambda) \cong \mathbf{A}^\omega \mathbf{e}(s_a\lambda).$$

if and only if $\omega_\lambda(a) + \omega_{s_a\lambda}(-a) \leq 0$.

Proof. The right multiplication by the element $\varphi_a \mathbf{e}(s_a\lambda) = \mathbf{e}(\lambda)\varphi_a \mathbf{e}(s_a\lambda)$ yields $\mathbf{A}^\omega \mathbf{e}(\lambda) \longrightarrow \mathbf{A}^\omega \varphi_a \mathbf{e}(s_a\lambda) \longrightarrow \mathbf{A}^\omega \mathbf{e}(s_a\lambda)$. Hence if $\varphi_a^2 \mathbf{e}(\lambda) = f\mathbf{e}(\lambda) \in \mathbf{C}[V]\mathbf{e}(\lambda)$ for $f \in \mathbf{C}[V]$ invertible, then φ_a^2 is an isomorphism. The condition that x be invertible is exactly as stated. □

2.2.3 Clan decomposition

As in § 2.1.5, we extend ω_{λ_0} to a W_S-invariant function $\tilde{\omega}_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$ and we suppose that the extension $\tilde{\omega}_{\lambda_0}$ has finite support. Consider the following sub-family of hyperplanes[3]

$$\mathfrak{D}^\omega = \{H_a \subset E \; ; \; a \in S, \; \tilde{\omega}_{\lambda_0}(a) \geq 1\}$$

The connected components of the following space

$$E_\circ^\omega = E \setminus \bigcup_{H \in \mathfrak{D}^\omega} H$$

are called clans. Since $\tilde{\omega}_{\lambda_0}$ is supposed to be finitely supported, the family $\mathfrak{D}^\omega$ is finite, the set of connected components $\pi_0\left(E_\circ^\omega\right)$ is finite and there are only a finite number of clans.

Let $\mathfrak{C} \subset E_\circ^\omega$ be a clan. Since $E_\circ^\omega$ is the complement of a finite hyperplane arrangement, $\mathfrak{C}$ is a convex polytope. The **salient cone** of $\mathfrak{C}$ is defined to be the convex polyhedral cone $\kappa \subset V$ whose dual cone $\kappa^\vee$ is the cone of linear functions which are bounded from below on $\mathfrak{C}$:

$$\kappa^\vee = \left\{ v \in V^* \; ; \; \inf_{x \in \mathfrak{C}} \langle v, x \rangle > -\infty \right\}, \quad \kappa = \kappa^{\vee\vee} = \{x \in V \; ; \; \langle v, x \rangle \geq 0, \; \forall v \in \kappa^\vee\}.$$

In fact, $\kappa^\vee$ is a convex polyhedral cone, and κ is strictly convex polyhedral generated by a finite subset of $P^\vee$.

We say that clan $\mathfrak{C} \subset E_\circ^\omega$ is **generic** if its salient cone is of maximal dimension.

Denote by $\nu_0 \in E$ the fundamental alcove.

Lemma 2.18. *Let $w \in W_S$ and $a \in \Delta$. Then $w^{-1}\nu_0$ and $w^{-1}s_a\nu_0$ are in the same clan if and only if the intertwiner φ_a induces an isomorphism of $\mathbf{A}^\omega$-modules*

$$\mathbf{A}^\omega \mathbf{e}(w\lambda_0) \cong \mathbf{A}^\omega \mathbf{e}(s_a w \lambda_0).$$

Proof.

$$\begin{aligned} \varphi_a^2 \mathbf{e}(w\lambda_0) = \mathbf{e}(w\lambda_0) &\Leftrightarrow \omega_{w\lambda_0}(a) + \omega_{s_a w \lambda_0}(-a) \leq 0 \\ &\Leftrightarrow \tilde{\omega}_{\lambda_0}(w^{-1}a) + \tilde{\omega}_{\lambda_0}(-w^{-1}a) \leq 0 \Leftrightarrow H_{wa} \notin \mathfrak{D}^\omega \end{aligned}$$

The last condition is equivalent to that $w^{-1}\nu_0$ and $w^{-1}s_a\nu_0$ belong to the same clan. □

The following proposition follows immediately from the lemma and the definition of clans.

Proposition 2.19. *If $w, w' \in W_S$ are such that $w^{-1}\nu_0$ and $w'^{-1}\nu_0$ lie in the same clan, then right multiplication by the intertwiner $\varphi_{w'w^{-1}}\mathbf{e}(w\lambda)$ yields an isomorphism $\mathbf{A}^\omega \mathbf{e}(w'\lambda) \longrightarrow \mathbf{A}^\omega \mathbf{e}(w\lambda)$.*

□

[3] "$\mathfrak{D}$" for "divâr"

Corollary 2.20. *Let $M \in \mathbf{A}^\omega$-gmod. If $w, w' \in W_S$ are such that $w\nu_0^{-1}$ and $w'\nu_0^{-1}$ lie in the same clan, then multiplication by the intertwiner $\varphi_{w'w^{-1}}\mathbf{e}(w\lambda)$ yields an isomorphism of graded $\mathcal{Z}$-modules $\mathbf{e}(w\lambda_0)M \cong \mathbf{e}(w'\lambda_0)M$. In particular, in this case there is an equality of graded dimensions*

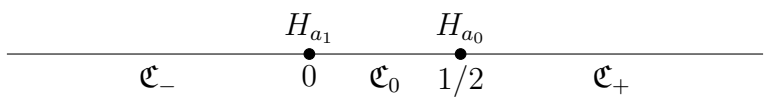
$$\operatorname{gdim} \mathbf{e}(w\lambda_0)M = \operatorname{gdim} \mathbf{e}(w'\lambda_0)M.$$

Proof. Indeed, we have

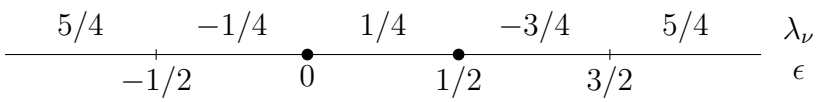
$$\mathbf{e}(w\lambda_0)M \cong \operatorname{Hom}_{\mathbf{A}^\omega}(\mathbf{A}^\omega\mathbf{e}(w\lambda_0), M) \xrightarrow{\varphi^*_{w'w^{-1}}} \operatorname{Hom}_{\mathbf{A}^\omega}(\mathbf{A}^\omega\mathbf{e}(w'\lambda_0), M) \cong \mathbf{e}(w\lambda_0)M.$$

□

Example 2.21. *Consider the example Example 2.3 (iii). The alcoves in E are of the form $]n, n+1/2[$ for $n \in (1/2)\mathbf{Z}$ and the fundamental alcove is $\nu_0 =]0, 1/2[$. We have $\mathfrak{D}^\omega = \{H_{a_0}, H_{a_1}\}$, with $\{a_0 = 1-2\epsilon, a_1 = 2\epsilon\} = \Delta$. The clan decomposition is depicted as follows:*



The clans $\mathfrak{C}_- =]-\infty, 0[$ and $\mathfrak{C}_+ =]1/2, +\infty[$ are generic while the clan $\mathfrak{C}_0 =]0, 1/2[= \nu_0$ is not generic. To each alcove $\nu = w^{-1}\nu_0$ with $w \in W_S$, we attach the element $\lambda_\nu = w\lambda_0 \in E$



In particular, the alcoves $\nu =]1/2, 3/2[$ and $\nu' =]3/2, 5/2[$ lie in the same clan $\mathfrak{C}_+$ with $\lambda_\nu = -3/4$ and $\lambda_{\nu'} = s_0\lambda_\nu = 5/4$. In this case Proposition 2.19 amount to the fact that the intertwiners $\varphi_{a_0}\mathbf{e}(\lambda_{\nu'}) : \mathbf{A}^\omega\mathbf{e}(\lambda_\nu) \longrightarrow \mathbf{A}^\omega\mathbf{e}(\lambda_{\nu'})$ and $\varphi_{a_0}\mathbf{e}(\lambda_\nu) : \mathbf{A}^\omega\mathbf{e}(\lambda_{\nu'}) \longrightarrow \mathbf{A}^\omega\mathbf{e}(\lambda_\nu)$ are isomorphisms and inverse to each other.

The projective $\mathbf{A}^\omega$-modules $\mathbf{A}^\omega\mathbf{e}(\lambda_\nu)$ are indecomposible and they are non-isomorphic for alcoves ν in the three different clans $\mathfrak{C}_-, \mathfrak{C}_0$ and $\mathfrak{C}^+$. Choose any alcoves $\nu_+ \subset \mathfrak{C}_+$, $\nu_- \subset \mathfrak{C}_-$ and denote $\lambda_+ = \lambda_{\nu_+}, \lambda_- = \lambda_{\nu_-}$, $P_+ = \mathbf{A}^\omega\mathbf{e}(\lambda_+)$,$P_0 = \mathbf{A}^\omega\mathbf{e}(\lambda_0)$ and $P_- = \mathbf{A}^\omega\mathbf{e}(\lambda_-)$. Their simple quotients, denoted by L_+, L_0 and L_-, form a complete collection of simple objects of $\mathbf{A}^\omega$-gmod up to grading shifts. The graded dimension is given by

$$\operatorname{gdim} \mathbf{e}(\lambda_\nu)L_* = \begin{cases} 1 & \nu \subseteq \mathfrak{C}_* \\ 0 & \nu \not\subseteq \mathfrak{C}_* \end{cases}, \quad * \in \{+, 0, -\}.$$

In particular, L_+ and L_- are infinite dimensional and L_0 is finite dimensional. The cosocle filtrations of P_+, P_0 and P_- are described as follows:

$$P_+ = \begin{bmatrix} L_+ & & \\ & L_0\langle -1\rangle & \\ L_+\langle -2\rangle & & L_-\langle -2\rangle \\ & L_0\langle -3\rangle & \\ L_+\langle -4\rangle & & L_-\langle -4\rangle \\ & \vdots & \end{bmatrix}, \quad P_0 = \begin{bmatrix} & L_0 & \\ L_+\langle -1\rangle & & L_-\langle -1\rangle \\ & L_0\langle -2\rangle & \\ L_+\langle -3\rangle & & L_-\langle -3\rangle \\ & L_0\langle -2\rangle & \\ & \vdots & \end{bmatrix}, \quad P_- = \begin{bmatrix} & & L_- \\ & L_0\langle -1\rangle & \\ L_+\langle -2\rangle & & L_-\langle -2\rangle \\ & L_0\langle -3\rangle & \\ L_+\langle -4\rangle & & L_-\langle -4\rangle \\ & \vdots & \end{bmatrix}.$$

2.2.4 Basic properties of graded modules of $\mathbf{A}^\omega$

We choose a finite subset $\Sigma \subset W_S$ such that for every clan $\mathfrak{C} \subset E_\circ^\omega$, there exists $w \in \Sigma$ with $w^{-1}\nu_0 \subset C$. Set $P_\Sigma = \bigoplus_{w\in\Sigma} \mathbf{A}^\omega \mathbf{e}(w\lambda_0)$.

Lemma 2.22. *The module P_Σ is a graded projective generator of the graded category $\mathbf{A}^\omega$-gmod.*

Proof. For any $y \in W_S$, we can find $w \in \Sigma$ such that $y^{-1}\nu_0$ and $w^{-1}\nu_0$ are in the same clan. By Proposition 2.19, there exists an isomorphism

$$\mathbf{A}^\omega \mathbf{e}(w\lambda_0) \cong \mathbf{A}^\omega \mathbf{e}(y\lambda_0)$$

Since the former is a direct factor of P_Σ, the above isomorphism yields a surjection $P_\Sigma \twoheadrightarrow \mathbf{A}^\omega \mathbf{e}(y\lambda_0)$. Combining this projection with Proposition 2.15, we see that P_Σ is a compact graded generator, which is clearly projective. □

Denote $A_\Sigma = (\mathrm{gEnd}_{\mathbf{A}^\omega\text{-gmod}} P_\Sigma)^{\mathrm{op}}$. It follows Lemma 2.22 that there is a graded equivalence

$$\mathrm{gHom}_{\mathbf{A}^\omega\text{-gmod}}(P_\Sigma, -) : \mathbf{A}^\omega\text{-gmod} \xrightarrow{\cong} A_\Sigma\text{-gmod} \tag{2.23}$$

Proposition 2.24. *Then the following properties are satisfied:*

(i) The category $\mathbf{A}^\omega$-gmod is noetherian and the subcategory $\mathbf{A}^\omega$-gmod$_0$ consists of objects of finite length.

(ii) For each $M \in \mathbf{A}^\omega$-gmod and each $\lambda \in W_S\lambda_0$, the graded dimension $\mathrm{gdim}\,\mathbf{e}(\lambda)M$ is in $\mathbf{N}(\!(v)\!)$. Moreover, $M \in \mathbf{A}^\omega$-gmod$_0$ if and only if $\mathrm{gdim}\,\mathbf{e}(\lambda)M \in \mathbf{N}[v, v^{\pm 1}]$ for all $\lambda \in W_S\lambda_0$.

(iii) Every object of $\mathbf{A}^\omega$-gmod admits a projective cover in the same category.

(iv) We have $\mathrm{Irr}(\mathbf{A}^\omega\text{-gmod}_0) \cong \mathrm{Irr}(\mathbf{A}^\omega\text{-gmod})$.

(v) The map (2.16) is an isomorphism $\mathcal{Z} \cong \mathrm{gEnd}\,(\mathrm{id}_{\mathbf{A}^\omega\text{-gmod}})$.

Proof. By the graded Morita equivalence (2.23), it suffices to show the corresponding statements for A_Σ-gmod.

Since A_Σ is of finite rank over the graded polynomial ring $\mathbf{C}[V]^{W_{\lambda_0}}$, it is laurentian (i.e. its graded dimension is in $\mathbf{N}(\!(v)\!)$) and thus graded semi-perfect. The statements (i)–(iv) result from the laurentian property.

We prove (v). Consider the $\mathbf{A}^\omega$-module $\mathrm{Pol}_{W_S\lambda_0} \in \mathbf{A}^\omega$-gmod. Since each factor $\mathrm{Pol}_\lambda = \mathbf{C}[V]$ is a free $\mathcal{Z}$-module of finite rank, the sum $\mathrm{Pol}_{W_S\lambda_0}$ is a free $\mathcal{Z}$-module of infinite rank. Inverting the maximal ideal $\mathfrak{m}_\mathcal{Z} \subset \mathcal{Z}$, we get a homomorphism

$$\rho : \mathbf{A}^{-\infty} \longrightarrow \bigoplus_{\lambda,\lambda' \in W_S\lambda_0} \mathrm{gHom}_{\mathfrak{m}_\mathcal{Z}^{-1}\mathcal{Z}}(\mathrm{Rat}_\lambda, \mathrm{Rat}_{\lambda'}),$$

We claim that ρ is an isomorphism. It is injective since $\mathrm{Pol}_{W_S\lambda_0}$ is a faithful $\mathbf{A}^\omega$-module by definition and it remains faithful after $\mathfrak{m}_\mathcal{Z}$ is inverted. It is easy to see from the

definition of $\mathbf{A}^\omega$ that for $\lambda \in W_S\lambda_0$ and $a \in \Delta$, the operator $s_a\mathbf{e}(\lambda) : \mathrm{Rat}_\lambda \longrightarrow \mathrm{Rat}_{s_a\lambda}$ is in the image of ρ. For any $\lambda, \lambda' \in W_S\lambda_0$, let $W_{\lambda,\lambda'} = \{w \in W_S \,;\, w\lambda = \lambda'\}$. The family $\{\mathbf{e}(\lambda')w\mathbf{e}(\lambda)\}_{w\in W_{\lambda,\lambda'}}$ is in the image of ρ. The graded ring Rat_λ is a graded Galois extension of $\mathfrak{m}_Z^{-1}\mathcal{Z}$ with Galois group W_λ. It follows from the Galois theory that

$$\mathrm{gEnd}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}(\mathrm{Rat}_\lambda) \cong \mathrm{Rat}_\lambda \rtimes \mathbf{C}W_\lambda.$$

We have already seen that $\{w\,\mathbf{e}(\lambda)\}_{w\in W_\lambda}$ is in $\operatorname{im}\rho$ the and $\mathfrak{m}_{\mathcal{Z}}^{-1}\mathbf{C}[V]\mathbf{e}(\lambda) = \mathrm{Rat}_\lambda$ is also in the image of ρ. It follows that $\mathrm{gEnd}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}(\mathrm{Rat}_\lambda) \subset \operatorname{im}\rho$. Let $\lambda, \lambda' \in W_S\lambda_0$ and choose $w \in W_{\lambda,\lambda'}$. Then $w\,\mathbf{e}(\lambda) \in \operatorname{im}\rho$ is an isomorphism $w\,\mathbf{e}(\lambda) : \mathrm{Rat}_\lambda \cong \mathrm{Rat}_{\lambda'}$ and the pre-composition yields

$$- \circ w\mathbf{e}(\lambda) : \mathrm{gEnd}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}(\mathrm{Rat}_\lambda) \cong \mathrm{gHom}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}\left(\mathrm{Rat}_\lambda, \mathrm{Rat}_{\lambda'}\right).$$

Thus $\mathrm{gHom}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}\left(\mathfrak{m}_{\mathcal{Z}}^{-1}\,\mathrm{Rat}_\lambda, \mathfrak{m}_{\mathcal{Z}}^{-1}\,\mathrm{Rat}_{\lambda'}\right) \subset \operatorname{im}\rho$. We see that ρ is surjective and the claim is proven.

Put $\mathrm{Rat}_\Sigma = \bigoplus_{w\in\Sigma}\mathrm{Rat}_{w\lambda_0}$. Now we have

$$\mathfrak{m}_{\mathcal{Z}}^{-1}A_\Sigma = \mathrm{gEnd}_{\mathfrak{m}_Z^{-1}\mathbf{A}^\omega}\left(\mathrm{Rat}_\Sigma\right) \cong \mathrm{gEnd}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}\left(\mathrm{Pol}_\Sigma\right).$$

Hence

$$\begin{aligned}\mathrm{gEnd}\left(\mathrm{id}_{\mathbf{A}^\omega\text{-gmod}}\right) &\cong \mathrm{gEnd}\left(\mathrm{id}_{A_\Sigma\text{-gmod}}\right) = \mathrm{Z}\left(A_\Sigma\right) = \mathrm{Z}\left(\mathfrak{m}_{\mathcal{Z}}^{-1}A_\Sigma\right) \cap A_\Sigma \\ &= \mathrm{Z}\left(\mathrm{gEnd}_{\mathfrak{m}_Z^{-1}\mathcal{Z}}\left(\mathrm{Pol}_\Sigma\right)\right) \cap A_\Sigma = \mathfrak{m}_{\mathcal{Z}}^{-1}\mathcal{Z} \cap A_\Sigma = \mathcal{Z}.\end{aligned}$$

□

2.2.5 Basic properties of ungraded $\mathbf{A}^\omega$-modules

Let $U : \mathbf{A}^\omega\text{-gmod}_0 \longrightarrow \mathbf{A}^\omega\text{-mod}_0$ be the grading-forgetting functor. We extends it to $U : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathrm{Pro}(\mathbf{A}^\omega\text{-mod}_0)$ by requiring U to preserve filtered inverse limits. The extended functor is exact. Define the subcategory $\mathbf{A}^\omega\text{-mod}^\wedge \subset \mathrm{Pro}(\mathbf{A}^\omega\text{-mod}_0)$ to be the essential image of this functor. Let $\mathcal{Z}^\wedge = \varprojlim_{N\longrightarrow\infty}\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}}^N$.

Proposition 2.25. *Then the following properties are satisfied:*

(i) The functor forgetting the grading $U : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{A}^\omega\text{-mod}$ *is exact and it induces* $\mathrm{Irr}(\mathbf{A}^\omega\text{-gmod})/\langle\mathbf{Z}\rangle \cong \mathrm{Irr}(\mathbf{A}^\omega\text{-mod}^\wedge)$. *Moreover, for all* $M, N \in \mathbf{A}^\omega\text{-gmod}$ *and* $n \in \mathbf{N}$ *we have*

$$\prod_{k\in\mathbf{Z}}\mathrm{Ext}^n(M, N\langle k\rangle) \cong \mathrm{Ext}^n(UM, UN).$$

(ii) The category $\mathbf{A}^\omega\text{-mod}^\wedge$ *is noetherian and the subcategory* $\mathbf{A}^\omega\text{-mod}_0$ *consists of objects of finite length.*

(iii) Every object of $\mathbf{A}^\omega\text{-mod}^\wedge$ *admits a projective cover in the same category.*

(iv) We have $\mathrm{Irr}(\mathbf{A}^\omega\text{-mod}_0) \cong \mathrm{Irr}(\mathbf{A}^\omega\text{-mod}^\wedge)$.

(v) The map (2.16) *induces an isomorphism* $\mathcal{Z}^\wedge \cong \mathrm{End}\left(\mathrm{id}_{\mathbf{A}^\omega\text{-mod}^\wedge}\right)$.

These statements follow from Proposition 2.24. □

2.2.6 Induction and restriction

Let $\mathbf{A}^\omega_R \subset \mathbf{A}^\omega$ be the subalgebra generated by $f\mathbf{e}(\lambda)$ and $\tau^\omega_a\mathbf{e}(\lambda)$ for $\lambda \in W_S\lambda_0$, $f \in \mathbf{C}[V]$ and $a \in \Delta_0$. For $\lambda_1 \in W_S\lambda_0$, denote $\mathbf{e}_{R,\lambda_1} = \sum_{\lambda\in W_R\lambda_1} \mathbf{e}(\lambda)$ and define $\mathbf{A}^\omega_{R,\lambda_1} = \mathbf{e}_{R,\lambda_1}\mathbf{A}^\omega_R\mathbf{e}_{R,\lambda_1}$ to be the idempotent subalgebra. In other words, $\mathbf{A}^\omega_{R,\lambda_1}$ is the subalgebra of $\mathbf{A}^\omega$ generated by $f\mathbf{e}(\lambda)$ and $\tau^\omega_a\mathbf{e}(\lambda)$ for $\lambda \in W_R\lambda_1$, $f \in \mathbf{C}[V]$ and $a \in \Delta_0$.

For each $\lambda_1 \in W_S\lambda_0$, we define the induction, restriction and co-induction functors

$$\begin{aligned}
&\mathrm{ind}^S_{R,\lambda_1} : \mathbf{A}^\omega_{R,\lambda_1}\text{-gmod} \longrightarrow \mathbf{A}^\omega\text{-gmod}, \quad N \mapsto \mathbf{A}^\omega\mathbf{e}_{R,\lambda_1} \otimes_{\mathbf{A}^\omega_{R,\lambda_1}} N\\
&\mathrm{res}^S_{R,\lambda_1} : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{A}^\omega_{R,\lambda_1}\text{-gmod}, \quad M \mapsto \mathbf{e}_{R,\lambda_1}M \cong \mathrm{gHom}_{\mathbf{A}^\omega}(\mathbf{A}^\omega\mathbf{e}_{R,\lambda_1}, M)\\
&\mathrm{coind}^S_{R,\lambda_1} : \mathbf{A}^\omega_{R,\lambda_1}\text{-gmod} \longrightarrow \mathbf{A}^\omega\text{-gmod}, \quad N \mapsto \bigoplus_{\lambda\in W_S\lambda_0} \mathrm{gHom}_{\mathbf{A}^\omega_{R,\lambda_1}}(\mathbf{e}_{R,\lambda_1}\mathbf{A}^\omega\mathbf{e}(\lambda), N).
\end{aligned}$$

They form a triplet of adjoint functors $\left(\mathrm{ind}^S_{R,\lambda_1}, \mathrm{res}^S_{R,\lambda_1}, \mathrm{coind}^S_{R,\lambda_1}\right)$

Proposition 2.26. *The functors* $\mathrm{ind}^S_{R,\lambda_1}, \mathrm{res}^S_{R,\lambda_1}$ *and* $\mathrm{coind}^S_{R,\lambda_1}$ *are exact.*

Proof. The functor $\mathrm{res}^S_{R,\lambda_1}$ is clearly exact. By Theorem 2.8, we have a decomposition of right $\mathbf{A}^\omega_{R,\lambda_1}$-module

$$\mathbf{A}^\omega\mathbf{e}_{R,\lambda_1} \cong \bigoplus_{w\in W^R} \tau^\omega_w\mathbf{A}^\omega_{R,\lambda_1} \tag{2.27}$$

where $W^R \subset W_S$ is the set of shortest representatives of the elements in W_S/W_R and $\tau^\omega_w = \bigoplus_{\lambda\in W_R\lambda_1} \tau^\omega_{a_l}\cdots\tau^\omega_{a_1}\mathbf{e}(\lambda)$ for any reduced expression $w = s_{a_l}\cdots s_{a_1}$. Therefore $\mathbf{A}^\omega\mathbf{e}_{R,\lambda_1}$ is a free right $\mathbf{A}^\omega_{R,\lambda_1}$-module, so $\mathrm{ind}^S_{R,\lambda_1}$ is exact. Similarly, $\mathrm{coind}^S_{R,\lambda_1}$ is also exact. □

2.3 Filtered $\mathbf{A}^\omega$-modules

We consider $\mathbf{A}^\omega$-modules equipped with filtrations which are compatible with the filtration by length F on $\mathbf{A}^\omega$. Most results in this section are non-unital version of the classical theory of filtered rings and filtered modules which one can find in [18].

2.3.1 Good filtrations on $\mathbf{A}^\omega$-modules

Let $M \in \mathbf{A}^\omega$-gmod.

Definition 2.28. *A good filtration F on M is a sequence $\{F_{\leq n}M\}_{n\in\mathbf{Z}}$ of graded $\mathbf{C}[V]$-submodules of M satisfying the following properties:*

(i) $F_{\leq n-1} \subseteq F_{\leq n}$ *for all* $n \in \mathbf{Z}$

(ii) For each $n \in \mathbf{Z}$*, there exists a finite subset* $\Sigma_n \subset W_S\lambda_0$ *such that*[4]

$$F_{\leq n}M = \bigoplus_{\lambda\in\Sigma_n} \mathbf{e}(\lambda)F_{\leq n}M$$

[4]We require this condition because we work with a non-unital associative algebra.

(iii) $F_n M = 0$ *for* $n \ll 0$

(iv) $\bigcup_{n\in\mathbf{Z}} F_n M = M$

(v)
$$(F_{\leq n}\mathbf{A}^\omega)\,(F_{\leq m}M) \subseteq F_{\leq n+m}M, \quad \forall n, m \in \mathbf{N}$$

(vi) There exists $m_0 \gg 0$ *satisfying*
$$(F_{\leq n}\mathbf{A}^\omega)\,(F_{\leq m}M) = F_{\leq n+m}M, \quad \forall n \geq 0, \forall m \geq m_0.$$

The following result is standard.

Proposition 2.29. *Good filtrations exist for the objects of* $\mathbf{A}^\omega$-gmod. *If* F *and* F' *are two good filtrations on* $M \in \mathbf{A}^\omega$-gmod, *then there exists* $i_0 \gg 0$ *such that*
$$F'_{\leq n-i_0}M \leq F_{\leq n}M \leq F'_{\leq n+i_0}M, \quad \forall n \in \mathbf{Z}.$$

□

2.3.2 Associated graded of good filtrations

Recall the monoid algebra $\mathbf{C}Q_+^\vee$ from §2.1.6. Given a good filtration F on a weight module M, the associated graded $\mathrm{gr}^F M = \bigoplus_{k\in\mathbf{Z}} F_{\leq k}M/F_{\leq k-1}M$ is a $\mathrm{gr}^F\mathbf{A}^\omega$-module. By Theorem 2.14, $\mathrm{gr}^F M$ acquires a $\mathbf{C}Q_+^\vee$-module structure. The following lemma is a direct consequence of Proposition 2.29.

Lemma 2.30. *If* F *and* F' *are good filtrations on* M, *then there exist*

(i) a finite filtration of $\mathrm{gr}^F\,\mathbf{A}^\omega$*-submodules* F' *on* $\mathrm{gr}^F\,M$,

(ii) a finite filtration of $\mathrm{gr}^F\,\mathbf{A}^\omega$*-submodules* F *on* $\mathrm{gr}^{F'}\,M$ *and*

(iii) an isomorphism of $\mathrm{gr}^F\,\mathbf{A}^\omega$*-modules* $\mathrm{gr}^{F'}\,\mathrm{gr}^F\,M \cong \mathrm{gr}^F\,\mathrm{gr}^{F'}\,M$.

Proof. By Proposition 2.29, there exists $i_0 \gg 0$ such that $F_{\leq n-i_0}M \leq F'_{\leq n}M \leq F_{\leq n+i_0}M$ for all $n \in \mathbf{Z}$. For $m \in [-i_0, i_0]$, define $F'_{\leq n,\leq m}M = \left(F'_{\leq n}M \cap F_{\leq n+m}M\right) + F'_{\leq n-1}M$. Then the quotient $\mathrm{gr}^{F'}\,M$ acquires a filtration
$$F_{\leq m}\,\mathrm{gr}_n^{F'}\,M = F'_{\leq n,m}M/F'_{\leq n-1}M \subseteq F'_{\leq n}M/F'_{\leq n-1}M = \mathrm{gr}_n^{F'}\,M,$$
which satisfies $\left(\mathrm{gr}_l^F\,\mathbf{A}^\omega\right)\left(F_{\leq m}\,\mathrm{gr}_n^{F'}\,M\right) \subseteq F_{\leq m}\,\mathrm{gr}_{n+l}^{F'}\,M$. Hence for each $m \in [-i_0, i_0]$, the quotient $\mathrm{gr}_m^F\mathrm{gr}^{F'}M = F_{\leq m}\mathrm{gr}^{F'}M/F_{\leq m-1}\mathrm{gr}^{F'}M$ is itself a $\mathrm{gr}\mathbf{A}^\omega$-module. Similarly, we put $F_{\leq m,\leq n}M = \left(F_{\leq m}M \cap F'_{\leq m+n}M\right) + F_{\leq m-1}M$ so that $\mathrm{gr}^F\,M$ acquires a filtration by $\mathrm{gr}^F\,\mathbf{A}^\omega$-modules.
However, since
$$\mathrm{gr}_m^F\,\mathrm{gr}_n^{F'}\,M = \frac{\left(F'_{\leq n,\leq m}M + F'_{\leq n-1}M\right)/F'_{\leq n-1}M}{\left(F'_{\leq n,\leq m-1}M + F'_{\leq n-1}M\right)/F'_{\leq n-1}M} \cong \frac{F'_{\leq n,\leq m}M + F'_{\leq n-1}M}{F'_{\leq n,\leq m-1}M + F'_{\leq n-1}M}$$
$$= \frac{\left(F'_{\leq n}M \cap F_{\leq n+m}M\right) + F'_{\leq n-1}M}{\left(F'_{\leq n}M \cap F_{\leq n+m-1}M\right) + F'_{\leq n-1}M} \cong \frac{F'_{\leq n}M \cap F_{\leq n+m}M}{\left(F'_{\leq n}M \cap F_{\leq n+m-1}M\right) + \left(F'_{\leq n-1}M \cap F_{\leq n+m}M\right)}$$

and similarly

$$\operatorname{gr}_n^{F'} \operatorname{gr}_m^F M \cong \frac{F_{\leq m}M \cap F'_{\leq m+n}M}{\left(F_{\leq m}M \cap F'_{\leq m+n-1}M\right) + \left(F_{\leq m-1}M \cap F'_{\leq m+n}M\right)},$$

We have $\operatorname{gr}_{m-n}^F \operatorname{gr}_n^{F'} M \cong \operatorname{gr}_{n-m}^{F'} \operatorname{gr}_m^F M$. Therefore,

$$\bigoplus_{n=-i_0}^{i_0} \operatorname{gr}_n^{F'} \operatorname{gr}^F M \cong \bigoplus_{m=-i_0}^{i_0} \operatorname{gr}_m^F \operatorname{gr}^{F'} M.$$

□

Proposition 2.31. *Let $M \in \mathbf{A}^\omega$-gmod and F a good filtration on M. Then the $\mathbf{C}Q_+^\vee$-module $\operatorname{gr}^F M$ is a coherent $\mathbf{C}Q_+^\vee \otimes \mathbf{C}[V]$-module. Moreover, if $M \in \mathbf{A}^\omega$-gmod_0, then $\operatorname{gr}^F M$ is a coherent $\mathbf{C}Q_+^\vee$-module.*

Proof. We observe that the coherence for $\operatorname{gr}^F M$ is independent of the choice of the good filtration F. Indeed, if F' is another good filtration on M, then by Lemma 2.30,

$$\operatorname{gr}^F M \text{ coherent} \Leftrightarrow \bigoplus_{n=-i_0}^{i_0} \operatorname{gr}_n^{F'} \operatorname{gr}^F M \cong \bigoplus_{m=-i_0}^{i_0} \operatorname{gr}_m^F \operatorname{gr}^{F'} M \text{ coherent} \Leftrightarrow \operatorname{gr}^{F'} M \text{ coherent}.$$

We prove the first assertion. By Proposition 2.15, there is a surjection of the form

$$\bigoplus_{j=1}^{r} \mathbf{A}^\omega \mathbf{e}(\lambda_j)\langle a_j \rangle \twoheadrightarrow M.$$

By pulling back the good filtration F on M along this surjection, we may suppose that M is of the form $M = \mathbf{A}^\omega \mathbf{e}(\lambda_j)$. By the independence of the coherence of with respect to the good filtrations, it suffices to prove the coherence for the length filtration F on $\mathbf{A}^\omega \mathbf{e}(\lambda_j)$. It follows from Theorem 2.14 that

$$\operatorname{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda_j) \cong \Theta_R^{\omega,\lambda}(\operatorname{gr}^F \mathbf{C}Q^\vee) \otimes \left(\bigoplus_{w \in W_R} \mathbf{C}[V]\tau_w^\omega \right) \mathbf{e}(\lambda).$$

Since $\operatorname{gr}^F \mathbf{C}Q^\vee$ is coherent over $\mathbf{C}Q_+^\vee$ by Lemma 2.11 (ii) and $\bigoplus_{w \in W_R} \mathbf{C}[V]\tau_w^\omega$ is free of finite rank over $\mathcal{Z}$, it follows that $\operatorname{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda_j)$ is coherent over $\mathbf{C}Q_+^\vee \otimes \mathcal{Z}$.

Suppose now $M \in \mathbf{A}^\omega$-gmod_0 so that $\mathcal{Z}$ acts via the quotient $\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}}^n$ for some $n \in \mathbf{N}$. Since $\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}}^n$ is finite-dimensional, M must be coherent over $\mathbf{C}Q_+^\vee$. □

2.3.3 Support of $\mathbf{A}^\omega$-modules of finite length

Let $M \in \mathbf{A}^\omega$-gmod_0. In view of Proposition 2.31, we can make the following definition:

Definition 2.32. *The support of M, denoted by $\operatorname{Supp} M$, is defined to be the support of $\operatorname{gr}^F M$ as coherent $\mathbf{C}Q_+^\vee$-module, for any choice of good filtration F on M.*

By Lemma 2.30, the definition of $\operatorname{Supp} M$ is independent of the choice of a good filtration.

We define the Gelfand–Kirillov dimension of a weight module M of $\mathbf{A}^\omega$ to be the following number: upon choosing a good filtration F on M,

$$\dim_{\mathrm{GK}} M = \limsup_{n\longrightarrow\infty} \frac{\log \dim F_{\leq n} M}{\log n}.$$

By Proposition 2.29, this number does not depend on the choice of F.

Proposition 2.33. *Let $M \in \mathbf{A}^\omega$-gmod_0. Then the Gelfand–Kirillov dimension $\dim_{\mathrm{GK}} M$ coincides with the Krull dimension of $\operatorname{Supp} M$.*

Proof. Taking the associated graded, we have

$$\dim F_{\leq n} M = \dim \bigoplus_{k=0}^{n} \mathrm{gr}_k^F M.$$

Notice that $\mathbf{C}Q_+^\vee$ is finitely generated graded ring where $\deg X^\mu = \ell(X^\mu)$ and $\mathrm{gr}^F M$ is a finitely generated graded module over it. Hence $\dim_{\mathrm{GK}} M$ is nothing but the degree of the Hilbert polynomial of $\mathrm{gr}^F M$, which is equal to the Krull dimension of $\operatorname{Supp} M$. □

2.3.4 Induction of filtered modules

Recall the subalgebra $\mathbf{A}^\omega_{R,\lambda_1} \subset \mathbf{A}^\omega$ of §2.2.6. Good filtrations on objects of $\mathbf{A}^\omega_{R,\lambda_1}$-gmod are defined in a similar manner.

Suppose $N \in \mathbf{A}^\omega_{R,\lambda_1}$-gmod is equipped with a good filtration F which satisfies $F_{\leq k} N = \left(F_{\leq k}\mathbf{A}^\omega_{R,\lambda_1}\right)\left(F_{\leq 0} N\right)$ for $k \geq 0$ and $F_{\leq -1} N = 0$.

Let $M = \mathrm{ind}^S_{R,\lambda_1} N$. The adjunction unit yields an inclusion of $\mathcal{Z}$-modules $N \hookrightarrow M$. Define a filtration $F_{\leq n} M = \left(F_{\leq n}\mathbf{A}^\omega\right)\left(F_{\leq 0} N\right)$.

Lemma 2.34. *The filtration F on M is good and satisfies*

$$\mathrm{gr}^F M \cong \left(\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1}\right) \otimes_{\mathrm{gr}^F \mathbf{A}^\omega_{R,\lambda_1}} \left(\mathrm{gr}^F N\right).$$

Proof. By the hypothesis on $F_{\leq n} N$, we have $\mathrm{gr}_n^F N = \left(\mathrm{gr}_n^F \mathbf{A}^\omega_{R,\lambda_1}\right)\left(\mathrm{gr}_0^F N\right)$ and $\mathrm{gr}_n^F M = \left(\mathrm{gr}_n^F \mathbf{A}^\omega\right)\left(\mathrm{gr}_0^F N\right)$. By the decomposition (2.27), we deduce

$$\mathrm{gr}_k^F \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1} = \bigoplus_{j=0}^{k} \bigoplus_{\substack{w\in W^R \\ \ell(w)=j}} \tau_w^\omega \mathrm{gr}_{k-j}^F \mathbf{A}^\omega_{R,\lambda_1},$$

from which

$$\begin{aligned}\left(\left(\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1}\right) \otimes_{\mathrm{gr}^F \mathbf{A}^\omega_{R,\lambda_1}} \left(\mathrm{gr}^F N\right)\right)_n &= \bigoplus_{k=0}^{n} \bigoplus_{\substack{w\in W^R \\ \ell(w)=j}} \tau_w^\omega \left(\mathrm{gr}_{n-k}^F \mathbf{A}^\omega_{R,\lambda_1}\right)\left(\mathrm{gr}_0^F N\right) \\ &= \left(\mathrm{gr}_n^F \mathbf{A}^\omega_n\right)\left(\mathrm{gr}_0^F N\right) = \mathrm{gr}_n^F M\end{aligned}$$

□

Proposition 2.35. *For any* $N \in \mathbf{A}^\omega\text{-gmod}_0$ *and* $0 \neq M' \subset \operatorname{ind}_{R,\lambda_1}^S N$, *we have* $\operatorname{Supp} M' = \operatorname{Spec} \mathbf{C}Q_+^\vee$.

Proof. Let F be a good filtration on N as above and denote $M = \operatorname{ind}_{R,\lambda_1}^S N$, so that by Lemma 2.34 $\operatorname{gr}^F M \cong \operatorname{gr}^F(\mathbf{A}^\omega \mathbf{e}_{R,\lambda_1}) \otimes (\operatorname{gr}^F N)$. By Theorem 2.14, we have

$$\operatorname{gr}^F M \cong \bigoplus_{\lambda \in W_R\lambda_1} \Theta_R^{\omega,\lambda}\left(\operatorname{gr}^F \mathbf{C}Q^\vee\right) \otimes_{\mathbf{C}} \mathbf{e}(\lambda)\operatorname{gr}^F N$$

and by Lemma 2.11 (ii), $\operatorname{gr}^F M$ is a Cohen–Macaulay module of maximal dimension over $\mathbf{C}Q_+^\vee$, so it is torsion-free. For any $0 \neq M' \subset M$, the restriction to M' of F is a good filtration and $\operatorname{gr}^F M' \subset \operatorname{gr}^F M$. Hence $\operatorname{Supp} M' = \mathbf{C}Q_+^\vee$. □

2.4 Quiver Hecke algebras

We keep the notations of root systems (E, S, Δ) and (V, R, Δ_0).

2.4.1 The algebra $\mathbf{B}^\omega$

Define the complex torus $T = Q^\vee \otimes \mathbf{C}^\times$ so that the ring of regular functions $\mathbf{C}[T]$ is isomorphic to the group algebra $\mathbf{C}P$. For any $\alpha \in P$, we denote by $Y^\alpha \in \mathbf{C}[T]$ the corresponding element.

Fix $\ell_0 \in T$. Define for each $\ell \in W_R\ell_0$ a polynomial ring $\operatorname{Pol}_\ell = \mathbf{C}[V]$ and let $\operatorname{Pol}_{W_R\ell_0} = \bigoplus_{\ell \in W_R\ell_0} \operatorname{Pol}_\ell$. For each ℓ, define $\mathbf{e}(\ell) : \operatorname{Pol}_{W_R\ell_0} \longrightarrow \operatorname{Pol}_\ell$ to be the idempotent linear endomorphism of projection onto the factor Pol_ℓ. Let $R_0 = R^+ \setminus 2R$ denote the set of indivisible roots and $R_0^+ = R_0 \cap R^+$ denote the set of indivisible positive roots.

Choose any $\lambda_0 \in \exp^{-1}(\ell_0)$. Then the algebra $\mathcal{Z}$ from §2.1.2 acts on Pol_ℓ: for any $w \in W_R$, the element $f \in \mathcal{Z} = \mathbf{C}[V]^{W_{\lambda_0}}$ acts on $\operatorname{Pol}_{w\ell_0}$ by multiplication by $w(f)$.

Let $\omega = \{\omega_\ell\}_{\ell \in W_R\ell_0}$ be a family of functions $\omega_\ell : R_0^+ \longrightarrow \mathbf{Z}_{\geq -1}$ satisfying the properties:

(i) If $2\alpha \notin R$, then $\omega_\ell(\alpha) = -1$ implies $Y^\alpha(\ell) = 1$.

(ii) If $2\alpha \in R$, then $\omega_\ell(\alpha) = -1$ implies $Y^\alpha(\ell) \in \{1, -1\}$.

(iii) For $w \in W_R$ and $\alpha \in R_0^+ \cap w^{-1}R_0^+$ we have $\omega_\ell(\alpha) = \omega_{w\ell}(w\alpha)$.

For each $\alpha \in \Delta_0$ and $\ell \in W_R\ell_0$, we define an operator $\tau_\alpha^\omega \mathbf{e}(\ell) : \operatorname{Pol}_\ell \longrightarrow \operatorname{Pol}_{s_\alpha \ell}$ by

$$\tau_\alpha^\omega \mathbf{e}(\ell) = \begin{cases} \alpha^{-1}(s_\alpha - 1) & \omega_\ell(\alpha) = -1 \\ \alpha^{\omega_\ell(\alpha)} s_\alpha & \omega_\ell(\alpha) \geq 0 \end{cases}.$$

Here $s_\alpha : \mathbf{C}[V] \longrightarrow \mathbf{C}[V]$ is the reflection with respect to α. We define $\mathbf{B}^\omega$ to be the subalgebra of $\operatorname{End}_{\mathcal{Z}}(\operatorname{Pol}_{W_R\ell_0})$ generated by $\mathbf{C}[V]\mathbf{e}(\ell)$ and $\tau_\alpha^\omega \mathbf{e}(\ell)$. All the statements of Proposition 2.24 hold for $\mathbf{B}^\omega$. In particular, the centre of $\mathcal{B}^\omega$ is equal to $\mathcal{Z}$.

Example 2.36.

(i) If $\ell_0 = 1 \in T$ and $\omega = \{-1\}_{\ell=\ell_0}$ is the -1 constant function, then $\mathbf{B}^\omega$ is the nil-Hecke algebra of type W_R.

(ii) If $\ell_0 = 1 \in T$ and $\omega = \{0\}_{\ell=\ell_0}$ is the zero constant function, then $\mathbf{B}^\omega = \mathbf{C}[V] \rtimes W_R$ is the skew tensor product.

2.4.2 Basis theorem

Theorem 2.37. *For any $w \in W_R$, choose a reduced expression $w = s_{a_1} \cdots s_{a_l}$ and put $\tau_w^\omega = \tau_{\alpha_l}^\omega \cdots \tau_{\alpha_1}^\omega$. Then there is a decomposition*

$$\mathbf{B}^\omega = \bigoplus_{\ell \in W_R \ell_0} \bigoplus_{\substack{w \in W_R \\ \ell(w)=n}} \mathbf{C}[V] \tau_w^\omega \mathbf{e}(\ell).$$

Proof. To prove it, we apply the results Theorem 2.55 and Theorem 2.43 which are proven independently of this one. In order to apply them, we choose a point $\lambda_0 \in \exp^{-1}(\ell_0) \subset V$. For all $\alpha \in R^+$ we construct a family $\{\omega'_\lambda\}_{\lambda \in W_S \lambda_1}$ satisfying the conditions of §2.1.3 such that $\partial\omega' = \omega$, where $\partial\omega'$ is as defined in §2.5.2. We first define a function $\tilde{\omega}_{\ell_0} : R \longrightarrow \mathbf{Z}_{\geq -1}$ as follows:

(i) For any $\alpha \in R_0^+$ such that $2\alpha \notin R$, we set $\tilde{\omega}_{\ell_0}(\alpha) = \omega_{\ell_0}(\alpha)$ and $\tilde{\omega}_{\ell_0}(-\alpha) = \tilde{\omega}_{w_0\ell_0}(-w_0\alpha)$.

(ii) For any $\alpha \in R_0^+$ such that $2\alpha \in R$, If $Y^\alpha(\ell) = -1$, we set $\tilde{\omega}_{\ell_0}(2\alpha) = \omega_{\ell_0}(\alpha)$, $\tilde{\omega}_{\ell_0}(-2\alpha) = \tilde{\omega}_{w_0\ell_0}(-w_0\alpha)$ and $\tilde{\omega}_{\ell_0}(\alpha) = \tilde{\omega}_{\ell_0}(-\alpha) = 0$. Otherwise, we set $\tilde{\omega}_{\ell_0}(\alpha) = \omega_{\ell_0}(\alpha)$, $\tilde{\omega}_{\ell_0}(-\alpha) = \tilde{\omega}_{w_0\ell_0}(-w_0\alpha)$ and $\tilde{\omega}_{\ell_0}(2\alpha) = \tilde{\omega}_{\ell_0}(-2\alpha) = 0$.

The function ω_{ℓ_0} is clearly W_{ℓ_0} invariant and has image in $\mathbf{Z}_{\geq -1}$. We choose a section of the projection $W_{\lambda_0}\backslash S \longrightarrow W_{\ell_0}\backslash R$, denoted $f : W_{\ell_0}\backslash R \longrightarrow W_{\lambda_0}\backslash S$ in such a way that $f(\alpha)(\lambda_0) = 0$ if $\alpha(\ell_0) = 0$ for each $\alpha \in R_0$. We set $\tilde{\omega}'_{\lambda_0} = f_*\tilde{\omega}_{\ell_0}$ so that $\tilde{\omega}'_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$ is a W_{λ_0}-invariant function of finite support. The family $\{\omega'_\lambda\}_{\lambda \in W_S \lambda_0}$ is then defined by $\omega'_{w\lambda_0}(a) = \tilde{\omega}'_{\lambda_0}(w^{-1}a)$ for all $w \in W_S$ and $a \in S^+$.

Theorem 2.43 implies that upon choosing a good $\gamma \in Q^\vee$, there is an isomorphism $\mathbf{B}^\omega \cong \mathbf{e}_\gamma \mathbf{A}^{\omega'} \mathbf{e}_\gamma$ identifying $\tau_\alpha^\omega \mathbf{e}(\ell)$ with $\sigma_\alpha \mathbf{e}({}^\gamma\ell)$ and by Theorem 2.55, the idempotent subalgebra $\mathbf{e}_\gamma \mathbf{A}^{\omega'} \mathbf{e}_\gamma$ has a decomposition in terms of $\sigma_\alpha \mathbf{e}({}^\gamma\ell)$. Hence $\mathbf{B}^\omega$ also has a decomposition as in the statement. □

2.4.3 Frobenius form on $\mathbf{B}^\omega$

As observed in [8], the basis theorem Theorem 2.37 implies that the algebra $\mathbf{B}^\omega$ is Frobenius over its centre $\mathcal{Z}$.

Lemma 2.38. *$\mathbf{B}^\omega$ is a Frobenius algebra over $\mathcal{Z}$.*

Proof. Consider the filtration by length

$$F_{\leq n}\mathbf{B}^\omega = \sum_{\ell \in W_R \ell_0} \sum_{k=0}^{n} \sum_{(\alpha_1,\ldots,\alpha_k) \in \Delta_0^k} \mathbf{C}[V] \tau_{\alpha_1}^\omega \cdots \tau_{\alpha_k}^\omega \mathbf{e}(\ell).$$

We set $N = \#R^+ = \ell(w_0)$ and let $w_0 = s_{\alpha_N} \cdots s_{\alpha_1}$ be any reduced expression for the longest element $w_0 \in W_R$ and set $\tau^\omega_{w_0}\mathbf{e}(\ell) = \tau^\omega_{\alpha_N} \cdots \tau^\omega_{\alpha_1}\mathbf{e}(\ell)$. By Theorem 2.37, we have $F_{\leq N}\mathbf{B}^\omega = \mathbf{B}^\omega$ and

$$\mathrm{gr}^F_N \mathbf{B}^\omega \cong \bigoplus_{\ell \in W_R \ell_0} \mathbf{C}[V]\tau^\omega_{w_0}\mathbf{e}(\ell).$$

Let $R_{\lambda_0} = \{\alpha \in R\,;\, \alpha(\lambda_0) = 0\}$ be the sub-root system associated with λ_0 and let $R^+_{\lambda_0} \subset R_{\lambda_0}$ be any positive system. Let $\mathfrak{d} = \prod_{\alpha \in R^+_{\lambda_0}} \alpha \in \mathbf{C}[V]$. It is well known that $\mathbf{C}[V]$ is a symmetric algebra over $\mathcal{Z}$ with the trace map $f \mapsto \vartheta_{w_0(W_{\lambda_0})}(f)$, where $\vartheta_{w_0(W_{\lambda_0})}$ is a composition of Demazure operators for the longest element $w_0(W_{\lambda_0})$ of W_{λ_0} with respect to $R^+_{\lambda_0}$. It is known that $\vartheta_{w_0(W_{\lambda_0})}(\mathfrak{d})$ is a non-zero constant. Let tr be the composition

$$\begin{aligned}\mathbf{B}^\omega \longrightarrow \mathrm{gr}^F_N\mathbf{B}^\omega = \bigoplus_\ell \mathbf{C}[V]\tau_{w_0}\mathbf{e}(\ell) &\xrightarrow{\tau_{w_0}\mathbf{e}(\ell)\mapsto 1} \bigoplus_\ell \mathbf{C}[V] \\ &\xrightarrow{\vartheta_{w_0(W_\ell)}} \bigoplus_\ell \mathbf{C}[V]^{W_\ell} \cong \bigoplus_\ell \mathcal{Z} \xrightarrow{\sum_{\ell \in W_R\ell_0}} \mathcal{Z}.\end{aligned}$$

Then tr is a Frobenius form. □

2.5 Knizhnik–Zamolodchikov functor V

We resume to the assumptions of §2.1.

In this section, we introduce a functor $\mathbf{V} : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{B}^\omega\text{-gmod}$, which is a quotient functor satisfying the double centraliser property. We construct it by choosing an idempotent element $\mathbf{e}_\gamma \in \mathbf{A}^\omega$ and establish an isomorphism $\mathbf{B}^\omega \cong \mathbf{e}_\gamma\mathbf{A}^\omega\mathbf{e}_\gamma$.

2.5.1 The idempotent construction

We consider the following exponential map

$$\begin{aligned} E \cong V = Q^\vee \otimes \mathbf{R} &\xrightarrow{\exp} Q^\vee \otimes \mathbf{C}^\times = T \\ \mu \otimes r &\mapsto \mu \otimes e^{2\pi i r}.\end{aligned}$$

Choose an element $\gamma \in Q^\vee$ such that

$$\langle \gamma, \alpha \rangle \ll 0 \quad \text{for all } \alpha \in R^+. \tag{2.39}$$

We define a section of the projection $\partial : W_S \longrightarrow W_S/Q^\vee = W_R$ by

$$\begin{aligned} {}^\gamma\bullet : W_R &\longrightarrow W_S \\ w &\mapsto X^\gamma\, w\, X^{-\gamma} = wX^{w^{-1}\gamma - \gamma}\end{aligned}$$

and a section of the exponential map $W_S\lambda_0 \xrightarrow{\exp} W_R\ell_0$ by

$$\begin{aligned} {}^\gamma\bullet : W_R\ell_0 &\longrightarrow W_S\lambda_0 \\ w\ell_0 &\mapsto X^\gamma\, w\lambda_0.\end{aligned}$$

It is clear that ${}^\gamma w\, {}^\gamma \ell = {}^\gamma(w\ell)$.

Given a family of order functions $\{\omega_\lambda : S^+ \longrightarrow \mathbf{Z}_{\geq -1}\}_{\lambda \in W_S\lambda_0}$ satisfying the axioms of §2.1.3, we can associate a family of order functions $\partial\omega = \left\{\partial\omega_\ell : R_0^+ \longrightarrow \mathbf{Z}_{\geq -1}\right\}_{\ell \in W_R\ell_0}$ by setting for each $\ell \in W_R\ell_0$

$$\partial\omega_\ell(\alpha) = \sum_{\substack{a \in S^+ \\ \partial a \in \{\alpha, 2\alpha\}}} \omega_{{}^\gamma\ell}(a). \tag{2.40}$$

The definition of $\partial\omega$ is independent of the choice of γ. This family of order functions gives rise to an algebra $\mathbf{B}^{\partial\omega}$ as defined in §2.4.1. We will abbreviate $\mathbf{B}^\omega = \mathbf{B}^{\partial\omega}$ and $\tau_\alpha^\omega = \tau_\alpha^{\partial\omega}$.

For any ℓ and $\alpha \in \Delta_0$, we define an operator $\sigma_\alpha \mathbf{e}({}^\gamma\ell) : \mathrm{Pol}_{{}^\gamma\ell} \longrightarrow \mathrm{Pol}_{{}^\gamma(s_\alpha\ell)}$ by

$$\sigma_\alpha \mathbf{e}({}^\gamma\ell) = \begin{cases} \alpha^{-1}(s_\alpha - 1) & \partial\omega_\ell(\alpha) = -1 \\ \alpha^{\partial\omega_\ell(\alpha)} s_\alpha & \partial\omega_\ell(\alpha) \geq 0 \end{cases}. \tag{2.41}$$

Define the idempotent[5]

$$\mathbf{e}_\gamma = \sum_{\lambda \in {}^\gamma(W_R\ell_0)} \mathbf{e}(\lambda) \in \mathbf{A}^\omega. \tag{2.42}$$

The main result is the following, which will be proven in §2.5.5:

Theorem 2.43. *Upon choosing $\gamma \in Q^\vee$ satisfying (2.39), there is an isomorphism of graded $\mathcal{Z}$-algebras*

$$\begin{aligned} i_\gamma : \mathbf{B}^\omega &\cong \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma \\ f\mathbf{e}(\ell) &\mapsto f\mathbf{e}({}^\gamma\ell) \\ \tau_\alpha^\omega \mathbf{e}(\ell) &\mapsto \sigma_\alpha({}^\gamma\ell)\mathbf{e}({}^\gamma\ell). \end{aligned}$$

Moreover, for any other choice γ', the intertwiner $\varphi_{X^{\gamma'-\gamma}}\mathbf{e}_{\gamma'} : \mathbf{A}^\omega\mathbf{e}_{\gamma'} \longrightarrow \mathbf{A}^\omega\mathbf{e}_\gamma$ yields a factorisation $i_\gamma = \varphi_{X^{\gamma'-\gamma}} \circ i_{\gamma'}$.

Example 2.44. *Resume to the setting of Example 2.3 (iii) and Example 2.21. The coroot lattice is given by $Q^\vee = \mathbf{Z} \subset \mathbf{R} = E$. Recall that $\lambda_0 = 1/4 \in E$. We may take $\gamma = s_1 s_0 = -1$ so that ${}^\gamma(W_R\ell_0) = \{\lambda_+, \lambda_-\}$, where $\lambda_+ = s_1 s_0 \lambda_0 = -3/4$ and $\lambda_- = s_1 s_0 s_1 \lambda_0 = -5/4$. It follows that $\lambda_- = s_1 s_0 s_1 s_0 s_1 \lambda_+$ and*

$$\mathbf{e}(\lambda_-)\mathbf{A}^\omega\mathbf{e}(\lambda_+) = \mathbf{C}[\epsilon]\tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\mathbf{e}(\lambda_+), \quad \mathbf{e}(\lambda_+)\mathbf{A}^\omega\mathbf{e}(\lambda_-) = \mathbf{C}[\epsilon]\tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\mathbf{e}(\lambda_-).$$

Denote by $s : \mathbf{C}[\epsilon] \longrightarrow \mathbf{C}[\epsilon]$ the automorphism $\epsilon \mapsto -\epsilon$. Calculate the products:

$$\begin{aligned} \tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\mathbf{e}(\lambda_+) &= \tau_{a_1}^\omega\mathbf{e}(5/4)\tau_{a_0}^\omega\mathbf{e}(-1/4)\tau_{a_1}^\omega\mathbf{e}(1/4)\tau_{a_0}^\omega\mathbf{e}(3/4)\tau_{a_1}^\omega\mathbf{e}(-3/4), \\ &= s \cdot s \cdot (\epsilon s) \cdot s \cdot s = \epsilon s \\ \tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\tau_{a_0}^\omega\tau_{a_1}^\omega\mathbf{e}(\lambda_-) &= \tau_{a_1}^\omega\mathbf{e}(3/4)\tau_{a_0}^\omega\mathbf{e}(1/4)\tau_{a_1}^\omega\mathbf{e}(-1/4)\tau_{a_0}^\omega\mathbf{e}(5/4)\tau_{a_1}^\omega\mathbf{e}(-5/4) \\ &= s \cdot (-\epsilon s) \cdot s \cdot s \cdot s = \epsilon s. \end{aligned}$$

Let $\alpha = \partial a_1 \in \Delta_0$ be the simple root for $(V, R) = A_1$. Denote $\ell_+ = \exp(2\pi i\lambda_+) = i$ and $\ell_- = \exp(2\pi i\lambda_-) = -i$. The derivative $\partial\omega$ is given by

$$\partial\omega_{\ell_+}(\alpha) = \sum_{k \in \mathbf{N}} \omega_{\lambda_+}(\alpha + k) = 1, \quad \partial\omega_{\ell_-}(\alpha) = \sum_{k \in \mathbf{N}} \omega_{\lambda_-}(\alpha + k) = 1.$$

[5] As we will see in Lemma 2.50, it amounts to choosing one alcove in every generic clan.

It follows that

$$\sigma_\alpha \mathbf{e}(\lambda_+) = \alpha^{\partial\omega_{\ell_+}(\alpha)} s = \tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\mathbf{e}(\lambda_+) \quad \sigma_\alpha \mathbf{e}(\lambda_-) = \alpha^{\partial\omega_{\ell_-}(\alpha)} s = \tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\mathbf{e}(\lambda_-)$$

and there is an isomorphism

$$\begin{aligned} \mathbf{B}^\omega &\xrightarrow{\cong} \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma \\ \mathbf{e}(\ell_+) &\mapsto \mathbf{e}(\lambda_+) \\ \mathbf{e}(\ell_-) &\mapsto \mathbf{e}(\lambda_-) \\ \tau^\omega_\alpha &\mapsto \tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\mathbf{e}_\gamma. \end{aligned}$$

2.5.2 A formula for order functions

Let $\gamma \in Q^\vee$ be an element satisfying (2.39). We prove a relation between the family $\{\omega_\lambda\}_{\lambda\in W_S\lambda_0}$ and its derivative $\{\partial\omega_\ell\}_{\ell\in W_R\ell_0}$ (2.40).

Lemma 2.45. *For any $\ell \in W_R\ell_0$ and $w \in W_R$, following formula holds in $\mathbf{C}(V)$:*

$$\prod_{b\in S^+\cap {}^\gamma w^{-1}S^-} (-\partial b)^{\omega_\lambda(b)} = \prod_{\beta\in R_0^+\cap w^{-1}R_0^-} (-\beta)^{\partial\omega_\ell(\beta)}.$$

Proof. Denote $\lambda = {}^\gamma\ell$. By definition, we have ${}^\gamma w = wX^{w^{-1}\gamma-\gamma}$. We calculate for $\beta \in R_0$

$$(w\beta)({}^\gamma(w\ell)) = \beta({}^\gamma\ell) - \langle\beta, \gamma - w^{-1}\gamma\rangle.$$

We first prove that

$$\prod_{\substack{b\in S^+\cap {}^\gamma w^{-1}S^- \\ \partial b\notin R^+\cap w^{-1}R^-}} (-\partial b)^{\omega_\lambda(b)} = 1. \tag{2.46}$$

Suppose that $\beta + k\delta \in S^+ \cap ({}^\gamma w)^{-1} S^-$, then $0 \le k < -\langle\beta, \gamma - w^{-1}\gamma\rangle$. Hence $\langle\beta, \gamma - w^{-1}\gamma\rangle < 0$. In the case $\beta \in R_0^+$ and $w\beta \in R_0^+$. Since our choice of γ makes that $(w\beta)({}^\gamma(w\ell)) \ll 0$, we deduce

$$(\beta + k\delta)({}^\gamma\ell) = (w\beta)({}^\gamma(w\ell)) + \langle\beta, \gamma - w^{-1}\gamma\rangle + k < (w\beta)({}^\gamma(w\ell)) \ll 0.$$

This implies that

$$\beta + k\delta \in S^+ \cap (w)^{-1}S^- \Rightarrow \omega_{{}^\gamma\ell}(\beta + k\delta) = 0.$$

if we choose γ carefully. Similar arguements can be used for $\beta \in R \setminus R_0$. Thus (2.46) holds.

Now we show that

$$\prod_{\substack{b\in S^+\cap {}^\gamma w^{-1}S^- \\ \partial b\in R^+\cap w^{-1}R^-}} (-\partial b)^{\omega_\lambda(b)} = \prod_{\beta\in R_0^+\cap w^{-1}R_0^-} (-\beta)^{\partial\omega_\ell(\beta)}. \tag{2.47}$$

Let $\beta \in R_0^+ \cap w^{-1}R_0^-$. In this case $M := -\langle\beta, \gamma - w^{-1}\gamma\rangle = -\langle\beta - w\beta, \gamma\rangle \gg 0$ by the hypothesis on γ. Therefore $\beta + k\delta \in S^+ \cap {}^\gamma w^{-1}S^-$ for all $0 \le k \le -\langle\beta - w\beta, \gamma\rangle = M$. We have

$$\sum_{\substack{b\in S^+\cap {}^\gamma w^{-1}S^- \\ \partial b=\beta}} \omega_\lambda(b) = \sum_{k=0}^{M} \omega_\lambda(\beta + k\delta)$$

By the hypothesis of finite support for ω_λ, we have

$$\sum_{k=0}^{M} \omega_\lambda(\beta + k\delta) = \sum_{k\in\mathbf{N}} \omega_\lambda(\beta + k\delta)$$

In the case where $2\beta \notin R$, we obtain

$$\sum_{\substack{b\in S^+\cap{}^\gamma w^{-1}S^- \\ \partial b=\beta}} \omega_\lambda(b) = \partial\omega_\ell(\beta). \tag{2.48}$$

For $\beta \in R_0^+$ such that $2\beta \in R$, we have similarly

$$\sum_{\substack{b\in S^+\cap{}^\gamma w^{-1}S^- \\ \partial b=2\beta}} \omega_\lambda(b) = \sum_{k=0}^{M-1} \omega_\lambda(2\beta + (2k+1)\delta) = \sum_{k\in\mathbf{N}} \omega_\lambda(2\beta + (2k+1)\delta)$$

Hence

$$\sum_{\substack{b\in S^+\cap{}^\gamma w^{-1}S^- \\ \partial b\in\{\beta,2\beta\}}} \omega_\lambda(b) = \sum_{k\in\mathbf{N}} \omega_\lambda(\beta + k\delta) + \sum_{k\in\mathbf{N}} \omega_\lambda(2\beta + (2k+1)\delta) = \partial\omega_\ell(\beta). \tag{2.49}$$

The summation formulae (2.48) and (2.49) yield (2.47). The two product formulae (2.46) and (2.47) together yield Lemma 2.45. □

2.5.3 Preparatory lemmas

Let $\gamma \in Q^\vee$ be an element satisfying (2.39). Recall the notion of clans and generic clans §2.2.3 and the fundamental alcove $\nu_0 \subset E$.

Lemma 2.50. *For $w \in W_R$, the alcoves $w^{-1}X^{-\gamma}\nu_0$ is in a generic clan and every generic clan contains at least one such alcove. Moreover, for a different choice $\gamma' \in Q^\vee$, the alcoves $w^{-1}X^{-\gamma}\nu_0$ and $w^{-1}X^{-\gamma'}\nu_0$ are in the same clan.*

Proof. Since the clans are connected components of the complement $E_\circ^\omega$ of the hyperplanes in $\mathfrak{D}^\omega = \{H_a \subset E\ ;\ a \in S,\ a(\lambda_0) = h_a\}$, any two points $x, y \in E_\circ^\omega$ are in the same clan if $a(x)a(y) > 0$ for all $a \in S$ with $H_a \in \mathfrak{D}^\omega$. Let $\mathfrak{C}_w \subset E_\circ^\omega$ be the clan such that $w^{-1}X^{-\gamma}\nu_0 \subset \mathfrak{C}_w$. Take any point $x \in \nu_0$. Set $x_w(t) = w^{-1}X^{-\gamma}(x - t\gamma)$ for $t \in \mathbf{R}_{\geq 0}$ so that $x_w(0) \in \mathfrak{C}_w$. Then for any $a \in S$ such that $H_a \in \mathfrak{D}^\omega$ and for all $t \in \mathbf{R}_{\geq 0}$, the following inequality holds:

$$\begin{cases} a(x_w(t)) > 0 & wa \in S^+ \\ a(x_w(t)) < 0 & wa \in S^- \end{cases} \tag{2.51}$$

Therefore, $x_w(t) \in \mathfrak{C}_w$. Let $t \longrightarrow \infty$, we see that for any $\alpha \in w^{-1}R^+$,

$$\alpha(w^{-1}X^{-\gamma}(x - t\gamma)) = (w\alpha)(x) - \langle w\alpha, (1+t)\gamma\rangle \longrightarrow +\infty.$$

similarly, for any $\alpha \in w^{-1}R^+$, $\alpha(w^{-1}X^{-\gamma}(x - t\gamma)) \longrightarrow -\infty$. Hence every affine root is unbounded on the $\mathfrak{C}_w$, from which the genericity of $\mathfrak{C}_w$. Clearly, the inequalities (2.51) do not depend on the choice of γ.

Let $\mathfrak{C}$ be a generic clan, consider the salient cone κ defined in §2.2.3. The genericity of $\mathfrak{C}$ means that κ is of full dimension $\dim V$. Let $\mathrm{C}_0 \subset V$ be the fundamental Weyl chamber and let $w^{-1}\mathrm{C}_0 \subset V$ be a Weyl chamber with $w \in W_R$ such that $\mathrm{Int}(\kappa) \cap w^{-1}\mathrm{C}_0 \neq \emptyset$. It is obvious that $w^{-1}X^{-\gamma}\nu_0 \subset \mathfrak{C}$. □

Recall the element σ_α of (2.41).

Lemma 2.52. *We have* $\sigma_\alpha \mathbf{e}({}^\gamma\ell) \in \mathbf{A}^\omega$.

Proof. Denote $\lambda = {}^\gamma\ell$. Let ${}^\gamma s_\alpha = s_{a_l} \cdots s_{a_1}$ be any reduced decompsition and denote $\sigma'_\alpha \mathbf{e}(\lambda) = \tau_{a_l} \cdots \tau_{a_1} \mathbf{e}(\lambda)$. We shall prove that $\sigma'_\alpha \mathbf{e}(\lambda) - \sigma_\alpha \mathbf{e}(\lambda) \in \mathbf{A}^\omega$.

Applying Proposition 2.5(ii), we see that

$$\sigma'_\alpha \mathbf{e}(\lambda) \equiv s_{a_l} \cdots s_{a_1} \left(\prod_{c \in S^+ \cap {}^\gamma s_\alpha S^-} (\partial c)^{\omega_\lambda(c)} \right) \mathbf{e}(\lambda) \mod F_{\leq l-1}\mathbf{A}^{-\infty} \tag{2.53}$$

and Lemma 2.45 yields

$$\prod_{c \in S^+ \cap {}^\gamma s_\alpha S^-} (\partial c)^{\omega_\lambda(c) + \delta_{c(\lambda)=0}} = (-\alpha)^{\omega_\ell(\alpha) + \delta_{Y^\alpha(\ell)=1}}.$$

Thus the right hand side of (2.53) is congruent to σ_α modulo $F_{\leq l-1}\mathbf{A}^{-\infty}$. Hence by the compatibility of the filtrations by length Corollary 2.9,

$$\sigma'_\alpha \mathbf{e}(\lambda) - \sigma_\alpha \mathbf{e}(\lambda) \in \mathbf{e}({}^\gamma s_\alpha \lambda) \left(F_{\leq l-1}\mathbf{A}^{-\infty} \cap \mathbf{A}^o \right) \mathbf{e}(\lambda) = \mathbf{e}({}^\gamma s_\alpha \lambda) \left(F_{\leq l-1}\mathbf{A}^o \right) \mathbf{e}(\lambda).$$

We show that in fact $\sigma'_\alpha \mathbf{e}(\lambda) - \sigma_\alpha \mathbf{e}(\lambda) \in \mathbf{A}^\omega$. For any different choice γ', by Lemma 2.50 and Proposition 2.19, the intertwiner $\varphi_{X^{\gamma'-\gamma}} : \mathbf{A}^\omega \mathbf{e}(\lambda) \longrightarrow \mathbf{A}^\omega \mathbf{e}({}^{\gamma'}\ell)$ is an isomorphism, so statement does not depend on the choice of γ. We claim that if we choose γ in such a way that $0 \ll \langle \alpha, \gamma \rangle \ll \langle \beta, \gamma \rangle$ for all $\beta \in \Delta_0 \setminus \{\alpha\}$, then there is an inequality of lengthes

$$\ell({}^\gamma s_\alpha) \leq \ell({}^\gamma w), \quad \forall w \in W_R \setminus \{1\} \tag{2.54}$$

We complete the proof provided (2.54). Note that the stabilisers satisfy ${}^\gamma W_\ell = W_\lambda$. There are two cases to be discussed:

(i) If $s_\alpha \ell \neq \ell$, then by (2.54) we have $\ell(w) \geq l$ for all $w \in W_S$ such that $w\lambda = {}^\gamma s_\alpha \lambda$. It follows from Theorem 2.8 that $\mathbf{e}({}^\gamma s_\alpha \lambda) \left(F_{\leq l-1}\mathbf{A}^o \right) \mathbf{e}(\lambda) = 0$. Hence $\sigma_\alpha \mathbf{e}(\lambda) = \sigma'_\alpha \mathbf{e}(\lambda) \in \mathbf{A}^\omega$.

(ii) If $s_\alpha \ell = \ell$, then by (2.54) we have $\ell(w) \geq l$ for all $1 \neq w \in W_\lambda$ and thus by Theorem 2.8, we see that $\mathbf{e}(\lambda) \left(F_{\leq l-1}\mathbf{A}^o \right) \mathbf{e}(\lambda) = \mathbf{C}[V]\mathbf{e}(\lambda) = \mathbf{e}(\lambda) \left(F_{\leq l-1}\mathbf{A}^\omega \right) \mathbf{e}(\lambda)$. Thus $\sigma_\alpha \mathbf{e}(\lambda) - \sigma'_\alpha \mathbf{e}(\lambda) \in \mathbf{A}^\omega$ and $\sigma_\alpha \mathbf{e}(\lambda) \in \mathbf{A}^\omega$. Hence the proof is completed.

We prove (2.54). Indeed by Proposition 1.1,

$$\begin{aligned} l = \ell({}^\gamma s_\alpha) &= \ell(s_\alpha) + \left| \sum_{\alpha' \in \{\alpha, 2\alpha\}} -\langle \alpha', \alpha^\vee \rangle \langle \alpha, \gamma \rangle + 1 \right| + \left| \sum_{\beta \in R^+ \setminus \{\alpha, 2\alpha\}} -\langle \beta, \alpha^\vee \rangle \langle \alpha, \gamma \rangle + 1 \right| \\ &\leq \#R + 1 + \langle 2\rho, \alpha^\vee \rangle \langle \alpha, \gamma \rangle \leq \ell({}^\gamma w), \quad \forall w \neq 1 \in W_R. \end{aligned}$$

while for any $w \in W_R \setminus \{1, s_\alpha\}$, there exists $\beta \in R_0^+ \setminus w^{-1}R_0^-$ with $\beta \neq \alpha$, so

$$\ell({}^\gamma w) \geq \left| \sum_{\beta' \in R^+ \cap w^{-1}R^-} \langle \beta', w^{-1}\gamma - \gamma \rangle \right| = \left| \sum_{\beta' \in R^+ \cap w^{-1}R^-} \langle w\beta' - \beta', \gamma \rangle \right| \geq \langle \beta, \gamma \rangle \geq l.$$

□

2.5.4 Basis theorem for generic clans

Let $\gamma \in Q^\vee$ be an element satisfying (2.39). Recall the idempotent of generic clans $\mathbf{e}_\gamma$ from (2.42).

Theorem 2.55. *The idempotent subalgebra $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ is generated by $\mathbf{C}[V]\mathbf{e}(\lambda)$ and $\sigma_\alpha \mathbf{e}(\lambda)$ for $\alpha \in \Delta_0$ and $\lambda \in {}^\gamma(W_R\ell_0)$. Moreover, if for any $w \in W_R$ we set $\sigma_w = \sigma_{\alpha_n} \cdots \sigma_{\alpha_1}$ by choosing any reduced expression $w = s_n \cdots s_1$, then there is a decomposition*

$$\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda) = \bigoplus_{\lambda \in {}^\gamma(W_R\ell_0)} \bigoplus_{w \in W_R} \mathbf{C}[V]\sigma_w \mathbf{e}(\lambda).$$

Proof. Let $\ell \in W_R\ell_0$ and $w \in W_R$. Denote $\lambda = {}^\gamma\ell$. Choose any reduced expressions $w = s_{\alpha_n} \cdots s_{\alpha_1}$ and ${}^\gamma w = s_{a_l} \cdots s_{a_1}$ for $\alpha_1, \cdots, \alpha_n \in \Delta_0$ and $a_1, \cdots, a_l \in \Delta$ and set

$$\sigma'_w \mathbf{e}(\lambda) = \tau_{a_l} \cdots \tau_{a_1} \mathbf{e}(\lambda) \in \mathbf{A}^\omega$$
$$\sigma_w \mathbf{e}(\lambda) = \sigma_{\alpha_n} \cdots \sigma_{\alpha_1} \mathbf{e}(\lambda) \in \mathbf{A}^\omega$$

We claim that

$$\sigma'_w \mathbf{e}(\lambda) \equiv \sigma_w \mathbf{e}(\lambda) \mod F_{\leq l-1}\mathbf{A}^\omega. \tag{2.56}$$

Recall the rational function matrix algebra $\mathbf{A}^{-\infty} = \mathfrak{m}_{\mathcal{Z}}^{-1}\mathbf{A}^o$. By Proposition 2.5 (ii) and Lemma 2.45, we have

$$\begin{aligned}\sigma' \mathbf{e}(\lambda) &\equiv s_{\partial a_l} \cdots s_{\partial a_1} \left(\prod_{b \in S^+ \cap {}^\gamma S^-} (-\partial b)^{\omega_\lambda(b)} \right) \mathbf{e}(\lambda) \mod F_{\leq l-1}\mathbf{A}^{-\infty} \\ &\equiv \sigma_w \mathbf{e}(\lambda) \mod F_{\leq n-1}\mathbf{A}^{-\infty}.\end{aligned}$$

As $n \leq l$, the above congruences yield $\sigma'_w \mathbf{e}(\lambda) - \sigma_w \mathbf{e}(\lambda) \in \mathbf{A}^\omega \cap F_{\leq l-1}\mathbf{A}^{-\infty}$. By Corollary 2.9, we have $\mathbf{A}^\omega \cap F_{\leq l-1}\mathbf{A}^{-\infty} = F_{\leq l-1}\mathbf{A}^\omega$, so the claim (2.56) is proven.

According to Theorem 2.8, the family $(\sigma'_w \mathbf{e}(\lambda))_{w \in W_R}$ form a basis for $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda)$. The decomposition of $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda)$ follows from the triangularity (2.56) of the transition matrix between the basis $(\sigma'_w \mathbf{e}(\lambda))_{w \in W_R}$ and the family $(\sigma_w \mathbf{e}(\lambda))_{w \in W_R}$. □

2.5.5 Proof of Theorem 2.43

Proof. We define an isomorphism of $\mathcal{Z}$-modules $\mathrm{Pol}_{W_R\ell_0} \cong \mathrm{Pol}_{W_S\lambda_0}$ straightforwardly by the identification:

$$\mathrm{Pol}_\ell = \mathbf{C}[V] = \mathrm{Pol}_{{}^\gamma\ell}.$$

It is a $\mathcal{Z}$-module isomorphism since $\mathcal{Z} \cong W_{\gamma_\ell} \cong W_\ell$. This isomorphism of $\mathcal{Z}$-modules yields a faithful representation of $\mathbf{B}^\omega$ on $\mathbf{e}_\gamma \operatorname{Pol}_{W_S\lambda_0}$, and by the definition of $\mathbf{B}^\omega$, this representation is described by the formula

$$f\mathbf{e}(\ell) \cdot g = f\mathbf{e}({}^\gamma\ell)g, \quad \tau_\alpha \mathbf{e}(\ell) \cdot g = \sigma_\alpha \mathbf{e}({}^\gamma\ell)g.$$

By Theorem 2.8, the image of $\mathbf{B}^\omega$ in $\operatorname{End}_{\mathcal{Z}}(\mathbf{e}_\gamma \operatorname{Pol}_{W_S\lambda_0})$ coincides with $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$. Notice that $\deg \tau_\alpha^\omega \mathbf{e}(\ell) = \omega_\ell(\alpha) = \deg \sigma_\alpha \mathbf{e}({}^\gamma\ell)$. Hence the map i_γ is an isomorphism of graded $\mathcal{Z}$-algebras.

For any other choice γ', since by Lemma 2.50, $w^{-1}X^\gamma$ and $w^{-1}X^{\gamma'}$ lie in the same generic clan, by Proposition 2.19, the intertwiner $\varphi_{X^{\gamma'-\gamma}}$ yields isomorphisms of $\mathbf{A}^\omega$-modules $\mathbf{A}^\omega \mathbf{e}_{\gamma'} \cong \mathbf{A}^\omega \mathbf{e}_\gamma$ and hence isomorphisms of algebras

$$\mathbf{e}_{\gamma'} \mathbf{A}^\omega \mathbf{e}_{\gamma'} \cong \operatorname{End}_{\mathbf{A}^\omega}(\mathbf{A}^\omega \mathbf{e}_{\gamma'}) \cong \operatorname{End}_{\mathbf{A}^\omega}(\mathbf{A}^\omega \mathbf{e}_\gamma) \cong \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma.$$

The factorisation $i_\gamma = \varphi_{X^{\gamma'-\gamma}} \circ i_{\gamma'}$ follows from the observation that $\partial(X^{\gamma'-\gamma}) = 1 \in W_R$.

□

2.5.6 The functor V

Choose a $\gamma \in Q^\vee$ such that $\langle R^+, \gamma \rangle \ll 0$ as in §2.5.2. With Theorem 2.43, we can make the following definition:

Definition 2.57. *The Knizhnik–Zamolodchikov (KZ) functor* **V** *is defined by*

$$\begin{aligned} \mathbf{V} : \mathbf{A}^\omega\text{-gmod} &\longrightarrow \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma\text{-gmod} \xrightarrow[\cong]{i_\gamma^*} \mathbf{B}^\omega\text{-gmod} \\ M &\mapsto \mathbf{e}_\gamma M. \end{aligned}$$

By the second assertion of Theorem 2.43, the definition of **V** is independent of the choice of γ up to canonical isomorphism.

Since **V** is defined as an idempotent truncation, it admits left and right adjoint functors

$$\mathbf{V}^\top : N \mapsto \bigoplus_{\lambda \in W_S\lambda_0} \operatorname{gHom}_{\mathbf{B}^\omega}(\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda), N) \quad \text{and} \quad {}^\top\mathbf{V} : N \mapsto \mathbf{A}^\omega \mathbf{e}_\gamma \otimes_{\mathbf{B}^\omega} N$$

and **V** is a quotient functor in the sense that the adjoint counit $\mathbf{V} \circ \mathbf{V}^\top \longrightarrow \mathrm{id}_{\mathbf{B}^\omega}$ is an isomorphism.

2.5.7 Support characterisation of V

For $M \in \mathbf{A}^\omega$-gmod, define the following subset of E:

$$\operatorname{Spec}_E M = \{\lambda \in W_S\lambda_0 \,;\, \mathbf{e}(\lambda)M \neq 0\}.$$

For any alcove $\nu \subset E$, there is a unique $w \in W_S$ such that $\nu = w^{-1}\nu_0$. We will denote $\lambda_\nu = w\lambda_0$. Recall the Gelfand–Kirillov dimension $\dim_{\mathrm{GK}} M$ and the support $\operatorname{Supp} M$ from §2.3.3.

Theorem 2.58. *Let $M \in \mathbf{A}^\omega$-gmod$_0$. The following conditions are equivalent:*

(i) $\mathbf{V}M = 0$

(ii) for every alcove ν lying in a generic clan, we have $\mathbf{e}(\lambda_\nu)M = 0$

(iii) the set $\operatorname{Spec}_E M$ *is contained in a finite union of affine hyperplanes of E*

(iv) $\dim_{\mathrm{GK}} M \leq \operatorname{rk} R - 1$

(v) $\operatorname{Supp} M \neq \operatorname{Spec} \mathbf{C}Q_+^\vee$

Proof. Since every object of the category $\mathbf{A}^\omega$-gmod is of finite length and all the conditions (i)–(v) are stable under extensions, we may suppose that M is simple.

(i) $\Leftrightarrow$ (ii) follows from the definition $\mathbf{V}M = \mathbf{e}_\gamma M$ and the invariance of dimension of $\mathbf{e}(\lambda)M$ for λ's in a same clan Corollary 2.20.

We prove (ii) $\Rightarrow$ (iii). By the finiteness of the clan decomposition, it suffices to show that for each non-generic clan $\mathfrak{C}$, the set $\{\lambda_\nu \,;\, \nu \subseteq \mathfrak{C}\}$ lies in a finite union of affine hyperplanes of E. By the non-genericity of $\mathfrak{C}$, there exists $\alpha \in R$ which is bounded on $\mathfrak{C}$. Let $\Lambda = \ker \alpha \cap Q^\vee$. Notice that $Q^\vee$ is a free $\mathbf{Z}$-module of rank $\operatorname{rk} R - 1$. Let $\mathfrak{A}_\mathfrak{C}$ be the set of alcoves contained in $\mathfrak{C}$. For $\nu, \nu' \in \mathfrak{A}_\mathfrak{C}$, we write $\nu \sim_\Lambda \nu'$ if there exists $\mu \in \Lambda$ such that $\nu + \mu = \nu'$. For any $\nu \in \mathfrak{A}_\mathfrak{C}$, since $X^\mu \lambda_\nu = \lambda_\nu + \mu$, the set $\{\lambda_{\nu'} \,;\, \nu' \sim_\Lambda \nu\}$ is contained in the hyperplane $w\,(\lambda_0 + \Lambda_\mathbf{R})$ for any $w \in W_S$ such that $\nu = w^{-1}\nu_0$. Since α is bounded on $\mathfrak{C}$, the quotient $\mathfrak{A}_\mathfrak{C}/\sim_\Lambda$ is a finite set and thus the set

$$\{\lambda_\nu \,;\, \nu \subset \mathfrak{C}\} \subset \bigcup_{\nu \in \mathfrak{A}_\mathfrak{C}/\Lambda} \{\lambda_{\nu'}\}_{\nu' \sim_\Lambda \nu}$$

is contained in a finite union of hyperplanes, whence (iii).

We prove (iii) $\Rightarrow$ (iv). Suppose that $\operatorname{Spec}_E M$ is contained in a finite number of hyperplanes. Choose any $\lambda_1 \in \operatorname{Spec}_E M$. Let $r = \dim E$. Via the identification $E \cong V$ induced by $\Delta_0 \subset \Delta$, we view E as an euclidean vector space. Since

$$\operatorname{Spec}_E M \subset \bigcup_{w \in W_R} (w\lambda_1 + Q^\vee)$$

is contained in a finite union of the intersection of lattices and hyperplanes, we have

$$\lim_{n \longrightarrow \infty} \frac{\#\{\lambda \in \operatorname{Spec}_E M \,;\, \|\lambda\| < n\}}{n^{r-1+\varepsilon}} = 0, \quad \forall \varepsilon > 0.$$

Let $a \in \Delta$ be an affine simple root and let $m \in \mathbf{e}(\lambda)M$ with $\mathbf{e}(\lambda)M \neq 0$. Then we have $\tau_a^\omega m \in \mathbf{e}(s_a\lambda)M$. Moreover, we have $\|s_a\lambda\| \leq \|\lambda\| + \delta$ for some constant δ which depends only on the affine root system (E, S). It follows that if we define for $t \in \mathbf{R}_{\geq 0}$ the subspace

$$M_{\leq t} = \sum_{\substack{\lambda \in \operatorname{Spec}_E M \\ \|\lambda\| \leq t}} \mathbf{e}(\lambda)M,$$

then $\tau_a^\omega M_{\leq t} \subset M_{\leq t+\delta}$, so $F_{\leq 1}\mathbf{A}^\omega M_{\leq t} \subset M_{\leq t+\delta}$. By induction on $n \in \mathbf{N}$, we see that $(F_{\leq n}\mathbf{A}^\omega)\, M_{\leq t} \subset M_{\leq t+n\delta}$. Since there is only a finite number of clans and since the

dimension of $\mathbf{e}(w\lambda_0)M_\lambda$ for $w^{-1}\nu_0$ in a fixed clan is constant by Corollary 2.20. In particular, the dimension of the components $\mathbf{e}(\lambda)M$ is bounded. Hence for any finite dimensional subspaces $L \subset M$, we have

$$\lim_{n\longrightarrow\infty} \frac{\dim\left(F_{\leq n}\mathbf{A}^\omega \cdot L\right)}{n^{r-1+\varepsilon}} = 0, \quad \forall \varepsilon > 0. \tag{2.59}$$

Indeed, let $t_0 \in \mathbf{R}$ be such that $L \subset M_{\leq t_0}$, then $\dim\left(F_{\leq n}\mathbf{A}^\omega \cdot L\right) \leq \dim L_{\leq t_0+n\delta} = o\left(n^{r-1+\varepsilon}\right)$. The majoration (2.59) implies (iv). The equivalence (iv) $\Leftrightarrow$ (v) results from Proposition 2.33.

We prove $\neg$(ii) $\Rightarrow$ $\neg$(iv). Suppose there exists a generic clan $\mathfrak{C}$ and an alcove $\nu \subset \mathfrak{C}$ such that $\mathbf{e}\left(\lambda_\nu\right)M \neq 0$. Let $\kappa \subset V$ be the salient cone of $\mathfrak{C}$ (*cf.* §2.2.3). For any $\mu \in \kappa \cap Q^\vee$, we have $X^{-\mu}\nu \in \mathfrak{C}$ and by Proposition 2.19, $\mathbf{e}(X^\mu\lambda_\nu)M \cong \mathbf{e}(\lambda_\nu)M \neq 0$. It follows that

$$\begin{aligned}\dim\left(F_{\leq n}\mathbf{A}^\omega\right)\left(\mathbf{e}(\lambda_\nu)M\right) &\geq \dim \sum_{\substack{\mu\in\kappa\cap Q^\vee\\ \ell(X^\mu)\leq n}} \mathbf{e}(X^\mu\lambda_\nu)M \\ &\geq \frac{\#\{\mu \in \kappa \cap Q^\vee\ ;\ \ell(X^\mu) \leq n\}}{\#W_{\lambda_0}} \dim \mathbf{e}(\lambda_\nu)M\end{aligned}$$

By the genericity of $\mathfrak{C}$, the salient cone κ contains an open subset of V, so its intersection with a full-ranked lattice $Q^\vee$ satisfies

$$\lim_{n\longrightarrow\infty} \frac{\#\{\mu \in \kappa \cap Q^\vee\ ;\ \ell(X^\mu) \leq n\}}{n^r} =: c > 0.$$

Hence

$$\dim_{\mathrm{GK}} M \geq \lim_{n\longrightarrow\infty} \frac{\log \dim (F_{\leq 1}\mathbf{A}^\omega)^n \mathbf{e}(\lambda_\nu)M}{\log n} \geq \lim_{n\longrightarrow\infty} \frac{\log\left(cn^r/\#W_{\lambda_0}\right)}{\log n} = r,$$

whence (iv) is not satisfied. □

2.5.8 Double centraliser property

Recall the parabolic subalgebra $\mathbf{A}^\omega_{R,\lambda_1}$ from §2.2.6.

Lemma 2.60. *Let* $\lambda_1 \in W_S\lambda_0$, $N \in \mathbf{A}^\omega_{R,\lambda_1}$-gmod *and* $L \in \mathbf{A}^\omega$-gmod. *Suppose that* $\mathbf{V}L = 0$, *then* $\mathrm{gHom}\left(L, \mathrm{ind}^S_{R,\lambda_1} N\right) = 0$.

Proof. It follows from Theorem 2.58 (i)$\Rightarrow$(v) and Proposition 2.35. □

Let $(\mathbf{A}^\omega/\mathfrak{m}_\mathcal{Z})$-gmod be the full subcategory of $\mathbf{A}^\omega$-gmod consisting of objects M such that $\mathfrak{m}_\mathcal{Z}M = 0$. The inclusion $(\mathbf{A}^\omega/\mathfrak{m}_\mathcal{Z})$-gmod $\hookrightarrow \mathbf{A}^\omega$-gmod has a left adjoint functor $-\otimes_\mathcal{Z}\mathbf{C}$, which is right exact. We denote by $-\otimes^{\mathrm{L}}_\mathcal{Z}\mathbf{C}$ its derived functor. The next lemma is the method of lifting faithfulness borrowed from [44, 4.42].

Lemma 2.61. *Let* $M \in \mathbf{A}^\omega$-gmod *be an object satisfying the following properties:*

(i) M *is free over the centre* $\mathcal{Z}$,

(ii) there exists $\lambda_1 \in W_S\lambda_0$ *and* $N \in \mathbf{A}^\omega_{R,\lambda_1}$-$\mathrm{gmod}_0$ *such that* $M/\mathfrak{m}_\mathcal{Z}M \cong \mathrm{ind}^S_{R,\lambda_1} N$.

Then for any $L \in \mathbf{A}^{\omega}$-gmod *such that* $\mathbf{V}L = 0$, *we have* $\mathrm{gHom}(L, M) = 0$ *and* $\mathrm{gExt}^1(L, M) = 0$.

Proof. We suppose that $M \neq 0$. Let $K = \mathrm{RgHom}(L, M)$ be in the derived category $\mathrm{D}^+(\mathcal{Z}\text{-gMod})$. We suppose that K is a minimal projective resolution. Since

$$K \otimes_{\mathcal{Z}} \mathbf{C} \cong \mathrm{RgHom}\left(L \otimes^{\mathrm{L}}_{\mathcal{Z}} \mathbf{C}, M \otimes^{\mathrm{L}}_{\mathcal{Z}} \mathbf{C}\right) \cong \mathrm{RgHom}\left(L \otimes^{\mathrm{L}}_{\mathcal{Z}} \mathbf{C}, M/\mathfrak{m}_{\mathcal{Z}} M\right)$$

by the flatness of M over $\mathcal{Z}$, so $K \otimes \mathbf{C} \in \mathrm{D}^{\geq 0}(\mathbf{C})$. By the second assumption and Lemma 2.60, we have

$$\mathrm{H}^0(K \otimes_{\mathcal{Z}} \mathbf{C}) = \mathrm{Hom}\left(L \otimes_{\mathcal{Z}} \mathbf{C}, M/\mathfrak{m}_{\mathcal{Z}} M\right) = 0.$$

Consequently $\mathrm{H}^{\leq 0}(K) = 0$ by Nakayama's lemma.

Suppose that $\mathrm{H}^1(K) \neq 0$. Since the localisation $\mathfrak{m}_{\mathcal{Z}}^{-1} M$ is a weight module over $\mathbf{A}^{-\infty}$, which is semisimple, $\mathrm{H}^1(K)$ must be a torsion module over $\mathcal{Z}$ so $K^0 \neq 0$. However, the minimality of K would imply $\mathrm{H}^0(K \otimes_{\mathcal{Z}} \mathbf{C}) \neq 0$, contradiction. Hence $\mathrm{H}^{\leq 1}(K) = 0$ and so $\mathrm{gHom}(L, M) = 0$ and $\mathrm{gExt}^1(L, M) = 0$ as asserted.

□

Proposition 2.62. *Let* $M \in \mathbf{A}^{\omega}$-gmod *be as in Lemma 2.61. Then the adjoint unit yields an isomorphism* $M \cong (\mathbf{V}^{\top} \circ \mathbf{V})M$.

Proof. Set $X = \mathrm{Cone}\left(M \longrightarrow (\mathrm{R}\mathbf{V}^{\top} \circ \mathbf{V})M\right) \in \mathrm{D}^+(\mathbf{A}^{\omega}\text{-gmod})$, so that there is a distinguished triangle

$$M \longrightarrow (\mathrm{R}\mathbf{V}^{\top} \circ \mathbf{V})M \longrightarrow X \longrightarrow M[1]. \tag{2.63}$$

By the adjunction and the exactness of $\mathbf{V}$, we have $\mathbf{V}X \cong \mathrm{Cone}(\mathbf{V}M \longrightarrow (\mathbf{V} \circ \mathrm{R}\mathbf{V}^{\top} \circ \mathbf{V})M) = 0$ and hence

$$\mathbf{V}\mathrm{H}^k(X) \cong \mathrm{H}^k(\mathbf{V}X) = 0, \quad k \in \mathbf{Z}.$$

Applying Lemma 2.61 with $L = \mathrm{H}^0(X)$ and $L = \mathrm{H}^{-1}(X)$, we deduce

$$\mathrm{gHom}\left(\mathrm{H}^0(X), M\right) = 0, \quad \mathrm{gHom}\left(\mathrm{H}^0(X)[-1], M\right) = \mathrm{gExt}^1\left(\mathrm{H}^0(X), M\right) = 0$$
$$\mathrm{gHom}\left(\mathrm{H}^{-1}(X), M\right) = 0,$$

whence

$$\mathrm{gHom}(\tau_{\leq 0} X, M) = 0, \quad \mathrm{gHom}(\tau_{\leq 0} X, M[1]) = \mathrm{gHom}(\tau_{\leq 0} X[-1], M) = 0. \tag{2.64}$$

Applying $\mathrm{RgHom}\left(\tau_{\leq 0} X, -\right)$ to the distinguished triangle (2.63), we obtain the long exact sequence

$$\mathrm{gHom}\left(\tau_{\leq 0} X, M\right) \longrightarrow \mathrm{gHom}\left(\tau_{\leq 0} X, (\mathrm{R}\mathbf{V}^{\top} \circ \mathbf{V})M\right) \longrightarrow \mathrm{gHom}\left(\tau_{\leq 0} X, X\right) \longrightarrow \mathrm{gHom}\left(\tau_{\leq 0} X, M[1]\right).$$

By (2.64), the first and the last term of the sequence vanish. Hence,

$$\mathrm{gHom}\left(\tau_{\leq 0} X, X\right) \cong \mathrm{gHom}\left(\tau_{\leq 0} X, (\mathrm{R}\mathbf{V}^{\top} \circ \mathbf{V})M\right) \cong \mathrm{gHom}\left(\tau_{\leq 0} \mathbf{V}X, \mathbf{V}M\right) = 0,$$

which implies that $\tau_{\leq 0} X = 0$. Applying H^0 to the distinguished triangle (2.63), we deduce that the adjunction unit $M \longrightarrow (\mathbf{V}^{\top} \circ \mathbf{V})M$ is an isomorphism. □

Theorem 2.65 (Double centraliser property[6]). *The canonical map*

$$\mathbf{A}^{\omega} \longrightarrow \bigoplus_{\lambda,\lambda' \in W_S\lambda_0} \mathrm{gHom}_{\mathbf{B}^{\omega}}\left(\mathbf{V}\mathbf{A}^{\omega}\mathbf{e}\left(\lambda\right), \mathbf{V}\mathbf{A}^{\omega}\mathbf{e}\left(\lambda'\right)\right)$$

is an isomorphism.

Proof. Observe that for each $\lambda \in W_S\lambda_0$, the module $\mathbf{A}^{\omega}\mathbf{e}(\lambda) \in \mathbf{A}^{\omega}$-gmod satisfies the conditions of Lemma 2.61. Indeed, $\mathbf{A}^{\omega}\mathbf{e}(\lambda)$ is flat over $\mathcal{Z}$ by Theorem 2.8. For the second condition, we have $\mathbf{A}^{\omega}\mathbf{e}(\lambda) \cong \mathrm{ind}_{R,\lambda_1}^{S} \mathbf{A}_{R,\lambda}^{\omega}\mathbf{e}(\lambda)$, so $\mathbf{A}^{\omega}\mathbf{e}(\lambda)/\mathfrak{m}_{\mathcal{Z}} \cong \mathrm{ind}_{R,\lambda_1}^{S}\left(\mathbf{A}_{R,\lambda}^{\omega}\mathbf{e}(\lambda)/\mathfrak{m}_{\mathcal{Z}}\right)$. Applying Proposition 2.62, we obtain

$$\mathbf{A}^{\omega} \cong \bigoplus_{\lambda,\lambda' \in W_S\lambda_0} \mathrm{gHom}_{\mathbf{A}^{\omega}}\left(\mathbf{A}^{\omega}\mathbf{e}\left(\lambda\right), \mathbf{A}^{\omega}\mathbf{e}\left(\lambda'\right)\right) \xrightarrow{\cong} \bigoplus_{\lambda,\lambda' \in W_S\lambda_0} \mathrm{gHom}_{\mathbf{A}^{\omega}}\left(\mathbf{V}\mathbf{A}^{\omega}\mathbf{e}\left(\lambda\right), \mathbf{V}\mathbf{A}^{\omega}\mathbf{e}\left(\lambda'\right)\right).$$

□

Corollary 2.66. *There are isomorphisms*

(i) of $(\mathbf{A}^{\omega}, \mathbf{B}^{\omega})$*-bimodules* $\bigoplus_{\lambda \in W_S\lambda_0} \mathrm{gHom}_{\mathbf{B}^{\omega}}(\mathbf{e}_{\gamma}\mathbf{A}^{\omega}\mathbf{e}(\lambda), \mathbf{B}^{\omega}) \cong \mathbf{A}^{\omega}\mathbf{e}_{\gamma}$ *and*

(ii) of $(\mathbf{B}^{\omega}, \mathbf{A}^{\omega})$*-bimodules* $\bigoplus_{\lambda \in W_S\lambda_0} \mathrm{gHom}_{(\mathbf{B}^{\omega})^{\mathrm{op}}}(\mathbf{e}(\lambda)\mathbf{A}^{\omega}\mathbf{e}_{\gamma}, \mathbf{B}^{\omega}) \cong \mathbf{e}_{\gamma}\mathbf{A}^{\omega}$.

In particular, the $\mathbf{B}^{\omega}$*-module* $\mathbf{e}_{\gamma}\mathbf{A}^{\omega}$ *is reflexive.*

Proof. Statement (i) is deduced from Theorem 2.65 by right-multiplication with $\mathbf{e}_{\gamma}$. Under the anti-involution $\mathbf{A}^{\omega} \cong (\mathbf{A}^{\omega})^{\mathrm{op}}$ (resp. $\mathbf{B}^{\omega} \cong (\mathbf{B}^{\omega})^{\mathrm{op}}$) given by $\tau_a^{\omega}\mathbf{e}(\lambda) \mapsto \tau_a^{\omega}\mathbf{e}(s_a\lambda)$ (resp. $\tau_{\alpha}^{\omega}\mathbf{e}(\ell) \mapsto \tau_{\alpha}^{\omega}\mathbf{e}(s_{\alpha}\ell)$), the isomorphism of (ii) is identified with that of (i). The validity of (ii) follows from (i). □

2.5.9 Categorical characterisation of V

Proposition 2.67. *Let* $L \in \mathbf{A}^{\omega}$-$\mathrm{gmod}_0$ *be a simple object. Then the following conditions are equivalent:*

(i) $\mathbf{V}L = 0$

(ii) there exists a projective object $\mathcal{P}$ *of the subcategory* $(\mathbf{A}^{\omega}/\mathfrak{m}_{\mathcal{Z}})$-gmod *such that*

$$\mathrm{gHom}\left(L, \mathcal{P}\right) \neq 0$$

(iii) the projective cover of L *in the subcategory* $(\mathbf{A}^{\omega}/\mathfrak{m}_{\mathcal{Z}})$-gmod *is injective.*

[6] Let A and B be unital associative rings. Usually, one says that an (A, B)-bimodule P satisfies the double centraliser property if the structural maps $A \longrightarrow \mathrm{End}_{B^{\mathrm{op}}}(P)$ and $B \longrightarrow \mathrm{End}_A(P)^{\mathrm{op}}$ are isomorphisms. The above theorem provides a graded, non-unital version of this property for the $(\mathbf{A}^{\omega}, \mathbf{B}^{\omega})$-bimodule $\mathbf{A}^{\omega}\mathbf{e}_{\gamma}$.

Proof. Since $\mathbf{B}^\omega$ is a Frobenius algebra over $\mathcal{Z}$ by Lemma 2.38, its quotient $\overline{\mathbf{B}}^\omega = \mathbf{B}^\omega/\mathfrak{m}_{\mathcal{Z}}\mathbf{B}^\omega$ is a Frobenius algebra over $\mathbf{C}$ and in particular, $\overline{\mathbf{B}}^\omega$ is self-injective.

We prove (i) $\Leftrightarrow$ (ii) and (i) $\Rightarrow$ (iii). Let $L \in \mathbf{A}^\omega\text{-gmod}_0$ be any simple object. If $\mathbf{V}L = 0$, then by Lemma 2.61, we have $\mathrm{gHom}\,(L, \mathbf{A}^\omega\mathbf{e}(\lambda)) = 0$ for all $\lambda \in W_S$. If $\mathbf{V}L \neq 0$, since $\mathbf{e}_\gamma$ is a quotient functor, $\mathbf{V}L \in \mathbf{B}\text{-gmod}_0$ must be simple. We have $\mathfrak{m}_{\mathcal{Z}}\mathbf{V}L = 0$, so we may view L as a $\overline{\mathbf{B}}^\omega$-module. By the self-injectivity, there exists a non-zero map $\iota : \mathbf{V}L \hookrightarrow \overline{\mathbf{B}}^\omega$ and the adjunction yields an injective map $L \longrightarrow \mathbf{V}^\top\overline{\mathbf{B}}^\omega$. Proposition 2.62 yields $\mathbf{A}^\omega\mathbf{e}_\gamma \cong \mathbf{V}^\top\mathbf{V}\mathbf{A}^\omega\mathbf{e}_\gamma = \mathbf{V}^\top B^\omega$, so $\mathbf{V}^\top\overline{\mathbf{B}}^\omega \cong \mathbf{A}^\omega\mathbf{e}_\gamma/\mathfrak{m}_{\mathcal{Z}}$, which implies that $L \hookrightarrow \mathbf{A}^\omega\mathbf{e}_\gamma/\mathfrak{m}_{\mathcal{Z}}$, whence (ii). Since $\mathbf{V}^\top$ preserves injective objects, we see that $\mathbf{A}^\omega\mathbf{e}_\gamma/\mathfrak{m}_{\mathcal{Z}}$ is injective-projective in $(\mathbf{A}^\omega/\mathfrak{m}_{\mathcal{Z}})\text{-gmod}$, whence (iii).

We prove (iii) $\Rightarrow$ (i). Suppose that the projective cover of L in $(\mathbf{A}^\omega/\mathfrak{m}_{\mathcal{Z}})\text{-gmod}$, denoted by $\mathcal{P}$, is injective, the socle $\operatorname{soc}\mathcal{P}$ satisfies $\mathbf{V}\operatorname{soc}\mathcal{P} \neq 0$ by Lemma 2.60. Let $\mathcal{I} \in (\mathbf{B}^\omega/\mathfrak{m}_{\mathcal{Z}})\text{-gmod}$ be the injective hull of $\mathbf{V}\operatorname{soc}\mathcal{P}$. Then $\mathbf{V}^\top\mathcal{I}$ is the injective hull of $\operatorname{soc}\mathcal{P}$, hence isomorphic to $\mathcal{P}$. It follows that $\mathcal{P}$ is a direct factor of $\mathbf{V}^\top\overline{\mathbf{B}}^\omega \cong \mathbf{A}^\omega\mathbf{e}_\gamma/\mathfrak{m}_{\mathcal{Z}}$, so $\mathbf{V}L \neq 0$. □

Example 2.68. *Resume to the setting of examples Example 2.3 (iii), Example 2.21 and Example 2.44. We have* $\mathbf{e}_\gamma\mathbf{A}^\omega = P_+ \oplus P_-$ *so* $\mathbf{V}L_0 = 0$*, while* $\mathbf{V}L_+ \neq 0$ *and* $\mathbf{V}L_- \neq 0$ *are simple objects in* $\mathbf{B}^\omega\text{-gmod}$*. In regard of Theorem 2.58, we have* $\dim_{\mathrm{GK}} L_+ = \dim_{\mathrm{GK}} L_- = 1$ *while* $\dim_{\mathrm{GK}} L_0 = 0$*. The cosocle filtration of* $\mathbf{V}P_+$*,* $\mathbf{V}P_0$ *and* $\mathbf{V}P_-$ *are described by the following:*

$$\mathbf{V}P_+ = \begin{bmatrix} \mathbf{V}L_+ \\ \mathbf{V}L_+\langle -2\rangle \ \ \mathbf{V}L_-\langle -2\rangle \\ \mathbf{V}L_+\langle -4\rangle \ \ \mathbf{V}L_-\langle -4\rangle \\ \mathbf{V}L_+\langle -6\rangle \ \ \mathbf{V}L_-\langle -6\rangle \\ \vdots \end{bmatrix}, \quad \mathbf{V}P_0 = \begin{bmatrix} \mathbf{V}L_+\langle -1\rangle \ \ \mathbf{V}L_-\langle -1\rangle \\ \mathbf{V}L_+\langle -3\rangle \ \ \mathbf{V}L_-\langle -3\rangle \\ \mathbf{V}L_+\langle -5\rangle \ \ \mathbf{V}L_-\langle -5\rangle \\ \vdots \end{bmatrix}, \quad \mathbf{V}P_- = \begin{bmatrix} \mathbf{V}L_- \\ \mathbf{V}L_+\langle -2\rangle \ \ \mathbf{V}L_-\langle -2\rangle \\ \mathbf{V}L_+\langle -4\rangle \ \ \mathbf{V}L_-\langle -4\rangle \\ \mathbf{V}L_+\langle -6\rangle \ \ \mathbf{V}L_-\langle -6\rangle \\ \vdots \end{bmatrix}.$$

From this description it is obvious that the functor $\mathbf{V}$ *is fully faithful on the projective objects, so* $\mathbf{V}$ *satisfies the double centraliser property Theorem 2.65.*

Consider the quotients

$$P_+/\mathfrak{m}_{\mathcal{Z}} = \begin{bmatrix} L_+ \\ L_0\langle -1\rangle \\ L_-\langle -2\rangle \end{bmatrix}, \quad P_0/\mathfrak{m}_{\mathcal{Z}} = \begin{bmatrix} L_0 \\ L_+\langle -1\rangle \ \ L_-\langle -1\rangle \end{bmatrix}, \quad P_-/\mathfrak{m}_{\mathcal{Z}} = \begin{bmatrix} L_- \\ L_0\langle -1\rangle \\ L_+\langle -2\rangle \end{bmatrix}.$$

It follows that $P_+/\mathfrak{m}_{\mathcal{Z}}$ *(resp.* $P_-/\mathfrak{m}_{\mathcal{Z}}$*) is the injective hull of* $L_-\langle -2\rangle$ *(resp.* $L_+\langle -2\rangle$*) in the category* $(\mathbf{A}^\omega/\mathfrak{m}_{\mathcal{Z}})\text{-gmod}$ *while* $P_0/\mathfrak{m}_{\mathcal{Z}}$ *is not injective. Hence* L_+ *and* L_- *satisfy the equivalent conditions of Proposition 2.67.*

Remark 2.69. *In view of the double centraliser property Theorem 2.65, we may view* $(\mathbf{A}^\omega\text{-gmod}, \mathbf{V})$ *as a partial resolution of the category* $\mathbf{B}^{\partial\omega}\text{-gmod}$*. The map* $\omega \mapsto \partial\omega$ *is far from being unique. Indeed, for any given* ω*, translating the multiplicity of any affine root by* $n\delta$ *for any* $n \in \mathbf{Z}$ *does not change the derivative* $\partial\omega$*. Therefore, a possible measure for the singularity of* $\mathbf{A}^\omega\text{-gmod}$ *would be the singularity of the hyperplane arrangement given by* $\omega - o$*, counted with multiplicities. There are a finite number of equivalent classes of pairs* $(\mathbf{A}^\omega\text{-gmod}, \mathbf{V})$*, determined by the configuration of this hyperplane arrangement.*

We expect that if the divisor associated to $\omega - o$ *is of normal crossing (a fortiori multiplicity-free) on the quotient space* E/W_{λ_0}*, then the global dimension of* $\mathbf{A}^\omega\text{-gmod}$ *should be finite (*$\mathbf{A}^\omega\text{-gmod}$ *is generally not affine quasi-hereditary, however.) Moreover, for*

any two such families ω and ω' such that $\partial\omega = \partial\omega'$, there should be a derived equivalence $\mathrm{D}^b(\mathbf{A}^\omega\text{-gmod}) \cong \mathrm{D}^b\left(\mathbf{A}^{\omega'}\text{-gmod}\right)$ compatible with the KZ functors $\mathbf{V}$. This would be a quiver-Hecke-algebraic analogue of Losev's derived equivalences [28] for rational Cherednik algebras.

A heuristic explanation of the derived equivalence: Corollary 2.66 shows that the $\mathbf{B}^\omega$-module $\mathbf{e}_\gamma\mathbf{A}^\omega$ is reflexive and Theorem 2.65 shows that $\mathbf{A}^\omega = \mathrm{gEnd}(\mathbf{e}_\gamma\mathbf{A}^\omega)$. If $\mathbf{A}^\omega$ happens to be of finite global dimension, then the pair $(\mathbf{A}^\omega, \mathbf{V})$ will be a non-commutative crepant resolution for $\mathbf{B}^\omega$ in the sense of Van den Bergh up to the Cohen–Macaulay condition, which needs a non-commutative analogue. According to one of the main conjectures in the (non-commutative) minimal model programme [5] [22], (non-commutative) crepant resolutions should be related by derived equivalences.

Chapter 3

Perverse sheaves on graded Lie algebras

Introduction

Let G be a reductive complex algebraic group and let $\mathfrak{g}$ be its Lie algebra. Let $m \in \mathbf{Z}_{>0}$ and let $\theta : \mu_m \longrightarrow \mathrm{Ad}(G)$ be a group homomorphism. Then θ gives rise to a $\mathbf{Z}/m$-grading on $\mathfrak{g}$, written $\mathfrak{g} = \bigoplus_{\underline{n} \in \mathbf{Z}/m} \mathfrak{g}_{\underline{n}}$ with $\mathfrak{g}_{\underline{n}} = \{X \in \mathfrak{g} \; ; \; \mu(\zeta)X = \zeta^n X\}$ and let $G_{\underline{0}} = G^\theta$ be the θ-fixed points. Let $d \in \mathbf{Z}$ and let $\underline{d} = d \mod m \in \mathbf{Z}/m$. One is interested in the $G_{\underline{0}}$-equivariant bounded derived category of constructible sheaves on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$, denoted by $\mathrm{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}})$.

In [38], Lusztig and Yun studied the $G_{\underline{0}}$-orbits in the nilpotent cone $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}} \subset \mathfrak{g}_{\underline{d}}$ and found a way to produce uniformly all the simple $G_{\underline{0}}$-equivariant perverse sheaves on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$. They remarked that the naive adaptation of parabolic induction from the $\mathbf{Z}$-graded case, which was studied in [37], can not produce all the simple perverse sheaves on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ as direct factor of induced sheaves. Instead, they introduced the notions of spiral induction and spiral restriction, which turn out to be the right notions of induction and restriction of sheaves on $\mathbf{Z}/m$-graded Lie algebras as they produce all simple perverse sheaves and share a large part of the features of parabolic induction and restriction on $\mathbf{Z}$-graded Lie algebras. In a way, the naive parabolic induction can also be realised as the induction through certain spirals, so the spiral induction is a generalisation of it.

According to [38], there is a decomposition of triangulated category

$$\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right) = \bigoplus_{\zeta \in \underline{\mathfrak{T}}(\mathfrak{g}_{\underline{d}})} \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta .$$

Let $\mathfrak{T}(\mathfrak{g}_{\underline{d}})$ be the collection of systems $(L, \mathfrak{l}_*, \mathrm{O}, \mathscr{C})$, called *admissible systems*, where L is a pseudo-Levi subgroup of G [1], $\mathfrak{l}_*$ is a $\mathbf{Z}$-grading on the Lie algebra $\mathfrak{l} = \mathrm{Lie}\, L$ such that $\mathfrak{l}_n \subset \mathfrak{g}_{\underline{n}}$ for all $n \in \mathbf{N}$, and $(\mathrm{O}, \mathscr{C})$ is a cuspidal pair on $\mathfrak{l}_d$ in the sense of [37]. The set above $\underline{\mathfrak{T}}(\mathfrak{g}_{\underline{d}})$ is the set of $G_{\underline{0}}$-conjugacy classes in $\mathfrak{T}(\mathfrak{g}_{\underline{d}})$. Each subcategory $\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$ is called a block. The block corresponding to the system $\zeta_0 = (T, \mathfrak{t}_*, \{0\}, \delta_0)$, where T is a maximal torus of $G_{\underline{0}}$ and δ_0 is the punctual sheaf supported on 0 with fibre $\overline{\mathbf{Q}}_\ell$, is called the principal series.

[1] Recall that a pseudo-Levi subgroup of G is the centraliser of a semisimple element $s \in G$.

Let $\mathbf{C}_q^\times$ be a one-dimensional torus acting linearly on $\mathfrak{g}$ by weight 2. The action of $\mathbf{C}_q^\times$ commutes with the adjoint action of G so that there is a $G_{\underline{0}} \times \mathbf{C}_q^\times$ action on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$. By the Jacobson–Morosov theorem, the $G_{\underline{0}}$-orbits in $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ are stable by the action of $\mathbf{C}_q^\times$. The functor of forgetting the $\mathbf{C}_q^\times$-action $\mathrm{D}_{G_{\underline{0}}\times\mathbf{C}_q^\times}(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}) \longrightarrow \mathrm{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}})$ induces

$$\mathrm{Irr}\,\mathrm{Perv}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}) \cong \mathrm{Irr}\,\mathrm{Perv}_{G_{\underline{0}}\times\mathbf{C}_q^\times}(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}})$$

on the isomorphism classes of equivariant simple perverse sheaves. Similarly, each admissible system $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ acquires automatically a $G_{\underline{0}} \times \mathbf{C}_q^\times$-equivariant structure. The spiral induction, spiral restriction and the decomposition of equivariant category into blocks also has a $G_{\underline{0}} \times \mathbf{C}_q^\times$-equivariant version.

From an algebraic point of view, the Springer correspondence of [53] and [40] tells that the simple perverse sheaves on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ correspond to simple modules of certain block of a dDAHA $\mathbb{H}$. Let $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ be an admissible system on $\mathfrak{g}_{\underline{d}}$. The sum of the spiral inductions of the perverse sheaf $\mathrm{IC}(\mathscr{C})$ is an (ind-)complex $\mathbf{I}_{\mathrm{aff}}$ in the equivariant category $\mathrm{D}_{G_{\underline{0}}\times\mathbf{C}_q^\times}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)$ and its extension algebra $\mathcal{H} = \mathrm{Hom}^\bullet_{G_{\underline{0}}\times\mathbf{C}_q^\times}(\mathbf{I}_{\mathrm{aff}}, \mathbf{I}_{\mathrm{aff}})$ realises a localised version of $\mathbb{H}$ (which means that $\mathcal{H}$ is Morita-equivalent to a block of the category $\mathcal{O}$ of $\mathbb{H}$). The classical techniques of extension algebras as described in Chriss–Ginzburg [13] allow one to analyse the structure of the block of $\mathbb{H}$ via the geometry of $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ and the complex $\mathbf{I}_{\mathrm{aff}}$. In particular, the simple $\mathbb{H}$-modules in the block correspond to simple direct factors of $\mathbf{I}_{\mathrm{aff}}$ and the dimension of a simple module is equal to the multiplicity of the corresponding simple director factor in $\mathbf{I}_{\mathrm{aff}}$. Moreover, for each simple module of $\mathbb{H}$ there is a standard module, defined in terms of the !-fibre of $\mathbf{I}_{\mathrm{aff}}$ at a nilpotent $G_{\underline{0}}$-orbit in $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$.

On the other hand, several decades before, Kazhdan and Lusztig [23] have already given a realisation of the affine Hecke algebras $\mathbb{K}$ via the equivariant K-theory on the Steinberg variety over $\mathfrak{g}$ and by localisation, a certain block of $\mathbb{K}$ can be realised with a certain class of perverse sheaves on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$. However, the perverse sheaves which show up in this construction are still not well understood. This block of $\mathbb{K}$ has some features of modular representation theory. Below we introduce a semisimple complex $\mathbf{I}_{\mathrm{fin}}$ whose extension algebra $\mathcal{K} = \mathrm{Hom}^\bullet_{G_{\underline{0}}\times\mathbf{C}_q^\times}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{fin}})$ which realises a localised version of $\mathbb{K}$.

In §3.1, we recall the notion of a $\mathbf{Z}/m$-grading on Lie algebras and the spiral inductions–restrictions introduced in [38]. We put accent on a number of immediate consequences of the main theorem thereof.

In §3.3, we introduce the notion of the parabolic inductions–restrictions in the $\mathbf{Z}/m$-graded setting. The definitions are similar to the parabolic inductions–restrictions in the $\mathbf{Z}$-graded setting considered in [37]. We also discuss some of their relations with the spiral inductions–restrictions.

In § 3.4, we introduce the supercuspidal pairs, in terms of parabolic inductions–restrictions. Briefly, a supercuspidal pair is an equivariant local system which is annihilated by every strict parabolic restriction. We deduce some properties of supercuspidal pairs which are in complete analogy with the "cuspidal pairs" of [37]. Their relation with the cuspidal pairs considered in [38] is also discussed.

In §3.5, we prove Theorem 3.20, which, very roughly, states that the local systems that one can obtain from $\mathbf{Z}/m$-graded parabolic inductions from supercuspidal pairs are the same as those that one can obtain from the spiral inductions.

The apparition of the dDAHA $\mathbb{H}$ and the AHA $\mathbb{K}$ indicates the properties of the

Knizhnik–Zamolodchikov monodromy functor $\mathbb{V}$ considered in §1.4 should be intimately related to the geometry of $\mathfrak{g}_{\underline{d}}$. In particular, the complex $\mathbf{I}_{\text{fin}}$ should be a direct factor of $\mathbf{I}_{\text{aff}}$ up to degree shifting and the deficit measures how singular the corresponding block of $\mathbb{K}$ is. Making analogy with the case of RDAHAs, one expects that the KZ functor does not annihilate the proper standard modules of $\mathbb{H}$. This non-vanishing property can be translated into a sheaf theoretic statement Theorem 3.20.

Conventions and notations

All algebraic varieties and schemes will be separated of finite type over $\mathbf{C}$.

Fix a prime number ℓ. For any scheme X, we denote $\mathrm{D}^b(X)$ the bounded derived category of constructible $\overline{\mathbf{Q}}_\ell$-sheaves on X.

For any algebraic group G acting on X (on the left), we denote $\mathrm{D}^b_G(X) = \mathrm{D}^b([G\backslash X])$ the G-equivariant bounded derived category of $\overline{\mathbf{Q}}_\ell$-sheaves on X, or equivalently bounded derived category of $\overline{\mathbf{Q}}_\ell$-sheaves on the quotient stack $[G\backslash X]$, as defined in Laszlo–Olsson [26]. When the symbol b is absent from D^b, it means the unbounded derived category. We denote $\mathrm{Perv}_G(X) \subseteq \mathrm{D}^b_G(X)$ the subcategory of complexes whose image in $\mathrm{D}^b(X)$ is perverse ; those are perverse sheaves on the stack $[G\backslash X]$ up to a shift.

On these derived categories, the six operations $\otimes, \mathcal{H}\mathrm{om}, f_*, f^*, f_!, f^!$ will be understood as derived functors. We suppress the symbols R and L from $\mathrm{R}f_*$, $\otimes^{\mathrm{L}}$, etc. The bi-duality functor will be denoted $\mathbf{D}$, the perverse cohomology functors will be denoted ${}^p\mathrm{H}^k$ for $k \in \mathbf{Z}$ and for any local system $\mathcal{L}$ supported on some locally closed subset, its intersection complex will be denoted $\mathrm{IC}(\mathcal{L}) \in \mathrm{Perv}_G(X)$. For any complexes $\mathscr{F}, \mathscr{G} \in \mathrm{D}^b_G(X)$ and $k \in \mathbf{Z}$, denote $\mathrm{Hom}^k_G(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathrm{D}^b_G(X)}(\mathscr{F}, \mathscr{G}[k])$ and $\mathrm{Hom}^\bullet_G(\mathscr{F}, \mathscr{G}) = \bigoplus_{k\in\mathbf{Z}} \mathrm{Hom}^k_G(\mathscr{F}, \mathscr{G})$.

For any algebraic group G, the set of one-parameter subgroups (or co-character) is denoted $\mathbf{X}_*(G)$. We will adopt the notion of *co-characters up to isogeny.* The set of co-characters up to isogeny is defined to be the quotient set

$$\mathbf{X}_*(G)_{\mathbf{Q}} = \mathbf{X}_*(G) \times \mathbf{N}^* / \left((\lambda, p) \sim (\mu, q) \Leftrightarrow q\lambda = p\mu\right)$$

and the elements are denoted by $\lambda/p = (\lambda, p)$. We will also consider weight spaces of elements of $\mathbf{X}_*(G)_{\mathbf{Q}}$. If $\rho : G \longrightarrow \mathrm{Aut}(V)$ is a rational representation and if $\lambda \in \mathbf{X}_*(G)_{\mathbf{Q}}$, then for each $r \in \mathbf{Q}$ the weight space of λ of weight r in V is denoted ${}^\lambda_r V$. Namely, let $k \in \mathbf{Z} \setminus \{0\}$ such that $k\lambda \in \mathbf{X}_*(G)$, then

$${}^\lambda_r V = \left\{ v \in V \ ; \ \mathrm{Ad}_{(k\lambda)(t)}(v) = t^{kr} v, \forall t \in \mathbf{G}_{\mathrm{m}} \right\}.$$

For any pair of adjoint functors (F, G), we denote by $\mathrm{id} \longrightarrow GF$ the adjunction unit and by $FG \longrightarrow \mathrm{id}$ the adjunction co-unit.

3.1 $\mathbf{Z}/m$-grading, spirals and splittings

We recall in this section the notion of spirals and splittings as defined in [38].

3.1.1 $\mathbf{Z}/m$-grading on G

Let G be a connected simply connected semisimple algebraic group over $\mathbf{C}$. The Lie algebra is denoted $\mathfrak{g} = \operatorname{Lie} G$.

We fix a positive integer $m \in \mathbf{N}^*$. For any integer $k \in \mathbf{Z}$, we will denote $\underline{k} = k \bmod m \in \mathbf{Z}/m$. Let

$$\mathfrak{g} = \bigoplus_{\underline{i} \in \mathbf{Z}/m} \mathfrak{g}_{\underline{i}}$$

be a $\mathbf{Z}/m$-grading on $\mathfrak{g}$ such that $[\mathfrak{g}_{\underline{i}}, \mathfrak{g}_{\underline{j}}] \subseteq \mathfrak{g}_{\underline{i}+\underline{j}}$ for all $\underline{i}, \underline{j} \in \mathbf{Z}/m$. We define a homomorphism $\theta : \mu_m \longrightarrow \operatorname{Aut}(\mathfrak{g})$ by setting

$$\theta(\zeta) \mid_{\mathfrak{g}_{\underline{j}}} = \zeta^j, \quad \forall j \in \mathbf{Z}.$$

The semisimplicity of G implies that $\operatorname{Aut}(G) \cong \operatorname{Aut}(\mathfrak{g})$, so we can write $\theta : \mu_m \longrightarrow \operatorname{Aut}(G)$.

We will assume throughout that θ is inner, meaning that the image of θ lands on the subgroup $\operatorname{Ad}(G) \subset \operatorname{Aut}(G)$.

We fix once and for all a non-zero integer $d \in \mathbf{Z}_{\neq 0}$. The adjoint action of G on $\mathfrak{g}$ restricts to an action of the fix-point subgroup $G_{\underline{0}} = G^\theta \subset G$ on the graded piece $\mathfrak{g}_{\underline{d}}$. Let $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}} = \mathfrak{g}_{\underline{d}} \cap \mathfrak{g}^{\mathrm{nil}}$ be the closed subvariety of nilpotent elements. The geometric objects of interest are the $G_{\underline{0}}$ orbits in $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ and the $G_{\underline{0}}$-equivariant derived category $\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)$.

Let $\Pi\left(\mathfrak{g}_{\underline{d}}\right)$ denote the set of pairs $(\mathrm{O}, \mathscr{L})$ where $\mathrm{O} \subset \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ is a nilpotent $G_{\underline{0}}$-orbit and $\mathscr{L} \in \operatorname{Loc}_{G_{\underline{0}}}(\mathrm{O})$ is an (isomorphism class of) irreducible $G_{\underline{0}}$-equivariant local system on O.

3.1.2 Jacobson–Morosov theorem

Recall the theorem of Jacobson–Morosov in the $\mathbf{Z}/m$-graded setting. Let $e \in \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$. According to [38, 2.3], we can complete e into an $\mathfrak{sl}_2$-triple $\phi = (e, h, f)$ with $h \in \mathfrak{g}_{\underline{0}}$ and $f \in \mathfrak{g}_{-\underline{d}}$. Consequently, there is a cocharacter $\lambda \in \mathbf{X}_*(G_{\underline{0}})$, the exponentiation of h, which we will denote by $\lambda : t \mapsto t^h$. Moreover, the set of such triple ϕ with a given e forms a principal homogeneous space under the action of the unipotent part of the stabiliser of e in $G_{\underline{0}}$.

Using the Jacobson–Morosov theorem, one can show that the number of $G_{\underline{0}}$-orbits in $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ is finite and moreover, each orbit is invariant by dilatation. We will also consier the action of the one-dimensional torus $\mathbf{C}_q^\times = \mathbf{C}^\times$ on $\mathfrak{g}_{\underline{d}}$, given by $(q, x) \mapsto q^{-2}x$.

3.1.3 Spirals, nilpotent radical and splittings

Let $\lambda \in \mathbf{X}_*\left(G_{\underline{0}}\right)_{\mathbf{Q}}$ be a co-character and let $\epsilon \in \{1, -1\}$. We associate to it a $\mathbf{Z}$-graded Lie algebra ${}^\epsilon\mathfrak{p}_*^\lambda = \bigoplus_{n \in \mathbf{Z}} {}^\epsilon\mathfrak{p}_n^\lambda$ where

$${}^\epsilon\mathfrak{p}_n^\lambda = \bigoplus_{\substack{r \in \mathbf{Q} \\ r \geq \epsilon n}} {}^\lambda_r\mathfrak{g}_{\underline{n}}.$$

We also define a $\mathbf{Z}$-graded Lie subalgebra of $\mathfrak{g}$

$${}^\epsilon\mathfrak{l}_*^\lambda = \bigoplus_{n \in \mathbf{Z}} {}^\epsilon\mathfrak{l}_n^\lambda, \quad {}^\epsilon\mathfrak{l}_n^\lambda = {}^\lambda_{\epsilon n}\mathfrak{g}_{\underline{n}}.$$

Note that they depend on the sign $\epsilon \in \{\pm 1\}$. Such a graded sub-space ${}^{\epsilon}\mathfrak{p}_*^{\lambda}$ is called an **ϵ-spiral** of $\mathfrak{g}$ and such an ${}^{\epsilon}\mathfrak{l}_*^{\lambda}$ is called a **splitting** of the spiral ${}^{\epsilon}\mathfrak{p}_*^{\lambda}$. We let ${}^{\epsilon}\mathfrak{l}^{\lambda}$ be the same Lie algebra as $\mathfrak{l}_*^{\lambda}$ with grading forgotten.

We define the $\mathbf{Z}$-graded Lie algebra ${}^{\epsilon}\mathfrak{u}_*^{\lambda}$ with

$$ {}^{\epsilon}\mathfrak{u}_*^{\lambda} = \bigoplus_{n\in\mathbf{Z}} {}^{\epsilon}\mathfrak{u}_n^{\lambda}, \quad {}^{\epsilon}\mathfrak{u}_n^{\lambda} = \bigoplus_{r>\epsilon n} {}^{\lambda}_{r}\mathfrak{g}_{\underline{n}} $$

to be the **nilpotent radical** of the spiral ${}^{\epsilon}\mathfrak{p}_*^{\lambda}$.

Then ${}^{\epsilon}\mathfrak{u}_*^{\lambda}$ forms a homogeneous ideal of the graded Lie algebra ${}^{\epsilon}\mathfrak{p}_*^{\lambda}$ and that for each $n \in \mathbf{Z}$ the subspace ${}^{\epsilon}\mathfrak{l}_n^{\lambda} \subseteq {}^{\epsilon}\mathfrak{p}_n^{\lambda}$ is mapped isomorphically onto the quotient ${}^{\epsilon}\mathfrak{p}_n^{\lambda}/{}^{\epsilon}\mathfrak{u}_n^{\lambda}$ via the obvious projection. The Lie algebra $\mathfrak{l}_*^{\lambda}$ is the Lie algebra of a pseudo-Levi subgroup of G, [40, 2.2.5].

We will write $\mathfrak{p}_*^{\lambda}$, $\mathfrak{l}_*^{\lambda}$ and $\mathfrak{u}_*^{\lambda}$ instead of ${}^{\epsilon}\mathfrak{p}_*^{\lambda}$, ${}^{\epsilon}\mathfrak{l}_*^{\lambda}$ and ${}^{\epsilon}\mathfrak{u}_*^{\lambda}$ in a context where the signature ϵ is clear.

3.2 Spiral induction and restriction

3.2.1 Spiral induction and restriction

With the datum $(\mathfrak{p}_*, \mathfrak{l}_*, \mathfrak{u}_*) = \left({}^{\epsilon}\mathfrak{p}_*^{\lambda}, {}^{\epsilon}\mathfrak{l}_*^{\lambda}, {}^{\epsilon}\mathfrak{u}_*^{\lambda}\right)$ of a spiral together with a splitting, we can define the functor of induction. Let $P_0 = \exp(\mathfrak{p}_0)$ and $L_0 = \exp(\mathfrak{l}_0)$.

Fix an integer $d \in \mathbf{Z} \setminus \{0\}$ and $\epsilon \in \{1, -1\}$. We consider the following diagram

$$ \mathfrak{g}_{\underline{d}} \xleftarrow{\alpha} G_{\underline{0}} \times^{P_0} \mathfrak{p}_d \xleftarrow{\beta} G_{\underline{0}} \times \mathfrak{p}_d \xrightarrow{\gamma} \mathfrak{l}_d, $$

where

$$ \alpha(g,x) = \mathrm{Ad}(g)x; \quad \beta(g,x) = (g,x); \quad \gamma(g,x) = x \mod \mathfrak{u}_d. $$

Then, the morphism α is proper whereas γ is a smooth fibration and β is a P_0-torsor.

We have the following sequences

$$ \mathfrak{l}_d \xleftarrow{\alpha} G_{\underline{0}} \times^{U_0} \mathfrak{p}_d \xrightarrow{\beta} G_{\underline{0}} \times^{P_0} \mathfrak{p}_d \xrightarrow{\gamma} \mathfrak{g}_{\underline{d}}, \qquad \mathfrak{g}_{\underline{d}} \xleftarrow{\delta} \mathfrak{p}_d \xrightarrow{\varepsilon} \mathfrak{l}_d. $$

The spiral induction and restriction are defined as

$$ \begin{aligned} \mathrm{Ind}_{\mathfrak{l}_d\subset\mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} &= \gamma_*(\beta^!)^{-1}\alpha^! : \mathrm{D}_{L_0}(\mathfrak{l}_d) \longrightarrow \mathrm{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{d}}) \\ \mathrm{Res}_{\mathfrak{l}_d\subset\mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} &= \varepsilon_!\delta^* : \mathrm{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{d}}) \longrightarrow \mathrm{D}_{L_0}(\mathfrak{l}_d). \end{aligned} $$

In fact, for the sign $\epsilon = d/|d|$ the functors induce

$$ \mathrm{Ind}_{\mathfrak{l}_d\subset\mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \mathrm{D}_{L_0}\left(\mathfrak{l}_d^{\mathrm{nil}}\right) \longrightarrow \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right), \quad \mathrm{Res}_{\mathfrak{l}_d\subset\mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} : \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right) \longrightarrow \mathrm{D}_{L_0}\left(\mathfrak{l}_d^{\mathrm{nil}}\right). $$

Since every variety in the sequence (3.2.1) can be equipped with a $\mathbf{C}_q^{\times}$-action (by weight -2 on the Lie-algebraic part), there is also a $G_{\underline{0}} \times \mathbf{C}_q^{\times}$-equivariant version of the induction and restriction

$$ \begin{aligned} \mathrm{Ind}_{\mathfrak{l}_d\subset\mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} &: \mathrm{D}_{L_0\times\mathbf{C}_q^{\times}}(\mathfrak{l}_d) \longrightarrow \mathrm{D}_{G_{\underline{0}}\times\mathbf{C}_q^{\times}}(\mathfrak{g}_{\underline{d}}) \\ \mathrm{Res}_{\mathfrak{l}_d\subset\mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} &: \mathrm{D}_{G_{\underline{0}}\times\mathbf{C}_q^{\times}}(\mathfrak{g}_{\underline{d}}) \longrightarrow \mathrm{D}_{L_0\times\mathbf{C}_q^{\times}}(\mathfrak{l}_d). \end{aligned} $$

3.2.2 Adjunction

We can rephrase the definition of spiral induction and restriction as follows: there is a diagram of algebraic stacks

$$\begin{tikzcd}
\mathfrak{l}_d \arrow[d, "\breve{\beta}"] & \mathfrak{p}_d \arrow[l, "\varepsilon"'] \arrow[d, "\bar{\beta}"] & G_{\underline{0}} \times \mathfrak{p}_d \arrow[l, "\alpha"'] \arrow[d, "\beta"] \\
{[\mathfrak{l}_d/P_0]} & {[\mathfrak{p}_d/P_0]} \arrow[l, "\bar{\varepsilon}"'] \arrow[d, "\bar{\gamma}"] & G_{\underline{0}} \times^{P_0} \mathfrak{p}_d \arrow[l, "\bar{\alpha}"'] \arrow[d, "\gamma"] \\
{[\mathfrak{l}_d/L_0]} \arrow[u, "q"] & {[\mathfrak{g}_{\underline{d}}/G_{\underline{0}}]} & \mathfrak{g}_{\underline{d}} \arrow[l, "\breve{\alpha}"]
\end{tikzcd}$$

Then $\mathrm{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} = \bar{\gamma}_* \bar{\varepsilon}^! q_*$ and $\mathrm{Res}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} = q^* \bar{\varepsilon}_! \bar{\gamma}^*$.

It is clear that $\left(\mathrm{Res}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}}, \mathrm{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}}\right)$ is a pair of adjoint functors.

3.2.3 Admissible systems and block decomposition

From now on, we suppose that $\epsilon = d/|d|$ unless it is stated otherwise.

Recall that an admissible system on $\mathfrak{g}_{\underline{d}}$, as defined in [38], is a datum $(M, \mathfrak{m}_*, \mathrm{O}, \mathscr{C})$, where $M \subset G$ is a subgroup whose Lie algebra is equipped with a $\mathbf{Z}$-grading $\mathfrak{m}_* = \bigoplus_{n \in \mathbf{Z}} \mathfrak{m}_n$ arising as $\mathfrak{l}_*^\lambda$ as splitting of a spiral, $\mathrm{O} \subset \mathfrak{m}_d^{\mathrm{nil}}$ is a M_0-orbit and $\mathscr{C} \in \mathrm{Loc}_{M_0}(\mathrm{O})$ is a cuspidal local system in the sense of [37, 4.2(c),4.4(a)].

An isomorphism of admissible systems $(M, \mathfrak{m}_*, \mathrm{O}, \mathscr{C}) \cong (M', \mathfrak{m}'_*, \mathrm{O}', \mathscr{C}')$ is a pair (g, η) of element $g \in G_{\underline{0}}$ such that $gMg^{-1} = M'$, $\mathrm{Ad}_g \mathfrak{m}_n = \mathfrak{m}'_n$, $\mathrm{Ad}_g \mathrm{O} = \mathrm{O}'$ and an isomorphism $\eta : g^*\mathscr{C}' \cong \mathscr{C}$. Let $\mathfrak{T}(\mathfrak{g}_{\underline{d}})$ denote the groupoid of admissible systems on $\mathfrak{g}_{\underline{d}}$ and let $\underline{\mathfrak{T}}(\mathfrak{g}_{\underline{d}})$ denote the set of isomorphism classes its objects.

Let $\zeta = (M, \mathfrak{m}_*, \mathrm{O}, \mathscr{C}) \in \underline{\mathfrak{T}}(\mathfrak{g}_{\underline{d}})$ be an admissible system on $\mathfrak{g}_{\underline{d}}$. We define $\mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$ to be the Serre subcategory of $\mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)$ generated by the constituents of $\mathrm{Ind}_{\mathfrak{m}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \mathrm{IC}(\mathscr{C})$ for various spiral $\mathfrak{p}_*$ which has $\mathfrak{m}_*$ as splitting. We define $\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$ to be the full subcategory of $\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)$ of objects $\mathscr{K}$ such that ${}^p\mathscr{H}^k\mathscr{K} \in \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$ for all $k \in \mathbf{Z}$. We also let $\Pi(\mathfrak{g}_{\underline{d}})_\zeta \subset \Pi(\mathfrak{g}_{\underline{d}})$ be the subsets of pairs $(\mathrm{O}, \mathscr{L})$ such that $\mathrm{IC}(\mathscr{L}) \in \mathrm{Perv}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$. We will call these subcategories and subsets the **blocks** of ζ.

According to [38, 0.6], there are decompositions into subcategories

$$\mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right) = \bigoplus_{\zeta \in \underline{\mathfrak{T}}(\mathfrak{g}_{\underline{d}})} \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta, \qquad \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right) = \bigoplus_{\zeta \in \underline{\mathfrak{T}}(\mathfrak{g}_{\underline{d}})} \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta. \tag{3.1}$$

3.2.4

The following is an immediate consequence of the decompositions (3.1), the adjunction of restriction of induction functors and the transitivity of them.

Lemma 3.2. *Let $\mathfrak{p}_* = \mathfrak{p}_*^\lambda$ and $\mathfrak{l}_* = \mathfrak{l}_*^\lambda$ be spiral and splitting as above. Let $\zeta = (M, \mathfrak{m}_*, \mathscr{C})$ be an admissible system on $\mathfrak{l}_d$. Then the spiral induction and restriction preserve series:*

$$\mathrm{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} : \mathrm{D}_{L_0}\left(\mathfrak{l}_d^{\mathrm{nil}}\right)_\zeta \longrightarrow \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$$
$$\mathrm{Res}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} : \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta \longrightarrow \mathrm{D}_{L_0}\left(\mathfrak{l}_d^{\mathrm{nil}}\right)_\zeta .$$

On the other hand, if $\zeta' = (M', \mathfrak{m}'_*, \mathscr{C}')$ is an admissible sysmtem on $\mathfrak{g}_{\underline{d}}$ such that no $G_{\underline{0}}$-conjugate of $(M', \mathfrak{m}'_*)$ is a $\mathbf{Z}$-graded Levi of $(L, \mathfrak{l}_*)$, then $\mathrm{Res}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_{\zeta'} = 0$.

□

3.2.5 Spiral attached to a nilpotent element

Let $\phi = (e, h, f)$ be an $\mathfrak{sl}_2$-triplet with $e \in \mathfrak{g}_{\underline{d}}$, $h \in \mathfrak{g}_{\underline{0}}$ and $f \in \mathfrak{g}_{-\underline{d}}$. We can attach to ϕ a spiral $\mathfrak{p}_*^\phi$ and a splitting $\mathfrak{l}_*^\phi$ as follows: ϕ can be integrated to a homomorphism of algebraic groups $\phi : \mathrm{SL}_2 \longrightarrow G$. The cocharacter $\lambda \in \mathbf{X}_*(G_{\underline{0}})$ defined by $\lambda(t) = \phi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ gives rise to a spiral $\mathfrak{p}_*^{(d/2)\lambda}$, denoted by $\mathfrak{p}_*^\phi$, and a splitting $\mathfrak{l}_*^{(d/2)\lambda}$, denoted by $\mathfrak{l}_*^\phi$. The algebraic subgroups $P_0^\phi = \exp\left(\mathfrak{p}_0^\phi\right)$, $L^\phi = \exp\left(\mathfrak{l}_*^\phi\right)$ and $L_0^\phi = \exp\left(\mathfrak{l}_0^\phi\right)$ of G are well-defined. They are called **spiral**, and respectively **splitting**, **attached to** ϕ.

It is clear that $e \in \mathfrak{l}_d^\phi$, $h \in \mathfrak{l}_0^\phi$ and $f \in \mathfrak{l}_{-d}^\phi$. We will need the following results, proven in [38, 2.9]:

Lemma 3.3. *(i) The image of γ is $\overline{\mathrm{C}}$, the Zariski closure of C. Moreover, the morphism γ restricts to an isomorphism $\gamma^{-1}(\mathrm{C}) \cong \mathrm{C}$.*

(ii) There is an isomorphism $\pi_0\left(Z_{L_0}(e)\right) \cong \pi_0\left(Z_{G_{\underline{0}}}(e)\right)$.

□

3.2.6 Parity of cohomology

Proposition 3.4. *Let $(\mathrm{O}, \mathscr{C})$ be a cuspidal pair on $\mathfrak{l}_d$, let $\mathfrak{p}_*$ be a spiral on $\mathfrak{g}$ which admits $\mathfrak{l}_*$ as splitting and let $j_{\mathrm{O}'} : \mathrm{O}' \hookrightarrow \mathfrak{g}_{\underline{d}}$ be a $G_{\underline{0}}$-orbit. Then the complexes $j_{\mathrm{O}'}^* \mathrm{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \mathrm{IC}(\mathscr{C})$ and $j_{\mathrm{O}'}^! \mathrm{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \mathrm{IC}(\mathscr{C})$ have no cohomology in odd degrees.*

Proof. By Verdier duality and the fact that the dual $\mathscr{C}^\vee$ is again cuspidal, it suffices to prove the statement for $j_{\mathrm{O}'}^*$ only. The statement for $j_{\mathrm{O}'}^*$ follows from [39, 14.10]. □

3.2.7 Preservation of series under extension of sheaves

Lemma 3.5. *Let $(\mathrm{O}, \mathscr{L}) \in \Pi(\mathfrak{g}_{\underline{d}})_\zeta$ for $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$. Then $j_{\mathrm{O}!}\mathscr{L}, j_{\mathrm{O}*}\mathscr{L} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$.*

Proof. The decomposition of equivariant derived category yields a decomposition of complex

$$j_{\mathrm{O}*}\mathscr{L} \cong \bigoplus_{\zeta' \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})} (j_{\mathrm{O}*}\mathscr{L})_{\zeta'} .$$

Let $\zeta' = (M', \mathfrak{m}'_*, \mathrm{O}', \mathscr{C}')$ be any admissible system on $\mathfrak{g}_{\underline{d}}$ which is not isomorphic to ζ', we show that $(j_{\mathrm{O}*}\mathscr{C})_{\zeta'} = 0$. Suppose that $(j_{\mathrm{O}*}\mathscr{L})_{\zeta'} \neq 0$. Let $\delta \in \mathbf{Z}$ be the smallest integer such that ${}^p\mathscr{H}^\delta (j_{\mathrm{O}*}\mathscr{L})_{\zeta'} \neq 0$. Let $\mathrm{IC}(\mathscr{F}) \in \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_{\zeta'}$ be a simple subobject of ${}^p\mathscr{H}^\delta (j_{\mathrm{O}*}\mathscr{L})_{\zeta'} \neq 0$ so that $\mathrm{Hom}\left(\mathrm{IC}(\mathscr{F}), {}^p\tau^{\leq\delta}(j_{\mathrm{O}*}\mathscr{L})_{\zeta'}[\delta]\right) \neq 0$. Using the distinguished triangle coming from the perverse t-structure

$$ {}^p\tau^{\leq\delta}(j_{\mathrm{O}*}\mathscr{L})_{\zeta'} \longrightarrow (j_{\mathrm{O}*}\mathscr{L})_{\zeta'} \longrightarrow {}^p\tau^{>\delta}(j_{\mathrm{O}*}\mathscr{L})_{\zeta'} \xrightarrow{[1]}, $$

we see that the obvious map

$$ \mathrm{Hom}\left(\mathrm{IC}(\mathscr{F}), {}^p\tau^{\leq\delta}(j_{\mathrm{O}*}\mathscr{L})_{\zeta'}[\delta]\right) \longrightarrow \mathrm{Hom}\left(\mathrm{IC}(\mathscr{F}), (j_{\mathrm{O}*}\mathscr{L})_{\zeta'}[\delta]\right) = \mathrm{Hom}\left(j_{\mathrm{O}}^*\,\mathrm{IC}(\mathscr{F}), \mathscr{L}[\delta]\right) $$

is injective, so the last space is non-zero, which contradicts the fact that $\zeta \ncong \zeta'$ by [39, 13.8(a)]. Thus $j_{\mathrm{O}*}\mathscr{L} = (j_{\mathrm{O}*}\mathscr{L})_\zeta \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$.

If $\mathrm{IC}(\mathscr{L}) \in \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$, then $\mathrm{IC}(\mathscr{L}^\vee) \in \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_{\zeta^\vee}$ for $\zeta^\vee$ dual to ζ. Thus $j_{\mathrm{O}*}\mathscr{L}^\vee \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_{\zeta^\vee}$ so $j_{\mathrm{O}!}\mathscr{L} \cong \mathbf{D}j_{\mathrm{O}*}\mathscr{L}^\vee[2\dim \mathrm{O}] \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$ since the Verdier duality $\mathbf{D}$ exchanges the blocks ζ and $\zeta^\vee$. □

3.2.8

Proposition 3.6. *Let $\mathrm{O} \subset \mathfrak{g}^{\mathrm{nil}}_{\underline{d}}$ be a $G_{\underline{0}}$ orbit and let $j : \mathrm{O} \hookrightarrow \mathfrak{g}^{\mathrm{nil}}_{\underline{d}}$ be the inclusion. Then the functors j_*j^*, $j_!j^*$, $j_*j^!$ and $j_!j^!$ preserve the subcategory $\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$ for each $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$.*

Proof. It is enough to show that for every $\mathscr{K} = \mathrm{IC}(\mathscr{L}) \in \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$, we have

$$ j_*j^*\mathscr{K}, j_!j^*\mathscr{K}, j_*j^!\mathscr{K}, j_!j^!\mathscr{K} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta. $$

Let $\mathscr{L} \subset \mathscr{H}^k j^*\mathscr{K}$ be a simple constituent. Then we have $\mathrm{IC}(\mathscr{L}) \in \mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$ by [39, 13.8(a)]. By the previous lemma, we have $j_*\mathscr{L} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$. By devissage, we see that $j_*j^*\mathscr{K} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$. The verification of the other three is similar. □

3.3 Parabolic induction and restriction

3.3.1 Parabolic induction and restriction

Let $M \subset G$ be a pseudo-Levi subgroup. We say that M is **θ-isotropic** if M contains a θ-fixed maximal torus. Since θ is supposed to be inner, G is a θ-isotropic pseudo-Levi subgroup of itself. Conversely, for any θ-isotropic pseudo-Levi M, the action θ on G restricts to an inner action $\theta : \mu_m \longrightarrow M^{\mathrm{ad}}$ on M. If M is a θ-isotropic Levi subgroup, then every parabolic of G which admits M as Levi subgroup is stable under θ. If $Q \subset G$ is a θ-stable parabolic subgroup, then since Q is its own normaliser in G, the image of θ must lie in Q/Z_G. Consequently, Q has a θ-isotropic Levi factor.

Let $M \subset G$ be a θ-isotropic Levi subgroup and let $Q \subset G$ be a parabolic subgroup having M as Levi factor. The Lie subalgebras $\mathfrak{m} = \operatorname{Lie} M$ and $\mathfrak{q} = \operatorname{Lie} Q$ of $\mathfrak{g}$ are $\mathbf{Z}/m$-graded. We have the following sequences

$$\mathfrak{m}_{\underline{d}} \xleftarrow{\alpha} G_{\underline{0}} \times^{U_{\underline{0}}} \mathfrak{q}_{\underline{d}} \xrightarrow{\beta} G_{\underline{0}} \times^{Q_{\underline{0}}} \mathfrak{q}_{\underline{d}} \xrightarrow{\gamma} \mathfrak{g}_{\underline{d}}, \qquad \mathfrak{g}_{\underline{d}} \xleftarrow{\delta} \mathfrak{q}_{\underline{d}} \xrightarrow{\varepsilon} \mathfrak{m}_{\underline{d}}.$$

The parabolic induction and restriction are defined as

$$\begin{aligned} \operatorname{Ind}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} &= \gamma_*(\beta^!)^{-1}\alpha^! : \mathrm{D}_{M_{\underline{0}}}\left(\mathfrak{m}_{\underline{d}}^{\mathrm{nil}}\right) \longrightarrow \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right) \\ \operatorname{Res}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} &= \varepsilon_!\delta^* : \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right) \longrightarrow \mathrm{D}_{M_{\underline{0}}}\left(\mathfrak{m}_{\underline{d}}^{\mathrm{nil}}\right). \end{aligned}$$

As in §3.2.2, there is a pair of adjoint functors $\left(\operatorname{Res}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}}, \operatorname{Ind}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}}\right)$.

3.3.2 Transitivity of parabolic inductions

We state two transitivity properties of the parabolic induction functor. In addition to $\mathfrak{q}$ and $\mathfrak{m}$ as above, let $\mathfrak{q}' \subset \mathfrak{m}$ be a θ-stable parabolic subalgebra and let $\mathfrak{m}' \subset \mathfrak{q}'$ be a θ-stable Levi factor. Let $\tilde{\mathfrak{q}}'$ be the inverse image of $\mathfrak{q}'$ under the quotient map $\mathfrak{q} \longrightarrow \mathfrak{m}$.

Proposition 3.7. *There is an isomorphism of functors*

$$\operatorname{Ind}_{\mathfrak{m}'_{\underline{d}} \subset \tilde{\mathfrak{q}}'_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} \cong \operatorname{Ind}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} \circ \operatorname{Ind}_{\mathfrak{m}'_{\underline{d}} \subset \mathfrak{q}'_{\underline{d}}}^{\mathfrak{m}_{\underline{d}}}.$$

Proof. By adjunction, it suffices to show the corresponding statement for restrictions. The diagram

$$\begin{tikzcd} \mathfrak{g} & \mathfrak{q} \arrow[l, "\delta"'] \arrow[d, "\epsilon"] & \tilde{\mathfrak{q}}' \arrow[l, "\delta''"'] \arrow[d, "\epsilon''"] \arrow[dl, phantom, "\urcorner"] \\ & \mathfrak{m} & \mathfrak{q}' \arrow[l, "\delta'"'] \arrow[d, "\epsilon'"] \\ & & \mathfrak{m}' \end{tikzcd}$$

induces

$$\operatorname{Res}_{\mathfrak{m}'_{\underline{d}} \subset \mathfrak{q}'_{\underline{d}}}^{\mathfrak{m}_{\underline{d}}} \circ \operatorname{Res}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} = \epsilon'_!\delta'^*\epsilon_!\delta^* \cong \epsilon_!\epsilon''_!\delta''^*\delta^* = \epsilon'_!\delta'^*\epsilon_!\delta^* \cong (\epsilon\epsilon'')_!(\delta\delta'')^* = \operatorname{Res}_{\mathfrak{m}'_{\underline{d}} \subset \tilde{\mathfrak{q}}'_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}}.$$

□

If now instead of parabolic algebra, let $\epsilon \in \{\pm 1\}$, let $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}_*$ be a spiral of $\mathfrak{m}$, let $\mathfrak{l}_* \subset \mathfrak{p}_*$ be a splitting and let $\tilde{\mathfrak{p}}_n$ be the inverse image of $\mathfrak{p}_n$ under the quotient $\mathfrak{q}_n \longrightarrow \mathfrak{m}_n$ for each $n \in \mathbf{Z}$, then $\tilde{\mathfrak{p}}_*$ is a spiral of $\mathfrak{g}$. The following is proven in the same way as Proposition 3.7.

Proposition 3.8. *There is an isomorphism of functors*

$$\operatorname{Ind}_{\mathfrak{l}_d \subset \tilde{\mathfrak{p}}_d}^{\mathfrak{g}_{\underline{d}}} \cong \operatorname{Ind}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} \circ \operatorname{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{m}_{\underline{d}}}.$$

□

3.3.3 Preservation of series under parabolic inductions

We assume that $\epsilon = d/|d|$. The following is an immediate consequence of the transitivity Proposition 3.8 and the adjunction of parabolic restriction and induction functors.

Lemma 3.9. *Let $\mathfrak{q}_* = \mathfrak{q}_*^\lambda$ and $\mathfrak{m}_* = \mathfrak{m}_*^\lambda$ be parabolic and Levi subalgebras as above. Let $\zeta \in \mathfrak{T}(\mathfrak{m}_{\underline{d}})$ be an admissible system on $\mathfrak{m}_{\underline{d}}$, which can also be viewed as on $\mathfrak{g}_{\underline{d}}$. Then the parabolic induction and restriction preserve the block of ζ:*

$$\mathrm{Ind}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} : \mathrm{D}_{L_{\underline{0}}}\left(\mathfrak{m}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta \longrightarrow \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$$
$$\mathrm{Res}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} : \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta \longrightarrow \mathrm{D}_{L_{\underline{0}}}\left(\mathfrak{m}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta .$$

□

3.3.4 Parabolic subalgebra given by a cocharacter

Let $\lambda \in \mathbf{X}_*(G_{\underline{0}})_{\mathbf{Q}}$ be a fractional cocharacter. We associate to it the following $\mathbf{Z}/m$-graded Lie subalgebras of $\mathfrak{g}_*$: $\mathfrak{q}_*^\lambda = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{q}_i^\lambda$, $\mathfrak{m}_*^\lambda = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{m}_i^\lambda$ and $\mathfrak{v}_*^\lambda = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{v}_i^\lambda$ by setting

$$\mathfrak{q}_i^\lambda = \bigoplus_{\substack{r \in \mathbf{Q} \\ r \geq 0}} {}_r^\lambda\mathfrak{g}_i, \quad \mathfrak{m}_i^\lambda = {}_0^\lambda\mathfrak{g}_i, \quad \mathfrak{v}_i^\lambda = \bigoplus_{\substack{r \in \mathbf{Q} \\ r > 0}} {}_r^\lambda\mathfrak{g}_i.$$

Clearly, $\mathfrak{q}_*^\lambda$ is a $\mathbf{Z}/m$-graded Levi subalgebra of $\mathfrak{g}_*$ and $\mathfrak{q}_*^\lambda = \mathfrak{m}_*^\lambda \oplus \mathfrak{v}_*^\lambda$ is a Levi decomposition. It is clear that $M = \exp(\mathfrak{m}_*^\lambda)$ is a θ-isotropic Levi subgroup. Conversely, given any θ-isotropic Levi subgroup $M \subset G$ and any parabolic $Q \subset G$ which admits M as Levi factor, there exists $\lambda \in \mathbf{X}_*(G_{\underline{0}})$ such that $\mathfrak{q}_* = \mathfrak{q}_*^\lambda$ and $\mathfrak{m}_* = \mathfrak{m}_*^\lambda$.

3.3.5 Distinguished nilpotent elements

A nilpotent element $e \in \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ is said to be **θ-distinguished**[2] if $\left(Z_{G_{\underline{0}}}(e)/Z_G\right)^\circ$ is unipotent and a nilpotent $G_{\underline{0}}$-orbit in $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ is called θ-distinguished if it consists of θ-distinguished elements.

Lemma 3.10. *Let $e \in \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ be θ-distinguished. Then any θ-isotropic pseudo-Levi subgroup $H \subset G$ such that $e \in \mathfrak{h}_{\underline{d}}$ is of same semisimple rank as G.*

Proof. Let $T \subset H$ be a θ-fixed maximal torus. Suppose that the semisimple rank of H is strictly smaller than G. Then there is a strict Levi subgroup $K \subsetneq G$ such that $H \subset K$ exists $\lambda \in \mathbf{X}_*(T)$ such that $K = Z_G(\varphi)$. In particular, $\operatorname{im}\varphi \subset Z_{G_{\underline{0}}}(e)$ is a torus not contained in Z_G, contradiction to the assumption on e. Therefore H is of same semisimple rank as G. □

[2]Recall that in the ungraded setting, a nilpotent element $e \in \mathfrak{g}^{\mathrm{nil}}$ is called distinguished if $(Z_G(e)/Z_G)^\circ$ is unipotent and a nilpotent orbit G-orbit in $\mathfrak{g}^{\mathrm{nil}}$ is called distinguished if it consists of distinguished elements.

3.4 Supercuspidal pairs

3.4.1 Supercuspidal pairs

Let $\underline{d} \in \mathbf{Z} \setminus \{0\}$ and let $(\mathrm{O}, \mathscr{C}) \in \Pi(\mathfrak{g}_{\underline{d}})$. In the rest of this chapter, we assume that $\epsilon = d/|d|$.

Definition 3.11. *We say that* $(\mathrm{O}, \mathscr{C})$ *is a* **supercuspidal pair** *on* $\mathfrak{g}_{\underline{d}}$ *if for every* θ*-stable proper parabolic subalgebra* $\mathfrak{q} \subsetneq \mathfrak{g}$ *with* θ*-isotropic Levi factor* $\mathfrak{m}$ *and unipotent radical* $\mathfrak{v}$ *and any* $z \in \mathfrak{m}_{\underline{d}}^{\mathrm{nil}}$*, we have* $\mathrm{R}\Gamma_c(\mathrm{O} \cap (z + \mathfrak{v}_{\underline{d}}), \mathscr{C}) = 0$.

3.4.2 Supercuspidal orbits are θ-distinguished

We prove a $\mathbf{Z}/m$-graded version of a result of Lusztig [30, 2.8][3].

Lemma 3.12. *Let* $0 \neq \mathscr{K} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)$ *be such that* $\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}} \mathscr{K} = 0$ *for every* θ*-stable proper parabolic subalgebra* $\mathfrak{q} \subsetneq \mathfrak{g}$ *with* θ*-isotropic Levi factor* $\mathfrak{m} \subset \mathfrak{q}$*. Then every* $G_{\underline{0}}$*-orbit* $\mathrm{O} \subset \mathrm{Supp}\,\mathscr{K}$ *open in* $\mathrm{Supp}\,\mathscr{K}$ *is* θ*-distinguished.*

Proof. Let $e \in \mathrm{O}$ and let $S \subset Z_{G_{\underline{0}}/Z_G}(e)$ be a maximal torus. Let $M = Z_G(S)$, which is θ-isotropic, let Q be a parabolic subgroup of G which has M as Levi subgroup and let $V \subset Q$ be the unipotent radical. Since θ fix S pointwise, the image of θ lies in M/Z_G. It follows that Q is θ-stable.

We claim that the $V_{\underline{0}}$-orbit $\mathrm{Ad}_{V_{\underline{0}}}(e)$ is a connected component of $\mathrm{O} \cap (e + \mathfrak{v}_{\underline{d}})$. By [30, 2.9(a)], $\mathrm{Ad}_V(e)$ is a connected component of $\mathrm{Ad}_G(e) \cap (e + \mathfrak{v})$. Since

$$\mathrm{Ad}_V(e) \cap (\mathrm{O} \cap (e + \mathfrak{v}_{\underline{d}})) = \mathrm{Ad}_V(e) \cap \mathfrak{g}_{\underline{d}},$$

the latter is open and closed in $\mathrm{O} \cap (e + \mathfrak{v}_{\underline{d}})$, so it is enough to show that the map

$$\eta_0 : V_{\underline{0}} \longrightarrow \mathrm{Ad}_V(e) \cap \mathfrak{g}_{\underline{d}}, \quad u \mapsto \mathrm{Ad}_u(e)$$

is surjective. Let

$$\eta : V \longrightarrow \mathrm{Ad}_V(e), \quad u \mapsto \mathrm{Ad}_u(e).$$

Suppose that $z \in \mathrm{Ad}_V(e) \cap \mathfrak{g}_{\underline{d}}$. Then $\eta^{-1}(z) \subset V$ is θ-stable. Since $\eta^{-1}(z)$ is a left coset of $Z_V(e)$, it is an affine space, acted on by a finite group μ_m through θ. It follows that there exist θ-fixed points in $\eta^{-1}(z)$, or in other words, $\eta_0(z) \neq \emptyset$. Hence η_0 is surjective and therefore $\mathrm{Ad}_{V_{\underline{0}}}(e)$ is a connected component of $\mathrm{O} \cap (e + \mathfrak{v}_{\underline{d}})$.

Since $\pi_0\left(Z_{V_{\underline{0}}}(e)\right) = 1$, the restriction $\mathscr{K}\,|_{\mathrm{Ad}_{V_{\underline{0}}}(e)}$, being $V_{\underline{0}}$-equivariant, must be constant on the connected component $\mathrm{Ad}_{V_{\underline{0}}}(e)$. It follows that $\mathrm{R}\Gamma_c\left(\mathrm{Ad}_{V_{\underline{0}}}(e), \mathscr{K}\,|_{\mathrm{Ad}_{V_{\underline{0}}}(e)}\right) \neq 0$ and hence $\mathrm{R}\Gamma_c\left(\mathrm{O} \cap (e + \mathfrak{v}_{\underline{d}}), \mathscr{K}\,|_{\mathrm{O} \cap (e + \mathfrak{v}_{\underline{d}})}\right) \neq 0$ since the former is a direct factor of the latter.

Let $\mathfrak{q}_{\underline{d}} \xrightarrow{\beta} \mathfrak{m}_{\underline{d}}$ be the projection onto Levi factor. Then, since $e \in \mathrm{O} \cap \mathfrak{m}_{\underline{d}}$ and since $\mathrm{O} \subset \mathrm{Supp}\,\mathscr{K}$ is open, we have $\mathrm{Supp}\,\mathscr{K} \cap \beta^{-1}(e) = \mathrm{O} \cap (\{e\} + \mathfrak{v}_{\underline{d}})$. Therefore the fiber in e of

[3] It states that if G is a reductive group and $\mathrm{O} \subset \mathfrak{g}^{\mathrm{nil}}$ is a nilpotent G-orbit supporting a cuspidal local system, then O is distinguished.

the complex $\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}}\subset\mathfrak{q}_{\underline{d}}}\mathscr{K}=\beta_!\left(\mathscr{K}\mid_{\mathfrak{q}_{\underline{d}}}\right)$ is isomorphic to $\mathrm{R}\Gamma_c\left(\mathrm{O}\cap(e+\mathfrak{v}_{\underline{d}}),\mathscr{K}\mid_{\mathrm{O}\cap(e+\mathfrak{v}_{\underline{d}})}\right)$, which is non-vanishing.

By the assumption on $\mathscr{K}$, we must have $G=M=Z_G(S)$. Thus $S=1$ and $Z_{G_{\underline{0}}/Z_G}(e)$ is unipotent. It follows that e is θ-distinguished. □

Proposition 3.13. *Let $(\mathrm{O},\mathscr{C})$ be a supercuspidal pair on $\mathfrak{g}_{\underline{d}}$. Then the $G_{\underline{0}}$-orbit $\mathrm{O}\subset\mathfrak{g}_{\underline{d}}$ is θ-distinguished.*

Proof. Apply Lemma 3.12 to $\mathscr{K}=j_{\mathrm{O}!}\mathscr{C}$, where $j_{\mathrm{O}}:\mathrm{O}\longrightarrow\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}$ is the inclusion. □

3.4.3 Primitive pairs

Let $\zeta=(L,\mathfrak{l}_*,\mathrm{O}_L,\mathscr{C}_L)\in\mathfrak{T}(\mathfrak{g}_{\underline{d}})$ be an admissible system on $\mathfrak{g}_{\underline{d}}$. Let $\mathrm{O}=\mathrm{Ad}_{G_{\underline{0}}}\mathrm{O}_L\subset\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}$ be the *extension* of O_L to $G_{\underline{0}}$-orbit. Let $x\in\mathrm{O}_L$. By [38, 3.8(a)], it is known that the inclusion $Z_{L_0}(x)\subset Z_{G_{\underline{0}}}(x)$ induces an isomorphism on groups of connected components. Therefore, there is a unique extension of $\mathscr{C}_L$ into a $G_{\underline{0}}$-local system on O, denoted by $\mathscr{L}$. The pair $(\mathrm{O},\mathscr{C})$ is called the **primitive pair** attached to ζ, according to *loc. cit.*. Below is a characterisation of the primitive pair.

Lemma 3.14. *Let $(\mathrm{O},\mathscr{C})$ denote the primitive pair on $\mathfrak{g}_{\underline{d}}$ attached to ζ. Let $(\mathrm{C},\mathscr{L})\in\Pi(\mathfrak{g}_{\underline{d}})_\zeta$ a pair in the block of ζ. Suppose that $\mathrm{C}\subset\overline{\mathrm{O}}$. Then $(\mathrm{C},\mathscr{L})\cong(\mathrm{O},\mathscr{C})$.*

Proof. Suppose first that $\mathrm{C}\neq\mathrm{O}$. Let $j_{\mathrm{C}}:\mathrm{C}\hookrightarrow\mathfrak{g}_{\underline{d}}$ denotes the inclusion of orbit. Let $\mathfrak{p}_*$ be any spiral in $\mathfrak{g}$ which admits $\mathfrak{l}_*$ as splitting. The support of the spiral restriction complex $\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}j_{\mathrm{C}!}\mathscr{L}$ is contained in $\overline{\mathrm{O}}_L\setminus\mathrm{O}_L$. Thus the cleaness [34] of the cuspidal local system $\mathscr{C}_L$ implies that

$$\mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(j_{\mathrm{C}!}\mathscr{L},\mathrm{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}\mathrm{IC}(\mathscr{C}_L)\right)\cong\mathrm{Hom}^\bullet_{L_0}\left(\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}j_{\mathrm{C}!}\mathscr{L},\mathrm{IC}\left(\mathscr{C}_L\right)\right)=0.$$

It follows that $j_{\mathrm{C}!}\mathscr{L}=0$ since the simple constituents of $\mathrm{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}\mathrm{IC}(\mathscr{C}_L)$ for all $\mathfrak{p}_*$ together generate the block $\mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$. The full-faithfullness of $j_{\mathrm{C}!}$ yields $\mathscr{L}=0$, contradiction.

We have seen that $\mathrm{C}=\mathrm{O}$. Let $\mathfrak{p}_*$ be as above. It follows that the complex $\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}j_{\mathrm{C}!}\mathscr{L}$ is isomorphic to $j_{L!}\left(\mathscr{L}\mid_{\mathrm{O}_L}\right)$, where $j_L:\mathrm{O}_L\longrightarrow\mathfrak{l}_d$ is the inclusion of orbit. If $\mathscr{L}\ncong\mathscr{C}$, then $\mathscr{L}\mid_{\mathrm{O}_L}\ncong\mathscr{C}_L$. Again, $\mathrm{Hom}^\bullet_{L_0}\left(\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}i_!\mathscr{L},\mathscr{C}_L\right)=0$ and the adjuction yields

$$\mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(i_!\mathscr{L},\mathrm{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d\subset\mathfrak{p}_d}\mathscr{C}_L\right)=0,$$

contradiction. It follows that $\mathscr{L}\cong\mathscr{C}$ and thus $\mathrm{IC}(\mathscr{L})\cong\mathrm{IC}\left(\mathscr{C}\right)$. □

3.4.4 Characterisation of supercuspidal pairs

Theorem 3.15. *Let $(\mathrm{O},\mathscr{C})\in\Pi(\mathfrak{g}_{\underline{d}})$. The following conditions are equivalent*

(i) The pair $(\mathrm{O},\mathscr{C})$ is supercuspidal

(ii) There exists an admissible system $\zeta=(L,\mathfrak{l}_*,\mathrm{O}_L,\mathscr{C}_L)\in\mathfrak{T}(\mathfrak{g}_{\underline{d}})$ such that L is of the same semisimple rank as G and $(\mathrm{O},\mathscr{C})$ is the primitive pair attached to ζ.

Proof. Suppose first that (ii) holds. Let $M \subsetneq G$ be any θ-isotropic strict Levi subgroup and let $Q \subset G$ be a parabolic subgroup which admits M as Levi factor. Let $j_{\mathrm{O}} : \mathrm{O} \longrightarrow \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ be the inclusion of orbit. It follows that $j_{\mathrm{O}!}\mathscr{C} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_{\zeta}$ and thus

$$\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}} j_{\mathrm{O}!}\mathscr{C} = 0 \in \mathrm{D}_{M_{\underline{0}}}\left(\mathfrak{m}_{\underline{d}}^{\mathrm{nil}}\right)_{\zeta}$$

as there is no conjugate of L contained in M (the semisimple rank of L being greater than that of M). It follows that $(\mathrm{O}, \mathscr{C})$ is supercuspidal.

Suppose then that (i) holds. Let $\zeta = (L, \mathfrak{l}_*, \mathrm{O}_L, \mathscr{C}_L) \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ such that $(\mathrm{O}, \mathscr{C}) \in \Pi(\mathfrak{g}_{\underline{d}})_{\zeta}$. By Lemma 3.16 below, $(\mathrm{O}, \mathscr{C})$ is isomorphic to the primitive pair attached to ζ. By Lemma 3.12, O is θ-distinguished, so by Lemma 3.10, H is of the same semisimple rank as G. □

Lemma 3.16. *Let $\zeta = (L, \mathfrak{l}_*, \mathrm{O}_L, \mathscr{C}_L) \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ and let $0 \neq \mathscr{K} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_{\zeta}$ be such that $\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}} \mathscr{K} = 0$ for every θ-isotropic Levi $M \subsetneq G$ and any parabolic Q having M as Levi factor.*

(i) There is an isomorphism

$$\mathscr{K} \cong \bigoplus_{n \in \mathbf{Z}} \mathrm{IC}(\mathscr{C})[-n]^{\oplus d_n},$$

where $d_n \in \mathbf{N}$ for each $n \in \mathbf{Z}$ and $(\mathrm{O}, \mathscr{C})$ is the primitive pair attached to ζ.

(ii) The semisimple rank of L is equal to that of G.

Proof. We first prove that $\mathscr{K}$ is supported on the closure of the primitive orbit $\mathrm{Ad}_{G_{\underline{0}}} \mathrm{O}_L$. Let $\mathrm{O} \subset \mathrm{Supp}\, \mathscr{K}$ be an open orbit. Let (e, h, f) be an $\mathfrak{sl}_2$-triplet with $e \in \mathrm{O}$, $h \in \mathfrak{g}_{\underline{0}}$ and $f \in \mathfrak{g}_{-\underline{d}}$ and define $\varphi \in \mathbf{X}_*(G_{\underline{0}})$ by $\varphi : t \mapsto t^h$. For $n \in \mathbf{Z}$, define

$$\mathfrak{p}_n = \bigoplus_{m \geq 2n\epsilon} {}^{d\epsilon \cdot \varphi}_{m}\mathfrak{g}_{\underline{n}}, \quad \mathfrak{l}_n = {}^{d\epsilon \cdot \varphi}_{2n\epsilon}\mathfrak{g}_{\underline{n}}$$

for $n \in \mathbf{Z}$ so that $e \in \mathfrak{l}_d$. Then $\mathfrak{l} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{l}_n$ is a $\mathbf{Z}$-graded pseudo-Levi subalgebra of $\mathfrak{g}$. Let $L_0 = \exp(\mathfrak{l}_0)$ and let $\mathrm{O}_L = \mathrm{Ad}_{L_0}(e)$ and let $\mathscr{C}_L \subset \mathscr{K}\,|_{\mathrm{O}_L}$ be a simple constituent. The result [38, 2.9(c)] says that $Z_{L_0}(e)$ is a maximal reductive subgroup of $Z_{G_{\underline{0}}}(e)$, so that the inclusion $L_0 \subset G_{\underline{0}}$ induces an isomorphism $\pi_0\left(Z_{L_0}(e)\right) \cong \pi_0\left(Z_{G_{\underline{0}}}(e)\right)$. Therefore the L_0-equivariant local system $\mathscr{C}_L$ is irreducible.

We show that $(\mathrm{O}_L, \mathscr{C}_L)$ is cuspidal. Suppose that $\mathfrak{q}_L \subsetneq \mathfrak{l}$ be a $\mathbf{Z}$-graded strict parabolic subalgebra given by

$$\mathfrak{q}_L = \bigoplus_{n \geq 0} {}^{\psi}_{n}\mathfrak{l}$$

for some $\psi \in \mathbf{X}_*(H_0)$. Let $\mathfrak{m}_L = {}^{\psi}_{0}\mathfrak{l}$ be the Levi factor and let $\mathfrak{v}_L \subset \mathfrak{q}_L$ be the unipotent radical. Similarly, let $\mathfrak{q}, \mathfrak{m}, \mathfrak{v}$ be the θ-stable subalgebras of $\mathfrak{g}$ is given by

$$\mathfrak{q} = \bigoplus_{n \geq 0} {}^{\psi}_{n}\mathfrak{g}, \quad \mathfrak{m} = {}^{\psi}_{0}\mathfrak{g}, \mathfrak{v} = \bigoplus_{n > 0} {}^{\psi}_{n}\mathfrak{g}.$$

We must have $\mathfrak{q} \subsetneq \mathfrak{g}$ since $\mathfrak{q}_L = \mathfrak{q} \cap \mathfrak{l} \subsetneq \mathfrak{l}$.

Put $\tilde{\varphi} : \mathbf{C}^\times_\varphi \xrightarrow{(\varphi,-1)} G_{\underline{0}} \times \mathbf{C}^\times_q$. Then $\mathfrak{l}_d = (\mathfrak{g}_{\underline{d}})^{\tilde{\varphi}}$ and $\mathrm{O}_L = \mathrm{O}^{\tilde{\varphi}}$ are the fixed points by $\tilde{\varphi}$. Fix any $z \in \mathfrak{l}_d$ and let

$$X = \mathrm{O} \cap (z + \mathfrak{v}_{\underline{d}}) \subset \mathfrak{g}_{\underline{d}} \quad \text{so that} \quad X^{\tilde{\varphi}} = \mathrm{O}_L \cap (z + \mathfrak{v}_{L,d}) .$$

By the assumption on $\mathscr{K}$, we have $\mathrm{H}^\bullet_c(X, \mathscr{K}\mid_X) = 0$, which implies $\mathrm{H}^\bullet_c\left(X^{\tilde{\varphi}}, \mathscr{K}\mid_{X^{\tilde{\varphi}}}\right) = 0$ by [34, lemma 2]. Since $\mathscr{C}_L$ is a direct factor of $\mathscr{K}\mid_{\mathrm{O}_L}$, we deduce $\mathrm{H}^\bullet_c\left(X^{\tilde{\varphi}}, \mathscr{C}_L\mid_{X^{\tilde{\varphi}}}\right) = 0$, whence the cuspidality of $(\mathrm{O}_L, \mathscr{C}_L)$.

Let $j_{\mathrm{O}} : \mathrm{O} \longrightarrow \mathfrak{g}_{\underline{d}}$ be the inclusion of orbit. It is shown in [38, 7.1(c)] that $j^*_{\mathrm{O}} \operatorname{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d \subset \mathfrak{p}_d} \mathrm{IC}(\mathscr{C}_L)$ is the unique extension of $\mathscr{C}_L$ to O, so

$$\operatorname{Hom}^\bullet_{G_{\underline{0}}}\left(\operatorname{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{l}_d \subset \mathfrak{p}_d} \mathrm{IC}(\mathscr{C}_L), j_{\mathrm{O}*} j^*_{\mathrm{O}} \mathscr{K}\right) \neq 0$$

and thus the admissible system $(L, \mathfrak{l}_*, \mathrm{O}_L, \mathscr{C}_L)$ is conjugate to ζ and in particular, $\mathrm{O} = \mathrm{Ad}_{G_{\underline{0}}} \mathrm{O}_L$ is the primitive orbit attached to ζ. It follows that $\mathscr{K}$ is supported on the closure of O.

Now Lemma 3.14 implies that for any $k \in \mathbf{Z}$ we have ${}^p\mathscr{H}^n\mathscr{K} \cong \mathrm{IC}(\mathscr{C})^{\oplus d_n}$ for some $d_n \in \mathbf{N}$. Since by the odd vanishing of [39, 14.8], there is no self-extension of $\mathrm{IC}(\mathscr{C})$ in the equivariant derived category, we conclude that

$$\mathscr{K} \cong \bigoplus_{n \in \mathbf{Z}} \mathrm{IC}(\mathscr{C})[-n]^{\oplus d_n}.$$

□

3.4.5 Cleaness of primitive pairs

Finally, we prove the cleaness for primitive pairs on $\mathfrak{g}_{\underline{d}}$. Let $\zeta = (M, \mathfrak{m}_*, \mathrm{O}_M, \mathscr{C}_M) \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ be a admissible system and let $(\mathrm{O}, \mathscr{C}) \in \Pi(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}})_\zeta$ be the primitive pair attached to ζ. Let $j : \mathrm{O} \hookrightarrow \mathfrak{g}_{\underline{d}}$ denote the inclusion of orbit and let $i : \overline{\mathrm{O}} \setminus \mathrm{O} \hookrightarrow \mathfrak{g}_{\underline{d}}$ denote the complement of O in its closure.

Theorem 3.17. *We have $i^! j_! \mathscr{C} = i^* j_* \mathscr{C} = 0$. Consequently, the natural morphisms $j_! \mathscr{C}[\dim \mathrm{O}] \longrightarrow \mathrm{IC}(\mathscr{C}) \longrightarrow j_* \mathscr{C}[\dim \mathrm{O}]$ are isomorphisms.*

Proof. Let $\mathscr{K} = i_* i^! j_! \mathscr{C} \in \mathrm{D}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)_\zeta$. Applying Lemma 3.14 to simple factors of the the perverse cohomologies of $\mathscr{K}$, we see that $\mathscr{K} = 0$ and thus $i^! j_! \mathscr{C} = 0$. Similarly, we have $i^* j_* \mathscr{C} = 0$.

Now it is clear from above that $j_! \mathscr{C}[\dim \mathrm{O}]$ and $j_* \mathscr{C}[\dim \mathrm{O}]$ are perverse and they are isomorphic via the natural morphism $j_! \mathscr{C}[\dim \mathrm{O}] \longrightarrow j_* \mathscr{C}[\dim \mathrm{O}]$. The isomorphism of $\mathrm{IC}(\mathscr{C})$ with the two complexes results from the definition of $\mathrm{IC}(\mathscr{C})$. □

According to Theorem 3.15, supercuspidal pairs are primitive pairs. Thus this theorem applies to supercuspidal pairs.

3.5 Supercuspidal systems and blocks

In this section, we give an alternative description of blocks of $\mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)$ in terms of supercuspidal local systems on θ-isotropic Levi subalgebras.

3.5.1 Supercuspidal systems

Definition 3.18. *A* **supercuspidal system** *on* $\mathfrak{g}_{\underline{d}}$ *is a triplet* $(M, \mathrm{O}, \mathscr{C})$*, where* $M \subset G$ *is a* θ*-isotropic Levi subgroup and* $(\mathrm{O}, \mathscr{C})$ *is a supercuspidal pair on* $\mathfrak{m}_{\underline{d}}$*.*

An isomorphism of supercuspidal systems $(M, \mathrm{O}, \mathscr{C}) \cong (M', \mathrm{O}', \mathscr{C}')$ is a pair (g, η) of element $g \in G_{\underline{0}}$ such that $\mathrm{Ad}_g M = M'$, $\mathrm{Ad}_g \mathrm{O} = \mathrm{O}'$ and an isomorphism $\eta : g^*\mathscr{C}' \cong \mathscr{C}$. Let $\mathfrak{S}(\mathfrak{g}_{\underline{d}})$ denote the groupoid of supercuspidal systems on $\mathfrak{g}_{\underline{d}}$ and let $\underline{\mathfrak{S}}(\mathfrak{g}_{\underline{d}})$ denote the set of isomorphism classes of its objects.

Let $\sigma = (M, \mathrm{O}, \mathscr{C}) \in \underline{\mathfrak{S}}(\mathfrak{g}_{\underline{d}})$ be a supercuspidal system on $\mathfrak{g}_{\underline{d}}$. We define $\Pi(\mathfrak{g}_{\underline{d}})_\sigma \subset \Pi(\mathfrak{g}_{\underline{d}})$ to be the subset of pairs $(\mathrm{C}, \mathscr{L})$ for which there exists a parabolic $Q \subset G$ which has M as Levi and that

$$\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}\left(\mathscr{L}, j^!_{\mathrm{C}} \mathrm{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}} \mathrm{IC}(\mathscr{C})\right) \neq 0.$$

3.5.2 Partition into supercuspidal series

Proposition 3.19. *For any pair* $(\mathrm{C}, \mathscr{L}) \in \Pi(\mathfrak{g}_{\underline{d}})$*. There exists a supercuspidal system* $\sigma = (M, \mathrm{O}, \mathscr{C})$ *such that* $(\mathrm{C}, \mathscr{L}) \in \Pi(\mathfrak{g}_{\underline{d}})_\sigma$*.*

Proof. Let (Q, M) be a pair of θ-stable parabolic subgroup subgroup Q of G and a θ-isotropic Levi factor M of Q which satisfies the following two conditions:

(i) if $V \subset Q$ denotes the unipotent radical and $\mathfrak{v} = \mathrm{Lie}\, V$, then for any $z \in \mathrm{C}$, we have

$$\mathrm{R}\Gamma_c\left(\mathrm{C} \cap (z + \mathfrak{v}_{\underline{d}}), \mathscr{L}\,|_{\mathrm{C} \cap (z + \mathfrak{v}_{\underline{d}})}\right) \neq 0$$

(ii) among those having the first property, the pair (Q, M) is minimal with respect to the partial order:

$$(Q, M) \leq (Q', M') \quad \Leftrightarrow \quad Q \subset Q' \quad \wedge \quad M \subset M'.$$

We have a diagram

$$\begin{tikzcd}
 & \mathrm{C} \cap \mathfrak{q}_{\underline{d}} \arrow[r, "i'_{\mathfrak{q}}"] \arrow[d, "j'_{\mathrm{C}}"] & \mathrm{C} \arrow[d, "j_{\mathrm{C}}"] \\
\mathfrak{m}_{\underline{d}} & \mathfrak{q}_{\underline{d}} \arrow[l, "\pi_{\mathfrak{q}}"] \arrow[r, "i_{\mathfrak{q}}"] & \mathfrak{g}_{\underline{d}}
\end{tikzcd}$$

Put $\mathscr{K} = (\pi_{\mathfrak{q}} j'_{\mathrm{C}})_! i'^*_{\mathfrak{q}} \mathscr{L}$, which is isomorphic to $\mathrm{Res}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}} j_{\mathrm{C},!} \mathscr{L}$ by the base change theorem for proper morphisms. For any $z \in \mathrm{C} \cap \mathfrak{q}_{\underline{d}}$, we have

$$(\pi_{\mathfrak{q}} j'_{\mathrm{C}})^{-1}(\pi_{\mathfrak{q}}(z)) = \mathrm{C} \cap (z + \mathfrak{v}_{\underline{d}}).$$

Thus by the assumption (i) on the pair (Q, M) and by the base change theorem for proper morphisms, the complex $\mathscr{K}$ is non-zero.

Let $\zeta = (M', \mathfrak{m}'_*, \mathrm{O}', \mathscr{C}') \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ be an admissible system such that $(\mathrm{C}, \mathscr{L}) \in \Pi(\mathfrak{g}_{\underline{d}})_\zeta$. Since $0 \neq \mathscr{K} \in \mathrm{D}_{M_{\underline{0}}}(\mathfrak{m}_{\underline{d}}^{\mathrm{nil}})_\zeta$, we may suppose that $\mathfrak{m}'_*$ is a $\mathbf{Z}$-graded pseudo-Levi subalgebra of $\mathfrak{m}$. Let $\mathrm{O} \subset \operatorname{Supp} \mathscr{K}$ be an $M_{\underline{0}}$-orbit which is open in $\operatorname{Supp} \mathscr{K}$.

Let $(\mathrm{O}_M, \mathscr{C}_M)$ be the primitive pair on $\mathfrak{m}_{\underline{d}}$ attached to ζ. Lemma 3.16(i) implies that for any $k \in \mathbf{Z}$ we have ${}^p\mathscr{H}^k\mathscr{K} \cong \mathrm{IC}(\mathscr{C}_M)^{\oplus m_k}$ for some $m_k \in \mathbf{N}$ so that

$$\operatorname{Hom}^\bullet_{G_{\underline{0}}}\left(j_{\mathrm{C}!}\mathscr{L}, \operatorname{Ind}^{\mathfrak{g}_{\underline{d}}}_{\mathfrak{m}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}} \mathrm{IC}(\mathscr{C}_M)\right) \cong \operatorname{Hom}^\bullet_{M_{\underline{0}}}(\mathscr{K}, \mathrm{IC}(\mathscr{C}_M)) \neq 0.$$

Lemma 3.16(ii) implies that M is of the same semisimple rank as G. Therefore $(\mathrm{O}_M, \mathscr{C}_M)$ supercuspidal by Theorem 3.15. We find that $(\mathrm{C}, \mathscr{L}) \in \Pi(\mathfrak{g}_{\underline{d}})_\sigma$ for $\sigma = (M, \mathrm{O}_M, \mathscr{C}_M)$. □

Theorem 3.20. *The following assertions hold:*

(i) There is an equivalence of groupoids $\mathfrak{T}(\mathfrak{g}_{\underline{d}}) \cong \mathfrak{S}(\mathfrak{g}_{\underline{d}})$, denoted by $\zeta \mapsto \sigma_\zeta$.

(ii) For each admissible system $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ with corresponding supercuspidal system $\sigma_\zeta \in \mathfrak{S}(\mathfrak{g}_{\underline{d}})$, we have $\Pi(\mathfrak{g}_{\underline{d}})_\zeta = \Pi(\mathfrak{g}_{\underline{d}})_{\sigma_\zeta}$.

□

Chapter 4

Generalised Springer correspondence for cyclicly graded Lie algebras

Introduction

In the present chapter, we establish a generalised Springer correspondence in the case where the Lie algebra is $\mathbf{Z}/m$-graded, which was conjectured by Lusztig–Yun [40]. The main result is Theorem 4.40, which confirms the multiplicity-one conjecture proposed in [40] and can be viewed as a generalised Springer correspondence in the sense of Lusztig [30][31], for certain degenerate double affine Hecke algebras (dDAHAs) with possibly unequal parameters.

In [40], the authors have attached for each admissible system ζ an affine root system and a dDAHA $\mathbb{H}_\zeta$. The main result of [40] is the construction of an action of $\mathbb{H}_\zeta$ on an infinite sum of the spiral induction from various spirals

$$\mathbf{I} = \bigoplus_{\mathfrak{p}_*} \bigoplus_{k} {}^p\mathscr{H}^k \operatorname{Ind}_{\mathfrak{l}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \mathscr{C}$$

which is an ind-object in the category of $G_{\underline{0}}$-equivariant perverse sheaves on $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$. They have conjectured that for each simple constituent $\mathcal{S}$ of $\mathbf{I}$, the $\mathbb{H}_\zeta$-module $\operatorname{Hom}(\mathcal{S}, \mathbf{I})$ is irreducible. Theorem 4.40 confirms this conjecture. The case where M is a maximal torus and $\mathscr{C}$ is trivial has earlier been studied by Vasserot in [53].

Our strategy is very close to that of [53]. We use the technique of convolution algebras developed in Chriss–Ginzburg [13], Lusztig [31],[36] and Evens–Mirković [15].

Firstly, we can attach to the datum $(G, G_{\underline{0}}, \mathfrak{g}, \mathfrak{g}_*)$ a loop group $\mathrm{L}G$ and a loop algebra $\mathrm{L}\mathfrak{g}$. There is an one-dimension torus $\mathbf{C}^\times$ acting on $\mathrm{L}G$ and $\mathrm{L}\mathfrak{g}$, compatible with the adjoint action of $\mathrm{L}G$ on $\mathrm{L}\mathfrak{g}$, such that the fixed points are given by $G_{\underline{0}}$ and $\mathfrak{g}_{\underline{d}}$. The quadruple $(M, M_0, \mathfrak{m}, \mathfrak{m}_*)$ defines a Levi subgroup M of the loop group $\mathrm{L}G$. The subgroup M and the $\mathbf{Z}$-graded Lie algebra $\mathfrak{m}$ is stable under the $\mathbf{C}^\times$-action and the fixed points are given by M_0 and $\mathfrak{m}_d$. Similarly, any spiral $\mathfrak{p}_*$ define a $\mathbf{C}^\times$-stable parahoric subgroup $P \subseteq \mathrm{L}G$.

One would then like to imitate the proof of Lusztig [31] in the case of graded affine Hecke algebras, with G replaced by $\mathrm{L}G$, parabolics replaced by parahorics and graded affine Hecke algebra's replaced by dDAHAs. One immediate obstacle is the infinite dimensionality of the geometric objects such as $\mathrm{L}G$, which prevents us from making free

use of six operations for ℓ-adic sheaves and perverse sheaves on related geometric objects, such as $\mathrm{L}\mathfrak{g}$, the affine Springer resolution $\mathrm{L}\dot{\mathfrak{g}} = \mathrm{L}G \times^P \mathfrak{p} \longrightarrow \mathfrak{g}$ and the affine Steinberg variety $\mathrm{L}\ddot{\mathfrak{g}} = \mathrm{L}\dot{\mathfrak{g}} \times_{\mathrm{L}\mathfrak{g}} \mathrm{L}\dot{\mathfrak{g}}$.

As shown in [53], the upshot is to consider directly the $\mathbf{C}^\times$-fixed points of those varieties. A suitable assumption on the $\mathbf{C}^\times$-actions makes the fixed points some infinite disjoint union of algebraic varieties of finite type. Moreover, the fixed point components can be described in terms of flag varieties, Springer resolutions and Steinberg varieties of some Levi subgroups. The fixed point components can be parametrised with combinatorial data, such as Coxeter complexes. The $\mathbb{H}_\zeta$-action is then constructed by reduction to its parabolic subalgebras. Each proper parabolic subalgebra of $\mathbb{H}_\zeta$ is a graded affine Hecke algebra, which is treated by Lusztig in [31] [36]. It turns out that it suffices to define the action on proper parabolic subalgebras of $\mathbb{H}_\zeta$, since the defining relations of $\mathbb{H}_\zeta$ involve at most two simple reflections in the Weyl group. We will construct a specialised convolution algebra $\widehat{\mathcal{H}}$, which governs the infinite sum of perverse sheaves $\mathbf{I}$, and a homomorphism $\Phi : \mathbb{H}_\zeta \longrightarrow \widehat{\mathcal{H}}$. We will also show that the image of Φ is dense in a suitable product topology of $\widehat{\mathcal{H}}$. The conjecture of multiplicity-one then results from the density of the image of Φ.

In §4.1, we recall the relative affine Weyl group and Coxeter complexes introduced in [40].

In §4.2, we describe our principal geometric objects: varieties $\mathcal{X}^{\nu,\nu'}$, $\mathcal{T}^\nu$ and $\mathcal{Z}^{\nu,\nu'}$, which play the role of fixed point components of $(\mathrm{L}G/P) \times (\mathrm{L}G/P)$, $\mathrm{L}\dot{\mathfrak{g}}$ and $\mathrm{L}\ddot{\mathfrak{g}}$, respectively.

In §4.3, we define and study the specialised convolution algebra $\widehat{\mathcal{H}}$, which is equipped with a product topology. It can be viewed as the full convolution algebra specialised at an infinitesimal character a.

In §4.4, we define a degenerate double affine Hecke algebra (dDAHA) $\mathbb{H} = \mathbb{H}_\zeta$ and we construct a homomorphism $\Phi : \mathbb{H} \longrightarrow \widehat{\mathcal{H}}$. We prove that the image of Φ is dense in §4.4.6.

In §4.5, we compare our construction with that of Lusztig–Yun [40] and prove the multiplicity-one conjecture of *loc. cit.* We classify irreducible $\mathbb{H}$-modules with prescribed eigenvalues in Theorem 4.40. We also relate irreducible modules with torus fixed points of affine Springer fibres in Theorem 4.42.

4.1 Affine root system attached to $\mathfrak{g}$

This section is a recall of the definition of the affine alcove complex defined in [40], which is isomorphic to the (geometric realisation of the) Coxeter complex for the affine Weyl group.

4.1.1 $\mathbf{Z}/m$-graded Lie algebras

We retain the setting of §3. Moreover, we suppose G is simple and simply connected. Its Lie algebra $\mathfrak{g}$ is $\mathbf{Z}/m$-graded.

We assume the $\mathbf{Z}/m$-grading is induced by some co-character $\tilde{\theta} \in \mathbf{X}_*\left(G^{\mathrm{ad}}\right)$ in the sense

that

$$\mathfrak{g}_{\underline{i}} = \bigoplus_{\substack{k\in\mathbf{Z}\\ \underline{i}=\underline{k}}} {}^{\tilde{\theta}}_{k}\mathfrak{g}.$$

We say that $\tilde{\theta}$ is a **lifting** of the $\mathbf{Z}/m$-grading on $\mathfrak{g}$ or that $\tilde{\theta}$ lifts the $\mathbf{Z}/m$-grading θ on $\mathfrak{g}$. Let $G_{\underline{0}} = \exp(\mathfrak{g}_{\underline{0}}) \subseteq G$, which is a reductive subgroup of the same rank as G. This assumption makes the affine root system, defined in §4.1.2, untwisted. We fix once and for all a lifting $\tilde{\theta}$.

The torus $\mathbf{C}_q^\times = \mathbf{C}^\times$ acts linearly on the Lie algebra $\mathfrak{g}$ by weight -2. Let $u \in \mathbf{X}^*\left(\mathbf{C}_q^\times\right)$ denote the defining character.

We will use a subscript q to indicate a direct product with $\mathbf{C}_q^\times$, such as $G_{\underline{0},q} = G_{\underline{0}} \rtimes \mathbf{C}_q^\times$.

Besides, we fix a maximal torus $T \subseteq G$ centralised by $\tilde{\theta}$ so that $\tilde{\theta} \in \mathbf{X}_*(T/Z_G)$. In particular, T is in $G_{\underline{0}}$.

4.1.2 Affine root hyperplane arrangement

Let $\mathbb{A}$ denote the vector space $\mathbf{X}_*(T) \otimes \mathbf{Q}$. Since T and the adjoint torus $T^{\mathrm{ad}} = T/Z_G$ are isogenous, we may regard $\tilde{\theta}/m$ as a point of $\mathbb{A}$. We will denote $\mathbf{x} = \tilde{\theta}/m$. The root system of G is a subset $R \subset \mathbf{X}^*(T) \subset \mathbb{A}^*$, viewed as linear functions on $\mathbb{A}$. We define $S = R \times \mathbf{Z}$ to be the affine root system of G. Each affine root $a = (\alpha, n) \in S$ is a non-constant affine function on $\mathbb{A}$. For $a \in S$, the zero of this linear function is affine hyperplane of $\mathbb{A}$, denoted by H_a. The affine hyperplanes yield a stratification of $\mathbb{A}$ into **facets** (i.e. subsets of $\mathbb{A}$ determined by a fintie number of equations $a = 0$ or $a > 0$ with $a \in S$). Let $\mathfrak{F}$ denote the collection of facets. The facets of maximal dimension are called **alcoves**. The set of alcoves is denoted by $\mathfrak{A}$.

The affine Weyl group (with respect to T) is given by $W_S = W(G,T) \ltimes \mathbf{X}_*(T)$, which acts on $\mathbb{A}$ by reflections and translations. This action preserves the set $\{H_a\}_{a\in S}$ of affine root hyperplanes, inducing thus an action on $\mathfrak{F}$. The restriction of the action to $\mathfrak{A} \subseteq \mathfrak{F}$ is simply transitive. We will abbreviate $W = W_S$.

Let $W_{\mathbf{x}} = \mathrm{Stab}_W(\mathbf{x})$ be the stabiliser of $\mathbf{x} \in \mathbb{A}$. The rational co-character $\mathbf{x}$ gives rise to an identification $W_{\mathbf{x}} \cong W(G_{\underline{0}}, T)$ by identifying $\mathbf{x}$ as origin.

4.1.3 Simple reflections and subsets

Given a alcove $\kappa \in \mathfrak{A}$, let $\Delta^\kappa \subseteq S$ be the corresponding set of affine simple roots. If we consider the injective map $\Delta^\kappa \hookrightarrow W, \quad a \mapsto s_a$, then, the pair (W, Δ^κ) forms a Coxeter system.

Simple W acts simply transitively on the set of alcoves $\mathfrak{A}$, given any two alcoves $\kappa, \kappa' \in \mathfrak{A}$, there is a unique $w \in W$ such that $\kappa' = w\kappa$. Then, w yields isomorphisms of Coxeter systems

$$(W, \Delta^\kappa) \cong \left(W, \Delta^{\kappa'}\right)$$
$$y \mapsto wyw^{-1}, \quad s_a \mapsto ws_aw^{-1} = s_{wa}.$$

Thus we can define the **canonical Weyl group** $(\mathcal{W}, \Delta)$ to be the inverse limit of

the pairs (W, Δ^κ) for $\kappa \in \mathfrak{A}$, so that for any $\kappa \in \mathfrak{A}$, there is a canonical isomorphism $(\mathcal{W}, \Delta) \cong (W, \Delta^\kappa)$.

For any subset $J \subseteq \Delta$, we will denote (W_J, J) the Coxeter sub-system of $(\mathcal{W}, \Delta)$ generated by J. For any alcove $\kappa \in \mathfrak{A}$, we denote $(W_{J^\kappa}, J^\kappa) \subseteq (W, \Delta^\kappa)$ the Coxeter sub-system which is the image of (W_J, J) in (W, Δ^κ) under the canonical isomorphism $(\mathcal{W}, \Delta) \cong (W, \Delta^\kappa)$.

Given any $\kappa \in \mathfrak{A}$, there is a natural identification of $(\mathfrak{F}, \leq)$ with the Coxeter complex of (W, Δ^κ). The cell yW_{J^κ} in the Coxeter complex is identified with the facet $y(\partial_J \kappa) \subseteq \mathbb{A}$, where

$$\partial_J \kappa = \left\{ y \in \mathbb{A} \; ; \; \begin{array}{ll} a(y) = 0, & \forall a \in J^\kappa \\ a(y) > 0, & \forall s_a \in \Delta^\kappa \setminus J^\kappa \end{array} \right\}.$$

Here s_a means the reflection with respect to the hyperplane $H = \{y \in \mathbb{A} \; ; \; a(y) = 0\}$.

Given two alcoves $\kappa, \kappa' \in \mathfrak{A}$. Let $w \in W$ be such that $w\kappa = \kappa'$. Then for any $J \subsetneq \Delta$ we have $y(\partial_J \kappa') = yw(\partial_J \kappa)$. Therefore, to each facet $y(\partial_J \kappa) \in \mathfrak{F}$, there is a proper subset $J \subsetneq \Delta$ attached, independent of the alcove $\kappa \in \mathfrak{A}$. The subset $J \subseteq \Delta$ is called the **type** of the facet $y(\partial_J \kappa') \in \mathfrak{F}$. Let $\mathfrak{F}_J \subseteq \mathfrak{F}$ denote the set of facets of type J.

For any pair of proper subsets $J \subseteq K \subsetneq \Delta$, there is a boundary map $\partial_K : \mathfrak{F}_J \longrightarrow \mathfrak{F}_K$ which sends any facet ν of type J to its sub-facet of type K. Then W_{J^κ} is equal to the stabiliser of $\partial_J \kappa$ in W.

4.1.4 Correspondence between spirals and facets

We fix the sign $\epsilon = d/|d|$ which is required in the definition of spirals. Let $\mathfrak{P}_T$ be the set of spirals $\mathfrak{p}_*$ such that $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^\lambda$ for some $\lambda \in \mathbf{X}_*(T)_{\mathbf{Q}}$, *cf.* §3.1.3. There is a bijection

$$\mathfrak{F} \cong \mathfrak{P}_T.$$

Given a facet $\nu \in \mathfrak{F}$, we choose a point $y \in \nu$ and set $\lambda_y = m\epsilon(\mathbf{x} - y) = \epsilon\left(\tilde{\theta} - my\right) \in \mathbf{X}_*(T)_{\mathbf{Q}}$. It gives rise to the spiral ${}^\epsilon\mathfrak{p}_*^{\lambda_y}$, which does not depend on the choice of $y \in \nu$, see [40, 3.4.4].

We will denote $\mathfrak{p}_*^\nu = {}^\epsilon\mathfrak{p}_*^{\lambda_y}$ as well as $\mathfrak{u}_*^\nu = {}^\epsilon\mathfrak{u}_*^{\lambda_y}$ and $\mathfrak{l}_*^\nu = {}^\epsilon\mathfrak{l}_*^{\lambda_y}$ for any $y \in \nu$. We will also denote $P_0^\nu = \exp(\mathfrak{p}_0^\nu)$, $L^\nu = \exp(\mathfrak{l}^\nu)$, $L_0^\nu = \exp(\mathfrak{l}_0^\nu)$ and $U_0^\nu = \exp(\mathfrak{u}_0^\nu)$. Those are subgroups of G.

4.1.5 Graded pseudo-Levi attached to relevant affine subspaces

Let $\mathfrak{E}$ be the collection of affine subspaces of $\mathbb{A}$ which are non-empty intersection of a finite subset of $\{H_a\}_{a \in S}$. Elements of $\mathfrak{E}$ are called **relevant subspaces**.

Let $\mathfrak{M}_T^{\mathbf{Z}-\mathrm{gr}}$ be the set of quadruplets $(M, M_0, \mathfrak{m}, \mathfrak{m}_*)$ with M a pseudo-Levi subgroup of G containing the maximal torus T and $\mathfrak{m}_*$ a $\mathbf{Z}$-grading on $\mathfrak{m} = \mathrm{Lie}\, M$ which makes $\mathfrak{m}$ a graded Lie algebra such that $T \subseteq M_0$, where $M_0 = \exp(\mathfrak{m}_0)$. There is an injection

$$\mathfrak{E} \hookrightarrow \mathfrak{M}_T^{\mathbf{Z}-\mathrm{gr}}$$

defined as follows: Given any relevant subspace $\mathbb{E} \in \mathfrak{E}$, choosing any facet $\nu \in \mathfrak{F}$ which spans $\mathbb{E}$ as affine subspace, if we set

$$\left(M^{\mathbb{E}}, M_0^{\mathbb{E}}, \mathfrak{m}^{\mathbb{E}}, \mathfrak{m}_*^{\mathbb{E}}\right) = \left(L^\nu, L_0^\nu, \mathfrak{l}^\nu, \mathfrak{l}_*^\nu\right),$$

then the splitting $\left(M^{\mathbb{E}}, M_0^{\mathbb{E}}, \mathfrak{m}^{\mathbb{E}}, \mathfrak{m}_*^{\mathbb{E}}\right)$ does not depends on the choice of ν, [40, 3.4.7].

4.1.6 Cuspidal local system and relative Weyl group

Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*) \in \mathfrak{M}_T^{\mathbf{Z}-\mathrm{gr}}$. Assume that there is an M-equivariant cuspidal local system $\mathscr{C}$ on a nilpotent orbit $\mathrm{O} \subseteq \mathfrak{m}$ such that $\mathrm{O}_d = \mathrm{O} \cap \mathfrak{m}_d \neq \emptyset$. Suppose that the sextuple $\zeta = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \mathrm{O}, \mathscr{C}) \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$ be an admissible system *cf.* §3.2.3. In particular, $\mathscr{C}\mid_{\mathfrak{m}_d}$ is an irreducible M_0-equivariant local system on O_d. Let (e, h, f) be an $\mathfrak{sl}_2$-triple with $e \in \mathrm{O}_d$, $h \in \mathfrak{m}_0$ and $f \in \mathfrak{m}_{-d}$ and let $\varphi \in \mathbf{X}_*(G_{\underline{0}})$ be given by the Jacobson–Morosov theorem so that φ acts on e with weight 2. Let $M_q = M \times \mathbf{C}_q^\times$. Consider the obvious inclusion $Z_M(e) \hookrightarrow Z_{M_q}(e)$, which induces $\pi_0(Z_M(e)) \longrightarrow \pi_0\left(Z_{M_q}(e)\right)$. The last map is an isomorphism, since

$$\begin{aligned} Z_M(e) \times \mathbf{C}^\times &\cong Z_{M\times\mathbf{C}_q^\times}(e) \\ (g, q) &\mapsto (g\varphi(q), q) \end{aligned}$$

and since $\pi_0(Z_M(e) \times \mathbf{C}^\times) \cong \pi_0(Z_M(e))$. It follows that the M-equivariant local system $\mathscr{C}$ is naturally equipped with an M_q-equivariant structure. Likewise, $\mathscr{C}\mid_{\mathfrak{m}_d}$ is equipped with an M_q-equivariant structure. We fix such a system for the rest of the article.

According to [38, 3.1(d)], there exists $\nu \in \mathfrak{F}$ such that $\mathfrak{m}_* = \mathfrak{l}_*^\nu$. Let $\mathbb{E}^M \in \mathfrak{E}$ be the relevant affine subspace spanned by ν and let $W_{\mathbb{E}^M} \subset W$ be the subgroup which fixes $\mathbb{E}^M$ pointwise.

The affine root hyperplanes $\{H_a\}_{a\in S}$ induce, by intersection with $\mathbb{E}^M$, a collection of **relative affine root hyperplanes**

$$\left\{H \cap \mathbb{E}^M; H \in \mathfrak{H},\ H \cap \mathbb{E}^M \text{ is a hyperplane in } \mathbb{E}^M\right\} \subseteq \mathfrak{E}.$$

One defines similarly the collection of **$\mathbb{E}^M$-facets** $\mathfrak{F}(\mathbb{E}^M) \subseteq \mathfrak{F}$, as well as **$\mathbb{E}^M$-alcoves** $\mathfrak{A}\left(\mathbb{E}^M\right) \subseteq \mathfrak{F}$ and **relevant $\mathbb{E}^M$-subspaces** $\mathfrak{E}\left(\mathbb{E}^M\right) \subseteq \mathfrak{E}$.

Let $I \subsetneq \Delta$ be the type of ν so that $\nu \in \mathfrak{F}_I$. Let $\kappa \in \mathfrak{A}$ be a alcove such that $\nu = \partial_I \kappa$.

Let $J \subsetneq \Delta$ be a proper subset containing I. We denote $\sigma = \partial_J \kappa \in \mathfrak{F}\left(\mathbb{E}^M\right)$, which is a sub-facet of ν. Consider the datum of graded pseudo-Levi $(L^\sigma, L_0^\sigma, \mathfrak{l}^\sigma, \mathfrak{l}_*^\sigma)$ such that $W(L^\sigma, T) \cong W_{J^\kappa} = \mathrm{Stab}_W(\sigma)$. Moreover, since $(M, M_0, \mathfrak{m}, \mathfrak{m}_*) = (L^\nu, L_0^\nu, \mathfrak{l}^\nu, \mathfrak{l}_*^\nu)$, M is a Levi subgroup of L^σ and there is a nilpotent M-orbit $\mathrm{O} \subseteq \mathfrak{m}$ which supports a cuspidal local system $\mathscr{C}$ and an isomorphism. Let $w_0^{J^\kappa} \in W_{J^\kappa}$ be the longest element. A theorem of Lusztig [30, 9.2], [31, 2.5] asserts that $w_0^{J^\kappa}$ stabilises the Levi subgroup M, the nilpotent orbit $\mathrm{O} \subseteq \mathfrak{m}$ as well as the cuspidal local system on $\mathscr{C}$. In particular, $w_0^{J^\kappa} \in N_W(W_{\mathbb{E}^M})$.

Under the canonical isomorphism $(W, \Delta^\kappa) \cong (\mathcal{W}, \Delta)$, $w_0^{J^\kappa}$ is sent to w_0^J. We see that $w_0^J \in N_{\mathcal{W}}(\mathcal{W}_I)$. Therefore, the results of §B.1 is applicable to $(\mathcal{W}, \Delta)$ and I. In particular, Theorem B.9 yields the following results.

Theorem 4.1. *The following assertions hold:*

(i) *In the Coxeter system* $(\mathcal{W}, \Delta)$*, the shortest coset representatives for the elements of the quotient* $N_{\mathcal{W}}(\mathcal{W}_I)/\mathcal{W}_I$ *form a subgroup* $\tilde{\mathcal{W}}$ *of* $N_{\mathcal{W}}(\mathcal{W}_I)$*, which maps isomorphically onto* $N_{\mathcal{W}}(\mathcal{W}_I)/\mathcal{W}_I$ *under the quotient map. Moreover, if we set*

$$\tilde{\Delta} = \begin{cases} \emptyset & \text{if } \#(\Delta \setminus I) = 1, \\ \left\{ w_0^{I \cup \{s\}} w_0^I \in \tilde{\mathcal{W}} \; ; \; s \in \Delta \setminus I \right\} & \text{if } \#(\Delta \setminus I) > 1, \end{cases}$$

then $\left(\tilde{\mathcal{W}}, \tilde{\Delta}\right)$ *forms a Coxeter system.*

(ii) *Let* $\tilde{\ell} : \tilde{\mathcal{W}} \longrightarrow \mathbf{N}$ *denote the length function on* $\left(\tilde{\mathcal{W}}, \tilde{\Delta}\right)$*. Then for any elements* $w, y \in \tilde{\mathcal{W}}$*, we have*

$$\ell(w) + \ell(y) = \ell(wy) \quad \text{if and only if} \quad \tilde{\ell}(w) + \tilde{\ell}(y) = \tilde{\ell}(wy).$$

□

The **canonical relative affine Weyl group** is defined to be the Coxeter system $\left(\tilde{\mathcal{W}}, \tilde{\Delta}\right)$ of the above theorem.

Then Theorem B.11 in the present situation can be restated as

Theorem 4.2. *The* $\mathbb{E}^M$*-alcoves in* $\mathfrak{A}\left(\mathbb{E}^M\right) \subseteq \mathfrak{F}$ *are of the same type* I*.* □

This theorem justifies the notation I without κ involved. Hereafter, we will denote $\Xi = \mathfrak{A}(\mathbb{E}^M)$.

The non-canonical **relative Weyl group** is defined to be $\tilde{W} = N_W\left(\mathbb{E}^M\right)/W_{\mathbb{E}^M}$, where $N_W\left(\mathbb{E}^M\right)$ is the subgroup of W which preserves $\mathbb{E}^M$. We choose an alcove $\kappa \in \mathfrak{A}$ such that $\partial_I \kappa = \nu$. The canonical isomorphism $(\mathcal{W}, \Delta) \cong (W, \Delta^\kappa)$ induces $\tilde{\mathcal{W}} \cong \tilde{W}$. It is easy to see that this isomorphism does not depend on the choice of κ. We denote by $\tilde{\Delta}^\nu \subseteq \tilde{W}$ the image of $\tilde{\Delta}$ under this isomorphism. Similarly, for each $\nu \in \Xi$, we the affine subspace $\mathbb{E}^\nu \subset \mathbb{A}$ spanned by ν coincides with $\mathbb{E}^M$ and for each pair $\nu, \nu' \in \Xi$, exists $w \in N_W(\mathbb{E}^M)$ such that $w\nu = \nu'$, which induces an isomorphism $\mathbb{E}^\nu \cong \mathbb{E}^{\nu'}$. Let $\mathbb{E}$ be the inverse limit of these $\mathbb{E}^\nu$.

4.1.7 Relative position

Proposition 4.3. *The following assertions hold:*

(i) *There is a canonical left* $\mathcal{W}$*-action on* $\mathfrak{F}_I$*, which commutes with the left* W*-action and induces a left* $\tilde{\mathcal{W}}$*-action on* $\mathfrak{A}(\mathbb{E}^M)$*.*

(ii) *There is a canonical inclusion* $\tilde{\mathcal{W}} \hookrightarrow \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$*.*

(iii) *There is a canonical bijection*

$$W \backslash (\mathfrak{F}_I \times \mathfrak{F}_I) \cong \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$$

which induces a bijection

$$\tilde{W} \backslash \left(\mathfrak{A}\left(\mathbb{E}^M\right) \times \mathfrak{A}\left(\mathbb{E}^M\right)\right) \cong \tilde{\mathcal{W}}.$$

Proof. For each $w \in \mathcal{W}$ and each $\kappa \in \mathfrak{A}$, we denote by $w^\kappa \in W$ the image of w under the canonical isomorphism $(\mathcal{W}, \Delta) \cong (W, \Delta^\kappa)$ *cf.* §4.1.3. We define the left action of $\mathcal{W}$ on the set of $\mathbb{A}$-alcoves $\mathfrak{A}$ by

$$\begin{aligned} \mathcal{W} \times \mathfrak{A} &\longrightarrow \mathfrak{A} \\ (w, \kappa) &\mapsto (w^\kappa)^{-1}\kappa. \end{aligned}$$

This action descends in a unique way via the boundary map $\partial_I : \mathfrak{A} \longrightarrow \mathfrak{F}_I$, giving a right $\mathcal{W}$-action on $\mathfrak{F}_I$. Obviously, the subgroup $N_{\mathcal{W}}(\mathcal{W}_I)$ stabilises the subset $\mathfrak{A}\left(\mathbb{E}^M\right) \subseteq \mathfrak{F}_I$ and induces a right action of $\tilde{\mathcal{W}} \cong N_{\mathcal{W}}(\mathcal{W}_I)/\mathcal{W}_I$ on $\mathfrak{A}\left(\mathbb{E}^M\right)$. This proves (i).

Choosing any $\kappa \in \mathfrak{A}$, we define

$$\begin{aligned} \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I &\longrightarrow W \backslash (\mathfrak{F}_I \times \mathfrak{F}_I) \\ \mathcal{W}_I w \mathcal{W}_I &\mapsto W\left(\partial_I \kappa, w^\kappa \partial_I \kappa\right). \end{aligned} \tag{4.4}$$

It is a bijection[1] and does not depend on the choice of κ. Now, suppose that $\nu = \partial_I \kappa$ is in $\mathfrak{A}(\mathbb{E}^M)$. By definition, if $w \in \tilde{\mathcal{W}}$, then w normalises $\mathcal{W}_I$, so $w^\kappa \in \tilde{W}$ preserves $\mathbb{E}^M$. From the fact that

$$\mathfrak{A}\left(\mathbb{E}^M\right) = \left\{\mu \in \mathfrak{F}_I \ ; \ \mu \subset \mathbb{E}^M\right\},$$

it follows that $w^{-1}\nu \in \mathfrak{A}\left(\mathbb{E}^M\right)$. Therefore the bijection (4.4) restricts to an injection

$$\tilde{\mathcal{W}} \longrightarrow \tilde{W} \backslash \left(\mathfrak{A}\left(\mathbb{E}^M\right) \times \mathfrak{A}\left(\mathbb{E}^M\right)\right)$$

It is also surjective since $(\tilde{\mathcal{W}}, \tilde{\Delta}) \cong (\tilde{W}, \Delta^\nu)$ and that the $\tilde{W}$-action on $\mathfrak{A}\left(\mathbb{E}^M\right)$ is transitive *cf.* Theorem 4.2.

This proves (ii) and (iii).

□

Definition 4.5. *For any two elements $\nu, \nu' \in \mathfrak{F}_I$, the image of $(\nu, \nu') \in \mathfrak{F}_I \times \mathfrak{F}_I$ under*

$$\mathfrak{F}_I \times \mathfrak{F}_I \longrightarrow W \backslash (\mathfrak{F}_I \times \mathfrak{F}_I) \cong \mathcal{W}_I \xrightarrow{(4.4)} \mathcal{W}/\mathcal{W}_I$$

is called the **relative position** *of ν and ν'. The relative position is called* **good** *if it is in the image of the inclusion $\tilde{\mathcal{W}} \hookrightarrow \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$,* **bad** *otherwise.*

The following criterion is immediate.

Lemma 4.6. *Two elements $\nu, \nu' \in \mathfrak{F}_I$ are in good relative position if and only if ν and ν' span the same affine subspaces of $\mathbb{A}$.* □

4.2 Steinberg type varieties

We keep the assumptions of the previous sections. In particular, there is a sextuple $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \mathrm{O}, \mathscr{C})$ *cf.* §4.1.6, a co-character $\tilde{\theta} \in \mathbf{X}_*\left(T^{\mathrm{ad}}\right)$ which lifts the $\mathbf{Z}/m$-grading on $\mathfrak{g}$ and $\mathbf{x} = \tilde{\theta}/m \in \mathbb{A}$.

In addition, we fix henceforth a non-zero integer $d \in \mathbf{Z} \setminus \{0\}$ and a sign $\epsilon \in \{1, -1\}$.

[1] Notice that there is a bijection $\mathcal{W}_I \backslash \mathcal{W} \longrightarrow \mathfrak{F}_I$ given by $w \mapsto w^\kappa \partial_I \kappa$, which depends on κ.

4.2.1 Varieties of pairs of flags

We denote $\underline{\Xi} = N_{W_{\mathbf{x}}}(\mathbb{E}^M)\backslash\Xi$. We will denote by $\underline{\nu}$ the image of ν in $\underline{\Xi}$ for each $\nu \in \Xi$. Since the left $\mathcal{W}$-action of Proposition 4.3(i) on Ξ commutes with the left $W_{\mathbf{x}}$-action, we get a canonical left $\mathcal{W}$-action on $\underline{\Xi}$.

For $\nu, \nu' \in \Xi$, we set

$$\mathcal{X}^{\nu,\nu'} = (G_{\underline{0}}/P_0^{\nu}) \times \left(G_{\underline{0}}/P_0^{\nu'}\right).$$

Then, $G_{\underline{0}}$ acts on $\mathcal{X}^{\nu,\nu'}$ by diagonal left translation.

Proposition 4.7. *Up to canonical isomorphism, The $G_{\underline{0}}$-variety $\mathcal{X}^{\nu,\nu'}$ depends only on the classes $\underline{\nu}, \underline{\nu}' \in \underline{\Xi}$.*

Proof. Indeed, if $y, y' \in W_{\mathbf{x}}$, choosing a lifting $\dot{y}, \dot{y}' \in N_{G_{\underline{0}}}(T)$ of y and y', we have a $G_{\underline{0}}$-equivariant isomorphism

$$\begin{aligned} \mathcal{X}^{\nu,\nu'} &\cong \mathcal{X}^{y\nu,y'\nu'} \\ \left(gP_0^{\nu}, g'P_0^{\nu'}\right) &\mapsto \left(g\dot{y}^{-1}P_0^{\nu}, g'\dot{y}'^{-1}P_0^{\nu'}\right), \end{aligned}$$

which is canonical. □

4.2.2 Orbits and double cosets

Let $\nu, \nu' \in \Xi$. The orbits of the $G_{\underline{0}}$-action on $\mathcal{X}^{\nu,\nu'}$ are identified with the orbits of diagonal left translation of $W_{\mathbf{x}}$ on $\left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right) \times \left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right)$, where $W_{\mathbf{x},\mathbb{E}^M} = W_{\mathbf{x}} \cap W_{\mathbb{E}^M}$. There is a canonical inclusion

$$\begin{aligned} \left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right) \times \left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right) &\hookrightarrow \Xi \times \Xi \\ (u, v) &\mapsto (u\nu, v\nu'), \end{aligned}$$

which induces a map

$$G_{\underline{0}}\backslash\mathcal{X}^{\nu,\nu'} \cong W_{\mathbf{x}}\backslash\left(\left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right) \times \left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right)\right) \to W\backslash\left(\mathfrak{F}_I \times \mathfrak{F}_I\right) \cong \mathcal{W}_I\backslash\mathcal{W}/\mathcal{W}_I$$

which we denote by $\Pi_{\nu,\nu'} : G_{\underline{0}}\backslash\mathcal{X}^{\nu,\nu'} \longrightarrow \mathcal{W}_I\backslash\mathcal{W}/\mathcal{W}_I$. The map $\Pi_{\nu,\nu'}$ depends only on the $W_{\mathbf{x}}$-orbits $\underline{\nu}, \underline{\nu}' \in \underline{\Xi}$ in the sense of proposition Proposition 4.7.

For $w \in \mathcal{W}$ we will denote $[w] = \mathcal{W}_I w \mathcal{W}_I \in \mathcal{W}_I\backslash\mathcal{W}/\mathcal{W}_I$ the double coset containing w.

Lemma 4.8. *Let $[w] \in \mathcal{W}_I\backslash\mathcal{W}/\mathcal{W}_I$.*

(i) The fibre $\Pi_{\nu,\nu'}^{-1}[w]$ is finite and the union of $G_{\underline{0}}$-orbits $\bigcup_{O \in \Pi_{\nu,\nu'}^{-1}[w]} O$ is locally closed.

(ii) If $w \in \tilde{\mathcal{W}}$, then $\#\left(\Pi_{\nu,\nu'}^{-1}[w]\right) \leq 1$. Moreover, $\#\left(\Pi_{\nu,\nu'}^{-1}[w]\right) = 1$ if and only if $\underline{\nu} = w\underline{\nu}'$.

Proof. The assertion (i) is standard and follows from the Bruhat decomposition.

Suppose now that $w \in \tilde{\mathcal{W}}$ and $\nu, \nu' \in \Xi$ such that $\underline{\nu} = w\underline{\nu}'$. We may assume that $\nu = w\nu'$ since $\mathcal{X}^{\nu,\nu'}$ and $\Pi_{\nu,\nu'}$ depends only on the $W_{\mathbf{x}}$-orbits $\underline{\nu}$ and $\underline{\nu}'$. Denote $w^{\nu} \in \tilde{W}^{\nu}$

the image of w under the isomorphism $\left(\tilde{\mathcal{W}}, \tilde{\Delta}\right) \cong \left(\tilde{W}^{\nu}, \tilde{\Delta}^{\nu}\right)$ so that $\nu' = w^{\nu}\nu$ *cf.* §4.1.7. It suffices to show that the map

$$\begin{aligned}\varphi : W_{\mathbf{x}} \backslash \left(\left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right) \times \left(W_{\mathbf{x}}/W_{\mathbf{x},\mathbb{E}^M}\right)\right) &\longrightarrow W \backslash \left(\mathfrak{F}_I \times \mathfrak{F}_I\right) \\ W_{\mathbf{x}} \cdot (u, u') &\mapsto W \cdot (u\nu, u'w^{\nu}\nu)\end{aligned}$$

is injective over $W \cdot (\nu, w^{\nu}\nu) \in W \backslash (\Xi \times \Xi)$. Assume we have $u, u', v, v' \in W_{\mathbf{x}}$ such that $\left(uW_{\mathbf{x},\mathbb{E}^M}, u'W_{\mathbf{x},\mathbb{E}^M}\right)$ and $\left(vW_{\mathbf{x},\mathbb{E}^M}, v'W_{\mathbf{x},\mathbb{E}^M}\right)$ are sent to $W \cdot (\nu, w^{\nu}\nu)$, in other words

$$W \cdot (u\nu, u'w^{\nu}\nu) = W \cdot (\nu, w^{\nu}\nu) = W \cdot (v\nu, v'w^{\nu}\nu)\,.$$

We shall prove that $W_{\mathbf{x}} \cdot (u\nu, u'w^{\nu}\nu) = W_{\mathbf{x}} \cdot (v\nu, v'w^{\nu}\nu)$.

Indeed, as

$$W \cdot (\nu, w^{\nu}\nu) = W \cdot (u\nu, u'w^{\nu}\nu) = W \cdot \left(\nu, u^{-1}u'w^{\nu}\nu\right)$$

there exists $q \in W_{\nu} = W_{\nu}$ such that $qw^{\nu}\nu = u^{-1}u'w^{\nu}\nu$, or equivalently

$$(w^{\nu})^{-1}\, q^{-1}u^{-1}u'w^{\nu} \in W_{\nu}.$$

Since w^{ν} normalises W_{ν}, we obtain $q^{-1}u^{-1}u' \in W_{\nu}$ and hence $u^{-1}u' \in W_{\nu} \cap W_{\mathbf{x}}$. Similarly, we have $v^{-1}v' \in W_{\nu} \cap W_{\mathbf{x}}$. Therefore $v'^{-1}vu^{-1}u' \in W_{\nu} \cap W_{\mathbf{x}}$. Finally, we conclude that

$$\begin{aligned}W_{\mathbf{x}} \cdot (u\nu, u'\nu') &= W_{\mathbf{x}}v^{-1}v'u'^{-1}u\left(\nu, u^{-1}u'\nu'\right) = W_{\mathbf{x}} \cdot \left(v^{-1}v'u'^{-1}u\nu, v^{-1}v'\nu'\right) \\ &= W_{\mathbf{x}} \cdot \left(\nu, v^{-1}v'\nu'\right) = W_{\mathbf{x}} \cdot (v\nu, v'\nu')\,.\end{aligned}$$

□

Recall that there is a canonical bijection $P_0^{\nu} \backslash G_{\underline{0}} / P_0^{\nu'} \cong G_{\underline{0}} \backslash \mathcal{X}^{\nu,\nu'}$.

Proposition 4.9. *Let $\Omega \in P_0^{\nu} \backslash G_{\underline{0}} / P_0^{\nu'}$ be a double coset and let $O_{\Omega} \in G_{\underline{0}} \backslash \mathcal{X}^{\nu,\nu'}$ be the corresponding orbit. Then $\Pi_{\nu,\nu'}(O_{\Omega}) \in \tilde{\mathcal{W}}$ if and only if for any $g \in \Omega$, the following natural inclusion*

$$\left(\mathfrak{p}_N^{\nu} \cap \operatorname{Ad}_g \mathfrak{p}_N^{\nu'}\right) / \left(\mathfrak{u}_N^{\nu} \cap \operatorname{Ad}_g \mathfrak{u}_N^{\nu'}\right) \longrightarrow \operatorname{Ad}_g \mathfrak{p}_N^{\nu'} / \operatorname{Ad}_g \mathfrak{u}_N^{\nu'}$$

is an isomorphism for each $N \in \mathbf{Z}$.

Proof. Let $g \in \Omega$. According to [38, 5.1], there is a splitting $\mathfrak{l}_*$ of $\mathfrak{p}_*^{\nu}$ and a splitting $\mathfrak{l}'_*$ of $\operatorname{Ad}_g \mathfrak{p}_*^{\nu'}$ such that $L_0^{\nu} = \exp(\mathfrak{l}_0)$ and $L_0^{\nu'} = \exp(\mathfrak{l}'_0)$ contain a common maximal torus T' of $G_{\underline{0}}$.

Let $g' \in G_{\underline{0}}$ be such that $g'T'g'^{-1} = T$. Then $\operatorname{Ad}_{g'} \mathfrak{p}_*^{\nu}$ and $\operatorname{Ad}_{g'g} \mathfrak{p}_*^{\nu'}$ are in $\mathfrak{P}_T$ and there are unique $y, w \in \mathcal{W}$ such that $\operatorname{Ad}_{g'} \mathfrak{p}_*^{\nu} = \mathfrak{p}_*^{\nu y}$ and $\operatorname{Ad}_{g'g} \mathfrak{p}_*^{\nu'} = \mathfrak{p}_*^{\nu yw}$. From the definition of $\Pi_{\nu,\nu'}$, we see that $\Pi_{\nu,\nu'}(O_{\Omega}) = [w]$. By Lemma 4.6, $[w] \in \tilde{\mathcal{W}}$ if and only if $\mathbb{E}^{\nu y} = \mathbb{E}^{\nu yw}$. Looking at the Levi decomposition $\mathfrak{p}_*^{\nu y} = \mathfrak{l}_*^{\nu y} \oplus \mathfrak{u}_*^{\nu y}$ and $\mathfrak{p}_*^{\nu yw} = \mathfrak{l}_*^{\nu yw} \oplus \mathfrak{u}_*^{\nu yw}$, we see that $\mathbb{E}^{\nu y} = \mathbb{E}^{\nu yw}$ if and only if

$$\left(\mathfrak{p}_N^{\nu y} \cap \mathfrak{p}_N^{\nu yw}\right) / \left(\mathfrak{u}_N^{\nu y} \cap \mathfrak{u}_N^{\nu yw}\right) \longrightarrow \mathfrak{p}_N^{\nu yw} / \mathfrak{u}_N^{\nu yw} \tag{4.10}$$

is an isomorphism for all $N \in \mathbf{Z}$. Taking into account the definitions $\operatorname{Ad}_{g'} \mathfrak{p}_*^{\nu} = \mathfrak{p}_*^{\nu y}$ and $\operatorname{Ad}_{g'g} \mathfrak{p}_*^{\nu'} = \mathfrak{p}_*^{\nu yw}$, the condition that the map (4.10) being an isomorphism is equivalent to that the map

$$\left(\operatorname{Ad}_{g'} \mathfrak{p}_N^{\nu} \cap \operatorname{Ad}_{g'g} \mathfrak{p}_N^{\nu'}\right) / \left(\operatorname{Ad}_{g'} \mathfrak{u}_N^{\nu} \cap \operatorname{Ad}_{g'g} \mathfrak{u}_N^{\nu'}\right) \longrightarrow \operatorname{Ad}_{g'g} \mathfrak{p}_N^{\nu'} / \operatorname{Ad}_{g'g} \mathfrak{u}_N^{\nu'} \tag{4.11}$$

being an isomorphism for all $N \in \mathbf{Z}$. We obtain the desired equivalent condition by applying $\operatorname{Ad}_{g'^{-1}}$ to the map (4.11). □

Remark 4.12. *In the terminologies of [38, 5.2], if a double coset $\Omega \in P_0^\nu \backslash G_{\underline{0}} / P_0^{\nu'}$ satisfies the condition of Proposition 4.9, it is called* **good**. *It is called* **bad** *if otherwise.*

4.2.3 Stratification of $\mathcal{X}^{\nu,\nu'}$

We define for each $w \in \mathcal{W}$ and each $\nu, \nu' \in \Xi$

$$\mathcal{X}_w^{\nu,\nu'} = \bigcup_{O \in \Pi_{\nu,\nu'}^{-1}[w]} O \subseteq \mathcal{X}^{\nu,\nu'}$$

to be a locally closed subset of $\mathcal{X}^{\nu,\nu'}$ equipped with the reduced subscheme structure. Obviously, $\mathcal{X}_w^{\nu,\nu'} = \mathcal{X}_{w'}^{\nu,\nu'}$ if $[w] = [w']$, so the variety $\mathcal{X}_w^{\nu,\nu'}$ does not depend on the representative w for the class $[w]$.

Proposition 4.13. *For each $\nu, \nu' \in \Xi$ and each $w \in \tilde{\mathcal{W}}$ such that $w\underline{\nu'} = \underline{\nu}$, there is an isomorphism of $G_{\underline{0}}$-schemes*

$$\begin{array}{ccccc} G_{\underline{0}}/P_0^\nu \cap P_0^{w^{-1}\nu} & \xrightarrow{\cong} & \mathcal{X}_w^{\nu,w^{-1}\nu} & \xrightarrow{\cong} & \mathcal{X}_w^{\nu,\nu'} \\ g \cdot \left(P_0^\nu \cap P_0^{w^{-1}\nu}\right) & \longmapsto & \left(gP_0^\nu, gP_0^{w^{-1}\nu}\right) & & \end{array}.$$

Proof. By Lemma 4.8, $\mathcal{X}_w^{\nu,\nu'}$ is a $G_{\underline{0}}$-orbit. The proposition results from the fact that

$$\mathrm{Stab}_{G_{\underline{0}}}\left(eP_0^\nu, eP_0^{\nu w}\right) = P_0^\nu \cap P_0^{w^{-1}\nu}.$$

□

4.2.4 Induction of cuspidal local system

Recall that we have for each $\nu \in \Xi$ a Levi decomposition of spiral $\mathfrak{p}_*^\nu = \mathfrak{m}_* \oplus \mathfrak{u}_*^\nu$, nilpotent M-orbit $\mathrm{O} \subset \mathfrak{m}$ and cuspidal local system $\mathscr{C}$ on O. By restriction, the local system $\mathscr{C}\,|_{\mathrm{O}_d}$ is M_0-equivariant.

For each $\nu \in \Xi$, we introduce a variety

$$\mathcal{T}^\nu = G_{\underline{0}} \times^{P_0^\nu} \left(\mathrm{O}_d \times \mathfrak{u}_d^\nu\right).$$

It is a smooth variety equipped with an action of $G_{\underline{0},q}$. By induction of groups, there is a $G_{\underline{0}}$-equivariant local system

$$\dot{\mathscr{C}} = G_{\underline{0}} \times^{P_0^\nu} \left(\mathscr{C}\,|_{\mathrm{O}_d} \boxtimes \overline{\mathbf{Q}}_\ell\right)$$

on $\mathcal{T}^\nu$. Let $\alpha^\nu : \mathcal{T}^\nu \longrightarrow \mathfrak{g}_{\underline{d}}$ be the natural morphism given by the adjoint action of $G_{\underline{0}}$.

If $\nu, \nu' \in \Xi$ are such that $\underline{\nu} = \underline{\nu'}$, then there is a $G_{\underline{0}}$-equivariant canonical isomorphism $\mathcal{T}^\nu \cong \mathcal{T}^{\nu'}$ which identifies α^ν with $\alpha^{\nu'}$ and the sheaf $\dot{\mathscr{C}}$ on $\mathcal{T}^\nu$ andbthe sheaf $\dot{\mathscr{C}}$ on $\mathcal{T}^{\nu'}$.

We define the **Lusztig sheaf** for the relative alcove $\nu \in \Xi$ to be $\mathbf{I}^\nu = \alpha_!^\nu \,\mathrm{IC}(\dot{\mathscr{C}}) \in \mathrm{D}^b_{G_{\underline{0}}}(\mathfrak{g}_{\underline{d}})$. For the same reason as in the last paragraph, $\mathbf{I}^\nu$ depends only on the $W_{\mathbf{x}}$-orbit $\underline{\nu}$.

Proposition 4.14. *For $\underline{\nu} \in \underline{\Xi}$, the following statements hold*

(i) If $\epsilon = d/|d|$, then the complex $\mathbf{I}^\nu$ is supported on the nilpotent cone, i.e. $\mathbf{I}^\nu = \alpha_!^\nu \operatorname{IC}(\dot{\mathscr{C}}) \in \mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}}\right)$

(ii) We have $\mathbf{I}^\nu \cong \bigoplus_{k\in\mathbf{Z}} {}^p\mathrm{H}^k\mathbf{I}^\nu[-k]$ and each factor ${}^p\mathrm{H}^k\mathbf{I}^\nu$ is a semi-simple perverse sheaves.

Proof. The (i) follows from [38, 7.1(a)]. The (ii) follows from the Beĭlinson–Bernstein–Deligne–Gabber decomposition theorem and that the purity of $\operatorname{IC}(\dot{\mathscr{C}})$ [37, 1.4].

□

4.2.5 Steinberg type varieties

For $\nu, \nu' \in \Xi$, we set

$$\mathcal{Z}^{\nu,\nu'} = \mathcal{T}^\nu \times_{\mathfrak{g}_{\underline{d}}} \mathcal{T}^{\nu'}$$

and we denote $q_1 : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^\nu$ and $q_2 : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^{\nu'}$ the canonical projections.

For the same reason as in §4.2.4, this variety is independent of the choice of ν and ν' in their $W_{\mathbf{x}}$-conjugacy classes.

We have a $G_{\underline{0},q}$-equivariant local system on $\mathcal{Z}^{\nu,\nu'}$

$$\mathscr{K}^{\nu,\nu'} = \mathcal{H}\mathrm{om}\left(q_2^*\dot{\mathscr{C}}, q_1^!\dot{\mathscr{C}}\right) \cong s^!\left(\dot{\mathscr{C}} \boxtimes \mathbf{D}\dot{\mathscr{C}}\right),$$

where $s = (q_1, q_2) : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^\nu \times \mathcal{T}^{\nu'}$. We shall simply write $\mathscr{K} = \mathscr{K}^{\nu,\nu'}$.

There is a canonical $G_{\underline{0},q}$-equivariant map which forgets the Lie algebra components

$$\mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{X}^{\nu,\nu'},$$

which depends only on the $W_{\mathbf{x}}$-orbits $\underline{\nu}$ and $\underline{\nu}'$. For $w \in \mathcal{W}$, we denote $\mathcal{Z}_w^{\nu,\nu'}$ the pre-image of $\mathcal{X}_w^{\nu,\nu'}$ under the above map.

Proposition 4.15. *For $\nu, \nu' \in \underline{\Xi}$ and $w \in \tilde{\mathcal{W}}$ such that $w\underline{\nu'} = \underline{\nu}$, there is an equivariant isomorphism of $G_{\underline{0}}$-schemes*

$$\begin{array}{ccccc} G_{\underline{0}} \times^{P_0^\nu \cap P_0^{w^{-1}\nu}} \left(\mathrm{O}_d^\nu \times \left(\mathfrak{u}_d^\nu \cap \mathfrak{u}_d^{w^{-1}\nu}\right)\right) & \xrightarrow{\cong} & \mathcal{Z}_w^{\nu,w^{-1}\nu} & \xrightarrow{\cong} & \mathcal{Z}_w^{\nu,\nu'} \\ (g,x) & \longmapsto & ((g,x),(g,x)) & & \end{array}.$$

4.3 Convolution algebra

We keep the assumptions of §4.2. In particular, there is an admissible system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \mathrm{O}, \mathscr{C})$ and the cuspidal local system $\mathscr{C}$ induces a sheaf $\dot{\mathscr{C}}$ on $\mathcal{T}^\nu$ for each $\underline{\nu} \in \Xi$.

4.3.1 Convolution product

Let $\nu, \nu' \in \Xi$. We have two projections $q_1 : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^{\nu}$ and $q_2 : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^{\nu'}$ *cf.* §4.2.5. We define

$$\mathcal{H}^{\nu,\nu'} = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}\left(q_2^*\dot{\mathscr{C}}, q_1^!\dot{\mathscr{C}}\right).$$

It is isomorphic to $\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}, \mathscr{K}\right)$ in a natural way. For the same reason as in §4.2.4, the definition of $\mathcal{H}^{\nu,\nu'}$ is independent of ν and ν' in their $W_{\mathbf{x}}$-conjugacy class.

By the formalism of [36] and [15], for $\nu, \nu', \nu'' \in \Xi$, there is a convolution morphism

$$\mathcal{H}^{\nu,\nu'} \otimes \mathcal{H}^{\nu',\nu''} \longrightarrow \mathcal{H}^{\nu,\nu''}.$$

The $\mathrm{H}^{\bullet}_{G_{\underline{0}}}$-module $\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}, \mathscr{K}\right)$ is isomorphic by Verdier duality to the extension space $\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}\left(\mathbf{I}^{\nu'}, \mathbf{I}^{\nu}\right)$ and the convolution product is identified with the Yoneda product.

We recall the construction of *loc. cit.* Let $\widehat{\mathrm{O}} \subseteq \mathfrak{l}^{\nu}$ be the closure of O in $\mathfrak{l}^{\nu}$ and let $\widehat{\mathrm{O}}_d = \widehat{\mathrm{O}} \cap \mathfrak{l}^{\nu}_d$. Since the orbit O is distinguished, O_d must be open dense in $\mathfrak{l}^{\nu}_d$ *cf.* [37, 4.4a]. This implies that $\widehat{\mathrm{O}}^{\nu}_d$ is either empty or equal to $\mathfrak{l}^{\nu}_d$. We set $\widehat{\mathcal{T}}^{\nu} = G_{\underline{0}} \times^{P^{\nu}_0} \left(\widehat{\mathrm{O}}_d \times \mathfrak{u}^{\nu}_d\right)$ and $\widehat{\mathcal{Z}}^{\nu,\nu'} = \widehat{\mathcal{T}}^{\nu} \times_{\mathfrak{g}_{\underline{d}}} \widehat{\mathcal{T}}^{\nu}$. We denote $u^{\nu} : \mathcal{T}^{\nu} \hookrightarrow \widehat{\mathcal{T}}^{\nu}$ and $u^{\nu,\nu'} : \mathcal{Z}^{\nu,\nu'} \hookrightarrow \widehat{\mathcal{Z}}^{\nu,\nu'}$ the open embeddings. We remark that $\widehat{\mathcal{T}}^{\nu}$ is smooth and is proper over $\mathfrak{g}_{\underline{d}}$. We denote $q_1 : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^{\nu}$, $q_2 : \mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{T}^{\nu'}$, $\widehat{q}_1 : \widehat{\mathcal{Z}}^{\nu,\nu'} \longrightarrow \widehat{\mathcal{T}}^{\nu}$ and $\widehat{q}_2 : \widehat{\mathcal{Z}}^{\nu,\nu'} \longrightarrow \widehat{\mathcal{T}}^{\nu'}$ the canonical projections.

By the cleaness of $\mathscr{C}$, we have $u^{\nu}_!\dot{\mathscr{C}} = u^{\nu}_*\dot{\mathscr{C}}$. We will thus identify $\dot{\mathscr{C}}$ with its direct image on $\widehat{\mathcal{T}}^{\nu}$ without any confusion. Define $\mathscr{K} = \mathcal{H}\mathrm{om}\left(q_2^*\dot{\mathscr{C}}, q_1^!\dot{\mathscr{C}}\right)$ on $\mathcal{Z}^{\nu,\nu'}$ and $\widehat{\mathscr{K}} = \mathcal{H}\mathrm{om}\left(\widehat{q}_2^*\dot{\mathscr{C}}, \widehat{q}_1^!\dot{\mathscr{C}}\right)$ on $\widehat{\mathcal{Z}}^{\nu,\nu'}$. We have

$$\begin{aligned}\widehat{\mathscr{K}} &\cong \mathcal{H}\mathrm{om}\left(\widehat{q}_2^* u^{\nu}_! \dot{\mathscr{C}}, \widehat{q}_1^! \dot{\mathscr{C}}\right) \cong \mathcal{H}\mathrm{om}\left(u^{\nu,\nu'}_! q_2^* \dot{\mathscr{C}}, \widehat{q}_1^! \dot{\mathscr{C}}\right) \\ &\cong u^{\nu,\nu'}_* \mathcal{H}\mathrm{om}\left(q_2^* \dot{\mathscr{C}}, u^{\nu,\nu'!} \widehat{q}_1^! \dot{\mathscr{C}}\right) \cong u^{\nu,\nu'}_* \mathcal{H}\mathrm{om}\left(q_2^* \dot{\mathscr{C}}, q_1^! u^{\nu'*} \dot{\mathscr{C}}\right) \cong u^{\nu,\nu'}_* \mathscr{K}.\end{aligned}$$

Therefore

$$\mathcal{H}^{\nu,\nu'} = \mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}, \mathscr{K}\right) \cong \mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\widehat{\mathcal{Z}}^{\nu,\nu'}, \widehat{\mathscr{K}}\right).$$

The cleaness assures that we can safely work on the non-proper version $\mathcal{T}^{\nu}$, $\mathcal{Z}^{\nu,\nu'}$ and $\mathscr{K}$.

Now we describe the convolution product. Given $\nu, \nu', \nu'' \in \Xi$, we consider the following diagram

$$\begin{tikzcd}
\mathcal{Z}^{\nu,\nu'} \times \mathcal{Z}^{\nu',\nu''} \arrow[d, "t"] & \mathcal{Z}^{\nu,\nu'} \times_{\mathcal{T}^{\nu'}} \mathcal{Z}^{\nu',\nu''} \arrow[l, hook'] \arrow[r, "\mu"] \arrow[d, "r"] & \mathcal{Z}^{\nu,\nu''} \arrow[d, "s"] \\
\left(\mathcal{T}^{\nu} \times \mathcal{T}^{\nu'}\right) \times \left(\mathcal{T}^{\nu'} \times \mathcal{T}^{\nu''}\right) & \mathcal{T}^{\nu} \times \mathcal{T}^{\nu'} \times \mathcal{T}^{\nu''} \arrow[l, "\gamma"'] \arrow[r, "u"] & \mathcal{T}^{\nu} \times \mathcal{T}^{\nu''}
\end{tikzcd}$$

There is a sequence of maps

$$\begin{aligned}
&\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'},\mathscr{K}\right)\otimes\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu',\nu''},\mathscr{K}\right)\longrightarrow\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}\times\mathcal{Z}^{\nu',\nu''},t^{!}\left(\dot{\mathscr{C}}\boxtimes\mathbf{D}\dot{\mathscr{C}}\boxtimes\dot{\mathscr{C}}\boxtimes\mathbf{D}\dot{\mathscr{C}}\right)\right)\\
&\xrightarrow{t_*t^{!}(\mathrm{id}\longrightarrow\gamma_*\gamma^*)}\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}\times\mathcal{Z}^{\nu',\nu''},t^{!}\gamma_*\left(\dot{\mathscr{C}}\boxtimes\left(\mathbf{D}\dot{\mathscr{C}}\otimes\dot{\mathscr{C}}\right)\boxtimes\mathbf{D}\dot{\mathscr{C}}\right)\right)\\
&\cong\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}\times_{\mathcal{T}^{\nu'}}\mathcal{Z}^{\nu',\nu''},r^{!}\left(\dot{\mathscr{C}}\boxtimes\left(\mathbf{D}\dot{\mathscr{C}}\otimes\dot{\mathscr{C}}\right)\boxtimes\mathbf{D}\dot{\mathscr{C}}\right)\right)\\
&\longrightarrow\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}\times_{\mathcal{T}^{\nu'}}\mathcal{Z}^{\nu',\nu''},r^{!}\left(\dot{\mathscr{C}}\boxtimes\mathbf{D}\overline{\mathbf{Q}}_{\ell}\boxtimes\mathbf{D}\dot{\mathscr{C}}\right)\right)\cong\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}\times_{\mathcal{T}^{\nu'}}\mathcal{Z}^{\nu',\nu''},r^{!}u^{!}\left(\dot{\mathscr{C}}\boxtimes\mathbf{D}\dot{\mathscr{C}}\right)\right)\\
&\cong\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}\times_{\mathcal{T}^{\nu'}}\mathcal{Z}^{\nu',\nu''},\mu^{!}\mathscr{K}\right)\longrightarrow\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu''},\mathscr{K}\right).
\end{aligned}$$

The convolution product is then defined to be the composite $\mathcal{H}^{\nu,\nu'}\otimes\mathcal{H}^{\nu',\nu''}\longrightarrow\mathcal{H}^{\nu,\nu''}$.

For any $G_{\underline{0}}$-stable closed subschemes $V\subset\mathcal{Z}^{\nu,\nu'}$, $V'\subset\mathcal{Z}^{\nu',\nu''}$ and $V''\subset\mathcal{Z}^{\nu,\nu''}$, we define similarly the convolution product so that the folloing diagram commutes

$$\begin{array}{ccc}
\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(V,i_V^{!}\mathscr{K}\right)\otimes\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(V',i_{V'}^{!}\mathscr{K}\right) & \longrightarrow & \mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(V'',i_{V''}^{!}\mathscr{K}\right)\\
\downarrow & & \downarrow\\
\mathcal{H}^{\nu,\nu'}\otimes\mathcal{H}^{\nu',\nu''} & \longrightarrow & \mathcal{H}^{\nu,\nu''}
\end{array},$$

where the vertical arrows are given by adjunction co-units $i_{V!}i_V^{!}\longrightarrow\mathrm{id}$, etc. Henceforth, we will simply denote $\mathscr{K}$ for $i_V^{!}\mathscr{K}$, for any such closed immersion i_V.

4.3.2 Polynomial action

For each $\nu\in\Xi$, let[2] $A^{\nu}=\mathrm{Z}^0\left(L^{\nu}\right)=\mathrm{Z}^0\left(M^{\nu}\right)$ be the neutral component of the centre of L^{ν}. We have $\mathbb{E}^{\nu}\cong\mathbf{X}_*\left(A^{\nu}\right)_{\mathbf{Q}}$. We define $\mathbf{S}^{\nu}$ to be the algebra of $\overline{\mathbf{Q}}_{\ell}$-valued polynomial functions on $\mathbb{E}^{\nu}$, so that $\mathbf{S}^{\nu}\cong\mathrm{H}^{\bullet}_{A^{\nu}}$. If $\nu,\nu'\in\Xi$, then the canonical isomorphism $\mathbb{E}^{\nu}\cong\mathbb{E}^{\nu'}$ induces a canonical isomorphism $\mathbf{S}^{\nu}\cong\mathbf{S}^{\nu'}$. We define equally the canonical object $\mathbf{S}$ such that $\mathbf{S}\cong\mathbf{S}^{\nu}$ for all $\nu\in\Xi$.

Lemma 4.16. *For any $\nu,\nu'\in\Xi$, the equivariant cohomology group $\mathcal{H}^{\nu,\nu'}$ has a $(\mathbf{S}^{\nu},\mathbf{S}^{\nu'})$-bimodule structure, which makes the convolution product $\mathcal{H}^{\nu,\nu'}\times\mathcal{H}^{\nu',\nu''}\longrightarrow\mathcal{H}^{\nu,\nu''}$ a $\mathbf{S}^{\nu'}$-bilinear map of $(\mathbf{S}^{\nu},\mathbf{S}^{\nu'})$-bimodules.*

More generally, given any $G_{\underline{0},q}$-invariant subvariety $i_V:V\hookrightarrow\mathcal{Z}$, the cohomology $\mathrm{H}^{\bullet}_{G_{\underline{0}}}(V,i_V^{!}\mathscr{K})$ has a structure of $(\mathbf{S}^{\nu},\mathbf{S}^{\nu'})$-bimodules.

Proof. We will only prove the statement for $\mathcal{H}^{\nu,\nu'}$. By the induction principle of equivariant cohomology, there are isomorphisms

$$\mathrm{H}^{\bullet}_{G_{\underline{0},q}}\left(\mathcal{T}^{\nu},\overline{\mathbf{Q}}_{\ell}\right)\cong\mathrm{H}^{\bullet}_{M_{0,q}}\left(\mathrm{O}_d,\overline{\mathbf{Q}}_{\ell}\right)\cong\mathrm{H}^{\bullet}_{\mathrm{Stab}_{L^{\nu}_{0,q}}(\mathbf{o})}.$$

[2]Since $L^{\nu}=M$ and $\mathbb{E}^{\nu}=\mathbb{E}^{M}$ for all $\nu\in\Xi$, the definition of A^{ν} does not depends on ν. However, we identify $\mathbb{E}^{\nu}$ and A^{ν} for different $\nu\in\Xi$ via non-trivial isomorphisms.

where $\mathbf{o}$ is any point in the L_0^ν-orbit O_d. By the Jacobson–Morosov theorem, there exists a cocharacter $\varphi \in X_*(L_0^\nu)$ which acts on $\mathbf{o}$ by weight 2. Hence there is a morphism

$$\begin{aligned} A^\nu \times \mathbf{C}_q^\times &\longrightarrow G_{\underline{0}} \times \mathbf{C}_q^\times \\ (g,u) &\mapsto (g\varphi(u), u)\,. \end{aligned} \tag{4.17}$$

whose image is a maximal reductive subgroup of $\mathrm{Stab}_{L^\nu_{0,q}}(\mathbf{o})$, [31, 2.3]. Thus we obtain an isomorphisms of rings

$$\mathbf{S}^\nu \otimes \overline{\mathbf{Q}}_\ell[u] \cong \mathrm{H}^\bullet_{A^\nu_q} \cong \mathrm{H}^\bullet_{G_{\underline{0},q}}(\mathcal{T}^\nu, \overline{\mathbf{Q}}_\ell),$$

Since the orbit O_d is distinguished, the cocharacter φ acts on $\mathfrak{l}^\nu_d$ by weight 2. Hence by the formula (4.17), the torus $\mathbf{C}_q^\times$ of left hand side acts trivially on $\mathfrak{l}^\nu_d$. Substituting $u \mapsto -d/m$ into the left-hand side, we obtain $\mathbf{S}^\nu \cong \mathrm{H}^\bullet_{G_{\underline{0}}}(\mathcal{T}^\nu, \overline{\mathbf{Q}}_\ell)$, which acts on $\mathcal{H}^{\nu,\nu'}$ by cup product via the pull-back:

$$q_1^* : \mathrm{H}^\bullet_{G_{\underline{0}}}(\mathcal{T}^\nu, \overline{\mathbf{Q}}_\ell) \longrightarrow \mathrm{H}^\bullet_{G_{\underline{0}}}(\mathcal{Z}^{\nu,\nu'}, \overline{\mathbf{Q}}_\ell)$$

with the map q_1 as defined in §4.2.5. Similarly, we have $\mathbf{S}^{\nu'} \cong \mathrm{H}^\bullet_{G_{\underline{0}}}\left(\mathcal{T}^{\nu'}, \overline{\mathbf{Q}}_\ell\right)$ which acts via q_2^*. Thus, the tensor product $\mathbf{S}^\nu \otimes_{\mathrm{H}_{G_{\underline{0}}}} \mathbf{S}^{\nu'}$ acts on $\mathrm{H}^\bullet_{G_{\underline{0}}}(\mathcal{Z}, \mathscr{K})$ via the cup product.

□

Remark 4.18. *The subtil point of this $\mathbf{S}^\nu$-action is that in the isomorphism 4.17, the $\mathbf{C}_q^\times$-part is not the identity map. In other words, the quotient map $\mathrm{H}^\bullet_{G_{\underline{0},q}}(\mathcal{T}^\nu, \overline{\mathbf{Q}}_\ell) \longrightarrow \mathrm{H}^\bullet_{G_{\underline{0}}}(\mathcal{T}^\nu, \overline{\mathbf{Q}}_\ell)$ implicit in the proof is not simply forgetting the $\mathbf{C}_q^\times$-part of the equivariance.*

4.3.3 Cohomology of components $\mathcal{Z}_w^{\nu,\nu'}$

We define for $\nu, \nu' \in \Xi$ and $[w] \in \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$

$$\mathcal{H}_w^{\nu,\nu'} = \mathrm{H}^\bullet_{G_{\underline{0}}}\left(\mathcal{Z}_w^{\nu,\nu'}, i_w^! \mathscr{K}\right),$$

where $i_w : \mathcal{Z}_w^{\nu,\nu'} \longrightarrow \mathcal{Z}^{\nu,\nu'}$ is the inclusion. By the construction of Lemma 4.16, $\mathcal{H}_w^{\nu,\nu'}$ comes equipped with an $(\mathbf{S}^\nu, \mathbf{S}^{\nu'})$-bimodule structure.

Proposition 4.19. *Let $\nu, \nu' \in \Xi$ and $[w] \in \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$.*

(iv) If $w\underline{\nu'} \neq \underline{\nu}$, then $\mathcal{H}_w^{\nu,\nu'} = 0$.

(v) If the relative position $[w]$ is bad, i.e. $[w] \notin \tilde{\mathcal{W}}$, then $\mathcal{H}_w^{\nu,\nu'} = 0$.

(vi) If $[w]$ is good, i.e. $[w] \in \tilde{\mathcal{W}}$, and if $w\underline{\nu'} = \underline{\nu}$, then $\mathcal{H}_w^{\nu,\nu'}$ is a free of rank 1 as left graded $\mathbf{S}^\nu$-module and as right graded $\mathbf{S}^{\nu'}$-module, vanishing in odd degrees.

Proof. If $w\underline{\nu'} \neq \underline{\nu}$, then $\mathcal{X}_w^{\nu,\nu'} = \emptyset$ and $\mathcal{Z}_w^{\nu,\nu'} = \emptyset$ so $\mathcal{H}_w^{\nu,\nu'} = 0$.

We turn to (ii). Suppose that $[w] \notin \tilde{\mathcal{W}}$. Recall that there is a left $G_{\underline{0}}$-action on $\mathcal{X}_w^{\nu,\nu'}$ with a finite number of orbits. Let

$$\mathcal{X}_w^{\nu,\nu'} = \bigsqcup_{\Omega} \mathcal{X}_\Omega$$

be a decomposition into $G_{\underline{0}}$ orbits, where Ω is taken over a finite subset of $P_0^{\nu}\backslash G_{\underline{0}}/P_0^{\nu'}$. We define $\mathcal{Z}_\Omega \subseteq \mathcal{Z}^{\nu,\nu'}$ to be the pre-image of $\mathcal{X}_\Omega$ under the projection $\mathcal{Z}^{\nu,\nu'} \longrightarrow \mathcal{X}^{\nu,\nu'}$. Notice that all double cosets Ω that appear in the decomposition are *bad* in the sense of [38, 5.2] since $[w]$ is not in $\tilde{\mathcal{W}}$.

Let

$$\dot{\mathfrak{p}}_d^{\nu} = (\mathrm{O}_d \times \mathfrak{u}_d^{\nu}) \times_{\mathfrak{g}_{\underline{d}}} \mathcal{T}^{\nu'}$$

so that $\mathcal{Z}^{\nu,\nu'} \cong G_{\underline{0}} \times^{P_0^{\nu''}} \dot{\mathfrak{p}}_d^{\nu'}$. Moreover, for each Ω if we denote

$$\dot{\mathfrak{p}}_{d,\Omega}^{\nu} = (\mathrm{O}_d \times \mathfrak{u}_d^{\nu}) \times_{\mathfrak{g}_{\underline{d}}} \left(\Omega \times^{P_0^{\nu'}} \left(\mathrm{O}_d \times \mathfrak{u}_d^{\nu'}\right)\right) \subseteq \dot{\mathfrak{p}}_d^{\nu}$$

then there is a diagram of cartesian squares

$$\begin{tikzcd}
\dot{\mathfrak{p}}_{d,\Omega}^{\nu} \arrow[r, "r'"] \arrow[d, "q_1'"] & \mathcal{Z}_\Omega \arrow[r, "q_2"] \arrow[d, "q_1"] & \mathcal{T}^{\nu'} \arrow[d] \\
\mathfrak{p}_d^{\nu} \arrow[r, "r"] & \mathcal{T}^{\nu} \arrow[r] & \mathfrak{g}_{\underline{d}}
\end{tikzcd}$$

Let $i : \mathcal{Z}_\Omega \hookrightarrow \mathcal{Z}^{\nu,\nu'}$ be the inclusion. Now

$$\begin{aligned}
\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}_\Omega, i^!\mathscr{K}\right) &\cong \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}\left(q_2^*\dot{\mathscr{C}}, q_1^!\dot{\mathscr{C}}\right) \cong \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}\left(q_{1!}q_2^*\dot{\mathscr{C}}, \dot{\mathscr{C}}\right) \\
&\cong \mathrm{Hom}^{\bullet}_{P_0}\left(r^*q_{1!}q_2^*\dot{\mathscr{C}}, r^*\dot{\mathscr{C}}\right) \cong \mathrm{Hom}^{\bullet}_{P_0}\left(q_{1!}'r'^*q_2^*\dot{\mathscr{C}}, r^*\dot{\mathscr{C}}\right).
\end{aligned}$$

Taking into account the cuspidality of $\mathscr{C}$ and the fact that Ω is bad, *cf.* Proposition 4.9 and Remark 4.12, it is shown in [38, 5.3] that $q_{1!}'r'^*q_2^*\dot{\mathscr{C}} = 0$. Thus $\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}_\Omega, i^!\mathscr{K}\right) = 0$ for each Ω. By an argument of long exact sequence of cohomoloy, we conclude that

$$\mathcal{H}_w^{\nu,\nu'} = \mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}_w^{\nu,\nu'}, \mathscr{K}\right) = 0.$$

This proves (ii).

We turn to (iii). Suppose that $w\nu' = \nu$ with $w \in \tilde{\mathcal{W}}$. Using the description Proposition 4.15, by the induction property of equivariant cohomology, we see that

$$\begin{aligned}
\mathcal{H}_w^{\nu,\nu'} &= \mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(G_{\underline{0}} \times^{P_0^{\nu}\cap P_0^{\nu'}} \left(\mathrm{O}_d \times \mathfrak{u}_d^{\nu} \cap \mathfrak{u}_d^{\nu'}\right), \mathscr{K}\right) \cong \mathrm{H}^{\bullet}_{P_0^{\nu}\cap P_0^{\nu'}}\left(\mathrm{O}_d \times \mathfrak{u}_d^{\nu} \cap \mathfrak{u}_d^{\nu'}, \mathscr{K}\mid_{\mathrm{O}_d\times\mathfrak{u}_d^{\nu}\cap\mathfrak{u}_d^{\nu'}}\right) \\
&\cong \mathrm{H}^{\bullet}_{L_0^{\nu}}\left(\mathrm{O}_d, \mathscr{K}\mid_{\mathrm{O}_d}\right).
\end{aligned}$$

From the definition of $\mathscr{K}$, we see that

$$\mathscr{K}\mid_{\mathrm{O}_d} = \mathcal{H}\mathrm{om}\left(\mathscr{C}, \mathscr{C}\right)\left[2\dim G_{\underline{0}}/P_0^{\nu}\cap P_0^{\nu'} - 2\dim \mathfrak{u}_d^{\nu}\cap\mathfrak{u}_d^{\nu'}\right]$$

According to a result of Lusztig [37, 4.4(a)], the subscheme $\mathrm{O}_d \subseteq \mathfrak{l}_d^{\nu}$ is the unique open M_0-orbit whenever it is non-empty, whence up to even shift of cohomological degree, there are isomorphisms

$$\mathrm{H}^{\bullet}_{M_0}\left(\mathrm{O}_d, \mathscr{K}\right) \cong \mathrm{H}^{\bullet}_{M_0}\left(\mathrm{O}_d, \mathcal{H}\mathrm{om}\left(\mathscr{C}, \mathscr{C}\right)\right) \cong \mathrm{H}^{\bullet}_{\mathrm{Stab}_{M_0}(\mathbf{o})}\left(\{\mathbf{o}\}, \mathscr{C}_{\mathbf{o}}^* \otimes \mathscr{C}_{\mathbf{o}}\right) \cong \mathbf{S}^{\nu}.$$

The last isomorphism is due to the fact that the fibre $\mathscr{C}_{\mathbf{o}}$ in $\mathbf{o}$ is a irreducible representation of $\pi_0\left(\mathrm{Stab}_{M_0}(\mathbf{o})\right)$. Similarly, $\mathcal{H}_w^{\nu,\nu'} \cong \mathbf{S}^{\nu'}$, *cf.* §4.1.6. See also [31, 4.2].

By construction, the $\mathbf{S}^\nu$-action is the same as the cup product on $\mathrm{H}^\bullet_{M_0}(\mathrm{O}_d, \mathscr{K})$. Therefore $\mathcal{H}^{\nu,\nu'}_w$ is free left graded $\mathbf{S}^\nu$-module of rank 1. Applying Proposition 4.15 to $\nu = w\nu'$ and reasoning similarly with the transposed $\mathcal{Z}^{\nu',\nu}_w \cong \mathcal{Z}^{\nu,\nu'}_{w^{-1}}$, we see that $\mathcal{H}^{\nu,\nu'}_w$ is free right graded $\mathbf{S}^{\nu'}$-module of rank 1. It clearly vanishes in odd degrees. This complete the proof.

□

Remark 4.20. *The special feature of convolution algebra with a cuspidal local system as coefficient is that not every Bruhat cell on the partial flag variety contributes to the cohomology, but only those corresponding to the subset $\tilde{W} \subset \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$ do.*

4.3.4 Filtration by length

Recall that for each $\nu, \nu' \in \Xi$, the cohomology $\mathcal{H}^{\nu,\nu'}_w = \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_w, \mathscr{K}\right)$ is equipped with an $(\mathbf{S}^\nu, \mathbf{S}^{\nu'})$-bimodule structure *cf.* §4.3.2.

Let ${}^I\mathcal{W}^I \subseteq \mathcal{W}$ be the set of representatives of minimal length of the double quotient $[w] \in \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$. Note that $\tilde{\mathcal{W}} \subseteq {}^I\mathcal{W}^I$. We introduce a partial order $\leq_\ell$ on $\mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$ in the following way: for $y, y' \in {}^I\mathcal{W}^I$, the relation $[y] \leq_\ell [y']$ holds if and only if $\ell(y) < \ell(y')$ or $y = y'$. Similarly, the relation $[y] \leq_\ell [y']$ means $\ell(y) < \ell(y')$.

For each $\nu, \nu' \in \Xi$, we filter the schemes $\mathcal{X}^{\nu,\nu'}, \mathcal{Z}^{\nu,\nu'}$ and the convolution algebras accordingly. Note that by Bruhat decomposition, for each pair of elements $w, y \in \tilde{\mathcal{W}}$, if the variety $\mathcal{X}^{\nu,\nu'}_w$ lies in the closure of $\mathcal{X}^{\nu,\nu'}_y$, then $w \leq_\ell y$. For $[w] \in \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_I$. We set

$$\mathcal{X}^{\nu,\nu'}_{\leq w} = \bigcup_{\substack{y \in {}^I\mathcal{W}^I \\ y \leq_\ell w}} \mathcal{X}^{\nu,\nu'}_y, \quad \mathcal{Z}^{\nu,\nu'}_{\leq w} = \bigcup_{\substack{y \in {}^I\mathcal{W}^I \\ y \leq_\ell w}} \mathcal{Z}^{\nu,\nu'}_y, \quad \mathcal{X}^{\nu,\nu'}_{<w} = \mathcal{X}^{\nu,\nu'}_{\leq w} \setminus \mathcal{X}^{\nu,\nu'}_w, \quad \mathcal{Z}^{\nu,\nu'}_{<w} = \mathcal{Z}^{\nu,\nu'}_{\leq w} \setminus \mathcal{Z}^{\nu,\nu'}_w.$$

Those are reduced closed subschemes of $\mathcal{X}^{\nu,\nu'}$ and $\mathcal{Z}^{\nu,\nu'}$.

For each $w \in \mathcal{W}$, corresponding to the triplet $\mathcal{Z}^{\nu,\nu'}_{<w} \subseteq \mathcal{Z}^{\nu,\nu'}_{\leq w} \supseteq \mathcal{Z}^{\nu,\nu'}_w$, there is a long exact sequence of graded $\mathbf{S}^\nu$-$\mathbf{S}^{\nu'}$-bimodules

$$\mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{<w}, \mathscr{K}\right) \longrightarrow \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{\leq w}, \mathscr{K}\right) \longrightarrow \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_w, \mathscr{K}\right) \xrightarrow{\partial} \mathrm{H}^{*+1}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{<w}, \mathscr{K}\right).$$

By Proposition 4.19, the cohomology group $\mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_w, \mathscr{K}\right)$ vanishes in odd degrees. Therefore, by induction on w, we have $\partial = 0$ in the above sequence. This yields the following short exact sequence

$$0 \longrightarrow \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{<w}, \mathscr{K}\right) \longrightarrow \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{\leq w}, \mathscr{K}\right) \longrightarrow \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_w, \mathscr{K}\right) \longrightarrow 0.$$

We define then

$$\mathcal{H}^{\nu,\nu'}_{\leq w} = \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{\leq w}, \mathscr{K}\right), \quad \mathcal{H}^{\nu,\nu'}_{<w} = \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'}_{<w}, \mathscr{K}\right).$$

By induction on w, we see that the natural morphism $\mathcal{H}^{\nu,\nu'}_{\leq w} \longrightarrow \mathcal{H}^{\nu,\nu'}$ is injective. Hence $\mathcal{H}^{\nu,\nu'}_{\leq w}$ is an $(\mathbf{S}^\nu, \mathbf{S}^{\nu'})$ sub-bimodule of $\mathcal{H}^{\nu,\nu'}$. The sub-quotient $\mathcal{H}^{\nu,\nu'}_{\leq w} / \mathcal{H}^{\nu,\nu'}_{<w}$ is isomorphic to the bimodule $\mathcal{H}^{\nu,\nu'}_w$. We denote by $\mathcal{H}^{\nu,\nu'}_* = \bigoplus_{w \in {}^I\mathcal{W}^I} \mathcal{H}^{\nu,\nu'}_w$ the associated graded $(\mathbf{S}^\nu, \mathbf{S}^{\nu'})$-bimodule.

4.3.5 Convolution product on graded pieces

For $y \in \tilde{\mathcal{W}}$, recall that there are two lengths attached to y: $\ell(y)$ and $\tilde{\ell}(y)$. The former is the length of the element y in the full affine Coxeter system $(\mathcal{W}, S)$, whereas the latter is the length of y in the relative Coxeter system $\left(\tilde{\mathcal{W}}, \tilde{S}\right)$. The Theorem 4.1 implies that $\ell(yw) = \ell(y) + \ell(w)$ if and only if $\tilde{\ell}(yw) = \tilde{\ell}(y) + \tilde{\ell}(w)$ for any $y, w \in \tilde{\mathcal{W}}$.

Lemma 4.21. *The convolution product on $\mathcal{H}^{\nu,\nu'}$ induces a product on the associated graded $(\mathbf{S}^\nu, \mathbf{S}^{\nu'})$-bimodules $\mathcal{H}_*^{\nu,\nu'}$.*

Proof. Let $w, w' \in \tilde{\mathcal{W}}$. We suppose that $\tilde{\ell}(ww') = \tilde{\ell}(w) + \tilde{\ell}(w')$, so that $\ell(ww') = \ell(w) + \ell(w')$. For any element $y \in Y$, let

$$\overline{C_y} = \bigsqcup_{y' \leq_\ell y} \mathcal{W}_I y' \mathcal{W}_I \subseteq \mathcal{W}$$

We then have $\overline{C_w} \cdot \overline{C_{w'}} \subseteq \overline{C_{ww'}}$.

Let $\nu \in \Xi$ and denote $\nu' = w^{-1}\nu$, $\nu'' = w'^{-1}\nu'$. The image of the convolution

$$\mathcal{Z}_{\leq w}^{\nu,\nu'} \times_{\mathcal{T}^{\nu'}} \mathcal{Z}_{\leq w'}^{\nu',\nu''} \longrightarrow \mathcal{Z}^{\nu,\nu''}$$

lies in the closed subscheme $\mathcal{Z}_{\leq ww'}^{\nu,\nu''}$. It follows that the restriction of the convolution product of cohomology *cf.* §4.3.1

$$\mathcal{H}_{\leq w}^{\nu,\nu'} \otimes \mathcal{H}_{\leq w'}^{\nu',\nu''} \longrightarrow \mathcal{H}^{\nu,\nu''}$$

factorises through the sub-bimodule $\mathcal{H}_{\leq ww'}^{\nu,\nu''} \subseteq \mathcal{H}^{\nu,\nu''}$. It follows that the convolution product respects the filtration by length, so it induces a product on the graded pieces. □

With , the following lemma of transversality follows easily from the Bruhat decomposition.

Lemma 4.22. *Given $\nu \in \Xi$, $w, w' \in \tilde{\mathcal{W}}$ such that $\tilde{\ell}(ww') = \tilde{\ell}(w) + \tilde{\ell}(w')$, let $\nu' = w^{-1}\nu$ and $\nu'' = w'^{-1}\nu'$.*

(i) The projection $\mathcal{Z}^{\nu,\nu'} \times_{\mathcal{T}^{\nu'}} \mathcal{Z}^{\nu',\nu''} \longrightarrow \mathcal{Z}^{\nu,\nu''}$ induces an isomorphism $\overline{\mu} : \mathcal{Z}_w^{\nu,\nu'} \times_{\mathcal{T}^{\nu'}} \mathcal{Z}_{w'}^{\nu',\nu''} \cong \mathcal{Z}_{ww'}^{\nu,\nu''}$.

(ii) The convolution product on graded pieces $\mathcal{H}_w^{\nu,\nu'} \otimes \mathcal{H}_{w'}^{\nu',\nu''} \longrightarrow \mathcal{H}_{ww'}^{\nu,\nu''}$ agrees with the Gysin map of the closed embedding

$$\mathcal{Z}_{ww'}^{\nu,\nu''} \cong \mathcal{Z}_w^{\nu,\nu'} \times_{\mathcal{T}^{\nu'}} \mathcal{Z}_{w'}^{\nu',\nu''} \hookrightarrow \mathcal{Z}_w^{\nu,\nu'} \times \mathcal{Z}_{w'}^{\nu',\nu''}.$$

(iii) The convolution product

$$\mathcal{H}_w^{\nu,\nu'} \otimes \mathcal{H}_{w'}^{\nu',\nu''} \longrightarrow \mathcal{H}_{ww'}^{\nu,\nu''}$$

is surjective.

(iv) The polynomial actions cf. §4.3.2 yields an isomorphism of algebras $\mathbf{S}^\nu \cong \mathcal{H}_e^{\nu,\nu}$.

Proof. The statement (i) follows easily from the equivalence $l(yw) = l(y) + l(w) \Leftrightarrow \tilde{l}(yw) = \tilde{l}(y) + \tilde{l}(w)$ and the Bruhat decomposition. The statement (ii) and (iii) are consequences of the "transversality" result (i), *cf.* *[13, 7.6.12]*. The statement (ii) follows from (iii) and Proposition 4.19 (iii). □

4.3.6 Completion

Let $\mathfrak{m}_0 \subset \mathrm{H}^\bullet_{G_{\underline{0}}}$ denote the graded maximal ideal (positive-degree part) and let $\mathrm{H}^\wedge$ be the $\mathfrak{m}_0$-adic completion, equipped with the $\mathfrak{m}_0$-adic topology. Define the following topological vector spaces

$$\widehat{\mathcal{H}}^{\nu,\nu'} = \mathcal{H}^{\nu,\nu'} \otimes_{\mathrm{H}^\bullet_{G_{\underline{0}}}} \mathrm{H}^\wedge, \quad \widehat{\mathcal{H}}^{\nu,\nu'}_{\leq w} = \mathcal{H}^{\nu,\nu'}_{\leq w} \otimes_{\mathrm{H}^\bullet_{G_{\underline{0}}}} \mathrm{H}^\wedge,$$
$$\widehat{\mathcal{H}} = \prod_{\underline{\nu}' \in \underline{\Xi}} \bigoplus_{\underline{\nu} \in \underline{\Xi}} \widehat{\mathcal{H}}^{\nu,\nu'}, \quad \widehat{\mathcal{H}}_{\leq w} = \prod_{\underline{\nu}' \in \underline{\Xi}} \bigoplus_{\underline{\nu} \in \underline{\Xi}} \widehat{\mathcal{H}}^{\nu,\nu'}_{\leq w}$$

(Recall that $\mathcal{H}^{\nu,\nu'}$ depends only on the orbits $\underline{\nu}, \underline{\nu}' \in \underline{\Xi}$ up to canonical isomorphism, *cf.* §4.3.1. In the category of topological abelian groups, the topology on the direct sum is the weak topology of the union of the finite sums, whereas the topology on the finite sum coincides with the product topology since the category is $\mathbf{Z}$-linear.)

Since the convolution product $\mathcal{H}^{\nu,\nu'} \times \mathcal{H}^{\nu',\nu''} \longrightarrow \mathcal{H}^{\nu,\nu''}$ (§4.3.1) is $\mathrm{H}_{G_{\underline{0}}}$-linear, the completion at $\mathfrak{m}_0$ yields a continuous linear map

$$\widehat{\mathcal{H}}^{\nu,\nu'} \times \widehat{\mathcal{H}}^{\nu',\nu''} \longrightarrow \widehat{\mathcal{H}}^{\nu,\nu''}.$$

Letting vary $\underline{\nu}, \underline{\nu}', \underline{\nu}'' \in \underline{\Xi}$, it induces on $\widehat{\mathcal{H}}$ a structure of topological ring.

(recall that $\mathbf{I}^\nu$ depends only on the class $\underline{\nu} \in \underline{\Xi}$) with kernel being topologically nilpotent and closed.

Definition 4.23. *A module M of $\widehat{\mathcal{H}}$ is called* **smooth** *if the action of $\widehat{\mathcal{H}}$ on M is continuous when M is equipped with the discrete topology. Equivalently, M is smooth if for each $m \in M$, the annihilator*

$$\mathrm{ann}(m) = \{h \in \mathcal{H}_a \; ; \; hm = 0\}$$

is open in $\widehat{\mathcal{H}}$.

Let $\widehat{\mathcal{H}}$-$\mathrm{mod}^{\mathrm{sm}}$ denote the category of smooth $\widehat{\mathcal{H}}$-modules. Recall the isomorphism $\mathcal{H}^{\nu,\nu'} \cong \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}^{\nu'}, \mathbf{I}^\nu)$. The following fact is standard about extension algebras.

Lemma 4.24. *There is a canonical bijection between the set of isomorphic classes of simple objects of $\widehat{\mathcal{H}}$-$\mathrm{mod}^{\mathrm{sm}}$ and the set of simple constituents of $\bigoplus_{\underline{\nu} \in \underline{\Xi}} \bigoplus_{k \in \mathbf{Z}} {}^p\mathrm{H}^k \mathbf{I}^\nu$.*

See [53, 6.1] for a proof. □

4.3.7 Spectra of the polynomial actions

Let $\nu \in \Xi$. We consider the folloing ring map

$$\Phi^{\nu,\nu} : \mathbf{S} \cong \mathbf{S}^\nu \cong \mathcal{H}^{\nu,\nu}_e \hookrightarrow \widehat{\mathcal{H}}^{\nu,\nu}_e,$$

The first map is the canonical isomorphism *cf.* §4.3.2, followed by the isomorphism Lemma 4.22 (iv), the obvious inclusion, the $\mathfrak{m}_0$-adic completion.

Let $\mathbf{x}^M \in \mathbb{E}^M$ be the image of the orthogonal projection (with respect to the Killing form) of $\mathbf{x} \in \mathbb{A}$ onto $\mathbb{E}^M$ and let $\mathbf{x}_\nu$ be the image of $\mathbf{x}^M$ under the canonical isomorphism $\mathbb{E}^M = \mathbb{E}^\nu \cong \mathbb{E}$ *cf.* §4.1.6.

The following results describe the spectrum of the $\mathbf{S}$-actions.

Lemma 4.25 ([40, 4.3.4]). *For any $\nu \in \Xi$, the map $\Phi^{\nu,\nu'}$ identify $\widehat{\mathcal{H}}_e^{\nu,\nu'}$ with the completion of $\mathbf{S}$ at $\mathbf{x}_\nu \in \mathbb{E}$.*

□

4.4 Morphism from the degenerate double affine Hecke algebra

Following the strategy of [53] and [40], we will construct a morphism $\mathbb{H} \longrightarrow \widehat{\mathcal{H}}$ from the degenerate double affine Hecke algebra (dDAHA) to the convolution algebra.

4.4.1 Degenerate double affine Hecke algebra

In order to describe the degenerate double affine Hecke algebra, we need to define an affine root system on the affine space $\mathbb{E}$ §4.1.6. The restriction of the affine root system S to the subspace $\mathbb{E}^M$ yield an affine root system $\tilde{S}'$ on $\mathbb{E}^M$, which may not be reduced. We define $\tilde{S}$ to be an reduced affine root system on $\mathbb{E}^M$ isogeneous to $\tilde{S}'$ such that $\tilde{S}$ is isomorphic to the affinisation of some finite root system R. The unbased affine root system $(\mathbb{E}^M, \tilde{S})$ induces a based[3] affine root system $(\mathbb{E}, \tilde{S}, \tilde{\Delta})$.

For each $a \in \tilde{\Delta}$, there is a positive integer $c_a \geq 2$ introduced in [40, 2.4.8]. This number is described in [35, §7]. The constants c_a need to be renormalised according to the isogeny that we have chosen for $\tilde{S}$. In the case of principal block (i.e. $M = T$ is a maximal torus), we have $c_a = 2$ for each $a \in \Delta$. In the case where R is of type B/C, we assume R to be of type BC, so that $c_a = c_b$ whenever $a, b \in \tilde{\Delta}$ are conjugate by the extended Weyl group.

Define the degenerate double affine Hecke algebra attached to the admissible system ζ as the one defined in §1.2 with the affine root system $(\mathbb{E}^M, \tilde{S}')$ and with parameters $\{(d/2m)c_a\}_{a\in\tilde{S}}$.

More explicitely, the dDAHA is defined to be $\mathbb{H} = \mathbf{C}\tilde{\mathcal{W}} \otimes \mathbf{S}$ with the following commutation laws:

$$\tilde{s}_a\, f - \tilde{s}_a(f)\, \tilde{s}_a = (d/2m)c_a \frac{f - \tilde{s}_a(f)}{a}, \quad a \in \tilde{\Delta}$$

where $\tilde{s}_a \in \tilde{\mathcal{W}}$ is the orthogonal reflection on $\mathbb{E}$ with respect to the relative affine hyperplane $\{a = 0\}$ on $\mathbb{E}$.

Remark 4.26. *The class of degenerate DAHAs which can be constructed in the present setting is limited for non-simply laced root systems. Since the constants c_a are certain integers determined by the cuspidal pair $(\mathrm{O}, \mathscr{C})$, only certain integral proportions between parameters can appear.*

[3] Any choice of $\nu \in \Xi$ corresponds to a choice of base $\tilde{\Delta}^\nu \subset \tilde{S}'$. The inverse limit $\mathbb{E} = \varprojlim_{\nu\in\Xi} \mathbb{E}^M$ then eliminates the non-canonicity of the choice of ν and gives a base $\tilde{\Delta}$.

4.4.2 Graded affine Hecke algebras

Let $J \subsetneq \Delta$ be a proper subset containing I *cf.* §4.1.6. We have a parabolic Coxeter subsystem $\left(\tilde{\mathcal{W}}_J, \tilde{J}\right)$ of the canonical relative Weyl group $\left(\tilde{\mathcal{W}}, \tilde{\Delta}\right)$, where $\tilde{J} = \{\tilde{s}_a \; ; \; a \in \Delta \setminus J\}$ *cf.* §B.1.3. We define $\mathbb{H}_J$ to the subalgebra of $\mathbb{H}$ generated by $\mathbf{S}$ and s_a for $a \in \tilde{\Delta}$. In the case where $J = I$, we recover the polynomial algebra

$$\mathbb{H}_I = \mathbf{S}.$$

Define $\Xi_J = \partial_J \Xi \subseteq \mathfrak{F}_J$. Let $\sigma \in \Xi_J$. The tuple $(H, H_0, \mathfrak{h}, \mathfrak{h}_*) = (L^\sigma, L_0^\sigma, \mathfrak{l}^\sigma, \mathfrak{l}_*^\sigma)$ is a graded pseudo-Levi subgroup of G *cf.* §4.1.4. We define $\mathfrak{F}_I^\sigma = \{\nu \in \mathfrak{F}_I ; \partial_J \nu = \sigma\}$. Then the stabiliser $\mathrm{Stab}_W(\sigma)$ acts transitively on $\mathfrak{F}_I^\sigma$.

Each $\nu \in \mathfrak{F}_I^\sigma$ gives rise to a parabolic subgroup $P^{\sigma \leq \nu} \subseteq H$ whose Lie algebra $\mathfrak{p}_*^{\sigma \leq \nu}$ is $\mathbf{Z}$-graded and that $\mathfrak{p}_n^{\sigma \leq \nu} = \mathfrak{p}_n^\nu \cap \mathfrak{h}_n$. We define also $\mathfrak{u}_n^{\sigma \leq \nu} = \mathfrak{u}_n^\nu \cap \mathfrak{h}_n$ so that $U^{\sigma \leq \nu} = \exp\left(\mathfrak{u}^{\sigma \leq \nu}\right)$ is the unipotent radical of $P^{\sigma \leq \nu}$. Notice that we have a Levi decomposition $\mathfrak{p}_*^{\sigma \leq \nu} = \mathfrak{l}_*^\nu \oplus \mathfrak{u}_*^{\sigma \leq \nu}$ which respects the gradings.

Choose any $\nu \in \mathfrak{F}_I^\sigma \cap \Xi$ so that $\mathfrak{l}^\nu = \mathfrak{m}$. Let $\dot{\mathfrak{h}}_{\mathrm{O}} = H \times^{P^{\sigma \leq \nu}} \left(\mathrm{O} \times \mathfrak{u}^{\sigma \leq \nu}\right)$ and let $\alpha : \dot{\mathfrak{h}}_{\mathrm{O}} \longrightarrow \mathfrak{h}$ be the natural morphism defined by the adjoint action. As before, we denote $H_q = H \times \mathbf{C}_q^\times$. The group H_q acts on $\dot{\mathfrak{h}}$ in a natural way. The M_q-equivariant cuspidal local system $\mathscr{C}$ on $\mathrm{O} \subseteq \mathfrak{p}^{\sigma \leq \nu}$ induces a H_q-equivariant local system $\dot{\mathscr{C}}$ on $\dot{\mathfrak{h}}_{\mathrm{O}}$. Put $\ddot{\mathfrak{h}}_{\mathrm{O}} = \dot{\mathfrak{h}}_{\mathrm{O}} \times_{\mathfrak{h}_{\mathrm{O}}} \dot{\mathfrak{h}}_{\mathrm{O}}$. There is a complex $\mathscr{K} = \mathcal{H}\mathrm{om}\left(q_2^* \dot{\mathscr{C}}, q_1^! \dot{\mathscr{C}}\right)$ on $\ddot{\mathfrak{h}}_{\mathrm{O}}$, where $q_1, q_2 : \ddot{\mathfrak{h}}_{\mathrm{O}} \longrightarrow \dot{\mathfrak{h}}_{\mathrm{O}}$ are the canonical projections. The cohomology $\mathrm{H}_{H_q}^\bullet\left(\ddot{\mathfrak{h}}_{\mathrm{O}}, \mathscr{K}\right)$, being isomorphic to $\mathrm{Hom}_{H_q}^\bullet\left(\alpha_* \dot{\mathscr{C}}, \alpha_* \dot{\mathscr{C}}\right)$, becomes a ring with the Yoneda product.

Theorem 4.27 (8.11,[36]; 4.4.5,[40]). *There is a canonical isomorphism of algebras*

$$\mathbb{H}_J \cong \mathrm{H}_{H_q}^\bullet\left(\ddot{\mathfrak{h}}_{\mathrm{O}}, \mathscr{K}\right) |_{u=d/2m} .$$

□

4.4.3 Localisation of graded affine Hecke algebras

The pseudo-Levi subalgebra $\mathfrak{h}$ arising from $\sigma \in \Xi_J$ is $\mathbf{Z}$-graded. Let $\mathbf{C}_{\mathrm{deg}}^\times = \mathbf{C}^\times$ be the torus which acts on $\mathfrak{h}_n$ by weight $2n$ for each $n \in \mathbf{Z}$. Define a cocharacter

$$\iota : \mathbf{C}^\times \longrightarrow \mathbf{C}_{\mathrm{deg}}^\times \times \mathbf{C}_q^\times, \quad t \mapsto \left(t, t^{-d}\right).$$

Then ι acts on the variety $\mathfrak{h}$ and all related geometric objects. Fixed points of ι are given by

$$(\mathfrak{h})^\iota = \mathfrak{h}_d, \quad H^\iota = H_0.$$

We apply a version of the equivariant localisation with respect to ι on the variety $\ddot{\mathfrak{h}}_{\mathrm{O}}$.

We will denote $\Xi_\sigma = \Xi \cap \mathfrak{F}_I^\sigma$ and $\underline{\Xi}_\sigma$ the image of Ξ_σ in $\underline{\Xi}$. For each $\nu, \nu' \in \Xi_\sigma$, we set

$$\mathcal{Z}_H^{\nu,\nu'} = \left(H_0 \times^{P_0^{\sigma \leq \nu}} \left(\mathrm{O}_d \times \mathfrak{u}_d^{\sigma \leq \nu}\right)\right) \times_{\mathfrak{h}_d} \left(H_0 \times^{P_0^{\sigma \leq \nu'}} \left(\mathrm{O}_d \times \mathfrak{u}_d^{\sigma \leq \nu'}\right)\right).$$

It is a variant of the Steinberg-type varieties $\mathcal{Z}^{\nu,\nu'}$. Following the same procedure as in §4.3.1, we define a $H_0 \times \mathbf{C}_q^\times$-equivariant complex $\mathscr{K}_H$ on $\mathcal{Z}_H^{\nu,\nu'}$ as well as a convolution product on the cohomology $\mathrm{H}^\bullet_{H_{0,q}}\left(\mathcal{Z}_H^{\nu,\nu'}, \mathscr{K}_H\right)$.

Lemma 4.28. *If $i : \ddot{\mathfrak{h}}_{\mathrm{O}}^\iota \longrightarrow \ddot{\mathfrak{h}}_{\mathrm{O}}$ denote the inclusion of fixed points, then there is a canonical isomorphism of algebras*

$$\bigoplus_{\underline{\nu},\underline{\nu}' \in \underline{\Xi}_\sigma} \mathrm{H}^\bullet_{H_0}\left(\mathcal{Z}_H^{\nu,\nu'}, \mathscr{K}_H\right) \cong \mathrm{H}^\bullet_{H_0}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}^\iota, i^*\mathscr{K}\right)$$

Proof. The ι-fixed points of $\ddot{\mathfrak{h}}_{\mathrm{O}}^\iota$ can be described as follows: for each $\nu \in \mathfrak{F}_I^\sigma$, the Levi subgroups $M \subset H$ and $L^\nu \subset H$ are H-conjugate; let $\mathrm{O}^\nu \subset \mathfrak{l}^\nu$ denote the image of $\mathrm{O} \subset \mathfrak{m}$ via under this conjugation and $\mathrm{O}_d^\nu = \mathrm{O}^\nu \cap \mathfrak{l}_d^\nu$; then

$$\ddot{\mathfrak{h}}_{\mathrm{O}}^\iota = \bigsqcup_{\underline{\nu},\underline{\nu}' \in W_{\mathbf{x}} \backslash \mathfrak{F}_I^\sigma} \left(H_0 \times^{P_0^{\sigma \leq \nu}} \left(\mathrm{O}_d^\nu \times \mathfrak{u}_d^{\sigma \leq \nu}\right)\right) \times_{\mathfrak{h}_d} \left(H_0 \times^{P_0^{\sigma \leq \nu'}} \left(\mathrm{O}_d^{\nu'} \times \mathfrak{u}_d^{\sigma \leq \nu'}\right)\right).$$

It is shown in the proof of [40, 4.5.4] that $\mathrm{O}_d^\nu = \emptyset$ if $\underline{\nu} \notin \underline{\Xi}$. Similarly $\mathrm{O}_d^{\nu'} = \emptyset$ if $\underline{\nu}' \notin \underline{\Xi}$. By the cleaness of $\mathscr{C}$ (§4.3.1), the restriction of cohomlogy yields an isomorphism

$$\bigoplus_{\underline{\nu},\underline{\nu}' \in \underline{\Xi}_\sigma} \mathrm{H}^\bullet_{H_0}\left(\mathcal{Z}_H^{\nu,\nu'}, \mathscr{K}_H\right) \cong \mathrm{H}^\bullet_{H_0}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}^\iota, i^*\mathscr{K}\right).$$

□

The cocharacter $\iota : \mathbf{C}^\times \longrightarrow H_{0,q}$ yields a closed point $\iota \in \mathrm{Spec}(\mathrm{H}^\bullet_{H_{0,q}})$. The localisation theorem (*cf.* Evens–Mirković [15, 4.10]) states that we have an isomorphism of algebras after the completion of equivariant cohomology groups at the maximal ideal $\mathfrak{m}_{\iota/2m} \subset \mathrm{H}^\bullet_{H_{0,q}}$:

$$\mathrm{H}^*_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}, \mathscr{K}\right)^\wedge_{\mathfrak{m}_{\iota/2m}} \cong \mathrm{H}^*_{H_{0,q}}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}^{\mathbf{C}_{\mathbf{x}}^\times}, i^*\mathscr{K}\right)^\wedge_{\mathfrak{m}_{\iota/2m}}. \tag{4.29}$$

Combining Theorem 4.27, Lemma 4.28 and (4.29), we obtain

$$\begin{aligned} \mathbb{H}_J &\cong \mathrm{H}^*_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}, \mathscr{K}\right)|_{u=d/2m} \longrightarrow \mathrm{H}^*_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}, \mathscr{K}\right)^\wedge_{\mathfrak{m}_{\iota/2m}}|_{u=d/2m} \\ &\cong \mathrm{H}^*_{H_{0,q}}\left(\ddot{\mathfrak{h}}_{\mathrm{O}}^\iota, i^*\mathscr{K}\right)^\wedge_{\mathfrak{m}_{\iota/2m}}|_{u=d/2m} \cong \bigoplus_{\underline{\nu}',\underline{\nu}'' \in \underline{\Xi}_\sigma} \mathrm{H}^*_{H_0}\left(\mathcal{Z}_H^{\nu',\nu''}, \mathscr{K}_H\right)^\wedge_{\mathfrak{m}_0}. \end{aligned} \tag{4.30}$$

The composition is injective and the image is dense in the $\mathfrak{m}_0$-adic topology on the last term.

4.4.4 Map from $\mathbb{H}_J$ to $\widehat{\mathcal{H}}$

We introduce intermediate morphisms which relate $\mathcal{Z}_H^{\nu,\nu'}$ *cf.* §4.4.3 and $\mathcal{Z}^{\nu,\nu'}$ *cf.* §4.3.1 and deduce a ring homomorphism on the convolution algebras.

We fix $\sigma \in \Xi_J$. For $\nu, \nu' \in \Xi_\sigma$, we put

$$\tilde{\mathcal{Z}}_H^{\nu,\nu'} = \left(P_0^\sigma \times^{P_0^\nu} \left(\mathrm{O}_d \times \mathfrak{u}_d^\nu\right)\right) \times_{\mathfrak{g}_{\underline{d}}} \left(P_0^\sigma \times^{P_0^{\nu'}} \left(\mathrm{O}_d \times \mathfrak{u}_d^{\nu'}\right)\right).$$

Lemma 4.31. *For any $\nu, \nu' \in \Xi_\sigma$, there is a diagram of canonical morphisms*

$$\mathcal{Z}_H^{\nu,\nu'} \xleftarrow[p]{} \tilde{\mathcal{Z}}_H^{\nu,\nu'} \xhookrightarrow{i} \mathcal{Z}^{\nu,\nu'}$$

in which

(i) p is an affine fibration with fibers isomorphic to $\mathfrak{u}_d^\sigma$, whereas

(ii) i is a closed immersion, which factorises as

$$\begin{tikzcd} \tilde{\mathcal{Z}}_H^{\nu,\nu'} \arrow[r, "i"] \arrow[dr, "(e,\mathrm{id})"'] & \bigcup_{[w]\in \mathcal{W}_I\backslash\mathcal{W}_J/\mathcal{W}_I} \mathcal{Z}_w^{\nu,\nu'} \arrow[r, "\subseteq"] & \mathcal{Z}^{\nu,\nu'} \\ & G_{\underline{0}} \times^{P_0^\sigma} \tilde{\mathcal{Z}}_H^{\nu,\nu'} \arrow[u, "\cong"] & \end{tikzcd}.$$

Proof. Since the adjoint action of P_0^ν on $\mathrm{O}_d \times (\mathfrak{u}_d^\nu/\mathfrak{u}_d^\sigma)$ factorises through the quotient $P_0^\nu \longrightarrow P_0^\nu/U_0^\sigma \cong P_0^{\sigma\leq\nu}$ *cf.* §4.4.2, we have the isomorphisms

$$P_0^\sigma \times^{P_0^\nu} (\mathrm{O}_d \times (\mathfrak{u}_d^\nu/\mathfrak{u}_d^\sigma)) \cong (P_0^\sigma/U_0^\sigma) \times^{P_0^\nu/U_0^\sigma} (\mathrm{O}_d \times (\mathfrak{u}_d^\nu/\mathfrak{u}_d^\sigma)) \cong H_0 \times^{P_0^{\sigma\leq\nu}} \left(\mathrm{O}_d \times \mathfrak{u}_d^{\sigma\leq\nu}\right)$$

which fits into the following sequence

$$P_0^\sigma \times^{P_0^\nu} (\mathrm{O}_d \times \mathfrak{u}_d^\nu) \xrightarrow{\text{projection}} P_0^\sigma \times^{P_0^\nu} (\mathrm{O}_d \times (\mathfrak{u}_d^\nu/\mathfrak{u}_d^\sigma)) \cong H_0 \times^{P_0^{\sigma\leq\nu}} \left(\widehat{\mathrm{O}}_d \times \mathfrak{u}_d^{\sigma\leq\nu}\right)$$

in which the first morphism is a locally trivial fibration with fibers $\mathfrak{u}_d^\sigma$. Substituting these morphisms into the definition of $\tilde{\mathcal{Z}}_H^{\nu,\nu'}$ yields morphism p. The morphism i comes from the inclusion $P_0^\sigma \hookrightarrow G_{\underline{0}}$. This proves (i).

The statement (ii) is due to the fact that the image of the inclusion of $G_{\underline{0}}$-schemes

$$G_{\underline{0}} \times^{P_0^\sigma} \left((P_0^\sigma/P_0^\nu) \times \left(P_0^\sigma/P_0^{\nu'}\right)\right) \longrightarrow (G_{\underline{0}}/P_0^\nu) \times \left(G_{\underline{0}}/P_0^{\nu'}\right). \tag{4.32}$$

identified with $\bigcup_{[w]\in \mathcal{W}_I\backslash\mathcal{W}_J/\mathcal{W}_I} \mathcal{X}_w^{\nu,\nu'}$. □

Convolution products can be defined similarly on $Z_H^{\nu,\nu'}$ and on $\tilde{Z}_H^{\nu,\nu'}$.

Proposition 4.33. *The morphisms i and p considered in Lemma 4.31 induce maps*

$$\mathrm{H}^\bullet_{H_0}\left(\mathcal{Z}_H^{\nu,\nu'}, \mathscr{K}_H\right) \cong \mathrm{H}^\bullet_{P_0^\sigma}\left(\tilde{\mathcal{Z}}_H^{\nu,\nu'}, \tilde{\mathscr{K}}_H\right) \hookrightarrow \mathrm{H}^\bullet_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu''}, \mathscr{K}\right),$$

which commute with the convolution products.

Proof. We have a commutative diagram

$$\begin{tikzcd}
(\mathcal{T}^{\nu}\times\mathcal{T}^{\nu'})\times(\mathcal{T}^{\nu'}\times\mathcal{T}^{\nu''}) & & \mathcal{T}^{\nu}\times\mathcal{T}^{\nu'}\times\mathcal{T}^{\nu''} \arrow[ll, "d"'] & & & \\
\mathcal{Z}^{\nu,\nu'}\times\mathcal{Z}^{\nu',\nu''} \arrow[u] \arrow[rr, phantom, "\square"] & & \mathcal{Z}^{\nu,\nu'}\times_{\mathcal{T}^{\nu'}}\mathcal{Z}^{\nu',\nu''} \arrow[ll, "\gamma"'] \arrow[u] \arrow[r, "\mu"] & \mathcal{Z}^{\nu,\nu''} & & \\
\tilde{\mathcal{Z}}_H^{\nu,\nu'}\times\tilde{\mathcal{Z}}_H^{\nu',\nu''} \arrow[u, hook, "i\times i"] \arrow[d, two heads, "p\times p"] \arrow[rr, phantom, "\square"] & & \tilde{\mathcal{Z}}_H^{\nu,\nu'}\times_{\mathcal{T}^{\nu'}}\tilde{\mathcal{Z}}_H^{\nu',\nu''} \arrow[ll, "\tilde{\gamma}_H"'] \arrow[u, hook, "i\times i"] \arrow[d, two heads, "p\times p"] \arrow[r, "\tilde{\mu}_H"] \arrow[dr, phantom, "\square"] & \tilde{\mathcal{Z}}_H^{\nu,\nu''} \arrow[u, hook, "i"] \arrow[d, two heads, "p"] \arrow[r] \arrow[dr, phantom, "\square"] & \mathfrak{p}_d^{\sigma} \arrow[d, two heads] & \\
\mathcal{Z}_H^{\nu,\nu'}\times\mathcal{Z}_H^{\nu',\nu''} & & \mathcal{Z}_H^{\nu,\nu'}\times_{\mathcal{T}_H^{\nu'}}\mathcal{Z}_H^{\nu',\nu''} \arrow[ll, "\gamma_H"'] \arrow[r, "\mu_H"] & \mathcal{Z}_H^{\nu,\nu''} \arrow[r] & \mathfrak{h}_d &
\end{tikzcd},$$

containing four cartesian squares as indicated by a box "$\square$". The second line of it defines the convolution product on $\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu,\nu'},\mathscr{K}\right)$, the third line defines the convolution product on $\mathrm{H}^{\bullet}_{P_0^{\sigma}}\left(\tilde{\mathcal{Z}}_H^{\nu,\nu'},\tilde{\mathscr{K}}_H\right)$, whereas the fourth line defines the convolution product on $\mathrm{H}^{\bullet}_{H_0}\left(\mathcal{Z}_H^{\nu,\nu'},\mathscr{K}_H\right)$. The diagram induces a corresponding commutative diagram of cohomology, thanks to base change and composability of units and co-units of adjuntion and the fact that there is a canonical isomorphism $i^*\mathscr{K}\cong\tilde{\mathscr{K}}_H$. $\square$

Let $\underline{\Xi}_J=W_{\mathbf{x}}\backslash\Xi_J$, so that there are partitions of sets

$$\Xi=\bigsqcup_{\sigma\in\Xi_J}\Xi_{\sigma},\quad \underline{\Xi}=\bigsqcup_{\sigma\in\underline{\Xi}_J}\underline{\Xi}_{\sigma}.$$

We obtain a sequence of injective maps of algebras

$$\begin{aligned}\mathbb{H}_J &\xrightarrow{(4.30)} \prod_{\underline{\sigma}\in\underline{\Xi}_J}\bigoplus_{\underline{\nu}',\underline{\nu}''\in\underline{\Xi}_{\sigma}}\mathrm{H}^{\bullet}_{H_0}\left(\mathcal{Z}_H^{\nu',\nu''},\mathscr{K}\right)^{\wedge}\xrightarrow{(4.33)}\prod_{\underline{\sigma}\in\underline{\Xi}_J}\bigoplus_{\underline{\nu}',\underline{\nu}''\in\underline{\Xi}_{\sigma}}\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu',\nu''},\mathscr{K}\right)^{\wedge}\\
&\hookrightarrow\prod_{\underline{\nu}''\in\underline{\Xi}}\bigoplus_{\underline{\nu}'\in\underline{\Xi}}\mathrm{H}^{\bullet}_{G_{\underline{0}}}\left(\mathcal{Z}^{\nu',\nu''},\mathscr{K}\right)^{\wedge}=\widehat{\mathcal{H}},\end{aligned}$$

cf. §4.4.3. We denote the composite of it by $\Phi_J:\mathbb{H}_J\longrightarrow\widehat{\mathcal{H}}$.

Proposition 4.34. *For $\nu,\nu'\in\Xi$ and $w\in\tilde{\mathcal{W}}_J$ such that $\underline{\nu}=w\underline{\nu}'$, the graded piece $\widehat{\mathcal{H}}^{\nu,\nu'}_{\leq w}/\widehat{\mathcal{H}}^{\nu,\nu'}_{<w}\cong\widehat{\mathcal{H}}^{\nu,\nu'}_{w}$ is a free of rank 1 as left $\widehat{\mathcal{H}}^{\nu,\nu}_{e}$-module and as right $\widehat{\mathcal{H}}^{\nu',\nu'}_{e}$-module generated by the image of $\Phi_J(w)$.*

Proof. By [31, 4.9], there is a decomposition of $\mathrm{H}^{\bullet}_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O},e},\mathscr{K}\right)$-bimodules

$$\mathrm{H}^{\bullet}_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O}},\mathscr{K}\right)\cong\bigoplus_{w\in\tilde{\mathcal{W}}_J}\mathrm{H}^{\bullet}_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O},w},\mathscr{K}\right),$$

By the construction of Lusztig's isomorphism Theorem 4.27, the graded piece $\mathrm{H}^{\bullet}_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O},w},\mathscr{K}\right)=\mathrm{H}^{\bullet}_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O},\leq w},\mathscr{K}\right)/\mathrm{H}^{\bullet}_{H_q}\left(\ddot{\mathfrak{h}}_{\mathrm{O},<w},\mathscr{K}\right)$ is freely generated by the image of $w\in\mathbb{H}_J$. The maps (4.30) and 4.33 then sends it to a generator of $\widehat{\mathcal{H}}^{\nu,\nu'}_{w}$.

□

4.4.5 Map from $\mathbb{H}$ to $\widehat{\mathcal{H}}$

We prove that the maps Φ_J are compatible one to another and thus define a morphism from the dDAHA to $\widehat{\mathcal{H}}$. The proof of the following lemma is straightfoward.

Lemma 4.35. *Given two proper subsets J, K of S such that $I \subseteq J \subseteq K \subsetneq S$, there is a canonical inclusion of graded affine Hecke algebras $\mathbb{H}_J \subseteq \mathbb{H}_K$ and the maps Φ_J and Φ_K are compatible in the sense that*

$$\Phi_K \mid_{\mathbb{H}_J} = \Phi_J.$$

□

Corollary 4.36. *There is a ring homomorphism*

$$\Phi : \mathbb{H} \longrightarrow \widehat{\mathcal{H}}.$$

Proof. From the definition of $\mathbb{H}$, we clearly have $\mathbb{H} \cong \varinjlim_{I \subseteq J \subsetneq S} \mathbb{H}_J$. By Lemma 4.35, the system $\{\Phi_J\}_J$ is compatible under restriction. We define $\Phi = \varinjlim_{I \subseteq J \subsetneq S} \Phi_J$, which is a ring homomorphism

$$\Phi : \mathbb{H} \cong \varinjlim_{I \subseteq J \subsetneq S} \mathbb{H}_J \longrightarrow \widehat{\mathcal{H}}.$$

□

4.4.6 Density of image

Following the line of [53, 4.8, 4.9], we prove that the image of Φ is dense in the product topology of the convolution algebra $\widehat{\mathcal{H}}$.

Recall that $\widehat{\mathcal{H}}$ defined in §4.3.6 is equipped with a topology.

Lemma 4.37. *Let $w \in \tilde{\mathcal{W}}$. For each $\nu \in \Xi$, letting $\nu' = w^{-1}\nu$, the graded piece $\widehat{\mathcal{H}}_w^{\nu,\nu'}$ is free of rank 1 as left $\widehat{\mathcal{H}}_e^{\nu,\nu}$-module and as right $\widehat{\mathcal{H}}_e^{\nu',\nu'}$-module, with generator the image of $\Phi(w)$ in $\mathcal{H}_{w,a}^{\nu,\nu'}$.*

Proof. We prove the assertion by induction on $\tilde{\ell}(w)$. When $\tilde{\ell}(w) = 0$, this is trivial. When $\tilde{\ell}(w) = 1$, $w = \tilde{s}_a \in \tilde{\mathcal{W}}$ is a relative simple reflection and the result follows from Proposition 4.34.

Suppose then $\tilde{\ell}(w) \geq 2$. We choose $s \in \tilde{\mathcal{W}}$ such that $\tilde{\ell}(s) = 1$ and $\tilde{\ell}(ws) = \tilde{\ell}(w) - 1$. Since the image of $\Phi(ws)$ in $\widehat{\mathcal{H}}_w^{\nu,(ws)^{-1}\nu}$ and the image of $\Phi(s)$ in $\widehat{\mathcal{H}}_s^{(ws)^{-1}\nu,\nu'}$ are free generators by inductive hypothesis, by the surjectivity Lemma 4.22 (iii), $\Phi(w) = \Phi(ws)\Phi(s)$ in turn generates freely $\widehat{\mathcal{H}}_{ws}^{\nu,\nu'}$. □

Lemma 4.38. *In the particular case where $J = I$, the image of the morphism*

$$\Phi_I : \mathbf{S} = \mathbb{H}_I \longrightarrow \widehat{\mathcal{H}}_e$$

is dense.

Proof. By definition $\widehat{\mathcal{H}}_e = \prod_{\underline{\nu}\in\underline{\Xi}} \widehat{\mathcal{H}}_e^{\nu,\nu}$. Since the composite of Φ_I with each projection $\prod_{\underline{\nu}\in\underline{\Xi}} \widehat{\mathcal{H}}_e^{\nu,\nu} \longrightarrow \widehat{\mathcal{H}}_e^{\nu,\nu}$ is equal to the map $\Phi^{\nu,\nu}$ of Lemma 4.25. By Lemma 4.25, $\widehat{\mathcal{H}}_e^{\nu,\nu}$ is a topological $\mathbf{S}$-module supported on $\mathbf{x}_\nu \in \mathbb{E}$ and that the image of the composition

$$\mathbf{S} \xrightarrow{\Phi_I} \widehat{\mathcal{H}}_e \xrightarrow{\text{proj}} \widehat{\mathcal{H}}_e^{\nu,\nu}$$

is dense. If $\nu, \nu' \in \Xi$ are such that $\mathbf{x}_\nu = \mathbf{x}_{\nu'}$, then ν and ν' are conjugate by $W_{\mathbf{x}}$, which implies that $\underline{\nu} = \underline{\nu}' \in \underline{\Xi}$. As there is a canonical isomorphism $\operatorname{Spec} \mathbb{H}_I \cong \mathbb{E} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}_\ell$, we may consider $\mathbf{x}_\nu$ as closed points of $\operatorname{Spec} \mathbf{S}$.

Since we have seen that $\mathbf{x}_\nu \neq \mathbf{x}_{\nu'}$ if $\underline{\nu} \neq \underline{\nu}'$, for any finite subset $\underline{\Xi}' \subseteq \underline{\Xi}$, the composite

$$\mathbf{S} \xrightarrow{\Phi_I} \prod_{\nu\in\underline{\Xi}} \widehat{\mathcal{H}}_e^{\nu,\nu} \to \prod_{\nu\in\underline{\Xi}'} \widehat{\mathcal{H}}_e^{\nu,\nu}$$

is dense as well, by the chinese remainder theorem. Hence, by the definition of product topology, the image of Φ_I is dense. □

Theorem 4.39. *The image of $\Phi : \mathbb{H} \longrightarrow \widehat{\mathcal{H}}$ is dense.*

Proof. Taking the subset $I = I_{\mathbb{E}} \subseteq S$, we have $\Phi_I : \mathbb{H}_I \longrightarrow \widehat{\mathcal{H}}_e$ with dense image, after Lemma 4.38, whence $\operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_e$ is dense in $\widehat{\mathcal{H}}_e$. We can prove by induction on $\ell(w)$ that $\operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_{\leq w}$ is dense in $\widehat{\mathcal{H}}_{\leq w}$ for each $w \in \tilde{\mathcal{W}}$. Indeed, assume that $\operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_{\leq w'}$ is dense in $\mathcal{H}_{\leq w',a}$ for all w' such that $\ell(w') < \ell(w)$. Then

$$\sum_{\ell(w')<\ell(w)} \left(\operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_{\leq w'}\right) \subseteq \operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_{<w}.$$

is dense in $\widehat{\mathcal{H}}_{<w}$. Consider following diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_{<w} \arrow[r] \arrow[d, "\subseteq"] & \operatorname{im}(\Phi) \cap \widehat{\mathcal{H}}_{\leq w} \arrow[r] \arrow[d, "\subseteq"] & \frac{\operatorname{im}(\Phi)\cap\widehat{\mathcal{H}}_{\leq w}}{\operatorname{im}(\Phi)\cap\widehat{\mathcal{H}}_{<w}} \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{H}}_{<w} \arrow[r] & \widehat{\mathcal{H}}_{\leq w} \arrow[r] & \widehat{\mathcal{H}}_w \arrow[r] & 0
\end{tikzcd}$$

The image of the right vertical arrow, denoted V, is an $\mathbb{H}_I$-submodule. By the density of $\operatorname{im}(\Phi_I) \subseteq \widehat{\mathcal{H}}_e$, the closure $\overline{V}$ is a $\widehat{\mathcal{H}}_e$-submodule of $\widehat{\mathcal{H}}_w$. By Lemma 4.37, since $\overline{V}$ contains $\Phi(w)$, we have $\overline{V} = \widehat{\mathcal{H}}_w$, so the right vertical arrow has dense image. Since the left arrow also has dense image, so is the middle arrow.

Since $\bigcup_{w\in\tilde{\mathcal{W}}} \widehat{\mathcal{H}}_{\leq w}$ is dense in $\widehat{\mathcal{H}}$, so is the image of Φ. □

4.5 Simple and proper standard modules

In this section, we suppose that $\epsilon = d/|d|$.

4.5.1 Geometric parametrisation of simple modules

Consider the block $\mathcal{O}_{\mathbf{x}_\nu}(\mathbb{H})$ of the category $\mathcal{O}$ of the dDAHA $\mathbb{H}$ associated to the $\mathcal{W}$-orbits $\{\mathbf{x}_\nu\}_{\nu\in\Xi}$ in $\mathbb{E}$, as defined in §1.2.3.

Recall the Lusztig sheaf $\mathbf{I}^\nu$ defined in §4.2.4. By the hypothesis $\epsilon = d/|d|$ and Proposition 4.14 (i), the complex $\mathbf{I}^\nu$ is supported in the nilpotent cone $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ for all $\nu \in \Xi$ and they generate the block $\mathrm{D}^b(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}})_\zeta$, *cf.* §3.2.3. Let $\mathbf{I} = \bigoplus_{\nu\in\underline{\Xi}} \mathbf{I}^\nu$ be the (infinite) sum. Recall the category of smooth modules $\widehat{\mathcal{H}}$-mod^{sm} defined in §4.3.6.

Theorem 4.40 (Springer correspondence)**.** *The following statements hold:*

(i) The pull-back via the homomorphism Φ of § 4.4.5, denoted by Φ^*, induces an equivalence of category

$$\Phi^* : \widehat{\mathcal{H}}\text{-mod}^{sm} \cong \mathrm{O}_{\mathbf{x}_\nu}(\mathbb{H}).$$

(ii) For any $\pi = (\mathrm{O}_\pi, \mathscr{L}_\pi) \in \Pi(\mathfrak{g}_{\underline{d}})$ (cf. §3.1.1), the $\mathbb{H}$-module

$$\mathbf{L}_\pi = \Phi^* \operatorname{Hom}_{G_{\underline{0}}}(\mathrm{IC}(\mathscr{L}_\pi), {}^p\mathscr{H}^\bullet\mathbf{I})$$

is in $\mathcal{O}_a(\mathbb{H})$ and irreducible if $\pi \in \Pi(\mathfrak{g}_{\underline{d}}\mathfrak{g})_\zeta$; otherwise $\mathbf{L}_\pi = 0$.

(iii) The simple objects in $\mathcal{O}_{\mathbf{x}_\nu}(\mathbb{H})$ are given by $\{\mathbf{L}_\pi\}_{\pi\in\Pi(\mathfrak{g}_{\underline{d}})_\zeta}$.

Remark 4.41. *(i) The assertion (ii) confirms the multiplicity-one conjecture in [40].*

(ii) We have supposed that the points $\mathbf{x}_\nu \in \mathbb{E}$ are rational. However, the general case can be easily reduced to the rational case.

(iii) With more care, one can remove the hypothesis that the grading on $\mathfrak{g}$ is inner by working with twisted affine root systems.

Proof of Theorem 4.40. The statement (i) is standard, as the map $\Phi : \mathbb{H} \longrightarrow \widehat{\mathcal{H}}$ is injective with dense image, see [53, 7.6]. The assertion (ii) follows from (i) together with Lemma 4.24 and Theorem 4.39.

Now let $\mathbf{L} \in \mathcal{O}_a(\mathbb{H})$ be irreducible. Using (i), $\mathbf{L}$ can be equipped with a smooth $\widehat{\mathcal{H}}$-module structure, which is simple. By Lemma 4.24, it must be isomorphic to some simple constituent of $\mathbf{I}$, thus isomorphic to $\mathbf{L}_\pi$ for some $\pi \in \Pi(\mathfrak{g}_{\underline{d}})$. This completes the proof. □

4.5.2 Proper standard modules

Let $z \in \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ be a nilpotent element. Let $\mathcal{T}_e^\nu$ be the fibre of $\mathcal{T}^\nu \longrightarrow \mathfrak{g}_{\underline{d}}$ at z and let $i_z : \mathcal{T}_z^\nu \longrightarrow \mathcal{T}^\nu$ be the closed inclusion. Consider the vector space

$$\overline{\Delta}_z = \bigoplus_{\underline{\nu}\in\underline{\Xi}} \mathrm{H}^\bullet\left(\mathcal{T}_z^\nu, i_z^!\dot{\mathscr{C}}\right).$$

By the formalism of convolution algebras, for each $\nu, \nu' \in \Xi$ there is a natural map

$$\mathcal{H}^{\nu,\nu'} \longrightarrow \operatorname{Hom}\left(\mathrm{H}^\bullet\left(\mathcal{T}_z^{\nu'}, i^!\dot{\mathscr{C}}\right), \mathrm{H}^\bullet\left(\mathcal{T}_z^\nu, i^!\dot{\mathscr{C}}\right)\right).$$

Taking the sum over $\underline{\nu} \in \underline{\Xi}$ and the product over $\underline{\nu}' \in \underline{\Xi}$, we obtain a smooth $\widehat{\mathcal{H}}$-action on $\overline{\Delta}_z$. Besides, there is a natural $\pi_0\left(G_{\underline{0},z}\right)$-action on $\overline{\Delta}_z$, which commutes with the $\widehat{\mathcal{H}}$-action. For any $\pi = (\mathrm{O}_\pi, \mathscr{L}_\pi) \in \Pi(\mathfrak{g}_{\underline{d}})$, define

$$\overline{\Delta}_\pi = \mathrm{Hom}_{\pi_0\left(G_{\underline{0},z}\right)}\left(\mathscr{L}_z, \overline{\Delta}_z\right)$$

for any $z \in \mathrm{O}_\pi$. to be the $\mathscr{L}_z$-isotypic component as $\mathbf{C}\pi_0(G_{\underline{0},z})$-module, which is a $\widehat{\mathcal{H}}$-submodule of $\overline{\Delta}_\pi$. We view $\overline{\Delta}_\pi$ as a $\mathbb{H}$-module via $\Phi : \mathbb{H} \longrightarrow \widehat{\mathcal{H}}$ *cf.* §4.4.5. We call $\overline{\Delta}_\pi$ the **proper standard module** of $\mathbb{H}$.

Theorem 4.42. *For each pair $\pi \in \Pi(\mathfrak{g}_{\underline{d}})$ as above, the following holds.*

(i) $\overline{\Delta}_\pi \in \mathcal{O}_{\mathbf{x}_\nu}(\mathbb{H})$.

(ii) $\overline{\Delta}_\pi \neq 0$ *if and only if* $\mathbf{L}_\pi \neq 0$ *if and only if* $\pi \in \Pi(\mathfrak{g}_{\underline{d}})_\zeta$.

(iii) For any pair $\pi' = (\mathrm{O}_{\pi'}, \mathscr{L}_{\pi'}) \in \Pi(\mathfrak{g}_{\underline{d}})_\zeta$, *the Jordan–Hölder multiplicity of* $\mathbf{L}_{z',\chi'}$ *in* $\overline{\Delta}_{z,\chi}$ *is given by*

$$\left[\overline{\Delta}_\pi : \mathbf{L}_{\pi'}\right] = \sum_k \dim \mathrm{Hom}_{\pi_0\left(G_{\underline{0},z}\right)}\left(z^*\mathscr{L}_\pi, \mathrm{H}^k\left(z^! \,\mathrm{IC}(\mathscr{L}_{\pi'})\right)\right).$$

for any $z \in \mathrm{O}_\pi$.

Proof. The assertion (i) results from the smoothness of the $\widehat{\mathcal{H}}$-action and Theorem 4.40 (i).

The assertion (ii) can be proven with the same arguments as [37, 8.17], using the block decomposition of the equivariant category of $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$, *cf.* §3.2.3. The assertion (iii) is standard, see [13, 8.6.23]. □

Chapter 5

Applications

In this chapter, we present some applications of the constructions from the previous chapters, in the hope of clarifying the relations between them.

5.1 Sheaf-theoretic interpretation of V

In this section, we explain a sheaf theoretic construction of the Knizhnik–Zamolodchikov functor $\mathbf{V} : \mathbf{H}_{\lambda_0}$ -gmod $\longrightarrow \mathbf{K}_{\ell_0}$ -gmod for degenerate double affine Hecke algebras, *cf.* §1.5.

Keep the notations from §4.1.1. For simplicity, in this section, we fix the principal block admissible system $\zeta_0 = (T, \mathfrak{t}_*, 0, \delta_0)$ on $\mathfrak{g}_{\underline{d}}$ (*cf.* §3.2.3) and let $\sigma_0 = (T, 0, \delta_0)$ be the corresponding principal supercuspidal system on $\mathfrak{g}_{\underline{d}}$ (*cf.* §3.5.2). Here δ_0 is the skyscraper sheaf supported on $0 \in \mathfrak{t}$ with fiber $\overline{\mathbf{Q}}_\ell$. We denote $G_{\underline{0},q} = G_{\underline{0}} \times \mathbf{C}_q^\times$. We also fix an isomorphism of fields $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$. We also fix a sign $\epsilon \in \{1, -1\}$.

5.1.1 The algebras $\mathbf{H}_{\lambda_0}$ and $\mathcal{H}$

We discuss here the relation between the idempotent form $\mathbf{H}_{\lambda_0}$ of the degenerate DAHA defined in §1.2.5 and the convolution algebra $\widehat{\mathcal{H}}$ defined in §4.3.

The idempotent algebra $\mathbf{H}_{\lambda_0}$ depends on an an affine root system (E, S), a family of parameters $(h_a)_{a\in\Delta}$ and a W_S-orbit $W_S\lambda_0 \subset E$. On the other hand, the convolution algebra $\widehat{\mathcal{H}}$ depends on the affine root system $(\mathbb{A}_{\mathbf{R}}, S)$, where $\mathbb{A}_{\mathbf{R}} = \mathbb{A} \otimes_{\mathbf{Q}} \mathbf{R}$, and the W_S-orbit $\{\mathbf{x}_\nu\}_{\nu\in\underline{\Xi}} \subset \mathbb{A}$ (*cf.* §4.3.7). The orbit $\tilde{W}\mathbf{x}_\nu \in \mathbb{A}$ coincides with the spectrum of the action of the polynomial subalgebra $\mathbf{S} \subset \mathbb{H}$. We define $\mathcal{H} = \bigoplus_{\nu,\nu'\in\underline{\Xi}} \mathcal{H}^{\nu',\nu}$ (recall that $\mathcal{H}^{\nu',\nu} \cong \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}^\nu, \mathbf{I}^{\nu'})$.) It is an non-unital associative algebra with the Yoneda product as multiplication. There is an obvious inclusion $\mathcal{H} \subset \widehat{\mathcal{H}}$ with dense image. Now we set $E = \mathbb{A}_{\mathbf{R}}$, $S = \tilde{S}$ and $h_a = c_a(-d/m), \lambda_0 = \mathbf{x}_{\nu_0}$ by choosing any $\nu_0 \in \Xi$.

Proposition 5.1. *The extension algebra $\mathcal{H}$ is isomorphic to the idempotent algebra $\mathbf{H}_{\lambda_0}$ defined with the datum $(E, S, (h_a)_{a\in\Delta}, W_S\lambda_0) = (\mathbb{A}_{\mathbf{R}}, (d/m)_{a\in\Delta}, W_S\mathbf{x}_\nu)$.*

Proof. We construct a representation of $\mathcal{H}$ on the polynomial algebra $\mathrm{Pol}_{W_S\lambda_0}$. For each $\nu \in \Xi$, consider the cohomology $\mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{T}^\nu, \dot{\mathscr{C}} \otimes \dot{\mathscr{C}}^*\right)$. Since the projection $[\mathcal{T}^\nu/G_{\underline{0}}] \longrightarrow$

$[\mathrm{O}_d^\nu/L_0^\nu]$ is a vector bundle with fiber $\mathfrak{u}_d^\nu$, the pulling back along the projection yields an isomorphism

$$\mathrm{H}^*_{L_0}\left(\mathrm{O}_d^\nu, \mathscr{C}\otimes\mathscr{C}^*\right) \cong \mathrm{H}^*_{G_{\underline{0}}}\left(\mathcal{T}^\nu, \dot{\mathscr{C}}\otimes\dot{\mathscr{C}}^*\right).$$

Since $\mathscr{C}$ is a cuspidal local system, the former equivariant cohomology group can be calculated with any chosen point $\mathbf{o}\in\mathrm{O}_d^\nu$:

$$\mathrm{H}^*_{L_0}\left(\mathrm{O}_d^\nu, \mathscr{C}\otimes\mathscr{C}^*\right) \cong \mathrm{H}^*_{\mathrm{Stab}_{L_0}(\mathbf{o})}\left(\{\mathbf{o}\}, (\mathscr{C}\otimes\mathscr{C}^*)_{\mathbf{o}}\right) \cong \mathrm{H}^*_T = \mathbf{S}.$$

The algebra $\mathbf{S}$ is, by definition, the ring of polynomial functions on the affine space $\mathbb{A}$. Upon choosing $\mathbf{x}_\nu\in\mathbb{A}$ as the origin, we obtain isomorphism $V\cong\mathbb{A}$, where V is the vector space of translations on $\mathbb{A}$. Combined with (5.1.1) and (5.1.1), this yields isomorphisms of graded algebras

$$\mathrm{Pol}_{\mathbf{x}_\nu} \cong \mathbf{C}[V] \cong \mathbf{S} \cong \mathrm{H}^*_{G_{\underline{0}}}(\mathcal{T}^\nu, \dot{\mathscr{C}}\otimes\dot{\mathscr{C}}^*).$$

By the formalism of convolution algebras, there is a convolution product for $\nu,\nu'\in\Xi$:

$$\mathcal{H}^{\nu',\nu}\otimes\mathrm{H}^*_{G_{\underline{0}}}(\mathcal{T}^\nu, \mathscr{C}\otimes\mathscr{C}^*) \longrightarrow \mathrm{H}^*_{G_{\underline{0}}}(\mathcal{T}^{\nu'}, \mathscr{C}\otimes\mathscr{C}^*),$$

which gives a $\mathcal{H}$-representation on $\mathrm{Pol}_{W_S\mathbf{x}_\nu}$.

Passing to the $\mathfrak{m}_0$-adic completion, we obtain a continuous $\widehat{\mathcal{H}}$-representation on $\widehat{\mathrm{Pol}}_{W_S\mathbf{x}_\nu}$ and hence a $\mathbb{H}$-module by pulling back along the map Φ of §4.4.5. By the same arguments as in §4.4.4, this $\mathbb{H}$-module comes from the graded-affine-Hecke-algebraic analogue, which can be identified with an infinite sum of the polynomial representation $\mathbb{H}_J/\mathbb{H}_J\cdot(s_a-1\ ;\ a\in J)$ suitably completed. We see that this $\mathbb{H}$-representation on $\widehat{\mathrm{Pol}}_{W_S\mathbf{x}_\nu}$ is faithful and agrees with the representation ψ defined in §1.2.4. Hence we obtain an isomorphism $\widehat{\mathcal{H}}\cong\widehat{\mathbb{H}}_{\lambda_0}$. Restricting this isomorphism to the subalgebra

$$\bigoplus_{\nu,\nu'\in\underline{\Xi}} \mathrm{gHom}_{\mathcal{Z}}(\mathrm{Pol}_{\mathbf{x}_\nu}, \mathrm{Pol}_{\mathbf{x}_{\nu'}}) \subset \mathrm{End}^{\mathrm{cont}}_{\widehat{\mathcal{Z}}}(\widehat{\mathrm{Pol}}_{W_S\mathbf{x}_\nu})$$

we obtain $\mathcal{H}\cong\mathbf{H}_{\lambda_0}$.

□

5.1.2 Spiral and parabolic inductions

We show that parabolic inductions from the principal supercuspidal system $(\mathfrak{t},0,\delta_0)$ are special cases of spiral inductions from the admissible system $(\mathfrak{t},\mathfrak{t}_*,0,\delta_0)$. We take the family of order functions $\omega_\lambda=\mathrm{ord}_\lambda$ as in §1.2.5 so that $\mathbf{H}_{\lambda_0}$ coincides with the algebra $\mathbf{A}^\omega$ defined in §2.1.3. Recall the notions of alcoves §1.1.4, clans and salient cones §2.2.3 and parabolic subalgebra attached to a cocharacter §3.3.4. The following lemma explains the geometric meaning of the clan decomposition:

Lemma 5.2. *Suppose $\nu,\nu'\subset\mathbb{A}_{\mathbf{R}}$ are two alcoves which lie in a same clan $\mathfrak{C}$. Then $\mathfrak{p}_d^\nu=\mathfrak{p}_d^{\nu'}$.*

Proof. From the definition of $\mathfrak{D}^\omega$, an affine root $a=\alpha-r$ belongs to $\mathfrak{D}^\omega$ if $\omega_{\mathbf{x}}(a)\geq 1$. For the order function $\omega_{\mathbf{x}}=\mathrm{ord}_{\mathbf{x}}$, the condition is given by $\langle\alpha,\mathbf{x}\rangle-r=d/m$. On the other hand, by the definition of §4.1.4, with any point $y\in\nu$ chosen, the spiral $\mathfrak{p}_d^\nu$ is given by

$$\mathfrak{p}_d^\nu=\mathfrak{p}_d^{\epsilon(\tilde{\theta}-my)}={}^{\epsilon(\tilde{\theta}-my)}_{\geq\epsilon d}\mathfrak{g}_{\underline{d}}.$$

If we let y varies continuous in $\mathbb{A}_{\mathbf{R}}$, then the subspace ${}^{\epsilon(\tilde{\theta}-my)}_{\geq \epsilon d}\mathfrak{g}_{\underline{d}}$ changes when there is some $\alpha \in R$ such that $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\underline{d}}$ and the value $\langle \alpha, \tilde{\theta} - my\rangle - d$ changes sign. However, the condition $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\underline{d}}$ is equivalent to that $r := \left(\langle \alpha, \tilde{\theta}\rangle - d\right)/m$ is an integer. In this case, put $a = \alpha - r \in S$ and we have $\langle \alpha, \tilde{\theta} - my\rangle = -ma(y)$, so the equality $\langle \alpha, \tilde{\theta} - my\rangle - d = 0$ happens if and only if y touch the wall $H_a \in \mathfrak{D}^\omega$. In other words, as long as y stay in the clan $\mathfrak{C}$, the value $\langle \alpha, \tilde{\theta} - my\rangle - d$ never changes sign, so $\mathfrak{p}_d^{\epsilon(\tilde{\theta}-my)}$ remains unchange. □

The geometric meaning of generic clans is clarified in the following lemma:

Lemma 5.3. *$w\,\nu \subset E$ be an alcove lying in some generic clan $\mathfrak{C} \subset E$. Let v' be a point in the interior of the salient cone κ of $\mathfrak{C}$. Set $v = -\epsilon v'$. Then there is an inclusion $\mathfrak{p}_d^\nu \subseteq \mathfrak{q}_{\underline{d}}^v$ of subspaces of $\mathfrak{g}_{\underline{d}}$ which induces an isomorphism of complexes*

$$\mathrm{Ind}_{\mathfrak{t}_d \subset \mathfrak{p}_d^\nu}^{\mathfrak{g}_{\underline{d}}} \delta_0 \cong \mathrm{Ind}_{\mathfrak{t}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}^v}^{\mathfrak{g}_{\underline{d}}} \delta_0.$$

Proof. By the definition of §4.1.4, with any point $y \in \nu$ chosen, the spiral $\mathfrak{p}_d^\nu$ is given by

$$\mathfrak{p}_d^\nu = \mathfrak{p}_d^{\epsilon(\tilde{\theta}-my)} = {}^{\epsilon(\tilde{\theta}-my)}_{\geq \epsilon d}\mathfrak{g}_{\underline{d}}$$

If v' is in the interior of κ, then $y + tv'$ remains in the same clan $\mathfrak{C}$ for any $t \in \mathbf{R}_{\geq 0}$. This implies that ${}^{\epsilon(\tilde{\theta}-m(y+tv'))}_{\geq \epsilon d}\mathfrak{g}_{\underline{d}} = \mathfrak{p}_d^\nu$ for each $t \in \mathbf{R}_{\geq 0}$. Let $t \longrightarrow +\infty$, we obtain

$$\mathfrak{p}_d^\nu = \lim_{t \longrightarrow +\infty} {}^{\epsilon(\tilde{\theta}-my)+tv}_{\geq \epsilon d}\mathfrak{g}_{\underline{d}} \subset {}^{v}_{\geq 0}\mathfrak{g}_{\underline{d}} = \mathfrak{q}_{\underline{d}}^v.$$

For the same reason, we have an equality on the respective nilpotent radicals $\mathfrak{u}_d^\nu = \mathfrak{v}_{\underline{d}}^v$. Thus there is an isomorphism of varieties

$$G_{\underline{0}} \times^{P_0^\nu} \mathfrak{u}_d^\nu \cong G_{\underline{0}} \times^{Q_{\underline{0}}^v} \mathfrak{v}_{\underline{d}}^v.$$

Note that since P_0^ν and $Q_{\underline{0}}^v$ are both Borel subgroups of $G_{\underline{0}}$, they are conjugate. The direct image of the constant sheaf $\overline{\mathbf{Q}}_\ell$ on these two varieties give $\mathrm{Ind}_{\mathfrak{t}_d \subset \mathfrak{p}_d^\nu}^{\mathfrak{g}_{\underline{d}}} \delta_0$ and $\mathrm{Ind}_{\mathfrak{t}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}^v}^{\mathfrak{g}_{\underline{d}}} \delta_0$ respectively. Hence the two complexes are isomorphic. □

5.1.3 Sheaf-theoretic interpretation of V

Let $\mathcal{P}_\zeta$ be the set of spirals $\mathfrak{p}_*$ which admits the maximal torus $\mathfrak{t}_*$ as splitting and let $\mathcal{P}_\sigma$ be the set of θ-stable Borel subalgebras having $\mathfrak{t}$ as Levi factor, *cf.* §3.3. The Weyl group $W_0 = Z_{G_{\underline{0}}}(\mathfrak{t})/T$ acts on $\mathcal{P}_\zeta$ and $\mathcal{P}_\sigma$ by adjoint action. Let $\underline{\mathcal{P}}_\zeta$ and $\underline{\mathcal{P}}_\sigma$ denote the respective sets of W_0-conjugacy classes.

Let

$$\mathbf{I}_{\mathrm{aff}} = \bigoplus_{\mathfrak{p}_* \in \underline{\mathcal{P}}_\zeta} \mathrm{Ind}_{\mathfrak{t}_d \subset \mathfrak{p}_d}^{\mathfrak{g}_{\underline{d}}} \delta_0, \quad \mathbf{I}_{\mathrm{fin}} = \bigoplus_{\mathfrak{q} \in \underline{\mathcal{P}}_\sigma} \mathrm{Ind}_{\mathfrak{t}_{\underline{d}} \subset \mathfrak{q}_{\underline{d}}}^{\mathfrak{g}_{\underline{d}}} \delta_0.$$

In virtue of the spiral-facet correspondence, the complex $\mathbf{I}_{\mathrm{aff}}$ is nothing but the Lusztig sheaf $\mathbf{I} = \bigoplus_{\nu \in \Xi} \mathbf{I}^\nu$ that we have introduced in §4.5. The complex $\mathbf{I}_{\mathrm{fin}}$ is a "parabolic-induction analogue" of the Lusztig sheaf $\mathbf{I}_{\mathrm{aff}}$, which has been studied in §3.5. Define $\mathcal{K} := \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{fin}})$, which is equipped with the Yoneda product.

Theorem 5.4. *There are isomorphisms:*

(i) $\mathbf{H}_{\lambda_0} \cong \mathcal{H}$ *as graded algebras*

(ii) $\mathbf{K}_{\ell_0} \cong \mathcal{K}$ *as graded algebras*

(iii) via the above two isomorphisms, $\mathbf{V} \cong \mathrm{gHom}_{\mathcal{H}}\left(\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{aff}}), -\right)$.

Proof. The statement (i) has been proven in Proposition 5.1. We prove (ii) and (iii). By Lemma 5.3, the complex $\mathbf{I}_{\mathrm{fin}}$ is subcomplex of $\mathbf{I}_{\mathrm{aff}}$, which takes one factor $\mathbf{I}^{\nu}$ for each $W_{\mathbf{x}}$-conjugacy class of clans. The extension space $\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{aff}})$ is thus isomorphic to $\mathcal{H}\mathbf{e}_{\gamma}$ for some idempotent element $\mathbf{e}_{\gamma} \in \mathcal{H}$. In view of Proposition 5.1 and Lemma 5.3, the idempotent $\mathbf{e}_{\gamma}$ can be chosen as one described in Theorem 2.55, i.e. an idempotent corresponding to generic clans. It follows that the extension algebra $\mathcal{K}$ is isomorphic to

$$\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{fin}}) \cong \mathrm{gEnd}_{\mathcal{H}}(\mathcal{H}\mathbf{e}_{\gamma}, \mathcal{H}\mathbf{e}_{\gamma}) \cong \mathbf{e}_{\gamma}\mathcal{H}\mathbf{e}_{\gamma} \cong \mathbf{e}_{\gamma}\mathbf{H}_{\lambda_0}\mathbf{e}_{\gamma} \cong \mathbf{K}_{\ell_0},$$

where the last isomorphism is due to Theorem 2.43, whence (ii). The module $\mathbf{H}_{\lambda_0}\mathbf{e}_{\gamma}$ which represents the functor $\mathbf{V}$ is isomorphic to $\mathcal{H}\mathbf{e}_{\gamma} = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{aff}})$, whence (iii). □

Remark 5.5.

(i) The algebras $\mathbf{K}_{\ell_0}$ *and* $\mathcal{K}$ *depend only on the congruence class of* d *modulo* m*. However, the isomorphism* $\mathbf{K}_{\ell_0} \cong \mathcal{K}$ *in (ii) depends on the sign* $\epsilon = \pm 1$*. This can be observed from Lemma 5.3, where the definition of* $\mathfrak{q}^{v}_{\underline{d}}$ *depends on* ϵ*.*

(ii) This theorem provides a Ginzburg–Kazhdan–Lusztig geometric realisation of the affine Hecke algebra without equivariant K-theory, at the price of the extreme obscurity of the isomorphism $\mathbf{e}_{\gamma}\mathbf{H}_{\lambda_0}\mathbf{e}_{\gamma} \cong \mathbf{K}_{\ell_0}$*.*

(iii) I could have proven this theorem for a general admissible system $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$*. However, the notational complexity kept me from doing so. Note that the general version of this theorem would be a new geometric construction of the affine Hecke algebra since so far there is no K-theoretic construction which could be applied with coefficient sheaves.*

5.2 Properly stratified categories

We explain the results of [21] on extension algebras, later axiomatised in [25]. All the results presented here are borrowed from *loc. cit.*

Fix an admissible system $\zeta \in \mathfrak{T}(\mathfrak{g}_{\underline{d}})$.

5.2.1 Set of parameters

As in §3.1.1, let $\Pi = \Pi(\mathfrak{g}_{\underline{d}})$ denote the set of isomorphism classes of pairs $\pi = (\mathrm{C}_{\pi}, \mathscr{L}_{\pi})$ on $\mathfrak{g}_{\underline{d}}$. For any $G_{\underline{0}}$-orbit $\mathrm{C} \subset \mathfrak{g}^{\mathrm{nil}}_{\underline{d}}$, let Π_{C} denote the isomorphism classes of pairs π for which $\mathrm{C}_{\pi} = \mathrm{C}$. Let $\Pi_{\zeta} \subset \Pi$ be the subset of Π consisting of $\pi \in \Pi$ such that $\mathrm{IC}(\mathscr{L}_{\pi}) \in \mathrm{Perv}_{G_{\underline{0}}}(\mathfrak{g}^{\mathrm{nil}}_{\underline{d}})_{\zeta}$.

We put the partial order of inverse inclusion of orbit closure on the set of orbits $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}/G_{\underline{0}}$: given orbits $\mathrm{C}, \mathrm{C}' \in \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}/G_{\underline{0}}$, we define $\mathrm{C} \leq \mathrm{C}'$ if $\mathrm{C}' \subset \overline{\mathrm{C}}$. It induces a pre-order on Π via the first projection $\pi \mapsto \mathrm{C}_\pi$, denoted by $\preceq$. In this section, we denote by $\mathbf{I}_\zeta = \bigoplus_{\pi \in \Pi_\zeta} \mathrm{IC}(\mathscr{L}_\pi)$ the sum of perverse sheaves in the block of ζ and by $\mathbf{A}_\zeta = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}_\zeta, \mathbf{I}_\zeta)$ the extension algebra, graded by the cohomological degree. The algebra $\mathbf{A}_\zeta$ is Morita-equivalent to (smooth modules of) the convolution algebra $\widehat{\mathcal{H}}$ for the sign $\epsilon = d/|d|$.

The isomorphism classes of simple $\mathbf{A}_\zeta$ are parametrised, up to shift of degree, by the set Π_ζ: for each $\pi = (\mathrm{C}_\pi, \mathscr{L}_\pi)$, the module $L_\pi = \mathrm{Hom}_{G_{\underline{0}}}(\mathrm{IC}(\mathscr{L}_\pi), \mathbf{I}_\zeta)$ is the simple module indexed by π and $P_\pi = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathrm{IC}(\mathscr{L}_\pi), \mathbf{I}_\zeta)$ is its projective cover.

5.2.2 Standard modules

Recall the notion of affine stratified categories of [25].

For any $\pi = (\mathrm{O}_\pi, \mathscr{L}_\pi) \in \Pi_\zeta$, define the **standard module** to be $\Delta_\pi = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(j_{\pi!}\mathscr{L}_\pi, \mathbf{I}_\zeta) \in \mathbf{A}_\zeta$-gmod. We say that a graded module $M \in \mathbf{A}_\zeta$-gmod is Δ-filtered if $\mathbf{A}_\zeta$ admits a finite filtration of $\mathbf{A}_\zeta$-submodules $M = M_l \supset \cdots \supset M_0 = 0$ such that $M_j/M_{j-1} = \Delta_{\pi_j}\langle n_j \rangle$ with some $\pi_j \in \Pi_\zeta$ and $n_j \in \mathbf{Z}$ for each $j \in [1, l]$.

Lemma 5.6. *If $\mathscr{K} \in \mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}\right)_\zeta$ satisfies the $*$-parity, then $\mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathscr{K}, \mathbf{I}_\zeta)$ admits a Δ-filtration.*

Proof. Let $\emptyset = U_0 \subset \cdots \subset U_N = \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ be a filtration by $G_{\underline{0}}$-stable open subset such that each $U_k \setminus U_{k-1}$ is a single $G_{\underline{0}}$-orbit. Let $j_k : U_k \hookrightarrow \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$ and $i_k : U_k \setminus U_{k-1} \hookrightarrow U_k$ denote the inclusions. Then we have exact sequences

$$0 \longrightarrow \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(j_{k!}i_{k*}i_k^* j_k^! \mathscr{K}, \mathbf{I}_\zeta\right) \longrightarrow \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(j_{k!}j_k^! \mathscr{K}, \mathbf{I}_\zeta\right) \longrightarrow \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(j_{(k-1)!}j_{k-1}^! \mathscr{K}, \mathbf{I}_\zeta\right) \longrightarrow 0$$

Since $i_k^* j_k^! \mathscr{K}$ has no odd cohomology, it is isomorphic to some finite sum

$$\bigoplus_{\substack{\pi \in \Pi \\ \mathrm{O}_\pi = U_k \setminus U_{k-1}}} \bigoplus_{l \in \mathbf{Z}} \mathscr{L}_\pi[l]^{\oplus m_{\pi,l}}$$

for some $m_{\pi,k} \in \mathbf{N}$. Thus

$$\mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(j_{k!}i_{k*}i_k^* j_k^! \mathscr{K}, \mathbf{I}_\zeta\right) \cong \bigoplus_{\pi, l} \Delta(\pi)\langle l \rangle^{\oplus m_{\pi,l}}.$$

By induction on k, it follows that $\mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathscr{K}, \mathbf{I}_\zeta)$ admits a Δ-filtration. □

In particular, the projectives Π_π are Δ-filtered.

Lemma 5.7. *The module Δ_π is a projective cover of L_π in $\mathbf{A}_\zeta$-gmod$^{\preceq \pi}$.*

Proof. We show that $\Delta_\pi \in \mathbf{A}_\zeta$-gmod$^{\preceq \pi}$. Let $\sigma \in \Pi\left(\mathfrak{g}_{\underline{d}}\right)_\zeta$. Then $\mathrm{gHom}\left(P_\sigma, \Delta_\pi\right) \cong \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(j_{\pi!}\mathscr{L}_\pi, \mathrm{IC}\left(\mathscr{L}_\sigma\right)\right)$, which vanishes if $\sigma \not\leq \pi$. Since P_σ is a projective cover of L_σ, we have $\mathrm{gHom}\left(P_\sigma, \Delta_\pi\right) \neq 0$ if and only if Δ_π admits L_σ as subquotient. Thus $\Delta_\pi \in \mathbf{A}_\zeta$-gmod$^{\preceq \pi}$.

□

5.2.3 Endomorphism algebra of standard modules

For $\mathrm{C} \in \mathfrak{g}_{\underline{d}}^{\mathrm{nil}}/G_{\underline{0}}$, put $\mathscr{L}_{\mathrm{C}} = \bigoplus_{\pi \in \Pi_{\mathrm{C}}} \mathscr{L}_\pi$ and $\mathscr{L}_{\mathrm{C},\zeta} = \bigoplus_{\pi \in \Pi_{\mathrm{C},\zeta}} \mathscr{L}_\pi$. Define

$$B_{\mathrm{C}} = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathscr{L}_{\mathrm{C}}, \mathscr{L}_{\mathrm{C}}), \quad B_{\mathrm{C},\zeta} = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathscr{L}_{\mathrm{C},\zeta}, \mathscr{L}_{\mathrm{C},\zeta}).$$

Then $B_{\mathrm{C},\zeta}$ is a direct factor of the ring B_{C} and B_{C} is Morita-equivalent to $\Gamma \ltimes \mathrm{H}^\bullet_{Z_{G_{\underline{0}}}(z)^0}$, where $z \in \mathrm{C}$ and $\Gamma = \pi_0(Z_{G_{\underline{0}}}(z))$. Put $\Delta_{\mathrm{C},\zeta} = \bigoplus_{\pi \in \Pi_{\zeta,\mathrm{C}}} \Delta_\pi$. We have $B_{\mathrm{C},\zeta} \cong \mathrm{gHom}_{\mathbf{A}_\zeta}(\Delta_{\mathrm{C},\zeta}, \Delta_{\mathrm{C},\zeta})$.

Lemma 5.8. *For any $\sigma \in \Pi_\zeta$, the graded $B_{\mathrm{C},\zeta}$-module $\mathrm{gHom}_{\mathbf{A}_\zeta}(P_\sigma, \Delta_{\mathrm{C},\zeta})$ is projective of finite type.*

Proof. We have

$$\mathrm{gHom}_{\mathbf{A}_\zeta}(P_\sigma, \Delta_{\mathrm{C},\zeta}) \cong \mathrm{Hom}^\bullet_{G_{\underline{0}}}(j_{\mathrm{C}!}\mathscr{L}_{\mathrm{C},\zeta}, \mathrm{IC}(\mathscr{L}_\sigma)) \cong \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(\mathscr{L}_{\mathrm{C},\zeta}, j^!_{\mathrm{C}}\,\mathrm{IC}(\mathscr{L}_\sigma)\right).$$

Since the complex $j^!_{\mathrm{C}}\,\mathrm{IC}(\mathscr{L}_\sigma)$ is semisimple, it lies in $\mathrm{add}(\mathscr{L}_{\mathrm{C},\zeta})$. Hence $\mathrm{gHom}_{\mathbf{A}_\zeta}(P_\sigma, \Delta_{\mathrm{C},\zeta})$ belongs to $\mathrm{add}(B_{\mathrm{C},\zeta})$. □

5.2.4 Proper standard modules

In § 4.5.2, we have introduced the proper standard modules $\overline{\Delta}_z$ and $\overline{\Delta}_\pi$ for $\widehat{\mathcal{H}}$. For any $\pi \in \Pi_\zeta$, choose any $z \in \mathrm{O}_\pi$. We define the **proper standard module** to be $\overline{\Delta}_\pi = \mathrm{Hom}^{\bullet+2\dim \mathrm{C}}_\Gamma(\chi_\pi, z^!\mathbf{I}_\zeta)$, where $\chi_\pi = (\mathscr{L}_\pi)_z$ is a simple $\mathbf{C}\Gamma$-module. We also introduce $\overline{\Delta}_{\mathrm{C},\zeta} = \bigoplus_{\pi \in \Pi_{\mathrm{C},\zeta}} \overline{\Delta}_\pi$. Then the image of $\overline{\Delta}_{\mathrm{C},\zeta}$ under the Morita equivalence between $\mathbf{A}_\zeta$ and $\widehat{\mathcal{H}}$ is isomorphic to $\overline{\Delta}_z$.

Proposition 5.9. *The following statements hold:*

(i) There is an isomorphism

$$\overline{\Delta}_{\mathrm{C},\zeta} \cong \Delta_{\mathrm{C},\zeta}/J \cdot \Delta_{\mathrm{C},\zeta},$$

where $J = J(B_{\mathrm{C},\zeta})$ is the graded Jacobson radical of $B_{\mathrm{C},\zeta}$.

(ii) For each $\pi \geq \Pi_{\mathrm{C},\zeta}$, there exists an separated descreasing filtration $\{\Delta_\pi^{\geq k}\}_{k \in \mathbf{N}}$ of graded $\mathbf{A}_\zeta$-submodules such that the successive quotients $\Delta_\pi^{\geq k}/\Delta_\pi^{\geq k+1}$ is a finite direct sum of $\mathbf{A}_\zeta$-modules of the form $\overline{\Delta}_\pi\langle n\rangle$ for $\pi \in \Pi_{\mathrm{C},\zeta}$ and $n \in \mathbf{Z}$.

Proof. Since

$$B_{\mathrm{C},\zeta} = \bigoplus_{\pi,\pi' \in \Pi_{\mathrm{C},\zeta}} \left(\mathrm{Hom}(\chi_\pi, \chi_{\pi'}) \otimes \mathrm{H}^\bullet_{Z_{G_{\underline{0}}}(z)^0}\right)^\Gamma$$

$$\Delta_{\mathrm{C},\zeta} = \bigoplus_{\pi \in \Pi_{\mathrm{C},\zeta}} \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathscr{L}_\pi, j^!_{\mathrm{C}}\mathbf{I}_\zeta) \cong \bigoplus_{\pi \in \Pi_{\mathrm{C},\zeta}} \left(\mathrm{Hom}^{\bullet+2\dim \mathrm{C}}(\chi_\pi, z^!\mathbf{I}_\zeta) \otimes \mathrm{H}^\bullet_{Z_{G_{\underline{0}}}(z)^0}\right)^\Gamma$$

we deduce

$$J^k = \bigoplus_{\pi,\pi'\in\Pi_{\mathrm{C},\zeta}} \left(\mathrm{Hom}(\chi_\pi,\chi_{\pi'})\otimes \mathrm{H}^{\geq k}_{Z_{G_{\underline{0}}}(z)^0}\right)^\Gamma$$

$$J^k\cdot\Delta_{\mathrm{C},\zeta} = \bigoplus_{\pi\in\Pi_{\mathrm{C},\zeta}} \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(\mathscr{L}_\pi, j^!_{\mathrm{C}}\mathbf{I}_\zeta\right) \cong \bigoplus_{\pi\in\Pi_{\mathrm{C},\zeta}} \left(\mathrm{Hom}^{\bullet+2\dim\mathrm{C}}\left(\chi_\pi, z^!\mathbf{I}_\zeta\right)\otimes \mathrm{H}^{\geq k}_{Z_{G_{\underline{0}}}(z)^0}\right)^\Gamma$$

$$\Delta_{\mathrm{C},\zeta}/J\cdot\Delta_{\mathrm{C},\zeta} = \bigoplus_{\pi\in\Pi_{\mathrm{C},\zeta}} \mathrm{Hom}^\bullet_{G_{\underline{0}}}\left(\mathscr{L}_\pi, j^!_{\mathrm{C}}\mathbf{I}_\zeta\right) \cong \bigoplus_{\pi\in\Pi_{\mathrm{C},\zeta}} \left(\mathrm{Hom}^{\bullet+2\dim\mathrm{C}}\left(\chi_\pi, z^!\mathbf{I}_\zeta\right)\right)^\Gamma \cong \overline{\Delta}_{\mathrm{C},\zeta}.$$

This proves (i). For (ii), we define the filtration on Δ_π by $\Delta_\pi^{\geq k} = J^k\cdot\Delta_{\mathrm{C},\zeta}\cap\Delta_\pi$, so that

$$\Delta_\pi^{\geq k}/\Delta_\pi^{\geq k+1} \cong \mathrm{Hom}^{\bullet+2\dim\mathrm{C}}_\Gamma\left(\left(\mathrm{H}^k_{Z_{G_{\underline{0}}}(z)^0}\right)^*\otimes\chi_\pi, z^!\mathbf{I}_\zeta\right).$$

Decomposing $\left(\mathrm{H}^k_{Z_{G_{\underline{0}}}(z)^0}\right)^*\otimes\chi_\pi$ into a sum of simple graded Γ-modules, we see that $J^k\Delta_{\mathrm{C},\zeta}/J^{k+1}\cdot\Delta_{\mathrm{C},\zeta}$ is a sum of proper standard modules. □

5.2.5 Properly stratified categories

Proposition 5.10. *The category* $\mathbf{A}_\zeta$-gmod *is properly stratified* [1].

Proof. It follows from Lemma 5.6, Lemma 5.7 and Lemma 5.8 that the axioms of [25, 5.1] are satisfied. □

In the case of principal block $\zeta = \zeta_0 = (T, \mathfrak{t}_*, 0, \delta_0)$, we have:

Corollary 5.11. *The category* $\mathbf{H}_{\lambda_0}$-gmod *is properly stratified.*

Proof. By Proposition 5.1, we have $\mathbf{H}\cong\mathcal{H}$. Since $\mathcal{H}$ and $\mathbf{A}_\zeta$ are both extension algebas of semisimle complexes whose set of simple constituents coincide up to shift of degree, there is an equivalence $\mathcal{H}$-gmod $\cong$ $\mathbf{A}_\zeta$-gmod, where $\mathcal{H}$-gmod is the the category of finitely generated graded weight $\mathcal{H}$-modules, *cf.* §2.2.1. Hence $\mathbf{H}_{\lambda_0}$-gmod is properly stratified. □

5.3 Examples from cyclic quivers

5.3.1 Cyclic quivers

Let $m\in\mathbf{Z}_{\geq 2}$. Consider the m-cyclic quiver $\Gamma = (I, H)$ where $I = \mathbf{Z}/m$ and $H = \{(i, i+1)\ ;\ i\in\mathbf{Z}/m\}$. Let $V = \mathbf{C}^n$, $G = \mathrm{GL}(V)$, $\mathfrak{g} = \mathfrak{gl}(V)$ and let $\theta : \mu_m \longrightarrow G$ be any homomorphism. Then θ gives rise to a $\mathbf{Z}/m$-grading $V = \bigoplus_{i\in\mathbf{Z}/m} V_i$ as well as $\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}/m}\mathfrak{g}_i$. In particular, $\mathfrak{g}_{\underline{1}} = \bigoplus_{i\in\mathbf{Z}/m}\mathrm{Hom}\left(V_i, V_{i+1}\right)$ is the space of representations of Γ on V and $G_{\underline{0}} = \prod_{i\in\mathbf{Z}/m}\mathrm{GL}(V_i)$ acts on $\mathfrak{g}_{\underline{1}}$ by conjugation.

[1] Although the endomorphism ring $\mathrm{gEnd}_{\mathbf{A}_\zeta}(\Delta_{\mathrm{C},\zeta})$ is not a graded polynomial ring, it is close to be one as we have already seen.

It is known that the isotropy subgroup $Z_{G_{\underline{0}}}(x)$ is connected for every $x \in \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$. Therefore, there is only one admissible system on $\mathfrak{g}_{\underline{1}}$, namely $\underline{\mathfrak{T}}(\mathfrak{g}_{\underline{1}}) = \{(T, \mathfrak{t}_*, 0, \delta_0)\}$, where T is a maximal torus of $G_{\underline{0}}$, $\mathfrak{t}_* = \mathfrak{t}_0$ and δ_0 is the constant sheaf on $\{0\} = \mathfrak{t}_1$. Therefore, we have $\mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right) = \mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)_{(T,\mathfrak{t}_*,0,\delta_0)}$, etc.

The nilpotent orbits of $\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$ have been well-studied, notably in [33], [2], [27]. We will put accent on the perspective of [48].

Let $\zeta = (T, \mathfrak{t}_*, 0, \delta_0) \in \mathfrak{T}(\mathfrak{g}_{\underline{1}})$ and $\sigma = (T, 0, \delta_0) \in \mathfrak{S}(\mathfrak{g}_{\underline{1}})$ be the only admissible system and supercuspidal system, respectively, on $\mathfrak{g}_{\underline{1}}$. Recall that in §5.1.3 we have defined $\underline{\mathcal{P}}_\zeta$ (resp. $\underline{\mathcal{P}}_\sigma$) to be the set of $W_\zeta = W(G_{\underline{0}}, T)$-conjugacy classes of spirals of $\mathfrak{g}$ having $\mathfrak{t}_*$ as splitting (resp. Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$). As in §5.1.3, we put

$$\mathbf{I}_{\mathrm{fin}} = \bigoplus_{\mathfrak{b} \in \underline{\mathcal{P}}_\sigma} \mathrm{Ind}_{\mathfrak{t}_{\underline{1}} \subset \mathfrak{b}_{\underline{1}}}^{\mathfrak{g}_{\underline{1}}} \delta_0 \qquad \mathbf{I}_{\mathrm{aff}} = \bigoplus_{\mathfrak{p}_* \in \underline{\mathcal{P}}_\zeta} \mathrm{Ind}_{\mathfrak{t}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_{\underline{1}}} \delta_0.$$

Let $\mathcal{K} = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{fin}})$ and $\mathcal{H} = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{aff}}, \mathbf{I}_{\mathrm{aff}})$. Since $\mathbf{I}_{\mathrm{aff}}$ is an infinite sum, here $\mathrm{Hom}^\bullet$ means endomorphisms of finite rank so that $\mathcal{H}$ is a graded non-unital algebra.

The set $\underline{\mathcal{P}}_\sigma$ has a combinatorial description. Let $\beta = \mathrm{gdim}\, V \in \mathbf{N}I = \mathrm{Map}(I, \mathbf{N})$ be the dimension vector of V. Note that the length of β is $|\beta| = \dim V = n$. For $i \in \mathbf{Z}/m$ let $\alpha_i \in \mathbf{N}I$ be the vector which is 1 at the i-th place and 0 elsewhere. Define

$$I^n = \left\{ (\nu_1, \ldots, \nu_n) \in I^n \; ; \; \sum_{k=1}^n \alpha_{\nu_k} = n \right\}.$$

The elements of I^β are called **complete sequences** of β. The bijection $\underline{\mathcal{P}}_\sigma \cong I^\beta$ is given by the following: if $\mathfrak{b} \in \mathcal{P}_\sigma$ fixes an I-graded complete flag $V = F^0 \supset F^1 \supset \cdots \supset F^n = 0$, then to $\mathfrak{b} \in \mathcal{P}_\sigma$ corresponds the complete sequence $(\nu_1, \ldots, \nu_n)$ where $\nu_k \in I$ is the support of F^{k-1}/F^k.

In order to define the quiver Schur algebra, we introduce a combinatorial description for all parabolic types. Let $\mathrm{Comp}(\beta)$ be the set of sequences $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_l)$ of non-zero elements in $\mathbf{N}I$ such that $\sum_k \gamma_k = \beta$. If $Q \subset G$ is the parabolic subgroup which stabilises an I-graded flag $V = F^0 \supset F^1 \supset \cdots \supset F^l = 0$, then to $\mathfrak{q} = \mathrm{Lie}\, Q$ one attaches the element $\gamma = (\gamma_1, \cdots, \gamma_l) \in \mathrm{Comp}(\beta)$ with $\gamma_k = \mathrm{gdim}\, F^{k-1}/F^k$. This induces a bijection between the $G_{\underline{0}}$-conjugacy classes of all θ-graded parabolic subgroups and the set $\mathrm{Comp}(\beta)$. For any $\gamma \in \mathrm{Comp}(\beta)$ we choose a parabolic subgroup Q^γ in the corresponding $G_{\underline{0}}$-conjugacy class which contains T. We will denote its Lie algebra by $\mathfrak{q}^\gamma$, its unipotent radical by $\mathfrak{v}^\gamma$ and its Levi factor containing $\mathfrak{t}$ by $\mathfrak{m}^\gamma$.

Set

$$\mathbf{I}_{\mathrm{sc}} = \bigoplus_{\gamma \in \mathrm{Comp}(\beta)} \mathrm{Ind}_{\mathfrak{m}^\gamma_{\underline{1}} \subset \mathfrak{q}^\gamma_{\underline{1}}}^{\mathfrak{g}_{\underline{1}}} \delta_0,$$

where δ_0 is the punctual sheaf supported on $0 \in \mathfrak{m}^\gamma_{\underline{1}}$ with fiber $\overline{\mathbf{Q}}_\ell$. We define an algebra $\mathbf{Sc} = \mathrm{Hom}^\bullet_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{sc}}, \mathbf{I}_{\mathrm{sc}})$, called the quiver Schur algebra in [47].

By [13, 8.6.2], the isomorphism classes of simple modules of $\mathbf{Sc}$ are in canonical bijection with the isomorphism classes of simple perverse sheaves which appear as direct factor of ${}^p\mathscr{H}^k\mathbf{I}_{\mathrm{sc}}$ for some $k \in \mathbf{Z}$. However, it is known by [52, Theorem 4] that the simple $\mathbf{Sc}$-modules are in bijection with the $G_{\underline{0}}$-orbits in $\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$. It follows that the semisimple complex $\mathbf{I}_{\mathrm{sc}}$ contains all the irreducible objects in $\mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)$ in its perverse cohomologies. Since

$\mathbf{I}_{\mathrm{aff}}$ also contains all the irreducible objects in $\mathrm{Perv}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)$, we can define an equivalence $\mathbf{Sc}$-gMod $\cong \mathcal{H}$-gMod^{sm} via the bimodule $F : \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{sc}}, \mathbf{I}_{\mathrm{aff}})$.

Let $P_\Sigma = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{sc}}, \mathbf{I}_{\mathrm{fin}})$. Then we have a functor

$$\Sigma : \mathbf{Sc}\text{-gmod} \longrightarrow \mathcal{K}\text{-gmod}, \quad \Sigma(M) = P_\Sigma \otimes_{\mathbf{Sc}} M.$$

Since $\Sigma = \mathbf{V} \circ F$, and $\mathbf{V}$ satisfies the double centraliser property by Theorem 2.65, it follows that $\mathcal{K} \cong \mathrm{End}_{\mathbf{Sc}^{\mathrm{op}}}(P_\Sigma)$ and $\mathbf{Sc}^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{K}}(P_\Sigma)$ and the functor Σ is also a quotient functor with the double centraliser property. In this sense, Σ can be viewed as a version of the Schur–Weyl duality.

5.3.2 Algebraic description

Using the functor Σ, one can describe the quiver Schur algebra $\mathbf{Sc}$ in algebraic terms. Let $Q_{i,j}(x,y)$ be a $\mathbf{Q}[x,y]$-valued matrix with $1 \le i,j \le n$ defined by

$$Q_{i,j}(x,y) = \begin{cases} 0 & i = j \\ (x-y)^{\delta_{i,j+1}}(y-x)^{\delta_{i+1,j}} & i \neq j \end{cases}.$$

The quiver Hecke algebra $\mathbf{R}(\beta)$ of the cyclic quiver Γ with dimension vector $\beta = \mathrm{gdim}\, V \in \mathbf{N}I$ is generated by $\mathrm{e}(\nu)$, r_t and x_k with $\nu \in I^\beta$ being a complete sequence of β, $1 \le t \le n-1$ and $1 \le k \le n$. It satisfies the following defining relations

$$\mathrm{e}(\nu)\mathrm{e}(\nu') = \delta_{\nu,\nu'}\mathrm{e}(\nu), \quad \sum_{\nu \in I^\beta} \mathrm{e}(\nu) = 1, \quad \mathrm{x}_k\mathrm{e}(\nu) = \mathrm{e}(\nu)\mathrm{x}_k, \quad \mathrm{x}_k\mathrm{x}_l = \mathrm{x}_l\mathrm{x}_k,$$

$$\mathrm{r}_t\mathrm{e}(\nu) = \mathrm{e}(s_t(\nu))\mathrm{r}_t, \quad \mathrm{r}_t\mathrm{r}_s = \mathrm{r}_s\mathrm{r}_t \quad \text{if } |t-s| > 1, \quad \mathrm{r}_t^2\mathrm{e}(\nu) = Q_{\nu_t,\nu_{t+1}}(\mathrm{x}_t, \mathrm{x}_{t+1})$$

$$(\mathrm{r}_t\mathrm{x}_k - \mathrm{x}_{s_t(k)}\mathrm{r}_t)\mathrm{e}(\nu) = \begin{cases} -\mathrm{e}(\nu) & (k = t) \wedge (\nu_t = \nu_{t+1}) \\ \mathrm{e}(\nu) & (k = t+1) \wedge (\nu_t = \nu_{t+1}) \\ 0 & \text{otherwise} \end{cases},$$

$$\left(\mathrm{r}_{t+1}\mathrm{r}_t\mathrm{r}_{t+1} - \mathrm{r}_t\mathrm{r}_{t+1}\mathrm{r}_t\right)\mathrm{e}(\nu) = \begin{cases} \frac{Q_{\nu_{t+1},\nu_t}(\mathrm{x}_{t+1},\mathrm{x}_t) - Q_{\nu_{t+1},\nu_t}(\mathrm{x}_{t+2},\mathrm{x}_{t+1})}{\mathrm{x}_{t+2} - \mathrm{x}_t}\mathrm{e}(\nu) & \nu_k = \nu_{k+2} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{P}(\beta) = \bigoplus_{\nu \in I^\beta} \mathbf{S}(\nu)$ where $\mathbf{S}(\nu) = \mathbf{C}[X_1, \cdots, X_n]$. We define the polynomial representation of $\mathbf{R}(\beta)$ on $\mathbf{P}(\beta)$. For $\nu \in I^\beta$ and for each $f \in \mathbf{C}[X_1, \cdots, X_n]$, denote by $f(\nu) \in \mathbf{S}(\nu)$ the corresponding element. The polynomial representation of $\mathbf{R}(\beta)$ on $\mathbf{P}(\beta)$ is defined by

$$\mathrm{e}(\nu')f(\nu) = \delta_{\nu,\nu'}f(\nu), \quad \mathrm{x}_k f(\nu) = X_k f(\nu),$$

$$\mathrm{r}_t f(\nu) = \begin{cases} (X_t - X_{t+1})^{\delta_{\nu_t+1,\nu_{t+1}}}(s_t f)(s_t\nu) & \nu_t \neq \nu_{t+1} \\ \partial_t f(\nu) & \nu_t = \nu_{t+1} \end{cases}$$

More generally, if $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_l) \in \mathrm{Comp}(\beta)$, then we have an inclusion of algebras

$$\mathbf{R}(\gamma_1) \otimes \cdots \otimes \mathbf{R}(\gamma_l) \subset \mathbf{R}(\beta).$$

Define

$$\mathbf{P}(\gamma) = \mathbf{R}(\beta) \otimes_{\mathbf{R}(\gamma_1)\otimes\cdots\otimes\mathbf{R}(\gamma_l)} \mathbf{P}(\gamma_1) \otimes \cdots \otimes \mathbf{P}(\gamma_l).$$

According to [51], there is an isomorphism $\mathbf{R}(\beta)$. Under this isomorphism the $\mathcal{K}$-module $\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\delta_0, \mathbf{I}_{\mathrm{fin}})$ is identified with the polynomial representaion $\mathbf{P}(\beta)$. More generally, for each $\gamma \in \mathrm{Comp}(\beta)$ the module $\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}\left(\mathrm{Ind}^{\mathfrak{g}_{\underline{1}}}_{\mathfrak{m}_{\underline{\gamma}} \subset \mathfrak{q}_{\underline{\gamma}}} \delta_0, \mathbf{I}_{\mathrm{fin}}\right)$ is identified with $\mathbf{P}(\gamma)$. Let $\mathbf{T} = \bigoplus_{\gamma \in \mathrm{Comp}(\beta)} \mathbf{P}(\gamma)$. It follows that $\mathbf{T}$ is identified with P_{Σ}. We deduce from the double centraliser property of the functor Σ that $\mathbf{Sc}^{\mathrm{op}} \cong \mathrm{End}^{\bullet}_{\mathcal{K}}(\mathbf{T})$.

5.3.3 Cyclic quivers with involution

Let $m \in \mathbf{Z}_{\geq 1}$. Consider the quiver $\Gamma = (I, H)$ where $I = \mathbf{Z}/2m$ and $H = \{(i, i+1) \ ; \ i \in \mathbf{Z}/2m\}$. Let (V, ω) be a symplectic vector space of dimension $2n$ for some $n \in \mathbf{N}$, let $G = \mathrm{Sp}(V, \omega)$, $\mathfrak{g} = \mathfrak{sp}(V, \omega)$ and let $\theta : \mu_{2m} \longrightarrow G$ be any homomorphism. Then θ gives rise to a $\mathbf{Z}/2m$-grading on V

$$V = \bigoplus_{i \in \mathbf{Z}/2m} V_i, \quad V_i = \left\{v \in V \ ; \ \theta(\zeta)v = \zeta^i v, \forall \zeta \in \mu_{2m}\right\}.$$

and on $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}/2m} \mathfrak{g}_i, \quad \mathfrak{g}_i = \left\{x \in \mathfrak{g} \ ; \ \mathrm{Ad}_{\theta(\zeta)}\, x = \zeta^i x, \forall \zeta \in \mu_{2m}\right\}.$$

In particular, $\mathfrak{g}_{\underline{1}}$ is the space of anti-self-adjoint representations of Γ on (V, ω). For any subspace $U \subset V$, we write $U^{\perp} = \{v \in V \ ; \ \omega(v, U) = 0\}$. Then it is clear that for each $i \in \mathbf{Z}/2m$, we have $V_i^{\perp} = \bigoplus_{i+j \neq 0} V_j$, so that ω restricts to a perfect pairing

$$\omega\,|_{V_i \times V_{-i}} \colon V_i \times V_{-i} \longrightarrow \mathbf{C}.$$

The θ-fixed points $G_{\underline{0}} = G^{\theta}$ is connected and acts on $\mathfrak{g}_{\underline{1}}$ by the adjoint action. We put as before $G_{\underline{0}} = G_{\underline{0}} \times \mathbf{C}_q^{\times}$ and we make $\mathbf{C}_q^{\times}$ act linearly on $\mathfrak{g}_{\underline{1}}$ by weight 2.

The isotropy subgroup $Z_{G_{\underline{0}}}(x)$ is connected for every $x \in \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$. Therefore, there is only one admissible system on $\mathfrak{g}_{\underline{1}}$, namely $\underline{\mathfrak{T}}(\mathfrak{g}_{\underline{1}}) = \{(T, \mathfrak{t}_*, 0, \delta)\}$, where T is a maximal torus of $G_{\underline{0}}$, $\mathfrak{t}_* = \mathfrak{t}_0$ and δ is the constant sheaf on $\{0\} = \mathfrak{t}_1$. We have

$$\mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right) = \mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)_{(T, \mathfrak{t}_*, 0, \delta_0)}.$$

As in §5.1.3, we put

$$\mathbf{I}_{\mathrm{fin}} = \bigoplus_{\mathfrak{b} \in \underline{\mathcal{P}}_{\sigma}} \mathrm{Ind}^{\mathfrak{g}_{\underline{1}}}_{\mathfrak{t}_{\underline{1}} \subset \mathfrak{b}_{\underline{1}}} \delta_0 \qquad \mathbf{I}_{\mathrm{aff}} = \bigoplus_{\mathfrak{p}_* \in \underline{\mathcal{P}}_{\zeta}} \mathrm{Ind}^{\mathfrak{g}_{\underline{1}}}_{\mathfrak{t}_1 \subset \mathfrak{p}_1} \delta_0.$$

Put $\mathcal{H} = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{aff}}, \mathbf{I}_{\mathrm{aff}})$, $\mathcal{K} = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{fin}}, \mathbf{I}_{\mathrm{fin}})$ and $P_{\mathbf{V}} = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}_{\mathrm{aff}}, \mathbf{I}_{\mathrm{fin}})$ as before.

We shall define a symplectic analogue of the quiver Schur algebra. To this end, we introduce a combinatorial description for a certain class of parabolic subgroups of G.

Let $\overline{\bullet} : I \longrightarrow I$, $\overline{\bullet} : H \longrightarrow H$ be the involution of the quiver Γ given by $\overline{i} = -i$ and $\overline{(i, i+1)} = (-i-1, -i)$ for $i \in \mathbf{Z}/2m = I$. Let $\mathbf{N}I = \mathrm{Map}\,(I, \mathbf{N})$ be the monoid of maps from I to $\mathbf{N}$ and let $\mathbf{N}I^{\times} = \mathbf{N}I \setminus \{0\}$. For any $\beta \in \mathbf{N}I$, let

$$\mathrm{Comp}\,(\beta) = \left\{(\gamma_{1-l}, \ldots, \gamma_l) \ ; \ l \in \mathbf{N}, \ \gamma_i \in \mathbf{N}I^{\times}, \ \sum_{i=1-l}^{l} \gamma_i = \beta \ , \ \gamma_{1-i} = \overline{\gamma_i}\right\}.$$

From now on, $\beta = \operatorname{gdim} V \in \mathbf{N}I$.

Let $\gamma = (\gamma_{1-l}, \cdots, \gamma_l) \in \operatorname{Comp}(\beta)$. An isotropic flag in V of type γ is a filtration $0 = V^l \subset V^{l-1} \subset \cdots \subset V^{-l} = V$ by I-graded subspaces of V such that $\operatorname{gdim} V^{i-1}/V^i = \gamma_i$ and that $V^{i\perp} = V^{-i}$. Define X_γ to be the varieties of isotropic flags in V of type γ and let $\tilde{X}_\gamma \subset \mathfrak{g}_{\underline{1}}^{\mathrm{nil}} \times X_\gamma$ be the subvariety of pairs $\left(x, \{V^i\}_{-l\leq i\leq l}\right)$ such that $xV^i \subset V^{i+1}$. Let $\{V^i\}_{-l\leq i\leq l} \in X_\gamma$ be any isotropic flag of type γ, let $Q^\gamma \subset G$ be the parabolic subgroup which stabilises the flag and let M^γ be any θ-stable Levi-factor of Q^γ. The pair (Q^γ, M^γ) is well-defined up to conjugation by $G_{\underline{0}}$. Then the direct image of the constant sheaf $\pi_*^\gamma \overline{\mathbf{Q}}_\ell$ via the projection $\pi^\gamma : \tilde{X}_\gamma \longrightarrow \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$ is equal to the parabolic induction $\operatorname{Ind}_{\mathfrak{m}_{\underline{1}}^\gamma \subset \mathfrak{q}_{\underline{1}}^\gamma}^{\mathfrak{g}_{\underline{1}}} \delta_0$ of the punctual sheaf on 0.

We put

$$\mathbf{I}_{\mathrm{sc}} = \bigoplus_{\gamma\in\mathrm{Comp}(\beta)} \operatorname{Ind}_{\mathfrak{m}_{\underline{1}}^\gamma \subset \mathfrak{q}_{\underline{1}}^\gamma}^{\mathfrak{g}_{\underline{1}}} \delta_0[d_\gamma],$$

where $d_\gamma = \dim \tilde{X}_\gamma$, and we define

$$\mathbf{Sc} = \operatorname{Hom}_{G_{\underline{0}}}^\bullet (\mathbf{I}_{\mathrm{sc}}, \mathbf{I}_{\mathrm{sc}}).$$

The algebra $\mathbf{Sc}$ can be viewed as a "quiver Schur algebra of type C". The main result here is the following, the proof of which is postponed to the next subsection.

Proposition 5.12. *The following statements hold:*

(i) Let $\mathrm{O} \subset \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$. *Then* $\mathrm{IC}(\mathrm{O})$ *is a direct factor of* ${}^p\mathscr{H}^* \operatorname{Ind}_{\mathfrak{m}_{\underline{1}}^\gamma \subset \mathfrak{q}_{\underline{1}}^\gamma}^{\mathfrak{g}_{\underline{1}}} \delta_0$ *for some* $\gamma \in \operatorname{Comp}(\beta)$.

(ii) The isomorphism classes of simple $\mathbf{Sc}$*-modules are in canonical bijection with* $G_{\underline{0}}$*-orbits in* $\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$.

We put $P_F = \operatorname{Hom}_{G_{\underline{0}}}^\bullet (\mathbf{I}_{\mathrm{sc}}, \mathbf{I}_{\mathrm{aff}})$ and $P_\Sigma = \operatorname{Hom}_{G_{\underline{0}}}^\bullet (\mathbf{I}_{\mathrm{sc}}, \mathbf{I}_{\mathrm{fin}})$. Since both $\mathbf{I}_{\mathrm{aff}}$ and $\mathbf{I}_{\mathrm{sc}}$ contain all simple objects of $\operatorname{Perv}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}})$, there is an equivalence of category

$$\begin{aligned} F : \mathbf{Sc}\text{-gmod} &\longrightarrow \mathcal{H}\text{-gmod} \\ N &\mapsto P_F \otimes_{\mathbf{Sc}} N \end{aligned}$$

and an exact functor

$$\begin{aligned} \Sigma : \mathbf{Sc}\text{-gmod} &\longrightarrow \mathcal{K}\text{-gmod} \\ N &\mapsto P_\Sigma \otimes_{\mathbf{Sc}} N \end{aligned}$$

such that $\Sigma \cong \mathbf{V} \circ F$. Thus $\mathbf{V}$ can be identified with the "Schur functor" Σ, which satisfies the double centraliser property. The algebra $\mathcal{K}$ is similar to the "quiver Hecke algebra of type B and C" of [50]. As in the previous example, the category $\mathbf{Sc}$-gmod is also affine quasi-hereditary with respect to the partial order or orbit closure.

It will be interesting to give an algebraic presentation of the algebras $\mathcal{K}$ and $\mathbf{Sc}$ as well as the induction/restriction functors between different dimension vector β.

5.3.4 Proof of Proposition 5.12

We begin with two lemmas for the preparation.

Lemma 5.13. *There exists $U \subset V$ such that $U^\perp = U$, $x(U) = U$ and $V \cong U \oplus U^\vee$ as anti-symmetric representation.*

Proof. We first prove that each indecomposable subrepresentation of (V, x) is isotropic with respect to ω. Let $M \subset V$ be indecomposable. Then there is a homogeneous element $a \in M_i$ such that $a, x(a), \ldots, x^l(a)$ form a basis of M for some $l \in \mathbf{N}$. Suppose that $x^j(a) \in M_{i'}$ and $x^k(a) \in M_{\theta(i')}$ for some $j, k \in \mathbf{N}$ and $i \in I$. Then $\omega(x^j(a), x^k(a)) = (-1)^{k+j}\omega(x^k(a), x^j(a)) = (-1)^{k+j}\omega(x^j(a), x^k(a))$ so that $\omega(x^j(a), x^k(a)) = 0$, whence M is isotropic as claimed.

Now we prove the lemma by induction on the number of indecomposable factors of (V, x). If $V = 0$, the statement is trivial. Suppose that $V \neq 0$. Then we can find an indecomposable summand $U \subset V$ and write $V = U \oplus U'$. Then we see that $V = U^\perp \oplus U'^\perp$ and so $U'^\perp \cong U^\vee$. It follows that $V = (U \oplus U^\vee) \oplus \left(U' \cap U^\perp\right)$ and that ω is non-degenerate on $U' \cap U^\perp$. Since $U' \cap U^\perp$ has strictly fewer indecomposable factors, the inductive hypothesis applies and the lemma follows. □

Let $U \subset V$ be an I-graded lagrangian subspace. Let $M^U = \mathrm{GL}(U)$, $\mathfrak{m} = \mathfrak{gl}(U)$ and $Q^U = \{g \in G \,;\, g(U) \subset U\}$. Then Q^U is a θ-stable parabolic subgroup of G having M^U as Levi-factor. The parabolic induction $\mathrm{Ind}_{\mathfrak{m}_{\underline{1}}^U \subset \mathfrak{q}_{\underline{1}}^U}^{\mathfrak{g}_{\underline{1}}}$ induces a map

$$K\left(\mathrm{D}^b_{M^U_{\underline{0}}}\left(\mathfrak{m}_{\underline{1}}^{U,\mathrm{nil}}\right)\right) \longrightarrow K\left(\mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)\right).$$

Taking over all such $U \subset V$ and tensorising with $\mathbf{Q}$, we obtain a map

$$\bigoplus_{U \subset V} K\left(\mathrm{D}^b_{M^U_{\underline{0}}}\left(\mathfrak{m}_{\underline{1}}^{U,\mathrm{nil}}\right)\right)_{\mathbf{Q}} \longrightarrow K\left(\mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)\right)_{\mathbf{Q}} \tag{5.14}$$

Lemma 5.15. *The map (5.14) is surjective.*

Proof. Let $j : \mathrm{O} \hookrightarrow \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$ be a $G_{\underline{0}}$-orbit and let $x \in \mathrm{O}$. We prove by induction on $N = \dim \mathrm{O}$ that $\left[j_!\overline{\mathbf{Q}}_\ell\right] \in K\left(\mathrm{D}^b_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)\right)$ is in the image of (5.14). We suppose that this statement has been proven for orbits of dimension $> N$.

Let $U \subset V$ be as in Lemma 5.13 so that $x \in \mathfrak{m}_{\underline{1}}^{\mathrm{nil}}$. Put $\mathrm{O}_U = M_{\underline{0}}^U x \overset{j_U}{\hookrightarrow} \mathfrak{m}_{\underline{1}}^{\mathrm{nil}}$ and

$$\mathscr{K} = \mathrm{Ind}_{\mathfrak{m}_{\underline{1}}^U \subset \mathfrak{q}_{\underline{1}}^U}^{\mathfrak{g}_{\underline{1}}} j_{U!}\overline{\mathbf{Q}}_\ell.$$

Then by proper base change theorem, any orbit $j_{\mathrm{C}} : \mathrm{C} \hookrightarrow \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$ such that $j_{\mathrm{C}}^*\mathscr{K} \not\cong 0$ must satisfies $\overline{\mathrm{C}} \supseteq \mathrm{O}$. Put

$$\widehat{\mathrm{O}} = \bigcup_{\substack{\mathrm{C} \subset \mathfrak{g}_{\underline{1}}^{\mathrm{nil}} \\ \overline{\mathrm{C}} \supset \mathrm{O} \\ \mathrm{C} \neq \mathrm{O}}} \mathrm{C} \overset{j_{\widehat{\mathrm{O}}}}{\hookrightarrow} \mathfrak{g}_{\underline{1}}^{\mathrm{nil}}$$

so that there is a distinguished triangle

$$j_{\widehat{\mathrm{O}}!}j_{\widehat{\mathrm{O}}}^*\mathscr{K} \longrightarrow \mathscr{K} \longrightarrow j_{\mathrm{O}!}j_{\mathrm{O}}^*\mathscr{K} \overset{[1]}{\longrightarrow},$$

which yields $[j_{\mathrm{O}!}j_{\mathrm{O}}^{*}\mathscr{K}] = [\mathscr{K}] - \left[j_{\widehat{\mathrm{O}}!}j_{\widehat{\mathrm{O}}^{*}}\mathscr{K}\right]$. The dimension of every $G_{\underline{0}}$-orbit $\mathrm{C} \subset \widehat{\mathrm{O}}$ must be strictly greater than N. By induction hypothesis and the above arguments, the class $\left[j_{\widehat{\mathrm{O}}!}j_{\widehat{\mathrm{O}}^{*}}\mathscr{K}\right]$ lies in the image of (5.14). On the other hand, $[\mathscr{K}]$ is also in the image by definition. We see that $[j_{\mathrm{O}!}j_{\mathrm{O}}^{*}\mathscr{K}]$ also lies in the image. Now, since $j_{\mathrm{O}}^{*}\mathscr{K}$ is an even complex, the class $[j_{\mathrm{O}!}j_{\mathrm{O}}^{*}\mathscr{K}]$ is a strictly positive multiple of $\left[j_{\mathrm{O}!}\overline{\mathbf{Q}}_{\ell}\right]$. Consequently, $\left[j_{\mathrm{O}!}\overline{\mathbf{Q}}_{\ell}\right]$ lies in the image of (5.14) as claimed.

As $\left\{\left[j_{\mathrm{O}!}\overline{\mathbf{Q}}_{\ell}\right]\right\}_{\mathrm{O}\subset\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}}$ forms a basis for $K\left(\mathrm{D}^{b}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)\right)_{\mathbf{Q}}$, it follows that the map (5.14) is surjective. □

Proof of Proposition 5.12. The statement (ii) follows from (i) by [13, 8.6.12]. It remains to prove (i).

Since $\{[\mathrm{IC}\,(\mathrm{O})]\}_{\mathrm{O}\subset\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}}$ forms a basis for $K\left(\mathrm{D}^{b}_{G_{\underline{0}}}\left(\mathfrak{g}_{\underline{1}}^{\mathrm{nil}}\right)\right)_{\mathbf{Q}}$, by the previous lemma, there exists a I-graded lagrangian subspace $U \subset V$ such that $\mathrm{IC}\,(\mathrm{O})$ appears in ${}^{p}\mathscr{H}^{*}\,\mathrm{Ind}_{\mathfrak{m}_{\underline{1}}^{U}\subset\mathfrak{q}_{\underline{1}}^{U}}^{\mathfrak{g}_{\underline{1}}}\mathscr{L}$ for some $\mathscr{L} \in \mathrm{Irr}\,\mathrm{Perv}_{M_{\underline{0}}^{U}}\left(\mathfrak{m}_{\underline{1}}^{U,\mathrm{nil}}\right)$. By the result in type A, there exists some parabolic $Q' \subset M^{U}$ and Levi factor $M \subset Q$ such that $\mathscr{L}$ appears in ${}^{p}\mathscr{H}^{*}\,\mathrm{Ind}_{\mathfrak{m}_{\underline{1}}\subset\mathfrak{q}_{\underline{1}}'}^{\mathfrak{m}_{\underline{1}}^{U}}\delta_{0}$. It follows from the transitivity of Ind that $\mathrm{IC}\,(\mathrm{O})$ appears in ${}^{p}\mathscr{H}^{*}\,\mathrm{Ind}_{\mathfrak{m}_{\underline{1}}\subset\mathfrak{q}_{\underline{1}}}^{\mathfrak{g}_{\underline{1}}}\delta_{0}$, where $\mathfrak{q}$ is the inverse image of $\mathfrak{q}'$ via the quotient map $\mathfrak{q}^{U} \longrightarrow \mathfrak{m}^{U}$. Since pair $(\mathfrak{q},\mathfrak{m})$ must be conjugate to $(\mathfrak{q}^{\gamma},\mathfrak{m}^{\gamma})$ for some $\gamma \in \mathrm{Comp}\,(\beta)$, the perverse sheaf $\mathrm{IC}(\mathrm{O})$ appears in ${}^{p}\mathscr{H}^{*}\,\mathrm{Ind}_{\mathfrak{m}_{\underline{1}}^{\gamma}\subset\mathfrak{q}_{\underline{1}}^{\gamma}}^{\mathfrak{g}_{\underline{1}}}\delta_{0}$. □

5.4 Concluding remarks

We list some perspectives of possible directions for future works.

5.4.1 Derived equivalence for dDAHAs

We indicate how to obtain certain derived equivalences for dDAHAs via perverse sheaves, *cf.* Remark 2.69. Take the setting of §4 and fix a sign $\epsilon \in \{1,-1\}$. Let $d, d' \in \mathbf{Z}\setminus\{0\}$ be integers such that $\underline{d} = \underline{d}'$, so that we have the Lusztig sheafs $\mathbf{I}$ and $\mathbf{I}'$. If $\epsilon = d/|d|$, then $\mathbf{I}$ is supported in the nilpotent cone $\mathfrak{g}_{\underline{d}}^{\mathrm{nil}}$; otherwise its image under the Fourier–Sato transform is supported in $\mathfrak{g}_{-\underline{d}}^{\mathrm{nil}}$. If $dd' > 0$, then the simple constituents of $\mathbf{I}$ and $\mathbf{I}'$ up to cohomological shifts coincide, so that the algebras $\widehat{\mathcal{H}} = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I},\mathbf{I})$ and $\widehat{\mathcal{H}}' = \mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}',\mathbf{I}')$ are Morita-equivalent via the $(\widehat{\mathcal{H}},\widehat{\mathcal{H}}')$-bimodule $\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}',\mathbf{I})$.

Suppose $dd' < 0$. We may assume that $\epsilon = d/|d| = -d'/|d'|$. Let $i : \mathfrak{g}_{\underline{d}}^{\mathrm{nil}} \hookrightarrow \mathfrak{g}_{\underline{d}}$ be the inclusion and let $\mathbb{T}$ denote the Fourier–Sato transform between $\mathrm{D}^{b}_{G_{\underline{0}}\times\mathbf{C}_{q}^{\times}}(\mathfrak{g}_{\underline{d}})$ and $\mathrm{D}^{b}_{G_{\underline{0}}\times\mathbf{C}_{q}^{\times}}(\mathfrak{g}_{-\underline{d}})$ so that $\mathbb{T}\mathbf{I}' \in \mathrm{D}^{b}_{G_{\underline{0}}\times\mathbf{C}_{q}^{\times}}(\mathfrak{g}_{-\underline{d}}^{\mathrm{nil}})$. In view of the result of Achar–Mautner [1], we expect that $i^{*}\circ\mathbb{T}\circ i_{*} : \mathrm{D}^{b}_{G_{\underline{0}}\times\mathbf{C}_{q}^{\times}}(\mathfrak{g}_{-\underline{d}}^{\mathrm{nil}}) \longrightarrow \mathrm{D}^{b}_{G_{\underline{0}}\times\mathbf{C}_{q}^{\times}}(\mathfrak{g}_{\underline{d}}^{\mathrm{nil}})$ is an equivalence. This will imply that $\mathrm{Hom}^{\bullet}_{G_{\underline{0}}}(\mathbf{I}',\mathbf{I})$ induces a derived equivalence between $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}'$.

These equivalences are compatible with the KZ functor $\mathbf{V}$ by Theorem 5.4 (iii). Via Theorem 4.40 (i), they will induce equivalences for dDAHAs with "opposite parameters".

5.4.2 Geometric construction of quiver double Hecke algebras

In view of the geometric construction of quiver Hecke algebras of Varagnolo–Vasserot [51], one can probably construct the quiver double Hecke algebras with Borel–Moore homology, at least in type A. One starts with a quiver $Q = (I, H)$ and a function $\eta : H \longrightarrow \mathbf{Z}$. Fixing a dimension vector $\beta \in \mathbf{N}I$, we consider the set Ξ of enhanced sequences

$$\Xi^\beta = \left\{ \nu = ((\nu_1, d_1), \cdots, (\nu_l, d_l)) \ ; \ \nu_j \in I, \ d_j \in \mathbf{Z}, \ \sum_j \nu_j = \beta \right\}.$$

The Coxeter group of type $A^{(1)}_{l-1}$ acts on Ξ^β: the generators $s_1, \cdots, s_{l-1}$ acts by permutation and s_0 acts by

$$((\nu_1, d_1), \cdots, (\nu_l, d_l)) \mapsto ((\nu_l, d_l - 1), (\nu_2, d_2) \cdots, (\nu_{l-1}, d_l), (\nu_1, d_1 + 1)).$$

Let $\mathcal{K} = \mathbf{C}[\varpi^{\pm 1}]$ and $\mathcal{O} = \mathbf{C}[\varpi]$. The rings $\mathcal{K}$ and $\mathcal{O}$ are $\mathbf{Z}$-graded in such a way that $\deg \varpi = 1$. Set $V = \mathcal{K}^\beta$, which is $I \times \mathbf{Z}$-graded. Let

$$E^\beta = \left\{ (x_h)_{h \in H} \ ; \ x_h \in \mathrm{Hom}^{\eta(h)}_{\mathcal{O}}(V_{h'}, V_{h''}) \right\}.$$

Clearly, there is an isomorphism $E^\beta \cong \mathrm{Rep}(Q, \mathbf{C}^\beta)$. A flag of type $\nu \in \Xi$ in V is an infinite sequence of $I \times \mathbf{Z}$-graded $\mathcal{O}$-lattices $\cdots \supset V^{-1} \supset V^0 \supset V^1 \supset \cdots$ in V such that $V^{j+l} = \varpi V^j$ for $j \in \mathbf{Z}$ and $\mathrm{gdim}\, V_{j-1}/V_j = (\nu_j, d_j)$ for $j \in [1, l]$. Let F^ν denote the variety of flags of type ν. We define

$$\tilde{F}^\nu = \left\{ (V^\bullet, x) \in F^\nu \times E^\beta \ ; \ xV^j \subset V^{j+1} \right\}.$$

Denote the second projection map by $p_\nu : \tilde{F}^\nu \longrightarrow E^\beta$. The affine analogue of Lusztig's sheaf will be the infinite sum $\mathbf{I}^\beta = \bigoplus_{\nu \in \Xi^\beta} p_{\nu *} \overline{\mathbf{Q}}_\ell [\dim \tilde{F}^\nu]$. We expect that the equivariant extension algebra of $\mathbf{I}^\beta$ for various quivers $Q = (I, H)$ and functions η should exhausts the class of quiver double Hecke algebras in type A (up to the nuance of GL and SL).

5.4.3 The quiver $D^{(1)}_4$

Consider the quiver $Q = (I, H) = D^{(1)}_4$ with the following orientation:

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 0 & \leftarrow & 4 \\ & & \uparrow & & \\ & & 3 & & \end{array}$$

We consider the primitive imaginary root $\delta = (2, 1, 1, 1) \in \mathbf{N}I$ for the dimension vector. Ringel's results on tame quivers imply that there is a stratification on the moduli stack $\mathrm{Rep}^\delta(Q)$ of representations of Q of dimension δ. There are three "periodic" strata in $\mathrm{Rep}^\delta(Q)$, given by $V_{(1,1,1,0,0)} \oplus V_{(1,0,0,1,1)}$, $V_{(1,1,0,1,0)} \oplus V_{(1,1,0,0,1)}$ and $V_{(1,1,0,0,1)} \oplus V_{(1,0,1,1,0)}$, where $V_{(1,1,1,0,0)}$ is the indecomposable $\mathbf{C}Q$-module of dimension $(1, 1, 1, 0, 0)$, etc. Denote them by S_1, S_2 and S_3, respectively.

Take the function $\eta : H \longrightarrow \mathbf{Z}$ with $\eta(h) = -1$ for each $h \in H$. We can see that $\mathrm{IC}(S_1)$, $\mathrm{IC}(S_2)$ and $\mathrm{IC}(S_3)$ appears in $\mathbf{I}^\delta$ as simple constituents. Indeed, for the stratum S_1, if we take the enhanced sequence $\nu = ((1,0),(2,0),(0,-1),(3,1),(4,1),(0,0))$, then the image of $\pi_\nu : \tilde{F}^\nu \longrightarrow E_\beta$ will be the closure of S_1. Similarly, $\mathrm{IC}(S_1)$ and $\mathrm{IC}(S_2)$ appear in $\mathbf{I}^\delta$. These are the only three sheaves absent from the Lusztig sheaf. The same arguments as in §5.2 show that the extension algebra $\mathrm{Hom}^\bullet(\mathbf{I}^\delta, \mathbf{I}^\delta)$ is affine quasi-hereditary, so its global dimension is finite.

This example shows that besides the cases of dDAHA's, one can make use of affinisation to "resolve the singularities" of quiver Hecke algebras, *cf.* Remark 2.69.

5.4.4 Parabolic induction for $\mathbf{A}^\omega$

The idempotent construction in §2.5.1 can probably be generalised, which will be an analogue for $\mathbf{A}^\omega$ of the Bezrukavnikov–Etingof parabolic induction for rational Cherednik algebras [3]. It will be interesting to employ the techniques of categorification to study the module category $\mathbf{A}^\omega$-gmod and the cyclotomic quotients of $\mathbf{A}^\omega$.

5.4.5 Deformations of $\mathbf{A}^\omega$ and faithfulness of $\mathbf{V}$

Deformations of $\mathbf{A}^\omega$ are easy to construct. One replaces in the definition of the operator $\tau_a^\omega \mathbf{e}(\lambda)$ in §2.1.3 by the following:

$$\tau_a^\omega \mathbf{e}(\lambda) = \begin{cases} (\partial a + \hbar_{\lambda,a})^{\omega_\lambda(a)+1}(\partial a)^{-1}(s_{\partial a} - 1) & a(\lambda) = 0 \\ (\partial a + \hbar_{\lambda,a})^{\omega_\lambda(a)} s_{\partial a} & a(\lambda) \neq 0 \end{cases}$$

where $(\hbar_{\lambda,a})_{\lambda \in W_S\lambda_0, a \in S^+}$ are parameters of deformation satisfying $\hbar_{w\lambda,wa} = \hbar_{\lambda,a}$ for all $w \in W_S$ and $a \in S^+ \cap w^{-1}S^+$. We expect the functor $\mathbf{V}$ defined in §2.5.6 to have stronger faithfulness for deformed algebra $\mathbf{A}^\omega$. This can be seen in the geometric situation of degenerate DAHAs. In this case, there is an action of torus $\mathbf{C}_q^\times$ which provides a one-parameter non-trivial deformation of the convolution algebra $\widehat{\mathcal{H}}$. The standard modules Δ_π are flat over the base of deformation. This allows one to apply Rouquier's lifting of faithfulness as in Lemma 2.61.

Appendix A

Category of pro-objects

A.1 Category of pro-objects

A.1.1

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Pro}(\mathcal{A})$ and $\mathrm{Ind}(\mathcal{A})$ the category of pro-objects and ind-objects. The basic reference for these is [19, 8.6]. All the results below are stated for $\mathrm{Pro}(\mathcal{A})$ while they all have a dual version for $\mathrm{Ind}(\mathcal{A})$. An object of $\mathrm{Pro}(\mathcal{A})$ is a filtered "projective limit" of objects of $\mathcal{A}$. If

$$M^{(i)} = \text{“}\varprojlim_{j\in\mathcal{I}^{(i)}}\text{”}\, M_j^{(i)}, \quad M_j^{(i)} \in \mathcal{A}, \quad i \in \{1,2\}$$

are two objects of $\mathrm{Pro}(\mathcal{A})$, where $\mathcal{I}^{(i)}$'s are filtrant diagram categories and $M^{(i)} : \mathcal{I}^{(i)\mathrm{op}} \longrightarrow \mathcal{A}$'s are functors, then the Hom-space between them is given by

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}\left(M^{(1)}, M^{(2)}\right) = \varprojlim_{j\in\mathcal{I}^{(2)}} \varinjlim_{i\in\mathcal{I}^{(1)}} \mathrm{Hom}_{\mathcal{A}}\left(M_i^{(1)}, M_j^{(2)}\right). \tag{A.1}$$

A.1.2

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $\mathcal{B}$ an abelian category which admits filtered projective limits and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. We define the extension of F:

$$F : \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{B}), \quad F(M) = \text{“}\varprojlim_{(M',a)\in\mathcal{A}^M_{\mathrm{epi}}}\text{”}\; F(M').$$

According to [19, 8.6.8], the extended functor $F : \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{B})$ is still exact.

A.1.3

For every $M \in \mathrm{Pro}(\mathcal{A})$, let $\mathcal{A}^M$ denotes the category whose objects are pairs (M', a) where $M' \in \mathcal{A}$ and $a \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(M, M')$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}^M}\left((M_1, a_1), (M_2, a_2)\right) = \left\{b \in \mathrm{Hom}_{\mathcal{A}}(M_1, M_2)\ ;\ a_2 = b \circ a_1\right\}.$$

Every object $M \in \mathrm{Pro}(\mathcal{A})$ can be expressed as the following filtered limit $\mathcal{A}^M$:

$$M \cong \text{“}\varprojlim_{(M',a)\in\mathcal{A}^M}\text{”} \; M'.$$

Let $\mathcal{A}^M_{\mathrm{epi}} \subset \mathcal{A}^M$ be the full subcategory whose objects are the pairs (M', q) with q being an epimorphism.

Lemma A.2. *Let $\mathcal{A}$ be an artinian abelian category. Then*

(i) $\mathcal{A}$ is a Serre subcategory of $\mathrm{Pro}(\mathcal{A})$*,*

(ii) any object $M \in \mathrm{Pro}(\mathcal{A})$ can be written as the following filtered projective limit

$$M \cong \text{“}\varprojlim_{(M',a)\in\mathcal{A}^M_{\mathrm{epi}}}\text{”} \; M'$$

(iii) $\mathcal{A}$ is the full subcategory of artinian objects in $\mathrm{Pro}(\mathcal{A})$*.*

(iv) if $\varphi : N \longrightarrow M$ is a morphism in $\mathcal{A}$ such that for every (M', q) in $\mathcal{A}^M_{\mathrm{epi}}$, the composite $q \circ \varphi$ is an epimorphism, then φ is an epimorphism.

Proof. We first prove that $\mathcal{A} \subset \mathrm{Pro}(\mathcal{A})$ is closed under taking sub-objects.

For every $M \in \mathrm{Pro}(\mathcal{A})$, let $\mathcal{A}^M$ denotes the category whose objects are pairs (M', a) where $M' \in \mathcal{A}$ and $a \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(M, M')$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}^M}\left((M_1, a_1), (M_2, a_2)\right) = \left\{b \in \mathrm{Hom}_{\mathcal{A}}(M_1, M_2) \; ; \; a_2 = b \circ a_1\right\}.$$

Let $M \in \mathrm{Pro}(\mathcal{A})$. Suppose that there exists $\tilde{M} \in \mathcal{A}$ and a monomorphism $\iota : M \rightarrowtail \tilde{M}$. We can consider the full subcategory $\mathcal{A}^M_1 \subset \mathcal{A}^M$ of pairs (M', a) with a being monomorphism. The subcategory $\mathcal{A}^M_1$ is cofinal. Indeed, if $(M', a) \in \mathcal{A}^M$, then $\left(M' \times \tilde{M}, (a, \iota)\right) \in \mathcal{A}^M_1$. Let $\mathcal{A}^M_2 \subset \mathcal{A}^M_1$ be the full subcategory of objects which are minimal, in the sense that if there is $(M'', b) \in \mathcal{A}^M_1$ with a monomorphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(M'', M')$ such that $\varphi \circ b = a$, then φ is an isomorphism. By the minimality of the objects of $\mathcal{A}^M_2$, it is easy to see that the Hom-space $\mathrm{Hom}_{\mathcal{A}^M_2}\left((M', a), (M'', b)\right)$ consists of exactly one element for every $(M', a), (M'', b) \in \mathcal{A}^M_2$. It follows that any object $(M', a) \in \mathcal{A}^M_2$ yields an isomorphism $a : M \cong M'$. As $\mathcal{A}$ is artinian, $\mathcal{A}^M_2$ cannot be empty, whence $M \in \mathcal{A}$.

To prove (ii), in view of (A.1.3), it suffices to show that $\mathcal{A}^M_{\mathrm{epi}}$ is cofinal. The previous paragraph shows that for $(M', a) \in \mathcal{A}^M$, the image $\mathrm{im}(a)$ is in $\mathcal{A}$. Consider the factorisation $M \xrightarrow{\pi_a} \mathrm{im}(a) \xrightarrow{\bar{a}} M'$. Then $(\mathrm{im}(a), \pi_a) \in \mathcal{A}^M_{\mathrm{epi}}$ and there is a morphism $\bar{a} : (\mathrm{im}(a), \pi_a) \longrightarrow (M', a)$ in $\mathcal{A}^M$. Thus $\mathcal{A}^M_{\mathrm{epi}}$ is cofinal in $\mathcal{A}^M$.

We prove (iii). Let $M \in \mathcal{A}$. Since $\mathcal{A} \subset \mathrm{Pro}(\mathcal{A})$ is closed under taking sub-objects, every descending chain of sub-objects of M is in the subcategory $\mathcal{A}$, which by assumption must stabilise. Thus M is artinian in $\mathrm{Pro}(\mathcal{A})$. Suppose that $M \in \mathrm{Pro}(\mathcal{A})$ is artinian. There must be a minimal sub-object $M' \subset M$ such that M/M' lies in $\mathcal{A}$, meaning that

the category $\mathcal{A}_3^M$ has an initial object. By (ii), M being the projective limit on $\mathcal{A}_3^M$ must lie in $\mathcal{A}$, whence (iii). The assertion (i) follows immediately from (iii).

We prove (iv). Let $c : M \longrightarrow \operatorname{coker}\varphi = C$ be the cokernel. Suppose that $C \neq 0$. Since $C \in \operatorname{Pro}(\mathcal{A})$, there exists an epimorphism $p : C \longrightarrow C'$ with $0 \neq C' \in \mathcal{A}$. Since $p \circ c : M \longrightarrow C'$ is epimorphism, the composite $p \circ c \circ \varphi$ is also an epimorphism by hypothesis. However, as $c \circ \varphi = 0$, we see that $C' = 0$, contradiction. Thus $C' = 0$ and φ is an epimorphism. □

A.1.4

Suppose that $\mathcal{A}$ is noetherian and artinian. We define an endo-functor

$$\operatorname{hd} : \operatorname{Pro}(\mathcal{A}) \longrightarrow \operatorname{Pro}(\mathcal{A}), \quad \operatorname{hd}(M) = \text{“}\varprojlim_{(M',q)\in\mathcal{A}_{\mathrm{epi}}^M}\text{”} \operatorname{hd}(M')$$

where $\operatorname{hd}(M')$ is the largest semisimple quotient of M' in $\mathcal{A}$. For every $M \in \operatorname{Pro}(\mathcal{A})$, there is a canonical map $\pi_M : M \longrightarrow \operatorname{hd}(M)$.

Lemma A.3. *Let $\mathcal{A}$ be a noetherian artinian abelian category. Let $\varphi : N \longrightarrow M$ be a morphism in* $\operatorname{Pro}(\mathcal{A})$*. Suppose that the composite $N \xrightarrow{\varphi} M \xrightarrow{\pi_M} \operatorname{hd}(M)$ is an epimorphism. Then φ is an epimophism.*

Proof. We first prove the statement in the case that $M \in \mathcal{A}$. In this case, since $\operatorname{coker}\varphi$ is a quotient of M, we have an epimorphism $\operatorname{hd}(M) \twoheadrightarrow \operatorname{hd}(\operatorname{coker}\varphi)$. As the composite $N \longrightarrow \operatorname{hd}(M) \longrightarrow \operatorname{hd}(\operatorname{coker}\varphi)$ is zero and is an epimorphism, it implies that $\operatorname{hd}(\operatorname{coker}\varphi) = 0$. As $\mathcal{A}$ is noetherian, it follows that $\operatorname{coker}\varphi = 0$, so φ is surjective.

In general, let $M \in \operatorname{Pro}(\mathcal{A})$. Let (M', q) be any object of $\mathcal{A}_{\mathrm{epi}}^M$. Then $\pi_{M'} \circ q \circ \varphi$ is an epimorphism. By the previous paragraph, $q \circ \varphi$ is also an epimorphism. Then Lemma A.2 (iv) implies that φ is an epimorphism. □

A.1.5

Lemma A.4. *Suppose that $\operatorname{Ext}_{\mathcal{A}}^1(M, N)$ is finite dimensional for M and N simple. Let $M \in \mathcal{A}$ be a simple object. Then there exists a projective cover $P_M \in \operatorname{Pro}(\mathcal{A})$.*

Proof. We construct an object $P^{(n)} \in \operatorname{Pro}(\mathcal{A})$ for any $n \in \mathbf{N}$ by induction. Let $P^{(n)} = M$. For $n > 0$, let

$$0 \longrightarrow \prod_{L\in\operatorname{Irr}(\mathcal{A})/\sim} \operatorname{Ext}_{\mathcal{A}}^1\left(P^{(n-1)}, L\right)^* \otimes L \longrightarrow P^{(n)} \longrightarrow P^{(n-1)} \longrightarrow 0$$

be the short exact sequence corresponding to the diagonal class

$$\Delta \in \prod_{L\in\in\operatorname{Irr}(\mathcal{A})/\sim} \operatorname{Ext}_{\mathcal{A}}^1\left(P^{(n-1)}, L\right)^* \otimes \operatorname{Ext}_{\mathcal{A}}^1\left(P^{(n-1)}, L\right).$$

Put $P = \text{“}\varprojlim_{n\longrightarrow\infty}\text{”} P^{(n)}$. Then P is a projective since we have

$$\operatorname{Ext}_{\operatorname{Pro}(\mathcal{A})}^1(P, L) = 0$$

by construction. Denote $p : P \longrightarrow M$ the obvious epimorphism.

Now, let $\mathcal{A}_M^P$ be the category whose objects are triples (π, Q, π'), where

- $Q \in \mathcal{A}$
- $\pi \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P, Q)$ is an epimorphism and
- $\pi' \in \mathrm{Hom}_{\mathcal{A}}(Q, M)$

such that

- $\pi' \circ \pi = p \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P, M)$ and
- π' induces an isomorphism $\mathrm{hd}(Q) \cong M$.

The morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}_M^P}\left((\pi_1, Q_1, \pi_1'), (\pi_2, Q_2, \pi_2')\right) = \left\{\varphi \in \mathrm{Hom}_{\mathcal{A}}\left(Q_1, Q_2\right) \ ; \ \varphi \circ \pi_1 = \pi_2\right\}.$$

Put

$$P_M = \text{“}\varprojlim_{(\pi, Q, \pi') \in \mathcal{A}_M^P}\text{”} \quad Q.$$

□

A.1.6

Let $\widehat{\mathcal{A}}$ be the full subcategory of $\mathrm{Pro}(\mathcal{A})$ consisting of objects $M \in \mathrm{Pro}(\mathcal{A})$ such that $\mathrm{hd}(M) \in \mathcal{A}$.

Let $\{L_i\}_{i \in I}$ be a complete collection of simple objects and for each $i \in I$, let $P_i \longrightarrow L_i$ be a projective cover. Suppose tha Put $P_{\mathcal{A}} = \bigoplus_{i \in I} P_i$ and $B_{\mathcal{A}} = \mathrm{End}_{\mathrm{Pro}(\mathcal{A})}(P_{\mathcal{A}})$.

Proposition A.5. *There is an equivalence of categories*

$$B_{\mathcal{A}}\text{-mod} \cong \widehat{\mathcal{A}}$$

which indentifies the subcategory of $B_{\mathcal{A}}$-modules of finite length with $\mathcal{A}$

A.1.7

Let $\mathcal{Z}_{\mathcal{A}} = \mathrm{End}(\mathrm{id}_{\mathcal{A}})$ be the categorical centre. It is obvious that the restriction map $\mathrm{End}(\mathrm{id}_{\mathrm{Pro}(\mathcal{A})}) \longrightarrow \mathcal{Z}_{\mathcal{A}}$ is an isomorphism. Suppose that the set of isomorphism classes of simple objects of $\mathcal{A}$ is finite. Let $\{L_i\}_{i \in I}$ be a complete collection of simple objects and for each $i \in I$, let $P_i \longrightarrow L_i$ be a projective cover. Put $P_{\mathcal{A}} = \bigoplus_{i \in I} P_i$.

Proposition A.6. *The canonical map*

$$\mathcal{Z}_{\mathcal{A}} \longrightarrow \mathcal{Z}(\mathrm{End}_{\mathrm{Pro}(\mathcal{A})}(P_{\mathcal{A}}))$$

is an isomorphism.

Appendix B

Relative Coxeter groups

B.1 Relative Coxeter groups

This section is independent of the rest of the article. We recollect certain results of Lusztig [29], [35] and [31] concerning the structure of relative Weyl groups. We generalise these results to a more general setting.

B.1.1 Set of reflections

Let (W, S) be a Coxeter system with $S \subseteq W$ the set of simple reflections. We follow Bourbaki [6, ch 4] and define $T = \{wsw^{-1} \in W \ ; \ s \in S, w \in W\}$ to be the subset of elements that are conjugate to some simple reflection. Elements of T are called **reflections**. Given any $w \in W$, let $w = w_1 w_2 \dots w_q$ be a reduced decomposition with $w_i \in S$ for all i. For each $t \in T$, we let

$$\eta\,(w,t) = \begin{cases} -1 & \text{if } t = w_1 w_2 \dots w_{j-1} w_j w_{j-1} \dots w_2 w_1 \text{ for some } 1 \leq j \leq q \\ 1 & \text{otherwise} \end{cases}.$$

It turns out that $\eta\,(w,t)$ is independent of the choice of reduced decomposition. Let $T(w) = \{t \in T; \eta(w,t) = -1\}$ and $D(w) = T(w) \cap S$. The following properties about $T(w)$ is standard.

Proposition B.1. *In the above notations,*

(i) There is a characterisation $T(w) = \{t \in T; \ell(tw) < \ell(w)\} \subseteq T$.

(ii) For each $w \in W$, we have $\ell(w) = \#T(w)$.

(iii) For each $w, y \in W$, we have $T(wy) \subseteq T(w) \cup wT(y)w^{-1}$.

□

B.1.2 Conjugation of parabolic subgroups

For any subset $\Sigma \subseteq S$, let (W_Σ, Σ) be the Coxeter subsystem of (W, S) generated by Σ. Accordingly, let T_Σ denote the set of reflections in the Coxeter system (W_Σ, Σ).

We define subsets ${}^{\Sigma}W = \{w \in W \ ; \ T(w) \cap T_{\Sigma} = \emptyset\}$, $W^{\Sigma} = \{w \in W \ ; \ T(w^{-1}) \cap T_{\Sigma} = \emptyset\} = \left({}^{\Sigma}W\right)^{-1}$. Then, W^{Σ} and ${}^{\Sigma}W$ form a full subset of representatives for the elements of W/W_{Σ} and $W_{\Sigma}\backslash W$, respectively. Moreover, the elements of W^{Σ} and ${}^{\Sigma}W$ are characterised by the property that they are of minimal length in their own classes.

For any two subsets $\Sigma, \Sigma' \subseteq S$, we define a subset

$$\mathrm{N}\left(\Sigma, \Sigma'\right) = \left\{y \in W^{\Sigma} \cap {}^{\Sigma'}W \ ; \ yW_{\Sigma}y^{-1} = W_{\Sigma'}\right\}$$

Proposition B.2. *The following statements hold:*

(i) For any subsets $\Sigma, \Sigma' \subseteq S$ and for any element $y \in \mathrm{N}\left(\Sigma, \Sigma'\right)$, the isomorphism defined by conjugation

$$\begin{aligned} \mathrm{Int}_y : W_{\Sigma} &\longrightarrow W_{\Sigma'} \\ w &\mapsto ywy^{-1} \end{aligned}$$

induces an isomorphism of Coxeter systems $(W_{\Sigma}, \Sigma) \cong (W_{\Sigma'}, \Sigma')$ and in particular a bijection on the reflections $T_{\Sigma} \cong T_{\Sigma'}$.

(ii) For any subsets $\Sigma, \Sigma', \Sigma'' \subseteq S$, the multiplication on W restricts to a map

$$\mathrm{N}\left(\Sigma', \Sigma''\right) \times \mathrm{N}\left(\Sigma, \Sigma'\right) \longrightarrow \mathrm{N}\left(\Sigma, \Sigma''\right).$$

Proof. For (i), it suffices to show that $\mathrm{Int}_y(\Sigma) = \Sigma'$, or equivalently that the isomorphism Int_y preserves the length. Indeed, we have for any $w \in W_{\Sigma}$

$$\ell(y) + \ell(w) = \ell\left(yw\right) = \ell\left(ywy^{-1}y\right) = \ell\left(ywy^{-1}\right) + \ell(y),$$

where first equality is due to the fact that $y \in W^{\Sigma}$ and the third is due to the fact that $y \in {}^{\Sigma'}W$. This proves (i).

For (ii), since $y'yW_{\Sigma}y^{-1}y'^{-1} = y'W_{\Sigma'}y'^{-1} = W_{\Sigma''}$, it remains to show that $y'y \in W^{\Sigma} \cap {}^{\Sigma''}W$. By symmetry, we need only to show that $y'y \in {}^{\Sigma''}W$. The assumption that $y \in {}^{\Sigma'}W$ and $y' \in {}^{\Sigma''}W$, implies $T(y) \cap T_{\Sigma''} =$ and $y'T(y)y'^{-1} \cap T_{\Sigma''} = y'\left(T(y) \cap T_{\Sigma'}\right)y'^{-1} = \emptyset$. Therefore, by Proposition B.1(iii), we deduce that

$$T(yy') \cap T_{\Sigma''} \subseteq \left(T(y) \cap T_{\Sigma''}\right) \cup \left(y'T(y)y'^{-1}T_{\Sigma''}\right) = \emptyset,$$

whence $yy' \in {}^{\Sigma''}W$.

□

Now we study the decompsition of elements in $\mathrm{N}\left(\Sigma, \Sigma'\right)$.

Proposition B.3. *Let $\Sigma, \Sigma' \subseteq S$ be subsets and let $w \in \mathrm{N}\left(\Sigma, \Sigma'\right)$. Suppose that W_{Σ} and $W_{\Sigma'}$ are finite. For each $s \in S \setminus \Sigma'$, the following conditions are equivalent:*

(i) $\ell(sw) = \ell(w) - 1$,

(ii) $s \in D\left(w_0^{\Sigma'}w\right) \setminus \Sigma' = D(w)$,

(iii) $W_{\Sigma' \cup \{s\}}$ *is finite and* $\ell\left(w_0^{\Sigma' \cup \{s\}}w_0^{\Sigma'}w\right) = \ell(w) - \ell\left(w_0^{\Sigma' \cup \{s\}}w_0^{\Sigma'}\right)$.

(iv) $T_{\Sigma'\cup\{s\}} \setminus T_{\Sigma'} \subseteq T\left(w_0^{\Sigma'} w\right) \setminus T_{\Sigma'} = T\left(w\right)$,

Proof. The conditions (i) and (ii) are equivalent by definition.

Suppose first the condition (i): $\ell(sw) = \ell(w) - 1$. We claim that for any element $x \in W_{\Sigma'\cup\{s\}}$, we have

$$w_0^{\Sigma'} w \geq x w_0^{\Sigma'} w \tag{B.4}$$

in the Bruhat order. We prove this by induction on $\ell(x)$. When $\ell(x) = 0$, the equation (B.4) holds trivially. Suppose that $\ell(x) > 0$, so that $\emptyset \neq D(x) \subseteq \Sigma' \cup \{s\}$. We choose any $s' \in D(x)$ so that $\ell(s'x) < \ell(x)$ and thus $w_0^{\Sigma'} w \geq s' x w_0^{\Sigma'} w$ by the inductive hypothesis. If $s' x w_0^{\Sigma'} w > x w_0^{\Sigma'} w$, then we are done. Otherwise we have $s' \notin D\left(s' x w_0^{\Sigma'} w\right)$. Since $w \in \mathrm{N}(\Sigma, \Sigma')$, we know that $w_0^{\Sigma'} w = w w_0^{\Sigma}$ so that $\Sigma' \cup \{s\} \subseteq D(w_0^{\Sigma'}) \cup D(w) = D(w_0^{\Sigma'} w)$, which implies $s' \in D(w_0^{\Sigma'} w) \setminus D(s' x w_0^{\Sigma'} w)$. In this case, the lifting property of the Bruhat order (see [4, 2.2.7]) implies that $w_0^{\Sigma'} w \geq x w_0^{\Sigma'} w$, so (B.4) is verified. In particular, the equation (B.4) immediately implies that $W_{\Sigma'\cup\{s\}}$ is finite and thus there exists a unique longest element $w_0^{\Sigma'\cup\{s\}}$. Now applying (B.4) to elements of $T_{\Sigma'\cup\{s\}} \subseteq W_{\Sigma'\cup\{s\}}$, we deduce that $T_{\Sigma'\cup\{s\}} \subseteq T(w_0^{\Sigma'} w)$. Therefore

$$T_{\Sigma'\cup\{s\}} \setminus T_{\Sigma'} \subseteq T\left(w_0^{\Sigma'} w\right) \setminus T_{\Sigma'} = T\left(w\right)$$

This implies (iv).

Assuming (iv), we see that $T_{\Sigma'\cup\{s\}} \subseteq T(w)$ must be finite, so is $W_{\Sigma'\cup\{s\}}$, and that

$$\ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} w\right) = \#T\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} w\right) = \#T\left(w\right) - \#T_{\Sigma'\cup\{s\}} + \#T_{\Sigma'} = \ell(w) - \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'}\right)$$

and we have thus (iii).

Assuming (iii), since $w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} \in W^{\Sigma'}$, every reduced decomposition of it ends with s. This implies

$$\begin{aligned}\ell\left(sw\right) &\leq \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} w\right) + \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} s\right) \\ &= \ell\left(w\right) - \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'}\right) + \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'}\right) - 1 = \ell(w) - 1,\end{aligned}$$

whence (i).

□

Proposition B.5. *For any $\Sigma, \Sigma' \subseteq S$ such that W_Σ and $W_{\Sigma'}$ are finite and for any $y \in \mathrm{N}(\Sigma, \Sigma')$, there exists a sequence $\{\Sigma_i\}_{i=1}^{q}$ of subsets of S and a sequence $\{y_i\}_{i=1}^{q-1}$ of elements of W such that*

(i) $\Sigma_1 = \Sigma$ *and* $\Sigma_q = \Sigma'$,

(ii) the subsets Σ_i and Σ_{i+1} differ by at most one element for all $i = 1 \ldots q-1$,

(iii) $y_i \in \mathrm{N}\left(\Sigma_i, \Sigma_{i+1}\right)$ *and there exists $s_i \in S \setminus \Sigma_i$ such that $y_i = w_0^{\Sigma_i\cup\{s\}} w_0^{\Sigma_i}$ for all $i = 1 \ldots q-1$,*

(iv) $\ell(y) = \ell(y_1) + \ldots + \ell(y_q)$.

Proof. We prove the statement by induction on $\ell(y)$. If $\ell(y) = 0$, it is trivial. Suppose that $\ell(y) > 0$. We choose an element $s \in D(y)$. Since $y \in {}^{\Sigma'}W$, we see that $s \in S \setminus \Sigma'$. By Proposition B.3, we have

$$\ell(y) = \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} y\right) + \ell\left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'}\right).$$

Now, the conjugation $\mathrm{Int}_{w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'}}$ sends Σ' to some subset $\Sigma'' \subseteq S$. Therefore,

$$w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} y \in \mathrm{N}\left(\Sigma, \Sigma''\right).$$

Applying the inductive hypothesis to $w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'} y$, there exists then $\Sigma_1 = \Sigma, \cdots, \Sigma_q = \Sigma''$ and $y_1, \cdots, y_{q-1}$ satisfying the desired properties. We define then $\Sigma_{q+1} = \Sigma'$, $\Sigma' \cup \{s\} = \Sigma'' \cup \{s_q\}$ and $y_q = \left(w_0^{\Sigma'\cup\{s\}} w_0^{\Sigma'}\right)^{-1} = w_0^{\Sigma''\cup\{s_q\}} w_0^{\Sigma''}$. Then the sequences $\{\Sigma_i\}_{i=1}^{q+1}$ and $\{y_i\}_{i=1}^{q}$ have the desired property. □

B.1.3 Relative Coxeter groups

Let (W, S) be a Coxeter system and let $\Sigma \subseteq S$ be a subset which generates a parabolic subgroup W_Σ.

We assume the following two conditions:

(i) The parabolic subgroup W_Σ is finite.

(ii) For each subset $\Sigma' \subseteq S$ containing Σ, if the parabolic subgroup $W_{\Sigma'}$ generated by Σ' is finite, then its longest element $w_0^{\Sigma'}$ normalises W_Σ.

By the first hypothesis, there is a longest element $w_0^\Sigma \in W_\Sigma$. It defines an automorphism of group

$$\begin{aligned} W_\Sigma &\longrightarrow W_\Sigma \\ w &\mapsto w_0^\Sigma w w_0^\Sigma \end{aligned}$$

As w_0^Σ is longest, for any element $w \in W_\Sigma$, we have $\ell\left(w w_0^\Sigma\right) = \ell\left(w_0^\Sigma w\right) = \ell\left(w_0^\Sigma\right) - \ell(w)$ and thus $\ell\left(w_0^\Sigma w w_0^\Sigma\right) = \ell(w)$. In particular, it induces an involution on Σ and on T_Σ.

Consider the quotient group $N_W\left(W_\Sigma\right)/W_\Sigma$. In each class $C \in N_W\left(W_\Sigma\right)/W_\Sigma$, there is exactly one element $w \in C$ of minimal length $\ell(w) = \min_{y\in C} \ell(y)$. Such an element is characterised by the property that $T(w) \cap T_\Sigma = \emptyset$. Let $W^\Sigma = \{w \in W; T(w) \cap T_\Sigma = \emptyset\}$ and $\tilde{W} = N_W\left(W_\Sigma\right) \cap W^\Sigma$.

Proposition B.6. *The following statements hold:*

(i) The normaliser $N_W(W_\Sigma)$ is isomorphic in the canonical way to the semi-direct product $\tilde{W} \ltimes W_\Sigma$.

(ii) For any $w \in \tilde{W}$, the conjugation

$$\begin{aligned} W_\Sigma &\longrightarrow W_\Sigma \\ y &\mapsto w y w^{-1} \end{aligned}$$

preserves the length function and thus induces a permutation of Σ and of T_Σ.

(iii) The element w_0^Σ centralises $\tilde{W}$.

(iv) For any $w \in \tilde{W}$, we have $T\left(w_0^\Sigma w\right) = T(w) \sqcup T_\Sigma$ and $D\left(w_0^\Sigma w\right) = D(w) \sqcup \Sigma$.

Proof. For (i), since $\tilde{W} \longrightarrow N_W(W_\Sigma)/W_\Sigma$ is a bijection, it suffices to show that $\tilde{W}$ is closed under multiplication. Indeed, if $w, y \in \tilde{W}$, then Int_w preserves the set T_Σ. Now $T_{wy} \subseteq T(w) \cup wT(y)w^{-1} \subseteq T_W \setminus T_\Sigma$, so $wy \in N_W(W_\Sigma) \cap {}^\Sigma W = \tilde{W}$. This proves (i).

For (ii), we notice that $\ell(w) + \ell(y) = \ell(wy) = \ell(wyw^{-1}w)$. Since $wyw^{-1} \in W_\Sigma$, we have $\ell(wyw^{-1}w) = \ell(wyw^{-1}) + \ell(w)$. Thus $\ell(y) = \ell(wyw^{-1})$. This proves (ii).

For each element $w \in \tilde{W}$, the conjugation Int_w preserves T_Σ, the length function on W_Σ, thus the longest element of (W_Σ, Σ). This proves (iii).

The statement (iv) results immediately from (ii) and (iii). □

Proposition B.3 applied to the case $\Sigma = \Sigma'$ can be restated as follows.

Proposition B.7. *Let $w \in \tilde{W}$. For each $s \in S \setminus \Sigma$, the following conditions are equivalent:*

(i) $\ell(sw) = \ell(w) - 1$,

(ii) $s \in D\left(w_0^\Sigma w\right) \setminus \Sigma = D(w)$,

(iii) $W_{\Sigma \cup \{s\}}$ is finite and $\ell\left(w_0^{\Sigma\cup\{s\}} w_0^\Sigma w\right) = \ell(w) - \ell\left(w_0^{\Sigma\cup\{s\}} w_0^\Sigma\right)$.

(iv) $T_{\Sigma\cup\{s\}} \setminus T_\Sigma \subseteq T\left(w_0^\Sigma w\right) \setminus T_\Sigma = T(w)$,

We define

$$\Sigma^{\complement} = \left\{s \in S \setminus \Sigma; \#W_{\Sigma\cup\{s\}} < \infty\right\}.$$

For each $s \in \Sigma$, we denote $\tilde{s} = w_0^{\Sigma\cup\{s\}} w_0^\Sigma \in W$. Clearly, $\tilde{s} \in \tilde{W}$. We set $\tilde{S} = \left\{\tilde{s} \in \tilde{W}; s \in \Sigma^{\complement}\right\}$. Let $\tilde{T} \subseteq \tilde{W}$ denote the subset of elements that are $\tilde{W}$-conjugate to some element of $\tilde{S}$. For each $w \in \tilde{W}$, we denote

$$\tilde{T}(w) = \left\{t \in \tilde{T}; \ell(tw) < \ell(w)\right\}, \quad \tilde{D}(w) = \tilde{T}(w) \cap \tilde{S}.$$

Proposition B.8. *Let $w \in \tilde{W}$ and let $\tilde{s} \in \tilde{D}(w)$. Suppose that we are given a decomposition $w = w_1 \ldots w_r$ with $w_i \in \tilde{S}$ for all i and that $\ell(w) = \ell(w_1) + \ldots + \ell(w_r)$. Then there exists an $1 \leq i \leq r$ such that*

$$\tilde{s} w_1 \ldots w_{i-1} = w_1 \ldots w_i.$$

Proof. From Proposition B.7 we deduce that $\ell(\tilde{s}w) + \ell(\tilde{s}) = \ell(w)$. Let $m \geq 1$ be the minimal integer such that

$$\ell(\tilde{s} w_1 \ldots w_m) < \ell(\tilde{s}) + \ell(w_1 \ldots w_m)$$

By Proposition B.7 again, we have

$$\begin{aligned}
&\ell(w_1 \ldots w_{m-1}) + \ell(w_m) - \ell(\tilde{s}) = \ell(w_1 \ldots w_m) - \ell(\tilde{s}) = \ell(\tilde{s} w_1 \ldots w_m) \\
&= \ell(\tilde{s} w_1 \ldots w_{m-1}) - \ell(w_m) = \ell(w_1 \ldots w_{m-1}) + \ell(\tilde{s}) - \ell(w_m).
\end{aligned}$$

Therefore $\ell(\tilde{s}) = \ell(w_m)$.

Now let $\tilde{s} = u_1 \dots u_r$, and $w_m = v_1 \dots v_r$ be reduced decompositions in (W, S) so that $r = \ell(\tilde{s}) = \ell(w_m)$ and $u_i, w_i \in S$ for all i. We also let $w_1 \dots w_{m-1} = x_1 \dots x_{r'}$ be a reduced decomposition for $w_1 \dots w_{m-1} \in W$. Now for each $j = 1 \dots r$, we have

$$\begin{aligned}\ell(w_1 \dots w_m) - (r - j + 1) &\leq \ell(u_j \dots u_r w_1 \dots w_m) = \ell(u_j \dots u_r x_1 \dots x_{r'} v_1 \dots v_r) \\ &= \ell(u_{j-1} \dots u_1 \tilde{s} w_1 \dots w_m) \leq \ell(\tilde{s} w_1 \dots w_m) + j - 1 = \ell(w_1 \dots w_m) - (r - j + 1)\end{aligned}$$

whence $\ell(u_1 \dots u_r x_1 \dots x_{r'} v_1 \dots v_r) = \ell(w_1 \dots w_m) - (r - j + 1)$. Applying the exchange condition for (W, S), there is either an $1 \leq i \leq r$ such that

$$u_r x_1 \dots x_{j-1} = x_1 \dots x_j$$

or there is an $1 \leq i \leq r'$ such that

$$u_r x_1 \dots x_{r'} v_1 \dots v_{i-1} = x_1 \dots x_{r'} v_1 \dots v_i.$$

The first case cannot really happen since $x_1 \dots x_{r'}$ is a reduced word. because otherwise $\ell(u_r x_1 \dots x_{r'}) = \ell(x_1 \dots \widehat{x}_i \dots x_{r'}) \leq r'$, absurd. Thus

$$u_r x_1 \dots x_{r'} v_1 \dots v_r = x_1 \dots x_{r'} v_1 \dots \widehat{v}_{i_1} \dots v_r.$$

with $i_1 = i$. Now Applying again the exchange condition and reasoning on the reducedness of $u_r x_1 \cdots x_r$, we have

$$u_{r-1} u_r x_1 \dots x_{r'} v_1 \dots v_r = u_{r-1} x_1 \dots x_{r'} v_1 \dots \widehat{v}_{i_1} \dots v_r = x_1 \dots x_{r'} v_1 \dots \widehat{v}_{i_1} \dots v_{i_2} \dots v_r$$

for some $1 \leq i_2 \leq m$ and $i_2 \neq i_1$. Continuing on, we arrive at

$$u_1 \dots u_r x_1 \dots x_{r'} v_1 \dots v_r = x_1 \dots x_{r'}$$

so that

$$\tilde{s} w_1 \dots w_m = u_1 \dots u_r x_1 \dots x_{r'} v_1 \dots v_r = x_1 \dots x_{r'} = w_1 \dots w_{m-1}$$

or equivalently $\tilde{s} w_1 \dots w_{m-1} = w_1 \dots w_m$, since $\tilde{s}^2 = e$. This proves the proposition.

□

For $w \in \tilde{W}$, we denote $\tilde{\ell}(w) \in \mathbf{N} \cup \{\infty\}$ the length of w in the Coxeter group $\left(\tilde{W}, \tilde{S}\right)$ (in contrary to the length $\ell(w)$ in the Coxeter group (W, S)). This is defined to be

$$\tilde{\ell}(w) = \inf\left\{q \in \mathbf{N}; \exists w_1 \dots w_q \in \tilde{S},\ w = w_1 \dots w_q\right\} \in \mathbf{N} \cup \{\infty\}.$$

Theorem B.9. *In the above notations,*

(i) the pair $\left(\tilde{W}, \tilde{S}\right)$ *forms a Coxeter system.*

(ii) Then for any $w, y \in \tilde{W}$*, we have*

$$\tilde{\ell}(y) + \tilde{\ell}(w) = \tilde{\ell}(yw) \quad \textit{if and only if} \quad \ell(y) + \ell(w) = \ell(yw).$$

Proof. Let us first verify that $\tilde{S}$ generates $\tilde{W}$. Let $w \in \tilde{W}$. We prove by induction on $\ell(w)$ that w is in the subgroup generated by $\tilde{S}$. When $\ell(w) = 0$, it is trivially true. Suppose that $w \neq e$. Take any $s \in D(w)$. Then $s \in S \setminus \Sigma$, and by Proposition B.7 we have

$$T_{\Sigma \cup \{s\}} \setminus T_\Sigma \subseteq T(w).$$

From this, we see that $T(w) = T(w\tilde{s}) \sqcup \left(T_{\Sigma \cup \{s\}} \setminus T_\Sigma\right)$, so

$$\ell(w\tilde{s}) = \ell(w) - \ell(\tilde{s}) < \ell(w)$$

By induction hypothesis, $w\tilde{s}$ belongs to the subgroup generated by $\tilde{S}$, and so is w. In particular, $\tilde{\ell}(w) < \infty$ for all $w \in \tilde{W}$.

We prove (ii). It suffices to treat the case where $y = \tilde{s} \in \tilde{S}$ since the general case follows by induction on $\tilde{\ell}(y)$. Let $w = w_1 \dots w_q \in \tilde{W}$ be a decomposition in $(\tilde{W}, \tilde{S})$ such that $w_i \in \tilde{S}$, and $q = \tilde{\ell}(w)$ is minimal.

Suppose first that $\ell(\tilde{s}w) < \ell(\tilde{s}) + \ell(w)$. Then Proposition B.8 implies that there is an index $1 \leq j \leq q$ such that $\tilde{s}w_1 \dots w_{j-1} = w_j \dots w_q$, and hence

$$\tilde{s}w = w_1 \dots w_{j-1} w_{j+1} \dots w_q$$

which implies $\tilde{\ell}(\tilde{s}w) = \tilde{\ell}(w) - 1$.

Similarly, if we replace w with $\tilde{s}w$ in the above argument, we will have that $\ell(w) < \ell(\tilde{s}) + \ell(\tilde{s}w)$ implies $\tilde{\ell}(w) = \tilde{\ell}(\tilde{s}w) - 1$.

According to Proposition B.7, either $\ell(\tilde{s}w) = \ell(w) + \ell(\tilde{s})$ or $\ell(\tilde{s}w) = \ell(w) - \ell(\tilde{s})$ should hold. Therefore we have the equivalences of conditions:

$$\begin{aligned} \tilde{\ell}(\tilde{s}w) = \tilde{\ell}(w) + 1 &\Longleftrightarrow \ell(\tilde{s}w) = \ell(w) + \ell(\tilde{s}) \\ \tilde{\ell}(\tilde{s}w) = \tilde{\ell}(w) - 1 &\Longleftrightarrow \ell(\tilde{s}w) = \ell(w) - \ell(\tilde{s}). \end{aligned} \tag{B.10}$$

This proves (ii).

In order to show that $\left(\tilde{W}, \tilde{S}\right)$ is a Coxeter system, it suffices to verify the exchange condition [6, 4.1.6, Théorème 1]. This follows immediately from the above equivalences of conditions Equation B.10 and Proposition B.8.

□

Since $\tilde{W} \cong N_W(W_\Sigma)/W_\Sigma$, the resulting Coxeter system $\left(\tilde{W}, \tilde{S}\right)$ or the quotient

$$\left(N_W(W_\Sigma)/W_\Sigma, \tilde{S} \cdot W_\Sigma / W_\Sigma\right)$$

is called the **relative Coxeter group**.

B.1.4 Coxeter complex

Let $\mathfrak{F}$ be the Coxeter complex of (W, S), which is defined by

$$\mathfrak{F} = \left\{ yW_\Sigma \subseteq W \ ; \ \Sigma \subsetneq S, \ y \in W^\Sigma \right\}$$

equipped with a partial order $(\mathfrak{F}, \leq)$ opposite to inclusions. Let $\mathfrak{C} \subseteq \mathfrak{F}$ be the set of maximal elements in this partial order, consisting of the singletons. The spherical part of the Coxeter complex to the subset

$$\mathfrak{F}^{\mathrm{sph}} = \{yW_\Sigma \in \mathfrak{F}\ ;\ \#W_\Sigma < \infty\}$$

equipped with the induced partial order. Then clearly $\mathfrak{C} \subseteq \mathfrak{F}^{\mathrm{sph}}$. Elements of $\mathfrak{C}$ are called **chambers**.

The group W acts on $\mathfrak{F}$ and $\mathfrak{F}^{\mathrm{sph}}$ by left translation. The restriction of this action on $\mathfrak{C}$ is simply transitive. Let $yW_\Sigma \in \mathfrak{F}$ be a facet, then the stabiliser of it is described by

$$\mathrm{Stab}_W\,(yW_\Sigma) = yW_\Sigma y^{-1}.$$

In particular, if $yW_\Sigma \in \mathfrak{F}^{\mathrm{sph}}$, then it has finite stabiliser.

We remark that in the case where (W, S) is of finite or affine type, $\mathfrak{F}$ and $\mathfrak{F}^{\mathrm{sph}}$ coincide. In the case of finite type, $\mathfrak{F}$ has a geometric realisation as a sphere of dimension $\#S - 1$, while in the case of affine type, $\mathfrak{F}$ can be realised as an euclidean space of dimension $\#S - 1$.

B.1.5 Fixed sub-complex

Now we fix $\Sigma \subsetneq S$ such that W_Σ is finite. Consider the sub-complex $\mathfrak{F}^{\mathrm{sph}}(\Sigma) = \left(\mathfrak{F}^{\mathrm{sph}}\right)^{W_\Sigma}$ consisting of the facets fixed by the subgroup W_Σ. The fixed facets $y'W_{\Sigma'} \in \mathfrak{F}^{\mathrm{sph}}\,(\Sigma)$ are characterised by the property that $W_\Sigma \subseteq \mathrm{Stab}_W(y'W_{\Sigma'}) = y'W_{\Sigma'}y'^{-1}$. Let $\mathfrak{C}(\Sigma) \subseteq \mathfrak{F}^{\mathrm{sph}}(\Sigma)$ be the set of maximal elements, called chambers in $\mathfrak{F}^{\mathrm{sph}}\,(\Sigma)$. Then the elements $y'W_{\Sigma'} \in \mathfrak{F}^{\mathrm{sph}}(\Sigma)$ are those such that $y'W_{\Sigma'}y'^{-1} = W_\Sigma$.

Theorem B.11. *Suppose that $\Sigma \subsetneq S$ satisfies the condtions of §B.1.3. Let $y'W_{\Sigma'} \in \mathfrak{C}(\Sigma)$ be any chamber. Then $\Sigma' = \Sigma$. Moreover, the relative Coxeter group $\tilde{W} = \mathrm{N}(\Sigma, \Sigma)/W_\Sigma$ acts simply transitively on $\mathfrak{C}\,(\Sigma)$.*

Proof. Applying Proposition B.5, we obtain sequences $\{\Sigma_i\}_{i=1}^q$ and $\{y_i\}_{i=1}^q$ which have the properties listed in the proposition. In particular, $y_i = w_0^{\Sigma_i \cup \{s\}} w_0^{\Sigma_i}$ holds. Since $w_0^{\Sigma_1 \cup \{s\}}$ normalises W_Σ, so does y_1. Hence $\Sigma_2 = \Sigma_1 = \Sigma$. By induction, we can show that $\Sigma_i = \Sigma$ and $y_i \in \mathrm{N}(\Sigma, \Sigma)$ for all i. Thus $\Sigma' = \Sigma$ and that $y'W_\Sigma$ and W_Σ are connected by $\mathrm{N}\,(\Sigma, \Sigma)$. □

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Bibliographie

[1] P. N. Achar et C. Mautner – « Sheaves on nilpotent cones, Fourier transform, and a geometric Ringel duality », *arXiv preprint arXiv :1207.7044* (2012).

[2] S. Ariki – « On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$ », *J. Math. Kyoto Univ.* **36** (1996), no. 4, p. 789–808.

[3] R. Bezrukavnikov et P. Etingof – « Parabolic induction and restriction functors for rational Cherednik algebras », *Selecta Mathematica* **14** (2009), no. 3-4, p. 397–425.

[4] A. Bjorner et F. Brenti – *Combinatorics of Coxeter groups*, vol. 231, Springer Science & Business Media, 2006.

[5] A. Bondal et D. Orlov – « Derived categories of coherent sheaves », *arXiv preprint math/0206295* (2002).

[6] N. Bourbaki – *éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV : Groupes de Coxeter et systèmes de Tits. Chapitre V : Groupes engendrés par des réflexions. Chapitre VI : systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[7] — , *Eléments de mathématique : Algèbre commutative*, Springer, 2006.

[8] A. Brown, I. Gordon et C. Stroppel – « Cherednik, Hecke and quantum algebras as free Frobenius and Calabi–Yau extensions », *Journal of Algebra* **319** (2008), p. 1007–1034.

[9] J. Brundan et A. Kleshchev – « Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras », *Inventiones mathematicae* **178** (2009), no. 3, p. 451.

[10] J. Brundan, A. Kleshchev et P. J. McNamara – « Homological properties of finite-type Khovanov–Lauda–Rouquier algebras », *Duke Mathematical Journal* **163** (2014), no. 7, p. 1353–1404.

[11] I. Cherednik – « Double affine Hecke algebras, Knizhnik–Zamolodchikov equations, and Macdonald's operators », *International Mathematics Research Notices* **1992** (1992), no. 9, p. 171–180.

[12] — , « Integration of quantum many-body problems by affine Knizhnik–Zamolodchikov equations. », *Adv. Math.* **106** (1994), no. 1, p. 65–95 (English).

[13] N. Chriss et V. Ginzburg – « Representation theory and complex geometry », (1997).

[14] P. ETINGOF et X. MA – « Lecture notes on Cherednik algebras », *arXiv preprint arXiv :1001.0432* (2010).

[15] S. EVENS et I. MIRKOVIĆ – « Fourier transform and the Iwahori–Matsumoto involution », *Duke Mathematical Journal* **86** (1997), no. 3, p. 435–464.

[16] P. GABRIEL – « Des catégories abéliennes », *Bull. Soc. Math. France* **90** (1962), p. 323–448.

[17] V. GINZBURG, N. GUAY, E. OPDAM et R. ROUQUIER – « On the category $\mathscr{O}$ for rational Cherednik algebras », *Invent. Math.* **154** (2003), no. 3, p. 617–651.

[18] R. HOTTA, K. TAKEUCHI et T. TANISAKI – *D-modules, perverse sheaves, and representation theory*, vol. 236, Springer Science & Business Media, 2007.

[19] M. KASHIWARA et P. SCHAPIRA – *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006.

[20] S. KATO – « Poincaré–Birkhoff–Witt bases and Khovanov–Lauda–Rouquier algebras », *Duke Mathematical Journal* **163** (2014), no. 3, p. 619–663.

[21] — , « An algebraic study of extension algebras », *Amer. J. Math.* **139** (2017), no. 3, p. 567–615.

[22] Y. KAWAMATA – « D-equivalence and K-equivalence », *Journal of Differential Geometry* **61** (2002), no. 1, p. 147–171.

[23] D. KAZHDAN et G. LUSZTIG – « Proof of the Deligne-Langlands conjecture for Hecke algebras », *Invent. Math.* **87** (1987), no. 1, p. 153–215.

[24] M. KHOVANOV et A. LAUDA – « A diagrammatic approach to categorification of quantum groups I », *Representation Theory of the American Mathematical Society* **13** (2009), no. 14, p. 309–347.

[25] A. S. KLESHCHEV – « Affine highest weight categories and affine quasihereditary algebras », *Proc. Lond. Math. Soc. (3)* **110** (2015), no. 4, p. 841–882.

[26] Y. LASZLO et M. OLSSON – « The six operations for sheaves on Artin stacks. II. Adic coefficients », *Publ. Math. Inst. Hautes Études Sci.* (2008), no. 107, p. 169–210.

[27] B. LECLERC, J.-Y. THIBON et E. VASSEROT – « Zelevinsky's involution at roots of unity », *J. Reine Angew. Math.* **513** (1999), p. 33–51.

[28] I. LOSEV – « Derived equivalences for rational Cherednik algebras », *Duke Mathematical Journal* **166** (2017), no. 1, p. 27–73.

[29] G. LUSZTIG – « Coxeter orbits and eigenspaces of frobenius », *Inventiones mathematicae* **38** (1976), no. 2, p. 101–159.

[30] — , « Intersection cohomology complexes on a reductive group », *Inventiones mathematicae* **75** (1984), no. 2, p. 205–272.

[31] — , « Cuspidal local systems and graded Hecke algebras, I », *Publications Mathématiques de l'IHÉS* **67** (1988), p. 145–202 (eng).

[32] — , « Affine Hecke algebras and their graded version », *Journal of the American Mathematical Society* **2** (1989), no. 3, p. 599–635.

[33] — , « Quivers, perverse sheaves, and quantized enveloping algebras », *J. Amer. Math. Soc.* **4** (1991), no. 2, p. 365–421.

[34] — , « Vanishing properties of cuspidal local systems », *Proceedings of the National Academy of Sciences* **91** (1994), no. 4, p. 1438–1439.

[35] — , « Classification of unipotent representations of simple p-adic groups », *International Mathematics Research Notices* **1995** (1995), no. 11, p. 517–589.

[36] — , « Cuspidal local systems and graded Hecke algebras, II », *Representations of groups*, 1995, p. 217–275.

[37] — , « Study of perverse sheaves arising from graded Lie algebras », *Advances in Mathematics* **112** (1995), no. 2, p. 147–217.

[38] G. Lusztig et Z. Yun – « $\mathbf{Z}/m$-graded Lie algebras and perverse sheaves, I », *Represent. Theory* **21** (2017), p. 277–321.

[39] — , « $\mathbf{Z}/m$-graded Lie algebras and perverse sheaves, II », *Represent. Theory* **21** (2017), p. 322–353.

[40] — , « $\mathbb{Z}/m\mathbb{Z}$-graded Lie algebras and perverse sheaves, III : Graded double affine Hecke algebra », *Represent. Theory* **22** (2018), p. 87–118.

[41] I. G. Macdonald – *Affine Hecke algebras and orthogonal polynomials*, vol. 157, Cambridge University Press, 2003.

[42] E. M. Opdam – *Lecture notes on Dunkl operators for real and complex reflection groups.*, vol. 8, Tokyo : Mathematical Society of Japan, 2000 (English).

[43] R. Rouquier – « 2-Kac-Moody algebras », *arXiv e-prints* (2008), p. arXiv :0812.5023.

[44] — , « q-Schur algebras and complex reflection groups », *Mosc. Math. J.* **8** (2008), no. 1, p. 119–158, 184.

[45] J.-P. Serre – *Galois cohomology. Transl. from the French by Patrick Ion. 2nd printing.*, 2nd printing éd., Berlin : Springer, 2002 (English).

[46] P. Shan, M. Varagnolo et E. Vasserot – « Canonical bases and affine Hecke algebras of type D », *Advances in Mathematics* **227** (2011), no. 1, p. 267–291.

[47] C. Stroppel et B. Webster – « Quiver Schur algebras and q-Fock space », *arXiv e-prints* (2011), p. arXiv :1110.1115.

[48] M. Varagnolo et E. Vasserot – « From double affine Hecke algebras to quantized affine Schur algebras », *Int. Math. Res. Not.* (2004), no. 26, p. 1299–1333.

[49] — , « Finite-dimensional representations of DAHA and affine Springer fibers : the spherical case », *Duke Mathematical Journal* **147** (2009), no. 3, p. 439–540.

[50] — , « Canonical bases and affine Hecke algebras of type B », *Inventiones mathematicae* **185** (2011), no. 3, p. 593–693.

[51] — , « Canonical bases and KLR-algebras. », *J. Reine Angew. Math.* **659** (2011), p. 67–100 (English).

[52] E. VASSEROT – « Affine quantum groups and equivariant K-theory », *Transformation Groups* **3** (1998), no. 3, p. 269–299.

[53] — , « Induced and simple modules of double affine Hecke algebras », *Duke Math. J.* **126** (2005), no. 2, p. 251–323.