



Topological holography

Songyuan Li

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Abstract

This thesis concentrates on the topologically twisted version of the AdS_3/CFT_2 correspondence. We start by studying the topologically twisted $N = 2$ supersymmetric Liouville theory with massive deformations. We determine the chiral ring elements and compute their correlations functions. We then turn to define a three-dimensional gravity theory that is the holographic dual of Liouville theory. The action of this gravity theory is the same as the one of classical general relativity while the path integral measure is taken to be the same as the Liouville theory. We study in detail the properties of this gravity theory and then generalize it to the supersymmetric case where by changing the boundary conditions we find it dual to the topologically twisted $N = 2$ supersymmetric Liouville theory mentioned above. The classical configurations corresponding to the chiral ring will be computed. Next, we turn to a string realization of the duality and analyze a string theory on the locally AdS_3 background with a conformally flat boundary. We find the boundary energy-momentum tensor becomes topological when we impose a specific coupling to the boundary metric given by the boundary conditions. Finally, we study the topological orbifold conformal field theories on the symmetric product of a complex surface M , which is the conjectured dual to the ten-dimensional string theory on $AdS_3 \times S^3 \times M$.

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Introduction

Since its birth, quantum mechanics has greatly improved our understanding of physical phenomena at microscopic level. Among the four fundamental interactions of nature, three (electromagnetism, weak interaction and strong interaction) have been successfully described by the Standard Model of particle physics which is under the framework laid out by quantum mechanics. The Standard Model is a huge success and has made various predictions later proved by the experiment. One recent example is the experimental discovery of the Higgs boson in 2012 [1]. However, there is one exception, the gravitational interaction, not described quantum mechanically. Our current theory for gravity is a classical theory, general relativity. It is not consistent with quantum mechanics since the theory one gets by naively quantizing general relativity in the same way one does for the three other interactions has divergences that cannot be removed by the renormalization technique. Pursuing of a quantum theory of gravity has become one of the most important tasks in theoretical physics for decades. In fact, it has a long history and the earliest attempt can be dated back to the days when quantum mechanics was newly born [2].

Our best candidate theory today for a quantum theory of gravity is string theory. In string theory, point-like particles considered in quantum field theory are replaced by one dimensional objects, strings. String theory describes how these strings propagate and interact. There are two type of strings called the open string and the closed string, depending on whether the string has two end points or forms a loop. In the simplest string theory, the bosonic string theory which contains only bosons, one can already finds in the closed string spectrum spin-2 massless states which correspond to gravitons. Therefore, string theory indeed describes a quantum theory of gravity. To avoid anomalies, bosonic string theory lives in 26-dimensional spacetime. The extra dimensions are compact and cannot be seen at large scale. However, bosonic string theory contain unstable states called tachyons. To eliminate them, one can add supersymmetry and turn the string theory into superstring theory, which now lives ten dimensions. There are five types of consistent superstring theory called type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. These five theories and M-theory, whose low energy effective theory is 11-dimensional supergravity theory, are related by duality relations [3]. After the fruitful results in past few decades, string theory has become the most promising theory to understand the quantum nature of gravity.

One important direction to study quantum gravity theories, especially string theory, is the celebrated *AdS/CFT* correspondence. It is a conjectured duality between gravity theories and quantum field theories first proposed by Maldacena in 1997 [4]. This duality relates two different types of theories. One is a gravity theory with negative

cosmological constant, in other words on the Anti-de Sitter (AdS) space. The other is a conformal field theory, which is a quantum field theory, living on the conformal boundary of the AdS space, which is one dimension less, and thus this duality is also referred to as the holographic duality. One well-known example is the duality between the type IIB string theory on $AdS_5 \times S^5$ and the $N = 4$ supersymmetric Yang-Mills theory in four dimensions. In this duality, the operators in the conformal field theory correspond to fields on the gravity side. The conjecture is that [5]

$$Z_{AdS}(\phi_0) = \langle \exp(\int \phi_0 O) \rangle_{CFT}. \quad (1)$$

The left hand side is the partition function of the gravity theory computed by integrating all configurations that the field ϕ goes to ϕ_0 at the boundary of the AdS space. The right hand side is a vacuum expectation value of the operator $\exp(\int \phi_0 O)$ in the dual conformal field theory, where O is the operator that is dual to the field ϕ . With this, one is able to get results of one theory by doing computation in the dual theory. This will be an advantage when the computation in one theory is much easier than the computation in the dual theory. Indeed, the AdS/CFT correspondence is a strong-weak duality [4]. This allows one to study the strongly-coupled regime of one theory by studying the weakly-coupled regime of the dual theory and it is easier than directly studying the strongly-coupled theory since it is in general much harder to study a theory non-perturbatively than perturbatively. It is also this feature that makes the AdS/CFT not only useful in the studies of quantum gravity, but also in the studies of strongly-coupled quantum field theories and even condensed matter physics.

Although there have been many computations that have found agreement between results from the gravity theory and dual conformal field theory (e.g. [6,7]), this duality remains a conjecture and is very hard to prove rigorously. A probably easier approach to prove the holographic duality that is studied in this thesis is to consider the topologically twisted version where theories on both sides undergo a topological twist. The topologically twisted theory is of the type constructed in [8,9] by Witten. The coupling between the metric and the matter fields is changed such that the correlation functions of observables no longer depend on the metric but only topological features. This method has the advantage that topological theories have less degrees of freedom and thus makes it possible to compare exact computations from the two sides of the holographic dualities and even prove some exact duality relations.

In this thesis, I will focus on topologically twisting the AdS_3/CFT_2 duality. This duality has the well-known application of the BTZ black hole entropy computed in the dual conformal field theory [10]. In fact, it has already been realized even before [4] of Maldacena that three-dimensional gravity theories with a negative cosmological constant are related two-dimensional conformal field theories. It was first pointed out by Brown and Henneaux that gravity in three dimensions with a negative cosmological constant with certain boundary conditions has an asymptotic Virasoro symmetry which is the same as the two-dimensional conformal symmetry [11], and it is later shown [12] that the classical gravity action with such boundary conditions can be reduced to a Liouville action on the boundary of the three-dimensional spacetime. This equivalence between actions is then generalized by adding supersymmetry [13]. This relation between the classical actions makes it tempting to assume that the quantum

theories are also equivalent. Unfortunately, this cannot be the case in a naive manner. First of all, Liouville theory does not have an $SL(2, \mathbb{C})$ invariant normalizable ground state which should correspond to the a normalizable global AdS_3 ground state. Another point is that Liouville theory does not have enough state to explain the BTZ black hole entropy. The logarithm of the density of states is given by the Cardy's formula [14]

$$\log \rho \approx 2\pi \sqrt{\frac{c_{Leff}}{6} L_0} + 2\pi \sqrt{\frac{c_{Reff}}{6} \bar{L}_0}. \quad (2)$$

For Liouville theory, $c_{eff} = 1$ for both left and right movers while to get the BTZ black hole entropy one expect $c_{eff} = \frac{3l}{2G_N}$, which is the Brown-Henneaux central charge, for both left and right movers [10]. Nevertheless, we will, in this thesis, make use of the equivalence between the actions and define a three-dimensional gravity theory to be the gravitational dual of Liouville conformal field theory by proposing the path integral measure to be the same. This has the advantage that the resulting theory is manifestly consistent and unitary, because Liouville theory is. We will find that the theory contains only BTZ geometries with zero angular momentum, which can be dressed with boundary gravitons. They correspond to the primary state and their descendants in the Liouville theory. We will then study the supersymmetric generalization of this correspondence and consider a topological twist. This is achieved by picking a different boundary condition for the R-symmetry gauge field in the bulk that depends on the boundary metric. Thus, the coupling to the boundary metric is changed to the same way in a topologically twisted theory. We will then determine the classical configurations in the bulk that correspond to spectrum of the topologically twisted boundary theory, the supersymmetric Liouville theory.

We will also consider the string theory description of the gravity side and study the topological twist. The string theory has a realization for this duality which is between the type IIB string theory on $AdS_3 \times S^3 \times M$, where the four dimensional manifold M can be T^4 or K_3 and a $N = (4, 4)$ supersymmetric two-dimensional conformal field theory [4, 15]. This duality can be applied to the study of the $D1 - D5$ system which is important in the study of black holes in string theory [4]. We will study a simpler string theory which contains AdS_3 factor in its background, which is on $AdS_3 \times S^1$. The twisting is achieved by specific couplings to the boundary metric inspired by the studies for the supergravity case mentioned above. We also study the twisting of the dual theory to the ten-dimensional string theory, which is a symmetric orbifold conformal field theory and compute some correlators that one can compare once the above mentioned twisting is generalized to the ten-dimensional string theory.

The organization of the thesis is as follows. In chapter 1, I will review some basic knowledge of the chiral ring and topological twisting of two-dimensional $N = 2$ supersymmetric conformal field theories, which are going to be the basis of the discussion in the following chapters. Chapter 2 will be a discussion on topologically twisting of two-dimensional $N = (2, 2)$ supersymmetric conformal theories with a non-compact space, especially focusing on $N = 2$ super-Liouville theory. We will give the elements of the chiral ring for the $N = 2$ super-Liouville theory with deformations of relevant operators. Then we will turn to the AdS_3 (super)gravity theory that is dual to the (super-)Liouville theory in chapter 3. We will show in this chapter that by picking a particular boundary condition, the dual conformal field theory becomes the twisted

super-Liouville theory. In chapter 4, we define a string theory in three-dimensional anti-de Sitter space-time with specific couplings to the boundary metric. We will then show that the boundary energy-momentum tensor, the functional derivative of the string partition with respect to the boundary metric, is equal to the energy-momentum tensor of the topologically twisted boundary theory. Thus, the string theory will be independent of the boundary metric. Then, we will consider the topologically twisted supersymmetric orbifold theory that is expected to be dual to the full string theory in chapter 5. Finally, we conclude the whole thesis and give some examples of follow up open problems of the discussion in this thesis.

Chapitre 1

Basics

We wish to study topologically twisted *AdS/CFT* correspondence. Therefore, we will in this chapter review a prototypical example of topological twisting on the conformal field theory side, which will find particular realizations in the following chapters. We will start by briefly reviewing some basic knowledge of the chiral ring, and then turn to topological twisting of two-dimensional $N = 2$ supersymmetric conformal field theories. We will show that the chiral ring of an untwisted theory corresponds to the spectrum of the twisted theory.

1.1 Chiral Ring

For an $N = 2$ supersymmetric conformal field theory, there exist two fermionic currents and a $U(1)$ current $J(z)$ corresponding to the R symmetry. The two fermionic currents are charged ± 1 under this $U(1)$ current and are denoted $G^\pm(z)$. The operator product expansion (OPE) of these currents and the energy-momentum tensor $T(z)$ gives the $N = 2$ superconformal algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right)G_{m+r}^\pm \\ [L_m, J_n] &= -nJ_{m+n} \\ [J_n, G_r^\pm] &= \pm G_{n+r}^\pm \\ [J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\ \{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0 \\ \{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \end{aligned} \tag{1.1}$$

where L_n , G_r^\pm and J_n are the Laurent expansion coefficients of $T(z)$, $G(z)$ and $J(z)$. $n, m, r - \nu, s - \nu \in \mathbb{Z}$ with $\nu = 0$ in the Ramond (R) sector and $\nu = 1/2$ in the Neveu-Schwarz (NS) sector. c is the central charge. Consider an arbitrary state $|\psi\rangle$ in the NS sector, whose conformal dimension is h and $U(1)$ R-charge is q , since we are only interested in unitary theories, one has, applying the last relation in (1.1) and also

$$G_{\frac{1}{2}}^- = (G_{-\frac{1}{2}}^+)^{\dagger},$$

$$0 \leq \langle \psi | \{G_{1/2}^-, G_{-1/2}^+\} | \psi \rangle = \langle \psi | (2L_0 - J_0) | \psi \rangle = (2h - q) \langle \psi | \psi \rangle, \quad (1.2)$$

and thus $h \geq \frac{q}{2}$. Exchanging $+$ and $-$, one has q changed into $-q$ and therefore unitarity tells us for an arbitrary state in the NS sector $h \geq \frac{|q|}{2}$.

A state $|\phi\rangle$ is called left chiral if [16]

$$G_{-1/2}^+ |\phi\rangle = 0, \quad (1.3)$$

and chiral primary if it furthermore satisfy

$$G_{n+1/2}^+ |\phi\rangle = G_{n+1/2}^- |\phi\rangle = 0 \quad \text{for } n \geq 0. \quad (1.4)$$

One can also define right chiral (primary) state in the same way for the anti-holomorphic part and anti-chiral (primary) states by interchanging G^+ and G^- in the above equations. With these constraints, the \leq in the unitary constraint (1.2) will now become an equality, i.e. $h = \frac{q}{2}$ for chiral primary states. One can further use the superconformal algebra to prove that

$$L_n |\phi\rangle = J_n |\phi\rangle = 0 \quad \text{for } n > 0, \quad (1.5)$$

where $L_{n>0} |\phi\rangle = 0$ indicates that chiral primary states are also conformal primary.

One can also prove that states with $h = \frac{q}{2}$ have to be chiral primary. Unitarity tells us for such states $|\phi\rangle$,

$$\begin{aligned} G_{-1/2}^+ |\phi\rangle &= 0 \\ G_{1/2}^- |\phi\rangle &= 0 \\ J_n |\phi\rangle &= 0 \quad \text{for } n > 0. \end{aligned} \quad (1.6)$$

The last equation is true since the left hand side has charge q and conformal dimension $\frac{q}{2} - n$ and thus violates the unitarity condition. Along with the superconformal algebra, one can use these three equations and show that $|\phi\rangle$ indeed satisfy the definition of the chiral primary states. Therefore, a state is a chiral primary state if and only if it satisfies $h = \frac{q}{2}$.

One can also further prove that any chiral state $|\phi\rangle$ can be written as the form

$$|\phi\rangle = |\phi_0\rangle + G_{-1/2}^+ |\phi_1\rangle, \quad (1.7)$$

where $|\phi_0\rangle$ is a chiral primary state. The proof is as follows [16]. One first needs to prove that any NS state $|\phi\rangle$ with conformal dimension h and R-charge q can be written as

$$|\phi\rangle = |\phi_0\rangle + G_{-1/2}^+ |\phi_1\rangle + G_{1/2}^- |\phi_2\rangle, \quad (1.8)$$

where $|\phi_0\rangle$ is a chiral primary state. Consider the norm

$$|(|\phi\rangle - G_{-1/2}^+ |\psi_1\rangle - G_{1/2}^- |\psi_2\rangle)|^2. \quad (1.9)$$

Its minimum exists since one can concentrate on the subsector where $|\psi_1\rangle$ and $|\psi_2\rangle$ has conformal dimension and R-charge $(h - 1/2, q - 1)$ and $(h + 1/2, q + 1)$ respectively. For

non-degenerate conformal field theories, this subspace is finite and thus the minimum exists. Let $|\phi_1\rangle$ and $|\phi_2\rangle$ be the $|\psi_1\rangle$ and $|\psi_2\rangle$ that minimize the norm, and write

$$|\phi_0\rangle = |\phi\rangle - G_{-1/2}^+ |\phi_1\rangle - G_{1/2}^- |\phi_2\rangle. \quad (1.10)$$

Since the norm reaches minimum, it should be invariant under arbitrary infinitesimal transformation

$$|\phi_0\rangle \rightarrow |\phi_0\rangle + G_{-1/2}^+ |\epsilon_1\rangle + G_{1/2}^- |\epsilon_2\rangle, \quad (1.11)$$

and thus $|\phi_0\rangle$ has to satisfy

$$G_{-1/2}^+ |\phi_0\rangle = G_{1/2}^- |\phi_0\rangle = 0. \quad (1.12)$$

This is sufficient to show that $h = \frac{q}{2}$ and therefore $|\phi_0\rangle$ is a chiral primary state, and (1.8) follows. If $|\phi\rangle$ is now chiral, one then finds

$$G_{-1/2}^+ G_{1/2}^- |\phi_2\rangle = 0. \quad (1.13)$$

Multiplying the left hand side with $\langle\phi_2|$ from left, unitarity then implies

$$G_{1/2}^- |\phi_2\rangle = 0, \quad (1.14)$$

and (1.7) is proved.

Unitarity can give also other constraints. Consider the anti-commutator

$$\{G_{-3/2}^+, G_{3/2}^-\} = 2L_0 - 3J_0 + \frac{2c}{3} \quad (1.15)$$

from the superconformal algebra, one finds that the conformal dimension of conformal primary states has to satisfy

$$0 \leq h \leq \frac{c}{6}. \quad (1.16)$$

If the spectrum of L_0 is discrete and non-degenerate, then there are only a finite numbers of chiral primary states.

These chiral primary states correspond to some operators in the conformal field theory through the operator-state correspondence. We define such operators to be chiral primary operators. Consider the OPE of the two chiral primary operators ϕ^{h_1, q_1} and ϕ^{h_2, q_2} , one has, in general,

$$\phi^{h_1, q_1}(z_1) \phi^{h_2, q_2}(z_2) = \sum C_{123} (z_1 - z_2)^{-h_1 - h_2 + h_3} \phi^{h_3, q_3}(z_2). \quad (1.17)$$

Since the $U(1)$ charge is additive, only terms with $q_3 = q_1 + q_2$ contribute on the right hand side. Using the fact that ϕ^{h_1, q_1} and ϕ^{h_2, q_2} are chiral primary operators, one has

$$\phi^{h_1, q_1}(z_1) \phi^{h_2, q_2}(z_2) = \sum_{q_3 = q_1 + q_2} C_{123} (z_1 - z_2)^{-\frac{q_3}{2} + h_3} \phi^{h_3, q_3}(z_2). \quad (1.18)$$

Now, the unitarity condition (1.2) tells us that all ϕ^{h_3, q_3} satisfy $h_3 \geq \frac{|q_3|}{2}$, and thus the OPE is regular when $z_1 \rightarrow z_2$. We thus define the product of two chiral primaries to be the naive product

$$\phi^{h_1, q_1} \phi^{h_2, q_2}(z) = \lim_{z' \rightarrow z} \phi^{h_1, q_1}(z') \phi^{h_2, q_2}(z). \quad (1.19)$$

Moreover, under the limit $z' \rightarrow z$, only terms with $h_3 = \frac{q_3}{2}$ survive on the right hand side of (1.18), and thus the product of two chiral primaries can only be a chiral primary if it is not zero. Since we have only a finite number of chiral primaries, this multiplication rule defines a finite ring of the chiral primaries, which is called the chiral ring.

We now take a look at the Ramond sector. Unitarity and anti-commutation relation

$$\{G_0^+, G_0^-\} = 2L_0 - \frac{c}{12} \quad (1.20)$$

in the superconformal algebra in the Ramond sector indicate that all states in the Ramond sector satisfy $h \geq \frac{c}{24}$. Equality is acquired when the state $|\phi\rangle$ satisfy

$$G_0^+|\phi\rangle = G_0^-|\phi\rangle. \quad (1.21)$$

For $N = 2$ superconformal algebra, the NS sector and R sector can be continuously connected by a spectral flow [17]. The operator for this spectral flow between Hilbert spaces with a parameter θ is denoted U_θ . The operators O will then be mapped to $U_\theta O U_\theta^{-1}$. The superconformal algebra is preserved if

$$\begin{aligned} U_\theta L_n U_\theta^{-1} &= L_n + \theta J_n + \frac{c}{6} \theta^2 \delta_{n,0} \\ U_\theta G_r^+ U_\theta^{-1} &= G_{r+\theta}^+ \\ U_\theta G_r^- U_\theta^{-1} &= G_{r-\theta}^- \\ U_\theta J_n U_\theta^{-1} &= J_n + \frac{c}{6} \theta \delta_{n,0}. \end{aligned} \quad (1.22)$$

For the flow with $\theta = \frac{1}{2}$, the NS sector will be mapped to the R sector. The chiral primary states will flow to states $|\phi\rangle$ that satisfy

$$G_{n \geq 0}^+ |\phi\rangle = G_{n \geq 0}^- |\phi\rangle = 0, \quad (1.23)$$

and by the anticommutator (1.20) has conformal dimension $\frac{c}{24}$. Thus, under the spectral flow, the chiral primary states in the NS sector flow to the supersymmetric ground states in the Ramond sector, which are precisely those states that contribute to the Witten index $\text{tr}(-1)^F$.

1.2 Topological Conformal Field Theories

Two dimensional quantum field theories with extended supersymmetry can undergo a topological twist and become a topological quantum theory of the type first constructed by Witten ([8] for four dimensions and [9] for two dimensions). Such type of topological quantum field theory has the following features.

- The action is invariant under a symmetry transformation δ that satisfies $\delta^2 = 0$.
- There exists a tensor $G^{\mu\nu}$ such that the energy-momentum tensor can be written as the form $T^{\mu\nu} = \delta G^{\mu\nu}$.
- The observables O satisfy $\delta O = 0$.

Since δ is a symmetry, one has for any W ¹

$$\langle \delta W \rangle = 0. \quad (1.24)$$

Now consider an arbitrary correlation function of observables

$$\langle O_1 O_2 \cdots O_n \rangle. \quad (1.25)$$

Under a variation $\delta g_{\mu\nu}$ of the metric, the variation of this correlation function is²

$$\int d^D x \sqrt{g} \delta g_{\mu\nu} \langle O_1 O_2 \cdots O_n T^{\mu\nu} \rangle. \quad (1.26)$$

Using $T^{\mu\nu} = \delta G^{\mu\nu}$ and $\delta O_i = 0$, one has

$$\langle O_1 O_2 \cdots O_n T^{\mu\nu} \rangle = \langle \delta(O_1 O_2 \cdots O_n G^{\mu\nu}) \rangle = 0, \quad (1.27)$$

and thus the correlation function is invariant under a variation $\delta g_{\mu\nu}$ of the metric. Such theory is indeed topological. Note that since $\delta^2 = 0$, for an observable O , $O + \delta\Lambda$ for arbitrary Λ also satisfy the condition observables have to satisfy. Moreover, the correlation function is invariant under the replacement $O_i \rightarrow O_i + \delta\Lambda$. This is because the variation of the correlation function under such replacement is

$$\langle O_1 O_2 \cdots O_{i-1} \delta\Lambda O_{i+1} \cdots O_n \rangle = \langle \delta(O_1 O_2 \cdots O_{i-1} \Lambda O_{i+1} \cdots O_n) \rangle = 0. \quad (1.28)$$

Therefore, O and $O + \delta\Lambda$ are equivalent in the topological theory. Let Q be the charge that generates the symmetry transformation δ . The above result indicates that the observables of the theory are described by the cohomology class of Q , $\text{Ker} Q / \text{Im} Q$.

We will now show how to realize the twist for $N = 2$ superconformal field theories in two dimensions. This can be done by adding a coupling between the $U(1)$ R-current and the spin connection ω , i.e. adding a term of the form [18]

$$\int d^2 x (J \pm \bar{J})^\mu \frac{\omega_\mu}{2} \quad (1.29)$$

to the action. Taking the functional derivative of the action near flat metric, one finds the energy-momentum tensor is now given by

$$T = T^{\text{phys}} + \frac{1}{2} \partial J, \quad (1.30)$$

where T^{phys} is the energy-momentum of the original untwisted theory. For the right mover, we have similar result with the sign in front of $\frac{1}{2} \partial \bar{J}$ depending on the sign in front of \bar{J} in (1.29), giving two types of twist often referred to as A-twist and B-twist

1. We assume that the path integral measure is invariant under δ .
 2. We assume that the operators O_i do not depend on the metric.

[19]. With the original superconformal algebra in the NS sector and this new energy-momentum tensor, one can deduce the topological $N = 2$ superconformal algebra [20]

$$\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n} \\
[L_m, G_n^+] &= -nG_{m+n}^+ \\
[L_m, G_n^-] &= (m - n)G_{m+n}^- \\
[L_m, J_n] &= -nJ_{m+n} - \frac{c}{6}m(m+1)\delta_{m+n,0} \\
[J_m, G_n^\pm] &= \pm G_{m+n}^\pm \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\
\{G_m^+, G_n^+\} &= \{G_m^-, G_n^-\} = 0 \\
\{G_m^+, G_n^-\} &= 2L_{m+n} + 2mJ_{m+n} - \frac{c}{3}n(n+1)\delta_{m+n,0}, \tag{1.31}
\end{aligned}$$

where L_n are now the Laurent expansion coefficients of this new energy-momentum tensor and G_n^\pm is equal to $G_{n\mp 1/2}^\pm$ of the original algebra. The bosonic Virasoro subalgebra now has central charge zero. On the other hand, the R-current now becomes anomalous, giving a selection rule that correlators on surface with genus g is non-vanishing only if the total R-charge of the operators in the correlator is equal to $\frac{c}{3}(1 - g)$ [21].

If we define a current

$$G^-(z) = \sum_n z^{-n-2} G_n^-, \tag{1.32}$$

then from the algebra, we find the energy-momentum tensor

$$T(z) = \{G_0^+, \frac{1}{2}G^-(z)\}. \tag{1.33}$$

Since the charge G_0^+ satisfy $(G_0^+)^2 = 0$, this theory indeed becomes a topological theory, with observables given by the cohomology classes of G_0^+ . Moreover, one can pick a unique representative of for each class that is also invariant under the charge G_0^- . Remember that G_0^\pm is equal to $G_{\mp 1/2}^\pm$ of the untwisted theory, this indicates that such representative are precisely the left chiral primary operators in the untwisted theory. If we also consider the right movers, then we can find that the observables of the two type of twisted theories correspond to chiral-chiral primary operators and chiral-anti-chiral primary operators respectively.

1.3 Summary

In this chapter, we have reviewed some basic knowledge of chiral primary operators, topologically twisted $N = 2$ supersymmetric conformal field theories and their relation, where the observables of the topologically twisted theory correspond to the chiral primary operators in the untwisted theory. The discussion in this chapter is going to be the important basis of the study of topological *AdS/CFT* correspondence in the following chapters.

Chapitre 2

Twisted Non-Compact Models

There have been a number of studies on the topological twist of massive $N = (2, 2)$ supersymmetric theories in two dimensions with discrete spectra (e.g. [20, 22–24]). In this chapter we will generalize these analyses to $N = (2, 2)$ supersymmetric two-dimensional theories to theories with a non-compact target space. We will focus on the $N = 2$ super-Liouville theory which is T-dual to an $SL(2, \mathbb{R})/U(1)$ coset superconformal field theory [25]. This is the conformal field theory side of the holographic pair that will be studied in the next chapter.

We will determine the chiral ring elements in the presence of relevant operator deformations and then compute the correlation functions to get the generalized Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. The topological-anti-topological sector of the deformed $N = 2$ Liouville theory will also be investigated, where we will study the tt^* equations for the deformed theory. The following main part of this chapter and appendices A and B will be contents extracted from the paper [26] I coauthored with J. Troost.

2.1 The Topological Conformal Field Theory

In this section, we describe topological conformal field theories. Firstly, we recall the topological conformal field theories that arise from twisting the $N = 2$ minimal model conformal field theories [20, 22–24]. Secondly, we introduce non-compact counterparts (see e.g. [27–32] for necessary background).

2.1.1 The Topological Compact Model

Quantum field theories in two dimensions with $N = (2, 2)$ supersymmetry are sufficiently constrained to frequently allow for the identification of their interacting infrared fixed points. An example of such a successful identification is the proposal that the $N = 2$ minimal model conformal field theory is the infrared fixed point of a $N = 2$ Landau-Ginzburg model with as field content an $N = (2, 2)$ chiral superfield Φ , and interactions dictated by the superpotential $W = \Phi^{k_c}/k_c$. The positive integer power k_c is related to the central charge of the $N = 2$ minimal model by the relation

$c = 3 - 6/k_c$. We will refer to the integer k_c as the level of the model.¹ Considerable evidence for this identification was amassed over the years [22–24].

Part of the evidence lay in the understanding of the topological subsector of the infrared fixed point, the topological conformal field theory [20]. We briefly review the properties of this topologically twisted compact $N = 2$ superconformal field theory. Our starting point is the $N = 2$ minimal model, which can be thought of as a supersymmetric sigma-model on the coset $SU(2)/U(1)$. We concentrate on the A-type diagonal modular invariant. We moreover focus on the (chiral, chiral) ring [16] of the minimal model at central charge $c = 3 - 6/k_c$. It is made up of the operators in the spectrum whose conformal dimension is half their R-charge, both for the left- and the right-movers. In the NSNS sector, they can be identified as arising from the parent $SU(2)$ operators with spin j which have angular momentum m equal to their spin, $j = m$. The spin j runs over the values $j = 0, 1/2, \dots, (k_c - 2)/2$ where $k_c - 2$ is the level of the bosonic $SU(2)$ current algebra of the parent theory on the group $SU(2)$. The chiral ring thus has $k_c - 1$ elements that have R-charges in the set $\{0, 1/k_c, 2/k_c, \dots, 1 - 2/k_c\}$ in the NSNS sector. The ring structure constants agree with those of the polynomial ring $\mathbb{C}[X]$ modded out by the ideal generated by X^{k_c-1} , where X has R-charge $1/k_c$. The ring has a unit. In the Ramond-Ramond sector, after spectral flow by half a unit in the appropriate direction, the chiral primary operators map to ground states as we have seen in chapter 1. These have R-charges shifted down by $c/6 = 1/2 - 1/k_c$ and therefore take values in the set $\{-1/2 + 1/k_c, -1/2 + 2/k_c, \dots, 1/2 - 1/k_c\}$ which runs from $-c/6$ to $+c/6$ with increments of $1/k_c$. The topological metric, defined to be the expectation value of two chiral operators, which we can identify with X^i and X^j , in the topologically twisted theory on the sphere equals (see e.g. [21])

$$\eta_{ij} = \langle X^i X^j \rangle_0 = \delta_{i+j, k_c-2}. \quad (2.1)$$

We note that the vacuum in this compact theory is normalizable, and that the vacuum state, with conformal dimension and R-charge both equal to zero, survives the topological twist. The structure constants of the ring are related to the three-point functions via the topological metric, and they equal

$$\begin{aligned} c_{ij}^l &= \delta_{i+j, l} \\ c_{ijl} &= \langle X^i X^j X^l \rangle_0 = c_{ij}^m \eta_{ml} = \delta_{i+j+l, k_c-2}, \end{aligned} \quad (2.2)$$

where i, j, l need to be in the spectrum in order for the structure constants or three-point function to be non-zero. This information is sufficient to describe all correlators of the topological conformal field theory on any Riemann surface, by cutting and sewing. The topological conformal field theory has an alternative description in terms of a twisted $N = 2$ Landau-Ginzburg model. The Landau-Ginzburg theory has a superpotential

$$W = \frac{X^{k_c}}{k_c} \quad (2.3)$$

and the chiral ring is again the polynomial ring $\mathbb{C}[X]$ modded out by $W_X = \frac{dW}{dX} = X^{k_c-1}$.

1. We denote the level of the compact model by k_c , to contrast with the level of the non-compact model which will be denoted k .

Establishing the dictionary for these supersymmetric quantum field theories took several years, and ingenious checks on the identification of the fixed points were performed. We refer to the extensive literature for detailed discussions. Our lightning review mostly serves as a point of reference for the models to come, which are of an entirely different nature, yet show remarkable resemblances.

2.1.2 The Topological Non-Compact Model

More than twenty years ago, it was suggested that the correspondence between Landau-Ginzburg models and conformal field theories could be usefully extended to non-compact models, with continuous spectrum and a central charge $c > 3$ [33]. See also [34–36]. The $N = 2$ Liouville theory can indeed be understood as a linear dilaton theory, with an exponential supersymmetric potential which is marginal. Importantly, the linear dilaton profile fixes part of the asymptotics [25] and renders the superpotential term consistent with conformal symmetry. These are two aspects that differ from the compact theories. It was also shown that the $N = 2$ Liouville theory is T-dual to the $N = 2$ coset conformal field theory on the cigar $SL(2, \mathbb{R})/U(1)$ [25, 37]. Earlier on, a more rudimentary probe of the $N = 2$ Liouville theory, namely the Witten index (defined as a periodic path integral on the torus), was computed in [38], and demonstrated to depend on the choice of asymptotic radius. Given an $N = 2$ Liouville theory of central charge $c = 3 + 6/k$, where k is a positive integer, the natural choices of asymptotic radii are multiples of $\sqrt{\alpha'/k}$. The Witten index is equal to the chosen multiple. Since then, our understanding of the spectrum of the $SL(2, \mathbb{R})/U(1)$ conformal field theory has progressed both through an analysis of the bosonic partition function [40] and its supersymmetric counterpart [29, 32], as well as through an analysis of the elliptic genus [31].

We will consider the topologically twisted $N = 2$ Liouville theory at central charge $c = 3 + 6/k$ and with an asymptotic radius equal to $\sqrt{k\alpha'}$. We choose the level k to be a positive integer. We make this choice of radius since we want the number of ground states to be equal to the level (minus one). The theory is a $N = (2, 2)$ theory with chiral superfield Φ and superpotential

$$W = \mu \exp(b\Phi) \tag{2.4}$$

where $b = \sqrt{k/2}$, and the theory is supplemented with a linear dilaton at infinity of slope $1/b$. Establishing the chiral ring of the theory is subtle. (See also [30] for a discussion.) The Witten index calculation [38] cleverly cancels a volume divergence (as detailed in [31]), and gives a result equal to k (namely, the number of coverings of the minimal circle radius $\sqrt{\alpha'/k}$). The potential Ramond-Ramond ground states are identified in the NSNS (chiral, chiral) ring of the conformal field theory as follows. The $N = 2$ Liouville theory under consideration is T-dual to a cigar coset superconformal field theory on the coset $SL(2, \mathbb{R})_k/U(1)$ [25, 37], modded out by \mathbb{Z}_k . In the coset conformal field theory we can identify the (chiral, chiral) states of the original Liouville theory as arising from discrete representations D_j^+ , with lowest weight state with spin component j . The list of possible values for the spin j contains at least the set $j = 1, 3/2, \dots, (k-1)/2, k/2$. The supplementary values $j = 1/2$ and $j = (k+1)/2$ correspond to almost, but not quite normalizable states, in the following sense. The

discrete lowest weight representation $D_{1/2}^+$ is sometimes referred to as mock discrete. It does not arise in the Plancherel formula for the representation space decomposition of quadratically integrable functions on the group manifold $SL(2, \mathbb{R})$. Intuitively, it is analogous to the ground state of conformal field theories in two dimensions with any non-compact target space, for instance a flat space \mathbb{R}^n . The ground state norm is infinite, and proportional to the volume of the target space. Similarly, in Liouville theory, the state at the end of the continuous spectrum (namely in the representation $D_{1/2}^+$) is almost, but not quite, normalizable. Spectral flow then implies that the same type of argument applies to the representation with $j = (k+1)/2$ [41]. The bounds on the spin were first suggested in [42] and firmly established in [41] and [36].

This subtlety is also manifest after regularizing the conformal field theory with a linear deformation in Φ , as analyzed in [38] :

$$W_m = \mu \exp(b\Phi) - m\Phi. \quad (2.5)$$

In [38] it is shown that after regularization with the mass term, only $k-1$ normalizable ground states can be identified. Relatedly, in the calculation of the non-compact elliptic genus [31] a regularization choice decides on whether the $j = 1/2$ (or $j = (k+1)/2$) state contributes to the holomorphic part of the elliptic genus, or not. Thus, the volume divergence contaminates the ground state counting and even the supersymmetric index must be interpreted with care. See [43] for a much more detailed discussion of closely related intricacies.

Despite the subtleties we encounter, we can draw a number of conclusions at this stage. First of all, the unit operator is not normalizable, and is not in the topological cohomology. The chiral ring is naturally without unit. Nevertheless, it is true that all rings can be rendered unital by adding a unit (operator) by hand. If we do, we must remember that there is no (normalizable) state-operator correspondence for the unit operator. Secondly, the ring must contain the elements with R-charge $\{2/k, 3/k, \dots, 1\}$ which are strictly normalizable. They correspond to the operators $\exp(\frac{n}{2b}\Phi)$ where $n = 2, 3, \dots, k$. There is another candidate operator at $j = 1/2$, namely $\exp(\frac{1}{2b}\Phi)$ corresponding to $(h, q) = (1/2k, 1/k)$, which permits the interpretation as the state in which to evaluate topological field theory correlators. It is a zero momentum state (since $j = 1/2$), much as the $SL(2, \mathbb{C})$ invariant vacuum in a conformal field theory with target space \mathbb{R}^p . Importantly, it does carry a conformal dimension and R-charge. Finally, it has a counterpart at $j = (k+1)/2$ (that arises from spectral flow of a $D_{1/2}^-$ representation in the parent $SL(2, \mathbb{R})$ theory [41]).

To further discuss the operator ring, it is handy to introduce the field theory variable

$$Y^{-1} = e^{\frac{1}{2b}\Phi}, \quad (2.6)$$

in terms of which the superpotential (2.4) reads

$$W = \frac{Y^{-k}}{k}, \quad (2.7)$$

for the choice $\mu = 1/k$. This superpotential has been proposed as a starting point for analysis a while back [33], but it was mostly made sense of directly at the conformal fixed point [25], in the field variable Φ . One of the goals in this chapter is to show

that indeed, the negative power superpotential (2.7) leads to a useful and efficient description of the topological model. The variable Y will be handy, despite having a non-canonical kinetic term, and various other perturbing features.

In these variables then, we will formulate and study two rings. The first ring is the (strict) chiral ring R , given by linear combinations of the set $\{Y^{-2}, Y^{-3}, \dots, Y^{-k}\}$. We can think of the chiral ring as a subspace of the ring $\mathbb{C}[Y^{-1}]$ of polynomials in Y^{-1} . Moreover, the chiral ring is a subspace of the ideal $Y^{-2}\mathbb{C}[Y^{-1}]$ which has no unit. In the latter ideal, the chiral ring is the quotient by the ideal $\langle Y^{-k+1} \rangle$ generated by $Y^{-k+1} = -Y^2 W_Y$. Indeed, note that we generate only powers Y^{-k-1} or higher by multiplying Y^{-k+1} by elements in $Y^{-2}\mathbb{C}[Y^{-1}]$. We can alternatively think of the chiral ring as being obtained by setting Y^{-k-1} to zero, and we denote these statements as $R = Y^{-2}\mathbb{C}[Y^{-1}]/\langle Y^{-k+1} \rangle = Y^{-2}\mathbb{C}[Y^{-1}]/\{Y^{-k-1} = 0\}$. The (strict) chiral ring is made up of all linear combinations of the $k-1$ monomials in the chiral ring that are strictly normalizable. The Ramond-Ramond sector R-charges of the operators are $\{-1/2 - 1/k, \dots, 1/2 - 1/k\}$.² The second ring is the extended chiral ring R_{ext} , which consists of operators that are linear combinations of the $k+1$ operators $\{Y^{-1}, Y^{-2}, \dots, Y^{-k}, Y^{-k-1}\}$, with a standard multiplication rule, and such that Y^{-k-2} is equivalent to zero. Again we can describe the extended chiral ring abstractly. The ring of polynomials has an ideal $Y^{-1}\mathbb{C}[Y^{-1}]$ generated from the monomial Y^{-1} . This ideal is a subring without unity. In the latter ring, we consider the ideal $\langle W_Y \rangle$ generated by $W_Y = -Y^{-k-1}$. We can then consider the quotient ring of $Y^{-1}\mathbb{C}[Y^{-1}]$ modded out by the ideal $\langle W_Y \rangle$. Again, note that the first element we will put to zero is Y^{-k-2} , since we multiply elements of the ring into the ideal, and the lowest order element in the ring is Y^{-1} . Thus, we can symbolically write the extended chiral ring as $R_{ext} = Y^{-1}\mathbb{C}[Y^{-1}]/\langle W_Y \rangle = Y^{-1}\mathbb{C}[Y^{-1}]/\{Y^{-k-2} = 0\}$. The R-charges of the extended chiral ring elements in the NS sector are $\{1/k, 2/k, \dots, 1, 1 + 1/k\}$. In the Ramond-Ramond sector the R-charges are $\{-1/2, -1/2 - 1/k, \dots, 1/2 - 1/k, +1/2\}$. The set of R-charges is symmetric under charge conjugation. In both rings, there is an operator that allows for supersymmetric marginal deformation of R-charge 1 – it is the original superpotential term. Both the strict and the extended chiral rings will be conceptually useful.

Our next step in defining the non-compact topological conformal field theory is to find a non-zero, non-degenerate symmetric bilinear form η on the ring in the guise of a spherical two-point function for the generating operators Y^{-i} . We wish to define the topological metric η_{ij} again as a two-point function

$$\eta_{ij} \stackrel{?}{=} \langle 0 | Y^{-i} Y^{-j} | 0 \rangle_0, \quad (2.8)$$

but there are several reasons why this is not entirely trivial to make sense of. The first reason is that the state $|0\rangle$ is the non-normalizable $SL(2, \mathbb{C})$ invariant state, which belongs neither to the original, nor to the twisted theory. Therefore, rather than take the expression (2.8) literally, we interpret the expression as referring to the expectation value

$$\eta_{ij} = \langle Y^{-1} | Y^{-i+1} Y^{-j+1} | Y^{-1} \rangle_0, \quad (2.9)$$

2. Note that the spectrum of R-charges for the states that are strictly normalizable is equal for a non-compact model at level k and a compact model at level $k_c = k$. We will make use of this remark in section 2.4.

where the state $|Y^{-1}\rangle$ in which we take the vacuum expectation value is the state corresponding to the operator Y^{-1} , namely the almost-normalizable state with conformal dimension and R-charge $(h, q) = (1/2k, 1/k)$. Secondly, we analyze R-charge conservation in the topologically twisted theory. The charge conservation equation for topological correlators evaluated in the $SL(2, \mathbb{C})$ invariant ground state is, as we have discussed in chapter 1,

$$\sum_{i=1}^r q_i + \sum_{i=r+1}^{r+n} (q_i - 1) = \frac{c}{3}(1 - g) \quad (2.10)$$

where g is the genus of the Riemann surface on which we compute the correlators, the charges q_1, \dots, q_r are the R-charges of the unintegrated vertex operators, and q_{r+1}, \dots, q_{r+n} are the R-charges of the integrated vertex operators. The R-charge at infinity on the right hand side arises from the twisting of the energy momentum tensor by the derivative of the R-current. In our context, the R-charge conservation must moreover be modified appropriately by the R-charge of the states in which we take the expectation value (see equation (2.9)).

In practice, for the spherical two-point function we need

$$q_i + q_j = c/3. \quad (2.11)$$

It is thus natural to propose the result

$$\eta_{ij} = \delta_{i+j, k+2}. \quad (2.12)$$

Let us briefly discuss this result. One way to view the result is as an expectation value for the operator Y^{-k-2} , which may be surprising since the operator is formally equivalent to zero in the chiral ring. An alternative view on the expectation value is that we evaluate the expectation value of the operators Y^{-k} in the (almost-normalizable) state corresponding to the operator Y^{-1} . Thus, the seemingly surprising feature of giving a vacuum expectation value to the operator Y^{-k-2} , which is set to zero in the ring under the constraint $-Y^{-1}W_Y = Y^{-k-2} \equiv 0$ can either be viewed as due to the strong divergence in the norm of the state $|0\rangle$ or, alternatively, can be read as corresponding to the vacuum expectation value of (non-zero) operator Y^{-k} in the state generated by Y^{-1} . We will provide more detail to the last interpretation shortly.

Meanwhile, we observe that the metric η_{ij} provides a mild cross check on the identification of the extended and strict rings. Indeed, the topological metric is a non-degenerate symmetric bilinear form on the extended ring R_{ext} as well as on the vectorial subspace corresponding to the ring R .

Finally, given the metric η and charge conservation, it is natural to also propose the three-point function

$$c_{ijl} = \delta_{i+j+l, k+2}. \quad (2.13)$$

We will further argue these proposals in subsection 2.1.3, and we will derive interesting consequences of accepting them in section 2.2.

2.1.3 Small Deformations

In this subsection, we study how the topological conformal field theories behave under a small deformation that gives rise to separated massive vacua. The analysis provides extra insight into the features typical of the non-compact model.

Near the Compact TCFT

To introduce our tools, we reconsider the topological minimal model correlators from a Landau-Ginzburg perspective. The correlators of the topological Landau-Ginzburg theory on a genus g Riemann surface are computed through the formula [44]

$$\langle X^{i_1} \dots X^{i_n} \rangle_g = \sum_{W_X=0} X^{i_1} \dots X^{i_n} H^{g-1}, \quad (2.14)$$

where $H = W_{XX}$ is the Hessian of the superpotential. We compute the correlator by first splitting the classical vacua by adding a regulator term ϵX to the superpotential, and afterwards take the regulator to zero to recover the conformal correlators. This is a well-defined procedure in this context, and the correlators agree with those that follow from the chiral ring relations provided in subsection 2.1.1. It is easy to show this explicitly. The calculation includes the steps :

$$\begin{aligned} W &= -\epsilon X + \frac{X^{k_c}}{k_c} & W_X &= -\epsilon + X^{k_c-1} \\ H &= W_{XX} = (k_c - 1)X^{k_c-2} & X_n &= \epsilon^{\frac{1}{k_c-1}} e^{\frac{2\pi i}{k_c-1}n} \end{aligned} \quad (2.15)$$

where the X_n label the $k - 1$ vacua over which we sum in formula (2.14), and $n = 0, 1, \dots, k_c - 2$. At small ϵ , we can then approximate e.g. the spherical one-point function by :

$$\langle X^j \rangle_0 = \epsilon^{\frac{j}{k_c-1}} \sum_{n=0}^{k_c-2} e^{\frac{2\pi i}{k_c-1}jn} (k_c - 1)^{-1} e^{\frac{-2\pi i}{k_c-1}(k_c-2)n} \epsilon^{-\frac{k_c-2}{k_c-1}}. \quad (2.16)$$

For a positive integer power j we find that this equals

$$\langle X^j \rangle_0 = \delta_{j, k_c-2}. \quad (2.17)$$

This agrees with the metric and three-point function on the sphere provided earlier. The regularization procedure is unambiguous, and finite. Let's explore where the analogous approach for the non-compact models leads us.

Near the Non-Compact TCFT

We wish to find a similar confirmation for the correlators proposed for the non-compact topological conformal field theory in subsection 2.1.2. The first issue to address for the non-compact models is how to regulate. We can imagine various regulators : the logarithmic regulator of [38], a mass term linear in Y , the addition of an almost-normalizable operator Y^{-1} term to the superpotential, or the addition of a normalizable Y^{-2} term. Let's discuss some of these regulators in detail.

The Y Deformation

The derivative of the superpotential is of order $k+1$ in Y^{-1} , and thus, we may expect a $k+1$ fold degeneracy of the vacuum state. If we add a term linear in the Y variable, this expectation is borne out, and we calculate

$$\begin{aligned} W &= \epsilon Y + \frac{Y^{-k}}{k} & W_Y &= \epsilon - Y^{-k-1} \\ H = W_{YY} &= (k+1)Y^{-k-2} & Y_n^{-1} &= \epsilon^{\frac{1}{k+1}} e^{\frac{2\pi i}{k+1}n} \end{aligned} \quad (2.18)$$

where n labels the vacua over which we sum in formula (2.14), and $n = 0, 1, \dots, k$. We then find the spherical one-point function

$$\begin{aligned} \langle Y^{-j} \rangle_0 &= \epsilon^{\frac{j}{k+1}} \sum_{n=0}^k e^{\frac{2\pi i}{k+1}jn} (k+1)^{-1} e^{-\frac{2\pi i}{k+1}(k+2)n} \epsilon^{-\frac{k+2}{k+1}} \\ &= \frac{1}{\epsilon} \delta_{j,1} + \delta_{j,k+2}. \end{aligned} \quad (2.19)$$

If we concentrate on two-point functions of operators that are in the spectrum, we find agreement with the topological metric η_{ij} proposed in subsection 2.1.2, as well as with the three-point functions. There is an important phenomenon to note here. A standard reasoning says that, since we sum over points where $W_Y = 0$ in the formula (2.14), we should expect that the vacuum expectation value of $W_Y = -Y^{-k-1}$ as well as its multiples equal zero. Again, naively, this simple reasoning contradicts the result (2.19). Let us explain then why on second thought, it does not. Consider the operator $-Y^{-1}W_Y = -\epsilon Y^{-1} + Y^{-k-2}$ after deformation by ϵ . The vacuum expectation value of this operator is indeed zero, in an interesting manner. The vacuum expectation value of Y^{-1} diverges and cancels the contribution of the Y^{-k-2} operator to the expectation value. The regularization and an intricate cancellation leads to the possibility of a non-zero expectation value for Y^{-k-2} . A similar remark applies to the regularization approaches that follow.

The Logarithmic Deformation

We consider another regularization, amongst other reasons to check the robustness of our conclusions. Let's review the approach of [38] in the Y -variables. The superpotential is deformed by a logarithmic term in the Y -variable. The calculation goes

$$\begin{aligned} W &= \epsilon \log Y + Y^{-k}/k \\ W_Y &= \epsilon Y^{-1} - Y^{-k-1} \\ W_{YY} &= -\epsilon Y^{-2} + (k+1)Y^{-k-2} \end{aligned} \quad (2.20)$$

and

$$W_Y = 0 \quad (2.21)$$

has solutions

$$\begin{aligned} Y^{-1} &= 0 \\ Y_n^{-1} &= \epsilon^{1/k} e^{2\pi i n/k}. \end{aligned} \quad (2.22)$$

We have brought only k of the vacuum solutions to a finite value and need to be careful about the contribution of the vacuum at infinity. We apply the formula (2.14) and find for the spherical one-point function, for $j > 2$

$$\begin{aligned}\langle Y^{-j} \rangle &= \sum_{n=0}^{k-1} \epsilon^{j/k} e^{2\pi i n(j-2)/k} \epsilon^{-1-2/k} \frac{1}{k} \\ &= \delta_{j,k+2}.\end{aligned}\tag{2.23}$$

For $j \leq 2$, the contribution at infinity makes for an ambiguous result. For the one-point function of most interest, the result agrees with the linear regularization.

The Φ -variables

Note that from the perspective of the Φ variables, the contribution of the point at $Y^{-1} = 0$ is spurious, namely, solely due to the change of variables. Indeed, let's illustrate this with an explicit calculation in the Φ variable. The superpotential becomes [39]

$$W = \mu e^{b\Phi} - m\Phi.\tag{2.24}$$

The computation goes

$$\begin{aligned}W_\Phi &= \mu b e^{b\Phi} - m \\ W_{\Phi\Phi} &= \mu b^2 e^{b\Phi}\end{aligned}\tag{2.25}$$

There are only k vacua solving the equation $\mu b(e^{\frac{1}{2b}\Phi})^k = m$, as in the Witten index calculation [39]. We find the topological metric for the operators $e^{\frac{j}{b}\Phi}$

$$\eta_{ij} = \frac{1}{mb} \sum_{n=0}^{k-1} e^{\frac{2\pi i n(i+j)}{k}} \left(\frac{m}{\mu b}\right)^{\frac{i+j}{k}} = 2k\delta_{i+j,k}.\tag{2.26}$$

The relative normalization of the bilinear form η_{ij} is $2k$ times larger in the Φ variables than in the Y -variables, merely due to a change of variables. The shift in the indices is due to the difference in measure (or Hessian). Indeed, we see that the topological metric is now automatically evaluated in the state with conformal dimension and R-charge equal to $(1/2k, 1/k)$ (from the shift in the indices i, j by one in the metric (2.26) compared to (2.12)). (See the discussion in subsection 2.1.2.) Other deformation proposals, based on the almost normalizable operator Y^{-1} or the normalizable operator Y^{-2} confirm the robust nature of the proposed topological conformal field theory correlators. We therefore posit these correlators as our starting point and move on to uncover interesting properties of these topological theories.

2.2 The Topological Massive Model

Superconformal field theories can be deformed to massive supersymmetric quantum field theories by adding an operator to the Lagrangian. When the deformations are consistent with $N = (2, 2)$ supersymmetry, the resulting theory can be topologically twisted to produce a topological quantum field theory. In this section, we study

relevant deformations of the $N = 2$ Liouville conformal field theory that preserve supersymmetry, and twist them into topological massive quantum field theories. The perturbed correlators are highly constrained, and governed by a free energy determined by WDVV equations. The deformation space is a Fröbenius manifold [45]. For compact models, the deformed operators with given R-charge, as well as the structure constants as a function of the deformation parameters were determined in [21]. We do refer to this reference for details on the solution of the compact model, which we only summarize below. The solution of the non-compact model is more intricate still, and we will describe it in detail.

2.2.1 The Compact Topological Field Theory

For the $N = 2$ minimal model and its relevant deformation, a solution for the topological theory was elegantly presented in [21]. The deformed structure constants c_{ijl} of the ring of the compact theory are defined in terms of the correlation functions of operators ϕ_i at criticality and with fixed R-charge, as follows :

$$c_{ijl}(t) = \langle \phi_i \phi_j \phi_l \exp(\sum_n t_n \int \phi_n) \rangle_0, \quad (2.27)$$

where the correlators are evaluated in the topological conformal theory and where the deformation of the action is taken into account through the exponentiation of the integrated operators [21]. The structure constants satisfy a number of consistency conditions that imply that they can be derived from a generating functional F which is the free energy at tree level [21]. Two paths to the calculation of the structure constants are laid out in [21]. The first is to define a deformed superpotential

$$W(X, t) = \frac{1}{k_c} X^{k_c} - \sum_{i=0}^{k_c-2} g_i(t_j) X^i, \quad (2.28)$$

where the g_i are equal to the deformation parameters t_i to first order in t_i . The chiral primaries ϕ_i are equal to the original chiral primaries X^i at zero deformation, and have a fixed R-charge $q_i = i/k$, as well as fixed leading order behavior X^i . They are proposed to become parameter dependent as follows

$$\phi_i(X; t) = -\partial_i W(X, t), \quad (2.29)$$

where ∂_i is a partial derivative with respect to t_i . The chiral ring coupling constants $c_{ij}^l(t)$ are then computed with respect to the basis ϕ_i , i.e. they satisfy $\phi_i \phi_j = c_{ij}^l \phi_l$ for all t_i .

On the ring $R_c = \mathbb{C}[X]/\langle W_X \rangle$ of polynomials modulo the derivative of the superpotential, a non-degenerate symmetric bilinear form is introduced which equals a contour integral of a ratio of polynomials

$$(\chi_1, \chi_2) = \sum \text{Res} \left(\frac{\chi_1 \chi_2}{W_X} \right) = \oint \frac{dX}{2\pi i} \frac{\chi_1 \chi_2}{W_X}. \quad (2.30)$$

The contour is taken to circle all zeroes of W_X , and can be taken large (compared to the zeros). The bilinear form is independent of the chosen representatives χ_i in the

quotient ring R_c . Given the leading behavior of the operators ϕ_i , and expanding near infinity, we can prove [21] that the metric equals

$$(\phi_i, \phi_j) = c_{ij}^l \sum \text{Res}(\phi_l/W_X) = c_{ij0} = \eta_{ij}. \quad (2.31)$$

The structure constants are then given in terms of the ϕ_i by the formula

$$c_{ijl}(t) = \sum \text{Res} \frac{\phi_i \phi_j \phi_l}{W_X}. \quad (2.32)$$

Crucially, by exploiting a parafermionic selection rule that is inherited from $SU(2)_{k_c}$ fusion rules, a recursion relation for the operators ϕ_i of fixed R-charge is proven in [21], which permits a closed form solution. It is given by the determinant formula

$$\phi_i(X, t) = (-1)^i \det \begin{pmatrix} -X & 1 & 0 & \dots & 0 \\ t_{k_c-2} & -X & 1 & \dots & \dots \\ t_{k_c-3} & t_{k_c-2} & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ t_{k_c-i} & \dots & t_{k_c-3} & t_{k_c-2} & -X \end{pmatrix}, \quad (2.33)$$

for $i = 0, 1, \dots, k_c - 2$. One also obtains

$$W_X(X, t) = \phi_{k_c-1}(X, t), \quad (2.34)$$

where the right hand side extends the definition (2.33) to $i = k_c - 1$. One can then integrate up to find $W(X, t)$, using also the differential equations $\phi_i = -\partial_i W$.

A second manner in which the solution is obtained is by defining the auxiliary (Lax) root L of the superpotential

$$\frac{L^{k_c}}{k_c} = W, \quad (2.35)$$

and to think of it as a formal power series

$$L(X) = X(1 + \sum_{j=2} b_j X^{-j}) \quad (2.36)$$

near large X . The fields ϕ_i are then proposed to be

$$\phi_i(X) = [L^i \partial_X L]_{\geq 0} \quad (2.37)$$

where the index on square brackets indicates which terms in the formal power series are to be taken. The residues of the root L are the parameters t_i :

$$t_{k_c-2-i} = -\frac{1}{i+1} \text{Res}(L^{i+1}). \quad (2.38)$$

Since we expanded at large X (e.g. by imagining a large contour, i.e. looking at the residues from afar), we can evaluate the residue as the coefficient of the $1/X$ term. The consistency of the first and the second method can be demonstrated using the Lagrange inversion formula for formal power series expansions, reviewed in appendix B.

Furthermore, the structure constants can then be computed to be [21]

$$c_{ijl} = \frac{1}{(l+1)(k_c+l+1)} \partial_i \partial_j \text{Res}(L^{k_c+l+1}), \quad (2.39)$$

which implies that the first derivatives of the free energy integral $F = \langle \exp(\sum_i t_i \phi_i) \rangle_0$ equal

$$\partial_i F = \frac{1}{(i+1)(k_c+i+1)} \text{Res}(L^{k_c+i+1}). \quad (2.40)$$

The proofs of this laundry list of formulas are non-trivial and interesting, and they are given in [21], when combined with the Lagrange inversion formula. We will provide proofs for the non-compact models below, in great detail. Our goal in reviewing the compact formulas is to have a simpler and useful point of comparison. Explicit examples of generators of correlation functions F are integrated in appendix A.1 for levels $k_c = 2, 3, 4$ and 5 .

2.2.2 The Non-Compact Topological Field Theory

In this section, we study relevant deformations of the topological non-compact conformal field theory that we introduced in section 2.1. We limit ourselves to strictly normalizable deformations, proportional to an operator in the ring R generated by the set of operators $\{Y^{-2}, Y^{-3}, \dots, Y^{-k}\}$. The selection rules on fusion are less strong in the non-compact model, and therefore we only follow the second path reviewed in subsection 2.2.1. We firstly propose a closed form solution, and then verify that it satisfies all necessary conditions. Our first goal is to find deformed operators ϕ_i of fixed R-charge i/k and with leading behavior Y^{-i} . The deformation parameters of the model will be denoted t_i . We also make use of the alternative notation $t_{k-i} = -s_{i=1, \dots, k-2}$ and $t_k = 1 - s_0$, where the label on s_i equals its R-charge (times the level k). Note that there is a marginal operator in the ring R , and there is therefore a parameter s_0 with zero R-charge. Moreover, we will take the parameter s_0 to have value one in the undeformed theory while the other parameters are zero. We impose the constraint that the parameter s_0 is always non-zero.

Secondly, we wish to determine the superpotential (or at least, all of its derivatives), and thirdly, we compute the structures constants c_{ijl} which are again generated by deformation

$$c_{ijl}(t) = \langle \phi_i \phi_j \phi_l \exp(\sum_n t_n \int \phi_n) \rangle_0, \quad (2.41)$$

where the operators on the right hand side are those at the conformal point. Finally, we calculate the free energy F from which the structure constants and all correlators can be derived.

The Operators

The bilinear form on the deformed ring, in analogy to equation (2.30), is given by

$$(\chi_1, \chi_2) = \sum \text{Res}_Y \left(\frac{\chi_1 \chi_2}{W_Y} \right), \quad (2.42)$$

where the sum is over all the zeros of the derivative of the superpotential W_Y . Since the zeros of W_Y are near $Y^{-1} = 0$, we rewrite the formula in terms of the inverse Y^{-1}

$$(\chi_1, \chi_2) = - \sum \text{Res}_{Y^{-1}} \left(\frac{Y^2 \chi_1 \chi_2}{W_Y} \right). \quad (2.43)$$

The symmetry and bi-linearity of the form (\cdot, \cdot) are obvious. The independence of the choice of representative follows from the fact that the ring elements have at least a Y^{-2} prefactor, and only positive powers of Y^{-1} . Non-degeneracy will follow from our explicit choice of generators of the ring, and the calculation of the non-degenerate metric η .

The deformed superpotential for the massive models can be generically parameterized as

$$W = \frac{Y^{-k}}{k} \left(1 + \sum_{i=0}^{k-2} h_i Y^i \right). \quad (2.44)$$

The functions h_i when expanded to first order in all t_j only have a non-zero coefficient in front of t_{k-i} . The leading coefficient $1 + h_0$ must remain non-zero in order to keep the asymptotics of the model intact. Following the lead of the compact model, we define the root L of the superpotential

$$L = (kW)^{\frac{1}{k}}, \quad (2.45)$$

which we think of as a formal power series in Y , and propose the equation

$$\phi_i = [-L^{i-2} \partial_Y L]_{\leq -2} \quad (2.46)$$

for the operators ϕ_i of definite charge $q_i = i/k$ in the deformed chiral ring. Moreover, we define the deformation parameters as

$$t_{k-i} - \delta_{i,0} \equiv -s_i = -\frac{1}{i+1} \text{Res}_{Y^{-1}} (Y^2 L^{i+1}). \quad (2.47)$$

We must prove the following properties. The operators ϕ_i have the appropriate charge. The metric η_{ij} is undeformed. The structure constants c_{ijl} associated to the operators ϕ_i obey that they are symmetric, and they can be integrated up to a free energy functional F . We will prove these properties one by one.

Firstly, the charge of the root L is $1/k$. The charge of the operator Y is $-1/k$. Thus, the field ϕ in equation (2.46) has charge i/k as desired. The leading behavior of ϕ_i is dictated by the leading behavior of L , and is indeed Y^{-i} , with lower powers of Y^{-1} trailing. The lower powers are all in the strict chiral ring R by the design of equation (2.46).

The calculation of the metric in the deformed chiral ring generated by the operators ϕ_i for $i = 2, 3, \dots, \leq k$ reads

$$\begin{aligned} (\phi_i, \phi_j) &= \text{Res}_{Y^{-1}} (Y^2 [L^{i-2} \partial_Y L]_{\leq -2} \phi_j / W_Y) \\ &= \text{Res}_{Y^{-1}} (Y^2 L^{i-k-1} \phi_j) \\ &= \text{Res}_{Y^{-1}} (Y^2 L^{i-k-1} [(-) L^{j-2} \partial_Y L]_{\leq -2}) \\ &= \text{Res}_{Y^{-1}} (-Y^2 L^{i+j-k-3} \partial_Y L) \\ &= \delta_{i+j, k+2}, \end{aligned} \quad (2.48)$$

where the first step is true due to the fact that we can drop the constraint on the terms we consider in the bracket since

$$Y^2[L^{i-2}\partial_Y L]_{>-2}\phi_j/W_Y = Y^{k-j+3} \times (\text{sum of non-negative powers of } Y) \quad (2.49)$$

does not contain a linear term in Y . We have used that $L^{i-2}\partial_Y L = \frac{1}{i-1}\partial_Y L^{i-1}$ does not have a Y^{-1} term for $i \neq 1$. The second step uses the definition (2.46) of the operator ϕ_j . The third step is justified by a similar argument as the first step. The last step is made while observing that only when $i + j - k - 3 = -1$ the argument of the residue is not a total derivative, and that in that case, the residue is one. Thus, the metric is invariant under deformation, and non-degenerate.

We also require that after deformation, the equation $\partial_i W = -\phi_i$ is still valid. By deriving the equation for W in terms of the root L , and plugging the resulting equation for $\phi_i = -L^{k-1}\partial_i L$ (where the derivative is with respect to t_i) into the equation for the metric η :

$$\eta_{ij} = \text{Res}_{Y^{-1}}(Y^2 L^{i-1-k} \phi_j) = -\frac{1}{i-1} \text{Res}_{Y^{-1}} \partial_i(Y^2 L^{i-1}), \quad (2.50)$$

we find after integration that equation (2.47) holds (where t_i are conveniently compared to the compact models, while the alternative parameterization in terms of s_i is handy for keeping track of R-charges).

The Explicit Operators

In principle, these equations are sufficient description of the solution of the model, but it is interesting to solve the model more explicitly. In particular, we are confronted with a considerably more intricate combinatorics problem than was the case in the compact model of subsection 2.2.1. We delve into this combinatorial quagmire, and compute the operators ϕ_i as explicit expressions in the parameters s_i . Our starting points are equations (2.46) and (2.47).

Firstly, notice that from the superpotential and the definition of the root L , we can write

$$YL = \sum_{n=0}^{\infty} b_n Y^n, \quad (2.51)$$

where the coefficient b_0 satisfies $b_0 \neq 0$. The equation (2.47) for the parameters s_i then becomes

$$\begin{aligned} s_j &= \frac{1}{j+1} [(YL)^{j+1}]_j \\ &= \frac{b_0^{j+1}}{j+1} \left[1 + \sum_{l=1}^{j+1} \binom{j+1}{l} \left(\sum_{n=1}^{\infty} \frac{b_n}{b_0} Y^n \right)^l \right]_j \\ &= \frac{b_0^{j+1}}{j+1} \sum_{l=1}^j \binom{j+1}{l} \frac{l!}{j!} B_{j,l} \left(1! \frac{b_1}{b_0}, 2! \frac{b_2}{b_0}, \dots \right) \\ &= \sum_{l=1}^j \frac{b_0^{j+1}}{(j+1-l)!} B_{j,l} \left(1! \frac{b_1}{b_0}, 2! \frac{b_2}{b_0}, \dots \right) \end{aligned} \quad (2.52)$$

for $j \neq 0$, where the polynomials $B_{j,l}$ are partial Bell Polynomials [50, 51] that arise from expanding powers of multinomials. Remark that for the special case $j = 0$, one finds $s_0 = b_0$. We then get for $j \neq 0$,

$$j! \frac{s_j}{s_0^{j+1}} = \sum_{l=1}^j \binom{j}{l-1} (l-1)! B_{j,l} \left(1! \frac{b_1}{b_0}, 2! \frac{b_2}{b_0}, \dots \right). \quad (2.53)$$

The combinatorial heart of the calculation occurs at the following step, which, given equation (2.53), relates the partial Bell polynomials in the parameters s with the partial Bell polynomials of the coefficients b of the operator YL . This step is given by Theorem 15 of [52], which states that for all integers $1 \leq p < i$, we now have

$$\sum_{l=1}^p \binom{i-p-1}{l-1} (l-1)! B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right) = \sum_{l=1}^p \binom{i-1}{l-1} (l-1)! B_{p,l} \left(1! \frac{b_1}{b_0}, 2! \frac{b_2}{b_0}, \dots \right). \quad (2.54)$$

Similar to the above computation for the parameters s_j , the operators (2.46) become

$$\begin{aligned} \phi_{i+1} &= -\frac{1}{i} [\partial_Y (Y^{-i} s_0^i (1 + \sum_{n=1}^{\infty} \frac{b_n}{b_0} Y^n)^i)]_{\leq -2} \\ &= s_0^i [Y^{-i-1} + \sum_{p=1}^{i-1} f_p^{(i)} Y^{-i-1+p}], \end{aligned} \quad (2.55)$$

where

$$\begin{aligned} f_p^{(i)} &= \frac{i-p}{i} [(1 + \sum_{n=1}^{\infty} \frac{b_n}{b_0} Y^n)^i]_p \\ &= \frac{i-p}{i} \sum_{l=1}^p \binom{i}{l} \frac{l!}{p!} B_{p,l} \left(1! \frac{b_1}{b_0}, 2! \frac{b_2}{b_0}, \dots \right) \\ &= \frac{i-p}{p!} \sum_{l=1}^p \binom{i-1}{l-1} (l-1)! B_{p,l} \left(1! \frac{b_1}{b_0}, 2! \frac{b_2}{b_0}, \dots \right). \end{aligned} \quad (2.56)$$

Multiplying by $\frac{i-p}{p!}$ on both sides of (2.54) one obtains

$$f_p^{(i)} = \sum_{l=1}^p \binom{i-p}{l} \frac{l!}{p!} B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right). \quad (2.57)$$

Now we use the definition of the Bell polynomials [51] and get

$$\frac{l!}{p!} B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right) = \sum \frac{l!}{r_1! \dots r_{p-l+1}!} \left(\frac{s_1}{s_0^2} \right)^{r_1} \dots \left(\frac{s_{p-l+1}}{s_0^{p-l+2}} \right)^{r_{p-l+1}}, \quad (2.58)$$

where the sum is taken over all non-negative integers $r_1, r_2, \dots, r_{p-l+1}$ such that $\sum_{n=1}^{p-l+1} r_n = l$ and $\sum_{n=1}^{p-l+1} n r_n = p$. The right hand side of equation (2.58) can be rewritten as

$$s_0^{-p-l} \times \sum_{j_1 + \dots + j_l = p, j_n \geq 1} s_{j_1} \dots s_{j_l}. \quad (2.59)$$

Therefore, we find the coefficients $f_p^{(i)}$ of the operators

$$f_p^{(i)} = s_0^{-i} \sum_{j_1 + \dots + j_{i-p} = p, j_n \geq 0} s_{j_1} \cdots s_{j_{i-p}}. \quad (2.60)$$

In summary, the explicit expression for the operators ϕ_{i+1} in term of the parameters s_j is

$$\phi_{i+1} = \sum_{p=0}^{i-1} \sum_{j_1 + \dots + j_{i-p} = p} s_{j_1} \cdots s_{j_{i-p}} Y^{-i-1+p}. \quad (2.61)$$

There is also an interesting recursion relation among the operators ϕ_i :

$$\phi_{i+2} = s_0 Y^{-1} \phi_{i+1} + \sum_{n=1}^i \left[- \sum_{l=1}^n (-s_0)^{-l} \sum_{j_1 + \dots + j_l = n, j_m \geq 1} s_{j_1} \cdots s_{j_l} \right] \phi_{i+2-n}. \quad (2.62)$$

We prove this recursion relation by comparing the coefficient of Y^{-i-2+p} for all p from 0 to i on both sides of equation (2.62). For $p = 0$ it is trivial that the two coefficients match. For $p > 0$, we proceed as follows. We first notice that the coefficient in front of the operator ϕ_{i+2-n} in the recursion relation (2.62) can be written in terms of the partial Bell polynomials as

$$s_0^n \sum_{l=1}^n (-s_0)^{l-1} \frac{l!}{n!} B_{n,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right). \quad (2.63)$$

The coefficient of Y^{-i-2+p} on the left hand side of equation (2.62) is then

$$s_0^{i+1} \sum_{l=1}^p \binom{i+1-p}{l} \frac{l!}{p!} B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right), \quad (2.64)$$

while the coefficient on the right hand side is the sum of

$$s_0^{i+1} \sum_{l=1}^p \binom{i-p}{l} \frac{l!}{p!} B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right) \quad (2.65)$$

and

$$s_0^{i+1} \sum_{n=1}^p \sum_{l_1=0}^{p-n} \sum_{l_2=1}^n \binom{i+1-p}{l_1} \frac{(-s_0)^{l_2-1} l_1! l_2!}{(p-n)! n!} B_{p-n, l_1} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right) B_{n, l_2} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right). \quad (2.66)$$

Using the formula $B_{p,l} = 0$ for $p < l$, we do not change the value of expression (2.66) if we change the range of the two sums over l_1 and l_2 into $\sum_{l_1=0}^{p-1}$ and $\sum_{l_2=1}^p$. We can then do the sum over n by applying equation (1.4) of [53] for the partial Bell polynomials and obtain

$$s_0^{i+1} \sum_{l_1=0}^{p-1} \sum_{l_2=1}^p \binom{i+1-p}{l_1} \frac{(-s_0)^{l_2-1} (l_1 + l_2)!}{p!} B_{p, l_1 + l_2} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right) \quad (2.67)$$

for this part of the coefficient of Y^{-i-2+p} on the right hand side of (2.62). Furthermore, denote $l = l_1 + l_2$ such that the expression (2.66) becomes

$$s_0^{i+1} \sum_{l=1}^p \left[(-1)^{l-1} \sum_{l_1=0}^{l-1} (-1)^{l_1} \binom{i+1-p}{l_1} \right] B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right). \quad (2.68)$$

Since we have the binomial identity

$$\sum_{l_1=0}^{l-1} (-1)^{l_1} \binom{i+1-p}{l_1} = (-1)^{l-1} \binom{i-p}{l-1}, \quad (2.69)$$

we find that expression (2.66) is equal to

$$s_0^{i+1} \sum_{l=1}^p \binom{i-p}{l-1} \frac{l!}{p!} B_{p,l} \left(1! \frac{s_1}{s_0^2}, 2! \frac{s_2}{s_0^3}, \dots \right). \quad (2.70)$$

Finally, adding the contribution (2.65) to the coefficient of the right hand side, we find that the total coefficient of Y^{-i-2+p} in the recursion relation (2.62) is indeed equal to the left hand side coefficient (2.64), which proves the recursion relation. We believe this relation is interesting by itself, since a comparison to the compact case suggests that they may follow from non-compact fusion rules.

The combinatorial theorems [52, 53] that are central to this subsection are surprisingly intricate and recent, but they are proven with elementary analytic means. It would be good to have a better conceptual understanding of these combinatorial theorems, and a clearer picture of what is counted e.g. in graph theoretical terms. Instead, we concentrate on the further description of our solution to the non-compact model.

The Superpotential

Similarly as in the compact case, the Y -derivative of the superpotential is given by the expression for the ϕ_i , where we equate the index to $i = k + 1$. We thus have the formula

$$W_Y = -\phi_{k+1}. \quad (2.71)$$

We can then integrate the equations $\partial_i W = -\phi_i$ as well as $W_Y = -\phi_{k+1}$ to obtain an expression for the superpotential (up to an arbitrary constant).

The Structure Constants and the Free Energy

The equations (2.61) also provide solutions for the structure constants $c_{ij}^l(s)$ as a function of the parameters s_i , through the equations

$$\phi_i \phi_j = c_{ij}^l(s) \phi_l. \quad (2.72)$$

We can prove that the definition of the parameters s_i is consistent with desired properties of the structure constants c_{ij}^l , namely, their symmetry and integrability, and we will be able to generically integrate for the first derivatives of the generating functional

F . Derivatives ∂_i will again be with respect to the variables t_i . As a preliminary, we derive the useful equation

$$\partial_j \phi_i = \partial_j \left[-\frac{1}{i-1} \partial_Y L^{i-1} \right]_{\leq -2} = -\frac{1}{i-1} \partial_Y ([\partial_j L^{i-1}]_{\leq -1}) = -\partial_Y ([L^{i-2} \partial_j L]_{\leq -1}).$$

Next, we demonstrate that the three-point function, which is manifestly symmetric, is a double derivative with respect to the variables t_i :

$$\begin{aligned} c_{ijl} &= -\text{Res}_{Y^{-1}} \left(\frac{Y^2}{W_Y} \phi_i \phi_j \phi_l \right) = \text{Res}_{Y^{-1}} \left(\frac{Y^2}{W_Y} \phi_i \phi_j [L^{l-2} \partial_Y L]_{\leq -2} \right) \\ &= \text{Res}_{Y^{-1}} (Y^2 \phi_i \phi_j L^{-k+l-1}) - \text{Res}_{Y^{-1}} \left(\frac{Y^2}{W_Y} \phi_i \phi_j [L^{l-2} \partial_Y L]_{\geq -1} \right) \\ &= \frac{1}{l-1} (-\text{Res}_{Y^{-1}} (Y^2 \phi_i \partial_j L^{l-1}) - \text{Res}_{Y^{-1}} (Y^2 [\frac{\phi_i \phi_j}{W_Y}]_{\leq -1} \partial_Y (L^{l-1}))) \\ &= \frac{1}{l-1} (-\text{Res}_{Y^{-1}} (Y^2 \phi_i \partial_j L^{l-1}) + \text{Res}_{Y^{-1}} (Y^2 \partial_Y [\frac{\phi_i \phi_j}{W_Y}]_{\leq -1} L^{l-1})) \\ &= \frac{1}{l-1} (-\text{Res}_{Y^{-1}} (Y^2 \phi_i \partial_j L^{l-1}) - \text{Res}_{Y^{-1}} (Y^2 \partial_Y [L^{i-k-1} \phi_j]_{\leq -1} L^{l-1})) \\ &= -\frac{1}{l-1} \text{Res}_{Y^{-1}} (Y^2 \partial_j (\phi_i L^{l-1})) = \frac{1}{(l-1)(l+k-1)} \partial_i \partial_j \text{Res}_{Y^{-1}} (Y^2 L^{k+l-1}) \end{aligned} \quad (2.73)$$

where we again restricted to $2 \leq i, j, l \leq k$. The justification for the steps in the proof are of the same type as those we detailed for the calculation of the topological metric η_{ij} . We then find that the first derivatives of the generator of correlation functions F is

$$\partial_l F = \frac{\text{Res}_{Y^{-1}} (Y^2 L^{k+l-1})}{(l-1)(l+k-1)}. \quad (2.74)$$

Crucially, the equation (2.74) is integrable. The proof of the integrability proceeds as follows :

$$\begin{aligned} \partial_j \partial_l F &= -\frac{\text{Res}_{Y^{-1}} (Y^2 L^{l-1} \phi_j)}{l-1} \\ &= \frac{\text{Res}_{Y^{-1}} (Y^2 L^{l-1} [\partial_Y L^{j-1}]_{\leq -2})}{(l-1)(j-1)} \\ &= \frac{\text{Res}_{Y^{-1}} (Y^2 [L^{l-1}]_{\geq 1} \partial_Y L^{j-1})}{(l-1)(j-1)} \\ &= -\frac{\text{Res}_{Y^{-1}} (Y^2 [\partial_Y L^{l-1}]_{\geq 0} L^{j-1})}{(l-1)(j-1)} \\ &= \frac{\text{Res}_{Y^{-1}} (Y^2 [\partial_Y L^{l-1}]_{\leq -2} L^{j-1})}{(l-1)(j-1)} \\ &= \partial_l \partial_j F. \end{aligned} \quad (2.75)$$

We have used the fact that the residue of a total derivative is zero twice in the previous to last step, once for L^{l-1} and once for L^{l+j-2} .

Finally, in appendix A.2 we provide explicitly integrated values of the generating function F for the non-compact models at levels 2, 3, 4 and 5 .

Final Remarks

The solution of the massive non-compact topological quantum field theory through the determination of its chiral ring reveals novel features. It generalizes the integrable structure that existed in the compact case in a beautiful and non-trivial manner. We needed to enlarge our combinatorial toolkit in order to find the explicit solution, in particular to invert a power series expansion in terms of partial Bell polynomials. It will be interesting to also manifestly link our solution to an integrable hierarchy.

2.3 The Conformal Metric

In this section we explore the extension of our analysis of the topological non-compact theories to the topological-anti-topological sector [39]. In particular, we calculate the overlap between states generated by chiral, and states generated by anti-chiral operators.

The overlap between chiral and anti-chiral states defines a metric $g_{m\bar{n}}$ which can be computed through localization at the conformal point. This fact was understood long time ago [39], and recently carefully confirmed using contemporary techniques [46, 47]. The final formula for the metric $g_{m\bar{n}}$ at the conformal point is

$$\begin{aligned} g_{m\bar{n}}(0) &= \frac{1}{\pi} \int dY d\bar{Y} Y^{-m} \bar{Y}^{-\bar{n}} \exp(W(Y) - \bar{W}(\bar{Y})) \\ &= \frac{1}{\pi} \int dX d\bar{X} X^{m-2} \bar{X}^{n-2} \exp(W(X^{-1}) - \bar{W}(\bar{X}^{-1})) \\ &= \delta_{m,n} k^{\frac{2m-2}{k}-1} \frac{\Gamma(\frac{m-1}{k})}{\Gamma(1 - \frac{m-1}{k})}, \end{aligned} \quad (2.76)$$

where we have used the result of the chiral-anti-chiral metric computation for a compact minimal model given in [39, 46, 47], and constrained ourselves to the operators in the spectrum only, namely to the quantum numbers $2 \leq m, n \leq k$. The change of variables $Y = X^{-1}$ reduces the problem to the calculation for $N = 2$ minimal models, in contrast to what we saw for the topological chiral ring. The normalization of the metric agrees with our normalization for the topological metric η , and the reality constraint to be discussed in section 2.4.

It is interesting to compare this calculation to the calculation in the variables Φ . It is crucial to note that even for constant field configurations, on the sphere, there can be a factor in the path integral that arises from a linear dilaton term in the action. The linear dilaton contribution in the radial direction ρ is

$$S_{l.d.} = \frac{1}{4\pi} \int d^2\sigma \frac{Q}{2} \rho \sqrt{\hat{g}} \mathcal{R}^{(2)} \quad (2.77)$$

where \hat{g} and $\mathcal{R}^{(2)}$ are the determinant of the sphere metric and the world sheet Ricci scalar. The coefficient Q is related to the central charge by the equation

$$Q = \sqrt{\frac{2}{k}} = \frac{1}{b}. \quad (2.78)$$

In the localization computation, only the zero modes of fields contribute. Therefore, we can take ρ out from the integral and use the Gauss-Bonnet theorem to find

$$S_{l.d.} = \frac{\rho}{b}. \quad (2.79)$$

The factor $e^{-S_{l.d.}}$ thus becomes part of the measure in the euclidean path integral. In the Φ variables then, the metric is

$$\begin{aligned} g_{m\bar{n}}(0) &= \frac{1}{\pi} \int_{0 \leq \theta \leq 4\pi b} d\phi d\bar{\phi} e^{-\frac{1}{b}\rho} e^{\frac{m}{2b}\phi} e^{\frac{\bar{n}}{2b}\bar{\phi}} \exp(W - \bar{W}) \\ &= \frac{1}{\pi} \int_{0 \leq \theta \leq 4\pi b} d\phi e^{-\frac{1}{2b}\phi} d\bar{\phi} e^{-\frac{1}{2b}\bar{\phi}} e^{\frac{m}{2b}\phi} e^{\frac{\bar{n}}{2b}\bar{\phi}} \exp(W - \bar{W}), \end{aligned} \quad (2.80)$$

where ρ and θ are the real and imaginary zero modes of the bosonic component ϕ of the superfield Φ . The range of θ is determined by our choice of Liouville theory, as discussed in section 2.1. By a change of variables $Y = e^{-\frac{1}{2b}\phi}$ and setting $\mu = \frac{1}{k}$, the metric equals the result in (2.76) multiplied by $(2b)^2 = 2k$. The change of normalization (also visible in the topological metric η in section 2.1) is due to the change of variables from Y to Φ , and can be absorbed in a redefinition of the path integral measure if so desired.

We can perform a consistency check on the localization result for the metric. The structure constants $c_{ij}^l(0)$ at the conformal point were computed in [38] by first determining the wave-functions of the Ramond-Ramond sector ground states in a quantum mechanics approach to the supersymmetric quantum field theory, and then calculating the overlap of an operator sandwiched between these wave-functions. The final result for the structure constants was

$$C_{q_1 q_2}^{\epsilon}{}^{q_3} = \left(\frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)} \frac{\Gamma(1-q_1-\epsilon)\Gamma(1-q_2-\epsilon)\Gamma(q_3+\epsilon)}{\Gamma(q_1+\epsilon)\Gamma(q_2+\epsilon)\Gamma(1-q_3-\epsilon)} \right)^{\frac{1}{2}}, \quad (2.81)$$

where q_i denote the R-charges of the operators and ϵ is a regulator. In [38], the regulator ϵ measures the difference in R-charge between a reference state and the (almost normalizable) state with R-charge $1/k$. The choice of normalization of operators in [38] was to take the metric $g_{m\bar{n}}(0) = 1$, which implies that we must renormalize our trivial structure constants $c_{i_1 i_2}^{i_3}(0)$ in order to compare. Moreover, we also wish to take as our evaluation state a state close to the almost normalizable state with R-charge $1/k$. Therefore, we normalize our metric by the norm squared of the state with label $m = 1 + \delta$, and we consider our operator charges (acting on this state) to be $q_i = (m_i - 1)/k$. See also the discussion of this point in section 2.1. Under these circumstances, we find, after adapting our normalization and using (2.76)

$$C_{q_1 q_2}^{\delta}{}^{q_3} = k^{-\delta/k} \left(\frac{\Gamma(\delta/k)}{\Gamma(1-\delta/k)} \right)^{1/2} \left(\frac{\Gamma(1-q_1)\Gamma(1-q_2)\Gamma(q_3)}{\Gamma(q_1)\Gamma(q_2)\Gamma(1-q_3)} \right)^{\frac{1}{2}}. \quad (2.82)$$

For δ/k and ϵ small and equal, we find agreement between the structure constants (2.81) and (2.82) in the functional dependence on the charges q_i , as well as for the normalization. This is good check on the localization argument we used to compute the metric. Indeed, since there can be subtleties in applying localization arguments to

non-compact models, as for instance in the case of the elliptic genus [31, 43, 48, 49], the check is useful. It would be interesting to understand the regularization of volume divergences in three-point correlators in non-compact models better still. For instance, it would be informative to derive the structure constants directly from $N = 2$ Liouville conformal field theory.

2.4 The Topological-Anti-Topological Massive Metric

In this section, we take first steps in exploring the chiral-anti-chiral metric $g_{m\bar{n}}$ away from the conformal point. In particular, we will study how it evolves when we add the almost normalizable deformation proportional to Y^{-1} . The crux of the section will be to argue that the latter problem in the non-compact model reduces to a problem in a compact model. In this section, we closely follow [39], to which we must refer for more details on the compact case.

2.4.1 The tt^* Equation

In [39] the dependence of the chiral-anti-chiral metric on deformation parameters t_i of the model was analyzed and shown to be governed by differential equations named the tt^* equations. These are flatness equations for a unitary gauge connection which can be written as differential equations for the metric $g_{m\bar{n}}$ (viewed as a matrix g) [39]

$$\begin{aligned}\bar{\partial}_i(g\partial_j g^{-1}) - [C_j, g(C_i)^\dagger g^{-1}] &= 0 \\ \partial_i C_j - \partial_j C_i + [g(\partial_i g^{-1}), C_j] - [g(\partial_j g^{-1}), C_i] &= 0\end{aligned}\tag{2.83}$$

where C_i is the matrix that captures the action of the operator coupling to t_i on the Ramond-Ramond vacuum states, or equivalently, it is the matrix of chiral ring structure constants. Moreover, from the relation between a basis generated by anti-chiral operators and a basis formed by acting with chiral operators, as well as from the CPT invariance of the quantum field theory, one obtains a reality condition relating the topological metric η_{mn} with the topological-anti-topological metric $g_{m\bar{n}}$ [39]

$$\eta^{-1}g(\eta^{-1}g)^* = 1.\tag{2.84}$$

2.4.2 A Family of Theories

In this subsection, we study a very particular one parameter family of non-compact theories for which the tt^* equations are solvable to a considerable extent.

The Spectrum and The Toda Equation

The tt^* formalism was developed for deformations of the theory, in particular by relevant and normalizable operators [39]. In the following, we will suppose that it also applies to a family of theories obtained by deformation with an almost normalizable relevant operator, and study to which conclusions this assumption leads. Note that the operator Y^{-1} belongs to the extend chiral ring defined in section 2.1.

Consider then the almost normalizable operator Y^{-1} , and (drastically) deform the superpotential of the superconformal theory with a term proportional to this operator :

$$W_t = \frac{Y^{-k}}{k} - tY^{-1}. \quad (2.85)$$

As in section 2.1, we consider the $k-1$ vacua associated to the operators Y^{-2}, \dots, Y^{-k} and, by using the operator constraint $Y^{-k-1} = tY^{-2}$, we note that the multiplication by the operator Y^{-1} acts on these vacua as the matrix

$$C_t = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \\ t & 0 & \dots & \dots & 0 \end{pmatrix}. \quad (2.86)$$

To render the tt^* equations more concrete, we also use the explicit form of the topological metric. Our metric reads $\eta_{i,k+2-i} = 1$. The reality constraint (2.84) then becomes

$$\langle i|i \rangle \langle k+2-i|k+2-i \rangle = 1 \quad (2.87)$$

where we used the fact that the metric g is diagonal in a basis of operators with fixed R-charges. We can then define the unknowns

$$\varphi_i = \log \langle i|i \rangle \quad (2.88)$$

for $i = 2, 3, \dots, k$. Using the explicit matrix (2.86), corresponding to a sum of positive roots of A_{k-2} as well as t times the affine root, we then find that the tt^* equation reduces to a set of \hat{A}_{k-2} Toda equations for φ_i , after a change of variables [39]. These reasonings are as in the compact case [39], with the important caveat that we are discussing a family of theories (and not a deformation of a given theory). To solve the tt^* differential equations explicitly, we preliminarily discuss the boundary conditions.

The Boundary Conditions

There are two limiting regimes that are under direct perturbative control. Firstly, when the parameter t is zero, we are at the conformal point, and the metric is equal to the conformal metric (2.76). This provides one boundary condition.

Secondly, when the parameter t is large, the infrared dynamics is governed by isolated massive vacua, in which one can also compute the metric [39]. The metric can be presented in a basis of operators which equal one in one vacuum and zero in all others. In this basis, the metric in the infrared is again diagonal and equals

$$g_{m\bar{n}} \approx \frac{\delta_{m\bar{n}}}{|H(Y_n)|}, \quad (2.89)$$

where $H(Y_n)$ is the Hessian evaluated at vacuum Y_n . The first (off-diagonal) correction to this classical limit is given in terms of the instanton/soliton action of the interpolating solution between two vacua :

$$\frac{g_{m\bar{l}}}{(g_{m\bar{m}}g_{\bar{l}l})^{1/2}} \approx e^{i\alpha} (4\pi z_{ml})^{-1/2} \exp(-2z_{ml}), \quad (2.90)$$

where the soliton action for the soliton interpolating between vacua m and l is

$$z_{ml} = |W(Y_m) - W(Y_l)|, \quad (2.91)$$

and α is an arbitrary phase. This establishes the second boundary condition.

A Parallel

In this subsection, we make a simple parallel with far reaching consequences. We recall that the most relevant deformation tX of a compact model with Landau-Ginzburg potential X^{k_c}/k_c gives rise to a tt^* equation which is an affine \hat{A}_{k_c-2} Toda equation [39]. This is also the case for the generalized tt^* equation for the family of non-compact models at level $k = k_c$ that we study. Moreover, the boundary condition in the ultraviolet of the compact model agrees with the initial condition for the non-compact model, because the respective metrics (2.76) are obtained from each other by a change of variables. Since the boundary conditions, the phase symmetries, as well as the differential equations governing the deformed metric $g_{m\bar{n}}(t)$ agree, the deformed metrics themselves must coincide.

A simple confirmation of this idea is found by observing, as in section 2.1, that the spectrum of Ramond sector charges in the spectrum for the non-compact model at level k is $\{-1/2 + 1/k, -1/2 + 2/k, \dots, 1/2 - 1/k\}$. This agrees on the nose with the spectrum of charges of a compact model with central charge $c^c = 3 - 6/k_c$ at the same value of the level $k_c = k$. Thus, the Ward identities imply that the behavior of the metric near the conformal point is also the same, as it must be due to the more powerful reasoning above. The metric in the infrared must then also match, when looked at from the right perspective. We'll see how this works out in practice in a specific example below.

A side remark is that in [39], the central charge of the parent conformal field theory is read off from the behavior of the metric, e.g. near the conformal point. We claim now that the metric is pertinent to models with differing central charges. These statements are reconciled by the fact that in [39] a direct relation between the maximal Ramond-Ramond R-charge and the central charge was exploited, which is only valid in a compact model. That link is severed in the non-compact model under consideration, once more because the $SL(2, \mathbb{C})$ invariant vacuum is not part of the normalizable state space. If one exploits the different connection between the maximal Ramond-Ramond R-charge, the gap between R-charges, and the central charge in the non-compact model (namely, $q_{max} + 2/k = c/6$), one recovers a consistent picture. We will illustrate this generic reasoning with the example of the non-compact model at level $k = 3$, where we can be very explicit.

Example : Level $k = 3$

In this subsection, we explicitly determine the metric for a family of superpotentials $W_t^{(3)}$ at level $k = 3$

$$W_t^{(3)} = \frac{Y^{-3}}{3} - tY^{-1}. \quad (2.92)$$

The classical vacua of the theory are at $Y^{-1} = 0, +\sqrt{t}$ and $-\sqrt{t}$. As argued previously, we will consider the vacua at 0 to be irrelevant (namely, at an infinite distance in field space, or corresponding to not strictly normalizable states), and concentrate on the vacua at $Y^{-1} = \pm\sqrt{t}$. The spectrum of Ramond-Ramond sector normalizable ground state charges is $\{-1/6, +1/6\}$. The solution for the metric must parallel the solution of the compact model at level $k_c = 3$, discussed in [39]. We follow their discussion with small but important modifications.

Firstly, we rewrite the tt^* equation, still under the crucial assumption that it is valid for the almost normalizable operator proportional to Y^{-1} . Since the metric does not depend on the phase of t , we can define $x = |t|^2$ and $y(x) = \langle Y^{-3} | Y^{-3} \rangle$. The reality condition (2.84) tells us that also $y(x)^{-1} = \langle Y^{-2} | Y^{-2} \rangle$. Then the tt^* equation becomes the Painlevé III equation [39]

$$\frac{d}{dx} \left(x \frac{d}{dx} \log y \right) = y^2 - \frac{x}{y^2}. \quad (2.93)$$

Using the initial condition of the metric (2.76) at the conformal point, one sets

$$y^2(t=0) = \frac{\langle Y^{-3} | Y^{-3} \rangle}{\langle Y^{-2} | Y^{-2} \rangle} \Big|_{t=0} = 3^{\frac{2}{3}} \left(\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \right)^2. \quad (2.94)$$

These equations already fix the metric, but we can moreover predict the behavior of the solution at large deformation parameter t , where the vacua become massive and well-separated. The behavior at large t is governed by the existence of the two vacua at $Y^{-1} = \pm\sqrt{t}$, and the metric is diagonal in a basis associated to these vacua (as described around equations (2.89) and (2.90)) and determined by the Hessian

$$|H|_{\pm\sqrt{t}} = 2t^{5/2}. \quad (2.95)$$

We note that the Hessian is distinct in the compact and the non-compact case. We will return to discuss this point shortly. The corrections (2.90) to the metric (2.89) are determined by the action of the soliton interpolating between these two vacua

$$z = |W(\sqrt{t}) - W(-\sqrt{t})| = 4/3|t|^{3/2} = 4/3x^{3/4}. \quad (2.96)$$

We still need to link the operators that create the separated vacua with the operators Y^{-2} and Y^{-3} that create the vacua with fixed R-charge in the conformal point. To that end, we first define the operators l_{\pm} that take value one on the $\pm\sqrt{t}$ quantum vacua and zero otherwise. As a consequence of this definition, we must have the operator relations

$$\begin{aligned} Y^{-2} &= t(l_+ + l_-) \\ Y^{-3} &= t^{3/2}(l_+ - l_-), \end{aligned} \quad (2.97)$$

valid at large t . These operator relations also differ between the non-compact and the compact model. See [39].

With these definitions for the operators l_{\pm} that create the separated vacua at large t , we have the leading metric behaviors

$$\begin{aligned}\langle l_{\pm}|l_{\pm}\rangle &= \frac{1}{2|t|^{5/2}} + \dots \\ \langle l_{\pm}|l_{\mp}\rangle &= \frac{\beta}{2|t|^{5/2}} z^{-1/2} e^{-2z} + \dots,\end{aligned}\tag{2.98}$$

where β is a numerical coefficient. For the ratio of the norms then, we find the large t asymptotics

$$\frac{\langle Y^{-3}|Y^{-3}\rangle}{\langle Y^{-2}|Y^{-2}\rangle} = t(1 - 2\beta z^{-1/2} e^{-2z} + \dots).\tag{2.99}$$

Crucially, we note that the different asymptotics for the individual norms of the states in the non-compact and the compact model drop out in this ratio. Secondly, the different behavior (2.98) of the metric near the individual large t vacuum states is due to the different link (2.97) between the operators l_{\pm} and the conformal field theory operators.

Finally, we recall that the regular solution to the Painlevé III differential equation (2.93) which satisfies the boundary condition (2.94) indeed has the desired asymptotics (2.99) [39]. Using the results in [39], one can compute the value of the numerical constant $\beta = -1/(2\sqrt{\pi})$. In short, the metric is identical to the metric in a compact model, as argued in more generality in subsection 2.4.2.

Chapitre 3

*AdS*₃/Liouville Duality

In this chapter, we will define a three-dimensional gravity theory to be the dual of Liouville theory. We will start from the bosonic case and then turn to the supersymmetric case. We will show how by picking a specific boundary condition the supergravity action reduces to the action of the topologically twisted $N = 2$ super-Liouville theory. We further determine the classical configurations that correspond to the chiral primary operators (i.e. the spectrum of the topological theory).

The discussion in this chapter on the bosonic case and appendices C and D comes from my paper [54] written with N. Toumbas and J. Troost with a slight reorganization. Discussion on the supersymmetric case is extracted from the paper [55] I coauthored with J. Troost.

3.1 Conformal Field Theory and Metrics

In the Euclidean, the bulk solution to Einstein's equation with negative cosmological constant is determined by the boundary data [5]. We consider various boundary set-ups, with operator insertions in different topologies. In the case of the *AdS*₃/*CFT*₂ correspondence, the metric is determined by the expectation value of the energy-momentum tensor in the boundary conformal field theory (in the presence of operator insertions). The expectation value of the energy-momentum tensor of a two-dimensional conformal field theory is heavily constrained, and we analyze consequences of these generic properties for the bulk metric in this section. Most of these properties are implicitly or explicitly known – we present an elementary review.

3.1.1 The Energy-momentum Tensor and the Metric

The Fefferman-Graham asymptotic expansion of the solution to Einstein gravity with negative cosmological constant and given boundary data terminates quickly in three dimensions [56]. The technique used to determine it is the integration of the Chern-Simons equations of motion [56] with Brown-Henneaux boundary conditions [11]. The resulting metric is :

$$ds^2 = l^2 d\rho^2 + 4Gl(-Tdz^2 - \bar{T}d\bar{z}^2) + (l^2 e^{2\rho} + 16G^2 T\bar{T}e^{-2\rho})dzd\bar{z}, \quad (3.1)$$

for a planar or spherical boundary parameterized by the coordinates z and \bar{z} . The boundary data are captured by the expectation values $T = T_{zz}(z)$ and $\bar{T} = T_{\bar{z}\bar{z}}(\bar{z})$ of the energy-momentum tensor components of the boundary conformal field theory. Under conformal transformations, the boundary coordinates transform in a standard way to leading order in the radial expansion. These standard transformations are accompanied by subleading corrections as well as a change in the radial coordinate, to preserve the form of the metric (3.1) [56]. Our goal in this section is to uncover generic properties of the metric (3.1) using the properties of the energy-momentum tensor components.

3.1.2 Energy-momentum Tensor Expectation Value

The energy-momentum tensor component $T(z)$ is meromorphic. When inserted in a two-dimensional Euclidean conformal field theory correlation function, it gives rise to meromorphic functions of the coordinate z of the point of insertion. We consider a planar boundary, which we compactify to a two-sphere, and assume the presence of n primary operator insertions. In order to determine the resulting bulk metric, we must compute the value of the energy-momentum tensor when inserted in the corresponding n -point function of the conformal field theory. The value of the energy-momentum tensor component T (and likewise of \bar{T}) is considerably constrained by the property of meromorphy. Its double and single poles are known, given the conformal weight of the primary operator insertions. It is also restricted to behave as z^{-4} at most for large z . This is for the point at infinity to be regular. Let us enumerate results for a low number of insertions n .

Firstly, the one-point function is zero because of conformal invariance. For the two-point function, with insertions at z_1 and z_2 , we expect to be able to expand the energy-momentum tensor expectation value in terms of a holomorphic function, and at most double pole singularities at z_1 and z_2 . Moreover, the residues of the double poles are prescribed. Imposing that the energy-momentum tensor behaves as z^{-4} for large z fixes its value uniquely to :

$$\frac{\langle T(z)O_1O_2 \rangle}{\langle O_1O_2 \rangle} \equiv T^{(2)}(z) = \frac{h}{(z-z_1)^2} + \frac{h}{(z-z_2)^2} - \frac{2h}{(z-z_1)(z-z_2)}. \quad (3.2)$$

We had to set $h_1 = h_2$ to obtain a non-zero two-point function. It is easy to check that also the subleading single pole behaviour is as expected for a conformal primary. There is an interesting special case, when one of the insertions is at infinity, and the other insertion is at zero :¹

$$T^{0,\infty}(z) = \frac{h}{z^2}. \quad (3.3)$$

Next, we record the expectation value of the energy-momentum tensor component $T(z)$ in the presence of three primary conformal field theory operators with conformal dimensions h_i :

$$T^{(3)}(z) = h_1\left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right)\left(\frac{1}{z-z_1} - \frac{1}{z-z_3}\right) + \text{cyclic}. \quad (3.4)$$

Higher point functions are also heavily constrained.

1. Regularity at infinity is now violated because of the insertion.

3.1.3 Energy-momentum Tensor on the Cylinder

In the following we also make use of a boundary manifold which is a cylinder. In two dimensions, it can be obtained by the conformal transformation $z = e^{-iw}$ from the plane. It is important that the conformal transformation maps the points $z = 0$ and $z = \infty$ to the open ends of the infinite cylinder. The expectation value of the energy-momentum tensor on the cylinder is determined from the value on the plane through the quasi-primary transformation property of the energy-momentum tensor :

$$(\partial_z w)^2 T'(w) = T(z) - \frac{c}{12} \{w, z\}, \quad (3.5)$$

where the Schwarzian $\{\cdot, \cdot\}$ for the map $z = e^{-iw}$ evaluates to :

$$\{w, z\} = \frac{1}{2z^2}. \quad (3.6)$$

Thus, the energy-momentum tensor component T_c on the cylinder equals

$$T_c(w) = -z^2 T(z) + \frac{c}{24}. \quad (3.7)$$

Using this map, we straightforwardly compute the expectation values of the energy-momentum tensor on the cylinder for the configurations discussed previously :

$$T_c^{0,\infty}(w) = -h + \frac{c}{24}, \quad (3.8)$$

$$T_c^{(2)}(w) = \frac{h}{4} \frac{\sin^2 \frac{w_1 - w_2}{2}}{\sin^2 \frac{w - w_1}{2} \sin^2 \frac{w - w_2}{2}} + \frac{c}{24}, \quad (3.9)$$

$$T_c^{(3)}(w) = h_1 \frac{\sin \frac{w_2 - w_1}{2}}{2 \sin \frac{w - w_1}{2} \sin \frac{w - w_2}{2}} \frac{\sin \frac{w_3 - w_1}{2}}{2 \sin \frac{w - w_1}{2} \sin \frac{w - w_3}{2}} + \text{cyclic} + \frac{c}{24}. \quad (3.10)$$

Thus, we have computed examples of energy-momentum tensor expectation values that universally arise in two-dimensional conformal field theories. We commence the study of the corresponding metrics in the next section, and continue the analysis in Appendix C.

3.2 Universal Metrics

In the previous section, we computed universal expectation values of the energy-momentum tensor. Through the elliptic boundary value problem of three-dimensional Euclidean Einstein gravity with negative cosmological constant [5], these translate into universal metrics [56] which we study in the present section.

3.2.1 The BTZ Black Hole Metric

We recorded the constant energy-momentum tensor expectation value $T_c^{0,\infty}$ (3.8) on the cylinder corresponding to a primary state of conformal dimension h , propagating from the infinite past to the infinite future. The resulting metric is straightforwardly

determined. If we take equal left and right conformal dimensions, then the metric can be transformed to the Euclidean BTZ black hole metric with zero spin :

$$ds_{BTZ}^2 = l^2 \left(\frac{r^2}{l^2} - 8GM \right) dt^2 + \left(\frac{r^2}{l^2} - 8GM \right)^{-1} dr^2 + r^2 d\phi^2. \quad (3.11)$$

The equivalence is demonstrated through the identifications and coordinate transformations [56] :

$$\begin{aligned} T_c^{0,\infty} = \bar{T}_c^{0,\infty} &= -\frac{Ml}{2} \\ w &= \phi + it \\ r_+^2 &= 8GMl^2 = 4l^2 e^{2\rho_0} \\ r^2 &= 8GMl^2 \cosh^2(\rho - \rho_0). \end{aligned} \quad (3.12)$$

In the classical theory we need to suppose that the conformal dimension of the operator on the plane satisfies the bound $h \geq c/24$. The Euclidean metric (3.11) can be analytically continued to the Lorentzian BTZ black hole. This whole analysis is easily generalized to the case of a BTZ black hole with non-zero angular momentum [56].

The Horizon

We can also study the metric if we start out with a planar boundary and the energy-momentum tensor $T^{0,\infty}$ (3.3). A useful and generic tool of analysis is to determine the location of the event horizon, where the metric (3.1) degenerates. It satisfies the equation :

$$e^{2\rho_{hor}} = 4Gl^{-1}(T\bar{T})^{\frac{1}{2}}. \quad (3.13)$$

Thus, for instance, for the planar energy-momentum tensor $T^{0,\infty}$, the horizon radius is

$$e^{2\rho_{hor}} = 4Gl^{-1}h \frac{1}{|z|^2}, \quad (3.14)$$

as a function of the planar coordinate z . The first thing to note is that the horizon now reaches radial infinity (at the boundary insertion). If we put an infrared cut-off R on the coordinate radius e^ρ of the bulk AdS_3 space, then we cut out a little hole from the surface of the event horizon, parameterized by the angle of z , where $R^2 = 4Gl^{-1}h|z|^{-2}$. There is an analogous singularity at $z = \infty$. Of course, the excisions correspond to the incoming and outgoing state that is present in the cylindrical set-up. In other words, the horizon surface is a sphere with two marked points, where the horizon reaches the boundary, or a sphere with two small disks cut out, when we introduce an infrared cut-off in the bulk. We recuperate the standard cylinder topology of the BTZ horizon in this manner.

3.2.2 Properties of the Metrics

It should be clear that this reasoning is very generic. The singular behaviour of the operator product expansion of the energy-momentum tensor with a (conformal primary) operator insertion will guarantee that near a local insertion with sufficiently

high conformal dimension, the metric can be locally transformed (through the exponential map) into a BTZ black hole metric with mass $Ml = 2h - c/12$ (where for simplicity only we assume zero spin). This is independent of the number and the precise nature of the insertions. Thus, for example, in the cases of the energy-momentum tensors $T^{(2)}$ and $T^{(3)}$, we can state that locally, near one of the insertions, the metric looks like a BTZ black hole metric with (left) conformal dimension h_i . The horizon looks like a sphere with two or three peaks, reaching the AdS_3 boundary.

On the cylinder, we locally find the same picture. Care should be taken because of the fact that the unit operator gives rise to a constant energy-momentum tensor T_c at infinity. The non-zero horizon radius for the cylindrical horizon should not be confused with a black hole insertion. Otherwise, the cylindrical picture is the same as the planar picture – the reasoning can be applied locally. Note that the singularities in the metric only depend on the planar or cylindrical coordinates on a radial slice. Thus, they propagate radially inward towards the center on straight lines.

Let us make a few more generic remarks on the universal metrics. We note that on the horizon surface defined by equation (3.13), the determinant of the metric has a double zero. Thus, when we cross the surface, the nature of the metric remains the same, in the sense that we are still outside the horizon. One can see this in more detail in e.g. the BTZ metric, in which case the radial coordinate ρ touches the horizon at ρ_0 , and then returns to infinity.

In appendix C, we start the exploration of the metrics associated to the energy-momentum tensors $T^{(2)}$ and $T^{(3)}$ after analytic continuation to the Lorentzian cylinder. We believe these Euclidean and Lorentzian metrics are interesting, and since they are universal by meromorphy, they deserve further analysis. As background for the following section, it is more useful to us to turn to a different characterization of the same class of metrics, in terms of monodromies.

3.2.3 On Monodromies

It is known how the above picture is reflected in the Chern-Simons formulation of three-dimensional gravity. The Banados metric (3.1) is associated to the (left) Chern-Simons connection [56] :

$$A = \begin{pmatrix} -\frac{1}{2}d\rho & -ie^\rho dw \\ -i\frac{4G\ell^2}{l}e^{-\rho}dw & \frac{1}{2}d\rho \end{pmatrix}. \quad (3.15)$$

This connection is the left combination of dreibein and spin connection (and there is an analogous right combination). The presence of BTZ sources in the metric is tracked by the monodromies of the connection around the singularity [57]. The simplest example is again the standard BTZ black hole, where the left connection reads

$$A = \begin{pmatrix} -\frac{1}{2}d\rho & -ie^\rho dw \\ 2iGM e^{-\rho}dw & \frac{1}{2}d\rho \end{pmatrix}. \quad (3.16)$$

We integrate it over the angular part of w , at the horizon radius $e^{\rho_0} = \sqrt{2GM}$. The matrix is then constant and gives rise to a monodromy trace

$$\text{tr } P \exp \int A = 2 \cosh 2\pi\sqrt{2GM}. \quad (3.17)$$

Our next example is slightly more involved. We compute the monodromy around a Euclidean insertion on the plane. The left-connection equals

$$A = \begin{pmatrix} -\frac{1}{2}d\rho & -ie^\rho dz \\ -i\frac{4G}{l}\frac{h}{z^2}e^{-\rho}dz & \frac{1}{2}d\rho \end{pmatrix}. \quad (3.18)$$

At fixed $e^{2\rho_{hor}} = \frac{4G}{l}h/|z|^2$, the connection reduces to

$$A = \begin{pmatrix} 0 & \sqrt{\frac{4G}{l}}he^{i\theta}d\theta \\ \sqrt{\frac{4G}{l}}he^{-i\theta}d\theta & 0 \end{pmatrix}. \quad (3.19)$$

We can gauge transform the connection to a constant matrix, and compute the resulting trace :

$$\text{tr } P \exp \int A = 2 \cosh 2\pi \sqrt{\frac{4G}{l}(h - c/24)}. \quad (3.20)$$

Thus we recognize Euclidean vertex operator insertions as BTZ geometries, locally. More generally, near a Euclidean insertion that gives rise to a double pole in the energy-momentum tensor component $T(z)$, the monodromy calculation and result reduces to the hyperbolic monodromy (3.20).

3.3 Liouville Quantum Gravity

In the previous sections, we discussed a few universal properties of the AdS_3/CFT_2 holographic duality. In this section, we commit to a very particular dual conformal field theory, namely Liouville theory. The boundary conformal field theory is then well studied. We take the holographic dictionary to *define* the bulk three-dimensional theory of quantum gravity, and call the resulting theory Liouville quantum gravity. In this section, we first recall properties of the unitary Liouville conformal field theory. We then provide a holographic gravitational interpretation of these basic properties, and critically discuss the ensuing theory of quantum gravity.

3.3.1 Properties of Liouville Theory

Liouville theory is a consistent unitary conformal field theory in two dimensions. See e.g. [58, 59] for an older and a more recent review. The spectrum and the three-point functions entirely determine the theory. The consistency of the theory hinges on crossing symmetry of the four-point function on the sphere and the covariant modularity of the torus one-point function. Both have been understood [60, 61]. Therefore, if we define three-dimensional quantum gravity as the dual of Liouville theory, it will be manifestly consistent as well as unitary. The question arises which properties the resulting theory of quantum gravity exhibits when expressed as a three-dimensional bulk theory through holographic duality. To answer the question, we firstly review various properties of Liouville theory and then translate them using a holographic dictionary.

The Liouville action S_L is

$$S_L = \frac{1}{4\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + \mu e^{2b\phi} + QR^{(2)}\phi \right), \quad (3.21)$$

where the parameter b is related to the bare central charge c of the theory by

$$c = 1 + 6Q^2 = 1 + (b + b^{-1})^2, \quad (3.22)$$

which ensures that the exponential interaction term in the action is marginal. The conformal dimensions of the Liouville vertex operator $e^{2\alpha\phi}$ is given by the formula

$$h = \alpha(Q - \alpha), \quad (3.23)$$

and the spectrum of momenta of the unitary Liouville theory is determined by the momenta $\alpha \in Q/2 + i\mathbb{R}^+$. Thus, the continuous spectrum has a lower limit conformal dimension $h_{min} = Q^2/4 = (c - 1)/24$. The effective central charge relevant to the Cardy formula is then

$$c_{eff} = c - 24h_{min} = 1. \quad (3.24)$$

The effective central charge reflects the fact that we have a single boson, and this is visible in the ultraviolet growth of the number of states as a function of their conformal dimension [14].

The Liouville action (3.21) in the presence of operator insertions needs to be regularized. In the semi-classical theory, the semi-classical Liouville field $\varphi = 2b\phi$ satisfies the boundary conditions

$$\begin{aligned} \varphi &= -2\eta_i \log |z - z_i|^2 + O(1) \quad \text{at } z \rightarrow z_i \\ \varphi &= -2 \log |z|^2 + O(1) \quad \text{at } z \rightarrow \infty, \end{aligned} \quad (3.25)$$

in the presence of operator insertions with $\alpha_i \approx \eta_i/b$ and a linear dilaton charge at infinity. Following e.g. [62], we cut out small disks D_i of radius ϵ_i around each insertion, as well as around infinity, and define the regularized Liouville action :

$$S_L^{reg} = S_L + \varphi_\infty + 2 \log r_\infty - \sum_i (\eta_i \varphi_i + 2\eta_i^2 \log \epsilon_i), \quad (3.26)$$

where we defined the boundary contributions

$$\varphi_\infty = \frac{1}{2\pi r_\infty} \int_{\partial D_\infty} \varphi dl, \quad (3.27)$$

and

$$\varphi_i = \frac{1}{2\pi \epsilon_i} \int_{\partial D_i} \varphi dl, \quad (3.28)$$

which cancel the divergences and insure that the boundary conditions are consistent.

The Liouville three-point functions can be guessed by computing their pole structure and their semi-classical behaviour [62, 63]. The three-point functions can also be derived by requiring that null vectors in degenerate representations (reached by analytic continuation from the unitary spectrum) decouple [64]. Furthermore, the three-point functions can be numerically bootstrapped [65]. Explicitly, they are given by :

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{Y_0 Y(2\alpha_1) Y(2\alpha_2) Y(2\alpha_3)}{Y(\alpha_1 + \alpha_2 + \alpha_3 - Q) Y(\alpha_1 + \alpha_2 - \alpha_3) Y(\alpha_2 + \alpha_3 - \alpha_1) Y(\alpha_3 + \alpha_1 - \alpha_2)} \quad (3.29)$$

where the definition of the special function Y is

$$\log Y(x) = \int_0^\infty \frac{dt}{t} \left(\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2} - x\right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right), \quad (3.30)$$

and we need :

$$Y_0 = \partial_x Y(x)|_{x=0}. \quad (3.31)$$

The reflection amplitude codes the fact that an operator in Liouville theory at momentum $\alpha = Q/2 + ip$, where p is a positive real number, creates the same state as the operator at momentum $Q/2 - ip$. In the mini-superspace approximation to Liouville theory, a normalizable state consists of a wave bouncing off an exponential potential, and returning with equal and opposite momentum. We have an equation of the type

$$|-p\rangle = R(-p)|p\rangle, \quad (3.32)$$

where the reflection amplitude R is related to the three-point function by taking a limit in one of the momenta. See e.g. [59] for the detailed formulas. These data fix the Liouville conformal field theory uniquely, and it can be proven to satisfy crossing symmetry as well as consistency of the torus one-point function. Thus, we have a two-dimensional conformal field theory that is consistent and unitary, and that to a large extent is unique, as seen both from the null vector decoupling argument [64], and from the numerical bootstrap approach [65].

3.3.2 The Holographic Interpretation

In this subsection, we holographically interpret properties of Liouville theory. We first discuss the big picture of three-dimensional Liouville gravity, and subsequent subsections fill in more details.

The Big Picture

The Liouville conformal field theory has peculiar properties that translate into unexpected properties of its proposed three-dimensional quantum gravity dual. Firstly, there is no normalizable $SL(2, \mathbb{C})$ invariant vacuum state in the Euclidean conformal field theory. This translates holographically into the absence of an $so(3, 1)$ invariant quantum gravity ground state in the theory. In other words, the path integral of our quantum theory of gravity is such that the Euclidean AdS_3 space-time is not normalizable. The spectrum of the theory consists of primaries with conformal dimensions which form a continuous spectrum starting at a gap $(c-1)/24$. These primaries create hyperbolic monodromies at their insertion, and translate holographically to BTZ space-times. Moreover, the descendants, generated by the action of the Fourier modes of the boundary energy-momentum tensor, can be interpreted as boundary gravitons localized at the boundary of the three-dimensional space-time.

Thus the spectrum of the boundary conformal field theory has the following bulk interpretation. The only excitations that we have in our three-dimensional theory of quantum gravity are primary BTZ black holes, dressed with boundary gravitons. The theory has no propagating gravitons or other particles. A naked black hole (i.e. with

no boundary gravitons) corresponds to a single quantum state. It has zero entropy. The boundary gravitons give entropy to a state at given high mass $Ml = L_0 + \bar{L}_0$ as dictated by the Cardy formula, at effective central charge $c_{eff} = 1$.^{2 3} The theory we define only has primary states of spin zero. Thus, there are no primary spinning black holes.

In our quantum theory of gravity, it is hard to detect the horizon of a black hole. That is because there are no propagating degrees of freedom. The only degrees of freedom are the primary states corresponding to the black holes themselves, as well as the boundary degrees of freedom that live at spatial infinity. Thus, it is impossible to derive the traditional macroscopic picture of the thermodynamic properties of the BTZ black holes [66,67] within our limited theory. This may be just as well, since we have just established that the microscopic entropy of the black holes we consider is zero. On the other hand, coupling to other degrees of freedom would re-establish a traditional thermodynamic picture, but it is challenging to do this consistently with the postulates of holographic duality and quantum gravity.^{4 5 6}

The Liouville theory has one parameter which is the bare central charge c . A perturbation expansion exists in one over the bare central charge c , the $1/c$ expansion. In the dual gravitational theory, the expansion is in G_N/l where G_N is the three-dimensional Newton constant and l is the radius of curvature of the locally AdS_3 space-time.

There is a detail that we skipped over in our explanation. The AdS_3 space-time corresponds to a state with (planar) conformal dimension h equal to zero both classically and quantum mechanically. It therefore has weight $-c/24$ with respect to the cylindrical Hamiltonian. The minimal conformal dimension in the spectrum of Liouville theory is $Q^2/4 = (c-1)/24$. In the cylinder frame, this becomes a minimal mass of $-1/24$. This is negative. Starting at (planar) conformal dimension $(c-1)/24$, we already identify the resulting states with (quantum) BTZ geometries because they have hyperbolic monodromy quantum mechanically. Note that this quantum correction to the classical picture is a one-loop effect. We will comment further on one-loop effects in subsection 3.3.4.

2. In no sense do these degrees of freedom account fully for the traditional semi-classical Bekenstein-Hawking entropy of the three-dimensional black holes.

3. We have divided by a space-time volume factor due to the continuous spectrum in making these statements.

4. Relatedly, in our theory, since there is no way we can detect the horizon of a black hole, there is little reason to call the black hole geometry black. We could therefore always refer to the BTZ black hole as the BTZ geometry. We will refrain from this heavy change of nomenclature, and keep referring to the BTZ geometry as being a black hole. The reader should keep the essential conceptual caveat in mind.

5. At least one way to consistently couple matter is provided by AdS_3 string theory. The resulting quantum theory of gravity has a multitude of extra degrees of freedom – enough to measure as well as account for the black hole entropy.

6. Note that in string theory too, it is impossible to implement the standard low energy operation of adding a single scalar field. Our theory of quantum gravity shares this feature of being rigid.

The Action

Since the holographic relation between Liouville theory and the bulk gravitational theory is crucial to us, we carefully review the map between the boundary and the bulk action [12]. We do modify the discussion of the zero modes of the bulk theory. In particular, we gauge a symmetry in order to guarantee that the left and right zero modes of the reduced, chiral Wess-Zumino-Witten models on the boundary are glued together, such that we reproduce the diagonal spectrum of the boundary Liouville theory. The details of the procedure are as follows.

The Einstein-Hilbert action in three dimensions with a negative cosmological constant $\Lambda = -l^{-2}$ can be written as the difference of two $sl(2, \mathbb{R})$ Chern-Simons actions [68, 69],

$$S_{EH}[A, \tilde{A}] = S_{CS}[A] - S_{CS}[\tilde{A}], \quad (3.33)$$

where the connection one-forms A and \tilde{A} are related to the dreibein e_μ^a and the (dualized) spin connection ω_μ^a through the formulas $A_\mu^a = \omega_\mu^a + \frac{1}{l}e_\mu^a$ and $\tilde{A}_\mu^a = \omega_\mu^a - \frac{1}{l}e_\mu^a$. The Chern-Simons action can be written in polar t, r, θ components as

$$S_{CS}[A] = \frac{l}{16\pi G_N} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad (3.34)$$

$$= \frac{l}{16\pi G_N} \int_M dt dr d\theta \text{tr}(A_\theta \dot{A}_r - A_r \dot{A}_\theta + 2A_0 F_{r\theta}). \quad (3.35)$$

The Brown-Henneaux boundary conditions at large radius r on the fields A and \tilde{A} are [11]

$$A \sim \begin{pmatrix} \frac{dr}{2r} & O(\frac{1}{r}) \\ \frac{r}{l} dx^+ & -\frac{dr}{2r} \end{pmatrix}, \quad \tilde{A} \sim \begin{pmatrix} -\frac{dr}{2r} & \frac{r}{l} dx^- \\ O(\frac{1}{r}) & \frac{dr}{2r} \end{pmatrix}, \quad (3.36)$$

where $x^\pm = t \pm \theta$. This implies that $A_- = \tilde{A}_+ = 0$ on the boundary. If one computes the variation of the action δS_{EH} when the equation of motion and this boundary condition are satisfied one obtains a surface term $\frac{l}{16\pi G_N} \delta \int_{\Sigma_2} dt d\theta \text{tr}(A_\theta^2 + \tilde{A}_\theta^2)$, where Σ_2 is the asymptotic surface at $r \rightarrow \infty$. In order to make the action and the boundary condition compatible, we add a surface term to the Einstein-Hilbert action. The action becomes

$$S[A, \tilde{A}] = S_{CS}[A] - \frac{l}{16\pi G_N} \int_{\Sigma_2} dt d\theta \text{tr}(A_\theta^2) - S_{CS}[\tilde{A}] - \frac{l}{16\pi G_N} \int_{\Sigma_2} dt d\theta \text{tr}(\tilde{A}_\theta^2). \quad (3.37)$$

The field components A_0 and \tilde{A}_0 are Lagrange multipliers that implement the constraints $F_{r\theta} = \tilde{F}_{r\theta} = 0$. Solving these constraints one gets

$$\begin{aligned} A_i &= G_1^{-1} \partial_i G_1, \\ \tilde{A} &= G_2^{-1} \partial_i G_2, \end{aligned} \quad (3.38)$$

for $i = r, \theta$, where $G_{1,2}$ are elements of $SL(2, \mathbb{R})$.⁷ On the boundary, the fields $G_{1,2}$ asymptotically behave as

$$G_1 \sim g_1(t, \theta) \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}, \quad G_2 \sim g_2(t, \theta) \begin{pmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{pmatrix}, \quad (3.39)$$

7. This is the solution only when one does not consider holonomies. For the case with non-zero holonomies, we refer to the Appendix of [13].

such that the Brown-Henneaux boundary condition (3.36) is satisfied. Exploiting this parameterization, the action (3.37) can be written as the difference of two chiral WZW actions [12]

$$S[A, \tilde{A}] = \frac{l}{16\pi G_N} (S_{WZW}^R[g_1] - S_{WZW}^L[g_2]). \quad (3.40)$$

Given the relation between the Einstein-Hilbert action and Chern-Simons theory on the one hand, and Chern-Simons theory and chiral Wess-Zumino-Witten models on the other hand, this is as expected. The two chiral WZW actions are given by

$$\begin{aligned} S_{WZW}^R &= \int_{\Sigma_2} dt d\theta \operatorname{tr} [g_1^{-1} \partial_t g_1 g_1^{-1} \partial_\theta g_1 - (g_1^{-1} \partial_\theta g_1)^2] + \Gamma[g_1] \\ S_{WZW}^L &= \int_{\Sigma_2} dt d\theta \operatorname{tr} [g_2^{-1} \partial_t g_2 g_2^{-1} \partial_\theta g_2 + (g_2^{-1} \partial_\theta g_2)^2] + \Gamma[g_2], \end{aligned} \quad (3.41)$$

where $\Gamma[g]$ is the three-dimensional part of the WZW action. Note that the chiral WZW actions are invariant under the transformations

$$\begin{aligned} g_1 &\rightarrow h_1(t) g_1 f_1(x^+) \\ g_2 &\rightarrow h_2(t) g_2 f_2(x^-), \end{aligned} \quad (3.42)$$

where $h_{1,2}, f_{1,2} \in SL(2, \mathbb{R})$.

We gauge a subgroup of the above symmetry group. This subgroup consists of transformations

$$\begin{aligned} g_1 &\rightarrow h(t) g_1 \\ g_2 &\rightarrow h(t) g_2, \end{aligned} \quad (3.43)$$

and will guarantee that the left and right zero modes of g_1 and g_2 are glued together. Since these transformations are already symmetries of the action, gauging the symmetry comes down to restricting the integral over configuration space – we only integrate over the zero modes a single time. This can be implemented by defining the fields

$$g = g_1^{-1} g_2, \quad u = -g_2^{-1} \partial_\theta g_1 g_1^{-1} g_2 - g_2^{-1} \partial_\theta g_2, \quad (3.44)$$

which are invariant under the gauged subgroup. In terms of these fields, we find that the action can be written as

$$S[g, u] = \frac{l}{16\pi G_N} \left[\int_{\Sigma_2} dt d\theta \operatorname{tr} (u g^{-1} \partial_t g - \frac{1}{2} u^2 - \frac{1}{2} (g^{-1} \partial_\theta g)^2) - \Gamma(g) \right]. \quad (3.45)$$

The measures are related by

$$Dg Du = \frac{1}{V} Dg_1 Dg_2, \quad (3.46)$$

where V is volume of the set $\{g(t) \in SL(2, \mathbb{R})\}$. Performing the functional integral over the field u , we get the non-chiral WZW action

$$S[g] = \frac{l}{16\pi G_N} \left[\int_{\Sigma_2} dt d\theta \operatorname{tr} (2g^{-1} \partial_+ g g^{-1} \partial_- g) - \Gamma(g) \right], \quad (3.47)$$

where $\partial_{\pm} = \frac{1}{2}(\partial_t \pm \partial_{\theta})$. Crucially, we have eliminated a zero mode.

The next step amounts to Drinfeld-Sokolov or Hamiltonian reduction, induced by the boundary conditions. Firstly, we apply Gauss decomposition to write the remaining field g as

$$g = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(\frac{1}{2}\varphi) & 0 \\ 0 & \exp(-\frac{1}{2}\varphi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}. \quad (3.48)$$

The action will then become

$$S[g] = \frac{l}{8\pi G_N} \int dt d\theta \left[\frac{1}{2} \partial_+ \varphi \partial_- \varphi + 2 \partial_- X \partial_+ Y \exp(-\varphi) \right]. \quad (3.49)$$

Consider now the action as defined on a cylinder $[t_1, t_2] \times S^1$ of finite height [12]. The boundary conditions (3.36) will impose the following constraints :

$$\partial_- X \exp(-\varphi) = 1, \quad \partial_+ Y \exp(-\varphi) = -1. \quad (3.50)$$

In order to make the variation of the action zero when the equation of motion is satisfied, and in the presence of such constraint on the boundary, one improves the action by adding a term. The action then reads

$$S_{\text{improved}}[g] = S[g] - \frac{l}{8\pi G_N} \oint d\theta (X \partial_+ Y + Y \partial_- X) \exp(-\varphi) \Big|_{t_1}^{t_2}. \quad (3.51)$$

Plugging in the constraints, the improved action becomes a Liouville action

$$S[\varphi] = \frac{l}{8\pi G_N} \int dt d\theta \left[\frac{1}{2} \partial_+ \varphi \partial_- \varphi + 2 \exp(\varphi) \right]. \quad (3.52)$$

Alternatively, one can determine the final action as follows. One derives the equation of motion for the field φ from the action (3.49), and finds

$$\partial_+ \partial_- \varphi = -2 \partial_- X \partial_+ Y \exp(-\varphi). \quad (3.53)$$

Eliminating the fields X and Y using the constraint (3.50), one obtains

$$\partial_+ \partial_- \varphi = 2 \exp(\varphi), \quad (3.54)$$

which is the same as the equation of motion one gets starting from (3.52). Therefore the boundary action is the Liouville action (3.52).

The Reflection Amplitude

Each Liouville state has a gravitational dual state. The Liouville state satisfies a reflection property (3.32). We can ask whether there is a bulk analogue of the reflection property. Firstly, we should note that the bulk geometry depends only on the boundary energy-momentum tensor, which is invariant under reflection. Thus, the bulk geometry is invariant under reflection. Thus, there is a pressing question as to how evaluating the bulk action can lead to a different result for on the one hand momentum p and on the other hand momentum $-p$.

As demonstrated in section above, the bulk action equals the boundary Liouville action. A subtlety arises when we allow primary operator insertions. This guarantees that the bulk and boundary action have additional divergences. Thus, we must regularize the bulk (and boundary) action in the presence of operator insertions. It would be interesting to analyze bulk regularization in the presence of operator insertions in generality. We concentrate on the following phenomenon.

When we introduce the microscopic picture for the boundary energy-momentum tensor provided by Liouville theory, there are natural candidate primary operator insertions, defined in terms of the Liouville field. In semi-classical Liouville theory, in order to evaluate the action, we regularize it. The regularization depends on the Liouville field, which in turn does depend on the choice of the sign of the momentum. (See e.g. equation (3.26).) Thus, the semi-classical (regularized Liouville, bulk, and boundary) action changes under a change of momentum. From the bulk perspective, the regularization is in terms of a field which is a non-local functional of the energy-momentum tensor. The Liouville regulator does have neat properties. It guarantees that independently of the choice of momentum (which is, roughly, a square root of the conformal dimension), the semi-classical actions will be related by a phase factor. Thus, they can be associated to the same (albeit reflected) state (as in equation (3.32)).⁸

The semi-classical actions for the three-point correlator (see also subsection 3.3.3), for instance, are related by a factor which is the semi-classical limit $R_{s.c.}$ of the reflection amplitude [62] :

$$e^{-\frac{1}{b^2}S_{cl}(p_1,p_2,p_3)} \approx R_{s.c.}(-p_1)e^{-\frac{1}{b^2}S_{cl}(-p_1,p_2,p_3)}. \quad (3.55)$$

Finally, quantum Liouville theory and the quantum reflection amplitude R extend the semi-classical discussion to a theory that is exact in G_N/l .

The States

The Liouville partition function is

$$Z = V \int \frac{dp}{2\pi} \frac{e^{-\pi\tau_2 p^2}}{|\eta(q)|^2} = \frac{V}{2\pi\sqrt{\tau_2}|\eta(q)|^2}, \quad (3.56)$$

where $q = e^{2\pi i\tau}$, τ is the modulus of the rectangular torus and V is an infinite volume factor due to the Liouville zero mode. Conformal invariance implies that the partition function will depend on the product LT only where L is the size of the boundary circle and T is the temperature. We can express this product in terms of the parameter τ_2 . The partition function coincides with the partition function of one free boson. This matches the fact that the effective number of boundary degrees of freedom is of finite central charge, equal to one. The partition function is the simplest holomorphically factorized partition function which satisfies the requirement of reproducing the spectrum of boundary gravitons as well as matching the bare Brown-Henneaux central charge. The continuous spectrum of momenta matches geometries in the bulk space-time which have singularities cloaked by horizons.

8. See also [70] for a detailed defense of the statement that natural regulators for the bulk gravity action lead to equality with the boundary Liouville action, albeit in the differing context of non-normalizable point particle excitations.

With a finite number of effective degrees of freedom, we do not expect a phase transition. The partition function is a continuous function of the temperature (times the volume) τ_2 . In this model there is no Hawking-Page transition. In fact, the partition function is independent of the bare central charge, and there is no semi-classical limit, nor a summed approximation over saddles.

3.3.3 Black Hole Correlators

We described how Liouville quantum gravity only has black hole states dressed with boundary gravitons. Moreover, the theory enjoys the full conformal invariance of a two-dimensional conformal field theory. Beyond the known spectrum of conformal dimensions, a natural observable to compute is therefore the correlation function of three operators that create (primary) black holes states. The correlators of descendent black holes, namely black holes dressed with boundary gravitons, are generated from these by the symmetries of the theory. If we consider the correlation function of more than three black holes, those can be reconstructed by factorization.

The three-point function respects global conformal invariance and is therefore proportional to

$$\langle O_1 O_2 O_3 \rangle = \frac{C_{123}}{|z_1 - z_2|^{2h_1+2h_2-2h_3} |z_2 - z_3|^{2h_2+2h_3-2h_1} |z_3 - z_1|^{2h_3+2h_1-2h_2}}. \quad (3.57)$$

In Liouville quantum gravity, the proportionality constant C_{123} is given by the Liouville three-point function. The semi-classical three-point function is again determined by the regularized Liouville action, which we interpreted as a regularization of the bulk action in the presence of operator insertions. The $1/c = 2G_N/(3l)$ corrections arise from integrating over the Liouville field configurations, which we consider to be the microscopic degrees of freedom of the theory. We study these statements in more detail below.

The Semi-Classical Liouville Three-point Function

In this subsection, we remind the reader of the calculation of the semi-classical approximation to the Liouville three-point function [62]. The boundary conditions on the semi-classical Liouville field $\varphi = 2b\phi$ are

$$\begin{aligned} \varphi &= -2\eta_i \log |z - z_i|^2 + O(1) \quad \text{at } z \rightarrow z_i \\ \varphi &= -2 \log |z|^2 + O(1) \quad \text{at } z \rightarrow \infty, \end{aligned} \quad (3.58)$$

while the action is regularized as before. In the semiclassical limit where $b \rightarrow 0$, the three point function in the Liouville theory with operator insertions $\exp(\eta_i \varphi(z_i))$ is given by $\exp(-\frac{1}{b^2} S_{cl})$ where S_{cl} is the classical Liouville action evaluated on the classical solution to the Liouville equation. This classical solution is given by [62]

$$\varphi_{\eta_1, \eta_2, \eta_3}(z|z_1, z_2, z_3) = -2 \log[a_1 \psi_1(z) \psi_1(\bar{z}) + a_2 \psi_2(z) \psi_2(\bar{z})], \quad (3.59)$$

where

$$\begin{aligned} \psi_1 &= (z - z_1)^{\eta_1} (z - z_2)^{1-\eta_1-\eta_3} (z - z_3)^{\eta_3} \times \\ &\quad {}_2F_1(\eta_1 + \eta_3 - \eta_2, \eta_1 + \eta_2 + \eta_3 - 1, 2\eta_1, x) \end{aligned} \quad (3.60)$$

$$\begin{aligned} \psi_2 &= (z - z_1)^{1-\eta_1} (z - z_2)^{\eta_1+\eta_3-1} (z - z_3)^{1-\eta_3} \times \\ &\quad {}_2F_1(1 + \eta_2 - \eta_1 - \eta_3, 2 - \eta_1 - \eta_2 - \eta_3, 2 - 2\eta_1, x), \end{aligned} \quad (3.61)$$

with the variable x defined as

$$x = \frac{(z - z_1)z_{32}}{(z - z_2)z_{31}}. \quad (3.62)$$

The constants a_1 and a_2 are

$$\begin{aligned} a_1^2 &= \frac{\pi \mu b^2}{|z_{13}|^{4\eta_3+4\eta_1-2} |z_{12}|^{2-4\eta_2} |z_{23}|^{2-4\eta_1}} \times \\ &\quad \frac{\gamma(\eta_1 + \eta_2 + \eta_3 - 1) \gamma(\eta_1 + \eta_3 - \eta_2) \gamma(\eta_1 + \eta_2 - \eta_3)}{\gamma^2(2\eta_1) \gamma(\eta_2 + \eta_3 - \eta_1)} \end{aligned} \quad (3.63)$$

$$a_2 = -\frac{\pi \mu b^2}{|z_{13}|^2 (1 - 2\eta_1)^2 a_1}, \quad (3.64)$$

where the special function γ is a ratio of Γ functions,

$$\gamma(x) = \frac{\Gamma(1-x)}{\Gamma(x)}. \quad (3.65)$$

The conformal dimensions of the operators in the semiclassical limit are given by ($\alpha_i = \eta_i/b$)

$$h_i = \alpha_i(Q - \alpha_i) \approx \frac{\eta_i(1 - \eta_i)}{b^2}, \quad (3.66)$$

and the energy-momentum tensor can indeed be computed to be

$$T(z) = \sum_{i=1}^3 \frac{h_i}{(z - z_i)^2} - \left[\frac{h_1 + h_2 - h_3}{(z - z_1)(z - z_2)} + \text{cyclic permutations} \right]. \quad (3.67)$$

We verify this prediction of holomorphy explicitly in appendix D using the semiclassical solution (3.59).

In summary, through the relation between the Einstein-Hilbert action in the bulk and the Liouville theory on the boundary [12], reviewed and complemented in section 3.3.2, and the regularization of the Liouville action (3.26) which we take to be valid for Liouville quantum gravity as well, we equate the semi-classical approximation to the quantum Liouville three-point function [62] with the semi-classical approximation to the gravity action evaluated on the geometry (3.1) determined by the energy-momentum tensor (3.67). In this manner, Liouville theory propels itself as a proposal for a quantization of this classical theory of gravity.

We recall that this proposal satisfies a non-trivial consistency check, in that three-point functions of states with opposite momentum both associated to a given space-time energy-momentum tensor and metric are related through a phase factor (essentially given by the reflection amplitude).

3.3.4 One Loop Correction

The semi-classical expansion of Liouville theory can be continued to higher orders. In particular, at one loop one can compute the exact quantum conformal dimension of the Liouville vertex operators [71]. The details are as follows. By regularizing the one-loop determinant around the classical three-point configuration in an $SL(2, \mathbb{C})$ covariant manner, and by analyzing the dependence of the resulting one-loop determinant on the insertion points of the three Liouville operators, one finds the $O(b^2)$ correction to the semi-classical conformal dimension of the operator insertions and reproduces the exact conformal dimension [71] :

$$h_i = \alpha_i(Q - \alpha_i) = \frac{1}{b^2} \eta_i(1 - \eta_i) + \eta_i. \quad (3.68)$$

It is proven explicitly in [71] that there is no further correction at two loop order, and, from our knowledge of Liouville conformal field theory, we know the property holds to all orders. Thus, the conformal dimension is one loop exact.

The holographic interpretation of the result is that integrating over gravitons gives a one loop correction to the mass, which is one loop exact in Liouville quantum gravity. Note that here, we again identify the bulk gravitational measure with the Liouville measure.⁹

Finally, we note the following phenomenon. When we compute the minimal (real) conformal dimension using the leading order formula, we find $1/(4b^2)$ which equals $c/24$ in the semi-classical limit $b \rightarrow 0$. When we use the exact formula (3.68), with the correction linear in the momentum η_i , then the exact minimal conformal dimension $(c-1)/24$ is again recuperated. Since the latter value for the minimal conformal dimension is the origin of the shift in the minimal mass of a black hole, we again recognize this as a one loop quantum gravity effect.¹⁰

3.4 Supersymmetric $AdS_3/Liouville$

We will now study the supersymmetric generalization of the above holographic pair. This is the starting point of performing the topological twisting on the gravity side by picking a specific boundary condition for the gauge fields in the next section.

3.4.1 Bulk and Boundary Actions

We consider the supergravity theory with $N = 2$ supersymmetry. The supergravity action S can be written as the difference of two $\mathfrak{osp}(2|2, \mathbb{R})$ Chern-Simons actions [12, 13, 68, 69]

$$S[\Gamma, \tilde{\Gamma}] = S_{CS}[\Gamma] - S_{CS}[\tilde{\Gamma}], \quad (3.69)$$

9. From section 3.1 and section 3.3.2 it is clear that this entails a choice of how to integrate over metrics determined by energy-momentum tensor components.

10. While formula (3.20) suggests a point particle spectrum of width $1/c$ for conformal dimensions between $(c-1)/24$ to $c/24$, the one loop correction due to integration over the Liouville, radial mode suggests the interpretation we provided here. This remains to be rigorously substantiated.

where the Chern-Simons action S_{CS} on a line times a disk equals

$$\begin{aligned} S_{CS}[\Gamma] &= \frac{k_R}{8\pi} \int_{\mathbb{R} \times \text{Disk}} \text{Str}(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma) \\ &= \frac{k_R}{8\pi} \int dt dr d\varphi \text{Str}(\Gamma_\varphi \partial_t \Gamma_r - \Gamma_r \partial_t \Gamma_\varphi + 2\Gamma_0 F_{r\varphi}), \end{aligned} \quad (3.70)$$

with the level k_R given in terms of the three-dimensional Newton constant G_N by $k_R = \frac{l}{2G_N}$. The $\text{osp}(2|2, \mathbb{R})$ valued connections Γ and $\tilde{\Gamma}$ can be decomposed in terms of the Lie algebra generators as

$$\Gamma = A^3 \frac{\sigma^3}{2} + A^+ \sigma^+ + A^- \sigma^- + A^{(R)} T + \psi_{+\alpha} R^{+\alpha} + \psi_{-\alpha} R^{-\alpha}, \quad (3.71)$$

where the index α takes the two values $\alpha = 1, 2$. These generators satisfy the $\text{osp}(2|2, \mathbb{R})$ commutation relations,

$$\begin{aligned} [\frac{\sigma^3}{2}, \sigma^\pm] &= \pm \sigma^\pm, & [\sigma^+, \sigma^-] &= \sigma^3, & [\frac{\sigma^3}{2}, R^{\pm\alpha}] &= \pm \frac{1}{2} R^{\pm\alpha} \\ [\sigma^\pm, R^{\mp\alpha}] &= R^{\pm\alpha}, & [T, R^{\pm\alpha}] &= -(\lambda)_\beta^{\alpha\beta} R^{\pm\beta}, \\ \{R^{\pm\alpha}, R^{\pm\beta}\} &= \pm \eta^{\alpha\beta} \sigma^\pm & \{R^{\pm\alpha}, R^{\mp\beta}\} &= -\eta^{\alpha\beta} \frac{\sigma^3}{2} \pm \frac{1}{2} \lambda^{\alpha\beta} T, \end{aligned} \quad (3.72)$$

where the metric $\eta_{\alpha\beta} = \delta_{\alpha\beta}$ and its inverse $\eta^{\alpha\beta}$ can be used to lower and raise the α indices, and the λ matrix is related to the two-dimensional epsilon symbol through the equation $\lambda^\alpha_\gamma \eta^{\gamma\beta} = \epsilon^{\alpha\beta}$. See [13] for more details. The $\text{sl}(2, \mathbb{R})$ components A^a and \tilde{A}^a of the connection are related to the dreibein e_μ^a and the Hodge dual of the spin connection ω_μ^a through the formulas $A_\mu^a = \omega_\mu^a + \frac{1}{l} e_\mu^a$ and $\tilde{A}_\mu^a = \omega_\mu^a - \frac{1}{l} e_\mu^a$. We pick a bulk space-time of the form of a real line times a disk, with a cylindrical boundary, and choose a radial coordinate r that increases towards the boundary. The connections Γ and $\tilde{\Gamma}$ satisfy generalized Brown-Henneaux boundary conditions at large radius r [13, 72]

$$\begin{aligned} \Gamma &\xrightarrow{r \rightarrow \infty} \left(\frac{4\pi L}{k_R} \frac{\sigma^+}{r} + r\sigma^- + \frac{1}{\sqrt{r}} \frac{4\pi Q_{+\alpha}}{k_R} R^{+\alpha} + \frac{2\pi B}{k_R} T \right) dx^+ + 0 dx^- + \frac{\sigma^3}{2} \frac{dr}{r} \\ \tilde{\Gamma} &\xrightarrow{r \rightarrow \infty} \left(\frac{4\pi \tilde{L}}{k_R} \frac{\sigma^-}{r} + r\sigma^+ + \frac{1}{\sqrt{r}} \frac{4\pi \tilde{Q}_{-\alpha}}{k_R} R^{-\alpha} + \frac{2\pi \tilde{B}}{k_R} T \right) dx^- + 0 dx^+ + \left(-\frac{\sigma^3}{2} \right) \frac{dr}{r}, \end{aligned} \quad (3.73)$$

where the boundary light cone coordinates are $x^\pm = t \pm \varphi$ and the coordinate φ is compact with identification $\varphi \equiv \varphi + 2\pi$. The fluctuating components of the metric, gravitinos and gauge field on the boundary are given by the quantities $L, \tilde{L}, Q_{+\alpha}, \tilde{Q}_{-\alpha}, B$ and \tilde{B} which are arbitrary functions of the boundary coordinates x^\pm . In order to make the action and the boundary conditions $\Gamma_- = 0 = \tilde{\Gamma}_+$ compatible, one has to add the term

$$S_{\text{extra}} = -\frac{k_R}{8\pi} \int_{\Sigma_2} dt d\varphi \text{Str}(\Gamma_\varphi^2 + \tilde{\Gamma}_\varphi^2) \quad (3.74)$$

to the supergravity action S , where $\Sigma_2 = \mathbb{R} \times S^1$ is the asymptotic cylinder at $r \rightarrow \infty$. The extra term ensures that the variation of the total action is zero when the equation

of motion and boundary conditions are satisfied. The total action (again denoted S) equals

$$S[\Gamma, \tilde{\Gamma}] = S_{CS}[\Gamma] - \frac{k_R}{8\pi} \int_{\Sigma_2} dt d\varphi \text{Str}(\Gamma_\varphi^2) - S_{CS}[\tilde{\Gamma}] - \frac{k_R}{8\pi} \int_{\Sigma_2} dt d\varphi \text{Str}(\tilde{\Gamma}_\varphi^2). \quad (3.75)$$

The time components Γ_0 and $\tilde{\Gamma}_0$ of the connections are Lagrange multipliers that implement the zero flux constraints $F_{r\varphi} = 0 = \tilde{F}_{r\varphi}$. By solving these constraints one finds the spatial components Γ_i of the gauge connections,

$$\begin{aligned} \Gamma_i &= G_1^{-1} \partial_i G_1, \\ \tilde{\Gamma}_i &= G_2^{-1} \partial_i G_2, \end{aligned} \quad (3.76)$$

for $i = r, \varphi$, where the elements $G_{1,2}$ are elements of the group $\text{OSp}(2|2, \mathbb{R})$ or a finite cover. Near the boundary the fields $G_{1,2}$ behave as

$$\begin{aligned} G_1 &\xrightarrow{r \rightarrow \infty} g_1(t, \varphi) \exp\left(\frac{\sigma^3}{2} \log r\right), \\ G_2 &\xrightarrow{r \rightarrow \infty} g_2(t, \varphi) \exp\left(-\frac{\sigma^3}{2} \log r\right), \end{aligned} \quad (3.77)$$

such that the boundary conditions (3.73) are indeed satisfied. Following the approach for the bosonic case [12], the action can be written as a difference of two chiral supersymmetric Wess-Zumino-Witten actions

$$S[\Gamma, \tilde{\Gamma}] = \frac{k_R}{8\pi} (S_{WZW}^R[g_1] - S_{WZW}^L[g_2]). \quad (3.78)$$

The two chiral actions are given by

$$\begin{aligned} S_{WZW}^R &= \int_{\Sigma_2} dt d\varphi \text{Str}[g_1^{-1} \partial_t g_1 g_1^{-1} \partial_\varphi g_1 - (g_1^{-1} \partial_\varphi g_1)^2] + \Gamma[g_1], \\ S_{WZW}^L &= \int_{\Sigma_2} dt d\varphi \text{Str}(g_2^{-1} \partial_t g_2 g_2^{-1} \partial_\varphi g_2 + (g_2^{-1} \partial_\varphi g_2)^2) + \Gamma[g_2], \end{aligned} \quad (3.79)$$

where $\Gamma[g]$ is the Wess-Zumino term in the Wess-Zumino-Witten action.

As we did in the bosonic case in the first half of this chapter, we gauge the action that rotates the zero modes of the fields anti-diagonally, such that the zero modes are locked and give rise to a single boundary field with one set of zero modes and both left moving and right moving oscillations. This is crucial to our purposes. The action is then a non-chiral supersymmetric Wess-Zumino-Witten action which only depends on a new group valued variable $g = g_1^{-1} g_2$,

$$S = \frac{k_R}{8\pi} \int dx^+ dx^- (g^{-1} \partial_+ g g^{-1} \partial_- g) - \Gamma[g]. \quad (3.80)$$

The boundary conditions induce a Drinfeld-Sokolov reduction of the degrees of freedom. Indeed, the boundary conditions in terms of the new variables read

$$\begin{aligned} (g^{-1} \partial_- g)^{(+)} &= 1, & (g^{-1} \partial_- g)^{(3)} &= 0, & (g^{-1} \partial_- g)^{(+\alpha)} &= 0, \\ (\partial_+ g g^{-1})^{(-)} &= -1, & (\partial_+ g g^{-1})^{(3)} &= 0, & (\partial_+ g g^{-1})^{(-\alpha)} &= 0, \end{aligned} \quad (3.81)$$

where $(\cdot \cdot \cdot)^{(a)}$ represents the component corresponding to the generator labeled by a . We apply a Gauss type decomposition to the group element g

$$g = g_+ g_0 g_- , \quad (3.82)$$

where the group element factors are

$$\begin{aligned} g_+ &= \exp(x\sigma^+ + \psi_{+\alpha}R^{+\alpha}) , \\ g_0 &= \exp(\rho\sigma^3 + \theta T) , \\ g_- &= \exp(y\sigma^- + \psi_{-\alpha}R^{-\alpha}) . \end{aligned} \quad (3.83)$$

In terms of these coordinates the action becomes

$$\begin{aligned} S &= \frac{k_R}{4\pi} \int d^2x [\partial_+\rho\partial_-\rho + \partial_+\theta\partial_-\theta - ie^{-\rho}\partial_-\psi_{+\alpha}u^{\alpha\beta}\partial_+\psi_{-\beta} \\ &\quad + e^{-2\rho}(\partial_-x - \frac{i}{2}\eta^{\alpha\beta}\psi_{+\alpha}\partial_-\psi_{+\beta})(\partial_+y - \frac{i}{2}\eta^{\gamma\delta}\psi_{-\gamma}\partial_+\psi_{-\delta})] , \end{aligned} \quad (3.84)$$

while the boundary conditions are

$$\begin{aligned} e^{-2\rho}(\partial_-x - \frac{i}{2}\eta^{\alpha\beta}\psi_{+\alpha}\partial_-\psi_{+\beta}) &= 1 , \quad y + \partial_-\rho + \frac{i}{2}e^{-\rho}u^{\alpha\beta}\psi_{-\beta}\partial_-\psi_{+\alpha} = 0 , \\ \eta^{\alpha\beta}\psi_{-\alpha} + e^{-\rho}u^{\alpha\beta}\partial_-\psi_{+\alpha} &= 0 , \quad e^{-2\rho}(\partial_+y - \frac{i}{2}\eta^{\alpha\beta}\psi_{-\alpha}\partial_+\psi_{-\beta}) = -1 , \\ -x + \partial_+\rho - \frac{i}{2}e^{-\rho}(u^{-1})^{\alpha\beta}\psi_{+\beta}\partial_+\psi_{-\alpha} &= 0 , \quad \eta^{\alpha\beta}\psi_{+\alpha} + e^{-\rho}(u^{-1})^{\alpha\beta}\partial_+\psi_{-\alpha} = 0 , \end{aligned} \quad (3.85)$$

where we have used the variable $u = \exp(\theta\lambda) = \cos\theta + \sin\theta\lambda$. The equations of motion for the fields ρ , θ and $\psi_{\pm\alpha}$ given by the action (3.84) are

$$\begin{aligned} \partial_+\partial_-\rho &= -2e^{-2\rho}(\partial_-x - \frac{i}{2}\eta^{\alpha\beta}\psi_{+\alpha}\partial_-\psi_{+\beta})(\partial_+y - \frac{i}{2}\eta^{\gamma\delta}\psi_{-\gamma}\partial_+\psi_{-\delta}) + ie^{-\rho}u^{\alpha\beta}\partial_-\psi_{+\alpha}\partial_+\psi_{-\beta} , \\ \partial_+\partial_-\theta &= -ie^{-\rho}\partial_-\psi_{+\alpha}(\lambda u)^{\alpha\beta}\partial_+\psi_{-\beta} , \\ \partial_-\psi_{+\alpha}e^{-2\rho}(\partial_+y - \frac{i}{2}\eta^{\gamma\delta}\psi_{-\gamma}\partial_+\psi_{-\delta}) &= \partial_-[e^{-\rho}(u^{-1})^{\beta}_{\alpha}\partial_+\psi_{-\beta} - e^{-2\rho}(\partial_+y - \frac{i}{2}\eta^{\gamma\delta}\psi_{-\gamma}\partial_+\psi_{-\delta})] , \\ \partial_+\psi_{-\alpha}e^{-2\rho}(\partial_-x - \frac{i}{2}\eta^{\gamma\delta}\psi_{+\gamma}\partial_-\psi_{+\delta}) &= \partial_+[-e^{-\rho}u^{\beta}_{\alpha}\partial_-\psi_{+\beta} - e^{-2\rho}(\partial_-x - \frac{i}{2}\eta^{\gamma\delta}\psi_{+\gamma}\partial_-\psi_{+\delta})] . \end{aligned} \quad (3.86)$$

Using the constraints deduced above from the boundary conditions, the equations of motion can be simplified to

$$\begin{aligned} \partial_+\partial_-\rho &= 2e^{2\rho} - ie^{\rho}\psi_{+\alpha}u^{\alpha\beta}\psi_{-\beta} , & \partial_+\partial_-\theta &= -ie^{\rho}\psi_{+\alpha}(\lambda u)^{\alpha\beta}\psi_{-\beta} , \\ \eta^{\alpha\beta}\partial_-\psi_{+\beta} &= -e^{\rho}u^{\alpha\beta}\psi_{-\beta} , & \eta^{\alpha\beta}\partial_+\psi_{-\beta} &= -e^{\rho}u^{\beta\alpha}\psi_{+\beta} . \end{aligned} \quad (3.87)$$

These equations are the same as the equations of motion one obtains starting from the action¹¹

$$\begin{aligned} S &= \frac{k_R}{4\pi} \int d^2x [\partial_+\rho\partial_-\rho + \partial_+\theta\partial_-\theta + \frac{i}{2}\psi_{+\alpha}\eta^{\alpha\beta}\partial_-\psi_{+\beta} - \frac{i}{2}\psi_{-\alpha}\eta^{\alpha\beta}\partial_+\psi_{-\beta} \\ &\quad + 2e^{2\rho} - ie^{\rho}\psi_{+\alpha}(e^{\theta\lambda})^{\alpha\beta}\psi_{-\beta}] . \end{aligned} \quad (3.88)$$

11. To obtain fields in a more standard normalization, with the self-dual radius $\sqrt{\alpha'}$ set equal to $\sqrt{\alpha'} = \sqrt{2}$, one would define new fields $\rho_{stan} = \sqrt{k_R}\rho$ and $\theta_{stan} = \sqrt{k_R}\theta$, and similarly for the fermions.

This is the action of the $N = 2$ supersymmetric Liouville theory studied in chapter 2, with $\rho + i\theta$ and $\psi_{+1} + i\psi_{+2}, \psi_{-1} - i\psi_{-2}$ being the lowest and fermionic components of the superfield Φ respectively.

We note that the bulk gravitational theory is characterized by two integers. The first integer identifying our theory is the level k_R . The fact that it is an integer follows from our choice of gauge group, which we take to have a compact $SO(2) = U(1)$ factor. Indeed, the level $k_{U(1)}$ of a $U(1)$ Chern-Simons theory with action

$$S_{CS}^{(U(1))} = \frac{k_{U(1)}}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad (3.89)$$

is an integer. The $U(1)_R$ part of the supersymmetric gravitational Chern-Simons action thus enforces the quantization of the R-symmetry level k_R . The second integer arises as follows. The angular coordinate θ can be chosen to be identified modulo $2\pi N$ where N is an integer. For simplicity, we take the integer N to be positive. Thus, our theory is characterized by the pair of positive integers (k_R, N) .

3.4.2 The Exact Solutions

For three-dimensional gravity with a negative cosmological constant, the Fefferman-Graham expansion for a metric solution to Einstein's equations terminates. The explicit all order solution in the bosonic case was determined in [56]. The method exploited the Chern-Simons formulation of the three-dimensional Einstein-Hilbert action. Presently, we demonstrate that this method can be extended to the supergravity case. We solve the Chern-Simons equations of motion with the boundary conditions given above, and thus provide an all order (truncating) solution to the Fefferman-Graham expansion in supergravity. Firstly, we impose the gauge condition

$$\Gamma_r = \frac{\sigma^3}{2r}. \quad (3.90)$$

This gauge condition is compatible with the boundary conditions (3.73). The Chern-Simons equations of motion or flatness conditions are

$$\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = 0. \quad (3.91)$$

The components with $\mu = -$ and $\nu = r$ combined with the boundary condition $\Gamma_- = 0$ indicate that the connection component Γ_- vanishes everywhere. The remaining equations read

$$\partial_- \Gamma_+ = 0, \quad (3.92)$$

$$\partial_r \Gamma_+ + [\Gamma_r, \Gamma_+] = 0. \quad (3.93)$$

The first equation implies that the connection component Γ_+ does not depend on the light-cone coordinate x^- , while the second equation fixes its radial dependence,

$$\Gamma_+ = b^{-1} \hat{\Gamma}(x^+) b, \quad (3.94)$$

where the group element b equals

$$b = \exp\left(\frac{\sigma^3}{2} \log r\right). \quad (3.95)$$

The function $\hat{\Gamma}(x^+)$ is related to the boundary fields L , $Q_{+\alpha}$ and B in equation (3.73) by

$$\hat{\Gamma} = \frac{4\pi L}{k_R} \sigma^+ + \sigma^- + \frac{4\pi Q_{+\alpha}}{k_R} R^{+\alpha} + \frac{2\pi B}{k_R} T. \quad (3.96)$$

The solution for the connection is thus

$$\Gamma = \left(\frac{1}{r} \frac{4\pi L}{k_R} \sigma^+ + r \sigma^- + \frac{1}{\sqrt{r}} \frac{4\pi Q_{+\alpha}}{k_R} R^{+\alpha} + \frac{2\pi B}{k_R} T \right) dx^+ + 0 dx^- + \left(\frac{1}{r} \frac{\sigma^3}{2} \right) dr. \quad (3.97)$$

One can similarly obtain the solution for the connection $\tilde{\Gamma}$:

$$\tilde{\Gamma} = \left(\frac{1}{r} \frac{4\pi \tilde{L}}{k_R} \sigma^- + r \sigma^+ + \frac{1}{\sqrt{r}} \frac{4\pi \tilde{Q}_{-\alpha}}{k_R} R^{-\alpha} + \frac{2\pi \tilde{B}}{k_R} T \right) dx^- + 0 dx^+ + \left(-\frac{1}{r} \frac{\sigma^3}{2} \right) dr. \quad (3.98)$$

The metric and the R-symmetry gauge fields are therefore given by

$$\begin{aligned} ds^2 &= l^2 \frac{dr^2}{r^2} + 4G_N l \left(2\pi L (dx^+)^2 + 2\pi \tilde{L} (dx^-)^2 \right) + \left(l^2 r^2 + 64\pi^2 G_N^2 L \tilde{L} r^{-2} \right) dx^+ dx^-, \\ A^{(R)} &= \frac{2\pi B}{k_R} dx^+, \quad \tilde{A}^{(R)} = \frac{2\pi \tilde{B}}{k_R} dx^-. \end{aligned} \quad (3.99)$$

The boundary energy-momentum tensor component T and the R-symmetry current J are related to the boundary functions L , \tilde{L} , B , \tilde{B} through the equations [13]

$$\begin{aligned} T(x^+) &= -2\pi \left(L + \frac{2\pi}{2k_R} B^2 \right), & \bar{T}(x^-) &= -2\pi \left(\tilde{L} + \frac{2\pi}{2k_R} \tilde{B}^2 \right), \\ J(x^+) &= -2\pi i B, & \bar{J}(x^-) &= -2\pi i \tilde{B}. \end{aligned} \quad (3.100)$$

These currents and the supercurrents satisfy an asymptotic $N = 2$ superconformal algebra [13]. Therefore, the metric and the R-symmetry gauge fields can be written in terms of the energy-momentum tensor and R-symmetry current as

$$\begin{aligned} ds^2 &= l^2 \frac{dr^2}{r^2} - 4G_N l \left(\left(T - \frac{J^2}{2k_R} \right) (dx^+)^2 + \left(\bar{T} - \frac{\bar{J}^2}{2k_R} \right) (dx^-)^2 \right) \\ &\quad + \left(l^2 r^2 + \frac{16G_N^2}{r^2} \left(T - \frac{J^2}{2k_R} \right) \left(\bar{T} - \frac{\bar{J}^2}{2k_R} \right) \right) dx^+ dx^-, \\ A^{(R)} &= \frac{iJ}{k_R} dx^+, \quad \tilde{A}^{(R)} = \frac{i\bar{J}}{k_R} dx^-. \end{aligned} \quad (3.101)$$

In similar fashion the gravitini are related to the boundary fields $Q_{+\alpha}$ and $\tilde{Q}_{-\alpha}$ as well as the boundary supercurrents.

3.4.3 Supersymmetric Liouville Quantum Gravity

In the previous discussion of this chapter, a bulk theory of quantum gravity in three-dimensional anti-de Sitter space-time was defined as the dual of the bosonic Liouville conformal field theory. The latter theory is unitary and consistent and the spectrum and three-point functions are explicitly known. These properties are thus inherited by

the dual bulk theory. Various original characteristics of the resulting bulk theory were discussed above to which we do refer for a broader discussion that also largely applies to the present generalization. Let us only mention the lack of a microscopic as well as a macroscopic picture of black hole thermodynamics in this theory.

We extend the approach above for bosonic theories to include supersymmetry. We consider the $N = 2$ superconformal Liouville theory on the two-dimensional boundary to be the definition of a quantum theory of supergravity in the anti-de Sitter three-dimensional bulk. The classical actions agree, as demonstrated in [13] and recalled in subsection 3.4.1. We also identify the measures of the quantum theories. Our discussion of the quantum mechanical model is mainly based on the boundary conformal field theory since it is considerably better understood than the (reduced) quantum Chern-Simons theory on the super group.

Matching Bulk and Boundary Parameters

We parameterize the central charge of the $N = 2$ Liouville conformal field theory as $c = 3 + 6/k$ where we will refer once more to the parameter k as the level.¹² Semi-classically, the gravitational level $k_R = l/(2G_N)$ is related to the central charge by the formula $c = 3k_R$ [13, 72]. At large central charge and large level k_R , we therefore have the relation $k_R \approx 2/k$.

In the quantum theory, when k_R^{ren} represents the level of the quantum $U(1)_R$ current and the renormalized value of the cosmological constant in Planck units, the central charge must still be related to the level through the relation $c = 3k_R^{ren}$ by the structure of the $N = 2$ superconformal algebra. Thus, the relation $k_R^{ren} = 1 + 2/k$ is exact.¹³ Thus, we have identified bulk and boundary central charges, and therefore a first parameter of both theories.

The level k of the Liouville theory and therefore its central charge can be arbitrary. In particular, the level k can be small and therefore we can reach the semi-classical regime where the central charge c is large. Note that when we take into account the quantization condition on the $U(1)$ Chern-Simons level k_R , we find that the level k is twice the inverse of an integer.

We remark in passing that the identification of the level k_R with (twice) the inverse of the level k is reminiscent of the FZZ or T-duality between Liouville theory and the cigar coset conformal field theory. This duality here obtains a holographic counterpart. The bulk is semi-classical when the central charge and level k_R are large, while the coset curvature interactions are small when the level k is large and the central charge c is close to three.

Secondly, we note that the radius of the $U(1)_R$ direction that corresponds to the

12. It is the level of the $sl(2, \mathbb{R})$ algebra that governs the parent Wess-Zumino-Witten model of the T-dual coset conformal field theory.

13. It is natural to ask how the renormalized radius k_R^{ren} is related to the classical coefficient k_R in the Chern-Simons action. The Chern-Simons level may be one-loop perturbatively renormalized by the dual Coxeter number which is 1 for $osp(2|2)$ [73]. If we assume this to be the case, we obtain $k_R^{ren} = k_R + 1$ for positive levels. Then $c = 3k_R^{ren} = 3k_R + 3 = 3 + 6/k$ and $k_R = 2/k$ is the exact relation between the classical coefficient k_R and the level k . While this is natural from a boundary current algebra perspective, it is hard to solidly justify from a three-dimensional path integral perspective on Chern-Simons theory on super groups. See Appendix E of [74] for a detailed critical discussion.

angular direction in the $N = 2$ Liouville conformal field theory is determined by the action (3.84) and the equivalence relation on the angular coordinate θ . Since we chose a radius which is N times the minimal radius in the bulk theory, we have a radius $N\sqrt{\alpha'/k}$ in the Liouville theory. This matches the semi-classical identification of the radius from the action in equation (3.88).

The Spectrum

Now that we matched the two parameters of the bulk and the boundary theories, we can exploit our knowledge of the quantum theory on the boundary to make statements about the holographic dual in the bulk. The spectrum of $N = 2$ Liouville theory as well as its three-point functions are known. We review only a few salient features of the conformal field theory and remark on their gravitational counterparts. It will certainly be interesting to explore the dictionary further.

In this subsection, we concentrate on the spectrum and remark on its gravitational counterpart. The spectrum of $N = 2$ Liouville theory operators is classified in terms of $N = 2$ superconformal primaries. Moreover, it is convenient to parameterize the spectrum of $N = 2$ superconformal primaries in terms of variables natural in the cigar coset model $SL(2, \mathbb{R})_k/U(1)$. Superconformal primary operators are characterized by their conformal dimension h and R-charge q . We parameterize these charges in the NS sector in terms of the $sl(2, \mathbb{R})$ spin j and $u(1)$ charge m as (see e.g. [29, 32, 37, 40] for background) :

$$h = -\frac{j(j-1)}{k} + \frac{m^2}{k}, \quad q = \frac{2m}{k}. \quad (3.102)$$

Since the theory has $N = 2$ superconformal symmetry, spectral flow will determine the spectrum in the R sector. The spectrum consists of continuous representations that have $sl(2, \mathbb{R})$ spin $j = 1/2 + is$ where s is a real number and m can take any value compatible with the choice of radius of the angular direction of $N = 2$ Liouville theory. The conformal dimensions of the continuous representations satisfy the inequality

$$h \geq \frac{1}{4k}. \quad (3.103)$$

These representations have equal left and right $sl(2, \mathbb{R})$ spin j , but they do allow for unbalanced left and right angular momentum on the compact direction when the excitation winds. We recall that in these conventions (namely with respect to the planar Hamiltonian) the mass of the anti-de Sitter space-time is zero. The continuum of bulk states has a mass (equal to the minimal left plus right conformal dimension) above $1/(2k) = (c-3)/12$. At large central charge (and small level), there is a considerable gap with respect to the non-normalizable $SL(2, \mathbb{C})$ invariant vacuum. We recall that the continuous states create hyperbolic monodromies [58] and correspond to black hole geometries in the bulk [57]. In contrast to the bosonic theory discussed above, the supersymmetric theory at hand contains black hole primary states with spin, when the boundary excitation has non-zero winding. Since the left and right conformal dimensions are positive in the unitary $N = 2$ Liouville theory, the cosmic censorship bound for spinning black holes is satisfied in the bulk.

Furthermore, there are discrete representations in the spectrum of the $N = 2$ Liouville conformal field theory [29, 32, 40]. These operators have quantized spins j

and angular momenta m whose absolute value ranges from j to $+\infty$. The range of allowed spins is $1/2 \leq j \leq (k+1)/2$. These discrete representations correspond to elliptic monodromies and can be identified with bulk geometries that correspond to particles with spin.¹⁴ This too contrasts with the bosonic set-up.

The Chiral/Chiral Spectrum

We argued above that the radius of the angular direction of the supersymmetric Liouville theory is $R = N\sqrt{\alpha'/k}$. The multiple N determines the Witten index [31,38] and therefore the number of chiral primaries. The left and right moving quantum numbers m and \bar{m} are quantized as

$$\begin{aligned} 2m &= \frac{k}{N}n + Nw, \\ 2\bar{m} &= \frac{k}{N}n - Nw. \end{aligned} \tag{3.104}$$

To list chiral-chiral primary operators, we impose the relation $h = q/2$ which in turn forces the equality $j = m$ for a chiral primary on the left. A similar reasoning holds on the right such that for chiral-chiral primary operators we have that $m = \bar{m}$ and the winding w equals zero. The number of allowed chiral/chiral primaries equals N since the range of allowed discrete spins j is of width k .¹⁵ We will study the bulk counterpart to the spectrum of chiral primaries in subsection 3.5.3.

3.5 A Twisted Holography

In this section, we topologically twist the two members of the duality given in the above section. In a first part, we twist the bulk supergravity theory by defining a twisted set of boundary conditions in the presence of a non-trivial boundary metric. We argue that the boundary conditions give rise to a topological $N = 2$ superconformal algebra as the asymptotic symmetry algebra. Secondly, we review how every theory with $N = 2$ superconformal symmetry in two dimensions gives rise to a topological quantum field theory. We recall properties of the observables, corresponding to chiral ring elements in the physical theory. Finally, we argue that the topologically twisted theories match in both their observables and their structure constants.

3.5.1 The Twisted Supergravity Theory

In this subsection, we describe the topologically twisted supergravity theory. In order to identify the twisted theory, we study the holographic duality in the presence of a non-trivial background boundary metric. Indeed, we know from the boundary perspective that the difference between the physical and the topological theory lies in the manner in which they couple to a boundary metric. We concentrate on the case where

14. To realize that the geometric monodromies are elliptic it is important to note that the geometry is determined by a combination of the energy-momentum and R-charge. See equation (3.101).

15. The boundary values correspond to almost-normalizable states (that effectively count for half each in the Witten index of the non-compact theory). See [26,31] for detailed discussions.

the boundary metric is conformally flat for simplicity. Importantly, we propose the boundary conditions that will give rise to the topologically twisted boundary theory. We then compute the action and verify that it is equivalent to the topologically twisted boundary action.

As a by-product, we make several observations. Firstly, the bulk action and boundary conditions in the presence of a conformally flat metric can be obtained by a formal gauge transformation from the standard case.¹⁶ The boundary Liouville action in the presence of a non-trivial conformally flat boundary metric is obtained by a field redefinition closely related to the formal bulk gauge transformation. Secondly, this observation holds both in the correspondence between pure gravity and bosonic Liouville theory and in the relation between pure supergravity and supersymmetric Liouville theory. Thirdly, we show that a further formal gauge transformation in the bulk transports us from the bulk extended supergravity theory to its topologically twisted version. The latter satisfies new boundary conditions.¹⁷ We confirm that the bulk theory gives rise to a boundary action which is topologically twisted, and of total central charge zero.

After this conceptual introduction, it is time to delve into the details. Concretely, we propose that the bulk supergravity theory after topological twisting and coupling to the conformally flat boundary metric $g_{(0)\mu\nu} = \exp(2\omega)\eta_{\mu\nu}$ corresponds to a bulk Chern-Simons theory of the type discussed in section 3.4, supplemented with the new boundary conditions :

$$\begin{aligned}
\Gamma &\xrightarrow{r \rightarrow \infty} \left(-\partial_+ \omega \frac{\sigma^3}{2} + \frac{4\pi L}{k_R} \frac{\sigma^+}{r} + r e^\omega \sigma^- + \frac{1}{\sqrt{r}} \frac{4\pi Q_{+\alpha}}{k_R} R^{+\alpha} + \frac{2\pi B}{k_R} T \right) dx^+ \\
&\quad + \left[\partial_- \omega \left(\frac{\sigma^3}{2} + \frac{iT}{2} \right) + \frac{1}{r e^\omega} \partial_+ \partial_- \omega \sigma^+ \right] dx^- + \frac{\sigma^3}{2} \frac{dr}{r} \\
\tilde{\Gamma} &\xrightarrow{r \rightarrow \infty} \left(\partial_- \omega \frac{\sigma^3}{2} + \frac{4\pi \tilde{L}}{k_R} \frac{\sigma^-}{r} + r e^\omega \sigma^+ + \frac{1}{\sqrt{r}} \frac{4\pi \tilde{Q}_{-\alpha}}{k_R} R^{-\alpha} + \frac{2\pi \tilde{B}}{k_R} T \right) dx^- \\
&\quad + \left[-\partial_+ \omega \left(\frac{\sigma^3}{2} + \frac{iT}{2} \right) + \frac{1}{r e^\omega} \partial_+ \partial_- \omega \sigma^- \right] dx^+ + \left(-\frac{\sigma^3}{2} \right) \frac{dr}{r}. \tag{3.105}
\end{aligned}$$

These boundary conditions are related to those described in equation (3.73) by a gauge transformation with gauge parameter :

$$\gamma = h^{-1} f_1 h, \tag{3.106}$$

and a similar gauge transformation on the right

$$\tilde{\gamma} = h f_2 h^{-1}, \tag{3.107}$$

16. The formal gauge transformation is non-trivial at large radius.

17. For discussions of the set of possible boundary conditions in the bosonic context, see e.g. [75, 76]. For generalizations in extended supergravity theories, see e.g. [77]. The fact that we mix with the R-current in a way prescribed by the twist, is novel.

where the group valued factors $f_{1,2}$ and h are given by

$$\begin{aligned} f_1 &= \exp(\partial_+ \omega \sigma^+) \exp[\omega(\frac{\sigma^3}{2} + \frac{iT}{2})], \\ f_2 &= \exp(\partial_- \omega \sigma^-) \exp[-\omega(\frac{\sigma^3}{2} + \frac{iT}{2})], \\ h &= \exp(\frac{\sigma^3}{2} \log r). \end{aligned} \quad (3.108)$$

The factor $\exp[\frac{iT}{2}]$ is responsible for the topological twist. If one wants to consider the untwisted theory with conformally flat boundary metric, this factor should be omitted. Under this gauge transformation, the Chern-Simons action now becomes a Chern-Simons action for the gauge transformed field plus a boundary term linear in the gauge transformed fields and a term only depends on the above gauge parameters, which we drop since it contains no dynamical degrees of freedom. To make the action compatible with the boundary condition, one now needs to add not only the extra boundary terms in section 3.4.1, but also another boundary term such that its variation cancels the variation of the additional boundary term in the action. Since the term is linear in the gauge fields, the extra term that needs to be added cancels it. Therefore, our action is of the same form as the one in 3.4.1, with the same boundary condition for the gauge transformed fields. We can then follow the steps in 3.4.1, and find that the action can be reduced to a Wess-Zumino-Witten action with the new glued field :

$$\hat{g} = f_1 g f_2^{-1}. \quad (3.109)$$

Once again one performs the Gauss decomposition of g and finds :

$$\begin{aligned} \rho &= \hat{\rho} - \omega, & \theta &= \hat{\theta} - \omega, \\ \psi_+ &= \hat{\psi}_+, & \bar{\psi}_+ &= e^\omega \hat{\bar{\psi}}_+, \\ \psi_- &= e^\omega \hat{\psi}_-, & \bar{\psi}_- &= \hat{\bar{\psi}}_-, \end{aligned} \quad (3.110)$$

where the hatted variables correspond to the Gauss decomposition (3.82) of the group valued field \hat{g} . The fermions ψ_\pm and $\bar{\psi}_\pm$ are defined by

$$\begin{aligned} \psi_\pm &= \psi_{\pm 1} - i\psi_{\pm 2}, \\ \bar{\psi}_\pm &= \psi_{\pm 1} + i\psi_{\pm 2}, \end{aligned} \quad (3.111)$$

and the same definition holds for their hatted counterparts. In the untwisted theory, the field θ corresponding to the R-current will remain invariant after introduction of the conformal factor while the fermions has the relation

$$\begin{aligned} \psi_\pm &= e^{\frac{\omega}{2}} \hat{\psi}_\pm \\ \bar{\psi}_\pm &= e^{\frac{\omega}{2}} \hat{\bar{\psi}}_\pm. \end{aligned} \quad (3.112)$$

The resulting boundary action can be acquired by replacing the unhatted variables in (3.88) by the hatted ones and takes the form

$$\begin{aligned} S_{Liouville}^{top} &= \frac{k_R}{4\pi} \int dx^+ dx^- \left(\partial_+ \rho \partial_- \rho + \partial_+ \theta \partial_- \theta + ie^\omega \bar{\psi}_+ \partial_- \psi_+ + ie^\omega \psi_- \partial_+ \bar{\psi}_- \right. \\ &\quad \left. + e^{2\omega} [e^{2\rho} - i(e^{\rho+i\theta} \bar{\psi}_+ \psi_- + e^{\rho-i\theta} \psi_+ \bar{\psi}_-) + \frac{1}{4}(\rho + i\theta)\mathcal{R}^{(2)}] \right). \end{aligned} \quad (3.113)$$

The ω dependence arises from the metric, its determinant and the two-dimensional Ricci scalar $\mathcal{R}^{(2)}$. This boundary action equals the action of the topologically twisted $N=2$ Liouville theory [78] after a field redefinition

$$\begin{aligned}\bar{\psi}_+ &\rightarrow e^\omega \bar{\psi}_+ \\ \psi_- &\rightarrow e^\omega \psi_-.\end{aligned}\tag{3.114}$$

The action is different from the physical action by a coupling between θ and the boundary curvature, giving the energy-momentum tensor

$$\hat{T} = T_{top} = T + \frac{1}{2} D_z J.\tag{3.115}$$

3.5.2 The Topological Conformal Field Theory on the Boundary

We turn to recall how to topologically twist the boundary theory as done in chapter 1. It is understood that a two-dimensional theory with $N = 2$ superconformal symmetry provides a starting point for defining a topological quantum field theory [9, 20]. Indeed, the twisted energy-momentum tensor component $T_{top} = T + \partial J/2$ gives rise to a zero central charge conformal field theory that is independent of the metric [9, 20]. The BRST charge whose cohomology defines the space of observables of the resulting topological quantum field theory is $Q = \oint G^+$ where the supercurrent G^+ of positive R-charge is a current of dimension one in the twisted theory. The current G^- of negative $U(1)_R$ charge becomes a current of dimension two after twisting, and is the pre-image of the BRST exact energy-momentum tensor :

$$\{Q, G^-\} = \left\{ \oint G^+, G^- \right\} \propto T_{top}.\tag{3.116}$$

The observables of the theory originate in the chiral-chiral ring elements of the physical theory when we assume that the left and right topological twist are of the same type. The energy-momentum tensor is BRST exact and this indeed implies that after coupling to the metric on the boundary, there is no metric dependence in the topological theory. In the topological Landau-Ginzburg model that we deal with presently, the action is in fact not BRST exact, but rescaling the boundary metric inside the action still leads to localization of the path integral on constant field configurations [44].¹⁸ The localization of correlators was thoroughly exploited in the solution to topological $N = 2$ Liouville theory in chapter 2. There, the theory with Witten index N equal to the positive integer level k was solved.

The topologically twisted $N = 2$ Liouville theory that we have at hand is a generalization of the one studied in chapter 2. We momentarily digress to sketch the generalization of the analysis there to include all cases of interest here. Firstly, we allow for a general positive level k . Secondly, we allow for any Witten index N . This has as a consequence that the Landau-Ginzburg superpotential of the $N = 2$ Liouville theory can be written as

$$W = Y^{-N},\tag{3.117}$$

18. We note that the metric on the boundary is determined by the boundary value of the bulk space-time metric.

where the chiral superfield Y is related to the Liouville superfield Φ by $Y^{-1} = \exp \frac{1}{N} \sqrt{\frac{k}{2}} \Phi$ (in the conventions of chapter 2 to which we must refer for background). This operator is well defined due to the particular choice of radius. The spectrum and superpotential lead to (strictly normalizable) chiral/chiral primary operators of the form $Y^{-2}, Y^{-3}, \dots, Y^{-N}$. This description agrees with the one given in subsection 3.4.3. To find agreement with the strictly normalizable NS-NS sector states of [38] these operators must act on the almost-normalizable chiral state at the bottom of the continuum with $j = 1/2 = m$. The resulting states have R-charges $n/N + 1/k$ where $n = 1, 2, \dots, N-1$. When we compute the two-point function in the chiral state at the bottom of the continuum, corresponding to the operator $e^{\frac{1}{\sqrt{2k}}\Phi}$, we find from anomalous R-charge conservation :¹⁹

$$\langle e^{\frac{1}{\sqrt{2k}}\Phi} | Y^{-j} Y^{-l} | e^{\frac{1}{\sqrt{2k}}\Phi} \rangle = \langle e^{\frac{1}{\sqrt{2k}}\Phi} | \exp \frac{j}{N} \sqrt{\frac{k}{2}} \Phi \exp \frac{l}{N} \sqrt{\frac{k}{2}} \Phi | e^{\frac{1}{\sqrt{2k}}\Phi} \rangle = \delta_{j+l, N}. \quad (3.118)$$

Because of the form of the superpotential (3.117) and the chiral ring operators Y^{-j} , we surmise that the resulting topological conformal field theory and its deformations are governed by the N -th reduced KdV integrable hierarchy. It would be good to substantiate this prediction along the lines of [26, 79].

3.5.3 The Gravitational Chiral Primaries

In subsections 3.4.3 and 3.5.2, we described the chiral ring elements from the perspective of the $N = 2$ superconformal Liouville theory on the boundary. In this subsection we investigate the chiral ring from the viewpoint of the bulk supergravity theory in three-dimensional anti-de Sitter space-time. We demonstrate that a quantization of the classical supergravity chiral solution set agrees with the space of observables in the twisted topological conformal field theory.

To set up the calculation, we remind the reader of a property of the chiral ring given in chapter 1. Chiral primary states are defined to be the states annihilated by all the operators $G_{r>0}^{\pm}$ as well as $G_{-1/2}^+$. Such states obey the equality $h = q/2$. In fact, the opposite is also true. A state $|\phi\rangle$ obeying the equality $h = q/2$ can be proven to be a chiral primary state. Indeed, first note that the condition $h = q/2$ is equivalent to the operator equalities $G_{-1/2}^+ |\phi\rangle = 0 = G_{1/2}^- |\phi\rangle$ since

$$\langle \phi | 2L_0 - J_0 | \phi \rangle = \langle \phi | \{G_{-1/2}^+, G_{1/2}^-\} | \phi \rangle = |G_{-1/2}^+ |\phi\rangle|^2 + |G_{1/2}^- |\phi\rangle|^2. \quad (3.119)$$

Consider the states $J_{n>0} |\phi\rangle$ which have conformal dimension $h-n$ and charge q . These states do not satisfy the unitarity bound $h \geq q/2$ and thus have to be zero. Then, one can use the commutation relation $[J_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}$ to obtain the vanishing of the states

$$G_{n-1/2}^+ |\phi\rangle = J_n G_{-1/2}^+ |\phi\rangle - G_{-1/2}^+ J_n |\phi\rangle = 0, \quad (3.120)$$

$$G_{n+1/2}^- |\phi\rangle = G_{1/2}^- J_n |\phi\rangle - J_n G_{1/2}^- |\phi\rangle = 0 \quad (3.121)$$

19. See subsection 3.5.4 for more explanation.

for $n > 0$. Therefore the state $|\phi\rangle$ is chiral primary. In summary, if we demand that a configuration is annihilated by the operators $G_{-1/2}^+$ and $G_{1/2}^-$, then it is a chiral primary solution. Thus, we look for the supergravity solutions that are annihilated by these two operations. We perform the calculations in the physical theory of section 3.4.

The variation of the supergravity fields under supersymmetric transformations with parameter ϵ_α is [13]

$$\delta L = -i\eta^{\alpha\beta}[\frac{1}{2}(Q_{+\alpha}\epsilon_\beta)' + Q_{+\alpha}\epsilon'_\beta] - \frac{2\pi i}{k_B}(\lambda)^{\alpha\beta}BQ_{+\alpha}\epsilon_\beta, \quad (3.122)$$

$$\delta B = i(\lambda)^{\alpha\beta}Q_{+\alpha}\epsilon_\beta, \quad (3.123)$$

$$\delta Q_{+\alpha} = -\frac{k_R}{4\pi}\epsilon''_\alpha + \frac{1}{2}(\lambda)_\alpha^\beta[(B\epsilon_\beta)' + B\epsilon'_\beta] + (L + \frac{2\pi}{2k_R}B^2)\epsilon_\alpha. \quad (3.124)$$

The variation generated by the operator $G_{-1/2}^+$ corresponds to the parameter choice $\epsilon_1 = i\epsilon_2 = \frac{1}{2}\exp(\frac{i}{2}\theta)$. Demanding that the variation δB equals zero then provides the constraint $Q_{+1} = iQ_{+2}$ on the gravitini. That constraint also guarantees that the variation δL vanishes. Finally, the demand $\delta Q_{+\alpha} = 0$ gives rise to only one independent equation,

$$\frac{k_R}{16\pi} + \frac{i}{2}(B' + iB) + L + \frac{2\pi}{2k_R}B^2 = 0. \quad (3.125)$$

Defining the tensor $\hat{L} = L + \frac{2\pi}{2k_R}B^2$ [13], the equation simplifies to

$$\frac{k_R}{16\pi} + \frac{i}{2}(B' + iB) + \hat{L} = 0. \quad (3.126)$$

The Fourier modes \hat{L}_n and B_n defined by $\hat{L} = \frac{1}{2\pi}\sum_n \hat{L}_n e^{in\theta}$, $B = \frac{1}{2\pi}\sum_n B_n e^{in\theta}$ then satisfy

$$\hat{L}_n + \frac{c}{24}\delta_{n,0} - \frac{1}{2}(n+1)B_n = 0, \quad (3.127)$$

where we have used the relation $c = 3k_R$ between the central charge and the level of the R-current. This equation is equivalent to the operator equation $T + \frac{1}{2}\partial J = 0$ in planar coordinates. The second supercharge $G_{1/2}^-$ corresponds to the choice of parameters $\epsilon_1 = -i\epsilon_2 = -\frac{1}{2}\exp(-\frac{i}{2}\theta)$. A similar analysis leads to the equations $Q_{+1} = -iQ_{+2}$ as well as

$$\frac{k_R}{16\pi} + \frac{i}{2}(-B' + iB) + \hat{L} = 0. \quad (3.128)$$

If we require the bulk solution to be invariant under both transformations generated by the charges $G_{-1/2}^+$ and $G_{1/2}^-$, the solutions satisfy

$$Q_{+1} = 0 = Q_{+2}, \quad B' = 0, \quad \hat{L} + \frac{k_R}{16\pi} - \frac{1}{2}B = 0, \quad (3.129)$$

which corresponds to the classical values

$$T = \frac{h}{z^2}, \quad J = \frac{2h}{z}, \quad G^\pm = 0 \quad (3.130)$$

in planar coordinates. These are the expectation values corresponding to the insertion of a state with conformal weight h at 0 and ∞ , and of charge $q = 2h$. In other

words, these classical values do correspond to chiral primary sources, as we wished to demonstrate.

The quantization of the space of chiral solutions can be understood as follows. The quantization of the parameter $q = 2h$ is implied by the quantization of $U(1)_R$ charge. The periodicity of the angular coordinate θ and the consequent periodicity of the $U(1)_R$ Wilson line prescribe that the coefficient $2h$ of the $U(1)_R$ connection $A^{(R)}$ is quantized in units of $1/N$. Otherwise, the source that generates the classical configuration (3.130) would not be gauge invariant in the quantum theory. The constraint of bulk gauge invariance agrees with the periodicity constraint in the dual theory. The bounds on the spin j are more intricate to argue. We provide a sketch of the reasoning that leads to those bounds. The lower bound is a consequence of only allowing normalizable discrete representations as sources. The upper bound can most easily be viewed as a consequence of spectral flow [41]. The action of spectral flow and its relation to boundary spectral flow was analyzed in [13]. We surmise that the boundary argument for the upper bound on j can be carried over to the bulk. Clearly, these arguments use in part the microscopic description of the bulk theory implied by its definition as the dual of $N = 2$ Liouville conformal field theory. Thus, the analysis of the chiral solution space illustrates well at which stage the definition of the bulk theory through the boundary quantum field theory intervenes.

3.5.4 The Bulk Chiral Ring

Chiral primary operators have an operator product at coinciding points which turns the set of chiral primaries into a ring. In the case where the level k is an integer and the Witten index N is chosen equal to the level, the topological Liouville theory and its deformations were carefully studied in chapter 2. The chiral ring structure constants at the conformal point were fixed by anomalous $U(1)_R$ charge conservation. We claimed that the same property holds in the theory with more general Witten index N in subsection 3.5.2. In the following, we stress that the bulk supergravity theory is subject to the same anomalous R-charge conservation argument.

Indeed, the anomalous R-charge conservation follows from contour deformation on the sphere, as well as the anomalous transformation of the R-current under conformal transformations. The latter is an immediate consequence of the form of the topological energy-momentum tensor :

$$T_{top} = T + \frac{1}{2}\partial J. \quad (3.131)$$

As a result of the twist, the conformal transformation property of the $U(1)_R$ current changes due to the third order pole in its operator product expansion with the energy-momentum tensor, proportional to $c/3$. As a consequence, the charge conservation rule on a boundary sphere reads

$$\sum_i q_i = \frac{c}{3}, \quad (3.132)$$

where the q_i are the R-charges of vertex operator insertions. Equivalently, the anomalous conservation rule comes from the term in the action of the form

$$S_R = \frac{ik_R}{16\pi} \int \mathcal{R}^{(2)} \theta, \quad (3.133)$$

which arose upon twisting. This term then feeds, on the sphere, into the conservation rule that follows from integrating over the zero mode of the field θ , again leading to anomalous R-charge conservation.

An important point to realize is that the same argument applies, *mutatis mutandis*, to the bulk supergravity theory. Indeed, the above reasoning depended only on the symmetry structure of the theory, which is known to be present in the bulk supergravity theory as well. Equivalently, we derived the term (3.133) from the twisted bulk supergravity theory in subsection 3.5.1. In summary, anomalous R-charge conservation fixes the structure constants at the conformal point both on the boundary and in the bulk.

It is important to note that in our duality, we assume that the bulk and boundary measures of integration in the path integral are the same. When applying localization in the style of [44], we assume this in particular for the zero mode integration over bosonic as well as fermionic zero modes.

Finally, we make the observation that when we talk about the gravitational side, we have made it manifest that what we mean by a topological theory of gravity in anti-de Sitter space-time is a theory that is twisted in such a manner as to become independent of the boundary value of the metric.

Chapitre 4

Twisted String Theory in Anti-de Sitter Space

In this chapter, we define a string theory in three-dimensional anti-de Sitter space-time with specific couplings to the boundary metric. The bulk generating function of correlation functions is defined as a functional of the boundary metric. Its derivative with respect to the boundary metric gives the boundary energy-momentum tensor. Our goal will be to demonstrate that with an appropriate twist in the boundary conditions, the boundary energy-momentum tensor is the energy-momentum tensor of the topologically twisted boundary theory. Since the latter is known to be cohomologically exact, that guarantees the independence of the bulk correlators from the boundary metric. We thus implement the idea of topologically twisting the *AdS/CFT* correspondence in three-dimensional string theory.

This chapter and appendices E and F will consist of contents from my paper [80] with J. Troost, with some reorganization and rephrasing to fit in this thesis.

4.1 The Generating Function

We define the string theory generating function Z of correlation functions :

$$Z[\phi^{(0)}, J] = \sum_{\text{genus}=0}^{\infty} Z_{\text{genus}}[\phi^{(0)}, J] = \sum_{\text{genus}=0}^{\infty} \int [DX]_{\text{genus}} \exp(-S[\phi^{(0)}, J, X]) . \quad (4.1)$$

The generating function Z has a perturbative definition as a sum over genera. We emphasized that the generating function of correlation functions depends on the background fields that we collectively denoted $\phi^{(0)}$. The generating function is also a functional of the sources J which couple to the normalizable string excitations. The correlation functions of normalizable string states in the background $\phi^{(0)}$ can be obtained genus by genus through derivation with respect to the sources J and by path integrating over the world sheet fields X and the moduli space of Riemann surfaces. In principle, we can shift the background fields $\phi^{(0)}$ by (for instance) coherent states of the normalizable modes, and obtain a different background which is part of the same quantum theory [81]. The non-normalizable part of the fields of the theory however are given once and for all, and they specify the quantum theory of gravity at hand. The

traditional definition of perturbative first quantized string theory makes sense when the background as well as the perturbative excitations are on-shell.

We will study the generating function $Z[\phi^{(0)}, J]$ and its derivatives in string theory in a three-dimensional space-time with a negative cosmological constant. The most frequently studied background in this category is the three-dimensional anti-de Sitter Poincaré patch with metric :

$$ds^2 = l^2 \left(\frac{dr^2}{r^2} + r^2 dx d\bar{x} \right). \quad (4.2)$$

The background is supported by a NS-NS three-form field strength $H_{(3)}$ which is proportional to the volume form on the AdS_3 space-time. The dilaton Φ is constant. The boundary metric in the Poincaré patch (4.2) is flat. One of our goals is to understand the bulk holographic counterpart to the distinction between a physical boundary theory with a standard energy-momentum tensor, and a topologically twisted boundary theory with an energy-momentum tensor which is trivial in cohomology. To make the distinction between energy-momentum tensors, we need to study their coupling to the boundary metric. We therefore analyze string theory on a broader set of backgrounds than the AdS_3 space-time (4.2) with flat boundary.

In field theory, we compute the energy-momentum tensor as the derivative of the action with respect to the metric. In our first quantized string theoretic approach, a good stand-in for the space-time effective action is the generator of correlation functions Z . Indeed, it has been argued that the string effective action is closely related to the generating function of correlation functions (see e.g. [82] for a review). Thus, the boundary energy-momentum tensor will measure the change of the generating function Z under perturbations of the boundary metric.

Therefore, we are motivated to study string theory in a bulk space-time metric $G_{\mu\nu}$ which depends on a boundary space-time metric $g_{ij}^{(0)}$. To leading order in a radial expansion, we study strings propagating in the background space-time :

$$ds^2 = l^2 \left(\frac{dr^2}{r^2} + r^2 g_{ij}^{(0)} dx^i dx^j + O(r^0) dx^i dx^j \right). \quad (4.3)$$

The boundary metric $g_{ij}^{(0)}$ is part of the non-normalizable boundary conditions in the bulk. It is important to note that its introduction will also lead to subleading terms in the metric since the total background must remain on-shell. Once we introduce the background boundary metric $g_{ij}^{(0)}$, the generating function $Z[g_{ij}^{(0)}, J]$ becomes a functional of the metric. We can differentiate the generating function Z with respect to the boundary metric to obtain a world sheet vertex operator that codes the space-time boundary energy-momentum tensor. This space-time energy-momentum tensor operator was determined in [83,84] for a flat boundary metric through slightly different means. A derivation that roughly matches the scheme outlined above can be found in [85]. We will generalize the derivation to the case of a generic conformally flat boundary.

Moreover, we wish to show that for AdS_3 backgrounds dual to $N = 2$ superconformal field theories on the boundary, there is a definition of bulk backgrounds depending

on boundary metrics $g_{ij}^{(0)}$ that leads to a topological boundary space-time energy-momentum tensor. Indeed, the boundary $N = 2$ superconformal field theory can be twisted to a topological conformal field theory [9, 20] (see also chapter 1). The energy-momentum tensor of the untwisted boundary $N = 2$ superconformal field theory we denote by \mathcal{T} , its left-moving $U(1)_R$ current by \mathcal{J}^R and the left-moving $N = 2$ supercurrents by $\mathcal{G}^{\pm 1}$. The topological conformal field theory is defined by restricting observables to a zero mode supercharge \mathcal{G}_0^+ cohomology, and by observing that the twisted energy-momentum tensor

$$\mathcal{T}_{xx}^{\text{top}} = \mathcal{T}_{xx} + \frac{1}{2} \nabla_x \mathcal{J}_x^R \quad (4.4)$$

is \mathcal{G}_0^+ exact as a consequence of the boundary $N = 2$ superconformal algebra [9, 20]. Thus, part of our task is to show that there is a string theory generating function of correlation functions whose derivative with respect to the boundary metric gives rise to the topological energy-momentum tensor \mathcal{T}^{top} . This is what we wish to demonstrate in this chapter for backgrounds with a conformally flat boundary.

4.2 The Three-Dimensional Stringy Geometries

In this section, we dress standard three-dimensional solutions of Einstein gravity with a negative cosmological constant and arbitrary boundary metric [56] with a NS-NS three-form flux $H_{(3)}$ and a constant dilaton Φ such that they become solutions of the space-time equations of motion of string theory.

Three-dimensional gravity with a negative cosmological constant has the following solution in Fefferman-Graham gauge [56]

$$ds^2 = l^2 \left(\frac{dr^2}{r^2} + (r^2 g_{ij}^{(0)} + g_{ij}^{(2)} + r^{-2} g_{ij}^{(4)}) dx^i dx^j \right), \quad (4.5)$$

where $g_{ij}^{(0)}(x^k)$ is a freely chosen boundary metric while $g^{(2,4)}$ will be largely determined by the Einstein equations :

$$g_{ij}^{(2)} = -\frac{1}{2} R^{(0)} g_{ij}^{(0)} - \frac{4G_N}{l} T_{ij} \quad (4.6)$$

$$g_{ij}^{(4)} = \frac{1}{4} g_{ik}^{(2)} g^{(0)kl} g_{lj}^{(2)}, \quad (4.7)$$

where T_{ij} is not uniquely determined. The Einstein equations only constrain it to be conserved, i.e. $\nabla^i T_{ij} = 0$, and satisfy the trace anomaly $T_i^i = -c R^{(0)}/12$ with Brown-Henneaux central charge $c = 3l/(2G_N)$ [11].

We now consider the string theory in the NS-NS sector with a background metric $G_{\mu\nu}$, a NS-NS three-form flux $H_{(3)}$ as well as a dilaton Φ . These fields should satisfy

1. We use a different notation for the energy-momentum tensor and the currents from the the previous chapters to avoid confusion with the worldsheet energy-momentum tensor and currents.

the following equations of motion

$$\begin{aligned}
\alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} &= 0 \\
-\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi H_{\omega\mu\nu} &= 0 \\
\frac{c_{\text{matter}} - c_{\text{critical}}}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} &= 0.
\end{aligned} \tag{4.8}$$

We consider the solution where the metric satisfies

$$R_{\mu\nu} = -\frac{2}{l^2} G_{\mu\nu}, \tag{4.9}$$

i.e. the 3-dimensional Einstein equation with a negative cosmological constant $-\frac{1}{l^2}$, and take the dilaton Φ to be constant such that the string coupling we get is constant. In this case, the above equations of motion are satisfied by the following NS-NS 3-form flux

$$H_{(3)} = \frac{2}{l} \sqrt{|G|} dx^\mu \wedge dx^\nu \wedge dx^\rho, \tag{4.10}$$

which is proportional to the volume form of the metric solution.

We summarize the stringy standard background solution :

$$\begin{aligned}
ds^2 &= l^2 \left(\frac{dr^2}{r^2} + (r^2 g_{ij}^{(0)} + g_{ij}^{(2)} + r^{-2} g_{ij}^{(4)}) dx^i dx^j \right) \\
H_{(3)} &= \frac{2}{l} \sqrt{|G|} dx^\mu \wedge dx^\nu \wedge dx^\rho \\
\Phi &= \text{constant}.
\end{aligned} \tag{4.11}$$

4.3 The twisted background

To topologically twist we need a boundary conformal field theory with at least $N = 2$ supersymmetry and therefore a $U(1)_R$ symmetry. The conditions on the bulk string theory in order to reach such boundary theories were carefully analyzed in [86]. If we assume that the boundary conformal field theory has both a left and a right $N = 2$ superconformal algebra, then a reasonably generic representation of backgrounds that allow for this boundary conformal symmetry are backgrounds with an extra circle direction. Thus, we generalize the standard string background (4.11) to include an extra circle.

We consider a four-dimensional spacetime which is the product of a three dimensional space-time with negative cosmological constant and a circle fibered over the locally AdS_3 factor. Again, we concentrate on the NS-NS sector. We pick a dilaton Φ which is constant and consider a four-dimensional metric G_{MN}^4 as well as a NS-NS two-form $B_{(2)}^4$ of the type :

$$\begin{aligned}
G_{MN}^4 dx^M dx^N &= G_{\mu\nu} dx^\mu dx^\nu + G_{44} (d\theta + A_\mu dx^\mu)^2 \\
B_{(2)}^4 = \frac{1}{2} B_{MN} dx^M \wedge dx^N &= \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + B_{\mu 4} dx^\mu \wedge d\theta,
\end{aligned} \tag{4.12}$$

where the indices M, N can take three-dimensional values μ as well as 4. We dimensionally reduce the space-time action

$$S_{\text{space-time}} = \frac{1}{2\kappa_0^2} \int d^4x (-G^4)^{\frac{1}{2}} e^{-2\Phi} (R + 4\nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} H_{MNL} H^{MNL}) \quad (4.13)$$

on the circle and get the three-dimensional action [87],

$$S_{\text{space-time}} = \frac{2\pi R}{2\kappa_0^2} \int d^3x (-G)^{\frac{1}{2}} e^{-2\Phi_3} \left(R_3 - \partial_\mu \sigma \partial^\mu \sigma + 4\partial_\mu \Phi_3 \partial^\mu \Phi_3 - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} (\tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} + 3e^{-2\sigma} H_{4\mu\nu} H_4^{\mu\nu}) \right) \quad (4.14)$$

where

$$\begin{aligned} G_{44} &= e^{2\sigma}, & \Phi_3 &= \Phi - \sigma/2, \\ \tilde{H}_{\mu\nu\lambda} &= (\partial_\mu B_{\nu\lambda} - A_\mu H_{4\nu\lambda}) + \text{cyclic}. \end{aligned} \quad (4.15)$$

If we take both the dilaton Φ and the metric component G_{44} constant, the action can be written as

$$S_{\text{space-time}} = \frac{2\pi R}{2\kappa_0^2} \int d^3x (-G)^{\frac{1}{2}} e^{-2\Phi_3} \left(R_3 - \frac{1}{12} \tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} e^{-2\sigma} H_{4\mu\nu} H_4^{\mu\nu} \right). \quad (4.16)$$

Thus, we have an almost standard three-dimensional Neveu-Schwarz-Neveu-Schwarz sector, as well as two Maxwell fields A_μ and $B_{4\mu}$. In the following, we set the metric component $G_{44} = 1$ by rescaling the coordinate θ , and redefine the Maxwell fields by

$$\begin{aligned} A^R &= \frac{1}{2l} (A_\mu + B_{\mu 4}) \\ \bar{A}^R &= \frac{1}{2l} (A_\mu - B_{\mu 4}). \end{aligned} \quad (4.17)$$

4.3.1 Degrees of Freedom

An understanding of the dynamics of the gauge field fluctuations around the solution (4.11) will be useful. To that end, we plug in the background NS-NS $H_{(3)}$ flux proportional to the volume form in (4.16). We concentrate on the fluctuations of the gauge fields A^R and \bar{A}^R only, and find the quadratic part of the effective action

$$S_{\text{gauge}}^{\text{quad}} = \frac{2\pi R e^{-2\Phi_3}}{2\kappa_0^2} \int d^3x \left(2l(A^R \wedge dA^R - \bar{A}^R \wedge d\bar{A}^R) - l^2(F^R * F^R + \bar{F}^R * \bar{F}^R) \right). \quad (4.18)$$

The free part of the equation of motion of the gauge field A^R reads :

$$d * F^R = -\frac{2}{l} dA^R. \quad (4.19)$$

As a consequence, the gauge field A^R can be split into two terms [88] :

$$A^R = A_{\text{flat}}^R - \frac{l}{2} * F^R. \quad (4.20)$$

The first term consists of a flat connection A_{flat}^R . The second term is proportional to the Hodge dual of the gauge invariant field strength and satisfies the equation of motion

$$l * d(A^R - A_{\text{flat}}^R) = -2(A^R - A_{\text{flat}}^R) \quad (4.21)$$

of a massive field dual to a $(2, 1)$ primary operator [89]. It is important to us that the flat part of the gauge field is dual to the current while its curvature is dual to a higher dimensional operator [89]. Moreover, note that the flat gauge field A_{flat}^R satisfies a first order differential equation and that we therefore expect to fix a single component of the flat gauge field at infinity when solving the equations of motion.

4.3.2 Flat Generalizations

The dimensionally reduced action (4.16) makes it manifest that we can generalize the stringy standard solution (4.11). Indeed, when the two Maxwell fields are flat, the NS-NS equations of motion reduce to the three-dimensional equations of motion (4.8). We conclude that the stringy standard background (4.11) can be augmented to a four-dimensional solution :

$$\begin{aligned} \Phi &: \text{constant} \\ G_{\mu\nu} &: \text{locally } AdS_3 \text{ with a non-trivial boundary metric} \\ H &: \text{proportional to the volume form} \\ G_{44} &: \text{constant} \\ A^R \text{ and } \bar{A}^R &: \text{flat.} \end{aligned} \quad (4.22)$$

4.3.3 The Explicit Boundary Gauge Field Dependence

Firstly, we consider a class of solutions that allows us to compute the boundary $U(1)_R$ current \mathcal{J}^R . The current couples to flat boundary gauge field fluctuations. Thus, we add a flat boundary fluctuation of the gauge field to the background solution (4.11). We parameterize the solution in terms of the fluctuation of the gauge field component δA_x^R and find from the flatness equation that the other component is given by :

$$\delta A_x^R = \int^{\bar{x}} \partial_x \delta A_{\bar{x}}^R. \quad (4.23)$$

This is a simple example of how the solution takes on a non-local character when we parameterize it in terms of boundary fluctuations that couple directly to conserved currents. It is important to note that parameterising the theory in terms of a given boundary component is tantamount to adding a particular boundary term to the action [90]. The resulting action should have an energy bounded from below. This requirement fixes the boundary component to be chosen in terms of the sign of the Chern-Simons level of the quadratic effective action (4.18) for the gauge fields [90]. We have chosen our boundary component accordingly.

4.3.4 The Explicit Boundary Metric Dependence

In the following, we choose to perturb the boundary metric around a conformally flat metric. Conformally flat background metrics have the advantage of allowing for

non-trivial boundary curvature $R^{(0)}$, while still retaining some of the simplicity of the background with a flat boundary metric.² Thus, we firstly note that the equation (4.6) for the subleading metric perturbation combined with the trace condition on the energy-momentum tensor permits a closed form local solution for the conformally flat boundary metric :

$$g_{ij}^{(0) \text{ conf. flat}} dx^i dx^j = e^{2\omega} dx d\bar{x}. \quad (4.24)$$

In the following we work with a background boundary metric which is a small perturbation of the conformally flat metric, and we shall solve for the bulk metric explicitly. In this generalization, non-local terms appear.

Thus, we wish to compute the explicit subleading metric dependence of the standard solution (4.11) for a boundary metric of the form :³

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2} + r^2 e^{2\omega} dx d\bar{x} + r^2 h_{xx} dx dx \\ &\quad + (\partial_x^2 \omega - (\partial_x \omega)^2) dx dx + 2\partial_x \partial_{\bar{x}} \omega dx d\bar{x} + (\partial_{\bar{x}}^2 \omega - (\partial_{\bar{x}} \omega)^2) d\bar{x} d\bar{x} \\ &\quad + \delta g_{xx}^{(2)} dx dx + 2\delta g_{x\bar{x}}^{(2)} dx d\bar{x} + \delta g_{\bar{x}\bar{x}}^{(2)} d\bar{x} d\bar{x} \\ &\quad + O(r^{-2}). \end{aligned} \quad (4.25)$$

As advertised, we have added a conformal factor $e^{2\omega}$ and have explicitly solved for the $O(r^0)$ subleading term as a function of the conformal factor ω . On top of this conformally flat boundary metric, we have added a leading $O(r^2)$ perturbation h_{xx} of the boundary metric component $g_{xx}^{(0)}$. Our task is to explicitly solve for the dependence of the subleading terms $\delta g_{ij}^{(2)}$ as a functional of the leading perturbation h_{xx} . Solving Einstein's equations to leading order in the metric perturbation h_{xx} leads to the equalities :

$$\delta g_{xx}^{(2)} = -\frac{1}{2} e^{-2\omega} (\partial_x \partial_{\bar{x}} h_{xx} - 2\partial_x \partial_{\bar{x}} \omega h_{xx} - 4\partial_x \omega \partial_{\bar{x}} h_{xx}), \quad (4.26)$$

$$\delta g_{x\bar{x}}^{(2)} = -\frac{1}{2} e^{-2\omega} [\partial_{\bar{x}}^2 h_{xx} - 2\partial_{\bar{x}} \omega \partial_{\bar{x}} h_{xx} - 2(\partial_{\bar{x}}^2 \omega - (\partial_{\bar{x}} \omega)^2) h_{xx}], \quad (4.27)$$

$$\delta g_{\bar{x}\bar{x}}^{(2)} = -\frac{1}{2} \int^x \partial_{\bar{x}}^3 (e^{-2\omega} h_{xx}). \quad (4.28)$$

The exact subleading $O(r^{-2})$ metric dependence on the perturbation is then easily obtained by plugging this perturbation for $g_{ij}^{(2)}$ into the consequence (4.7) of Einstein's equations. The determinant of the metric will be given by

$$\sqrt{G} = r \sqrt{g^{(0)}} \left(1 + \frac{g^{(0)ij} g_{ji}^{(2)}}{2r^2} + \frac{1}{4r^4} \frac{g^{(2)}}{g^{(0)}} \right). \quad (4.29)$$

and one can thus radially integrate the NS-NS three-form to obtain the two-form B fields :

$$B_{(2)} = \sqrt{g^{(0)}} (r^2 + \log r) g^{(0)ij} g_{ji}^{(2)} - \frac{1}{4r^2} \frac{g^{(2)}}{g^{(0)}} dx^i \wedge dx^j. \quad (4.30)$$

2. The curvature is crucial, to give but one example, to understand the boundary theory on a two-sphere. Indeed, while the boundary theory we are aiming for may be locally independent of the metric, it still depends on topological curvature invariants like the Euler number of the boundary Riemann surface. See e.g. the discussion at the end of chapter 3.

3. We often set the cosmological constant length scale $l = 1$ from now on.

We keep our string background on the mass shell, and compute the perturbation of the NS-NS two-form potential as a consequence of the metric perturbation h_{xx} to zeroth (or logarithmic) order in the radial coordinate r . We note that the metric perturbation h_{xx} does not change (the square root of the absolute value of) the determinant of the boundary metric $g_{ij}^{(0)}$ – it remains $\sqrt{g^{(0)}} = \frac{1}{2}e^{2\omega}$. Furthermore, we recall that the trace of the second order of the boundary metric equals half the boundary Ricci scalar, $g^{(0)ij}g_{ji}^{(2)} = -R^{(0)}/2$, such that we need the dependence of the Ricci-scalar on the perturbation h_{xx} to first order :

$$R^{(0)} = 4e^{-4\omega}(-2\partial_{\bar{x}}h_{xx}\partial_{\bar{x}}\omega + \partial_{\bar{x}}^2h_{xx} - 2e^{2\omega}\partial_x\partial_{\bar{x}}\omega). \quad (4.31)$$

Combining equations (4.30) and (4.31), we know explicitly the NS-NS two-form potential $B_{(2)}$ to first order in the metric perturbation h_{xx} .

4.3.5 The Asymptotic Twisted Generalization

Finally, we generalize the solution (4.11) to the background central to our intent. We introduce a dependence of the $U(1)_R$ gauge field on the asymptotic boundary metric $g_{ij}^{(0)}$ in order to couple the boundary metric non-trivially to the R-current. We draw inspiration from the analogous exercise performed in supergroup Chern-Simons theory in chapter 3 as well as from the literature on topological quantum field theories [8]. We wish to introduce a coupling $(\mathcal{J}^R \pm \bar{\mathcal{J}}^R)_\mu \omega^\mu$ between the R-currents and the spin connection one-form $\omega = \omega_\mu dx^\mu$ on the boundary, where the relative sign depends on the twist we perform.

Near the conformally flat backgrounds, the boundary conditions for the gauge field are

$$A_{\bar{x}}^R = -\frac{i}{4}\omega_{\bar{x}}^{+-}, \quad \bar{A}_x^R = \mp \frac{i}{4}\omega_x^{+-} \quad (4.32)$$

where ω^{+-} is the spin connection of the boundary metric. For the boundary metric perturbed by h_{xx} , the zweibeins are given by

$$e^+ = e^\omega dx, \quad e^- = e^\omega d\bar{x} + e^{-\omega} h_{xx} dx. \quad (4.33)$$

The variation of the spin connections under this h_{xx} perturbation is chiral and equal to

$$\delta\omega_x^{+-} = -2e^{-2\omega}\partial_{\bar{x}}h_{xx} = -2\nabla_{\bar{x}}(g^{\bar{x}x}h_{xx}), \quad \delta\omega_{\bar{x}}^{+-} = 0. \quad (4.34)$$

Therefore, the response of the gauge field component \bar{A}_x^R to the perturbation equals

$$\delta\bar{A}_x^R = \pm \frac{i}{2}\nabla_{\bar{x}}(g^{\bar{x}x}h_{xx}), \quad (4.35)$$

while $A_{\bar{x}}^R$ does not vary. Since we restrict ourselves to flat gauge fields which couple to the boundary R-current, the variation of the other components follow from the flatness condition.

4.4 The Space-Time Energy-Momentum Operator

In this section we use the perturbed background metrics and gauge potentials of the previous section and plug them into the generating function reviewed in section 4.1. We then differentiate with respect to the metric and gauge field perturbations to obtain the world sheet vertex operators that correspond to the boundary space-time energy-momentum tensor and R-current in backgrounds with conformally flat boundary. We will prove that the twisted string theory background leads to a topological energy-momentum tensor.

4.4.1 The Energy-Momentum

The worldsheet action we consider is

$$S = \frac{1}{2\pi\alpha'} \int d^2z (G_{MN} + B_{MN}) \partial X^N \bar{\partial} X^M, \quad (4.36)$$

where the metric G_{MN} and the NS-NS two-form B_{MN} are the ones given in the previous section. Differentiating with respect to boundary metric, one finds the boundary energy-momentum tensor to $O(r^0)$ in the r^{-2} expansion :

$$\begin{aligned} \mathcal{T}_{\bar{x}\bar{x}}(x', \bar{x}') &= \frac{l^2 e^{2\omega(x')}}{\alpha'} \int d^2z \left[r^2 \delta^{(2)}(x - x') \partial x \bar{\partial} x \right. \\ &\quad - \frac{1}{2} [(e^{-2\omega} \partial_x \partial_{\bar{x}} \delta^{(2)}(x - x')) - 2 \partial_x \partial_{\bar{x}} \omega \delta^{(2)}(x - x') - 4 \partial_x \omega \partial_{\bar{x}} (\delta^{(2)}(x - x'))] \partial x \bar{\partial} x \\ &\quad - \frac{1}{2} [\partial_{\bar{x}} (e^{-2\omega(x)} \partial_{\bar{x}} \delta^{(2)}(x' - x)) - 2e^{-2\omega} (\partial_{\bar{x}}^2 \omega - (\partial_{\bar{x}} \omega)^2) \delta^{(2)}(x - x')] (\partial x \bar{\partial} \bar{x} + \partial \bar{x} \bar{\partial} x) \\ &\quad - \frac{1}{2} \int^x \partial_{\bar{x}}^3 (e^{-2\omega} \delta^{(2)}(x - x')) \partial \bar{x} \bar{\partial} \bar{x} \\ &\quad \left. - \log r \partial_{\bar{x}} (e^{-2\omega} \partial_{\bar{x}} (\delta^{(2)}(x - x'))) (\partial \bar{x} \bar{\partial} x - \partial x \bar{\partial} \bar{x}) \right]. \end{aligned} \quad (4.37)$$

The equations of motion can be used to simplify this form. They imply the following order estimate

$$\bar{\partial} x = O(r^{-2}), \quad \partial \bar{x} = O(r^{-2}), \quad \partial \bar{\partial} (\log r + \omega) = O(r^{-2}), \quad (4.38)$$

and thus the $O(r^2)$ term collapses to a term of order $O(r^0)$ on shell while some of the other terms including all the non-local ones vanish to $O(r^0)$. The energy-momentum tensor after this simplification can be written as

$$\begin{aligned} \mathcal{T}_{\bar{x}\bar{x}}(x', \bar{x}') &= \frac{l^2 e^{2\omega(x')}}{\alpha'} \int d^2z \left[r^2 \delta^{(2)}(x - x') \partial x \bar{\partial} x \right. \\ &\quad - \frac{1}{2} [\partial_{\bar{x}} (e^{-2\omega(x)} \partial_{\bar{x}} \delta^{(2)}(x' - x)) - 2e^{-2\omega} (\partial_{\bar{x}}^2 \omega - (\partial_{\bar{x}} \omega)^2) \delta^{(2)}(x - x')] \partial x \bar{\partial} \bar{x} \\ &\quad \left. + \log r \partial_{\bar{x}} (e^{-2\omega} \partial_{\bar{x}} (\delta^{(2)}(x - x'))) \partial x \bar{\partial} \bar{x} \right]. \end{aligned} \quad (4.39)$$

One can also compute the component $T_{x\bar{x}}$, which is related to the trace of the energy-momentum tensor, in the same way. This component is given by

$$\mathcal{T}_{x\bar{x}} = \frac{e^{4\omega}}{4} \mathcal{T}^{x\bar{x}} = \frac{e^{4\omega}}{4} \frac{1}{2} \frac{4\pi \delta S}{\sqrt{g^{(0)}} \delta h_{x\bar{x}}} = \frac{\pi \delta S}{\delta \omega}. \quad (4.40)$$

The leading order in this vertex operator equals

$$\begin{aligned}\mathcal{T}_{x\bar{x}} &= \frac{l^2}{\alpha'} \int d^2z \left[\partial_x \partial_{\bar{x}} \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x} - \log r \partial_x \partial_{\bar{x}} \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x} \right] \\ &= \frac{l^2}{\alpha'} \int d^2z \left[\partial_x \partial_{\bar{x}} \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x} + \partial_x \delta^{(2)}(x' - x) \partial x \bar{\partial} \log r \right].\end{aligned}\quad (4.41)$$

Upon partial integration, the first term is subleading, while the second term, using the equations of motion, can be rewritten as :

$$\mathcal{T}_{x\bar{x}} = \frac{l^2}{\alpha'} \int d^2z \partial \delta^{(2)}(x' - x) \bar{\partial} \log r = -\frac{l^2}{\alpha'} \int d^2z \partial \delta^{(2)}(x' - x) \bar{\partial} \omega = \frac{l^2}{\alpha'} \int d^2z \delta^{(2)}(x' - x) \partial \bar{\partial} \omega.$$

One therefore finds the trace of the energy momentum tensor equal to

$$4e^{-2\omega(x')} \mathcal{T}_{x\bar{x}} = -\frac{l^2 R^{(0)}}{2\alpha'} \int d^2z \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x}. \quad (4.42)$$

Recall that the space-time central charge operator C^{st} from [84] :

$$C^{\text{st}} = 6 \frac{l^2}{\alpha'} \int d^2z \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x}, \quad (4.43)$$

one concludes that the trace of the energy momentum tensor \mathcal{T} satisfies the operator relation

$$\mathcal{T}_i^i = -\frac{C^{\text{st}}}{12} R^{(0)}. \quad (4.44)$$

We will prove that the energy momentum tensor is conserved in appendix E.

Compare with Literature

We will now consider a special simple case where $\omega = \text{constant}$, i.e. the boundary metric is flat, and prove that the energy momentum tensor agrees with the one determined by the exact conformal field theory approach of [83, 84]. For comparison purpose, we will temporarily use the coordinates $\phi, \gamma, \bar{\gamma}$ which is related to r, x, \bar{x} by

$$\begin{aligned}r &\rightarrow e^\phi \\ x &\rightarrow \gamma \\ \bar{x} &\rightarrow \bar{\gamma}.\end{aligned}\quad (4.45)$$

We also pick a different gauge for the NS-NS 2-form field B that will give the same NS-NS 3-form flux, which is now simplified to

$$H_{(3)} = (2e^{2\phi} \sqrt{g^{(0)}} - \partial_\gamma^2 h_{\bar{\gamma}\bar{\gamma}} + \dots) d\phi \wedge d\gamma \wedge d\bar{\gamma}. \quad (4.46)$$

The choice of the gauge is as follows :

$$\begin{aligned}B_{\gamma\bar{\gamma}} &= e^{2\phi} \sqrt{g^{(0)}} + O(e^{-2\phi}) \\ B_{\bar{\gamma}\phi} &= -\partial_\gamma h_{\bar{\gamma}\bar{\gamma}} + O(e^{-2\phi}).\end{aligned}\quad (4.47)$$

Taking $\omega = 0$, the holomorphic analog of (4.37) can be further simplified to

$$\begin{aligned} \mathcal{T}_{xx}(x, \bar{x}) \approx & \frac{l^2}{\alpha'} \int d^2 z \left[e^{2\phi} \delta^{(2)}(\gamma - x) \partial \bar{\gamma} \bar{\partial} \bar{\gamma} - \frac{1}{2} \partial_x^2 (\delta^{(2)}(\gamma - x)) \partial \gamma \bar{\partial} \bar{\gamma} \right. \\ & \left. + \partial_x (\delta^{(2)}(\gamma - x)) \bar{\partial} \bar{\gamma} \partial \phi \right]. \end{aligned} \quad (4.48)$$

Applying the property of the delta function

$$\delta^{(2)}(x) = \frac{1}{\pi} \partial_{\bar{x}} \frac{1}{x}, \quad (4.49)$$

the above energy-momentum tensor component can be rewritten as

$$\begin{aligned} \mathcal{T}_{xx}(x, \bar{x}) &= \frac{l^2}{\alpha'} \int d^2 z \left[e^{2\phi} \partial \bar{\gamma} \bar{\partial} \left(-\frac{1}{\pi(x - \gamma)} \right) - \frac{1}{2} \partial_x^2 \bar{\partial} \left(-\frac{1}{\pi(x - \gamma)} \right) \partial \gamma \right. \\ &\quad \left. + \partial_x \bar{\partial} \left(-\frac{1}{\pi(x - \gamma)} \right) \partial \phi + O(e^{-2\phi}) \right] \\ &= \frac{l^2}{\alpha'} \int d^2 z \left[e^{2\phi} \partial \bar{\gamma} \bar{\partial} \left(-\frac{1}{\pi(x - \gamma)} \right) - \bar{\partial} \left(-\frac{1}{\pi(x - \gamma)^3} \right) \partial \gamma - \bar{\partial} \left(-\frac{1}{\pi(x - \gamma)^2} \right) \partial \phi \right. \\ &\quad \left. + O(e^{-2\phi}) \right]. \end{aligned} \quad (4.50)$$

The equations of motion imply the following order estimate

$$\begin{aligned} \bar{\partial}(e^{2\phi} \partial \bar{\gamma}) &= O(e^{-2\phi}) \\ \partial(e^{2\phi} \bar{\partial} \gamma) &= O(e^{-2\phi}) \\ \partial \bar{\partial} \phi &= O(e^{-2\phi}) \end{aligned}$$

and therefore $e^{2\phi} \partial \bar{\gamma}$, $\partial \gamma$ and $\partial \phi$ are all holomorphic up to $O(e^{-2\phi})$ terms. One can then partially integrate the above form and get

$$\mathcal{T}_{xx}(x, \bar{x}) = \frac{l^2}{\alpha'} \oint \frac{dz}{2\pi i} \left[\frac{e^{2\phi} \partial \bar{\gamma}}{x - \gamma} - \frac{\partial \gamma}{(x - \gamma)^3} - \frac{\partial \phi}{(x - \gamma)^2} \right] + O(e^{-2\phi}) \text{ terms}. \quad (4.51)$$

This agrees (6.2) of [84], which is the $\phi \rightarrow \infty$ limit of the energy-momentum tensor. Thus, the result (4.37) at flat boundary matches with the literature.

4.4.2 The R-Symmetry Current

The $U(1)_R$ currents \mathcal{J}^R and $\bar{\mathcal{J}}^R$ couples to the perturbation of the flat boundary gauge field. They are chiral and the non-vanishing components \mathcal{J}_x^R and $\bar{\mathcal{J}}_{\bar{x}}^R$ are given by functionally differentiating the generating function with respect to A_x^R and $\bar{A}_{\bar{x}}^R$ respectively. Note that the gauge fields we consider always need to satisfy the flatness condition, one has to take into account the dependence of A_x^R and $\bar{A}_{\bar{x}}^R$ in the other gauge field components. Taking everything into account, the R-current \mathcal{J}^R is given by

$$\begin{aligned} \mathcal{J}_x^R(x', \bar{x}') &= 2\pi i \frac{\delta S}{\delta A_{\bar{x}}^R} \\ &= \frac{2il}{\alpha'} \int d^2 z [\delta^{(2)}(x - x') \bar{\partial} \bar{x} \partial \theta + \frac{\delta A_x^R(x, \bar{x})}{\delta A_{\bar{x}}^R(x', \bar{x}')} \bar{\partial} x \partial \theta] \\ &= \frac{2il}{\alpha'} \int d^2 z [\delta^{(2)}(x - x') \bar{\partial} \bar{x} \partial \theta + \int^{\bar{x}} \partial_x \delta^{(2)}(x - x') \bar{\partial} x \partial \theta], \end{aligned} \quad (4.52)$$

while $\bar{\mathcal{J}}^R$ follows analogously. We can show that the current component \mathcal{J}_x^R is holomorphic in conformally flat backgrounds and with vanishing gauge fields, $\partial_{\bar{x}}\mathcal{J}_x^R = 0$, up to contact terms in the quantum theory. The proof is as follows. First, one can use the flatness condition of the gauge fields to obtain

$$\partial_{\bar{x}} \frac{\delta A_x^R(x, \bar{x})}{\delta A_{\bar{x}}^R(x', \bar{x}')} = \partial_x \delta^{(2)}(x' - x). \quad (4.53)$$

Then, one uses this relation, the chain rule of functional derivative and that the world sheet current $\partial\theta$ is holomorphic when the gauge field vanishes, to obtain

$$\partial_{\bar{x}'} \mathcal{J}_x^R(x', \bar{x}') = -\frac{2il}{\alpha'} \int d^2z \bar{\partial}(\delta^{(2)}(x' - x)) \partial\theta \quad (4.54)$$

which is zero after partial integration.

4.4.3 The Topological Energy-Momentum Tensor

Recall that we work under the assumption that we have a string theory with space-time boundary $N = 2$ superconformal symmetry. Topologically twisting that superconformal field theory gives rise to a topological energy-momentum tensor of the form [9, 20]

$$\mathcal{T}_{xx}^{\text{top}} = \mathcal{T}_{xx} + \frac{1}{2} \nabla_x \mathcal{J}_x^R. \quad (4.55)$$

The topological energy-momentum tensor is \mathcal{G}_0^+ exact because it follows from the boundary $N = 2$ superconformal algebra that :

$$\mathcal{T}_{xx}^{\text{top}} = [\mathcal{G}_0^+, \mathcal{G}^-]. \quad (4.56)$$

We now functionally differentiate the generating function $Z[g^{(0)}]$ with respect to the boundary metric. Since the gauge fields now depend on the boundary metric, one gets not only the physical energy-momentum tensor, but also extra terms from this dependence. Using

$$\bar{\mathcal{J}}_{\bar{x}}^R = 2\pi i \frac{\delta S}{\delta \bar{A}_{\bar{x}}^R} \quad (4.57)$$

and the chain rule, the topological energy-momentum tensor is

$$\mathcal{T}_{\bar{x}\bar{x}}^{\text{top}} = \mathcal{T}_{\bar{x}\bar{x}} + 2\pi e^{2\omega} \int d^2x' \frac{\delta S}{\delta \bar{A}_{\bar{x}}^R(x', \bar{x}')} \frac{\delta \bar{A}_{\bar{x}}^R(x', \bar{x}')}{\delta h_{xx}} = \mathcal{T}_{\bar{x}\bar{x}} - i e^{2\omega} \int d^2x' \bar{\mathcal{J}}_{\bar{x}}^R(x', \bar{x}') \frac{\delta \bar{A}_{\bar{x}}^R(x', \bar{x}')}{\delta h_{xx}}. \quad (4.58)$$

Applying the gauge field variation (4.35), one finds the topological energy-momentum tensor

$$\begin{aligned} \mathcal{T}_{\bar{x}\bar{x}}^{\text{top}} &= \mathcal{T}_{\bar{x}\bar{x}} \mp \frac{1}{2} e^{2\omega} \partial_{\bar{x}} (e^{-2\omega} \bar{\mathcal{J}}_{\bar{x}}^R) \\ &= \mathcal{T}_{\bar{x}\bar{x}} \mp \frac{1}{2} \nabla_{\bar{x}} \bar{\mathcal{J}}_{\bar{x}}^R, \end{aligned} \quad (4.59)$$

in the conformally flat background (4.24). The holomorphic component can be acquired with the exact same reasoning and takes the form

$$\mathcal{T}_{xx}^{\text{top}} = \mathcal{T}_{xx} + \frac{1}{2} \nabla_x \mathcal{J}_x^R. \quad (4.60)$$

The extra terms agree with half the covariant derivative of the boundary R-current, thus proving the desired formula (4.55) for our twisted string theory.^{4, 5} The final result is closely tied to our choice of twisted asymptotic boundary conditions (4.32). Indeed, the final equation follows from the twisted boundary condition combined with the definition of the R-symmetry current. Our analysis has shown that we can embed the whole of this reasoning in on-shell string theory.

The expression (4.59) for the boundary energy-momentum tensor, combined with the boundary $N = 2$ superconformal algebra, guarantees an anomalous R-charge conservation rule for the correlators in the topologically twisted string theory. It will be interesting to analyze its consequences, for instance by computing twisted bulk string theory correlation functions from first principles.

4. We have proven the anti-holomorphic counterpart of the formula. The holomorphic version is proven analogously.

5. In Appendix F we briefly discuss how the trace of the energy-momentum tensor is modified in the twisted theory.

Chapitre 5

Topological Orbifold Theory

In this chapter, we will explore what is known about the correlators of the topologically twisted symmetric product conformal field theory. To understand the answer to this question, we reformulate both the results that are present on this problem in the mathematics literature, as well as the results in the physics literature. While the two have been rather convincingly matched where it concerns the spectrum of the topological conformal field theory, they have developed largely independently in as far as the structure constants of the ring of operators are concerned. Thus, we wish to simplify and connect these results such that they may shine light on both the mathematical as well as the physical side of the problem. We believe that the efficient mathematical description of the operator ring of the boundary theory also provides a scheme for the bulk analysis, and therefore for a proof of a topological subsector of an AdS/CFT correspondence.

This chapter and appendix G will be an interpolation of my paper [91] written with J. Troost.

5.1 The Complex Plane

A standard example of a symmetric product space in the mathematics literature is the symmetric product of the two-dimensional complex plane $M = \mathbb{C}^2$, or more precisely, the Hilbert scheme of points on the complex plane. An underlying reason is that the Hilbert scheme of points is both non-trivial and non-singular for two-dimensional complex surfaces [92–94]. A thorough understanding of this space [95–101] has lead to significant progress both in describing the cohomology of Hilbert schemes of points on generic complex surfaces, as well as their cohomology rings [102–106]. See e.g. [107] for an introduction. On the physics side of this domain, the discussion of the Hilbert scheme of points on the complex plane \mathbb{C}^2 is largely absent in the two-dimensional conformal field theory literature (although it is present in the literature on moduli spaces of instantons) since the more standard set-up is to study conformal field theories with a discrete spectrum, which requires the complex target space M to be compact. The compact surfaces $K3$ as well as T^4 are omnipresent in string theory compactifications, and the corresponding physical conformal field theories are well-studied. These studies have been extended to their symmetric orbifold products. In this

first section, we wish to bridge the gap between the standard model in mathematics, namely the Hilbert scheme of points on the complex plane \mathbb{C}^2 and the topologically twisted symmetric orbifold conformal field theory on the complex plane \mathbb{C}^2 .

Thus, we start out with a mathematical description of the cohomology of the Hilbert scheme of points on the plane, and the cohomology ring. Next, we define the symmetric orbifold conformal field theory, the operators in the twisted version of the theory, the cohomology, as well as the operator product of two cohomology elements. The conclusion will be that the two rings naturally match. Along the way, we will have learned a dictionary, as well as various subtleties that arise when discussing non-compact models. Moreover, the isomorphism permits us to obtain a large number of new results on the physical model.

5.1.1 The Hilbert Scheme of Points on the Complex Plane

We recall the definition of the Hilbert scheme of points on the plane \mathbb{C}^2 and its relation to the symmetric orbifold space. We review the calculation of its cohomology as well as the cup product in the cohomology ring. See [97, 107] for pedagogical introductions to the Hilbert scheme of points.

The Hilbert Scheme

The Hilbert scheme of points $\text{Hilb}^n(\mathbb{C}^2)$ on the complex plane is the space of ideals of co-dimension n in the space of polynomials $\mathbb{C}[x, y]$ [92]. The co-dimension one ideals are seen to be parameterized by the two-plane itself. The co-dimension two ideals correspond to either two distinct points or coinciding points together with an orientation. In general, the Hilbert scheme of points corresponds to a cover of the orbifold space $(\mathbb{C}^2)^n/S_n$ where S_n is the group that permutes the n copies of the complex two-plane. The projection is the Hilbert-Chow morphism. An important point is that the cover is regular when one studies the Hilbert scheme of points of a complex surface M [93]. The extra orientation data captured by the ideals of the polynomial ring desingularize the orbifold variety, and at the same time, correspond to the twisted sectors of the conformal field theory which is also regular when defined in accord with the axioms of two-dimensional conformal field theory [103, 104, 108, 109]. It is often handy to treat the Hilbert schemes for all values of n simultaneously. Physically, this is akin to studying a second quantized string theory [110].

The Cohomology

The final result for the cohomology of the Hilbert scheme of points on the complex two-plane is simple [96, 97, 107, 111]. One method to compute the cohomology is to introduce a perfect Morse function that projects contributions to the cohomology to the fixed points under a $\mathbb{C}^* \times \mathbb{C}^*$ two-torus action that multiplies the variables (x, y) by a non-zero complex number each [97]. The points of the Hilbert scheme that are invariant under the action are the monomial ideals, namely those ideals that are generated by monomials $x^a y^b$. When we divide the polynomial ring by the monomial ideal we find a vector space that has a basis of monomials that we can capture in a Young diagram. For a given value of n , we choose a partition $[\lambda] = [1^{m_1} 2^{m_2} \dots] = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0)$

of $n = \sum_{i=1}^l i m_i$ with length $l(\lambda) = \sum_i m_i$. The corresponding Young diagram is given by putting λ_1 boxes on the bottom row, λ_2 boxes on the second row and so on. The corresponding basis of the ring divided by the ideal is given by monomials $x^p y^q$ where (p, q) take values in the set $M_\lambda = \{(p, q) \in \mathbb{N}^2 : 0 \leq q < \lambda_{p+1}\}$. The contribution of a given fixed point to the cohomology is determined by the character of the torus action on the tangent space to the fixed point [97, 107]. After a calculation [97, 111], one finds that each fixed point partition $\mu \vdash n$ contributes to the Poincaré polynomial $P(t)$ as :

$$P(t)((\mathbb{C}^2)^{[n]}) = \sum_{\mu \vdash n} t^{2(n-l(\mu))}, \quad (5.1)$$

where the power of t keeps track of the degree of the cohomology elements. When we sum the Poincaré polynomials of all Hilbert schemes over n with dummy variable q to the power n , then we find the generator of Poincaré polynomials [97]

$$P(t, q) = \sum_{n \geq 0} q^n P_t((\mathbb{C})^{[n]}) = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m-2} q^m}. \quad (5.2)$$

We have $b_{2i}((\mathbb{C}^2)^{[n]}) = p(n, n-i)$ [96], namely the number of partitions of n into $n-i$ parts. Indeed, when we expand the denominator in the generating function (5.2), we need to match the sum of the powers of q to the number of copies n . Thus, we count partitions of n . Moreover, the power of t is twice the power of q , except that we subtract two for each factor and therefore in total twice the length of the partition as in equation (5.1).

The Cup Product

While the Betti numbers of the Hilbert scheme have been known for quite some time, the algebraic geometric understanding of the cohomology as a Fock space is more recent [97, 98]. This insight was further exploited to describe the ring structure constants of the cohomology ring [99–101] in elementary terms. The final result of the algebraic geometric analyses is as follows [100].

To each partition μ , one associates a conjugacy class of permutations $[\pi]$ which has cycle lengths given by the partition μ . We equip the space of functions on the permutations S_n with a convolution product $*$:

$$(f * g)(\pi) = \sum_{\sigma \in S_n} f(\pi \sigma^{-1}) g(\sigma), \quad (5.3)$$

that is inherited by the space of functions on the conjugacy classes. Moreover, each permutation π is assigned a degree $|\pi|$ which is equal to the minimal number of transpositions necessary to build it. For example, a single cycle permutation of length k has degree $k-1$. The ring structure of the cohomology is then described as follows. Associate a function on the conjugacy class to each cohomology class. Convolute the functions, under the condition that a term contributes if and only if the degrees of the permutations add. That convolution product on the class functions gives the structure constants in the cohomology ring of the Hilbert scheme of points on the plane [100].

A First Example

We will compute many structure constants in the following, and we start with a simple example. We denote the functions on the permutations in terms of the value the function f takes at a given permutation times that permutation : $f = \sum_{\pi \in S_n} f(\pi)\pi$. A first basic observation about the convolution product (5.3) is that if $f = \pi_1$ and $g = \pi_2$, then the convolution product at $\pi_1\pi_2$ equals one, and is zero otherwise. Consider a class function $\chi_{[\lambda]}$ on a conjugacy class $[\lambda]$:

$$\chi_{[\lambda]} = \sum_{\pi \in [\lambda]} \pi. \quad (5.4)$$

We wish to partially compute the convolution product of class functions $\chi_{[n_1]} * \chi_{[n_2]}$. In particular, we calculate a coefficient of interest, namely $(\chi_{[n_1]} * \chi_{[n_2]})(\chi_{[n_1+n_2-1]})$ where we have dropped the cycles of length 1 in the partition for the time being. We have chosen an example that satisfies the requirement that the degrees of the permutations add.

Firstly, note that we have $n!/((n-n_1)!n_1!) \times n_1!/n_1 = n!/((n-n_1)!n_1)$ n_1 -cycles. This is because we choose n_1 out of n elements, and then decide on their order. However, cyclically reordering them gives the same n_1 -cycle, hence the final division by the factor n_1 . For a given n_1 -cycle, to obtain a (n_1+n_2-1) -cycle, it is clear that we must have exactly one element in common between the n_1 -cycle and the n_2 -cycle. Thus, out of n_2 elements in the n_2 -cycle, we pick one to be one out of n_1 , and n_2-1 random out of the remaining $n-n_1$. We can choose to put the element that we chose out of the first n_1 first in the n_2 -cycle, and we then have $(n_2-1)!$ inequivalent orderings for the other elements. Thus, we had $n_1 \times (n-n_1)!/((n-n_1-n_2+1)!(n_2-1)!) \times (n_2-1)! = n_1 \times (n-n_1)!/(n-n_1-n_2+1)!$ options. Multiplying our options, we obtain a total of $n!/(n-n_1-n_2+1)!$ (n_1+n_2-1) -cycles. Since there are a total of $n!/((n-n_1-n_2+1)!(n_1+n_2-1))$ (n_1+n_2-1) -cycles, each of those cycles obtains a coefficient n_1+n_2-1 . We can write the final result in the self-evident notation :

$$[1^{n-n_1}n_1] * [1^{n-n_2}n_2] = (n_1+n_2-1)[1^{n-n_1-n_2+1}n_1+n_2-1] + \dots \quad (5.5)$$

Thus, the structure constant for the multiplication of the corresponding cohomology elements is n_1+n_2-1 , since the degrees of the permutations add. Calculating the structure constant is an exercise in combinatorics. We have performed a simple such exercise and will encounter more intricate examples in due course.

5.1.2 The Topological Conformal Field Theory

In the previous subsection, we have reviewed the mathematical description of the cohomology ring of the Hilbert scheme of points on the complex plane. In this subsection, we want to describe the relation between the mathematics and the topological conformal field theory on the symmetric orbifold of the complex plane. The Hilbert-Chow projection of the Hilbert scheme of points maps the Hilbert scheme onto the symmetric orbifold space. The latter can be thought off as the target space of the symmetric orbifold conformal field theory. However, the conformal field theory, similarly to the scheme, desingularizes the target space [103, 104]. Twisted sectors are necessarily

added to the theory and they capture new directions in the configuration space that render the theory consistent [108, 109].

The Conformal Field Theory

Let us describe the field content of the seed conformal field theory on \mathbb{C}^2 . We consider a $N = (4, 4)$ supersymmetric conformal field theory in two dimensions with four real scalars X^i and four Majorana fermions ψ^i . The complex scalars $X = (X^1 + iX^2)/\sqrt{2}$ and $Y = (X^3 + iX^4)/\sqrt{2}$ parameterize the complex two-plane \mathbb{C}^2 . The fermions $\psi_X^\pm = (\psi^1 \pm i\psi^2)/\sqrt{2}$ and $\psi_Y^\pm = (\psi^3 \pm i\psi^4)/\sqrt{2}$ live in the tangent bundle. The (left and right) fermions can be bosonized : $\psi_X^\pm = e^{\pm iH_X}$ and $\psi_Y^\pm = e^{\pm iH_Y}$. The theory enjoys a $N = 4$ superconformal symmetry in both the left- and right-moving sector with central charge $c = 6$. The symmetry of the seed and orbifold theory is directly related to the hyperkähler geometry of the target space.

The superconformal field theory has a continuous spectrum and the target space is non-compact. By picking a complex structure (or a preferred $N = 2$ superconformal subalgebra), we can define a chiral ring of operators. The superconformal generators of the $N = 2$ superconformal algebra can be chosen to be – see e.g. [112] :

$$\begin{aligned} G^+ &= i\sqrt{2}(\psi_X^+ \partial X + \psi_Y^+ \partial Y) \\ G^- &= i\sqrt{2}(\psi_X^- \partial \bar{X} + \psi_Y^- \partial \bar{Y}). \end{aligned} \quad (5.6)$$

When we compute the chiral ring cohomology, we need to decide in which space we compute it. Let us consider the space of polynomials in the scalar as well as the fermion fields. We moreover concentrate on chiral primaries such that we need to consider polynomial combinations of these fields only, and not their derivatives. Note that the operators \bar{X}, \bar{Y} and $\psi_{X,Y}^-$ are not annihilated by the cohomological operator $\oint G^+$, while the operators $\psi_{X,Y}^+$ are exact. Thus, we are left with an operator ring of chiral primaries generated by the complex fields X, Y . These operators have non-singular operator products – the logarithmic singularity that condemns these operators to play a marginal role in the description of the conformal field theory cancels in all calculations inside the ring. The chiral operator ring that we defined coincides with the polynomial ring $\mathbb{C}[X, Y]$ which was the starting point for the description of the Hilbert scheme.

The de Rham cohomology of the complex plane, on the other hand, is concentrated in degree zero, and it is of dimension one. The cohomology coincides with the cohomology of the $N = 1$ superconformal generator :

$$\oint G = \oint \delta_{ij} \psi^i \partial X^j, \quad (5.7)$$

along with its right-moving counterpart. The charge $\oint G$ acts as the differential operator d on differential forms represented as polynomials in the ψ^i differentials, depending on coefficients which are functions of the coordinates X^i . Thus, the $\oint G$ cohomology can be represented by the unit operator 1 only.¹

1. We note that for the non-compact manifold at hand, the de Rham and the Dolbeault cohomology do not coincide.

The symmetric product conformal field theory $Sym_N(\mathbb{C}^2)$ is defined by taking the tensor product of n copies of the $N = (4, 4)$ conformal field theory and dividing by the symmetric group S_n that permutes the copies. There is a list of definitions and prescriptions that determines the spectrum and correlation functions of the conformal field theory uniquely [108, 109]. We wish to explain why the topologically twisted orbifold conformal field theory gives rise to the same cohomology ring as the de Rham cohomology ring of the Hilbert scheme of points equipped with the cup product.

The Cohomology

Firstly, we want to match the operators that span the cohomology rings. The relevant operators have been described in the literature in great detail [113]. The single cycle elements of the cohomology ring of the Hilbert scheme map onto certain elements of the chiral ring in the orbifold twisted sectors. They are denoted as the operators $\sigma_{(n_1)}^{--}$ in [113] and can be described in terms of the twist operators $\tau_{(n_1)}$ of the lowest conformal dimension

$$h(\tau_{(n_1)}) = \frac{c}{24} \left(n_1 - \frac{1}{n_1} \right) \quad (5.8)$$

in the single cycle twisted sector, combined with an exponential in the bosonized fermions :

$$\sigma_{(n_1)}^{--} = e^{i \sum_{I=1}^{n_1} \frac{n_1-1}{2n_1} (H_X^I + H_Y^I)} \tau_{(n_1)}. \quad (5.9)$$

The fermions give total $U(1)_R$ charge $q = n_1 - 1$ to the operator which is of conformal dimension

$$h = \frac{1}{4} \left(n_1 - \frac{1}{n_1} \right) + \frac{n_1}{4} \frac{(n_1 - 1)^2}{n_1^2} = \frac{n_1 - 1}{2} = \frac{q}{2}, \quad (5.10)$$

and similarly in the right-moving sector. We have assumed that n_1 out of n copies of the symmetric product are involved in the operator. Permutations that consist of multiple cycles correspond to non-singular multiplications of operators $\sigma_{(n_i)}^{--}$. We still need to render the operator gauge invariant by conjugating with permutations. For each given permutation conjugacy class, we have precisely one chiral-chiral ring element of the type we discussed. These represent the cohomology elements one to one. While other chiral primaries exist, they will be trivial in de Rham cohomology, similarly to what we saw above for the seed theory. Indeed, if the original (de Rham) cohomology contains a single identity operator, the second quantized string perspective of [110] or the mathematics result [102] shows that the de Rham cohomology of the symmetric product conformal field theory is captured by the Poincaré polynomial (5.2). The crucial question becomes whether the operator product of these operators in the chiral ring agrees with the cup product of cohomology elements described in subsection 5.1.1.

The Chiral Ring

In the chiral ring, the product of operators can only be non-zero when R-charge conservation is satisfied. Moreover, two operators that are in twisted sectors labelled by the permutations π_1 and π_2 respectively, produce an operator in the twisted sector labelled by $\pi_1 \pi_2$. Consider for starters two operators in twisted sectors corresponding

to permutations that consist of a single cycle $\pi_1 = (n_1)$ and $\pi_2 = (n_2)$. Their R-charges are $n_1 - 1$ and $n_2 - 1$. Thus, if they produce a third operator with a single cycle, it must have length $n_1 + n_2 - 1$. The operator product coefficient of the operators $\sigma_{(n_i)}^{--}$ was computed in [113] and equals

$$|C_{n_1, n_2, n_1+n_2-1}^{---}|^2 = \frac{n_1 + n_2 - 1}{n_1 n_2} \quad (5.11)$$

If we normalize

$$\sigma_{(n_i)} = n_i \sigma_{(n_i)}^{--}, \quad (5.12)$$

the operator product produces precisely the corresponding operator $\sigma_{(n_1+n_2-1)}$ with pre-factor equal to one. This fact should be compared to the basic observation above equation (5.4). Finally, we note that to preserve R-charge we can allow for at most zero or one elements to overlap when taking the product of individual cycles. When we have zero overlap, the product is trivial. When we have an overlap of one, the product is as above.

Next, consider the generic case of a product of operators $\sigma_1 = \sigma_{(n_1^1)} \sigma_{(n_2^1)} \sigma_{(n_3^1)} \dots$ with $\sigma_2 = \sigma_{(n_1^2)} \sigma_{(n_2^2)} \dots$. The $U(1)_R$ charge of the initial operators is $\sum_i (n_i^a - 1)$. This is the total number of transpositions in the initial twisted sector permutation labels. Charge conservation guarantees that the total number of transpositions is conserved in the operator product expansion. Moreover, permutations that label twisted sectors compose (uniquely), matching the basic observation above equation (5.4). Finally, gauge invariant operators are conjugation invariant and therefore correspond to class functions on the permutation group. Thus, we reproduce the convolution product (5.3) with the additional requirement of degree conservation from the product of the chiral ring operators in the symmetric orbifold conformal field theory. This is in full accord with an alternative route, which is to mathematically abstract [103, 104] the cohomology coded in orbifold conformal field theory [108, 109], to then prove that it is equivalent to the Hilbert scheme cohomology [106].

5.1.3 A Plethora of Results on the Cohomology Ring

We have established that calculations in the cohomology ring of the symmetric orbifold conformal field theory on the complex two-plane \mathbb{C}^2 reduce to calculations in the symmetric group S_n . The latter can be notoriously hard, and it is a domain of mathematics in itself to push the frontiers of what we can concretely compute in permutation groups and which combinatorics we can compactly count. We believe it is useful to review a bit of combinatorial knowledge relevant to the cohomology ring structure constants. We start by recalling what is known about the connection coefficients (i.e. structure constants) of conjugacy classes in general, and then restrict to the case where they satisfy the additive degree condition or equivalently, R-charge conservation.²

2. One reason for discussing the combinatorics more generally is that the other connection coefficients are relevant to models beyond the complex two-plane \mathbb{C}^2 on which we concentrate in this section.

The Product of Conjugacy Classes of S_3 and S_4

The permutation group of one element is trivial. The permutation group of two elements $S_2 = \mathbb{Z}_2$ has two elements, each in its conjugacy class, and these trivially compose. Only the composition $(12)(12) = 1$ violates the degree condition.

The first slightly non-trivial case is the permutation group S_3 . It contains $3! = 6$ elements, and three conjugacy classes denoted $[1^3], [12], [3]$ in terms of partitions of three. The first has one element, the second three, and the third two. The multiplication table for S_3 is

Element	$()$	$(1\ 2)$	$(2\ 3)$	$(1\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$()$	$()$	$(1\ 2)$	$(2\ 3)$	$(1\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1\ 2)$	$(1\ 2)$	$()$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(2\ 3)$	$(1\ 3)$
$(2\ 3)$	$(2\ 3)$	$(1\ 3\ 2)$	$()$	$(1\ 2\ 3)$	$(1\ 3)$	$(1\ 2)$
$(1\ 3)$	$(1\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$()$	$(1\ 2)$	$(2\ 3)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(1\ 3)$	$(1\ 2)$	$(2\ 3)$	$(1\ 3\ 2)$	$()$
$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(2\ 3)$	$(1\ 3)$	$(1\ 2)$	$()$	$(1\ 2\ 3)$

where we act first with the column element, and then with the row element. The convolution algebra on conjugacy classes is

Conj. Class	$[()]$	$[(12)]$	$[(123)]$
$[()]$	$[()]$	$[(12)]$	$[(123)]$
$[(12)]$	$[(12)]$	$3[()] + 3[(123)]$	$2[(12)]$
$[(123)]$	$[(123)]$	$2[(12)]$	$2[()] + [(123)]$

As an example, consider the conjugacy class of transpositions $[(12)]$ and compose it with the conjugacy class of transpositions $[(12)]$. The conjugacy classes contain three elements each, and we convolute their sums. We obtain nine terms, three of which are the identity, and six of which are cyclic permutations of order three (of which there are two). This result is indicated in the third row, third column of the table.

In the symmetric group S_4 , one has permutations that are the product of two independent non-trivial cycles.³ The multiplication table for S_4 already becomes a bit cumbersome. We can still easily list the five conjugacy classes in partition or cycle notation : $[1^4] = [()]$, $[1^2 2] = [(12)]$, $[2^2] = [(12)(34)]$, $[13] = [(123)]$, $[4] = [(1234)]$. These have 1, 6, 3, 8 and 6 elements respectively. Their convolution algebra is :

Conj. Class	$[()]$	$[(12)]$	$[(12)(34)]$	$[(123)]$	$[(1234)]$
$[()]$	$[()]$	$[(12)]$	$[(12)(34)]$	$[(123)]$	$[(1234)]$
$[(12)]$	$[(12)]$	$6[()] + 2[(12)(34)] + 3[(123)]$	\dots	\dots	\dots
$[(12)(34)]$	$[(12)(34)]$	$[(12)] + 2[(1234)]$	\dots	\dots	\dots
$[(123)]$	$[(123)]$	$4[(12)] + 4[(1234)]$	\dots	\dots	\dots
$[(1234)]$	$[(1234)]$	$4[(12)(34)] + 3[(123)]$	\dots	\dots	\dots

where the remaining entries in the table are still fairly straightforward to compute. Indeed, these data have been compiled in Appendix I.B of [114] for $S_{n \leq 8}$. They can also

3. We used that $2 + 2 = 4$.

be reproduced using the symbolic manipulation program GAP [115]. Each coefficient in these tables that satisfies the degree condition is a finite n structure constant of the cohomology ring.

The Independence of the Order of the Group

Importantly, when we take a particular perspective, the structure constants are independent of the order of the permutation group, as long as the order is large enough [116]. Let us illustrate this theorem on a simple example. We have class convolution formulas for $n = 2, 3, 4, 5, 6$:

$$\begin{aligned}
 [2] * [2] &= 1[1^2] \\
 [12] * [12] &= 3[1^3] + 3[3] \\
 [1^2 2] * [1^2 2] &= 6[1^4] + 2[2^2] + 3[13] \\
 [1^3 2] * [1^3 2] &= 10[1^5] + 2[12^2] + 3[1^2 3] \\
 [1^4 2] * [1^4 2] &= 15[1^6] + 2[1^2 2^2] + 3[1^3 3].
 \end{aligned} \tag{5.13}$$

Consider the renormalized class sums :

$$A[\lambda] = \binom{n - r + m_1(\lambda)}{m_1(\lambda)} [\lambda], \tag{5.14}$$

where we first pick a value n for the group S_n in which we embed our partitions of the positive integer r by adding 1's as necessary. The original partition of the integer r has $m_1(\lambda)$ 1's to start with. Consider the left hand side of (5.13). Our original partition is $[2]$ and $r = 2$. Since $m_1(\lambda) = 0$, we do not renormalize the left hand side characters at all. On the right hand side, we need to renormalize. For instance, when we embed the $[1^2]$ partition into the $[1^3]$ partition, we need to pick 2 out of $3 - 2 + 2 = 3$ elements, and we pick up a factor of 3. For the next lines, we need to pick two out of 4,5,6 et cetera. For the other terms, the renormalization factors are trivial. We see that when we write :

$$\begin{aligned}
 A[2] * A[2] &= A[1^2] \\
 A[12] * A[12] &= A[1^3] + 3A[3] \\
 A[1^2 2] * A[1^2 2] &= A[1^4] + 2A[2^2] + 3A[13] \\
 A[1^3 2] * A[1^3 2] &= A[1^5] + 2A[12^2] + 3A[1^2 3] \\
 A[1^4 2] * A[1^4 2] &= A[1^6] + 2A[1^2 2^2] + 3A[1^3 3],
 \end{aligned} \tag{5.15}$$

the structure constants stabilize at the order which is the sum of the orders of the original partitions (namely four). This is generically true [116], and it shows that the convolution of symmetric group class functions is independent of the order of the group (at finite order, as long as it is large). Thus, taking the large n limit on the formulas for these renormalized structure constants is trivial – they do not depend on n at large n .

Partially General Results

While there is no general closed form expression that is an efficient rewriting of the original combinatorial problem of determining the multiplication of conjugacy classes, there are partially general and compact results for the connection coefficients.⁴ Especially regarding single cycle permutations and transpositions, there are more compact formulas. Let us for instance mention that the multiplication of the n -cycle conjugacy class with any conjugacy class has a reasonably compact expression [118]. Here we concentrate on the top connection coefficients, namely those connection coefficients that satisfy the R-charge conservation condition :

$$n - l(\pi_1) + n - l(\pi_2) = n - l(\pi_3) \quad (5.16)$$

where $l(\pi_i)$ is still the length of the permutation π_i which equals the number of cycles that make up the permutation (or the number of parts of the corresponding partition). When the structure constant labels satisfy this condition, the connection coefficients in the symmetric group are called top coefficients (since the condition (5.16) is extremal). Determining the top connection coefficients of the symmetric group is a hard and interesting problem about which many partial results are known.

Let us review such a result. Suppose we have two conjugacy classes labelled by $[\pi_{1,2}]$, and we wish to know the structure constant multiplying $[\pi_3]$, then we denote the number $c_{[\pi_1][\pi_2]}^{[\pi_3]}$. This number is symmetric in the lower indices and satisfies the property $h_{\pi_3} c_{[\pi_1][\pi_2]}^{[\pi_3]} = h_{\pi_2} c_{[\pi_1][\pi_3]}^{[\pi_2]}$ where h_{π_i} is the number of elements in the conjugacy class $[\pi_i]$. In [119–121], the top connection coefficients for two permutations that multiply into a single cycle were determined. Let the permutation conjugacy classes $[\pi_1] = [1^{i_1} 2^{i_2} \dots]$ and $[\pi_2] = [1^{k_1} \dots]$ correspond to two partitions of $n = n_3$ of lengths $l_1 = l(\pi_1) = i_1 + i_2 + \dots$ and $l_2 = l(\pi_2) = k_1 + k_2 + \dots$. With the degree restriction $l_1 + l_2 = n_3 + 1$ we have the top connection coefficient

$$c_{[\pi_1][\pi_2]}^{(n_3)} = n_3 \frac{(l_1 - 1)!(l_2 - 1)!}{i_1! i_2! \dots k_1! k_2! \dots}. \quad (5.17)$$

This result has a direct connection to two-coloured plane rooted-trees on n_3 edges [121] and has been further understood using Lagrange's inversion theorem and Macdonald's theory of symmetric functions in [122]. It gives an idea of the power of combinatorial theorems in the context of computing structure constants of the cohomology ring.

For illustration purposes, we apply the result (5.17) to our favorite case once more, namely $[\pi_i] = [(n_i)]$ with $n_3 = n$. We find the partition $i_1 = n - n_1$ and $i_{n_1} = 1$ with length $l_1 = n - n_1 + 1$ and similarly the partition $k_1 = n - n_2$, $k_{n_2} = 1$ with length $l_2 = n - n_2 + 1$. We obtain once more the structure constant (5.5) :

$$c_{[(n_1)][(n_2)]}^{(n_3)} = n_1 + n_2 - 1, \quad (5.18)$$

but now as a very simple example of a much more general formula. We stress that combinatorics theorems like (5.17) provide new results on operator product expansions of chiral operators in symmetric orbifold conformal field theories. The underlying maps to e.g. rooted trees or cacti [121] provide an efficient diagrammatics for the operator product expansions.

4. See e.g. the introduction of [117] for an overview.

5.1.4 The Interaction

It is interesting to delve a little deeper into how the isomorphism between the Hilbert scheme cohomology ring and the degree preserving convolution ring of class functions on the symmetric group is established [100,101]. In both rings one can identify a basic building block for the product which is the multiplication by the conjugacy class of transpositions (or its appropriate algebraic geometric dual). In fact, there is an interesting graded poset structure on the space of conjugacy classes of S_n which can be used to compute structure constants of the ring [120]. We reach the next level in the poset by composing with transpositions. (See Figure 2 of [120] for a neat illustration.) This structure played a role in the construction of a differential operator that cuts and joins cycles through multiplication by transpositions [117]. Firstly, one associates polynomials to conjugacy classes of permutations

$$\Phi(\pi) = p_{[\pi]} \quad (5.19)$$

where $p_\lambda = p_{\lambda_1} \dots p_{\lambda_i}$ is a product of power sum symmetric functions⁵ associated to a partition λ equal to the cycle distribution $[\pi]$ of the permutation π . We then have a differential operator H_{int} which satisfies [117] :

$$\Phi([1^{n-2}2] * \pi) = H_{int}\Phi(\pi) \quad (5.20)$$

for all permutations π . The differential operator H_{int} equals [117] :

$$H_{int} = \frac{1}{2} \sum_{i,j \geq 1} ij p_{i+j} \partial_{p_i} \partial_{p_j} + (i+j) p_i p_j \partial_{p_{i+j}}. \quad (5.21)$$

The first term in the operator joins two cycles while the second term cuts one cycle into two. This is the most elementary (cubic) interaction term in the cutting and joining of the second quantized string theory described in [110,123,124], and it is familiar from (topological) string theory. In [100], the operator H_{int} is restricted to interactions that preserve the degree. It is not hard to see that the only terms that respect the degree are the terms of the first (join) type. Using the transposition as a stepping stone, it is shown in [100] that the alternating character generates the whole ring (both on the combinatorial and the algebraic geometric side) and establishes an isomorphism between the top degree class multiplication and the Hilbert scheme cohomology ring. The (cutting and) joining method of proof is akin to reasonings in second quantized string theory, matrix string theory or two-dimensional topological gravity.

5.1.5 The Structure Constants are Hurwitz Numbers

In this subsection, we wish to recall the relation between the Hilbert scheme of the plane, the counting of permutations and the Hurwitz numbers [125]. We define the Hurwitz numbers $H_n^{\mathbb{P}^1}(\lambda_0, \dots, \lambda_k)$ as the number of $(k+1)$ tuples of permutations $(\pi_0, \pi_1, \dots, \pi_k)$ in S_n such that π_i is of cycle type λ_i and $\pi_0 \pi_1 \dots \pi_k = 1$, divided by $n!$. This counts the number of (possibly disconnected) covers of the sphere with $k+1$

5. We have $p_0 = 1$ and $p_{\lambda_i} = x_1^{\lambda_i} + \dots + x_n^{\lambda_i}$.

branching points of branching types λ_i [126]. When we restrict to counting only those covers that satisfy the additive degree relation, then we impose that

$$n - l(\lambda_0) = \sum_{i=1}^k (n - l(\lambda_i)), \quad (5.22)$$

where $l(\lambda_i)$ are the lengths of the partitions λ_i . For ease of notation, we introduce the normalization factor $\zeta(\lambda) = \zeta([1^{m_1} 2^{m_2} \dots])$:

$$\zeta(\lambda) = \prod_{i \geq 1} (i^{m_i} m_i!). \quad (5.23)$$

We then have, following [125], that the generalized structure constant $c_{\lambda_1 \dots \lambda_k}^{\lambda_0}$ is the (top) coefficient of the conjugacy class $[\lambda_0]$ in the product of conjugacy classes $[\lambda_i]$:

$$\prod_{i=1}^k [\lambda_i] = c_{\lambda_1 \dots \lambda_k}^{\lambda_0} [\lambda_0] + \dots \quad (5.24)$$

We rewrite this by using the number of elements in the given conjugacy class $[\lambda_0]$ and find :

$$\begin{aligned} c_{\lambda_1 \dots \lambda_k}^{\lambda_0} &= \frac{\zeta_{\lambda_0}}{n!} c_{\lambda_0 \dots \lambda_k}^{[1^n]} \\ &= \zeta_{\lambda_0} H_n^{\mathbb{P}^1}(\lambda^0, \dots, \lambda^k), \end{aligned} \quad (5.25)$$

where the final equation follows by the property of the Hurwitz numbers reviewed above. Thus, top connection coefficients for the multiplication of any number of conjugacy classes are equal to the Hurwitz numbers that count the number of (possible disconnected) branched covers of the sphere with the appropriate extremal branching behaviour. In the end, this is a direct consequence of description of the cohomology ring in terms of the symmetric group class functions [100]

5.1.6 The Broader Context and a Proof of a Conjecture

Let us put the results we obtained in a broader context. The first remark we wish to make is that we studied a ring which is a canonical quotient ring of the cohomology ring for a Hilbert scheme of points on a (quasi-projective) surface M . Indeed, the latter cohomology ring has an ideal generated by all operators that have a factor that takes a non-trivial value (i.e. not the identity) in the seed cohomology ring of M . If we divide the symmetric orbifold cohomology ring by that ideal, the quotient ring is isomorphic to the cohomology ring of the Hilbert scheme of the complex plane [127]. Thus, the latter captures part of the ring structure for any manifold M .

The second remark is that up till now, we have restricted ourselves to an analysis of the operator algebra and its structure constants. This is because in the non-compact case, the topological orbifold conformal field theory has no natural seed two-point function – in the compact case the two-point function is given by a (finite) integral over the compact manifold M .

Nevertheless, we can at this stage make a useful connection with the physical extremal correlators computed for three- and four-point functions in [131] and conjectured for any number of operator insertions in equation (4.58) of [131].⁶ In the physical theory, there are diagonalizable two-point functions. These can be used to trivially lower an index on the structure constants of the cohomology ring (5.25). This simple remark shows that the Hurwitz number structure constants (5.25) coincide with the correlators (4.58) in [131] when they pertain to the identity cohomology element, and proves the conjecture in that case. In fact, we make the sub-statement considerably more powerful in that we showed that the statement is valid at large enough but finite order n , with appropriate normalizations of the operators (as in (5.14)), and moreover extends to a large class of surfaces M .

5.1.7 Summary and Lessons

We briefly summarize this section and prepare the ground for the next one. When we divide the cohomology ring of the Hilbert scheme of points on a surface by the ideal generated by all non-zero elements in the cohomology of the original manifold, we obtain the cohomology ring of the Hilbert scheme of points of the plane. The cohomology ring of the plane has a cup product which agrees with the top coefficients of the convolution of the conjugacy classes in the symmetric group S_n . The convolution product counts permutations in given conjugacy classes that multiply to one. The Hurwitz numbers as well count these permutations. Therefore, the extremal operator product expansion, which satisfies the condition that they correspond to top coefficients, agree with the Hurwitz numbers. This statement is true at order n larger than the sum of the size of the permutations involved in the operator product, and the structure constants are independent of n at large enough, but finite n .

We have studied a non-compact symmetric orbifold conformal field theory. The ring structure of the original manifold \mathbb{C}^2 is trivial, and this simplifies the cohomology ring of the symmetric orbifold. In the next section, we will work with a compact complex surface M and thus add to the mix both a non-trivial seed cohomology ring, as well as a topological two-point function.

5.2 Compact Surfaces

In the previous section, we discussed the cohomology ring of the Hilbert scheme of points on the complex two-plane $M = \mathbb{C}^2$ and how the ring relates to a ring of chiral operators in the corresponding topological symmetric orbifold conformal field theory. In this section, we wish to extend that analysis to the case where the manifold M is a (quasi-projective) compact complex surface (with trivial canonical bundle), for example $M = K3$ or $M = T^4$.⁷ We wish to review that the chiral ring of the topological symmetric orbifold conformal field theory has a neat mathematical description in terms of the cohomology ring of the seed manifold M , as well as the combinatorics of permutations [105]. From that description, we derive old and new correlators

6. See also the end of section 5 of [132] for a discussion.

7. Surfaces with non-trivial canonical class can also be studied using similar techniques, but here we concentrate on the case most frequently encountered in string theory.

in the (topological) symmetric orbifold conformal field theory, as well as their generic characteristics.

5.2.1 The Orbifold Frobenius Algebra

The starting point of the construction in [105] is the cohomology ring of the manifold M , and a linear form T on the ring. The latter is the integral on the compact manifold M .⁸ The combined structure forms a graded Frobenius algebra A . The results of [105] then construct from this algebra, and the combinatorics of the symmetric group, a symmetric product Frobenius algebra $A^{[n]}$ that coincides with the chiral ring of the topological symmetric orbifold conformal field theory [103, 104, 106]. The construction of the algebra $A^{[n]}$ is lengthy – we will provide only the gist of the construction and refer to [105] for further details.

The Tensor Frobenius Algebra and Permutations

Following [105] we start with a graded commutative, associative algebra A over \mathbb{C} , with a unit and a linear form T such that the bilinear form $T(ab)$ induced by T is non-degenerate.⁹ Our graded Frobenius algebra is a finite dimensional graded vector space with grades ranging from $-d$ to d , with a multiplication of degree d and a unity element (necessarily of degree $-d$). These grades are the degrees of the cohomology elements on a manifold of complex dimension d , shifted by $-d$.^{10, 11} The linear form T is of degree $-d$.

The tensor product $A^{\otimes n}$ is a Frobenius algebra of degree nd . The product of tensor product elements is the (graded) tensor product of the individual products in the factors :

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = \epsilon(a, b) (a_1 b_1) \otimes \cdots \otimes (a_n b_n), \quad (5.26)$$

where the sign $\epsilon(a, b)$ keeps track of the number of odd elements (i.e. fermions) we had to exchange in order to perform the multiplication. The bi-linear form on the tensor product algebra $A^{\otimes n}$ is defined by

$$T(a_1 \otimes \cdots \otimes a_n) = T(a_1) \dots T(a_n). \quad (5.27)$$

It is of degree $-nd$ and is again non-degenerate. The symmetric group acts on the tensor product factors and the action takes into account the grading of the factors. For

8. From now on, will have in mind compact Kähler manifolds M . Note that in this case the Dolbeault cohomology is a refinement of the de Rham cohomology. Thus, for the compact case at hand, we no longer have to worry about the distinction between these cohomologies, as we did in section 5.1. The chiral ring will match perfectly onto the de Rham cohomology, both for the seed theory and for the Hilbert scheme.

9. We encourage the reader to already imagine that the graded associative algebra A is the algebra of cohomology elements of the manifold M and to think of the bi-linear form T as the topological conformal field theory two-point function, given by integrating the wedge product of two cohomology elements over the manifold M .

10. Our surface has dimension $d = 2$. We will nevertheless keep the dimension d general for a while.

11. The reader familiar with conformal field theory is invited to simultaneously think of the grade as the sum of the Ramond-Ramond sector $U(1)_R$ charges, and the shift $-d = -c/3$ as arising from spectral flow.

each bijection $f : \{1, 2, \dots, n\} \rightarrow I$, there is a canonical isomorphism $A^{\otimes n} \equiv A^{\otimes I}$. We will often implicitly make use of this in the following.¹²

Next, we define maps between tensor products of the original Frobenius algebra with a different number of factors. First, we reduce the number of factors. Suppose we have a partition $n = n_1 + n_2 + \dots + n_k$. We then define a ring homomorphism through multiplication :

$$\phi_{n,k} : A^{\otimes n} \rightarrow A^{\otimes k} : a_1 \otimes \dots \otimes a_n \mapsto (a_1 a_2 \dots a_{n_1}) \otimes \dots \otimes (a_{n_1+\dots+n_{k-1}+1} \dots a_n). \quad (5.28)$$

This generalizes to a map ϕ^* associated to any surjective map of indices $\phi : I \rightarrow J$. Moreover, we have the adjoint map $\phi_* : A^{\otimes J} \rightarrow A^{\otimes I}$ with respect to the bilinear forms (5.27) induced on the tensor products.

A particular example of these maps that we will encounter is the multiplication $\phi_{2,1} = \Delta^* : A \otimes A \rightarrow A$ as well as its adjoint, the co-multiplication $\Delta_* : A \rightarrow A \otimes A$. The image of the unit under the combined map is called the Euler class $e(A)$ of the algebra A . Let us compute the Euler class in more familiar terms. Consider a (graded) basis e_i of the algebra A . We introduce an expression for the structure constants c_{ij}^k of the algebra A as well as for the metric η_{ij} that arises from the bilinear form T :

$$\begin{aligned} e_i e_j &= c_{ij}^k e_k = (-1)^{ij} c_{ji}^k e_k \\ T(e_i e_j) &= \eta_{ij} = (-1)^{ij} \eta_{ji} = (-1)^i \eta_{ji} = (-1)^j \eta_{ji}. \end{aligned} \quad (5.29)$$

We drew attention to (anti-)symmetry properties of the structure constants and the metric (exploiting an obvious notation for the grading of the basis elements). We have a unique element in the cohomology that is the unit element and it has grade $-d$. By Poincaré duality, there is a unique volume element of grade $+d$ that evaluates under the linear form T to $T(\text{vol}) = 1$. The co-multiplication map maps the unity element to $c^{ji}_0 e_i \otimes e_j$ since it is the adjoint of the multiplication map captured by the structure constants. We denoted the unity basis element $e_0 = 1$ by the index zero. If we then multiply e_i with e_j , we obtain $1 \xrightarrow{\Delta^* \circ \Delta_*} c^{ji}_0 c_{ij}^k e_k = c^{ji}_0 c_{ij0} \text{vol} = \chi(A) \text{vol}$ where $\chi(A) = \sum_i (-1)^i \dim(A^i) = \eta_{ij} \eta^{ji}$ is the Euler-Poincaré characteristic of the algebra A . We thus map unity to the Euler class $e(A) = \chi(A) \text{vol}$.¹³ For future purposes, we introduce the notation

$$e^g = \otimes_{i \in I} e^{g(i)} \in A^{\otimes I} \quad (5.30)$$

for the tensor product of Euler classes associated to a function $g : I \rightarrow \mathbb{N}_0$. Next, we need to add ingredients that will be necessary to treat the combinatorial aspects of the symmetric orbifold theory. We recall that every permutation π has a degree $|\pi|$ given by the minimal number of transpositions necessary to construct it. For any subgroup $H \subset S_n$ and an H -stable subset $B \subset \{1, 2, \dots, n\}$, we write $H \setminus B$ for the space of orbits in the set B under the action of the group H . We note that the degree $|\pi|$ of a permutation can be identified as $|\pi| = n - |\langle \pi \rangle \setminus [n]|$ where $\langle \pi \rangle$ is the subgroup of S_n

12. At this stage, we have the topological tensor product conformal field theory Hilbert space with an action of the permutation group, the seed ring structure and a two-point function that extends to the tensor product.

13. For the four-torus, the Euler number $\chi(T^4)$ is zero, and therefore $e(A_{T^4}) = 0$. For the $K3$ surface, the Euler number equals $\chi(K3) = 24$.

generated by the permutation π . Indeed, this is the total number of elements minus the number of cycles, and thus the sum of cycle lengths minus one. This notation allows us to define the graph defect g for two permutations π and ρ evaluated on a set B :

$$g(\pi, \rho)(B) = \frac{1}{2}(|B| + 2 - |\langle \pi \rangle \setminus B| - |\langle \rho \rangle \setminus B| - |\langle \pi \rho \rangle \setminus B|). \quad (5.31)$$

The graph defect is a positive integer that has an interpretation as the genus of a Riemann surface [105].

The Twisted Sectors, the Multiplication and the Gauge Invariants

We introduce two further algebras $A\{S_n\}$ and $A^{[n]}$ [105]. The first is the analogue of the unprojected space of operators in the topological orbifold theory, while the second is the subspace of S_n gauge invariants. We need to define the first space, an action of the permutation group on the space, and most importantly, the equivariant product in the algebra. We define the algebra $A\{S_n\}$ for starters as the (graded) vector space of tensor products of operators multiplying permutations :

$$A\{S_n\} = \bigoplus_{\pi \in S_n} A^{\otimes \langle \pi \rangle \setminus [n]} \pi. \quad (5.32)$$

Each orbit of the permutation is assigned a single operator. The grading of an element $a\pi$ is the sum of the grades in the tensor algebra, namely $|a\pi| = |a|$. The action of the group S_n on the set $\{1, 2, \dots, n\} = [n]$ gives rise to a bijection between the orbits of a permutation π and its conjugates :

$$\sigma : \langle \pi \rangle \setminus [n] \rightarrow \langle \sigma \pi \sigma^{-1} \rangle \setminus [n] : x \mapsto \sigma x. \quad (5.33)$$

Thus we have an isomorphism $\tilde{\sigma}$

$$\tilde{\sigma} : A\{S_n\} \rightarrow A\{S_n\} : a\pi \mapsto \sigma(a)\sigma\pi\sigma^{-1}, \quad (5.34)$$

which defines the action of the symmetric group S_n on the vector space $A\{S_n\}$. The vector space $A^{[n]}$ is the subspace of symmetric group invariants $A^{[n]} = (A\{S_n\})^{S_n}$.

Finally, we come to the crucial definition of the product on $A\{S_n\}$ [105]. Firstly, note that any inclusion of subgroups $H \subset K$ leads to a surjection of orbit spaces $H \setminus [n] \rightarrow K \setminus [n]$ and by the previous constructions to a map

$$f^{H,K} : A^{\otimes H \setminus [n]} \rightarrow A^{\otimes K \setminus [n]} \quad (5.35)$$

and its adjoint $f_{K,H}$. These maps allow us to bring operators that lie in different twisted sectors into a common space in which we are able to multiply them, then to bring them back into the product twisted sector. Indeed, we define the operator multiplication map $m_{\pi,\rho}$ [105] :

$$m_{\pi,\rho} : A^{\otimes \langle \pi \rangle \setminus [n]} \otimes A^{\otimes \langle \rho \rangle \setminus [n]} \rightarrow A^{\otimes \langle \pi \rho \rangle \setminus [n]} \quad (5.36)$$

by

$$m_{\pi,\rho}(a \otimes b) = f_{\langle \pi, \rho \rangle, \pi \rho}(f^{\pi, \langle \pi, \rho \rangle}(a) \cdot f^{\rho, \langle \pi, \rho \rangle}(b) \cdot e^{g(\pi, \rho)}). \quad (5.37)$$

Note that the middle multiplication takes place in the tensor product associated to the subgroup $\langle \pi, \rho \rangle$ generated by both permutations π, ρ separately. The multiplication map $m_{\pi, \rho}$ not only prescribes how to sensibly combine operators in different tensor product factors, it also tells us that we need to take into account a correction factor that depends on the graph defect as well as on the Euler class $e(A)$ of the seed algebra A . It moreover prescribes how to co-multiply from the smaller set of orbits of the subgroup $\langle \pi, \rho \rangle$ to the orbits of the subgroup $\langle \pi \rho \rangle$. Finally, the cup product is given by :

$$a\pi \cdot b\rho = m_{\pi, \rho}(a \otimes b)\pi\rho, \quad (5.38)$$

namely, twists compose. The product is associative, S_n equivariant and homogeneous of degree nd , as proven in [105]. In other words, we have $|a\pi \cdot b\rho| = |a\pi| + |b\rho| + nd$.

The equivariant ring structure on $A\{S_n\}$ induces a ring structure on the space of invariants $A^{[n]}$ and moreover, the latter ring is a subring of the centre of $A\{S_n\}$ [105]. Thus, elements (anti-)commute in the orbifold ring $A^{[n]}$, as expected. The algebra $A^{[n]}$ is again a graded Frobenius algebra.

Remarks

- Let us describe briefly how the generic construction in this section relates to the case of the cohomology ring of the Hilbert scheme of the complex plane \mathbb{C}^2 in section 5.1. In that case, the seed Frobenius algebra A is trivial. Permutations multiply as in the multiplication formula (5.38). What remains to show is that the multiplication (5.38) of degree nd preserves the degree of the permutations. We will show that it does. The degree of the unit operator in A is equal to $-d$. We moreover recall the relation between the degree of a permutation and the number of orbits :

$$|\pi| = n - |\langle \pi \rangle \setminus [n]|. \quad (5.39)$$

If we consider the element 1π as an element of $A^{\otimes \langle \pi \rangle \setminus [n]}\pi$ then its degree is equal to $|\langle \pi \rangle \setminus [n]|$ times $-d$. Thus, if we have cohomology elements $a = 1 = b$, the degree condition for the product $m_{\pi, \rho}$ reduces to :

$$|1\pi \cdot 1\rho| = -d|\langle \pi \rangle \setminus [n]| - d|\langle \rho \rangle \setminus [n]| + dn = d(|\pi| + |\rho|) - dn = -d|\langle \pi \rho \rangle \setminus [n]| \quad (5.40)$$

and therefore $|\pi\rho| = n - |\langle \pi\rho \rangle \setminus [n]| = |\pi| + |\rho|$. This is indeed the degree or R-charge conservation condition familiar from section 5.1. More generally, the degree condition on the product agrees with total R-charge conservation in the conformal field theory.

- An isomorphism between the algebra of gauge invariants $A^{[n]}$ and a Fock space as for a second quantized string [110] can be established [102, 105].
- In [105] the cup product on Hilbert schemes of surfaces was proven to coincide with the efficient description reviewed above. Moreover, it was proven to coincide with the mathematical reformulation of the symmetric product cohomology [103, 104, 106].
- Note that since the Euler class $e(A(T^4))$ of T^4 is zero, because the Euler number is, the correction associated to the Euler class is absent for the T^4 topological symmetric orbifold conformal field theory. Moreover, the multiplication map will set to zero any multiplication that has non-zero genus or graph defect.

- For the case of $M = K3$, the Euler class correction is important.
- Whenever the graph defect (or genus) of two permutations π, ρ we multiply is larger or equal to two in a given orbit of the group $\langle \pi, \rho \rangle$ generated by the two permutations, then the multiplication $m_{\pi, \rho}$ contains a factor of $e(M)^2$ and is therefore equal to zero. In this sense, the multiplication formula contains one-loop corrections only. Provided we can identify the covering surface with the bulk string world sheet (see e.g. [130]), there are at most one loop corrections in the bulk topological string theory.
- Finally, we summarize in geometric terms, familiar from generic two-dimensional topological field theories, the mechanics behind the results we found. Recall that these theories are solvable in terms of their (pants) three-point functions, combined with cutting and gluing. From the three-point functions, we can compute the operator that inserts a handle in a Riemann surface. We have computed the handle operator $e = c_i^{ij} e_j$ to be (multiplication by) the Euler class. Since the Euler class squares to zero, the insertion of two handles makes for a vanishing amplitude.

We have reviewed the topological symmetric orbifold conformal field theory cohomology ring for a (quasi-projective) compact manifold M (with trivial canonical class) [105], and added some remarks that serve to guide the physics reader through the construction. We will next employ the efficient description to recompute a few structure constants and explicitly match them to the physics literature on the symmetric orbifold conformal field theory. These calculations confirm the general picture drawn above, as they must.

5.2.2 Cup Products of Single Cycle Elements

We compute the operator product of two operators associated to two permutations that are each single cycles with either zero elements overlapping, one, or two. One reason for performing these calculations is that they can be compared to calculations in the physical conformal field theory [113, 123, 128]. A second reason is that these correlators have been compared to bulk three-point functions [15, 129] and thus provide a concrete holographic bridge to string theory in three-dimensional anti-de Sitter space.

Without Overlap

In the case where we consider two operators $\alpha\sigma_{(n_1)}$ and $\beta\sigma_{(n_2)}$ in the algebra $A\{S_n\}$ concentrated in n_i factors of the symmetric product $A^{\otimes n}$ with zero overlap in the respective factors, the product is trivial, namely it is the ordinary tensor product of the individual operators.^{14, 15}

14. In the algebra of gauge invariants $A^{[n]}$ this situation will hardly occur. After making gauge invariants, various cycles are bound to overlap. In that circumstance, we have zero overlap for certain terms only.

15. In the following we will often use the notation α, β, \dots for operators in A , associated to a single orbit of the permutation. For example, we have $\alpha\sigma_{(n_1)} = \dots \otimes 1 \otimes \alpha \otimes 1 \otimes \dots \sigma_{(n_1)}$ where $\alpha \in A$ corresponds to the orbit of $\sigma_{(n_1)}$ of length n_1 .

A Single Overlap

We discuss a second case of a product of operators $\alpha\pi$ and $\beta\rho$ in the algebra $A\{S_n\}$, associated to two single cycle permutations $\pi = \sigma_{(n_1)}$, $\rho = \sigma_{(n_2)}$ with a single element in common. As a result, we have the multiplication of permutations $\pi\rho = \sigma_{(n_1+n_2-1)}$ and we act on the minimal space of indices $\{1, 2, 3, \dots, n_1 + n_2 - 1\} \equiv [n_1 + n_2 - 1]$.¹⁶ To compute the product of such operators in the algebra $A\{S_n\}$ according to the rules laid out in subsection 5.2.1, we establish a few preliminaries. The space of orbits of the various subgroups generated by the permutations is – with mild abuses of notation, and a particular choice of permutation action – :

$$\begin{aligned} \langle \pi \rangle \setminus [n_1 + n_2 - 1] &= \{[n_1], n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 1\} \\ \langle \rho \rangle \setminus [n_1 + n_2 - 1] &= \{1, 2, \dots, n_1 - 1, \{n_1, n_1 + 1, \dots, n_1 + n_2 - 1\}\} \\ \langle \pi\rho \rangle \setminus [n_1 + n_2 - 1] &= \{[n_1 + n_2 - 1]\} \\ \langle \pi, \rho \rangle \setminus [n_1 + n_2 - 1] &= \{[n_1 + n_2 - 1]\}. \end{aligned} \quad (5.41)$$

We have the graph defect g (5.31) for the stable subset $B = [n_1 + n_2 - 1]$:

$$g(\pi, \rho)([n_1 + n_2 - 1]) = \frac{1}{2}(n_1 + n_2 - 1 + 2 - n_1 - n_2 - 1) = 0. \quad (5.42)$$

We then apply maps $f^{\pi, \langle \pi, \rho \rangle}$ and $f^{\rho, \langle \pi, \rho \rangle}$ to elements α and β which are elements of the cohomology associated to the orbits of $\sigma_{(n_1)}$ and $\sigma_{(n_2)}$ respectively, and the identity otherwise, such that they become elements of a single algebra $A^{\langle \pi, \rho \rangle \setminus [n_1 + n_2 - 1]}$ associated to the orbit of the subgroup $\langle \sigma_{(n_1)}, \sigma_{(n_2)} \rangle$. In this single factor, we multiply α and β according to formula (5.37), and we receive no extra factor from the Euler class of the algebra A since the graph defect g (5.42) is zero. Moreover, for this case, we have that the orbits of $\langle \pi\rho \rangle$ and $\langle \pi, \rho \rangle$ are the same, and thus that the map $f_{\langle \pi, \rho \rangle, \pi\rho}$ is trivial. Thus the product $m_{\pi, \rho}(\alpha \otimes \beta) = \alpha\beta$ is merely the multiplication of cohomology elements, which we combine with the non-trivial composition of permutations to obtain the final result for the multiplication in $A\{S_n\}$:

$$\alpha\sigma_{(n_1)} \cdot \beta\sigma_{(n_2)} = \alpha\beta\sigma_{(n_1+n_2-1)}. \quad (5.43)$$

To obtain a product in $A^{[n]}$ in which we concentrate on the right hand side on a single cycle permutation generated by a single overlap, we merely need to count elements in conjugacy classes, as we did in subsection 5.1.1. We therefore find the structure constants of the original ring, captured by the product $\alpha\beta$, times the combinatorial coefficient $n_1 + n_2 - 1$. Thus, we have computed a subset of three-point functions. In this example, the ring structure factorized into the seed ring structure times combinatorics.

An Overlap of Two

In the next example we study two operators multiplying single cycles, overlapping in two factors of the tensor product space. The composition of permutations on which

16. This notation for a set conflicts with the notation for conjugacy classes. The context should be sufficient to distinguish the concepts.

we focus is $\sigma_{(n_1)}\sigma_{(n_2)} = \sigma_{n_1+n_2-3}$ where we have an overlap of two entries in the cyclic permutations that are in reverse order and therefore generate a single longer cycle. We concentrate on the example $\pi = \sigma_{(n_1)} = (12 \dots n_1)$ and $\rho = \sigma_{(n_2)} = (2, 1, n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 2)$ which composes to $\pi\rho = \sigma_{n_1+n_2-3} = (1, n_1 + 1, n_1 + 1, \dots, n_1 + n_2 - 2, 3, \dots, n_1)$. We apply again the iron logic of [105] and first compute the orbits of the permutations in the set $\{1, 2, \dots, n_1 + n_2 - 2\}$:

$$\begin{aligned} \langle \pi \rangle \setminus [n_1 + n_2 - 2] &= \{[n_1], n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 2\} \\ \langle \rho \rangle \setminus [n_1 + n_2 - 2] &= \{3, 4, \dots, n_1, \{1, 2, n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 2\}\} \\ \langle \pi\rho \rangle \setminus [n_1 + n_2 - 2] &= \{2, \{1, 3, 4, \dots, n_1 + n_2 - 2\}\} \\ \langle \pi, \rho \rangle \setminus [n_1 + n_2 - 2] &= [n_1 + n_2 - 2], \end{aligned} \quad (5.44)$$

as well as the graph defect g :

$$g(\pi, \rho)([n_1 + n_2 - 2]) = \frac{1}{2}(n_1 + n_2 - 2 + 2 - (n_1 - 1) - (n_2 - 1) - 2) = 0 \quad (5.45)$$

which is once again zero. As in the previous subsection, after projection and multiplication, we have the product $\alpha\beta$ of the elements α and β associated to the individual factors. However, now we have a non-trivial map $f_{\langle \pi, \rho \rangle, \pi\rho}$ which maps from a space with a single factor A back to the tensor product $A^{\otimes 2}$ associated to the two orbits of the product $\pi\rho$. This adjoint map back is nothing but the co-multiplication Δ_* . In slightly more detail than before, we have the maps :

$$\Delta(e_i \otimes e_j) = f^{\pi\rho, \langle \pi, \rho \rangle}(e_i \otimes e_j) = e_i e_j = c_{ij}^k e_k \quad (5.46)$$

as well as the adjoint condition :

$$T(e_l, f^{\pi\rho, \langle \pi, \rho \rangle}(e_i \otimes e_j)) = c_{ij}^k \eta_{lk} = (-1)^l c_{ijl} = T(f_{\langle \pi, \rho \rangle, \pi\rho} e_l, e_i \otimes e_j). \quad (5.47)$$

The unique solution is

$$\Delta_*(e_l) = f_{\langle \pi, \rho \rangle, \pi\rho} e_l = (-1)^l c^{mk}_l e_k \otimes e_m, \quad (5.48)$$

since

$$\begin{aligned} T((-1)^l c^{mk}_l e_k \otimes e_m, e_i \otimes e_j) &= (-1)^l c^{mk}_l (-1)^{mi} \eta_{ki} \eta_{mj} \\ &= (-1)^l c_{ijl}. \end{aligned} \quad (5.49)$$

Thus, we find the final product :

$$\alpha\sigma_{(n_1)} \cdot \beta\sigma_{(n_2)} = \Delta_*(\alpha\beta)\sigma_{(n_1+n_2-3)}, \quad (5.50)$$

which we can render explicit in terms of the structure constants c_{ij}^k and the metric η_{ij} of the seed theory :

$$\Delta_*(\alpha^i \beta^j c_{ij}^k e_k) = \alpha^i \beta^j c_{ij}^k (-1)^k c^{nm}_k e_m \otimes e_n. \quad (5.51)$$

Next, we wish to add in the gauge invariance combinatorics that arises when exploiting this result to obtain a structure constant in the orbifold algebra of gauge invariants

$A^{[n]}$. We count the number of such double overlap multiplications that appear when multiplying conjugacy invariant operators. We again start by picking n_1 elements out of n elements (say), and ordering them in $(n_1 - 1)!$ ways, to obtain $n!/(n - n_1)!/n_1$ different operators $\sigma_{(n_1)}$. We pick two consecutive elements out of the n_1 elements, which we can do in n_1 ways – we must put them in opposite order in the second permutation –, and moreover, we pick $(n_2 - 2)$ elements out of $n - n_1$, and order them in $(n_2 - 2)!$ ways, which we can do in $(n - n_1)!/(n - n_1 - n_2 + 2)!$ ways. Thus, this gives a total of $n!/(n - n_1 - n_2 + 2)!$ of combinations of the type we are looking for. We have $n!/((n - n_1 - n_2 + 3)!(n_1 + n_2 - 3))$ elements in the conjugacy class of $\sigma_{n_1+n_2-3}$, and we then still have $n - n_1 - n_2 + 3$ choices for the extra lonely element we need, for a total of $n!/((n - n_1 - n_2 + 2)!(n_1 + n_2 - 3))$ choices. Thus we have a prefactor of

$$n_1 + n_2 - 3 \quad (5.52)$$

that arises from the combinatorics. Apart from the structure constants and adjoint map that arises from the original cohomology ring, this is the structure constant of the multiplication of the classes of $\sigma_{(n_1)}$ with $\sigma_{(n_2)}$ into $\sigma_{(n_1+n_2-3)}$. If we denote the gauge invariant operator sum (without extra normalization factor) by $O_{(n_i)}(\alpha)$, we find the operator product coefficient :

$$O_{(n_1)}(\alpha)O_{(n_2)}(\beta) = (n_1 + n_2 - 3)O_{(1)(n_1+n_2-3)}(\Delta_*(\alpha\beta)) + \dots \quad (5.53)$$

where on the right hand side, we need to split the co-product over the two tensor factors associated to the cycle of length one and the cycle of length $n_1 + n_2 - 3$.

Thus, we computed a few structure constants in the orbifold cohomology ring of the Hilbert schemes of points on $M = T^4$ and $M = K3$ among others. We wish to compare these results to operator product expansions of chiral primary operators in the physical symmetric orbifold conformal field theory. Those were obtained using rather different techniques.

To Match the Physical Correlators

We wish to confirm that the mathematical construction of subsection 5.2.1 indeed captures the physical operator product expansions that survive topological twisting. From the references [123,128] and in particular [113], we glean a few structure constants in the symmetric orbifold conformal field theory chiral ring. The appropriate extraction from [113] is performed in Appendix G. As in [113], we use the notation $\sigma_{(n_i)}^{\pm,\pm}$ to indicate a chiral ring primary associated to the $(-, -) = (0, 0)$ Dolbeault cohomology, the $(+, -) = (2, 0)$, $(-, +) = (0, 2)$ and the $(+, +) = (2, 2)$ Dolbeault cohomology of the manifolds $M = K3$ or $M = T^4$. Here, we focus on these manifolds and classes only. Moreover, following [113] we factorize the structure constants into a structure constant associated to two-dimensional massless left-movers and a structure constant associated to right-movers in the symmetric orbifold conformal field theory. The structure constants from [113] can then be summarized by – see Appendix G :

$$C_{n_1 n_2; n_3}^{1_{n_1}, 1_{n_2}; 1_{n_3}} = \left(\frac{(1_{n_1} n_1 + 1_{n_2} n_2 + 1_{n_3} n_3 + 1)^2}{4 n_1 n_2 n_3} \right)^{\frac{1}{2}} \quad (5.54)$$

where the individual operators carry $U(1)_R$ charge $n_i + 1_{n_i}$ and 1_{n_i} is ± 1 depending on the (left or right) upper index of the operator $\sigma_{n_i}^{\pm, \pm}$, and we assume the R-charge constraint $n_1 + 1_{n_1} + n_2 + 1_{n_2} = n_3 + 1_{n_3}$ [113]. The (left or right) structure constants that preserve $U(1)_R$ charge are the cases [123] :

$$\begin{aligned} C_{n_1, n_2, n_1+n_2-1}^{---} &= \left(\frac{n_1 + n_2 - 1}{n_1 n_2} \right)^{\frac{1}{2}} \\ C_{n_1, n_2, n_1+n_2-1}^{-++} &= \left(\frac{n_2}{n_1(n_1 + n_2 - 1)} \right)^{\frac{1}{2}} \\ C_{n_1, n_2, n_1+n_2-3}^{--+} &= \left(\frac{1}{n_1 n_2 (n_1 + n_2 - 3)} \right)^{\frac{1}{2}}, \end{aligned} \quad (5.55)$$

and these chiral structure constants lead to the non-chiral operator product terms :

$$\begin{aligned} \sigma_{n_1}^{--} \sigma_{n_2}^{--} &= \frac{n_1 + n_2 - 1}{n_1 n_2} \sigma_{n_1+n_2-1}^{--} + \dots \\ \sigma_{n_1}^{--} \sigma_{n_2}^{++} &= \frac{n_2}{n_1(n_1 + n_2 - 1)} \sigma_{n_1+n_2-1}^{++} + \dots \\ \sigma_{n_1}^{--} \sigma_{n_2}^{--} &= \frac{1}{n_1 n_2 (n_1 + n_2 - 3)} \sigma_{n_1+n_2-3}^{++} + \dots \end{aligned} \quad (5.56)$$

$$\begin{aligned} \sigma_{n_1}^{--} \sigma_{n_2}^{+-} &= \frac{1}{n_1} \sigma_{n_1+n_2-1}^{+-} + \dots \\ \sigma_{n_1}^{-+} \sigma_{n_2}^{+-} &= \frac{1}{n_1 + n_2 - 1} \sigma_{n_1+n_2-1}^{++} + \dots \end{aligned} \quad (5.57)$$

We now perform the renormalization we already saw for the quotient ring

$$\sigma_{n_i} = n_i \sigma_{n_i}^{--} \quad (5.58)$$

as well as the renormalizations :

$$\frac{1}{n_i} \sigma_{n_i}^{++} = \text{vol } \sigma_{n_i}, \quad \sigma_{n_i}^{+-} = \alpha^{(2,0)} \sigma_{n_i} \quad \sigma_{n_i}^{-+} = \alpha^{(0,2)} \sigma_{n_i}. \quad (5.59)$$

We note that as long as the volume operator and the unit operator scale oppositely (and the middle cohomology elements do not scale) we keep their two-point functions T invariant. All the correlation functions (5.57) then precisely match those computed in subsection 5.2.2. For the special case of the correlator (5.56), this is the case because we concentrate on the term in the co-multiplication that associates the identity operator to the cycle of length one. We note that the correlators computed by elementary means in subsection 5.2.2 already contain a few more results than those that we could extract from the physics results.

5.2.3 Cup Products at Low and High Order

In this section, we compute products of operators at a few low orders n . This serves firstly as a small catalogue of results. Secondly, we will soon see that these results contain glimpses of results at large n , as well as examples that inspire. Thus, we compute the cup product in the algebra $A\{S_n\}$ and therefore $A^{[n]}$ for various values of the order n of the permutation group S_n .

Orders One and Two

For $n = 1$, the cup product equals the ordinary product in the cohomology. This defines the original Frobenius algebra A . For $n = 2$, there are two elements in S_2 . They give rise to the algebras

$$\begin{aligned} A\{S_2\} &= A^{\otimes 2}() + A(12) \\ A^{[2]} &= S^2 A[()] + A[(12)] \end{aligned} \quad (5.60)$$

with the multiplication rule :

$$\begin{aligned} \alpha_1 \otimes \alpha_2() \cdot \beta_1 \otimes \beta_2() &= (-1)^{\beta_1 \alpha_2} \alpha_1 \beta_1 \otimes \alpha_2 \beta_2() \\ \alpha_1 \otimes \alpha_2() \cdot \beta(12) &= \alpha_1 \alpha_2 \beta(12) \\ \alpha(12) \beta(12) &= \Delta_*(\alpha \beta)() \end{aligned} \quad (5.61)$$

in $A\{S_2\}$. An operator $\gamma_1 \otimes \gamma_2$ multiplying the unit element $()$ in the permutation group will be symmetrized in $A^{[2]}$, and otherwise the algebra is similar.

Order Three

At order three, we have the algebras :

$$\begin{aligned} A\{S_3\} &= A^{\otimes 3}() + A^{\otimes 2}(12) + A(123) + \dots \\ A^{[2]} &= S^3 A[()] + A \otimes A[(12)] + A[(123)] \end{aligned} \quad (5.62)$$

and in the algebra $A\{S_3\}$ the multiplication rule is (as in example 2.17 of [105]) :

$$\begin{aligned} (\alpha_1 \otimes \alpha_2)(12) \cdot (\beta_1 \otimes \beta_2)(13) &= \alpha_1 \alpha_2 \beta_1 \beta_2(132) \\ (\alpha_1 \otimes \alpha_2)(12) \cdot (\beta_1 \otimes \beta_2)(12) &= (-1)^{\beta_1 \alpha_2} \Delta_*(\alpha_1 \beta_1) \otimes (\alpha_2 \beta_2)() \\ \alpha(123) \cdot \beta(123) &= (\alpha \beta e)(132) \\ \alpha(123) \cdot \beta(132) &= \Delta_*^{(3)}(\alpha \beta)(). \end{aligned} \quad (5.63)$$

We used the symbol $\Delta_*^{(3)}$ for the adjoint of the multiplication map $\phi_{3,1} = \Delta^{*(3)} : A \otimes A \otimes A \rightarrow A : \alpha_1 \otimes \alpha_2 \otimes \alpha_3 \rightarrow \alpha_1 \alpha_2 \alpha_3$. Note that for the first time, we have a multiplication with a graph defect equal to one, leading to a multiplication with the Euler class e . Our interpretation of this structure constant as an operator product structure constant makes it the first such three-point function in AdS_3/CFT_2 calculated for a covering surface of genus one. We will generalize this example soon.

Orders Four and Six

We have the algebra

$$A^{[4]} = A \oplus A \otimes A \oplus S^2 A \oplus S^2 A \otimes A \oplus S^4 A \quad (5.64)$$

corresponding to the partitions $[4], [13], [2^2], [1^2 2], [1^4]$ of $n = 4$, and a similar decomposition of the algebra $A^{[6]}$ for $n = 6$. We wish to make a few illustrative and useful

calculations in these algebras using the multiplications in $A\{S_n\}$. We find the $n = 4$ products :

$$\begin{aligned}\alpha_1 \otimes \alpha_2 \otimes \alpha_3(12) \cdot \beta_1 \otimes \beta_2 \otimes \beta_3(12) &= (-1)^{\beta_1(\alpha_2+\alpha_3)+\beta_2\alpha_3} \Delta_*(\alpha_1\beta_1) \otimes \alpha_2\beta_2 \otimes \alpha_3\beta_3(12) \\ \alpha_1 \otimes \alpha_2 \otimes \alpha_3(12) \cdot \beta_1 \otimes \beta_2 \otimes \beta_3(13) &= (-1)^{\alpha_3(\beta_1+\beta_2)} \alpha_1\alpha_2\beta_1\beta_2 \otimes \alpha_3\beta_3(132) \\ \alpha_1 \otimes \alpha_2 \otimes \alpha_3(12) \cdot \beta_1 \otimes \beta_2 \otimes \beta_3(34) &= (-1)^{(\alpha_2+\alpha_3)(\beta_1+\beta_2)} \alpha_1\beta_1\beta_2 \otimes \alpha_2\alpha_3\beta_3(12)(34).\end{aligned}\quad (5.65)$$

If we compute the analogues in S_5 , we multiply the fifth wheel on the first wagon with the fifth wheel on the second, but learn little extra. In the algebra $A\{S_6\}$ we multiply :

$$\begin{aligned}\alpha_1 \otimes \dots (123)\beta_1 \otimes \beta_2 \otimes \dots (234) &= \epsilon(\alpha_i, \beta_i) \Delta_*(\alpha_1\alpha_2\beta_1\beta_2) \otimes \alpha_3\beta_3 \otimes \alpha_4\beta_4(12)(34) \\ \alpha_1 \otimes \dots (123)\beta_1 \otimes \beta_2 \otimes \dots (324) &= \epsilon \Delta_*(\alpha_1\alpha_2\beta_1\beta_2) \otimes \dots (124) \\ \alpha_1 \otimes \dots (123)\beta_1 \otimes \dots (345) &= \epsilon \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \otimes \alpha_4\beta_4(12345) \\ \alpha_1 \otimes \alpha_2 \otimes \dots (123)\beta_1 \otimes \dots (456) &= \epsilon \alpha_1\beta_1\beta_2\beta_3 \otimes \alpha_2\alpha_3\alpha_4\beta_4(123)(456).\end{aligned}\quad (5.66)$$

In these results, we have gradually suppressed details (like the sign ϵ depending on the grades of the elements α_i and β_i and spectator factors) in order to make them more compact.

The results for the $A\{S_4\}$ algebra lead to the gauge invariant $A^{[4]}$ multiplication rule for the transposition conjugation class :

$$\begin{aligned}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 + \alpha_1 \otimes \alpha_3 \otimes \alpha_2)[(12)] \cdot (\beta_1 \otimes \beta_2 \otimes \beta_3 + \beta_1 \otimes \beta_3 \otimes \beta_2)[(12)] &= \\ 6\left((-1)^{\beta_1(\alpha_2+\alpha_3)+\beta_2\alpha_3} \Delta_*(\alpha_1\beta_1) \otimes \alpha_2\beta_2 \otimes \alpha_3\beta_3 + \text{three symm.}\right)[()] & \\ + 2\left((-1)^{(\alpha_2+\alpha_3)(\beta_1+\beta_2)} \alpha_1\beta_1\beta_2 \otimes \alpha_2\alpha_3\beta_3 + \text{three symm.}\right)[(12)][(34)] & \\ + 3\left(\alpha_1\alpha_2\beta_1\beta_2 + \text{three symmetrization terms}\right)[(123)]. &\end{aligned}\quad (5.67)$$

It is clear that the structure constants depend on the seed theory only, though the combinatorics depends on the order of the symmetric group. In the algebra of gauge invariants $A^{[6]}$ we recall that the $[1^33]$ partition corresponds to a term $S^3A \otimes A$ in the direct sum $A^{[6]}$. When we compute the product of three-cycle conjugacy classes, we therefore symmetrize the results (5.66) as well as (5.63) over the factors associated to the cycles of length one, giving rise to six terms for each element of the conjugacy class $[1^33]$, of which there are 40. Thus, a naive counting gives 240^2 terms. While this can be written significantly more efficiently, we still refrain from writing out the final result. We remark that the conjugacy class combinatorics is captured by the product :

$$[1^33] * [1^33] = 40[1^6] + 8[1^22^2] + 10[1^33] + 2[3^2] + 5[15]. \quad (5.68)$$

Remarks

- We note that when we express the ring structure in an appropriate set of cohomology elements, the structure constants become independent of the order of the orbifold n [133], such that even low order n results capture structure constants of the large order ring (as long as the order is large enough to accommodate all possible conjugacy classes that can occur in the product). We refer to [133] for the precise statement. This property holds in the presence of genus one Euler class corrections.

- The combinatorial description of top connection coefficients and sub-top coefficients of the symmetric group [120, 121] may be useful to efficiently compute structure constants of the Hilbert scheme of points for compact surfaces.

5.2.4 Genus One and an Overlap of Three

In this subsection, we compute a simple yet new class of structure constants. We consider two operators $\alpha\pi$ and $\beta\rho$, proportional to permutations $\pi = \sigma_{(n_1)} = (1234 \dots n_1)$ and $\rho = \sigma_{(n_2)} = (1, 2, 3, n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 3)$ which overlap in the three elements 1, 2, 3. We list the orbits :

$$\begin{aligned} \langle \pi \rangle \setminus [n_1 + n_2 - 3] &= \{[n_1], n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 3\} \\ \langle \rho \rangle \setminus [n_1 + n_2 - 3] &= \{4, 5, \dots, n_1, \{1, 2, 3, n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 3\}\} \\ \langle \pi\rho \rangle \setminus [n_1 + n_2 - 3] &= \{1, 2, \dots, n_1 + n_2 - 3\} \\ \langle \pi, \rho \rangle \setminus [n_1 + n_2 - 3] &= [n_1 + n_2 - 3]. \end{aligned} \quad (5.69)$$

Importantly, the graph defect equals

$$g = \frac{1}{2}(n_1 + n_2 - 1 - (n_2 - 2) - (n_1 - 2) - 1) = 1. \quad (5.70)$$

We find the operator product :

$$\alpha\pi \cdot \beta\rho = (\alpha\beta e)\pi\rho. \quad (5.71)$$

This is a generalization of the product we found for equal three-cycles in S_3 . It is a simple and generic genus one result for the operator product expansion in the topological orbifold conformal field theory that forms a clear target for the holographically dual bulk AdS_3 topological string theory.

5.2.5 Further Examples

It is straightforward to generate further classes of examples using the tools we have laid out.

- Note that for $2l + 1$ identical (consecutive) elements in two cyclic permutations, the graph defect equals $g = l$, and therefore the product vanishes for $l = g \geq 2$. This provides an example of how to generate (vanishing) higher genus amplitudes of arbitrarily high genus.
- An example that involves the co-multiplication map $\Delta_*^{(n)}$, namely the adjoint of $\phi_{n,1}$, is found in the multiplication of two elements associated to inverse n -cycles :

$$\alpha(12 \dots n) \cdot \beta(n, n-1, \dots, 1) = \Delta_*^{(n)}(\alpha\beta)().$$
- Co-multiplication can also act on a graph defect one contribution :

$$\alpha(123 \dots n) \cdot \beta(1, 2, n, n-1, \dots, 4, 3) = \Delta_*^{(n-2)}(\alpha\beta e)(132).$$
- Finally, we provide an example involving a permutation that is the product of two cycles in the algebra $A\{S_4\}$:

$$\alpha_1 \otimes \alpha_2(12)(34) \cdot \beta_1 \otimes \beta_2 \otimes \beta_3(23) = \alpha_1 \alpha_2 \beta_1 \beta_2 \beta_3(1243).$$

We leave room to further combine and invent examples in creative ways.

5.2.6 Remarks

We have shown that the correlators that one computes in the symmetric orbifold conformal field theory as well as the bulk string theory agree with the mathematical description of the orbifold cohomology ring. The calculations are very efficient in the mathematical formalism, which uses only the structure constants of the seed cohomology ring and combinatorics. We stressed that the structure constants in the topologically twisted theory can be rendered independent of the order of the orbifold as long as one picks it to be (finite but) large enough. We moreover computed a set of genus one correlators and observed that connected higher genus correlators vanish. This simplification of the topological symmetric orbifold theory is a good preparation for proving a topological AdS/CFT correspondence. The task is to bijectively map the bulk gauge invariants to the operator algebra $A^{[n]}$ that captures the boundary chiral ring.

Conclusion

In this thesis, we mostly focused on studying the topologically twisted version of the holographic duality between three-dimensional gravity theories with negative cosmological constant and two-dimensional conformal field theories.

In chapter 2, we studied the topological twist of a non-compact $N = 2$ superconformal field theory, the $N = 2$ super-Liouville theory, with massive deformations. This theory is conformal field theory side of the holographic duality considered in chapter 3. We determined the chiral ring elements and the WDVV equations in the similar way for the $N = 2$ superconformal field theory with compact target spaces. We then studied the topological-anti-topological sector and investigated the tt^* equations of the deformed theory.

In chapter 3, inspired by the fact that the action three-dimensional gravity theory with negative cosmological constant can be reduced to a two-dimensional Liouville action on the boundary [12], we defined a three-dimensional quantum gravity theory that is the dual to Liouville theory by setting the path integral measures equal. Such defined gravity has peculiar properties, where there is no normalizable global AdS_3 states but only BTZ black hole states with zero angular momentum that can be dressed with boundary gravitons. Nevertheless, the theory is consistent and unitary since Liouville theory is. Moreover, it was found in [134] that the Liouville Hilbert space can be acquired by performing a projection procedure after directly quantizing three-dimensional gravity theories. An interesting open problem is to understand how this projection is related to taking the Liouville measure in the gravity path integral.

We then generalized this duality to the supersymmetric case and considered a topological twist in the second half of chapter 3. By picking a specific boundary condition for the gauge field in the supergravity theory, we found the supergravity action reduces to a boundary topologically twisted $N = 2$ super-Liouville theory. We then determined the classical configurations that correspond to the chiral ring elements which are determined by generalizing results of chapter 2. It would be interesting to apply the techniques of [135] to the topologically twisted supergravity theory and attempt to do further computation using localization.

Then, in chapter 4, we studied a string theory in locally three-dimensional anti-de Sitter space-time with a general boundary metric. By setting a twisting boundary condition for the background of the bulk string theory, we found the boundary energy-momentum tensor becomes topological. This is because the space-time $N = 2$ superconformal algebra can be derived from the bulk AdS_3 string theory, and the symmetry algebra implies that the topological energy-momentum tensor is \mathcal{G}_0^+ exact. Based on this, one can attempt to compute the worldsheet operator that correspond to the spa-

cetime chiral primary operators or prove that the string theory path integral for this theory localizes on some certain configurations in future research.

Finally, in chapter 5, we studied the topological superconformal theory with target space being a symmetric product of a complex surface M , which is the conjectured dual to the full ten-dimensional string theory on $AdS_3 \times S^3 \times M$. We have shown, by exploiting math literature, that the structure constants of a canonical quotient of the operator ring are given by Hurwitz numbers, proving a conjecture in the physics literature on extremal correlators. We then studied the full chiral ring given by a symmetric orbifold Frobenius algebra, where one can compute the correlators of chiral ring elements. A direct follow up work would be to study the ten-dimensional topologically twisted string theory using the method in chapter 4 and compare the correlators computed there with the ones in chapter 5.

Annexe A

Examples

In this appendix, we provide explicit formulas for the topological quantum field theories that we study in chapter 2, for low levels. Firstly, we recall the solutions of the compact models up to level $k_c = 5$. Secondly, we discuss the non-compact models, also up to level $k = 5$. We compute the superpotential W by integration from its various derivatives, and integrate the linear differential equations for the free energy F , which is the generator of correlation functions.

A.1 Compact Examples

We start out by describing the compact models. We start from the most trivial model, at level $k_c = 2$, and work our way up to level $k_c = 5$. For easy reference we provide many formulas, and comment little.

Level $k_c = 2$

At level two, we have the formulas

$$\phi_0 = 1 \tag{A.1}$$

and

$$\phi_1 = W_X = X \quad - \phi_0 = \partial_0 W = -1, \tag{A.2}$$

and therefore

$$W = X^2/2 - t_0. \tag{A.3}$$

The ring consists of the unit operator only and is undeformed. The structure constants are trivial and follow from the free energy

$$F(t_0) = t_0^3/3!. \tag{A.4}$$

Level $k_c = 3$

At level three, we find

$$\phi_0 = 1 \quad \phi_1 = X \quad W_X = X^2 - t_1. \tag{A.5}$$

We integrate to

$$W = X^3/3 - t_1 X - t_0. \quad (\text{A.6})$$

The ring is two-dimensional. We can potentially deform by deformation parameters t_0, t_1 of R-charge $1, 2/3$. The operators in the ring have R-charge $0, 1/3$ and therefore they can not mix (polynomially). The function $F(t_0, t_1)$ is equal to the triple integral of the structure constants and is equal to

$$F(t_0) = t_1 t_0^2/2 + t_1^4/4!. \quad (\text{A.7})$$

Level $k_c = 4$

In this case, the calculations run

$$\begin{aligned} \phi_0 &= 1 & \phi_1 &= X \\ \phi_2 &= X^2 - t_2 & W_X &= X^3 - 2t_2 X - t_1, \end{aligned} \quad (\text{A.8})$$

as well as

$$W = X^4/4 - t_2 X^2 - t_1 X + t_2^2/2 - t_0. \quad (\text{A.9})$$

The function $F(t_0, t_1, t_2)$ is

$$F(t_0, t_1, t_2) = t_0 t_1^2/2 + t_0^2 t_2/2 + t_2^2 t_1^2/4 + t_2^5/60. \quad (\text{A.10})$$

Level $k_c = 5$

The formulas are

$$\begin{aligned} \phi_0 &= 1 & \phi_1 &= X \\ \phi_2 &= X^2 - t_3 & \phi_3 &= X^3 - 2t_3 X - t_2 \\ W_X &= X^4 - 3t_3 X^2 - 2t_2 X - t_1 + t_3^2, \end{aligned} \quad (\text{A.11})$$

as well as

$$W = X^5/5 - t_3 X^3 - t_2 X^2 - t_1 X + t_3^2 X - t_0 + t_2 t_3. \quad (\text{A.12})$$

The free energy works out to be

$$F(t_i) = t_0 t_1 t_2 + t_0^2 t_3/2 + t_1^3/3! + t_1 t_2^2 t_3/2 + t_1^2 t_3^2/4 + t_2^4/12 + t_2^2 t_3^3/6. \quad (\text{A.13})$$

A.2 Non-Compact Examples

We turn to non-compact examples, and integrate up all the way to the generating function F . We list the formulas, and only make a few remarks.

A.2.1 Level $k = 2$

The model has only one operator, which squares to zero. The superpotential term demonstrates the non-linearity in the parameter s_0 .

$$\begin{aligned} \phi_2 &= s_0 Y^{-2} & W_Y &= -s_0^2 Y^{-3} \\ W &= s_0^2 \frac{Y^{-2}}{2} & F &= 0. \end{aligned} \quad (\text{A.14})$$

A.2.2 Level $k = 3$

There are two operators, and the ring is already non-trivially deformed at this level. The generator of correlation functions has a denominator which is a monomial in the parameter s_0 :

$$\begin{aligned}
 \phi_2 &= s_0 Y^{-2} & \phi_3 &= s_0^2 Y^{-3} + s_1 Y^{-2} \\
 W_Y &= -s_0^3 Y^{-4} - 2s_1 s_0 Y^{-3} & W &= \frac{Y^{-3}}{3} (s_0^3 + 3s_1 s_0 Y) \\
 F &= \frac{1}{12} \frac{s_1^4}{s_0^2}.
 \end{aligned} \tag{A.15}$$

A.2.3 Level $k = 4$

The homogeneity properties of the free energy are obvious from the following example

$$\begin{aligned}
 \phi_2 &= s_0 Y^{-2} & \phi_3 &= s_0^2 Y^{-3} + s_1 Y^{-2} \\
 \phi_4 &= s_0^3 Y^{-4} + 2s_0 s_1 Y^{-3} + s_2 Y^{-2} & W &= \frac{Y^{-4}}{4} (s_0^4 + 4s_0^2 s_1 Y + 2(s_1^2 + 2s_0 s_2) Y^2) \\
 F &= \frac{s_1^6 - 6s_2 s_1^4 s_0 + 12s_2^2 s_1^2 s_0^2 - 4s_2^3 s_0^3}{24s_0^4}.
 \end{aligned} \tag{A.16}$$

A.2.4 Level $k = 5$

For good measure, we throw in a final polynomial coding the correlation functions of the level 5 non-compact model :

$$\begin{aligned}
 F &= \frac{1}{120s_0^6} \left(20s_1^2 s_0^3 (3s_3^2 s_0 - 4s_2^3) + 15s_2 s_0^4 (s_2^3 - 4s_3^2 s_0) + 120s_3 s_2^2 s_1 s_0^4 - 120s_2 s_3 s_1^3 s_0^3 \right. \\
 &\quad \left. + 24s_3 s_1^5 s_0^2 + 90s_2^2 s_1^4 s_0^2 - 36s_2 s_1^6 s_0 + 5s_1^8 \right).
 \end{aligned} \tag{A.17}$$

Annexe B

A Few Properties of Formal Power Series

Formal power series are a useful tool in the fields of combinatorics, vertex operator algebras and others. For our purposes, it is useful to recall some rules of calculus valid for formal power series f (with finitely many non-zero negative power coefficients) :

$$f = \sum_{n \in \mathbb{Z}} f_n x^n. \quad (\text{B.1})$$

We define the bracket operation that picks out certain coefficients of a formal power series¹

$$[f]_n = f_n \quad (\text{B.2})$$

and has the property that the residue of the derivative of a formal power series is zero

$$\text{Res}(f') = [f']_{-1} = 0, \quad (\text{B.3})$$

which has many non-trivial consequences when cleverly applied. A formal power series whose constant coefficient f_0 is zero and whose linear coefficient f_1 is non-zero has a formal inverse. Suppose that g is the inverse of f (where $f_0 = 0$ and $f_1 \neq 0$). We then have the Lagrange inversion formula relating the coefficients of powers of these power series :

$$m[g^n]_m = n[f^{-m}]_{-n}. \quad (\text{B.4})$$

For $m = -1$, we find the equation relating the residue of the power of the inverse series g to the coefficients of the original series f

$$- [g^n]_{-1} = n[f]_{-n}. \quad (\text{B.5})$$

The latter Lagrangian inversion formula is useful in the solution of the compact topological quantum field theory. The solution of the non-compact topological quantum field theory requires more advanced techniques, described in the chapter 2.

1. In chapter 2, we also use the bracket operation to pick a subset of terms in a formal power series – we trust the context will provide sufficient guidance to distinguish these two uses.

Annexe C

Lorentzian Conformal Field Theory and Metrics

In this appendix, we discuss Lorentzian metrics that solve Einstein's equation with negative cosmological constant. We obtain the metrics through analytic continuation from the Euclidean metrics described in section 3.2. In particular, we again exploit the Banados form of the metric (3.1), now in Lorentzian signature. We determine Lorentzian energy-momentum tensor components through analytic continuation. Plugging those into the Lorentzian Banados metric, we obtain interesting solutions to Einstein's equations. We discuss properties of these solutions, but leave their full investigation for future work.

The Lorentzian Banados solution reads :

$$ds^2 = l^2 d\rho^2 + 4Gl(-T(dw^+)^2 - \bar{T}(dw^-)^2) + (l^2 e^{2\rho} + 16G^2 T \bar{T} e^{-2\rho}) dw^+ dw^- \quad (\text{C.1})$$

where T and \bar{T} indicate real and independent components of the Lorentzian energy-momentum tensor, and w^\pm are real coordinates. The first example metric is when T and \bar{T} are constant. We then reproduce the Lorentzian BTZ black hole metric. A second example is given by plugging in the Lorentzian version of the planar energy-momentum tensor :

$$T_L^{(0,\infty)} = \frac{h}{(z^+)^2} \quad \bar{T}_L^{(0,\infty)} = \frac{h}{(z^-)^2}. \quad (\text{C.2})$$

We obtain a horizon :

$$e^{4\rho_{hor}} = (4G/l)^2 h^2 (z^+ z^-)^{-2}. \quad (\text{C.3})$$

The horizon radius goes to infinity on the lines $z^+ = 0$ and $z^- = 0$. Where the lines intersect, the singularity is the product of the individual quadratic singularities, and becomes quartic. The horizon therefore forms Lorentzian mountain ranges with extreme peaks where the ranges cross. The singularities are located where the horizon intersects the boundary. When we introduce the cut-off on the radius, a ribbon in the form of a cross is cut out from the horizon surface.

Generically, it is subtle to analytically continue a complex valued energy-momentum tensor T_{zz} to a real-valued energy-momentum tensor component T_{++} . However, we have readied the expressions for the energy-momentum tensor components in section

3.2 in order to render this task eminently feasible. Thus, the energy-momentum tensors $T_c^{(2)}$ and $T_c^{(3)}$ become real when we replace the coordinates (w, \bar{w}) by real coordinates (w^+, w^-) , and choose the source points w_i^\pm to be real. This gives rise to energy-momentum tensors

$$\begin{aligned}
T_{c,L}^{(2)}(w^+) &= \frac{h}{4} \frac{\sin^2 \frac{w_1^+ - w_2^+}{2}}{\sin^2 \frac{w^+ - w_1^+}{2} \sin^2 \frac{w^+ - w_2^+}{2}} + \frac{c}{24}, \\
T_{c,L}^{(3)} &= h_1 \frac{\sin \frac{w_2^+ - w_1^+}{2}}{2 \sin \frac{w^+ - w_1^+}{2} \sin \frac{w^+ - w_2^+}{2}} \frac{\sin \frac{w_3^+ - w_1^+}{2}}{2 \sin \frac{w^+ - w_1^+}{2} \sin \frac{w^+ - w_3^+}{2}} + \text{cyclic} + \frac{c}{24} \quad (\text{C.4})
\end{aligned}$$

We offer a few generic observations :

- When one light-cone coordinate coincides between two insertions, e.g. $w_1^+ = w_2^+$, then the light-cone energy-momentum tensor component $T(w^+)$ does not depend on the conformal dimensions h_1 nor h_2 (but only on h_3), and only on the difference $w_2^+ - w_3^+ = w_1^+ - w_3^+$, and in precisely the way of a two-point energy-momentum tensor expectation value.
- When three insertions have the same light-cone coordinate, the energy-momentum tensor component is constant. This follows from translational invariance.

These facts are a consequence of the operator product expansion.

Naively, the metric associated to the energy-momentum tensor $T_{c,L}^{(3)}$ should correspond to three continuously interacting black hole states, but in trying to substantiate this claim, we encountered difficulties. To identify the precise physical interpretation of these interesting Lorentzian metrics requires considerably more work.

Annexe D

The Three-point Energy-momentum Tensor

In this appendix, we demonstrate explicitly that the semiclassical limit of the energy-momentum tensor in the presence of three insertions is given by formula (3.67), as it must be. The energy-momentum tensor of Liouville theory in the semiclassical limit is [62]

$$\begin{aligned} T(z) &= -(\partial\phi)^2 + Q\partial^2\phi \\ &= \frac{1}{2b^2}[-\frac{1}{2}(\partial\varphi)^2 + \partial^2\varphi], \end{aligned} \quad (\text{D.1})$$

where we have used that $\varphi = 2b\phi$ and that $Q = \frac{1}{b} + b \rightarrow \frac{1}{b}$ in the semiclassical limit $b \rightarrow 0$. Plugging in the classical solution (3.59), the energy-momentum tensor can be written as

$$T(z) = -\frac{a_1\psi_1''(z)\psi_1(\bar{z}) + a_2\psi_2(z)''\psi_2(\bar{z})}{b^2(a_1\psi_1(z)\psi_1(\bar{z}) + a_2\psi_2(z)\psi_2(\bar{z}))}. \quad (\text{D.2})$$

We will show that

$$\frac{\psi_1''}{\psi_1} = \frac{\psi_2''}{\psi_2}, \quad (\text{D.3})$$

and therefore

$$T(z) = -\frac{\psi_1''}{b^2\psi_1}. \quad (\text{D.4})$$

Using the explicit form of the field ψ_1 given in (3.60), we find

$$\begin{aligned} \frac{\psi_1''}{\psi_1} &= \frac{\eta_1(\eta_1 - 1)}{(z - z_1)^2} + \frac{(1 - \eta_1 - \eta_3)(-\eta_1 - \eta_3)}{(z - z_2)^2} + \frac{\eta_3(\eta_3 - 1)}{(z - z_3)^2} \\ &\quad + \frac{2\eta_1(1 - \eta_1 - \eta_3)}{(z - z_1)(z - z_2)} + \frac{2\eta_1\eta_3}{(z - z_1)(z - z_3)} + \frac{2(1 - \eta_1 - \eta_3)\eta_3}{(z - z_2)(z - z_3)} \\ &\quad + 2\left[\left(\frac{\eta_1}{z - z_1} + \frac{1 - \eta_1 - \eta_3}{z - z_2} + \frac{\eta_3}{z - z_3}\right)\frac{\partial x}{\partial z} + \frac{\partial^2 x}{\partial z^2}\right]\frac{{}_2F_1'}{{}_2F_1} \\ &\quad + \left(\frac{\partial x}{\partial z}\right)^2\frac{{}_2F_1''}{{}_2F_1}, \end{aligned} \quad (\text{D.5})$$

where the symbol ${}_2F_1$ stands for the hypergeometric function ${}_2F_1$ with arguments ${}_2F_1(\eta_1 + \eta_3 - \eta_2, \eta_1 + \eta_2 + \eta_3 - 1, 2\eta_1, x)$. Using the differential equation the hypergeometric function satisfies, we can express the second order derivative of the hypergeometric function ${}_2F_1$ in terms of its first order derivative and itself, which yields

$$\begin{aligned} \frac{\psi_1''}{\psi_1} = & \frac{\eta_1(\eta_1 - 1)}{(z - z_1)^2} + \frac{(1 - \eta_1 - \eta_3)(-\eta_1 - \eta_3)}{(z - z_2)^2} + \frac{\eta_3(\eta_3 - 1)}{(z - z_3)^2} \\ & + \frac{2\eta_1(1 - \eta_1 - \eta_3)}{(z - z_1)(z - z_2)} + \frac{2\eta_1\eta_3}{(z - z_1)(z - z_3)} + \frac{2(1 - \eta_1 - \eta_3)\eta_3}{(z - z_2)(z - z_3)} \\ & + 2\left(\frac{\eta_1}{z - z_1} + \frac{1 - \eta_1 - \eta_3}{z - z_2} + \frac{\eta_3}{z - z_3}\right) \frac{\partial x}{\partial z} \frac{{}_2F_1'}{{}_2F_1} \\ & + \left[\frac{\partial^2 x}{\partial z^2} + \left(\frac{\partial x}{\partial z}\right)^2 \left(\frac{2(\eta_1 + \eta_3)}{1 - x} - \frac{2\eta_1}{x(1 - x)}\right)\right] \frac{{}_2F_1'}{{}_2F_1} \\ & + \left(\frac{\partial x}{\partial z}\right)^2 \frac{(\eta_1 + \eta_3 - \eta_2)(\eta_1 + \eta_2 + \eta_3 - 1)}{x(1 - x)}. \end{aligned} \quad (\text{D.6})$$

Applying the definition of the quantity x given in (3.62), we obtain :

$$\frac{\partial x}{\partial z} = \frac{z_{12}z_{32}}{(z - z_2)^2 z_{31}}, \quad (\text{D.7})$$

$$\frac{\partial^2 x}{\partial z^2} = -\frac{2z_{12}z_{32}}{(z - z_2)^3 z_{31}}, \quad (\text{D.8})$$

$$1 - x = \frac{(z - z_3)z_{21}}{(z - z_2)z_{31}}, \quad (\text{D.9})$$

$$x(1 - x) = \frac{(z - z_1)(z - z_3)z_{21}z_{32}}{(z - z_2)^2 z_{31}^2}. \quad (\text{D.10})$$

We therefore find

$$\begin{aligned} & \frac{\partial^2 x}{\partial z^2} + \left(\frac{\partial x}{\partial z}\right)^2 \left(\frac{2(\eta_1 + \eta_3)}{1 - x} - \frac{2\eta_1}{x(1 - x)}\right) \\ = & -2 \frac{\partial x}{\partial z} \left[\frac{1}{z - z_2} + \frac{(\eta_1 + \eta_3)z_{32}}{(z - z_2)(z - z_3)} - \frac{\eta_1 z_{31}}{(z - z_1)(z - z_3)}\right] \\ = & -2 \frac{\partial x}{\partial z} \left(\frac{\eta_1}{z - z_1} + \frac{1 - \eta_1 - \eta_3}{z - z_2} + \frac{\eta_3}{z - z_3}\right), \end{aligned} \quad (\text{D.11})$$

and thus the total coefficient in front of $\frac{{}_2F_1'}{{}_2F_1}$ on the right hand side of (D.6) is zero. One can also simplify the remaining terms and get

$$\begin{aligned} \frac{\psi_1''}{\psi_1} = & \frac{\eta_1(\eta_1 - 1)}{(z - z_1)^2} + \frac{\eta_2(\eta_2 - 1)}{(z - z_2)^2} + \frac{\eta_3(\eta_3 - 1)}{(z - z_3)^2} \\ & - \left[\frac{\eta_1(\eta_1 - 1) + \eta_2(\eta_2 - 1) - \eta_3(\eta_3 - 1)}{(z - z_1)(z - z_2)} + \text{cyclic permutations}\right]. \end{aligned} \quad (\text{D.12})$$

Mutadis mutandis, equation (D.3) follows. The final result for the energy-momentum tensor then agrees with equation (3.67).

Annexe E

Conservation of the Energy-Momentum Tensor

We will switch back to the original coordinates while using $\phi = \log r$ when convenient.

$$\int_{\partial} d^2 x' \sqrt{g^{(0)}}(x') \nabla^i{}^{(0)} \mathcal{T}_{ij}(x') \xi^j(x') = \int d^2 z \xi^i (EOM)_i. \quad (\text{E.1})$$

We work in the gauge

$$\begin{aligned} B_{x\bar{x}} &= \frac{1}{2} e^{2\phi+2\omega} + O(e^{-2\phi}) \\ B_{\phi x} &= 2\partial_x \omega. \end{aligned} \quad (\text{E.2})$$

for the NS-NS 2-form B-field.

$$\sqrt{g^{(0)}}(x') \nabla^i{}^{(0)} T_{i\bar{x}}(x') = \int d^2 z \delta(x - x') (EOM)_{\bar{x}}, \quad (\text{E.3})$$

$$(EOM)_{\bar{x}} = \frac{l^2}{\alpha'} \int d^2 z \delta(x - x') \left(\partial(e^{2\phi+2\omega} \bar{\partial} x) + (\partial_x \partial_{\bar{x}}^2 \omega - 2\partial_x \partial_{\bar{x}} \omega \partial_{\bar{x}} \omega) \bar{\partial} \bar{x} \partial x - 2\partial_{\bar{x}} \partial_x \omega \bar{\partial} \phi \partial x \right). \quad (\text{E.4})$$

We expect our energy-momentum tensor determined above to be conserved, i.e.

$$\nabla^i \mathcal{T}_{i\bar{x}}(x', \bar{x}') = \partial_{x'} T_{\bar{x}\bar{x}}(x', \bar{x}') + \partial_{\bar{x}'} T_{x\bar{x}} - 2\partial_{\bar{x}'} \omega T_{x\bar{x}}. \quad (\text{E.5})$$

is equal to zero. We will prove this now in a gauge for the B-field similar to .

$$H = (2e^{2\phi} \sqrt{g^{(0)}} - \frac{1}{4} e^{2\omega} R^{(0)} + O(e^{-2\phi})) d\phi \wedge dx \wedge d\bar{x} \quad (\text{E.6})$$

Under a h_{xx} perturbation of the boundary metric, the gauge we choose for the B-field is

$$\begin{aligned} B_{x\bar{x}} &= e^{2\phi} \sqrt{g^{(0)}} + O(e^{-2\phi}) \\ B_{\phi x} &= 2\partial_x \omega - e^{-2\omega} \partial_{\bar{x}} h_{xx}. \end{aligned} \quad (\text{E.7})$$

Applying the order estimate given by the equations of motion, the energy-momentum tensor in this gauge is up to $O(e^{-2\phi})$ terms

$$\begin{aligned} \mathcal{T}_{\bar{x}\bar{x}}(x') &= \frac{l^2 e^{2\omega(x')}}{\alpha'} \int d^2 z \left[e^{2\phi} \delta^{(2)}(x' - x) \partial x \bar{\partial} x - \frac{1}{2} \partial_{\bar{x}} (e^{-2\omega(x)} \partial_{\bar{x}} \delta^{(2)}(x' - x)) \partial x \bar{\partial} x \right. \\ &\quad \left. + e^{-2\omega(x)} (\partial_{\bar{x}}^2 \omega - (\partial_{\bar{x}} \omega)^2) \delta^{(2)}(x' - x) \partial x \bar{\partial} x - e^{-2\omega(x)} \partial_{\bar{x}} \delta^{(2)}(x' - x) \partial x \bar{\partial} x \right] \end{aligned} \quad (E.8)$$

One can distribute the factor $e^{2\omega(x')}$ on each term and use that $e^{2\omega(x')} \delta^{(2)}(x' - x) = e^{2\omega(x)} \delta^{(2)}(x' - x)$ to simplify this energy momentum component :

$$\begin{aligned} \mathcal{T}_{\bar{x}\bar{x}}(x') &= \frac{l^2}{\alpha'} \int d^2 z \left[e^{2\phi+2\omega(x)} \delta^{(2)}(x' - x) \partial x \bar{\partial} x - \frac{1}{2} \partial_{\bar{x}}^2 (\delta^{(2)}(x' - x)) \partial x \bar{\partial} x \right. \\ &\quad \left. - \partial_{\bar{x}} (\delta^{(2)}(x' - x)) \partial x \bar{\partial} (\phi + \omega) - (\partial_{\bar{x}} \omega)^2 \delta^{(2)}(x' - x) \partial x \bar{\partial} x - 2 \partial_{\bar{x}} \omega \delta^{(2)}(x' - x) \partial x \bar{\partial} \phi \right]. \end{aligned} \quad (E.9)$$

Using the properties of the delta function, one can write

$$\begin{aligned} \mathcal{T}_{\bar{x}\bar{x}}(x') &= \frac{-l^2}{\pi \alpha'} \int d^2 z \left[\partial \left(\frac{1}{x' - \bar{x}} \right) e^{2\phi+2\omega(x)} \bar{\partial} x - \frac{1}{2} \partial_{x'}^2 \partial \left(\frac{1}{x' - \bar{x}} \right) \bar{\partial} x \right. \\ &\quad \left. + \partial_{x'} \partial \left(\frac{1}{x' - \bar{x}} \right) \bar{\partial} (\phi + \omega) + (\partial_{x'} \omega)^2 \partial \left(\frac{1}{x' - \bar{x}} \right) \bar{\partial} x - 2 \partial_{x'} \omega \partial \left(\frac{1}{x' - \bar{x}} \right) \bar{\partial} (\phi + \omega) \right]. \end{aligned} \quad (E.10)$$

Then, the x' derivative of the energy momentum tensor which appears in the covariant derivative (E.5) is

$$\begin{aligned} \partial_{x'} \mathcal{T}_{\bar{x}\bar{x}}(x') &= \frac{-l^2}{\pi \alpha'} \int d^2 z \left[\partial (\partial_{x'} \frac{1}{x' - \bar{x}}) e^{2\phi+2\omega(x)} \bar{\partial} x - \frac{1}{2} \partial (\partial_{x'} \partial_{x'}^2 \frac{1}{x' - \bar{x}}) \bar{\partial} x \right. \\ &\quad \left. + \partial (\partial_{x'} \partial_{x'} \frac{1}{x' - \bar{x}}) \bar{\partial} (\phi + \omega) + \partial \partial_{x'} \left(\frac{(\partial_{x'} \omega)^2}{x' - \bar{x}} \right) \bar{\partial} x - 2 \partial \partial_{x'} \left(\frac{\partial_{x'} \omega}{x' - \bar{x}} \right) \bar{\partial} (\phi + \omega) \right]. \end{aligned} \quad (E.11)$$

The second and third terms are trivial. They are of the form and therefore they are akin to pure gauge vector fields with well-defined gauge parameters, proportional to δ functions.¹ Of the remaining terms, we also consider only those that are not of this type and find :

$$\begin{aligned} \partial_{x'} \mathcal{T}_{\bar{x}\bar{x}}(x') &= \frac{-l^2}{\pi \alpha'} \int d^2 z \left[\partial (\partial_{x'} \frac{1}{x' - \bar{x}}) e^{2\phi+2\omega(x)} \bar{\partial} x + \partial \left(\frac{\partial_{x'} (\partial_{x'} \omega)^2}{x' - \bar{x}} \right) \bar{\partial} x - 2 \partial \left(\frac{\partial_{x'} \partial_{x'} \omega}{x' - \bar{x}} \right) \bar{\partial} (\phi + \omega) \right] \\ &= \frac{-l^2}{\pi \alpha'} \int d^2 z \left[\partial (\partial_{x'} \frac{1}{x' - \bar{x}}) e^{2\phi+2\omega(x)} \bar{\partial} x + 2 \partial_{x'} \partial_{x'} \omega \partial \left(\frac{1}{x' - \bar{x}} \right) (\partial_{x'} \omega) \bar{\partial} x \right. \\ &\quad \left. - 2 \partial_{x'} \partial_{x'} \omega \partial \left(\frac{1}{x' - \bar{x}} \right) \bar{\partial} (\phi + \omega) \right] \\ &= \frac{-l^2}{\alpha'} \int d^2 z \partial (\delta^{(2)}(x' - x)) e^{2\phi+2\omega(x)} \bar{\partial} x \\ &\quad + \frac{2l^2}{\alpha'} \partial_{x'} \partial_{x'} \omega \int d^2 z \left[\delta^{(2)}(x' - x) \partial_{x'} \omega \partial x \bar{\partial} x - \delta^{(2)}(x' - x) \partial x \bar{\partial} (\phi + \omega) \right] \\ &= \frac{l^2}{\alpha'} \int d^2 z \left[\delta^{(2)}(x' - x) (\partial (e^{2\phi+2\omega(x)} \bar{\partial} x) - 2 \partial_{x'} \partial_{x'} \omega \partial x \bar{\partial} \phi) \right. \\ &\quad \left. - \partial (\delta^{(2)}(x' - x) e^{2\phi+2\omega(x)} \bar{\partial} x) \right]. \end{aligned} \quad (E.12)$$

1. See e.g. [84] for an elaborate discussion of this point.

The last term in the last line is a well-defined total derivative term that we can neglect.

The other two terms in () are :

$$\partial_{x'} \mathcal{T}_{x\bar{x}} - 2\partial_{\bar{x}'} \omega \mathcal{T}_{x\bar{x}} = \frac{l^2}{\alpha'} \partial_{x'} \partial_{\bar{x}'} \partial_{\bar{x}'} \omega \int d^2 z \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x} - 2\partial_{\bar{x}'} \omega \frac{l^2}{\alpha'} \partial_{x'} \partial_{\bar{x}'} \omega \int d^2 z \delta^{(2)}(x' - x) \partial x \bar{\partial} \bar{x}. \quad (\text{E.13})$$

Combining the terms (E.12) and (E.13), we find that the covariant derivative (E.5) is equal to

$$\nabla^i \mathcal{T}_{i\bar{x}} = \frac{l^2}{\alpha'} \int d^2 z \delta^{(2)}(x' - x) \left[\partial (e^{2\phi + 2\omega(x)} \bar{\partial} x) + \partial_x \partial_{\bar{x}}^2 \omega \partial x \bar{\partial} \bar{x} - 2\partial_x \partial_{\bar{x}} \omega \partial_{\bar{x}} \omega \partial x \bar{\partial} \bar{x} - 2\partial_x \partial_{\bar{x}} \omega \partial x \bar{\partial} \phi \right].$$

This matches the variation (E.4) of the action under an infinitesimal field redefinition. Thus we have shown that the energy-momentum tensor is indeed conserved.

Annexe F

The Twisted Trace

The flat gauge fields that satisfy the boundary conditions (4.32) for conformally flat metric (for a given choice of twist) are locally :

$$A^R = \frac{i}{2}d\omega, \quad \bar{A}^R = \frac{i}{2}d\omega. \quad (\text{F.1})$$

The action becomes

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left[(G_{\mu\nu} + B_{\mu\nu}) \bar{\partial} X^\mu \partial X^\nu + i l \bar{\partial} \omega \partial \theta + i l \partial \omega \bar{\partial} \theta + \partial \theta \bar{\partial} \theta \right]. \quad (\text{F.2})$$

The equation of motion for the angle θ is

$$\partial \bar{\partial} (\theta + i l \omega) = 0. \quad (\text{F.3})$$

The variation of the conformal factor in the boundary condition gives rise to an extra term in the energy-momentum tensor component $T_{x\bar{x}}^{top}$:

$$T_{x\bar{x}}^{top} = T_{x\bar{x}}^{phys} + \frac{i l}{2\alpha'} \int d^2z \left[\bar{\partial} (\delta^{(2)}(x' - x)) \partial \theta + \partial (\delta^{(2)}(x' - x)) \bar{\partial} \theta \right]. \quad (\text{F.4})$$

Using the equation of motion (F.3), and up to BRST exact terms, we can write

$$\begin{aligned} T_{x\bar{x}}^{top} &= T_{x\bar{x}}^{phys} + \frac{l^2}{2\alpha'} \int d^2z \left[\bar{\partial} (\delta^{(2)}(x' - x)) \partial \omega + \partial (\delta^{(2)}(x' - x)) \bar{\partial} \omega \right] \\ &= T_{x\bar{x}}^{phys} - \frac{l^2}{\alpha'} \int d^2z \delta^{(2)}(x' - x) \partial \bar{\partial} \omega \\ &= 0, \end{aligned} \quad (\text{F.5})$$

up to subleading terms, of order $O(e^{-2\phi}) = O(r^{-2})$. Finally, let us remark that the twisted action (F.2) shows that the left-moving and right-moving part of the circular direction θ is shifted by ω upon twisting. When we perform the T-dual topological twist, it is convenient to perform the above reasoning using the T-dual coordinate $\theta_{dual} = \theta_L - \theta_R$. The underlying reason is that it is the T-dual circle that renders the corresponding (axial) $U(1)_R$ symmetry geometrical.

Annexe G

Operator Product Coefficients

In this appendix only, for easier comparison, we use the notation of [113], and we extract from that paper a number of structure constants that we use in chapter 5. The paper defines and uses operators $\sigma_n^{\pm, \pm}$ of (left, right) R-charge $n \pm 1$ depending on the sign of the upper index on the operator. The structure constants in the operator product expansion of two of these operators giving rise to a third are computed separately for right- and left-movers. Moreover, the paper [113] separates out a reduced structure constant where one has factored out an $su(2)_R$ Clebsch-Gordan coefficient. This is sensible if one wishes to capture the full $N = (4, 4)$ superconformal structure of the correlators. In twisting the topological theory, we pick a particular $N = (2, 2)$ superconformal subalgebra, and we break the $su(2)_R$ symmetry. We therefore wish to write the structure constants rather in a $u(1)_R$ language (and restore the Clebsch-Gordan coefficients). Firstly, we recall from [113] the $su(2)_R$ reduced chiral structure constants :

$$\begin{aligned}\hat{C}_{nmq}^{1n, 1m, 1q} &= \left(\frac{(1_n n + 1_m m + 1_q q + 1)^2}{4mnq} \frac{\Sigma! \alpha_n! \alpha_m! \alpha_q!}{(n+1_n)!(m+1_m)!(q+1_q)!} \right)^{\frac{1}{2}} \\ \Sigma &= \frac{1}{2}(n+1_n+m+1_m+q+1_q)+1 \\ \alpha_n &= \Sigma - n - 1_n - 1.\end{aligned}\tag{G.1}$$

If we delve into appendix C of [113], we can reconstruct the $N = 2$ structure constants from these $N = 4$ structure constants by multiplying back in the Clebsch-Gordan coefficient :

$$\left(\begin{array}{ccc} \frac{n+1_n}{2} & \frac{m+1_m}{2} & \frac{q+1_q}{2} \\ \frac{q+1_q-m-1_m}{2} & \frac{m+1_m}{2} & -\frac{q+1_q}{2} \end{array} \right) = \left(\frac{(m+1_m)!(q+1_q)!}{\Sigma! \alpha_n!} \right)^{\frac{1}{2}}\tag{G.2}$$

We then obtain the formula :

$$C_{nmq}^{1n, 1m, 1q} = \left(\frac{(1_n n + 1_m m + 1_q q + 1)^2}{4mnq} \frac{\alpha_m! \alpha_q!}{(n+1_n)!} \right)^{\frac{1}{2}}\tag{G.3}$$

$$\begin{aligned}\Sigma &= \frac{1}{2}(n+1_n+m+1_m+q+1_q)+1 \\ \alpha_n &= \Sigma - n - 1_n - 1.\end{aligned}\tag{G.4}$$

Since we have the $U(1)_R$ charge constraint $n + 1_n + m + 1_m = q + 1_q$, we can simplify further and obtain the equation (5.54) for the chiral structure constants in chapter 5.

Bibliographie

- [1] G. Aad *et al.* [ATLAS], “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” *Phys. Lett. B* **716** (2012), 1-29 doi :10.1016/j.physletb.2012.08.020 [arXiv :1207.7214 [hep-ex]].
- [2] L. Rosenfeld, “ Zur quantelung der wellenfelder" *Ann. der Physik* **5** (1930),113.
- [3] E. Witten, *Nucl. Phys. B* **443** (1995), 85-126 doi :10.1016/0550-3213(95)00158-O [arXiv :hep-th/9503124 [hep-th]].
- [4] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] doi :10.1023/A :1026654312961, 10.4310/ATMP.1998.v2.n2.a1 [hep-th/9711200].
- [5] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998), 253-291 doi :10.4310/ATMP.1998.v2.n2.a2 [arXiv :hep-th/9802150 [hep-th]].
- [6] A. Strominger and C. Vafa, *Phys. Lett. B* **379** (1996) 99 doi :10.1016/0370-2693(96)00345-0 [hep-th/9601029].
- [7] I. R. Klebanov, *Nucl. Phys. B* **496** (1997), 231-242 doi :10.1016/S0550-3213(97)00235-6 [arXiv :hep-th/9702076 [hep-th]].
- [8] E. Witten, “Topological Quantum Field Theory,” *Commun. Math. Phys.* **117** (1988), 353 doi :10.1007/BF01223371
- [9] E. Witten, “Topological Sigma Models,” *Commun. Math. Phys.* **118** (1988), 411 doi :10.1007/BF01466725
- [10] A. Strominger, “Black hole entropy from near horizon microstates,” *JHEP* **02** (1998), 009 doi :10.1088/1126-6708/1998/02/009 [arXiv :hep-th/9712251 [hep-th]].
- [11] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries : An Example from Three-Dimensional Gravity,” *Commun. Math. Phys.* **104** (1986), 207-226 doi :10.1007/BF01211590
- [12] O. Coussaert, M. Henneaux and P. van Driel, “The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant,” *Class. Quant. Grav.* **12** (1995), 2961-2966 doi :10.1088/0264-9381/12/12/012 [arXiv :gr-qc/9506019 [gr-qc]].

- [13] M. Henneaux, L. Maoz and A. Schwimmer, “Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity,” *Annals Phys.* **282** (2000), 31-66 doi :10.1006/aphy.2000.5994 [arXiv :hep-th/9910013 [hep-th]].
- [14] J. L. Cardy, “Operator Content of Two-Dimensional Conformally Invariant Theories,” *Nucl. Phys. B* **270** (1986), 186-204 doi :10.1016/0550-3213(86)90552-3
- [15] A. Dabholkar and A. Pakman, “Exact chiral ring of AdS(3) / CFT(2),” *Adv. Theor. Math. Phys.* **13** (2009) no.2, 409-462 doi :10.4310/ATMP.2009.v13.n2.a2 [arXiv :hep-th/0703022 [hep-th]].
- [16] W. Lerche, C. Vafa and N. P. Warner, “Chiral Rings in N=2 Superconformal Theories,” *Nucl. Phys. B* **324** (1989), 427-474 doi :10.1016/0550-3213(89)90474-4
- [17] A. Schwimmer and N. Seiberg, “Comments on the N=2, N=3, N=4 Superconformal Algebras in Two-Dimensions,” *Phys. Lett. B* **184** (1987), 191-196 doi :10.1016/0370-2693(87)90566-1
- [18] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” *Commun. Math. Phys.* **165** (1994) 311 doi :10.1007/BF02099774 [hep-th/9309140].
- [19] E. Witten, “Mirror manifolds and topological field theory,” *AMS/IP Stud. Adv. Math.* **9** (1998), 121-160 [arXiv :hep-th/9112056 [hep-th]].
- [20] T. Eguchi and S. K. Yang, “N=2 superconformal models as topological field theories,” *Mod. Phys. Lett. A* **5** (1990) 1693. doi :10.1142/S0217732390001943
- [21] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, “Topological strings in $d < 1$,” *Nucl. Phys. B* **352** (1991) 59. doi :10.1016/0550-3213(91)90129-L
- [22] E. J. Martinec, “Algebraic Geometry and Effective Lagrangians,” *Phys. Lett. B* **217** (1989) 431. doi :10.1016/0370-2693(89)90074-9
- [23] C. Vafa and N. P. Warner, “Catastrophes and the Classification of Conformal Theories,” *Phys. Lett. B* **218** (1989) 51. doi :10.1016/0370-2693(89)90473-5
- [24] E. Witten, “On the Landau-Ginzburg description of N=2 minimal models,” *Int. J. Mod. Phys. A* **9** (1994) 4783 doi :10.1142/S0217751X9400193X [hep-th/9304026].
- [25] K. Hori and A. Kapustin, “Duality of the fermionic 2-D black hole and N=2 liouville theory as mirror symmetry,” *JHEP* **0108** (2001) 045 doi :10.1088/1126-6708/2001/08/045 [hep-th/0104202].
- [26] S. Li and J. Troost, “Twisted Massive Non-Compact Models,” *JHEP* **07** (2018), 166 doi :10.1007/JHEP07(2018)166 [arXiv :1803.08710 [hep-th]].
- [27] T. Eguchi and Y. Sugawara, “Modular invariance in superstring on Calabi-Yau n fold with ADE singularity,” *Nucl. Phys. B* **577** (2000) 3 doi :10.1016/S0550-3213(00)00150-4 [hep-th/0002100].
- [28] T. Eguchi and Y. Sugawara, “Conifold type singularities, N=2 Liouville and $SL(2, \mathbb{R})/U(1)$ theories,” *JHEP* **0501** (2005) 027 doi :10.1088/1126-6708/2005/01/027 [hep-th/0411041].

- [29] T. Eguchi and Y. Sugawara, “ $SL(2, \mathbb{R}) / U(1)$ supercoset and elliptic genera of noncompact Calabi-Yau manifolds,” JHEP **0405** (2004) 014 doi :10.1088/1126-6708/2004/05/014 [hep-th/0403193].
- [30] S. K. Ashok, R. Benichou and J. Troost, “Non-compact Gepner Models, Landau-Ginzburg Orbifolds and Mirror Symmetry,” JHEP **0801** (2008) 050 doi :10.1088/1126-6708/2008/01/050 [arXiv :0710.1990 [hep-th]].
- [31] J. Troost, “The non-compact elliptic genus : mock or modular,” JHEP **1006** (2010) 104 doi :10.1007/JHEP06(2010)104 [arXiv :1004.3649 [hep-th]].
- [32] D. Israel, C. Kounnas, A. Pakman and J. Troost, “The Partition function of the supersymmetric two-dimensional black hole and little string theory,” JHEP **0406** (2004) 033 doi :10.1088/1126-6708/2004/06/033 [hep-th/0403237].
- [33] H. Ooguri and C. Vafa, “Two-dimensional black hole and singularities of CY manifolds,” Nucl. Phys. B **463** (1996) 55 doi :10.1016/0550-3213(96)00008-9 [hep-th/9511164].
- [34] D. Ghoshal and C. Vafa, “ $C = 1$ string as the topological theory of the conifold,” Nucl. Phys. B **453** (1995) 121 doi :10.1016/0550-3213(95)00408-K [hep-th/9506122].
- [35] D. Ghoshal and S. Mukhi, “Topological Landau-Ginzburg model of two-dimensional string theory,” Nucl. Phys. B **425** (1994) 173 doi :10.1016/0550-3213(94)90177-5 [hep-th/9312189].
- [36] A. Hanany, Y. Oz and M. R. Plesser, “Topological Landau-Ginzburg formulation and integrable structure of 2-d string theory,” Nucl. Phys. B **425** (1994) 150 doi :10.1016/0550-3213(94)90176-7 [hep-th/9401030].
- [37] D. Israel, A. Pakman and J. Troost, “D-branes in $N=2$ Liouville theory and its mirror,” Nucl. Phys. B **710** (2005) 529 doi :10.1016/j.nuclphysb.2005.01.014 [hep-th/0405259].
- [38] L. Girardello, A. Pasquinucci and M. Porrati, “ $N=2$ Morse-Liouville theory and nonminimal superconformal theories,” Nucl. Phys. B **352** (1991) 769. doi :10.1016/0550-3213(91)90106-8
- [39] S. Cecotti and C. Vafa, “Topological antitopological fusion,” Nucl. Phys. B **367** (1991) 359. doi :10.1016/0550-3213(91)90021-O
- [40] A. Hanany, N. Prezas and J. Troost, “The Partition function of the two-dimensional black hole conformal field theory,” JHEP **0204** (2002) 014 doi :10.1088/1126-6708/2002/04/014 [hep-th/0202129].
- [41] J. M. Maldacena and H. Ooguri, “Strings in $AdS(3)$ and $SL(2, \mathbb{R})$ WZW model 1. : The Spectrum,” J. Math. Phys. **42** (2001) 2929 doi :10.1063/1.1377273 [hep-th/0001053].
- [42] A. Giveon and D. Kutasov, “Little string theory in a double scaling limit,” JHEP **9910** (1999) 034 doi :10.1088/1126-6708/1999/10/034 [hep-th/9909110].
- [43] J. Troost, “An Elliptic Triptych,” JHEP **1710** (2017) 078 doi :10.1007/JHEP10(2017)078 [arXiv :1706.02576 [hep-th]].
- [44] C. Vafa, “Topological Landau-Ginzburg models,” Mod. Phys. Lett. A **6** (1991) 337. doi :10.1142/S0217732391000324

- [45] B. Dubrovin, “Geometry of 2-D topological field theories,” *Lect. Notes Math.* **1620** (1996) 120 doi :10.1007/BFb0094793 [hep-th/9407018].
- [46] N. Ishtiaque, “2D BPS Rings from Sphere Partition Functions,” arXiv :1712.02551 [hep-th].
- [47] J. Chen, “On exact correlation functions of chiral ring operators in $2d \mathcal{N} = (2, 2)$ SCFTs via localization,” arXiv :1712.01164 [hep-th].
- [48] S. K. Ashok, N. Doroud and J. Troost, “Localization and real Jacobi forms,” *JHEP* **1404** (2014) 119 doi :10.1007/JHEP04(2014)119 [arXiv :1311.1110 [hep-th]].
- [49] S. Murthy, “A holomorphic anomaly in the elliptic genus,” *JHEP* **1406** (2014) 165 doi :10.1007/JHEP06(2014)165 [arXiv :1311.0918 [hep-th]].
- [50] E. Bell, “Exponential polynomials,” *Ann. of Math.* **35** (1934) 258.
- [51] L. Comtet, “Advanced Combinatorics : The Art of Finite and Infinite Expansions,” D. Reidel Publishing Co., Dordrecht, 1974.
- [52] D. Birmajer, J. Gil, M. Weiner, “Some convolution identities and an inverse relation involving partial Bell polynomials,” *The Electronic Journal of Combinatorics* **19(4)** (2012) 34.
- [53] D. Cvijović, “New identities for the partial Bell polynomials,” *Applied Mathematics Letters* **24(9)** (2011) 1544.
- [54] S. Li, N. Toubas and J. Troost, “Liouville Quantum Gravity,” *Nucl. Phys. B* **952** (2020), 114913 doi :10.1016/j.nuclphysb.2019.114913 [arXiv :1903.06501 [hep-th]].
- [55] S. Li and J. Troost, “Pure and Twisted Holography,” *JHEP* **03** (2020), 144 doi :10.1007/JHEP03(2020)144 [arXiv :1911.06019 [hep-th]].
- [56] M. Banados, “Three-dimensional quantum geometry and black holes,” *AIP Conf. Proc.* **484** (1999) no.1, 147 doi :10.1063/1.59661 [hep-th/9901148].
- [57] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, “Geometry of the (2+1) black hole,” *Phys. Rev. D* **48** (1993) 1506 Erratum : [*Phys. Rev. D* **88** (2013) 069902] doi :10.1103/PhysRevD.48.1506, 10.1103/PhysRevD.88.069902 [gr-qc/9302012].
- [58] N. Seiberg, “Notes on quantum Liouville theory and quantum gravity,” *Prog. Theor. Phys. Suppl.* **102** (1990) 319. doi :10.1143/PTPS.102.319
- [59] J. Teschner, “Liouville theory revisited,” *Class. Quant. Grav.* **18** (2001) R153 doi :10.1088/0264-9381/18/23/201 [hep-th/0104158].
- [60] B. Ponsot and J. Teschner, “Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U(q)(sl(2, R))$,” *Commun. Math. Phys.* **224** (2001) 613 doi :10.1007/PL00005590 [math/0007097 [math-qa]].
- [61] J. Teschner, “From Liouville theory to the quantum geometry of Riemann surfaces,” hep-th/0308031.
- [62] A. B. Zamolodchikov and A. B. Zamolodchikov, “Structure constants and conformal bootstrap in Liouville field theory,” *Nucl. Phys. B* **477** (1996) 577 doi :10.1016/0550-3213(96)00351-3 [hep-th/9506136].

- [63] H. Dorn and H. J. Otto, “Two and three point functions in Liouville theory,” Nucl. Phys. B **429** (1994) 375 doi :10.1016/0550-3213(94)00352-1 [hep-th/9403141].
- [64] J. Teschner, “On the Liouville three point function,” Phys. Lett. B **363** (1995) 65 doi :10.1016/0370-2693(95)01200-A [hep-th/9507109].
- [65] S. Collier, P. Kravchuk, Y. H. Lin and X. Yin, “Bootstrapping the Spectral Function : On the Uniqueness of Liouville and the Universality of BTZ,” JHEP **1809** (2018) 150 doi :10.1007/JHEP09(2018)150 [arXiv :1702.00423 [hep-th]].
- [66] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. **43** (1975) 199 Erratum : [Commun. Math. Phys. **46** (1976) 206]. doi :10.1007/BF02345020, 10.1007/BF01608497
- [67] S. Carlip and C. Teitelboim, “Aspects of black hole quantum mechanics and thermodynamics in (2+1)-dimensions,” Phys. Rev. D **51** (1995) 622 doi :10.1103/PhysRevD.51.622 [gr-qc/9405070].
- [68] A. Achucarro and P. K. Townsend, “A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories,” Phys. Lett. B **180** (1986) 89. doi :10.1016/0370-2693(86)90140-1
- [69] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” Nucl. Phys. B **311** (1988) 46. doi :10.1016/0550-3213(88)90143-5
- [70] K. Krasnov, “3-D gravity, point particles and Liouville theory,” Class. Quant. Grav. **18** (2001) 1291 doi :10.1088/0264-9381/18/7/311 [hep-th/0008253].
- [71] P. Menotti and G. Vajente, “Semiclassical and quantum Liouville theory on the sphere,” Nucl. Phys. B **709** (2005) 465 doi :10.1016/j.nuclphysb.2004.12.014 [hep-th/0411003].
- [72] J. D. Brown and M. Henneaux, “On the Poisson Brackets of Differentiable Generators in Classical Field Theory,” J. Math. Phys. **27** (1986) 489. doi :10.1063/1.527249
- [73] V. G. Kac and M. Wakimoto, “Integrable highest weight modules over affine superalgebras and number theory,” hep-th/9407057.
- [74] V. Mikhaylov and E. Witten, “Branes And Supergroups,” Commun. Math. Phys. **340** (2015) no.2, 699 doi :10.1007/s00220-015-2449-y [arXiv :1410.1175 [hep-th]].
- [75] G. Compère, W. Song and A. Strominger, “New Boundary Conditions for AdS₃,” JHEP **1305** (2013) 152 doi :10.1007/JHEP05(2013)152 [arXiv :1303.2662 [hep-th]].
- [76] D. Grumiller and M. Riegler, “Most general AdS₃ boundary conditions,” JHEP **1610** (2016) 023 doi :10.1007/JHEP10(2016)023 [arXiv :1608.01308 [hep-th]].
- [77] C. E. Valcárcel, “New boundary conditions for (extended) AdS₃ supergravity,” Class. Quant. Grav. **36** (2019) 065002 doi :10.1088/1361-6382/ab04da [arXiv :1812.02799 [hep-th]].
- [78] J. M. F. Labastida and P. M. Llatas, “Topological matter in two-dimensions,” Nucl. Phys. B **379** (1992) 220 doi :10.1016/0550-3213(92)90596-4 [hep-th/9112051].

- [79] S. K. Ashok and J. Troost, “A Duality in Two-Dimensional Gravity,” JHEP **1905** (2019) 111 doi :10.1007/JHEP05(2019)111 [arXiv :1812.05822 [hep-th]].
- [80] S. Li and J. Troost, “Twisted String Theory in Anti-de Sitter Space,” JHEP **11** (2020), 047 doi :10.1007/JHEP11(2020)047 [arXiv :2005.13817 [hep-th]].
- [81] A. Sen and B. Zwiebach, “Quantum background independence of closed string field theory,” Nucl. Phys. B **423** (1994) 580 doi :10.1016/0550-3213(94)90145-7 [hep-th/9311009].
- [82] A. A. Tseytlin, “Sigma Model Approach To String Theory,” Int. J. Mod. Phys. A **4** (1989) 1257. doi :10.1142/S0217751X8900056X
- [83] A. Giveon, D. Kutasov and N. Seiberg, “Comments on string theory on $AdS(3)$,” Adv. Theor. Math. Phys. **2** (1998) 733 doi :10.4310/ATMP.1998.v2.n4.a3 [hep-th/9806194].
- [84] D. Kutasov and N. Seiberg, “More comments on string theory on $AdS(3)$,” JHEP **9904** (1999) 008 doi :10.1088/1126-6708/1999/04/008 [hep-th/9903219].
- [85] J. Troost, “The AdS_3 boundary energy momentum tensor, exact in the string length over the curvature radius,” Phys. Lett. B **689** (2010) 201 doi :10.1016/j.physletb.2010.04.075 [arXiv :1006.4223 [hep-th]].
- [86] A. Giveon and A. Pakman, “More on superstrings in $AdS(3) \times N$,” JHEP **0303** (2003) 056 doi :10.1088/1126-6708/2003/03/056 [hep-th/0302217].
- [87] J. Polchinski, “String theory. Vol. 1 : An introduction to the bosonic string,” Cambridge University Press, 1998. doi :10.1017/CBO9780511816079
- [88] T. Andrade, J. I. Jottar and R. G. Leigh, “Boundary Conditions and Unitarity : the Maxwell-Chern-Simons System in AdS_3/CFT_2 ,” JHEP **05** (2012), 071 doi :10.1007/JHEP05(2012)071 [arXiv :1111.5054 [hep-th]].
- [89] S. Gukov, E. Martinec, G. W. Moore and A. Strominger, “The Search for a holographic dual to $AdS(3) \times S^{*3} \times S^{*3} \times S^{*1}$,” Adv. Theor. Math. Phys. **9** (2005) 435 doi :10.4310/ATMP.2005.v9.n3.a3, 10.1142/97898127753440035 [hep-th/0403090].
- [90] P. Kraus, “Lectures on black holes and the $AdS(3) / CFT(2)$ correspondence,” Lect. Notes Phys. **755** (2008), 193-247 [arXiv :hep-th/0609074 [hep-th]].
- [91] S. Li and J. Troost, “The Topological Symmetric Orbifold,” accepted for publication in JHEP [arXiv :2006.09346 [hep-th]].
- [92] A. Grothendieck, “Fondements de la Géométrie Algébrique, Séminaire Bourbaki,” IHP, Paris (1957-1964)
- [93] J. Fogarty, “Algebraic families on an Algebraic Surface,” Amer. Jour. of Math. bf 13, No 2, (1968), 511–521
- [94] A. Beauville, “Variétés Kähleriennes dont la première classe de Chern est nulle,” Journal of Differential Geometry, **18**(4), 755-782 (1983).
- [95] J. Briançon, “Description of $Hilb^n \mathbb{C}[x, y]$, Invent. Math. **41** (1977), 45–89
- [96] G. Ellingsrud, S. A. Strømme, “On the homology of the Hilbert scheme of points in the plane,” Inventiones mathematicae, **87**(2), 343-352 (1987).

- [97] H. Nakajima, "Lectures on Hilbert schemes of points on surfaces (No. 18). American Mathematical Soc. (1999).
- [98] I. Grojnowski, "Instantons and affine algebras I : the Hilbert scheme and vertex operators," arXiv preprint alg-geom/9506020.
- [99] M. Lehn, "Chern classes of tautological sheaves on Hilbert schemes of points on surfaces," *Inventiones mathematicae*, **136**(1), 157-207 (1999).
- [100] M. Lehn, C. Sorger, "Symmetric groups and the cup product on the cohomology of Hilbert schemes," *Duke Mathematical Journal* **110.2** (2001) : 345-357.
- [101] E. Vasserot, "Sur l'anneau de cohomologie du schéma de Hilbert de C^2 ," *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, **332**(1), 7-12 (2001).
- [102] L. Göttsche, "The Betti numbers of the Hilbert scheme of points on a smooth projective surface," *Mathematische Annalen*, **286**(1-3), 193-207 (1990).
- [103] W. m. Chen and Y. b. Ruan, "A New cohomology theory for orbifold," *Commun. Math. Phys.* **248** (2004), 1-31 doi :10.1007/s00220-004-1089-4 [arXiv :math/0004129 [math.AG]].
- [104] Y. Ruan, "Stringy geometry and topology of orbifolds," [arXiv :math/0011149 [math.AG]].
- [105] M. Lehn, C. Sorger, "The cup product of Hilbert schemes for K3 surfaces," *Inventiones mathematicae* **152.2** (2003) : 305-329.
- [106] B. Fantechi, L. Göttsche, "Orbifold cohomology for global quotients," *Duke Mathematical Journal* **117.2** (2003) : 197-227.
- [107] J. Bertin, "The punctual Hilbert scheme : an introduction," (2008).
- [108] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, "Strings on Orbifolds," *Nucl. Phys. B* **261** (1985), 678-686 doi :10.1016/0550-3213(85)90593-0
- [109] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, "Strings on Orbifolds. 2.," *Nucl. Phys. B* **274** (1986), 285-314 doi :10.1016/0550-3213(86)90287-7
- [110] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," *Commun. Math. Phys.* **185** (1997), 197-209 doi :10.1007/s002200050087 [arXiv :hep-th/9608096 [hep-th]].
- [111] M. Haiman, "t, q-Catalan numbers and the Hilbert scheme," *Discrete Mathematics*, **193** (1-3), 201-224 (1998).
- [112] J Polchinski, "String theory : Volume 2, superstring theory and beyond," Cambridge university press (1998).
- [113] O. Lunin and S. D. Mathur, "Three point functions for $M(N) / S(N)$ orbifolds with $N=4$ supersymmetry," *Commun. Math. Phys.* **227** (2002), 385-419 doi :10.1007/s002200200638 [arXiv :hep-th/0103169 [hep-th]].
- [114] G. James, A. Kerber, "The Representation Theory of the Symmetric Group," *Encyclopedia Math. Appl.* (1981).
- [115] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.11.0 ; 2020. (<https://www.gap-system.org>)

- [116] V. Ivanov, S. Kerov, "The algebra of conjugacy classes in symmetric groups and partial permutations," *Journal of Mathematical Sciences*, **107** (5), 4212-4230 (2001).
- [117] I. Goulden, "A differential operator for symmetric functions and the combinatorics of multiplying transpositions," *Transactions of the American Mathematical Society*, **344**(1), 421-440 (1994).
- [118] A. Goupil, "On products of conjugacy classes of the symmetric group," *Discrete mathematics*, **79** (1), 49-57 (1990).
- [119] H. Farahat, G. Higman, "The centres of symmetric group rings," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **250** (1261), 212-221 (1959).
- [120] F. Bédard, A. Goupil, "The poset of conjugacy classes and decomposition of products in the symmetric group," *Canadian mathematical bulletin*, **35** (2), 152-160(1992).
- [121] I. Goulden, D. Jackson, "The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group," *European Journal of Combinatorics*, **13**(5), 357-365 (1992).
- [122] I. Goulden, D. Jackson, "Symmetrical Functions and Macdonald's Result for Top Connexion Coefficients in the Symmetrical Group," *Journal of Algebra*, **166** (2), 364-378 (1994).
- [123] A. Jevicki, M. Mihailescu and S. Ramgoolam, "Gravity from CFT on $S^*N(X)$: Symmetries and interactions," *Nucl. Phys. B* **577** (2000), 47-72 doi :10.1016/S0550-3213(00)00147-4 [arXiv :hep-th/9907144 [hep-th]].
- [124] R. Dijkgraaf and L. Motl, "Matrix string theory, contact terms, and superstring field theory," [arXiv :hep-th/0309238 [hep-th]].
- [125] W. P. Li, Z. Qin, W. Wang, "Hilbert scheme intersection numbers, Hurwitz numbers, and Gromow-Witten invariants", In *Infinite-Dimensional Aspects of Representation Theory and Applications : International Conference on Infinite-Dimensional Aspects of Representation Theory and Applications*, May 18-22, 2004, University of Virginia, Charlottesville, Virginia (Vol. 392, p. 67). American Mathematical Soc.
- [126] A. Hurwitz, "Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten," *Mathematische Annalen*, **39** (1), 1-60 (1891).
- [127] W. P. Li, Z. Qin, W. Wang, "Ideals of the cohomology rings of Hilbert schemes and their applications," *Transactions of the American Mathematical Society*, **356** (1), 245-265.
- [128] O. Lunin and S. D. Mathur, "Correlation functions for $M^*N / S(N)$ orbifolds," *Commun. Math. Phys.* **219** (2001), 399-442 doi :10.1007/s002200100431 [arXiv :hep-th/0006196 [hep-th]].
- [129] M. R. Gaberdiel and I. Kirsch, "Worldsheet correlators in $AdS(3)/CFT(2)$," *JHEP* **04** (2007), 050 doi :10.1088/1126-6708/2007/04/050 [arXiv :hep-th/0703001 [hep-th]].

- [130] A. Pakman, L. Rastelli and S. S. Razamat, “Diagrams for Symmetric Product Orbifolds,” JHEP **10** (2009), 034 doi :10.1088/1126-6708/2009/10/034 [arXiv :0905.3448 [hep-th]].
- [131] A. Pakman, L. Rastelli and S. S. Razamat, “Extremal Correlators and Hurwitz Numbers in Symmetric Product Orbifolds,” Phys. Rev. D **80** (2009), 086009 doi :10.1103/PhysRevD.80.086009 [arXiv :0905.3451 [hep-th]].
- [132] L. Rastelli and M. Wijnholt, “Minimal AdS(3),” Adv. Theor. Math. Phys. **11** (2007) no.2, 291 doi :10.4310/ATMP.2007.v11.n2.a4 [hep-th/0507037].
- [133] W. P. Li, Z. Qin, W. Wang, “Stability of the cohomology rings of Hilbert schemes of points on surfaces," arXiv preprint math/0107139.
- [134] J. Kim and M. Porrati, “On a Canonical Quantization of 3D Anti de Sitter Pure Gravity,” JHEP **1510** (2015) 096 doi :10.1007/JHEP10(2015)096 [arXiv :1508.03638 [hep-th]].
- [135] B. de Wit, S. Murthy and V. Reys, “BRST quantization and equivariant cohomology : localization with asymptotic boundaries,” JHEP **1809** (2018) 084 doi :10.1007/JHEP09(2018)084 [arXiv :1806.03690 [hep-th]].

RÉSUMÉ

Cette thèse porte sur la version topologiquement twistée de la correspondance AdS_3/CFT_2 . Nous commençons par étudier la théorie de Liouville $N = 2$ supersymétrique topologiquement twistée avec des déformations massives. Nous déterminons les éléments de l'anneau chirale et nous calculons leurs fonctions de corrélations. Nous définissons ensuite une théorie de la gravité à trois dimensions qui est le dual holographique de la théorie de Liouville. L'action de cette théorie de la gravité est la même que celle de la relativité générale classique tandis que la mesure de l'intégrale de chemin est la même que celle de la théorie de Liouville. Nous étudions en détail les propriétés de cette théorie de la gravité avant de généraliser au cas supersymétrique pour lequel, en changeant les conditions aux limites, nous observons qu'il est dual à la théorie de Liouville $N = 2$ supersymétrique topologiquement twistée mentionnée précédemment. Les configurations classiques correspondant à l'anneau chirale seront calculées. Nous nous intéresserons ensuite à une réalisation de la dualité en termes de cordes et analysons une théorie de cordes sur un fond localement AdS_3 avec une frontière conformalement plate. Nous observons que le tenseur énergie-impulsion de la frontière devient topologique lorsque l'on impose un couplage spécifique à la métrique de la frontière donné par les conditions aux limites. Nous étudions finalement les théories topologiques conformes de champs d'orbifolds sur le produit symétrique d'une surface complexe M , qui est le dual conjecturé de la théorie des cordes à dix dimensions sur $AdS_3 \times S^3 \times M$.

MOTS CLÉS

Gravité quantique, théorie des cordes, correspondance AdS/CFT , théorie quantique topologique des champs

ABSTRACT

This thesis concentrates on the topologically twisted version of the AdS_3/CFT_2 correspondence. We start by studying the topologically twisted $N = 2$ supersymmetric Liouville theory with massive deformations. We determine the chiral ring elements and compute their correlations functions. We then turn to define a three-dimensional gravity theory that is the holographic dual of Liouville theory. The action of this gravity theory is the same as the one of classical general relativity while the path integral measure is taken to be the same as the Liouville theory. We study in detail the properties of this gravity theory and then generalize it to the supersymmetric case where by changing the boundary conditions we find it dual to the topologically twisted $N = 2$ supersymmetric Liouville theory mentioned above. The classical configurations corresponding to the chiral ring will be computed. Next, we turn to a string realization of the duality and analyze a string theory on the locally AdS_3 background with a conformally flat boundary. We find the boundary energy-momentum tensor becomes topological when we impose a specific coupling to the boundary metric given by the boundary conditions. Finally, we study the topological orbifold conformal field theories on the symmetric product of a complex surface M , which is the conjectured dual to the ten-dimensional string theory on $AdS_3 \times S^3 \times M$.

KEYWORDS

Quantum gravity, string theory, AdS/CFT correspondence, topological quantum field theory