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Ziad Kobeissi

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UNIVERSITÉ DE PARIS

École doctorale Sciences Mathématiques de Paris Centre (ED 386)

Laboratoire Jacques-Louis Lions

**QUELQUES CONTRIBUTIONS À LA THÉORIE DES JEUX À CHAMP
MOYEN**

CONTRIBUTIONS TO THE THEORY OF MEAN FIELD GAMES

Thèse de doctorat présentée par

Ziad Kobeissi

Sous la direction d'Yves Achdou et de Pierre Cardaliaguet

Présentée et soutenue à Paris le 02/10/2020, devant le jury composé de

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Introduction

La théorie des jeux à champ moyen permet l'étude de systèmes dans lesquels une infinité de joueurs interagissent. Ces joueurs sont supposés indiscernables, interchangeables et parfaitement rationnels. c'est-à-dire que chaque joueur possède un but et agit toujours de manière optimale pour l'atteindre, malgré la présence d'autres joueurs qui sont interchangeables de son point de vue. Leurs interactions ont lieu à travers des quantités moyennes agrégées qui dépendent symétriquement et faiblement de chaque joueur. De telles interactions sont appelées à *champ moyen*. Cette terminologie vient de la physique statistique et plus particulièrement de l'étude de systèmes mécaniques dans lesquels un grand nombre de particules interagissent (comme les particules subatomiques, ou les étoiles d'une galaxie).

Lorsque le nombre de joueurs est fini, la théorie des jeux s'intéresse souvent aux équilibres de Nash dont la théorie est antérieure à celle des jeux à champ moyen. Les applications des équilibres de Nash avec un grand nombre de joueurs sont nombreuses et variées. En économie, le comportement d'un agent dépend souvent de quantités macro-économiques dont les valeurs résultent des habitudes de consommation de l'ensemble des agents de manière statistique à travers la loi de l'offre et la demande. En dynamique des populations (humaines, animales, bactériennes . . .), des phénomènes de foules naissent du mouvement individuel de chaque agent. D'autres exemples viennent des réseaux sociaux, de la finance, de l'économie des énergies fossiles et renouvelables . . . L'apparition de situations d'équilibre dans de tels cas est observée. Cependant, l'analyse théorique de ces problèmes s'avère souvent difficile et leur résolution numérique totalement impossible à cause du grand nombre de joueurs. C'est ici que la théorie des jeux à champ moyen apporte une réponse. Le passage à une infinité de joueurs engendre des simplifications permettant l'étude du comportement statistique des joueurs. La solution d'un jeu à champ moyen peut ensuite être utilisée comme une bonne approximation d'un équilibre de Nash avec un nombre fini de joueurs, lorsque ce nombre est très grand.

La théorie des jeux à champ moyen est assez récente puisqu'elle date des années 2005-2006, et des travaux indépendants de Jean-Michel Lasry et Pierre-Louis Lions [82, 83, 84], et de Minyi Y. Huang, Peter E. Caines et Roland E. Malhamé [69, 70].

1.1 La théorie des jeux à champ moyen

1.1.1 Le jeu à N joueurs

Nous introduisons ici le cadre mathématique le plus élémentaire possible. Nous considérons des jeux différentiels, dont la théorie a été introduite par R. Isaac [72] et N.S. Pontryagin

[93]. Dans un jeu différentiel, les variables de temps et d'espace sont continues. On note $N > 1$ le nombre de joueurs. L'horizon de temps est noté $T > 0$ et on suppose pour éviter des détails technique que l'état des joueurs évolue dans $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ le tore de dimension $d \geq 1$. L'état du joueur d'indice $i \in \{1, 2, \dots, N\}$ varie selon un processus aléatoire noté $(X_t^i)_{t \in [0, T]}$ qui satisfait l'équation différentielle stochastique suivante,

$$dX_t^i = \alpha_t^i dt + \sqrt{2\nu} dW_t^i, \quad X_0^i = x^i \in \mathbb{T}^d,$$

où $\nu > 0$ est le coefficient de diffusion, $(W^j)_{1 \leq j \leq N}$ sont N mouvements browniens indépendants, et α^i est le contrôle le contrôle du joueur. Nous supposons que α^i est un processus markovien dépendant de l'état de tous les joueurs ; en particulier il est de la forme $\alpha_t^i = \alpha^i \left(t, (X_t^j)_{1 \leq j \leq N} \right)$, où α^i est maintenant une fonction mesurable par rapport à la tribu borélienne de $[0, T] \times (\mathbb{T}^d)^N$ dans \mathbb{T}^d . Le joueur d'indice i cherche à minimiser son coût individuel donné par

$$J \left(\alpha^i, (\alpha^j)_{j \neq i} \right) = \mathbb{E} \left[\int_0^T L(t, X_t^i, \alpha_t^i, m_t^i) dt + g(X_T^i, m_T^i) \right],$$

où $L : [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \rightarrow \mathbb{R}^d$ est le Lagrangien, g est le coût terminal, $m_t^i = \sum_{j \neq i} \delta_{X_t^j}$ est la mesure empirique des autres joueurs, $\mathcal{P}(\mathbb{T}^d)$ est l'espace des mesures de probabilité sur \mathbb{T}^d , et δ est la mesure de Dirac. On fait l'hypothèse de coercivité suivante:

$$(1.1.1) \quad \lim_{|\alpha| \rightarrow \infty} \frac{|L(t, x, \alpha, m)|}{|\alpha|} = +\infty,$$

pour tout $(t, x, m) \in [0, T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$. Rappelons la définition d'un équilibre de Nash :

Définition 1.1.1. *Un équilibre de Nash (en boucle fermée) du système à N joueurs décrit ci-dessus est un N -uplet $(\alpha^1, \dots, \alpha^N)$ de fonctions mesurables de $[0, T] \times (\mathbb{T}^d)^N$ dans \mathbb{R}^d , tel que*

$$J \left(\alpha^i, (\alpha^j)_{j \neq i} \right) \leq J \left(\alpha, (\alpha^j)_{j \neq i} \right),$$

pour tout $i \in \{1, \dots, N\}$ et $\alpha : [0, T] \times (\mathbb{T}^d)^N \mapsto \mathbb{R}^d$ mesurable.

Définissons H le Hamiltonien du système à N joueurs comme la transformée de Legendre-Fenchel de L par rapport à α , c.-à-d.

$$H(t, x, p, m) = \sup_{\alpha \in \mathbb{R}^d} -p \cdot \alpha - L(t, x, \alpha, m).$$

pour $(t, x, p, m) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$. Nous définissons le système de Nash à N joueurs par les équations suivantes:

$$(1.1.2) \quad \begin{cases} -\partial_t v^{N,i} - \nu \sum_{j=1}^N \Delta_{x^j} v^{N,i} + H(t, x^i, \nabla_{x^i} v^{N,i}, m_x^{N,i}) + \sum_{j \neq i} \nabla_{x^j} v^{N,i} \cdot H_p(t, x^j, \nabla_{x^j} v^{N,j}, m_x^{N,j}) = 0, \\ v^{N,i}(T, x) = g(x^i, m_x^{N,i}), \end{cases}$$

où $v^{N,i} : [0, T] \times (\mathbb{T}^d)^N \rightarrow \mathbb{R}$ et $m_x^{N,i} = \sum_{j \neq i} \delta_{x^j}$, pour $i \in \{1, \dots, N\}$. Lorsqu'une solution au système précédent existe et est régulière, un équilibre de Nash est donné par les fonctions

de contrôle $\alpha^i(t, x) = -H_p(t, x, \nabla_{x^i} v^i(t, x), m_x^i)$. Dans ce cas, l'équation stochastique satisfaite par l'état d'un joueur devient :

$$(1.1.3) \quad dX_t^i = -H_p\left(t, X_t^i, \nabla_{x^i} v^{N,i}(t, X_t^i), m_{X_t^i}^{N,i}\right) dt + \sqrt{2\nu} dW_t^i, \quad X_0^i = x^i \in \mathbb{T}^d.$$

Lorsque les joueurs utilisent ces contrôles, $v^{N,i}$ est appelée fonction valeur du joueur i , et représente le coût le plus faible en moyenne que le joueur i puisse atteindre entre les temps t et T si son état est donné par x au temps t .

Le système (1.1.2) est un système de N équations de Hamilton-Jacobi couplées. Nous renvoyons aux travaux de M.G. Crandall et P.L. Lions [51], pour une étude approfondie des équations de Hamilton-Jacobi et du cadre dans lequel elles sont bien posées: la théorie des solutions de viscosité. Nous renvoyons aux travaux de A. Bensoussan et J. Frehse [18, 19] pour une étude du système (1.1.2).

La principale difficulté dans l'étude du système (1.1.2) consiste en le fait qu'un joueur, lorsqu'il change de stratégie, affecte l'ensemble des autres joueurs. Ainsi l'existence et l'unicité de solutions de (1.1.2) sont en général des problèmes compliqués. Sous des hypothèses de croissance sur le Hamiltonien H , le livre d'O.A. Ladyzenskaja, V.A. Solonnikov, et N.N. Ural'ceva [81] apporte des éléments de réponse. Cependant, lorsque le nombre de joueurs est grand, toute étude qualitative ou quantitative du comportement des solutions de (1.1.2) devient impossible à cause du nombre d'équations et de leur fort couplage.

Contrairement à ce que laisse penser la remarque précédente, la structure des équilibres de Nash devrait se simplifier lorsque le nombre de joueurs devient très grand puisque l'effet de chaque joueur sur le système devient négligeable et que les lois de la statistique devraient s'appliquer. C'est l'idée fondatrice de la théorie des jeux à champ moyen, développée en même temps dans les travaux initiaux indépendants de Jean-Michel Lasry et Pierre-Louis Lions [82, 83, 84], et de Minyi Y. Huang, Peter E. Caines et Roland E. Malhamé [69, 70].

1.1.2 Le système de jeu à champ moyen

Pour commencer, plaçons nous directement à la limite $N = \infty$, lorsqu'une infinité de joueurs interagissent à travers leur distribution statistique. Un joueur représentatif a alors une influence nulle sur le système total. Son état $(X_t)_{t \in [0, T]}$ est donné par

$$(1.1.4) \quad dX_t = \alpha_t dt + \sqrt{2\nu} dW_t,$$

où W est un mouvement brownien, α est le contrôle, et X_0 est un processus aléatoire indépendant de W et à valeur dans \mathbb{T}^d de loi $\mathcal{L}(X_0) = m_0$. Le joueur représentatif cherche à minimiser son coût, donné par

$$(1.1.5) \quad J(\alpha, m) = \mathbb{E} \left[\int_0^T L(t, X_t, \alpha_t, m(t)) dt + g(X_T, m(T)) \right],$$

où $t \mapsto m(t) \in \mathcal{P}(\mathbb{T}^d)$ est une prédiction de la distribution statistique des autres joueurs à tous les temps $t \in [0, T]$; $\mathcal{P}(\mathbb{T}^d)$ est l'ensemble des mesures de probabilité sur \mathbb{T}^d . La loi des grands nombres et la limite de champ moyen donne la condition d'équilibre $m(t) = \mathcal{L}(X_t)$ pour tout $t \in [0, T]$. Dans ce cas, le système d'EDPs prend la forme suivante :

$$(1.1.6) \quad \begin{cases} -\partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x), m(t)) = 0 & \text{sur } (0, T) \times \mathbb{T}^d, \\ \partial_t m(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(t, x, \nabla_x u(t, x), m(t)) m(t, x)) = 0 & \text{sur } (0, T) \times \mathbb{T}^d, \\ u(T, x) = g(x, m(T)) & \text{sur } \mathbb{T}^d, \\ m(0) = m_0. \end{cases}$$

La fonction valeur du joueur représentatif est ici notée u . Elle satisfait une équation de Hamilton-Jacobi-Bellman rétrograde en temps qui rappelle celle du système à N joueurs (1.1.2). Cette dernière équation est cependant plus simple puisqu'elle est posée sur $(0, T) \times \mathbb{T}^d$ au lieu de $(0, T) \times (\mathbb{T}^d)^N$; de plus le terme de couplage faisant intervenir les $v^{N,j}$ a disparu. La seconde équation est une équation de Fokker-Planck-Kolmogorov qui traduit l'évolution de la distribution statistique des joueurs lorsqu'ils suivent la stratégie optimale donnée par $\alpha(t, x) = -H_p(t, x, \nabla_x u(t, x), m(t))$.

Le système (1.1.6) n'est plus un système d'évolution (c.-à-d. le temps s'écoule dans une direction), puisqu'il contient une équation rétrograde en temps et une équation posée dans le sens naturel du temps. Les systèmes de ce type sont en général difficiles à étudier. Cependant, (1.1.6) a une structure particulière : l'équation de Fokker-Planck-Kolmogorov est le dual de l'équation linéarisée de Hamilton-Jacobi-Bellman. Cette structure simplifie l'étude de (1.1.6), permettant notamment d'obtenir des estimations a priori sur les solutions. L'existence de solutions est traitée dans de nombreux travaux [17, 20, 32, 34, 36, 37, 39, 41, 61, 64, 69, 71, 68, 85, 87]. Un critère d'unicité pour la solution du système de jeu à champ moyen a été présenté par J.M. Lasry et P.L. Lions dans leurs premiers travaux sur la théorie des jeux à champ moyen [85] Théorème 2.5 :

Théorème 1.1.2. *Supposons que H est séparé, c'est-à-dire de la forme*

$$(1.1.7) \quad H(t, x, p, m) = \tilde{H}(t, x, p) - f(t, x, m),$$

et que les fonctions f et g sont monotones au sens de Lasry-Lions, c'est-à-dire qu'elles vérifient

$$(1.1.8) \quad \begin{aligned} \int_{\mathbb{T}^d} f(t, x, m^1) - f(t, x, m^2) d(m^1 - m^2)(x) &\geq 0, \\ \int_{\mathbb{T}^d} g(x, m^1) - g(x, m^2) d(m^1 - m^2)(x) &\geq 0, \end{aligned}$$

pour tout $t \in [0, T]$ et $m^1, m^2 \in \mathcal{P}(\mathbb{T}^d)$. Il existe au plus une solution à (1.1.6).

Les inégalités (1.1.8) peuvent s'interpréter comme une aversion de la part de chaque joueur aux zones à forte densité de population. Dans ses cours au Collège de France [87], P.L. Lions étend ce résultat d'unicité au cas où H est non-séparé et dépend localement de m ; remplaçant la première inégalité de (1.1.8) par l'hypothèse que la matrice M , définie par

$$M = \begin{pmatrix} mH_{pp}(t, x, p, m) & \frac{m}{2}H_{pm}(t, x, p, m) \\ \frac{m}{2}H_{pm}(t, x, p, m) & -H_m(t, x, p, m) \end{pmatrix},$$

soit symétrique définie positive pour tout $(t, x, p, m) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}_+$.

De tels résultats d'unicité ne sont pas transposables au système de Nash à N joueurs (1.1.2). De plus, la résolution d'approximation numérique de (1.1.6) paraît possible, contrairement à celle de (1.1.2) si le nombre de joueurs est grand. Nous renvoyons à la littérature sur les applications numériques des MFGs [3, 4, 5, 6, 9, 14, 27, 29, 31, 44, 45].

Il faut aussi comprendre en quoi le système 1.1.6 est bien la limite en un certain sens du système de Nash 1.1.2 lorsque le nombre de joueurs tend vers l'infini. On sépare les résultats dans cette direction en deux catégories :

- (a) le fait qu'une solution du système de jeu à champ moyen (1.1.6) soit une bonne approximation d'un équilibre de Nash à N joueurs, lorsque N est grand ;

- (b) la convergence des équilibres de Nash vers des solutions du système de jeu à champ moyen, lorsque le nombre de joueurs tend vers l'infini.

Le premier point a été établi dès les prémisses de la théorie des jeux à champ moyen [69], et semble maintenant bien compris et appréhendé par la littérature existante [39, 71, 68, 77]. Le second point (b) reste quant à lui un problème ouvert. Citons quelques travaux récents qui en donnent des réponses partielles : le livre de P. Cardaliaguet, F. Delarue, J-M. Lasry et P-L. Lions [35] ; le livre de R. Carmona et F. Delarue [42] ; et l'article de D. Lacker [80]. Dans [35], les auteurs s'intéressent à la master equation, qui est une EDP (équation aux dérivées partielles) non-locale non-linéaire posée sur l'espace infini-dimensionnel des mesures de probabilités. Le système (1.1.6) peut-être dérivé de la master equation par une méthode des caractéristiques en dimension infini le long des mesures de distribution des états. Par conséquent, l'existence et l'unicité d'une solution régulière de la master equation sont très liées à celles du système (1.1.6) ainsi qu'à sa stabilité par rapport à la condition initiale m_0 . Il est démontré dans [35] que, sous certaines hypothèses, une solution de la master equation existe, elle est unique et régulière, et c'est la limite des solutions du système (1.1.2), voir [35] théorème 2.4.8. Un résultat similaire est présenté dans [42], théorème 6.28, permettant de traiter le cas linéaire-quadratique. Dans [53, 54], F. Delarue, D. Lacker et R. Kavita poursuivent les mêmes idées pour obtenir un théorème central limite et un principe de grande déviation. Nous renvoyons à [87] pour plus de résultats sur la master equation.

Cependant, les résultats précédents dépendent de l'unicité des solutions du système (1.1.6), et donc de l'hypothèse de monotonie de Lasry-Lions (1.1.8). En particulier, dans le cas non-monotone, l'approche par la master equation échoue. D. Lacker [80], propose une définition de solution faible pour le système de jeu à champ moyen, et montre que lorsque l'espace des contrôles est compact, alors les équilibres de Nash à N joueurs convergent vers de telles solutions faibles.

1.1.3 Jeux à champ moyen de contrôle

Dans les deux parties précédentes, nous avons considéré des interactions de type champ moyen qui font uniquement intervenir la distribution statistique de l'état des joueurs. Pourtant, la majeure partie des résultats démontrés dans cette thèse porte sur des jeux dont les interactions peuvent faire intervenir la distribution jointe des états et des contrôles des joueurs. Nous utiliserons les dénominations *jeux de contrôles* et *jeux à champ moyen de contrôles* (introduite dans [38]) pour faire référence à de tels jeux.

Pour introduire la théorie des jeux à champ moyen de contrôle, nous nous plaçons encore une fois dans le cadre mathématique le plus élémentaire. Ainsi nous considérons toujours des jeux posés dans le tore de dimension d , \mathbb{T}^d ; nous nous plaçons à la limite $N = \infty$; l'état d'un joueur représentatif est donné par (1.1.4) ; on remplace le coût d'un joueur (1.1.5) par le coût :

$$(1.1.9) \quad J(\alpha, \mu) = \mathbb{E} \left[\int_0^T L(t, X_t, \alpha_t, \mu(t)) dt + g(X_T, m(T)) \right],$$

où $\mu(t)$ est la loi jointe de l'état et du contrôle des joueurs au temps $t \in [0, T]$, c.-à-d. $\mu(t) = \mathcal{L}(X_t, \alpha_t)$; et $m(t)$ est la première marginale de $\mu(t)$, c'est-à-dire la loi des états des joueurs $m(t) = \mathcal{L}(X_t)$. Dans ce cas le système de jeux à champ moyen 1.1.6 est

remplacé par le système de jeu à champ moyen de contrôle,

$$\begin{aligned}
 (1.1.10a) \quad & -\partial_t u - \nu \Delta u + H(x, \nabla_x u(t, x), \mu(t)) = 0 && \text{sur } (0, T) \times \mathbb{T}^d, \\
 (1.1.10b) \quad & \partial_t m - \nu \Delta m - \operatorname{div}(H_p(x, \nabla_x u(t, x), \mu(t))m) = 0 && \text{sur } (0, T) \times \mathbb{T}^d, \\
 (1.1.10c) \quad & \mu(t) = \left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) && \text{sur } [0, T], \\
 (1.1.10d) \quad & u(T, x) = g(x, m(T)) && \text{sur } \mathbb{T}^d, \\
 (1.1.10e) \quad & m(0) = m_0,
 \end{aligned}$$

où H est la transformée de Legendre de L par rapport à α , et $\left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t)$ est la mesure image de $m(t)$ par l'application $x \mapsto -H_p(x, \nabla_x u(t, x), \mu(t))$. Les applications des jeux à champ moyen de contrôles sont nombreuses et variées dans des domaines tels que l'économie, la finance, l'énergie, les mouvements de population. . . Citons par exemple [38, 43, 41, 75] pour un modèle de stratégie optimale de trading à haute fréquence (appelé parfois modèle d'Almgren et Chriss) ; [25, 47, 62, 65, 73, 75, 76] pour le modèle de Bertrand et Cournot de production de ressources épuisables ; [11] pour une application au marché de l'électricité ; [2, 75, 76] pour des modèles de mouvement de population ; [66] pour une application à la vente d'espaces publicitaires sur internet utilisant l'apprentissage par renforcement.

Malgré l'intérêt que présentent les jeux à champ moyen de contrôle dans de nombreuses applications, le nombre de résultats et de travaux théoriques dans ce domaine est faible. La majorité des questions qui ont été résolues pour les jeux à champ moyen sans interaction par le contrôle, sont encore des questions ouvertes pour les jeux à champ moyen de contrôle.

En effet, l'étude du système (1.1.10) s'avère en général bien plus compliquée que celle de (1.1.6). Il existe deux difficultés majeures inhérentes à (1.1.10) et qui sont absentes de (1.1.6) :

Pb1 la loi jointe de l'état et du contrôle des joueurs est donnée par une équation de point fixe (même lorsque u et m sont donnés) ;

Pb2 l'équation de Hamilton-Jacobi-Bellman dépend de manière non-locale du contrôle optimal, les estimées a priori sont alors plus difficiles à obtenir.

Pour répondre au premier point, il faut supposer que le problème de point fixe (1.1.10c) satisfait par μ est bien posé. Nous distinguons trois cas étudiés dans la littérature, qui fournissent une telle propriété d'inversion :

H1 H est séparé, c.-à-d.

$$(1.1.11) \quad H(t, x, p, \mu) = \tilde{H}(t, x, p) - f(t, x, \mu),$$

pour tout $(t, x, p, \mu) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$;

H2 la fonction L est monotone, au sens où elle vérifie l'inégalité suivante

$$(1.1.12) \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} L(t, x, \alpha, \mu^1) - L(t, x, \alpha, \mu^2) d(\mu^1 - \mu^2)(x, \alpha) \geq 0,$$

pour tout $(t, x) \in [0, T] \times \mathbb{T}^d$ et $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$;

H3 pour tout $(t, x, p, m) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$, l'application $\alpha \mapsto -H_p(t, x, p, (I_d, \alpha) \# m)$ est une contraction pour une certaine métrique sur l'espace des fonctions de contrôle.

En fait, **H1** peut-être vue comme un cas particulier de chacun des deux autres cas. Cette hypothèse est utilisée dans [41, 43] ; l'existence et l'unicité dans ce cas, lorsque l'ensemble des contrôle est supposé compact, sont démontrées dans [43]. L'hypothèse **H2** est étudiée dans [38, 41, 76] : l'unicité des solutions est démontrée [41, 76] ; l'existence de solutions faibles est étudiée dans [38], lorsque l'opérateur de diffusion est potentiellement dégénéré, et la dépendance de H en μ est uniformément borné ; lorsque l'opérateur de diffusion est non-dégénéré, le résultat d'existence le plus général se trouve dans [76], où L est supposé similaire à une fonction puissance en α et μ d'exposant quelconque dans $(1, \infty)$. L'hypothèse **H3** est introduite dans [75] et des résultats d'existence sont prouvés dans différentes situations. L'unicité paraît peu probable si aucune des hypothèses **H1** et **H2** n'est satisfaite ; un exemple où une approximation numérique de (1.1.10) admet plusieurs solutions est présenté dans [2].

Pour illustrer la difficulté **Pb2**, comparons les résultats obtenus lorsque le principe du maximum est appliqué aux systèmes de jeux à champ moyen avec et sans interaction par la loi du contrôle : si le Hamiltonien H de (1.1.6) est suffisamment régulier alors $H(t, x, 0, m)$ est uniformément borné en (t, x, m) (puisque $\mathcal{P}(\mathbb{T}^d)$ est compact pour la plupart des topologies usuelles), on obtient ainsi une borne uniforme sur $|u|$; sous des hypothèses de régularité similaires sur H pour le système (1.1.10), on obtient en général uniquement une estimation de $|u|$ qui dépend de μ . Les autres estimées usuelles sur u et m souffrent du même problème dans le système (1.1.10). Cette difficulté n'a pu être résolue pour l'instant qu'en ajoutant des hypothèses supplémentaires qui ne sont pas nécessaires pour (1.1.6). Le seul cas où les résultats des systèmes (1.1.6) et (1.1.10) semblent similaires (mais cela demande plus de travail pour (1.1.10)) est le cas monotone, c'est-à-dire lorsque L satisfait **H2**, comme démontré dans [76].

Ajoutons qu'il existe d'autres résultats sur les jeux à champ moyen de contrôle. Le cas stationnaire est étudié dans [59], l'existence de solutions est démontrée lorsque certains paramètres de couplage en la loi de contrôle sont petits. Dans [60], le système du premier ordre (c.-à-d. lorsque $\nu = 0$) est étudié par des méthodes probabilistes, et un résultat d'existence est démontré, sous des conditions de croissances de le Hamiltonien. Un résultat d'unicité est énoncé dans [24], pour un cas particulier de (1.1.10) où l'interaction par le contrôle est simple (elle dépend de la moyenne des contrôles).

1.2 Organisation de la thèse

Les résultats de cette thèse sont divisés en quatre chapitres qui correspondent tous à un article de recherche. Les chapitres 2 et 4 sont des pré-prints disponibles sur arXiv, ils correspondent respectivement à [75] et [76]. Le chapitre 3 est un travail en collaboration avec Y. Achdou accepté pour publication dans le journal *AIMS Mathematics in Engineering* [2]. Le chapitre 5 a été réalisé en collaboration avec L.M. Briceño, D. Kalise, M. Laurière, A. Matéos Gonzalez et F.J. Silva ; il est publié dans *ESAIM : Proc. Surveys* [27].

Le chapitre 2 introduit une nouvelle hypothèse de structure sur le Hamiltonien sous laquelle plusieurs résultats d'existence pour (1.1.10) sont démontrés. Cette hypothèse de structure a déjà été mentionnée formellement plus haut, voir : **H3**. Nous ne faisons pas d'hypothèse de monotonie dans ce chapitre. Dans ce cas, l'unicité n'est pas vérifiée en général (un exemple de non-unicité est montré dans le chapitre 3). Nous supposons que le Hamiltonien se comporte asymptotiquement comme une fonction puissance en ses arguments, dont l'exposant est quelconque. Nous démontrons alors l'existence de solutions à (1.1.10) sous des jeux d'hypothèses différentes, détaillées plus loin. L'unicité est démontrée uniquement lorsque T est assez petit. Ces résultats peuvent ensuite être étendus au

système de jeux à champ moyen de contrôle posé sur \mathbb{R}^d plutôt que sur \mathbb{T}^d par la méthode introduite au chapitre 4. Nous présentons aussi quatre applications des résultats énoncés.

Le chapitre 3 reprend une application introduite dans le chapitre 2 dans le cas quadratique et est consacré à des simulations numériques. Pour cela nous adaptons, aux jeux à champs moyen de contrôle, un schéma numérique inspiré de celui introduit par Y. Achdou et I. Capuzzo-Dolcetta [3, 6] pour le système de jeux à champ moyen sans interaction par le contrôle. Dans cette application, les agents essaient d’aller dans la même direction que la moyenne des agents autour d’eux. Cela rappelle le modèle de Cucker et Smale [52], et semble très bien adapté à des modèles de mouvement de foules de piétons ou au trafic routier. L’hypothèse de monotonie n’est pas vérifiée, et nous exhibons, dans un cas particulier, plusieurs solutions au même système de jeu à champ moyen de contrôle discret. Nous présentons un autre exemple dans lequel notre modèle permet de simuler un phénomène de queue (les joueurs ralentissent en amont d’un obstacle et forme une queue), qui est difficile à obtenir avec d’autres modèles. Nous faisons aussi une analyse de l’influence des différents paramètres de notre modèle.

Dans le chapitre 4, nous nous intéressons à une extension du système (1.1.10) posé sur \mathbb{R}^d plutôt que sur \mathbb{T}^d , et où l’on autorise les agents à avoir des fonctions de drift générales. Nous nous concentrons sur le cas monotone, c.-à-d. lorsque le Lagrangien satisfait l’hypothèse H2. Nous supposons que le Hamiltonien se comporte asymptotiquement comme une fonction puissance en ses arguments, dont l’exposant est supérieur à un. Dans ce cas, nous sommes capables d’obtenir des résultats d’existence et d’unicité similaires à ceux obtenus pour 1.1.6, au prix d’efforts supplémentaires. Nous introduisons aussi une méthode permettant d’étendre les résultats d’existence sur le tore à l’existence sur l’espace euclidien ; et une méthode pour généraliser l’existence de solution à des fonction de drifts générales.

Dans le chapitre 5, nous étudions une approche numérique pour le système (1.1.6) avec couplage local. La discrétisation que nous considérons résulte d’une approche variationnelle décrite, pour le problème stationnaire, dans [29] et mène au schéma aux différences finies introduit par Achdou et Capuzzo-Dolcetta dans [3, 6]. Dans le but de résoudre des problèmes variationnels en dimension finie, les auteurs de [29] implémentent un algorithme primal-dual introduit par Chambolle et Pock dans [46], qui consiste à résoudre itérativement des systèmes linéaires et à appliquer un opérateur proximal. Nous appliquons cette méthode à un jeu à champ moyen dépendant du temps et, lorsque le paramètre de viscosité est assez grand, nous améliorons la résolution du système linéaire en remplaçant l’approche directe utilisée dans [29] par des algorithmes itératifs préconditionnés. L’algorithme qui semble donner les meilleurs résultats numériques ici utilise un préconditionnement par une méthode de multigrille.

1.2.1 À propos des solutions classiques du système de jeu à champ moyen de contrôle

Nous présentons ici les résultats principaux et les idées du chapitre 2, [75]. Nous donnons des résultats d’existence de solution au système (1.1.10) sous l’hypothèse H3 qui sera écrite rigoureusement ci-dessous.

Dans ce chapitre, une solution du système (1.1.10) est un triplet (u, m, μ) tel que :

- u est une solution classique de l’équation de Hamilton-Jacobi-Bellman (1.1.10a) ;
- m est uniformément continue et sa densité par rapport à la mesure de Lebesgue est une solution au sens des distributions de l’équation de Fokker-Planck-Kolmogorov (1.1.10b) ;

- $\mu : [0, T] \mapsto \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfait (1.1.10c) en tout $t \in [0, T]$.

Remarquons que si (u, m, μ) est une solution de (1.1.10), alors le flot de mesures $\mu(t)$ satisfaisant (1.1.10c) a une forme particulière : chaque mesure $\mu(t)$ est la mesure image d'une mesure de probabilité définie sur \mathbb{T}^d , et sa seconde marginale est supportée sur un compacte.

L'hypothèse de structure centrale (**FP1** et **FP2**) dans ce chapitre est qu'il existe $\lambda_0 \in [0, 1)$ tel que :

$$(1.2.1) \quad \begin{aligned} & |H_p(x, p, (I_d, \alpha) \# m) - H_p(x, p, (I_d, \tilde{\alpha}) \# m)| \leq \lambda_0 \|\alpha - \tilde{\alpha}\|_{L^{q_0}(m)} \\ & |H_p(x, p, (I_d, \alpha) \# m)| \leq C_0 (1 + |p|^{q-1}) + \lambda_0 \|\alpha\|_{L^{q_0}(m)}, \end{aligned}$$

pour tout $(x, p, m) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$ et $\alpha, \tilde{\alpha} \in L^\infty(m)$; où C_0 est une constante positive, $q \in (1, \infty)$ est l'exposant de croissance de le Hamiltonien, $q_0 \in [1, \infty]$ est l'exposant d'intégration de l'interaction à champ moyen de contrôle.

Une analyse rapide des termes dans (1.2.1) nous donne que $|H_p|$, $|p|^{q-1}$ et $\|\alpha\|_{L^{q_0}(m)}$ sont tous trois homogènes à la norme euclidienne d'un contrôle ; ainsi les exposants dans (1.2.1) sont critiques, au sens où aucun d'eux ne peut être pris plus grand sans rendre le système inhomogène. La même analyse dimensionnelle implique que si q est l'exposant de croissance asymptotique de H en p , alors $q' = \frac{q}{q-1}$ sera l'exposant de croissance asymptotique de H en μ . De telles considérations apparaissent dans les hypothèses de croissance de le Hamiltonien qui sont détaillées dans la partie 2.2.2 du chapitre 2.

L'hypothèse $\lambda_0 < 1$ est ici essentielle pour prouver que (1.1.10c) admet un point fixe. Nous renvoyons à la remarque 3.4.3 au chapitre 3 pour un exemple où il n'existe pas de solution à 1.1.10, ni à son équivalent à N joueurs, dans le cas $\lambda_0 = 1$.

Nous faisons également des hypothèses de régularité sur H et nous supposons que la dépendance de g et H , respectivement en m et la première marginale de μ , est non-locale et uniformément borné. Dans ce cadre, il existe une solution au système de jeu à champ moyen sans interaction de contrôle (1.1.6). Cependant, pour le système (1.1.10), des hypothèses supplémentaires sont nécessaires. Nous montrons alors l'existence de solution dans chacun des cas suivant :

- si l'exposant asymptotique de H en μ est sous-critique ;
- si l'exposant asymptotique de H_x en μ est faible ;
- lorsque certains paramètres sont petits ;
- pour un horizon de temps T court.

L'une des principales difficultés pour prouver l'existence de solutions vient du fait qu'on n'arrive pas à obtenir les mêmes estimations a priori que dans les cas plus classiques (voir **Pb2**). En particulier, nous n'avons pas d'estimation a priori uniforme sur $|u|$ en général. Une contribution notable est que nous introduisons alors une méthode de type Bernstein pour l'équation de Hamilton-Jacobi (1.1.10a) qui ne demande aucune estimée sur $|u|$. Nous sommes alors capables de donner une estimation de $|\nabla_x u|$ qui dépend de manière au plus linéaire de $\sup |u|$. Une telle inégalité nous semble optimale, puisque pour des raisons d'homogénéité évidentes, la dépendance ne devrait pas être plus faible que linéaire.

La question de l'unicité est abordée rapidement : nous montrons que la solution de (1.1.10), dans le cas d'un horizon de temps court, est unique. L'unicité ne devrait pas subsister pour des horizons de temps arbitrairement grands comme le laisse présager certains

résultats numériques réalisés dans le chapitre 3 (où plusieurs solutions du même système discrétisé sont exhibées).

Tous les résultats de ce chapitre traitent du système (1.1.10) qui est posé sur le tore \mathbb{T}^d . Il serait plus réaliste pour beaucoup d'applications de considérer plutôt un système de jeu à champ moyen posé sur \mathbb{R}^d . Nous insistons sur le fait qu'il est possible d'étendre les résultats de ce chapitre à \mathbb{R}^d , en utilisant la méthode introduite dans le chapitre 4.

Enfin, nous donnons quatre applications des résultats théoriques de ce chapitre ; chacune s'inspire de modèles existants dans la littérature et en propose une extension qui peut s'avérer souhaitable :

- un modèle de production de ressources épuisables, dans lequel un agent produit plusieurs types de ressources ; notre contribution consiste à autoriser le prix de ces ressources à être négativement corrélé ;
- un modèle de trading à haute fréquence dans lequel on introduit un prix de type bid-and-ask ;
- un modèle de déplacement des oiseaux migrateurs (le flocking model en anglais introduit par Cucker et Smale [52]) du premier ordre, c.-à-d. sans accélération et avec une interaction par les vitesses ;
- un modèle de déplacement de population, aussi inspiré du modèle de Cucker et Smale.

1.2.2 Approximation par la méthode des différences finies du système de jeu à champ moyen de contrôle

Ce chapitre est issu d'un travail en collaboration avec Y. Achdou. Il est consacré à l'étude numérique d'un modèle de dynamique des populations introduit dans le chapitre 2. Pour qu'un tel modèle soit plus réaliste, l'espace des états est ici Ω un compact à bord régulier inclus dans \mathbb{R}^d .

Si $\mu(t)$ est la loi jointe de l'état et du contrôle des joueurs au temps $t \in [0, T]$, on définit $V(t, x)$ le déplacement moyen en $x \in \Omega$ par

$$(1.2.2) \quad V(t, x) = \frac{1}{Z(t, x)} \int_{(y, \alpha) \in \Omega \times \mathbb{R}^d} \alpha K(x, y) d\mu(t, y, \alpha),$$

où $K : \Omega \times \Omega \rightarrow \mathbb{R}_+$ est un noyau markovien et $Z(t, x)$ est une constante de normalisation définie par $Z(t, x) = \int_{(y, \alpha) \in \Omega \times \mathbb{R}^d} K(x, y) d\mu(t, y)$. Nous prendrons ensuite $K(x, \cdot)$ à support compact dans une boule centrée en x . Formellement, $V(t, x)$ est une moyenne pondérée des contrôles des joueurs autour de x . Nous définissons le Lagrangien $L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ par

$$L(\alpha, V) = \frac{\theta}{2} |\alpha - \lambda V|^2 + \frac{1 - \theta}{2} |\alpha|^2, \quad (\alpha, V) \in \mathbb{R}^d \times \mathbb{R}^d,$$

où λ et θ sont deux constantes telles que $\lambda < 1$ et $\theta \in [0, 1]$. Heuristiquement, si $\lambda > 0$ alors chaque joueur préfère aller dans une direction similaire à la direction moyenne des joueurs autour de lui (représenté par V) ; si $\lambda < 0$, alors chaque joueur préfère s'opposer à la foule en allant dans la direction contraire à V .

Le système de jeu à champ moyen de contrôle est donné par :

$$(1.2.3) \quad \begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{2} |\nabla_x u - \lambda \theta V|^2 - \frac{\lambda^2 \theta}{2} |V|^2 = f(x, m), & \text{in } (0, T) \times \Omega, \\ \partial_t m - \nu \Delta m - \operatorname{div}((\nabla_x u - \lambda \theta V) m) = 0, & \text{in } (0, T) \times \Omega, \\ V(t, x) = -\frac{1}{Z(t, x)} \int_{\Omega} (\nabla_x u(t, y) - \lambda \theta V(t, y)) K(x, y) dm(t, y), & \text{in } (0, T) \times \Omega, \\ Z(t, x) = \int_{\Omega} K(x, y) dm(t, y), & \text{in } (0, T) \times \Omega, \\ u(T, x) = \phi(x), & \text{in } \Omega, \\ m(0, x) = m_0(x), & \text{in } \Omega, \end{cases}$$

où $f : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ est une fonction locale définie par $f(x, m) = cm + f_0(x)$, pour $c \geq 0$ et f_0 une fonction de Ω dans \mathbb{R} ; $\phi : \Omega \rightarrow \mathbb{R}$ est le coût terminal. Les conditions au bord sur $\partial\Omega$ peuvent être des conditions de Neumann ou de Dirichlet.

Pour faire des simulations numériques de ce modèle, nous adaptons une méthode des différences finies, introduite par Y. Achdou et I. Capuzzo-Dolcetta [3, 6], à ce modèle de jeu à champ moyen de contrôle. Ceci est ici possible car la dynamique du système ne dépend de la loi jointe de l'état et du contrôle μ (qui est un objet de dimension infinie) que de manière indirecte à travers le déplacement moyen $V(t, x)$ qui est un objet fini-dimensionnel.

L'algorithme de résolution numérique consiste en une méthode de continuation sur une suite décroissante de coefficients de viscosités $\nu^1 > \nu^2 > \dots > \nu^s > 0$, pour $s \geq 1$. Pour $\tilde{s} \in \{2, \dots, s\}$, nous résolvons le système de jeu à champ moyen avec $\nu^{\tilde{s}}$ par une méthode de Newton en se servant de la solution obtenue avec $\nu^{\tilde{s}-1}$ comme choix initial. L'itération de la méthode de Newton est faite sur le couple (f, V) , où f est le terme de droite dans l'équation de Hamilton-Jacobi de (1.2.3), de la manière suivante :

1. nous obtenons u la solution de l'équation de Hamilton-Jacobi discrétisée avec V comme déplacement moyen et f comme second membre ;
2. nous résolvons l'équation de Fokker-Planck discrétisée en utilisant le triplet (u, f, V) ;
3. nous mettons à jour f avec la formule $f = cm + f_0$;
4. nous mettons à jour V en résolvant le problème de point fixe en V par une méthode d'itération de Picard (V est le point fixe d'un opérateur contractant).

Nous analysons deux exemples dans lesquels nous prenons $\lambda > 0$ (c.-à-d. un joueur préfère aller dans le sens de la foule autour de lui) :

- un domaine carré ; les joueurs sont initialement regroupés dans le coin inférieur gauche et le coin supérieur droit ; leur objectif est d'arriver dans l'un des deux autres coins avant en un temps donné ;
- un hall rectangulaire que doivent traverser les joueurs, avec une entrée à gauche et une sortie à droite.

Nous exhibons plusieurs solutions différentes dans le premier exemple. Nous pouvons alors imaginer que le système (1.2.3) possède aussi plusieurs solutions. Un phénomène de rassemblement a lieu, comme dans le modèle de Cucker et Smale [52]. Dans le second exemple, nous mettons en lumière un autre effet intéressant de notre modèle avec interaction par la vitesse : les joueurs anticipent leur ralentissement (ou accélération) futur, créant

un phénomène de queue qu'il est difficile de faire émerger d'un modèle mathématique en général.

Nous nous intéressons aussi au nombre d'itérations nécessaires à notre algorithme pour différents choix de paramètres et différents pas de discrétisation. En particulier, nous constatons que le nombre d'itérations semble dépendre faiblement du nombre total de points de discrétisation.

1.2.3 Jeux à champ moyen avec interactions monotones par la loi des états et des contrôles des joueurs

Contrairement aux systèmes considérés jusque là, nous nous plaçons cette fois dans l'espace euclidien \mathbb{R}^d , et nous autorisons la dynamique des joueurs à avoir une forme générale. Plus précisément, l'état d'un joueur, $X_t \in \mathbb{R}^d$, satisfait maintenant l'équation stochastique suivante :

$$dX_t = b(t, X_t, \alpha_t) dt + \sqrt{2\nu} dW_t,$$

où $b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$. Nous renvoyons à la remarque 4.2.9iv) pour le cas où l'on autorise b à dépendre de $\mu_\alpha(t)$ la loi jointe de l'état et du contrôle, c.-à-d. $\mu_\alpha(t) = \mathcal{L}(X_t, \alpha_t)$ pour $t \in [0, T]$. Nous introduisons $\mu_b(t)$ la loi jointe de l'état et du drift.

Nous supposons que b est inversible par rapport à α , et nous appelons son inverse $\alpha^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$. Le système que l'on considère est alors :

$$(1.2.4) \quad \begin{cases} -\partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x), \mu_\alpha(t)) = f(t, x, m(t)) & \text{in } (0, T) \times \mathbb{R}^d, \\ \partial_t m(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(t, x, \nabla_x u(t, x), \mu_\alpha(t)) m) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \mu_\alpha(t) = \left[(x, \tilde{b}) \mapsto (x, \alpha^*(t, x, \tilde{b})) \right] \# \mu_b(t) & \text{in } [0, T], \\ \mu_b(t) = \left(I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu_\alpha(t)) \right) \# m(t) & \text{in } [0, T], \\ u(T, x) = g(x, m(T)) & \text{in } \mathbb{R}^d, \\ m(0, x) = m_0(x) & \text{in } \mathbb{R}^d. \end{cases}$$

L'hypothèse principale est une hypothèse de monotonie, voir **H2**. Nous supposons de plus que

- la dynamique b et le Lagrangien L se comportent asymptotiquement comme des puissances de leurs arguments,
- f et g sont non-locales, régulières et uniformément bornées ainsi que leurs dérivées.

Sous de telles hypothèses, nous prouvons l'existence et l'unicité de la solution de (1.2.4). Nous obtenons de ce fait des résultats comparables à ceux contenus dans la littérature sur le système (1.1.6) sous des hypothèses similaires.

L'unicité est obtenue par la même méthode que pour le système (1.1.6). La majeure partie du chapitre est alors consacrée à la preuve de l'existence de solution. Les grandes étapes de cette preuve sont:

- nous commençons par regarder le système (1.1.10) posé sur le tore \mathbb{T}^d et avec $b = \alpha$;
- nous montrons qu'il existe un unique point fixe (1.1.10c) en utilisant principalement l'hypothèse de monotonie **H2** et le théorème de point fixe de Leray-Schauder ;
- nous obtenons des estimations a priori uniformes sur les solutions de (1.2.4) ;

- le théorème de point fixe de Leray-Schauder permet de prouver l'existence de solution à (1.2.4) ;
- toujours lorsque $b = \alpha$, nous étendons le résultat d'existence précédent du tore à l'espace euclidien en faisant tendre le diamètre du tore vers l'infini, grâce à des propriétés de compacité induites par les estimations a priori obtenues précédemment ;
- nous généralisons au cas où b est quelconque, la preuve consiste à : exprimer le problème de contrôle optimal du système (1.2.4) comme un nouveau problème d'optimisation sur le drift plutôt que sur le contrôle ; puis appliquer les résultats des points précédents.

Il est important de préciser que les deux méthodes, introduites dans ce travail, qui nous permettent d'étendre notre résultat d'existence (d'abord à \mathbb{R}^d puis à des dynamiques générales) préservent la monotonie du Lagrangien. Nous pensons qu'elles peuvent être adaptées à d'autres situations. En particulier, elles permettent d'étendre les résultats du chapitre 2.

Nous présentons deux applications :

- un modèle de production de ressources épuisables en dimensions $d \geq 1$: nous généralisons substantiellement les résultats d'existence et d'unicité démontrés dans [25] ;
- le modèle de mouvement de population introduit dans le chapitre 2 et étudié numériquement dans le chapitre 3, dans le cas où un agent cherche à aller dans la direction opposée à la direction moyenne de la foule.

1.2.4 Un algorithme primal-dual pour des jeux à champ moyen dynamique du second-ordre et à couplage local

Ce chapitre est issu d'un travail en collaboration avec L.M. Briceño, D. Kalise, M. Laurière, A. Matéos Gonzalez et F.J. Silva. Contrairement aux précédents travaux, nous nous intéressons ici à un système de jeu à champ moyen sans interaction de contrôle. Nous supposons que H est à variables séparées, c.-à-d. de la forme (1.1.7) ; la fonction de couplage $f : \mathbb{T}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}$ et le coût terminal $g : \mathbb{T}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}$ sont des fonctions locales et croissantes.

Nous considérons l'interprétation variationnelle du système (1.1.6),

$$(1.2.5) \quad \begin{cases} \inf_{(m,w)} \int_0^T \int_{\mathbb{T}^d} [B(x, m(t, x), w(t, x)) + F(x, m(t, x))] dx + \int_{\mathbb{T}^d} G(x, m(T, x)) dx \\ \text{tel que} \quad \partial_t m - \nu \Delta m + \operatorname{div}(w) = 0 \quad \text{in } \mathbb{T}^d \times (0, T), \\ m(\cdot, 0) = m_0(\cdot) \quad \text{in } \mathbb{T}^d, \end{cases}$$

où $B : \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ et $F, G : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ sont définies par

$$B(x, m, w) = \begin{cases} mL(x, -\frac{w}{m}) & \text{si } m > 0, \\ 0 & \text{si } (m, w) = (0, 0), \\ +\infty & \text{sinon,} \end{cases}$$

$$F(x, m) = \begin{cases} \int_0^m f(x, m') dm' & \text{si } m \geq 0, \\ +\infty & \text{sinon,} \end{cases} \quad G(x, m) = \begin{cases} \int_0^m g(x, m') dm' & \text{si } m \geq 0, \\ +\infty & \text{sinon.} \end{cases}$$

Si L est convexe, le système (1.2.5) est convexe. Par conséquent, nous pouvons utiliser des méthodes de dualité pour prouver l'existence et l'unicité de la solution. Nous renvoyons à [37] pour une analyse rigoureuse.

Dans ce chapitre, nous discrétisons (1.2.5) par la méthode des différences finies introduites dans [3, 6]. Ensuite, nous résolvons numériquement le système discrétisé par l'algorithme de Chamboll-Pock [46]. Cette méthode a déjà été utilisée dans le cas stationnaire dans [29]. Cependant, dans le cas dynamique (c.-à-d. nous ajoutons une dimension temporelle supplémentaire), la complexité numérique du calcul demande parfois un nombre d'itérations très grand qui rend prohibitif l'utilisation de cette méthode. Nous proposons ici des méthodes de préconditionnement pour passer outre cette difficulté. En particulier, nous adaptons une stratégie de préconditionnement reposant sur une méthode multigrille, introduite dans [8]. Lorsque le paramètre de viscosité n'est pas très faible, nous arrivons ainsi à résoudre rapidement le problème discrétisé. Les vitesses de convergence pour plusieurs choix de paramètres et plusieurs coefficients de viscosité sont présentées.

1.2.5 Perspectives et travaux futurs

Comme nous l'avons décrit dans la partie 1.1, l'une des questions fondamentales de la théorie des jeux à champ moyen est la convergence des équilibres de Nash à N joueurs vers les solutions du système de jeu à champ moyen. Si cette question a déjà obtenu des réponses partielles dans le cas de jeux sans interactions de contrôle ; aucun travail n'a été réalisé à notre connaissance dans le cas de jeux à champ moyen de contrôle (dans le cas closed-loop). J'effectue actuellement un tel travail de recherche dans un cas particulier, lorsque l'horizon est court. L'idée que la convergence des équilibres de Nash devrait avoir lieu en temps court (au moins pour les jeux sans interaction de contrôle) a été évoquée par P.L. Lions dans ses cours au collège de France [87].

Il serait aussi intéressant d'étudier la master equation avec interactions de contrôle. Les résultats du chapitre 4 laissent penser que si les interactions sont monotones, les résultats sur les systèmes de jeu à champ moyen avec et sans interaction de contrôle sont similaires. Dans ce cas, nous pouvons nous demander dans quelle mesure il est possible d'étendre les résultats existants sur la master equation au cas des jeux avec interactions monotones par le contrôle.

J'aimerais aussi étendre l'étude des systèmes de jeu à champ moyen de contrôle étudiés dans cette thèse au cas où l'interaction par la loi du contrôle n'est pas markovienne. À notre connaissance, aucun résultat théorique n'existe dans ce cas dans la littérature EDP. Pourtant, il existe des applications en finance (et certainement dans d'autres domaines) d'un tel système : par exemple le modèle d'Almgren et Chriss de trading haute fréquence, dont la version à champ moyen est discutée dans la partie 2.6.2, fait intervenir le prix d'actifs financiers dépendant de la loi des contrôles des joueurs à tous les instants antérieurs. Dans la littérature disponible, on a toujours fait une simplification permettant de ne considérer que des interactions markoviennes. Cependant, il paraît peu probable que cette simplification reste pertinente dans des applications plus réalistes. Les résultats du chapitre 2 semblent pouvoir être étendus à cette situation. Une étude approfondie fait partie de mes projets de recherche.

Enfin, j'aimerais étudier les liens entre la théorie des jeux à champ moyen et le machine learning. Les raisons pour lesquelles les méthodes récentes de deep learning sont aussi efficaces sont encore mal connues à l'heure actuelle. Il semble que l'une des principales difficultés dans l'étude théorique de ces méthodes vienne du grand nombre de paramètres dans les réseaux de neurones. L'architecture des réseaux de neurones modernes fait naturellement intervenir des interactions de type champ moyen entre les neurones. Nous pouvons alors imaginer que certains outils de la théorie des jeux à champ moyen peuvent s'appliquer à des questions actuelles de la recherche en machine learning et deep learning. Nous renvoyons à [49, 67, 74, 88, 99, 98], pour des travaux sur le comportement

asymptotique de réseaux de neurones, lorsque le nombre de neurones tend vers l'infinie.

On Classical Solutions to the Mean Field Game System of Controls

We consider a class of mean field games in which the optimal strategy of a representative agent depends on the statistical distribution of the states and controls.

We prove some existence results for the forward-backward system of PDEs under rather natural assumptions. The main step of the proof consists of obtaining a priori estimates on the gradient of the value function by Bernstein's method. Uniqueness is also proved under more restrictive assumptions.

Finally, we discuss some examples to which the previously mentioned existence (and possibly uniqueness) results apply.

2.1 Introduction

The theory of Mean Field Games (MFG for short) has been introduced in the independent works of J.M. Lasry and P.L. Lions [83, 84, 85], and of M.Y. Huang, P.E. Caines and R. Malhamé [69, 70]. It aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of agents tends to infinity. The agents are supposed to be rational (given a cost to be minimized, they always choose the optimal strategies), and indistinguishable. Furthermore, the agents interact via some empirical averages of quantities which depend on the state variable.

At the limit when $N \rightarrow +\infty$, the game may be modeled by a system of two coupled partial differential equations (PDEs), which is named the MFG system. On the one hand, there is a Fokker-Planck-Kolmogorov equation describing the evolution of the statistical distribution m of the state variable; this equation is a forward in time parabolic equation, and the initial distribution at time $t = 0$ is given. On the other hand, the optimal value of a generic agent at some time t and state x is noted $u(t, x)$ and is defined as the lowest cost that a representative agent can achieve from time t to T if it is at state x at time t . The value function satisfies a Hamilton-Jacobi-Bellman equation posed backward in time with a terminal condition involving a terminal cost. In the present work, we will restrict our attention to the case when the costs and the dynamics are periodic in the state variable, and we will work in the d -dimensional torus \mathbb{T}^d (as it is often done in the MFG literature for simplicity). We will take a finite horizon time $T > 0$, and will only consider second-order non-degenerate MFG systems. In this case, the MFG system is often written as:

$$\begin{aligned}
 (2.1.1a) \quad & -\partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x)) = f(x, m(t)) && \text{in } (0, T) \times \mathbb{T}^d, \\
 (2.1.1b) \quad & \partial_t m(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(t, x, \nabla_x u(t, x))m) = 0 && \text{in } (0, T) \times \mathbb{T}^d, \\
 (2.1.1c) \quad & u(T, x) = g(x, m(T)) && \text{in } \mathbb{T}^d, \\
 (2.1.1d) \quad & m(0, x) = m_0(x) && \text{in } \mathbb{T}^d.
 \end{aligned}$$

We refer the reader to [35] for some theoretical results on the convergence of the N -agents Nash equilibrium to the solutions of the MFG system. For a thorough study of the well-posedness of the MFG system, see the videos of P.L. Lions' lecture at the Collège de France, and some lecture notes [33].

There is also an important literature on the probabilistic aspects of MFGs, see [39, 79] for some examples and [41, 42] for a detailed presentation of the probabilistic viewpoint.

For applications of MFGs, numerical simulations are crucial because it is most often impossible to find explicit or semi-explicit solutions to the MFG system. We refer to [3] for a survey on finite difference methods and to [7] for applications to crowd motion.

Most of the literature on MFGs is focused on the case when the mean field interactions only involves the distributions of states. Here we will consider a more general situation in which the cost of an individual agent depends on the joint distribution μ of states and optimal strategies. To underline this, we choose to use the terminology *Mean Field Games of Controls (MFGCs)* for this class of MFGs; the latter terminology was introduced in [38]. Within this framework, the usual MFG system (2.1.1) is replaced by the following MFGC

system,

$$\begin{aligned}
 (2.1.2a) \quad & -\partial_t u(t, x) - \nu \Delta u(t, x) + H(x, \nabla_x u(t, x), \mu(t)) = 0 && \text{in } (0, T) \times \mathbb{T}^d, \\
 (2.1.2b) \quad & \partial_t m(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(x, \nabla_x u(t, x), \mu(t))m) = 0 && \text{in } (0, T) \times \mathbb{T}^d, \\
 (2.1.2c) \quad & \mu(t) = \left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) && \text{in } [0, T], \\
 (2.1.2d) \quad & u(T, x) = g(x, m(T)) && \text{in } \mathbb{T}^d, \\
 (2.1.2e) \quad & m(0, x) = m_0(x) && \text{in } \mathbb{T}^d.
 \end{aligned}$$

We would like to point out two of the main difficulties that one may encounter when studying (2.1.2) and which are not present in the study of (2.1.1).

- 1) The joint law of states and controls satisfies a fixed point relation described by (2.1.2c).
- 2) The HJB equation (2.1.2a) is non-local with respect to $\nabla_x u$. Consequently, it is much more difficult to obtain uniform a priori estimates on u and its derivatives.

Difficulty **1**) is in general not straightforward and one needs to make assumptions for the fixed point in μ to have a unique solution when $(\nabla_x u, m)$ are given. An example in which this fixed point relation does not admit any solution is given in [2] Remark 4.3.

Let us provide a simple illustration for describing difficulty **2**) by comparing the results obtained when we apply the maximum principle on parabolic equations to (2.1.1a) and (2.1.2a) respectively: if u satisfies (2.1.1a) where f and g are assumed to be uniformly bounded with respect to m , then u is uniformly bounded; under the same assumption on g , if u is a solution to (2.1.2a) and H is not uniformly bounded with respect to μ , we can only say that u is bounded in absolute value by a constant depending on μ . The other estimates used in the usual arguments of existence in MFG systems suffer the same lack of uniformity with respect to μ . Conversely, the estimates of μ depend on $\nabla_x u$. It is not obvious a priori how to combine the estimates on μ and (u, m) in order to obtain uniform estimates on u . Consequently, compactness results are harder to obtain for (2.1.2) than for (2.1.1).

The main assumption of this paper, namely **FP1** and **FP2** described below, is an original structural assumption designed to address difficulty **1**). In particular, it implies that the map

$$\mu \mapsto \tilde{\mu} = \left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot), \mu) \right) \# m,$$

is a contraction in a convenient metric space, when (t, u, m) are given.

Moreover, we also assume that the Hamiltonian $H(x, p, \mu)$ behaves like a power function when p tends to infinity. See paragraph 2.2.2 for more details.

The main objective of this work is to discuss existence of the solutions of the MFGC system (2.1.2) within this framework. We will also give a uniqueness result under a short time horizon assumption. We refer to [2] for a numerical application with multiple solutions. Indeed, uniqueness does not hold in general for arbitrary time horizon. It can be obtained though, under a monotonicity assumption which is investigated in the companion paper [76]. In [76], existence and uniqueness of solutions of the MFGC system are proved under the above-mentioned monotonicity assumption and with Hamiltonian having similar growth as in the present paper. This monotonicity condition implies that the agents favor moving in a direction opposite to the mainstream. Such an assumption is adapted to some models coming from finance or economy; and may be unrealistic in several situations, in

particular in models of crowd motions. This explains why here we introduce a new structural assumption and refrain from assuming monotonicity or investigating uniqueness in the general case.

Related literature

In the first articles devoted to MFGCs, [59, 60], D. Gomes and his collaborators have given several existence results for MFGCs in various cases, using the terminology *extended MFGs* instead of *MFGCs*. For instance, [60] contains existence results for stationary games (infinite horizon) under the assumption that some of the parameters involved in the models are small. We refer to [24, 38, 43, 41, 76] for other existence and uniqueness results for MFGC systems.

Uniqueness is a major issue in MFG theory, it has been proved for (2.1.1) in [85, 87] under an assumption called the Lasry-Lions monotonicity on the coupling function f and the terminal cost g in the case of non-local coupling. This assumption has been extended to MFGC and discussed in [59, 41, 76] in which uniqueness is proved. It translates the fact that the agents prefer directions opposite to the mainstream direction; therefore it is not adapted to a large class of MFGC systems like crowd motion models in which an agent is more likely to go in the mainstream direction.

The latter example of population dynamic is the typical application we had in mind when writing the assumptions in the present paper, see paragraphs 2.6.3 and 2.6.4. To our knowledge, existence results for such MFGC systems have not been discussed in the literature before. Uniqueness should not hold in general but under a short-time assumption. We refer to [2] in which the MFGC system is discretized using a finite-difference scheme and simulations are provided where the approximating discrete MFGC system admits several different solutions.

For other applications of MFGCs we refer to [38] for an model of optimal trading, [25, 47, 62, 65, 73] in the case of competition between firms producing the same goods, or [11] for energy storage.

Organization of the paper

Section 2.2 describes the notations, assumptions and main results in this paper. In Section 2.3, we address difficulty 1) which consists of inverting the fixed point relation in μ (2.1.2c) and providing estimates on the resulting flow of measures. Section 2.4 is devoted to proving a priori estimates on the solutions to (2.1.2) and addresses difficulty 2). Section 2.5 contains the proofs of the main results. Finally, we discuss several applications in Section 2.6. Namely, we study

- the Bertrand and Cournot competition for exhaustible resources and introduce an extension to negatively correlated resources (for instance gold and other raw materials);
- a model of price impact for high-frequency trading by Almgren and Chriss in which we discuss the possibility for the bid and ask prices to be different;
- a first-order flocking model;
- a crowd motion model.

2.2 Notations and assumptions

2.2.1 Notations and definitions

The spaces of probability measures are equipped with the weak* topology. We denote by $\mathcal{P}_\infty(\mathbb{T}^d \times \mathbb{R}^d)$ the subset of measures μ in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ with a second marginal compactly supported. For $\mu \in \mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\tilde{q} \in [1, \infty)$, we define the quantities $\Lambda_{\tilde{q}}(\mu)$ and $\Lambda_\infty(\mu)$ by,

$$(2.2.1) \quad \begin{aligned} \Lambda_{\tilde{q}}(\mu) &= \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |\alpha|^{\tilde{q}} d\mu(x, \alpha) \right)^{\frac{1}{\tilde{q}}}, \\ \Lambda_\infty(\mu) &= \sup \{ |\alpha|, (x, \alpha) \in \text{supp } \mu \}. \end{aligned}$$

Jensen inequality states that,

$$(2.2.2) \quad \Lambda_{q_1}(\mu) \leq \Lambda_{q_2}(\mu),$$

for any $1 \leq q_1 \leq q_2 \leq \infty$.

For $R > 0$, we denote by $\mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$ the subset of measures μ in $\mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\Lambda_\infty(\mu) \leq R$. The probability measures μ involved in (2.1.2) have a particular form, since they are the images of a measure m on \mathbb{T}^d by (I_d, α) , where α is a bounded measurable functions from \mathbb{T}^d to \mathbb{R}^d ; in particular they are supported on the graph of α . For $m \in \mathcal{P}(\mathbb{T}^d)$, we call $\mathcal{P}_m(\mathbb{T}^d \times \mathbb{R}^d)$ the set of such measures. For $\mu \in \mathcal{P}_m(\mathbb{T}^d \times \mathbb{R}^d)$, we set α^μ to be the unique element of $L^\infty(m)$ such that $\mu = (I_d, \alpha^\mu) \# m$. Here, $\Lambda_{\tilde{q}}(\mu)$ and $\Lambda_\infty(\mu)$ defined in (2.2.1) are given by

$$(2.2.3) \quad \begin{aligned} \Lambda_{\tilde{q}}(\mu) &= \|\alpha^\mu\|_{L^{\tilde{q}}(m)}, \\ \Lambda_\infty(\mu) &= \|\alpha^\mu\|_{L^\infty(m)}. \end{aligned}$$

If X is a normed space and $|\cdot|_X$ is its norm, for $n \geq 1$ we denote by $C^0(X; \mathbb{R}^n)$ the set of bounded continuous functions from X to \mathbb{R}^n ; it is endowed with the norm $\|v\|_\infty = \sup_{x \in X} |v(x)|_X$.

We define $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$ as the set of the functions $v \in C^0([0, T] \times \mathbb{T}^d; \mathbb{R})$ differentiable at any point with respect to the state variable, and such that its gradient satisfies $\nabla_x v \in C^0([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$. This is a Banach space equipped with the norm $\|v\|_{C^{0,1}} = \|v\|_\infty + \|\nabla_x v\|_\infty$.

For $\beta \in (0, 1)$ and $n \geq 1$, we denote by $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R}^n)$ the parabolic space of Hölder continuous functions which is commonly defined by

$$C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R}^n) = \left\{ v \in C^0([0, T] \times \mathbb{T}^d; \mathbb{R}^n), \exists C > 0 \text{ s.t. } \forall (t_1, x_1), (t_2, x_2) \in [0, T] \times \mathbb{T}^d, \right. \\ \left. |v(t_1, x_1) - v(t_2, x_2)| \leq C (|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{\beta}{2}} \right\}.$$

This is a Banach space equipped with the norm,

$$\|v\|_{C^{\frac{\beta}{2}, \beta}} = \|v\|_\infty + \sup_{(t_1, x_1) \neq (t_2, x_2)} \frac{|v(t_1, x_1) - v(t_2, x_2)|}{(|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{\beta}{2}}}.$$

The space $C^{\frac{1+\beta}{2}, 1+\beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ is defined as the set of the functions $v \in C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$ such that $\nabla_x v \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R}^n)$ and which admits a finite norm defined by,

$$\|v\|_{C^{\frac{1+\beta}{2}, 1+\beta}} = \|v\|_\infty + \|\nabla_x v\|_{C^{\frac{\beta}{2}, \beta}} + \sup_{(t_1, x) \neq (t_2, x) \in [0, T] \times \mathbb{T}^d} \frac{|v(t_1, x) - v(t_2, x)|}{|t_1 - t_2|^{\frac{1+\beta}{2}}}.$$

We set $C^{1,2}([0, T] \times \mathbb{T}^d; \mathbb{R})$ to be the set of functions which admit first derivative with respect to time and second derivatives with respect to the state variables, such that these derivatives are continuous with respect to time and state.

Throughout the paper, what we call a solution to (2.1.2) is precisely defined by the following definition.

Definition 2.2.1. *The triple (u, m, μ) is a solution to (2.1.2) if $u \in C^{1,2}([0, T] \times \mathbb{T}^d)$ is a pointwise solution to the Hamilton-Jacobi-Bellman equation (2.1.2a) with terminal condition (2.1.2d), $m \in C^0([0, T] \times \mathbb{T}^d; \mathbb{R})$ is solution to the Fokker-Planck-Kolmogorov equation (2.1.2b) in the sense of distribution with initial condition (2.1.2e), and $\mu \in C^0([0, T]; \mathcal{P}_\infty(\mathbb{T}^d \times \mathbb{R}^d))$ satisfies (2.1.2c) at any $t \in [0, T]$.*

A simple way to overcome difficulty 2) is to assume that the Hamiltonian H and some of its derivatives admit uniform bounds with respect to μ . In this case, the well-posedness of the MFGC system with a possibly degenerate diffusion is investigated in [38]. Here we avoid such an assumption for (2.1.2) but we introduce the following approximating system which satisfies it,

$$\begin{aligned}
 (2.2.4a) \quad & -\partial_t u^M - \nu \Delta u^M + H(x, \nabla_x u^M(t, x), \mu^M(t)) = 0 && \text{in } (0, T) \times \mathbb{T}^d, \\
 (2.2.4b) \quad & \partial_t m_t^M - \nu \Delta m^M - \operatorname{div}(H_p(x, \nabla_x u^M(t, x), \mu^M(t)) m^M) = 0 && \text{in } (0, T) \times \mathbb{T}^d, \\
 (2.2.4c) \quad & \mu^M(t) = [I_d, T_M(-H_p(\cdot, \nabla_x u^M(t, \cdot), \mu^M(t)))] \# m^M(t) && \text{in } [0, T], \\
 (2.2.4d) \quad & u^M(T, x) = g(x, m^M(T)) && \text{in } \mathbb{T}^d, \\
 (2.2.4e) \quad & m^M(0) = m_0,
 \end{aligned}$$

where M is a positive constant and T_M is a truncation map defined by

$$T_M(v) = \begin{cases} v & \text{if } |v| \leq M, \\ \frac{M}{|v|} v & \text{otherwise.} \end{cases}$$

The latter definition can be naturally extended to the case when $M = \infty$ by taking $T_\infty = Id_{\mathbb{R}^d}$. In this case systems (2.1.2) and (2.2.4) coincide. A solution to (2.2.4) is defined by replacing (2.1.2) by (2.2.4) in Definition 2.2.1.

If $M < \infty$ and (u^M, m^M, μ^M) is a solution to (2.2.4), $\mu^M(t)$ is compactly supported in $B_{\mathbb{R}^d}(0, M)$ the closed ball in \mathbb{R}^d centered at 0 with radius M , for any $t \in [0, T]$. Consequently, u^M and m^M should satisfy estimates depending on M and uniform with respect to μ^M . Therefore, compactness results for (2.2.4) should be less demanding than for (2.1.2) and difficulty 2) should vanish.

2.2.2 Assumptions

Let us start with some reasonable assumptions about the regularity and the boundedness of the Hamiltonian, the terminal cost and the initial distribution of agents. We introduce two constants: $C_0 > 0$ and $\beta_0 \in (0, 1)$.

- A1** $H = H(x, p, \mu)$ is convex with respect to p , and differentiable with respect to (x, p) ; H_p is locally β_0 -Hölder continuous with respect to p ; H and H_p are continuous with respect to μ on $\mathcal{P}_{\infty, R}(\mathbb{T}^d \times \mathbb{R}^d)$ for any $R > 0$, where $\mathcal{P}_{\infty, R}(\mathbb{T}^d \times \mathbb{R}^d)$ is defined in paragraph 2.2.1 and equipped with the weak* topology.

A2 $g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \rightarrow \mathbb{R}$ is continuous, and we suppose that $x \mapsto g(x, m)$ is in $C^{2+\beta_0}(\mathbb{T}^d)$, with a norm bounded uniformly with respect to m , i.e.

$$\|g(\cdot, m)\|_{C^{2+\beta_0}} \leq C_0, \quad \forall m \in \mathcal{P}(\mathbb{T}^d).$$

A3 $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{T}^d and we also name m_0 its density (abuse of notation). Assume that $m_0 \in C^{\beta_0}(\mathbb{T}^d)$ and is positive (see Remark 2.2.2 below to drop out the positivity assumption).

These assumptions are not restrictive when looking for solutions with the regularity given in Definition 2.2.1. However, they can be relaxed if we are interested in weaker solutions of systems (2.1.2).

In this paper we consider nonlocal coupling through the controls. More precisely, we assume that these interactions involve the quantity $\Lambda_{q_0}(\mu)$ defined in (2.2.1).

Let us introduce the assumptions used to address difficulty 1) which consists of solving the fixed point relations in μ given in (2.1.2c) and (2.2.4c), when (u, m) are fixed and have the same regularity as in Definition 2.2.1. We introduce $\lambda_0 \in [0, 1)$, for all $(x, p, m) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$, and $\mu, \mu^1, \mu^2 \in \mathcal{P}_m(\mathbb{T}^d \times \mathbb{R}^d)$, we assume that,

$$\mathbf{FP1} \quad |H_p(x, p, \mu)| \leq C_0(1 + |p|^{q-1}) + \lambda_0 \Lambda_{q_0}(\mu).$$

$$\mathbf{FP2} \quad |H_p(x, p, \mu^1) - H_p(x, p, \mu^2)| \leq \lambda_0 \left\| \alpha^{\mu^1} - \alpha^{\mu^2} \right\|_{L^{q_0}(m)}.$$

These structural assumptions for MFGC are new in the literature and participate to the originality and novelty of the results presented in this paper. Moreover they do not seem to be restrictive as it is explained in what follows.

We recall that the optimal control of a representative agent is given by $\alpha = -H_p(x, \nabla_x u, \mu)$. Since $\Lambda_{q_0}(\mu)$ is homogeneous to the norm of a control, we cannot expect the dependency of H_p upon μ to involve an exponent larger than one. Moreover if m is the first marginal of μ , taking the $L^{q_0}(m)$ -norm in **FP1** makes $\Lambda_{q_0}(\mu)$ appear in both sides of the resulting inequality; this explains the form of the right-hand side in **FP1** and the necessity of choosing λ_0 smaller than 1. Similar arguments can provide insights on **FP2**, by noticing that if Λ_{q_0} was seen as a norm on $\mathcal{P}_m(\mathbb{T}^d \times \mathbb{R}^d)$ then $\left\| \alpha^{\mu^1} - \alpha^{\mu^2} \right\|_{L^{q_0}(m)}$ would be the associated distance. We refer to Remark 4.3 in [2] for a concrete example of a MFGC system which does not admit solution if $\lambda_0 = 1$.

As in a large part of the literature on MFG or HJB equations, we consider Hamiltonians that are power-like functions in p at least asymptotically. Let $q \in (1, \infty)$ be this asymptotic exponent, and q' the conjugate exponent of q defined by $q' = \frac{q}{q-1}$. Namely, we assume that H satisfies the following inequalities, for all $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{T}^d)$, and $\mu \in \mathcal{P}_m(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\mathbf{B1} \quad |H(x, 0, \mu)| \leq C_0 + \lambda_2 \Lambda_{q_0}(\mu)^{q'}, \quad \text{with } \lambda_2 \geq 0.$$

$$\mathbf{B2} \quad |H_x(x, p, \mu)| \leq C_0 \left(1 + |p|^q + \Lambda_{q_0}(\mu)^{q'} \right).$$

$$\mathbf{B3} \quad H_p(x, p, \mu) \cdot p - H(x, p, \mu) \geq C_0^{-1} \left(|p|^q - \lambda_1 \Lambda_{q_0}(\mu)^{q'} \right) - C_0, \quad \text{where } \lambda_1 \text{ is a nonnegative constant satisfying } 0 \leq \lambda_1 < \frac{(1-\lambda_0)^{q'}}{C_0^{q'}}.$$

One may notice that the dependencies of H upon p and μ involve different exponents (which happen to be equal when $q = 2$). Indeed the Legendre transform applied to a power-like function make the exponent change into its conjugate. Since H is defined as the Legendre transform of the Lagrangian L , the exponent in the dependency of L upon α should be q' . Moreover, $\Lambda_{q_0}(\mu)$ is homogeneous to the norm of a control, therefore L should at most involve $\Lambda_{q_0}(\mu)^{q'}$. Going back to the Hamiltonian by the Legendre transform, the exponent on $\Lambda_{q_0}(\mu)$ stays the same which explains the right-hand side in **B1-B3**. One may find the above-mentioned growth conditions on L in [76].

Assumption **B3** is a convexity property of H with respect to p . In MFG without coupling through the controls, such an assumption is common, the only difference is that the term in $\Lambda_{q_0}(\mu)$ does not appear. This assumption will be particularly useful to obtain energy integral estimates by taking advantage of the duality properties of the forward-backward systems (2.1.2a), (2.1.2b) and (2.2.4a), (2.2.4b). The inequality satisfied by λ_1 is needed in the calculation for getting these estimates. Let us mention that the right-hand side in this inequality comes from the estimates in Lemma 2.3.1, and that the constant C_0 can be identified with the one in **FP1**.

In order to obtain classical solutions of the HJB equations (2.1.2a) and (2.2.4a), we need Hölder continuity of $(t, x) \mapsto H(x, \nabla_x u(t, x), \mu(t))$. While the space regularity of the latter map is straightforward here, its time regularity may be more demanding and we need assumptions which allow one to compare H at different measures $\mu^1, \mu^2 \in \mathcal{P}_\infty(\mathbb{T}^d \times \mathbb{R}^d)$. Assumption **FP2** is not enough since it requires μ^1 and μ^2 to share the same marginal with respect to \mathbb{T}^d .

T For $R > 0$, there exists a constant $C_R > 0$ such that

$$\begin{aligned} |H(x, p, \mu^1) - H(x, p, \mu^2)| &\leq C_R \left(\|m^1 - m^2\|_\infty^{\beta_0} + \|\alpha^{\mu^1} - \alpha^{\mu^2}\|_\infty \right), \\ |H_p(x, p, \mu^1) - H_p(x, p, \mu^2)| &\leq C_R \left(\|m^1 - m^2\|_\infty^{\beta_0} + \|\alpha^{\mu^1} - \alpha^{\mu^2}\|_\infty \right), \end{aligned}$$

for (x, p, m^i, μ^i) such that $(x, p) \in \mathbb{T}^d \times \mathbb{R}^d$ with $|p| \leq R$, $m^i \in \mathcal{P}(\mathbb{T}^d) \cap C^0(\mathbb{T}^d)$ with $m^i \geq R^{-1}$, $\mu^i \in \mathcal{P}_{m^i}(\mathbb{T}^d \times \mathbb{R}^d)$ with $\alpha^{\mu^i} \in C^0(\mathbb{T}^d \times \mathbb{R}^d)$ and $\|\alpha^{\mu^i}\|_\infty \leq R$, $i = 1, 2$.

One may notice that when μ^1 and μ^2 have the same first marginal with respect to \mathbb{T}^d the second inequality in **T** is implied by **FP2**. If one is only interested in weak solution to (2.1.2), **T** can be removed.

Remark 2.2.2. Letting C_R depends on $\| (m^i)^{-1} \|_\infty$ was motivated by models of population dynamics which are discussed in paragraphs 2.6.3 and 2.6.4. The drawback of this assumption is that we have to assume that the initial distribution of agents m_0 is positive.

All the results in this paper hold if we do not assume m_0 to be positive in **A3**, and we remove the condition $m^i \geq R^{-1}$ in **T**.

2.2.3 Main results

We recall that assumptions **FP1** and **FP2** are designed to address difficulty **1**), and **T** to obtain time regularity of the fixed point μ in 2.1.2c or 2.2.4c. More precisely, we state the following lemma that will be proved in Section 2.3.

Lemma 2.2.3. Assume **A1**, **FP1**, **FP2** and **T**. Take $p \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$ and $m \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d)$ such that $m \geq R^{-1}$ and $m(t) \in \mathcal{P}(\mathbb{T}^d)$ for $t \in [0, T]$, where $\beta \in (0, 1)$

and $R > 0$ are constants. For any $t \in [0, T]$, there exists a unique $\mu(t) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfying

$$\mu^M(t) = [I_d, T_M(-H_p(\cdot, p(t, \cdot), \mu^M(t)))] \# m(t),$$

where $M \in (0, \infty]$. Moreover, the map $(t, x) \mapsto \alpha^{\mu^M(t)}(x)$ is in $C^{\frac{\beta\beta_0}{2}, \beta\beta_0}([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$, and its associated norm can be estimated from above by a constant which depends on $\|p\|_{C^{\frac{\beta}{2}, \beta}}$, $\|m\|_{C^{\frac{\beta}{2}, \beta}}$, and the constants in the assumptions.

In Section 2.4, we prove the a priori estimates stated in the following lemma.

Lemma 2.2.4. *Assume **A1-A3**, **B2**, **B3**, **FP1**, **FP2** and **T**. If (u, m, μ) is a solution to (2.2.4) for $M \in (0, \infty]$, then*

- $\|\nabla_x u\|_\infty \leq C(1 + \|u\|_\infty)$ and $\|u\|_\infty \leq C(1 + \|\nabla_x u\|_\infty^q)$, where C is independent of M and depends only on the constants in the assumptions,
- m is positive,
- $u \in C^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{T}^d)$,
- $m \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$,
- $(t, x) \mapsto \alpha^{\mu(t)}(x)$ is in $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$,

where $\beta \in (0, \beta_0^2)$. Moreover, $\|m^{-1}\|_\infty$ and the norms associated with the last three items above depend only on $\|u\|_\infty$, $\|m_0^{-1}\|_\infty$, β and the constants in the assumptions.

These estimates are weaker than their equivalents for MFG systems without interaction through controls. In particular, u is not uniformly bounded in $\|\cdot\|_\infty$ -norm. However, we believe that our estimate of $\|\nabla_x u\|_\infty$ is the best that we can achieve in our framework since its right-hand side should be at least linear with respect to $\|u\|_\infty$. To our knowledge, such an estimate for systems of MFG with nonlocal dependency on $\nabla_x u$ (or more generally for MFG systems in which we do not have a uniform a priori estimate on u) is new in the literature.

Here, these a priori estimates are not sufficient to address the difficulty **2)** and to obtain existence of solutions. However, existence can be obtained under several different kinds of assumptions; below, we supply a list of existence results under various assumptions:

Theorem 2.2.5. *Assume **A1-A3**, **B1-B3**, **FP1**, **FP2**, **T**. There exists a solution to (2.1.2) if one of the following assertions is satisfied*

- a) $q_0 \leq q'$ and $|H(x, 0, \mu)| \leq C_0 \left(1 + \Lambda_{q_0}(\mu)^{\tilde{q}}\right)$, where \tilde{q} is a constant satisfying $\tilde{q} < q'$ (Proposition 2.5.4),
- b) $q_0 \leq q'$ and $\lambda_1 + C_0 \lambda_2 < \frac{(1-\lambda_0)^{q'}}{C_0^{q'}}$ where λ_1 and λ_2 are respectively defined in **B3** and **B1**, the C_0 on the left-hand side comes from C_0^{-1} in **B3** and the C_0 on the right-hand side comes from **FP1** (Proposition 2.5.3),
- c) $|H(x, 0, \mu)| \leq C_0 \left(1 + \Lambda_{q_0}(\mu)^{q'-1}\right)$, for any $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (Proposition 2.5.5),
- d) $|H_x(x, p, \mu)| \leq C_0 \left(1 + |p| + \Lambda_{q_0}(\mu)^{q'-1}\right)$, for any $(x, p, \mu) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (Proposition 2.5.6),

e) $T \leq T_0$, where T_0 is a constant depending on the constants in the assumptions (Proposition 2.5.8).

An other additional assumption under which existence holds is the monotonicity condition addressed in [76].

We also give a uniqueness result under a short time horizon assumption.

Theorem 2.2.6 (*Uniqueness with short time horizon*). Assume **A1**, **A2**, **A3**, **B1**, **B2**, **B3**, **FP1**, **FP2**, and that the following three assumptions are satisfied,

- H_p is locally Lipschitz continuous with respect to p ,
- g satisfies

$$(2.2.5) \quad \|g(\cdot, m^1) - g(\cdot, m^2)\|_{C^{1+\beta}} \leq C_0 W_{q_1}(m^1, m^2),$$

for any $m^1, m^2 \in \mathcal{P}(\mathbb{T}^d)$, where $q_1 \in [1, \infty)$ and W_{q_1} is the q_1 -Wassertein distance on measures,

- the two inequalities in **T** hold when we replace $\|m^1 - m^2\|_\infty^{\beta_0}$ by $W_{q_1}(m^1, m^2)$.

There exists $T_1 > 0$ such that if $T < T_1$ then there is at most one solution to (2.1.2).

We believe that this uniqueness result can be easily extended to more general Hamiltonians, but that the short-time assumption is essential. Indeed numerical examples in which non-uniqueness occurs are presented in [2]. In these examples, we consider groups of agents who start from some crowded areas at time $t = 0$, and travel through the domain to arrive at some target areas. Imposing a short time assumption in such an example results in the agents not trying to reach the targets at all. Indeed in this case the kinetic cost makes it more expensive for them to cross the domain very quickly before the end of the game than to do nothing and just wait passively at their starting point. For this reason we were not interested in finding less restrictive assumptions in Theorem 2.2.6. This theorem should be only seen as an example of uniqueness result with a short time horizon assumption. In particular we wanted the proof in paragraph 2.5.4 to stay simple.

Remark 2.2.7. i) In this work, we only consider MFGC systems in the d -dimensional torus \mathbb{T}^d . However, we believe that our existence results (Theorem 2.2.5) hold under the same assumptions on the Euclidean space \mathbb{R}^d , and that the method introduced in [76] to pass from \mathbb{T}^d to \mathbb{R}^d can be applied here.

ii) We did not include the case $q = 1$ in this work (i.e. when the Hamiltonian is Lipschitz continuous in p). In this case, systems (2.1.2) and (2.2.4) coincide when M is large enough, therefore there exists a solution to (2.1.2) under assumptions **A1-A3**, **B1-B3**, **FP1**, **FP2** and **T**, by the same arguments as in Lemma 2.5.1.

2.3 The fixed point relation in μ and the proof of Lemma 2.2.3

We recall that (2.1.2) and (2.2.4) coincide when $M = \infty$. Here, we take $M \in (0, \infty]$.

The following lemma takes advantage of the structural assumptions **FP1** and **FP2** to solve the fixed point relations (2.1.2c) and (2.2.4c) which consists of difficulty **1**). It also states a priori estimates on μ which will be of great use in the next section to obtain a priori estimates on u and its derivatives.

Lemma 2.3.1. *Assume **A1**, **FP1** and **FP2**. Take $p \in C^0(\mathbb{T}^d; \mathbb{R}^d)$, and $m \in \mathcal{P}(\mathbb{T}^d)$. The following two assertions are satisfied.*

(i) *There exists a unique $\mu^M \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ such that*

$$(2.3.1) \quad \mu^M = [I_d, T_M(-H_p(\cdot, p(\cdot), \mu^M))] \# m.$$

For any $\tilde{q} \in [1, \infty]$, it satisfies

$$(2.3.2) \quad \Lambda_{\tilde{q}}(\mu^M) \leq \frac{C_0}{1 - \lambda_0} \left(1 + \left\| |p|^{q-1} \right\|_{L^{\max(q_0, \tilde{q})}(m)} \right).$$

(ii) *The map $(p, m) \mapsto \mu^M$ given by (2.3.1), is continuous from $C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$ to $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. We recall that the spaces of measures are equipped with the weak-* topology.*

Proof. (i) Let us define the following map,

$$\Phi_{(p,m)}^M : \begin{array}{l} C^0(\mathbb{T}^d; \mathbb{R}^d) \rightarrow C^0(\mathbb{T}^d; \mathbb{R}^d) \\ \alpha \mapsto \begin{cases} \mathbb{T}^d \rightarrow \mathbb{R}^d \\ x \mapsto T_M(-H_p(x, p(x), (I_d, \alpha) \# m)). \end{cases} \end{array}$$

This map is well defined by **(A1)**. It is λ_0 -Lipschitz continuous by **FP2** and the fact that T_M is 1-Lipschitz continuous, we recall that $\lambda_0 < 1$. Therefore it admits a unique fixed point by the Banach fixed point theorem. If $\mu^M \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfies (2.3.1) then α^{μ^M} is the only fixed point of $\Phi_{(p,m)}^M$. Conversely, if we denote by α the fixed point of $\Phi_{(p,m)}^M$, then μ^M defined by $\mu^M = (I_d, \alpha) \# m$ satisfies (2.3.1). This implies that (2.3.1) admits a unique fixed point that we name μ^M in what follows. From **FP1**, μ^M satisfies,

$$\begin{aligned} \Lambda_{\tilde{q}}(\mu^M) &= \left\| \alpha^{\mu^M} \right\|_{L^{\tilde{q}}(m)} \\ &\leq \left\| C_0 \left(1 + |p|^{q-1} \right) + \lambda_0 \Lambda_{q_0}(\mu^M) \right\|_{L^{\tilde{q}}(m)} \\ &\leq C_0 \left(1 + \left\| |p|^{q-1} \right\|_{L^{\tilde{q}}(m)} \right) + \lambda_0 \Lambda_{\tilde{q}}(\mu^M), \end{aligned}$$

for $\tilde{q} \geq q_0$, where we obtained the last line by using the triangle inequality for the $L^{\tilde{q}}$ -norm, and (2.2.2). This implies (2.3.2) for any $\tilde{q} \geq q_0$. Then we extend this result to $1 \leq \tilde{q} < q_0$ by combining (2.2.2) and (2.3.2) applied to q_0 .

(ii) Let $(p^n, m^n)_{n \in \mathbb{N}} \in (C^0(\mathbb{T}^d \times \mathbb{R}^d); \mathcal{P}(\mathbb{T}^d))^{\mathbb{N}}$ be a convergent sequence to (p, m) in $C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$. We define μ^N as before, and $(\mu^{N,n})_{n \in \mathbb{N}}$ the fixed points satisfying

$$(2.3.3) \quad \mu^{N,n} = [I_d, T_M(-H_p(\cdot, p^n(\cdot), \mu^{N,n}))] \# m^n$$

for $n \in \mathbb{N}$. The sequence $(p^n)_{n \in \mathbb{N}}$ is bounded in $C^0(\mathbb{T}^d; \mathbb{R}^d)$, thus (2.3.2) with $\tilde{q} = \infty$ yields that $(\mu^{N,n})_{n \in \mathbb{N}}$ are uniformly compactly supported. The sequence $(\mu^{N,n})$ is compact in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ endowed with the weak-* topology. Let $\tilde{\mu}$ be the limit of a subsequence $(\mu^{N, \varphi(n)})_{n \in \mathbb{N}}$, for $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ an increasing function. By continuity

of H_p and T_M , we can pass to the limit in (2.3.3) taken at $\varphi(n)$ when n tends to infinity, this gives that $\tilde{\mu}$ satisfies the same fixed point relation as μ . By uniqueness of this fixed point, we deduce that $\tilde{\mu} = \mu$. This implies that the entire sequence $(\mu^{N,n})$ tends to μ .

Therefore the map $(p, m) \mapsto \mu^M$ is continuous from $C^0(\mathbb{T}^d; \mathbb{R}^d) \times \mathcal{P}(\mathbb{T}^d)$ to $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. \square

In particular if $q_0 \leq q'$, (2.2.2) and (2.3.2) yield

$$(2.3.4) \quad \Lambda_{q_0}(\mu) \leq \Lambda_{q'}(\mu) \leq \frac{C_0}{1 - \lambda_0} \left(1 + \|p\|_{L^q(m)}^{q-1} \right),$$

and then we use the inequality $(a + b)^{q'} \leq \frac{a^{q'}}{\theta^{q'-1}} + \frac{b^{q'}}{(1-\theta)^{q'-1}}$ which holds for $a, b > 0$ and for any $\theta \in (0, 1)$, to obtain

$$(2.3.5) \quad \Lambda_{q_0}(\mu)^{q'} \leq \frac{C_0^{q'}}{(1 - \lambda_0)^{q'}} \left(\theta^{1-q'} + (1 - \theta)^{1-q'} \|p\|_{L^q(m)}^q \right).$$

If $q \in [1, \infty]$ without restriction, we obtain

$$(2.3.6) \quad \Lambda_{q_0}(\mu)^{q'} \leq \frac{C_0^{q'}}{(1 - \lambda_0)^{q'}} \left(\theta^{1-q'} + (1 - \theta)^{1-q'} \|p\|_{\infty}^q \right).$$

These latter three inequalities will be of great use in Section 2.4 for getting a priori estimates.

Given (u, m) as regular as in definition 2.2.1, we can use Lemma 2.3.1 to prove that the fixed point relations (2.1.2c) and (2.2.4c) are well-posed, and that if (u, m, μ) is a solution to (2.1.2) or (2.2.4) then μ is continuous with respect to time. However, we need a better regularity in time to get classical solution of the HJB equations (2.1.2a) and (2.2.4a). In Lemma 2.3.2, we use **T** to obtain an estimate of the distance between two fixed points of (2.3.1) associated with different (u, m) . We will be particularly interested in using this estimate on a solution to (2.2.4) at different times.

Lemma 2.3.2. *Assume **A1**, **FP1**, **FP2** and **T**. Take $p^1, p^2 \in C^0(\mathbb{T}^d; \mathbb{R}^d)$, and $m^1, m^2 \in \mathcal{P}(\mathbb{T}^d) \cap C^0(\mathbb{T}^d; \mathbb{R})$ some positive probability measures. We define $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ as the fixed point in (i) in Lemma 2.3.1 associated with (p^1, m^1) and (p^2, m^2) , respectively. There exists a constant C such that*

$$(2.3.7) \quad \left\| \alpha^{\mu^1} - \alpha^{\mu^2} \right\|_{\infty} \leq C \left(\|p^1 - p^2\|_{\infty}^{\beta_0} + \|m^1 - m^2\|_{\infty}^{\beta_0} \right),$$

where C depends on $\|p^i\|_{\infty}$, $\|(m^i)^{-1}\|_{\infty}$, for $i = 1, 2$, and the constants in the assumptions.

Proof. We define $\tilde{\mu}$ by $\tilde{\mu} = (I_d, \alpha^{\mu^1}) \# m^2$. The triangle inequality and the fact that T_M is a contraction imply that for any $x \in \mathbb{T}^d$,

$$\begin{aligned} \left| \alpha^{\mu^1}(x) - \alpha^{\mu^2}(x) \right| &\leq \left| H_p(x, p^1(x), \mu^1) - H_p(x, p^2(x), \mu^2) \right| \\ &\leq \left| H_p(x, p^1(x), \mu^1) - H_p(x, p^1(x), \tilde{\mu}) \right| \\ &\quad + \left| H_p(x, p^1(x), \tilde{\mu}) - H_p(x, p^1(x), \mu^2) \right| \\ &\quad + \left| H_p(x, p^1(x), \mu^2) - H_p(x, p^2(x), \mu^2) \right|. \end{aligned}$$

The measures μ^1 and $\tilde{\mu}$ are the image measures by the same function (I_d, α^{μ^1}) , of m^1 and m^2 respectively. From **T**, we obtain

$$|H_p(x, p^1(x), \mu^1) - H_p(x, p^1(x), \tilde{\mu})| \leq C_R \|m^1 - m^2\|_\infty^{\beta_0},$$

where $R = \max\left(\|p^i\|_\infty, \|(m^i)^{-1}\|_\infty\right)$ and C_R is the constant defined in **T**. We recall that $\Lambda_\infty(\mu^i)$ can be estimated from above by a quantity which only depends on $\|p^i\|_\infty$ and the constants in the assumptions, by (2.3.2).

Since $\tilde{\mu}$ and μ^2 have the same marginal with respect to \mathbb{T}^d , **FP2** yields that,

$$|H_p(x, p^1(x), \tilde{\mu}) - H_p(x, p^1(x), \mu^2)| \leq \lambda_0 \|\alpha^{\mu^1} - \alpha^{\mu^2}\|_\infty.$$

Then H_p is locally β_0 -Hölder continuous by **A1** so,

$$|H_p(x, p^1(x), \mu^2) - H_p(x, p^2(x), \mu^2)| \leq C \|p^1 - p^2\|_\infty^{\beta_0},$$

for some constant C . Combining the latter four inequalities, we obtain,

$$\|\alpha^{\mu^1} - \alpha^{\mu^2}\|_\infty \leq C_1 \left(\|p^1 - p^2\|_\infty^{\beta_0} + \|m^1 - m^2\|_\infty^{\beta_0} \right) + \lambda_0 \|\alpha^{\mu^1} - \alpha^{\mu^2}\|_\infty,$$

which implies (2.3.7) up to replacing C with $(1 - \lambda_0)^{-1} \max(C, C_R)$. \square

Lemma 2.2.3 is a straightforward consequence of Lemmas 2.3.1 and 2.3.2.

2.4 A priori estimates and the proof of Lemma 2.2.4

Here we take $M \in (0, \infty]$, and (u, m, μ) a solution to (2.2.4) defined in Definition 2.2.1. We will look for estimates independent of M which allow us to address difficulty **2**). These a priori estimates imply compactness results and play an essential role in the proofs of existence in Section 2.5.

2.4.1 A priori estimates on u

When we consider MFG without interactions through controls and with bounded coupling function and terminal cost, we can apply the maximum principle on parabolic differential equations to (2.4.3) below and get an a priori estimates of $\|u\|_\infty$ which only depends on the constants in the assumptions. However, for MFGC systems and more generally for HJB equations with non-local interactions in $\nabla_x u$, it is not possible to get such a strong a priori estimate directly from the maximum principle. Instead we get (2.4.1) and (2.4.2) which involve non-local quantities depending on $\nabla_x u$.

Lemma 2.4.1. *Under assumptions **A1**, **A2**, **B1**, **FP1**, **FP2**, and $q_0 \leq q'$, for $\theta \in (0, 1)$ u satisfies,*

(2.4.1)

$$\|u\|_\infty \leq C_0(1+T) + \frac{\lambda_2 C_0^{q'}}{(1-\lambda_0)^{q'}} \left(\theta^{1-q'} T + (1-\theta)^{1-q'} \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^q dm(t, x) dt \right),$$

where λ_2 is defined in **B1**. More generally, for any $q_0 \in [1, \infty]$ u satisfies,

$$(2.4.2) \quad \|u\|_\infty \leq C_0(1+T) + \frac{\lambda_2 C_0^{q'}}{(1-\lambda_0)^{q'}} \left(\theta^{1-q'} T + (1-\theta)^{1-q'} \|\nabla_x u\|_\infty^q \right).$$

Proof. Here, we can rewrite (2.2.4a) in the following way,

$$(2.4.3) \quad -\partial_t u(t, x) - \nu \Delta u(t, x) + \left[\int_0^1 H_p(x, s \nabla_x u, \mu(t)) ds \right] \cdot \nabla_x u(t, x) = -H(x, 0, \mu(t)),$$

for $(t, x) \in (0, T) \times \mathbb{T}^d$. The maximum principle for parabolic second-order equation applies to u and $-u$,

$$(2.4.4) \quad \|u\|_\infty \leq \|u(T, \cdot)\|_\infty + \int_0^T \|H(\cdot, 0, \mu(t))\|_\infty dt.$$

Moreover, $|H(x, 0, \mu(t))| \leq C_0 + \lambda_2 \Lambda_{q_0}(\mu(t))^{q'}$ and $|u(T, x)| \leq C_0$ come from **B1** and **A2**, respectively. We combine the latter inequalities with (2.3.5) and (2.3.6) to get (2.4.1) when $q_0 \leq q'$, and (2.4.2) respectively. \square

The non-local term in (2.4.1) involving $\nabla_x u$ corresponds roughly speaking to an energy. Moreover this is a quantity that naturally appears in MFG literature thanks to duality properties in the forward-backward systems (2.1.1), (2.1.2), or (2.2.4). More precisely, the FPK equations is the dual equation of the linearized HJB equation with respect to u . Lemma 2.4.2 provides an a priori estimate of this quantity.

Lemma 2.4.2. *Under assumptions **A1**, **A2**, **B3**, **FP1**, **FP2**, and $q_0 \leq q'$, the following inequality is satisfied,*

$$(2.4.5) \quad \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^q dm(t, x) dt \leq \left(1 - \frac{\lambda_1 C_0^{q'}}{(1-\theta)^{q'-1} (1-\lambda_0)^{q'}} \right)^{-1} \left(C_0 \|u\|_\infty + C_0^2 (1+T) + \frac{\lambda_1 C_0^{q'} T}{\theta^{q'-1} (1-\lambda_0)^{q'}} \right),$$

for any $\theta \in (0, 1)$ such that $\lambda_1 < \frac{(1-\theta)^{q'-1} (1-\lambda_0)^{q'}}{C_0^{q'}}$.

Proof. We multiply (2.2.4a) by $-m$ and (2.2.4b) by u ; we add up and integrate over $(0, T) \times \mathbb{T}^d$ the resulting quantities; after performing some integrations by part, we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} [H_p(x, \nabla_x u(t, x), \mu(t)) \cdot \nabla_x u - H(x, \nabla_x u(t, x), \mu(t))] dm(t, x) dt \\ = \int_{\mathbb{T}^d} u(0, x) dm^0(x) - \int_{\mathbb{T}^d} g(x, m(T)) dm(T, x), \end{aligned}$$

that we can combine with **B3** and **A2** to get,

$$C_0^{-1} \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^q dm(t, x) dt \leq \|u\|_\infty + C_0(1+T) + C_0^{-1} \lambda_1 \int_0^T \Lambda_{q_0}(\mu(t))^{q'} dt.$$

We integrate (2.3.5) over $(0, T)$,

$$\int_0^T \Lambda_{q_0}(\mu(t))^{q'} dt \leq \frac{C_0^{q'}}{(1-\lambda_0)^{q'}} \left(\theta^{1-q'} T + (1-\theta)^{1-q'} \int_0^T \int_{\mathbb{T}^d} |\nabla_x u|^q dm(t, x) dt \right),$$

where we can choose $\theta \in (0, 1)$ such that $\lambda_1 < \frac{(1-\theta)^{q'-1} (1-\lambda_0)^{q'}}{C_0^{q'}}$, since λ_1 satisfies the inequality in **B3**. The latter three inequalities imply (2.4.5). \square

Roughly speaking, Lemma 2.4.1 with $q_0 \leq q'$ and Lemma 2.4.2 provide opposite inequalities which may become complementary under a smallness condition on the parameters, implying a uniform estimate on $\|u\|_\infty$. This condition is explicitly given in the following corollary.

Corollary 2.4.3. *Under Assumptions **A1**, **A2**, **B1**, **B3**, **FP1**, **FP2**, $q_0 \leq q'$, and $\lambda_1 + C_0\lambda_2 < \frac{(1-\lambda_0)^{q'}}{C_0^{q'}}$, u is bounded by a quantity which only depends on the constants in the assumptions.*

Proof. Combing (2.4.1) and (2.4.5) results in,

$$\begin{aligned} \|u\|_\infty &\leq \frac{\lambda_2 C_0^{q'}}{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}} C_0 \left(1 - \frac{\lambda_1 C_0^{q'}}{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}}\right)^{-1} \|u\|_\infty + C_\theta \\ &\leq C_0 \lambda_2 \left(\frac{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}}{C_0^{q'}} - \lambda_1\right)^{-1} \|u\|_\infty + C_\theta \end{aligned}$$

where $\theta \in (0, 1)$ may be chosen such that $\lambda_1 + C_0\lambda_2 < \frac{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}}{C_0^{q'}}$, and C_θ is a positive constant depending on the constants in the assumptions and θ . This implies

$$\|u\|_\infty \leq \left(1 - C_0 \lambda_2 \left(\frac{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}}{C_0^{q'}} - \lambda_1\right)^{-1}\right) C_\theta,$$

where $C_0 \lambda_2 \left(\frac{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}}{C_0^{q'}} - \lambda_1\right)^{-1} < 1$, which concludes the proof. \square

Let us mention that in the assumption $\lambda_1 + C_0\lambda_2 < \frac{(1-\lambda_0)^{q'}}{C_0^{q'}}$ in Corollary 2.4.3, the constant C_0 in the left-hand side comes from the C_0^{-1} in **B3**, and the C_0 in the right-hand side comes from **FP1**.

2.4.2 A priori estimates on m

In order for the HJB equations (2.1.2a) and (2.2.4a) to admit classical solutions, we want μ to be regular in time. Since m is the marginal of μ with respect to \mathbb{T}^d , we first prove that m is regular in the following lemma. Moreover, we also prove that m stays positive, which is required in Lemma 2.2.3 to obtain time regularity on μ .

Lemma 2.4.4. *Under assumptions **A1**, **A3**, **FP1**, **FP2**, m is in $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ for $\beta \in (0, \beta_0)$ and its $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ -norm can be estimated from above by a constant which depends on $\|m_0\|_{C^{\beta_0}}$, $\|\nabla_x u\|_\infty$, β and the constants in the assumptions.*

Furthermore, m is positive everywhere and admits a positive lower bound which only depends on $\|m_0^{-1}\|_\infty$, $\|\nabla_x u\|_\infty$ and the constants in the assumptions.

Proof. The distribution of agents m satisfies the second-order parabolic FPK equation (2.2.4b), which is supplemented with a β_0 -Hölder continuous initial condition. Theorem 2.1 section V.2 in [81] states that m is uniformly bounded by a constant which depends on $\|m_0\|_\infty$ and $\|H_p(\cdot, \nabla_x u, \mu)\|_\infty$. This, (**FP1**) and (2.3.4) yield that $m H_p(\cdot, \nabla_x u, \mu, m)$ is bounded by a constant which depends on $\|m_0\|_\infty$, $\|\nabla_x u\|_\infty$ and the constant of the assumptions. Finally, Theorem 6.29 in [86] yields that $m \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d)$ for $\beta \in$

$(0, \beta_0)$, and its associated norm can be estimated from above by a constant which depends on $\|m_0\|_{C^{\beta_0}}$, $\|\nabla_x u\|_\infty$, β and the constants in the assumptions.

We define $T_\varepsilon = \inf(\{t \in [0, T], \|m(t)^{-1}\|_\infty \geq \varepsilon^{-1}\} \cup \{T\})$, for $0 < \varepsilon < \|m_0^{-1}\|_\infty$. In particular T_ε is positive, since we proved in the latter paragraph that m is continuous. On $[0, T_\varepsilon] \times \mathbb{T}^d$ we define the function n by $n = m^{-1}$, it satisfies the following partial differential equation in the sense of viscosity,

$$\partial_t n - \nu \Delta n - \operatorname{div}(\alpha n) + 2\alpha \cdot \nabla_x n = -2\nu \frac{|\nabla_x n|^2}{n},$$

supplemented with the initial condition $n(0) = m_0^{-1}$, where $\alpha(t, x) = -H_p(x, \nabla_x u(t, x), \mu(t))$ for $(t, x) \in [0, T] \times \mathbb{T}^d$. We define \tilde{n} as the unique weak solution of the following partial differential equation defined on $[0, T] \times \mathbb{T}^d$,

$$(2.4.6) \quad \partial_t \tilde{n} - \nu \Delta \tilde{n} - \operatorname{div}(\alpha \tilde{n}) + 2\alpha \cdot \nabla_x \tilde{n} = 0,$$

supplemented with the initial condition $\tilde{n}(0) = m_0^{-1}$. Theorem 2.1 section V.2 in [81] states that \tilde{n} is bounded from above by a constant which depends on $\|m_0^{-1}\|_\infty$, $\|\alpha\|_\infty$ and T . Moreover, n is a subsolution of the restriction of (2.4.6) to $[0, T_\varepsilon] \times \mathbb{T}^d$, with the same initial condition as \tilde{n} . Therefore, by a comparison argument for second-order parabolic equations in divergence form (Theorem 9.7 in [86] for instance), n and \tilde{n} satisfy $n \leq \tilde{n}$. This implies that there exists C a positive constant independent of T_ε , such that $\|n\|_\infty \leq C$. We conclude the proof by taking $\varepsilon = 2^{-1}C^{-1}$ and recalling that $\|\alpha\|_\infty$ can be estimated from above using FP1 and (2.3.4). \square

2.4.3 A priori estimates on derivatives of u

Bernstein methods are useful tools when studying HJB equations or MFG systems. They allow one to obtain a priori estimates on $\nabla_x u$ by considering the partial differential equations satisfied by some well-chosen functions depending on u and $\nabla_x u$. See for example the video of the lecture of P.L. Lions on November the 23rd 2018 [87], in which Bernstein estimates are derived for MFG systems without interactions through controls. More precisely, P.L. Lions used the function defined by $|\nabla_x u|^2 e^{-\eta u}$, for small η . Here this method might work only if we knew a uniform estimates on $\|u\|_\infty$ and if $q = 2$. After significant changes in the latter method, we can derive an estimate on u which is weaker than the one for MFG without interactions through controls. Namely, we state that $\|\nabla_x u\|_\infty$ is bounded by a quantity that depends linearly on $\|u\|_\infty$ by studying the functions w and φ defined in (2.4.11) below. To our knowledge, such estimates for systems of MFG with nonlocal dependency on $\nabla_x u$ (or more generally for MFG systems in which we do not have a uniform a priori estimate on u) are new in the literature. We believe that this result may hold for more general HJB equations with nonlocal dependency on $\nabla_x u$.

Lemma 2.4.5. *Under assumptions A1, A2, B2, B3, FP1 and FP2, there exists $C > 0$ depending only on the constants of the assumptions, such that*

$$(2.4.7) \quad \|\nabla_x u(t)\|_\infty \leq C \left(1 + \max_{t \leq s \leq T} \|u\|_\infty \right),$$

for any $t \in [0, T]$.

Proof. In what follows, we only prove that (2.4.7) holds for $t = 0$, however the proof does not use additional information available at $t = 0$ (the initial condition on m for example), so it can be repeated for any $t \in [0, T]$ and the constant C in (2.4.7) does not depend on t .

Here we wish to differentiate (2.2.4a) with respect to x ; however we did not assume in Definition 2.2.1 enough regularity on u for such an operation to have sense pointwisely on $(0, T) \times \mathbb{T}^d$. Especially the time derivative of $\nabla_x u$ and the third derivatives of u with respect to x are not required to exist. This leads us to introducing $\rho \in C^\infty([-\frac{1}{2}, \frac{1}{2}]^d)$ a non-negative mollifier such that $\rho(x) = 0$ if $|x| \geq \frac{1}{4}$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. We introduce $\rho^\delta = \delta^{-d} \rho(\frac{\cdot}{\delta})$ and $u^\delta(t) = \rho^\delta \star u(t)$, for any $0 < \delta < 1$ and $t \in [0, T]$, where \star denotes the convolution operator.

Thus u^δ depends smoothly on the state variable and its partial derivatives in space at any order have the same regularity in time as u , moreover it solves the following partial differential equation with final condition,

$$(2.4.8) \quad \begin{cases} -\partial_t u^\delta(t, x) - \nu \Delta u^\delta(t, x) + \rho^\delta \star (H(\cdot, \nabla_x u^\delta(t, \cdot), \mu(t))) (x) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ u^\delta(T, x) = \rho^\delta \star (g(\cdot, m(T, \cdot))) (x) & \text{in } \mathbb{T}^d, \end{cases}$$

Let us take the gradient with respect to the state variable of the latter equation and the scalar product of the resulting equality with $\nabla_x u^\delta$,

$$(2.4.9) \quad -\frac{1}{2} \partial_t \left| \nabla_x u^\delta \right|^2 - \nu \nabla_x u^\delta \cdot \Delta \left(\nabla_x u^\delta \right) + \nabla_x u^\delta \cdot D_{x,x}^2 u^\delta H_p \left(x, \nabla_x u^\delta, \mu \right) + \nabla_x u^\delta \cdot H_x^\delta \left(x, \nabla_x u, \mu \right) = \nabla_x u^\delta \cdot R^\delta(t, x),$$

where H^δ and R^δ are defined by

$$\begin{aligned} H^\delta(x, p, \mu) &= \rho^\delta \star (H(\cdot, p(\cdot), \mu)) (x), \\ R^\delta(t, x) &= D_{x,x}^2 u^\delta H_p \left(x, \nabla_x u^\delta, \mu \right) - \rho^\delta \star (D_{x,x}^2 u H_p(\cdot, \nabla_x u, \mu)). \end{aligned}$$

By simple calculus, we notice that

$$\begin{aligned} \nabla_x \left| \nabla_x u^\delta \right|^2 &= 2 D_{x,x}^2 u^\delta \nabla_x u^\delta, \\ \Delta \left| \nabla_x u^\delta \right|^2 &= 2 \nabla_x u^\delta \cdot \Delta \left(\nabla_x u^\delta \right) + 2 \left| D_{x,x}^2 u^\delta \right|^2, \end{aligned}$$

that we can combine with (2.4.9) and obtain

$$(2.4.10) \quad -\frac{1}{2} \partial_t \left| \nabla_x u^\delta \right|^2 - \frac{\nu}{2} \Delta \left| \nabla_x u^\delta \right|^2 + \nu \left| D_{x,x}^2 u^\delta \right|^2 + \frac{1}{2} \nabla_x \left| \nabla_x u^\delta \right|^2 \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) = -\nabla_x u^\delta \cdot H_x^\delta \left(x, \nabla_x u, \mu \right) + \nabla_x u^\delta \cdot R^\delta(t, x).$$

We define the functions φ and w^δ by

$$(2.4.11) \quad \begin{aligned} \varphi(v) &= \exp \left(\exp \left(- \left(a + b \|u\|_\infty^{-1} v \right) \right) \right), \text{ for } |v| \leq \|u\|_\infty, \\ w^\delta(t, x) &= \varphi(u^\delta(T-t, x)) \left| \nabla_x u^\delta \right|^2(T-t, x), \end{aligned}$$

where $a > 1$ and $b > 0$ are constants that will be defined below. The derivatives of φ are given by

$$(2.4.12) \quad \begin{aligned} \varphi'(v) &= -b \|u\|_\infty^{-1} e^{-(a+b\|u\|_\infty^{-1}v)} \varphi(v), \\ \varphi''(v) &= b^2 \|u\|_\infty^{-2} e^{-(a+b\|u\|_\infty^{-1}v)} \left(1 + e^{-(a+b\|u\|_\infty^{-1}v)} \right) \varphi(v), \end{aligned}$$

which implies that φ and φ' satisfy,

$$(2.4.13) \quad \begin{aligned} 1 &\leq \varphi(v) \leq e^{e^{-a+b}}, \\ b\|u\|_\infty^{-1}e^{-a-b} &\leq \frac{|\varphi'(v)|}{\varphi(v)} \leq b\|u\|_\infty^{-1}e^{-a+b}. \end{aligned}$$

Roughly speaking, we introduced a and b in order to have $\|\varphi\|_\infty\|\varphi^{-1}\|_\infty$ and $\|\varphi'\|_\infty\|(\varphi')^{-1}\|_\infty$ as close as possible to 1. This will be achieved by taking a large enough, and b small enough.

For simplicity of the notations, we will omit to write the argument of φ since it is always u^δ .

The derivatives of w^δ verify the following equalities,

$$\begin{aligned} -\varphi\partial_t\left|\nabla_x u^\delta\right|^2 &= \partial_t w^\delta + \frac{\varphi'}{\varphi}w^\delta\partial_t u^\delta, \\ \varphi\nabla_x\left|\nabla_x u^\delta\right|^2 &= \nabla_x w^\delta - \frac{\varphi'}{\varphi}w^\delta\nabla_x u^\delta, \\ \varphi\Delta\left|\nabla_x u^\delta\right|^2 &= \Delta w^\delta - \frac{\varphi'}{\varphi}w^\delta\Delta u^\delta - 2\frac{\varphi'}{\varphi}\nabla_x w^\delta \cdot \nabla_x u^\delta - \frac{\varphi''\varphi - 2(\varphi')^2}{\varphi^3}(w^\delta)^2. \end{aligned}$$

We multiply (2.4.10) by 2φ and use the latter equalities in the resulting relation,

$$(2.4.14) \quad \begin{aligned} \partial_t w^\delta - \nu\Delta w^\delta + \nabla_x w^\delta \cdot H_p(x, \nabla_x u^\delta, \mu) + 2\nu\frac{\varphi'}{\varphi}\nabla_x w^\delta \cdot \nabla_x u^\delta + 2\nu\varphi\left|D_{x,x}^2 u^\delta\right|^2 \\ = \frac{\varphi'}{\varphi}w^\delta\left[-\partial_t u^\delta - \nu\Delta u^\delta + \nabla_x u^\delta \cdot H_p(x, \nabla_x u^\delta, \mu)\right] - \nu\frac{\varphi''\varphi - 2(\varphi')^2}{\varphi^3}(w^\delta)^2 \\ - 2\varphi\nabla_x u^\delta \cdot H_x^\delta(x, \nabla_x u, \mu) + 2\varphi\nabla_x u^\delta \cdot R^\delta(t, x). \end{aligned}$$

We can rewrite the first line of (2.4.8) in the following way,

$$-\partial_t u^\delta - \nu\Delta u^\delta = -H(x, \nabla_x u^\delta, \mu) - Q^\delta,$$

where Q^δ is defined by,

$$Q^\delta(t, x) = H^\delta(x, \nabla_x u(t), \mu(t)) - H(x, \nabla_x u^\delta(t, x), \mu(t)).$$

This and (2.4.14) imply that

$$(2.4.15) \quad \begin{aligned} \partial_t w^\delta - \nu\Delta w^\delta + \nabla_x w^\delta \cdot H_p(x, \nabla_x u^\delta, \mu) + 2\nu\frac{\varphi'}{\varphi}\nabla_x w^\delta \cdot \nabla_x u^\delta + 2\nu\varphi\left|D_{x,x}^2 u^\delta\right|^2 \\ = \frac{\varphi'}{\varphi}w^\delta\left[\nabla_x u^\delta \cdot H_p(x, \nabla_x u^\delta, \mu) - H(x, \nabla_x u^\delta, \mu)\right] - \nu\frac{\varphi''\varphi - 2(\varphi')^2}{\varphi^3}(w^\delta)^2 \\ - 2\varphi\nabla_x u^\delta \cdot H_x^\delta(x, \nabla_x u, \mu) + 2\varphi\nabla_x u^\delta \cdot R^\delta(t, x) - \varphi'\left|\nabla_x u^\delta\right|^2 Q^\delta(t, x). \end{aligned}$$

In the following we will estimate from above the right-hand side of the latter expression. We notice that the second term of the right-hand side is negative since

$$(2.4.16) \quad \varphi''\varphi - 2(\varphi')^2 \geq 0.$$

We notice that R^δ and Q^δ are uniformly convergent to 0 as δ tends to 0, so we can assume that,

$$(2.4.17) \quad \left\|R^\delta\right\|_\infty + \left\|Q^\delta\right\|_\infty \leq \frac{\varepsilon}{2e\|\nabla_x u\|_\infty + \|\varphi'\|_\infty\|\nabla_x u\|_\infty^2},$$

for δ small enough and depending on $\varepsilon > 0$.

The first term in the last line of (2.4.15) can be bounded using **B2**,

$$(2.4.18) \quad -2\varphi \nabla_x u^\delta \cdot H_x^\delta(x, \nabla_x u, \mu) \leq 2C_0 \varphi \left\| \nabla_x u^\delta \right\|_\infty \left(1 + \|\nabla_x u\|_\infty^q + \Lambda_{q_0}(\mu(t))^{q'} \right).$$

In fact we are going to use the latter inequality to obtain (2.4.19) below, only by noticing that using (2.3.6), the right-hand side involves only terms with exponents in $\|w^\delta\|_\infty$ or $\|w^0\|_\infty$ not larger than $\frac{1+q}{2}$.

Then we use **B3** on the first term of the right-hand side of (2.4.15) since $\varphi' < 0$,

$$\begin{aligned} \frac{\varphi'}{\varphi} w^\delta \left[\nabla_x u^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) - H \left(x, \nabla_x u^\delta, \mu \right) \right] &\leq -C_0^{-1} \frac{|\varphi'|}{\varphi^{1+\frac{q}{2}}} \left(w^\delta \right)^{1+\frac{q}{2}} \\ &\quad + C_0 \frac{|\varphi'|}{\varphi} w^\delta + C_0^{-1} \lambda_1 \frac{|\varphi'|}{\varphi} w^\delta \Lambda_{q_0}(\mu(t))^{q'}. \end{aligned}$$

The term involving $-(w^\delta)^{1+\frac{q}{2}}$ is a key element in this proof. On the one hand, it will allow us to cancel the term in $w^\delta \Lambda_{q_0}(\mu(t))^{q'}$. On the other hand, we will use the fact that it has a larger exponent than any of the remaining terms.

From (2.3.6) and (2.4.13), we obtain

$$\frac{|\varphi'|}{\varphi} \Lambda_{q_0}(\mu(t))^{q'} \leq b \|u\|_\infty^{-1} e^{-a+b} \frac{C_0^{q'}}{(1-\lambda_0)^{q'}} \left(\theta^{1-q'} + (1-\theta)^{1-q'} \|w^0\|_\infty^{\frac{q}{2}} \right),$$

where $\theta \in (0, 1)$ will be defined below. Then (2.4.13) implies,

$$\frac{|\varphi'|}{\varphi^{1+\frac{q}{2}}} \geq b \|u\|_\infty^{-1} e^{-a-b} e^{-\frac{q}{2}e^{-a+b}}$$

Combining the latter six inequalities, (2.4.15), and the fact that $\|w^\delta\|_\infty \leq \|\varphi\|_\infty \|\varphi^{-1}\|_\infty \|w^0\|_\infty \leq e^{e^{-a}(e^b - e^{-b})} \|w^0\|_\infty$, we obtain the following partial differential inequality,

$$(2.4.19) \quad \begin{aligned} \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) + 2\nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta \\ \leq -C_0^{-1} b \|u\|_\infty^{-1} e^{-a-b} e^{-\frac{q}{2}e^{-a+b}} \left(w^\delta \right)^{1+\frac{q}{2}} + b \|u\|_\infty^{-1} e^{-a+b} \frac{\lambda_1 C_0^{q'-1} e^{e^{-a}(e^b - e^{-b})}}{(1-\theta)^{1-q'}(1-\lambda_0)^{q'}} \|w^0\|_\infty^{1+\frac{q}{2}} \\ \quad + \varepsilon + C_{a,b,\theta} \left(1 + \|u\|_\infty^{-1} \right) \left(1 + \|w^0\|_\infty^{\frac{1+q}{2}} \right) \end{aligned}$$

where $C_{a,b,\theta}$ is a positive constant which only depends on the constants in the assumptions and in (a, b, θ) . We systematically used the inequality $\|w^0\|_\infty^r \leq 1 + \|w^0\|_\infty^{\frac{1+q}{2}}$ on every term of the form $\|w^0\|_\infty^r$ with $0 < r < \frac{1+q}{2}$.

Let us mention the following result: the function y^+ defined by $y^+ = \max \left(y_0, K^{-\frac{1}{k}} \|f\|_\infty^{\frac{1}{k}} \right)$ is a super-solution of the following differential equation,

$$\begin{cases} y'(t) = -Ky(t)^k + f(t) \\ y(0) = y_0 \end{cases}$$

posed on $[0, T]$, where k and y_0 are positive constants and f is a bounded positive function.

This and $\|w(0)\|_\infty \leq eC_0^2$ which comes from **A2** and (2.4.13), yield that a super-solution to (2.4.19) is given by

$$\left[\frac{\lambda_1 C_0^{q'} e^{2b} e^{\frac{q}{2} e^{-a+b}} e^{e^{-a}(e^b - e^{-b})}}{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}} \|w^0\|_\infty^{1+\frac{q}{2}} + C_{a,b,\theta} (1 + \|u\|_\infty) \left(1 + \|w^0\|_\infty^{\frac{1+q}{2}} + \varepsilon \right) \right]^{\frac{2}{2+q}},$$

where we replace $C_{a,b,\theta}$ with $C_{a,b,\theta} + (eC_0^2)^{1+\frac{q}{2}}$.

From a comparison argument for parabolic second-order equation, w^δ is not larger than the latter expression. This result holds for w^0 by letting δ and ε tend to 0, thus w^0 verifies the following inequality,

$$\|w^0\|_\infty^{1+\frac{q}{2}} \leq \frac{\lambda_1 C_0^{q'} e^{2b} e^{\frac{q}{2} e^{-a+b}} e^{e^{-a}(e^b - e^{-b})}}{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}} \|w^0\|_\infty^{1+\frac{q}{2}} + C_{a,b,\theta} (1 + \|u\|_\infty) \left(1 + \|w^0\|_\infty^{\frac{1+q}{2}} \right).$$

By **B3**, we can choose $a > 1$ large enough, $b > 0$ and $\theta \in (0, 1)$ small enough such that $\frac{\lambda_1 C_0^{q'} e^{2b} e^{\frac{q}{2} e^{-a+b}} e^{e^{-a}(e^b - e^{-b})}}{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}} < 1$. This implies

$$(2.4.20) \quad \|w^0\|_\infty^{1+\frac{q}{2}} \leq C_{a,b,\theta} (1 + \|u\|_\infty) \left(1 + \|w^0\|_\infty^{\frac{1+q}{2}} \right),$$

where we increased $C_{a,b,\theta}$ into $\left(1 - \frac{\lambda_1 C_0^{q'+1} e^{2b} e^{\frac{q}{2} e^{-a+b}} e^{e^{-a}(e^b - e^{-b})}}{(1-\theta)^{q'-1}(1-\lambda_0)^{q'}} \right)^{-1} C_{a,b,\theta}$.

We make out two cases: the first case is when $\|w^0\|_\infty^{\frac{1}{2}} \leq 2C_{a,b,\theta} (1 + \|u\|_\infty)$. The second case is when $\|w^0\|_\infty^{\frac{1}{2}} > 2C_{a,b,\theta} (1 + \|u\|_\infty)$. In the latter case, (2.4.20) implies that $\|w^0\|_\infty^{1+\frac{q}{2}} \leq \frac{1}{2} \|w^0\|_\infty^{\frac{1}{2}} \left(1 + \|w^0\|_\infty^{\frac{1+q}{2}} \right)$, which implies that $\|w^0\|_\infty \leq 1$. Therefore, in any of the two latter cases we obtain

$$\|w^0\|_\infty^{\frac{1}{2}} \leq 1 + 2C_{a,b,\theta} (1 + \|u\|_\infty).$$

This and (2.4.13) yield (2.4.7) when $t = 0$, this concludes the proof. \square

Now, we can combine the estimates obtained in this section with classical results on parabolic second-order equations and get further estimates of u and its derivatives and on m .

Lemma 2.4.6. *Assume **A1**, **A2**, **B2**, **B3**, **FP1**, **FP2** and **T**. The function u is in $C^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{T}^d)$ for any $\beta \in (0, \beta_0^2)$, where β_0 was introduced in the assumptions. Its $C^{1+\frac{\beta}{2}, 2+\beta}$ -norm can be bounded by a quantity depending only on $\|u\|_\infty$, β , and the constants in the assumptions.*

Proof. Lemma 2.4.5 states that $\|\nabla_x u\|_\infty$ is bounded by a quantity which depends on $\|u\|_\infty$ and the constants in the assumptions. So is $\Lambda_{q_0}(\mu)$ by (2.3.2). Then u is the solution of the heat equation with a right-hand side equal to $-H(x, \nabla_x u, \mu)$ which is bounded in L^∞ . Classical results (see for example Theorem 6.48 in [86]) state that for any $\beta \in (0, 1)$, the $C^{\frac{1}{2}+\frac{\beta}{2}, 1+\beta}$ -norm of u is bounded by a constant which depends on the L^∞ -norm of the right-hand side, the terminal condition, and β .

Lemma 2.4.4 yields that m is in $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d)$ for $\beta \in (0, \beta_0)$, is positive, and that both its $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}^d)$ -norm and its lower bound depend on $\|u\|_\infty$, $\|m_0^{-1}\|_\infty$, β , and the constant of the assumptions.

Therefore, Lemma 2.2.3 yields that $[(t, x) \mapsto \alpha^{\mu(t)}(x)] \in C^{\frac{\beta\beta_0}{2}, \beta\beta_0}([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$.

From **A1**, H is locally Lipschitz continuous with respect to (x, p) . This and **T** imply that $[(t, x) \mapsto H(t, \nabla_x u(t, x), \mu(t))] \in C^{\frac{\beta\beta_0}{2}, \beta\beta_0}([0, T] \times \mathbb{T}^d)$. Thus u is the solution of the backward heat equation with a right-hand side in $C^{\frac{\beta\beta_0}{2}, \beta\beta_0}$ supplemented with terminal condition in $C^{2+\beta_0}$. Classical results (see for instance Theorem 4.9 in [86]) yield that u is in $C^{1+\frac{\beta\beta_0}{2}, 2+\beta\beta_0}$, and its $C^{1+\frac{\beta\beta_0}{2}, 2+\beta\beta_0}$ -norm depends on $\|g(\cdot, m(T))\|_{C^{2+\beta_0}}$ and the $C^{\frac{\beta\beta_0}{2}, \beta\beta_0}$ -norm of the right-hand side. We recall that β is any constant in $(0, \beta_0)$. The proof of the lemma is complete.

Following precisely the dependencies in the above estimates, we obtain that the $C^{1+\frac{\beta\beta_0}{2}, 2+\beta\beta_0}$ norm of u can be estimated from above by a constant which depends on $\|u\|_\infty$, $\|m_0^{-1}\|_\infty$, β , and the constants in the assumptions. \square

The conclusions of Lemmas 2.4.1, 2.4.4, 2.4.5 and 2.4.6 are summarized in Lemma 2.2.4.

2.5 Existence and uniqueness results under additional assumptions

2.5.1 Solving the MFGC systems for $M < \infty$

Lemma 2.5.1. *Under assumptions **A1-A3**, **B2**, **B3**, **FP1**, **FP2**, **T** and $M \in (0, \infty)$, there exists at least one solution to (2.2.4).*

Proof. For $(u, m) \in C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$, we define $\mu^M \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ by

$$\mu^M(t) = [I_d, T_M(-H_p(\cdot, \nabla_x u(t, \cdot), \mu^M(t)))] \# m(t) \text{ in } [0, T],$$

using Lemma 2.2.3. Then we define u^M as the viscosity solution of the following backward HJB equation with a final condition,

$$(2.5.1) \quad \begin{cases} -\partial_t u^M(t, x) - \nu \Delta u^M(t, x) + H(x, \nabla_x u^M(t, x), \mu^M(t)) = 0 \\ u^M(T, x) = g(x, m(t)). \end{cases}$$

We can rewrite the first line of the latter system in the following way,

$$-\partial_t u^M - \nu \Delta u^M + \nabla_x u^M \cdot \int_0^1 H_p(x, s \nabla_x u^M, \mu^M(t)) ds = -H(x, 0, \mu^M(t)),$$

where the right-hand side is bounded using $\Lambda_\infty(\mu(t)) \leq M$, **B1** and (2.2.2). The maximum principle for second-order parabolic equation provides that u^M is bounded. Here, the proof of Lemma 2.4.5 can be repeated to prove that $\|\nabla_x u\|_\infty$ is bounded by a constant which depends on M and the constants in the assumptions. Then with the same argument as in Lemma 2.4.6, u^M is bounded in $C^{\frac{1}{2}+\frac{\beta}{2}, 1+\beta}$ -norm, for all $\beta \in (0, 1)$.

We define m^M as the solution in the sense of distributions of the following Fokker-Planck-Kolmogorov equation with an initial condition,

$$\begin{cases} \partial_t m_t^M(t, x) - \nu \Delta m^M(t, x) + \operatorname{div}(b(t, x)m^M) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ m^M(0) = m_0, \end{cases}$$

with $b(t, x) = -H_p(x, \nabla_x u^M(t, x), \mu^M(t))$ which is a continuous function with respect to (t, x) . Using the same arguments as in Lemma 2.4.4, we get that $m \in C^{\frac{\beta}{2}, \beta}(\mathbb{T}^d; \mathbb{R})$ for $\beta \in (0, \beta_0)$.

Moreover, $\|u\|_{C^{\frac{1}{2} + \frac{\beta}{2}, 1 + \beta}}$, $\|m\|_{C^{\frac{\beta}{2}, \beta}}$ are bounded by a constant which depends on M , β and the constants in the assumptions. The map $(u, m) \mapsto \mu^M$ is continuous from $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ to $C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ by Lemma 2.3.1. The map $(m, \mu^M) \mapsto u^M$ is continuous from $C^0([0, T]; \mathcal{P}(\mathbb{T}^d)) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ to $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R})$ by the stability of the solutions of viscosity. The map $(u^M, \mu^M) \mapsto m^M$ is continuous from $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ to $C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ by linearity of the FPK equation.

Thus the map $(u, m) \mapsto (u^M, m^M)$ is continuous from $C^{0,1}([0, T] \times \mathbb{T}^d; \mathbb{R}) \times C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ to itself. Its fixed points are exactly the solutions to (2.2.4). The image of this map is a subset of a convex compact set. Therefore, there exists a fixed point by Schauder theorem, see [56] Corollary 11.2.

Using the same arguments as in the proof of Lemma 2.4.6, such a fixed point u satisfies $u \in C^{1 + \frac{\beta}{2}, 2 + \beta}([0, T] \times \mathbb{T}^d; \mathbb{R})$ for any $\beta \in (0, \beta_0^2)$. \square

Considering $M < \infty$ in (2.2.4) consists of enforcing the condition $\Lambda_\infty(\mu(t)) \leq M$, i.e. the fact that the support of $\mu(t)$ is embedded in the compact set $\mathbb{T}^d \times B_{\mathbb{R}^d}(0, M)$, for $t \in [0, T]$. Therefore, the interactions through controls are uniformly bounded. Lemma 2.5.1 relies on that to state the existence of solutions to (2.2.4). For $M = \infty$, we can not obtain such a uniform estimate by combining only the results of Section 2.4. However if such an estimate exists, the result of Lemma 2.5.1 holds for $M = \infty$ and yields the existence of solutions to (2.1.2). More precisely, if a solution to (2.2.4) satisfies $\Lambda_\infty(\mu(t)) < M$ for any $t \in [0, T]$, then it is also a solution to (2.1.2). This is summarized in the following Corollary.

Corollary 2.5.2. *Under the same assumptions as in Lemma 2.5.1, if, for any $M > 0$, any solution (u, m, μ) to (2.2.4) satisfies $\|u\|_\infty \leq C$, or $\|\nabla_x u\|_\infty \leq C$, for some $C > 0$, then there exists at least one solution to (2.1.2).*

Proof. By Lemma 2.5.1, we define (u, m, μ) as a solution to (2.2.4) for $M \in (0, \infty)$ that will be defined later. By Lemma 2.2.4, assuming that $\|u\|_\infty$ is bounded is equivalent to assuming that $\|\nabla_x u\|_\infty$ is bounded. Therefore, without loss of generality, we can assume that $\|\nabla_x u\|_\infty \leq C$. From FP1 and (2.3.4), we obtain

$$\|H_p(x, \nabla_x u, \mu)\|_\infty \leq C_0(1 + C^{q-1}) + \frac{\lambda_0 C_0}{1 - \lambda_0}(1 + C^{q-1}).$$

We define $M = 1 + C_0(1 + C^{q-1}) + \frac{\lambda_0 C_0}{1 - \lambda_0}(1 + C^{q-1})$, then the truncation T_M leaves $-H_p(\cdot, \nabla_x u^M, \mu)$ unchanged. Hence (u, m, μ) is a solution to (2.1.2). \square

2.5.2 Existence results when $q_0 \leq q'$

When $q_0 \leq q'$, we can use integral energy estimates. More precisely, inequalities (2.4.1) and (2.4.5) hold. Therefore, the assumptions under which we can prove existence should be weaker than in the case $q_0 > q'$ in which we have less estimates at our disposal.

In particular, Corollary 2.4.3 provides a uniform estimate on $\|u\|_\infty$ under suitable assumptions. Corollary 2.5.2 then yields the existence of a solution to (2.1.2): hence we may state the following theorem:

Proposition 2.5.3 (*Existence of solution with small non-linearities*). Under assumptions **A1-A3**, **B1-B3**, **FP1**, **FP2**, **T**, $q_0 \leq q'$, and $\lambda_1 + C_0\lambda_2 < \frac{(1-\lambda_0)^{q'}}{C_0^{q'}}$, there exists at least one solution to (2.1.2).

Instead of assuming that the multiplicative parameters are small like in Proposition 2.5.3; we suppose in Propositions 2.5.5 below the exponent for the interactions through controls is in fact smaller than the one appearing in **B1**.

Proposition 2.5.4. Assume **A1-A3**, **B2**, **B3**, **FP1**, **FP2**, **T**, $q_0 \leq q'$, and that H satisfies

$$|H(x, 0, \mu)| \leq C_0 \left(1 + \Lambda_{q_0}(\mu)^{\tilde{q}}\right),$$

for $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, where $\tilde{q} \in [0, q')$ is a constant. There exists a solution to (2.1.2).

Proof. Let (u, m, μ) be a solution to (2.2.4) for $M \in (0, \infty)$. From **A2**, (2.4.4) and the new assumption, we obtain that,

$$\begin{aligned} \|u\|_\infty &\leq C_0(1+T) + C_0 \int_0^T \Lambda_{q_0}(\mu(t))^{\tilde{q}} dt \\ &\leq C_0(1+T) + C_0 T^{\frac{q'-\tilde{q}}{q'}} \left(\int_0^T \Lambda_{q_0}(\mu(t))^{q'} dt \right)^{\frac{\tilde{q}}{q'}}, \end{aligned}$$

where the second line is obtained by a Hölder inequality, since $\tilde{q} < q'$. Let us recall that the inequality $(a+b)^{\frac{\tilde{q}}{q'}} \leq a^{\frac{\tilde{q}}{q'}} + b^{\frac{\tilde{q}}{q'}}$ holds for any $a, b > 0$. The latter two inequalities and (2.3.5) with $\theta = \frac{1}{2}$ imply,

$$\|u\|_\infty \leq C + C \left(\int_0^T \int_{\mathbb{T}^d} |\nabla_x u(t, x)|^q dm(t, x) dt \right)^{\frac{\tilde{q}}{q'}},$$

where $C > 0$ is a constant which depends on the constants in the assumptions. This and (2.4.5) yield that,

$$\|u\|_\infty \leq C + C \|u\|_\infty^{\frac{\tilde{q}}{q'}},$$

up changing the value of C . Let us make out two cases: the first case is when $\|u\|_\infty \leq (2C)^{\frac{q'}{q'-\tilde{q}}}$. The second case is when $\|u\|_\infty > (2C)^{\frac{q'}{q'-\tilde{q}}}$, which implies $\|u\|_\infty \leq C + \frac{1}{2}\|u\|_\infty$. In any of the two cases, u is uniformly bounded with respect to M . The desired result then stems from Corollary 2.5.2. \square

2.5.3 Existence results which do not need the assumption $q_0 < q'$

Here, we do not make the assumption $q_0 \leq q'$. We can still obtain an existence result in the same spirit as the one provided in Proposition 2.5.4. In the following proposition, the exponent for the interactions through controls is assumed to be smaller than the one appearing in **B1** or in Proposition 2.5.4.

Proposition 2.5.5. Assume **A1-A3**, **B2**, **B3**, **FP1**, **FP2**, **T**, and that H satisfies

$$(2.5.2) \quad |H(x, 0, \mu)| \leq C_0 \left(1 + \Lambda_{q_0}(\mu)^{q'-1}\right)$$

for any $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. There exists a solution to (2.1.2).

Proof. Take (u, m, μ) a solution to (2.2.4) for $M \in (0, \infty)$. Let us combine (2.4.3), (2.3.6) for $\theta = \frac{1}{2}$, (2.4.7), (2.5.2), and the inequality $(a + b)^{\frac{1}{q}} \leq a^{\frac{1}{q}} + b^{\frac{1}{q}}$ which holds for $a, b > 0$; this yields

$$(2.5.3) \quad -\partial_t u - \nu \Delta u + \nabla_x u \cdot \int_0^1 H_p(x, s \nabla_x u(t, x), \mu(t)) ds \leq C \left(1 + \max_{t \leq s \leq T} \|u(s)\|_\infty \right),$$

for a constant $C > 0$ which depends only on the constants in the assumptions. We recall that $\|u(T)\|_\infty \leq C_0$ by **A2**. We consider $y_+, y_- \in C^1([0, T]; \mathbb{R})$ defined as $y_+(t) = Ct + C_0 e^{Ct}$ and $y_-(t) = -Ct - C_0 e^{Ct}$ such that they are solution to the following differential equations

$$\begin{cases} y'_+(t) = C(1 + y_+(t)) \\ y_+(0) = C_0, \\ y'_-(t) = C(-1 + y_-(t)) \\ y_-(0) = -C_0. \end{cases}$$

By a comparison argument for second-order parabolic equation we obtain,

$$-CT - C_0 e^{CT} \leq y_-(T - t) \leq u(t, x) \leq y_+(t) \leq CT + C_0 e^{CT},$$

for $(t, x) \in [0, T] \times \mathbb{T}^d$. Therefore u is uniformly bounded with respect to M . The desired result then stems from Corollary 2.5.2. \square

In Propositions 2.5.4 and 2.5.5, we changed the exponent appearing in **B1**. In the following proposition, we assume a smaller exponent than the one appearing in **B2** instead.

Proposition 2.5.6 (*Existence with more restrictive assumptions on H_x*). Assume **A1-A3**, **B1**, **B3**, **FP1**, **FP2**, **T**, and the following inequality,

$$(2.5.4) \quad |H_x(x, p, \mu)| \leq C_0 \left(1 + |p| + \Lambda_{q_0}(\mu)^{q'-1} \right),$$

for any $(x, p, \mu) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. There exists at least one solution to (2.1.2).

Proof. Take (u, m, μ) a solution to (2.2.4), for $M \in (0, \infty)$.

First step: we prove the following inequality,

$$(2.5.5) \quad \|\nabla_x u(t)\|_\infty^q \leq C \left(1 + \sup_{t \leq s \leq T} \|u(s)\|_\infty \right),$$

for any $t \in [0, T]$, where $C > 0$ is a constant depending only on the constants in the assumptions. We will only prove this inequality for $t = 0$, however the proof does not use the additional information available at $t = 0$ (the initial condition on m for example), so it can be repeated for any $t \in [0, T]$ and the constant C in (2.5.5) does not depend on t .

We introduce $\varphi, w^\delta, a, b, \delta$ and ε as in the proof of Lemma 2.4.5. Using (2.5.4) instead of **B2**, we obtain

$$-2\varphi \nabla_x u^\delta \cdot H_x^\delta(x, \nabla_x u, \mu) \leq 2C_0 \varphi \left\| \nabla_x u^\delta \right\|_\infty \left(1 + \left\| \nabla_x u^\delta \right\|_\infty + \Lambda_{q_0}(\mu(t))^{q'-1} \right).$$

From this and (2.3.2), one may notice that the right-hand side of the latter inequality only involves terms with exponents in $\|w^\delta\|_\infty$ or $\|w^0\|_\infty$ nor larger than $\frac{1}{2}(1 + (q-1)(q'-1)) =$

1. This and the same arguments as in the proof of Lemma 2.4.5 between (2.4.18) and (2.4.19), lead to the following inequality,

$$\begin{aligned} & \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p \left(x, \nabla_x u^\delta, \mu \right) - 2\nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta \\ & \leq -C_0^{-1} b \|u\|_\infty^{-1} e^{-a-b} e^{-\frac{q}{2} e^{-a+b}} \left(w^\delta \right)^{1+\frac{q}{2}} + b \|u\|_\infty^{-1} e^{-a+b} \frac{\lambda_1 C_0^{q'} e^{e^{-a}(e^b - e^{-b})}}{(1-\theta)^{1-q'} (1-\lambda_0)^{q'}} \|w^0\|_\infty^{1+\frac{q}{2}} \\ & \quad + \varepsilon + C_{a,b,\theta} \left(1 + \|u\|_\infty^{-1} \right) \left(1 + \|w^0\|_\infty \right), \end{aligned}$$

instead of (2.4.19), where the novelty is the exponent on $\|w^0\|_\infty$ at the last line which changed from $\frac{1+q}{2}$ to 1. Then following the same steps as in the proof of Lemma 2.4.5 until the end, we obtain that,

$$\|\nabla_x u\|_\infty^q \leq C_{a,b,\theta} (1 + \|u\|_\infty).$$

This concludes the first step of the proof.

Second step: obtaining a uniform estimate on u .

Using B1, (2.3.6) with $\theta = \frac{1}{2}$ and (2.5.5), we obtain that,

$$\begin{aligned} |H(x, 0, \mu(t))| & \leq C_0 \left(1 + \Lambda_{q_0} (\mu(t))^{q'} \right) \\ & \leq 2C_0 + \frac{C_0^{q'+1} 2^{q'-1}}{(1-\lambda)^{q'}} (1 + \|\nabla_x u(t)\|_\infty^q) \\ & \leq C \left(1 + \max_{t \leq s \leq T} \|u(s)\|_\infty \right), \end{aligned}$$

where the constant C from the previous step may have been increased. This implies that u satisfies the same partial differential inequality as in the proof of Proposition 2.5.5, namely (2.5.3). Therefore the same arguments as in Proposition 2.5.5 apply and we conclude that there exists a solution to (2.1.2). \square

Remark 2.5.7. Note that the exponent $q' - 1$ actually appears in several applications: for instance, the price impact model described in paragraph 2.6.2 in the quadratic case (i.e. $q = 2$) with $\varepsilon = 0$ (i.e. when the bidding and asking prices are equal), satisfies the assumptions in both Propositions 2.5.5 and 2.5.6 with an exponent exactly equal to $q' - 1$.

2.5.4 Existence and uniqueness results with a short-time horizon assumption

Under a short-time horizon assumption, existence and even uniqueness of solutions are well-known in the MFG literature. Indeed, when the time horizon is small, one may obtain strong a priori estimates under non-restrictive assumptions. These estimates combined with Corollary 2.5.2 yield existence of solution to (2.1.2) as stated in the following proposition.

Proposition 2.5.8 (*Existence with short time horizon*). Assume A1, A2-B1, B2-FP1, FP2, and T. There exists $T_0 > 0$ such that, if $T \leq T_0$ then there exists a solution to (2.1.2).

Proof. Take (u, m, μ) a solution to (2.2.4) for $M \in (0, \infty)$. We combine (2.4.3), FP1, (2.3.6), (2.4.7), and the convex inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$, and we obtain

$$(2.5.6) \quad -\partial_t u - \nu \Delta u + \nabla_x u \cdot \int_0^1 H_p(x, s \nabla_x u(t, x), \mu(t)) ds \leq C \left(1 + \max_{t \leq s \leq T} \|u(s)\|_\infty^q \right),$$

where C is a positive constant which depends only on the constants in the assumptions. We recall that $\|u(T)\|_\infty \leq C_0$ by **A2**. Let us consider the following differential equation,

$$\begin{cases} y'(t) = C(1 + y^q) \\ y(0) = C_0. \end{cases}$$

There exists $T_0 > 0$ such that the latter differential equation admits a bounded solution on $[0, T_0]$. We suppose that $T \leq T_0$, then $(t, x) \mapsto y(T - t)$ is a super-solution to (2.5.6). Hence by a comparison principle, we get that $u \leq y$. The same argument applies in order to prove that $u \geq -y$. Therefore u is uniformly bounded with respect to M , and there exists a solution to (2.1.2) by Corollary 2.5.2. \square

We will now prove Theorem 2.2.6 which states that uniqueness is achieved under a short-time horizon assumption. We believe that this uniqueness result can be easily extended to more general Hamiltonians, but that the short-time assumption is essential. Indeed, numerical simulations in [2] show that uniqueness does not hold for the discrete MFGC system obtained by approximating (2.1.2) with finite differences; we believe that uniqueness does not hold for (2.1.2) either. Theorem 2.2.6 should be interpreted only as a simple example of uniqueness result with a short-time horizon assumption.

Proof of Theorem 2.2.6. We suppose that $T_1 \leq T_0$, where T_0 was defined in Proposition 2.5.8, so that a solution to (2.1.2) satisfies uniform estimates on $\|u\|_\infty$, $\|u^i\|_{C^{1,2}}$ and $\|m^i\|_{C^0}$ by Lemma 2.4.6, for $i = 1, 2$. Take (u^1, m^1, μ^1) and (u^2, m^2, μ^2) two solutions to (2.1.2). We define $u = u^1 - u^2$, $m = m^1 - m^2$ and $\alpha = \alpha^{\mu^1} - \alpha^{\mu^2}$.

In this proof $C > 0$ is a constant which may differ from line to line and depends only on the constants in the assumptions, $\|u^i\|_{C^{1,2}}$ and $\|m^i\|_{C^0}$, for $i = 1, 2$.

We can repeat the proof of Lemma 2.3.2 replacing $\|m^1 - m^2\|_\infty^{\beta_0}$ and $\|p^1 - p^2\|_\infty^{\beta_0}$ respectively with $W_{q_1}(m^1, m^2)$ and $\|p^1 - p^2\|_\infty^{\beta_0}$ everywhere and we obtain that,

$$(2.5.7) \quad \|\alpha(t)\|_\infty \leq C (\|\nabla_x u(t)\|_\infty + W_{q_1}(m^1(t), m^2(t))),$$

for any $t \in [0, T]$. Let us consider X^1 and X^2 two random processes defined by

$$\begin{aligned} dX_t^1 &= \alpha^{\mu^1}(t, X_t^1)dt + \sqrt{2\nu}dW_t \\ dX_t^2 &= \alpha^{\mu^2}(t, X_t^2)dt + \sqrt{2\nu}dW_t \\ X_0^1 &= X_0^2 = X^0, \end{aligned}$$

where X^0 is a random variable on \mathbb{T}^d with law m^0 and W is a Brownian motion independent of X^0 . The respective laws of $(X_t^1, \alpha^{\mu^1}(t, X_t^1))$ and $(X_t^2, \alpha^{\mu^2}(t, X_t^2))$ are $\mu^1(t)$ and $\mu^2(t)$. Then we obtain,

$$\begin{aligned} \mathbb{E} [|X_t^1 - X_t^2|^{q_1}]^{\frac{1}{q_1}} &= \mathbb{E} \left[\left| \int_0^t \alpha^{\mu^1}(s, X_s^1) - \alpha^{\mu^2}(s, X_s^2) ds \right|^{q_1} \right]^{\frac{1}{q_1}} \\ &\leq \int_0^t \mathbb{E} \left[\left| \alpha^{\mu^1}(s, X_s^1) - \alpha^{\mu^2}(s, X_s^2) \right|^{q_1} \right]^{\frac{1}{q_1}} ds \\ &\leq \int_0^t \mathbb{E} \left[\left| \alpha^{\mu^1}(s, X_s^1) - \alpha^{\mu^1}(s, X_s^2) \right|^{q_1} \right]^{\frac{1}{q_1}} + \mathbb{E} \left[\left| \alpha^{\mu^1}(s, X_s^2) - \alpha^{\mu^2}(s, X_s^2) \right|^{q_1} \right]^{\frac{1}{q_1}} ds, \end{aligned}$$

where we used the triangle inequality for the L^{q_1} -norm twice. By the first additional assumption of the theorem and **A1**, α^{μ^1} is Lipschitz continuous with respect to x and its

Lipschitz constant depends on $\|u^i\|_{C^{0,1}}$ and $\Lambda_\infty(\mu^1)$. Using the estimates from the proof of Proposition 2.5.8, it only depends on the constants in the assumptions. This, the latter inequality and (2.5.7) imply

$$\mathbb{E} [|X_t^1 - X_t^2|^{q_1}]^{\frac{1}{q_1}} \leq C \int_0^t \mathbb{E} [|X_s^1 - X_s^2|^{q_1}]^{\frac{1}{q_1}} ds + CT \sup_{0 \leq t' \leq T} (\|\nabla_x u(t')\|_\infty + W_{q_1}(m^1(t'), m^2(t'))).$$

This and Gronwall's inequality yield that,

$$\sup_{0 \leq t \leq T} \mathbb{E} [|X_t^1 - X_t^2|^{q_1}]^{\frac{1}{q_1}} \leq CT \sup_{0 \leq t \leq T} (\|\nabla_x u(t)\|_\infty + W_{q_1}(m^1(t), m^2(t))).$$

From now on, we assume that $T \leq \frac{1}{2C}$, so that $(1 - CT) \geq \frac{1}{2}$. Since $W_{q_1}(m^1(t), m^2(t)) \leq \mathbb{E} [|X_t^1 - X_t^2|^{q_1}]^{\frac{1}{q_1}}$, we obtain:

$$(2.5.8) \quad \sup_{0 \leq t \leq T} W_{q_1}(m^1(t), m^2(t)) \leq CT \sup_{0 \leq t \leq T} \|\nabla_x u(t)\|_\infty.$$

Hence u satisfies the following equation,

$$\begin{cases} -\partial_t u - \nu \Delta u = -H(x, \nabla_x u^1, \mu^1) + H(x, \nabla_x u^2, \mu^2), \\ u(T, x) = g(x, m^1(T)) - g(x, m^2(T)). \end{cases}$$

The right-hand side of the first line can be estimated in absolute value from above as follows:

$$|H(x, \nabla_x u^1, \mu^1) - H(x, \nabla_x u^2, \mu^2)| \leq C \sup_{0 \leq t' \leq T} \|\nabla_x u(t')\|_\infty,$$

by **T**, (2.5.7) and (2.5.8). Since $u(T, \cdot) \in C^{1+\beta}(\mathbb{T}^d)$, Theorem 6.48 in [86] yields that $u \in C^{\frac{1}{2} + \frac{\beta}{2}, 1+\beta}([0, T] \times \mathbb{T}^d)$ and it satisfies:

$$\begin{aligned} \sup_{t \in [0, T]} \|\nabla_x u(t)\|_\infty &\leq \|\nabla_x (g(\cdot, m^1(T)) - g(\cdot, m^2(T)))\|_\infty \\ &\quad + CT^{\frac{\beta}{2}} \sup_{t \in [0, T]} (\|\nabla_x u(t)\|_\infty + \|g(\cdot, m^1(T)) - g(\cdot, m^2(T))\|_{C^{1+\beta}}). \end{aligned}$$

This, (2.2.5) and (2.5.8) yield,

$$\sup_{t \in [0, T]} \|\nabla_x u(t)\|_\infty \leq CT^{\frac{\beta}{2}} \sup_{t \in [0, T]} \|\nabla_x u(t)\|_\infty.$$

Thus if we suppose furthermore that $T < C^{-\frac{2}{\beta}}$, then $\nabla_x u = 0$, so $m = 0$ by (2.5.8), then $\mu^1 = \mu^2$ by (2.5.7), and finally u^1 and u^2 solve the same Hamilton-Jacobi-Bellman equation with the same terminal condition, so by uniqueness $u = 0$.

Therefore, we proved the uniqueness for $T < T_1$ where T_1 is defined by $T_1 = \min T_0, \left(C^{-\frac{2}{\beta}}, C^{-1}\right)$. \square

2.6 Applications

Here, we are going to work on \mathbb{T}^d , while it would be more realistic to work in the whole space \mathbb{R}^d for the applications considered below. We would like to recall that the existence results contained in the present work hold for MFGC systems on \mathbb{R}^d using the method introduced in [76] to pass from the torus to the whole Euclidean space. Therefore, the conclusions of this section may be adapted to treat the same applications on \mathbb{R}^d .

2.6.1 Exhaustible resource model with nonpositively correlated resources

This model is often referred to as Bertrand and Cournot competition model for exhaustible resources, introduced in the independent works of Cournot [50] and Bertrand [23]; its mean field game version in dimension one was introduced in [65] and numerically analyzed in [47]; for theoretical results see [25, 62, 73, 63]. We consider a continuum of producers selling exhaustible resources. The production of a representative agent is $(q_t)_{t \in [0, T]}$; the agents differ in their production capacities $X_t \in \mathbb{T}$ (the state variable), that satisfies,

$$dX_t = -q_t dt + \sqrt{2\nu} dW_t,$$

where $\nu > 0$ and W is a Brownian motion. Each producer is selling a different resource and has her own consumers. However, the resources are substitutable and any consumer may change her mind and buy from a competitor depending on the degree of competition in the game (which is characterized by ε in the linear demand case below for instance). Therefore, the selling price per unit of resource that a producer can make when she sales q units of resource, depends naturally on q and on the quantity produced by the other agents. The price satisfies a supply-demand relationship, and is given by $P(q, \bar{q})$, where \bar{q} is the aggregate demand which depends on the overall distribution of productions of the agents. A producer tries to maximize her profit, or equivalently to minimize the following quantity,

$$\mathbb{E} \left[\int_0^T -P(q_t, \bar{q}_t) \cdot q_t dt + g(X_T) \right],$$

where g is a terminal cost which often penalizes the producers who have non-zero production capacities at the end of the game. In the Cournot competition, see [50], a producer is controlling her production q . Like the MFG version of the Bertrand and Cournot competition introduced in [47], here we consider the Bertrand formulation [23], where an agent directly controls her selling price $\alpha = P(q, \bar{q})$. After inverting the latter equality, the production can be viewed as a function of the price and the mean field. Mathematically this corresponds to writing $q = Q(\alpha, \bar{\alpha})$.

In [47], the authors considered a linear demand system depending on $\bar{q}_{\text{lin}} = \int_{\mathbb{T}} q(x) dm(x)$, and a price satisfying $\alpha = P_{\text{lin}}(q, \bar{q}_{\text{lin}}) = 1 - q - \varepsilon \bar{q}_{\text{lin}}$. In this case, the running cost L^{lin} and its Legendre transform H^{lin} are defined by

$$\begin{aligned} L^{\text{lin}}(\alpha, \mu) &= \alpha^2 + \frac{\varepsilon}{1+\varepsilon} \alpha \bar{\alpha} - \frac{1}{1+\varepsilon} \alpha, \\ H^{\text{lin}}(p, \mu) &= \frac{1}{4} \left(p + \frac{\varepsilon}{1+\varepsilon} \bar{\alpha} - \frac{1}{1+\varepsilon} \right)^2, \end{aligned}$$

where $\alpha, p \in \mathbb{R}$, $\mu \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ and $\bar{\alpha}$ is defined by $\bar{\alpha} = \int_{\mathbb{T} \times \mathbb{R}} \tilde{\alpha} d\mu(y, \tilde{\alpha})$. Therefore the system of MFGC has the following form,

$$(2.6.1) \quad \begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{4} \left(\nabla_x u + \frac{\varepsilon}{1+\varepsilon} \bar{\alpha} - \frac{1}{1+\varepsilon} \right)^2 = 0, \\ \partial_t m - \nu \Delta m - \operatorname{div} \left(\frac{1}{2} \left(\nabla_x u + \frac{\varepsilon}{1+\varepsilon} \bar{\alpha} - \frac{1}{1+\varepsilon} \right) m \right) = 0, \\ \bar{\alpha}(t) = - \int_{\mathbb{T}} \frac{1}{2} \left(\nabla_x u + \frac{\varepsilon}{1+\varepsilon} \bar{\alpha}(t) - \frac{1}{1+\varepsilon} \right) dm(t, x), \\ u(T, x) = g(x), \\ m(0, x) = m_0(x), \end{cases}$$

for $(t, x) \in [0, T] \times \mathbb{T}$. Roughly speaking, $\varepsilon = 0$ corresponds to a monopoly in which a producer does not suffer from competition, and she plays as if she was alone in the game. Conversely, $\varepsilon = \infty$ stands for all the producers selling the same resource and the consumers not having any preference.

Here, Theorem 2.2.5 d) implies the following existence result.

Proposition 2.6.1. *If m_0 and g satisfy **A2** and **A3**, there exists a solution to (2.6.1) for any $\varepsilon \in (0, \infty)$.*

To prove it, we may take $q = 2$, $q_0 = 1$, $\lambda_0 = \frac{\varepsilon}{2(1+\varepsilon)}$, $\lambda_1 = 1$, and $C_0 = \frac{1}{2}$ in **FP1**; then we check the assumptions of Theorem 2.2.5 d). In this case, the inequality in **B3** has the form $1 < \left(\frac{2+\varepsilon}{1+\varepsilon}\right)^2$, and is satisfied for any $\varepsilon \in (0, \infty)$.

Here, the Lagrangian L^{lin} satisfies a monotonicity assumption, but the latter existence result does not take advantage of it. We refer to [76] for a uniqueness result and an other existence result for the solution to (2.6.1) using this monotonicity assumption. Generalizations of (2.6.1) to larger dimensions with more general Hamiltonians and prices are also discussed in [76] under the monotonicity assumption.

In what follows, we provide a simple example of a generalization of (2.6.1) in which the monotonicity assumption does not hold and the results in [76] do not apply anymore. However, the results in the present work may hold in some cases even without the monotonicity assumption.

Let us consider a model in which every producer sells d different kinds of resources. The price of each resource depends on the mean field like in (2.6.1). Namely, we take $Q = M\bar{\alpha} - \alpha$ which is now a d -dimensional vector and where $M \in \mathbb{R}^{d \times d}$ is a given matrix. This leads to the following MFGC system,

$$(2.6.2) \quad \begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{4} (\nabla_x u + M\bar{\alpha})^2 = f(x, m), \\ \partial_t m - \nu \Delta m - \operatorname{div} \left(\frac{1}{2} (\nabla_x u + M\bar{\alpha}) m \right) = 0, \\ \bar{\alpha}(t) = - \left(I_d + \frac{1}{2} M \right)^{-1} \int_{\mathbb{R}^d} \nabla_x u(t, x) dm(t, x), \\ u(T, x) = g(x), \\ m(0, x) = m_0(x), \end{cases}$$

Proposition 2.6.2. *Assume **A2**, **A3**, that M has an operator norm smaller than 1, and that f is continuous, and differentiable with respect to x with continuous derivatives. There exists a solution to (2.6.3).*

The proof consists in taking $q = 2$, $q_0 = 1$, $\lambda_1 = 1$, $C_0 = \frac{1}{2}$ in **FP1**, and $\lambda_0 = \frac{\|M\|}{2}$, where $\|M\|$ is the operator norm of M ; and we check the assumptions of Theorem 2.2.5 d).

The monotonicity assumption discussed in [76] is equivalent to assuming that M is a positive semi-definite matrix. Here, we do not make such an assumption.

What we have in mind in the latter example is the case where the prices of the different resources may be negatively correlated, like cars and oil (if the production of cars increases, then the demand for oil also increases and the price of oil rises while the price of cars decreases), or pesticides and medicines, or gold and other raw materials. To our knowledge, such a generalization of the exhaustible resource model to negatively correlated resources is new in the MFG literature.

More generally, we believe that our results hold for the following MFGC system under various different sets of assumptions that we will not detailed here,

$$(2.6.3) \quad \begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla_x u + Q(t, x, \mu)) = f(t, x, m(t)), \\ \partial_t m - \nu \Delta m - \operatorname{div}(H_p(x, \nabla_x u + Q(t, x, \mu)) m) = 0, \\ \mu(t) = \left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot) + Q(t, \cdot, \mu(t))) \# m(t) \right) \\ u(T, x) = g(x, m(T)), \\ m(0, x) = m_0(x), \end{cases}$$

where $Q : [0, T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a vector characterizing the mean field interactions.

2.6.2 Price impact models with bid and ask prices

The price impact model without bid and ask prices is inspired by the Almgren and Chriss's model [13], and was introduced in the MFG literature in [38] and [43] where existence and uniqueness results are proved when the admissible controls stay in a compact set. Here we consider an extension with bid and ask prices.

We suppose that a continuum of agents are trading an asset, the state of a representative agent is X_t the amount of this asset she owns. Her control α is the quantity she buys (if $\alpha \geq 0$) or sell (if $\alpha < 0$). The state space is the one-dimensional torus \mathbb{T} , and X_t is given by,

$$dX_t = \alpha_t dt + \sigma dW_t,$$

where W is a Brownian motion, and $\sigma > 0$ is a real constant. We define S_t as the asking price of the asset, and $\varepsilon(\mu(t))$ as the difference between the bidding and asking prices, where $\mu(t)$ is the law of (X_t, α_t) . The agent buys at the bidding price $S_t + \varepsilon(\mu_t)$, thus her cash is given by

$$dK_t = -(\alpha_t S_t + \alpha_t \varepsilon(\mu(t)) + \ell(\alpha_t)) dt,$$

where ℓ is a differentiable function standing for the transaction cost. The price S_t evolves accordingly with the amount of transactions at time t , it satisfies the following SDE,

$$dS_t = A(\mu(t)) dt, \\ \text{where } A(\mu(t)) = \int_{\mathbb{T} \times \mathbb{R}} \ell'(\alpha) d\mu(t, x, \alpha),$$

The wealth of a representative agent is given by $V_t = V_0 + X_t S_t + K_t$ and it satisfies the following SDE,

$$(2.6.4) \quad dV_t = (X_t A(\mu(t)) - \ell(\alpha_t) - \varepsilon(\mu(t)) \alpha_t) dt + \sigma S_t dW_t.$$

The objective function that she will try to maximize is given by,

$$\mathbb{E} \left[V_T - \int_0^T f(X_t) dt - g(X_T) \right],$$

where f and g are penalization costs for holding stocks. Here, the Lagrangian and Hamiltonian are given by,

$$L^{\text{PI}}(x, \alpha, \mu) = \ell(\alpha) + \alpha \varepsilon(\mu) - x A(\mu), \\ H^{\text{PI}}(x, p, \mu) = h(p + \varepsilon(\mu)) + x A(\mu),$$

for $(x, \alpha, \mu) \in \mathbb{T} \times \mathbb{R} \times \mathcal{P}(\mathbb{T} \times \mathbb{R})$, where h is the Legendre transform of ℓ .

The linear-quadratic case with $\varepsilon = 0$ is treated in [41]. Here, taking $\varepsilon = 0$ corresponds to assuming that the bidding and asking prices coincide. In this case the optimal control is given by $-h_p(p)$ and does not depend explicitly on μ . If $\varepsilon \neq 0$, the optimal control depends explicitly on μ and L^{PI} is not separable in α and μ , this prevents us from using the results in [41].

Let us give an example of choices for the functions ℓ and ε under which our result apply and a solution of the MFGC price impact model exists.

Proposition 2.6.3. *Assume **A2**, **A3**, that f is C^1 , and that c and ε are respectively given by $\ell(\alpha) = \frac{|\alpha|^2}{2}$ and $\varepsilon(\mu) = \tilde{\varepsilon} \left(\int_{\mathbb{T} \times \mathbb{R}} |\alpha|^2 d\mu(x, \alpha) \right)^{\frac{1}{2}}$, where $0 < \tilde{\varepsilon} < \frac{1}{2}$. There exists a solution to (2.1.2) with H^{PI} .*

This existence result is a consequence of 2.2.5 c), where the assumptions are satisfied for $q = q_0 = 2$, $\lambda_0 = \varepsilon$, $\lambda_1 = \frac{1}{4}$ and $C_0 = 1$ in **FP1**. We would like to insist on the fact that Theorem 2.2.5 c) provides the existence of solutions for a wild class of Hamiltonian, larger than the one of the latter proposition and which goes beyond the linear-quadratic case.

Let us mention that we would be interested in defining the bidding price by $(1 + \tilde{\varepsilon})S_t$, where $\tilde{\varepsilon} > 0$. The associated MFGC system cannot be using the conclusions of the present work because the mean field interaction at time t would depend not only on μ_t but on $(\mu_s)_{s \in [0, t]}$. However, we believe that existence holds under similar assumptions as here, and we plan to prove it in forthcoming works.

2.6.3 First-order flocking model with velocity as controls

Cucker and Smale proposed a form of Vicsek model in [52] to illustrate the behavior of flocks of birds. This model is of second-order in the sense that the state of an agent is given by a couple (x, v) standing for her position and velocity respectively, and the equation of evolution of her state involves considering her acceleration.

A game version of this model in which an agent controls her acceleration has been introduced in [91], the authors derived a MFG formulation in the infinite horizon case. Here we are interested in the finite horizon problem which was studied in [43, 41]. This model is still of second-order. More precisely the state of an agent is given by $(X_t, V_t)_{t \in [0, T]}$ respectively her position and velocity, two random processes which satisfy the following system of stochastic differential equations,

$$\begin{aligned} dX_t &= V_t dt, \\ dV_t &= a_t dt + \sigma dW_t, \end{aligned}$$

where a_t is the individual's acceleration vector and her control, W is a d -dimensional Brownian motion, and $\sigma \in \mathbb{R}^{d \times d}$ is a positive definite matrix. The cost that a representative agent tries to minimize is given by

$$\mathbb{E} \left[\int_0^T \frac{|a_t|^2}{2} + \frac{1}{2} \left| \int_{\mathbb{T}^d} (v - V_t^i) \varphi(|x - X_t^i|) d\mu(t, x, v) \right|^2 + f(X_t) dt \right],$$

where $\mu(t) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ is the joint distribution of states and velocities of the agents, φ is a C^1 nonincreasing function, and f is a C^1 function modeling the spatial preferences of the agents (for instance, we can take f significantly smaller in some areas which corresponds to where the food is).

Here we consider an alternative viewpoint in which an agent directly controls her velocity. This is a first-order model since the state of an agent is now given by a vector of \mathbb{T}^d , and the acceleration does not appear anymore in the dynamics of a given agent, which is given by

$$dX_t = \alpha_t dt + \sigma dW_t.$$

Here, the cost that an agent tries to minimize is given by

$$\mathbb{E} \left[\int_0^T \frac{|\alpha_t|^2}{2} + \frac{1}{2} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} (\tilde{\alpha} - \alpha_t) \varphi(|x - X_t^i|) d\mu(t, x, \tilde{\alpha}) \right|^2 + f(X_t) dt \right].$$

First-order physical models are generally easier to study than second-order models. However the price we paid here to go from a second-order model to a first-order model is to consider a MFGC system instead of a MFG system without interaction through the controls.

If $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ and $m \in \mathcal{P}(\mathbb{T}^d)$ are such that m is the marginal of μ with respect to \mathbb{T}^d , we define $A(x, \mu)$ and $Z(x, \mu)$ by,

$$\begin{cases} A(x, \mu) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \tilde{\alpha} \varphi(|x - y|) d\mu(y, \tilde{\alpha}), \\ Z(x, \mu) = \int_{\mathbb{T}^d} \varphi(|x - y|) dm(y), \end{cases}$$

for $x \in \mathbb{T}^d$. We define the Lagrangian of the first-order flocking model by,

$$L^{\text{FM}}(x, \alpha, \mu) = \frac{|\alpha|^2}{2} + \frac{1}{2} |Z(x, \mu)\alpha - A(x, \mu)|^2 + f(x),$$

for $(x, \alpha, \mu) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, and the Hamiltonian by,

$$H^{\text{FM}}(x, p, \mu) = \frac{1}{2(1 + Z(x, \mu)^2)} \left(|p|^2 - 2Z(x, \mu)A(x, \mu) \cdot p - |A(x, \mu)|^2 \right) - f(x),$$

for $p \in \mathbb{R}^d$, such that H^{FM} is the Legendre's transform of L^{FM} .

Proposition 2.6.4. *Under assumptions **A2** and **A3**, there exists $T_0 > 0$ such that if $T < T_0$, there exists a unique solution to (2.1.2) with H^{FM} .*

Hereafter, we present an other model for crowd motion which is very similar to the first-order flocking model discussed above. The main difference between these two models is the normalization constants. However, the assumptions and conclusions of this work are more adapted to the following crowd motion model and we can derive more existence results for it. We believe that these results can be adapted to the first-order Cucker-Smale system.

2.6.4 A model of crowd motion

This model of crowd motion has been numerically studied in [2] in the quadratic case, and has some similarities with the first-order flocking model presented in the previous paragraph. For $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, we define $V(x, \mu)$ and $Z_{q_0}(x, \mu)$ by

$$\begin{cases} V(x, \mu) = \frac{1}{Z_{q_0}(x, \mu)} \int_{\mathbb{T}^d \times \mathbb{R}^d} \tilde{\alpha} k(x, y) d\mu(y, \tilde{\alpha}), \\ Z_{q_0}(x, \mu) = \left(\int_{\mathbb{T}^d} k(x, y)^{q_0} dm(y) \right)^{\frac{1}{q_0}}, \end{cases}$$

where $q_0 \in (1, \infty]$, q'_0 is the conjugate exponent of q_0 , $k : \mathbb{T}^d \times \mathbb{T}^d \rightarrow \mathbb{R}_+$ is a nonnegative C^1 kernel, and $m \in \mathcal{P}(\mathbb{T}^d)$ is the marginal of μ with respect to \mathbb{T}^d . The quantity $V(x, \mu)$ is called the average drift.

The state of a representative agent is given by her position $X_t \in \mathbb{T}^d$ and she controls her velocity α_t ,

$$dX_t = \alpha_t dt + \sqrt{2\nu} dW_t.$$

Her objective is to minimize the cost given by,

$$\mathbb{E} \left[\int_0^T \frac{\theta}{a'} |\alpha_t - \tilde{\lambda} V(X_t, \mu(t))|^{a'} + \frac{1-\theta}{b'} |\alpha_t|^{b'} + f(X_t) dt + g(X_T) \right],$$

where $-1 < \tilde{\lambda} < 1$ and $0 \leq \theta \leq 1$ are two constants standing for the preference of an individual to have a similar (resp. opposite) control as the mainstream when $\tilde{\lambda} > 0$ (resp. $\tilde{\lambda} < 0$), f and g are respectively the running cost and the terminal cost which encode the spatial preferences of the agents, and $a', b' > 1$ are exponents.

Here, we take $q = \min(a, b)$. In this model we define the Lagrangian by,

$$(2.6.5) \quad L(x, \alpha, \mu) = \frac{\theta}{a'} |\alpha - \tilde{\lambda} V(x, \mu)|^{a'} + \frac{1-\theta}{b'} |\alpha|^{b'},$$

and the Hamiltonian as its Legendre transform. If $a = b = 2$, H is given by

$$H(x, p, \mu) = \frac{|p|^2}{2} - \tilde{\lambda} \theta p \cdot V(x, \mu) - \frac{\tilde{\lambda}^2 \theta (1-\theta)}{2} |V(x, \mu)|^2.$$

If $\theta = 1$, H satisfies

$$H(x, p, \mu) = \frac{1}{a} |p|^a - \tilde{\lambda} p \cdot V(x, \mu).$$

For other choices of the parameters a, b and θ , H does not admit an explicit form.

Proposition 2.6.5. *Assume that g and m_0 satisfy **A2** and **A3** respectively. There exists a solution to (2.1.2) where H is the Legendre transform of L given in (2.6.5), under one of the following assertions,*

- a) $q_0 \leq q'$ and $a \neq b$,
- b) $q_0 \leq q'$ and one of the following assertions is satisfied,
 - i) $\theta < \theta_0$,
 - ii) $\theta > 1 - \theta_0$,
 - iii) $|\tilde{\lambda}| < \lambda_0$,

where $\theta_0, \lambda_0 \in (0, 1)$ are constants coming from Theorem 2.2.5 c),

- c) $\theta = 1$,
- d) $k(x, y)$ is constant,
- e) $T < T_0$, where T_0 is a positive constant coming from Theorem 2.2.5 e).

Proof. We refer to the appendix, Lemma 2.6.7 for the proof that H satisfies **A1-A3**, **B1-B3**, **FP1-FP2**, and **T**. The existence results **c)**, **d)** and **e)** are direct consequences of Theorem 2.2.5 **c)**, **d)** and **e)** respectively.

We define $\tilde{L}(\alpha, V)$ by

$$\tilde{L}(\alpha, V) = \frac{\theta}{a'} |\alpha - \tilde{\lambda}V|^{a'} + \frac{1-\theta}{b'} |\alpha|^{b'},$$

for $\alpha, V \in \mathbb{R}^d$, $\tilde{H}(p, V)$ as the Legendre transform of \tilde{L} with respect to its first argument, and $\alpha(p, V)$ as the unique control which achieves the maximum in the definition of \tilde{H} (it is unique because \tilde{L} is strictly convex with respect to α).

Proof of a). Take $V \in \mathbb{R}^d$ and $\alpha = \alpha(0, V)$, since α achieves the maximum in the definition of $\tilde{H}(0, V)$, we know that

$$0 = \theta |\alpha - \tilde{\lambda}V|^{a'-2} (\alpha - \tilde{\lambda}V) + (1-\theta) |\alpha|^{b'-2} \alpha,$$

which implies

$$(2.6.6) \quad \theta |\alpha - \tilde{\lambda}V|^{a'-1} = (1-\theta) |\alpha|^{b'-1},$$

and then

$$(2.6.7) \quad \left(\theta^{a-1} (1-\theta)^{2-a} |\alpha|^{\frac{(a'-2)(b'-1)}{a'-1}} + (1-\theta) |\alpha|^{b'-2} \right) \alpha = \tilde{\lambda} \theta^{a-1} (1-\theta)^{2-a} |\alpha|^{\frac{(a'-2)(b'-1)}{a'-1}} V.$$

The two latter equalities yield $\lim_{V \rightarrow +\infty} |\alpha(0, V)| = +\infty$. We make out two cases:

- if $a > b$ then we have $\frac{(a'-2)(b'-1)}{a'-1} < b' - 2$, and $|\alpha| = o_{+\infty}(|V|)$. Therefore, (2.6.7) yields

$$|\alpha|^{b'-1 - \frac{(a'-2)(b'-1)}{a'-1}} = o_{+\infty}(|V|),$$

and $b' - 1 - \frac{(a'-2)(b'-1)}{a'-1} = \frac{a-1}{b-1} > 1$, so we obtain

$$|\alpha| = o_{+\infty}\left(|V|^{\frac{b-1}{a-1}}\right),$$

which yields

$$\tilde{H}(0, V) = o_{+\infty}\left(|V|^{a'}\right) + o_{+\infty}\left(|V|^{\frac{b-1}{a-1}b'}\right),$$

with $a' < b'$, and $\frac{b-1}{a-1}b' < b'$, and $b = q$.

- if $a < b$ then we have $\frac{(a'-2)(b'-1)}{a'-1} > b' - 2$, and $\alpha = \tilde{\lambda}V + o_{+\infty}(|V|)$. Therefore, (2.6.7) yields

$$\left(1 + o_{+\infty}\left(|V|^{b'-2 - \frac{(a'-2)(b'-1)}{a'-1}}\right)\right) \alpha = \tilde{\lambda}V.$$

We notice that $b' - 2 - \frac{(a'-2)(b'-1)}{a'-1} = \frac{b'-a'}{a'-1} < 0$, and we obtain

$$\alpha = \tilde{\lambda}V + o_{+\infty}\left(|V|^{1 + \frac{b'-a'}{a'-1}}\right) = \tilde{\lambda}V + o_{+\infty}\left(|V|^{\frac{a-1}{b-1}}\right).$$

This implies

$$\tilde{H}(0, V) = o_{+\infty}\left(|V|^{\frac{b-1}{a-1}a'}\right) + o_{+\infty}\left(|V|^{b'}\right),$$

with $b' < a'$, and $\frac{a-1}{b-1}a' < a'$, and $a = q$.

We conclude by (2.6.8) and Theorem 2.2.5 a).

Proof of b)

Here, we assume that $a = b$ since the case $a \neq b$ is addressed in a).

Take $V \in \mathbb{R}^d$, and $\alpha = \alpha(0, V)$. In this case, $\tilde{H}(0, V)$ admits an explicit form given by

$$\tilde{H}(0, V) = -\frac{|\tilde{\lambda}|^{a'}}{a'} \frac{\theta(1-\theta)^a + (1-\theta)\theta^a}{((1-\theta)^{a-1} + \theta^{a-1})^{a'}} |V|^{a'}.$$

Therefore, taking $\tilde{\lambda}$, θ or $(1-\theta)$ small enough allows one to conclude by (2.6.8) and Theorem 2.2.5 b). \square

Appendix: verification of the assumptions for the model of crowd motion

We start by establishing some properties of the function V in the following lemma.

Lemma 2.6.6. *The function V is C^1 with respect to x and it satisfies*

$$(2.6.8) \quad \|V(\cdot, \mu)\|_{\infty} \leq \Lambda_{q_0}(\mu),$$

where $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$.

For $m \in \mathcal{P}(\mathbb{T}^d)$ and $\mu^1, \mu^2 \in \mathcal{P}_m(\mathbb{T}^d \times \mathbb{R}^d)$, the following inequality is satisfied,

$$(2.6.9) \quad \|V(\cdot, \mu^1) - V(\cdot, \mu^2)\|_{\infty} \leq \left\| \alpha^{\mu^1} - \alpha^{\mu^2} \right\|_{L^{q_0}(m)}.$$

For $R > 0$, there exists $C_R > 0$ a constant such that,

$$(2.6.10) \quad \|V(\cdot, \mu^1) - V(\cdot, \mu^2)\|_{\infty} \leq C_R \left(\left\| \alpha^{\mu^1} - \alpha^{\mu^2} \right\|_{\infty} + \|m^1 - m^2\|_{\infty} \right),$$

for (m^i, μ^i) such that $m^i \in \mathcal{P}(\mathbb{T}^d) \cap C^0(\mathbb{T}^d)$ with $m^i \geq R^{-1}$, $\mu^i \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ with $\alpha^{\mu^i} \in C^0(\mathbb{T}^d \times \mathbb{R}^d)$ and $\left\| \alpha^{\mu^i} \right\|_{\infty} \leq R$, $i = 1, 2$.

Proof. The function V has at least the same regularity as k with respect to the state variable since V is the convolution product of k with a probability measure. Then (2.6.8) and (2.6.9) are straightforward using Hölder inequality. Let us take the same notation as in (2.6.10), for $x \in \mathbb{T}^d$ we get

$$\begin{aligned} |V(x, \mu^1) - V(x, \mu^2)| &\leq \frac{1}{Z_{q_0}(x, \mu^1)} \int_{\mathbb{T}^d} k(x, y) \left| \alpha^{\mu^1}(y) - \alpha^{\mu^2}(y) \right| dm^1(y) \\ &\quad + \frac{1}{Z_{q_0}(x, \mu^1)} \int_{\mathbb{T}^d} k(x, y) \left| \alpha^{\mu^2}(y) \right| |m^1(y) - m^2(y)| dy \\ &\quad + \left| \frac{1}{Z_{q_0}(x, \mu^1)} - \frac{1}{Z_{q_0}(x, \mu^2)} \right| \int_{\mathbb{T}^d} k(x, y) \left| \alpha^{\mu^2}(y) \right| dm^2(y) \\ &\leq \left\| \alpha^{\mu^1} - \alpha^{\mu^2} \right\|_{\infty} + \frac{1}{Z_{q_0}(x, \mu^1)} \int_{\mathbb{T}^d} k(x, y) dy \left\| \alpha^{\mu^2} \right\|_{\infty} \|m^1 - m^2\|_{\infty} \\ &\quad + \frac{1}{Z_{q_0}(x, \mu^1)} \left\| \alpha^{\mu^2} \right\|_{L^{q_0}(m^2)} |Z_{q_0}(x, \mu^1) - Z_{q_0}(x, \mu^2)|. \end{aligned}$$

Moreover, we know that $Z_{q_0}(x, \mu^1) \geq R^{-\frac{1}{q'_0}} \left(\int_{\mathbb{T}^d} k(0, y)^{q'_0} dy \right)^{\frac{1}{q'_0}} > 0$ where the right-hand side does not depend on x , and

$$\begin{aligned} |Z_{q_0}(x, \mu^1) - Z_{q_0}(x, \mu^2)| &\leq \max_{i=1,2} \left(\frac{1}{q'_0} \left(Z_{q_0}(x, \mu^i)^{q'_0} \right)^{\frac{1}{q'_0}-1} \right) \left| Z_{q_0}(x, \mu^1)^{q'_0} - Z_{q_0}(x, \mu^2)^{q'_0} \right| \\ &\leq \frac{1}{q'_0} \left(\min_{i=1,2} Z_{q_0}(x, \mu^i) \right)^{1-q'_0} \int_{\mathbb{T}^d} k(0, y)^{q'_0} dy \|m^1 - m^2\|_\infty \\ &\leq \frac{1}{q'_0} R^{\frac{1}{q_0}} \left(\int_{\mathbb{T}^d} k(0, y)^{q'_0} dy \right)^{\frac{1}{q_0}} \|m^1 - m^2\|_\infty. \end{aligned}$$

The latter two chains of inequalities imply (2.6.10) with $C_R = 1 + R^{1+\frac{1}{q_0}} + \frac{1}{q'_0} R^2$. \square

Here, we assume $\theta \in (0, 1)$. Indeed, H admits an explicit form when $\theta = 0$ or $\theta = 1$, then checking **A1-A3**, **B1-B3**, **FP1-FP2**, and **T** is straightforward.

Lemma 2.6.7. *Assumptions **A1**, **B1-B3**, **FP1**, **FP2**, and **T** are satisfied when L is defined in (2.6.5).*

Proof. We define \tilde{L} , \tilde{H} and α as in the proof of 2.6.5.

*Checking **A1**, **B1** and **B2**.*

The Legendre transform of a function is convex, therefore H is convex with respect to p . Since L is strictly convex, H is differentiable with respect to p . Moreover, $\alpha = -H_p$ thus H_p is continuous by the Maximum theorem. Then $H(x, p, \mu) = p \cdot H_p(x, p, \mu) - L(x, -H_p(x, p, \mu), \mu)$, so H is continuous. Finally, H is differentiable with respect to x by the envelop theorem and

$$(2.6.11) \quad H_x(x, p, \mu) = -L_x(x, -H_p(x, p, \mu), \mu),$$

for $(x, p, \mu) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$.

Using the growth properties of L , we can prove that there exists $C_0 > 0$ such that

$$(2.6.12) \quad |H_p(t, x, p, \mu)| \leq C_0 (1 + |p|^{q-1} + \Lambda_{q'}(\mu)),$$

$$(2.6.13) \quad |H(t, x, p, \mu)| \leq C_0 (1 + |p|^q + \Lambda_{q'}(\mu)^{q'}),$$

$$(2.6.14) \quad |H_x(t, x, p, \mu)| \leq C_0 (1 + |p|^q + \Lambda_{q'}(\mu)^{q'}),$$

for any $(x, p, \mu) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. We refer to [76] Lemma 2.5 for a complete proof.

One may prove that the function $h : z \in \mathbb{R}^d \mapsto |z|^{a'} \in \mathbb{R}$ satisfies $h(z) - h(y) - \nabla h(y) \cdot (y - x) \geq C_R^{-1} |y - z|^{\max(a', 2)}$ for $y, z \in \mathbb{R}^d$ such that $|y| \leq R$, $|z| \leq R$, where $C_R > 0$ is a constant. This implies that for $R > 0$ there exists $C_R > 0$ a constant such that L satisfies

$$L(x, \alpha^2, \mu) - L(x, \alpha^1, \mu) - (\alpha^2 - \alpha^1) \cdot L_\alpha(x, \alpha^1, \mu) \geq C_R^{-1} |\alpha^2 - \alpha^1|^{\max(q', 2)},$$

for $(\alpha^1, \alpha^2, \mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_\infty(\mathbb{T}^d \times \mathbb{R}^d)$, such that $|\alpha^i| \leq R$ and $\Lambda_{q_0}(\mu) \leq R$. This implies

$$(\alpha^2 - \alpha^1) \cdot (L_\alpha(x, \alpha^2, \mu) - L_\alpha(x, \alpha^1, \mu)) \geq 2C_R^{-1} |\alpha^2 - \alpha^1|^{\max(q', 2)}.$$

Take $p^i \in \mathbb{R}^d$ and $\alpha^i = -H_p(x, p^i, \mu)$, $i = 1, 2$. Recalling the conjugacy relation $p^i = -L_\alpha(x, \alpha^i, \mu)$ we obtain that H_p is locally Hölder continuous with respect to p .

Checking B3.

Take $(p, V) \in \mathbb{R}^{2d}$ and $\alpha = \alpha(p, V)$, the optimal control α satisfies

$$(2.6.15) \quad p = -D_\alpha \tilde{L}(\alpha, V) = -\theta |\alpha - \tilde{\lambda}V|^{a'-2} (\alpha - \tilde{\lambda}V) - (1 - \theta) |\alpha|^{b'-2} \alpha,$$

If $(p, V) \neq (0, 0)$, this implies

$$(2.6.16) \quad \alpha = \frac{-p + \tilde{\lambda}\theta |\alpha - \tilde{\lambda}V|^{a'-2} V}{\theta |\alpha - \tilde{\lambda}V|^{a'-2} + (1 - \theta) |\alpha|^{b'-2}},$$

and

$$(2.6.17) \quad \alpha - \tilde{\lambda}V = \frac{-p + \tilde{\lambda}(1 - \theta) |\alpha|^{b'-2} V}{\theta |\alpha - \tilde{\lambda}V|^{a'-2} + (1 - \theta) |\alpha|^{b'-2}}.$$

From (2.6.15), we deduce that

$$\theta |\alpha - \tilde{\lambda}V|^{a'-1} \geq \frac{1}{2} |p|, \text{ or } (1 - \theta) |\alpha|^{b'-1} \geq \frac{1}{2} |p|.$$

We recall that $\alpha = -H_p(p, V)$, hence

$$\begin{aligned} \tilde{H}_p(p, V) \cdot p - \tilde{H}(p, V) &= \tilde{L}(\alpha, V) \\ &= \frac{\theta}{a'} |\alpha - \tilde{\lambda}V|^{a'} + \frac{1 - \theta}{b'} |\alpha|^{b'}, \\ &\geq \min \left(\frac{|p|^a}{2^a a' \theta^{a-1}}, \frac{|p|^b}{2^b b' (1 - \theta)^{b-1}} \right), \end{aligned}$$

which implies **B3**.

Proof that α is differentiable with respect to V at $(0, 0)$.

Take $V \in \mathbb{R}^d$ that will eventually tend to 0 and $\alpha = \alpha(0, V)$. From (2.6.15) we obtain

$$0 = \theta |\alpha - \tilde{\lambda}V|^{a'-2} (\alpha - \tilde{\lambda}V) + (1 - \theta) |\alpha|^{b'-2} \alpha,$$

Let us recall inequalities (2.6.6) and (2.6.7).

- if $a > b$ then $\frac{(a'-2)(b'-1)}{a'-1} < b' - 2$, and we obtain the following expansion as $|V|$ tends to 0,

$$\alpha = \tilde{\lambda}V + o(|V|).$$

- if $a = b$ we obtain,

$$(2.6.18) \quad \alpha = \tilde{\lambda} \frac{\theta^{a-1}}{\theta^{a-1} + (1 - \theta)^{a-1}} V.$$

- if $a < b$ then $\frac{(a'-2)(b'-1)}{a'-1} > b' - 2$, and we obtain the following estimate as $|V|$ tends to 0,

$$\alpha = o(|V|).$$

Therefore the derivatives of α with respect to V in any of the above three cases are:

$$(2.6.19) \quad D_V \alpha(0, 0) = \begin{cases} \tilde{\lambda} I_d & \text{if } b < a \\ \tilde{\lambda} \frac{\theta^{a-1}}{\theta^{a-1} + (1 - \theta)^{a-1}} I_d & \text{if } b = a \\ 0 & \text{if } b > a. \end{cases}$$

Proof that the operator norm of $D_V \alpha = (\partial_{V^j} \alpha^i)_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d}$ is not larger than λ .

Here, the norm of a square matrix $A \in \mathbb{R}^{d \times d}$ is defined by $\|A\| = \sup_{X \neq 0} \frac{|AX|}{|X|}$. Let us introduce

$$\begin{aligned} v_1 &= \mathbf{1}_{\alpha - \tilde{\lambda}V \neq 0} \frac{\alpha - \tilde{\lambda}V}{|\alpha - \tilde{\lambda}V|}, & B &= I_d + (a' - 2)v_1 v_1^T, \\ v_2 &= \mathbf{1}_{\alpha \neq 0} \frac{\alpha}{|\alpha|}, & C &= I_d + (b' - 2)v_2 v_2^T. \end{aligned}$$

We recall that if $v_i \neq 0$, then $v_i v_i^T$ is the orthogonal projection onto $\mathbb{R}v_i$ for $i = 1, 2$.

If $\alpha = \tilde{\lambda}V = 0$ then $(p, V) = (0, 0)$, we see on (2.6.19) that $D_V \alpha$ is a positive semi-definite matrix with eigenvalues in $[-\lambda, \lambda]$. Therefore, we can now assume that $(\alpha, V) \neq (0, 0)$.

Let us assume temporarily that $a' \neq 2, b' \neq 2, \alpha - \tilde{\lambda}V \neq 0, \alpha \neq 0$. Then we differentiate the i -th component of (2.6.15) with respect to V^j ,

$$\begin{aligned} 0 &= \theta |\alpha - \tilde{\lambda}V|^{a'-2} \left(\partial_{V^j} \alpha^i - \tilde{\lambda} \delta_{i,j} \right) \\ &\quad + \theta (a' - 2) |\alpha - \tilde{\lambda}V|^{a'-4} \sum_{k=1}^d \left(\partial_{V^j} \alpha^k - \tilde{\lambda} \delta_{k,j} \right) \left(\alpha^i - \tilde{\lambda}V^i \right) \left(\alpha^k - \tilde{\lambda}V^k \right) \\ &\quad + (1 - \theta) |\alpha|^{b'-2} \partial_{V^j} \alpha^i + (1 - \theta) (b' - 2) |\alpha|^{b'-4} \sum_{k=1}^d \partial_{V^j} \alpha^k \alpha^i \alpha^k. \end{aligned}$$

This implies

$$0 = \theta |\alpha - \tilde{\lambda}V|^{a'-2} B \left(D_V \alpha - \tilde{\lambda} I_d \right) + (1 - \theta) |\alpha|^{b'-2} C D_V \alpha,$$

and thus

$$(2.6.20) \quad D_V \alpha = \tilde{\lambda} \left[I_d + \frac{(1 - \theta) |\alpha|^{b'-2}}{\theta |\alpha - \tilde{\lambda}V|^{a'-2}} B^{-1} C \right]^{-1}.$$

We can check that this last equation holds in the general case for any $(\alpha, V) \neq (0, 0), a', b'$.

- If $(a' - 2)v_1 = 0$ (i.e. $B = I_d$) or $(b' - 2)v_2 = 0$ (i.e. $C = I_d$), then (2.6.20) yields that $D_V \alpha$ is a positive definite matrix with eigenvalues in $(-\lambda, \lambda)$.
- If $(a' - 2)v_1 \neq 0, (b' - 2)v_2 \neq 0$ and v_1, v_2 are aligned, Then B and C commute and $B^{-1}C$ is a positive definite matrix. Then (2.6.20) yields that $D_V \alpha$ is a positive definite matrix with eigenvalues in $(-\lambda, \lambda)$.
- The last case consists of assuming that $(a' - 2)v_1 \neq 0, (b' - 2)v_2 \neq 0$, and v_1, v_2 are linearly independent. We define k by $k = \frac{(1 - \theta) |\alpha|^{b'-2}}{\theta |\alpha - \tilde{\lambda}V|^{a'-2}} > 0$. The two orthogonal subspaces $\text{Span}(v_1, v_2)$ and $\{v_1, v_2\}^\perp$ are stable by $D_V \alpha, B, C$. The restriction of $D_V \alpha$ to $\{v_1, v_2\}^\perp$ is positive definite with eigenvalues in $(-\lambda, \lambda)$.

Let us denote by $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{M}_{2 \times 2}(\mathbb{R})$ respectively the restriction of $D_V \alpha, B$ and C to $\text{Span}(v_1, v_2)$. We notice that

$$\tilde{B}^{-1} = I_d + ((a' - 1)^{-1} - 1) v_1 v_1^\perp,$$

thus the eigenvalues of \tilde{B}^{-1} are 1 and $(a'-1)^{-1} \leq 1$ since $a' \geq 2$. The eigenvalues of \tilde{C} are 1 and $(b'-1) \geq 1$. Lemma 2.6.8 below yields that $M = (I_d + k\tilde{B}^{-1}\tilde{C})(I_d + k\tilde{C}\tilde{B}^{-1})$ is a positive definite matrix with eigenvalues not smaller than 1. This implies

$$\begin{aligned} \|\tilde{A}X\|^2 &= \lambda^2 \langle M^{-1}X, X \rangle \\ &\leq \lambda^2 \|X\|^2. \end{aligned}$$

This concludes the proof that the norm of $D_V \alpha$ is not larger than λ .

Proof of FP2.

Take $(p, V^1, V^2) \in \mathbb{R}^{3d}$ and $\alpha^i = -\tilde{H}_p(p, V^i)$, $i = 1, 2$, then

$$\begin{aligned} \left| \tilde{H}_p(p, V_1) - \tilde{H}_p(p, V_2) \right| &\leq \sup_{s \in [0,1]} \{ \|D_V \alpha(p, sV_1 + (1-s)V_2)\| \} |V^1 - V^2| \\ &\leq \lambda |V^1 - V^2|. \end{aligned}$$

Combining the latter inequality and (2.6.9), we conclude that FP2 is satisfied.

Proof of FP1.

Let $(p, V) \in \mathbb{R}^{2d}$, we take $\alpha = -H_p(p, V)$.

- We suppose $b' \geq a'$, we make out two cases: the first case is when $|\alpha| \leq |p|^{b-1}$; the second case is when $|\alpha| > |p|^{b-1} = |p|^{\frac{1}{b'-1}}$ which implies

$$\begin{aligned} |\alpha| &\leq \left| \frac{-p + \tilde{\lambda}\theta|\alpha - \tilde{\lambda}V|^{a'-2}V}{\theta|\alpha - \tilde{\lambda}V|^{a'-2} + (1-\theta)|\alpha|^{b'-2}} \right| \\ &\leq \frac{|p|}{(1-\theta)|\alpha|^{b'-2}} + \lambda|V| \\ &\leq (1-\theta)^{-1}|p|^{1-\frac{b'-2}{b'-1}} + \lambda|V|, \end{aligned}$$

using (2.6.16). We recall that $1 - \frac{b'-2}{b'-1} = b - 1$, hence

$$(2.6.21) \quad \left| \tilde{H}_p(p, V) \right| = |\alpha| \leq (1-\theta)^{-1}|p|^{b-1} + \lambda|V|.$$

- We suppose that $b' < a'$, we make out two cases: the first case is when $|\alpha - \tilde{\lambda}V| \leq |p|^{a-1}$; the second case is when $|\alpha - \tilde{\lambda}V| > |p|^{\frac{1}{a'-1}}$ which implies

$$\begin{aligned} |\alpha| &\leq \left| \frac{-p + \tilde{\lambda}\theta|\alpha - \tilde{\lambda}V|^{a'-2}V}{\theta|\alpha - \tilde{\lambda}V|^{a'-2} + (1-\theta)|\alpha|^{b'-2}} \right| \\ &\leq \frac{|p|}{\theta|\alpha - \tilde{\lambda}V|^{a'-2}} + \lambda|V| \\ &\leq \theta^{-1}|p|^{1-\frac{a'-2}{a'-1}} + \lambda|V|, \end{aligned}$$

where we used (2.6.16). From the equality $1 - \frac{a'-2}{a'-1} = a - 1$, we deduce

$$(2.6.22) \quad |H_p(p, V)| = |\alpha| \leq \theta^{-1}|p|^{a-1} + \lambda|V|.$$

This concludes the proof of **FP1**.

Proof of T.

We proved above that $\tilde{\alpha}$ is locally Lipschitz continuous with respect to V and we recall that \tilde{L} is C^1 . Therefore \tilde{H} is also locally Lipschitz with respect to V . This and (2.6.10) implies that **T** holds. \square

Lemma 2.6.8. *Let $B, C \in \mathcal{M}_{2 \times 2}(\mathbb{R})$ be two positive definite matrices with eigenvalues $(1, r)$ and $(1, s)$ respectively, and $0 < r \leq 1, s \geq 1$. Then for any $k > 0$ the matrix M defined by*

$$M = I_d + k(BC + CB) + k^2 BC^2 B,$$

is positive definite with eigenvalues not smaller than 1.

Proof. We can assume that B, C have the following form:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad B = U \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} U^T, \quad \text{with } U \in \mathcal{O}_2(\mathbb{R}),$$

since the eigenvalues of M are invariant by taking the conjugate of B and C by the same orthogonal matrix. The same argument and noticing that C commutes with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, imply that we can assume that U admits a positive determinant, and thus we can write it as

$$U = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix},$$

with $\chi \in [0, 2\pi)$. In this case, M is given by

$$\begin{aligned} M &= I_d + k(BC + CB) + k^2 BC^2 B \\ &\sim I_d + kU^T \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} + k \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} U^T \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} U + k^2 \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} U^T \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}. \end{aligned}$$

We name \tilde{M} the matrix in the last line of the latter calculation, M and \tilde{M} have the same eigenvalues. Let us compute \tilde{M}

$$\tilde{M} = \begin{pmatrix} \cos^2 \chi (1+k)^2 + \sin^2 \chi (1+ks)^2 & -k(s-1)[1+r+kr(1+s)] \cos \chi \sin \chi \\ -k(s-1)[1+r+kr(1+s)] \cos \chi \sin \chi & \cos^2 \chi (1+krs)^2 + \sin^2 \chi (1+kr)^2 \end{pmatrix},$$

its trace is given by

$$\text{tr}(\tilde{M}) = \cos^2 \chi (1+k)^2 + \sin^2 \chi (1+kr)^2 + \cos^2 \chi (1+krs)^2 + \sin^2 \chi (1+ks)^2,$$

and its determinant by

$$\begin{aligned} \det(\tilde{M}) &= (1+k)^2 (1+krs)^2 \cos^4 \chi + (1+kr)^2 (1+ks)^2 \sin^4 \chi \\ &\quad + 2(1+k)(1+kr)(1+ks)(1+krs) \cos^2 \chi \sin^2 \chi \\ &= [(1+k)(1+krs) \cos^2 \chi + (1+kr)(1+ks) \sin^2 \chi]^2. \end{aligned}$$

The eigenvalues of \tilde{M} are the roots of the following second-order polynomial function,

$$X^2 - \text{tr}(\tilde{M})X + \det(\tilde{M}),$$

its smallest root is

$$\frac{1}{2} \left(\text{tr}(\tilde{M}) - \sqrt{\text{tr}^2(\tilde{M}) - 4 \det(\tilde{M})} \right),$$

which is not smaller than 1 if and only if

$$\operatorname{tr}^2(\widetilde{M}) - 4 \det(\widetilde{M}) \leq \left(\operatorname{tr}(\widetilde{M}) - 2 \right)^2.$$

Therefore, it is sufficient to check that $\operatorname{tr}(\widetilde{M}) \leq \det(\widetilde{M}) + 1$ to conclude. We define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} f(x) = & (1+k)^2(1+krs)^2x^2 + (1+kr)^2(1+ks)^2(1-x)^2 \\ & + 2(1+k)(1+kr)(1+ks)(1+krs)x(1-x) \cos^2 \chi (1+k)^2 \\ & + \sin^2 \chi (1+kr)^2 + \cos^2 \chi (1+krs)^2 + \sin^2 \chi (1+ks)^2 + 1. \end{aligned}$$

This is a second-order polynomial in x with

$$\begin{aligned} f(0) &= ((1+kr)^2 - 1) ((1+ks)^2 - 1) \geq 0 \\ f(1) &= ((1+k)^2 - 1) ((1+krs)^2 - 1) \geq 0, \\ f''(x) &= 2[(1+k)(1+krs) - (1+kr)(1+ks)]^2. \end{aligned}$$

If $(1+k)(1+krs) - (1+kr)(1+ks) = 0$, then f is linear and thus $f(x) \geq 0$ for all $x \in [0, 1]$.

If $(1+k)(1+krs) - (1+kr)(1+ks) \neq 0$, then the minimum of this polynomial function on \mathbb{R} is obtained at x_{\min} defined as

$$\begin{aligned} x_{\min} &= \frac{(1+k)^2 + (1+krs)^2 - (1+ks)^2 - (1-kr)^2}{2[(1+k)(1+krs) - (1+kr)(1+ks)]^2} \\ &= \frac{(1-r^2)(1-s^2)k^2 + 2(1-r)(1-s)k}{2[(1+k)(1+krs) - (1+kr)(1+ks)]^2} \leq 0, \end{aligned}$$

since $0 < r \leq 1, s \geq 1$ and $k > 0$. Thus f has no local minimum on $[0, 1]$, then $f(x) \geq 0$ for all $x \in [0, 1]$ since $f(0) \geq 0$ and $f(1) \geq 0$.

Since $\det(\widetilde{M}) - \operatorname{tr}(\widetilde{M}) + 1 = f(\cos^2 \chi) \geq 0$, this concludes the proof of the lemma. \square

Mean Field Games of Controls: Finite Difference Approximations

Joint work with Yves Achdou.

We consider a class of mean field games in which the agents interact through both their states and controls, and we focus on situations in which a generic agent tries to adjust her speed (control) to an average speed (the average is made in a neighborhood in the state space). In such cases, the monotonicity assumptions that are frequently made in the theory of mean field games do not hold, and uniqueness cannot be expected in general.

Such model lead to systems of forward-backward nonlinear nonlocal parabolic equations; the latter are supplemented with various kinds of boundary conditions, in particular Neumann-like boundary conditions stemming from reflection conditions on the underlying controlled stochastic processes.

The present work deals with numerical approximations of the above mentioned systems. After describing the finite difference scheme, we propose an iterative method for solving the systems of nonlinear equations that arise in the discrete setting; it combines a continuation method, Newton iterations and inner loops of a bigradient like solver.

The numerical method is used for simulating two examples. We also make experiments on the behaviour of the iterative algorithm when the parameters of the model vary.

3.1 Introduction

The theory of mean field games, (*MFGs* for short), aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of agents tends to infinity. It supposes that the rational agents are indistinguishable and individually have a negligible influence on the game, and that each individual strategy is influenced by some averages of quantities depending on the states (or the controls as in the present work) of the other agents. MFGs have been introduced in the pioneering works of J-M. Lasry and P-L. Lions [83, 84, 85]. Independently and at approximately the same time, the notion of mean field games arose in the engineering literature, see the works of M.Y. Huang, P.E. Caines and R.Malhamé [69, 70].

The present work deals with numerical approximations of mean field games in which the agents interact through both their states and controls; it follows a more theoretical work by the second author, [75], which is devoted to the mathematical analysis of the related systems of nonlocal partial differential equations. There is not much literature on MFGs in which the agents also interact through their controls, see [60, 59, 38, 41, 25, 75]. To stress the fact that the latter situation is considered, we will sometimes use the terminology *mean field games of control* and the acronym *MFGC*.

The MFGC that is considered in the present work is described by the following system of nonlocal partial differential equations:

$$(3.1.1) \quad \begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla_x u(t, x), \mu(t)) = f(x, m(t)) & \text{in } [0, T] \times \Omega, \\ \partial_t m - \nu \Delta m - \operatorname{div}(H_p(x, \nabla_x u(t, x), \mu(t)) m) = 0 & \text{in } (0, T] \times \Omega, \\ \mu(t) = \left(I_d, -H_p(\cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) & \text{in } [0, T], \\ u(T, x) = \phi(x) & \text{in } \Omega, \\ m(0, x) = m_0(x) & \text{in } \Omega. \end{cases}$$

Here, the state space Ω is a bounded domain of \mathbb{R}^d with a piecewise smooth boundary, while the controls are vectors of \mathbb{R}^d . The time horizon is $T > 0$, and the parameter $\nu > 0$ is linked to the level of noise in the trajectories (the dynamics of a given agent is described by $dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$ where (B_t) is a standard d -dimensional Brownian motion and $\alpha_t \in \mathbb{R}^d$ is the control at time t). A special feature of the present model is that the third argument of the Hamiltonian H is a probability measure $\mu(t) \in \mathcal{P}(\Omega \times \mathbb{R}^d)$, which stands for the joint probability of the states and optimal controls of the agents at time t . In (3.1.1), $u : [0, T] \times \Omega \rightarrow \mathbb{R}$ and $m : [0, T] \times \Omega \rightarrow \mathbb{R}_+$ respectively stand for the value function of a representative agent and the distribution of states. The first, respectively second line in (3.1.1) is the Hamilton-Jacobi-Bellman equation (*HJB* for short) which leads to the optimal control of a representative agent, respectively the Fokker-Planck-Kolmogorov equation (*FP* for short) which describes the transport-diffusion of the distribution of states by the optimal control law. The HJB equation is a backward w.r.t. time parabolic equation and is supplemented with a condition at $t = T$ which involves the terminal cost function $\phi : \Omega \rightarrow \mathbb{R}$, whereas the FP equation is forward w.r.t. time and is supplemented with an initial condition, which translates the fact that the initial distribution of states is known. The HJB equation also involves the so-called *coupling* function $f : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$, or, in other words, $f(X_t, m(t, X(t)))$ is part of the running cost of a representative agent. We shall discuss later the boundary conditions on $(0, T) \times \partial\Omega$ associated with the first two equations in (3.1.1). The third equation in (3.1.1) gives the connection between $\mu(t)$ and $m(t)$, and shows in particular that $\mu(t)$ is supported by a d -dimensional geometrical object, namely the graph of the feedback law: $\Omega \rightarrow \mathbb{R}^d, x \mapsto -H_p(x, \nabla u(t, x), \mu(t))$.

3.1.1 A brief discussion on the mathematical analysis of (3.1.1)

Recall that the Hamiltonian of the problem is $(x, p, \mu) \mapsto H(x, p, \mu)$, $(x, p, \mu) \in \Omega \times \mathbb{R}^d \times \mathcal{P}(\Omega \times \mathbb{R}^d)$.

From the viewpoint of mathematical analysis, a priori estimates for (3.1.1) are more difficult to obtain than in the case when the agents interact only via the distribution of states m . Indeed, in the latter case, if for example the costs f and ϕ are uniformly bounded, then a priori estimates on $\|u\|_\infty$ stem from the maximum principle for second-order parabolic equations. By contrast, since the Hamiltonian in (3.1.1) depends non-locally on $\nabla_x u$, the maximum principle applied to the HJB equation only permits to bound $\|u\|_\infty$ by a quantity which depends (quadratically under standard assumptions on H) on $\|\nabla_x u\|_\infty$, and this information may be useless without additional arguments.

If the agents interact only through the distribution of states and if the Hamiltonian depends separately on p and m , a natural assumption is that the latter is monotone with respect to m , see [83, 84, 85]; it implies existence and uniqueness of solutions, see in [87, 85]. Such an assumption is quite sensible in many situations, since it models the aversion of the agents to highly crowded regions of the state space. It is possible to extend these arguments to MFGCs, see [41] for a probabilistic point of view and [38] for a PDE point of view, and the monotonicity assumption then means that the agents favor controls that are opposite to the main stream. In [75] and in the present work, we prefer to avoid such an assumption, because it is generally not satisfied, at least in models of crowd motion: indeed, in models of traffic or pedestrian flows, a generic agent would rather try to adjust her speed (control) to the average speed in a neighborhood of her position.

The third equation in (3.1.1) can be seen as a fixed point problem for μ given u and m , which turns to be well-posed under the Lasry-Lions monotonicity assumption adapted to MFGC, provided that u and m are smooth enough. We shall replace this assumption by a new structural condition which has been introduced in [75], namely that H_p depends linearly on the variable μ and is a contraction with respect to μ (using a suitable distance on probability measures). In the context of crowd motion, this structural condition is satisfied if the representative agent targets controls that are proportional to an average of the controls chosen by the other agents nearby, with a positive proportionality coefficient smaller than one. Were this coefficient equal to or larger than one, it would be easy to cook up examples in which there is no solution to (3.1.1) or even to the N -agent game, see Remark 3.4.3 below.

In [75], the focus is put on the existence of solutions rather than on their uniqueness. Indeed, without the monotonicity condition, uniqueness is unlikely in general if T is not small. Consider for example a game in which the function ϕ has two perfectly symmetrical minima (the targets), $f = 0$ and where the initial distribution of states has the same symmetry. If H depends on $|\nabla u(t, x)|$ only (no interaction through the controls), then a representative agent will simply travel to the minimum which is closest to her initial position. On the contrary, if the representative agent favors a control close to the average one, then there are at least two symmetrical solutions in which the whole distribution moves toward one of the two minima.

Going back to existence results, it is now frequent in the MFG literature to obtain energy estimates by testing the HJB equation by m , the FP equation by u , summing the resulting equations, integrating in the time and state variables then making suitable integrations by parts. If the value function is uniformly bounded from below (which is often the case even if there are interactions through the controls), this results in a relationship between the $L^\infty([0, T]; L^1(\Omega))$ -norm of the positive part of u and the $L_m^2([0, T] \times \Omega; \mathbb{R}^d)$ -norm of $\nabla_x u$; this observation can then be used to obtain additional a priori estimates,

which may be combined with the ones obtained from the maximum principle and discussed above, and finally with Bernstein method. This strategy has been implemented in [75] and leads to the existence of a solution to (3.1.1) under suitable assumptions. Existence was proved in [75] for periodic problems in any of the following cases:

- short time horizon
- monotonicity
- small enough parameters (in particular the contraction factor mentioned above, see the parameters λ and θ in (3.1.7)-(3.1.10) below)
- weak dependency of the average drift on the state variable

Note that in [43], existence and uniqueness have been proved with probabilistic arguments in the case where the Hamiltonian depends separately on p and μ .

3.1.2 A more detailed description of the considered class of MFGCs

For any $x \in \partial\Omega$, let $n(x)$ be the outward pointing unit normal vector to $\partial\Omega$ at x . The dynamics of a representative agent is given by

$$(3.1.2) \quad dX_t = \alpha_t dt + \sqrt{2\nu} dB_t - 2\nu n(X_t) dL_t, \quad X_t \in \bar{\Omega}, \quad 0 \leq t \leq T,$$

where $(B_t)_{t \in [0, T]}$ is a standard d -dimensional Brownian motion, $(\alpha_t)_{t \in [0, T]}$ is the control, a stochastic process adapted to (B_t) with values in \mathbb{R}^d , L_t is the local time of the process X_t on $\partial\Omega$. We assume that $X_{|t=0}$ is a random variable in $\bar{\Omega}$, independent of (B_t) for all $t > 0$, whose law is $\mathcal{L}(X_{|t=0}) = m_0$.

In what follows, the part of the running cost which models the interactions via the controls will involve an average drift $V \in \mathbb{R}^d$:

$$(3.1.3) \quad V(t, x) = \frac{1}{Z(t, x)} \int_{(y, \alpha) \in \Omega \times \mathbb{R}^d} \alpha K(x, y) d\mu(t, y, \alpha),$$

for $(t, x) \in [0, T] \times \Omega$, where K is a nonnegative kernel, $Z(t, x) = \int_{\Omega} K(x, y) dm(t, y)$ is a normalization factor, $\mu(t, \cdot, \cdot)$ is the law of the joint distribution of the states and the controls, and $m(t, \cdot)$ is the law of the distribution of states.

Hereafter, we are going to focus in MFGCs in which the cost to be minimized by a representative agent is

$$(3.1.4) \quad J(\alpha) = \mathbb{E} \left[\int_0^T \left(\frac{\theta}{2} |\alpha_t - \lambda V(t, X_t)|^2 + \frac{1-\theta}{2} |\alpha_t|^2 + cm(t, X_t) + f_0(X_t) \right) dt + \phi(X_T) \right],$$

where λ, θ are real numbers such that $-1 < \lambda < 1$ and $0 < \theta \leq 1$. This leads us to define the Lagrangian

$$(3.1.5) \quad L(\alpha, V) = \frac{\theta}{2} |\alpha - \lambda V|^2 + \frac{1-\theta}{2} |\alpha|^2, \quad (\alpha, V) \in \mathbb{R}^d \times \mathbb{R}^d,$$

which is convex with respect to α , and its Legendre transform (with respect to α):

$$(3.1.6) \quad H(p, V) = \frac{1}{2} |p - \lambda\theta V|^2 - \frac{\lambda^2\theta}{2} |V|^2, \quad (p, V) \in \mathbb{R}^d \times \mathbb{R}^d.$$

With this definition of the running cost, the first three lines of (3.1.1) can be written as follows:

$$(3.1.7) \quad -\partial_t u - \nu \Delta u + \frac{1}{2} |\nabla_x u - \lambda \theta V|^2 - \frac{\lambda^2 \theta}{2} |V|^2 = cm + f_0(x), \quad \text{in } [0, T) \times \Omega,$$

$$(3.1.8) \quad \partial_t m - \nu \Delta m - \operatorname{div}((\nabla_x u - \lambda \theta V) m) = 0, \quad \text{in } (0, T] \times \Omega,$$

$$(3.1.9) \quad V(t, x) = -\frac{1}{Z(t, x)} \int_{\Omega} (\nabla_x u(t, y) - \lambda \theta V(t, y)) K(x, y) dm(t, y), \quad \text{in } (0, T) \times \Omega,$$

$$(3.1.10) \quad Z(t, x) = \int_{\Omega} K(x, y) dm(t, y), \quad \text{in } [0, T) \times \Omega,$$

$$(3.1.11) \quad u(T, x) = \phi(x), \quad \text{in } \Omega,$$

$$(3.1.12) \quad m(0, x) = m_0(x), \quad \text{in } \Omega.$$

We can now specify the boundary conditions on $(0, T) \times \partial\Omega$. First, assuming that u is smooth enough, the optimal control of a representative player is given by the feedback law:

$$\alpha^*(t, x) = -\left(\nabla_x u(t, x) - \lambda \theta V(t, x)\right).$$

The reflection condition at the boundary translates into the Neumann boundary conditions:

$$(3.1.13) \quad \frac{\partial u}{\partial n}(t, x) = 0, \quad \text{on } [0, T) \times \partial\Omega.$$

Now, from the definition of m , $\int_{\Omega} m(t, x) \psi(x) dx = \mathbb{E} \left(\int_{\Omega} m_0(x) \psi(X_{t,x}) dx \right)$ for all smooth enough test-function ψ such that $\frac{\partial \psi}{\partial n}(x) = 0$ on $\partial\Omega$. Taking the time derivative of the latter equality, and using (3.1.13), we deduce that

$$(3.1.14) \quad \nu \frac{\partial m}{\partial n}(t, x) - \lambda \theta m(t, x) V(t, x) \cdot n(x) = 0, \quad \text{on } (0, T] \times \partial\Omega.$$

Note that (3.1.8), (3.1.13) and (3.1.14) imply that the total mass $\int_{\Omega} m(t, x) dx$ is conserved.

We have studied or will sometimes consider other kinds of boundary conditions on some parts of the boundary, for example:

- periodic conditions, i.e. we set (3.1.7)-(3.1.12) in the flat torus $\Omega = \mathbb{R}^d / \mathbb{Z}^d$
- At a part of the boundary standing for an exit, there is an exit cost. This yields a Dirichlet condition on u . The Dirichlet condition on m : $m = 0$ at exits, means that the agents stop taking part in the mean field game as soon as they reach the exit. Such boundary conditions will be simulated numerically in paragraph 3.4.2 below.
- If a part of the boundary stands for an entrance, then it is natural to impose a Dirichlet condition on u to prevent the agents from crossing the entrance outward, and a flux-like condition on m to specify the entering flux of agents, see also paragraph 3.4.2 below.

3.1.3 Organization of the paper

Section 3.2 is devoted to the description of the finite difference scheme; it is based on a monotone upwind scheme for the HJB equation, and a scheme for the FP equation which is obtained by differentiating the discrete HJB equation and taking the adjoint. It is an extension of the finite difference schemes proposed and studied by the first author and I.

Capuzzo-Dolcetta in [6, 3] for simpler MFGs. Designing an efficient strategy for solving the system of non linear equations arising in the discrete version of MFGCs is a challenging task, in particular because

- the underlying system of PDEs is forward-backward and cannot be solved by simply marching in time
- there is no underlying variational structure
- Equation (3.1.9) is non local.

In Section 3.3, we propose a strategy for solving the system of non linear equations: it is a continuation method in which the viscosity parameter (for instance) is progressively decreased to the desired value, and for each value of the latter parameter, the system of non linear equations is solved by a Newton method with inner iterations of a bi-gradient like algorithm.

Finally, in Section 3.4, we discuss the results of numerical simulations in two cases. In the first example, we show in particular that there exist multiple solutions and we discuss how the iteration count of the solver is affected by the variation of the parameters in the model. The second case is a model for a crowd of agents crossing a rectangular hall from an entrance to an exit: we consider situations in which queues occur, and we show how the solution depends on the parameters.

3.2 Finite difference methods

In order to approximate (3.1.7)-(3.1.12) numerically, we propose a finite difference method reminiscent of that introduced in [6, 3] for MFGs without coupling through the controls. The important features of such methods are as follows

- they are based on monotone finite difference schemes for (3.1.7). Hence, a comparison principle still holds at the discrete level.
- the special structure of (3.1.7)-(3.1.8) is preserved at the discrete level, namely that the FP equation can be obtained by differentiating the HJB equation w.r.t. u and by taking the adjoint of the resulting equation. This results in a monotone approximation of (3.1.8), which ensures that the discrete version of m remains non negative at all time if m_0 is non negative. The discrete FP equation will also preserve mass.

To simplify the discussion, let us focus on the case when $d = 2$ and $\Omega = (0, 1)^2$. More complex domains (even with holes) can be handled by the present method, but an additional effort would be needed if the domain had boundaries not aligned with the axes of the underlying grid that will be introduced soon. We also assume for simplicity that the boundary conditions are of the type (3.1.13)-(3.1.14) on the whole $\partial\Omega$.

3.2.1 Notations and definitions

For two positive integers N_T, N_h , we set $\Delta t = T/N_T$, the time step, and $h = 1/N_h$, the step related to the state variables. Consider the set of discrete times $\mathcal{T}_{\Delta t} = \{t_k = k\Delta t, k = 0, \dots, N_T\}$ and the grid $\Omega_h = \{x_{i,j} = (ih, jh), i, j = 0, \dots, N_h\}$.

The goal will be to approximate $u(t_k, x_{i,j})$ and $m(t_k, x_{i,j})$ respectively by $u_{i,j}^k$ and $m_{i,j}^k$, for all $k \in \{0, \dots, N_T\}$ and $(i, j) \in \{0, \dots, N_h\}^2$, by solving the discrete version of

(3.1.7)-(3.1.14) arising from the finite difference scheme.

Let us define the discrete time derivative of $y : \mathcal{T}_{\Delta t} \times \Omega_h \rightarrow \mathbb{R}$ by the collection of real numbers:

$$(3.2.1) \quad D_t y_{i,j}^k = \frac{y_{i,j}^{k+1} - y_{i,j}^k}{\Delta t},$$

for $k = 0, \dots, N_T - 1$, $i, j = 0, \dots, N_h$.

Given a grid function $y : \Omega_h \rightarrow \mathbb{R}$, we introduce the first order right sided difference operators

$$(3.2.2) \quad \begin{aligned} (D_1^+ y)_{i,j} &= \frac{y_{i+1,j} - y_{i,j}}{h}, \\ (D_2^+ y)_{i,j} &= \frac{y_{i,j+1} - y_{i,j}}{h}, \end{aligned}$$

and the five point discrete laplace operator:

$$(3.2.3) \quad (\Delta_h y)_{i,j} = -\frac{1}{h^2} (4y_{i,j} - y_{i+1,j} - y_{i-1,j} - y_{i,j+1} - y_{i,j-1}),$$

for all (i, j) such that $0 \leq i, j \leq N_h$. Note that the definition of these operators at boundary nodes of Ω_h needs to extend the grid function y on a layer of nodes outside Ω_h . This is done by using discrete versions of the Neumann conditions (3.1.13)- (3.1.14): assume that the boundary condition for y is of the type $\frac{\partial y}{\partial n} = z$, where z is a given continuous function. Let $(z_{i,j}^k)$ is a suitable grid function approximating z . We then choose the discrete version of the latter equation (first order scheme):

$$(3.2.4) \quad \begin{aligned} y_{-1,j}^k &= y_{0,j}^k + h z_{0,j}^k & \text{and} & & y_{N_h+1,j}^k &= y_{N_h,j}^k + h z_{N_h,j}^k, \\ y_{i,-1}^k &= y_{i,0}^k + h z_{i,0}^k & \text{and} & & y_{i,N_h+1}^k &= y_{i,N_h}^k + h z_{i,N_h}^k, \end{aligned}$$

for $i, j = 0, \dots, N_h$ and $k = 0, \dots, N_T$.

Remark 3.2.1. *We have chosen a first order scheme for the boundary condition in order to preserve the monotonicity of the discrete Hamiltonian at boundary nodes, see later. Since the overall scheme is monotone as it was already mentioned above, thus first order, it would be pointless to choose a higher order scheme for the boundary conditions.*

Remark 3.2.2. *Note that the previously mentioned discrete operators are not needed at the nodes of $\partial\Omega_h$ at which Dirichlet boundary conditions are imposed.*

For a grid function $V : \Omega_h \rightarrow \mathbb{R}$, we define the value of V at half-integer indices by linear interpolation:

$$(3.2.5) \quad \begin{aligned} V_{i+\frac{1}{2},j} &= \frac{V_{i+1,j} + V_{i,j}}{2}, & V_{-\frac{1}{2},j} &= V_{0,j}, & \text{and} & & V_{N_h+\frac{1}{2},j} &= V_{N_h,j}, & 0 \leq i \leq N_h - 1, 0 \leq j \leq N_h, \\ V_{i,j+\frac{1}{2}} &= \frac{V_{i,j+1} + V_{i,j}}{2}, & V_{i,-\frac{1}{2}} &= V_{i,0}, & \text{and} & & V_{i,N_h+\frac{1}{2}} &= V_{i,N_h}, & 0 \leq i \leq N_h, 0 \leq j \leq N_h - 1. \end{aligned}$$

In order to define the Godunov scheme for (3.1.7), we introduce the map Φ : for two grid functions $y : \Omega_h \rightarrow \mathbb{R}$ and $V : \Omega_h \rightarrow \mathbb{R}^2$, the grid function $\Phi(y, V) : \Omega_h \rightarrow \mathbb{R}^4$ is

defined by

$$(3.2.6) \quad \Phi(y, V)_{i,j} = \left\{ \begin{array}{l} \left[(D_1^+ y)_{i,j} - \lambda \theta V_{i+\frac{1}{2},j,1} \right]_- \\ - \left[(D_1^+ y)_{i-1,j} - \lambda \theta V_{i-\frac{1}{2},j,1} \right]_+ \\ \left[(D_2^+ y)_{i,j} - \lambda \theta V_{i,j+\frac{1}{2},2} \right]_- \\ - \left[(D_2^+ y)_{i,j-1} - \lambda \theta V_{i,j-\frac{1}{2},2} \right]_+ \end{array} \right\} \quad \text{for } i, j = 0, \dots, N_h.$$

Finally, let us introduce the function $g : \mathbb{R}^4 \times \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$(3.2.7) \quad g(q, V) = \frac{1}{2} |q|^2 - \frac{\lambda^2 \theta}{2} |V|^2.$$

3.2.2 The scheme

With the ingredients defined in paragraph 3.2.1, we are ready to propose the discrete version of (3.1.7)- (3.1.10):

$$(3.2.8) \quad -D_t u_{i,j}^k - \nu \left(\Delta_h u^k \right)_{i,j} + g \left(\Phi \left(u^k, V^k \right)_{i,j}, V_{i,j}^k \right) = c m_{i,j}^{k+1} + f_0(x_{i,j}),$$

$$(3.2.9) \quad D_t m_{i,j}^k - \nu \left(\Delta_h m^{k+1} \right)_{i,j} - \mathcal{T} \left(\Phi \left(u^k, V^k \right), m^{k+1} \right)_{i,j} = 0,$$

$$(3.2.10) \quad V_{i,j,1}^k = -\frac{1}{Z_{i,j}^k} \sum_{r,s} \left(\frac{u_{r+1,s}^k - u_{r-1,s}^k}{2h} - \lambda \theta V_{r,s,1}^k \right) K(x_{i,j}, x_{r,s}) m_{r,s}^{k+1},$$

$$(3.2.11) \quad V_{i,j,2}^k = -\frac{1}{Z_{i,j}^k} \sum_{r,s} \left(\frac{u_{r,s+1}^k - u_{r,s-1}^k}{2h} - \lambda \theta V_{r,s,2}^k \right) K(x_{i,j}, x_{r,s}) m_{r,s}^{k+1},$$

$$(3.2.12) \quad Z_{i,j}^k = \sum_{r,s} K(x_{i,j}, x_{r,s}) m_{r,s}^{k+1},$$

for $i, j = 0, \dots, N_h$, $k = 0, \dots, N_T - 1$, where the discrete transport operator \mathcal{T} is defined by

$$(3.2.13) \quad \mathcal{T}(q, m)_{i,j} = \frac{1}{h} \left(-q_{i,j,1} m_{i,j} + q_{i-1,j,1} m_{i-1,j} + q_{i,j,2} m_{i,j} - q_{i+1,j,2} m_{i+1,j} \right. \\ \left. - q_{i,j,3} m_{i,j} + q_{i,j-1,3} m_{i,j-1} + q_{i,j,4} m_{i,j} - q_{i,j+1,4} m_{i,j+1} \right),$$

for any $i, j = 0, \dots, N_h$. Note that (3.2.6) is not enough in order to fully determine $\mathcal{T}(\Phi(u, V), m)$ at the boundary nodes. We also need the following quantities:

$$(3.2.14) \quad \begin{aligned} \Phi(u, V)_{-1,j,1} &= \left[(D_1^+ u)_{-1,j} - \lambda \theta V_{0,j,1} \right]_- = [\lambda \theta V_{0,j,1}]_+, \\ \Phi(u, V)_{N_h+1,j,2} &= - \left[(D_1^+ u)_{N_h,j} - \lambda \theta V_{N_h,j,1} \right]_+ = - [\lambda \theta V_{N_h,j,1}]_-, \\ \Phi(u, V)_{i,-1,3} &= \left[(D_2^+ u)_{i,-1} - \lambda \theta V_{i,0,2} \right]_- = [\lambda \theta V_{i,0,2}]_+, \\ \Phi(u, V)_{i,N_h+1,4} &= - \left[(D_2^+ u)_{i,N_h} - \lambda \theta V_{i,N_h,2} \right]_+ = - [\lambda \theta V_{i,N_h,2}]_-, \end{aligned}$$

for $i, j = 0, \dots, N_h$, where the last identity in each line comes from (3.2.16) below. The discrete version of (3.1.11) and (3.1.12) is

$$(3.2.15) \quad u_{i,j}^{N_T} = \phi(x_{i,j}), \text{ and } m_{i,j}^0 = m_0(x_{i,j}), \quad 0 \leq i, j \leq N_h.$$

The discrete version of (3.1.13)- (3.1.14) is

$$(3.2.16) \quad \begin{cases} u_{-1,j}^k = u_{0,j}^k & \text{and} & u_{N+1,j}^k = u_{N,j}^k, \\ u_{i,-1}^k = u_{i,0}^k & \text{and} & u_{i,N+1}^k = u_{i,N}^k, \end{cases}$$

$$(3.2.17) \quad \begin{cases} \nu \left(m_{-1,j}^{k+1} - m_{0,j}^{k+1} \right) - h \left[\lambda \theta V_{0,j,1}^k \right]_+ m_{0,j}^{k+1} + h \left[\lambda \theta V_{0,j,1}^k \right]_- m_{-1,j}^{k+1} = 0, \\ \nu \left(m_{N,j}^{k+1} - m_{N+1,j}^{k+1} \right) + h \left[\lambda \theta V_{N,j,1}^k \right]_+ m_{N,j}^{k+1} - h \left[\lambda \theta V_{N,j,1}^k \right]_- m_{N+1,j}^{k+1} = 0, \\ \nu \left(m_{i,-1}^{k+1} - m_{i,0}^{k+1} \right) - h \left[\lambda \theta V_{i,0,1}^k \right]_+ m_{i,0}^{k+1} + h \left[\lambda \theta V_{i,0,1}^k \right]_- m_{i,-1}^{k+1} = 0, \\ \nu \left(m_{i,N}^{k+1} - m_{i,N+1}^{k+1} \right) + h \left[\lambda \theta V_{i,N,1}^k \right]_+ m_{i,N}^{k+1} - h \left[\lambda \theta V_{i,N,1}^k \right]_- m_{i,N+1}^{k+1} = 0, \end{cases}$$

for $i, j = 0, \dots, N_h$ and $k = 0, \dots, N_T - 1$.

Note that (3.2.8) is an implicit scheme for (3.1.7), (recall that (3.1.7) is backward w.r.t. time), whereas (3.2.9) is an implicit scheme for (3.1.8), (recall that (3.1.8) is forward w.r.t. time). This explains why no restriction is made on the time step.

The discrete Hamiltonian introduced in (3.2.7) $g : \mathbb{R}^4 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $(q, V) \mapsto g(q, V)$, has the following properties:

H1 (monotonicity) if $q = (q_1, q_2, q_3, q_4)$, then g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to q_2 and q_4

H2 (consistency) $g(q_1, q_1, q_3, q_3, V) = H(q_1, q_3, V)$

H3 (regularity) g is C^1 .

Remark 3.2.3. We aim at using Newton iterations in order to solve the system of nonlinear equations arising from the discrete scheme. For that purpose, we need to linearize the discrete version of the Fokker-Planck equation. Therefore, for another small positive parameter ε , we may replace the definition (3.2.6) of Φ by the following:

$$(3.2.18) \quad \Phi(y, V)_{i,j} = \begin{cases} \left[(D_1^+ y)_{i,j} - \lambda \theta V_{i+\frac{1}{2},j,1} \right]_{-\varepsilon} \\ - \left[(D_1^+ y)_{i-1,j} - \lambda \theta V_{i-\frac{1}{2},j,1} \right]_{+\varepsilon} \\ \left[(D_2^+ y)_{i,j} - \lambda \theta V_{i,j+\frac{1}{2},2} \right]_{-\varepsilon} \\ - \left[(D_2^+ y)_{i,j-1} - \lambda \theta V_{i,j-\frac{1}{2},2} \right]_{+\varepsilon} \end{cases} \quad \text{for } i, j = 0, \dots, N_h,$$

where the C^1 approximation $a \mapsto a_{+,\varepsilon}$ of $a \mapsto a_+ = a \mathbf{1}_{a>0}$ is defined by

$$(3.2.19) \quad a_{+,\varepsilon} = \frac{1}{2} \left(e^{-\frac{|a|}{\varepsilon}} - 1 \right) \varepsilon + \mathbf{1}_{a>0} a,$$

and $a_{-,\varepsilon} = a_{+,\varepsilon} - a$. We may modify (3.2.14) and (3.2.17) accordingly.

Lemma 3.2.4. Assume that (u, m, V) satisfies (3.2.8)-(3.2.12), (3.2.15), and (3.2.16), (3.2.17).

The total mass of m is preserved, i.e.

$$(3.2.20) \quad \sum_{1 \leq i,j \leq N_h} m_{i,j}^k = \sum_{1 \leq i,j \leq N_h} m_{i,j}^0,$$

for any $k = 0, \dots, N_T$.

Proof. From (3.2.3), we deduce that

$$(3.2.21) \quad \sum_{0 \leq i, j \leq N_h} \nu \left(\Delta m^{k+1} \right)_{i,j} = \frac{\nu}{h^2} \left[\sum_{j=0}^{N_h} \left(m_{N_h+1,j}^{k+1} - m_{N_h,j}^{k+1} - m_{-1,j}^{k+1} + m_{0,j}^{k+1} \right) + \sum_{i=0}^{N_h} \left(m_{i,N_h+1}^{k+1} - m_{i,N_h}^{k+1} - m_{i,-1}^{k+1} + m_{i,0}^{k+1} \right) \right].$$

Then we sum (3.2.13) for $i, j = 0, \dots, N_h$:

$$(3.2.22) \quad \sum_{0 \leq i, j \leq N_h} \mathcal{J}(q, m)_{i,j} = \frac{1}{h} \sum_{j=1}^{N_h} -q_{N_h,j,1} m_{N_h,j} + q_{-1,j,1} m_{-1,j} + q_{0,j,2} m_{0,j} - q_{N_h+1,j,2} m_{N_h+1,j} + \frac{1}{h} \sum_{i=1}^{N_h} -q_{i,N_h,3} m_{i,N_h} + q_{i,-1,3} m_{i,-1} + q_{i,0,4} m_{i,0} - q_{i,N_h+1,4} m_{i,N_h+1}.$$

From (3.2.6), (3.2.14), (3.2.16) and (3.2.17), the sums of the right hand sides of (3.2.22) and (3.2.21) vanish if $q = \Phi(u^k, V^k)$, for any $k = 0, \dots, N_T - 1$.

From the observations above, summing (3.2.9) on $i, j = 0, \dots, N_h$ yields

$$\sum_{0 \leq i, j \leq N_h} m_{i,j}^{k+1} = \sum_{0 \leq i, j \leq N_h} m_{i,j}^k,$$

then the desired result. \square

3.2.3 Solving the discrete version of the Hamilton-Jacobi-Bellman equation

For brevity, we will use the notation $(y_{i,j}^k)$ for a grid function, omitting that the indices i, j take their values in $\{0, \dots, N_h\}$ and that k takes its values in $\{0, \dots, N_T\}$. This paragraph is devoted to solving the system of nonlinear equations satisfied by the grid function $(u_{i,j}^k)$, given the grid functions $(m_{i,j}^k)$ and $(V_{i,j}^k)$. Let $\tilde{f} = (\tilde{f}_{i,j}^k)$ denote the grid function defined by

$$(3.2.23) \quad \tilde{f}_{i,j}^k = c m_{i,j}^{k+1} + f_0(x_{i,j}).$$

We may then rewrite (3.2.8), (3.2.16) and the first identity in (3.2.15) in the compact form

$$(3.2.24) \quad u = \mathcal{F}_u(\tilde{f}, V).$$

Finding u given m and V amounts to solving a discrete version of a nonlinear parabolic equation posed backward in time with Neumann boundary conditions. This is much simpler than solving the complete forward-backward system for (u, m, V) , because a backward time-marching procedure can be used. Since, as it was already observed above, the scheme is implicit, each time step consists of solving the discrete version of a nonlinear elliptic partial differential equation which is local because V is given. Starting from the terminal time step N_T for which (3.2.15) gives an explicit formula for u^{N_T} , the k -th step of the backward loop consists of computing u^k by solving (3.2.8) and (3.2.16) given u^{k+1} , m^{k+1} and V^k . This is done by means of Newton iterations.

For completeness, let us give a few details on Newton iterations. We introduce the operator \mathbb{R}_u ,

$$(3.2.25) \quad \mathbb{R}_u(u', v', f', V')_{i,j} = u'_{i,j} - v'_{i,j} + \Delta t \left[-\nu (\Delta_h u')_{i,j} + g \left(\Phi(u', V')_{i,j}, V'_{i,j} \right) - f'_{i,j} \right],$$

for $u', v', f' : \Omega_h \rightarrow \mathbb{R}$, $V' : \Omega_h \rightarrow \mathbb{R}^2$, and $i, j = 0, \dots, N_h$. Note that in (3.2.25), u' is extended outside Ω_h thanks to (3.2.16). We aim at approximating the solution of

$$(3.2.26) \quad \mathbb{R}_u(u', u^{k+1}, \tilde{f}^k, V^k) = 0.$$

Starting from an initial guess noted $u^{k,0} : \Omega_h \rightarrow \mathbb{R}$, the Newton iterations consist of computing by induction a sequence $u^{k,\ell}$ of approximations of the solution to (3.2.26): given $u^{k,\ell}$, $u^{k,\ell+1}$ is found by solving the system of linear equations

$$d_u \mathbb{R}_u \left(u^{k,\ell}, u^{k+1}, \tilde{f}^k, V^k \right) (u^{k,\ell+1} - u^{k,\ell}) = -\mathbb{R}_u \left(u^{k,\ell}, u^{k+1}, \tilde{f}^k, V^k \right).$$

In the latter equation, the Jacobian matrix $d_u \mathbb{R}_u \left(u^{k,\ell}, u^{k+1}, \tilde{f}^k, V^k \right)$ is sparse since the PDE is local, and invertible since the scheme is monotone. Note that $d_u \mathbb{R}_u \left(u^{k,\ell}, u^{k+1}, \tilde{f}^k, V^k \right)$ depends neither on u^{k+1} nor on \tilde{f}^k , so we can write it $d_u \mathbb{R}_u \left(u^{k,\ell}, \cdot, \cdot, V^k \right)$. The system of linear equations is solved by using special algorithms for sparse matrices. In the numerical simulations presented below, we use the C-library UMFPACK, see [1], implementing unsymmetric multifrontal method and direct sparse LU factorization.

Note that one may choose the initial guess $u^{k,0} = u^{k+1}$. The Newton iterations are stopped when the residual $|\mathbb{R}_u(u^{k,\ell}, u^{k+1}, \tilde{f}^k, V^k)|$ is small enough, say for $\ell = \ell_0$. Then we set $u^k = u^{k,\ell_0}$. It is well known that the Newton algorithm for (3.2.8) is equivalent to an optimal policy iteration algorithm, that it is convergent for any initial guess, and that the convergence is quadratic.

3.2.4 Solving the discrete version of the Fokker-Planck-Kolmogorov equation

This paragraph is devoted to solving the system of linear equations satisfied by the grid function $(m^k_{i,j})$, given the grid functions $(u^k_{i,j})$ and $(V^k_{i,j})$. We may then rewrite (3.2.9), (3.2.17) and the second identity in (3.2.15) in the compact form

$$(3.2.27) \quad m = \mathcal{F}_m(u, V).$$

Finding m given u and V amounts to solving a discrete version of a linear parabolic equation posed forward in time with Neumann boundary conditions. Since the scheme is implicit, each time step consists of solving the discrete version of a linear elliptic partial differential equation which is local. Starting from the terminal time step 0 for which (3.2.15) gives an explicit formula for m^0 , the k -th step of the forward loop consists of computing m^{k+1} by solving (3.2.9), (3.2.17) given m^k , u^{k+1} and V^k . It is easy to see that the matrix of the latter system of linear equations is exactly the transposed of $d_u \mathbb{R}_u \left(u^k, \cdot, \cdot, V^k \right)$ which has been introduced in the previous paragraph. Therefore, it is sparse and invertible. In the numerical simulations presented below, we use the C-library UMFPACK again.

3.3 Newton algorithms for solving the whole system (3.2.8)-(3.2.17)

The solution of the system of nonlinear equations (3.2.8)-(3.2.17) is not easy because

1. the system is the discrete version of a system of forward-backward equations, which precludes a simple time marching algorithm
2. The easiest instances of MFGs can be interpreted as optimal control problems driven by a partial differential equation, which opens the way to algorithms based on the variational structure. In the MFGC considered here, there is no underlying variational principle and these algorithms cannot be used.

Following previous works of the first author, see [3], we choose to use a continuation method (for example with respect to the viscosity parameter ν) in which every system of nonlinear equations (given the parameter of the continuation method) is solved by means of Newton iterations. With Newton algorithm, it is important to have a good initial guess of the solution; for that, we take advantage of the continuation method by choosing the initial guess as the solution obtained with the previous value of the parameter. Alternatively, we have sometimes taken the initial guess from the simulation of the same problem on a coarser grid, using interpolation.

It is also important to implement the Newton algorithm on a “well conditioned” system. Therefore, we shall not directly address (3.2.8)-(3.2.17), but we shall rather eliminate the unknowns u and m by using the time-marching loops described in paragraphs 3.2.3 and 3.2.4 and see (3.2.8)-(3.2.17) as a fixed point problem for (\tilde{f}, V) .

Before describing the algorithm, we need to provide a numerical method for obtaining the average drift given u and m , i.e. for solving (3.2.10)-(3.2.11) at least approximately.

3.3.1 The coupling cost and the average drift

Let us introduce \mathcal{F}_f which maps the grid function m defined on $\mathcal{T}_{\Delta t} \times \Omega_h$ to the grid function \tilde{f} given in (3.2.23). We also define the maps \mathbb{Z} and \mathcal{V} by

$$(3.3.1) \quad \mathbb{Z}(m')_{i,j} = \sum_{r,s} K(x_{i,j}, x_{r,s}) m'_{r,s},$$

$$(3.3.2) \quad \mathcal{V}(u', m', V')_{i,j,1} = - \sum_{r,s} \left(\frac{u'_{r+1,s} - u'_{r-1,s}}{2h} - \lambda \theta V'_{r,s,1} \right) K(x_{i,j}, x_{r,s}) \frac{m'_{r,s}}{\mathbb{Z}(m')_{i,j}},$$

$$(3.3.3) \quad \mathcal{V}(u', m', V')_{i,j,2} = - \sum_{r,s} \left(\frac{u'_{r,s+1} - u'_{r,s-1}}{2h} - \lambda \theta V'_{r,s,2} \right) K(x_{i,j}, x_{r,s}) \frac{m'_{r,s}}{\mathbb{Z}(m')_{i,j}},$$

for $u', m' : \Omega_h \rightarrow \mathbb{R}$, $V' : \Omega_h \rightarrow \mathbb{R}^2$ and $i, j = 0, \dots, N_h$.

It can be proved exactly in the same manner as in the continuous case, see [75, Lemma 2.4], that $V' \mapsto \mathcal{V}(u', m', V')$ is a contraction in the maximum-norm for instance, with a contraction factor $|\lambda|\theta$.

For a positive integer L , we define the map \mathcal{F}_V by :

$$(3.3.4) \quad \mathcal{F}_V(u, m, V) = \tilde{V},$$

$$(3.3.5) \quad \tilde{V}^k = \tilde{V}^{k,L}, \quad \text{for } k = 0, \dots, N_T - 1,$$

where $\tilde{V}^{k,0} = V^k$ and the sequence $\tilde{V}^{k,\ell}$ is defined by the following induction:

$$(3.3.6) \quad \tilde{V}^{k,\ell} = \mathcal{V}(u^k, m^{k+1}, \tilde{V}^{k,\ell-1}), \quad 1 \leq \ell \leq L.$$

Remark 3.3.1. 1. With a slight abuse of notation, if $L = \infty$, then \tilde{V}^k is the fixed point of the map $V' \mapsto \mathcal{V}(u^k, m^{k+1}, V')$.

2. The induction (3.3.6) corresponds to Jacobi iterations and can be easily parallelised. It is also possible to implement Gauss-Seidel iterations, which are a little more complex to write. They consist of using the components of $\tilde{V}^{k,\ell}$ as soon as they are obtained (instead of those of $\tilde{V}^{k,\ell-1}$) in order to compute the components of $\tilde{V}^{k,\ell}$ which have not been obtained yet. In our implementation, we have in fact used the Gauss-Seidel iterations with a lexicographic ordering of the components of the grid functions.

Remark 3.3.2. As we shall see in paragraph 3.4.1.4, the choice of L does not impact the convergence of the overall iterative algorithm. Therefore, a good choice turns out to be $L = 1$.

3.3.2 The linearized operators

3.3.2.1 Notation

Let $u, m, f : \mathcal{T}_{\Delta h} \times \Omega_h \rightarrow \mathbb{R}$ be generic grid functions standing respectively for discrete versions of the value function, the law of the distribution of states, the right hand side of the discrete HJB equation (3.2.8). Let $V : \mathcal{T}_{\Delta h} \times \Omega_h \rightarrow \mathbb{R}^2$ be a generic grid function standing for the average drift. Let us introduce the operators obtained by differentiation of the maps \mathcal{F}_u , \mathcal{F}_m and \mathcal{F}_V :

$$(3.3.7) \quad \begin{aligned} B_{u,f} &= D_f \mathcal{F}_u(f, V), & B_{u,V} &= D_V \mathcal{F}_u(f, V), \\ B_{m,V} &= D_u \mathcal{F}_m(u, V), & B_{m,u} &= D_u \mathcal{F}_m(u, V), \\ C_{f,m} &= D_m \mathcal{F}_f(m), & C_{V,u} &= D_u \mathcal{F}_V(u, m, V), \\ C_{V,m} &= D_m \mathcal{F}_V(u, m, V), & C_{V,V} &= D_V \mathcal{F}_V(u, m, V). \end{aligned}$$

In the three paragraphs below, we explain how these differential operators can be computed.

3.3.2.2 Linearized Hamilton-Jacobi-Bellman equation

The variation du of $u = \mathcal{F}_u(f, V)$ induced by variations df and dV of f and V is given by

$$(3.3.8) \quad du = \begin{pmatrix} B_{u,f} & B_{u,V} \end{pmatrix} \begin{pmatrix} df \\ dV \end{pmatrix}.$$

More explicitly, du is obtained by linearizing (3.2.8), (3.2.16) and (3.2.15). It satisfies

$$(3.3.9) \quad -D_i du_{i,j}^k - \nu \left(\Delta_h du^k \right)_{i,j} + \Phi \left(u^k, V^k \right)_{i,j} \cdot \left(d\Phi \left(u^k, V^k \right) \left(du^k, dV^k \right) \right)_{i,j} - \lambda^2 \theta V_{i,j}^k \cdot dV_{i,j}^k = df_{i,j}^k,$$

for $i, j = 0, \dots, N_h$ and $k = 0, \dots, N_T$. The first (respectively second) inner product appearing in (3.3.9) takes place in \mathbb{R}^4 (respectively \mathbb{R}^2). The set of equations (3.3.9) is supplemented with the terminal condition

$$(3.3.10) \quad du_{i,j}^{N_T} = 0,$$

and the boundary conditions

$$(3.3.11) \quad \begin{cases} du_{-1,j}^k = du_{0,j}^k, & du_{N_h+1,j}^k = du_{N_h,j}^k, \\ du_{i,-1}^k = du_{i,0}^k, & du_{i,N_h+1}^k = du_{i,N_h}^k. \end{cases}$$

Given df and dV , the variation du is found by solving (3.3.9), (3.3.10) and (3.3.11). This is done by marching backward in time and at each time step solving a system of linear equations with a sparse and invertible matrix (of the same form as in the Newton iterations described in 3.2.3). Again, we use the library UMFPACK for that.

3.3.2.3 Linearized Kolmogorov-Fokker-Planck equation

The variation dm of $m = \mathcal{F}_m(u, V)$ induced by variations du and dV of u and V is given by

$$(3.3.12) \quad dm = (B_{m,u} \quad B_{m,V}) \begin{pmatrix} du \\ dV \end{pmatrix}.$$

The grid function dm is obtained by linearizing (3.2.9), (3.2.17) and (3.2.15). It satisfies

$$(3.3.13) \quad D_t dm_{i,j}^k - \nu \left(\Delta_h dm^{k+1} \right)_{i,j} - \mathcal{T}_{i,j} \left(d\Phi \left(u^k, V^k \right) \left(du^k, dV^k \right) \right), m^{k+1} - \mathcal{T}_{i,j} \left(\Phi \left(u^k, V^k \right), dm^{k+1} \right) = 0,$$

for $i, j = 0, \dots, N_h$ and $k = 0, \dots, N_T$. The set of equations (3.3.13) is supplemented with the initial condition

$$(3.3.14) \quad dm_{i,j}^0 = 0, \quad i, j = 0, \dots, N_h,$$

and the boundary conditions

$$(3.3.15) \quad \begin{cases} \nu \left(dm_{-1,j}^{k+1} - dm_{0,j}^{k+1} \right) - h \left[\lambda \theta V_{0,j,1}^k \right]_+ dm_{0,j}^{k+1} + h \left[\lambda \theta V_{0,j,1}^k \right]_- dm_{-1,j}^{k+1} = 0, \\ \nu \left(dm_{N,j}^{k+1} - dm_{N+1,j}^{k+1} \right) + h \left[\lambda \theta V_{N,j,1}^k \right]_+ dm_{N,j}^{k+1} - h \left[\lambda \theta V_{N,j,1}^k \right]_- dm_{N+1,j}^{k+1} = 0, \\ \nu \left(dm_{i,-1}^{k+1} - dm_{i,0}^{k+1} \right) - h \left[\lambda \theta V_{i,0,1}^k \right]_+ dm_{i,0}^{k+1} + h \left[\lambda \theta V_{i,0,1}^k \right]_- dm_{i,-1}^{k+1} = 0, \\ \nu \left(dm_{i,N}^{k+1} - dm_{i,N+1}^{k+1} \right) + h \left[\lambda \theta V_{i,N,1}^k \right]_+ dm_{i,N}^{k+1} - h \left[\lambda \theta V_{i,N,1}^k \right]_- dm_{i,N+1}^{k+1} = 0. \end{cases}$$

Here again, (3.3.13), (3.3.14) and (3.3.15) is solved by marching in time and solving a system of linear equations at each time step (using UMFPACK).

3.3.2.4 Linearized coupling costs and average drifts

The variation of $\tilde{f} = \mathcal{F}_f(m)$ induced by a variation dm of m is obviously given by $d\tilde{f}_{i,j}^k = c dm_{i,j}^k$, hence $C_{f,m} dm = c dm$.

Let us turn to the variation of $\tilde{V} = \mathcal{F}_V(u, m, V)$ induced by the variations du , dm and dV of u , m and V . Differentiating (3.3.4)-(3.3.6) leads to

$$\begin{aligned} d\tilde{V}^{k,0} &= dV^k, \\ d\tilde{V}^{k,\ell} &= \mathcal{V} \left(du^k, m^{k+1}, d\tilde{V}^{k,\ell-1} \right) + d_m \mathcal{V} \left(u^k, m^{k+1}, \tilde{V}^{k,\ell-1} \right) dm^{k+1}, \\ d\tilde{V}^k &= d\tilde{V}^{k,L}, \end{aligned}$$

see (3.3.6) for the definition of $\tilde{V}^{k,L}$.

This permits to compute

$$d\tilde{V} = (C_{V,u} \quad C_{V,m} \quad C_{V,V}) \begin{pmatrix} du \\ dm \\ dV \end{pmatrix}.$$

To summarize,

$$\begin{pmatrix} d\tilde{f} \\ d\tilde{V} \end{pmatrix} = C \begin{pmatrix} du \\ dm \\ dV \end{pmatrix}, \quad \text{with} \quad C = \begin{pmatrix} 0 & C_{f,m} & 0 \\ C_{V,u} & C_{V,m} & C_{V,V} \end{pmatrix}.$$

3.3.3 The algorithm for solving (3.2.8)-(3.2.17)

We see (3.2.8)-(3.2.17) as a fixed point problem for the pair (f, V) , which we write

$$(3.3.16) \quad \mathcal{G}_f(f, V) = 0, \quad \text{and} \quad \mathcal{G}_V(f, V) = 0,$$

where $\mathcal{G}_f, \mathcal{G}_V$ are defined by

$$(3.3.17) \quad \mathcal{G}_f(f, V) = f - \mathcal{F}_f \circ \mathcal{F}_m(\mathcal{F}_u(f, V), V),$$

$$(3.3.18) \quad \mathcal{G}_V(f, V) = V - \mathcal{F}_V(\mathcal{F}_u(f, V), \mathcal{F}_m(\mathcal{F}_u(f, V), V), V).$$

Remark 3.3.3. Given u and m , $V \mapsto W = \mathcal{F}_V(u, m, V)$ is obtained as follows: for each k , computing W^k consists of iterating $W^k \times \mathcal{V}^L(u^k, m^{k+1}, W^k)$ L times starting from $W^k = V^k$. Recall that $\mathcal{V}(u^k, m^{k+1}, \cdot)$ is in (3.3.2) and (3.3.3), and is a contraction with a unique fixed point. Therefore, $\mathcal{F}_V(u, m, \cdot)$ has also a unique fixed point which does not depend on L . Since \mathcal{F}_u and \mathcal{F}_m do not depend on L , neither do the solutions of (3.3.16).

The Newton iterations involve the Jacobian $A(f, V)$ of the map $(f, V) \mapsto (\mathcal{G}_f(f, V), \mathcal{G}_V(f, V))$.

We set

$$(3.3.19) \quad \begin{cases} A_{f,f}(f, V) = D_f \mathcal{G}_f(f, V), & A_{f,V}(f, V) = D_V \mathcal{G}_f(f, V), \\ A_{V,f}(f, V) = D_f \mathcal{G}_V(f, V), & A_{V,V}(f, V) = D_V \mathcal{G}_V(f, V), \end{cases}$$

and the blocks of the Jacobian $A(f, V)$ are defined by:

$$(3.3.20) \quad A_{f,f}(f, V) = I_N - C_{f,m} B_{m,u} B_{u,f},$$

$$(3.3.21) \quad A_{f,V}(f, V) = -C_{f,m} (B_{m,u} B_{u,V} + B_{m,V}),$$

$$(3.3.22) \quad A_{V,f}(f, V) = -C_{V,u} B_{u,f} - C_{V,m} B_{m,u} B_{u,f},$$

$$(3.3.23) \quad A_{V,V}(f, V) = I_{2N} - C_{V,u} B_{u,V} - C_{V,m} (B_{m,u} B_{u,V} + B_{m,V}) - C_{V,V},$$

where $B_{u,f}, B_{u,V}, B_{m,u}, B_{m,V}, C_{V,u}, C_{V,m}, C_{V,V}$ are given by (3.3.7) with $u = \mathcal{F}_u(f, V)$ and $m = \mathcal{F}_m(u, V)$. In an equivalent manner, we can write

$$(3.3.24) \quad A(f, V) = I_{3N} - \begin{pmatrix} C_{f,m} B_{m,u} B_{u,f} & C_{f,m} (B_{m,u} B_{u,V} + B_{m,V}) \\ C_{V,u} B_{u,f} + C_{V,m} B_{m,u} B_{u,f} & C_{V,u} B_{u,V} + C_{V,m} (B_{m,u} B_{u,V} + B_{m,V}) + C_{V,V} \end{pmatrix},$$

or

$$(3.3.25) \quad A(f, V) = I_{3N} - \begin{pmatrix} 0_N & C_{f,m} & 0_{N,2N} \\ C_{V,u} & C_{V,m} & C_{V,V} \end{pmatrix} \begin{pmatrix} I_N & 0_{N,2N} \\ B_{m,u} & B_{m,V} \\ 0_{2N,N} & I_{2N} \end{pmatrix} \begin{pmatrix} B_{u,f} & B_{u,V} \\ 0_{2N,N} & I_{2N} \end{pmatrix}.$$

Every Newton iteration for solving (3.3.16) consists of solving a system of linear equations of the form

$$(3.3.26) \quad A(f^\ell, V^\ell) \begin{pmatrix} f^{\ell+1} - f^\ell \\ V^{\ell+1} - V^\ell \end{pmatrix} = - \begin{pmatrix} \mathcal{G}_f(f^\ell, V^\ell) \\ \mathcal{G}_V(f^\ell, V^\ell) \end{pmatrix}.$$

In our implementation, this system is solved iteratively by BiCGStab algorithm, see [101]. Note that BiCGStab algorithm only requires a function that computes

$$A(f^\ell, V^\ell) \begin{pmatrix} \hat{f} \\ \hat{V} \end{pmatrix},$$

for any grid functions $\hat{f} : \mathcal{T}_{\Delta t} \times \Omega_h \rightarrow \mathbb{R}$ and $\hat{V} : \mathcal{T}_{\Delta t} \times \Omega_h \rightarrow \mathbb{R}^2$. This is done using (3.3.25) and combining the methods described in paragraph 3.3.2. The assembly of the Jacobian matrix is not needed.

3.4 Numerical simulations

We are going to discuss the results of numerical simulations in two cases, both related to crowd motion.

3.4.1 First example

3.4.1.1 Description of the model

The state space is the square $\Omega = (-0.5, 0.5)^2$ and the time horizon is $T = 1$.

We consider the MFGC described by (3.1.7)-(3.1.14), in which

1. The kernel k in (3.1.9) is radial, i.e. of the form $K(x, y) = K_\rho(|x - y|)$. Here $r \mapsto K_\rho(r)$ is a non-increasing C^1 function defined on \mathbb{R}_+ , with

$$(3.4.1) \quad K_\rho(r) = 0, \quad \text{if } r \geq \rho, \quad \text{and} \quad K_\rho(r) = 1, \quad \text{if } r \leq 0.9\rho,$$

for a positive number $\rho > 0$, which is the radius of the disc in the state space that a representative agent uses for computing the average of the controls.

2. The parameter c in (3.1.7) is chosen as $c = 10^{-3}$. Recall that the cost $cm(t, x)$ reflects the aversion of a representative player to crowded regions of the state space.
3. At time $t = 0$, the agents are distributed in the top-right and the bottom-left corners of the domain Ω , see the left part of Figure 3.1. The density is piecewise constant, with values appearing on Figure 3.1.
4. The terminal cost is also piecewise constant. It takes a small value in the top-left and the bottom-right corners of the domain Ω , see the right part of Figure 3.1 for the chosen values. This cost attracts the agents to the latter two corners of Ω .
5. The function f_0 in (3.1.7) is chosen to be proportional to the terminal cost, with a factor 0.1. This term is linked to the running cost and has the same effect as the terminal cost.
6. Recall that the boundary conditions (3.1.13)-(3.2.17) rule out the entry or exit of agents. The total mass of m is conserved.

Remark 3.4.1. *Note that the problem is symmetric with respect to the two diagonals of the square domain. The grid of the domain is chosen to have the same symmetry.*

Remark 3.4.2. *The two parts of the running cost: $cm(t, X_t)$ and $\frac{\theta}{2} |\alpha - \lambda V(t, X_t)|^2$ may have competing effects: indeed, the former cost may incitate the agents to spread all over the state space, whereas the latter may result in the agents selecting the same controls therefore staying grouped. Although the constant $c = 10^{-3}$ seems small, it is chosen in such a way that the above-mentioned two costs have the same orders of magnitude for e.g. $\theta = 1$ and $\lambda = 0.9$.*

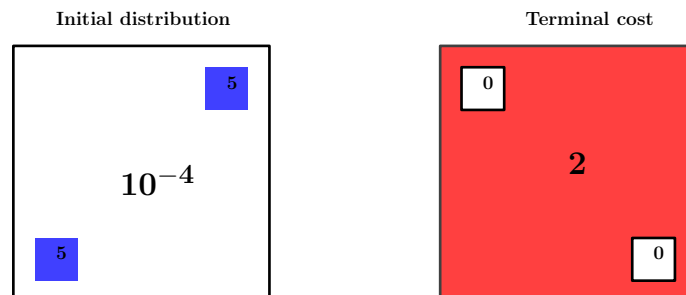


Figure 3.1 – Example 1. Left: the initial distribution of states. Right: the terminal cost.

3.4.1.2 The results of the simulations: discussion

In the present simulation, we choose the parameters as follows:

$$\nu = 10^{-3}, \quad \theta = 1, \quad \lambda = 0.9, \quad \rho = 0.2, \quad c = 10^{-3}.$$

On Figure 3.2, we display snapshots of m , u and the optimal feedback law at several times. Since both the problem and the grid are symmetric with respect to the two diagonals, m , u and the feedback have the same symmetry for all times. Let us describe the evolution of the distribution of states, that we can name “*gathering-kissing-splitting*” referring to the “*drafting-kissing-tumbling*” phenomenon in fluid mechanics (for the interaction of particles in a fluid).

- The term $cm(t, X_t)$ in the running cost prevents the part of the distribution initially supported in one of the opposite corners (say the bottom-left corner) to travel directly to one of the targets (say the bottom-right corner). On the other hand, the interaction through controls prevents this part of the distribution to split into two equal parts which would travel directly to the two targets, because the agents favor controls close to the local average. Therefore, the part of the distribution initially supported in one of the opposite corners first travels to the center of the domain. This is the *gathering* phase of the evolution.
- The two parts of the distribution reach the center of the domain, where the local average of the velocity becomes small; the dominating cost then becomes the one which attracts the agents to the bottom-right and top-left corners. Because of the repulsive effect due to the part $cm(t, X_t)$ of the running cost, the part of the distribution initially supported in one of the opposite corners (say the bottom-left corner) and having traveled toward the center of the domains splits into two parts which make each a ninety-degrees turn. This is the *kissing* phase of the evolution.
- After the *kissing* phase, the distribution of states *splits* into parts which travel to the targets.

Finally, we see that the paths followed by the agents is far from being a shortest path to the targets.

Let us compare these results with a simulation of the MFG obtained by cancelling either θ or λ in (3.1.7)-(3.1.14) while keeping all the other parameters and the grid unchanged. Since the problems remains symmetric with respect to the two diagonals, the obtained results have the same symmetry. On Figure 3.3, we display snapshots of m , u and the optimal feedback law at several times. We see that the evolution is quite different from that displayed on Figure 3.2, since the two parts of the distribution initially supported in two opposite corners of the domain symmetrically split into two parts each, which travels directly to the targets. The initial splitting phase is due to the coupling cost $cm(t, X_t)$ which favors the spread of the distribution. The path followed by the agents is close to a shortest path to the targets.

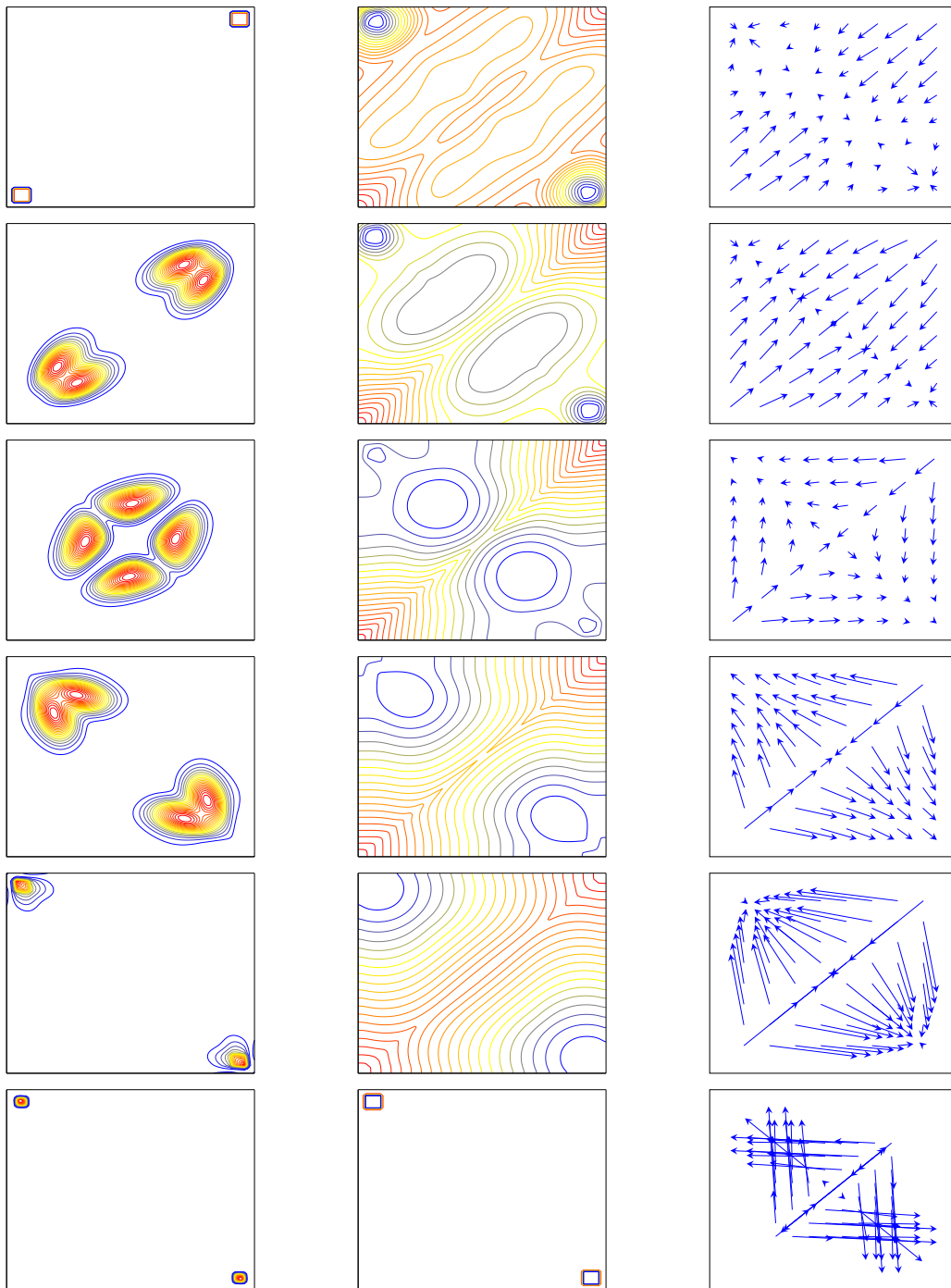


Figure 3.2 – Example 1. Gathering-kissing-splitting; snapshots at $t = 0, 0.2, 0.4, 0.6, 0.8, 1$. Left: contours of m . Center: contours of u . Right: optimal feedback α .

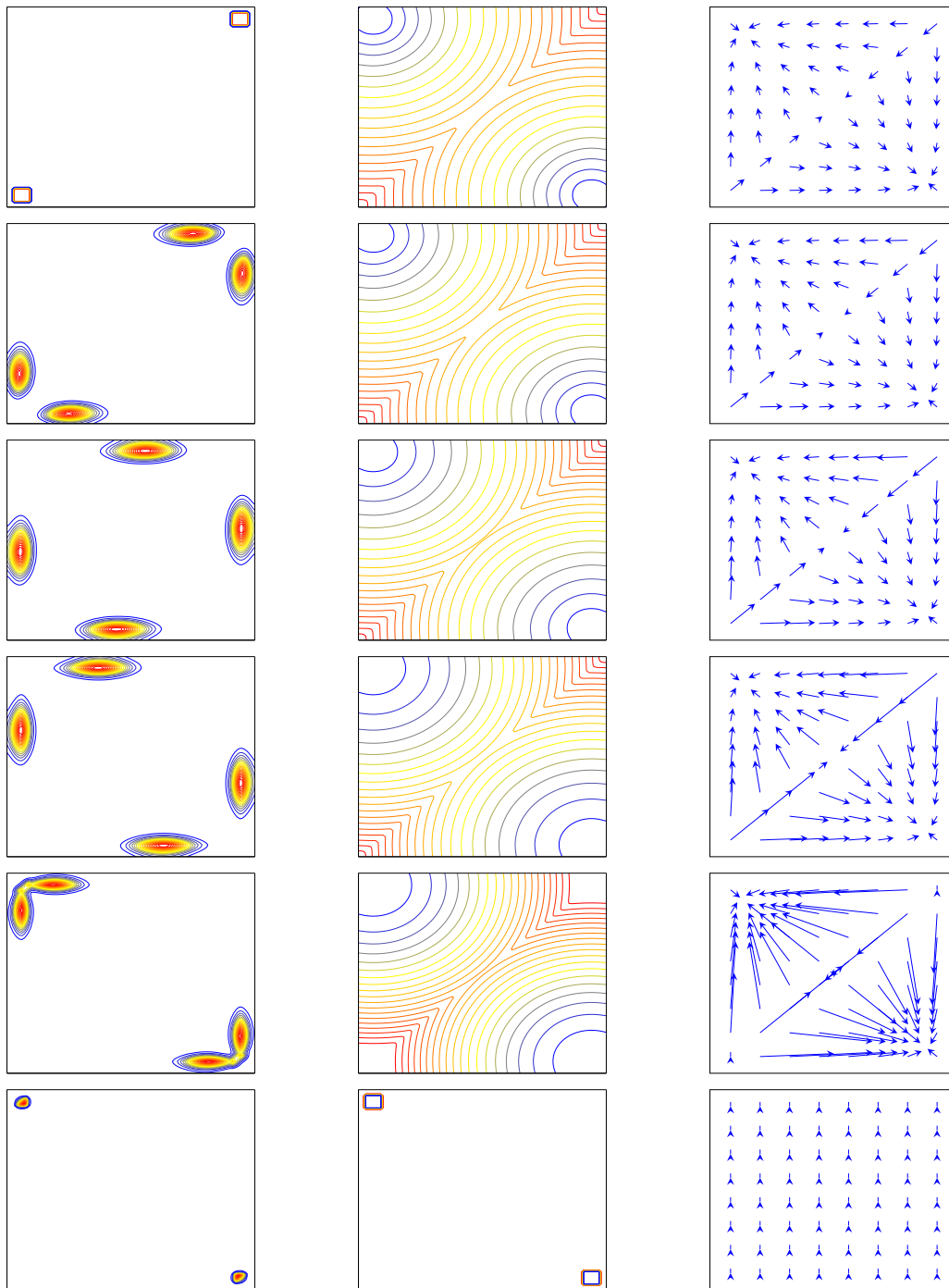


Figure 3.3 – Example 1. Same parameters except $\theta\lambda = 0$; snapshots at $t = 0, 0.2, 0.4, 0.6, 0.8, 1$. Left: contours of m . Center: contours of u . Right: optimal feedback α .

3.4.1.3 Non-uniqueness of solutions

The solution of the MFGC displayed on Figure 3.2 is likely to be unstable because the paths followed by the agents are significantly longer than the shortest paths to the targets. We expect that there are other solutions. We are going to see that this is indeed the case. For that purpose, we are going to introduce a vanishing perturbation of the initial distribution which breaks the symmetry of the problem. This will lead to different

solutions to (3.1.7)-(3.1.14). An example of perturbation is displayed on Figure 3.4. It consists of adding very little mass at the top and the bottom of the domain; the perturbed distribution is no longer symmetric with respect to the diagonals. We expect that the agents initially distributed near the bottom target will go to the right, and that all the agents initially distributed at the bottom of the domain will follow them, because of the interaction through controls. Similarly, we expect that all the agents initially located at the top of the domain will go to the left.

In our simulations, we use a continuation method, i.e. we consider perturbations corresponding to a decreasing sequence of nonnegative parameters $(\pi_n)_{n \in 0, \dots, N}$. The last value $\pi_N = 0$ corresponds to the distribution displayed on the left of Figure 3.1. For each new value π_{n+1} of the parameter, we initialize the Newton iterations described in Section 3.3 by the solution corresponding to the preceding value π_n . For positive values of π_n , the simulated solution is not symmetric with respect to the diagonals, and this property is preserved for the last value $\pi_N = 0$.

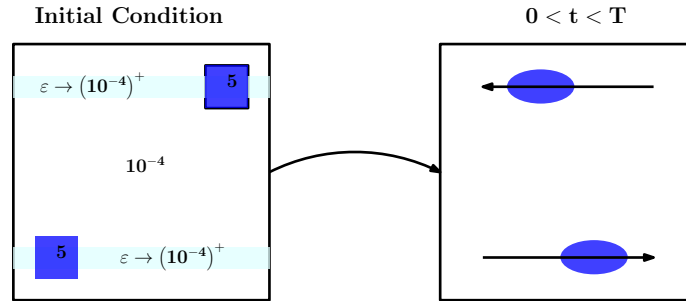


Figure 3.4 – Example 1. A continuation method leading to a different solution. Left: a vanishing perturbation of the initial distribution. Right: the expected evolution of the distribution.

On Figure 3.5, we display three different solutions (the distribution of states at different times) obtained with three different sequences of vanishing perturbations of the initial distribution displayed on Figure 3.1: the solution displayed on the left corresponds to the sequence of perturbations displayed on Figure 3.4; applying to the perturbation a symmetry with respect to a diagonal, we obtain the solution displayed on the center; the solution displayed on the right is obtained by another kind of perturbation symmetric with respect to one diagonal only, located at the top and at the left of the domain: for this solution, we again notice gathering and kissing phases, and that the paths followed by the agents deviate from the shortest ones.

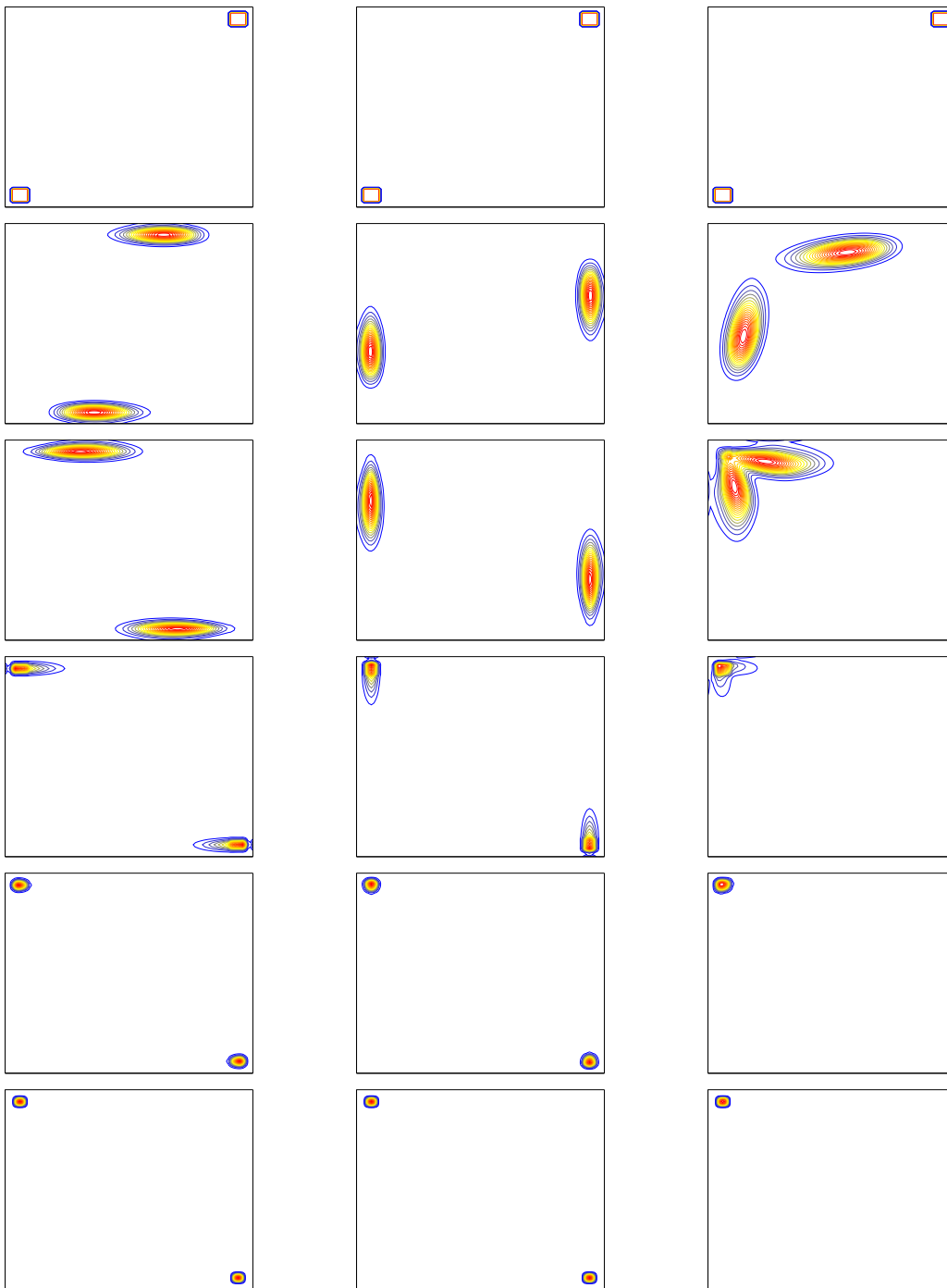


Figure 3.5 – Example 1. Different solutions obtained by adding vanishing perturbations to the initial distribution of states. The distribution at time $t = 0, 0.2, 0.4, 0.6, 0.8, 1$. The three columns correspond to three different sequences of perturbations.

3.4.1.4 Behaviour of the algorithm

In this paragraph, we investigate how the behaviour of the algorithm described in Section 3.3 is affected by the variations of the parameters in the model. Recall that the algorithm is based on Newton iterations with a BiCGStab inner loop at each step.

We start by the grid size and the viscosity parameter ν : in the experiments reported below, the stopping criterion for the outer Newton iteration is that the normalized Eu-

clidean norm of the residual is smaller than 10^{-8} . The stopping criterion for the inner BiCGStab iterations is that the ratio of the norms of the current and the initial residuals is smaller than 10^{-7} (hence, it involves a relative error). The parameters of the model are set to

$$\theta = 1, \quad \lambda = 0.9, \quad c = 10^{-3}, \quad \rho = 0.2, \quad L = 1.$$

Table 3.1 displays the iterations counts for the outer Newton and the inner BiCGStab loops for different viscosities and grid sizes. We notice that the number of iterations is not very sensitive to the grid size, as expected because the map $\mathcal{G} - I_d$, with \mathcal{G} defined in (3.3.17) (3.3.18), is the discrete version of a map which has compactness properties. The iteration counts increase as ν is decreased.

Table 3.1 – Number of outer Newton iterations and average number of inner BiCGStab iterations per Newton step (the stopping criteria are given in the text) with different viscosities and grid sizes.

ν	$26 \times 26 \times 26$		$51 \times 51 \times 51$		$76 \times 76 \times 76$		$101 \times 101 \times 101$	
	Newton	BiCGStab	Newton	BiCGStab	Newton	BiCGStab	Newton	BiCGStab
0.5	3	13.67	3	12.67	3	15	3	14.33
0.1	9	14.67	10	21.9	12	15.08	12	18.25
0.05	4	16.5	4	17.25	4	16.25	4	19.25
0.01	4	42.5	4	29	5	30.4	5	28.8

Table 3.2 displays the iteration counts for different choices of λ and θ , and fixed values the other parameters:

$$\nu = 10^{-3}, \quad c = 10^{-3}, \quad \rho = 0.2, \quad L = 1,$$

with a 101×101 nodes grid and 101 time steps.

In fact, the numbers in Table 3.2 are obtained by running a continuation method in θ and λ : for each cell of the table, the solution corresponding either to the left or the upper neighboring cell is used as an initial guess for the Newton iterations. If a cell has two such neighbors, the choice of the initial guess is made as follows: in the top-right triangular part of the table, strictly above the diagonal, we choose the initial guess corresponding to the left neighboring cell. In the bottom-left triangular part of the table, including the diagonal, we choose the initial guess corresponding to the cell immediately above.

We see that changing λ and θ mostly impacts the number of iterations of the inner loop. A reason for that is that V is obtained by solving a linear fixed point problem whose contraction factor is $\lambda\theta$. Hence, it is sensible that the number of iterations necessary to solve the systems of linear equations arising in the Newton steps increases as $\lambda\theta$ tends to 1.

Recall that we use a continuation method, consisting of decreasing progressively ν until it reaches the desired value; in Figure 3.6 (respectively Figure 3.7), we plot the average number of BiCGStab iterations versus ν , (respectively the number of Newton iterations versus ν) for different values of λ , θ and c . The stopping criteria have been described above. The grid contains 101×101 nodes and there are 101 time steps. We recover the information contained in Tables 3.1 and 3.2, i.e. that ν mostly impacts the number of iterations in the inner loop. We also notice that when ν is large, the number of iterations seems to depend more on λ and θ than on c and that it becomes highly sensitive to c when ν is small, but we do not really know how to explain this. The number of iterations in the outer Newton loop seems much less sensitive than the number of inner BiCGStab iterations. Looking

Table 3.2 – Sensitivity to λ and θ .

(a) Average number of BiCGStab iterations. (b) number of Newton iterations.

$\lambda \backslash \theta$	0.2	0.4	0.6	0.8	1	$\lambda \backslash \theta$	0.2	0.4	0.6	0.8	1
0.2	3.5	4.5	7	8	8	0.2	2	2	2	2	2
0.4	4.5	7	8	9	10.67	0.4	2	3	2	3	3
0.6	5	8	9	12	15.67	0.6	2	3	3	3	3
0.8	7	8.67	10.67	15.67	27.25	0.8	3	3	3	3	4
0.9	7.33	8.67	12	18	40	0.9	3	3	3	3	6

at the overall number of BiCGStab iterations, they are not very different for the choices $(\lambda, \theta, c) = (0.9, 1, 0)$ and $(\lambda, \theta, c) = (0, 0, 0.1)$, since the latter case needs more Newton iterations but less BiCGStab iterations per Newton steps.

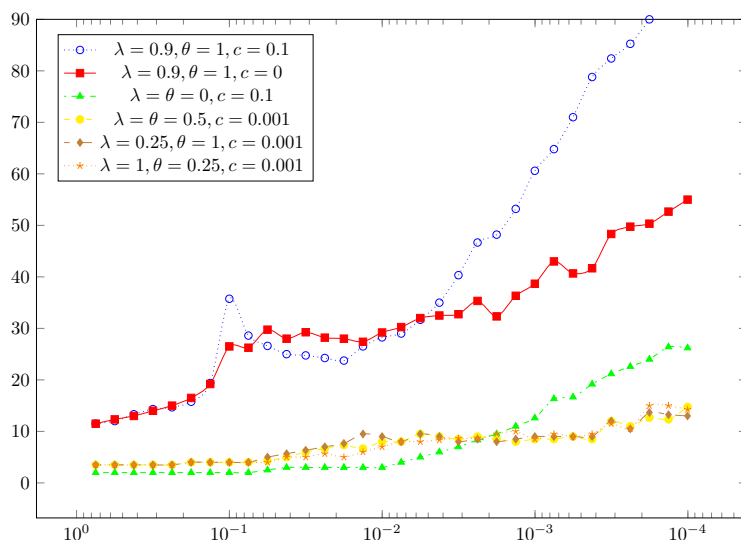


Figure 3.6 – Average number of iterations of BiCGStab iterations per Newton step versus ν for different choices of λ , θ and c .

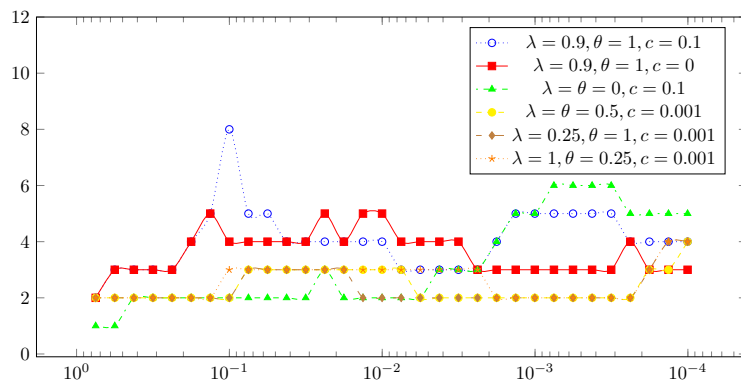


Figure 3.7 – Number of iterations of Newton iterations versus ν for different choices of λ , θ and c .

Finally, we investigate the sensitivity of the algorithm to L , the number of fixed point

iterations yielding a proxy of V , see paragraph 3.3.1. Table 3.3 contains the iteration counts of the inner and outer loops for different values of ν and L , and two choices of (λ, θ) (all the other parameters are fixed $c = 10^{-3}$, $\rho = 0.2$). We notice that the choice of L seems to have little impact on the iteration count. For these reasons, $L = 1$ appears to be a good choice.

Table 3.3 – Number of outer Newton iterations and average number of inner BiCGStab iterations per Newton step (the stopping criteria are given in the text) with different viscosities and L .

$\lambda = \theta = 0.8$								
$\nu \backslash L$	1		2		3		5	
	Newton	BiCGStab	Newton	BiCGStab	Newton	BiCGStab	Newton	BiCGStab
0.1	3	7.33	3	6	3	6	3	5.33
0.01	3	13.67	3	13	3	15.33	3	13.67
0.001	2	14.5	2	14.5	2	17.5	2	16
0.0001	3	22	3	22.67	3	22.33	3	24.67
$\lambda = 0.9, \theta = 1$								
$\nu \backslash L$	1		2		3		5	
	Newton	BiCGStab	Newton	BiCGStab	Newton	BiCGStab	Newton	BiCGStab
0.1	4	26.75	4	19.75	4	19.5	4	19.25
0.01	4	29	4	25.75	5	28	5	29
0.001	3	36.67	3	34	3	38.67	3	37.67
0.0001	3	55	3	58.33	3	64	3	63.33

3.4.2 Second example

As a second example, we consider a model for a crowd crossing a hall, with an entrance at the left and an exit at the right. Roughly speaking, a generic agent will interact with those located in a cone ahead of her.

3.4.2.1 Description of the model

The state space is the rectangle $\Omega = (-1, 1) \times (-0.1, 0.1)$ and the time horizon is $T = 8$. Let ℓ denote the length of the bottom and upper sides of $\partial\Omega$: $\ell = 2$.

The boundary of Ω is split into three parts Γ_{ND} , Γ_{NN} and Γ_{DD} which respectively stand for an entrance, some walls and an exit:

$$(3.4.2) \quad \Gamma_{DD} = \{1\} \times [-0.05, 0.05], \quad \Gamma_{ND} = \{-1\} \times [-0.05, 0.05], \quad \Gamma_{NN} = \partial\Omega \setminus (\Gamma_{DD} \cup \Gamma_{ND}),$$

and we consider the following boundary conditions:

$$(3.4.3) \quad \left. \begin{aligned} \frac{\partial u}{\partial n}(t, x) &= 0 \\ \nu \frac{\partial m}{\partial n}(t, x) - \lambda \theta m(t, x) V(t, x) \cdot n(x) &= 0 \end{aligned} \right\} \quad \text{on } \Gamma_{NN},$$

$$(3.4.4) \quad \left. \begin{aligned} u(t, x) &= 6 \\ \nu \frac{\partial m}{\partial n}(t, x) + m(t, x) H_p(\nabla u(t, x), V(t, x)) \cdot n(x) &= -2 \end{aligned} \right\} \quad \text{on } \Gamma_{ND},$$

$$(3.4.5) \quad \left. \begin{aligned} u(t, x) &= -4 \\ m(t, x) &= 0 \end{aligned} \right\} \quad \text{on } \Gamma_{DD},$$

where H is defined below.

The system of PDEs is still (3.1.7)-(3.1.10), with the following new features:

- compared to (3.1.5), the part of the running cost accounting for the interaction through controls is multiplied by a positive normalization factor a :

$$(3.4.6) \quad L(\alpha, V) = a \frac{\theta}{2} |\alpha - \lambda V|^2 + a \frac{1 - \theta}{2} |\alpha|^2,$$

and the associated Hamiltonian by the Legendre's transform is

$$(3.4.7) \quad H(p, V) = \frac{a}{2} \left| \frac{p}{a} - \lambda \theta V \right|^2 - a \frac{\lambda^2 \theta}{2} |V|^2.$$

We choose the normalization factor a as follows:

$$(3.4.8) \quad a = \tilde{a}(1 - \lambda^2 \theta)^{-1},$$

where \tilde{a} is a positive constant (independent of λ or θ). As we shall see later, the normalization allows us to compare the solutions obtained with different values of λ and θ . We shall also see that this normalization is specific to the example under consideration and may not be relevant in other situations.

- The kernel K is given by

$$(3.4.9) \quad K(x, y) = \mathbf{1}_{y_1 \geq x_1} K_\rho(|x - y|) \kappa(|\omega|),$$

where ω is the angle made by the vector $y - x$ with the vector $(1, 0)$ and

1. k_ρ is defined as in the first example
2. κ is a nonincreasing C^1 function defined such that

$$\kappa = \begin{cases} 1, & \text{if } 0 \leq \omega \leq 0.9\omega_0, \\ 0, & \text{if } \omega \geq \omega_0, \end{cases}$$

for a given angle $\omega_0 \in (0, \frac{\pi}{2})$.

- The parameter c in (3.1.7) is chosen to be zero
- We shall consider various functions f_0 modeling the cost for staying at a given point in the domain.

The initial condition is $m_0(x) = 10^{-4}$ for all $x \in \Omega$, and the terminal cost is 0.

3.4.2.2 The case when f is constant: comparisons with a one-dimensional problem

In this paragraph, we choose $f_0(x) = F$ for all $x \in \Omega$.

Simplification: a one-shot one-dimensional game We approximate the MFGC by a one-shot one-dimensional game. The state space is the interval $[0, \ell]$. The points $x = 0$ and $x = \ell$ respectively stand for an entrance and an exit. When she enters the domain, a representative agent chooses her drift $\alpha \in \mathbb{R}_+$ once and for all, and her dynamics is given by $x'(t) = \alpha$ for $t = \ell/\alpha$. The distribution of drifts is a probability measure μ on \mathbb{R}_+ and the average drift is $V = \int_{\alpha \in \mathbb{R}_+} \alpha d\mu(\alpha)$. There is no entry or exit costs. With the running cost $L(\alpha, V) + F$, where L is given by (3.4.6), the mean field Nash equilibrium reads

$$\text{support}(\mu) \subset \operatorname{argmin}_{\alpha} \frac{\ell}{\alpha} \left(a \left(\frac{\theta}{2} (\alpha - \lambda V)^2 + \frac{1-\theta}{2} \alpha^2 \right) + F \right).$$

Given V , one checks that the unique solution of the minimization problem in α is

$$(3.4.10) \quad \alpha^*(V) = \left(\frac{2F + a\theta\lambda^2 V^2}{a} \right)^{\frac{1}{2}}.$$

Hence $\mu = \delta_{\alpha^*(V)}$, and the static mean field Nash equilibrium reads

$$V = \left(\frac{2F + a\theta\lambda^2 V^2}{a} \right)^{\frac{1}{2}},$$

which yields

$$(3.4.11) \quad \alpha^* = V = \left(\frac{2F}{a(1-\theta\lambda^2)} \right)^{\frac{1}{2}}.$$

This leads us to choose $a = \tilde{a}/(1-\theta\lambda^2)$, which yields $\alpha^* = V = \sqrt{\frac{2F}{\tilde{a}}}$. Note that the value corresponding to the mean field Nash equilibrium is

$$(3.4.12) \quad J_{\text{MFG}} = \frac{\ell}{\alpha^*} \left(a \left(\frac{\theta}{2} (\alpha^* - \lambda \alpha^*)^2 + \frac{1-\theta}{2} (\alpha^*)^2 \right) + F \right) = \ell \frac{1-\lambda\theta}{1-\lambda^2\theta} \sqrt{2F\tilde{a}}.$$

Remark 3.4.3. Note that if $\theta = \lambda = 1$, then (3.4.10) implies that $\alpha^*(V) > V$, while an equilibrium corresponds to $V = \alpha^*(V)$. Thus, there is no mean field equilibrium if $\theta = \lambda = 1$.

Remark 3.4.4 (Comparison with the one-shot mean field type control problem). The Nash equilibrium must be distinguished from the situation in which the drift of the agents is determined in order to minimize the global cost; the latter problem belongs to the class of mean field type control problems (MFTC for short) studied by A. Bensoussan and his collaborators, [21] and R. Carmona and F. Delarue, [40].

In this case, the problem reads

$$\inf_{\mu} \int_{\alpha \in \mathbb{R}_+} \frac{\ell}{\alpha} \left(a \left(\frac{\theta}{2} \left(\alpha - \lambda \int_{\beta \in \mathbb{R}_+} \beta d\mu(\beta) \right)^2 + \frac{1-\theta}{2} \alpha^2 \right) + F \right) d\mu(\alpha).$$

Looking for the solution as the Dirac mass $\delta_{\alpha^{**}}$, we see that α^{**} minimizes

$$\alpha \mapsto \frac{\ell}{\alpha} \left(a \left(\frac{\theta}{2} (\alpha - \lambda \alpha)^2 + \frac{1-\theta}{2} \alpha^2 \right) + F \right),$$

thus

$$\alpha^{**} = \left(\frac{2F}{a(1 - 2\theta\lambda + \theta\lambda^2)} \right)^{\frac{1}{2}}.$$

The value of the problem is

$$J_{MFTC} = \ell \sqrt{2Fa(1 - \lambda\theta(2 - \lambda))} \leq \frac{\ell}{2} \sqrt{2Fa} \frac{1 - \lambda\theta(2 - \lambda)}{\sqrt{1 - \lambda^2\theta}} + \frac{\ell}{2} \sqrt{2Fa(1 - \lambda^2\theta)} = J_{MFG},$$

from Young's inequality $y \leq \frac{y^2}{2z} + \frac{z}{2}$, with $y = \sqrt{1 - \lambda\theta(2 - \lambda)}$ and $z = \sqrt{1 - \lambda^2\theta}$. The value of the MFTC is lower than the value of the MFG, which is natural since in the former case, the agents are collaborating to minimize the global cost. Moreover, the values of the two problems coincide only if $\lambda\theta = 0$, i.e. when there are no interactions through controls.

Comparison with the one-shot one-dimensional problem We have made numerical simulations of the MFGC described in paragraph 3.4.2.1 with $f_0(x) = F = 1$ for all $x \in \Omega$, $a = \tilde{a}(1 - \lambda^2\theta)^{-1}$ and $\tilde{a} = 2$. We wish to compare the results with the explicit formulas obtained above for the one-shot one-dimensional MFG. If the latter problem is a good approximation of the former one, we should find α_1 close to $\sqrt{\frac{2F}{a}} = 1$, therefore mostly independent of λ and θ . Note that the approximation by the one-shot one-dimensional mean field game is sensible only if the time remaining to the horizon is large enough, i.e. significantly larger than $\ell/\alpha^* = \ell$. If $T - t$ is small, then the optimal strategy for a representative player located on the left part of the domain is not to move.

On Figure 3.8, we display the norm of the optimal feedback for several values of λ and θ . We see that the optimal drift is close to 1 in agreement with the explicit formula found for the simplified one-shot one-dimensional MFG. We also see that the agents located at the bottom and top of Ω head toward the center of the domain (i.e. $x_2 = 0$) before reaching the exit. Thus, the second coordinate of α gets large and there are singularities on both sides of the exit, whose amplitude increases as $\lambda\theta$ tends to 1.

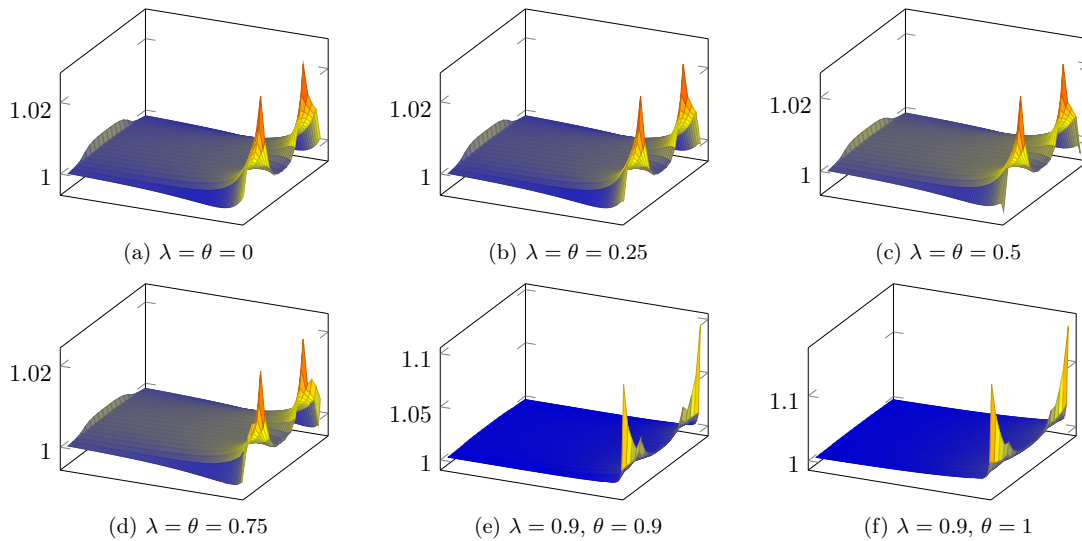


Figure 3.8 – Example 2, $f_0(x) = 1$. The optimal feedback α for different choices of λ and θ at $t = T/2$.

3.4.2.3 Queues

Here, we choose f as follows:

$$f_0(x) = \begin{cases} 4, & \text{if } x \leq -0.1, \\ 1, & \text{if } x \geq 0.1, \\ 2.5 - 15x, & \text{otherwise;} \end{cases}$$

hence, an agent pays a cost for staying in Ω which is higher in the left part of the domain.

The parameters are

$$\nu = 0.001, \quad \tilde{a} = 2, \quad \lambda = 0.95, \quad \theta = 0.95, \quad \rho = 0.25, \quad \omega_0 = \frac{\pi}{3}.$$

The grid has 101×21 nodes and there are 101 time steps.

The evolution of the distribution of states and of the optimal feedback is displayed on Figure 3.9. We compare these results with a simulation of the MFG obtained by cancelling either θ or λ while keeping all the other parameters and the grid unchanged, see Figure 3.10. On Figure 3.9, we see that a well distributed queue takes place from the entrance to the exit, by contrast with Figure 3.10 where we see that the agent rush and accumulate in the right part of the domain. Similarly, the deceleration is much stiffer in the latter case. It is not surprising that the interactions through controls have the effect of smoothing the distribution of states and the optimal feedback law. On the bottom of Figure 3.10, we see that when t is close to the horizon, the distribution is mainly concentrated near the middle of the domain but slightly on the right: this corresponds to the agents that have reached the zone where f_0 has the smaller value, i.e. 1, but for which reaching the exit before T becomes too costly. There is also a smaller bump near the entrance corresponding to agents that would pay too high a cost to reach the right part of the domain before T . These phenomena are clearly a side effects due to the finite horizon. These side effects are also present on Figure 3.9, but they are attenuated by the interaction through the controls.

Remark 3.4.5. *In models accounting for congestion effects, the cost of motion depends on the density of the distribution of states and gets higher in the crowded regions. We refer to [87] for a pioneering discussion of MFG models including congestion, to [10] for the analysis of the system of PDEs that arise with these models, and to [7] for numerical simulations. Such models also permit to describe queueing phenomena, because the agents located in crowded regions pay a large cost for moving. In [7], since the congestion effects are taken into account in a local manner, the queues take place in small regions of the state domain. By contrast, the present nonlocal model accounts for the fact that the agents anticipate low speed regions, making the traffic more fluid and the distribution of state smoother. Although it is quite possible to do, we have not incorporated congestion effects in the present model. We plan to do it in forthcoming works.*

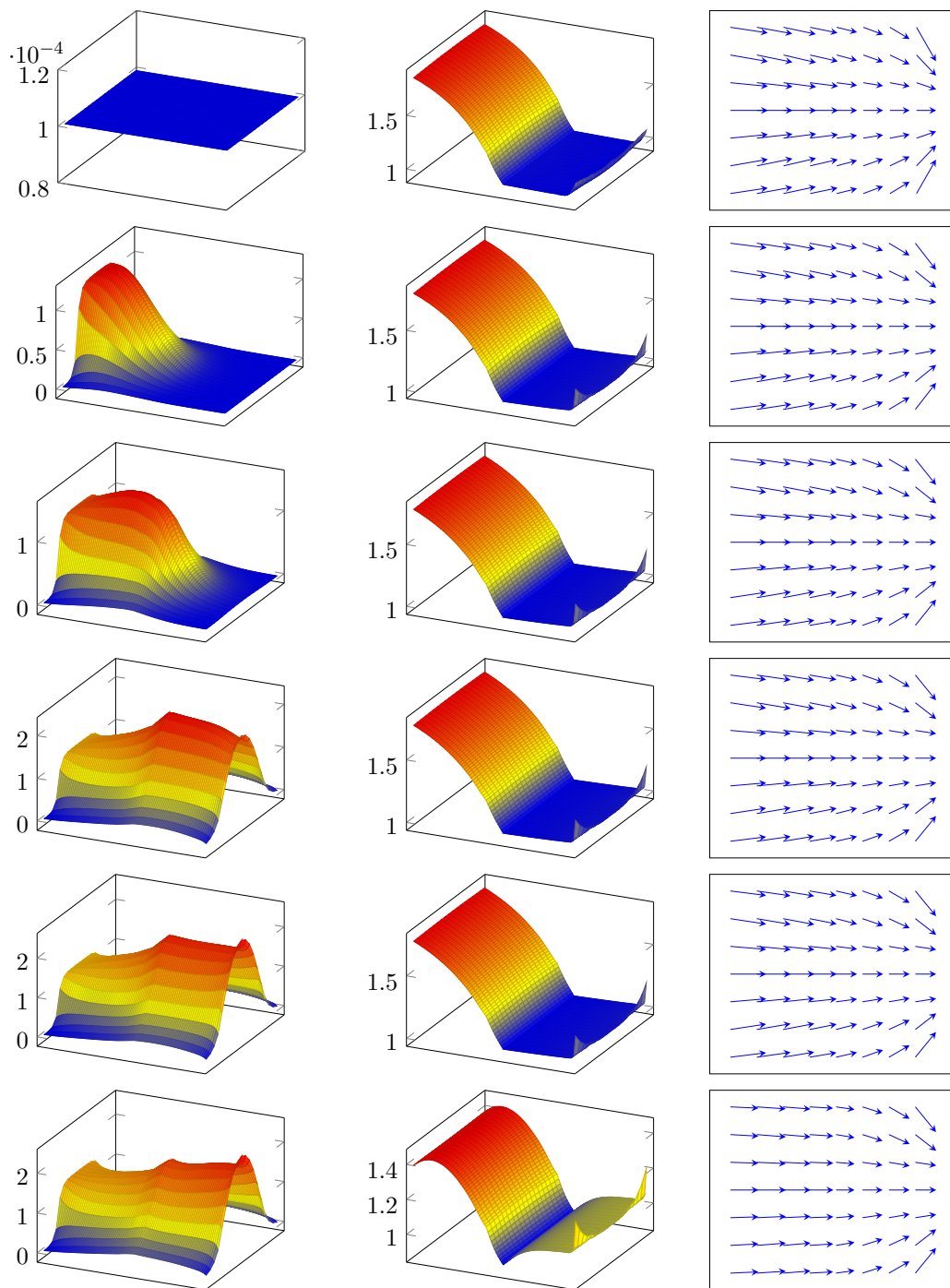


Figure 3.9 – Example 2. Interaction via controls: snapshots at $t = 0, 0.4, 0.8, 2, 4, 7$. Left: the distribution of states. Center: the norm of the optimal feedback. Right: the optimal feedback.

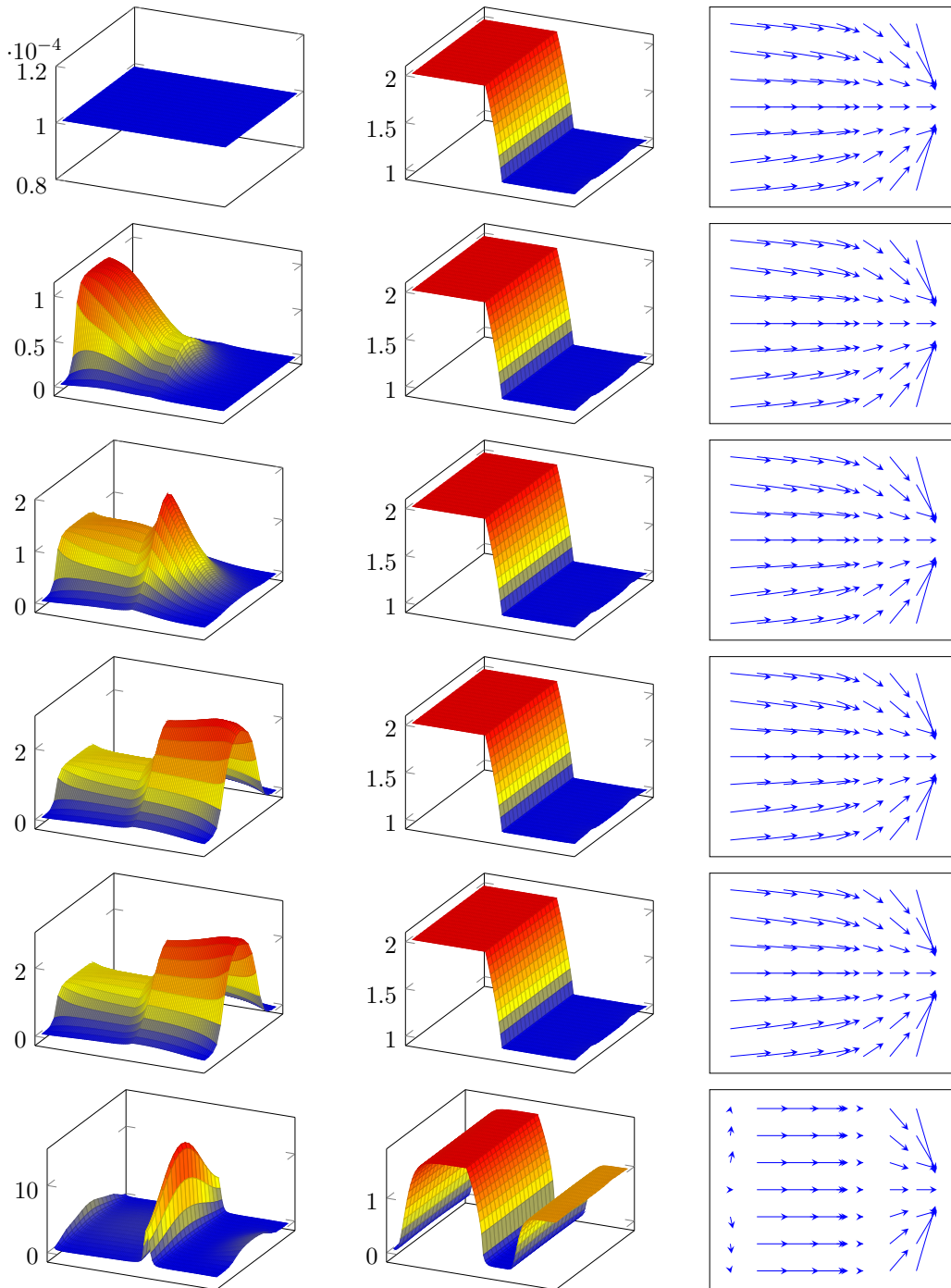


Figure 3.10 – Example 2. Same parameters except $\lambda\theta = 0$: snapshots at $t = 0, 0.4, 0.8, 2, 4, 7$. Left: the distribution of states. Center: the norm of the optimal feedback. Right: the optimal feedback.

3.4.2.4 Stationary regime

We now look for a stationary equilibrium. For that, we use an iterative method in order to progressively diminish the effects of the initial and terminal conditions: starting from (u^0, m^0, V^0) , the numerical solution of the finite horizon problem described above, we construct a sequence of approximate solutions $(u^\ell, m^\ell, V^\ell)_{\ell \geq 1}$ by the following induction: $(u^{\ell+1}, m^{\ell+1}, V^{\ell+1})$ is the solution of the finite horizon problem with the same system of

PDEs in $(0, T) \times \Omega$, the same boundary conditions on $(0, T) \times \partial\Omega$, and the new initial and terminal conditions as follows:

$$(3.4.13) \quad u^{\ell+1}(T, x) = u^\ell\left(\frac{T}{2}, x\right), \quad x \in \Omega,$$

$$(3.4.14) \quad m^{\ell+1}(0, x) = m^\ell\left(\frac{T}{2}, x\right), \quad x \in \Omega.$$

As ℓ tends to $+\infty$, we observe that (u^ℓ, m^ℓ, V^ℓ) converge to time-independent functions. At the limit, we obtain a steady solution of

$$\begin{aligned} -\nu\Delta u + \frac{a}{2} \left| \frac{1}{a} \nabla_x u - \lambda\theta V \right|^2 - a \frac{\lambda^2 \theta}{2} |V|^2 &= cm + f_0(x), & \text{in } \Omega, \\ -\nu\Delta m - \operatorname{div} \left(\left(\frac{1}{a} \nabla_x u - \lambda\theta V \right) m \right) &= 0, & \text{in } \Omega, \\ V(x) &= -\frac{1}{Z(x)} \int_{\Omega} \left(\frac{1}{a} \nabla_x u(y) - \lambda\theta V(y) \right) K(x, y) dm(y), & \text{in } \Omega, \\ Z(x) &= \int_{\Omega} K(x, y) dm(y), & \text{in } \Omega, \end{aligned}$$

with the boundary conditions on $\partial\Omega$ given by (3.4.3)-(3.4.5).

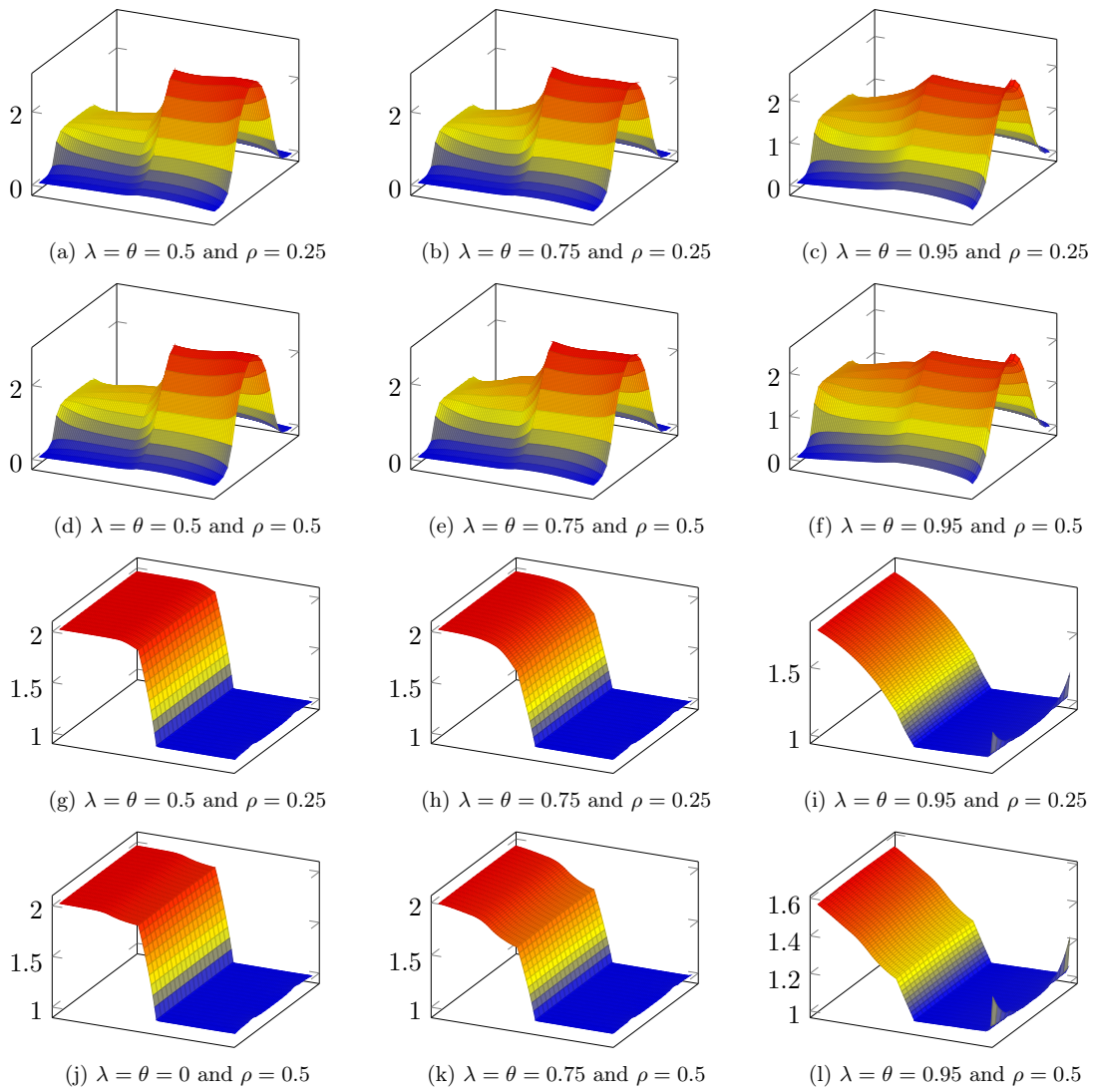


Figure 3.11

Example 2. Solutions of the stationary problem for different choices of λ, θ and ρ . Top six figures: the distribution of states m . Bottom six figures: norm of the optimal controls.

Mean Field Games with monotonous interactions through the law of states and controls of the agents

We consider a class of Mean Field Games in which the agents may interact through the statistical distribution of their states and controls. It is supposed that the Hamiltonian behaves like a power of its arguments as they tend to infinity, with an exponent larger than one. A monotonicity assumption is also made. Existence and uniqueness are proved using a priori estimates which stem from the monotonicity assumptions and Leray-Schauder theorem. Applications of the results are given.

4.1 Introduction

The theory of Mean Field Games (MFG for short) aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of agents tends to infinity. It has been introduced in the independent works of J.M. Lasry and P.L. Lions [83, 84, 85], and of M.Y. Huang, P.E. Caines and R. Malhamé [69, 70]. The agents are supposed to be rational (given a cost to be minimized, they always choose the optimal strategies), and indistinguishable. Furthermore, the agents interact via some empirical averages of quantities which depend on the state variable.

The most common Mean Field Game systems, in which the agents may interact only through their states can often be summarized by a system of two coupled partial differential equations which is named the MFG system. On the one hand, the optimal value of a generic agent at some time t and state x is denoted by $u(t, x)$ and is defined as the lowest cost that a representative agent can achieve from time t to T if it is at state x at time t . The value function satisfies a Hamilton-Jacobi-Bellman equation posed backward in time with a terminal condition involving a terminal cost. On the other hand, there is a Fokker-Planck-Kolmogorov equation describing the evolution of the statistical distribution m of the state variable; this equation is a forward in time parabolic equation, and the initial distribution at time $t = 0$ is given. Here we take a finite horizon time $T > 0$, and we only consider second-order nondegenerate MFG systems. In this case, the MFG system is often written as:

$$(4.1.1) \quad \begin{cases} -\partial_t u(t, x) - \nu \Delta u(t, x) + H(t, x, \nabla_x u(t, x)) = f(t, x, m(t)) & \text{in } (0, T) \times \mathbb{R}^d, \\ \partial_t m(t, x) - \nu \Delta m(t, x) - \operatorname{div}(H_p(t, x, \nabla_x u(t, x))m) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ u(T, x) = g(x, m(T)) & \text{in } \mathbb{R}^d, \\ m(0, x) = m_0(x) & \text{in } \mathbb{R}^d. \end{cases}$$

We refer the reader to [35] for some theoretical results on the convergence of the N -agent Nash equilibrium to the solutions of the MFG system. For a thorough study of the well-posedness of the MFG system, see the videos of P.L. Lions' lecture at the Collège de France, and the lecture notes [33].

In this paper we are considering a class of Mean Field Games in which agents may interact through their states and controls. To underline this, we choose to use the terminology *Mean Field Games of Controls (MFGCs)*; this terminology was introduced in [38].

Since the agents are assumed to be indistinguishable, a representative agent may be described by her state, which is a random process with value in \mathbb{R}^d denoted by $(X_t)_{t \in [0, T]}$ and satisfying the following stochastic differential equation,

$$(4.1.2) \quad dX_t = b(t, X_t, \alpha_t) dt + \sqrt{2\nu} dW_t,$$

where X_0 is a random process whose law is denoted by m_0 , $(W_t)_{t \in [0, T]}$ is a Brownian motion on \mathbb{R}^d independent with X_0 , and α_t is the control chosen by the agent at time t . The diffusion coefficient ν is assumed to be uncontrolled, constant and positive. The drift b naturally depends on the control, and may also depend on the time and the state; see Remark 4.2.9iv) below for allowing b to depend on μ_α the joint distribution of states and controls of the agents. At the equilibrium μ_α should be the law of the state and the control of the representative agent, i.e. $\mu_\alpha(t) = \mathcal{L}(X_t, \alpha_t)$, for $t \in [0, T]$. The aim of an agent is to minimize the functional given by,

$$(4.1.3) \quad \mathbb{E} \left[\int_0^T L(t, X_t, \alpha_t, \mu_\alpha(t)) + f(t, X_t, m(t)) dt + g(X_T, m(T)) \right],$$

where $m(t)$ is the distribution of agents at time t , which should satisfy $m(t) = \mathcal{L}(X_t)$ at the equilibrium. The coupling function f and the terminal cost g depend on m in a nonlocal manner. From L the Lagrangian and b the drift, we define H the Hamiltonian by,

$$(4.1.4) \quad H(t, x, p, \mu_\alpha) = \sup_{\alpha \in \mathbb{R}^d} -p \cdot b(t, x, \alpha) - L(t, x, \alpha, \mu_\alpha),$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$, $p \in \mathbb{R}^d$ and $\mu_\alpha \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, where $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$. Under some assumptions on b and L that will be introduced later, there exists a unique α which achieves the supremum in the latter equality and it also satisfies,

$$b(t, x, \alpha) = -H_p(t, x, p, \mu_\alpha).$$

In an attempt to keep this paper easy to read, we introduce μ_b as the joint law of states and drifts defined by

$$(4.1.5) \quad \mu_b(t) = \left(\pi_x, b(t) \right) \# \mu_\alpha,$$

where $\pi_x : (x, \alpha) \mapsto x$ is the projection onto the first variable, $b(t)$ is the map $(x, \alpha) \mapsto b(t, x, \alpha)$, and $\left(\pi_x, b(t) \right) \# \mu_\alpha$ is the pushforward measure of μ_α by the map $(x, \alpha) \mapsto (x, b(t, x, \alpha))$. We believe that the fixed point relation satisfied by μ_α at equilibrium is more clear if we distinguish μ_b from μ_α . We assume that b is invertible with respect to α , we denote $\alpha^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ its inverse map, see Assumption **B1** below. This implies that the equality (4.1.5) can be inverted to express μ_α in term of μ_b and we obtain (4.1.6c) below. Within this framework, the usual MFG system (4.1.1) is replaced by the following Mean Field Game of Controls (MFGC for short) system,

$$(4.1.6a) \quad -\partial_t u - \nu \Delta u + H(t, x, \nabla_x u(t, x), \mu_\alpha(t)) = f(t, x, m(t)) \quad \text{in } (0, T) \times \mathbb{R}^d,$$

$$(4.1.6b) \quad \partial_t m - \nu \Delta m - \operatorname{div}(H_p(t, x, \nabla_x u(t, x), \mu_\alpha(t)) m) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d,$$

$$(4.1.6c) \quad \mu_\alpha(t) = \left(\pi_x, \alpha^*(t) \right) \# \mu_b(t) \quad \text{in } [0, T],$$

$$(4.1.6d) \quad \mu_b(t) = \left(I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu_\alpha(t)) \right) \# m(t) \quad \text{in } [0, T],$$

$$(4.1.6e) \quad u(T, x) = g(x, m(T)) \quad \text{in } \mathbb{R}^d,$$

$$(4.1.6f) \quad m(0, x) = m_0(x) \quad \text{in } \mathbb{R}^d,$$

where $\alpha^*(t)$ is the map $(x, \tilde{b}) \mapsto \alpha^*(t, x, \tilde{b})$. The structural assumption under which we prove existence and uniqueness of the solution to (4.1.6) is that L satisfies the following inequality,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (L(t, x, \alpha, \mu^1) - L(t, x, \alpha, \mu^2)) d(\mu^1 - \mu^2)(x, \alpha) \geq 0.$$

for any $t \in [0, T]$, $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. This is the Lasry-Lions monotonicity assumption extended to MFGC that will be referred to as **A3**. This assumption is particularly adapted to applications in economics or finance.

This work follows naturally the analysis in [75] in which a MFGC system in the d -dimensional torus and with $b = \alpha$ is considered. In [75], the monotonicity assumption is replaced by another structural assumption, namely that the optimal control $-H_p$ is a contraction with respect to the second marginal of μ (when the other arguments and the first marginal are fixed) and that it is bounded by a quantity that depends linearly on the second marginal of μ with a coefficient smaller than 1.

Related works

Monotonicity assumptions for MFGC like **A3** have already been discussed in [38, 41, 59]. In [59], the authors proved uniqueness of the solution to (4.1.6) with $b = \alpha$ and $\nu = 0$ when it exists. In [41] Section 4.6, existence and uniqueness are proved in the quadratic case with a uniformly convex Lagrangian and under an additional linear growth assumption on H_x . In [38], the existence of weak solutions to a MFGC system with a possibly degenerate diffusion operator is proved assuming that the inequalities satisfied by H , its derivatives or the optimal control (here defined in **B1** as α^*), are uniform with respect to the joint law of states and controls μ_α .

A particular application of MFGC satisfying **A3**, namely the Bertrand and Cournot competition for exhaustible resource described in paragraph 4.3.1, has been broadly investigated in the literature. Let us mention a non exhaustive list of such works: [25, 47, 62, 65, 73]. Its mean field version has been introduced in [65], and obtained from the N -agent game in [47] in the case of a linear supply-demand function. A generalization to the multi-dimensional case is discussed in [25], and an extension to negatively correlated resources is addressed in [75].

A class of MFGC in which the Lagrangian depends separately on α and μ , has been investigated in [43] and [41]. In this case, **A3** is naturally satisfied since the left-hand side of the inequality is identically equal to 0. An existence result is proved in [43] under the additional assumption that the set of admissible controls is compact. The existence of solution is also proved in [41] Theorem 4.65 when the dependency of L upon μ is uniformly bounded with respect to μ .

The non-monotone case has been studied in [60, 75]. In [60], an existence result is proved in the stationary setting and under the assumption that the dependence of H on μ is small. In [75], the existence of solutions to the MFGC system in the d -dimensional torus and with $b = \alpha$ is discussed under similar growth assumptions as here. Existence of solutions to a MFGC system posed on the d -dimensional torus and with $b = \alpha$ was proved in [75] in any of the following cases:

- short time horizon,
- small enough parameters,
- weak dependency of H upon μ ,
- weak dependency of H_x upon μ ,

and uniqueness is proved only for a short time horizon. Indeed without a monotonicity assumption, it is unlikely that uniqueness holds in general, numerical examples of non-uniqueness of solutions to discrete approximations of (4.1.6) with $b = \alpha$ and in a bounded domain are showed in [2].

Organization of the paper

In Section 4.2, the notations and the assumptions are described, the case when the control is equal to the drift is discussed. The main results of the paper, namely the existence and uniqueness of solution to (4.1.6), are stated in paragraph 4.2.3. We give some insights on our strategy for proving the main results in paragraph 4.2.3. Two applications of the MFGC system (4.1.6) are presented in Section 4.3. Section 4.4 is devoted to solving the fixed point relation in the joint law of state and control in the particular case when the drift is equal to the control. Section 4.5 consists in giving a priori estimates for a MFGC system

posed on the d -dimensional torus. In Section 4.6, we prove existence and uniqueness of the solution to (4.1.6) and of an intermediate MFGC system.

4.2 Assumptions

4.2.1 Notations

The spaces of probability measures are equipped with the weak* topology. We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the subset of $\mathcal{P}(\mathbb{R}^d)$ of probability measures with finite second moments, and $\mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ the subset of measures μ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with a second marginal compactly supported. For $\mu \in \mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\tilde{q} \in [1, \infty)$, we define the quantities $\Lambda_{\tilde{q}}(\mu)$ and $\Lambda_\infty(\mu)$ by,

$$(4.2.1) \quad \begin{aligned} \Lambda_{\tilde{q}}(\mu) &= \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |\alpha|^{\tilde{q}} d\mu(x, \alpha) \right)^{\frac{1}{\tilde{q}}}, \\ \Lambda_\infty(\mu) &= \sup \{ |\alpha|, (x, \alpha) \in \text{supp } \mu \}. \end{aligned}$$

For $R > 0$, we denote by $\mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$ the subset of measures μ in $\mathcal{P}_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\Lambda_\infty(\mu) \leq R$. The probability measures μ_α and μ_b involved in (4.1.6), have a particular form, since they are the images of a measure m on \mathbb{R}^d by (I_d, α) and (I_d, b) respectively, where α and b are bounded measurable functions from \mathbb{R}^d to \mathbb{R}^d ; in particular they are supported on the graph of α and b respectively. For $m \in \mathcal{P}(\mathbb{R}^d)$, we call $\mathcal{P}_m(\mathbb{R}^d \times \mathbb{R}^d)$ the set of such measures. For $\mu \in \mathcal{P}_m(\mathbb{R}^d \times \mathbb{R}^d)$, we set α^μ to be the unique element of $L^\infty(m)$ such that $\mu = (I_d, \alpha^\mu) \# m$. Here, $\Lambda_{\tilde{q}}(\mu)$ and $\Lambda_\infty(\mu)$ defined in (4.2.1) are given by

$$(4.2.2) \quad \begin{aligned} \Lambda_{q'}(\mu) &= \|\alpha^\mu\|_{L^{q'}(m)}, \\ \Lambda_\infty(\mu) &= \|\alpha^\mu\|_{L^\infty(m)}. \end{aligned}$$

Let $C^0([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$ be the set of bounded continuous functions from $[0, T] \times \mathbb{R}^d$ to \mathbb{R}^n , for n a positive integer. We define $C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R})$ as the set of the functions $v \in C^0([0, T] \times \mathbb{R}^d; \mathbb{R})$ differentiable at any point with respect to the state variable, and whose its gradient $\nabla_x v$ is in $C^0([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ the set of continuous functions from $[0, T] \times \mathbb{R}^d$ to \mathbb{R}^d . We shall have the use of the parabolic spaces of Hölder continuous functions $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$ defined for any $\beta \in (0, 1)$ and $n \geq 1$ by,

$$C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{R}^d; \mathbb{R}^n) = \left\{ v \in C^0([0, T] \times \mathbb{R}^d; \mathbb{R}^n), \exists C > 0 \text{ s.t. } \forall (t_1, x_1), (t_2, x_2) \in [0, T] \times \mathbb{R}^d, \right. \\ \left. |v(t_1, x_1) - v(t_2, x_2)| \leq C (|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{\beta}{2}} \right\}.$$

This is a Banach space equipped with the norm,

$$\|v\|_{C^{\frac{\beta}{2}, \beta}} = \|v\|_\infty + \sup_{(t_1, x_1) \neq (t_2, x_2)} \frac{|v(t_1, x_1) - v(t_2, x_2)|}{(|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{\beta}{2}}}.$$

Then we introduce the Banach space $C^{\frac{1+\beta}{2}, 1+\beta}([0, T] \times \mathbb{R}^d; \mathbb{R})$ for $\beta \in (0, 1)$ as the set of the functions $v \in C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R})$ such that $\nabla_x v \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and which admits a finite norm defined by,

$$\|v\|_{C^{\frac{1+\beta}{2}, 1+\beta}} = \|v\|_\infty + \|\nabla_x v\|_{C^{\frac{\beta}{2}, \beta}} + \sup_{(t_1, x) \neq (t_2, x) \in [0, T] \times \mathbb{R}^d} \frac{|v(t_1, x) - v(t_2, x)|}{|t_1 - t_2|^{\frac{1+\beta}{2}}}.$$

When the drift b is equal to the control α , (4.1.6) can be simplified in the following system,

$$(4.2.3a) \quad -\partial_t u - \nu \Delta u + H(t, x, \nabla_x u(t, x), \mu(t)) = f(t, x, m(t)) \quad \text{in } (0, T) \times \mathbb{R}^d,$$

$$(4.2.3b) \quad \partial_t m - \nu \Delta m - \operatorname{div}(H_p(t, x, \nabla_x u(t, x), \mu(t)) m) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d,$$

$$(4.2.3c) \quad \mu(t) = \left(I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) \quad \text{in } [0, T],$$

$$(4.2.3d) \quad u(T, x) = g(x, m(T)) \quad \text{in } \mathbb{R}^d,$$

$$(4.2.3e) \quad m(0, x) = m_0(x) \quad \text{in } \mathbb{R}^d.$$

Here, making a distinction between μ_α and μ_b is pointless since they coincide. Therefore, we simply use the notation μ . For the system (4.2.3), the Hamiltonian is defined as the Legendre transform of L ,

$$(4.2.4) \quad H(t, x, p, \mu) = \sup_{\alpha \in \mathbb{R}^d} -p \cdot \alpha - L(t, x, \alpha, \mu).$$

Definition 4.2.1. *We say that $(u, m, \mu_\alpha, \mu_b)$ is a solution to (4.1.6) if*

- $u \in C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R})$ is a solution to the heat equation in the sense of distributions with a right-hand side equal to $(t, x) \mapsto f(t, x, m(t)) - H(t, x, \nabla_x u, \mu(t))$, and satisfies the terminal condition (4.1.6e),
- $m \in C^0([0, T]; \mathcal{P}(\mathbb{R}^d))$ is a solution to (4.1.6b) in the sense of distributions, and satisfies the initial condition (4.1.6f),
- $\mu_\alpha(t), \mu_b(t) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfy (4.1.6c) and (4.1.6d) for any $t \in [0, T]$.

We say that (u, m, μ) is a solution to (4.2.3) if u and m respectively satisfy the first two points of the latter definition with (4.2.3) instead of (4.1.6), and if $\mu(t) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies (4.2.3c) for any $t \in [0, T]$.

4.2.2 Hypotheses

The monotonicity assumption made in this paper concerns the Lagrangian. For this reason and the fact that sometimes it may be hard to obtain an explicit form of the Hamiltonian (like in the example of paragraph 4.3.2 below), all the assumptions will be formulated in term of the Lagrangian and never in term of the Hamiltonian. In particular, working with the Lagrangian gives more flexibility in the arguments of the proofs.

The constants entering the assumptions are C_0 a positive constant, $q \in (1, \infty)$ an exponent, $q' = \frac{q}{q-1}$ its conjugate exponent, and $\beta_0 \in (0, 1)$ a Hölder exponent.

A1 $L : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$ is differentiable with respect to (x, α) ; L and its derivatives are continuous on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$ for any $R > 0$; we recall that $\mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$ is endowed with the weak* topology on measures; we use the notation L_x, L_α and $L_{(x, \alpha)}$ for respectively the first-order derivatives of L with respect to x, α and (x, α) .

A2 The maximum in (4.1.4) is achieved at a unique $\alpha \in \mathbb{R}^d$.

A3 L satisfies the following monotonicity condition,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (L(t, x, \alpha, \mu^1) - L(t, x, \alpha, \mu^2)) d(\mu^1 - \mu^2)(x, \alpha) \geq 0.$$

for any $t \in [0, T], \mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$.

A4 $L(t, x, \alpha, \mu) \geq C_0^{-1} |\alpha|^{q'} - C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right)$, where $\Lambda_{q'}$ is defined in (4.2.1),

A5 $|L(t, x, \alpha, \mu)| \leq C_0 \left(1 + |\alpha|^{q'} + \Lambda_{q'}(\mu)^{q'}\right)$, and $|L_x(t, x, \alpha, \mu)| \leq C_0 \left(1 + |\alpha|^{q'} + \Lambda_{q'}(\mu)^{q'}\right)$,

A6 $\int_{\mathbb{R}^d} |x|^2 dm^0(x) \leq C_0$, $\|m^0\|_{C^{\beta_0}} \leq C_0$, $\|f(t, \cdot, m)\|_{C^1} \leq C_0$, $\|g(\cdot, m)\|_{C^{2+\beta_0}} \leq C_0$, for any $t \in [0, T]$ and $m \in \mathcal{P}(\mathbb{R}^d)$.

Assumption **A3** can be interpreted as a natural extension of the Lasry-Lions monotonicity condition to MFGC. Roughly speaking, the Lasry-Lions monotonicity condition in the case of MFG without interaction through controls, translates the fact that the agents have aversion for crowded regions of the state space. In the case of MFGC, the monotonicity condition implies that the agents favor moving in a direction opposite to the mainstream. Such an assumption is adapted to models of agents trading goods or financial assets. Indeed in most of the models coming from economics or finance, a buyer may prefer to buy when no one else is buying, and conversely a seller may prefer to sell when no one else is selling.

Assumptions **A4** and **A5** imply that at least asymptotically when α tends to infinity, L behaves like a power of α of exponent q' . Under the monotonicity assumption **A3**, uniqueness is in general easier to obtain than existence. For uniqueness, we assume that f and g are also monotonous, this is the purpose of the following assumption.

U For $m^1, m^2 \in \mathcal{P}(\mathbb{R}^d)$, and $t \in [0, T]$, assume that,

$$\begin{aligned} \int_{\mathbb{R}^d} (f(t, x, m^1) - f(t, x, m^2)) d(m^1 - m^2)(x) &\geq 0, \\ \int_{\mathbb{R}^d} (g(x, m^1) - g(x, m^2)) d(m^1 - m^2)(x) &\geq 0. \end{aligned}$$

In fact, assuming that f satisfies the inequality in **U**, implies that we can take $f = 0$ up to replacing L by $L + f$ and H by $H - f$. However, since **U** is not assumed for proving the existence of solutions, we have chosen to write this assumption explicitly, and keeping $f \neq 0$ is not pointless.

Let us now make assumptions on the drift function b , which concern the system (4.1.6),

B1 There exists a function $\alpha^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that the map $\tilde{b} \mapsto \alpha^*(t, x, \tilde{b})$ is the inverse map of $\alpha \mapsto b(t, x, \alpha)$, for any $(t, x) \in [0, T] \times \mathbb{R}^d$. Moreover α^* is continuous, differentiable with respect to x and \tilde{b} , with continuous derivatives.

B2 b and α^* satisfy

$$\begin{aligned} |b(t, x, \alpha, \mu_\alpha)| &\leq C_0 (1 + |\alpha|^{q_0}), \\ \left| \alpha^*(t, x, \tilde{b}, \mu_b) \right|^{q_0} + \left| \alpha_x^*(t, x, \tilde{b}, \mu_b) \right|^{q_0} &\leq C_0 (1 + |\tilde{b}|), \end{aligned}$$

for some exponent q_0 such that $0 \leq q_0 \leq q'$.

The inequalities in **B2** mean that $|b|$ behaves asymptotically like a power of α with exponent q_0 , when $|\alpha|$ is large.

4.2.3 Main results

The two main results in this work are Theorems 4.2.2 and 4.2.3 below, which respectively state the existence and uniqueness of the solution to (4.1.6).

Theorem 4.2.2. *Under assumptions A1-A3, U and B1 there is at most one solution to (4.1.6).*

Uniqueness results for MFGC systems with a monotonicity assumption have been proved in [59] and [41]. In [59], uniqueness is proved when the diffusion coefficient is equal to 0 and the drift is equal to the control, i.e. $\nu = 0$ and $b = \alpha$. In [41] Section 4.6, the authors stated uniqueness in the quadratic case. Theorem 4.2.2 is new in the sense that it yields uniqueness for a large new class of Lagrangians and drift functions. Indeed, beside the monotonicity assumption A3 and U, we only assume that L satisfies A1 and A2, and that the drift b is invertible in the sense of B1.

Theorem 4.2.3. *Under assumptions A1-A6 and B1-B2, there exists a solution to (4.1.6).*

The existence of solutions of the MFGC system is in general much more demanding than for MFG systems without interactions through the controls. Under monotonicity assumptions similar to A3, existence has been proved in [41] Section 4.6, for quadratic and uniform convex Lagrangians with a growth condition on the derivatives of the Hamiltonian. In [38], the existence of weak solutions of the monotonous MFGC system is discussed with a possibly degenerate diffusion operator, under assumptions which are uniform with respect to the joint law of states and controls.

Here, we prove existence of solutions of the monotonous MFGC system for a large class of Lagrangians and the drifts. Namely, we assume that the Lagrangians and drifts behave asymptotically like a power of α ; we allow them to have a growth in the law of the controls of at most the same order as the order of dependency upon α .

Before starting the discussion on existence of solutions to the MFGC systems (4.1.6) and (4.2.3), we introduce a new MFGC system set in the torus, so that the solutions should have more compactness properties. We define $\mathbb{T}_a^d = \mathbb{R}^d / (a\mathbb{Z}^d)$ the d -dimensional torus of radius $a > 0$. Namely, we consider:

$$(4.2.5a) \quad -\partial_t u - \nu \Delta u + H(t, x, \nabla_x u(t, x), \mu(t)) = f(t, x, m(t)) \quad \text{in } (0, T) \times \mathbb{T}_a^d,$$

$$(4.2.5b) \quad \partial_t m - \nu \Delta m - \operatorname{div}(H_p(t, x, \nabla_x u(t, x), \mu(t)) m) = 0 \quad \text{in } (0, T) \times \mathbb{T}_a^d,$$

$$(4.2.5c) \quad \mu(t) = \left(I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) \quad \text{in } [0, T],$$

$$(4.2.5d) \quad u(T, x) = g(x, m(T)) \quad \text{in } \mathbb{T}_a^d,$$

$$(4.2.5e) \quad m(0, x) = m_0(x) \quad \text{in } \mathbb{T}_a^d.$$

All the assumptions in paragraph 4.2.2 are stated in \mathbb{R}^d . When considering that $L : [0, T] \times \mathbb{T}_a^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$ (like in (4.2.5)) satisfies one of those assumptions, we shall simply replace \mathbb{R}^d by \mathbb{T}_a^d as the state set in the chosen assumption.

The fixed point satisfied by the joint law of states and controls, namely (4.1.6c)-(4.1.6d), (4.2.3c) or (4.2.5c), may be a difficult issue for MFGC systems. Here, using mainly the monotonicity assumption A3 and the compactness of the state space of (4.2.5), we prove in Section 4.4 the following lemma which states well-posedness for the fixed point (4.2.5c), and ensures continuity with respect to time.

Lemma 4.2.4. *Assume A1-A5. Let $p \in C^0([0, T] \times \mathbb{T}_a^d; \mathbb{R}^d)$ and $m \in C^0([0, T]; \mathcal{P}(\mathbb{T}_a^d))$ be such that $t \mapsto p(t, \cdot)$ is continuous with respect to the topology of the local uniform*

convergence and $m(t)$ admits a finite second moment uniformly bounded with respect to $t \in [0, T]$. For any $t \in [0, T]$, there exists a unique $\mu(t) \in \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ such that

$$\mu(t) = (I_d, -H_p(t, \cdot, p(t, \cdot), \mu(t))) \# m(t).$$

Moreover, the map $t \mapsto \mu(t)$ is continuous where $\mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ is equipped with the weak* topology.

The next step in our strategy for proving existence is to look for a priori estimates for the solutions of the MFGC systems and obtain compactness results to use a fixed point theorem. In section 4.5, we prove the a priori estimates stated in the following lemma for solutions to (4.2.5).

Lemma 4.2.5. *Assume **A1-A6**. If (u, m, μ) is a solution to (4.2.5), then $\|u\|_\infty$, $\|\nabla_x u\|_\infty$ and $\sup_{t \in [0, T]} \Lambda_\infty(\mu(t))$ are uniformly bounded by a constant independent of a .*

Let us mention that the a priori estimates of Lemma 4.2.5 rely on the monotonicity assumption on L and a Bernstein method introduced in [75]. To our knowledge, these are the first results in the literature of MFGC which use the monotonicity assumption for getting a priori estimates. They are the key ingredients of the proof of the existence of solutions to (4.2.5) in the following theorem, proved in paragraph 4.6.1.

Theorem 4.2.6. *Under assumptions **A1-A6**, there exists a solution to system (4.2.5).*

Therefore, for any $a > 0$ we can construct a solution to (4.2.5) which satisfies uniform estimates with respect to a . This allows us to construct a compact sequence of approximating solutions to (4.2.3). Passing to the limit for a subsequence allows us to generalize the conclusion of Theorem 4.2.6 to system (4.2.3). This leads to the following theorem proved in paragraph 4.6.2.

Theorem 4.2.7. *Under assumptions **A1-A6**, there exists a solution to (4.2.3).*

Uniqueness relies on the monotonicity assumptions **A3** and **U**, the following theorem is proved in paragraph 4.6.3.

Theorem 4.2.8. *Under assumptions **A1-A3** and **U**, there is at most one solution to (4.2.3) or (4.2.5).*

The idea to pass from (4.2.3) to (4.1.6), is to change the optimization problem in α into a new optimization problem expressed in term of b . In paragraph 4.6.4, we prove the equivalence between the solutions of these two optimization problems. A first existence results for (4.1.6) is stated in Corollary 4.6.3 which uses this equivalence. Theorem 4.2.3 is a consequence of Corollary 4.6.3 with more tractable assumptions. Let us mention that for proving Theorem 4.2.3, the structure of the Lagrangian should be invariant when passing from one optimization problem to the other. In particular, one may figure out that the assumptions on the Lagrangian behaving asymptotically like a power of α are preserved under our assumptions on the drift function b .

Finally, Theorem 4.2.2 is a consequence of Theorem 4.2.8 and the above-mentioned equivalence between the two optimization problems.

Remark 4.2.9. *i) If the Lagrangian admits the following form,*

$$L(t, x, \alpha, \mu) = L^0(t, x, \alpha) + L^1(t, \mu),$$

we say that the Lagrangian is separated. Then **A3** is automatically satisfied since the left-hand side of the inequality is identically equal to 0. In this case, the assumptions on L are satisfied if L^0 behaves asymptotically like a power of α of exponent q' , and L^1 at most involves $\Lambda_{q_0}(\mu)^{q'}$.

Here, we do not provide an explicit application in which the Lagrangian is separated, however this is a general hypothesis in the MFGC literature. Therefore, our framework in the present paper can be seen as an extension of the case when L is separated.

- ii) All our assumptions are uniform with respect to the state variable x . In particular, we restrain from considering more general functions f and g since this topic has been investigated in the literature devoted to MFG systems without interaction through controls; we believe that the same tools can be applied to the present case, and that our results may be extended so.
- iii) We did not address the case without diffusion, i.e. $\nu = 0$. However, all the a priori estimates of Sections 4.4 and 4.5 are uniform with respect to ν . Here, the diffusion is used to easily obtain compactness results which are central for proving our existence results since the proofs rely on a fixed point theorem and approximating sequences of solutions. Using weaker topological spaces and tools from the literature devoted to weak solutions of systems of MFGs without interaction through controls, we believe that we can extend our results to weak solutions to MFGC systems without diffusion or with possibly degenerate diffusion operators. We plan to address this question in forthcoming works.
- iv) We may also assume that b and α^* depend on μ_α and μ_b respectively, i.e. $b, \alpha^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \mapsto \mathbb{R}^d$. In this case, Theorems 4.2.2 and 4.2.3 hold if we replace **A3** by the following monotonicity condition:

A'3 For any $t \in [0, T]$ and $\mu^i \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, $i = 1, 2$, we set $\mu_\alpha^i = (\pi_x, \alpha^*(t, \mu^i)) \# \mu^i$ and $\tilde{\mu}_\alpha^i = (\pi_x, \alpha^*(t, \mu^{3-i})) \# \mu^i$. We assume that L satisfies

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} (L(t, x, \alpha, \mu_\alpha^1) - L(t, x, \alpha, \tilde{\mu}_\alpha^2)) d\mu_\alpha^1(x, \alpha) \\ & \quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} (L(t, x, \alpha, \mu_\alpha^2) - L(t, x, \alpha, \tilde{\mu}_\alpha^1)) d\mu_\alpha^2(x, \alpha) \geq 0. \end{aligned}$$

In this case, we may relax Assumption **B2** by adding the terms $C_0 \Lambda_{q'}(\mu_\alpha)^{q_0}$ and $C_0 \Lambda_{\frac{q'}{q_0}}(\mu_b)$ to the right-hand sides of the first and second inequalities respectively.

When **B1** holds and α^* does not depend on μ_b , Assumptions **A'3** and **A3** are equivalent.

General outline

The present work aims at proving Theorems 4.2.2 and 4.2.3. We list below the main steps of our analysis to make it easier for the reader to understand the structure of the proofs.

I We solve the fixed point (4.2.5c) in μ , which proves Lemma 4.2.4, in three steps:

I.a in Lemma 4.4.1 we state a priori estimates for a solution of (4.2.5c);

- I.b** using the Leray-Schauder fixed point theorem (Theorem 4.4.2), we solve the fixed point (4.2.5c) at any time $t \in [0, T]$, in Lemma 4.4.3;
 - I.c** we prove that the fixed point $\mu(t)$ defined at any $t \in [0, T]$ in step **I.b**, is continuous with respect to time (Lemma 4.4.4); this implies lemma 4.2.4.
- II** We prove the existence of a solution to (4.2.5), stated in Theorem 4.2.6, in two steps:
- II.a** we obtain a priori estimates for solutions to (4.2.5) (Lemmas 4.2.5 and 4.5.1);
 - II.b** in paragraph 4.6.1, we use Leray-Schauder fixed point theorem (Theorem 4.4.2) and the estimates of step **II.a** to conclude.
- III** We prove existence and uniqueness of the solution to (4.2.3) (Theorems 4.2.7 and 4.2.8):
- III.a** the proof of Theorem 4.2.7 is given in paragraph 4.6.2;
 - III.b** the proof of Theorem 4.2.8 is given in paragraph 4.6.3;
- IV** The proof of existence and uniqueness of the solution to (4.1.6) (Theorems 4.2.2 and 4.2.3) is given in paragraph 4.6.4.

Contribution

An important novelty in the present work comes from the assumptions we are considering. On the one hand, we consider a general class of monotonous Lagrangians which behave asymptotically like a power of α with any exponent in $(1, \infty)$ (while most of the results in the literature only address the quadratic case with uniformly convex Lagrangian); they may depend on moments of μ_α at most of the same order as the above-mentioned exponent of L in α ; we do not require them to depend separately on (x, α) and μ_α . On the other hand, the drift functions are also general since we allow them to behave like power functions and to be not separated too. See the assumptions in paragraph 4.2.2 for more details.

Moreover, most contributions focus on MFG systems stated on \mathbb{T}^d for simplicity. Here, we introduce a method to extend an existence result for a MFGC system stated on the torus to its counterpart on the whole Euclidean space. In particular, this method holds for MFG system without interaction through controls and the proof becomes easier. See paragraph 4.6.2. We also introduce a method to extend the well-posedness of MFGC (or MFG) systems to general drift functions, see paragraph 4.6.4. We would like to insist on the fact that our techniques are designed in order to preserve the structure of the Lagrangian when passing from one setting to another. Here, namely it preserves the monotonicity assumption **A3**. Furthermore, these methods apply to the conclusions of [75] and consequently generalize them.

4.2.4 Properties of the Lagrangian and the Hamiltonian in (4.2.3) and (4.2.5)

Here, we write the results and the proofs for the Lagrangian and Hamiltonian involved in system (4.2.3). However, none of the arguments below is specific to the domain \mathbb{R}^d , therefore the conclusions hold and the proofs can be repeated for the Lagrangian and Hamiltonian involved in (4.2.5).

We start by proving that under the assumptions of paragraph 4.2.2, when $b = \alpha$, L is strictly convex.

Lemma 4.2.10. *If L is coercive and differentiable with respect to α , and $b = \alpha$, assuming that L is strictly convex is equivalent to [A2](#).*

Proof. If L is strictly convex and coercive, it is straightforward to check [A2](#).

Conversely, we take $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. We set $\ell(\alpha) = L(t, x, \alpha, \mu)$. It is sufficient to prove that ℓ is strictly convex.

First step: proving that ℓ is convex.

We define ℓ^{**} as the biconjugate of ℓ , ℓ^{**} is in particular the Legendre transform of $H(t, x, \cdot, \mu)$. The map ℓ^{**} is convex and continuous since ℓ is coercive, and it satisfies $\ell^{**} \leq \ell$. In what follows, we will prove that $\ell^{**} = \ell$.

We assume by contradiction that $\ell^{**} \neq \ell$: there exists $\alpha^0 \in \mathbb{R}^d$ such that $\ell^{**}(\alpha^0) < \ell(\alpha^0)$. We recall that ℓ and ℓ^{**} admit the same convex envelope, therefore by Carathéodory's theorem, there exists $(\alpha^i)_{1 \leq i \leq d+1} \in (\mathbb{R}^d)^{d+1}$ and $(\lambda^i)_{1 \leq i \leq d+1} \in (\mathbb{R}_+)^{d+1}$ such that

$$\alpha^0 = \sum_{i=1}^{d+1} \lambda^i \alpha^i, \quad \ell^{**}(\alpha^0) = \sum_{i=1}^{d+1} \lambda^i \ell(\alpha^i), \quad \text{and} \quad \sum_{i=1}^{d+1} \lambda^i = 1.$$

Using the inequality $\ell^{**} \leq \ell$, we obtain that

$$\ell^{**}(\alpha) = \sum_{i=1}^{d+1} \lambda^i \ell(\alpha^i) \geq \sum_{i=1}^{d+1} \lambda^i \ell^{**}(\alpha^i).$$

This inequality is in fact an equality since ℓ^{**} is convex, which implies that $\ell^{**}(\alpha^i) = \ell(\alpha^i)$ for any $i \in \{1, 2, \dots, d+1\}$. Take $p \in \partial \ell^{**}(\alpha^0)$, where $\partial \ell^{**}(\alpha^0)$ is the subdifferential of ℓ^{**} at α^0 . For $i \in \{1, \dots, d+1\}$, this implies

$$\ell(\alpha^i) = \ell^{**}(\alpha^i) \geq \ell^{**}(\alpha^0) + p \cdot (\alpha^i - \alpha^0).$$

Multiplying the latter inequality by λ^i and taking the sum over i , yield that it is in fact an equality. Then, it is straightforward to check that $p \in \partial \ell^{**}(\alpha^i)$ for any i ; this implies that $p = \nabla_{\alpha} \ell(\alpha^i)$, since $\ell^{**}(\alpha^i) = \ell(\alpha^i)$ and ℓ is differentiable with respect to α . The maximum in the definition of $H(t, x, -p, \mu)$ is achieved at any α^i , this is a contradiction with [A2](#). Therefore $\ell = \ell^{**}$ and ℓ is convex.

Second step: ℓ is strictly convex.

By definition of the subdifferential of a convex function, $\alpha \in \mathbb{R}^d$ achieves the maximum in the definition of $H(t, x, -\nabla_{\alpha} \ell(\alpha), \mu)$. Using the fact that this maximum is unique by [A2](#), we obtain the strict convexity of ℓ , and the one of L with respect to α . \square

In paragraph [4.2.2](#), we assume that L behaves at infinity as a power of α of exponent q' . Because of the conjugacy relation between L and H , it implies that H behaves at infinity like a power of p of exponent q .

Lemma 4.2.11. *Under assumptions [A1](#), [A2](#), [A4](#) and [A5](#), the map H , defined in [4.2.4](#), is differentiable with respect to x and p , H and its derivatives are continuous on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$ for any $R > 0$. Moreover there exists $\tilde{C}_0 > 0$ a constant which only depends on C_0 and q such that*

$$(4.2.6) \quad |H_p(t, x, p, \mu)| \leq \tilde{C}_0 (1 + |p|^{q-1} + \Lambda_{q'}(\mu)),$$

$$(4.2.7) \quad |H(t, x, p, \mu)| \leq \tilde{C}_0 (1 + |p|^q + \Lambda_{q'}(\mu)^{q'}),$$

$$(4.2.8) \quad p \cdot H_p(t, x, p, \mu) - H(t, x, p, \mu) \geq \tilde{C}_0^{-1} |p|^q - \tilde{C}_0 (1 + \Lambda_{q'}(\mu)^{q'}),$$

$$(4.2.9) \quad |H_x(t, x, p, \mu)| \leq \tilde{C}_0 (1 + |p|^q + \Lambda_{q'}(\mu)^{q'}),$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$, $p \in \mathbb{R}^d$ and $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$.

Up to replacing C_0 with $\max(C_0, \tilde{C}_0)$, we can assume that the inequalities in Lemma 4.2.11 are satisfied with C_0 instead of \tilde{C}_0 .

Let us notice that it is possible and not more difficult to extend the results stated in Lemma 4.2.11 to the Hamiltonian used in (4.1.6) and defined in (4.1.4), however we will not have any use of such results in the present paper.

Proof. First step: differentiability of H in p , and continuity of H and H_p .

For $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, the map $\alpha \mapsto L(t, x, \alpha, \mu)$ is strictly convex by Lemma 4.2.10 and coercive by A4; Theorem 26.6 in [97] implies that H is differentiable with respect to p , the map $\alpha \mapsto -L_\alpha(t, x, \alpha, \mu)$ is invertible; its inverse map is $p \mapsto -H_p(t, x, p, \mu)$ by [97] Theorem 26.5. Theorem 26.6 in [97] also implies that the maximum in 4.2.4 is achieved by a unique control given by $-H_p(t, x, p, \mu)$. In the next step, we prove 4.2.6 which implies that H_p maps the bounded subsets of $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$ for $R > 0$ into relatively compact subspaces of \mathbb{R}^d ; we recall that L_α is continuous on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_{\infty, R}(\mathbb{R}^d \times \mathbb{R}^d)$; therefore H_p is likewise continuous on the same space. We recall that H satisfies

$$H(t, x, p, \mu) = p \cdot H_p(t, x, p, \mu) - L(t, x, -H_p(t, x, p, \mu), \mu),$$

therefore H is also continuous on the same spaces.

Second step: proving the first three inequalities of the Lemma. Using the growth assumptions on L , we first prove (4.2.6). On the one hand we have that

$$H(t, x, p, \mu) \geq -L(t, x, 0, \mu) \geq -C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right),$$

by A4 and the condition of optimality in (4.2.4). On the other hand, A5, the fact that $-H_p(t, x, p, \mu)$ satisfies the optimality condition in (4.2.4), and the Young inequality $y \cdot z \leq \frac{|y|^q}{q} + \frac{|z|^{q'}}{q'}$ for $y, z \in \mathbb{R}^d$, yield that,

$$\begin{aligned} H(t, x, p, \mu) &= p \cdot H_p(t, x, p, \mu) - L(t, x, -H_p(t, x, p, \mu), \mu) \\ &\leq \frac{1}{qC_0} |H_p(t, x, p, \mu)|^{q'} + \frac{C_0^{\frac{q}{q'}}}{q} |p|^q - C_0^{-1} |H_p(t, x, p, \mu)|^{q'} + C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right) \\ &\leq -\frac{1}{qC_0} |H_p(t, x, p, \mu)|^{q'} + \frac{C_0^{\frac{q}{q'}}}{q} |p|^q + C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right). \end{aligned}$$

Therefore, using the latter two chains of inequalities, and the fact that $\frac{q}{q'} = q - 1$, we obtain that,

$$(4.2.10) \quad \frac{1}{qC_0} |H_p(t, x, p, \mu)|^{q'} \leq \frac{C_0^{q-1}}{q} |p|^q + 2C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right).$$

This and the inequality $|y + z|^{\frac{1}{q'}} \leq |y|^{\frac{1}{q'}} + |z|^{\frac{1}{q'}}$ for $y, z \in \mathbb{R}$, imply that

$$|H_p(t, x, p, \mu)| \leq C_0^{q-1} |p|^{q-1} + (2qC_0^2)^{\frac{1}{q'}} \left(1 + \Lambda_{q'}(\mu)\right).$$

From A5 and (4.2.10), we obtain that,

$$\begin{aligned} |H(t, x, p, \mu)| &= |p \cdot H_p(t, x, p, \mu) - L(t, x, -H_p(t, x, p, \mu), \mu)| \\ &\leq \frac{|p|^q}{q} + \frac{|H_p(t, x, p, \mu)|^{q'}}{q'} + C_0 \left(1 + |H_p(t, x, p, \mu)|^{q'} + \Lambda_{q'}(\mu)^{q'}\right) \\ &\leq \left(\frac{1}{q} + \frac{C_0^q}{q'}\right) |p|^q + C_0 \left(1 + 2qC_0 \left(\frac{1}{q'} + C_0\right)\right) \left(1 + \Lambda_{q'}(\mu)^{q'}\right). \end{aligned}$$

We still have to prove (4.2.8). Let ε be a positive constant depending only on C_0 such that $\varepsilon - C_0\varepsilon^{q'} \geq \frac{\varepsilon}{2}$, by the optimality condition in (4.2.4) used with $\alpha = -\varepsilon|p|^{q-2}p$, we have,

$$\begin{aligned} H(t, x, p, \mu) &\geq \varepsilon|p|^q - L(t, x, -\varepsilon|p|^{q-2}p, \mu) \\ &\geq \varepsilon|p|^q - C_0 \left(1 + \varepsilon^{q'} |p|^{(q-1)q'} + \Lambda_{q'}(\mu)^{q'}\right) \\ &\geq \frac{\varepsilon}{2}|p|^q - C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right). \end{aligned}$$

Then from A5,

$$\begin{aligned} H(t, x, p, \mu) &= p \cdot H_p(t, x, p, \mu) - L(t, x, -H_p(t, x, p, \mu), \mu) \\ &\leq \frac{\varepsilon}{4}|p|^q + \frac{(4q\varepsilon^{-1})^{\frac{q'}{q}}}{q'} |H_p(t, x, p, \mu)|^{q'} + C_0 \left(1 + |H_p(t, x, p, \mu)|^{q'} + \Lambda_{q'}(\mu)^{q'}\right). \end{aligned}$$

Combining the latter two chains of inequalities, there exists C a positive constant depending only on C_0 such that

$$|H_p(t, x, p, \mu)|^{q'} \geq C^{-1}|p|^q - C \left(1 + \Lambda_{q'}(\mu)^{q'}\right).$$

This and A4 yield that

$$\begin{aligned} p \cdot H_p(t, x, p, \mu) - H(t, x, p, \mu) &= L(t, x, -H_p(t, x, p, \mu), \mu) \\ &\geq C_0^{-1} |H_p(t, x, p, \mu)|^{q'} - C_0 \left(1 + \Lambda_{q'}(\mu)^{q'}\right) \\ &\geq (C_0 C)^{-1} |p|^q - (C_0 + C_0^{-1} C) \left(1 + \Lambda_{q'}(\mu)^{q'}\right). \end{aligned}$$

Third step: the smoothness properties and the last inequality.

From (4.2.6), $-H_p(t, x, p, \mu)$ is locally uniformly bounded, therefore we can reduce the set of admissible controls α in (4.2.4) from \mathbb{R}^d to a compact subset of \mathbb{R}^d . Within this framework, the envelop theorem states that H is differentiable in x and its derivatives are defined by,

$$H_x(t, x, p, \mu) = -L_x(t, x, -H_p(t, x, p, \mu), \mu).$$

The continuity property of H_x relies on the ones of L_x and H_p . Moreover, from A5 and (4.2.10), we obtain

$$|H_x(t, x, p, \mu)| \leq C_0^{q+1}|p|^q + C_0 (1 + 2qC_0^2) \left(1 + \Lambda_{q'}(\mu)^{q'}\right).$$

This concludes the proof. □

4.3 Applications

4.3.1 Exhaustible resource model

This model is often referred to as Bertrand and Cournot competition model for exhaustible resources, introduced in the independent works of Cournot [50] and Bertrand [23]; its mean field game version in dimension one was introduced in [65] and numerically analyzed in [47]; for theoretical results see [25, 62, 73, 63]. We consider a continuum of producers selling exhaustible resources. The production of a representative agent A representative

agent controls her production $\alpha_t \geq 0$; the agents differ in their production capacity $X_t \in \mathbb{R}$ (the state variable), that satisfies,

$$dX_t = -\alpha_t dt + \sqrt{2\nu} dW_t,$$

where $\nu > 0$ and W is a d -dimensional Brownian motion. Each producer is selling a different resource and has her own consumers. However, the resources are substitutable and any consumer may change her mind and buy from a competitor depending on the degree of competition in the game (which stands for ε in the linear demand case below for instance). Therefore the selling price per unit of resource that a producer can make when she sales α units of resource, depends naturally on α and on the quantity produced by the other agents. The price satisfies a supply-demand relationship, and is given by $P(\alpha, \bar{\alpha})$, where $\bar{\alpha}$ is the accumulated demand which depends on the overall distribution of productions of the agents. A producer tries to maximize her profit, or equivalently to minimize the following quantity,

$$\mathbb{E} \left[\int_0^T -P(\alpha_t, \bar{\alpha}_t) \cdot q_t dt + g(X_T) \right],$$

where g is a terminal cost which often penalizes the producers who have non-zero production capacity at the end of the game. In [47], the authors considered a linear demand system depending on $\bar{\alpha}_{\text{lin}} = \int_{\mathbb{R} \times \mathbb{R}} \alpha d\mu(x, \alpha)$ where μ is the joint law of states and controls, and a price defines by $P_{\text{lin}}(\alpha, \bar{\alpha}_{\text{lin}}) = 1 - \alpha - \varepsilon \bar{\alpha}_{\text{lin}}$. In this case, L^{lin} the running cost, and H_{lin} its Legendre transform are defined by

$$\begin{aligned} L^{\text{lin}}(\alpha, \mu) &= \alpha^2 + \varepsilon \alpha \bar{\alpha} - \alpha, \\ H^{\text{lin}}(p, \mu) &= \frac{1}{4} (p + \varepsilon \bar{\alpha} - 1)^2, \end{aligned}$$

where $\alpha, p \in \mathbb{R}$, $\mu \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ and $\bar{\alpha}$ is defined by $\bar{\alpha} = \int_{\mathbb{R} \times \mathbb{R}} \tilde{\alpha} d\mu(y, \tilde{\alpha})$. Therefore the system of MFGC has the following form,

$$\begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{4} (\nabla_x u + \varepsilon \bar{\alpha} - 1)^2 = 0, \\ \partial_t m - \nu \Delta m - \operatorname{div} \left(\frac{1}{2} (\nabla_x u + \varepsilon \bar{\alpha} - 1) m \right) = 0, \\ \bar{\alpha}(t) = - \int_{\mathbb{R}^d} \frac{1}{2} (\nabla_x u + \varepsilon \bar{\alpha}(t) - 1) dm(t, x), \\ u(T, x) = g(x), \\ m(0, x) = m_0(x), \end{cases}$$

for $(t, x) \in [0, T] \times \mathbb{R}$. Roughly speaking, $\varepsilon = 0$ corresponds to a monopolist who does not suffer from competition, and she plays as if she was alone in the game. Conversely, $\varepsilon = \infty$ stands for all the producers selling the same resource and the consumers not having any a priori preference.

Let us consider the following generalization of the latter system to the d -dimensional case with a more general Hamiltonian and interaction through controls,

$$(4.3.1) \quad \begin{cases} -\partial_t u - \nu \Delta u + H(t, x, \nabla_x u + \varphi(x)^T P(t)) = f(t, x, m(t)), \\ \partial_t m - \nu \Delta m - \operatorname{div} (H_p(t, x, \nabla_x u + \varphi(x)^T P(t)) m) = 0, \\ P(t) = \Psi \left(t, - \int_{\mathbb{R}^d} \varphi(x) H_p(t, x, \nabla_x u + \varphi(x)^T P(t)) dm(t, x) \right), \\ u(T, x) = g(x, m(T)), \\ m(0, x) = m_0(x), \end{cases}$$

where $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$ and $\Psi : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$ are given functions. The counterpart of the latter system posed on \mathbb{T}^d has been introduced in [25]. Theorem 4.2.2 and 4.2.3 provide the existence and the uniqueness respectively of the solution of this MFGC system.

Proposition 4.3.1. *Assume **A1**, **A2**, **U**. If the function Ψ is continuous, $\Psi(t, \cdot)$ is monotone, locally Lipschitz continuous, and admits at most a power-like growth of exponent $q' - 1$ with a coefficient uniform in $t \in [0, T]$, there exists at most one solution to (4.3.1).*

Proposition 4.3.2. *Assume **A1**, **A2**, **A4-A6**, and that Ψ satisfies the same assumptions as in Proposition 4.3.1. There exists a solution to (4.3.1).*

Proof. Take the drift function as $b = \alpha$. We define the Lagrangian ℓ by

$$\ell(t, x, \alpha, \mu) = L(t, x, \alpha) + \varphi(x)\alpha \cdot P(t, \mu) + f(t, x, m),$$

where L is the Legendre transform of the map H in (4.3.1), and $P(t, \mu)$ is defined by $P(t, \mu) = \Psi\left(t, \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x)\alpha d\mu(x, \alpha)\right)$, for $(t, x, \alpha, m, \mu) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ such that m is the first marginal of μ . We take h as the Legendre transform of ℓ with respect to α .

If Ψ satisfies the assumptions in 4.3.1, any of the assumptions **A1**, **A2**, **A4**, or **A5** is preserved by replacing L by ℓ . Moreover, a straightforward calculation yields that

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\ell(t, x, \alpha, \mu^1) - \ell(t, x, \alpha, \mu^2)) d(\mu^1 - \mu^2)(x, \alpha) \\ = (P(t, \mu^1) - P(t, \mu^2)) \cdot \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x)\alpha d(\mu^1 - \mu^2)(x, \alpha), \end{aligned}$$

for $t \in [0, T]$ and $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. This and the monotonicity of Ψ implies that ℓ satisfies **A3**. Therefore, Propositions 4.3.1 and 4.3.2 are direct consequences of Theorems 4.2.2 and 4.2.3 respectively. \square

In [25], similar existence and uniqueness results for the counterpart of (4.3.1) posed on \mathbb{T}^d are given in the quadratic setting, with a uniformly convex Lagrangian and Ψ being the gradient of a convex map. Here, we generalize their results to a wider class of Lagrangians and functions Ψ .

For an extension of this model to the case when Ψ is non-monotone, see [75].

4.3.2 A model of crowd motion

This model of crowd motion has been introduced in [75] in the non-monotone setting. It has been numerically studied in [2] in the quadratic non-monotone case. For $\mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ we define $V(\mu)$ the average drift by,

$$V(\mu) = \frac{1}{Z(\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \alpha k(x) d\mu(x, \alpha),$$

where $Z(\mu)$ is a normalization constant defined by $Z(\mu) = \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} k(x)^{q_1} d\mu(x, \alpha)\right)^{\frac{1}{q_1}}$, for some constant $q_1 \in [q, \infty]$ where q is defined below. To be consistent with the notations used in [75], $k : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a non-negative kernel. By convention, if $Z(\mu) = 0$, we take $V(\mu) = 0$.

The state of a representative agent is given by her position $X_t \in \mathbb{R}^d$ which she controls through her velocity α via the following stochastic differential equation,

$$dX_t = \alpha_t dt + \sqrt{2\nu} dW_t.$$

Her objective is to minimize the cost functional given by,

$$\mathbb{E} \left[\int_0^T \frac{\theta}{2} |\alpha_t + \lambda V(\mu(t))|^2 + \frac{1-\theta}{a'} |\alpha_t|^{a'} + f(t, X_t, m(t)) dt + g(X_T, m(T)) \right],$$

where $\lambda \geq 0$ and $0 \leq \theta \leq 1$ are two constants standing for the intensity of the preference of an individual to have an opposite control as the stream one, and $a' > 1$ is an exponent. In this model we define the Lagrangian L by,

$$L(x, \alpha, \mu) = \frac{\theta}{2} |\alpha + \lambda V(\mu)|^2 + \frac{1-\theta}{a'} |\alpha|^{a'},$$

and the Hamiltonian H as its Legendre transform. The map H does not admit an explicit form for every choice of the parameters a' . We take $q' = \max(2, a')$, and $q = \frac{q'}{q'-1}$ its conjugate exponent.

Here, since the control is equal to the drift, the MFGC system is of the form of (4.2.3). Therefore, the following proposition is a consequence of Theorems 4.2.2 and 4.2.3.

Proposition 4.3.3. *Under assumption A6, there exists a solution to the above MFGC system of crowd motion.*

Under assumption U, this solution is unique.

The proof is straightforward, it consists in checking that L satisfies A1-A5.

For existence results of the MFGC system of this model with $\lambda < 0$, see [75].

4.4 The fixed point (4.2.5c) and the proof of Lemma 4.2.4

This section is devoted to step I. In paragraph 4.4.1, we state a priori estimates on a fixed point of (4.2.5c) (Lemma 4.4.1); then we use these estimates and Leray-Schauder fixed point theorem (Theorem 4.4.2) and obtain the existence of a fixed point (4.2.5c) at any time $t \in [0, T]$ (Lemma 4.4.3). We address the continuity with respect to time of the fixed point, i.e. step I.c, in Lemma 4.4.4.

In this section and the next one, we work on $\mathbb{T}_a^d = \mathbb{R}^d / (a\mathbb{Z}^d) \cong \mathbb{T}_a^d$ the d -dimensional torus of radius $a > 0$. Here we take $L : [0, T] \times \mathbb{T}_a^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$. All the assumptions in paragraph 4.2.2 are stated in \mathbb{R}^d , but, when considering that L satisfies one of those assumptions, we shall simply replace \mathbb{R}^d by \mathbb{T}_a^d as the state set in the chosen assumption (note that we keep \mathbb{R}^d as the set of admissible controls). The initial distribution m_0 is now in $\mathcal{P}(\mathbb{T}_a^d)$. The Hamiltonian H is still defined as the Legendre transform of L , i.e. it satisfies (4.2.4).

4.4.1 Leray-Schauder Theorem for solving the fixed point in μ

We start by stating a priori estimates for solutions of the fixed point in μ (4.2.5c), involving $\Lambda_{q'}(\mu)$ and $\Lambda_\infty(\mu)$ defined in (4.2.1).

Lemma 4.4.1. *Assume that L satisfies A1-A5. For any $t \in [0, T]$, $m \in \mathcal{P}(\mathbb{T}_a^d)$ and $p \in C^0(\mathbb{T}_a^d, \mathbb{R}^d)$, if there exists $\mu \in \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ such that*

$$(4.4.1) \quad \mu = (I_d, -H_p(t, \cdot, p(\cdot), \mu)) \# m,$$

then it satisfies

$$(4.4.2) \quad \Lambda_{q'}(\mu)^{q'} \leq 4C_0^2 + \frac{(q')^{q'-1} (2C_0)^q}{q} \|p\|_{L^q(m)}^q,$$

$$(4.4.3) \quad \Lambda_\infty(\mu) \leq C_0 (1 + \|p\|_\infty + \Lambda_{q'}(\mu)).$$

Proof. We use **A3** with $m \otimes \delta_0$ and μ satisfying (4.4.1),

$$(4.4.4) \quad \int_{\mathbb{T}_a^d \times \mathbb{R}^d} (L(t, x, \alpha, \mu) - L(t, x, \alpha, m \otimes \delta_0)) d\mu(x, \alpha) + \int_{\mathbb{T}_a^d} (L(t, x, 0, m \otimes \delta_0) - L(t, x, 0, \mu)) dm(x) \geq 0.$$

From **A5**, we obtain

$$(4.4.5) \quad \int_{\mathbb{T}_a^d} L(t, x, 0, m \otimes \delta_0) dm(x) \leq C_0.$$

The latter two inequalities, **A4** and the convexity of L (stated in Lemma 4.2.10) yield

$$\begin{aligned} C_0^{-1} \int_{\mathbb{T}_a^d \times \mathbb{R}^d} |\alpha|^{q'} d\mu(x, \alpha) - C_0 &\leq C_0 + \int_{\mathbb{T}_a^d} (L(t, x, \alpha^\mu(x), \mu) - L(t, x, 0, \mu)) dm(x) \\ &\leq C_0 + \int_{\mathbb{T}_a^d \times \mathbb{R}^d} \alpha \cdot L_\alpha(t, x, \alpha, \mu) d\mu(x, \alpha), \end{aligned}$$

where α^μ is defined in paragraph 4.2.1. We recall that $p(x) = -L_\alpha(t, x, \alpha^\mu(x), \mu)$. Using the inequality $yz \leq \frac{y^{q'}}{c^{q'} q'} + \frac{c^q z^q}{q}$ which holds for any $y, z, c > 0$, we obtain

$$\frac{1}{C_0} \int_{\mathbb{T}_a^d \times \mathbb{R}^d} |\alpha|^{q'} d\mu(x, \alpha) \leq 2C_0 + \frac{(2C_0 q')^{\frac{q}{q'}}}{q} \int_{\mathbb{T}_a^d} |p(x)|^q dm(x) + \frac{1}{2C_0} \int_{\mathbb{T}_a^d \times \mathbb{R}^d} |\alpha|^{q'} d\mu(x, \alpha).$$

This, the equality $\frac{q}{q'} + 1 = q$, and (4.2.6) imply (4.4.2) and (4.4.3), we recall that we assume $C_0 = \tilde{C}_0$. \square

Here, we shall use Leray-Schauder fixed point theorem as stated in [56] Theorem 11.6.

Theorem 4.4.2 (Leray-Schauder fixed point theorem). *Let \mathcal{B} be a Banach space and let Ψ be a compact mapping from $[0, 1] \times \mathcal{B}$ into \mathcal{B} such that $\Psi(0, x) = 0$ for all $x \in \mathcal{B}$. Suppose that there exists a constant C such that*

$$\|x\|_{\mathcal{B}} \leq C,$$

for all $(\theta, x) \in [0, 1] \times \mathcal{B}$ satisfying $x = \Psi(\theta, x)$. Then the mapping $\Psi(1, \cdot)$ of \mathcal{B} into itself has a fixed point.

From Lemma 4.4.1 and Theorem 4.4.2, we obtain the following existence result for a fixed point (4.2.5c).

Lemma 4.4.3. *Assume **A1-A5**. For $t \in [0, T]$, $m \in \mathcal{P}(\mathbb{T}_a^d)$ and $p \in C^0(\mathbb{T}_a^d; \mathbb{R}^d)$, there exists a unique $\mu \in \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ such that $\mu = (I_d, -H_p(t, \cdot, p(\cdot), \mu)) \# m$. Moreover, μ satisfies the inequality stated in Lemma 4.4.1.*

In the following proof, we will take advantage of the flexibility offered when making all assumptions on the Lagrangian, instead of the Hamiltonian. We will introduce a sequence of new Lagrangians. The associated Hamiltonians may not admit explicit form; therefore it would be difficult to check assumptions on them. Here on the one hand, checking the assumptions on the new Lagrangians is straightforward. On the other hand, we obtain the same conclusions on the new Hamiltonian as stated in Lemma 4.2.11.

Proof. Take (t, \bar{p}, m) satisfying the same assumptions as (t, p, m) in Lemma 4.4.3. In order to use the Leray-Schauder fixed point theorem later, we introduce the following family of Lagrangians indexed by $\lambda \in [0, 1]$,

$$(4.4.6) \quad L^{\bar{p}, \lambda}(x, \alpha, \mu) = \lambda L(t, x, \alpha, \mu) + (1 - \lambda) \left(\frac{|\alpha|^{q'}}{q'} - \alpha \cdot \bar{p}(x) \right),$$

for $(x, \alpha, \mu) \in \mathbb{T}_a^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$. We denote by $H^{\bar{p}, \lambda}$ the Legendre transform of $L^{\bar{p}, \lambda}$. For $\lambda = 0$ it satisfies

$$(4.4.7) \quad H^{\bar{p}, 0}(x, p, \mu) = \frac{1}{q} |p - \bar{p}(x)|^q.$$

From Young inequality, we obtain that

$$|\alpha \cdot \bar{p}(x)| \leq \frac{|\alpha|^{q'}}{2q'} + \frac{2^{q-1}}{q} \|p\|_\infty.$$

Therefore, up to changing C_0 into $\max\left(\frac{1}{2q'}, \frac{2^{q-1}}{q} \|p\|_\infty, C_0\right)$, we may assume that $L^{\bar{p}, \lambda}$ satisfies **A1-A5**, with the same constant C_0 for any $\lambda \in [0, 1]$. The map $(\lambda, x, p, \mu) \mapsto -H_p^{\bar{p}, \lambda}(x, p, \mu)$ is continuous on $[0, 1] \times \mathbb{T}_a^d \times \mathbb{R}^d \times \mathcal{P}_{\infty, R}(\mathbb{T}_a^d \times \mathbb{R}^d)$, for any $R > 0$, by the same arguments as in the proof of Lemma 4.2.11.

For $\alpha \in C^0(\mathbb{T}_a^d; \mathbb{R}^d)$, we set $\mu = (I_d, \alpha) \# m \in \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ and $\bar{\alpha}(x) = -H_p^{\bar{p}, \lambda}(x, \bar{p}(x), \mu)$, for $x \in \mathbb{T}_a^d$. We define the map Ψ , from $[0, 1] \times C^0(\mathbb{T}_a^d; \mathbb{R}^d)$ to $C^0(\mathbb{T}_a^d; \mathbb{R}^d)$, by $\Psi(\lambda, \alpha) = \bar{\alpha}$. If α is a fixed point of $\Psi(1, \cdot)$, then μ defined as above satisfies the fixed point in Lemma 4.4.3. Conversely, if μ satisfies the fixed point in Lemma 4.4.3, then α^μ (defined in paragraph 4.2.1) is a fixed point of $\Psi(1, \cdot)$.

The map Ψ is continuous by the continuity of $(\lambda, x, p, \mu) \mapsto -H_p^{\bar{p}, \lambda}(x, p, \mu)$. For $R > 0$, the set A_R , defined by $A_R = [0, 1] \times \mathbb{T}_a^d \times B_{\mathbb{R}^d}(0, R) \times \mathcal{P}_{\infty, R}(\mathbb{T}_a^d \times \mathbb{R}^d)$, is compact. By Heine theorem, the map $(\lambda, x, p, \mu) \mapsto -H_p^{\bar{p}, \lambda}(x, p, \mu)$ is uniformly continuous on A_R . Here, note that we use the fact that $\mathcal{P}_{\infty, R}(\mathbb{T}_a^d \times \mathbb{R}^d)$ is a metric space since the weak* topology coincides with the topology induced by the 1-Wassertein distance on $\mathcal{P}_{\infty, R}(\mathbb{T}_a^d \times \mathbb{R}^d)$. Heine theorem also implies that \bar{p} is uniformly continuous. Therefore, Ψ is a compact mapping from $[0, 1] \times C^0(\mathbb{T}_a^d; \mathbb{R}^d)$ to $C^0(\mathbb{T}_a^d; \mathbb{R}^d)$, i.e. it maps bounded subsets of $[0, 1] \times C^0(\mathbb{T}_a^d; \mathbb{R}^d)$ into relatively compact subsets of $C^0(\mathbb{T}_a^d; \mathbb{R}^d)$: this comes from the latter observation and Arzelà-Ascoli theorem.

Take α a fixed point of $\Psi(\lambda, \cdot)$, for $\lambda \in [0, 1]$, Lemma 4.4.1 implies that $\|\alpha\|_\infty$ is bounded by a constant C which does not depend on λ .

Moreover, it is straightforward to check that $\Psi(0, \cdot) = 0$. Leray-Schauder Theorem 4.4.2 implies that there exists a fixed point of the map $\alpha \mapsto \Psi(1, \alpha)$, which concludes the existence part of the proof.

The proof of uniqueness relies on **A3** and the strict convexity of L , see [38] Lemma 5.2 for the detailed proof. \square

4.4.2 The continuity of the fixed point in time

The fixed point result stated in Lemma 4.4.3 yields the existence of a map $(t, p, m) \mapsto \mu$. The continuity of this map is addressed in the following lemma:

Lemma 4.4.4. *Assume **A1-A5**. Let $(t^n, m^n, p^n)_{n \in \mathbb{N}}$ be a sequence in $[0, T] \times \mathcal{P}(\mathbb{T}_a^d) \times C^0(\mathbb{T}_a^d; \mathbb{R}^d)$. Assume that*

- $t^n \rightarrow_{n \rightarrow \infty} t \in [0, T]$,
- $(p^n)_{n \in \mathbb{N}}$ is uniformly convergent to $p \in C^0(\mathbb{T}_a^d; \mathbb{R}^d)$,
- $(m^n)_{n \in \mathbb{N}}$ tends to m in the weak* topology.

We define μ^n and μ as the unique solutions of the fixed point relation of Lemma 4.4.3 respectively associated to (t^n, m^n, p^n) and (t, m, p) , for $n \in \mathbb{N}$. Then the sequence $(\mu^n)_{n \in \mathbb{N}}$ tends to μ in $\mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ equipped with the weak* topology.

Proof. The sequence $(p^n)_{n \in \mathbb{N}}$ is uniformly bounded in the norm $\|\cdot\|_\infty$. Therefore $(\mu^n)_{n \in \mathbb{N}}$ is uniformly compactly supported by Lemma 4.4.1. Thus we can extract a subsequence $(\mu^{\varphi(n)})_{n \in \mathbb{N}}$ convergent to some limit $\tilde{\mu} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ in the weak* topology on measures.

We recall that α^μ is defined in paragraph 4.2.1. Here, since $\mu^{\varphi(n)}$ and μ are fixed points like in Lemma 4.4.3, they satisfy:

$$\begin{aligned}\alpha^{\mu^{\varphi(n)}}(x) &= -H_p(t^{\varphi(n)}, x, p^{\varphi(n)}(x), \mu^{\varphi(n)}), \\ \alpha^\mu(x) &= -H_p(t, x, p(x), \mu),\end{aligned}$$

for $x \in \mathbb{T}_a^d$ and $n \in \mathbb{N}$. From the continuity of H_p stated in Lemma 4.2.11, $(\alpha^{\varphi(n)})_{n \in \mathbb{N}}$ tends uniformly to the function $\tilde{\alpha} : x \mapsto -H_p(t, x, p, \tilde{\mu})$. Then $((I_d, \alpha^{\varphi(n)}) \# m^n)_{n \in \mathbb{N}}$ tends to $(I_d, \tilde{\alpha}) \# m$ in the weak* topology. Hence $\tilde{\mu}$ satisfies the same fixed point relation as μ ; by uniqueness we deduce that $\tilde{\mu} = \mu$. This implies that all the convergent subsequences of $(\mu^n)_{n \in \mathbb{N}}$ have the same limit μ , thus the whole sequence converges to μ . \square

Lemma 4.4.3 states that for all time the fixed point (4.2.5c) has a unique solution. Then Lemma 4.4.4 yields the continuity of the map defined by the fixed point under suitable assumptions. Therefore, the conclusion of step I.c and the Lemma 4.2.4 are straightforward consequences of these two lemmas.

Remark 4.4.5. *All the conclusions of this section hold when we relax Assumption A3, assuming that the inequality holds only when μ^1 and μ^2 have the same first marginal. Some applications of MFGC do not satisfy A3, but satisfy the above-mentioned relaxed monotonicity assumption. This is the case of the MFG version of the Almgren and Chriss' model for price impact and high-frequency trading, discussed in [38, 41, 43, 75].*

However, the a priori estimates in the next section do not hold under this relaxed monotonicity assumption. We refer to [75] for estimates which do not rely on A3 (Assumptions FP1 and FP2 in [75] are unnecessary if L satisfies the relaxed monotonicity assumption).

4.5 A priori estimates for the solutions to (4.2.5)

In order to use the Leray-Schauder fixed point theorem later, we introduce the following family of Lagrangians indexed by $\theta \in (0, 1]$,

$$(4.5.1) \quad L^\theta(t, x, \alpha, \mu) = \theta L(t, x, \theta^{-1}\alpha, \Theta(\mu)),$$

where the map $\Theta : \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d) \rightarrow \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$ is defined by $\Theta(\mu) = (I_d \otimes \theta^{-1}I_d) \# \mu$. Then the Hamiltonian defined as the Legendre transform of L^θ is given by

$$(4.5.2) \quad H^\theta(t, x, p, \mu) = \theta H(t, x, p, \Theta(\mu)).$$

The definition of the Hamiltonian can naturally be extended to $\theta = 0$ by $H^0 = 0$, the associated Lagrangian is $L^0 = 0$ if $\alpha = 0$ and $L^0 = \infty$ otherwise. We introduce the following system of MFGC,

$$(4.5.3a) \quad -\partial_t u - \nu \Delta u + H^\theta(t, x, \nabla_x u(t, x), \mu(t)) = \theta f(t, x, m(t)) \quad \text{in } (0, T) \times \mathbb{T}_a^d,$$

$$(4.5.3b) \quad \partial_t m - \nu \Delta m - \operatorname{div} \left(H_p^\theta(t, x, \nabla_x u(t, x), \mu(t)) m \right) = 0 \quad \text{in } (0, T) \times \mathbb{T}_a^d,$$

$$(4.5.3c) \quad \mu(t) = \left(I_d, -H_p^\theta(t, \cdot, \nabla_x u(t, \cdot), \mu(t)) \right) \# m(t) \quad \text{in } [0, T],$$

$$(4.5.3d) \quad u(T, x) = \theta g(x, m(T)) \quad \text{in } \mathbb{T}_a^d,$$

$$(4.5.3e) \quad m(0, x) = m_0(x) \quad \text{in } \mathbb{T}_a^d.$$

When $\theta = 1$, the latter system coincides with (4.2.5). When $\theta = 0$, (4.5.3) consists in a situation in which the state of a representative agent satisfies a non-controlled stochastic differential equation. Alternatively it can be interpreted as a game in which the agents pay an infinite price as soon as they try to use a control different than 0. In particular the case $\theta = 0$ is specific and easier than the case when $\theta > 0$. Therefore, in the rest of this section, we only consider $\theta \in (0, 1]$.

Let us mention that assumptions **A1-A3** are preserved when replacing L and H by L^θ and H^θ respectively. Moreover the inequalities from **A4, A5**, become respectively

$$(4.5.4) \quad L^\theta(t, x, \alpha, \mu) \geq C_0^{-1} \theta^{1-q'} |\alpha|^{q'} - C_0 \theta - C_0 \theta^{1-q'} \Lambda_{q'}(\mu)^{q'},$$

$$(4.5.5) \quad \left| L^\theta(t, x, \alpha, \mu) \right| \leq C_0 \theta + C_0 \theta^{1-q'} \left(|\alpha|^{q'} + \Lambda_{q'}(\mu)^{q'} \right),$$

since $\Lambda_{q'}(\Theta(\mu)) = \theta^{-1} \Lambda_{q'}(\mu)$. Furthermore, the conclusions of Lemma 4.2.11 hold and the inequalities become respectively

$$(4.5.6) \quad \left| H_p^\theta(t, x, p, \mu) \right| \leq C_0 \theta (1 + |p|^{q-1}) + C_0 \Lambda_{q'}(\mu),$$

$$(4.5.7) \quad \left| H^\theta(t, x, p, \mu) \right| \leq C_0 \theta (1 + |p|^q) + C_0 \theta^{1-q'} \Lambda_{q'}(\mu)^{q'},$$

$$(4.5.8) \quad p \cdot H_p^\theta(t, x, p, \mu) - H^\theta(t, x, p, \mu) \geq C_0^{-1} \theta |p|^q - C_0 \theta - C_0 \theta^{1-q'} \Lambda_{q'}(\mu)^{q'},$$

$$(4.5.9) \quad \left| H_x^\theta(t, x, p, \mu) \right| \leq C_0 \theta (1 + |p|^q) + C_0 \theta^{1-q'} \Lambda_{q'}(\mu)^{q'}.$$

We recall that without loss of generality, we assumed $\tilde{C}_0 = C_0$ where \tilde{C}_0 is defined in Lemma 4.2.11.

Instead of proving Lemma 4.2.5 and step **II.a**, we address the more general following lemma which provides a priori estimates not only for solutions to (4.2.5) but also for solutions to (4.5.3). This will help to use the Leray-Schauder theorem in the next section.

Lemma 4.5.1. *Under assumptions **A1-A6**, there exists a positive constant C which only depends on the constants in the assumptions and not on a or θ , such that the solution to (4.5.3) satisfies: $\|u\|_\infty \leq C\theta$, $\|\nabla_x u\|_\infty \leq C\theta^{\frac{1}{2}}$ and $\sup_{t \in [0, T]} \Lambda_\infty(\mu(t)) \leq C\theta$.*

Proof. First step: controlling $\int_0^T \Lambda_{q'}(\mu(t))^{q'} dt$

Let us take (X, α) defined by

$$\begin{cases} \alpha_t = \alpha^{\mu(t)}(t, X_t) = -H_p^\theta(t, X_t, \nabla_x u(t, X_t), \mu(t)), \\ dX_t = \alpha_t dt + \sqrt{2\nu} dB_t, \\ X_0 = \xi \sim m_0, \end{cases}$$

where $(B_t)_{t \in [0, T]}$ is a Brownian motion independent of ξ .

The function u is the value function of an optimization problem, i.e. the lowest cost that a representative agent can achieve from time t to T if $X_t = x$, when the probability measures m and μ are fixed, i.e.

$$(4.5.10) \quad \alpha_{|s \in [t, T]} = \operatorname{argmin}_{\alpha'} \mathbb{E} \left[\int_t^T L^\theta \left(s, X_s^{\alpha'}, \alpha'_s, \mu(s) \right) + \theta f \left(s, X_s^{\alpha'}, m(s) \right) ds + \theta g \left(X_T^{\alpha'}, m(T) \right) \right],$$

where for a control α' , we define

$$\begin{cases} dX_t^{\alpha'} = \alpha'_t dt + \sqrt{2\nu} dB'_t, \\ X_0^{\alpha'} = \xi' \sim m_0, \end{cases}$$

and $(B'_t)_{t \in [0, T]}$ is a Brownian motion independent of ξ' . Let us recall that for any $t \in [0, T]$, $m(t)$ is the law of X_t , and $\mu(t)$ is the law of (X_t, α_t) . We introduce \tilde{X} the stochastic process defined by

$$\begin{cases} d\tilde{X}_t = \sqrt{2\nu} dB_t, \\ \tilde{X}_0 = \xi \sim m_0. \end{cases}$$

We set $\tilde{m}(t) = \mathcal{L}(\tilde{X}_t)$ and $\tilde{\mu}(t) = \mathcal{L}(\tilde{X}_t) \otimes \delta_0$ for $t \in [0, T]$. For the strategy consisting in taking $\alpha' = 0$, (4.5.10) yields the inequality:

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu(t)) d\mu(t, x, \alpha) dt + \int_0^T \int_{\mathbb{T}_a^d} \theta f(t, x, m(t)) dm(t, x) dt + \int_{\mathbb{T}_a^d} \theta g(x, m(T)) dm(T, x) \\ & \leq \int_0^T \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu(t)) d\tilde{\mu}(t, x, \alpha) dt + \int_0^T \int_{\mathbb{T}_a^d} \theta f(t, x, m(t)) d\tilde{m}(t, x) dt + \int_{\mathbb{T}_a^d} \theta g(x, m(T)) d\tilde{m}(T, x). \end{aligned}$$

This and **A6** imply that,

$$(4.5.11) \quad \int_0^T \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu(t)) d\mu(t, x, \alpha) dt \leq \int_0^T \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu(t)) d\tilde{\mu}(t, x, \alpha) dt + 2C_0\theta(1+T).$$

Assumption **A3** with $(\mu(t), \tilde{\mu}(t))$ yields

$$(4.5.12) \quad \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu(t)) d\tilde{\mu}(t, x, \alpha) + \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \tilde{\mu}(t)) d\mu(t, x, \alpha) \leq \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \mu(t)) d\mu(t, x, \alpha) + \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \tilde{\mu}(t)) d\tilde{\mu}(t, x, \alpha).$$

Moreover, from **A5** we obtain that,

$$(4.5.13) \quad \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \tilde{\mu}(t)) \tilde{\mu}(t, d(x, \alpha)) = \int_{\mathbb{T}_a^d} \theta L(t, x, 0, \tilde{\mu}(t)) \tilde{m}(t, dx) \leq C_0\theta.$$

Therefore from (**A4**), (4.5.11), (4.5.12) and (4.5.13), we obtain,

$$\begin{aligned} \int_0^T \int_{\mathbb{T}_a^d} \left(C_0^{-1} \theta^{1-q'} |\alpha|^{q'} - C_0\theta \right) d\mu(t, x, \alpha) dt & \leq \int_0^T \int_{\mathbb{T}_a^d \times \mathbb{R}^d} L^\theta(t, x, \alpha, \tilde{\mu}(t)) d\mu(t, x, \alpha) dt \\ & \leq C_0\theta(2+3T). \end{aligned}$$

This implies

$$(4.5.14) \quad \int_0^T \Lambda_{q'}(\mu(t))^{q'} dt \leq 2C_0^2 \theta^{q'} (1 + 2T).$$

Second step: the uniform estimate on $\|u\|_\infty$

Let us rewrite (4.5.3a) in the following way,

$$-\partial_t u - \nu \Delta u + \left[\int_0^1 H_p^\theta(t, x, s \nabla_x u, \mu(t)) ds \right] \cdot \nabla_x u = H^\theta(t, x, 0, \mu(t)) + \theta f(t, x, m(t)),$$

for $(t, x) \in (0, T) \times \mathbb{T}_a^d$. The maximum principle for second-order parabolic equation, A6, and (4.2.7) yield that

$$\|u\|_\infty \leq C_0 \theta (1 + 2T) + C_0 \theta^{1-q'} \int_0^T \Lambda_{q'}(\mu(t))^{q'} dt,$$

which implies that u is uniformly bounded using the conclusion of the previous step.

Third step: the uniform estimate on $\|\nabla_x u\|_\infty$.

The proof of this step relies on the same Bernstein-like method introduced in [75] Lemma 6.5. We refer to the proof of the latter results for more details in the derivation of the equations below.

Let us introduce $\rho \in C^\infty\left(\left[-\frac{a}{2}, \frac{a}{2}\right]^d\right)$ a nonnegative mollifier such that $\rho(x) = 0$ if $|x| \geq \frac{a}{4}$ and $\int_{\left[-\frac{a}{2}, \frac{a}{2}\right]^d} \rho(x) dx = 1$. For any $0 < \delta < 1$ and $t \in [0, T]$, we introduce $\rho^\delta = \delta^{-d} \rho\left(\frac{\cdot}{\delta}\right)$ and $u^\delta(t) = \rho^\delta \star u(t)$ with \star being the convolution operator with respect to the state variable.

Possibly after modifying the constant C appearing in the first step, we can assume that $\|u\|_\infty + (1 + C_0) \theta^{1-q'} \int_0^T \Lambda_{q'}(\mu(s))^{q'} ds \leq C$ using the first two steps in such a way that C depends only on the constants in the assumptions, and not on θ . Then we introduce $\varphi : [-C, C] \rightarrow \mathbb{R}_+^*$ and w^δ defined by

$$(4.5.15) \quad \begin{aligned} \varphi(v) &= \exp(\exp(-v)), \\ w^\delta(t, x) &= \varphi\left(u^\delta(T-t, x) + (1 + C_0) \theta^{1-q'} \int_{T-t}^T \Lambda_{q'}(\mu(s))^{q'} ds\right) \left|\nabla_x u^\delta\right|^2(T-t, x), \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbb{T}_a^d$, $v \in B_{\mathbb{R}^d}(0, C)$. In particular $\varphi' < 0$, and φ , $1/\varphi$, $-\varphi'$ and $-1/\varphi'$ are uniformly bounded. We refer to the proof of Lemma 6.5 in [75] for the derivation of the following partial differential equation satisfied by w^δ ,

$$(4.5.16) \quad \begin{aligned} \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p^\theta(x, \nabla_x u^\delta, \mu) &+ 2\nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta + 2\nu \varphi \left|D_{x,x}^2 u^\delta\right|^2 \\ &= \frac{\varphi'}{\varphi} w^\delta \left[\nabla_x u^\delta \cdot H_p^\theta(x, \nabla_x u^\delta, \mu) - H^\theta(x, \nabla_x u^\delta, \mu) + (1 + C_0) \theta^{1-q'} \Lambda_{q'}(\mu)^{q'} \right] \\ &- \nu \frac{\varphi'' \varphi - 2(\varphi')^2}{\varphi^3} (w^\delta)^2 - 2\varphi \nabla_x u^\delta \cdot H_x^\theta(x, \nabla_x u^\delta, \mu) + 2\theta \varphi \nabla_x u^\delta \cdot f_x^\delta(x, m) + R^\delta(t, x) \end{aligned}$$

in which H^θ , f , f^δ , u , u^δ and μ are taken at time $T-t$ and w^δ at time t , and where f^δ

and R^δ , are defined by,

$$\begin{aligned} f^\delta(x, m) &= \rho^\delta \star (f(\cdot, m))(x), \\ R^\delta(t, x) &= -\varphi' \left| \nabla_x u^\delta \right|^2 \left[\rho^\delta \star \left(H^\theta(\cdot, \nabla_x u, \mu) \right) (x) - H^\theta(x, \nabla_x u^\delta, \mu) \right] \\ &\quad - 2\varphi \nabla_x u^\delta \cdot \left[\left(\rho^\delta \star H_x^\theta(\cdot, \nabla_x u(\cdot), \mu) \right) (x) - H_x^\theta(x, \nabla_x u^\delta, \mu) \right], \\ &\quad + 2\varphi \nabla_x u^\delta \cdot \left[D_{x,x}^2 u^\delta H_p^\theta(x, \nabla_x u^\delta, \mu) - \rho^\delta \star \left(D_{x,x}^2 u H_p^\theta(\cdot, \nabla_x u, \mu) \right) \right]. \end{aligned}$$

From (4.5.8), we obtain that

$$\nabla_x u^\delta \cdot H_p^\theta(x, \nabla_x u^\delta, \mu) - H^\theta(x, \nabla_x u^\delta, \mu) + (1 + C_0) \theta^{1-q'} \Lambda_{q'}(\mu)^{q'} \geq C_0^{-1} \theta \left| \nabla_x u^\delta \right|^q + \theta^{1-q'} \Lambda_{q'}(\mu)^{q'} - C_0 \theta.$$

Therefore, using **A6**, (4.5.16), (4.5.9), the facts that $\varphi' < 0$, that $\varphi''\varphi - 2(\varphi')^2 \geq 0$, that $\varphi, \varphi^{-1}, \varphi', (\varphi')^{-1}$ are bounded, and the latter inequality, we get

$$\begin{aligned} (4.5.17) \quad \partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p^\theta(x, \nabla_x u^\delta, \mu) + 2\nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta \\ \leq -C^{-1} \left(\theta (w^\delta)^{\frac{q}{2}} + \theta^{1-q'} \Lambda_{q'}(\mu)^{q'} \right) w^\delta \\ + C (w^\delta)^{\frac{1}{2}} \left[\theta + \theta (w^\delta)^{\frac{1}{2}} + \theta (w^\delta)^{\frac{q}{2}} + \theta^{1-q'} \Lambda_{q'}(\mu)^{q'} \right] + \|R^\delta\|_\infty, \end{aligned}$$

up to updating C . We notice that the terms with the highest exponents in w^δ and $\Lambda_{q'}(\mu)^{q'}$ in the right-hand side of the latter inequality is non-positive. Let us use Young inequalities and obtain

$$\begin{aligned} (w^\delta)^{\frac{1}{2}} \Lambda_{q'}(\mu)^{q'} &\leq \varepsilon w^\delta \Lambda_{q'}(\mu)^{q'} + \frac{1}{4\varepsilon} \Lambda_{q'}(\mu)^{q'}, \\ (w^\delta)^{\tilde{q}} &\leq \varepsilon (w^\delta)^{1+\frac{q}{2}} + \frac{q+2-2\tilde{q}}{q+2} \left(\frac{\varepsilon(q+2)}{2\tilde{q}} \right)^{-\frac{2}{q+2-2\tilde{q}}}, \end{aligned}$$

for any $\tilde{q} < 1 + \frac{q}{2}$ and $\varepsilon > 0$. Using systematically these two inequalities in (4.5.17) and taking ε small enough we finally obtain,

$$\partial_t w^\delta - \nu \Delta w^\delta + \nabla_x w^\delta \cdot H_p^\theta(x, \nabla_x u^\delta, \mu) - 2\nu \frac{\varphi'}{\varphi} \nabla_x w^\delta \cdot \nabla_x u^\delta \leq C_\varepsilon \left(\theta + \theta^{1-q'} \Lambda_{q'}(\mu)^{q'} \right) + \|R^\delta\|_\infty,$$

where C_ε is a constant which depends on ε and the constants in the assumptions. From **A6**, the initial condition of w^δ is bounded. Therefore the maximum principle for second-order parabolic equations implies that

$$(4.5.18) \quad \|w^\delta\|_\infty \leq C_\varepsilon \left(\theta + \theta T + \theta^{1-q'} \int_0^T \Lambda_{q'}(\mu(t))^{q'} dt \right) + T \|R^\delta\|_\infty.$$

Let us point out that $\nabla_x u$ is the solution of the following backward d -dimensional parabolic equation,

$$-\partial_t \nabla_x u - \nu \Delta \nabla_x u + D_{x,x}^2 u H_p(x, \nabla_x u, \mu) = \nabla_x f(x, m) - H_x(x, \nabla_x u, \mu),$$

which has bounded coefficients and right-hand side, and a terminal condition in $C^{1+\beta_0}(\mathbb{T}_a^d)$. Theorem 6.48 in [86] states that $\nabla_x u$ and $D_{x,x}^2 u$ are continuous. This and the continuity of

H^θ and H_x^θ stated in Lemma 4.2.11 imply that R^δ is uniformly convergent to 0 when δ tends to 0. We conclude this step of the proof by passing to the limit in (4.5.18) as δ tends to 0, using the estimate on $\int_0^T \Lambda_{q'}(\mu(t))^{q'} dt$ computed in the first step. We obtain that $\nabla_x u$ is uniformly bounded by a constant which depends on the constants in the assumptions, and depends linearly on $\theta^{\frac{1}{2}}$.

Fourth step: obtaining uniform estimates on $\Lambda_{q'}(\mu)$ and $\Lambda_\infty(\mu)$.

Repeating the calculation in the proof of Lemma 4.4.1 with L satisfying (4.5.4) and (4.5.5), we obtain:

$$(4.5.19) \quad \Lambda_{q'}(\mu(t))^{q'} \leq 4C_0^2 \theta^{q'} + \frac{(q')^{q-1} (2C_0)^q}{q} \theta^{q'} \|\nabla_x u(t)\|_{L^q(m(t))}^q.$$

This and the third step of this proof yield that $\sup_{t \in [0, T]} \Lambda_{q'}(\mu(t)) \leq C\theta$ for some C depending only on the constants of the assumptions. We conclude that $\sup_{t \in [0, T]} \Lambda_\infty(\mu(t))$ satisfies a similar inequality using (4.5.6). \square

4.6 Existence and Uniqueness Results

Paragraph 4.6.1 is devoted to proving the existence of solutions to (4.2.5), which is step II.b. In paragraph 4.6.2, we propose a method to extend the existence result to system (4.2.3) which is stated on \mathbb{R}^d ; this concludes step III.a. This method relies on compactness results using the uniform estimates of $\nabla_x u$ that we obtained in Lemma 4.5.1. In paragraph 4.6.3, we prove step III.b, namely the uniqueness of the solution to (4.2.3) and (4.2.5). Then the main results of the paper and step IV are addressed in paragraph 4.6.4. We introduce a one-to-one correspondance between solutions to (4.1.6) and (4.2.3), which allows us to obtain directly the existence and the uniqueness of the solution to (4.1.6) from the ones to (4.2.3).

4.6.1 Proof of Theorem 4.2.6: existence of solutions to (4.2.5)

We will use the a priori estimates stated in Section 4.5 and the latter fixed point theorem, in order to achieve step II.b and prove the existence of solutions to (4.2.5).

Proof of Theorem 4.2.6. We would like to use the Leray-Schauder theorem 4.4.2 on a map which takes a flow of measures $(\tilde{m}_t)_{t \in [0, T]} \in (\mathcal{P}(\mathbb{T}_a^d))^{[0, T]}$ as an argument. However, $\mathcal{P}(\mathbb{T}_a^d)$ is not a Banach space. A way to go through this difficulty is to compose the latter map with a continuous map from a convenient Banach space to the set of such flows of measures. Here, we consider the map introduced in [25], namely $\rho : C^0([0, T] \times \mathbb{T}_a^d; \mathbb{R}) \rightarrow C^0([0, T] \times \mathbb{T}_a^d; \mathbb{R})$ defined by

$$\rho(\tilde{m})(t, x) = \frac{\tilde{m}_+(t, x) - a^{-d} \int \tilde{m}_+(t, y) dy}{\max(1, \int \tilde{m}_+(t, y) dy)} + a^{-d},$$

where $\tilde{m}_+(t, x) = \max(0, \tilde{m}(t, x))$. We will also have the use of \tilde{m}^0 defined as the unique weak solution of

$$(4.6.1) \quad \partial_t \tilde{m}^0 - \nu \Delta \tilde{m}^0 = 0 \text{ on } (0, T) \times \mathbb{T}_a^d, \quad \text{and } \tilde{m}^0(0, \cdot) = m^0.$$

We are now ready to construct the map Ψ on which we will use the Leray-Schauder theorem 4.4.2. Take $\theta \in [0, 1]$, $u \in C^{0,1}([0, T] \times \mathbb{T}_a^d; \mathbb{R})$ and $\tilde{m} \in C^0([0, T] \times \mathbb{T}_a^d; \mathbb{R})$. We define

$m = \rho(\tilde{m} + \tilde{m}^0)$ and $(\mu, \alpha) \in C^0([0, T]; \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)) \times C^0([0, T] \times \mathbb{T}_a^d; \mathbb{R}^d)$ by,

$$\begin{aligned}\alpha(t, x) &= -H_p^\theta(t, x, \nabla_x u(t, x), \mu(t)) \\ \mu(t) &= (I_d, \alpha(t, \cdot)) \# m(t).\end{aligned}$$

This definition comes from the conclusions of Lemma 4.2.4 when $\theta > 0$. For $\theta = 0$, it simply consists in taking $\alpha = 0$ and $\mu(t) = m(t) \otimes \delta_0$. Here we can repeat the calculation and obtain inequality (4.5.19). This and (4.5.6) implies that $\|\alpha\|_\infty$ is bounded by $C\theta$ for some constant $C > 0$ which depends on $\|\nabla_x u\|_\infty$ and is independent of θ and a .

Then we define \bar{m} the solution in the sense of distributions of

$$\partial_t \bar{m} - \nu \Delta \bar{m} + \operatorname{div}(\alpha \bar{m}) = 0,$$

supplemented with the initial condition $\bar{m}(0, \cdot) = m^0$, with m^0 being β_0 -Hölder continuous. Theorem 2.1 section V.2 in [81] states that \bar{m} is uniformly bounded by a constant which depends on $\|m_0\|_\infty$ and $\|\alpha\|_\infty$. Theorem 6.29 in [86] yields that $m \in C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}_a^d)$ for $\beta \in (0, \beta_0)$, and that its associated norm can be estimated from above by a constant which depends on $\|\nabla_x u\|_\infty$, β , a and the constants in the assumptions. The same arguments applied to \tilde{m}^0 defined in (4.6.1) imply that \tilde{m}^0 is in $C^{\frac{\beta}{2}, \beta}([0, T] \times \mathbb{T}_a^d)$ and its associated norm is bounded.

Then we take $\bar{\mu}(t) = (I_d, \alpha(t, \cdot)) \# \bar{m}(t)$ for any $t \in [0, T]$, and $\bar{u} \in C^{0,1}([0, T] \times \mathbb{T}_a^d; \mathbb{R})$ the unique solution in the sense of distributions of the following heat equation with bounded right-hand side,

$$-\partial_t \bar{u} - \nu \Delta \bar{u} = -H^\theta(t, x, \nabla_x u, \bar{\mu}(t)) + \theta f(x, \bar{m}(t)),$$

supplemented with the terminal condition $\bar{u}(T, \cdot) = \theta g(\cdot, \bar{m}(T))$ which is in $C^{1+\beta_0}(\mathbb{T}_a^d)$. Classical results (see for example Theorem 6.48 in [86]) state that u is in $C^{\frac{1}{2} + \frac{\beta}{2}, 1+\beta}$ and its associated norm is bounded by a constant which depends on $\|\nabla_x u\|_\infty$, β , a and the constants in the assumptions.

We can now construct the map $\Psi : (\theta, u, \tilde{m}) \mapsto (\bar{u}, \bar{m} - \tilde{m}^0)$, from $C^{0,1}([0, T] \times \mathbb{T}_a^d; \mathbb{R}) \times C^0([0, T] \times \mathbb{T}_a^d; \mathbb{R}^d)$ into itself. This map is continuous and compact, it satisfies $\Psi(0, u, \tilde{m}) = 0$ for any (u, \tilde{m}) . In particular, the fact that $\|\alpha\|_\infty \leq C\theta$ in the previous paragraph, implies that \bar{m} tends to \tilde{m}^0 and \bar{u} tends to 0 as θ tends to 0. This gives the continuity of Ψ at $\theta = 0$. Moreover the fixed points of $\Psi(\theta)$ are exactly the solutions to (4.5.3), which are uniformly bounded by Lemma (4.5.1). Therefore, by the Leray-Schauder fixed point theorem 4.4.2, there exists a solution to (4.2.5). \square

4.6.2 Proof of Theorem 4.2.7: passing from the torus to \mathbb{R}^d

The purpose of this paragraph is to extend the existence result to the system (4.2.3) and achieve step III.a.

Proof of Theorem 4.2.7. First step: constructing a sequence of approximate solutions.

For $a > 0$ we define $\tilde{m}^{0,a} = \pi^a \# m^0$, where $\pi^a : \mathbb{R}^d \rightarrow \mathbb{T}_a^d$ is the quotient map. Let $\chi^a : \mathbb{T}_a^1 \rightarrow \mathbb{R}$ be the canonical injection from the one-dimensional torus of radius a to \mathbb{R} , which image is $[-\frac{a}{2}, \frac{a}{2}]$. Take $\tilde{\psi} \in C^2(\mathbb{R}; \mathbb{R})$ periodic with a period equal to 1 and such that,

$$(4.6.2) \quad \begin{aligned}\tilde{\psi}(x) &= x, \quad \text{if } |x| \leq \frac{1}{4}, \\ |\tilde{\psi}(x)| &\leq |x|, \quad \text{for any } x \in \left[-\frac{1}{2}, \frac{1}{2}\right],\end{aligned}$$

We define $\psi^a : \mathbb{T}_a^d \rightarrow \mathbb{R}^d$ by $\psi^a(x)_i = a\tilde{\psi}(a^{-1}\chi^a(x_i))$ for $i = 1, \dots, d$, this is a C^2 function. Since $\tilde{\psi}(\frac{\cdot}{a})$ has a period of a , the function $\psi^a \circ \pi^a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

$$(4.6.3) \quad \psi^a \circ \pi^a(x)_i = a\tilde{\psi}\left(\frac{x_i}{a}\right),$$

for $i = 1, \dots, d$ and $x \in \mathbb{R}^d$, and is a C^2 function. We are ready to construct periodic approximations of L , f and g defined by,

$$\begin{aligned} L^a(t, x, \alpha, \mu) &= L(t, \psi^a(x), \alpha, (\psi^a \otimes I_d) \# \mu), \\ f^a(t, x, m) &= f(t, \psi^a(x), \psi^a \# m), \\ g^a(x, m) &= g(\psi^a(x), \psi^a \# m), \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbb{T}_a^d$, $\alpha \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{T}_a^d \times \mathbb{R}^d)$. Let H^a be the periodic Hamiltonian associated with L^a by the Legendre transform:

$$H^a(t, x, p, \mu) = H(t, \psi^a(x), p, (\psi^a \otimes I_d) \# \mu).$$

Let us point out that the fact that L , H , f and g satisfy **A1-A6**, implies that L^a , H^a , f^a and g^a satisfy these assumptions too with $C_0 \|\tilde{\psi}'\|_\infty$ instead of C_0 . So we can define $(\tilde{u}^a, \tilde{m}^a, \tilde{\mu}^a)$ a solution to (4.2.5) with H^a , f^a , g^a and $\tilde{m}^{0,a}$ instead of H , f , g and m^0 . We define $u^a \in C^0([0, T] \times \mathbb{R}^d; \mathbb{R})$, $m^a \in C^0([0, T]; \mathcal{P}(\mathbb{R}^d))$ and $\mu^a \in C^0([0, T]; \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$ respectively by

$$u^a(t, x) = \tilde{u}^a(t, \pi^a(x)), \quad m^a(t) = \psi^a \# \tilde{m}^a(t), \quad \text{and} \quad \mu^a(t) = (\psi^a \otimes I_d) \# \tilde{\mu}^a(t),$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$.

Second step: Proving that m^a is compact.

We are going to use the Arzelà-Ascoli Theorem on $C^0([0, T]; (\mathcal{P}(\mathbb{R}^d), W_1))$ ($\mathcal{P}(\mathbb{R}^d)$ is endowed with the 1-Wassertein distance). First we prove that for any $t \in [0, T]$, the sequence $(m^a(t))_{a>1}$ is compact with the 1-Wassertein distance, by proving that $\int_{\mathbb{R}^d} |x|^2 dm^a(t, x)$ is uniformly bounded in a . At time $t = 0$, we have

$$\int_{\mathbb{R}^d} |x|^2 dm^a(0, x) = \int_{\mathbb{T}_a^d} |\psi^a(x)|^2 d\tilde{m}^{a,0}(x) = \int_{\mathbb{R}^d} |\psi^a \circ \pi^a(x)|^2 dm^0(x) \leq \int_{\mathbb{R}^d} |x|^2 dm^0(x) \leq C_0,$$

using (4.6.2), (4.6.3) and **A6**. Let us differentiate $\int_{\mathbb{R}^d} |x|^2 dm^a(t, x)$ with respect to time, perform some integrations by part and obtain that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 dm^a(t, x) &= \frac{d}{dt} \int_{\mathbb{T}_a^d} |\psi^a(x)|^2 d\tilde{m}^a(t, x) \\ &= \int_{\mathbb{T}_a^d} |\psi^a(x)|^2 \left(\nu \Delta \tilde{m}^a(t, x) - \operatorname{div} \left(\alpha^{\tilde{\mu}^a(t)}(x) \tilde{m}^a(t, x) \right) \right) dx \\ &= 2 \int_{\mathbb{T}_a^d} \sum_{i=1}^d \left[\nu \tilde{\psi}'' \left(\frac{\chi^a(x^i)}{a} \right) \tilde{\psi} \left(\frac{\chi^a(x^i)}{a} \right) + \nu \tilde{\psi}' \left(\frac{\chi^a(x^i)}{a} \right)^2 \right. \\ &\quad \left. + \psi^a(x) \tilde{\psi}' \left(\frac{\chi^a(x^i)}{a} \right) \alpha^{\tilde{\mu}^a(t), i}(x) \right] d\tilde{m}^a(t, x) \\ &\leq 2\nu d \|\tilde{\psi}''\|_\infty \|\tilde{\psi}\|_\infty + 2\nu d \|\tilde{\psi}'\|_\infty^2 + \|\psi'\|_\infty^2 \|\alpha^{\tilde{\mu}^a(t)}\|_\infty^2 + \int_{\mathbb{T}_a^d} |\psi^a(x)|^2 d\tilde{m}^a(t, x) \\ &\leq 2\nu d \|\tilde{\psi}''\|_\infty \|\tilde{\psi}\|_\infty + 2\nu d \|\tilde{\psi}'\|_\infty^2 + \|\tilde{\psi}'\|_\infty^2 \|\alpha^{\tilde{\mu}^a(t)}\|_\infty^2 + \int_{\mathbb{R}^d} |x|^2 dm^a(t, x). \end{aligned}$$

We recall that $(t, x) \mapsto \alpha^{\tilde{\mu}^a(t)}(x)$ is uniformly bounded with respect to t and a by Lemma 4.5.1. Therefore, the latter two inequalities and a comparison principle for ordinary differential equation imply that $\int_{\mathbb{R}^d} |x|^2 dm^a(t, x)$ is uniformly bounded with respect to a and t .

We define \bar{X}^a a random process on \mathbb{R}^d by

$$d\bar{X}_t^a = \alpha^{\mu^a(t)}(\pi^a(\bar{X}_t^a)) dt + \sqrt{2\nu d} dB_t, \quad \text{and } \mathcal{L}(\bar{X}_0^a) = m^0,$$

where B is a Brownian motion on \mathbb{R}^d independent of \bar{X}_0^a . For $t, s \in [0, T]$, we have that,

$$\begin{aligned} \mathbb{E}[|\bar{X}_t^a - \bar{X}_s^a|] &\leq \mathbb{E}\left[|\bar{X}_t^a - \bar{X}_s^a|^2\right]^{\frac{1}{2}} \\ &\leq \mathbb{E}\left[\left|\int_s^t \sqrt{2\nu d} dW_r\right|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\left|\int_s^t \alpha^{\mu^a(r)} dr\right|^2\right]^{\frac{1}{2}} \\ &\leq \sqrt{2\nu d}|t-s|^{\frac{1}{2}} + |t-s| \sup_{r \in [0, T]} \left\|\alpha^{\mu^a(r)}\right\|_{\infty}. \end{aligned}$$

We define $\tilde{X}_t^a = \pi^a(\bar{X}_t^a) \in \mathbb{T}_a^d$ and $X_t^a = \psi^a(\tilde{X}_t^a) \in \mathbb{R}^d$, for $t \in [0, T]$. One may check that the law of \tilde{X}_t^a satisfies the same Fokker-Planck equation in the sense of distributions as $\tilde{m}^a(t)$ by testing it with $C^\infty((0, T) \times \mathbb{T}^d)$ test functions. Therefore, the law of \tilde{X}_t^a is $\tilde{m}^a(t)$ and the law of X_t^a is $m^a(t)$. By definition of the 1-Wassertein distance, we obtain

$$\begin{aligned} W_1(m^a(t), m^a(s)) &\leq \mathbb{E}[|X_t^a - X_s^a|] \\ &\leq \mathbb{E}[|\psi^a \circ \pi^a(\bar{X}_t^a) - \psi^a \circ \pi^a(\bar{X}_s^a)|] \\ &\leq \left\|\tilde{\psi}'\right\|_{\infty} \mathbb{E}[|\bar{X}_t^a - \bar{X}_s^a|] \\ &\leq \left\|\tilde{\psi}'\right\|_{\infty} \left(\sqrt{2\nu d}|t-s|^{\frac{1}{2}} + |t-s| \sup_{r \in [0, T]} \left\|\alpha^{\mu^a(r)}\right\|_{\infty}\right), \end{aligned}$$

where we used (4.6.3) and the mean value theorem to pass from the second to the third line in the latter chain of inequalities. Therefore by the Arzelà-Ascoli theorem, $(m^a)_{a>0}$ is relatively compact in $C^0([0, T]; (\mathcal{P}(\mathbb{R}^d), W_1))$.

Third Step: passing to the limit for a subsequence.

We recall that \tilde{u}^a and $\nabla_x \tilde{u}^a$ are uniformly bounded with respect to a , so are u^a and $\nabla_x u^a$. Moreover u^a satisfies the following PDE,

$$-\partial_t u^a - \nu \Delta u^a + H(t, \psi^a \circ \pi^a(x), \nabla_x u(t, x), \mu^a(t)) = f(t, \psi^a \circ \pi^a(x), m^a(t)),$$

for $(t, x) \in (0, T) \times B_{\mathbb{R}^d}(0, a)$, we recall that $\psi^a \circ \pi^a(x) = x$ if $|x| \leq \frac{a}{4}$. For $a_0 > 0$, we choose a such that $a > 4(a_0 + 1)$, this implies that u^a satisfies a backward heat equation on $B_{\mathbb{R}^d}(0, a_0 + 1)$ with a bounded right-hand side, a bounded terminal condition, and bounded boundary conditions. Classical results on the heat equation (see for example Theorem 6.48 in [86]) state that u^a is in $C^{\frac{1}{2} + \frac{\beta}{2}, 1 + \beta}([0, T] \times B_{\mathbb{R}^d}(0, a_0); \mathbb{R})$ and that its associated norm is bounded by a constant which depends on the constants in the assumptions and a_0 , but not on a . Therefore $(u^a|_{B_{\mathbb{R}^d}(0, a_0)})_{a>1}$ is a compact sequence in $C^{0,1}([0, T] \times B_{\mathbb{R}^d}(0, a_0); \mathbb{R})$ for any $a_0 > 0$. Then by a diagonal extraction method, there exists a_n an increasing sequence tending to $+\infty$ in \mathbb{R}_+ such that

$$\begin{aligned} m^{a_n} &\rightarrow m && \text{in } C^0\left([0, T], \left(\mathcal{P}(\mathbb{R}^d), W_1\right)\right), \\ u^{a_n} &\rightarrow u && \text{locally in } C^{0,1}, \end{aligned}$$

for some $(u, m) \in C^{0,1}([0, T] \times \mathbb{R}^d; \mathbb{R}) \times C^0([0, T]; (\mathcal{P}(\mathbb{R}^d), W_1))$. Let us prove that for $t \in [0, T]$, $\mu^{a_n}(t)$ converges to a fixed point of (4.2.3c) when n tends to infinity; indeed we notice that

$$\begin{aligned} \mu^{a_n}(t) &= (\psi^{a_n} \otimes I_d) \# \tilde{\mu}^{a_n}(t) \\ &= (\psi^{a_n} \otimes I_d) \# [(I_d, -H_p^{a_n}(t, \cdot, \nabla_x \tilde{u}^{a_n}(t, \pi^{a_n} \circ \psi^{a_n}(\cdot)), \tilde{\mu}^{a_n}(t))) \# \tilde{m}^{a_n}] \\ &= (\psi^{a_n}, -H_p(t, \psi^{a_n}(\cdot), \nabla_x u^{a_n}(t, \psi^{a_n}(\cdot)), \mu^{a_n}(t))) \# \tilde{m}^{a_n} \\ &= (I_d, -H_p(t, \cdot, \nabla_x u^{a_n}(t, \cdot), \mu^{a_n}(t))) \# m^{a_n}. \end{aligned}$$

In particular, $\alpha^{\tilde{\mu}^{a_n}(t)} = \alpha^{\mu^{a_n}(t)} \circ \psi^{a_n}$ so $\|\alpha^{\mu^{a_n}(t)}\|_{L^\infty(m)}$ is not larger than $\|\alpha^{\tilde{\mu}^{a_n}(t)}\|_{L^\infty(\tilde{m})}$ since the support of m^{a_n} is contained in the image of the support of \tilde{m}^{a_n} by ψ^{a_n} . We proved in the previous step that $(m^a(t))_{a \geq 1}$ is compact in $(\mathcal{P}(\mathbb{R}^d), W_1)$, and so is $(\mu^{a_n}(t))_{n \geq 1}$ in $(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), W_1)$, since they are the pushforward measures of $(m^a(t))_{n \geq 1}$ by $(I_d, \alpha^{\mu^{a_n}(t)})$. Let $\mu(t) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be the limit of a convergent subsequence of $(\mu^{a_n}(t))_{n \geq 1}$. Passing to the limit in the weak* topology in the latter chain of equalities implies that

$$\mu(t) = (I_d, -H_p(t, \cdot, \nabla_x u(t, \cdot), \mu(t))) \# m(t).$$

Moreover, the uniqueness of the fixed point 4.2.3c holds here, see [38] Lemma 5.2 for the proof. We obtained that there exists a unique fixed point satisfying 4.2.3c, and that it is the limit of any convergent subsequence of $(\mu^{a_n}(t))$. This implies that the whole sequence $(\mu^{a_n}(t))_{n \geq 1}$ tends to $\mu(t)$ in $(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), W_1)$.

Let us point out that m^{a_n} satisfies

$$\partial_t m^{a_n} - \nu \Delta m^{a_n} - \operatorname{div}(H_p(t, x, \nabla_x u^{a_n}, \mu^{a_n}) m^{a_n}) = 0$$

in the sense of distributions on $(0, T) \times B(0, \frac{a_n}{4})$, by the definitions of ψ^a and $\tilde{\psi}$. Furthermore, at time $t = 0$ we know that $m^{a_n}(0) = (\psi^{a_n} \circ \pi^{a_n}) \# m^0$. We recall that $\psi^{a_n} \circ \pi^{a_n}(x) = x$ for $x \in B_{\mathbb{R}^d}(0, \frac{a_n}{4})$. This implies that $m^{a_n}(0)$ tends to m^0 in the weak* topology of $\mathcal{P}(\mathbb{R}^d)$.

Finally we obtain that (u, m, μ) is a solution to (4.2.3), by passing to the limit as n tends to infinity in the equations satisfied by $(u^{a_n}, m^{a_n}, \mu^{a_n})$. \square

Remark 4.6.1. *In the above proof, we obtain that there exists a unique fixed point satisfying 4.2.3c. We have thereby extended the conclusions of Lemma 4.4.3 to system 4.2.3. Similarly, one may extend the conclusions of Lemma 4.2.4 to system (4.2.3).*

4.6.3 Proof of Theorem 4.2.8: uniqueness of the solutions to (4.2.3) and (4.2.5)

Step III.b, namely the uniqueness of the solution to (4.2.3), is obtained from the monotonicity assumptions A3 and U, and the same arguments as in the case of MFG without interaction through controls.

Proof of Theorem 4.2.8. Here, we write the proof for the system (4.2.3). However, none of the arguments below is specific to the domain \mathbb{R}^d , therefore this proof can be repeated for (4.2.5).

We suppose that (u^1, m^1, μ^1) and (u^2, m^2, μ^2) are two solutions to (4.2.3). Now standard arguments (see [85]) lead to

$$(4.6.4) \quad \begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^d} [\nabla_x(u^1 - u^2) \cdot H_p(t, x, \nabla_x u^1, \mu^1) - H(t, x, \nabla_x u^1, \mu^1) + H(t, x, \nabla_x u^2, \mu^2)] dm^1(t, x) \\ &+ \int_0^T \int_{\mathbb{R}^d} [\nabla_x(u^2 - u^1) \cdot H_p(t, x, \nabla_x u^2, \mu^2) - H(t, x, \nabla_x u^2, \mu^2) + H(t, x, \nabla_x u^1, \mu^1)] dm^2(t, x) \\ &\quad + \int_0^T \int_{\mathbb{R}^d} (f(t, x, m^1(t)) - f(t, x, m^2(t))) d(m^1(t, x) - m^2(t, x)) dt \\ &\quad \quad + \int_{\mathbb{R}^d} (g(x, m^1(T)) - g(x, m^2(T))) d(m^1(T, x) - m^2(T, x)). \end{aligned}$$

Recall that

$$(4.6.5) \quad \begin{aligned} L(t, x, \alpha^{\mu^i}, \mu^i) &= \nabla_x u^i \cdot H_p(t, x, \nabla_x u^i, \mu^i) - H(t, x, \nabla_x u^i, \mu^i), \\ \nabla_x u^i &= -L_\alpha(t, x, \alpha^{\mu^i}, \mu^i), \end{aligned}$$

because L is the Legendre transform of H . From **U**, (4.6.4) and (4.6.5), we obtain that,

$$(4.6.6) \quad \begin{aligned} 0 &\geq \int_0^T \int_{\mathbb{R}^d} \left[L(t, x, \alpha^{\mu^1}, \mu^1) - L(t, x, \alpha^{\mu^2}, \mu^2) - (\alpha^{\mu^1} - \alpha^{\mu^2}) \cdot L_\alpha(t, x, \alpha^{\mu^2}, \mu^2) \right] dm^1(t, x) dt \\ &+ \int_0^T \int_{\mathbb{R}^d} \left[L(t, x, \alpha^{\mu^2}, \mu^2) - L(t, x, \alpha^{\mu^1}, \mu^1) - (\alpha^{\mu^2} - \alpha^{\mu^1}) \cdot L_\alpha(t, x, \alpha^{\mu^1}, \mu^1) \right] dm^2(t, x) dt \end{aligned}$$

The function L is strictly convex in α by Lemma 4.2.10, which implies that,

$$(4.6.7) \quad \begin{aligned} L(t, x, \alpha^{\mu^1}, \mu^2) - L(t, x, \alpha^{\mu^2}, \mu^2) - (\alpha^{\mu^1} - \alpha^{\mu^2}) \cdot L_\alpha(t, x, \alpha^{\mu^2}, \mu^2) &\geq 0, \\ L(t, x, \alpha^{\mu^2}, \mu^1) - L(t, x, \alpha^{\mu^1}, \mu^1) - (\alpha^{\mu^2} - \alpha^{\mu^1}) \cdot L_\alpha(t, x, \alpha^{\mu^1}, \mu^1) &\geq 0, \end{aligned}$$

and (4.6.7) turn to identities if and only if $\alpha^{\mu^1} = \alpha^{\mu^2}$. The latter inequalities and (4.6.6) yield

$$\begin{aligned} 0 &\geq \int_0^T \int_{\mathbb{R}^d} \left[L(t, x, \alpha^{\mu^1}, \mu^1) - L(t, x, \alpha^{\mu^1}, \mu^2) \right] dm^1 dt + \int_0^T \int_{\mathbb{R}^d} \left[L(t, x, \alpha^{\mu^2}, \mu^2) - L(t, x, \alpha^{\mu^2}, \mu^1) \right] dm^2 dt \\ &= \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} [L(t, x, \alpha, \mu^1) - L(t, x, \alpha, \mu^2)] d(\mu^1 - \mu^2)(t, x, \alpha) dt. \end{aligned}$$

Assumption **A3** turns the latter inequality into an equality. This, the case of equality in (4.6.7) and the continuity of α^{μ^1} and α^{μ^2} yield that $\alpha^{\mu^1} = \alpha^{\mu^2}$. This implies that $m^1 = m^2$ by the uniqueness of the solution to (4.2.3b), (4.2.3e). Therefore, we obtain $\mu^1 = \mu^2$, and then $u^1 = u^2$ by the uniqueness of the solution to (4.2.3a), 4.2.3d. \square

4.6.4 Theorems 4.2.2 and 4.2.3: existence and uniqueness of the solution to (4.1.6)

So far, no distinction has been made between μ_b and μ_α , because they coincide for (4.2.3) and (4.2.5). Now they may differ since the drift function and the control may be different.

In this case μ_b defined by $\mu_b(t) = (\pi_x, b(t)) \# \mu_\alpha(t)$ is naturally the joint law of the states and the drifts. The idea here to pass from (4.2.3) to (4.1.6), is to assume that b is invertible with respect to α , which changes the optimization problem in α into a new optimization problem expressed in term of b . This consists in changing the Lagrangian from $L(t, x, \alpha, \mu_\alpha)$ into

$$L^b(t, x, b, \mu_b) = L\left(t, x, \alpha^*(t, x, b, \mu_b), (\pi_x, \alpha^*(t)) \# \mu_b\right).$$

The Hamiltonian H^b defined as the Legendre transform of L^b is given by

$$(4.6.8) \quad H^b(t, x, p, \mu_b) = H\left(t, x, p, (\pi_x, \alpha^*(t)) \# \mu_b\right).$$

Conversely, we can obtain L and H from L^b and H^b with the following relations,

$$\begin{aligned} L(t, x, \alpha, \mu_\alpha) &= L^b\left(t, x, b(t, x, \alpha, \mu_\alpha), (\pi_x, b(t)) \# \mu_\alpha\right), \\ H(t, x, p, \mu_\alpha) &= H^b\left(t, x, p, (\pi_x, b(t, \mu_\alpha)) \# \mu_\alpha\right). \end{aligned}$$

Now we can state the following lemma which allows us to pass from (4.2.3) to (4.1.6), or vice versa.

Lemma 4.6.2. *Under assumption **B1**, $(u, m, \mu_\alpha, \mu_b)$ is a solution to (4.1.6) if and only if (u, m, μ_b) is a solution to (4.2.3) with H^b instead of H .*

The proof is straightforward and only consists in checking on the one hand, that (4.1.6a) and (4.1.6b) are respectively equivalent to (4.2.3a) and (4.2.3b) with H^b instead of H ; on the other hand, that (4.1.6c) and (4.1.6d) are equivalent to (4.2.3c) with H^b , where we take $\mu_b = \mu$ and μ_α defined by (4.1.6c).

The following existence theorem is a direct consequence of Lemma 4.6.2, and Theorem 4.2.7.

Corollary 4.6.3. *If L^b satisfies **A1-A6**, and b satisfies **B1**, there exists a solution to (4.1.6).*

Theorem 4.2.3, i.e. the existence part of step **IV**, is a consequence of the latter existence result in which the assumptions on L^b are stated on L instead, which makes them more tractable. However, we have to make the additional **B2**.

If L and b satisfy the assumptions of Theorem 4.2.3, it is straightforward to check that L^b satisfies **A1-A5**. Therefore, Theorem 4.2.3 is a consequence of Corollary 4.6.3. Finally, Theorem 4.2.2 and the uniqueness part of step **IV** are direct consequences of Theorem 4.2.8 and Lemma 4.6.2.

On the implementation of a primal-dual algorithm for second order time-dependent mean field games with local couplings

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We study a numerical approximation of a time-dependent Mean Field Game (MFG) system with local couplings. The discretization we consider stems from a variational approach described in [29] for the stationary problem and leads to the finite difference scheme introduced by Achdou and Capuzzo-Dolcetta in [6]. In order to solve the finite dimensional variational problems, in [29] the authors implement the primal-dual algorithm introduced by Chambolle and Pock in [46], whose core consists in iteratively solving linear systems and applying a proximity operator. We apply that method to time-dependent MFG and, for large viscosity parameters, we improve the linear system solution by replacing the direct approach used in [29] by suitable preconditioned iterative algorithms.

Nous étudions une approche numérique pour un système de jeu à champ moyen avec couplage local. La discrétisation que nous considérons résulte d'une approche variationnelle décrite, pour le problème stationnaire, dans [29] et mène au schéma aux différences finies introduit par Achdou et Capuzzo-Dolcetta dans [6]. Dans le but de résoudre des problèmes variationnels en dimension finie, dans [29] les auteurs implémentent un algorithme primal-dual introduit par Chambolle et Pock dans [46], dont l'essence consiste à résoudre itérativement des systèmes linéaires et à appliquer un opérateur proximal. Nous appliquons cette méthode à un jeu à champ moyen dépendant du temps et, lorsque le paramètre de viscosité est assez grand, nous améliorons la résolution du système linéaire en remplaçant l'approche directe utilisée dans [29] par des algorithmes itératifs préconditionnés.

5.1 Introduction

In this work we consider the following MFG system with local couplings

$$(MFG) \quad \begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u) = f(x, m(x, t)) & \text{in } \mathbb{T}^d \times [0, T], \\ \partial_t m - \nu \Delta m - \operatorname{div}(\nabla_p H(x, Du)m) = 0 & \text{in } \mathbb{T}^d \times [0, T], \\ m(\cdot, \cdot) = m_0(\cdot), \quad u(\cdot, T) = g(\cdot, m(\cdot, T)) & \text{in } \mathbb{T}^d. \end{cases}$$

In the notation above $\nu \geq 0$, $d \in \mathbb{N}$, \mathbb{T}^d is the d -dimensional torus, $H : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is jointly continuous and convex with respect to its second variable, $f, g : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and $m_0 \in L^1(\mathbb{T}^d)$ satisfies $m_0 \geq 0$ and $\int_{\mathbb{T}^d} m_0(x) dx = 1$.

System (MFG) has been introduced by J.-M. Lasry and P.-L. Lions in [84, 85] in order to describe the asymptotic behaviour of symmetric stochastic differential games as the number of players tends to infinity. Several analytical techniques can be used to prove the existence of solutions to (MFG) under various assumptions on the data. Despite the recent introduction of the MFG system, the literature dedicated to its theoretical study is already too rich to be covered exhaustively in this introduction. The interested reader may refer to the monographs [21, 61], the surveys [33, 58] and the references therein for the state of the art of the subject.

A useful approach that can be used to establish the existence of solutions to (MFG) is the variational one, already presented in [85]. The main idea behind is that, at least formally, system (MFG) can be seen as the first order optimality condition associated to minimizers of the following optimization problem

$$(P) \quad \begin{aligned} \inf_{(m,w)} \quad & \int_0^T \int_{\mathbb{T}^d} [b(x, m(x, t), w(x, t)) + F(x, m(x, t))] dx + \int_{\mathbb{T}^d} G(x, m(x, T)) dx \\ \text{subject to} \quad & \partial_t m - \nu \Delta m + \operatorname{div}(w) = 0 \quad \text{in } \mathbb{T}^d \times (0, T), \\ & m(\cdot, 0) = m_0(\cdot) \quad \text{in } \mathbb{T}^d, \end{aligned}$$

(provided that they exist). In (P), the functions $b : \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ and $F, G : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ are defined as follows

(5.1.1)

$$b(x, m, w) := \begin{cases} mH^*(x, -\frac{w}{m}) & \text{if } m > 0, \\ 0 & \text{if } (m, w) = (0, 0), \\ +\infty & \text{otherwise,} \end{cases}$$

$$F(x, m) := \begin{cases} \int_0^m f(x, m') dm' & \text{if } m \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \quad G(x, m) := \begin{cases} \int_0^m g(x, m') dm' & \text{if } m \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where, in the definition of b , $H^*(x, \cdot)$ denotes the Legendre-Fenchel conjugate of $H(x, \cdot)$. Under the assumption that $f(x, \cdot)$ and $g(x, \cdot)$ are non-decreasing, problem (P) is shown to be a convex optimization problem and convex duality techniques can be successfully applied in order to provide existence and uniqueness results to (MFG). This argument has been made rigorous in several articles: let us mention [34, 36] in the context of first order MFGs ($\nu = 0$), the paper [37] for degenerate second order MFGs, and finally [89, 90] for ergodic second order MFGs.

The variational approach described above has also been successful in the numerical resolution of system (MFG). In this direction, we mention the article [78] dealing with applications in economics, the paper [4] concerned with the so-called planning problem in MFGs, the works [16, 14] focused on the resolution of a discretization of (P) by the

Alternating Direction Method of Multipliers (ADMM) and [29] where several first order methods are implemented and compared for the stationary version of (MFG). Let us mention that the variational approach is closely related to the so-called mean field optimal control problem, for which numerical methods have been studied in [30, 12], among others.

In this paper we consider a finite difference discretization of problem (P). Assuming that $f(x, \cdot)$ and $g(x, \cdot)$ are non-decreasing, the discretization that we consider is such that it preserves the convexity properties of problem (P) and the first order optimality conditions for its solutions, which are shown to exist, coincide with the finite difference scheme for MFGs introduced in [6]. A very nice feature of this approach is that the solutions of the resulting discretized MFGs are shown to converge to the solutions of (MFG). We refer the reader to [5], where the convergence result is obtained under the assumption that (MFG) admits a unique classical solution, and to [9] in the framework of weak solutions (see [95] for the definition of this notion). We solve the discrete convex optimization problem by using the primal-dual algorithm introduced in [46]. As was pointed out in [29] (see also [92] in the context of transport problems), the primal-dual algorithm we consider seems to be faster than the ADMM when ν in (MFG) is small (or null). On the other hand, the efficiency of both methods is arguable when ν is large. This is due to the fact that, in both algorithms, at each iteration one has to invert a matrix whose condition number importantly increases as the viscosity parameter increases. Naturally, preconditioning strategies (see e.g. [22]) can then be used in order to improve the efficiency of both algorithms. This strategy has been already successfully implemented in [14] for the ADMM.

Our main objective in the present work is to take a closer look at the phenomenon described at the end of previous paragraph when considering the primal-dual algorithm. Therefore, we focus our analysis in the case where $\nu > 0$. We have implemented standard indirect methods for solving the linear systems appearing in the computation of the iterates of the primal-dual algorithm. As our numerical results suggest, it is very important to design suitable preconditioning strategies in order to be able to find solutions of the discretization of problem (P) efficiently, and in a robust way with respect to the viscosity parameter. For this, we explore different preconditioning strategies, and in particular, multigrid preconditioning (see also [8, 14], where multigrid strategies have been implemented for other solution methods).

The article is organized as follows. In section 5.2 we introduce some standard notation and we recall the finite difference scheme for (MFG) introduced in [6]. The variational interpretation of this finite difference scheme is discussed in section 5.3. Next, in section 5.4, we recall the primal-dual algorithm introduced in [46] and we consider its application to the discretization of (P). In section 5.5, we summarize the preconditioning strategies that we consider and we discuss a numerical example, which is the time-dependent version of one of the examples treated in [6, 29].

5.2 Preliminaries and the finite difference scheme

In this section we introduce some basic notation and present the finite difference scheme introduced in [6], whose efficient resolution will be the main subject of this article. For the sake of simplicity, we will assume that $d = 2$ and that given $q > 1$, with conjugate exponent denoted by $q' = q/(q - 1)$, the Hamiltonian $H : \mathbb{T}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ has the form

$$H(x, p) = \frac{1}{q'} |p|^{q'} \quad \forall x \in \mathbb{T}^2, p \in \mathbb{R}^2.$$

In this case the function b defined in (5.1.1) takes the form

$$b(x, m, w) = \begin{cases} \frac{|w|^q}{qm^{q-1}} & \text{if } m > 0, \\ 0 & \text{if } (m, w) = (0, 0), \\ +\infty & \text{otherwise.} \end{cases}$$

Let N_T, N_h be positive integers and set $\Delta t = T/N_T$, the time step, and $h = 1/N_h$, the space step. We associate to these steps a time grid $\mathcal{T}_{\Delta t} := \{t_k = k\Delta t ; k = 0, \dots, N_T\}$ and a space grid $\mathbb{T}_h^2 := \{x_{i,j} = (ih, jh) ; i, j \in \mathbb{Z}\}$. Since \mathbb{T}_h^2 intends to discretize \mathbb{T}^2 , we impose the identification $z_{i,j} = z_{(i \bmod N_h), (j \bmod N_h)}$, which allows to assume that $i, j \in \{0, \dots, N_h - 1\}$. A function $y := \mathbb{T}^2 \times [0, T] \rightarrow \mathbb{R}$ is approximated by its values at $(x_{i,j}, t_k) \in \mathbb{T}_h^2 \times \mathcal{T}_{\Delta t}$, which we denote by $y_{i,j}^k := y(x_{i,j}, t_k)$. Given $y : \mathbb{T}_h^2 \rightarrow \mathbb{R}$ we define the first order finite difference operators

$$(5.2.1) \quad \begin{aligned} (D_1 y)_{i,j} &:= \frac{y_{i+1,j} - y_{i,j}}{h}, \quad \text{and } (D_2 y)_{i,j} := \frac{y_{i,j+1} - y_{i,j}}{h}, \\ [D_h y]_{i,j} &:= ((D_1 y)_{i,j}, (D_1 y)_{i-1,j}, (D_2 y)_{i,j}, (D_2 y)_{i,j-1}), \\ \widehat{[D_h y]}_{i,j} &:= ((D_1 y)_{i,j}^-, -(D_1 y)_{i-1,j}^+, (D_2 y)_{i,j}^-, -(D_2 y)_{i,j-1}^+), \end{aligned}$$

where, for every $a \in \mathbb{R}$, we set $a^+ := \max(a, 0)$ and $a^- := a^+ - a$. The discrete Laplacian operator $\Delta_h y : \mathbb{T}_h^2 \rightarrow \mathbb{R}$ is defined by

$$(\Delta_h y)_{i,j} := -\frac{1}{h^2}(4y_{i,j} - y_{i+1,j} - y_{i-1,j} - y_{i,j+1} - y_{i,j-1}).$$

For $y : \mathcal{T}_{\Delta t} \rightarrow \mathbb{R}$ we define the discrete time derivative

$$D_t y^k := \frac{y^{k+1} - y^k}{\Delta t}.$$

The Godunov-type finite difference discretization of (MFG) introduced in [6] is as follows: find $u, m : \mathbb{T}_h^2 \times \mathcal{T}_{\Delta t} \rightarrow \mathbb{R}$ such that for all $0 \leq i, j \leq N_h - 1$ and $0 \leq k \leq N_T - 1$ we have

$$(MFG_{h,\Delta t}) \quad \begin{cases} -D_t u_{i,j}^k - \nu(\Delta_h u^k)_{i,j} + \frac{1}{q'} |\widehat{[D_h u^k]}_{i,j}|^{q'} = f(x_{i,j}, m_{i,j}^{k+1}), \\ D_t m_{i,j}^k - \nu(\Delta_h m^{k+1})_{i,j} - \mathcal{J}_{i,j}(u^k, m^{k+1}) = 0, \\ m_{i,j}^0 = \bar{m}_{i,j}, \quad u_{i,j}^{N_T} = g(x_{i,j}, m_{i,j}^{N_T}), \end{cases}$$

where

$$(5.2.2) \quad \bar{m}_{i,j} := \int_{|x-x_{i,j}|_\infty \leq \frac{h}{2}} m_0(x) dx \geq 0,$$

and the operator $\mathcal{J}(u', m') : \mathbb{T}_h^2 \rightarrow \mathbb{R}$, with $u', m' : \mathbb{T}_h^2 \rightarrow \mathbb{R}$, is defined by

$$\begin{aligned} \mathcal{J}_{i,j}(u', m') &:= \frac{1}{h} \left(-m'_{i,j} \frac{1}{q'} |\widehat{[D_h u']}_{i,j}|^{\frac{2-q}{q-1}} (D_1 u')_{i,j}^- + m'_{i-1,j} \frac{1}{q'} |\widehat{[D_h u']}_{i-1,j}|^{\frac{2-q}{q-1}} (D_1 u')_{i-1,j}^- \right. \\ &\quad + m'_{i+1,j} \frac{1}{q'} |\widehat{[D_h u']}_{i+1,j}|^{\frac{2-q}{q-1}} (D_1 u')_{i,j}^+ - m'_{i,j} \frac{1}{q'} |\widehat{[D_h u']}_{i,j}|^{\frac{2-q}{q-1}} (D_1 u')_{i-1,j}^+ \\ &\quad - m'_{i,j} \frac{1}{q'} |\widehat{[D_h u']}_{i,j}|^{\frac{2-q}{q-1}} (D_2 u')_{i,j}^- + m'_{i,j-1} \frac{1}{q'} |\widehat{[D_h u']}_{i,j-1}|^{\frac{2-q}{q-1}} (D_2 u')_{i,j-1}^- \\ &\quad \left. + m'_{i,j+1} \frac{1}{q'} |\widehat{[D_h u']}_{i,j+1}|^{\frac{2-q}{q-1}} (D_2 u')_{i,j}^+ - m'_{i,j} \frac{1}{q'} |\widehat{[D_h u']}_{i,j}|^{\frac{2-q}{q-1}} (D_2 u')_{i,j-1}^+ \right), \end{aligned}$$

with the convention:

$$(5.2.3) \quad |[\widehat{D_h u'}]_{i,j}|^{\frac{2-q}{q-1}} [\widehat{D_h u'}]_{i,j} = 0 \text{ if } q > 0 \text{ and } [\widehat{D_h u'}]_{i,j} = 0.$$

The existence of a solution $(u^{h,\Delta t}, m^{h,\Delta t})$ of system $(\text{MFG}_{h,\Delta t})$ is proved in [6, Theorem 6] as a consequence of Brouwer fixed point theorem. Furthermore, if we assume that f and g are increasing with respect to their second argument, and one of them is strictly increasing, this solution is unique when h is small enough (see [6, Theorem 7]). As we will see in the next section, these results can also be obtained by variational arguments. The convergence, as h and Δt tend to 0, of suitable extensions of $u^{h,\Delta t}$ and $m^{h,\Delta t}$ to $\mathbb{T}^2 \times [0, T]$ to a solution (u, m) of (MFG) is proved in [5] under the assumption that (u, m) is unique and sufficiently regular. The later smoothness assumption has been relaxed in [9].

5.3 The finite dimensional variational problem and the discrete MFG system

Following [29] in the stationary case and [4] for the planning problem, we introduce some finite-dimensional operators that will allow us to write easily a finite dimensional version of problem (P) . Denoting by \mathbb{R}_+ the set of non-negative real numbers and by \mathbb{R}_- the set of non-positive real numbers, we define $K := \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_-$ and for $v = (v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}) \in \mathbb{R}^4$ we denote by $P_K(v) = ((v^{(1)})^+, -(v^{(2)})^-, (v^{(3)})^+, -(v^{(4)})^-)$ its orthogonal projection onto K . Let $\mathcal{M} := \mathbb{R}^{(N_T+1) \times N_h \times N_h}$, $\mathcal{W} := (\mathbb{R}^4)^{N_T \times N_h \times N_h}$ and $\mathcal{U} := \mathbb{R}^{N_T \times N_h \times N_h}$. Let $A : \mathcal{M} \rightarrow \mathcal{U}$ and $B : \mathcal{W} \rightarrow \mathcal{U}$ be the linear operators defined by

$$(5.3.1) \quad \begin{aligned} (Am)_{i,j}^k &:= D_t m_{i,j}^k - \nu(\Delta_h m^{k+1})_{i,j}, \\ (Bw)_{i,j}^k &:= (D_1 w^{k,(1)})_{i-1,j} + (D_1 w^{k,(2)})_{i,j} + (D_2 w^{k,(3)})_{i,j-1} + (D_2 w^{k,(4)})_{i,j}, \end{aligned}$$

for all $0 \leq i, j \leq N_h - 1$ and $0 \leq k \leq N_T - 1$. One can easily check (see e.g. [6]) that the corresponding dual operators are given by

$$(5.3.2) \quad \begin{aligned} (B^* u)_{i,j}^k &= -[D_h u^k]_{i,j} \quad \text{for all } 0 \leq k \leq N_T - 1, \\ (A^* u)_{i,j}^k &= -D_t u_{i,j}^{k-1} - \nu(\Delta_h u^{k-1})_{i,j}, \quad \text{if } 1 \leq k \leq N_T - 1, \\ (A^* u)_{i,j}^0 &= -\frac{1}{\Delta t} u_{i,j}^0, \\ (A^* u)_{i,j}^{N_T} &= \frac{1}{\Delta t} u_{i,j}^{N_T-1} - \nu(\Delta_h u^{N_T-1})_{i,j}, \end{aligned}$$

for all $u \in \mathcal{U}$. For later use, notice that

$$\text{Ker}(B^*) = \{u \in \mathcal{U} \mid \forall k = 0, \dots, N_T - 1 \text{ there exists } c_k \in \mathbb{R} \text{ such that } u_{i,j}^k = c_k \ \forall i, j\},$$

and so

$$(5.3.3) \quad \text{Im}(B) = \text{Ker}(B^*)^\perp = \left\{ u \in \mathcal{U} \mid \sum_{i,j} u_{i,j}^k = 0 \ \forall k = 0, \dots, N_T - 1 \right\}.$$

Let us define $\widehat{b} : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R} \cup \{+\infty\}$

$$(5.3.4) \quad \widehat{b}(m, w) := \begin{cases} \frac{|w|^q}{qm^{q-1}}, & \text{if } m > 0, w \in K, \\ 0, & \text{if } (m, w) = (0, 0), \\ +\infty, & \text{otherwise,} \end{cases}$$

and the functions $\mathcal{B}, \mathcal{F} : \mathcal{M} \times \mathcal{W} \rightarrow \mathbb{R}$, $\mathcal{G} : \mathcal{M} \times \mathcal{W} \rightarrow \mathcal{M} \times \mathbb{R}^{N_h \times N_h}$ as

$$\begin{aligned}
 \mathcal{B}(m, w) &:= \sum_{\substack{1 \leq k \leq N_T, \\ 0 \leq i, j \leq N_h - 1}} \widehat{b}(m_{i,j}^k, w_{i,j}^{k-1}), \\
 \mathcal{F}(m) &:= \sum_{\substack{1 \leq k \leq N_T, \\ 0 \leq i, j \leq N_h - 1}} F(x_{i,j}, m_{i,j}^k) + \frac{1}{\Delta t} \sum_{0 \leq i, j \leq N_h - 1} G(x_{i,j}, m_{i,j}^{N_T}), \\
 \mathcal{G}(m, w) &:= (Am + Bw, m^0).
 \end{aligned}
 \tag{5.3.5}$$

Note that if $(m, w) \in \mathcal{M} \times \mathcal{W}$ is such that $\mathcal{G}(m, w) = (0, \bar{m})$, where we recall that \bar{m} is defined in (5.2.2), then

$$h^2 \sum_{i,j} m_{i,j}^k = 1 \quad \forall k = 0, \dots, N_T.
 \tag{5.3.6}$$

Indeed, by periodicity, $-\sum_{i,j} (\Delta_h m^{k+1})_{i,j} = 0$ and $\sum_{i,j} (Bw)_{i,j}^k = 0$ for all $k = 0, \dots, N_T - 1$. This implies that

$$0 = \sum_{i,j} (Am + Bw)_{i,j}^k = \frac{\sum_{i,j} m_{i,j}^{k+1}}{\Delta t} - \frac{\sum_{i,j} m_{i,j}^k}{\Delta t},$$

and so $h^2 \sum_{i,j} m_{i,j}^k = h^2 \sum_{i,j} \bar{m}_{i,j} = 1$ for all $k = 0, \dots, N_T$.

The discretization of the variational problem (P) that we consider is

$$(\mathbf{P}_{h,\Delta t}) \quad \inf_{(m,w) \in \mathcal{M} \times \mathcal{W}} \mathcal{B}(m, w) + \mathcal{F}(m), \quad \text{subject to } \mathcal{G}(m, w) = (0, \bar{m}),$$

where we recall that F and G in (5.3.5) are defined in (5.1.1).

We have the following result

Theorem 5.3.1. *For any $\nu > 0$ problem $(\mathbf{P}_{h,\Delta t})$ admits at least one solution $(m^{h,\Delta t}, w^{h,\Delta t})$ and associated to it there exists $u^{h,\Delta t} : \mathcal{M} \times \mathcal{W} \rightarrow \mathbb{R}$ such that $(\text{MFG}_{h,\Delta t})$ holds true. Moreover, $(m^{h,\Delta t})_{i,j}^k > 0$ for all $k = 1, \dots, N_T$, $i, j = 0, \dots, N_h - 1$.*

In order to prove the result above, let us first show a lemma that implies the feasibility of the constraints in $(\mathbf{P}_{h,\Delta t})$.

Lemma 5.3.2. *There exists $(\tilde{m}, \tilde{w}) \in \mathcal{M} \times \mathcal{W}$ such that*

$$\begin{aligned}
 \mathcal{G}(\tilde{m}, \tilde{w}) &= (0, \bar{m}), \quad \tilde{w}_{i,j}^k \in \text{int}(K) \quad \forall i, j = 1, \dots, N_h - 1, \quad k = 1, \dots, N_T - 1, \\
 \tilde{m}_{i,j}^k &> 0, \quad \forall i, j = 1, \dots, N_h - 1, \quad k = 1, \dots, N_T.
 \end{aligned}
 \tag{5.3.7}$$

Proof. Let us define $\tilde{m}_{i,j}^0 := \bar{m}_{i,j}$ and $\tilde{m}_{i,j}^k := 1$ for all $k = 1, \dots, N_T$ and i, j . Since $h^2 \sum_{i,j} \tilde{m}_{i,j}^k = 1$ for all $k = 0, \dots, N_T$, by (5.3.3) and the definition of A we easily get that $A\tilde{m} \in \text{Im}(B)$. Therefore, there exists $\hat{w} \in \mathcal{W}$ satisfying $\mathcal{G}(\tilde{m}, \hat{w}) = (0, \bar{m})$. Then, given $\delta > 0$, we set for all $k = 0, \dots, N_T - 1$ and i, j

$$\tilde{w}_{i,j}^k := \left(\hat{w}_{i,j}^{k,(1)} + \max_{i,j} \hat{w}_{i,j}^{k,(1)} + \delta, \hat{w}_{i,j}^{k,(2)} - \max_{i,j} \hat{w}_{i,j}^{k,(2)} - \delta, \hat{w}_{i,j}^{k,(3)} + \max_{i,j} \hat{w}_{i,j}^{k,(3)} + \delta, \hat{w}_{i,j}^{k,(4)} - \max_{i,j} \hat{w}_{i,j}^{k,(4)} - \delta \right),$$

which satisfies $\tilde{w}_{i,j}^k \in \text{int}(K)$ and $(B\tilde{w})^k = (B\hat{w})^k$. The result follows. \square

Now, we prove the existence of solutions to $(P_{h,\Delta t})$.

Lemma 5.3.3. *Problem $(P_{h,\Delta t})$ admits at least one solution $(m^{h,\Delta t}, w^{h,\Delta t})$ and every such solution satisfies $(m^{h,\Delta t})_{i,j}^k > 0$ for all $k = 1, \dots, N_T$, $i, j = 0, \dots, N_h - 1$.*

Proof. Let (m^n, w^n) be a minimizing sequence for $(P_{h,\Delta t})$. Lemma 5.3.2 implies that $\mathcal{B}(\bar{m}, \bar{w}) + \mathcal{F}(\bar{m}) < +\infty$. Therefore, there exists a constant $C_1 > 0$ such that

$$(5.3.8) \quad \mathcal{B}(m^n, w^n) + \mathcal{F}(m^n) \leq C_1 \quad \text{for all } n \in \mathbb{N}.$$

As a consequence, by definition of \hat{b} , $(m^n)_{i,j}^k \geq 0$ for all i, j and k and $(w^n)^k \in K$ for all k . Since $Am^n + Bw^n = 0$, relation (5.3.6) implies that $h^2 \sum_{i,j} (m^n)_{i,j}^k = 1$. In particular, there exists $C_2 > 0$ (independent of n) such that $\sup_{i,j,k} (m^n)_{i,j}^k \leq C_2$. Using that, if $(m^n)_{i,j}^k > 0$,

$$\hat{b}((m^n)_{i,j}^k, (w^n)_{i,j}^k) \geq \frac{|(w^n)_{i,j}^k|^q}{qC_2^{q-1}},$$

and that $\mathcal{F}(m^n)$ is uniformly bounded (because F and G are continuous and m^n is bounded), relation (5.3.8) yields the existence of $C_3 > 0$ (independent of n) such that $\sup_{i,j,k} |(w^n)_{i,j}^k| \leq C_3$. Thus, there exists $(m^{h,\Delta t}, w^{h,\Delta t}) \in \mathcal{M} \times \mathcal{W}$ such that, up to some subsequence, $m^n \rightarrow m^{h,\Delta t}$ and $w^n \rightarrow w^{h,\Delta t}$ as $n \rightarrow \infty$. Since $\mathcal{G}(m^n, w^n) = (0, \bar{m})$ we obtain that $\mathcal{G}(m^{h,\Delta t}, w^{h,\Delta t}) = (0, \bar{m})$. The lower semicontinuity of $\mathcal{B} + \mathcal{F}$ implies that

$$\mathcal{B}(m^{h,\Delta t}, w^{h,\Delta t}) + \mathcal{F}(m^{h,\Delta t}) \leq \lim_{n \rightarrow \infty} \mathcal{B}(m^n, w^n) + \mathcal{F}(m^n),$$

which implies that $(m^{h,\Delta t}, w^{h,\Delta t})$ solves $(P_{h,\Delta t})$. Finally, if $(m, w) \in \mathcal{M} \times \mathcal{W}$ solves $(P_{h,\Delta t})$ and $m_{i,j}^k = 0$ for some i, j and $k = 1, \dots, N_T$, then, by the definition of \mathcal{B} , we must have that $w_{i,j}^{k-1} = 0$. Thus, the constraint $(Am + Bw)_{i,j}^{k-1} = 0$ can be written as

$$\begin{aligned} & -\frac{m_{i,j}^{k-1}}{\Delta t} - \frac{\nu}{h^2} (m_{i+1,j}^k + m_{i-1,j}^k + m_{i,j+1}^k + m_{i,j-1}^k) \\ & = \frac{w_{i-1,j}^{k-1,(1)}}{h} - \frac{w_{i+1,j}^{k-1,(2)}}{h} + \frac{w_{i,j-1}^{k-1,(3)}}{h} - \frac{w_{i,j+1}^{k-1,(4)}}{h}. \end{aligned}$$

Since the left hand side above is non-positive and the right hand side is non-negative (by definition of K), we deduce that all the terms above are zero. By repeating the argument at the indexes neighboring (i, j) , we deduce that $m^k \equiv 0$ and so $h^2 \sum_{i,j} m_{i,j}^k = 0$ which, by (5.3.6), contradicts $\mathcal{G}(m, w) = (0, \bar{m})$. The result follows. \square

Remark 5.3.4. *Notice that the proof of the existence of a solution to $(P_{h,\Delta t})$ also works when $\nu = 0$.*

Proof of Theorem 5.3.1. By Lemma 5.3.3 we know that there exists a solution $(m^{h,\Delta t}, w^{h,\Delta t})$ to $(P_{h,\Delta t})$ and $m_{i,j}^{h,\Delta t} > 0$ for all i, j . Thus, in order to conclude it suffices to show the existence of $u^{h,\Delta t}$ such that $(MFG_{h,\Delta t})$ holds true. For notational convenience we will omit the superindexes h and Δt . Define the *Lagrangian* $\mathcal{L} := \mathcal{M} \times \mathcal{W} \times \mathcal{U} \times \mathbb{R}^{N_h \times N_h} \rightarrow \mathbb{R} \cup \{+\infty\}$, associated to $(P_{h,\Delta t})$, as

$$(5.3.9) \quad \begin{aligned} \mathcal{L}(m, w, u, \lambda) & := \mathcal{B}(m, w) + \mathcal{F}(m) - \langle u, Am + Bw \rangle - \langle \lambda, m^0 - \bar{m} \rangle \\ & = \mathcal{B}(m, w) + \mathcal{F}(m) - \langle A^*u, m \rangle - \langle B^*u, w \rangle - \langle \lambda, m^0 - \bar{m} \rangle. \end{aligned}$$

Note that the linear mapping $\mathcal{M} \ni m \mapsto (Am, m) \in \mathcal{U} \times \mathbb{R}^{N_h \times N_h}$ is invertible as it is shown by its matrix representation (see (5.4.7) in the next section). As a consequence \mathcal{G} is surjective and, hence, by standard arguments, there exists $(u, \lambda) \in \mathcal{U} \times \mathbb{R}^{N_h \times N_h}$ such that (5.3.10)

$$\begin{aligned} 0 &= \partial_{m_{i,j}^k} \mathcal{L}(m, w, u, \lambda) = -\frac{1}{q'} \frac{|w_{i,j}^{k-1}|^q}{(m_{i,j}^k)^q} + f(x_{i,j}, m_{i,j}^k) - [A^*u]_{i,j}^k \quad \forall k = 1, \dots, N_T - 1, \quad \forall i, j, \\ 0 &= \partial_{m_{i,j}^0} \mathcal{L}(m, w, u, \lambda) = -\lambda_{i,j} - [A^*u]_{i,j}^0 \quad \forall i, j, \\ 0 &= \partial_{m_{i,j}^{N_T}} \mathcal{L}(m, w, u, \lambda) = -\frac{1}{q'} \frac{|w_{i,j}^{N_T-1}|^q}{(m_{i,j}^{N_T})^q} + f(x_{i,j}, m_{i,j}^{N_T}) + \frac{1}{\Delta t} g(x_{i,j}, m_{i,j}^{N_T}) - [A^*u]_{i,j}^{N_T} \quad \forall i, j, \\ 0 &\in \partial_{w_{i,j}^{k-1}} \mathcal{L}(m, w, u, \lambda) = |w_{i,j}^{k-1}|^{q-2} \frac{w_{i,j}^{k-1}}{(m_{i,j}^k)^{q-1}} - [B^*u]_{i,j}^{k-1} + N_K(w_{i,j}^{k-1}) \quad \forall k = 1, \dots, N_T, \quad \forall i, j, \end{aligned}$$

where we have used definition (5.3.4) and that $m_{i,j}^k > 0$ for all $k = 1, \dots, N_T$ and all i, j . Defining $u_{i,j}^{N_T} := g(x_{i,j}, m_{i,j}^{N_T})$, by the last relation in (5.3.2), the third relation in (5.3.10) can be rewritten as

$$-D_t u_{i,j}^{N_T-1} - \nu(\Delta_h u^{N_T-1})_{i,j} + \frac{1}{q'} \frac{|w_{i,j}^{N_T-1}|^q}{(m_{i,j}^{N_T})^q} = f(x_{i,j}, m_{i,j}^{N_T}),$$

and hence, by the second relation in (5.3.2) and the first relation in (5.3.10), we have that

$$(5.3.11) \quad -D_t u_{i,j}^k - \nu(\Delta_h u^k)_{i,j} + \frac{1}{q'} \frac{|w_{i,j}^k|^q}{(m_{i,j}^{k+1})^q} = f(x_{i,j}, m_{i,j}^{k+1}) \quad \forall k = 0, \dots, N_T - 1, \quad \forall i, j.$$

The last relation in (5.3.10) yields that for all $k = 1, \dots, N_T$ and all i, j

$$\begin{cases} \frac{(m_{i,j}^k)^{q-1}}{|w_{i,j}^{k-1}|^{q-2}} [B^*u]_{i,j}^{k-1} \in w_{i,j}^{k-1} + N_K(w_{i,j}^{k-1}) & \text{if } w_{i,j}^{k-1} \neq 0, \\ [B^*u]_{i,j}^{k-1} \in N_K(0) & \text{if } w_{i,j}^{k-1} = 0, \end{cases}$$

which, by (5.3.2) and under the convention (5.2.3), is equivalent to

$$(5.3.12) \quad w_{i,j}^{k-1} = m_{i,j}^k |P_K(-[D_h u]_{i,j}^{k-1})|^{\frac{2-q}{q-1}} P_K(-[D_h u]_{i,j}^{k-1}) = m_{i,j}^k |[\widehat{D_h u^{k-1}}]_{i,j}|^{\frac{2-q}{q-1}} [\widehat{D_h u^{k-1}}]_{i,j}.$$

Shifting the index k , the expression above yields

$$\frac{1}{q'} \frac{|w_{i,j}^k|^q}{(m_{i,j}^{k+1})^q} = \frac{1}{q'} |[\widehat{D_h u^k}]_{i,j}|^{q'} \quad \forall k = 0, \dots, N_T - 1, \quad \forall i, j,$$

which, combined with (5.3.11), implies the first equation in (MFG_{h,Δt}). The second equation in (MFG_{h,Δt}) is a consequence of $Am + Bw = 0$ and the fact that (5.3.12) provides the identity

$$(Bw)_{i,j}^k = -\mathcal{J}_{i,j}(u^k, m^{k+1}) \quad \forall k = 0, \dots, N_T - 1, \quad \forall i, j.$$

The result follows. \square

Remark 5.3.5. (i) *The proof of the existence of solutions to (MFG_{h,Δt}) in Theorem 5.3.1 provides an alternative argument to the one in [6], based on Brouwer fixed-point theorem.*

(ii)(Uniqueness) *If $f(x, \cdot)$ and $g(x, \cdot)$ are increasing, with one of them being strictly increasing, then (MFG_{h,Δt}) has a unique solution. Indeed, under this assumption, the cost functional in (P_{h,Δt}) is convex w.r.t. (m, w) and strictly convex w.r.t. m . It is easy to check that this implies that if (m_1, w_1) and (m_2, w_2) are two solutions of (P_{h,Δt}) then $m_1 = m_2$. Using this fact and the definition of \hat{b} (see (5.3.4)), we also get that $w_1 = w_2$. Thus, under this monotonicity assumption, the solution $(m^{h,\Delta t}, w^{h,\Delta t})$ to (P_{h,Δt}) is unique. Having this result, the uniqueness of $u^{h,\Delta t}$ follows directly from [6, Lemma 1].*

5.4 A primal-dual algorithm

As discussed in [29], for solving the optimization problem

$$(5.4.1) \quad \min_{y \in \mathbb{R}^N} \varphi(y) + \psi(y),$$

and its dual

$$(5.4.2) \quad \min_{\sigma \in \mathbb{R}^N} \varphi^*(-\sigma) + \psi^*(\sigma),$$

where $\varphi: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\psi: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ are convex l.s.c. proper functions, methods in [28, 46, 48, 55, 57] can be applied with guaranteed convergence under mild assumptions. In [29], devoted to the stationary case, the method proposed in [46] has the best performance when the viscosity parameter is small or zero. This method is inspired by the first-order optimality conditions satisfied by a solution $(\hat{y}, \hat{\sigma})$ to (5.4.1)-(5.4.2) under standard qualification conditions, which reads (see [96, Theorem 8])

$$(5.4.3) \quad \begin{cases} -\hat{\sigma} \in \partial\varphi(\hat{y}) \\ \hat{y} \in \partial\psi^*(\hat{\sigma}) \end{cases} \Leftrightarrow \begin{cases} \hat{y} - \tau\hat{\sigma} \in \tau\partial\varphi(\hat{y}) + \hat{y} \\ \hat{\sigma} + \gamma\hat{y} \in \gamma\partial\psi^*(\hat{\sigma}) + \hat{\sigma} \end{cases} \Leftrightarrow \begin{cases} \text{prox}_{\tau\varphi}(\hat{y} - \tau\hat{\sigma}) = \hat{y} \\ \text{prox}_{\gamma\psi^*}(\hat{\sigma} + \gamma\hat{y}) = \hat{\sigma}, \end{cases}$$

where $\gamma > 0$ and $\tau > 0$ are arbitrary and, given a l.s.c. convex proper function $\phi: \mathbb{R}^N \rightarrow]-\infty, +\infty]$,

$$\text{prox}_{\gamma\phi} x := \operatorname{argmin}_{y \in \mathbb{R}^N} \left\{ \phi(y) + \frac{|y - x|^2}{2\gamma} \right\} = (I + \partial(\gamma\phi))^{-1}(x) \quad \forall x \in \mathbb{R}^N.$$

Given $\theta \in [0, 1]$, τ and γ satisfying $\tau\gamma < 1$, and starting points $(y^0, \tilde{y}^0, \sigma^0) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M$, the iterates $\{(y^k, \sigma^k)\}_{k \in \mathbb{N}}$ generated by

$$(5.4.4) \quad \begin{aligned} \sigma^{k+1} &:= \text{prox}_{\gamma\psi^*}(\sigma^k + \gamma\tilde{y}^k), \\ y^{k+1} &:= \text{prox}_{\tau\varphi}(y^k - \tau\sigma^{k+1}), \\ \tilde{y}^{k+1} &:= y^{k+1} + \theta(y^{k+1} - y^k) \end{aligned}$$

converge to a primal-dual solution $(\hat{y}, \hat{\sigma})$ to (5.4.1)-(5.4.2) (see, e.g., [46]).

In the case under study, the equations of the time-dependent discretization are very similar to their stationary counterparts (see [29]). Specifically, the discrete linear operators A and B defined in (5.3.1), by an abuse of notation, are represented by real matrices A and B , of dimensions $(N_T \times N_h^2) \times ((N_T + 1) \times N_h^2)$ and $(N_T \times N_h^2) \times (N_T \times 4N_h^2)$, respectively, given by

$$(5.4.5) \quad A := \begin{pmatrix} -\frac{1}{\Delta t} \text{Id}_{N_h^2} & \nu L + \frac{1}{\Delta t} \text{Id}_{N_h^2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{\Delta t} \text{Id}_{N_h^2} & \nu L + \frac{1}{\Delta t} \text{Id}_{N_h^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{1}{\Delta t} \text{Id}_{N_h^2} & \nu L + \frac{1}{\Delta t} \text{Id}_{N_h^2} \end{pmatrix},$$

and

$$(5.4.6) \quad B := \begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & M \end{pmatrix},$$

where $L \in \mathcal{M}_{N_h^2, N_h^2}(\mathbb{R})$ is the matrix that represents $-\Delta_h$ on the torus \mathbb{T}_h^2 and $M \in \mathcal{M}_{N_h^2, 4N_h^2}(\mathbb{R})$ is the matrix representing the discrete divergence. Denoting by \tilde{A} and \tilde{B} the $((N_T + 1) \times N_h^2) \times ((N_T + 1) \times N_h^2)$ and $((N_T + 1) \times N_h^2) \times (N_T \times 4N_h^2)$ real matrices

$$(5.4.7) \quad \tilde{A} := \begin{pmatrix} \text{Id}_{N_h^2} & 0 & \cdots & 0 \\ & A & & \end{pmatrix} \quad \text{and} \quad \tilde{B} := \begin{pmatrix} 0 & \cdots & 0 \\ & B & \end{pmatrix},$$

the constraint $\mathcal{G}(m, w) = (0, \bar{m})$ in $(P_{h, \Delta t})$ can be rewritten as $C(m, w) = (\bar{m}, 0)$, where $C := [\tilde{A} \mid \tilde{B}]$.

Remark 5.4.1. (i) *The matrix \tilde{A} is block lower triangular with invertible diagonal blocks and, hence, it is invertible. Indeed, the first diagonal block $\text{Id}_{N_h^2}$ is obviously invertible and the other blocks, given by $\nu L + \frac{1}{\Delta t} \text{Id}_{N_h^2}$, are also invertible because they are strictly diagonally dominant.*

(ii) *Since \tilde{A} is invertible, the matrix*

$$(5.4.8) \quad Q := CC^* = \tilde{A}\tilde{A}^* + \tilde{B}\tilde{B}^*$$

is positive definite and, hence, invertible.

Therefore, $(P_{h, \Delta t})$ is a particular instance of (5.4.1) with

$$\varphi(m, w) := \mathcal{B}(m, w) + \mathcal{F}(m), \quad \psi(m, w) := \iota_{\ker C + \{(m_f, w_f)\}}(m, w),$$

where (m_f, w_f) is a feasible vector (provided for instance by Lemma 5.3.2), and $\iota_{\ker C + \{(m_f, w_f)\}}$ is the function defined as 0 for all $(m, w) \in \ker C + \{(m_f, w_f)\}$ and $+\infty$, otherwise.

Since $\text{prox}_{\gamma\psi^*} = \text{Id} - \gamma \text{prox}_{\psi/\gamma} \circ (\text{Id}/\gamma) = \text{Id} - \gamma \text{prox}_{\psi} \circ (\text{Id}/\gamma)$ (see e.g. [15, Section 24.2]) and

$$\text{prox}_{\psi} : (m, w) \mapsto (m, w) - C^*Q^{-1}(C(m, w) - (\bar{m}, 0)),$$

we have

$$\text{prox}_{\gamma\psi^*} : (m, w) \mapsto C^*Q^{-1}(C(m, w) - \gamma(\bar{m}, 0)),$$

where Q is defined in (5.4.8). By setting $y^0 = (m^0, w^0)$, $\tilde{y}^0 = (\tilde{m}^0, \tilde{w}^0)$, $\sigma^0 = (n^0, v^0) \in \mathbb{R}^{N_T \times N_h^2} \times \mathbb{R}^{N_T \times (4N_h^2)}$, (5.4.4) becomes

$$(5.4.9) \quad \begin{cases} z^{[l+1]} = -Q^{-1} \left(\tilde{A}(n^{[l]} + \gamma\tilde{m}^{[l]}) + \tilde{B}(v^{[l]} + \gamma\tilde{w}^{[l]}) - \gamma(\bar{m}, 0) \right), \\ \begin{pmatrix} n^{[l+1]} \\ v^{[l+1]} \end{pmatrix} = \begin{pmatrix} \tilde{A}^* z^{[l+1]} \\ \tilde{B}^* z^{[l+1]} \end{pmatrix}, \\ \begin{pmatrix} m^{[l+1]} \\ w^{[l+1]} \end{pmatrix} = \text{prox}_{\tau\varphi} \left(\begin{pmatrix} m^{[l]} + \tau n^{[l+1]} \\ w^{[l]} + \tau v^{[l+1]} \end{pmatrix} \right), \\ \begin{pmatrix} \tilde{m}^{[l+1]} \\ \tilde{w}^{[l+1]} \end{pmatrix} = \begin{pmatrix} m^{[l+1]} + \theta(m^{[l+1]} - m^{[l]}) \\ w^{[l+1]} + \theta(w^{[l+1]} - w^{[l]}) \end{pmatrix}, \end{cases}$$

and, if $\gamma\tau < 1$, the convergence of $(m^{[l]}, w^{[l]})$ to a solution (\hat{m}, \hat{w}) to $(P_{h, \Delta t})$ is guaranteed together with the convergence of $(n^{[l]}, v^{[l]})$ to some (\hat{n}, \hat{v}) as $l \rightarrow \infty$. In order to compute the Lagrange multiplier $\hat{u} \in \mathcal{U}$, which solves the first equation in $(\text{MFG}_{h, \Delta t})$, note that (5.3.9) can be written equivalently as

$$(5.4.10) \quad \begin{aligned} \mathcal{L}(m, w, u, \lambda) &:= \varphi(m, w) - \langle (\lambda, u), \tilde{A}m + \tilde{B}w \rangle + \langle \lambda, \bar{m} \rangle \\ &= \varphi(m, w) - \left\langle \begin{pmatrix} \tilde{A}^* \\ \tilde{B}^* \end{pmatrix} (\lambda, u), \begin{pmatrix} m \\ w \end{pmatrix} \right\rangle + \langle \lambda, \bar{m} \rangle, \end{aligned}$$

and the optimality condition yields

$$(5.4.11) \quad \begin{pmatrix} \tilde{A}^* \\ \tilde{B}^* \end{pmatrix} \hat{z} \in \partial\varphi(\hat{m}, \hat{w}),$$

where (\hat{m}, \hat{w}) is the primal solution and $\hat{z} = (\hat{\lambda}, \hat{u})$. Therefore, in order to approximate \hat{z} , note that from (5.4.9) we have

$$(5.4.12) \quad \begin{pmatrix} \frac{m^{[l]} - m^{[l+1]}}{\tau} + \tilde{A}^* z^{[l+1]} \\ \frac{w^{[l]} - w^{[l+1]}}{\tau} + \tilde{B}^* z^{[l+1]} \end{pmatrix} \in \partial\varphi(m^{[l+1]}, w^{[l+1]})$$

and, hence, since the algorithm generates converging sequences $m^{[l]} \rightarrow \hat{m}$ and $w^{[l]} \rightarrow \hat{w}$ and $z^{[l]} \rightarrow \hat{z} := -Q^{-1}(\tilde{A}(\hat{n} + \gamma\hat{m}) + \tilde{B}(\hat{v} + \gamma\hat{w}) - \gamma(\bar{m}, 0))$, the closedness of the graph of $\partial\varphi$ [15, Proposition 20.38] yields (5.4.11) and, hence, a good approximation of \hat{z} is $z^{[l]}$ for l large enough. For obtaining $[u^*]_{i,j}^{N_T}$, a good approximation is $[u^{[l]}]_{i,j}^{N_T} = g(x_{i,j}, [m^{[l]}]_{i,j}^{N_T})$.

Remark 5.4.2. (i) *In order to obtain an efficient algorithm, the computation of $\text{prox}_{\tau\varphi}$ in (5.4.9) should be fast. A complete study of $\text{prox}_{\tau\varphi}$ is presented in [29, Section 3.2] showing that its computation depends on the resolution of a real equation, which can be efficiently solved.*

(ii) *An important step in (5.4.9) is the efficient computation of the inverse of Q . Different preconditioning strategies to tackle this issue will be presented in the following section.*

5.5 Preconditioning strategies

At the beginning of each iteration of the primal-dual algorithm (5.4.9), we require the solution of a linear system

$$(5.5.1) \quad Qz = b.$$

The purpose of this section is to discuss preconditioning strategies for the solution of this linear system. For the stationary setting discussed in [29], the solution of such a system via direct methods such as the *backlash* (`mldivide`) command in MATLAB ¹ was feasible for relatively fine meshes (up to the order of 100 nodes per space dimension). However, as shown in Table 5.1, introducing a temporal dimension and thus increasing the degrees of freedom to $N_h^2 \times N_T$ significantly increases the computation time. Indeed, the use of *backlash* on fine space and time grids – e.g. 128^2 space grid points and 40 time steps – requires an amount of RAM that is prohibitive on the machine used for our performance tests ², leading to “out of memory” errors. We mitigate this problem by exploring the solution of (5.5.1) via preconditioned iterative methods, which perform efficiently for finer space and time subdivisions and different viscosities.

We begin by illustrating the difficulties associated to the conditioning of the system in (5.5.1). Table 5.2 shows the condition number of the system for different space-time discretizations and viscosity values. Without any preconditioner, the condition numbers of different discretizations scale up to 10^8 . The same Table shows that by selecting a suitable preconditioner, such as the *modified incomplete Cholesky factorization* [22] (`michol` in MATLAB), the conditioning of the system is improved by 4 orders of magnitude.

¹<http://uk.mathworks.com/help/matlab/ref/mldivide.html>

²Intel Core i7-4600U @ 2.7GHz, 16GB RAM

(a) $N_T = 10$ (b) $N_T = 40$

$\nu \backslash N_h$	16	32	64	128	$\nu \backslash N_h$	16	32	64	128
5×10^{-4}	7.12	62.7	452	4720	5×10^{-4}	24.2	569	15600	[OOM]
5×10^{-3}	6.29	60.6	345	3690	5×10^{-3}	18.8	496	14200	[OOM]
5×10^{-2}	1.96	18.3	113	1340	5×10^{-2}	8.10	145	5000	[OOM]
0.5	1.18	9.41	56.1	660	0.5	4.50	72.3	2510	[OOM]

Table 5.1 – MATLAB’s *backslash* computation times (seconds) for a single linear system solved in (5.4.9) within the Chambolle-Pock algorithm under a tolerance equal to 10^{-4} in normalized ℓ^2 -norm. For fine meshes [OOM] indicates an out of memory error for the tested architecture.

We have tested different choices of preconditioners and iterative methods for our problem. Since the matrix Q in our setting is sparse, symmetric, and positive-definite, we have implemented an *incomplete Cholesky factorization* with diagonal scaling, a *modified incomplete Cholesky factorization*, and multigrid preconditioning. As for the choice of the iterative method, our tests included both *preconditioned conjugate gradient* (`pcg`), and the *biconjugate gradient stabilized method* (`BiCGStab`). The interested reader will find in [102, Chapters 6 and 8] a thorough description of the aforementioned methods, and in the Appendix of this article performance tables for the different methods.

Our findings suggest that the use of an iterative `pcg` method, preconditioned by modified incomplete Cholesky factorization is satisfactory for small viscosities ($\nu \leq 0.05$). However, this algorithm fails to converge for high viscosity systems on refined grids ($\nu = 0.5$, $N_T = 40$, $N_h \in \{64, 128\}$). Exchanging the `pcg` method by a `BiCGStab` algorithm preconditioned by modified incomplete Cholesky factorization slows down the process on finer grids, but allows for convergence in the failure cases of `pcg`: $\nu = 0.5$, $N_T = 40$, $N_h \in \{64, 128\}$.

In order to deal with (and exploit) the anisotropy of the system introduced by high viscosities, we have devised an algorithm consisting in a multigrid preconditioner with `BiCGStab` iterations akin to that described in Algorithm 1. It is the only among the tested methods which performs consistently for different viscosities and space-time discretizations. We discuss its implementation and assess its performance in the following section 5.5.1.

Table 5.2 – Condition numbers for Q without preconditioning (a), and with modified incomplete Cholesky factorization preconditioning (b).

(a) No preconditioner (scaling 10^4)(b) `michol` preconditioning (scaling 10^4)

$\nu \backslash DoF$	$32^2 \times 1$	$32^2 \times 10$	$32^2 \times 20$
5×10^{-5}	4.290	19.06	38.43
5×10^{-4}	4.296	19.25	39.10
5×10^{-3}	4.751	22.77	48.90
5×10^{-2}	48.43	227.7	466.0
0.5	4399	19250	36540

$\nu \backslash DoF$	$32^2 \times 1$	$32^2 \times 10$	$32^2 \times 20$
5×10^{-5}	0.04217	0.08535	0.1361
5×10^{-4}	0.04218	0.08612	0.1370
5×10^{-3}	0.04272	0.09325	0.1446
5×10^{-2}	0.1025	0.2501	0.3579
0.5	1.255	2.743	3.770

Algorithm 1 Preconditioned BiCGStab

```

 $x_l \leftarrow \text{BiCGStab}(Q_l, b_l, P_L, P_R, x_0, tol)$ 
procedure BiCGSTAB( $Q_l, b_l, P_L, P_R, x_0, tol$ )
   $r_0 := p_0 := Q_l x_0 - b_l; \quad \hat{r}_0 := \hat{p}_0 := P_l r_0; \quad \hat{\rho}_0 := \langle r_0, r_0 \rangle; \quad k := 0$ 
  while  $\|r_k\| \geq tol$  do
     $v_k := Q P_R \hat{p}_k$ 
     $\hat{v}_k := P_L v_k$ 
     $\hat{\alpha}_k := \hat{\rho}_k / \langle \hat{v}_k, \hat{r}_0 \rangle$ 
     $s_{k+1} := r_k - \hat{\alpha}_k v_k$ 
     $\hat{s}_{k+1} := P_L s_{k+1}$ 
     $t_{k+1} := A P_R \hat{s}_{k+1}$ 
     $\hat{t}_{k+1} := P_L t_{k+1}$ 
     $\hat{\omega}_{k+1} := \langle \hat{s}_{k+1}, \hat{t}_{k+1} \rangle / \langle \hat{t}_{k+1}, \hat{t}_{k+1} \rangle$ 
     $\hat{x}_{k+1} := \hat{x}_k + \hat{\alpha}_k \hat{p}_k + \hat{\omega}_{k+1} \hat{s}_{k+1}$ 
     $r_{k+1} := s_{k+1} - \hat{\omega}_{k+1} t_{k+1}$ 
     $\hat{r}_{k+1} := \hat{s}_{k+1} - \hat{\omega}_{k+1} \hat{t}_{k+1}$ 
     $\hat{\rho}_{k+1} := \langle \hat{r}_{k+1}, \hat{r}_0 \rangle$ 
     $\hat{\beta}_{k+1} := (\hat{\alpha}_k / \hat{\omega}_{k+1})(\rho_{k+1} / \rho_k)$ 
     $\hat{p}_{k+1} := \hat{r}_{k+1} + \hat{\beta}_{k+1}(\hat{p}_k - \hat{\omega}_{k+1} \hat{v}_k)$ 
     $k := k + 1$ 
  return  $x_k := P_R \hat{x}_k$ 

```

5.5.1 Multigrid preconditioner

We implement a multigrid preconditioned algorithm for solving (5.5.1). We refer the reader to [100] for an introduction and an overview of multigrid methods. We briefly review the main concepts behind the method. Consider two linear systems $A_1 \bar{x}_1 = b_1$ and $A_0 \bar{x}_0 = b_0$, stemming from two discretizations of a linear PDE over the grids G_1 and G_0 , respectively. Assume also that G_1 is a refinement of G_0 . Loosely speaking, the main idea of the method is that in order to find a good approximation of the solution \bar{x}_1 on the finer grid, we first consider what is known as a *smoothing* step. This step consists in computing a few iterates $x_1^1, \dots, x_1^{\eta_1}$ with a standard indirect method, such as Jacobi or Gauss-Seidel, and to define the *residual* $r_1 := b_1 - A_1 x_1^{\eta_1}$, which is shown to be *smoother* (less oscillatory) than the first residual $b_1 - A_1 x_1^1$. Then, we consider in the coarser grid G_0 the second system $A_0 \bar{x}_0 = b_0$ with $b_0 = \hat{r}_1$, where \hat{r}_1 is the restriction of r_1 to G_0 . Assuming that we can compute a good approximation of \bar{x}_0 , which we still denote by \bar{x}_0 , we then extend this solution to G_1 by using a linear interpolation. Calling e_1 the resulting vector, we update $x_1^{\eta_1}$ by redefining it as $x_1^{\eta_1} + e_1$ and we end the procedure by applying again a few iterations, say η_2 , of a smoothing method initialized at $x_1^{\eta_1}$. This last step is called *post smoothing*.

The previous paragraph introduced what is known as a *two grid iteration*. If we consider more grids G_0, G_1, \dots, G_ℓ , where for each $k = 0, \dots, \ell - 1$, $G_k \subseteq G_{k+1}$, we can proceed similarly and obtain a better approximation of the solution to $A_\ell \bar{x}_\ell = b_\ell$. As in the previous case, we begin with the finest grid G_ℓ and we perform η_1 smoothing steps to obtain the residual $r_\ell := b_\ell - A_\ell x_\ell^{\eta_1}$ whose restriction to $G_{\ell-1}$ is denoted by \hat{r}_ℓ . In this grid we consider the system $A_{\ell-1} x_{\ell-1} = \hat{r}_\ell$ and we perform again a smoothing step and a restriction of the residual to $G_{\ell-2}$. The procedure continues until we get to the coarsest grid G_0 , where the solution e_0 to the corresponding linear system can be found easily (typically using a direct method). Next, the solution e_1 on the grid G_1 is corrected with the interpolation of e_0 . Another post smoothing is performed to the corrected solution on G_1 and using its

interpolation in the grid G_2 we correct the previous solution on this grid. The smoothing, interpolation and correction iterations end once we arrive to the finest grid G_ℓ to obtain the final approximation of \bar{x}_ℓ . The previous procedure is called a multigrid method with a *V-cycle*. An alternative, to obtain a more accurate solution, is to proceed as before going from G_ℓ to $G_{\ell-1}$ and then for $k = \ell - 1, \dots, 1$ to perform two consecutive coarse-grid corrections, instead of one as in the *V-cycle*. The resulting procedure is known as multigrid with a *W-cycle*. Finally, in between the *V-cycle* and the *W-cycle*, we have the *F-cycle*, where in the process of going from the coarsest grid to the finest one, if a grid has been reached for the first time, another correction with the coarser grids using a *V-cycle* is performed.

In our context, we use one cycle of the multigrid algorithm, which is a linear operator as a function of the residual on the finest grid, as a preconditioner for solving (5.5.1) with the BiCGStab method. Since Q is related to the finite difference discretization of the operator $-\partial_{tt}^2 + \nu^2 \Delta^2 - \Delta$ and ν is not necessarily small, as in [8], it is natural to consider the refinements of the grid only in the space variable (we refer the reader to [100] for semi-coarsening multigrid methods in the context of anisotropic operators). We suppose that the spatial mesh is such that $N_h = H2^\ell$, with $H > 1$ and ℓ is a positive integer (in the numerical example in the next section H will be equal to 2 or 3, H^2 being the number of spatial points in the coarsest grid).

Let us specify the main steps of the multigrid method we use as a preconditioner.

- ◇ *Hierarchy of Grids*: Semi-coarsened grids G_k with size $(N_T+1)H^22^{2k}$ for all $k = 0 \dots \ell$.
- ◇ *Cycle*: We use the F-cycle.
- ◇ *Restriction operator*: As in [8], in order to restrict the residual on the grid G_k to the grid G_{k-1} , we use the second-order operator $R_k : \mathbb{R}^{(2^k H)^2(N_T+1)} \rightarrow \mathbb{R}^{(2^{k-1} H)^2(N_T+1)}$ defined by

$$(R_k X)_{i,j}^n := \frac{1}{16} \begin{pmatrix} 4X_{2i,2j}^n + 2(X_{2i+1,2j}^n + X_{2i-1,2j}^n + X_{2i,2j+1}^n + X_{2i,2j-1}^n) \\ X_{2i-1,2j-1}^n + X_{2i-1,2j+1}^n + X_{2i+1,2j-1}^n + X_{2i+1,2j+1}^n \end{pmatrix},$$

for $n = 0, \dots, N_T$, $i, j = 1, \dots, 2^{k-1}H$.

- ◇ *Interpolation operator*: We denote by $I_k : \mathbb{R}^{(2^{k-1} H)^2(N_T+1)} \rightarrow \mathbb{R}^{(2^k H)^2(N_T+1)}$ the interpolation operator from the grid G_{k-1} to the grid G_k . We have chosen a standard bilinear interpolation operator in the space variable, which is also a second-order operator and dual to the restriction operator ($I_k = 4R_k^*$). According to [26], the sum of the orders of R_k and I_k has to be at least equal to the degree of the differential operator. In our context, both are equal to 4.
- ◇ *Linear systems on the different grids*: The linear systems are defined by the matrices

$$Q_k := A_k A_k^* + B_k B_k^*, \quad k = 0, \dots, \ell,$$

where we recall that A_k and B_k are the finite difference discretizations of $\partial_t - \nu \Delta$ and $\text{div}(\cdot)$, respectively, on the grid G_k (see (5.3.1)).

- ◇ *Smoother*: Here we have used Gauss-Seidel iterations in the lexicographic order. There is no reason for choosing the lexicographic order, other than its simplicity.

- ◇ *Solving the system on the coarsest grid G_0* : We can use an exact solver such as *backlash* in MATLAB. Indeed, in G_0 the size of the system is really small with respect to the size of the system on the grid G_ℓ (in G_0 , we can even store the inverse of Q_0 and inversion at this level just becomes a matrix multiplication).

The multigrid preconditioning procedure is summarized in Algorithm 2.

Algorithm 2 Multigrid Preconditioner for $Q_\ell x_\ell = b_\ell$

```

 $P_L : y \mapsto \text{MultigridSolver}(\ell, 0, y, \text{cycle})$ 
 $x_l \leftarrow \text{BiCGStab}(Q_\ell, b_\ell, P_L, I_d, x_0, \text{tol})$ 
procedure MULTIGRID SOLVER( $k, x_k, b_k, \text{cycle}$ )
  if  $k = 0$  then
     $x_k \leftarrow Q_0^{-1} b_k$ 
  else
     $x_k \leftarrow$  Perform  $\eta_1$  steps of Gauss-Seidel from  $x_k$  with  $b_k$  as second member.
     $x_{k-1} \leftarrow 0$ 
     $x_{k-1} \leftarrow \text{MultigridSolver}(k-1, x_{k-1}, R_k(b_k - Q_k x_k), \text{cycle})$ 
    if cycle is W then
       $x_{k-1} \leftarrow \text{MultigridSolver}(k-1, x_{k-1}, R_k(b_k - Q_k x_k), \text{cycle})$ 
    if cycle is F then
       $x_{k-1} \leftarrow \text{MultigridSolver}(k-1, x_{k-1}, R_k(b_k - Q_k x_k), V)$ 
     $x_k \leftarrow x_k + I_k x_{k-1}$ 
     $x_k \leftarrow$  Perform  $\eta_2$  steps of Gauss-Seidel from  $x_k$  with  $b_k$  as second member.
  return  $x_k$ 

```

5.5.2 Numerical Tests

In this section we present a test case considered in [6], for which the stationary solution has been computed numerically in [29] using the primal-dual algorithm presented above.

The setting is as follows: we consider system (MFG) with $g \equiv 0$ and

$$f(x, y, m) := m^2 - \bar{H}(x, y), \quad \bar{H}(x, y) = \sin(2\pi y) + \sin(2\pi x) + \cos(2\pi x),$$

for all $(x, y) \in \mathbb{R}^2$ and $m \in \mathbb{R}_+$. This means that in the underlying differential game modelled by (MFG), a typical agent aims to get closer to the maxima of \bar{H} and, at the same time, he/she is adverse to crowded regions (because of the presence of the m^2 term in f).

We first validate the dynamic behavior of our solution. Figure 5.1 shows the evolution of the mass at four different time steps. Starting from a constant initial density, the mass converges to a steady state, and then, when t gets close to the final time T , the mass is influenced by the final cost and converges to a final state. This behavior is referred to as *turnpike phenomenon* in the literature [94]. It is illustrated by Figure 5.2, which displays as a function of time t the distance of the mass at time t to the stationary state computed as in [29]. In other words, denoting by $m^\infty \in \mathbb{R}^{N_h \times N_h}$ the solution to the discrete stationary problem and by $m \in \mathcal{M}$ the solution to the discrete evolutive problem, Figure 5.2 displays the graph of $k \mapsto \|m_\infty - m^k\|_{\ell_2} = \left(h^2 \sum_{i,j} (m_{i,j}^\infty - m_{i,j}^k)^2 \right)^{1/2}$, $k \in \{0, \dots, N_T\}$.

For the multi-grid preconditioner, Table 5.3 shows the computation times for different discretizations. It can be observed that finer meshes with 128^3 degrees of freedom are

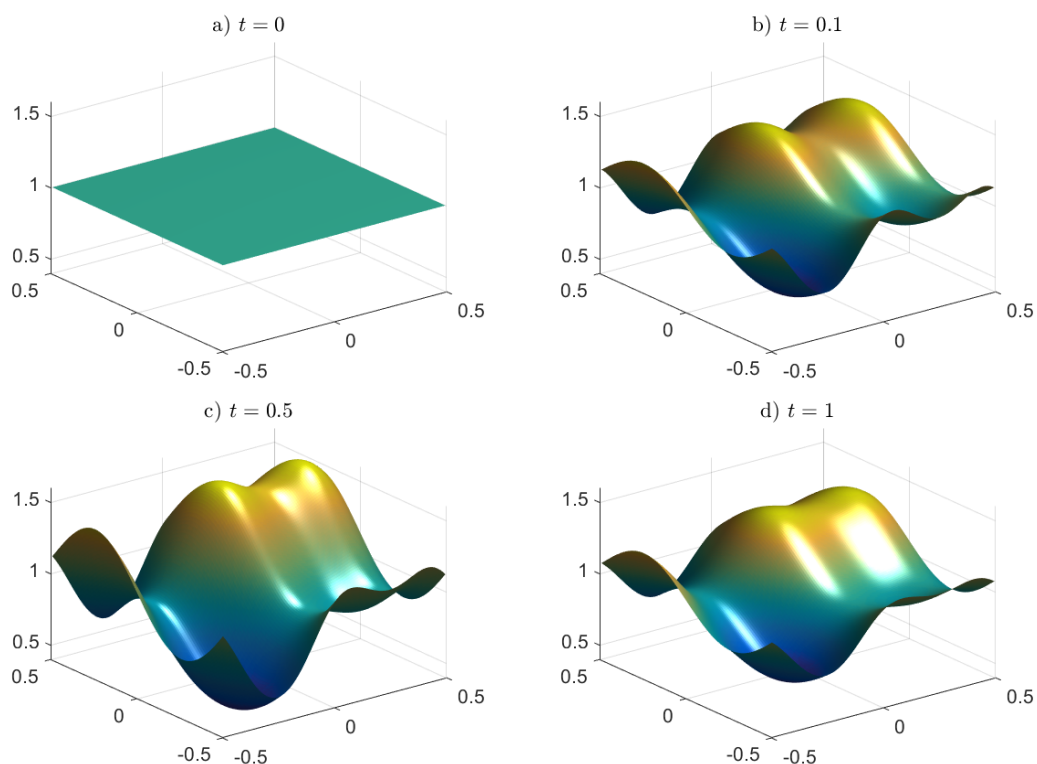


Figure 5.1 – Evolution of the density m obtained with the multi-grid preconditioner for $\nu = 0.5, T = 1, N_T = 200$ and $N_h = 128$. At $t = 0.12$ the solution is close to the solution of the associated stationary MFG.

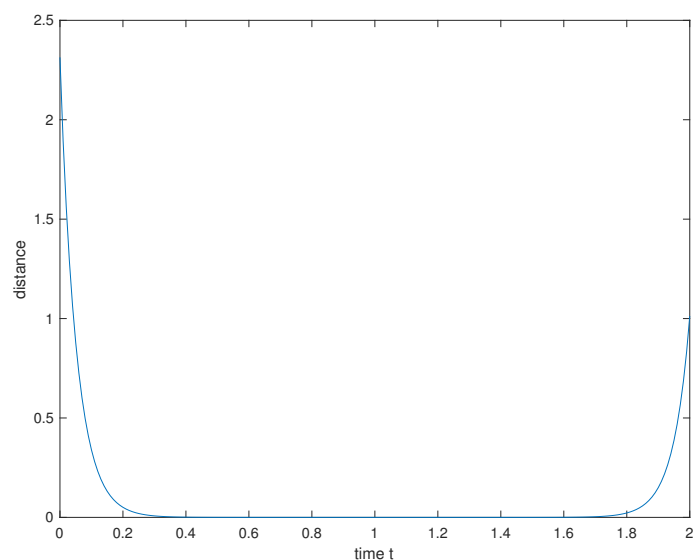


Figure 5.2 – Distance to the stationary solution at each time $t \in [0, T]$, for $\nu = 0.5, T = 2, N_T = 200$ and $N_h = 128$. The distance is computed using the ℓ^2 norm as explained in the text. The turnpike phenomenon is observed as for a long time frame the time-dependent mass approaches the solution of the stationary MFG.

solvable within CPU times which outperform others methods shown in the Appendix and in [29]. Furthermore, the method is robust with respect to different viscosities.

From Table 5.3 we observe that most of the computational time is used for solving the second proximal operator (the third equality of (5.4.9)), which does not use a multigrid strategy but which is a pointwise operator (see Proposition 3.1 of [29]) and thus could be fully paralellizable.

(a) Grid with $64 \times 64 \times 64$ points. (b) Grid with $128 \times 128 \times 128$ points.

ν	Total time	Time first prox	Iterations	ν	Total time	Time first prox	Iterations
0.6	116.3 [s]	11.50 [s]	20	0.6	921.1 [s]	107.2 [s]	20
0.36	120.4 [s]	11.40 [s]	21	0.36	952.3 [s]	118.0 [s]	21
0.2	119.0 [s]	11.26 [s]	22	0.2	1028.8 [s]	127.6 [s]	22
0.12	129.1 [s]	14.11 [s]	22	0.12	1036.4 [s]	135.5 [s]	23
0.046	225.0 [s]	23.28 [s]	39	0.046	1982.2 [s]	260.0 [s]	42

Table 5.3 – Time (in seconds) for the convergence of the Chambolle-Pock algorithm, cumulative time of the first proximal operator with the multigrid preconditioner, and number of iterations, for different viscosity values ν and two types of grids. Here we used $\eta_1 = \eta_2 = 2, T = 1$ and a tolerance between two iterations of the Chambolle-Pock algorithm equal to 10^{-6} in normalized ℓ^2 -norm.

Unlike the stationary case, low viscosities seem to make the algorithm be slightly slower. However, Table 5.4 shows that the average number of iterations of BiCGStab stays low regardless of the viscosity. Indeed Table 5.3 shows that more Chambolle-Pock iterations are needed to converge. The same behavior happens when we use a direct exact solver instead of the multi-grid preconditioned BiCGStab algorithm.

(a) iterations to decrease the residual by a factor 10^{-3} . (b) iterations to solve the system with an error of 10^{-8} .

ν	$32 \times 32 \times 32$	$64 \times 64 \times 64$	$128 \times 128 \times 128$	ν	$32 \times 32 \times 32$	$64 \times 64 \times 64$	$128 \times 128 \times 128$
0.6	1.65	1.86	2.33	0.6	3.33	3.40	3.38
0.36	1.62	1.90	2.43	0.36	3.10	3.21	3.83
0.2	1.68	1.93	2.59	0.2	3.07	3.31	4.20
0.12	1.84	2.25	2.65	0.12	3.25	3.73	4.64
0.046	1.68	2.05	2.63	0.046	2.88	3.59	4.67

Table 5.4 – Average number of iterations of the preconditioned BiCGStab with $\eta_1 = \eta_2 = 2, T = 1$ and a tolerance between two iterations of the Chambolle-Pock algorithm equal to 10^{-6} in normalized ℓ^2 -norm.

Concluding Remarks. In this work we have developed a first-order primal-dual algorithm for the solution of second-order, time-dependent mean field games. The procedure consists of: a variational formulation for the MFG, its discretization via finite differences, the application Chambolle-Pock algorithm to the resulting minimization. While this method has been studied for stationary MFG in [29], its numerical realization for time-dependent MFGs was prohibitive in terms of computing time, as the Chambolle-Pock iteration requires the solution of a large-scale linear system at each iteration. We have overcome this difficulty by studying different preconditioning strategies for the associated

linear system. Overall, the multigrid preconditioner with a **BiCGStab** iteration performs satisfactorily for different discretizations and viscosity values.

Appendix

(a) Unpreconditioned

$\nu \backslash N_h$	16	32	64	128
5×10^{-4}	18,5	87,6	448	1720
5×10^{-3}	22,1	98,6	392	1750
5×10^{-2}	20,7	93,4	607	8240
0.5	24,9	113	[X]	[X]

(b) michol

$\nu \backslash N_h$	16	32	64	128
5×10^{-4}	15,8	77,8	346	1390
5×10^{-3}	12,4	80,9	325	1244
5×10^{-2}	5,47	26,3	138	636
0.5	3,69	16,5	[X]	[X]

Table 5.5 – Conjugate Gradient computation times (s). (a) Unpreconditioned. (b) Preconditioned with modified incomplete Cholesky factorization. Time discretization: $N_T = 40$. [X] indicates no convergence.

(a) Unpreconditioned

$\nu \backslash N_h$	8	16	32	64
5×10^{-4}	2.42	14.2	69.0	294
5×10^{-3}	3.09	16.3	63.9	270
5×10^{-2}	1.41	12.9	61.3	389
0.5	3.41	16.5	98.9	[X]

(b) michol

$\nu \backslash N_h$	128	N_h	16	32	64	128
5×10^{-4}	12,4	10^{-4}	15.7	80.4	412	1890
5×10^{-3}	12,4	10^{-3}	12.2	82.2	369	1650
5×10^{-2}	5,47	10^{-2}	5.25	27.2	174	894
0.5	[X]	0.5	3.53	18.8	122	2120

Table 5.6 – **BiCGStab** computation times (s). (a) Unpreconditioned. (b) Preconditioned with modified incomplete Cholesky factorization. Time discretization: $N_T = 40$. [X] indicates no convergence.

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Quelques contributions à la théorie des jeux à champ moyen

Résumé

Cette thèse a pour objet d'étude la théorie des jeux à champs moyen. La majeure partie est consacrée à des jeux à champ moyen dans lesquels les joueurs peuvent interagir à travers la loi de leur état et de leur contrôle ; nous utiliserons la terminologie *jeu à champ moyen de contrôle* pour désigner de tels jeux. Dans un premier temps, nous faisons une hypothèse structure, qui consiste essentiellement à dire que la dynamique optimale dépend de la loi de contrôle de façon lipschitzienne avec une constante inférieure à un. Dans ce cas, nous prouvons plusieurs résultats d'existence de solutions au système de jeu à champ moyen de contrôle, et un résultat d'unicité en temps court. Dans un second temps, nous mettons en place un schéma numérique et faisons des simulations pour des modèles de mouvement de populations. Dans un troisième temps, nous montrons l'existence et l'unicité lorsque l'interaction par le contrôle satisfait une condition de monotonie. Le dernier chapitre concerne un algorithme de résolution numérique pour des jeux à champ moyen de type variationnel et sans interaction via la loi du contrôle ; nous utilisons une stratégie de préconditionnement par une méthode de multi-grille pour obtenir une convergence rapide.

Mots-clefs: Jeux à champ moyen, système d'équations aux dérivées partielles, simulations numériques.

Contributions to the theory of Mean Field Games

Abstract

This thesis deals with the theory of mean field games (MFG for short). The main part is dedicated to a class of games in which agents may interact through their law of states and controls; we use the terminology *mean field games of controls* (MFGC for short) to refer to this class of games. First, we assume that the optimal dynamics depends upon the law of controls in a Lipschitz way, with a Lipschitz constant smaller than one. In this case, we give several existence results on the solutions of the MFGC system, and one uniqueness result under a short-time horizon assumption. Second, we introduce a scheme and make simulations for a model of crowd motion. Third, under a monotonicity assumption on the interactions through the law of controls, we prove existence and uniqueness of the solution of the MFGC system. Finally, we introduce an algorithm for solving MFG systems of variational type, we use a preconditioned strategy based on a multigrid method.

Keywords: Mean Field Games, system of partial differential equations, numerical simulations.