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# Some problem about the control and analysis for the equations of fluid dynamics

Nicolás Molina

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**THÈSE DE DOCTORAT**  
**DE L'UNIVERSITÉ PSL**

Préparée à Université Paris Dauphine PSL

**Quelques problèmes de contrôle et d'analyse pour des équations de la dynamique des fluides**

Soutenue par

**Nicolás MOLINA**

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des Organisations,  
de la Société et de  
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Composition du jury :

Muriel BOULAKIA Maître de Conférences HDR, Sorbonne Université	<i>Rapporteur</i>
Sylvain ERVEDOZA Directeur de recherches, Institut de Mathématiques de Bordeaux	<i>Rapporteur</i>
Enrique FERNANDEZ-CARA Professeur des Universités, Universidad de Sevilla	<i>Examineur</i>
Olivier GLASS Professeur des Universités, Université Paris Dauphine PSL	<i>Directeur</i>
Céline GRANDMONT Directrice des Recherches, Centre de Recherche Inria de Paris	<i>Examineur</i>
Marius TUCSNACK Professeur des Universités, Institut de Mathématiques de Bordeaux	<i>Président du jury</i>



# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	Control theory . . . . .	4
1.1.1	Basic concepts . . . . .	4
1.1.2	Exact controllability . . . . .	7
1.1.3	Approximate controllability . . . . .	8
1.1.4	Null local controllability . . . . .	9
1.1.5	Optimal control . . . . .	9
1.1.6	Feedback stabilization problems . . . . .	10
1.1.7	Some results in control theory . . . . .	13
1.2	Some problems in fluid mechanics . . . . .	17
1.2.1	Incompressible models . . . . .	19
1.2.2	Compressible models . . . . .	22
1.3	Fluid-Structure Interaction Problems . . . . .	23
1.3.1	Rigid Solid . . . . .	24
1.3.2	Deformable solid . . . . .	25
1.3.3	Boundary conditions . . . . .	26
1.3.4	Added mass effect . . . . .	27
1.4	Results of the thesis . . . . .	30
1.4.1	Controllability of Navier-Stokes equations . . . . .	30
1.4.2	Stabilization of Navier-Stokes equations . . . . .	33
1.4.3	Cauchy problem for a fluid-structure interaction model . . . . .	36
<b>2</b>	<b>Control of Navier-Stokes equations</b>	<b>39</b>
2.1	Introduction . . . . .	39
2.2	Preliminary steps . . . . .	43
2.3	Carleman inequalities . . . . .	46
2.3.1	Construction of the weight function . . . . .	46
2.3.2	Carleman estimates for the heat equation . . . . .	47
2.3.3	Controllability of the transport equation . . . . .	50
2.4	The controllability problem (2.24) . . . . .	54
2.5	Controllability of system (2.19) . . . . .	59

2.6	Definition of the fixed point map . . . . .	62
2.7	Proof of Theorem 2.1.1 . . . . .	67
2.7.1	The map $\mathcal{F}$ in (2.109) is well-defined on $\mathcal{C}_R$ and taking values in $\mathcal{C}_R$ . . . . .	67
2.7.2	The fixed point argument . . . . .	72
<b>3</b>	<b>Stabilization of Navier-Stokes equations</b>	<b>76</b>
3.1	Introduction . . . . .	76
3.2	Linearized System around the steady state . . . . .	78
3.3	Stabilization of Transport equation . . . . .	80
3.3.1	$H^1$ estimates . . . . .	80
3.3.2	Higher Order estimates . . . . .	82
3.4	Stabilization of Transport equation with average term . . . . .	85
3.5	Stability of Parabolic equations . . . . .	86
3.6	Stabilization of the linear system . . . . .	87
3.7	Extra Regularity . . . . .	89
3.8	Non-linear system . . . . .	91
3.8.1	Estimates of the non-linear terms . . . . .	91
3.8.2	Small time existence of solutions . . . . .	92
3.8.3	Exponential decay of the complete system . . . . .	93
<b>4</b>	<b>Fluid-structure interaction</b>	<b>94</b>
4.1	Introduction . . . . .	94
4.2	Definition of solution . . . . .	97
4.3	Main result . . . . .	102
4.4	Added mass matrix . . . . .	103
4.5	Fixed Point Argument . . . . .	104
4.6	Estimates . . . . .	106
4.6.1	Estimates on the Kirchhoff potentials . . . . .	106
4.6.2	Estimates on the fluid velocity . . . . .	108
4.6.3	Spatial regularity of the flow and its inverse . . . . .	108
4.6.4	Regularity of the flow $X_{\alpha,\delta}$ and $Y_{\alpha,\delta}$ . . . . .	110
4.6.5	Regularity of $c_i(\alpha, \delta)$ . . . . .	111
4.6.6	Time derivative of the Kirchhoff potentials . . . . .	112
4.6.7	Holder estimates on the flow of the velocity . . . . .	114
4.6.8	Time derivative of the Biot-Savart Operator . . . . .	114
4.6.9	Time derivative of $\nabla^\perp \hat{\Phi}_i$ . . . . .	115
4.6.10	Estimates on the solid displacement . . . . .	116
4.6.11	Compactness . . . . .	117
4.6.12	Continuity of the fixed point map $\mathcal{G}$ . . . . .	117
4.7	Proof of the main theorem . . . . .	118

4.8 Conclusions . . . . . 120

# Chapter 1

## Introduction

### 1.1 Control theory

#### 1.1.1 Basic concepts

In control theory we are interested in the evolution of a dynamical system, in which we exert some influence. Even though dynamical systems can be of many types (difference equations, probabilistic, etc), here we are interested in systems modeled by a differential equation, possibly (and mainly) in infinite dimension. These systems can be written as

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), \\ x(0) &= x_0,\end{aligned}\tag{1.1}$$

where  $x : [0, T] \rightarrow A \subseteq X$  is the state of the system, that is,  $x(t)$  represents the state of the system at time  $t$ , taking values in a set  $A$ . The control function  $u : [0, T] \rightarrow U$  represents our influence over the system. For instance, if we consider the driving of a car,  $f$  would represent the Newton's laws that govern the car, and  $u$  would be the acceleration, direction of the steering wheel, etc. Normally the set  $U$  would represent some physical restriction over the control, for instance a maximum acceleration. The aim of control theory is to study the existence and characterization of control  $u$  satisfying certain goals, for instance attain certain state at certain time, minimizing certain cost along the trajectory, avoiding certain states, etc.

#### Some examples

In this part we present some example control systems.

**Example 1.1.1** (Spring or linearized pendulum). *As an easy concrete example we can consider the controllability of a mass attached to a spring. In*

this case the dynamic is described by

$$\ddot{x} + kx = u, \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

$$\dot{x}(0) = \dot{x}_0. \tag{1.4}$$

where  $k$  is the constant of elasticity of the spring. In this case the control  $u$  is the acceleration that we apply to the system. The controllability in this case is whether we can attain any final state  $x_f, \dot{x}_f$  in a given time  $T$ . For this system the answer to the controllability question is positive, independently on the control time  $T$ . In this simple case, this is easy to see, as we can choose any regular function  $x \in C^2([0, T])$  satisfying the conditions

$$x(0) = x_0,$$

$$\dot{x}(0) = \dot{x}_0,$$

$$x(T) = x_f,$$

$$\dot{x}(T) = \dot{x}_f.$$

This can be done for instance by considering a degree 3 polynomial and choosing the coefficients to satisfy the previous conditions, and then define the control as

$$u = \ddot{x} + kx,$$

so  $(x, u)$  satisfies the system by construction, and by uniqueness, this is **the** solution.

**Example 1.1.2** (Finite dimensional linear control system). *The fact that the previous example was controllable in any time is not an accident, this is the case more generally for linear autonomous system,*

$$\dot{x} = Ax + Bu, \tag{1.5}$$

where  $x : [0, T] \rightarrow \mathbb{R}^n$ ,  $u : [0, T] \rightarrow \mathbb{R}^m$ , and  $A, B$  are matrices of the corresponding dimensions ( $A \in \mathcal{M}_{n,n}, B \in \mathcal{M}_{n,m}$ ).

This type of system are of great importance because often for a nonlinear system we can study the local controllability properties around a equilibrium point by studying the linearized system around said point. In section 1.1.7 we will provide a general result to study the controllability of this type of systems, namely, the Kalman condition which gives sufficient and necessary conditions for the controllability of system (1.5) in terms of the matrices  $A$  and  $B$ .



Among the nonlinear control problems, an important family is the affine control systems where  $f$  is affine with respect to  $u$ :

**Example 1.1.3** (Affine system). *Let us consider a control system of the form*

$$\dot{y}(t) = f_0(y(t)) + \sum_i f_i(y(t))u_i(t), \quad (1.6)$$

where  $f_i$  are vector fields

$$f_i : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

of class  $C^\infty(\Omega)$ . If we let  $u_i \equiv 0$ , that is, the uncontrolled case, then we are only left with the term

$$f_0(y),$$

which is known as the drift of the system.

**Example 1.1.4** (Driftless affine system). *Of particular importance is the case where  $f_0 \equiv 0$  which is known as the driftless affine system*

$$\dot{y}(t) = \sum_i f_i(y(t))u_i(t), \quad (1.7)$$

which is one of the few general nonlinear problems where we have a general global controllability result.

Now we present two particular cases of the previous type of system in order to show how this type of system appears in practical problems.

**Example 1.1.5** (Baby stroller). *Let us consider a control system of a baby stroller (see for instance [14]: The stroller moves at speed  $u$  and angle  $\theta$  (that we control through the turning speed  $\omega$ ).*

$$\dot{x} = u \cos(\theta), \quad (1.8)$$

$$\dot{y} = u \sin(\theta), \quad (1.9)$$

$$\dot{z} = \omega. \quad (1.10)$$

In this case  $f_1(x) = (\cos(\theta), \sin(\theta), 0)$  and  $f_2(x) = (0, 0, 1)$ .

**Example 1.1.6** (Non holonomic integrator). *Let us consider the following control system known as the non holonomic integrator:*

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v, \\ \dot{z} = xv - yu. \end{cases} \quad (1.11)$$

In this case  $f_1(x) = (1, 0, -x)$  and  $f_2(x) = (0, 1, x)$ . we will see later that this system is globally controllable.

Finally, we show an example that does not fall in the previous categories.

**Example 1.1.7** (Zermelo control system). *As a fully nonlinear example, we consider the Zermelo control system. We consider a boat that wants to cross a river of width  $\ell$ . We describe the river bounded by the lines  $y = 0$  and  $y = \ell$ , the boat moves with constant speed  $v$  and we can control the angle  $u(t)$  of the boat. Also the current has speed  $c(y)$ .*

$$\begin{aligned}\dot{x} &= v \cos(u) + c(y), \\ \dot{y} &= v \sin(u).\end{aligned}$$

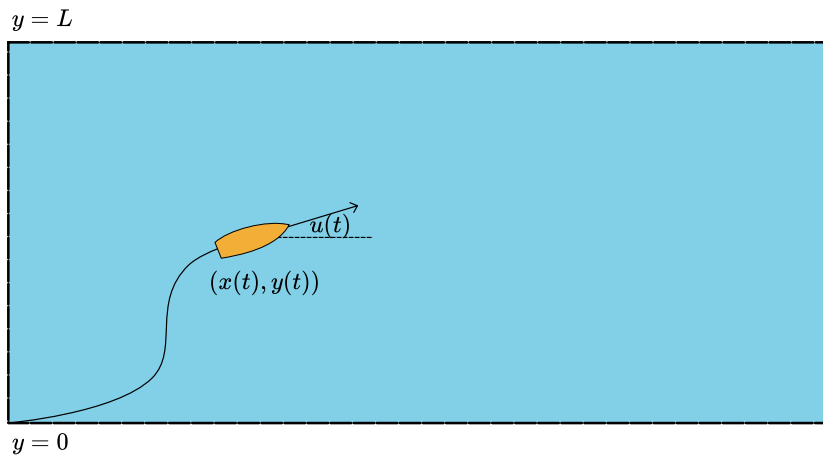


Figure 1.1: Zermelo control system

### 1.1.2 Exact controllability

The simplest control problem that we can consider is the controllability problem: Given an initial state  $x_0$ , and a target state  $x_f$ , can we find  $u$  such that the solution to (1.1) satisfies  $x(T) = x_f$ . More generally, given sets  $X_0$  of initial states, and  $X_f$  of final or target states, we say that system (1.1) is exactly controllable from  $X_0$  to  $X_f$  in time  $T$  if

$$\forall x_0 \in X_0, x_f \in X_f, \exists u : [0, T] \rightarrow U, \text{ s.t. } x_u(T) = x_f,$$

where  $x_u$  is the solution of (1.1) with  $u$  as a control. In most cases, we will ask some regularity of the control  $u$ , for instance  $u \in L^2((0, T), U)$  or  $u \in L^\infty((0, T), U)$  depending on the problem.

### 1.1.3 Approximate controllability

For some systems, we cannot expect to reach every final state (even if we restrict ourselves to an open neighborhood of a given state). In this case, we can ask for a weaker condition instead, that is, to only attain states in a dense subset. Let us see some examples.

**Example 1.1.8** (Interior controllability of the heat equation). *We consider interior controllability of the heat equation:*

$$\begin{cases} \partial_t y - \Delta y = u \chi_\omega & \text{in } \Omega, \\ y = 0 & \text{in } \partial\Omega, \end{cases} \quad (1.12)$$

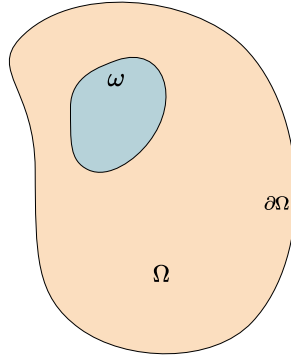


Figure 1.2: Interior control of heat equation

where  $\chi_\omega$  is the indicator function of the set  $\omega$  where  $\omega$  is a set contained in  $\Omega$ . The classical theory of regularity of the heat equation tells us that the solutions will be regular outside of  $\omega$ , so we cannot hope to control to any function in  $L^2(\Omega)$  for instance, but we can hope to attain a dense subset of  $L^2(\Omega)$ . In fact, as pointed out in [27], the appropriate idea of controllability is to control to final states that come from trajectories of (1.12) (but not necessarily the same initial condition). Thanks to the linearity of the equation this is equivalent to the null controllability described in the next section.

**Example 1.1.9** (Boundary controllability of the heat equation). *We consider boundary controllability of the heat equation:*

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega, \\ y = 0 & \text{in } \partial\Omega \setminus \Gamma, \\ y = u & \text{in } \Gamma, \end{cases} \quad (1.13)$$

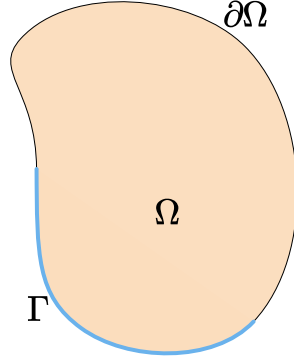


Figure 1.3: Boundary control of the heat equation

where  $\Gamma$  is a measurable subset of the boundary  $\partial\Omega$ . In this case, instead of a distributed control inside the domain  $\Omega$ , the control acts only on (a part of) the boundary of  $\Omega$ .

#### 1.1.4 Null local controllability

In particular in this thesis we will be concerned with the null controllability, that is, given a particular state  $\bar{x}$  of interest (in general 0), we are interested in knowing where we can reach that state from a set of initial states, more precisely,

$$\forall x \in \mathcal{I}, \exists u \in U, x_u(T) = \bar{x}.$$

In the particular case of local null controllability we only restrict ourselves to initial conditions close enough to the state  $\bar{x}$ , that is

$$\mathcal{I} = B(\bar{x}, \varepsilon),$$

for some  $\varepsilon > 0$  and an appropriate metric.

**Remark 1.1.1.** We should mention that the idea of local control also applies to other types of control problem, for instance we can consider local exact controllability, local approximate controllability etc. We focus on this one because it is the one that we study in the corresponding article.

#### 1.1.5 Optimal control

Another problem in control theory is the optimal control problem: In the typical case, when there is controllability, there is not one but many controls

$u$  that satisfy the control problem. This gives the possibility to choose among them the one best suited for our necessities. More concretely, we can consider a cost functional

$$\mathcal{J}(u, x) = \int_0^T f_0(x(t), u(t)) dt + g(x(T)),$$

and minimize it:

$$\min_{x_u(T)=\bar{x}} \mathcal{J}(u, x_u).$$

**Example 1.1.10.** *In the same spirit of the linear problem for controllability, an important problem in optimal control is the linear-quadratic problem:*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ \mathcal{J}(u, x) &= \int_0^T x(t)^* W x(t) + u(t)^* U u(t) dt + x(T)^* G x(T). \end{aligned} \quad (1.14)$$

For more details on the subject of optimal control see for instance [64] and [60], and for the infinite dimensional case, see [15].

### 1.1.6 Feedback stabilization problems

Another important problem in control theory is the problem of feedback stabilization. Here, instead of reaching a target equilibrium point  $\bar{x}$  exactly in a given time, we are interested in converging towards it in the limit when  $t \rightarrow \infty$ , (ideally at an exponential rate), with a control that only depends on the state at a given time, that is

$$u(t) = \mathcal{K}(x(t)),$$

where  $\mathcal{K}$  is a functional operator defined in appropriate spaces. This is equivalent to say that  $\bar{x}$  is a stable equilibrium point of the system

$$\dot{x} = f(x, \mathcal{K}(x)).$$

A generalization of this property that must be mention (although we will not study this case in this thesis), is the feedback stabilization towards a trajectory. Here, instead of taking an equilibrium point  $\bar{x}$ , we consider a whole trajectory of the system  $\tilde{x} : [0, \infty] \rightarrow A$ , and ask ourselves whether we can find a feedback function  $\mathcal{K}$  such that the solution stays close to  $\tilde{x}$ , more precisely, the solution to

$$\dot{x} = f(x, \mathcal{K}(x)).$$

should satisfy

$$d(x(t), \tilde{x}(t)) \xrightarrow[t \rightarrow 0]{} 0.$$

This kind of stabilization problem, is very important in the context of robust control, where we want to design a control not only that satisfies the objective, but also its stable to perturbation of the system. In this case,  $\tilde{x}(t)$  is the controlled trajectory, and we wish to add a feedback to stabilized to the controlled trajectory.

**Example 1.1.11** (Pendulum system). *Let us consider a pendulum of mass  $m$ , length  $\ell$  in a constant gravitational field  $g$ :*

$$\ddot{\phi} = -\frac{g}{\ell} \sin(\phi) + u. \quad (1.15)$$

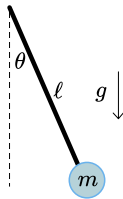


Figure 1.4: Pendulum system

We have the equilibrium trajectories  $\phi = 0$  and  $\phi = \pi$ . Classical stability analysis shows that  $0$  is a stable equilibrium but  $\pi$  is unstable. The stabilization problem then is to find a feedback law  $u = \mathcal{K}(\phi, \dot{\phi})$  to make the equilibrium  $\phi = \pi$  stable.

Now we will see two infinite dimensional stabilization problems.

**Example 1.1.12** (Stabilization of parabolic systems). *Let us consider the problem of stabilizing the heat equation with an interior feedback that is*

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega \mathcal{K}(y) & \text{in } [0, \infty) \times \Omega, \\ y(0) = y_0 & \text{in } \Omega, \\ y = 0 & \text{in } \partial\Omega, \end{cases} \quad (1.16)$$

where  $\mathcal{K}$  is a continuous (in an appropriate topology) function of the state  $y$ , and we have a similar setup to the example 1.1.9, but in this case the control takes the form  $u = \mathcal{K}(y)$ . We know that the solution to this system already decay exponentially at a rate  $\lambda_1$ , the first eigenvalue of the Laplace operator, that is

$$\|u\|_{L^2} \leq \|u_0\|_{L^2} e^{-\lambda_1 t}.$$

If we want to have exponential decay with an arbitrary decay rate  $\mu$  instead, we can use spectral decomposition. We know that there exists an orthonormal basis of  $L^2(\Omega)$ , such that

$$\begin{aligned} -\Delta e_k &= \lambda_k e_k && \text{in } \Omega, \\ e_k &= 0 && \text{in } \partial\Omega, \end{aligned}$$

where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots$$

If we express our solution as

$$y(t, x) = \sum_{k=1}^{\infty} y_k(t) e_k(x)$$

then the system (1.16) is written as the infinite system of ordinary differential equations

$$\dot{y}_k + \lambda_k y_k = \langle \chi_\Omega \mathcal{K}(y), e_k \rangle,$$

we see that for big enough  $k$ , the equation decay sufficiently fast, so we only need to make sure that the first  $N$  components decay as fast as we want. This is a linear finite dimensional control system, so we can apply the known result of this theory to find conditions under which the system is stabilizable with an arbitrary constant. For further details in this kind of technique see for instance [2].

**Example 1.1.13** (Stabilization of transport type equations). We consider now the problem of stabilizing the transport equation with periodic boundary conditions that is

$$\begin{cases} \partial_t y + c \partial_x y = \chi_\omega \mathcal{K}(y), \\ y(t, 0) = y(t, L), \\ y(0, x) = y_0(x), \end{cases} \quad (1.17)$$

in this case we can prove directly that a feedback on the form  $\mathcal{K}(y) = -\gamma y$  stabilizes the equation, thanks to the fact that we can compute the solution explicitly as

$$y(t, x) = y_0(x - ct) e^{-\gamma \int_0^t \chi_\omega(x - cs) ds},$$

where we consider the arguments modulo  $L$ . As long as  $\omega$  has positive measure, we will have

$$\int_0^t \chi_\omega(x - cs) ds \geq At - B,$$

for some positive constants  $A, B$ , hence we obtain the exponential decay

$$\|y(t)\|_L^\infty \leq C \|y_0\|_{L^2} e^{-\gamma At}.$$

### Output stabilization

In some cases, we don't have access to the whole state  $x$  but to a part  $y$  of it. More precisely,

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ y(t) = g(x(t), u(t)), \\ x(0) = x_0. \end{cases} \quad (1.18)$$

In this case we wish to find a feedback law  $u = \mathcal{K}(y)$  such that a certain equilibrium point  $\bar{x}$  becomes stable (or more generally a trajectory).

### 1.1.7 Some results in control theory

In this section we present some results about the controllability of certain problems described in the previous section, like the linear systems in infinity and finite dimensions and the driftless affine system. We start with the linear finite dimensional case.

#### Controllability of finite dimensional linear systems

Now we present some result that give us sufficient (and in some cases necessary) condition for controllability for several systems. Regarding to systems of the form (1.5), we have a general result to decide whether the system is controllable, the Kalman condition: If we repeatedly apply the matrix  $A$  to the matrix  $B$ , until  $A^{n-1}$  and put the resulting columns together in a matrix

$$K = [A^{n-1}B | A^{n-2}B | \dots | AB | B],$$

then the system (1.5) is controllable if and only if  $K$  is of full rank. The reason for considering only iterations until  $A^{n-1}$  is due to Cayley-Hamilton's theorem, which allow us to express  $A^n$  as a linear combination of the previous powers of  $A$ .



**Example 1.1.14.** *Going back to the spring System (1.2), we have*

$$A = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

*so in this case the Kalman matrix is*

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

*which is clearly invertible, hence of full rank. Thanks to the Kalman criterion, we have that the system (1.2) is controllable for any initial state to any final state in any time  $T$ .*

Now we present two result about controllability in the non linear case, one is for the case where the linearized system is controllable (in which case we have local controllability of the non linear system), and the second is the driftless affine case where we have sufficient conditions for global (and local with local controls).

### **Controlability of finite dimensional nonlinear systems**

As we mentioned earlier, one of the main tools to study the controllability of nonlinear systems, at least in the local case, is to study the linearized system, so we have the following linear test:

**Theorem 1.1.1.** *Let  $\tilde{x}(t), \tilde{u}(t)$  a trajectory of the control system*

$$\dot{x} = f(x, u). \tag{1.19}$$

*If the linearized system around  $\tilde{x}(t)$ ,*

$$\dot{x} = \partial_x f(\tilde{x}, \tilde{u})x + \partial_u f(\tilde{x}, \tilde{u})u,$$

*is controllable, then the system (1.19) is locally controllable around  $(\tilde{x}, \tilde{u})$ .*

**Example 1.1.15.** *Going back to system (1.15),*

$$\ddot{\phi} = -\frac{g}{\ell} \sin(\phi) + u. \tag{1.20}$$

*We have the equilibrium trajectory  $\phi = 0$ . The linearized system around 0 is*

$$\ddot{\phi} = -\frac{g}{\ell} \phi + u,$$

*which we saw that is controllable, hence we obtain that (1.15) is locally controllable around 0.*

Another important result for non linear systems is the Rashevski-Chow theorem for driftless affine systems. In order to introduce it we need to define the concepts of Lie bracket and Lie algebra: Given two functions  $f, g$  we define the lie bracket  $[f, g]$  as

$$[f, g](x) = \sum_i g_i(x) \partial_i f(x) - \sum_i f_i(x) \partial_i g(x).$$

The lie algebra generated by a set of functions  $\mathcal{F}$ , denoted  $\text{Lie}_x(\mathcal{F})$  is the smallest linear space that contains  $\mathcal{F}$  and is closed to the Lie bracket (this is well defined because the set of all smooth functions is closed to the Lie bracket).

**Theorem 1.1.2.** *The system (1.7) is globally controllable if*

$$\text{Lie}_x(f_1, f_2, \dots, f_m) = \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n.$$

For a proof see for instance [15].

**Example 1.1.16.** *For the system (1.8), we have that*

$$[f_1, f_2] = (\sin \theta, -\cos \theta, 2)^t,$$

hence  $f_1, f_2, [f_1, f_2]$  form an invertible matrix

$$\begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ \sin(\theta) & 0 & -\cos(\theta) \\ 0 & 1 & 2 \end{pmatrix},$$

So we have at every point  $(x, y, \theta)$

$$\text{Lie}_x(f_1, f_2) = \mathbb{R}^3.$$

From Theorem 1.1.2, we have that system (1.11) is globally exact controllable.

**Example 1.1.17.** *For the system (1.11), we have that*

$$[f_1, f_2] = (0, 0, 2),$$

hence  $f_1, f_2, [f_1, f_2]$  form an invertible matrix. So we have at every point  $x$

$$\text{Lie}_x(f_1, f_2) = \mathbb{R}^3.$$

From Theorem 1.1.2, we have that system (1.11) is globally exact controllable.

Now we see the case of infinity dimensions where the main difference is that the linear operator are not necessarily bounded.

## Controlability linear system in infinite dimensions

We are interested in the following type of systems

$$\partial_t y - Ay = Bu, \quad (1.21)$$

where in this case we do not ask  $A$  and  $B$  to be continuous (as differential operators usually are not), but we only ask to be densely defined unbounded operators. More precisely, we have

$$A : D(A) \subseteq X \rightarrow X$$

and

$$B : D(B) \subseteq U \rightarrow X$$

where  $X$  is the linear space of states of the system and  $U$  is the linear space of possible values to the control. Because of this we cannot simply define the exponential of the operator  $A$  as we would do in the finite dimensional case, instead we use semigroup theory, where the Hille-Yosida theorem tells us that if we assume that  $A$  has dense range and for every  $\lambda > 0$ ,  $I - \lambda A$  is invertible and satisfying

$$\|x - \lambda Ax\|_X \geq \|x\|_X \quad , \forall x \in D(A),$$

Then there is a function  $S : [0, \infty) \times X \rightarrow X$  with the semigroup property

$$S(t + s) = S(t) \circ S(s)$$

$$S(0) = I$$

and such that if  $x \in D(A)$ ,  $z(t) = S(t)x$  satisfies the uncontrolled differential equation, that is

$$\partial_t z(t) = Az(t), \forall t \in [0, \infty).$$

For some references on semigroup theory and in particular, the Hille-Yosida theorem, see for instance [3], [66] or [9].

In order to consider the controllability of such systems in infinite dimension, we recall the following result from functional analysis that generalizes the null-rank theorem:

**Theorem 1.1.3.** *Let  $T : X \rightarrow Y$  be a linear continuous operator, then we have the following:*

- $T$  is surjective if and only if the adjoint operator  $T^* : Y^* \rightarrow X^*$  is injective and we have the inequality for some constant  $C > 0$ :

$$\|T^*(x)\|_{X^*} \geq C\|x\|_{Y^*} \quad \forall x \in Y^*.$$

- $T$  has dense range if and only if the adjoint operator  $T^*$  is injective.

For a proof of this result see for instance [3]. To apply this to our system, we consider the input to state operator

$$\begin{aligned} T : L^2([0, T], U) &\rightarrow Y \\ u &\rightarrow y(T), \end{aligned}$$

where  $y$  is the solution to the system (1.21) with initial condition  $y(0) = 0$ .

We see that controllability from zero of the system is equivalent to the surjectivity of  $T$  and that approximate controllability is equivalent to dense range of  $T$ .

**Example 1.1.18.** *In the case of the controllability of the heat equation (1.12), the adjoint system is given by*

$$\begin{cases} -\partial_t \phi - \Delta \phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{in } \partial\Omega, \\ \phi(T) = \psi & \text{in } \Omega, \end{cases} \quad (1.22)$$

then the adjoint operator  $T^*(\psi)$  is given by  $\psi \in L^2(\Omega) \mapsto \phi(t, x)\chi(t, x) \in L^2([0, T], \omega)$ . Hence the approximate controllability is equivalent to the injectivity of  $T^*$  that is: if  $\phi = 0$  in  $[0, T] \times \omega$ , then  $\phi \equiv 0$  in all of  $[0, T] \times \Omega$ . For this system in fact we can prove null controllability which is equivalent to the inequality

$$\int_{\Omega} \phi(0, x)^2 dx \leq C \int_0^T \int_{\omega} \phi^2(t, x) dx.$$

The main tools to prove such result is the use of Carleman inequalities: the idea is to perform estimates with a carefully chosen weight function. This idea was originally introduced by Fursikov and Imanuvilov [26]. See also the article of Lebeau and Robbiano [40]

## 1.2 Some problems in fluid mechanics

Now we describe some models for the dynamics of fluid under different conditions. We assume that the fluid occupies a fixed domain  $\Omega$ . When it comes to modeling the evolution of fluids a very important and well known system of equations is the Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div}(\sigma), \end{cases} \quad (1.23)$$

where  $\rho : [0, T] \times \Omega \rightarrow \mathbb{R}$  is the density field of the fluid,  $u : [0, T] \times \Omega \rightarrow \mathbb{R}^n$  is the velocity field and  $p : [0, T] \times \Omega \rightarrow \mathbb{R}$  is the pressure field. Here  $\sigma$  is the stress vector field which expresses how the fluid particles interact with themselves. This field normally include the pressure and dissipative forces (viscosity). The first equation expresses the conservation of mass while the second one expresses the conservation of linear momentum. This equation is in conservative form (or divergence form) because expresses the change in the quantities involves in term of the div operator of some field. The second equation is usually written in terms of  $u$  rather than the momentum  $\rho u$ , i.e, in non conservative form as follows

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) = \operatorname{div}(\sigma). \end{cases} \quad (1.24)$$

In most cases we can assume that  $\sigma$  takes the following form

$$\sigma = -pI + \lambda \operatorname{div}(u)I + 2\mu D(u),$$

where  $p$  is the pressure and  $\lambda, \mu$  are known as the Lamé constant of the fluid and  $D(u)$  is the symmetrical gradient

$$D(u) = \frac{\nabla u + \nabla u^t}{2}.$$

In this case we say that the fluid is Newtonian. We restrict ourselves to Newtonian fluids in the rest of the thesis. In this case the system reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) - \lambda \Delta u - (\mu + \lambda) \nabla \operatorname{div} u = -\nabla p. \end{cases} \quad (1.25)$$

In the case where the fluid domain is no the entire space, we must provide boundary conditions saying what happens there. In the case where we have viscosity we normaly have the non slipping boudnary condition

$$u = 0 \quad \text{in } \partial\Omega,$$

in the case where the boundary is fixed, and

$$u = v \quad \text{in } \partial\Omega(t),$$

for a moving domain where  $v$  is the velocity of the boundary at the given point. On the other hand, is there is not viscosity, the fluid is allow to move

tangentially to the boundary so we have a condition only on the normal part of the velocity, known as an impermeability condition:

$$u \cdot n = 0 \quad \text{in } \partial\Omega,$$

or

$$u \cdot n = v \cdot n \quad \text{in } \partial\Omega(t),$$

in the fixed or moving case respectively. In this general case, because we have more unknowns than equations, we must supply an extra condition that relates the different fields, a constituent equation. Here we make the difference between two kinds of models, incompressible models where we assume that the fluid flow preserves volume, in which case the pressure acts as a Lagrange multiplier of this condition and compressible models, where we provide the pressure in terms of the other quantities like the density and temperature (alternatively we can use the internal energy or entropy as well). For a short mathematical introduction to fluid systems see for instance [11] For a more comprehensive mathematical study of the fluid equations see for instance [63], [45], [44] and for the physical side, [38].

### 1.2.1 Incompressible models

**Example 1.2.1** (Incompressible Navier-Stokes equations). *One important case of these equations is the incompressible case, where we assume that the flow of the fluid preserves volume locally, which is expressed in mathematical terms as*

$$\operatorname{div} u = 0,$$

*in the fluid domain. In this case we obtain the so called, incompressible Navier-Stokes equations:*

$$\begin{cases} (\partial_t u + (u \cdot \nabla)u) - \lambda \Delta u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases} \quad (1.26)$$

*We notice that in this case, we do not need a constituent equation, instead the unknown  $p$  acts as a Lagrange multiplier of the restriction  $\operatorname{div} u = 0$ . The existence for smooth solutions for these equations in the 2-dimensional case was established by Leray in [41]. See also the article of Ladyzhenskaya in [37] in the same topic. In the 3-dimensional case we only have local in time existence of strong solutions by Kato in [36] and global in time existence of some weak solutions by Leray in [42]. In this case weak solutions means in the sense of distribution with satisfying an energy inequality. The global existence of strong solutions is one of the 7 millennium problems defined by*

the Clay Mathematics Institute (of which only one has been solve to this date).

From this model, there are two limiting cases of importance:

**Example 1.2.2** (Euler System for an ideal fluid). *In the case where the viscosity effects can be neglected, that is the contribution of the term  $\lambda\Delta u$ , we obtain the Euler system:*

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{in } \partial\Omega. \end{cases} \quad (1.27)$$

*The global existence of solutions for this system was establish by Yudovich in [69] in the 2 dimensional case. See also [71] for the global existence of smooth solutions. One important quantity in the study of fluid is the vorticity defined as (for smooth enough flows)*

$$\omega(t, x) = (\operatorname{curl} u)(t, x).$$

*If we take the curl at both sides of (1.27), remembering that the curl of a gradient is always 0, we obtain*

$$\partial_t \omega + \operatorname{curl}(\omega \times u) = 0.$$

*This equation can be simplified further depending on the dimension. In dimension 2, it is equivalent to*

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

*so the vorticity is just transported by the fluid flow, in particular all the  $L^p$  norms of  $\omega$  remain constant ( $u$  is divergence free so its flow is volume preserving). This plays a key roll in the proof and will also appear in our study of the coupling of this equations with a deformable structure. In the 3 dimensional case we obtain the equation*

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = 0,$$

*instead, so  $\omega$  gets stretched by  $u$  as well as transported.*

*For the smooth solutions in dimension 3 is still an important open problem as the case of incompressible Navier-Stokes, although recently blow up of solutions in an intermediate class of Hölder continuous functions have been proved in [22].*

**Example 1.2.3** (Stokes system). *In the case where the convection term is considerably smaller than the diffusion term, we can approximate the Navier-Stokes equations by the Stokes system*

$$\begin{cases} \partial_t u - \lambda \Delta u = -\nabla p & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \quad (1.28)$$

*One key difference with respect to the previous systems is that this system is linear and can be studied in a similar way to the heat equation, but working in the space of divergence free functions. We can accomplish this in the following way: It can be prove that every function in  $L^2(\Omega)$  can be decomposed as*

$$f = \mathbb{P}(u) + \nabla q,$$

*where  $q \in L^2(\Omega)$ ,  $\mathbb{P}$  is divergence free and  $\mathbb{P}(u) \cdot n = 0$  on the boundary. This decomposition is unique up to an additive constant on  $q$ . The projection  $\mathbb{P}$  is called the Leray projection and has the following properties,*

- $$\operatorname{div} \mathbb{P}(u) = 0$$

*in the sense of distributions*

- $$\mathbb{P}(u) \cdot n = 0$$

*, in the trace sense. Even if in general we cannot define a trace operator in  $L^2(\Omega)$ , we can define the normal trace in the space of  $L^2(\Omega)$  functions with divergence in  $L^2(\Omega)$ , so due to the free divergence condition we can make sense of the trace.*

- *The Leray projector is orthogonal in  $L^2(\Omega)$  to gradient fields, that is*

$$\int_{\Omega} \mathbb{P}(u) \nabla q \, dx = 0 \quad \forall q \in H^1(\Omega)$$

*. This is just a restatement of the first condition.*

*If we apply the Leray projector to (1.28)*

$$\begin{cases} \partial_t u - \lambda \mathbb{S}u = 0 & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{in } \partial\Omega \end{cases} \quad (1.29)$$



were the stokes operator  $\mathbb{S}(u) = \mathbb{P}(\Delta u)$  ( even considering that  $\Delta u$  is divergence free, it doesn't necessarily satisfy the condition  $\Delta u \cdot n = 0$  on the boundary, so the Leray projection doesn't necessarily commute with the Laplace operator). It can be proved that  $\mathbb{S}$  is a dissipative linear (unbounded) operator in the space  $L^2_{\text{div}}(\Omega)$  so from the classical semigroup theory we obtain the existence of global solutions. For a more detailed exposition see for instance [63].

## 1.2.2 Compressible models

Without the incompressibility hypothesis we must provide an expression for the pressure in terms of other variables, typically, density and temperature. Now we see some examples of this.

**Example 1.2.4** (Compressible non-isentropic Navier-Stokes equations). *The most simple of this relations is the ideal gas law*

$$p = C\theta\rho,$$

where  $\theta$  is the temperature of the fluid. In this case, we introduce another unknown so we must provide an equation describing its evolution, in this case is the conservation of energy:

$$\rho(\partial_t\theta + (u \cdot \nabla)\theta) - k\Delta\theta = -p(\theta, \rho) \operatorname{div}(u) + \lambda \operatorname{div}(u)^2 + 2\mu D(u) : D(u),$$

where the product  $A : B$  is defined as

$$A : B = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij}.$$

So the complete system in this case is

$$\begin{cases} \partial_t\rho + \operatorname{div}(u\rho) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) = \lambda\Delta u + (\mu + \lambda)\nabla \operatorname{div} u - \nabla p, \\ \rho(\partial_t\theta + (u \cdot \nabla)\theta) - k\Delta\theta = -p \operatorname{div}(u) + \lambda \operatorname{div}(u)^2 + 2\mu D(u) : D(u). \end{cases} \quad (1.30)$$

**Example 1.2.5** (Compressible isentropic Navier-Stokes equations). *In some cases, we can assume that the pressure depends only on the density. From thermodynamics we know that this is equivalent to the dynamics being adiabatic, that is there is no heat transfer, which in turns is equivalent to the*

entropy being constant (hence the name *isentropic*). More over, in this case the pressure necessarily has the form

$$p(\rho) = C\rho^\gamma,$$

with  $C > 0$  and  $\gamma \in (1, \frac{5}{3})$  is the heat capacity ratio or adiabatic index. Then the density and velocity form a closed system as follows

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) = \lambda \Delta u + (\mu + \lambda) \nabla \operatorname{div} u - \nabla(C\rho^\gamma). \end{cases} \quad (1.31)$$

The existence of solutions for this system has been established for instance in Tani [62] or Valli [70]. Also for local existence around an stationary solution see the article of Nishida [49].

As we did in the incompressible case, we can also consider the situation without viscosity or the limit of vanishing viscosity.

**Example 1.2.6** (Compressible Euler equations). *As a limit case of the previous example, we can consider the dynamics when there is no viscosity, that is*

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) = -\nabla(C\rho^\gamma). \end{cases} \quad (1.32)$$

*In this case, we have a system of only hyperbolic equations, known as a system of conservation laws, as opposed to having a parabolic part due to the viscosity. One important feature of this type of equation is the development of shock waves. Even if the start with a very smooth initial condition, and the flows in the equation are smooth, at some finite time, the solutions can develop jump discontinuities of jumps in the derivatives known as shock waves.*

For some background on systems of conservation laws in general see for instance [58] or [11]. Also for the controllability problems related to this equations we can mention [17], [29], [30] for the one dimensional case and [54].

### 1.3 Fluid-Structure Interaction Problems

An important family of problems in continuous mechanics is the interaction between different continuous media (with different properties), in particular when one is a fluid and other is a solid. As in the case of a fluid alone, we have plenty of models to choose for the dynamics of the solid depending on

the conditions of the particular problem (rigid, moving parts, elastic, plastic, viscoelastic, etc ...).

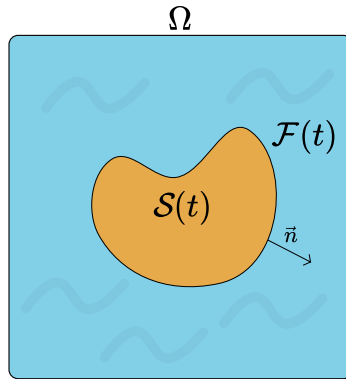


Figure 1.5: Fluid-Structure interaction problem

In the following subsection we present two cases, the rigid case, and the linear elastic case.

### 1.3.1 Rigid Solid

Now we describe as an example the case when a rigid solid is immersed in a perfect fluid.

**Example 1.3.1** (Rigid solid inside an ideal fluid). *The simplest case is the case of a rigid solid deformation. In this case we can describe the dynamics by the movement of the center of mass  $\ell(t)$  and the angular velocity  $\omega(t)$ . In this case the total force over the solid produced by the fluid pressure is given by*

$$F = - \int_{\partial S(t)} p(t, x) n(t, x) dS,$$

where  $n$  is the outer normal to the solid  $S(t)$ . Then Newtons law reads

$$m\ddot{\ell}(t) = - \int_{\partial S(t)} p(t, x) n(t, x) dS.$$

Similarly, the total torque produced by the pressure over the solid is

$$\tau = - \int_{\partial\mathcal{S}(t)} p(t, x)(x - \ell(t)) \wedge n(t, x) dS,$$

so

$$(J\omega)' = - \int_{\partial\mathcal{S}(t)} p(t, x)(x - \ell(t)) \wedge n(t, x) dS,$$

where  $J$  is the moment matrix of the solid. In the case where the fluid is viscous, the interaction with the fluid is described as the continuity of the velocity across the boundary: The total speed (normal and tangential) must be equal (no slipping condition).

$$u = \dot{\ell} + \omega \wedge (x - \ell(t)) \text{ in } \partial\mathcal{S}(t).$$

For an inviscid fluid, only the normal component must be continuous (impermeability condition):

$$u \cdot n = \left[ \dot{\ell} + \omega \wedge (x - \ell(t)) \right] \cdot n \text{ in } \partial\mathcal{S}(t).$$

For some reference on the study of the Cauchy problem for this type of system, see for instance [55],[56], [35], and for some other results [32] and [33].

### 1.3.2 Deformable solid

On the other hand, we can consider a deformable solid, which could be either a part of the boundary, for instance if we consider the flow of the blood in a blood vessel. For an example of this case, see for instance [67] where a viscous fluid is where the boundary obeys a damped Euler-Bernoulli beam equation. Even if we restrict ourselves to the case where the deformable body is at the interior of the fluid domain, still there is a variety of possible systems depending on the modeling of the structure, the fluid, and the interaction between them. We can mention for instance, in the case of a linear elastic structure inside a viscous compressible fluid the work of Desjardins et al [19]. Another example that we can mention the case of deformable bodies in an irrotational perfect fluid studied by Munnier in [52]. In this case the structures can modify their shape to attain certain goal, rather than just passively evolve from the motion of the fluid. In this thesis we will only focus on the linear elasticity equation:

$$\partial_{tt}\xi - \lambda\Delta\xi - (\mu + \lambda)\nabla \operatorname{div} \xi = 0, \quad (1.33)$$

where in this case  $\xi$  is the displacement of the solid with respect to the initial position, that is, the particle that starts at  $x$  in time  $t = 0$ , is at position  $\xi(t, x) + x$  at a given time  $t$ . We notice that this equation is similar to the wave equation

$$\partial_{tt}\phi - c^2\Delta\phi = 0, \quad (1.34)$$

except that instead of the Laplacian, we have the so called Lamé operator

$$\mathcal{L}(\phi) = \lambda\Delta\phi + (\mu + \lambda)\nabla\operatorname{div}\phi,$$

which is also an elliptic differential operator provided that  $\mu > 0$  and  $\mu + \lambda \geq 0$ . The difference is due to the fact that (1.34) only takes into account the effects of compression, whereas the linear elasticity equation (1.33) takes into account possible shear deformations.

### 1.3.3 Boundary conditions

Going back to system (1.33), we need to provide some boundary condition that describe the interaction between the structure and the fluid. In this case we generally have to provide two types of boundary conditions, the first related to the continuity of the velocity as we cross the boundary as we saw in the rigid solid case. The second one is the continuity of the stress tensor across the boundary. For a Newtonian fluid as in (1.25) the stress tensor is given by

$$\sigma_{\mathcal{F}} = -pI + \mu\frac{\nabla u + \nabla^t u}{2} + \lambda\operatorname{div}uI, \quad (1.35)$$

where  $I$  is the identity tensor. In particular for a inviscid fluid we have

$$\sigma_{\mathcal{F}} = -pI.$$

In the case of a deformable solid described by (1.33), we consider the first Piola tensor as the stress tensor. To define it, we begin by considering the flow  $X$  of the solid given by

$$X(t, x) = x + \xi(t, x),$$

which gives the position at time  $t$  that starts at  $x$  at time 0. The first Piola tensor is defined as

$$\sigma_S = \left(\mu\frac{\nabla\xi + \nabla^t\xi}{2} + \lambda\operatorname{div}\xi I\right)\frac{\nabla X}{\det(\nabla X)}.$$

If we compare this to the viscous part of (1.35), the factor  $\frac{\nabla X}{\det(\nabla X)}$  comes from the fact that we are describing the solid in Lagrangian coordinates, that is,

we follow the particles, as opposed to the case of (1.25) which describes the fluid in Eulerian coordinates, that is, in a fixed reference frame. Then the stress boundary condition is expressed as

$$\sigma_{\mathcal{F}}n = \sigma_S n.$$

where  $n$  is the normal to the solid. This means that the normal component of the stress tensor is continuous across the boundary. It is important to notice that, even though the equations for the dynamics (1.33) are linear, the boundary conditions make it a highly non linear system which poses a complication that we need to handle in the study of the equation.

An important concept in the study of this type of systems, and that becomes important in the proof of our article on this subject, is the concept of the added mass effect that we explain in the following subsection.

### 1.3.4 Added mass effect

As our experience with fluids would tell us, when we try to move a solid object inside a fluid, we must displace the fluid around it. As a consequence, the apparent mass of the object increases, which is known as the added mass effect. The reason for this is that the forces over the solid in general are related to the fluid pressure over the boundary which in turn depends on the first derivative of the fluid velocity because we generally have an equation of the form

$$\partial_t u + \dots = -\nabla p + \dots$$

Due to the impermeability boundary condition, part of  $u$  depends on the velocity of the fluid, so in turn, part of the force over the fluid depends on the acceleration of the solid. Rather than trying to explain the general situation, let us consider a simple example to explain the ideas:

**Example 1.3.2** (Added mass effect). *Let us consider a rigid solid sphere inside a fluid which is only allowed to move in the horizontal ( $x$ -axis) direction, and it is attached to a spring of elasticity constant  $k$ .*

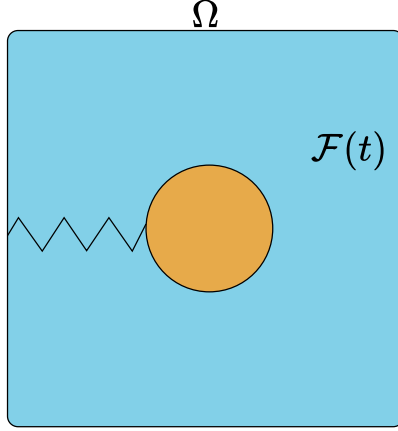


Figure 1.6: Mass attached to a spring inside a perfect fluid

Then the dynamics are given by Newton's law

$$\begin{aligned} \ddot{m}r(t) &= -kr(t) - \int_{\partial B(r(t),R)} p(t,x)n(t,x) \cdot e_1 dx, \\ &= -kr(t) - \int_{\partial B(0,R)} p(t,x+r(t)e_1)x_1 dx. \end{aligned}$$

As for the fluid, we consider the Euler equations for perfect fluids as before. Assuming that we know the vorticity  $\omega$ , we can recover  $u$  as

$$\left\{ \begin{array}{ll} \operatorname{div} u = 0 & \text{in } \mathcal{F}(t) \\ \operatorname{curl} u = \omega & \text{in } \mathcal{F}(t) \\ u \cdot n = 0 & \text{in } \partial\Omega \\ u \cdot n = \dot{r}(t)x_1 & \text{in } \partial\mathcal{S}(t) \\ \int_{\gamma} u\tau dx = \int_{\gamma} u_0\tau dx \end{array} \right. \quad (1.36)$$

In the last equation,  $\gamma$  is a curve that encloses the solid. The reason we add this condition is due to Kelvin theorem, the circulation over any closed curve inside the fluid remains constant as the flow evolves and we need this condition in order to have well-posedness of (1.36) because the fluid domain is not simply connected. This is a linear system in  $u$ , so we can decompose the solution as

$$u = \dot{r}(t)\nabla\Phi + K[\omega] + \nabla^\perp\Psi$$

where  $\Phi$  is the Kirchhoff potential satisfying the system

$$\begin{cases} \Delta\Phi = 0 & \text{in } \mathcal{F}(t), \\ \frac{\partial\Phi}{\partial n} = x_1 & \text{in } \partial\mathcal{S}(t), \\ \frac{\partial\Phi}{\partial n} = 0 & \text{in } \partial\Omega. \end{cases} \quad (1.37)$$

As for the rest of the terms,  $K[\omega]$  is called the Biot-Savart operator (By analogy to the Biot-Savart law in electromagnetism where the magnetic field satisfies a similar system with the vorticity  $\omega$  replaced by the electric current density) and satisfies the system

$$\begin{cases} \operatorname{div} K[\omega] = 0 & \text{in } \mathcal{F}(t), \\ \operatorname{curl} K[\omega] = \omega & \text{in } \mathcal{F}(t), \\ K[\omega] \cdot n = 0 & \text{in } \partial\mathcal{F}(t), \\ \int_{\gamma} K[\omega] \cdot \tau \, d\ell = 0 \end{cases} \quad (1.38)$$

and the function  $\Psi$  takes care of the circulation condition and satisfies

$$\begin{cases} \operatorname{curl} \nabla^{\perp} \Psi = 0 & \text{in } \mathcal{F}(t) \\ \Psi = 0 & \text{in } \partial\mathcal{F}(t) \\ \int_{\gamma} \nabla^{\perp} \Psi \cdot \tau \, dx = \int_{\gamma} u_0 \cdot \tau \, dx. \end{cases} \quad (1.39)$$

Here we only care about the fact that they do not depend on the velocity  $\dot{r}$  or higher derivatives at this point. Using this in the dynamics of the rigid solid,

$$\begin{aligned} m\ddot{r}(t) &= -kr(t) - \int_{\partial B(0,R)} p \frac{\partial\Phi}{\partial n} \, dx \\ &= -kr(t) + \int_{\mathcal{F}(t)} \nabla p \cdot \nabla\Phi \, dx \\ &= -kr(t) - \int_{\mathcal{F}(t)} (\partial_t u + (u \cdot \nabla)u) \cdot \nabla\Phi \, dx \\ &= -kr(t) - \int_{\mathcal{F}(t)} (\ddot{r}(t)\nabla\Phi + \dot{r}(t)\partial_t\Phi) \cdot \nabla\Phi \, dx \\ &\quad - \int_{\mathcal{F}(t)} (\partial_t K[\omega] + \partial_t \nabla^{\perp} \Psi + (u \cdot \nabla)u) \cdot \nabla\Phi \, dx. \end{aligned}$$



The important thing here is to notice that we have a second derivative on the right hand side that we have to put on the left hand side.

$$\begin{aligned} (m + \int_{\mathcal{F}(t)} |\nabla\Phi|^2 dx) \ddot{r}(t) &= -kr(t) \\ &- \int_{\mathcal{F}(t)} (\dot{r}(t) \partial_t \Phi + \partial_t K[\omega] + \partial_t \nabla^\perp \Psi + (u \cdot \nabla)u) \cdot \nabla \Phi dl. \end{aligned} \quad (1.40)$$

So we see that the system evolves with the apparent mass

$$m + \int_{\mathcal{F}(t)} |\nabla\Phi|^2 dx,$$

and the term

$$m_a(t) = \int_{\mathcal{F}(t)} |\nabla\Phi|^2 dx,$$

is known as the added mass matrix. In this one dimensional case we obtain just a scalar, but in a system with many degrees of freedom it is a matrix and in infinite dimensional systems (such as deformable solids) it is a (potentially unbounded) operator. It is important to notice that the added mass is always positive in this case. In general it is a semi positive definite matrix. The importance of identifying the added mass is of theoretical and practical importance. In numerical methods, usually it is important to have all the higher order derivatives at one side to have stability of the solutions. In the case of the theoretical study of the systems, it is important to define properly the fixed point argument and have the appropriate regularity of the solutions.

## 1.4 Results of the thesis

Now we present the main result that we prove in this thesis.

### 1.4.1 Controllability of Navier-Stokes equations

Our First results is about the controllability of the non-isentropic Navier-Stokes system

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } (0, T) \times \Omega, \\ \rho(\partial_t u + (u \cdot \nabla)u) + \nabla(p(\rho, \theta)) \\ \quad - \mu \Delta u - (\mu + \lambda) \nabla(\operatorname{div} u) = 0 \quad \text{in } (0, T) \times \Omega, \\ C_v \rho(\partial_t \theta + u \cdot \nabla \theta) - \lambda \operatorname{div}(u)^2 \\ - 2\mu D(u) : D(u) - p(\theta, \rho) \operatorname{div}(u) - \kappa \Delta \theta = 0 \quad \text{in } (0, T) \times \Omega. \end{array} \right. \quad (1.41)$$

We control the equation on the boundary, that is, for the velocity and the temperature, we prescribe the values on the boundary of the domain  $\partial\Omega$ . In the case of the density, due to the fact that satisfies a transport like equation, we are only allow to prescribe its values on a subset of the boundary, in this case, in the set where the flow enters the domain, which we can write as

$$\Gamma = \{(t, x) \in [0, T] \times \partial\Omega \mid u(t, x) \cdot n(t, x) < 0\}$$

where  $n$  is the outer normal to the fluid domain. We are interested in studying the system around constant states  $\bar{\rho} > 0$ ,  $\bar{u} \in \mathbb{R}^n$  and  $\bar{\theta} \in \mathbb{R}$ . For technical reasons, we will add need to add the condition on the velocity  $\bar{u} \neq 0$ .

**Theorem 1.4.1.** *Let  $d \in \{1, 2, 3\}$ ,  $\bar{\rho} > 0$  and  $\bar{u} \in \mathbb{R}^d \setminus \{0\}$ . Let  $L > 0$  be larger than the thickness of  $\Omega$  in the direction  $\bar{u}/|\bar{u}|$ , and assume*

$$T > L/|\bar{u}|. \quad (1.42)$$

*Then there exists  $\delta > 0$  such that for all  $(\rho_0, u_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$  satisfying*

$$\|(\rho_0, u_0, \theta_0)\|_{H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)} \leq \delta, \quad (1.43)$$

*there exists a solution  $(\rho_S, u_S, \theta_S)$  of (1.41) satisfying the initial condition*

$$\rho(0, x) = \bar{\rho} + \rho_0(x), \quad u(0, x) = \bar{u} + u_0(x), \quad \theta(0, x) = \bar{\theta} + \theta_0(x) \quad \text{in } \Omega, \quad (1.44)$$

*and the final condition*

$$\rho(T, x) = \bar{\rho}, \quad u(T, x) = \bar{u}, \quad \theta(T, x) = \bar{\theta} \quad \text{in } \Omega. \quad (1.45)$$

*Furthermore, we can choose the control so that for the controlled trajectory  $(\rho_S, u_S, \theta_S)$  one has the following regularity:*

$$\begin{aligned} \rho &\in C([0, T]; H^2(\Omega)) \\ u &\in L^2(0, T; H^3(\Omega)) \cap C^0([0, T]; H^2(\Omega)) \\ \theta &\in L^2(0, T; H^3(\Omega)) \cap C^0([0, T]; H^2(\Omega)). \end{aligned} \quad (1.46)$$

In this proof, we follow the ideas of [23], which deals with the same controllability problem in the isentropic case (i.e. where the pressure depends only on the density, hence there is no need to consider the temperature in the model). The first part of the proof consists of proving that certain linearized system is null controllable. For this we use a fixed point argument relying on Carleman inequalities, using the same inequalities as in [23] and Banach

fixed point theorem. Once we have established the null controllability of this linearized system, we perform a second fixed point argument to include the nonlinear terms, this time using Schauder fixed point theorem. In the study of the controllability of the linear system, we use as is usual, the equivalence between the controllability of the system and the observability of the adjoint system. The linear system is

$$\left\{ \begin{array}{l} \partial_t \rho + \bar{u} \cdot \nabla \rho + \bar{\rho} \operatorname{div}(u) = v_\rho \chi + \widehat{f}_\rho \quad \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u} \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u \\ \quad + \bar{p}_\rho \nabla \rho + \bar{p}_\theta \nabla \theta = v_u \chi + \widehat{f}_u \quad \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho}(\partial_t \theta + \bar{u} \cdot \nabla \theta) - \kappa \Delta \theta + \bar{p} \operatorname{div}(u) = v_\theta \chi + \widehat{f}_\theta \quad \text{in } (0, T) \times \mathbb{T}_L, \end{array} \right. \quad (1.47)$$

where  $f_\rho, f_u, f_\theta$  are the non linear terms of the equations that for the study of the linear system, we consider as a fixed function. Now the adjoint system is given by

$$\left\{ \begin{array}{l} -(\partial_t \sigma + \bar{u} \cdot \nabla \sigma) - \bar{p}_\rho \operatorname{div}(z) = g_\sigma \quad \text{in } (0, T) \times \mathbb{T}_L, \\ -\bar{\rho}(\partial_t z + (\bar{u} \cdot \nabla)z) - \mu \Delta z - (\mu + \lambda) \nabla(\operatorname{div} z) \\ \quad - \bar{\rho} \nabla \sigma - \bar{p} \nabla \eta = g_z \quad \text{in } (0, T) \times \mathbb{T}_L, \\ -C_v \bar{\rho}(\partial_t \eta + \bar{u} \cdot \nabla \eta) - \kappa \Delta \eta - \bar{p}_\theta \operatorname{div}(z) = g_\eta \quad \text{in } (0, T) \times \mathbb{T}_L. \end{array} \right. \quad (1.48)$$

where  $\sigma, z, \eta$  are the adjoint variables associated with  $\rho, u, \theta$  respectively, and  $g_\sigma, g_z, g_\eta$  are dual functions in negative sobolev spaces associated to  $f_\rho, f_u, f_\theta$ . A key point in the study of the adjoint is the use of a quantity

$$q := \operatorname{div}(z) + \frac{\bar{\rho}}{\nu} \sigma. \quad (1.49)$$

This quantity is analogous to the effective flux introduced by Lions in [45], except in this case we use it to study the adjoint system rather than the primal system. An important property of this quantity is that allow us to write a close system between  $z$  and  $\sigma$ , but with a coupling of zero order, that is, in the equation for  $z$ ,  $\sigma$  only appears without derivatives. The system written in terms of  $q$  is

$$\left\{ \begin{array}{l} -(\partial_t \sigma + \bar{u} \cdot \nabla \sigma) = \bar{p}_\rho(q - \frac{\bar{\rho}}{\nu} \sigma) + g_\sigma \quad \text{in } (0, T) \times \mathbb{T}_L, \\ -\bar{\rho}(\partial_t q + \bar{u} \cdot \nabla q) - \nu \Delta q = \bar{p} \Delta \eta \\ \quad + \frac{\bar{\rho}^2}{\nu} \bar{p}_\rho(q - \frac{\bar{\rho}}{\nu} \sigma) + \operatorname{div}(g_z) + \frac{\bar{\rho}^2}{\nu} g_\sigma \quad \text{in } (0, T) \times \mathbb{T}_L, \\ -C_v \bar{\rho}(\partial_t \eta + \bar{u} \cdot \nabla \eta) - \kappa \Delta \eta = \bar{p}_\theta(q - \frac{\bar{\rho}}{\nu} \sigma) + g_\eta \quad \text{in } (0, T) \times \mathbb{T}_L. \end{array} \right. \quad (1.50)$$

In comparison with (1.41), where we have the gradient of the density in the equation for  $u$ , in (2.23) we have only a zeroth order coupling. This difference allow us to get better estimates to perform the fixed point argument. The idea is that the weaker coupling allow us to use the controllability of the diagonal system (that is the system without the coupling) to obtain the controllability of the complete system choosing the parameters of the Carleman weights appropriately. The diagonal system consists then on two coupled parabolic equations ( $q$  and  $\theta$ ) which we can control using the Carleman weight defined in [23]. Two important features of this weights are that, unlike most Carleman weights for parabolic equations, this does not blow up at the final time  $T$ . The other important feature is that the weight is transported in some same by the constant speed  $\bar{u}$ .

Then, for the density equation we perform explicit estimates (in this case we have a transport equation in which we can work with the solutions directly) in order to obtain the appropriate Carleman estimates with the same weight.

### Open problems

An interesting generalization of this result would be to study what happens around solutions of (1.41) other than the non zero constants, in particular, what happens around zero solutions and around evolving solutions. On the other hand we could study the same problem coupling the non isentropic viscous fluid equations with other equations, for instance, with Maxwell equations to study Magnetohydrodynamics (MHD) control problems or with Newton's Gravitational laws to study The effects of gravity on the system.

### 1.4.2 Stabilization of Navier-Stokes equations

Our second result is about the stabilization by state feedback for the isentropic Navier-Stokes equations with a feedback only on the density equation

$$\begin{cases} \partial_t \rho_S + \operatorname{div}(\rho_S u_S) = K_\rho(\rho_S - \bar{\rho}, u_S - \bar{u}, \sigma) & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \rho_S(\partial_t u_S + (u_S \cdot \nabla)u_S) + \nabla(p(\rho_S)) \\ - \mu \Delta u_S - (\mu + \lambda) \nabla(\operatorname{div} u_S) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \end{cases} \quad (1.51)$$

Where again  $\bar{u}$  and  $\bar{\rho} > 0$  are constant velocity and density and we are interested in the evolution of the system around this constant states. The control  $K_\rho$  is the state feedback supported in the control set  $\omega \subseteq \mathbb{T}^d$  and depending on a one dimensional integrator  $\sigma$  depending on  $\rho_S$  and  $u_S$ . We

have the following assumptions on the geometry of the control domain  $\omega$  and on the parameters of the equation:

**Assumption 1** We assume the pressure function to be at least  $C^{2, \frac{1}{2}}$  on a neighborhood of  $\bar{\rho}$ . We will assume that all the characteristics of the flow given by  $\bar{u}$  intersect  $\omega'$ , in particular, the interesting case is when  $\bar{u} \neq 0$  otherwise we would require  $\omega = \mathbb{T}^d$ . Then we can choose  $\omega$  with the same property and  $\bar{\omega} \subset \omega'$ .

**Assumption 2** We also will assume that the quantity

$$\frac{\bar{\rho}\bar{p}_\rho}{\nu} > 0$$

is small enough. More specifically, we assume that it is smaller than the first nonzero eigenvalue of laplacian in the torus.

$$0 < \frac{\bar{\rho}\bar{p}_\rho}{\nu} < \lambda_1$$

In order to quantify the exponential decay, we consider  $M$  such that,

$$0 < M < \lambda_1 - \frac{\bar{p}_\rho\bar{\rho}}{\nu^2}. \quad (1.52)$$

where  $\lambda_1$  is the first non zero eigenvalue of the Laplacian on the torus and  $\bar{p}_\rho = p'(\bar{\rho})$ ,  $\gamma > 0$  is a constant related to the decay of the transport equation. Then, we have the following result:

**Theorem 1.4.2.** *Suppose that Assumption 1 and Assumption 2 are satisfied and that  $M, \bar{p}_\rho$  and  $\nu$  satisfy (1.52), and take any  $\epsilon \in (0, \frac{1}{2})$ . Then there are constants  $\delta > 0$  and  $C > 0$  such that if  $(\rho_0, u_0) \in H^{\frac{3}{2}+2\epsilon} \times H^{\frac{5}{2}+2\epsilon}$  and  $\|(\rho_0, u_0)\|_{H^{\frac{3}{2}+2\epsilon} \times H^{\frac{5}{2}+2\epsilon}} \leq \delta$ , then the solution of (1.51) satisfies the following exponential decay estimate*

$$\|(e^{Mt}\rho, e^{Mt}(u - \langle u \rangle), e^{Mt}\sigma)\|_{L^\infty(0, T; H^{\frac{3}{2}+2\epsilon} \times H^{\frac{5}{2}+2\epsilon} \times \mathbb{R})} \leq C \|(\rho_0, u_0)\|_{H^{\frac{3}{2}+2\epsilon} \times H^{\frac{5}{2}+2\epsilon}}.$$

The basic idea of the proof is similar to the previous result. We try to study the stabilization of a linearized system, and then we use the stabilization of this system to prove the stabilization of the full non linear problem. In the study of the linear system, for the density equation we needed to include a one dimensional integrator in order to stabilize the average value of the velocity. This need of an integrator is not unusual in the context of stabilization of this type of systems as explained by Coron in [18]. To put this

result in perspective, we can mention the work of [12] for the stabilizability (and controllability) of the Navier-Stokes System in the isentropic case with control on the velocity, [50] for a study of the same system in a more regular functional setting and [13] for a study of stabilizability on the zero velocity case. The feedback of the system takes the following form

$$K_\rho(\rho, \hat{u}, \sigma) = -\gamma\chi_\omega\rho + \sigma(\bar{u} + \hat{u}) \cdot \nabla\varphi + \sigma\frac{\bar{\rho}\bar{p}_\rho}{\nu}\varphi + \dot{\sigma}\varphi. \quad (1.53)$$

where  $\sigma$  is a one dimensional integrator satisfying

$$\dot{\sigma} = \sigma\langle \operatorname{div} u_S\varphi \rangle - \sigma\frac{\bar{\rho}\bar{p}_\rho}{\nu}\langle \operatorname{div} u_S\rho \rangle + \langle \gamma\chi_\omega(\rho_S - \bar{\rho}) \rangle - \gamma\langle \rho_S - \bar{\rho} \rangle,$$

and the function  $\varphi$  is a function supported on the control zone that has the property

$$\operatorname{supp} \varphi \cap \operatorname{supp} \omega' = \emptyset$$

and where  $\omega'$  is an open set contained in  $\omega$ . To construct such feedback law we begin by the simple feedback law of the transport equation

$$K_{\rho,1} = -\gamma\chi_\omega(\rho),$$

which suffices to stabilize the transport equation but we still need to deal with the average part of the velocity which does not decay in the torus (0 is a multiple eigenvalue of the Laplacian and Lamé operators). Hence we modify the feedback in order to stabilize this average term. When we do this, we obtain a system of the form

$$\partial_t\rho + (\bar{u} + \hat{u}) \cdot \nabla\rho - \bar{\rho}\frac{\bar{p}_\rho}{\nu}\langle \rho \rangle + \frac{\bar{\rho}\bar{p}_\rho}{\nu}\rho = K_\rho(\rho, \hat{u}, \sigma) + f. \quad (1.54)$$

where  $\langle \rho \rangle$  is the average of  $\rho$  over the torus. Hence this is a non local PDE. In order to stabilize this equation we define

$$\tilde{\rho} = \rho - \sigma\varphi.$$

and we study it as an stabilization problem in  $\tilde{\rho}$  and  $\langle \rho \rangle$ .

## Open Problems

We still do not know about stabilization with an arbitrary decay rate. Also, of course, it would be of interest to lift the small parameter restriction. Even if we consider (localized) controls on all the equations, it is still open to find a state feedback in Eulerian coordinates that stabilizes the system at an arbitrary decay rate. Yet another direction in which we could generalize the result is to stabilize around other type of solutions, not only non-zero constants, and in particular, around the zero velocity case.

### 1.4.3 Cauchy problem for a fluid-structure interaction model

Finally, we have a result about the existence of a Cauchy problem for a fluid-structure system (see figure 1.3). It consists of a deformable body following the system (1.33), but for a finite number of modes.

$$\left\{ \begin{array}{ll} \rho_S(0)\partial_{tt}\xi = \operatorname{div}(\pi(\xi)) = \mu\Delta\xi + (\lambda + \mu)\nabla\operatorname{div}\xi & \text{in } ]0, T] \times \mathcal{S}(0) \\ \partial_t u + (u \cdot \nabla)u = -\nabla p & \text{in } [0, T] \times \mathcal{F}(t) \\ \operatorname{div} u = 0 & \text{in } [0, T] \times \mathcal{S}(t) \\ u(t, x) \cdot n(t, x) = 0 & \text{in } [0, T] \times \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t) \\ (u(t, x) - \partial_t\xi(t, X^{-1}(t, x))) \cdot n(t, x) = 0 & \text{in } [0, T] \times \partial\mathcal{S}(t) \cap \partial\mathcal{F}(t) \\ \left( (\pi(\xi)(t, X^{-1}(t, x))) \frac{\nabla X(t, X^{-1}(t, x))}{\det(\nabla X(t, X^{-1}(t, x)))} + pI \right) \cdot n = 0 & \text{in } [0, T] \times \partial\mathcal{S}(t) \end{array} \right. \quad (1.55)$$

where as we said earlier,  $\xi$  represent the displacement of the solid with respect to the initial position (Lagrangian coordinates),  $X(t, x) = x + \xi(t, x)$  is the flow of the elastic solid,  $\mu$  and  $\lambda$  are the Lamé constant that describe the elasticity of the body. On the other hand  $u$  is the velocity of the fluid and  $p$  it is pressure. We add also the incompressibility condition and an impermeability condition with the wall of the fluid.

In order to couple the fluid and the elastic solid, we have two boundary conditions, one the impermeability with the solid, expressed as the continuity of the normal component of the velocity, as well as the continuity of the stress tensor that for the fluid is given by the pressure and for the elastic solid by the Piola tensor  $\pi(\xi)$  together with factor because of the mixture of Lagrangian and Eulerian coordinates. With a finite modes solution we mean that we express the displacement  $\xi$  as a finite sum of some time independent functions:

$$\xi(t, x) = \sum_{i=1}^{N+1} \alpha_i(t) \eta_i(x).$$

In this case we take  $\eta_i$  as the first eigenvalues of the Lamé operator with homogeneous Neumann boundary conditions. The reason we take  $N + 1$  modes is to take care of the incompressibility of the fluid. Because the fluid preserves the volume, and the total volume is also preserved, this forces the volume of the solid to remain constant as well, so we choose its coefficient

$\alpha_{N+1}(t) = \delta(t)$  where

$$\begin{aligned}\dot{\delta}(t) &= - \sum_i^N c_i(\alpha_j(t), \delta(t)) \dot{\alpha}_i(t) \\ c_i(\alpha, \delta) &= \frac{\int_{\partial S(\alpha, \delta)} \eta_i \cdot \hat{n}_{(\alpha, \delta)} dS}{\int_{\partial S(\alpha, \delta)} \eta_{N+1} \cdot \hat{n}_{(\alpha, \delta)} dS}.\end{aligned}$$

This ensures that  $\xi$  preserves volume. So we ask for the displacement to satisfy the linear lame equation on the space span by this eigenvalues, that is

$$\begin{aligned}\sum_j \int_{S(0)} \ddot{\alpha}_i(\eta_i - c_i \eta_{N+1})(\eta_j - c_j \eta_{N+1}) - \sum_j \int_{S(0)} \dot{\alpha}_i \dot{c}_i \eta_{N+1}(\eta_j - c_j \eta_{N+1}) \\ - \alpha_i \lambda_i - \lambda_{N+1} \delta c \\ = \int_{\partial S(t)} np(t, x)(\eta_i(X^{-1}(t, x)) - c_i(t) \eta_{N+1}(X^{-1}(t, x))) dS.\end{aligned}\quad (1.56)$$

for every  $i \in \{1, \dots, N\}$ . In order to explain our result we need to introduce the following functional space: For a set  $\Omega$  we denote by  $\mathcal{LL}(\Omega)$  the set of log-Lipschitz functions, that is, functions  $f : \Omega \rightarrow \mathbb{R}^n$  such that

$$\|f\|_{\mathcal{LL}} = \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{1 + \log(|x - y|)^+} < \infty$$

The importance of this space is that we can still ensure the uniqueness of the flow of  $f$  for a log-Lipschitz function. Our result is the following:

**Theorem 1.4.3.** *If the boundary of  $\mathcal{F}(0)$  is of class  $C^{2, \alpha}$  and  $\omega_0 \in L^\infty(\mathcal{F}(0))$ ,  $u_0 \in \mathcal{LL}(\mathcal{F}(0))$  and  $\int_{\partial S(0)} \eta_{N+1}(x) \cdot \hat{n} dS \neq 0$ , then there is  $T > 0$  such that there is a finite modes solution to the system (1.56) satisfying*

$$\alpha_i \in C^2([0, T]),$$

$$u \in L^\infty([0, T], \mathcal{LL}(\mathcal{F}(t))),$$

and

$$\omega \in L^\infty([0, T], L^\infty(\mathcal{F}(t))).$$

Moreover  $T$  satisfies

$$T > T_0(\|\alpha_0\|, \|X^{-1}\|, \int_{\partial S(0)} \eta_{N+1}(x) \cdot \hat{n} dS)$$

that is, the minimal time only depends uniformly on  $\alpha_0, X^{-1}$  and  $\gamma = \int_{\partial S(0)} \eta_{N+1}(x) \cdot \hat{n} dS$ .



Also the added mass effect matrix discussed in 1.3.4 plays an important role in the proof. The idea of the proof is to perform a fixed point argument as well: Given a possible displacement

$$\xi(t, x)$$

for the solid body, we can find the solution of the fluid around it considering the movement of the solid as prescribed, obtaining

$$\xi \rightarrow u$$

as the solution to the Euler equation with said movement. Then, from the fluid solution, we can compute the action over the solid to compute a new displacement. In general this displacement will not be equal to the initial displacement, but if we manage to prove that this map has a fixed point, we find a solution to our system.

$$\xi \rightarrow u \rightarrow \hat{\xi} = \xi$$

As it is common in fluid-structure problems, in order to study the system we need to introduce an added mass matrix. The reason for this is that in the equation for the displacement we have the pressure  $p$  at the right hand side.

### **Open Problems**

The obvious open problem that we could not tackle is what happens in the case for the complete linear elasticity equation, i.e. without considering the finite number of modes approximation. In top of that, we can consider what happens when we consider other equations for the deformable solid, after all, the linear elasticity equation is just an approximation of more general non linear equations.

# Chapter 2

## Local exact boundary controllability for the compressible Navier-Stokes equations

### 2.1 Introduction

We will consider the problem of controlling a viscous and compressible fluid contained in a bounded domain  $\Omega \subseteq \mathbb{R}^d$ , where the dimension  $d \in \{1, 2, 3\}$  and the control will act on the whole boundary, that is, we prescribe the velocity and the temperature of the fluid passing through the boundary, and the density of the incoming fluid. The fluid is modeled here by the compressible Navier-Stokes equations with temperature, and we assume that the pressure is a function of the density and the temperature. Because of this, we need to consider also the equation for the energy. More concretely, we consider the following:

$$\left\{ \begin{array}{ll} \partial_t \rho_S + \operatorname{div}(\rho_S u_S) = 0 & \text{in } (0, T) \times \Omega \\ \rho_S (\partial_t u_S + (u_S \cdot \nabla) u_S) = -\nabla(p(\rho_S, \theta_S)) \\ \quad + \mu \Delta u_S + (\mu + \lambda) \nabla(\operatorname{div} u_S) & \text{in } (0, T) \times \Omega \\ C_v \rho_S (\partial_t \theta_S + u_S \cdot \nabla \theta_S) - \kappa \Delta \theta_S = p(\theta_S, \rho_S) \operatorname{div}(u_S) \\ \quad + \lambda \operatorname{div}(u_S)^2 + 2\mu D(u_S) : D(u_S) & \text{in } (0, T) \times \Omega, \end{array} \right. \quad (2.1)$$

where  $\rho_S$  is the density,  $u_S$  the velocity,  $\theta_S$  is the temperature, and  $p$  is the pressure which is a function of  $\rho_S$  and  $\theta_S$  obeying, for instance, the ideal gas

law

$$p(\theta, \rho) = R\theta\rho. \quad (2.2)$$

Here, for square matrices  $A$  and  $B$ ,  $A : B$  denotes the Frobenius product defined by

$$A : B := \text{tr}(A^t B),$$

and for any vector field  $f$ ,  $D(f)$  is the symmetric gradient given by the formula

$$D(f)_{ij} = \frac{1}{2} \left( \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \right).$$

The parameters  $\mu$  and  $\lambda$  correspond to constant viscosity parameters and are assumed to satisfy  $\mu > 0$  and  $d\lambda + 2\mu \geq 0$  (the only condition required for our result is  $\mu > 0$  and  $\nu = \lambda + 2\mu > 0$ ). The constant  $C_v$  is the heat capacity which we assume  $C_v > 0$ .

We will consider the local exact controllability around constant states  $(\bar{\rho}, \bar{u}, \bar{\theta}) \in \mathbb{R}_+^* \times (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}$ . The controls applied to the system (2.1) in velocity and temperature are the values of  $u_S$  and  $\theta_S$  on the whole boundary

$$\theta|_{\partial\Omega}(t, x) = f_\theta(t, x), \quad (2.3)$$

$$u|_{\partial\Omega}(t, x) = f_u(t, x). \quad (2.4)$$

For what concerns the density we control only on the part of the region where the flow is inward  $\Gamma = \{(t, x) \in [0, T] \times \partial\Omega \mid u(t, x) \cdot n(x) < 0\}$  where  $\vec{n}$  is the exterior unit normal to  $\partial\Omega$

$$\rho|_\Gamma(t, x) = f_\rho(t, x). \quad (2.5)$$

Here the local exact controllability means that for every initial conditions  $(\rho_0, u_0, \theta_0)$  close enough to  $(\bar{\rho}, \bar{u}, \bar{\theta})$  (in the  $H^2$  norm in this case) there are controls such that the solution to (2.1) reaches the state  $(\bar{\rho}, \bar{u}, \bar{\theta})$  in certain given final time  $T > 0$ . Because we are controlling in the whole boundary, instead of considering the control explicitly, we may also consider (2.1) as an under-determined system (without boundary condition) and we have to find a trajectory satisfying certain initial and final conditions.

To state our main result we make the following definition, see [23]: Given  $e$  an unit vector in the sphere  $\mathbb{S}^{d-1}$  of  $\mathbb{R}^d$ , we define the *thickness* of a nonempty set  $A \subset \mathbb{R}^d$  in the direction  $e$  as follows

$$\sup \{s \geq 0 \mid \exists x \in A, x + se \in A\}.$$

The principal result of this article is the following theorem.

**Theorem 2.1.1.** *Let  $d \in \{1, 2, 3\}$ ,  $\bar{\rho} > 0$  and  $\bar{u} \in \mathbb{R}^d \setminus \{0\}$ . Let  $L > 0$  be larger than the thickness of  $\Omega$  in the direction  $\bar{u}/|\bar{u}|$ , and assume*

$$T > L/|\bar{u}|. \quad (2.6)$$

*Then there exists  $\delta > 0$  such that for all  $(\rho_0, u_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$  satisfying*

$$\|(\rho_0, u_0, \theta_0)\|_{H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)} \leq \delta, \quad (2.7)$$

*there exists a solution  $(\rho_S, u_S, \theta_S)$  of (2.1) satisfying the initial condition*

$$\rho_S(0, x) = \bar{\rho} + \rho_0(x), \quad u_S(0, x) = \bar{u} + u_0(x), \quad \theta_S(0, x) = \bar{\theta} + \theta_0(x) \quad \text{in } \Omega, \quad (2.8)$$

*and the final condition*

$$\rho_S(T, x) = \bar{\rho}, \quad u_S(T, x) = \bar{u}, \quad \theta_S(T, x) = \bar{\theta} \quad \text{in } \Omega. \quad (2.9)$$

*Furthermore, we can choose the control so that for the controlled trajectory  $(\rho_S, u_S, \theta_S)$  one has the following regularity:*

$$\begin{aligned} \rho_S &\in C([0, T]; H^2(\Omega)) \\ u_S &\in L^2(0, T; H^3(\Omega)) \cap C^0([0, T]; H^2(\Omega)) \\ \theta_S &\in L^2(0, T; H^3(\Omega)) \cap C^0([0, T]; H^2(\Omega)). \end{aligned} \quad (2.10)$$

Theorem 2.1.1 extends the results in [23] and [24] to the non isentropic case, i.e. when the pressure also depends on the temperature. As in the constant temperature case, our result proves local controllability to constant states having non-zero velocity. This restriction appears explicitly in the condition (2.6). As expected, this condition is remnant from the transport equation satisfied by the density which allows the information to travel at a velocity (close to)  $\bar{u}$ . Recently [47] studied the controllability of the linearized compressible Navier-Stokes equations in the case where  $\bar{u} = 0$  where it was proved that in the one dimensional case, the system is not null-controllable in the space  $L^2 \times L^2 \times L^2$  if we don't control the density on the whole domain but the system is null-controllable if we restrict the density to the  $H^1$  function with zero mean (and in fact, in this case there is no need of a control on the density) if we control the speed and the temperature on the whole domain (and controlling the velocity on the whole domain is a necessary condition).

This non-zero velocity condition for the controllability of compressible Navier-Stokes equations has been studied in [68, 12] focusing on the linearized equations in the case of zero-velocity. Concerning the closely related problem of stabilization of compressible Navier-Stokes equations, we can mention

the work of Chowdhury, Ramswamy, Raymond [12] and Chowdhury, Maity, Ramaswamy, Raymond [13].

Concerning the incompressible case of the Navier-Stokes equations, Fursikov and Imanuvilov [26] provide a local exact controllability result. On the other hand, for Euler equations, global exact controllability has been proved in dimension 2 by Coron [16], and in dimension 3 by Glass [31].

Our strategy will be to treat (2.1) as a coupling of parabolic and transport equations, hence we will adapt the techniques used in [23] to this case, which uses Carleman estimates in negative Sobolev spaces to prove the controllability of the parabolic and hyperbolic equations.

**Remark 2.1.1.** *In dimension  $d = 1$  the required regularity on the initial can be relaxed as follows*

**Theorem 2.1.2.** *Let  $d = 1$ ,  $\bar{\rho} > 0$  and  $\bar{u} \in \mathbb{R}^d \setminus \{0\}$ . Let  $L > 0$  be larger than the thickness of  $\Omega$  in the direction  $\bar{u}/|\bar{u}|$ , and assume*

$$T > L/|\bar{u}|. \quad (2.11)$$

*Then there exists  $\delta > 0$  such that for all  $(\rho_0, u_0, \theta_0) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$  satisfying*

$$\|(\rho_0, u_0, \theta_0)\|_{H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)} \leq \delta, \quad (2.12)$$

*there exists a solution  $(\rho_S, u_S, \theta_S)$  of (2.1) satisfying the initial condition*

$$\rho_S(0, x) = \bar{\rho} + \rho_0(x), \quad u_S(0, x) = \bar{u} + u_0(x), \quad \theta_S(0, x) = \bar{\theta} + \theta_0(x) \quad \text{in } \Omega, \quad (2.13)$$

*and the final condition*

$$\rho_S(T, x) = \bar{\rho}, \quad u_S(T, x) = \bar{u}, \quad \theta_S(T, x) = \bar{\theta} \quad \text{in } \Omega. \quad (2.14)$$

*Furthermore, we can choose the control so that for the controlled trajectory  $(\rho_S, u_S, \theta_S)$  one has the following regularity:*

$$\begin{aligned} \rho_S &\in C([0, T]; H^1(\Omega)) \\ u_S &\in L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \\ \theta_S &\in L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)). \end{aligned} \quad (2.15)$$

*For sake of simplicity, we will focus on the proof in the case  $d \in \{2, 3\}$ . The proof in the case  $d = 1$  can be easily deduced from the general case. In this regard, the regularity obtained is the same as in the article of Ervedoza and Savel [25], where they prove controllability to smooth trajectories for the isentropic Navier-Stokes equations.*

*Outline.* This paper is organized as follows. Section 2.3 introduces the Carleman estimates that we will need to prove controllability. Sections 2.4 and 2.5 study the controllability of the linearized problem. Section 2.6 then explains how to perform a fixed point argument using the controllability results developed before. Finally, in Section 2.7 we present the proof of Theorem 2.1.1.

## 2.2 Preliminary steps

First, as in [23], we reduce the boundary control problem to an interior control problem in the torus. The domain  $\Omega$  can be embedded into some torus  $\mathbb{T}_L$ , where  $\mathbb{T}_L$  is identified with  $[0, L]^d$  with periodic conditions. In order to construct the Carleman weight, we need to consider the length  $L$  large enough and we may consider the control problem in the cube  $\mathbb{T}_L$  with controls appearing as source terms supported in  $\mathbb{T}_L \setminus \bar{\Omega}$ . The new control system is the following:

$$\begin{cases} \partial_t \rho_S + \operatorname{div}(\rho_S u_S) = \check{v}_\rho & \text{in } (0, T) \times \mathbb{T}_L \\ \rho_S(\partial_t u_S + (u_S \cdot \nabla)u_S) + \nabla(p(\rho_S, \theta_S)) \\ \quad - \mu \Delta u_S - (\mu + \lambda) \nabla \operatorname{div} u_S = \check{v}_u & \text{in } (0, T) \times \mathbb{T}_L \\ C_v \rho_S(\partial_t \theta_S + u_S \cdot \nabla \theta_S) - \lambda \operatorname{div}(u_S)^2 - 2\mu D(u_S) : \nabla u_S \\ \quad - p(\theta_S, \rho_S) \operatorname{div}(u_S) - \kappa \Delta \theta_S = \check{v}_\theta & \text{in } (0, T) \times \mathbb{T}_L \end{cases} \quad (2.16)$$

with  $\check{v}_\rho$ ,  $\check{v}_u$  and  $\check{v}_\theta$  the control functions which are supported in  $[0, T] \times (\mathbb{T}_L \setminus \bar{\Omega})$ . Because we intend to study the linearized system around  $(\bar{\rho}, \bar{u}, \bar{\theta})$ , we consider the following translation in order to consider the system around  $(0, 0, 0)$ ,

$$\check{\rho} := \rho_S - \bar{\rho}, \quad \check{u} := u_S - \bar{u}, \quad \check{\theta} := \theta_S - \bar{\theta}.$$

Accordingly, we extend the initial conditions  $(\rho_0, u_0, \theta_0)$  to the torus  $\mathbb{T}_L$  in such a way that

$$\|(\check{\rho}_0, \check{u}_0, \check{\theta}_0)\|_{H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)} \leq C_L \|(\rho_0, u_0, \theta_0)\|_{H^2(\Omega) \times H^2(\Omega) \times H^2(\mathbb{T}_L)}. \quad (2.17)$$

We introduce the following function to make explicit the support of the controls  $\check{v}_\rho$ ,  $\check{v}_u$  and  $\check{v}_\theta$ . We take  $\chi \in C^\infty(\mathbb{T}_L; [0, 1])$  satisfying

$$\begin{cases} \chi(x) = 0 & \text{for all } x \text{ such that } d(x, \Omega) \leq \varepsilon, \\ \chi(x) = 1 & \text{for all } x \text{ such that } d(x, \Omega) \geq 2\varepsilon, \end{cases} \quad (2.18)$$

where  $\varepsilon > 0$  is small enough. How small will be explained in section 2.3.1 when we introduce the weights for the Carleman estimates. We will look for

$\check{v}_\rho$ ,  $\check{v}_u$  and  $\check{v}_\theta$  in the form

$$\check{v}_\rho = v_\rho \chi, \quad \check{v}_u = v_u \chi \text{ and } \check{v}_\theta = v_\theta \chi.$$

We will start by studying the linearized system around  $(\bar{\rho}, \bar{u}, \bar{\theta})$  which is

$$\begin{cases} \partial_t \rho + \bar{u} \cdot \nabla \rho + \bar{\rho} \operatorname{div}(u) = v_\rho \chi + \hat{f}_\rho & \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u} \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u \\ \quad + \bar{p}_\rho \nabla \rho + \bar{p}_\theta \nabla \theta = v_u \chi + \hat{f}_u & \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho}(\partial_t \theta + \bar{u} \cdot \nabla \theta) - \kappa \Delta \theta + \bar{p} \operatorname{div}(u) = v_\theta \chi + \hat{f}_\theta & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (2.19)$$

where

$$\bar{p} = p(\bar{\rho}, \bar{\theta}), \quad \bar{p}_\rho = \frac{\partial p}{\partial \rho}(\bar{\rho}, \bar{\theta}), \quad \bar{p}_\theta = \frac{\partial p}{\partial \theta}(\bar{\rho}, \bar{\theta}).$$

We will use the equivalence between the controllability of a linear system and the observability property for the adjoint system. In this case, the adjoint system of (2.19) is given by

$$\begin{cases} -(\partial_t \sigma + \bar{u} \cdot \nabla \sigma) - \bar{p}_\rho \operatorname{div}(z) = g_\sigma & \text{in } (0, T) \times \mathbb{T}_L, \\ -\bar{\rho}(\partial_t z + (\bar{u} \cdot \nabla)z) - \mu \Delta z - (\mu + \lambda) \nabla(\operatorname{div} z) - \bar{\rho} \nabla \sigma - \bar{p} \nabla \eta = g_z & \text{in } (0, T) \times \mathbb{T}_L, \\ -C_v \bar{\rho}(\partial_t \eta + \bar{u} \cdot \nabla \eta) - \kappa \Delta \eta - \bar{p}_\theta \operatorname{div}(z) = g_\eta & \text{in } (0, T) \times \mathbb{T}_L. \end{cases} \quad (2.20)$$

We notice that ignoring the coupling between the equations,  $\sigma$  satisfies a transport equation and  $z$  and  $\eta$  satisfy a parabolic equation. The main idea now is to use the controllability of those indual equations inside a fixed point scheme, but first we need to reduce to a system in which the coupling does not involve derivatives. If we take the divergence of the equation of  $z$ , we obtain the following system.

$$\begin{cases} -(\partial_t \sigma + \bar{u} \cdot \nabla \sigma) - \bar{p}_\rho \operatorname{div}(z) = g_\sigma & \text{in } (0, T) \times \mathbb{T}_L, \\ -\bar{\rho}(\partial_t \operatorname{div}(z) + \bar{u} \cdot \nabla \operatorname{div}(z)) - \nu \Delta(\operatorname{div}(z) + \frac{\bar{\rho}}{\nu} \sigma + \frac{\bar{p}}{\nu} \eta) = \operatorname{div}(g_z) & \text{in } (0, T) \times \mathbb{T}_L, \\ -C_v \bar{\rho}(\partial_t \eta + \bar{u} \cdot \nabla \eta) - \kappa \Delta \eta - \bar{p}_\theta \operatorname{div}(z) = g_\eta & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (2.21)$$

where

$$\nu := \lambda + 2\mu > 0.$$

Which is a closed system in  $\sigma, \operatorname{div} z, \eta$ . Now we introduce a new variable  $q$  in order to eliminate the higher order coupling:  $\Delta \frac{\bar{\rho}}{\nu} \sigma + \Delta \frac{\bar{p}}{\nu} \eta$  as follows:

$$q := \operatorname{div}(z) + \frac{\bar{\rho}}{\nu} \sigma. \quad (2.22)$$

This variable  $q$  was already introduced in [23] and was compared with the effective viscous flux introduced in [45]. In the new unknown  $(\sigma, q, \eta)$  the system (2.21) is as follows:

$$\begin{cases} -(\partial_t \sigma + \bar{u} \cdot \nabla \sigma) = \bar{p}_\rho(q - \frac{\bar{\rho}}{\nu} \sigma) + g_\sigma & \text{in } (0, T) \times \mathbb{T}_L, \\ -\bar{\rho}(\partial_t q + \bar{u} \cdot \nabla q) - \nu \Delta q = \bar{p} \Delta \eta \\ \quad + \frac{\bar{\rho}^2}{\nu} \bar{p}_\rho(q - \frac{\bar{\rho}}{\nu} \sigma) + \operatorname{div}(g_z) + \frac{\bar{\rho}^2}{\nu} g_\sigma & \text{in } (0, T) \times \mathbb{T}_L, \\ -C_v \bar{\rho}(\partial_t \eta + \bar{u} \cdot \nabla \eta) - \kappa \Delta \eta = \bar{p}_\theta(q - \frac{\bar{\rho}}{\nu} \sigma) + g_\eta & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (2.23)$$

Now that we got rid of the higher order coupling we use again the duality to obtain that the observability property of (2.23) is equivalent to the controllability of the following system in  $(r, y, h)$ , where the adjoint is taken with respect to the new variables  $(\sigma, q, \eta)$ .

$$\begin{cases} \partial_t r + \bar{u} \cdot \nabla r = -\frac{\bar{\rho}}{\nu}(\bar{p}_\rho r + \frac{\bar{\rho}^2}{\nu} \bar{p}_\rho y + \bar{p}_\theta h) + f_r + v_r \chi_0 & \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho}(\partial_t y + \bar{u} \cdot \nabla y) - \nu \Delta y = (\bar{p}_\rho r + \frac{\bar{\rho}^2}{\nu} \bar{p}_\rho y + \bar{p}_\theta h) + f_y + v_y \chi_0 & \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho}(\partial_t h + \bar{u} \cdot \nabla h) - \kappa \Delta h = \bar{p} \Delta y + f_h + v_h \chi_0 & \text{in } (0, T) \times \mathbb{T}_L, \\ (r(0, \cdot), y(0, \cdot), h(0, \cdot)) = (r_0, y_0, h_0) & \text{in } \mathbb{T}_L, \\ (r(T, \cdot), y(T, \cdot), h(T, \cdot)) = (0, 0, 0) & \text{in } \mathbb{T}_L, \end{cases} \quad (2.24)$$

where we have introduced a new smooth function  $\chi_0$  in order to have some margin. It satisfies

$$\operatorname{Supp} \chi_0 \subset \subset \{\chi = 1\} \text{ and } \chi_0(x) = 1 \text{ for all } x \in \mathbb{T}_L \text{ such that } d(x, \Omega) \geq 3\varepsilon. \quad (2.25)$$

Now in order to solve the controllability problem (2.24), we use a fixed point argument, and begin by considering the following decoupled controllability problem:

$$\begin{cases} \partial_t r + \bar{u} \cdot \nabla r + \frac{\bar{\rho} \bar{p}_\rho}{\nu} r = \tilde{f}_r + v_r \chi_0 & \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho} \partial_t y - \nu \Delta y = \tilde{f}_y + v_y \chi_0 & \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho} \partial_t h - \kappa \Delta h = \tilde{f}_h + v_h \chi_0 & \text{in } (0, T) \times \mathbb{T}_L, \\ (r(0, \cdot), y(0, \cdot), h(0, \cdot)) = (r_0, y_0, h_0) & \text{in } \mathbb{T}_L, \\ (r(T, \cdot), y(T, \cdot), h(T, \cdot)) = (0, 0, 0) & \text{in } \mathbb{T}_L. \end{cases} \quad (2.26)$$



## 2.3 Carleman inequalities

In this section, we will present the Carleman estimates that we will use to control in each equation of (2.26), and it is mainly borrowed from [1].

### 2.3.1 Construction of the weight function

Let us define the Carleman weight that we will use, which was introduced in [1] and has the particularity that it does not blow up in the initial time  $t = 0$ . Let us introduce a function  $\psi = \psi(t, x) \in C^2([0, T] \times \mathbb{T}_L, [6, 7])$  that satisfies the transport equation

$$\partial_t \psi + \bar{u} \cdot \nabla \psi = 0 \text{ in } (0, T) \times \mathbb{T}_L, \quad (2.27)$$

and for which, there exists a subset  $\omega \subset\subset \{\chi_0 = 1\}$  such that  $\psi$  does not have critical points in  $[0, T] \times (\mathbb{T}_L \setminus \omega)$ . Now we choose  $T_0 > 0$ ,  $T_1 > 0$  and  $\varepsilon > 0$  small enough so that

$$T_0 + 2T_1 < T - \frac{L_0 + 12\varepsilon}{|\bar{u}|}. \quad (2.28)$$

Now for any real number  $\alpha \geq 2$ , we introduce the weight function in time  $\zeta(t)$  defined by

$$\zeta = \zeta(t) \text{ such that } \begin{cases} \forall t \in [0, T_0], \zeta(t) = 1 + \left(1 - \frac{t}{T_0}\right)^\alpha, \\ \forall t \in [T_0, T - 2T_1], \zeta(t) = 1, \\ \forall t \in [T - T_1, T), \zeta(t) = \frac{1}{T - t}, \\ \zeta \text{ is increasing on } [T - 2T_1, T - T_1], \\ \zeta \in C^2([0, T]). \end{cases} \quad (2.29)$$

Then we consider the following weight function  $\varphi = \varphi(t, x)$ :

$$\varphi(t, x) = \zeta(t) \left( \lambda_0 e^{12\lambda_0} - \exp(\lambda_0 \psi(t, x)) \right), \quad (2.30)$$

where  $s, \lambda_0$  are positive parameters with  $s \geq 1, \lambda_0 \geq 1$ . The parameter  $\alpha$  is chosen as

$$\alpha = s\lambda_0^2 e^{2\lambda_0} \geq 2, \quad (2.31)$$

We point out that we have the following bounds, for all  $(t, x) \in [0, T] \times \mathbb{T}_L$ ,

$$\frac{14}{15}\Phi(t) \leq \varphi(t, x) \leq \Phi(t), \quad (2.32)$$

where

$$\Phi(t) := \zeta(t)\lambda_0 e^{12\lambda_0}. \quad (2.33)$$

**Notations.** In order to simplify the notation we will consider in what follows the following shortcuts

$$\|\cdot\|_{L^2(L^2)} = \|\cdot\|_{L^2(0,T;L^2(\mathbb{T}_L))}, \quad \|\cdot\|_{L^2(H^2)} = \|\cdot\|_{L^2(0,T;H^2(\mathbb{T}_L))},$$

and similarly for other spaces.

### 2.3.2 Carleman estimates for the heat equation

In this paragraph we will present the Carleman estimates for the following controllability problem: given  $y_0$  and  $\tilde{f}_y$ , find a control function  $v_y$  such that the solution  $y$  of

$$\begin{cases} \frac{\bar{\rho}}{\nu} \partial_t y - \Delta y = \tilde{f}_y + v_y \chi_0, & \text{in } (0, T) \times \mathbb{T}_L, \\ y(0, \cdot) = y_0, & \text{in } \mathbb{T}_L, \end{cases} \quad (2.34)$$

satisfies

$$y(T, \cdot) = 0, \quad \text{in } \mathbb{T}_L. \quad (2.35)$$

The following inequality was proved in [1] and the extra regularity estimates were proved in [23].

**Theorem 2.3.1.** *There exist constants  $C_0 > 0$  and  $s_0 \geq 1$  and  $\lambda_0 \geq 1$  large enough such that for all smooth functions  $w$  on  $[0, T] \times \mathbb{T}_L$  and for all  $s \geq s_0$ ,*

$$\begin{aligned} & s^{\frac{3}{2}} \lambda_0^2 \left\| \xi^{\frac{3}{2}} w e^{-s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{1/2} \lambda_0 \left\| \xi^{1/2} \nabla w e^{-s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ & \quad + s \lambda_0^{\frac{3}{2}} e^{7\lambda_0} \left\| w(0) e^{-s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)} \\ & \leq C_0 \left\| \left( -\frac{\bar{\rho}}{\nu} \partial_t - \Delta \right) w e^{-s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + C_0 s^{\frac{3}{2}} \lambda_0^2 \left\| \xi^{\frac{3}{2}} \chi_0 w e^{-s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))}. \end{aligned} \quad (2.36)$$

where we have set

$$\xi(t, x) = \zeta(t) \exp(\lambda_0 \psi(t, x)). \quad (2.37)$$

In the rest of the article,  $\lambda_0$  will be fixed so that Theorem 2.3.1 holds.

**Theorem 2.3.2.** *There exist positive constants  $C > 0$  and  $s_0 \geq 1$  such that for all  $s \geq s_0$ , for all  $\tilde{f}_y$  satisfying*

$$\left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} < \infty \quad (2.38)$$

and  $y_0 \in L^2(\mathbb{T}_L)$ , there exists a solution  $(y, v_y)$  of the control problem (2.34)–(2.35) which furthermore satisfies the following estimate:

$$\begin{aligned} & s^{\frac{3}{2}} \|y e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-\frac{3}{2}} \chi_0 v_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ & \quad + s^{1/2} \left\| \zeta^{-1} \nabla y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ & \leq C \left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + C s^{1/2} \|y_0 e^{s\varphi(0)}\|_{L^2(\mathbb{T}_L)}. \end{aligned} \quad (2.39)$$

Besides, this solution  $(y, v_y)$  can be obtained through a linear operator in  $(y_0, \tilde{f}_y)$ .

If  $y_0 \in H^1(\mathbb{T}_L)$ , we also have

$$\begin{aligned} & s^{-1/2} \left\| \zeta^{-2} \nabla^2 y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \leq C \left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ & \quad + C s^{1/2} \|y_0 e^{s\varphi(0)}\|_{L^2(\mathbb{T}_L)} + C s^{-1/2} \left\| \nabla y_0 e^{s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)}. \end{aligned} \quad (2.40)$$

Furthermore we have the following extra regularity depending on the regularities of the initial condition and source term:

1.  $v_y \in L^2(0, T; H^2(\mathbb{T}_L))$  and

$$\begin{aligned} \left\| \chi_0 v_y e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} & \leq C \left( \left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right. \\ & \quad \left. + \left\| y_0 e^{s\Phi(0)} \right\|_{L^2(\mathbb{T}_L)} \right). \end{aligned}$$

2. If furthermore  $y_0 \in H^2(\mathbb{T}_L)$ ,  $\tilde{f}_y e^{6s\Phi/7} \in L^2(0, T; H^1(\mathbb{T}_L))$  and  $\zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))$ ,

$$\begin{aligned} \left\| y e^{6s\Phi/7} \right\|_{L^2(0,T;H^3(\mathbb{T}_L))} & \leq C \left( \left\| \tilde{f}_y e^{6s\Phi/7} \right\|_{L^2(0,T;H^1(\mathbb{T}_L))} \right. \\ & \quad \left. + \left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| y_0 e^{s\Phi(0)} \right\|_{H^2(\mathbb{T}_L)} \right). \end{aligned}$$

3. If furthermore  $y_0 \in H^3(\mathbb{T}_L)$  and  $\tilde{f}_y e^{6s\Phi/7} \in L^2(0, T; H^2(\mathbb{T}_L))$ ,  $\zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))$ ,

$$\begin{aligned} \|y e^{6s\Phi/7}\|_{L^2(0, T; H^4(\mathbb{T}_L))} &\leq C \left( \left\| \tilde{f}_y e^{6s\Phi/7} \right\|_{L^2(0, T; H^2(\mathbb{T}_L))} \right. \\ &\quad \left. + \left\| \zeta^{-\frac{3}{2}} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + \|y_0 e^{s\Phi(0)}\|_{H^3(\mathbb{T}_L)} \right). \end{aligned}$$

The proof is done in [23]. Because of the presence of the higher order coupling  $\Delta y$  on the equation for  $h$  in (2.24), we will require to have estimates on  $\zeta^{-\frac{3}{2}} \Delta y$  instead of  $\zeta^{-2} \Delta y$  as in Theorem 2.3.2, hence we will need a shifted version of the estimates as in the following theorem

**Theorem 2.3.3.** *In the same context of Theorem 2.3.2, there exist positive constants  $C > 0$  and  $s_0 \geq 1$  such that for all  $s \geq s_0$ , for all  $\tilde{f}_y$  satisfying*

$$\left\| \zeta^{-1} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} < \infty \quad (2.41)$$

and  $y_0 \in H^1(\mathbb{T}_L)$ , then the solution  $(y, v_y)$  of the control problem satisfies

$$\begin{aligned} &s^{\frac{3}{2}} \left\| \zeta^{\frac{1}{2}} y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \chi_0 v_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} \\ &\quad + s^{1/2} \left\| \zeta^{-\frac{1}{2}} \nabla y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + s^{-1/2} \left\| \zeta^{-\frac{3}{2}} \nabla^2 y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} \\ &\leq C \left\| \zeta^{-1} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + C s^{1/2} \|y_0 e^{s\varphi(0)}\|_{L^2(\mathbb{T}_L)} + C s^{-1/2} \|\nabla y_0 e^{s\varphi(0)}\|_{L^2(\mathbb{T}_L)}. \end{aligned} \quad (2.42)$$

Besides, this solution  $(y, v_y)$  can be obtained through a linear operator in  $(y_0, \tilde{f}_y)$ .

Furthermore we have the following extra regularity depending on the regularities of the initial condition and source term:

1.  $v_y \in L^2(0, T; H^2(\mathbb{T}_L))$  and

$$\begin{aligned} \left\| \chi_0 v_y e^{6s\Phi/7} \right\|_{L^2(0, T; H^2(\mathbb{T}_L))} &\leq C \left( \left\| \zeta^{-1} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} \right. \\ &\quad \left. + \|y_0 e^{s\Phi(0)}\|_{L^2(\mathbb{T}_L)} \right). \end{aligned}$$

2. If furthermore  $y_0 \in H^2(\mathbb{T}_L)$ ,  $\tilde{f}_y e^{6s\Phi/7} \in L^2(0, T; H^1(\mathbb{T}_L))$  and  $\zeta^{-1} \tilde{f}_y e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))$ ,

$$\begin{aligned} \|ye^{6s\Phi/7}\|_{L^2(0, T; H^3(\mathbb{T}_L))} &\leq C \left( \left\| \tilde{f}_y e^{6s\Phi/7} \right\|_{L^2(0, T; H^1(\mathbb{T}_L))} \right. \\ &\quad \left. + \left\| \zeta^{-1} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + \|y_0 e^{s\Phi(0)}\|_{H^2(\mathbb{T}_L)} \right). \end{aligned}$$

3. If furthermore  $y_0 \in H^3(\mathbb{T}_L)$  and  $\tilde{f}_y e^{6s\Phi/7} \in L^2(0, T; H^2(\mathbb{T}_L))$ ,  $\zeta^{-1} \tilde{f}_y e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))$ ,

$$\begin{aligned} \|ye^{6s\Phi/7}\|_{L^2(0, T; H^4(\mathbb{T}_L))} &\leq C \left( \left\| \tilde{f}_y e^{6s\Phi/7} \right\|_{L^2(0, T; H^2(\mathbb{T}_L))} \right. \\ &\quad \left. + \left\| \zeta^{-1} \tilde{f}_y e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + \|y_0 e^{s\Phi(0)}\|_{H^3(\mathbb{T}_L)} \right). \end{aligned}$$

The proof is done by considering the equation satisfied by  $\zeta^{\frac{1}{2}}y$  and applying the same estimates of Theorem 2.3.2. The only difference is the appearance of the source term  $\dot{\zeta}\zeta^{-1}y$  which can be absorbed by the term  $s^{\frac{3}{2}} \left\| \zeta^{\frac{1}{2}}ye^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))}$  for  $s$  big enough.

**Remark 2.3.1.** When  $\nu \neq \frac{\kappa}{C_v}$ , we can consider instead of

$$q := \operatorname{div}(z) + \frac{\bar{\rho}}{\nu}\sigma$$

the dependent variable

$$q := \operatorname{div}(z) + \frac{\bar{\rho}}{\nu}\sigma + \frac{\bar{p}}{\nu - \frac{\kappa}{C_v}}\eta.$$

In this case, we obtain a similar system, with the exception that the term  $\bar{p}\Delta\eta$  does not appear. Without this higher order coupling, the previous shifted Carleman inequality is not necessary, which simplifies the proof.

### 2.3.3 Controllability of the transport equation

In this paragraph, we will present Carleman Inequalities for the following transport equation: Given  $\tilde{f}_r$  and  $r_0$ , find a control function  $v_r$  such that the solution  $r$  of

$$\begin{cases} \partial_t r + \bar{u} \cdot \nabla r + \frac{\bar{p}_\rho \bar{\rho}}{\nu} r = \tilde{f}_r + v_r \chi_0, & \text{in } (0, T) \times \mathbb{T}_L, \\ r(0, \cdot) = r_0, & \text{in } \mathbb{T}_L, \end{cases} \quad (2.43)$$

satisfies the controllability requirement

$$r(T, \cdot) = 0 \text{ in } \mathbb{T}_L. \quad (2.44)$$

The following Theorem is a minor modification of Theorem 3.5 in [23], the only difference being the Carleman weight being  $\zeta^{-1}$  instead of  $\zeta^{-\frac{3}{2}}$  ( $\theta^{-\frac{3}{2}}$  in the notation of [23])

**Theorem 2.3.4.** *Let  $(\bar{u}, T, \varepsilon)$  be as in (2.28). For all  $\tilde{f}_r$  with*

$$\left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} < \infty \quad (2.45)$$

and  $r_0 \in L^2(\mathbb{T}_L)$ , there exists a function  $v_r \in L^2(0,T;L^2(\mathbb{T}_L))$  such that the solution  $r$  of (2.43) satisfies the control requirement (2.44). Besides, the controlled trajectory  $r$  and the control function  $v_r$  satisfy

$$\begin{aligned} & \left\| \zeta^{-1} r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} v_r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ & \leq C \left( \left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| r_0 e^{s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)} \right). \end{aligned} \quad (2.46)$$

If  $r_0 \in H^1(\mathbb{T}_L)$  and  $\tilde{f}_r$  satisfies (2.43) and  $\tilde{f}_r e^{6s\Phi/7} \in L^2(0,T;H^1(\mathbb{T}_L))$ , then  $r$  furthermore belongs to  $L^2(0,T;H^1(\mathbb{T}_L))$  and satisfies

$$\begin{aligned} & \left\| r e^{6s\Phi/7} \right\|_{L^2(0,T;H^1(\mathbb{T}_L))} + \left\| v_r e^{6s\Phi/7} \right\|_{L^2(0,T;H^1(\mathbb{T}_L))} \\ & \leq C \left( \left\| \tilde{f}_r e^{6s\Phi/7} \right\|_{L^2(0,T;H^1(\mathbb{T}_L))} + \left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| r_0 e^{s\Phi(0)} \right\|_{H^1(\mathbb{T}_L)} \right). \end{aligned} \quad (2.47)$$

If  $r_0 \in H^2(\mathbb{T}_L)$  and  $\tilde{f}_r$  satisfies (2.43) and  $\tilde{f}_r e^{6s\Phi/7} \in L^2(0,T;H^2(\mathbb{T}_L))$ , then  $r$  belongs to  $L^2(0,T;H^2(\mathbb{T}_L))$  and satisfies

$$\begin{aligned} & \left\| r e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} + \left\| v_r e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} \\ & \leq C \left( \left\| \tilde{f}_r e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| r_0 e^{s\Phi(0)} \right\|_{H^2(\mathbb{T}_L)} \right). \end{aligned} \quad (2.48)$$

Besides, this solution  $(r, v_r)$  can be obtained through a linear operator in  $(r_0, \tilde{f}_r)$ .

*Proof.* The proof of Theorem 2.3.4 consists in an explicit construction solving the control problem (2.43)–(2.44) and then on suitable estimates on it.

*An explicit construction.* Let  $\eta_0$  be a smooth cut-off function taking value 1 on  $\{x \in \mathbb{T}_L, \text{ with } d(x, \bar{\Omega}) \leq 5\varepsilon + |\bar{u}|T_0\}$  and vanishing on  $\{x \in \mathbb{T}_L, d(x, \Omega) \geq 6\varepsilon + |\bar{u}|T_0\}$  with  $\varepsilon$  as in section 2.3.1 . We then introduce  $\eta$  the solution of

$$\begin{cases} \partial_t \eta + \bar{u} \cdot \nabla \eta = 0, & \text{in } (0, T) \times \mathbb{T}_L, \\ \eta(0, \cdot) = \eta_0 & \text{in } \mathbb{T}_L, \end{cases} \quad (2.49)$$

and the solutions  $r_f$  and  $r_b$  (here ‘ $f$ ’ stands for forward, ‘ $b$ ’ for backward) of

$$\begin{cases} \partial_t r_f + \bar{u} \cdot \nabla r_f + ar_f = \tilde{f}_r, & \text{in } (0, T) \times \mathbb{T}_L, \\ r_f(0, \cdot) = r_0 & \text{in } \mathbb{T}_L, \end{cases} \quad (2.50)$$

and

$$\begin{cases} \partial_t r_b + \bar{u} \cdot \nabla r_b + ar_b = \tilde{f}_r, & \text{in } (0, T) \times \mathbb{T}_L, \\ r_b(T, \cdot) = 0 & \text{in } \mathbb{T}_L, \end{cases} \quad (2.51)$$

where  $a$  denotes the constant

$$a = \frac{\bar{p}\bar{\rho}}{\nu}.$$

We then set

$$r = \eta_2(x) (\eta r_f + (1 - \eta)r_b) + (1 - \eta_2(x))\eta_1(t)r_f, \quad \text{in } (0, T) \times \mathbb{T}_L, \quad (2.52)$$

where  $\eta_1(t)$  is a smooth cut-off function taking value 1 on  $[0, T_0/2]$  and vanishing for  $t \geq T_0$  and  $\eta_2 = \eta_2(x)$  is a smooth cut-off function taking value 1 for  $x$  with  $d(x, \Omega) \leq 3\varepsilon$  and vanishing for  $x$  with  $d(x, \Omega) \geq 4\varepsilon$ . One easily checks that  $r$  solves

$$\begin{aligned} \partial_t r + \bar{u} \cdot \nabla r + ar &= \eta_2 \tilde{f}_r + (1 - \eta_2)\eta_1 \tilde{f}_r + \bar{u} \cdot \nabla \eta_2 (\eta r_f + (1 - \eta)r_b) \\ &\quad - \eta_1 \bar{u} \cdot \nabla \eta_2 r_f + (1 - \eta_2) \partial_t \eta_1 r_f \quad \text{in } (0, T) \times \mathbb{T}_L, \end{aligned} \quad (2.53)$$

thus corresponding to a control function

$$v_r = (\eta_2 - 1) \tilde{f}_r + (1 - \eta_2) \eta_1 \tilde{f}_r + \bar{u} \cdot \nabla \eta_2 (\eta r_f + (1 - \eta)r_b) - \eta_1 \bar{u} \cdot \nabla \eta_2 r_f + (1 - \eta_2) \partial_t \eta_1 r_f, \quad (2.54)$$

localized in the support of  $\chi_0$  due to the condition on the support of  $\eta_2$ . Besides,  $r$  given by (2.52) satisfies

$$r(0, \cdot) = r_0 \text{ in } \mathbb{T}_L, \quad r(T, \cdot) = 0 \text{ in } \mathbb{T}_L$$

due to the conditions on the support of  $\eta_0, \eta_1, \eta_2$  and the condition (2.28) on the flow corresponding to  $\bar{u}$ .

Actually, thanks to the choice of  $\varepsilon > 0, T_0 > 0$  and  $T_1 > 0$  in (2.28) we have

$$\eta_2(1 - \eta) = 0 \quad \text{for all } (t, x) \in [0, T_0] \times \mathbb{T}_L, \quad (2.55)$$

and

$$\eta_2 \eta = 0 \quad \text{for all } (t, x) \in [T - 2T_1, T] \times \mathbb{T}_L. \quad (2.56)$$

*Estimates on  $r$ .* Let us start with estimates on  $r_f$ . To get estimates on  $r_f$ , we perform weighted energy estimates on (2.50) on the time interval  $(0, T - 2T_1)$ . Multiplying (2.50) by  $\zeta^{-2} r_f e^{2s\varphi}$ , we obtain

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{T}_L} \zeta^{-2} |r_f|^2 e^{2s\varphi} \right) &\leq \frac{1}{2} \int_{\mathbb{T}_L} |r_f|^2 (-2a\zeta^{-2} e^{2s\varphi} + (\partial_t + \bar{u} \cdot \nabla)(\zeta^{-2} e^{2s\varphi})) \\ &\quad + \left( \int_{\mathbb{T}_L} \zeta^{-2} |r_f|^2 e^{2s\varphi} \right)^{1/2} \left( \int_{\mathbb{T}_L} \zeta^{-2} |\tilde{f}_r|^2 e^{2s\varphi} \right)^{1/2}. \end{aligned} \quad (2.57)$$

But, for all  $t \in (0, T - 2T_1)$  and  $x \in \mathbb{T}_L$ ,

$$(\partial_t + \bar{u} \cdot \nabla)(\zeta^{-2} e^{2s\varphi}) \leq 0.$$

We thus conclude

$$\|\zeta^{-1} r_f e^{s\varphi}\|_{L^\infty(0, T-2T_1; L^2)} \leq C \left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(0, T-2T_1; L^2)} + C \|r_0 e^{s\varphi(0)}\|_{L^2}.$$

Similarly, one can show that  $r_b$  satisfies

$$\|\zeta^{-1} r_b e^{s\varphi}\|_{L^\infty(T_0, T; L^2)} \leq C \left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(T_0, T; L^2)}.$$

To conclude that

$$\|\zeta^{-1} r e^{s\varphi}\|_{L^\infty(0, T; L^2)} \leq C \left\| \zeta^{-1} \tilde{f}_r e^{s\varphi} \right\|_{L^2(0, T; L^2)} + C \|r_0 e^{s\varphi(0)}\|_{L^2},$$

we use the explicit definition of  $r$  in (2.52) and identity (2.55), and notice that  $\eta_0, \eta_1$  and  $\eta_2$  belong to  $L^\infty$ , and  $(\eta, \hat{u})$  to  $L^\infty(L^\infty)$ .

The estimate on  $v_r$  in (2.46) is also a simple consequence of its explicit form in (2.54) and of the fact that  $\eta_0 \in W^{1, \infty}$ ,  $\eta_1 \in W^{1, \infty}$ ,  $\eta \in L^\infty(L^\infty)$ ,  $\eta_2 \in L^\infty$ .

*Regularity results.* To obtain regularity results on  $r$  and  $v_r$ , it is then sufficient to get regularity estimates on  $r_f$  solution of (2.50) on the time interval  $(0, T - 2T_1)$  and on  $r_b$  solution of (2.51) on the time interval  $(T_0, T)$ . These estimates are of the same nature, so we only focus on  $r_f$ , the other case being completely similar.



To get weighted estimates in higher norms, we do higher order energy estimates on (2.50). For instance,  $\nabla r_f$  satisfies the equation

$$\begin{cases} \partial_t \nabla r_f + (\bar{u} \cdot \nabla) \nabla r_f + a \nabla r_f = \nabla \tilde{f}_r, & \text{in } (0, T) \times \mathbb{T}_L, \\ \nabla r_f(0, \cdot) = \nabla r_0 & \text{in } \mathbb{T}_L, \end{cases} \quad (2.58)$$

Hence, using that  $\partial_t \Phi \leq 0$  on  $(0, T - 2T_1)$ , energy estimates directly provide

$$\|\nabla r_f e^{6s\Phi/7}\|_{L^\infty(0, T-2T_1; L^2)} \leq C \left( \|\nabla \tilde{f}_r e^{6s\Phi/7}\|_{L^1(L^2)} + \|\nabla r_0 e^{6s\Phi/7}\|_{L^2} \right). \quad (2.59)$$

This implies (2.47).

The equation of  $\nabla^2 r_f$  has the same form. For all  $(i, j) \in \{1, \dots, d\}^2$ ,

$$\begin{cases} \partial_t \partial_{i,j} r_f + (\bar{u} \cdot \nabla) \partial_{i,j} r_f + a \partial_{i,j} r_f = \partial_{i,j} \tilde{f}_r, & \text{in } (0, T) \times \mathbb{T}_L, \\ \partial_{i,j} r_f(0, \cdot) = \partial_{i,j} r_0 & \text{in } \mathbb{T}_L. \end{cases} \quad (2.60)$$

An energy estimate for  $\nabla^2 r_f$  on  $(0, T - 2T_1)$  directly yields

$$\|\nabla^2 r_f e^{6s\Phi/7}\|_{L^\infty(0, T-2T_1; L^2)} \leq C \left( \|\nabla^2 \tilde{f}_r e^{6s\Phi/7}\|_{L^1(L^2)} + \|\nabla^2 r_0 e^{6s\Phi/7}\|_{L^2} \right),$$

thus concluding the proof of Theorem 2.3.4.  $\square$

## 2.4 The controllability problem (2.24)

Using the results of the previous section about the controllability of the individual systems we will prove the controllability of system (2.24). More precisely we have the following theorem:

**Theorem 2.4.1.** *Let  $(\bar{u}, T, \varepsilon)$  be as in (2.28).*

*Let us fix an initial condition  $(r_0, y_0, h_0) \in L^2(\mathbb{T}_L) \times L^2(\mathbb{T}_L) \times L^2(\mathbb{T}_L)$ . There exist  $C > 0$  and  $s_0 \geq 1$  large enough such that for all  $s \geq s_0$ , if  $f_r, f_y$  and  $f_h$  satisfy the estimates*

$$\|\zeta^{-1} f_r e^{s\varphi}\|_{L^2(0, T; L^2(\mathbb{T}_L))} + \|\zeta^{-1} f_y e^{s\varphi}\|_{L^2(0, T; L^2(\mathbb{T}_L))} + \|\zeta^{-\frac{3}{2}} f_h e^{s\varphi}\|_{L^2(0, T; L^2(\mathbb{T}_L))} < \infty, \quad (2.61)$$

*there exists a controlled trajectory  $(r, y, h)$  solving (2.24) and satisfying the*

following estimate:

$$\begin{aligned}
& s^{\frac{2}{3}} \left\| \zeta^{-1} r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s \left\| h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \nabla h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad + s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad \quad + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C \left( s^{\frac{2}{3}} \left\| \zeta^{-1} f_r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{1}{3}} \left\| \zeta^{-1} f_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right. \\
& \quad \quad \left. + s^{-1/2} \left\| \zeta^{-\frac{3}{2}} f_h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right) \\
& + C \left( s^{\frac{2}{3}} \left\| r_0 e^{s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)} + s^{\frac{5}{6}} \left\| y_0 e^{s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)} + \left\| h_0 e^{s\varphi(0)} \right\|_{L^2(\mathbb{T}_L)} \right). \quad (2.62)
\end{aligned}$$

Furthermore, if the initial condition is more regular, say  $(r_0, y_0, h_0) \in H^2(\mathbb{T}_L) \times H^3(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$ , and  $f_r, f_y$  and  $f_h$  satisfy

$$f_r e^{s\Phi} \in L^2(0, T; H^2(\mathbb{T}_L)), \quad f_y e^{s\Phi} \in L^2(0, T; H^2(\mathbb{T}_L)), \quad f_h e^{s\Phi} \in L^2(0, T; H^1(\mathbb{T}_L)), \quad (2.63)$$

we furthermore have the following estimate:

$$\begin{aligned}
& \left\| r e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} + \left\| y e^{6s\Phi/7} \right\|_{L^2(0,T;H^4(\mathbb{T}_L))} + \left\| h e^{6s\Phi/7} \right\|_{L^2(0,T;H^3(\mathbb{T}_L))} \\
& \quad + \left\| \chi_0 v_r e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} + \left\| \chi_0 v_y e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} \\
& \quad \quad + \left\| \chi_0 v_h e^{6s\Phi/7} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} \\
& \leq C \left( \left\| f_r e^{s\Phi} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} + \left\| f_y e^{s\Phi} \right\|_{L^2(0,T;H^2(\mathbb{T}_L))} + \left\| f_h e^{s\Phi} \right\|_{L^2(0,T;H^1(\mathbb{T}_L))} \right. \\
& \quad \quad \left. + \left\| r_0 e^{s\Phi(0)} \right\|_{H^2(\mathbb{T}_L)} + \left\| y_0 e^{s\Phi(0)} \right\|_{H^3(\mathbb{T}_L)} + \left\| h_0 e^{s\Phi(0)} \right\|_{H^2(\mathbb{T}_L)} \right), \quad (2.64)
\end{aligned}$$

for some constant  $C$  independent of  $s \geq s_0$ .

*Proof.* We construct the controlled trajectory by using a fixed point argument.

Considering the Carleman estimates, we introduce the following sets:

$$\begin{aligned}
\mathcal{C}_s^r &= \{r \in L^2(0, T; L^2(\mathbb{T}_L)) \text{ s.t. } \zeta^{-1} r e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))\}, \\
\mathcal{C}_s^y &= \{y \in L^2(0, T; H^1(\mathbb{T}_L)) \text{ s.t. } \zeta^{\frac{1}{2}} y e^{s\varphi}, \zeta^{-\frac{1}{2}} \nabla y e^{s\varphi}, \\
& \quad \zeta^{-\frac{3}{2}} \Delta y e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))\}, \\
\mathcal{C}_s^h &= \{h \in L^2(0, T; H^1(\mathbb{T}_L)) \text{ s.t. } h e^{s\varphi}, \zeta^{-1} \nabla h e^{s\varphi} \in L^2(0, T; L^2(\mathbb{T}_L))\}. \quad (2.65)
\end{aligned}$$

For  $\tilde{r} \in \mathcal{C}^r$ ,  $\tilde{y} \in \mathcal{C}_s^y$  and  $\tilde{h} \in \mathcal{C}_s^h$ , we define

$$\begin{aligned}\tilde{f}_r &:= \tilde{f}_r(\tilde{y}, \tilde{h}) = f_r - \frac{\bar{\rho}}{\nu} \left( \frac{\bar{\rho}^2}{\nu} \bar{p}_\rho \tilde{y} + \bar{p}_\theta \tilde{h} \right), \\ \tilde{f}_y &:= \tilde{f}_y(\tilde{r}, \tilde{y}, \tilde{h}) = f_y - \bar{\rho} \bar{u} \cdot \nabla \tilde{y} + (\bar{p}_\rho \tilde{r} + \frac{\bar{\rho}^2}{\nu} \bar{p}_\rho \tilde{y} + \bar{p}_\theta \tilde{h}), \\ \tilde{f}_h &:= \tilde{f}_h(\tilde{r}, \tilde{y}, \tilde{h}) = f_h - C_v \bar{\rho} \bar{u} \cdot \nabla \tilde{h} + \bar{p} \Delta \tilde{y}.\end{aligned}$$

As  $f_r$ ,  $f_y$  and  $f_h$  satisfy (2.61), for  $(\tilde{r}, \tilde{y}, \tilde{h}) \in \mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h$ ,  $\tilde{f}_r$  satisfies (2.45),  $\tilde{f}_y$  satisfies (2.41) and  $\tilde{f}_h$  satisfies (2.38).

We then define a map  $\Lambda_s$  on  $\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h$  which to a data  $(\tilde{r}, \tilde{y}, \tilde{h}) \in \mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h$  associates  $(r, y, h)$ , where  $r$  is the solution of the controlled problem

$$\begin{cases} \partial_t r + \bar{u} \cdot \nabla r + \frac{\bar{p}_\rho \bar{\rho}}{\nu} r = \tilde{f}_r + v_r \chi_0, & \text{in } (0, T) \times \mathbb{T}_L, \\ r(0, \cdot) = r_0(\cdot), \quad r(T, \cdot) = 0, & \text{in } \mathbb{T}_L, \end{cases} \quad (2.66)$$

given by Theorem 2.3.4,  $y$  is the solution of the controlled problem

$$\begin{cases} \bar{\rho} \partial_t y - \nu \Delta y = \tilde{f}_y + v_y \chi_0, & \text{in } (0, T) \times \mathbb{T}_L, \\ y(0, \cdot) = y_0(\cdot), \quad y(T, \cdot) = 0, & \text{in } \mathbb{T}_L, \end{cases} \quad (2.67)$$

given by Theorem 2.3.3, and  $h$  is the solution of the controlled problem

$$\begin{cases} C_v \bar{\rho} \partial_t h - \kappa \Delta h = \tilde{f}_h + v_h \chi_0, & \text{in } (0, T) \times \mathbb{T}_L, \\ y(0, \cdot) = y_0(\cdot), \quad y(T, \cdot) = 0, & \text{in } \mathbb{T}_L, \end{cases} \quad (2.68)$$

given by Theorem 2.3.2.

We notice that in Theorems 2.3.2, 2.3.3 and 2.3.4 the maps  $(y_0, \tilde{f}_y) \mapsto (y, v_y)$  and  $(r_0, \tilde{f}_r) \mapsto (r, v_r)$  are linear. We now will prove that for a choice of the parameter  $s$  large enough,  $\Lambda_s$  is a contraction and then, by the Banach fixed point theorem, it has a fixed point.

Let  $(\tilde{r}_a, \tilde{y}_a, \tilde{h}_a), (\tilde{r}_b, \tilde{y}_b, \tilde{h}_b)$  be elements of  $\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h$  and consider their image under  $\Lambda_s$   $(r_a, y_a, h_a) = \Lambda_s(\tilde{r}_a, \tilde{y}_a, \tilde{h}_a)$  and  $(r_b, y_b, h_b) = \Lambda_s(\tilde{r}_b, \tilde{y}_b, \tilde{h}_b)$ . Then we consider the differences  $\mathcal{R} = r_a - r_b$ ,  $\mathcal{Y} = y_a - y_b$ ,  $\mathcal{H} = h_a - h_b$ ,  $\tilde{\mathcal{R}} = \tilde{r}_a - \tilde{r}_b$ ,  $\tilde{\mathcal{Y}} = \tilde{y}_a - \tilde{y}_b$ ,  $\tilde{\mathcal{H}} = \tilde{h}_a - \tilde{h}_b$ ,  $\tilde{\mathcal{F}}_r = \tilde{f}_r(\tilde{y}_a, \tilde{h}_a) - \tilde{f}_r(\tilde{y}_b, \tilde{h}_b)$ ,  $\tilde{\mathcal{F}}_y = \tilde{f}_y(\tilde{r}_a, \tilde{y}_a, \tilde{h}_a) - \tilde{f}_y(\tilde{r}_b, \tilde{y}_b, \tilde{h}_b)$  and  $\tilde{\mathcal{F}}_h = \tilde{f}_h(\tilde{r}_a, \tilde{y}_a, \tilde{h}_a) - \tilde{f}_h(\tilde{r}_b, \tilde{y}_b, \tilde{h}_b)$ , by Theorem 2.3.4 we have

$$\begin{aligned} s^{\frac{2}{3}} \left\| \zeta^{-1} \mathcal{R} e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} &\leq C s^{\frac{2}{3}} \left\| \zeta^{-1} \tilde{\mathcal{F}}_r e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} \\ &\leq C s^{-\frac{1}{3}} \left( s \left\| \zeta^{-1} \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} + s \left\| \zeta^{-1} \tilde{\mathcal{H}} e^{s\varphi} \right\|_{L^2(0, T; L^2(\mathbb{T}_L))} \right) \end{aligned} \quad (2.69)$$

while Theorem 2.3.3 applied to  $\mathcal{Y}$  implies (after multiplying (2.42) by  $s^{\frac{1}{3}}$ )

$$\begin{aligned}
& s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} \mathcal{Y} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla \mathcal{Y} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta \mathcal{Y} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C s^{\frac{1}{3}} \left\| \zeta^{-1} \tilde{\mathcal{F}}_y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C s^{-\frac{1}{3}} \left( s^{\frac{2}{3}} \left\| \zeta^{-1} \tilde{\mathcal{R}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{11}{6}} \left\| \zeta^{-1} \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right. \\
& \quad \left. + s^{\frac{2}{3}} \left\| \zeta^{-2} \dot{\zeta} \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right. \\
& \quad \left. + s \left\| \zeta^{-1} \tilde{\mathcal{H}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-1} \nabla \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right) \quad (2.70)
\end{aligned}$$

and applying the estimate (2.40) to  $\mathcal{H}$ ,

$$\begin{aligned}
& s \left\| \mathcal{H} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \nabla \mathcal{H} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C s^{-\frac{1}{2}} \left\| \zeta^{-\frac{3}{2}} \tilde{\mathcal{F}}_h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C s^{-\frac{1}{3}} \left( s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \nabla \tilde{\mathcal{H}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right). \quad (2.71)
\end{aligned}$$

In particular, we have

$$\begin{aligned}
& s^{\frac{2}{3}} \left\| \zeta^{-1} \mathcal{R} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s \left\| \mathcal{H} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \nabla \mathcal{H} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad + s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} \mathcal{Y} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla \mathcal{Y} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta \mathcal{Y} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C s^{-\frac{1}{3}} \left( s^{\frac{2}{3}} \left\| \zeta^{-1} \tilde{\mathcal{R}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right. \\
& \quad + s \left\| \tilde{\mathcal{H}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad \left. + \left\| \zeta^{-1} \nabla \tilde{\mathcal{H}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta \tilde{\mathcal{Y}} e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \right) \quad (2.72)
\end{aligned}$$

We define the following norm in  $\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h$

$$\begin{aligned} \|(r, y, h)\|_{\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h} &:= s^{\frac{2}{3}} \|\zeta^{-1} r e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s \|h e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ &\quad + \|\zeta^{-1} \nabla h e^{s\varphi}\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\ &\quad + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \end{aligned} \quad (2.73)$$

for which the map  $\Lambda_s$  satisfies

$$\begin{aligned} \left\| \Lambda_s(\tilde{r}_a, \tilde{y}_a, \tilde{h}_a) - \Lambda_s(\tilde{r}_b, \tilde{y}_b, \tilde{h}_b) \right\|_{\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h} \\ \leq C s^{-1/3} \|(r_a, y_a, h_a) - (r_b, y_b, h_b)\|_{\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h}. \end{aligned} \quad (2.74)$$

So, if we take  $s > C^3$  (and also  $s \geq s_0$ ) with  $C$  as in (2.74), the map  $\Lambda_s$  is a contractive mapping and by Banach's fixed point theorem,  $\Lambda_s$  has a unique fixed point  $(r, y, h)$  in  $\mathcal{C}_s^r \times \mathcal{C}_s^y \times \mathcal{C}_s^h$ . It is clear now that  $(r, y, h)$  solves the controllability problem (2.24). Finally, in order to prove the inequality, we estimate  $\tilde{f}_r(y, h)$ ,  $\tilde{f}_y(r, y, h)$  and  $\tilde{f}_h(r, y, h)$  by

$$\left\| \zeta^{-1} \tilde{f}_r(y, h) e^{s\varphi} \right\|_{L^2(L^2)} \leq C \|y e^{s\varphi}\|_{L^2(L^2)} + C \|h e^{s\varphi}\|_{L^2(L^2)} + C \|\zeta^{-1} f_r e^{s\varphi}\|_{L^2(L^2)}, \quad (2.75)$$

$$\begin{aligned} \left\| \zeta^{-1} \tilde{f}_y(r, y, h) e^{s\varphi} \right\|_{L^2(L^2)} &\leq C \|\zeta^{-1} f_y e^{s\varphi}\|_{L^2(L^2)} \\ &\quad + C \left( \|\zeta^{-1} r e^{s\varphi}\|_{L^2(L^2)} + s \|\zeta^{-1} y e^{s\varphi}\|_{L^2(L^2)} + \|\zeta^{-1} h e^{s\varphi}\|_{L^2(L^2)} \right. \\ &\quad \left. + \|\zeta^{-1} \nabla y e^{s\varphi}\|_{L^2(L^2)} \right), \end{aligned} \quad (2.76)$$

and

$$\begin{aligned} \left\| \zeta^{-\frac{3}{2}} \tilde{f}_h(r, y, h) e^{s\varphi} \right\|_{L^2(L^2)} &\leq C \left\| \zeta^{-\frac{3}{2}} f_h e^{s\varphi} \right\|_{L^2(L^2)} \\ &\quad + C \left( \left\| \zeta^{-\frac{3}{2}} \Delta y e^{s\varphi} \right\|_{L^2(L^2)} + \left\| \zeta^{-\frac{3}{2}} \nabla h e^{s\varphi} \right\|_{L^2(L^2)} \right). \end{aligned} \quad (2.77)$$

One gets with Theorems 2.3.2, 2.3.3 and 2.3.4 that  $(r, y, h)$  solution of (2.23)

satisfies

$$\begin{aligned}
& s^{\frac{2}{3}} \left\| \zeta^{-1} r e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s \left\| h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + \left\| \zeta^{-1} \nabla h e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad + s^{\frac{11}{6}} \left\| \zeta^{\frac{1}{2}} y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} + s^{\frac{5}{6}} \left\| \zeta^{-\frac{1}{2}} \nabla y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \quad \quad + s^{-\frac{1}{6}} \left\| \zeta^{-\frac{3}{2}} \Delta y e^{s\varphi} \right\|_{L^2(0,T;L^2(\mathbb{T}_L))} \\
& \leq C \left( s^{\frac{2}{3}} \left\| \zeta^{-1} f_r e^{s\varphi} \right\|_{L^2(L^2)} + s^{\frac{1}{3}} \left\| \zeta^{-1} f_y e^{s\varphi} \right\|_{L^2(L^2)} + s^{-1/2} \left\| \zeta^{-\frac{3}{2}} f_h e^{s\varphi} \right\|_{L^2(L^2)} \right) \\
& \quad + C \left( s^{\frac{2}{3}} \left\| r_0 e^{s\varphi(0)} \right\|_{L^2} + s^{\frac{5}{6}} \left\| y_0 e^{s\varphi(0)} \right\|_{L^2} + \left\| h_0 e^{s\varphi(0)} \right\|_{L^2} \right),
\end{aligned}$$

that is, the estimate (2.62).

*Regularity estimates on the solution of the control problem.* If we have the extra regularity for the initial condition  $(r_0, y_0, h_0) \in H^2(\mathbb{T}_L) \times H^3(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$  and the source terms  $f_r, f_y, f_h$  satisfy (2.63) then using a typical bootstrap argument together with the extra regularity estimates with Carleman weights of Theorem 2.3.2 item 2 and 3, Theorem 2.3.3 item 2 and 3 and Theorem 2.3.4 estimates (2.47) and (2.48), we can obtain (2.64).  $\square$

## 2.5 Controllability of system (2.19)

In this section, we recover the controllability of (2.19) from the controllability of (2.24). This is given in the following statement.

**Theorem 2.5.1.** *There exists  $s_0 \geq 1$ , such that for all  $s \geq s_0$ , for all  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0) \in H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$ ,  $\widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta$  such that  $\widehat{f}_\rho e^{s\Phi} \in L^2(0, T; H^2(\mathbb{T}_L))$  and  $\widehat{f}_u e^{7s\Phi/6}, \widehat{f}_\theta e^{s\Phi} \in L^2(0, T; H^1(\mathbb{T}_L))$ , there exist control functions  $v_\rho, v_u, v_\theta$  and a corresponding controlled trajectory  $(\rho, u, \theta)$  solving (2.19) with initial data  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$ , satisfying the controllability requirement (2.96), and depending linearly on the data  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, \widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta)$ . Besides, we have the estimate:*

$$\begin{aligned}
& \left\| (\rho e^{6s\Phi/7}, u e^{6s\Phi/7}, \theta e^{6s\Phi/7}) \right\|_{L^2(0,T;H^2(\mathbb{T}_L)) \times L^2(0,T;H^3(\mathbb{T}_L)) \times L^2(0,T;H^3(\mathbb{T}_L))} \\
& + \left\| (\chi v_\rho e^{6s\Phi/7}, \chi v_u e^{6s\Phi/7}, \chi v_\theta e^{6s\Phi/7}) \right\|_{L^2(0,T;H^2(\mathbb{T}_L)) \times L^2(0,T;H^1(\mathbb{T}_L)) \times L^2(0,T;H^1(\mathbb{T}_L))} \\
& \leq C \left\| (\widehat{f}_\rho e^{s\Phi}, \widehat{f}_u e^{7s\Phi/6}, \widehat{f}_\theta e^{s\Phi}) \right\|_{L^2(0,T;H^2(\mathbb{T}_L)) \times L^2(0,T;H^1(\mathbb{T}_L)) \times L^2(0,T;H^1(\mathbb{T}_L))} \\
& \quad + C \left\| (\widehat{\rho}_0 e^{s\Phi(0)}, \widehat{u}_0 e^{7s\Phi(0)/6}, \widehat{\theta}_0 e^{s\Phi(0)}) \right\|_{H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)}. \quad (2.78)
\end{aligned}$$

*Proof.* Our first step will be to recover observability estimates for the system (2.23). In order to simplify the notation, let us introduce the following spaces

$$\begin{aligned}\mathcal{I} &= L^2(H^2) \times L^2(H^2) \times L^2(H^1) \\ \mathcal{J} &= H^2 \times H^3 \times H^2\end{aligned}$$

By definition of the dual norm

$$\begin{aligned}& \left\| (\sigma e^{-s\Phi}, qe^{-s\Phi}, \eta e^{-s\Phi}) \right\|_{\mathcal{I}'} + \left\| (\sigma(0)e^{-s\Phi(0)}, q(0)e^{-s\Phi(0)}, \eta(0)e^{-s\Phi(0)}) \right\|_{\mathcal{J}'} \\ &= \sup_{\|(f_r e^{s\Phi}, f_y e^{s\Phi}, f_h e^{s\Phi})\|_{\mathcal{I}} \leq 1} \langle (f_r, f_y, f_h), (\sigma, q, \eta) \rangle_{\mathcal{I}', \mathcal{I}} \\ &+ \sup_{\|(r_0 e^{s\Phi(0)}, y_0 e^{s\Phi(0)}, h_0 e^{s\Phi(0)})\|_{\mathcal{J}} \leq 1} \langle (r_0, y_0, h_0), (\sigma(0), q(0), \eta(0)) \rangle_{\mathcal{J}, \mathcal{J}'}\end{aligned}$$

Now, for  $(r_0, y_0, h_0) \in H^2(\mathbb{T}_L) \times H^3(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$  and  $f_r, f_y, f_h$  satisfying the inclusion  $f_r e^{s\Phi}, f_y e^{s\Phi} \in L^2(0, T; H^2(\mathbb{T}_L))$ ,  $f_h e^{s\Phi} \in L^2(0, T; H^1(\mathbb{T}_L))$ , we consider the controlled trajectory of (2.24) in Theorem 2.4.1 in order to obtain

$$\begin{aligned}& \langle (f_r, f_y, f_h), (\sigma, q, \eta) \rangle_{\mathcal{I}', \mathcal{I}} + \langle (r_0, y_0, h_0), (\sigma(0), q(0), \eta(0)) \rangle_{\mathcal{J}, \mathcal{J}'} \\ &= \langle (g_\sigma, \operatorname{div} g_z + \frac{\bar{\rho}^2}{\nu} g_\sigma, g_\eta), (r, y, h) \rangle_{L^2(H^{-2}) \times L^2(H^{-4}) \times L^2(H^{-3}), L^2(H^2) \times L^2(H^4) \times L^2(H^3)} \\ &+ \langle (\sigma, q, \eta), \chi_0(v_r, v_y, v_h) \rangle_{L^2(H^{-2}), L^2(H^2)}.\end{aligned}$$

Using (2.64), we get:

$$\begin{aligned}& \left\| (\sigma e^{-s\Phi}, qe^{-s\Phi}, \eta e^{-s\Phi}) \right\|_{L^2(H^{-2}) \times L^2(H^{-2}) \times L^2(H^{-1})} \\ &+ \left\| (\sigma(0)e^{-s\Phi(0)}, q(0)e^{-s\Phi(0)}, \eta(0)e^{-s\Phi(0)}) \right\|_{H^{-2} \times H^{-3} \times H^{-2}} \\ &\leq C \left( \left\| (g_\sigma e^{-6s\Phi/7}, g_z e^{-6s\Phi/7}, g_\eta e^{-6s\Phi/7}) \right\|_{L^2(H^{-2}) \times L^2(H^{-3}) \times L^2(H^{-3})} \right. \\ &\quad \left. + \left\| \chi_0(\sigma e^{-6s\Phi/7}, q e^{-6s\Phi/7}, \eta e^{-6s\Phi/7}) \right\|_{L^2(H^{-2}) \times L^2(H^{-2}) \times L^2(H^{-1})} \right). \quad (2.79)\end{aligned}$$

The next step is to recover estimates for  $z$  in (2.20). In order to do that, we notice that  $z$  satisfies the following equation in terms of  $(\sigma, q, \eta)$

$$-\bar{\rho}(\partial_t z + \bar{u} \cdot \nabla z) - \mu \Delta z = g_z + \bar{\rho} \frac{\mu}{\nu} \nabla \sigma + (\lambda + \mu) \nabla q + \bar{p} \nabla \eta, \quad \text{in } (0, T) \times \mathbb{T}_L. \quad (2.80)$$

Applying Theorem 2.3.3 items 1 & 2 to the dual system of (2.80) we get the following Carleman inequality,

$$\begin{aligned}
& \left\| z e^{-7s\Phi/6} \right\|_{L^2(H^{-1})} + \left\| z(0) e^{-7s\Phi(0)/6} \right\|_{H^{-2}} \\
& \leq C \left( \left\| \chi_0 z e^{-s\Phi} \right\|_{L^2(H^{-1})} + \left\| (\sigma, q, \eta) e^{-s\Phi} \right\|_{L^2(H^{-2})} + \left\| g_z e^{-s\Phi} \right\|_{L^2(H^{-3})} \right) \\
& \leq C \left( \left\| (g_\sigma e^{-6s\Phi/7}, g_z e^{-6s\Phi/7}, g_\eta e^{-6s\Phi/7}) \right\|_{L^2(H^{-2}) \times L^2(H^{-3}) \times L^2(H^{-3})} \right. \\
& \quad \left. + \left\| \chi_0 (\sigma e^{-6s\Phi/7}, q e^{-6s\Phi/7}, \eta e^{-6s\Phi/7}) \right\|_{L^2(H^{-2}) \times L^2(H^{-2}) \times L^2(H^{-1})} \right. \\
& \quad \left. + \left\| \chi_0 z e^{-s\Phi} \right\|_{L^2(H^{-1})} \right).
\end{aligned}$$

As  $\chi = 1$  in  $\text{Supp } \chi_0$  (recall (2.25)), we have  $\chi_0 \chi = \chi_0$  and  $\chi_0 \text{div } z = \chi_0 \text{div}(\chi z)$ . Now using that  $\chi_0$  is a multiplier on  $H^{-2}$  we get

$$\begin{aligned}
\left\| \chi_0 q e^{-6s\Phi/7} \right\|_{L^2(H^{-2})} & \leq C \left\| \chi_0 \text{div } z e^{-6s\Phi/7} \right\|_{L^2(H^{-2})} + C \left\| \chi_0 \sigma e^{-6s\Phi/7} \right\|_{L^2(H^{-2})} \\
& \leq C \left\| \chi z e^{-6s\Phi/7} \right\|_{L^2(H^{-1})} + C \left\| \chi \sigma e^{-6s\Phi/7} \right\|_{L^2(H^{-2})},
\end{aligned}$$

and combining the above results, we obtain

$$\begin{aligned}
& \left\| \sigma e^{-s\Phi} \right\|_{L^2(H^{-2})} + \left\| \sigma(0) e^{-s\Phi(0)} \right\|_{H^{-2}} + \left\| z e^{-7s\Phi/6} \right\|_{L^2(H^{-1})} + \left\| z(0) e^{-7s\Phi(0)/6} \right\|_{H^{-2}} \\
& \quad + \left\| \eta e^{-s\Phi} \right\|_{L^2(H^{-1})} + \left\| \eta(0) e^{-s\Phi(0)} \right\|_{H^{-2}} \\
& \leq C \left\| (g_\sigma e^{-6s\Phi/7}, g_z e^{-6s\Phi/7}, g_\eta e^{-6s\Phi/7}) \right\|_{L^2(H^{-2}) \times L^2(H^{-3}) \times L^2(H^{-3})} \\
& \quad + C \left\| \chi(\sigma, z, \eta) e^{-6s\Phi/7} \right\|_{L^2(H^{-2}) \times L^2(H^{-1}) \times L^2(H^{-1})}. \quad (2.81)
\end{aligned}$$

Using that  $(\sigma, z, \eta)$  satisfies Equation (2.20), we again argue by duality to deduce that System (2.19) is controllable and the estimate (2.78) follows immediately.  $\square$

From the previous theorem it follows that

$$\begin{aligned}
& \left\| \rho e^{5s\Phi/6} \right\|_{C^0([0,T]; H^2(\mathbb{T}_L)) \cap H^1(0,T; L^2(\mathbb{T}_L))} \\
& \quad + \left\| u e^{5s\Phi/6} \right\|_{L^2(0,T; H^3(\mathbb{T}_L)) \cap C^0([0,T]; H^2(\mathbb{T}_L)) \cap H^1(0,T; H^1(\mathbb{T}_L))} \\
& \quad + \left\| \theta e^{5s\Phi/6} \right\|_{L^2(0,T; H^3(\mathbb{T}_L)) \cap C^0([0,T]; H^2(\mathbb{T}_L)) \cap H^1(0,T; H^1(\mathbb{T}_L))} \\
& \leq C \left\| (\widehat{f}_\rho e^{s\Phi}, \widehat{f}_u e^{7s\Phi/6}, \widehat{f}_\theta e^{s\Phi}) \right\|_{L^2(0,T; H^2(\mathbb{T}_L)) \times L^2(0,T; H^1(\mathbb{T}_L)) \times L^2(0,T; H^1(\mathbb{T}_L))} \\
& \quad + C \left\| (\widehat{\rho}_0 e^{s\Phi(0)}, \widehat{u}_0 e^{7s\Phi(0)/6}, \widehat{\theta}_0 e^{s\Phi(0)}) \right\|_{H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)}. \quad (2.82)
\end{aligned}$$



which can be obtained by applying directly the classical regularity estimates at the equations satisfy by  $\rho e^{5s\Phi/6}$ ,  $ue^{5s\Phi/6}$  and  $\theta e^{5s\Phi/6}$ . Considering this, we can define the following linear operator: Given  $\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, \widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta$  in the vector space

$$\left\{ \begin{aligned} &(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, \widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta) \in H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \\ &\quad \times L^2(0, T; H^2(\mathbb{T}_L)) \times L^2(0, T; H^1(\mathbb{T}_L)) \times L^2(0, T; H^1(\mathbb{T}_L)) \\ &\text{with } \widehat{f}_\rho e^{s\Phi} \in L^2(0, T; H^2(\mathbb{T}_L)) \text{ and } \widehat{f}_u e^{7s\Phi/6}, \widehat{f}_\theta e^{s\Phi} \in L^2(0, T; H^1(\mathbb{T}_L)) \end{aligned} \right\},$$

we consider  $(\rho, u, \theta)$  the controlled trajectory given by 2.5.1 solving (2.19) with the initial condition  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$ . Then we define  $\mathcal{G}$  as

$$\mathcal{G}(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, \widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta) = (\rho, u, \theta).$$

## 2.6 Definition of the fixed point map

We recall that the control problem that we consider is: Given  $(\check{\rho}_0, \check{u}_0, \check{\theta}_0)$  small in  $H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$ , find control functions  $\check{v}_\rho$ ,  $\check{v}_u$  and  $\check{v}_\theta$  supported in  $[0, T] \times (\mathbb{T}_L \setminus \overline{\Omega})$  such that the solution of

$$\left\{ \begin{aligned} &\partial_t \check{\rho} + (\bar{u} + \check{u}) \cdot \nabla \check{\rho} + \bar{\rho} \operatorname{div}(\check{u}) = \check{v}_\rho + \check{f}_\rho(\check{\rho}, \check{u}, \check{\theta}) && \text{in } (0, T) \times \mathbb{T}_L \\ &\check{\rho}(\partial_t \check{u} + (\bar{u} + \check{u}) \cdot \nabla \check{u}) - \mu \Delta \check{u} - (\mu + \lambda) \nabla \operatorname{div} \check{u} \\ &\quad + \bar{p}_\rho \nabla \check{\rho} + \bar{p}_\theta \nabla \check{\theta} = \check{v}_u + \check{f}_u(\check{\rho}, \check{u}, \check{\theta}) && \text{in } (0, T) \times \mathbb{T}_L \\ &C_v \bar{\rho}(\partial_t \check{\theta} + (\bar{u} + \check{u}) \cdot \nabla \check{\theta}) - \kappa \Delta \check{\theta} + \bar{p} \operatorname{div}(\check{u}) = \check{v}_\theta + \check{f}_\theta(\check{\rho}, \check{u}, \check{\theta}) && \text{in } (0, T) \times \mathbb{T}_L \end{aligned} \right. \quad (2.83)$$

with initial data

$$\check{\rho}(0, x) = \check{\rho}_0(x), \quad \check{u}(0, x) = \check{u}_0(x), \quad \check{\theta}(0, x) = \check{\theta}_0(x) \quad \text{in } \mathbb{T}_L, \quad (2.84)$$

and source terms

$$\check{f}_\rho(\check{\rho}, \check{u}, \check{\theta}) = -\check{\rho} \operatorname{div}(\check{u}) \quad (2.85)$$

$$\check{f}_u(\check{\rho}, \check{u}, \check{\theta}) = -\check{\rho}(\partial_t \check{u} + (\bar{u} + \check{u}) \cdot \nabla \check{u}) - \nabla(p(\bar{\rho} + \check{\rho}, \bar{\theta} + \check{\theta}) - \bar{p}_\rho \rho - \bar{p}_\theta \theta) \quad (2.86)$$

$$\begin{aligned} \check{f}_\theta(\check{\rho}, \check{u}, \check{\theta}) = & -C_v \check{\rho}(\partial_t \check{\theta} + (\bar{u} + \check{u}) \cdot \nabla \check{\theta}) + \lambda \operatorname{div}(\check{u})^2 + 2\mu D(\check{u}) : \nabla \check{u} \\ & - (p(\bar{\theta} + \check{\theta}, \bar{\rho} + \check{\rho}) - \bar{p}) \operatorname{div}(\check{u}) \end{aligned} \quad (2.87)$$

satisfies

$$\check{\rho}(T) = 0, \quad \check{u}(T) = 0, \quad \check{\theta}(T) = 0 \quad \text{in } \mathbb{T}_L. \quad (2.88)$$

Now we will use a fixed point argument to prove controllability of (2.83)–(2.88). We notice that the left hand side of (2.83) is linear in  $(\check{\rho}, \check{u}, \check{\theta})$  except for the convective terms

$$\check{u} \cdot \nabla \check{\rho}, \quad \check{u} \cdot \nabla \check{u}, \quad \check{u} \cdot \nabla \check{\theta}$$

which are quadratic. We could try to put them in the source terms, but the problem is that the term  $\check{u} \cdot \nabla \check{\rho}$  in  $\check{f}_\rho$  causes a loss of regularity in the fixed point and we do not have a regularization effect in this equation. Instead, we consider a change of coordinates close to the identity that will take care of this term. The idea of this change of variables is similar to the Lagrangian coordinates but instead of making disappear completely the convective term, it will leave only the constant part. Concretely, we define the flow  $X_{\check{u}} = X_{\check{u}}(t, \tau, x)$  corresponding to  $\check{u}$  and defined for  $(t, \tau, x) \in [0, T] \times [0, T] \times \mathbb{T}_L$  by the equation

$$\frac{dX_{\check{u}}}{dt}(t, \tau, x) = \bar{u} + \check{u}(t, X_{\check{u}}(t, \tau, x)), \quad t \in [0, T], \quad X_{\check{u}}(\tau, \tau, x) = x. \quad (2.89)$$

In order to be well defined, we will assume that  $\check{u} \in L^2(0, T; H^3(\mathbb{T}_L))$ , in which case, because of the inclusion

$$L^2(0, T; H^3(\mathbb{T}_L)) \subseteq L^1(0, T; Lip(\mathbb{T}_L)),$$

the flow  $X_{\check{u}}$  is well-defined classically by Cauchy-Lipschitz's theorem. We then set, for  $(t, x) \in [0, T] \times \mathbb{T}_L$ ,

$$Y_{\check{u}}(t, x) = X_{\check{u}}(t, T, X_0(T, t, x)), \quad Z_{\check{u}}(t, x) = X_0(t, T, X_{\check{u}}(T, t, x)), \quad (2.90)$$

which are inverse one from another, i.e.  $Y_{\check{u}}(t, Z_{\check{u}}(t, x)) = Z_{\check{u}}(t, Y_{\check{u}}(t, x)) = x$  for all  $(t, x) \in [0, T] \times \mathbb{T}_L$ . For  $\check{u}$  suitably small, both transformations  $Y_{\check{u}}(t, \cdot)$  and  $Z_{\check{u}}(t, \cdot)$ ,  $t \in [0, T]$ , are diffeomorphisms of  $\mathbb{T}_L$  which are close to the identity map on the torus.

We thus set, for  $(t, x) \in [0, T] \times \mathbb{T}_L$ ,

$$\rho(t, x) = \check{\rho}(t, Y_{\check{u}}(t, x)), \quad u(t, x) = \check{u}(t, Y_{\check{u}}(t, x)), \quad \theta(t, x) = \check{\theta}(t, Y_{\check{u}}(t, x)). \quad (2.91)$$

In these new coordinates, the problem is reduced to the controllability of (See the appendix for the computations)

$$\begin{cases} \partial_t \rho + \bar{u} \cdot \nabla \rho + \bar{\rho} \operatorname{div}(u) = v_\rho \chi + f_\rho(\rho, u, \theta) & \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u} \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u \\ \quad + \bar{p}_\rho \nabla \rho + \bar{p}_\theta \nabla \theta = v_u \chi + f_u(\rho, u, \theta) & \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho}(\partial_t \theta + \bar{u} \cdot \nabla \theta) - \kappa \Delta \theta + \bar{p} \operatorname{div}(u) = v_\theta \chi + f_\theta(\rho, u, \theta) & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (2.92)$$

where  $\chi$  as defined in (2.18), with initial data given by

$$\rho(0, x) = \check{\rho}_0(Y_{\check{u}}(0, x)), \quad u(0, x) = \check{u}_0(Y_{\check{u}}(0, x)), \quad \theta(0, x) = \check{\theta}_0(Y_{\check{u}}(0, x)) \quad \text{in } \mathbb{T}_L, \quad (2.93)$$

and source terms  $f_\rho(\rho, u)$  given by

$$f_\rho(\rho, u) = -\rho DZ_{\check{u}}^t(t, Y_{\check{u}}(t, x)) : Du - \bar{p}(DZ_{\check{u}}^t(t, Y_{\check{u}}(t, x)) - I) : Du,$$

$f_u(\rho, u, \theta)$  by

$$\begin{aligned} f_{i,u}(\rho, u, \theta) &= -\rho(\partial_t u_i + \bar{u} \cdot \nabla u_i) + \sum_{j=1}^d \partial_i Z_{j,\check{u}}(t, Y_{\check{u}}(t, x)) \partial_j (p(\bar{\rho} + \rho, \bar{\theta} + \theta) - \bar{p}_\rho \rho - \bar{p}_\theta \theta) \\ &+ \mu \left( \sum_{j,k,\ell=1}^d \partial_{k,\ell} u_i (\partial_j Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{j,k}) (\partial_j Z_{\ell,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{j,\ell}) \right. \\ &\quad \left. + \sum_{k=1}^d \partial_k u_i \Delta Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) \right) \\ &+ (\lambda + \mu) \left( \sum_{j,k,\ell=1}^d \partial_{k,\ell} u_j (\partial_j Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{j,k}) (\partial_i Z_{\ell,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{i,\ell}) \right) \\ &+ (\lambda + \mu) \left( \sum_{j,k=1}^d \partial_{i,j} Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) \partial_k u_j \right) - \bar{p}_\rho \left( \sum_{j=1}^d (\partial_i Z_{j,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{i,j}) \partial_j \rho \right) \\ &\quad - \bar{p}_\theta \left( \sum_{j=1}^d (\partial_i Z_{j,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{i,j}) \partial_j \theta \right), \quad (2.94) \end{aligned}$$

and  $f_\theta(\rho, u, \theta)$  by

$$\begin{aligned} f_\theta(\rho, u, \theta) &= -C_v \rho (\partial_t \theta + \bar{u} \cdot \nabla \theta) - \bar{p}(DZ_{\check{u}}^t(t, Y_{\check{u}}(t, x)) - I) : Du \\ &\quad + \lambda (DZ_{\check{u}}^t(t, Y_{\check{u}}(t, x)) : Du)^2 \\ &+ \kappa \left( \sum_{j,k,\ell=1}^d \partial_{k,\ell} \theta (\partial_j Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{j,k}) (\partial_j Z_{\ell,\check{u}}(t, Y_{\check{u}}(t, x)) - \delta_{j,\ell}) \right. \\ &\quad \left. + \sum_{k=1}^d \partial_k \theta \Delta Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) \right) \\ &+ \mu \sum_{i,j,k,\ell=1}^d (\partial_k u_j \partial_i Z_{k,\check{u}}(t, Y_{\check{u}}(t, x)) + \partial_k u_i \partial_j Z_{k,\check{u}}(t, Y_{\check{u}}(t, x))) (\partial_\ell u_i \partial_j Z_{\ell,\check{u}}(t, Y_{\check{u}}(t, x))) \\ &\quad - (p(\bar{\rho} + \rho, \bar{\theta} + \theta) - \bar{p}) DZ_{\check{u}}^t(t, Y_{\check{u}}(t, x)) : Du, \quad (2.95) \end{aligned}$$

where  $\delta_{j,k}$  is the Kronecker symbol, and satisfying the final condition

$$\rho(T) = 0, \quad u(T) = 0, \quad \theta(T) = 0 \quad \text{in } \mathbb{T}_L. \quad (2.96)$$

The corresponding control functions in (2.92) will then be given for  $(t, x) \in [0, T] \times \mathbb{T}_L$  by

$$\begin{aligned} \check{v}_\rho(t, x) &= \chi(Z_{\bar{u}}(t, x))v_\rho(t, Z_{\bar{u}}(t, x)) \\ \check{v}_u(t, x) &= \chi(Z_{\bar{u}}(t, x))v_u(t, Z_{\bar{u}}(t, x)) \\ \check{v}_\theta(t, x) &= \chi(Z_{\bar{u}}(t, x))v_\theta(t, Z_{\bar{u}}(t, x)), \end{aligned} \quad (2.97)$$

which are supported in  $[0, T] \times (\mathbb{T}_L \setminus \bar{\Omega})$  provided that

$$\chi(Z_{\bar{u}}(t, x)) = 0 \text{ for all } (t, x) \in [0, T] \times \bar{\Omega}. \quad (2.98)$$

We notice that the change of coordinates satisfies the following transport equation

$$\partial_t Y_{\bar{u}}(t, x) + \bar{u} \cdot \nabla Y_{\bar{u}}(t, x) = \bar{u} + u(t, x).$$

This has two advantages. First, it allows to obtain  $Y_u$  directly from  $u$ . Second, it allows to find estimates for  $Y_u$  in terms of  $u$  which will be helpful when proving the convergence of the fixed point.

Now we introduce the map  $\mathcal{F} : (\hat{\rho}, \hat{u}, \hat{\theta}) \mapsto (\rho, u, \theta)$  that we will use in the fixed point argument, defined on a convex subset of some weighted Sobolev spaces, corresponding to some Carleman estimate described later. This map is constructed as follows. Given  $(\hat{\rho}, \hat{u}, \hat{\theta})$  small in a suitable norm, we first define  $\hat{Y} = \hat{Y}(t, x)$  as the solution of

$$\partial_t \hat{Y} + \bar{u} \cdot \nabla \hat{Y} = \bar{u} + \hat{u}, \quad \text{in } (0, T) \times \mathbb{T}_L, \quad \hat{Y}(T, x) = x, \quad \text{in } \mathbb{T}_L, \quad (2.99)$$

Then we define  $\hat{Z} = \hat{Z}(t, x)$  as follows: for all  $t \in [0, T]$ ,  $\hat{Z}(t, \cdot)$  is the inverse of  $\hat{Y}(t, \cdot)$  on  $\mathbb{T}_L$ . In other words, for all  $(t, x) \in [0, T] \times \mathbb{T}_L$ ,

$$\hat{Z}(t, \hat{Y}(t, x)) = x, \quad \hat{Y}(t, \hat{Z}(t, x)) = x. \quad (2.100)$$

If  $\hat{u}$  is small enough,  $\hat{Y}(t, \cdot)$  is invertible for all  $t \in [0, T]$ , see Proposition 2.7.1.

Corresponding to the initial data, we introduce

$$\hat{\rho}_0(x) = \check{\rho}_0(\hat{Y}(0, x)), \quad \hat{u}_0(x) = \check{u}_0(\hat{Y}(0, x)), \quad \hat{\theta}_0(x) = \check{\theta}_0(\hat{Y}(0, x)), \quad \text{in } \mathbb{T}_L, \quad (2.101)$$

and, corresponding to the source terms,

$$f_\rho(\hat{\rho}, \hat{u}) = -\hat{\rho} D \hat{Z}^t(t, \hat{Y}(t, x)) : D \hat{u} - \bar{\rho} (D \hat{Z}^t(t, \hat{Y}(t, x)) - I) : D \hat{u}, \quad (2.102)$$

$$\begin{aligned}
f_{i,u}(\widehat{\rho}, \widehat{u}, \widehat{\theta}) &= -\widehat{\rho}(\partial_t \widehat{u}_i + \bar{u} \cdot \nabla \widehat{u}_i) + \sum_{j=1}^d \partial_i \widehat{Z}_{j,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) \partial_j (p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p}_\rho \widehat{\rho} - \bar{p}_\theta \widehat{\theta}) \\
&+ \mu \left( \sum_{j,k,\ell=1}^d \partial_{k,\ell} \widehat{u}_i \left( \partial_j \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{j,k} \right) \left( \partial_j \widehat{Z}_{\ell,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{j,\ell} \right) \right. \\
&\quad \left. + \sum_{k=1}^d \partial_k \widehat{u}_i \Delta \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) \right) \\
&+ (\lambda + \mu) \left( \sum_{j,k,\ell=1}^d \partial_{k,\ell} \widehat{u}_j \left( \partial_j \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{j,k} \right) \left( \partial_i \widehat{Z}_{\ell,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{i,\ell} \right) \right) \\
&+ (\lambda + \mu) \left( \sum_{j,k=1}^d \partial_{i,j} \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) \partial_k \widehat{u}_j \right) - \bar{p}_\rho \left( \sum_{j=1}^d (\partial_i \widehat{Z}_{j,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{i,j}) \partial_j \widehat{\rho} \right) \\
&\quad - \bar{p}_\theta \left( \sum_{j=1}^d (\partial_i \widehat{Z}_{j,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{i,j}) \partial_j \widehat{\theta} \right), \quad (2.103)
\end{aligned}$$

and

$$\begin{aligned}
f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta}) &= -C_v \widehat{\rho}(\partial_t \widehat{\theta} + \bar{u} \cdot \nabla \widehat{\theta}) - \bar{p}(D \widehat{Z}_{\bar{u}}^t(t, \widehat{Y}_{\bar{u}}(t, x)) - I) : D \widehat{u} \\
&\quad + \lambda \left( D \widehat{Z}_{\bar{u}}^t(t, \widehat{Y}_{\bar{u}}(t, x)) : D \widehat{u} \right)^2 \\
&+ \kappa \left( \sum_{j,k,\ell=1}^d \partial_{k,\ell} \widehat{\theta} \left( \partial_j \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{j,k} \right) \left( \partial_j \widehat{Z}_{\ell,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) - \delta_{j,\ell} \right) \right. \\
&\quad \left. + \sum_{k=1}^d \partial_k \widehat{\theta} \Delta \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) \right) \\
&+ \mu \sum_{i,j,k,\ell=1}^d \left( \partial_k \widehat{u}_j \partial_i \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) + \partial_k \widehat{u}_i \partial_j \widehat{Z}_{k,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) \right) \left( \partial_\ell \widehat{u}_i \partial_j \widehat{Z}_{\ell,\bar{u}}(t, \widehat{Y}_{\bar{u}}(t, x)) \right) \\
&\quad - (p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p}) D \widehat{Z}_{\bar{u}}^t(t, \widehat{Y}_{\bar{u}}(t, x)) : D \widehat{u}, \quad (2.104)
\end{aligned}$$

We then look for  $(\rho, u, \theta)$  solving the controllability problem

$$\begin{cases} \partial_t \rho + \bar{u} \cdot \nabla \rho + \bar{p} \operatorname{div}(u) = v_\rho \chi + f_\rho(\widehat{\rho}, \widehat{u}, \widehat{\theta}) & \text{in } (0, T) \times \mathbb{T}_L, \\ \bar{\rho}(\partial_t u + \bar{u} \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \bar{p}_\rho \nabla \rho \\ + \bar{p}_\theta \nabla \theta = v_u \chi + f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}) & \text{in } (0, T) \times \mathbb{T}_L, \\ C_v \bar{\rho}(\partial_t \theta + \bar{u} \cdot \nabla \theta) - \kappa \Delta \theta + \bar{p} \operatorname{div}(u) = v_\theta \chi + f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta}) & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (2.105)$$

with initial data

$$\rho(0, x) = \widehat{\rho}_0(x), \quad u(0, x) = \widehat{u}_0(x), \quad \theta(0, x) = \widehat{\theta}_0(x) \quad \text{in } \mathbb{T}_L, \quad (2.106)$$

with source terms  $f_\rho(\widehat{\rho}, \widehat{u}), f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}), f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  as in (2.102)–(2.104), and satisfying the controllability objective (2.96).

## 2.7 Proof of Theorem 2.1.1

First of all, we now take  $s = s_0$  so that theorem 2.5.1 applies. In order to apply the fixed point, we consider the following spaces:

$$E = L^\infty(0, T; H^2(\mathbb{T}_L)) \cap H^1(0, T; L^2(\mathbb{T}_L)) \quad (2.107)$$

$$F = L^2(0, T; H^3(\mathbb{T}_L)) \cap L^\infty(0, T; H^2(\mathbb{T}_L)) \cap H^1(0, T; H^1(\mathbb{T}_L)) \quad (2.108)$$

and we consider the set

$$\mathcal{C}_R = \{(\rho, u, \theta) \text{ with } \rho \in E, u \in F, \theta \in F \\ \|\rho e^{5s\Phi/6}, u e^{5s\Phi/6}, \theta e^{5s\Phi/6}\|_{E \times F \times F} \leq R\}.$$

Now we consider the map

$$\mathcal{F}(\widehat{\rho}, \widehat{u}, \widehat{\theta}) = \mathcal{G}(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, f_\rho(\widehat{\rho}, \widehat{u}), f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}), f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})), \quad (2.109)$$

where  $\mathcal{G}$  is the control map constructed in Theorem 2.5.1,  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$  is defined in (2.101) and  $f_\rho(\widehat{\rho}, \widehat{u}), f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}), f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  are defined in (2.102)–(2.104). As we noticed earlier, a fixed point of  $\mathcal{F}$  will be a solution to (2.92). Now, we will prove that the map  $\mathcal{F}$  is in fact well-defined on from  $\mathcal{C}_R$  to  $\mathcal{C}_R$ . This will involve having estimates for  $\widehat{\rho}_0 e^{s\Phi(0)}, \widehat{u}_0 e^{7s\Phi(0)/6}, \widehat{\theta}_0 e^{s\Phi(0)}, f_\rho(\widehat{\rho}, \widehat{u}) e^{s\Phi}, f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}) e^{7s\Phi/6}$  and  $f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta}) e^{s\Phi}$ . In particular we will need estimates for  $\widehat{Y}$  and  $\widehat{Z}$  that we will borrow from [23].

### 2.7.1 The map $\mathcal{F}$ in (2.109) is well-defined on $\mathcal{C}_R$ and taking values in $\mathcal{C}_R$

We start by presenting the results about  $\widehat{Y}$  and  $\widehat{Z}$  defined in (2.99)–(2.100) and prove some of their properties, in particular that they are close to the identity map. We can then define the source term  $f_\rho(\widehat{\rho}, \widehat{u}), f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}), f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  and the initial data  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$ . Accordingly, we will deduce that the map  $\mathcal{F}$  in (2.109) is well-defined on  $\mathcal{C}_R$  for  $R > 0$  small enough.

### Estimates on $\widehat{Y}$ and $\widehat{Z}$ in (2.99)–(2.100)

The following result concerns to estimates of  $Y$  and  $Z$ , in particular that  $Y$  is close to the identity henceforth  $Z$  is well defined.

**Proposition 2.7.1.** *Let  $\widehat{u} \in L^2(0, T; H^3(\mathbb{T}_L)) \cap H^1(0, T; H^1(\mathbb{T}_L))$  with*

$$\left\| \widehat{u} e^{5s\Phi/6} \right\|_{L^2(0, T; H^3(\mathbb{T}_L)) \cap H^1(0, T; H^1(\mathbb{T}_L))} \leq R. \quad (2.110)$$

*Then the map  $\widehat{Y}$  defined in (2.99) satisfies, for some constant  $C$  independent of  $R > 0$ :*

$$\begin{aligned} & \left\| (\widehat{Y}(t, x) - x) e^{5s\Phi(t)/12} \right\|_{C^0([0, T]; H^3(\mathbb{T}_L))} \\ & + \left\| (\widehat{Y}(t, x) - x) e^{s\Phi(t)/3} \right\|_{C^1([0, T]; H^2(\mathbb{T}_L))} \leq CR. \end{aligned} \quad (2.111)$$

*Therefore, there exists  $R_0 \in (0, 1)$  such that for all  $R \in (0, R_0)$  the map  $\widehat{Z}$  defined in (2.100) is well-defined and satisfies*

$$\left\| (D\widehat{Z}(t, \widehat{Y}(t, x)) - I) e^{5s\Phi(t)/12} \right\|_{C^0([0, T]; H^2(\mathbb{T}_L))} \leq CR, \quad (2.112)$$

$$\left\| (D\widehat{Z}(t, \widehat{Y}(t, x)) - I) e^{s\Phi(t)/3} \right\|_{W^{1/4, 5}(0, T; H^{7/4}(\mathbb{T}_L))} \leq CR, \quad (2.113)$$

$$\left\| D^2\widehat{Z}(t, \widehat{Y}(t, x)) e^{5s\Phi(t)/12} \right\|_{L^\infty(0, T; H^1(\mathbb{T}_L))} \leq CR, \quad (2.114)$$

and

$$\chi(\widehat{Z}(t, x)) = 0 \text{ for all } (t, x) \in [0, T] \times \overline{\Omega}, \quad (2.115)$$

where  $\chi$  is defined by (2.18).

See [23] for the proof.

### Estimates on $f_\rho(\widehat{\rho}, \widehat{u})$ , $f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta})$ , $f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})$

Now we present estimates concerning the non linear terms in (2.105).

**Lemma 2.7.1.** *Let  $(\widehat{\rho}, \widehat{u}, \widehat{\theta}) \in \mathcal{C}_R$  for some  $s \geq s_0$  and  $R \in (0, R_0)$ , where  $R_0$  is given by Proposition 2.7.1. Then we have the following estimate*

$$\begin{aligned} & \left\| (f_\rho(\widehat{\rho}, \widehat{u}) e^{s\Phi}, f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}) e^{7s\Phi/6}, f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta}) e^{s\Phi}) \right\|_{L^2(H^2(\mathbb{T}_L)) \times L^2(H^1(\mathbb{T}_L)) \times L^2(H^1(\mathbb{T}_L))} \\ & \leq CR^2, \end{aligned} \quad (2.116)$$

where  $f_\rho(\widehat{\rho}, \widehat{u})$ ,  $f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta})$ ,  $f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  are defined in (2.102)–(2.104).

*Proof.* Now we will estimate each of the terms that appear in the non linear part.

We start by considering  $f_\rho(\widehat{\rho}, \widehat{u})$ . We perform the following estimates term by term: as  $H^2(\mathbb{T}_L)$  is an algebra in dimension  $d \leq 3$ ,

$$\begin{aligned} & \left\| (\widehat{\rho} D \widehat{Z}^t(t, \widehat{Y}(t, x)) : D \widehat{u}) e^{s\Phi} \right\|_{L^2(H^2)} \\ & \leq C \left\| \widehat{\rho} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| D \widehat{Z}(t, \widehat{Y}(t, x)) \right\|_{L^\infty(H^2)} \left\| \widehat{u} e^{5s\Phi/6} \right\|_{L^2(H^3)} \leq CR^2, \end{aligned}$$

and

$$\begin{aligned} & \left\| \bar{\rho} (D \widehat{Z}^t(t, \widehat{Y}(t, x)) - I) : D \widehat{u} e^{s\Phi} \right\|_{L^2(H^2)} \\ & \leq C \left\| (D \widehat{Z}(t, \widehat{Y}(t, x)) - I) e^{5s\Phi/12} \right\|_{L^\infty(H^2)} \left\| D \widehat{u} e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq CR^2. \end{aligned}$$

Now we continue with  $f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta})$ . We estimate each term in (2.94). Using that the product is continuous from  $H^2(\mathbb{T}_L) \times H^1(\mathbb{T}_L)$  into  $H^1(\mathbb{T}_L)$ ,

$$\left\| \widehat{\rho} \partial_t \widehat{u} e^{7s\Phi/6} \right\|_{L^2(H^1)} \leq \left\| \widehat{\rho} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| \partial_t \widehat{u} e^{5s\Phi/6} \right\|_{L^2(H^1)} \leq CR^2.$$

Using again that  $H^2(\mathbb{T}_L)$  is an algebra,

$$\left\| \widehat{\rho} \widehat{u} \cdot \nabla \widehat{u} e^{7s\Phi/6} \right\|_{L^2(H^1)} \leq C \left\| \widehat{\rho} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| \widehat{u} e^{5s\Phi/6} \right\|_{L^2(H^3)} \leq CR^2.$$

For  $i, j, k, \ell \in \{1, \dots, d\}$ , we get

$$\begin{aligned} & \left\| \partial_{k,\ell} \widehat{u}_j (\partial_j \widehat{Z}_k(t, \widehat{Y}(t, x)) - \delta_{j,k}) (\partial_i \widehat{Z}_\ell(t, \widehat{Y}(t, x)) - \delta_{i,\ell}) e^{7s\Phi/6} \right\|_{L^2(H^1)} \\ & \leq C \left\| D^2 \widehat{u} e^{5s\Phi/6} \right\|_{L^2(H^1)} \left\| (D \widehat{Z}(t, Y(t, x)) - I) e^{5s\Phi/12} \right\|_{L^\infty(H^2)}^2 \leq CR^3. \end{aligned}$$

Similarly, for  $i, j, k \in \{1, \dots, d\}$ ,

$$\begin{aligned} & \left\| \partial_{i,j} \widehat{Z}_k(t, \widehat{Y}(t, x)) \partial_k \widehat{u}_j e^{7s\Phi/6} \right\|_{L^2(H^1)} \\ & \leq C \left\| D^2 \widehat{Z}(t, Y(t, x)) e^{5s\Phi/12} \right\|_{L^\infty(H^1)} \left\| D \widehat{u} e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq CR^2. \quad (2.117) \end{aligned}$$

In order to estimate the terms coming from the pressure, we write

$$p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p} - \bar{p}_\rho \widehat{\rho} - \bar{p}_\theta \widehat{\theta} = \widehat{\rho}^2 f(\widehat{\rho}, \widehat{\theta}) + \widehat{\theta}^2 g(\widehat{\rho}, \widehat{\theta}) + \widehat{\rho} \widehat{\theta} h(\widehat{\rho}, \widehat{\theta})$$



where  $f, g, h$  are  $C^1$  functions depending on the pressure law (here we use that the pressure law  $p$  belongs to  $C^3$  locally around  $\bar{\rho}, \bar{\theta}$ ), so we have

$$\begin{aligned} & \nabla(p(\bar{\rho} + \hat{\rho}, \bar{\theta} + \hat{\theta}) - \bar{p}_\rho \hat{\rho} - \bar{p}_\theta \hat{\theta}) \\ &= (2\hat{\rho}f(\hat{\rho}, \hat{\theta}) + \hat{\rho}^2 \partial_\rho f(\hat{\rho}, \hat{\theta}) + \hat{\theta}^2 \partial_\rho g(\hat{\rho}, \hat{\theta}) + \hat{\theta}h(\hat{\rho}, \hat{\theta}) + \hat{\rho}\hat{\theta} \partial_\rho h(\hat{\rho}, \hat{\theta})) \nabla \hat{\rho} \\ & \quad + (2\hat{\theta}g(\hat{\rho}, \hat{\theta}) + \hat{\theta}^2 \partial_\theta g(\hat{\rho}, \hat{\theta}) + \hat{\rho}^2 \partial_\theta f(\hat{\rho}, \hat{\theta}) + \hat{\rho}h(\hat{\rho}, \hat{\theta}) + \hat{\rho}\hat{\theta} \partial_\theta h(\hat{\rho}, \hat{\theta})) \nabla \hat{\theta}. \end{aligned}$$

As  $\|\hat{\rho}\|_{L^\infty(L^\infty)} \leq CR \leq C$ , we thus obtain, for  $i, j \in \{1, \dots, d\}$ ,

$$\begin{aligned} & \left\| \partial_i \hat{Z}_j(t, \hat{Y}(t, x)) \partial_j (p(\bar{\rho} + \hat{\rho}, \bar{\theta} + \hat{\theta}) - \bar{p}_\rho \hat{\rho} - \bar{p}_\theta \hat{\theta}) e^{7s\Phi/6} \right\|_{L^2(H^1)} \\ & \leq C \left\| D\hat{Z}(t, \hat{Y}) \right\|_{L^\infty(H^2)} \left\| \hat{\rho} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| \nabla \hat{\rho} e^{5s\Phi/6} \right\|_{L^2(H^1)} \\ & \quad + C \left\| D\hat{Z}(t, \hat{Y}) \right\|_{L^\infty(H^2)} \left\| \hat{\theta} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| \nabla \hat{\theta} e^{5s\Phi/6} \right\|_{L^2(H^1)} \leq CR^2. \end{aligned}$$

For  $i, j \in \{1, \dots, d\}$ ,

$$\begin{aligned} & \left\| \bar{p}_\rho (\partial_i \hat{Z}_j(t, \hat{Y}(t, x)) - \delta_{i,j}) \partial_j \hat{\rho} e^{7s\Phi/6} \right\|_{L^2(H^1)} \\ & \leq C \left\| (D\hat{Z}(t, \hat{Y}(t, x)) - I) e^{5s\Phi/12} \right\|_{L^\infty(H^2)} \left\| \hat{\rho} e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq CR^2. \end{aligned}$$

$$\begin{aligned} & \left\| \bar{p}_\theta (\partial_i \hat{Z}_j(t, \hat{Y}(t, x)) - \delta_{i,j}) \partial_j \hat{\theta} e^{7s\Phi/6} \right\|_{L^2(H^1)} \\ & \leq C \left\| (D\hat{Z}(t, \hat{Y}(t, x)) - I) e^{5s\Phi/12} \right\|_{L^\infty(H^2)} \left\| \hat{\theta} e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq CR^2. \end{aligned}$$

Finally we perform estimates in  $f_\theta(\hat{\rho}, \hat{u}, \hat{\theta})$ . Similarly to the first two estimates for  $f_u$ , we have

$$\begin{aligned} & \left\| C_v \hat{\rho} \partial_t \hat{\theta} e^{s\Phi} \right\|_{L^2(H^1)} \leq \left\| \hat{\rho} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| \partial_t \hat{\theta} e^{5s\Phi/6} \right\|_{L^2(H^1)} \leq CR^2. \\ & \left\| \hat{\rho} \hat{u} \cdot \nabla \hat{\theta} e^{s\Phi} \right\|_{L^2(H^1)} \leq C \left\| \hat{\rho} e^{5s\Phi/6} \right\|_{L^\infty(H^2)} \left\| \hat{\theta} e^{5s\Phi/6} \right\|_{L^2(H^3)} \leq CR^2. \end{aligned}$$

For  $i, j, k, \ell \in \{1, \dots, d\}$ , we get

$$\begin{aligned} & \left\| \partial_{k,\ell} \hat{\theta} (\partial_j \hat{Z}_k(t, \hat{Y}(t, x)) - \delta_{j,k}) (\partial_j \hat{Z}_\ell(t, \hat{Y}(t, x)) - \delta_{j,\ell}) e^{s\Phi} \right\|_{L^2(H^1)} \\ & \leq C \left\| D^2 \hat{\theta} e^{5s\Phi/6} \right\|_{L^2(H^1)} \left\| (D\hat{Z}(t, Y(t, x)) - I) e^{5s\Phi/12} \right\|_{L^\infty(H^2)}^2 \leq CR^3. \end{aligned}$$

$$\begin{aligned} & \left\| \bar{p}(D\widehat{Z}^t(t, \widehat{Y}(t, x)) - I) : D\widehat{u}e^{s\Phi} \right\|_{L^2(H^1)} \\ & \leq C \left\| (D\widehat{Z}(t, \widehat{Y}(t, x)) - I)e^{5s\Phi/12} \right\|_{L^\infty(H^2)} \left\| D\widehat{u}e^{5s\Phi/6} \right\|_{L^2(H^1)} \leq CR^2. \end{aligned}$$

$$\begin{aligned} & \left\| \lambda \left( D\widehat{Z}_{\check{u}}^t(t, \widehat{Y}_{\check{u}}(t, x)) : D\widehat{u} \right)^2 e^{s\Phi} \right\|_{L^2(H^1)} \\ & \leq C \left\| D\widehat{Z}(t, \widehat{Y}(t, x)) \right\|_{L^\infty(H^2)}^2 \left\| D\widehat{u}e^{5s\Phi/6} \right\|_{L^\infty(H^1)} \left\| D\widehat{u}e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq CR^2. \end{aligned}$$

$$\begin{aligned} & \left\| \partial_k \widehat{u}_j \partial_i \widehat{Z}_{k, \check{u}}(t, Y_{\check{u}}(t, x)) \partial_\ell \widehat{u}_i \partial_j \widehat{Z}_{\ell, \check{u}}(t, Y_{\check{u}}(t, x)) e^{s\Phi} \right\|_{L^2(H^1)} \\ & \leq C \left\| D\widehat{Z}(t, \widehat{Y}(t, x)) \right\|_{L^\infty(H^2)}^2 \left\| D\widehat{u}e^{5s\Phi/6} \right\|_{L^\infty(H^1)} \left\| D\widehat{u}e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq CR^2. \end{aligned}$$

$$\begin{aligned} & \left\| (p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p}) D\widehat{Z}_{\check{u}}^t(t, Y_{\check{u}}(t, x)) : D\widehat{u}e^{s\Phi} \right\|_{L^2(H^1)} \\ & \leq \left\| D\widehat{Z}(t, \widehat{Y}(t, x)) \right\|_{L^\infty(H^2)} \left\| (p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p}) e^{5s\Phi/6} \right\|_{L^2(H^2)} \left\| D\widehat{u}e^{5s\Phi/6} \right\|_{L^\infty(H^1)}, \end{aligned}$$

and using the fact that  $p$  is  $C^3$

$$\left\| (p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p}) e^{5s\Phi/6} \right\|_{L^2(H^2)} \leq C \left\| \widehat{\rho} e^{5s\Phi/6} \right\|_{L^2(H^2)} + C \left\| \widehat{\theta} e^{5s\Phi/6} \right\|_{L^2(H^2)},$$

hence

$$\left\| (p(\bar{\rho} + \widehat{\rho}, \bar{\theta} + \widehat{\theta}) - \bar{p}) D\widehat{Z}_{\check{u}}^t(t, Y_{\check{u}}(t, x)) : D\widehat{u}e^{s\Phi} \right\|_{L^2(H^1)} \leq CR^2.$$

Combining all the above estimates yields Lemma 2.7.1.  $\square$

### Estimates on $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$

We finally state the following estimates on  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$  defined in (2.101):

**Lemma 2.7.2.** *Let  $\widehat{u} \in L^2(0, T; H^3(\mathbb{T}_L)) \cap H^1(0, T; H^1(\mathbb{T}_L))$  satisfying (2.110) for some  $R \leq R_0$  given by Proposition 2.7.1. Let  $\delta \in (0, 1)$  and  $(\check{\rho}_0, \check{u}_0, \check{\theta}) \in H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$  with*

$$\left\| (\check{\rho}_0, \check{u}_0, \check{\theta}) \right\|_{H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)} \leq C_L \delta. \quad (2.118)$$

Define  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0)$  as in (2.101). Then there exists a constant  $C > 0$  independent of  $R$  such that

$$\left\| (\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0) \right\|_{H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)} \leq C\delta. \quad (2.119)$$

*Proof.* It is a straightforward consequence of the estimate (2.111) derived in Proposition 2.7.1.  $\square$

## Conclusion

Gathering together all the estimates obtained in Proposition 2.7.1, Lemmas 2.7.1 and 2.7.2 and using Theorem 2.5.1, we get the following theorem, that will be the key to apply the fixed point argument ( in particular that  $\mathcal{C}_R$  is invariant under  $\mathcal{F}$  for  $R$  small enough).

**Proposition 2.7.2.** *Let  $(\check{\rho}_0, \check{u}_0, \check{\theta}_0)$  in  $H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$  satisfying (2.118) for some  $\delta > 0$ ,  $(\hat{\rho}, \hat{u}, \hat{\theta}) \in \mathcal{C}_R$  for some  $R \in (0, R_0)$  with  $R_0$  given by Proposition 2.7.1. Then the map  $\mathcal{F}$  in (2.109) is well-defined, and there exist a constant  $C$  such that  $(\rho, u, \theta) = \mathcal{F}(\hat{\rho}, \hat{u}, \hat{\theta})$  satisfies*

$$\begin{aligned} \|\rho e^{5s\Phi/6}\|_{L^\infty(H^2) \cap H^1(L^2)} &\leq CR^2 + C\delta. & (2.120) \\ \|ue^{5s\Phi/6}\|_{L^2(H^3) \cap L^\infty(H^2) \cap H^1(H^1)} &\leq CR^2 + C\delta. \\ \|\theta e^{5s\Phi/6}\|_{L^2(H^3) \cap L^\infty(H^2) \cap H^1(H^1)} &\leq CR^2 + C\delta. \end{aligned}$$

Besides, the condition (2.115) is satisfied for  $\chi$  defined in (2.18).

## 2.7.2 The fixed point argument

Let's consider an initial condition  $(\rho_0, u_0, \theta_0)$  in  $H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$  small enough in the sense of (2.7) with  $\delta > 0$ . Then we extend the initial condition to  $(\check{\rho}_0, \check{u}_0, \check{\theta}_0)$  satisfying (2.17) and (2.118). Therefore, from Proposition 2.7.2, the map  $\mathcal{F}$  in (2.109) is well-defined for  $(\hat{\rho}, \hat{u}, \hat{\theta}) \in \mathcal{C}_R$  for  $R \in (0, R_0)$ , with  $R_0 > 0$  given by Proposition 2.7.1, and  $(\rho, u, \theta) = \mathcal{F}(\hat{\rho}, \hat{u}, \hat{\theta})$  satisfies (2.120) with some constant  $C > 0$ . We now choose  $R \in (0, R_0)$  such that  $CR < 1/2$  and  $\delta = R/(2C)$ , so that as a consequence of Proposition 2.7.2,  $\mathcal{F}$  maps  $\mathcal{C}_R$  into itself.

Now we intend to apply Schauder's fixed point theorem to the map  $\mathcal{F}$ . First, we notice that the set  $\mathcal{C}_R$  is convex and compact when endowed with the  $(L^2(0, T; L^2(\mathbb{T}_L)))^3$  topology, as a simple consequence of Aubin-Lions' Lemma, see e.g. [59].

Now the only hypothesis left of the Schauder's fixed point theorem is the continuity of the map  $\mathcal{F}$  on  $\mathcal{C}_R$  endowed with the  $(L^2(0, T; L^2(\mathbb{T}_L)))^3$  topology. Let us consider a sequence  $(\hat{\rho}_n, \hat{u}_n, \hat{\theta}_n)$  in  $\mathcal{C}_R$  converging strongly in  $(L^2(0, T; L^2(\mathbb{T}_L)))^3$  to some element  $(\hat{\rho}, \hat{u}, \hat{\theta})$ . The set  $\mathcal{C}_R$  is closed under the topology of  $(L^2(0, T; L^2(\mathbb{T}_L)))^3$  hence  $(\hat{\rho}, \hat{u}, \hat{\theta}) \in \mathcal{C}_R$ . Also, because of

Banach-Alaoglu's Theorem we have the weak-\* convergence of  $(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n)$  towards  $(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  in  $(L^\infty(H^2) \cap H^1(L^2)) \times (L^2(H^3) \cap H^1(H^1)) \times (L^2(H^3) \cap H^1(H^1))$ . Furthermore, because of the compactness given by Aubin-Lions' lemma, we also have the following strong convergences to  $(\widehat{\rho}, \widehat{u}, \widehat{\theta})$ :

$$\begin{aligned}
\widehat{\rho}_n &\xrightarrow[n \rightarrow \infty]{} \widehat{\rho} \quad \text{strongly in } L^\infty(0, T; L^\infty(\mathbb{T}_L)), \\
\widehat{\rho}_n &\xrightarrow[n \rightarrow \infty]{} \widehat{\rho} \quad \text{strongly in } L^2(0, T; H^1(\mathbb{T}_L)), \\
\widehat{u}_n &\xrightarrow[n \rightarrow \infty]{} \widehat{u} \quad \text{strongly in } L^2(0, T; H^2(\mathbb{T}_L)), \\
\widehat{\theta}_n &\xrightarrow[n \rightarrow \infty]{} \widehat{\theta} \quad \text{strongly in } L^2(0, T; H^2(\mathbb{T}_L)), \\
\widehat{\theta}_n &\xrightarrow[n \rightarrow \infty]{} \widehat{\theta} \quad \text{strongly in } L^\infty(0, T; L^\infty(\mathbb{T}_L)).
\end{aligned} \tag{2.121}$$

To every term of the sequence  $\widehat{u}_n$ , we associate the corresponding flow  $\widehat{Y}_n$  solving the transport equation,

$$\partial_t \widehat{Y}_n + \bar{u} \cdot \nabla \widehat{Y}_n = \bar{u} + \widehat{u}_n \quad \text{in } (0, T) \times \mathbb{T}_L, \quad \widehat{Y}_n(T, x) = x \quad \text{in } \mathbb{T}_L, \tag{2.122}$$

and the inverse  $\widehat{Z}_n$  as defined in (2.100). Using the estimate (2.111) we obtain that the sequence  $\widehat{Y}_n$  is bounded in  $L^\infty(H^3) \cap W^{1, \infty}(H^2)$  therefore it converges weakly-\* in  $L^\infty(H^3) \cap W^{1, \infty}(H^2)$ . Now we take the limit in Equation (2.122) to obtain that the weak limit  $\widehat{Y}$  satisfies (2.99). Also, because of the compactness given by Aubin-Lions' lemma we also have the strong convergence of  $\widehat{Y}_n$  to  $\widehat{Y}$  in  $W^{1/4, 5}(0, T; H^{11/4}(\mathbb{T}_L))$  and in  $C^0([0, T]; C^1(\mathbb{T}_L))$ . Now that we have convergence in classical spaces, we use the inverse function theorem to obtain that the sequence  $\widehat{Z}_n$  strongly converges to  $\widehat{Z}$  in  $C^0([0, T]; C^1(\mathbb{T}_L))$ . Furthermore

$$\begin{aligned}
D\widehat{Z}_n(t, \widehat{Y}_n(t, x)) &\xrightarrow[n \rightarrow \infty]{} D\widehat{Z}(t, \widehat{Y}(t, x)) \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}_L), \\
(\widehat{\rho}_{0,n}, \widehat{u}_{0,n}, \widehat{\theta}_{0,n}) &\xrightarrow[n \rightarrow \infty]{} (\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0) \quad \text{in } (\mathcal{D}'((0, T) \times \mathbb{T}_L))^3,
\end{aligned} \tag{2.123}$$

where  $(\widehat{\rho}_{0,n}, \widehat{u}_{0,n}, \widehat{\theta}_{0,n}) = (\check{\rho}_0(\widehat{Y}_n(0, x)), \check{u}_0(\widehat{Y}_n(0, x)), \check{\theta}_0(\widehat{Y}_n(0, x)))$ . From the uniform bounds (2.112)–(2.113) on the quantity  $D\widehat{Z}_n(t, \widehat{Y}_n(t, x)) - I$  and Aubin-Lions' Lemma, we also deduce that

$$D\widehat{Z}_n(t, \widehat{Y}_n(t, x)) - I \xrightarrow[n \rightarrow \infty]{} D\widehat{Z}(t, \widehat{Y}(t, x)) - I \quad \text{strongly in } L^2(0, T; H^1(\mathbb{T}_L)). \tag{2.124}$$

Using then the uniform bound (2.114), the identity

$$D^2\widehat{Z}_n(t, \widehat{Y}_n(t, x)) = D(D\widehat{Z}_n(t, \widehat{Y}_n(t, x)))D\widehat{Z}_n(t, \widehat{Y}_n(t, x)),$$

and the convergence (2.124), we also conclude that

$$D^2 \widehat{Z}_n(t, \widehat{Y}_n(t, x)) \rightharpoonup_{n \rightarrow \infty} D^2 \widehat{Z}(t, \widehat{Y}(t, x)) \text{ weakly in } L^2(0, T; H^1(\mathbb{T}_L)). \quad (2.125)$$

Considering all the previous convergences, we can prove that the functions  $f_\rho(\widehat{\rho}_n, \widehat{u}_n)$ ,  $f_u(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n)$  and  $f_\theta(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n)$  weakly converge to  $f_\rho(\widehat{\rho}, \widehat{u})$ ,  $f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  and  $f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  in  $L^1(0, T; H^1(\mathbb{T}_L))$ , and with Lemma 2.7.1, weakly in the weighted Sobolev space described by (2.116). As the map  $\mathcal{G}$  in Theorem 2.5.1 is a linear continuous operator in  $(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, \widehat{f}_\rho, \widehat{f}_u, \widehat{f}_\theta)$ , it is weakly continuous as well. Hence

$$\begin{aligned} (\rho_n, u_n, \theta_n) &= \mathcal{F}(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n) \\ &= \mathcal{G}(\widehat{\rho}_{0,n}, \widehat{u}_{0,n}, \widehat{\theta}_{0,n}, f_\rho(\widehat{\rho}_n, \widehat{u}_n), f_u(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n), f_\theta(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n)) \end{aligned} \quad (2.126)$$

weakly converges to

$$(\rho, u, \theta) = \mathcal{F}(\widehat{\rho}, \widehat{u}, \widehat{\theta}) = \mathcal{G}(\widehat{\rho}_0, \widehat{u}_0, \widehat{\theta}_0, f_\rho(\widehat{\rho}, \widehat{u}), f_u(\widehat{\rho}, \widehat{u}, \widehat{\theta}), f_\theta(\widehat{\rho}, \widehat{u}, \widehat{\theta}))$$

in  $\mathcal{D}'((0, T) \times \mathbb{T}_L)$ . But we know that  $\mathcal{C}_R$  is stable by the map  $\mathcal{F}$ , so that  $(\rho_n, u_n, \theta_n)$  all belong to the set  $\mathcal{C}_R$ , which is compact for the  $(L^2(0, T; L^2(\mathbb{T}_L)))^3$  topology. Therefore,  $(\rho_n, u_n, \theta_n) = \mathcal{F}(\widehat{\rho}_n, \widehat{u}_n, \widehat{\theta}_n)$  strongly converges to  $(\rho, u, \theta) = \mathcal{F}(\widehat{\rho}, \widehat{u}, \widehat{\theta})$  in  $(L^2(0, T; L^2(\mathbb{T}_L)))^3$ .

Now that we verified all the hypothesis of Schauder's fixed point theorem we obtain the existence of a fixed point  $(\rho, u, \theta) = \mathcal{F}(\rho, u, \theta)$  which by construction solves the control problem (2.92)–(2.93)–(2.96).

To go back to the original system (2.83), we define  $Y$  as the solution of

$$\partial_t Y + \bar{u} \cdot \nabla Y = \bar{u} + u \text{ in } (0, T) \times \mathbb{T}_L, \quad Y(T, x) = x \text{ in } \mathbb{T}_L,$$

and  $Z = Z(t, x)$  such that for all  $t \in [0, T]$ ,  $Z(t, \cdot)$  is the inverse of  $Y(t, \cdot)$ , which is well-defined according to Proposition 2.7.1. We then simply set, for all  $(t, x) \in [0, T] \times \mathbb{T}_L$ ,

$$\check{\rho}(t, x) = \rho(t, Z(t, x)), \quad \check{u}(t, x) = u(t, Z(t, x)), \quad \check{\theta}(t, x) = \theta(t, Z(t, x)). \quad (2.127)$$

By construction,  $(\check{\rho}, \check{u}, \check{\theta})$  solves (2.83)–(2.84) and the controllability requirement (2.88) with control functions  $(\check{v}_\rho, \check{v}_u, \check{v}_\theta)$  defined for  $(t, x) \in [0, T] \times \mathbb{T}_L$  by

$$\check{v}_\rho(t, x) = \chi(Z(t, x))v_\rho(t, Z(t, x)), \quad \check{v}_u(t, x) = \chi(Z(t, x))v_u(t, Z(t, x)),$$

$$\check{v}_\theta(t, x) = \chi(Z(t, x))v_\theta(t, Z(t, x)).$$

These control functions are supported in  $[0, T] \times (\mathbb{T}_L \setminus \overline{\Omega})$  thanks to (2.115), so that by restriction on  $\Omega$ , we get a solution  $(\rho_S, u_S, \theta_S) = (\bar{\rho}, \bar{u}, \bar{\theta}) + (\check{\rho}, \check{u}, \check{\theta})$  of (2.1) satisfying (2.8)–(2.9).

To get the regularity estimate in (2.10), we first show that the fixed point  $(\rho, u, \theta)$  of  $\mathcal{F}$  satisfies

$$\begin{aligned} (\rho, u, \theta) \in & C^0([0, T]; H^2(\mathbb{T}_L)) \times (L^2(0, T; H^3(\mathbb{T}_L)) \cap C^0([0, T]; H^2(\mathbb{T}_L))) \\ & \times (L^2(0, T; H^3(\mathbb{T}_L)) \cap C^0([0, T]; H^2(\mathbb{T}_L))), \end{aligned} \quad (2.128)$$

which is a consequence of (2.82). From these regularity results on  $(\rho, u, \theta)$ , (2.127) and the regularity estimates obtained on  $Z$  in Proposition 2.7.1, we deduce that

$$\begin{aligned} (\check{\rho}, \check{u}, \check{\theta}) \in & C^0([0, T]; H^2(\mathbb{T}_L)) \times (L^2(0, T; H^3(\mathbb{T}_L)) \cap C^0([0, T]; H^2(\mathbb{T}_L))) \\ & \times (L^2(0, T; H^3(\mathbb{T}_L)) \cap C^0([0, T]; H^2(\mathbb{T}_L))), \end{aligned} \quad (2.129)$$

from which (2.10) follows.

# Chapter 3

## Asymptotic Stabilization of the compressible Navier-Stokes equations with an interior feedback on the density

### 3.1 Introduction

The aim of this article is to study the asymptotic stabilization of the compressible Navier-Stokes equations on the torus  $\mathbb{T}^d$  of dimension  $d \leq 3$ , namely the following system

$$\partial_t \rho_S + \operatorname{div}(\rho_S u_S) = 0 \text{ in } \mathbb{R}_+ \times \mathbb{T}^d \quad (3.1)$$

$$\rho_S(\partial_t u_S + (u_S \cdot \nabla)u_S) + \nabla(p(\rho_S)) - \mu \Delta u_S - (\mu + \lambda) \nabla(\operatorname{div} u_S) = 0 \text{ in } \mathbb{R}_+ \times \mathbb{T}^d \quad (3.2)$$

where  $\rho_S$  is the density of the fluid,  $u_S$  the velocity and  $p$  is the pressure which is a function of  $\rho_S$ , typically  $p(\rho_S) = \kappa \rho_S^\gamma$ .

We consider the problem of stabilizing the system around a constant homogeneous state  $(\bar{u}, \bar{\rho})$  with an interior feedback control supported inside an open set  $\omega' \subseteq \mathbb{T}^d$  and acting on the density only. This means finding a feedback function  $K_\rho$  such that the following system is exponentially stable around  $(\bar{u}, \bar{\rho})$ :

$$\begin{cases} \partial_t \rho_S + \operatorname{div}(\rho_S u_S) = K_\rho(\rho_S - \bar{\rho}, u_S - \bar{u}, \sigma) & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \rho_S(\partial_t u_S + (u_S \cdot \nabla)u_S) + \nabla(p(\rho_S)) \\ \quad - \mu \Delta u_S - (\mu + \lambda) \nabla(\operatorname{div} u_S) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \end{cases} \quad (3.3)$$

Where  $\bar{p}_\rho = p'(\bar{\rho})$ ,  $\gamma > 0$  is a constant related to the decay of the transport equation.

### Assumptions

**Assumption 1** We assume the pressure function to be at least  $C^{2, \frac{1}{2}}$  on a neighborhood of  $\bar{\rho}$ . We will assume that all the characteristics of the flow given by  $\bar{u}$  intersect  $\omega'$ , in particular, the interesting case is when  $\bar{u} \neq 0$  otherwise we would require  $\omega = \mathbb{T}^d$ . Then we can choose  $\omega$  with the same property and  $\bar{\omega} \subset \omega'$ .

**Assumption 2** We also will assume that the quantity

$$\frac{\bar{\rho}\bar{p}_\rho}{\nu} > 0$$

is small enough. More especially, we assume that it is smaller than the first nonzero eigenvalue of laplacian in the torus.

$$0 < \frac{\bar{\rho}\bar{p}_\rho}{\nu} < \lambda_1$$

In order to construct the feedback operator we introduce a one dimensional integrator  $\sigma$  as the solution of the ODE:

We consider a function  $\varphi : \mathbb{T}^d \rightarrow \mathbb{R}$  supported in  $\omega' \setminus \omega$  and with  $\langle \phi \rangle = 1$  and a function  $\sigma : [0, \infty] \rightarrow \mathbb{R}$  given by the differential equation:

$$\dot{\sigma} = \sigma \langle \operatorname{div} u_S \varphi \rangle - \sigma \frac{\bar{\rho}\bar{p}_\rho}{\nu} \langle \operatorname{div} u_S \rho \rangle + \langle \gamma \chi_\omega (\rho_S - \bar{\rho}) \rangle - \gamma \langle \rho_S - \bar{\rho} \rangle,$$

where  $\langle \cdot \rangle$  denotes the average of a function on the torus

$$\langle f \rangle = \int_{\mathbb{T}^d} f(x) dx.$$

Then we will consider the feedback operator:

$$K_\rho(\rho, \hat{u}, \sigma) = -\gamma \chi_\omega \rho + \sigma(\bar{u} + \hat{u}) \cdot \nabla \varphi + \sigma \frac{\bar{\rho}\bar{p}_\rho}{\nu} \varphi + \dot{\sigma} \varphi. \quad (3.4)$$

This way our system reads:

$$\begin{cases} \partial_t \rho_S + \operatorname{div}(\rho_S u_S) = K_\rho(\rho_S - \bar{\rho}, u_S - \bar{u}, \sigma) & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \rho_S (\partial_t u_S + (u_S \cdot \nabla) u_S) + \nabla(p(\rho_S)) \\ \quad - \mu \Delta u_S - (\mu + \lambda) \nabla(\operatorname{div} u_S) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \dot{\sigma} = \sigma \langle \operatorname{div} u_S \varphi \rangle - \sigma \frac{\bar{\rho}\bar{p}_\rho}{\nu} \langle \operatorname{div} u_S \rho \rangle + \langle \gamma \chi_\omega (\rho_S - \bar{\rho}) \rangle - \gamma \langle \rho_S - \bar{\rho} \rangle & \text{in } \mathbb{R}_+ \end{cases} \quad (3.5)$$



In [18], Coron explains the need of adding an integrator for the stabilization of certain systems. In the topic of stabilization of Navier Stokes, we can remark the work of [12] for the stabilizability (and controllability) of the Navier-Stokes System in the isentropic case with control on the velocity, [50] for a study of the same system in a more regular functional setting and [13] for a study of stabilizability on the zero velocity case. For the related problem of controllability we can mention [23], in the case of boundary controls in both equations (but it is reduced to the internal controllability in the torus as in this article).

The main result of this article is the following theorem:

**Theorem 3.1.1.** *There is a constant  $\delta$  such that if  $(\rho_0, u_0) \in H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}$  and  $\|(\rho_0, u_0)\| \leq \delta$ , then the solution of (3.5) satisfies the following exponential decay estimate, for  $M$  satisfying (3.7)*

$$\|(e^{Mt}\rho, e^{Mt}(u - \langle u \rangle), e^{Mt}\sigma)\|_{L^\infty(0,T;H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon} \times \mathbb{R})} \leq C\|(\rho_0, u_0)\|_{H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}}$$

**Theorem 3.1.2.** *If Assumption 1 and Assumption 2 are satisfied and  $M, \bar{p}_\rho$  and  $\nu$  satisfy (3.7), then for any  $\varepsilon \in (0, \frac{1}{2})$ , there are constants  $\delta > 0$  and  $C > 0$  such that if  $(\rho_0, u_0) \in H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}$  and  $\|(\rho_0, u_0)\|_{H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}} \leq \delta$ , then the solution of (3.3) satisfies the following exponential decay estimate*

$$\|(e^{Mt}\rho, e^{Mt}(u - \langle u \rangle), e^{Mt}\sigma)\|_{L^\infty(0,T;H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon} \times \mathbb{R})} \leq C\|(\rho_0, u_0)\|_{H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}}.$$

## 3.2 Linearized System around the steady state

We start by studying the stabilization of the linearized system:

$$\begin{cases} \partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho = -\bar{\rho} \operatorname{div} u + K_\rho(\rho, u) + f_\rho & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho}(\partial_t u + (\bar{u} + \hat{u}) \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla(\operatorname{div} u) \\ \quad = -\bar{p}_\rho \nabla \rho + f_u & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \end{cases} \quad (3.6)$$

Here,  $f_\rho, f_u$  are source terms for the density and velocity respectively. We assume that  $\hat{u}$  satisfies

$$\|e^{\frac{M}{2}t}\hat{u}\|_{L^\infty(H^1)} \leq R,$$

with  $R$  a constant that we will choose small enough later and where the exponential decay constant  $M$  that we choose so that

$$M < \lambda_1 - \frac{\bar{p}_\rho \bar{\rho}}{\nu^2}. \quad (3.7)$$

where  $\lambda_1$  is the first nonzero eigenvalue of the laplacian on the torus. As it is done in the study of the controllability of the same system in [23] and in the non-isentropic case in [51], we introduce the effective flux:

$$q := \operatorname{div} u - \frac{\bar{p}_\rho}{\nu} \rho.$$

The idea of considering this quantity was introduced by Lions in [45]. With this new variable the linear system becomes:

$$\left\{ \begin{array}{ll} \partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho = -\bar{\rho} \left( q + \frac{\bar{p}_\rho}{\nu} \rho \right) + K_\rho(\rho, u) + f_\rho & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho} (\partial_t q + (\bar{u} + \hat{u}) \nabla q) - \nu \Delta q = -\bar{\rho} \varepsilon(\hat{u}) : \varepsilon(u)^t + \frac{\bar{p}_\rho \bar{\rho}}{\nu} \bar{\rho} \left( q + \frac{\bar{p}_\rho}{\nu} \rho \right) \\ \quad - \frac{\bar{\rho} \bar{p}_\rho}{\nu} K_\rho(\rho, u) + \operatorname{div} f_u - \frac{\bar{\rho} \bar{p}_\rho}{\nu} f_\rho & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho} (\partial_t u + (\bar{u} + \hat{u}) \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla (\operatorname{div} u) - \bar{p}_\rho \nabla \rho + f_u & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \end{array} \right. \quad (3.8)$$

Here  $\varepsilon(u) = \frac{\nabla u + \nabla u^t}{2}$  is the symmetric gradient of the velocity, and for two matrices we define the tensor scalar product

$$A : B = \sum_{i,j} A_{i,j} B_{i,j}$$

We notice that the equation for  $q$  is not necessarily stable as the constant part  $\langle q \rangle$  in principle could diverge exponentially because we are on the torus  $\mathbb{T}^d$  (0 is always an eigenvalue of the Laplace operator). To deal with this we notice that the constant part of  $q$  comes only from  $\langle \rho \rangle$  as  $\langle \operatorname{div} u \rangle = 0$ , so we use the fact that

$$\langle q \rangle = -\frac{\bar{p}_\rho}{\nu} \langle \rho \rangle$$

to modify the equation for  $\rho$  as follows

$$\left\{ \begin{array}{ll} \partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle = -\bar{\rho} (q - \langle q \rangle) + \frac{\bar{p}_\rho}{\nu} \rho \\ \quad + \chi_\omega K_\rho(\rho, \hat{u}) + f_\rho & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho} (\partial_t q + (\bar{u} + \hat{u}) \nabla q) - \nu \Delta q \\ \quad = -\bar{\rho} \varepsilon(\hat{u}) : \varepsilon(u)^t + \frac{\bar{p}_\rho \bar{\rho}}{\nu} \bar{\rho} \left( q + \frac{\bar{p}_\rho}{\nu} \rho \right) \\ \quad - \frac{\bar{\rho} \bar{p}_\rho}{\nu} \chi_\omega K_\rho(\rho, q) + \operatorname{div} f_u - \frac{\bar{\rho} \bar{p}_\rho}{\nu} f_\rho & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho} (\partial_t u + (\bar{u} + \hat{u}) \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla (\operatorname{div} u) \\ \quad = -\bar{p}_\rho \nabla \rho + f_u & \text{in } \mathbb{R}_+ \times \mathbb{T}^d \end{array} \right. \quad (3.9)$$

In the following sections, we will split the study of the asymptotic stability of system (3.3) in the study of the stability of the uncoupled equations, that is, the transport equations, the heat equations and the Lamé system.

### 3.3 Stabilization of Transport equation

In this section we study the estimates necessary for the transport equations. We will begin by the  $H^1$  estimates required for the study of the linear system and later the estimates in the fractional exponent Sobolev space  $H^s(\mathbb{T}^d)$ , for  $s \in \mathbb{R}_+$ .

#### 3.3.1 $H^1$ estimates

For the stabilization of the transport equation we will show that we can take a feedback of the form  $-\gamma\rho\chi_\omega$ , hence we consider the system

$$\partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \rho = -\gamma \chi_\omega \rho + f. \quad (3.10)$$

We will consider  $\Lambda : \mathbb{T}^d \rightarrow \mathbb{R}$  such that

$$\nabla \Lambda(x) \cdot \bar{u} < 0, x \notin \omega$$

The following lemma establishes that such function always exists.

**Lemma 3.3.1.** *There is a  $C^\infty(\mathbb{T}^d)$  function  $\Lambda : \mathbb{T}^d \rightarrow \mathbb{R}$  such that*

$$\nabla \Lambda(x) \cdot \bar{u} < 0, x \notin \omega.$$

*Proof.* For every point  $p$  in the torus we can find an open neighborhood  $V$  of  $p$  that satisfies the property: if  $p \in \omega$  we can take any neighborhood contained in  $\omega$  and define  $\Lambda_p \equiv 0$ . If  $p \in \mathbb{T}^d \setminus \omega$ , we consider the curve of slope  $\bar{u}$  that goes through  $p$  and starts and finishes at points  $i(p), f(p) \in \omega$  then, in a tubular neighborhood of the curve (a rotation of  $B(0, \varepsilon)_{d-1} \times [0, L]$ ), we can define  $\Lambda_p$  solving  $\bar{u} \cdot \nabla \Lambda_p = -1$  in  $\mathbb{T}^d \setminus \omega$ . By compactness of the torus, we can cover it with finite many of these neighborhoods,  $V_{p_1}, \dots, V_{p_n}$ .

Moreover, we can take a partition of unity  $\varphi_{p_i}$  with the property  $\bar{u} \cdot \nabla \varphi_{p_i} = 0$  in  $\mathbb{T}^d \setminus \omega$ . To prove that this is possible, let us consider  $\psi \in C^3(\mathbb{R}^{d-1})$  supported in  $B(0, 1)$ . Let us consider the orthogonal projection  $P_{\bar{u}}^i$  orthogonal to  $\bar{u}$ . Then we consider  $\phi_i$  defined outside  $\omega$  by

$$\varphi_i = \psi(P_{\bar{u}}^i(\frac{x}{\varepsilon})).$$

Finally we consider

$$\tilde{\varphi}_i = \frac{\varphi_i}{\sum_j \varphi_j},$$

which satisfies

$$\bar{u} \cdot \nabla \tilde{\varphi}_i = \frac{\bar{u} \cdot \nabla \varphi_i}{\sum_j \varphi_j} - \frac{\varphi_i \sum_j \bar{u} \cdot \nabla \varphi_j}{(\sum_j \varphi_j)^2} = 0,$$

in  $\mathbb{T}^d \setminus \omega$ . With this we can define

$$\Lambda = \sum_{i=1}^n \Lambda_{p_i} \tilde{\varphi}_{p_i},$$

which satisfies the property:

$$\bar{u} \cdot \nabla \Lambda = \sum_{i=1}^n \bar{u} \cdot \nabla \Lambda_{p_i} \tilde{\varphi}_{p_i} + \sum_{i=1}^n \Lambda_{p_i} \bar{u} \cdot \nabla \tilde{\varphi}_{p_i} = -1.$$

□

With this lemma we can prove the following stabilization result for the transport equation:

**Theorem 3.3.1.** *There exists  $\mu > 0$  depending only on  $\gamma$  and the geometry and parameters of the equation such that if  $\rho_0 \in L^2(\mathbb{T}^d)$  and  $e^{Mt} f \in L^\infty(0, T; H^1(\mathbb{T}^d))$  then the solution  $\rho$  of (3.10) satisfies the following estimates:*

$$\|e^{Mt} e^{\mu\Lambda} \rho\|_{L^\infty(L^2)} \leq \|e^{\mu\Lambda} \rho_0\|_{L^2} + \frac{1}{\gamma - M} \|e^{Mt} e^{\mu\Lambda} f\|_{L^\infty(L^2)}, \quad (3.11)$$

$$\begin{aligned} \|e^{Mt} e^{\mu\Lambda} \nabla \rho\|_{L^\infty(L^2)} &\leq \|e^{\mu\Lambda} \nabla \rho_0\|_{L^2} + \frac{1}{\gamma - M} \|e^{Mt} e^{\mu\Lambda} \nabla f\|_{L^\infty(L^2)} \\ &\quad + C\gamma \|e^{Mt} e^{\mu\Lambda} \rho\|_{L^\infty(L^2)}. \end{aligned} \quad (3.12)$$

*Proof.* In order to prove the stability estimates we use a technique introduced by J.-M. Coron in [17]. Let us consider the Lyapunov function

$$V[\rho](t) = \frac{1}{2} \int_{\mathbb{T}^d} e^{2\mu\Lambda} \rho^2 dx.$$

Then we compute

$$\begin{aligned}
\dot{V}[\rho](t) &= \int_{\mathbb{T}^d} e^{2\mu\Lambda} \rho \dot{\rho} \, dx = \int_{\mathbb{T}^d} e^{2\mu\Lambda} \rho (-(\bar{u} + \hat{u}) \cdot \nabla \rho + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \rho + -\gamma \chi_\omega \rho + f) \, dx \\
&= \frac{1}{2} \int_{\mathbb{T}^d} e^{2\mu\Lambda} \rho^2 (\operatorname{div} \hat{u} + \frac{\bar{\rho} \bar{p}_\rho}{\nu}) \, dx + \frac{\mu}{2} \int_{\mathbb{T}^d} e^{2\mu\Lambda} \rho^2 \Lambda(x) \cdot (\bar{u} + \hat{u}) \, dx \\
&\quad - \gamma \int_{\omega} \chi_\omega \rho^2 \, dx + \langle e^{\mu\Lambda t} \rho, e^{\mu\Lambda t} f \rangle \\
&\leq \frac{1}{2} \int_{\mathbb{T}^d \setminus \omega} e^{2\mu\Lambda} \rho^2 (\operatorname{div} \hat{u} + \frac{\bar{\rho} \bar{p}_\rho}{\nu} + \mu \Lambda(x) \cdot (\bar{u} + \hat{u})) \, dx \\
&\quad - \gamma \int_{\omega} e^{2\mu\Lambda} \rho^2 \, dx + \langle e^{\mu\Lambda t} \rho, e^{\mu\Lambda t} f \rangle.
\end{aligned}$$

Because of the condition  $\Lambda \cdot \bar{u} < 0$  in  $\mathbb{T}^d \setminus \omega$ , we can always choose  $\mu$  large enough and  $R$  small enough so that

$$(\operatorname{div} \hat{u} + \frac{\bar{\rho} \bar{p}_\rho}{\nu} + \mu \Lambda(x) \cdot (\bar{u} + \hat{u})) < -\gamma,$$

so we obtain

$$\dot{V}[\rho](t) \leq -\gamma V[\rho](t) + \sqrt{V[\rho](t)} \|e^{\mu\Lambda} f\|_{L^2},$$

from which we deduce the estimate (3.11). For estimate (3.12) we consider the equation solved by the derivatives of  $\rho$  and proceed in similar fashion. Differentiating with respect to the  $i$ -th variable:

$$\partial_t \partial_i \rho + (\bar{u} + \hat{u}) \cdot \nabla \partial_i \rho + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \partial_i \rho = -\gamma \chi_\omega \partial_i \rho + \partial_i f - \partial_i \hat{u}_j \partial_j \rho - \gamma \partial_i \chi_\omega \rho. \quad (3.13)$$

and we use the previous estimate with  $\rho$  replaced by  $\partial_i \rho$ .  $\square$

### 3.3.2 Higher Order estimates

When dealing with the non-linear system we will need higher order estimates. We begin with an  $H^2$  estimate, given in the following statement.

**Theorem 3.3.2.** *There exists  $C > 0$  depending on  $M$  and the parameters of the equation, such that if  $\rho_0 \in H^2(\mathbb{T}^d)$  and  $e^{Mt} f \in L^\infty(0, T; H^2(\mathbb{T}^d))$ , then the solution  $\rho$  of (3.10) satisfies the following estimates:*

$$\|e^{Mt} e^{\mu\Lambda} D^2 \rho\|_{L^\infty(L^2)} \leq \|e^{\mu\Lambda} D^2 \rho_0\|_{L^2} + C \|e^{Mt} e^{\mu\Lambda} D^2 f\|_{L^\infty(L^2)} + C \|e^{Mt} e^{\mu\Lambda} \rho\|_{L^\infty(H^1)}. \quad (3.14)$$

*Proof.* We begin by differentiating the equation (3.10) to obtain

$$\begin{aligned} \partial_t D_{ij}\rho + (\bar{u} + \hat{u}) \cdot \nabla D_{ij}\rho + \frac{\bar{\rho}\bar{p}_\rho}{\nu} D_{ij}\rho + D_i \hat{u}_k D_{kj}\rho \\ + D_j \hat{u}_k D_{ki}\rho + D_{ij} \hat{u}_k D_{k\rho} = -\gamma \chi_\omega D_{ij}\rho \\ - \gamma D_{ij} \chi_\omega \rho - \gamma D_i \chi_\omega D_j \rho - \gamma D_j \chi_\omega D_i \rho + D_{ij} f. \end{aligned} \quad (3.15)$$

Consider the Lyapunov function

$$V_2[\rho](t) = \sum_{i,j} \int_{\mathbb{T}^d} e^{\mu\Lambda(x)} (D_{ij}\rho)^2 dx.$$

With similar computations we obtain

$$\dot{V}_2[\rho](t) \leq -\gamma V_2[\rho](t) + \sqrt{V_2[\rho](t)} \|e^{\mu\Lambda} D_{ij} f\|_{L^2} - \int_{\mathbb{T}^d} D_{ij} \hat{u} \nabla \rho D_{ij} \rho dx,$$

and consequently

$$\dot{V}_2[\rho](t) \leq -\gamma V_2[\rho](t) + \sqrt{V_2[\rho](t)} \|e^{\mu\Lambda} D_{ij} f\|_{L^2} + \|D_{ij} \hat{u}\|_{L^3} \|\nabla \rho\|_{L^6(\mathbb{T}^d)} \sqrt{V_2[\rho]}. \quad (3.16)$$

Using the Sobolev embedding  $e^{Mt} \nabla f \in L^\infty(0, T; H^1(\mathbb{T}^d)) \subseteq L^\infty(0, T; L^6(\mathbb{T}^d))$ , we can obtain similar estimates to the ones in Theorem 3.3.1 for the Lyapunov function

$$V_{1,6}[\rho](t) = \sum_i \int_{\mathbb{T}^d} e^{\mu\Lambda(x)} (D_i \rho)^6 dx,$$

We obtain that

$$\begin{aligned} e^{Mt} \|\nabla \rho\|_{L^6(\mathbb{T}^d)} \leq \|\nabla \rho_0\|_{L^6(\mathbb{T}^d)} + C \|e^{Mt} e^{\mu\Lambda} \nabla f\|_{L^\infty(0, T; L^6(\mathbb{T}^d))} \\ + C \|e^{Mt} e^{\mu\Lambda} f\|_{L^\infty(0, T; H^1(\mathbb{T}^d))}. \end{aligned} \quad (3.17)$$

Applying this to (3.16) we obtain

$$\begin{aligned} \dot{V}_2[\rho](t) \leq -\gamma V_2[\rho](t) + C \sqrt{V_2[\rho](t)} \|e^{\mu\Lambda} f\|_{H^2} + R \sqrt{V_2[\rho](t)} e^{-Mt} \|\nabla \rho_0\|_{L^6(\mathbb{T}^d)} \\ CR \sqrt{V_2[\rho](t)} e^{-Mt} \|e^{Mt} e^{\mu\Lambda} \nabla f\|_{L^\infty(0, T; L^6(\mathbb{T}^d))} \\ + CR \sqrt{V_2[\rho](t)} e^{-Mt} \|e^{Mt} e^{\mu\Lambda} f\|_{L^\infty(0, T; H^1(\mathbb{T}^d))} \end{aligned} \quad (3.18)$$

from which, together with the Gronwall inequality, (3.14) follows.  $\square$

Using interpolation theory (see for instance [65]) we obtain the following estimate

**Theorem 3.3.3.** *if  $\rho_0 \in H^{\frac{3}{2}+\varepsilon}(\mathbb{T}^d)$  and  $e^{Mt}f \in L^\infty(0, T; H^{\frac{3}{2}+\varepsilon}(\mathbb{T}^d))$  then we have the estimate*

$$\|e^{Mt}e^{\mu\Lambda}\rho\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})} \leq \|e^{\mu\Lambda}\rho_0\|_{H^2} + C\|e^{Mt}e^{\mu\Lambda}f\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})}.$$

*Proof.* Let us consider the linear operator

$$\begin{aligned} X : L^\infty(H^1(\mathbb{T}^d) + H^2(\mathbb{T}^d)) &\rightarrow L^\infty(H^1(\mathbb{T}^d) + H^2(\mathbb{T}^d)) \\ e^{Mt}e^{\mu\Lambda}f &\rightarrow e^{Mt}e^{\mu\Lambda}\tilde{\rho}, \end{aligned}$$

where  $\tilde{\rho}$  is the solution of (3.10) with zero initial conditions. Then this operator is continuous in the spaces

$$X : L^\infty(H^1(\mathbb{T}^d)) \rightarrow L^\infty(H^1(\mathbb{T}^d)),$$

and

$$X : L^\infty(H^2(\mathbb{T}^d)) \rightarrow L^\infty(H^2(\mathbb{T}^d)),$$

from the previous results. From interpolation theory we obtain that the operator is continuous in the interpolation space

$$X : L^\infty(H^1(\mathbb{T}^d)) \rightarrow L^\infty(H^1(\mathbb{T}^d)),$$

and

$$X : (L^\infty(H^1(\mathbb{T}^d)), L^\infty(H^2(\mathbb{T}^d)))_{\lambda, \theta} \rightarrow (L^\infty(H^1(\mathbb{T}^d)), L^\infty(H^2(\mathbb{T}^d)))_{\lambda, \theta},$$

as well.

So we have the inequality

$$\|e^{Mt}e^{\mu\Lambda}Xf\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})} \leq \|e^{Mt}e^{\mu\Lambda}f\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})}.$$

Finally, we can write  $\rho$  as

$$e^{Mt}e^{\mu\Lambda}\rho = Xf + e^{Mt}e^{\mu\Lambda}S(t)\rho_0,$$

where  $S(t)\rho_0$  is the solution to the homogeneous equation. Combining (3.11), (3.12) and (3.14), we have

$$\|e^{Mt}e^{\mu\Lambda}S\rho\| \leq C\|e^{\mu\Lambda}\rho_0\|_{H^2},$$

and then

$$\|e^{Mt}e^{\mu\Lambda}\rho\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})} \leq \|e^{\mu\Lambda}\rho_0\|_{H^2} + C\|e^{Mt}e^{\mu\Lambda}f\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})}.$$

□

### 3.4 Stabilization of Transport equation with average term

In this section we see how to stabilize the equation with the extra average term:

$$\partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \rho = K_\rho(\rho, \hat{u}, \sigma) + f. \quad (3.19)$$

We recall that  $\varphi : \mathbb{T}^d \rightarrow \mathbb{R}$  supported in  $\omega' \setminus \omega$  and with  $\langle \phi \rangle = 1$  and a function  $\sigma : [0, \infty] \rightarrow \mathbb{R}$  given by the differential equation:

$$\dot{\sigma} - \sigma \langle \operatorname{div} \hat{u} \varphi \rangle + \sigma \frac{\bar{\rho} \bar{p}_\rho}{\nu} = -\langle \operatorname{div} \hat{u} \rho \rangle + \langle \gamma \chi_\omega \rho \rangle - \gamma \langle \rho \rangle,$$

and we let

$$\tilde{\rho} = \rho - \sigma \varphi.$$

Moreover we consider the feedback

$$K_\rho(\rho, \hat{u}, \sigma) = -\gamma \chi_\omega \rho + \sigma (\bar{u} + \hat{u}) \cdot \nabla \varphi + \sigma \frac{\bar{\rho} \bar{p}_\rho}{\nu} \varphi + \dot{\sigma} \varphi. \quad (3.20)$$

We deduce that  $\tilde{\rho}$  satisfies

$$\partial_t \tilde{\rho} + (\bar{u} + \hat{u}) \cdot \nabla \tilde{\rho} - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \tilde{\rho} = -\dot{\sigma} \varphi - \sigma (\bar{u} + \hat{u}) \cdot \nabla \varphi - \sigma \frac{\bar{\rho} \bar{p}_\rho}{\nu} \varphi + K_\rho(\rho, \hat{u}, \sigma) + f.$$

Hence it solves the equation

$$\partial_t \tilde{\rho} + (\bar{u} + \hat{u}) \cdot \nabla \tilde{\rho} - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \tilde{\rho} = -\gamma \chi_\omega \tilde{\rho} + f,$$

where we used that  $-\gamma \chi_\omega \rho = -\gamma \chi_\omega \tilde{\rho}$ . We also consider the equation satisfied by the average of  $\rho$

$$\partial_t \langle \rho \rangle + \langle \hat{u} \cdot \nabla \rho \rangle = \langle K_\rho(\rho, \hat{u}) \rangle + \langle f \rangle.$$

Therefore

$$\partial_t \langle \rho \rangle = \langle \operatorname{div} \hat{u} \rho \rangle - \langle \gamma \chi_\omega \rho \rangle - \sigma \langle \operatorname{div} \hat{u} \varphi \rangle + \sigma \frac{\bar{\rho} \bar{p}_\rho}{\nu} + \dot{\sigma} + \langle f \rangle.$$

Consequently

$$\partial_t \langle \rho \rangle = -\gamma \langle \rho \rangle + \langle f \rangle.$$

So we end up with the system

$$\begin{cases} \partial_t \tilde{\rho} + (\bar{u} + \hat{u}) \cdot \nabla \tilde{\rho} - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle + \frac{\bar{\rho} \bar{p}_\rho}{\nu} \tilde{\rho} = -\gamma \chi_\omega \tilde{\rho} + f. \\ \partial_t \langle \rho \rangle = -\gamma \langle \rho \rangle + \langle f \rangle. \end{cases} \quad (3.21)$$



From the second equation of(3.21) is clear that

$$|e^{Mt}\langle\rho\rangle| \leq \frac{1}{\gamma - M} (|\langle\rho_0\rangle| + \|e^{Mt}\langle f\rangle\|_{L^\infty}) \leq \frac{1}{\gamma - M} (|\langle\rho_0\rangle| + \|e^{Mt}f\|_{L^\infty(L^2)}).$$

Using this on the first equation we find

$$\begin{aligned} \|e^{Mt}\tilde{\rho}\|_{L^\infty(L^2)} &\leq \|e^{\mu\Lambda}\rho_0\|_{L^2} + \frac{1}{\gamma - M} \|e^{Mt}f + e^{Mt}\bar{\rho}\frac{\bar{p}\bar{\rho}}{\nu}\langle\rho\rangle\|_{L^\infty(L^2)} \\ &\leq \|e^{\mu\Lambda}\rho_0\|_{L^2} + \frac{1}{\gamma - M} |\langle\rho_0\rangle| + \frac{1}{\gamma - M} \|e^{Mt}f\|_{L^\infty(L^2)} + \frac{1}{\gamma - M} \|e^{Mt}\frac{\bar{\rho}\bar{p}\bar{\rho}}{\nu}\langle\rho\rangle\|_{L^\infty(L^2)} \end{aligned} \quad (3.22)$$

It follows that

$$\|e^{Mt}\tilde{\rho}\|_{L^\infty(L^2)} \leq \|e^{\mu\Lambda}\rho_0\|_{L^2} + \frac{1}{\gamma - M} |\langle\rho_0\rangle| + \frac{1}{\gamma - M} \left(1 + \frac{\bar{\rho}\bar{p}\bar{\rho}}{\nu(\gamma - M)}\right) \|e^{Mt}f\|_{L^\infty(L^2)}.$$

So we obtain finally for  $\rho$

$$\begin{aligned} \|e^{Mt}\rho\|_{L^\infty(L^2)} = \|e^{Mt}(\tilde{\rho} + \sigma\varphi)\|_{L^\infty(L^2)} &\leq \|e^{\mu\Lambda}\rho_0\|_{L^2} + \frac{1}{\gamma - M} |\langle\rho_0\rangle| + \frac{1}{\gamma - M} (1 + 2\|\varphi\|_{L^2} \\ &\quad + (1 + \|\varphi\|_{L^2}) \frac{\bar{\rho}\bar{p}\bar{\rho}}{\nu(\gamma - M)}) \|e^{Mt}f\|_{L^\infty(L^2)}, \end{aligned} \quad (3.23)$$

which proves the exponential decay.

### 3.5 Stability of Parabolic equations

For the following parabolic equation,

$$\bar{\rho}(\partial_t q + \bar{u}\nabla q) - \nu\Delta q - \frac{\bar{p}\bar{\rho}\bar{p}}{\nu}\bar{\rho}q = f_q,$$

we can prove the following stability estimates based on spectral decomposition: given  $M < \lambda_1 - \frac{\bar{p}\bar{\rho}\bar{p}}{\nu^2}$  where  $\lambda_1$  is the first non-zero eigenvalue of the Laplacian on the torus.

$$\|e^{Mt}(q - \langle q\rangle)\|_{L^\infty(L^2)} \leq \|(q_0 - \langle q_0\rangle)\|_{L^2} + C_M \|e^{Mt}f_q\|_{L^\infty(L^2)},$$

$$\|e^{\frac{3M}{4}t}\nabla q\|_{L^\infty(L^2)} \leq \|\nabla q_0\|_{L^2} + C_M \|e^{Mt}f_q\|_{L^\infty(L^2)}.$$

We have similar properties for the Lamé system.

### 3.6 Stabilization of the linear system

Using the previous results we now prove that the linear system is exponentially stable.

$$\left\{ \begin{array}{l} \partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle = -\bar{\rho}(q - \langle q \rangle) + \frac{\bar{p}_\rho}{\nu} \rho + \chi_\omega K_\rho(\rho, \hat{u}) + f_\rho \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho}(\partial_t q + (\bar{u} + \hat{u}) \nabla q) - \nu \Delta q = -\bar{\rho} D(\hat{u}) : D(u)^t + \frac{\bar{p}_\rho \bar{\rho}}{\nu} \bar{\rho} (q + \frac{\bar{p}_\rho}{\nu} \rho) \\ \quad - \frac{\bar{\rho} \bar{p}_\rho}{\nu} \chi_\omega K_\rho(\rho, q, \sigma) + \operatorname{div} f_u - \frac{\bar{\rho} \bar{p}_\rho}{\nu} f_\rho \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho}(\partial_t u + (\bar{u} + \hat{u}) \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla(\operatorname{div} u) \\ \quad = -\bar{p}_\rho \nabla \rho + f_u \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \dot{\sigma} - \sigma \langle \operatorname{div} \hat{u} \varphi \rangle + \sigma \frac{\bar{\rho} \bar{p}_\rho}{\nu} = \langle \operatorname{div} \hat{u} \rho \rangle - \langle \gamma \chi_\omega \rho \rangle + \gamma \langle \rho \rangle \end{array} \right. \quad (3.24)$$

Let  $T^* > 0$  such that a solution to the system exists in  $[0, T^*]$  for a given initial condition  $(u_0, \rho_0)$  and take  $A, B, D, E \in \mathbb{R}_+$  fixed to be determined. For a given  $T < T^*$  let us consider the norm

$$\begin{aligned} \|(\rho, q, u)\|_{X,T} &= \|e^{Mt} \rho\|_{L^\infty([0,T], L^2)} + A \|e^{\frac{3M}{4}t} \nabla \rho\|_{L^\infty([0,T], L^2)} \\ &\quad + B \|e^{Mt} q - \langle q \rangle\|_{L^\infty([0,T], L^2)} + D \|e^{\frac{3M}{4}t} \nabla q\|_{L^\infty([0,T], L^2)} \\ &\quad + E \|e^{\frac{M}{2}t} u - \langle u \rangle\|_{L^\infty([0,T], L^2)} + F \|e^{\frac{M}{2}t} Du\|_{L^\infty(L^2)}. \end{aligned} \quad (3.25)$$

Let us consider  $\hat{T}$  as

$$\hat{T} = \sup\{T \in [0, T^*] \mid \|(\rho, q, u)\| \leq K(\|(\rho_0, q_0, u_0)\| + \|(f_\rho, f_u))\|\}.$$

Using the previous estimates we obtain that

$$\|e^{Mt} \rho\| \leq \frac{C}{\gamma - M} \|e^{Mt}((q - \langle q \rangle))\|_{L^\infty([0,T], L^2(\mathbb{T}^d))} + C\|(\rho_0)\| + C\|f_\rho\|, \quad (3.26)$$

and

$$\|e^{\frac{3M}{4}t} \nabla \rho\| \leq \frac{C\gamma}{\gamma - M} \|e^{Mt} \nabla q\|_{L^\infty([0,T], L^2(\mathbb{T}^d))} + C\|(\nabla \rho_0)\| + C\|\nabla f_\rho\|. \quad (3.27)$$

For  $q$  we have the estimate

$$\begin{aligned} \|e^{Mt} q - \langle q \rangle\|_{L^\infty([0,T], L^2)} + \|e^{\frac{3M}{4}t} \nabla q\|_{L^\infty([0,T], L^2)} &\leq C \frac{\bar{p}_\rho \bar{\rho}}{\nu} (1 + \frac{\bar{p}_\rho \bar{\rho}}{\nu}) \gamma \|e^{Mt} \rho\|_{L^\infty(L^2)} \\ &\quad + C(B + D)R\gamma \|e^{\frac{M}{2}t} Du\|_{L^\infty(L^2)} + C\|q_0\| + C\|\operatorname{div} f_u - \frac{\bar{\rho} \bar{p}_\rho}{\nu} f_\rho\|, \end{aligned} \quad (3.28)$$

and similarly for  $u$

$$\begin{aligned} \|e^{\frac{M}{2}t}u - \langle u \rangle\|_{L^\infty([0,T],L^2)} + \|e^{\frac{M}{2}t}Du\|_{L^\infty(L^2)} &\leq C\|e^{\frac{3M}{4}t}(-\bar{\rho}\hat{u}\nabla u + \bar{p}_\rho\nabla\rho + f_u)\|_{L^\infty([0,T],L^2)} + C\|u_0\| \\ &\leq CR\|\nabla u\|_{L^\infty(L^2)} + C\|e^{\frac{3M}{4}t}\nabla\rho\|_{L^\infty(L^2)} + C\|u_0\| + C\|f_u\|. \end{aligned} \quad (3.29)$$

Multiplying with the respective constants and adding everything we obtain the following inequality:

$$\begin{aligned} \|(\rho, q, u)\|_{X,T} &\leq \frac{C}{\gamma - M}\|e^{Mt}(q - \langle q \rangle)\|_{L^\infty([0,T],L^2(\mathbb{T}^d))} \\ &\quad + \frac{AC\gamma}{\gamma - M}\|e^{\frac{3M}{4}t}\nabla q\|_{L^\infty([0,T],L^2(\mathbb{T}^d))} \\ &\quad + (B + D)C\frac{\bar{p}_\rho\bar{\rho}}{\nu}(1 + \frac{\bar{p}_\rho\bar{\rho}}{\nu})\gamma\|e^{Mt}\rho\|_{L^\infty([0,T],L^2(\mathbb{T}^d))} \\ &+ C(B + D + F + E)R\gamma\|e^{\frac{M}{2}t}Du\|_{L^\infty([0,T],L^2(\mathbb{T}^d))} + (E + F)C\|e^{\frac{3M}{4}t}\nabla\rho\|_{L^\infty([0,T],L^2(\mathbb{T}^d))} \\ &\quad + C\|(\rho_0, u_0)\| + C\|(f_\rho, f_u)\|. \end{aligned} \quad (3.30)$$

where the constant  $C$  is independent of  $T$  and it is bounded if we assume that  $\frac{\bar{p}_\rho\bar{\rho}}{\nu}$  belongs to a fixed interval. Now we choose  $A, B, D$  and  $E$  such that

$$\begin{aligned} C(B + D)\gamma\frac{\bar{p}_\rho\bar{\rho}}{\nu}(1 + \frac{\bar{p}_\rho\bar{\rho}}{\nu}) &< 1, \\ (E + F)C &< A, \\ \frac{C}{\gamma - M} &< B, \\ \frac{AC\gamma}{\gamma - M} &< D, \\ C(B + D + E + F)R\gamma &< F. \end{aligned}$$

This can be achieved in the following way: first we take  $\frac{\bar{p}_\rho\bar{\rho}}{\nu}$  small enough so

$$\frac{C^2}{\gamma - M}\gamma\frac{\bar{p}_\rho\bar{\rho}}{\nu}(1 + \frac{\bar{p}_\rho\bar{\rho}}{\nu}) < 1,$$

This allow to choose  $B$  so that

$$\frac{C}{\gamma - M} < B,$$

and

$$C(B)\gamma\frac{\bar{p}_\rho\bar{\rho}}{\nu}(1 + \frac{\bar{p}_\rho\bar{\rho}}{\nu}) < 1.$$

Next we choose  $D > 0$  small enough so that

$$C(B + D)\gamma\frac{\bar{p}_\rho\bar{\rho}}{\nu}(1 + \frac{\bar{p}_\rho\bar{\rho}}{\nu}) < 1.$$

After that, we take  $A > 0$  small enough so that

$$\frac{AC\gamma}{\gamma - M} < D,$$

$E > 0$  and  $F > 0$  small enough so that

$$(E + F)C < A,$$

and finally  $R$  so that

$$C(B + D + E + F)R\gamma < F.$$

With this choice of  $A, B, D, E, F$  and  $R$  we will have that for some  $L < 1$

$$\|(p, q, u)\|_{X,T} \leq L\|(p, q, u)\| + C\|f\| + C\|\rho_0, u_0\| < (LK + C)(\|(f_\rho, f_u)\| + \|\rho_0, u_0\|).$$

Because  $L < 1$  we can always chose  $K$  large enough such that

$$LK + C < K.$$

So

$$\|(p, q, u)\|_{X,T} < K(\|(f_\rho, f_u)\| + \|\rho_0, u_0\|),$$

which implies that  $\hat{T}$  was not the maximum so we conclude that

$$\|(\rho, q, u)\|_{X,T} \leq K\|(\rho_0, u_0)\|_{H^1, H^2} + K\|(f_\rho, f_u)\| \quad \forall T > 0,$$

so we obtain the exponential decay by taking the supremum over  $T > 0$ ,

$$\|(\rho, q, u)\|_{X,\infty} \leq K\|(\rho_0, u_0)\|_{H^1, H^2} + K\|(f_\rho, f_u)\|.$$

### 3.7 Extra Regularity

In order to perform the fixed point argument for studying the non linear system, we will need higher regularity estimates of the solutions assuming better regularity of the source terms. In this line we have the following result:

**Theorem 3.7.1.** *We assume that for  $\varepsilon \in (0, \frac{1}{2})$ ,*

$$\begin{aligned}\rho_0 &\in H^{\frac{3}{2}+2\varepsilon}, \\ u_0 &\in H^{\frac{5}{2}+2\varepsilon}, \\ e^{Mt} f_u &\in L^\infty(H^{\frac{1}{2}+3\varepsilon}), \\ e^{Mt} f_\rho &\in L^\infty(H^{\frac{3}{2}+3\varepsilon}),\end{aligned}$$

*then the solutions satisfies*

$$\begin{aligned}\|e^{Mt} u - \langle u \rangle\|_{H^{\frac{5}{2}+\varepsilon}} + \|e^{Mt} \rho\|_{H^{3/2+\varepsilon}} &\leq C \|e^{Mt} f_u\|_{L^\infty(H^{\frac{1}{2}+3\varepsilon})} + C \|e^{Mt} f_\rho\|_{L^\infty(H^{\frac{3}{2}+3\varepsilon})} \\ &\quad + \|u_0 - \langle u_0 \rangle\|_{H^{\frac{5}{2}+\varepsilon}} + \|\rho_0\|_{H^{3/2+\varepsilon}}.\end{aligned}\quad (3.31)$$

*Proof.* Using the following bootstrap type argument we can obtain higher regularity estimates that we will need in order to study the complete non-linear system:

$$q_0 \in H^{\frac{3}{2}+2\varepsilon}, e^{Mt} \rho \in L^\infty(L^2), e^{Mt} \operatorname{div} f_u \in L^\infty(H^{-\frac{1}{2}+3\varepsilon}) \Rightarrow e^{Mt} q \in L^\infty(H^{\frac{3}{2}+2\varepsilon}),$$

$$\rho_0 \in H^{\frac{3}{2}+2\varepsilon}, e^{Mt} q \in L^\infty(H^{\frac{3}{2}+2\varepsilon}), e^{Mt} f_\rho \in L^\infty(H^{\frac{3}{2}+3\varepsilon}) \Rightarrow e^{Mt} \rho \in L^\infty(H^{\frac{3}{2}+2\varepsilon}),$$

$$u_0 \in H^{\frac{3}{2}+\varepsilon}, e^{Mt} \rho \in L^\infty(H^{\frac{3}{2}+2\varepsilon}), e^{Mt} f_u \in L^\infty(H^{\frac{1}{2}+2\varepsilon}) \Rightarrow e^{Mt}(u - \langle u \rangle) \in L^\infty(H^{\frac{5}{2}+\varepsilon}).$$

we also have the corresponding estimates

$$\|e^{Mt} q\|_{L^\infty(H^{\frac{3}{2}+2\varepsilon})} \leq C \|q_0\|_{H^{\frac{3}{2}+2\varepsilon}} + \|e^{Mt} \rho\|_{L^\infty(L^2)} + \|e^{Mt} \operatorname{div} f_u\|_{L^\infty(H^{-\frac{1}{2}+3\varepsilon})},$$

$$\|e^{Mt} \rho\|_{L^\infty(H^{\frac{3}{2}+2\varepsilon})} \leq C \|q_0\|_{H^{\frac{3}{2}+2\varepsilon}} + \|e^{Mt} \rho\|_{L^\infty(L^2)} + \|e^{Mt} f_\rho\|_{L^\infty(H^{-\frac{1}{2}+3\varepsilon})},$$

$$\|e^{Mt}(u - \langle u \rangle)\|_{L^\infty(H^{\frac{5}{2}+\varepsilon})} \leq C \|u_0\|_{H^{\frac{5}{2}+2\varepsilon}} + \|e^{Mt} \rho\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})} + \|e^{Mt} f_u\|_{L^\infty(H^{\frac{1}{2}+2\varepsilon})}.$$

□

## 3.8 Non-linear system

In this part we set the following quantities:

$$\rho = \rho_S - \bar{\rho},$$

$$u = u_S - \bar{u},$$

We will consider the following functional setting. Let

$$X_\rho(T) = \{e^{Mt}\rho \in L^\infty([0, T], H^{\frac{3}{2}+2\varepsilon}), e^{Mt}\rho_t \in L^\infty([0, T], H^{\frac{1}{2}+2\varepsilon})\},$$

$$X_u(T) = \{e^{\frac{M}{2}t}u \in L^\infty([0, T], H^{\frac{5}{2}+2\varepsilon}), e^{\frac{M}{2}t}u_t \in L^\infty([0, T], H^{\frac{1}{2}+2\varepsilon})\},$$

with the obvious norms. And for given  $R > 0, K > 0$  consider the set

$$\mathcal{C}_{R,T} = \{(\rho, u, \sigma) \in X_\rho(T) \times X_u(T) \times L^\infty([0, T], e^{Mt}), \|\rho, u, \sigma\| \leq R\},$$

### 3.8.1 Estimates of the non-linear terms

In this part we estimate the non-linear terms to use in the continuous induction argument.

**Theorem 3.8.1.** *If  $(\rho, u, \sigma) \in \mathcal{C}_{R,T}$  then*

$$\|f_\rho, f_u, f_\sigma\| \leq CR^2.$$

*Proof.* The non-linear terms are given by:

$$f_\rho = -\rho \operatorname{div}(u),$$

$$f_u = -\rho \partial_t u - \rho(\bar{u} + u) \cdot \nabla u - (p'(\bar{\rho} + \rho) - p'(\bar{\rho})) \nabla \rho.$$

For  $f_\rho$  we have:

using the fact that  $H^{\frac{3}{2}+\varepsilon}$  is an algebra

$$\|e^{Mt}\rho \operatorname{div}(u)\|_{L^\infty(H^{\frac{3}{2}+3\varepsilon})} \leq C \|e^{Mt}\rho\|_{L^\infty(H^{\frac{3}{2}+2\varepsilon})} \|e^{Mt}\operatorname{div}(u)\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})} \leq CR^2.$$

Similarly for  $f_u$ : Using the fact that the products are continuous

$$H^{\frac{1}{2}+\varepsilon} \times H^{\frac{3}{2}+2\varepsilon} \rightarrow H^{\frac{1}{2}+3\varepsilon}$$

$$\|e^{Mt}\rho \partial_t u\|_{L^\infty(H^{\frac{1}{2}+3\varepsilon})} \leq C \|e^{Mt}\rho\|_{L^\infty(H^{\frac{3}{2}+2\varepsilon})} \|e^{Mt}\partial_t u\|_{L^\infty(H^{\frac{1}{2}+\varepsilon})} \leq CR^2.$$

$$\|e^{Mt}\rho(\bar{u} + u) \cdot \nabla u\|_{L^\infty(H^{\frac{1}{2}+3\varepsilon})} \leq C\|e^{Mt}\rho\|_{L^\infty(H^{\frac{3}{2}+2\varepsilon})}\|e^{Mt}Du\|_{L^\infty(H^{\frac{1}{2}+\varepsilon})} \leq CR^2.$$

Since the product is also continuous

$$H^{\frac{1}{2}+2\varepsilon} \times H^{\frac{3}{2}+\varepsilon} \rightarrow H^{\frac{1}{2}+3\varepsilon}$$

we have

$$\begin{aligned} & \|e^{Mt}(p'(\bar{\rho} + \rho) - p'(\bar{\rho}))\nabla\rho\|_{L^\infty(H^{\frac{1}{2}+3\varepsilon})} \\ & \leq C\|e^{Mt}\nabla\rho\|_{L^\infty(H^{\frac{1}{2}+2\varepsilon})}\|e^{Mt}(p'(\bar{\rho} + \rho) - p'(\bar{\rho}))\|_{L^\infty(H^{\frac{3}{2}+\varepsilon})} \leq CR^2. \end{aligned} \quad (3.32)$$

□

### 3.8.2 Small time existence of solutions

In this section we will proof that of small enough  $T > 0$ , solutions of (3.1)-(3.2) exist. In order to do that, we notice that we have the following version of the previous inequalities for small  $T$  and without the exponential weight.

**Theorem 3.8.2.** *Let  $T^* > 0$ , then there exists  $C_{T^*} > 0$  such that for  $T < T^*$  and  $(\rho_0, u_0) \in H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}$*

$$\begin{aligned} & \|(\rho, u, \sigma)\|_{L^\infty(0, T; H^{\frac{5}{2}+2\varepsilon} \times H^{\frac{3}{2}+2\varepsilon}, \mathbb{R})} \leq C\|(\rho_0, u_0)\|_{H^{\frac{5}{2}+2\varepsilon} \times H^{\frac{3}{2}+2\varepsilon}} \\ & + C_{T^*}T\|f_\rho, f_u, f_\sigma\|_{L^\infty(0, T; H^{\frac{1}{2}+2\varepsilon} \times H^{\frac{3}{2}+2\varepsilon} \times \mathbb{R})} \end{aligned}$$

with this inequality we can obtain the following theorem using a classical Banach fixed point argument

**Theorem 3.8.3.** *If  $(\rho_0, u_0) \in H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}$  then system (3.1)-(3.2) has a solution in  $[0, T]$  for some  $T$  with*

$$T > C \frac{1}{\|(\rho_0, u_0)\|_{H^{\frac{3}{2}+2\varepsilon} \times H^{\frac{5}{2}+2\varepsilon}}}$$

*Proof.* We define the following fixed point map: To  $(\hat{\rho}, \hat{u}, \hat{\sigma})$  we associate

$(\rho, u, \sigma) \in$  solution of

$$\left\{ \begin{array}{l} \partial_t \rho + (\bar{u} + \hat{u}) \cdot \nabla \rho - \bar{\rho} \frac{\bar{p}_\rho}{\nu} \langle \rho \rangle = -\bar{\rho}(q - \langle q \rangle) + \frac{\bar{p}_\rho}{\nu} \rho + \chi_\omega K_\rho(\rho, \hat{u}) + \hat{f}_\rho \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho}(\partial_t q + (\bar{u} + \hat{u}) \nabla q) - \nu \Delta q = -\bar{\rho} D(\hat{u}) : D(u)^t + \frac{\bar{p}_\rho \bar{\rho}}{\nu} \bar{\rho} (q + \frac{\bar{p}_\rho}{\nu} \rho) \\ \quad - \frac{\bar{\rho} \bar{p}_\rho}{\nu} \chi_\omega K_\rho(\rho, q, \sigma) + \operatorname{div} f_u - \frac{\bar{\rho} \bar{p}_\rho}{\nu} \hat{f}_\rho \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \bar{\rho}(\partial_t u + (\bar{u} + \hat{u}) \nabla u) - \mu \Delta u \\ \quad - (\mu + \lambda) \nabla(\operatorname{div} u) = -\bar{p}_\rho \nabla \rho + \hat{f}_u \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^d \\ \hat{\sigma} - \sigma \langle \operatorname{div} \hat{u} \varphi \rangle + \sigma \frac{\bar{\rho} \bar{p}_\rho}{\nu} = -\langle \operatorname{div} \hat{u} \rho \rangle + \langle \gamma \chi_\omega \rho \rangle - \gamma \langle \rho \rangle \end{array} \right. \quad (3.33)$$

where  $\hat{f}_u, \hat{f}_\rho$  are the non-linear terms computed with  $(\hat{\rho}, \hat{u}, \hat{\sigma})$ . Then if we consider to elements  $(\hat{\rho}_i, \hat{u}_i, \hat{\sigma}_i)$ , with  $i = 1, 2$  with solutions  $(\rho_i, u_i, \sigma_i)$  then the difference satisfies

$$\|(\rho_2 - \rho_1, u_2 - u_1, \sigma_2 - \sigma_1)\| \leq CT \|(\hat{\rho}_2 - \hat{\rho}_1, \hat{u}_2 - \hat{u}_1, \hat{\sigma}_2 - \hat{\sigma}_1)\|.$$

Taking  $T$  small enough, we have have a contraction, and we conclude by Banach fixed point.  $\square$

### 3.8.3 Exponential decay of the complete system

We finally prove the exponential decay for the feedback system (3.1)-(3.2) As we did for the linear system, we use continuous induction to provide the exponential estimates. We define

$$\hat{T} = \sup\{T \in \mathbb{R} \mid t > 0, (\rho, u, \sigma) \in \mathcal{C}_{R,T}\}$$

Thanks to Theorem 3.8.3, we have that  $\hat{T} > 0$ . Moreover, for  $T < \hat{T}$

$$\|(\rho, u, \sigma)\| \leq CR^2 + C\|\rho_0, u_0\|$$

Let us assume by contradiction that  $\hat{T} < \infty$ . Taking  $R$  and  $\|\rho_0, u_0\|$  small enough

$$\begin{aligned} R &< \frac{1}{2C}, \\ \|\rho_0, u_0\| &< \frac{1}{2CR} \end{aligned}$$

we obtain

$$\|(\rho, u, \sigma)\| < R$$

contradicting the minimality of  $\hat{T}$ , hence

$$(\rho, u, \sigma) \in \mathcal{C}_{R,T}, \quad \forall T > 0$$

so we obtain the exponential decay for the solutions.



# Chapter 4

## Cauchy problem for the Fluid-Structure interaction for the Euler and linear elasticity equations in dimension 2

### 4.1 Introduction

The aim of this article is to study the Cauchy problem for a system that models the interaction between an elastic solid contained in a fluid, which in turns is also contained in a fixed domain  $\Omega$  in dimension 2, that we will assume simply connected. There are several results in the literature depending on the specific model that it is used for the solid and the fluid. In our case, we consider an incompressible inviscid fluid, that is, we consider the Euler equations

$$\begin{aligned}\partial_t u + (u \cdot \nabla)u &= -\nabla p, \\ \operatorname{div} u &= 0,\end{aligned}$$

that is, the conservation of momentum and incompressibility respectively. In the case of the solid, we will consider a linear approximation of the elasticity equation, namely

$$\rho_S(0)\partial_{tt}\xi = \mu\Delta\xi + (\lambda + \mu)\nabla \operatorname{div} \xi,$$

where  $\xi(t, x)$  is the displacement of the element that starts at  $x$  in time  $t = 0$ . Even though this equation is linear, when we consider the interaction with the fluid, the boundary conditions depends non-linearly on  $\xi$ . To express

this interaction, we will impose the continuity of the normal component of the velocity across the boundary of the solid. We can contrast this with the viscous case, where we have continuity of the velocity, not only the normal component. Finally we will require that the normal component of the stress tensor is continuous across the boundary as well.

More precisely, we are interested in solutions for the following system of equations

$$\left\{ \begin{array}{ll} \rho_S(0)\partial_{tt}\xi = \operatorname{div}(\pi(\xi)) = \mu\Delta\xi + (\lambda + \mu)\nabla\operatorname{div}\xi & \text{in } ]0, T] \times \mathcal{S}(0) \\ \partial_t u + (u \cdot \nabla)u = -\nabla p & \text{in } [0, T] \times \mathcal{F}(t) \\ \operatorname{div} u = 0 & \text{in } [0, T] \times \mathcal{F}(t) \\ u(t, x) \cdot n(t, x) = 0 & \text{in } [0, T] \times \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t) \\ (u(t, x) - \partial_t \xi(t, X^{-1}(t, x))) \cdot n(t, x) = 0 & \text{in } [0, T] \times \partial\mathcal{S}(t) \cap \partial\mathcal{F}(t) \\ (\pi(\xi)(t, X^{-1}(t, x)) \frac{\nabla X(t, X^{-1}(t, x))}{\det(\nabla X(t, X^{-1}(t, x)))} + p(t, x)I) \cdot n(t, x) = 0 & \text{in } [0, T] \times \partial\mathcal{S}(t) \end{array} \right. \quad (4.1)$$

where  $\mathcal{S}(t)$  denotes the domain of the elastic solid,  $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$  is the domain of the fluid. Here and throughout the article, we use the abuse of notation

$$[0, T] \times A(t) = \{(t, x) \in [0, T] \times \mathbb{R}^2, x \in A(t)\},$$

where  $A(t)$  is a set that depends on time.

The displacement of the solid with respect to rest is denoted  $\xi(t, x)$ , and  $X$  is its flow, that is,  $X(t, x) = x + \xi(t, x)$ . The first equation describes the elastic deformation of the solid. The operator  $\pi$  is the first Piola stress tensor given by

$$\pi(\xi) = \lambda \operatorname{tr}(\varepsilon) + 2\mu\varepsilon,$$

where  $\varepsilon(\xi) = \frac{1}{2}(\nabla\xi + \nabla\xi^t)$  is the symmetric gradient.

The second and third equations describe the evolution of the inviscid incompressible fluid in  $\mathcal{F}(t)$  where  $u$  is the velocity of the fluid and  $p$  the pressure.

Finally to describe the interaction between the fluid and the solid we have the two boundary condition. The first one describes the continuity of the normal component of the speed

$$u(t, x) \cdot n(t, x) = \partial_t \xi(t, X^{-1}(t, x)) \cdot n(t, x).$$

On the other hand we have the continuity of the normal component of the Stress tensor

$$\sigma_{\mathcal{F}} \cdot n = \sigma_{\mathcal{S}} \cdot n,$$

where  $\sigma_{\mathcal{F}}$  is the Cauchy stress tensor given by

$$\sigma_{\mathcal{S}} = \pi(\xi) \frac{\nabla X}{\det(\nabla X)},$$

and for the fluid

$$\sigma_{\mathcal{F}} = -pI.$$

For technical reasons, we will consider a finite number of modes approximation for the movement of the solid, that is, we will assume a solution on the form

$$\xi(t, x) = \sum_{i=1}^{N+1} \alpha_i(t) \eta_i(x),$$

for some fixed functions  $\eta_i$ . The reason to take  $N + 1$  modes is due to the fact that we will need to ensure that the volume of the solid is constant, which is a consequence from the fact that the volume of the fluid remains constant (due to the incompressibility condition). So we will use the  $(N + 1)$ -th to maintain this condition. We could consider any orthonormal family  $\eta_i$  but in order to simplify the computations, we will take them as the eigenvalues of the Lamé equations

$$\begin{aligned} \mathcal{L}\eta_i(x) &= \lambda_i \eta_i(x) \quad x \in \mathcal{S}(0), \\ \pi(\eta_i)n &= 0 \quad x \in \partial\mathcal{S}(0), \end{aligned}$$

where the Lamé operator  $\mathcal{L}$  is defined as

$$\mathcal{L}\xi = \mu\Delta\xi + (\lambda + \mu)\nabla \operatorname{div} \xi.$$

Concerning previous related works, we have in the case for an linear elastic solid in a viscous incompressible fluid, the work of Desjardins et al [19] and in the compressible case [20], [21]. We also can mention the work of Boulakia [4] in the viscous case with a regularized elasticity equation.

For incompressible Euler equations we have results for a rigid solid, for instance, Glass and Sueur[32], [33] and the work of Glass, Sueur and Takahashi [34] for a more regular result. We can also mention the works of Ortega, Rosier and Takahashi [55], [56] for a incompressible perfect fluid. Finally we mention the works of Boulakia and Guerrero, [5], [6], [7]. We can also mention the case of the Cauchy problem for rigid swimmers in a viscous fluid the work of San Martin et al in [57] and also of Munnier in [53] and [10] as a middle ground between rigid and elastic solids.

An interesting problem involving solid fluid interactions is the controllability of such systems, in which case we can mention for instance the work of Boulakia and Osses [8] and the references within.

*Outline.* In section 2 we will define exactly what do we mean by a solution of the system (4.1). In section 3 we establish the main result of the article. Then, in section 4 we introduce the concept of added mass which we will need in order to properly define the fixed point scheme in section 5. In section 6 we prove all the technical results like regularity, continuity and compactness necessary to apply Schauder fixed point theorem and finally in section 7 we finish the proof of the main result.

## 4.2 Definition of solution

We will prove the existence of solution of (4.1) à la Yudovich, that is, if we consider the vorticity of the fluid  $\omega$ , then it satisfies the transport equation

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad (t, x) \in [0, T] \times \Omega,$$

where we have extended the vorticity by 0 inside the solid. This does not affect the solutions thanks to the condition  $u \cdot n = 0$  so the characteristic curves of  $u$  never enter the solid.

If we consider that  $\Theta$  the flow of  $u$  then

$$\omega(t, x) = \omega_0(\Theta(t, x)).$$

Reciprocally, given  $\omega$  we can recover the velocity as the solution of the system

$$\begin{aligned} \operatorname{div} u &= 0 && \text{in } \mathcal{F}(t) \\ \operatorname{curl} u &= \omega && \text{in } \mathcal{F}(t) \\ u \cdot n &= 0 && \text{in } \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t) \\ u \cdot n &= \partial_t \xi(t, X^{-1}(t, x)) && \partial\mathcal{S}(t) \end{aligned} \tag{4.2}$$

$$\int_{\gamma_i} \tau \cdot u dl = \int_{\gamma_i} \tau \cdot u_0 dl \quad i \in \{1, \dots, m\}$$

so for the fluid a solution would be  $(u, \omega, \Psi)$  satisfying where  $\gamma_i$  are curves around the holes of  $\mathcal{F}(t)$ . The last condition is due to the fact that the circulation around a closed curve is preserved due to Kelvin circulation theorem, and we need to impose it in order to have uniqueness of the solution due to the fact that we are working in a non simply connected domain which is necessary to have a solid inside the fluid in dimension 2. With this a equivalent formulation is (at least for strong solutions):

$$\operatorname{div} u = 0,$$

$$\Theta = x + \int_0^t u(t, \Theta(t, x)) dx,$$

$$\omega = \omega_0 \circ \Theta.$$

Concerning the solid, we decompose the displacement in a finite number of modes.

Let us take  $\alpha_i \in C^1([0, T])$  and  $\omega \in L^\infty(\Omega)$ . Then we associate the displacement

where  $\delta$  is the solution to the ODE

$$\dot{\delta}(t) = - \sum_i^N c_i(\alpha_j(t), \delta(t)) \dot{\alpha}_i(t),$$

and the function  $c$  is given by

$$c_i(\alpha, \delta) = \frac{\int_{\partial S_{(\alpha, \delta)}} \eta_i \cdot \hat{n}_{(\alpha, \delta)} dS}{\int_{\partial S_{(\alpha, \delta)}} \eta_{N+1} \cdot \hat{n}_{(\alpha, \delta)} dS},$$

and  $S_{(\alpha, \delta)} = (Id + \sum_i^N \alpha_i \eta_i + \delta \eta_{N+1}) \mathcal{S}(0)$ .

$$\xi(t, x) = \sum_i^N \alpha_i(t) \eta_i(x) + \delta(t) \eta_{N+1}(x),$$

where

$$\operatorname{div}(\pi_1(\eta_i)) = \lambda_i \eta_i,$$

and  $\delta$  is the solution to the ODE

$$\dot{\delta}(t) = - \sum_i^N c_i(\alpha_j(t), \delta(t)) \dot{\alpha}_i(t),$$

and the function  $c$  is given by

$$c_i(\alpha, \delta) = \frac{\int_{\partial S_{(\alpha, \delta)}} \eta_i \cdot \hat{n}_{(\alpha, \delta)} dS}{\int_{\partial S_{(\alpha, \delta)}} \eta_{N+1} \cdot \hat{n}_{(\alpha, \delta)} dS},$$

and  $S_{(\alpha, \delta)} = (Id + \sum_i^N \alpha_i \eta_i + \delta \eta_{N+1}) \mathcal{S}(0)$ . With this the derivative of the displacement is

$$\partial_t \xi = \sum_{i=1}^N \dot{\alpha}_i \eta_i + \dot{\delta} \eta_{N+1} = \sum_{i=1}^N \dot{\alpha}_i (\eta_i - c_i \eta_{N+1}),$$

hence the second derivative is

$$\partial_{tt}\xi = \sum_{i=1}^N \ddot{\alpha}_i(\eta_i - c_i\eta_{N+1}) - \dot{\alpha}_i\dot{c}_i\eta_{N+1},$$

We then project the equation of the solid over the space generated by the  $\eta_i - c_i\eta_{N+1}$ , for  $i \in \{1, \dots, N\}$ .

$$\begin{aligned} & \int_{S(0)} \ddot{\xi}(\eta_j - c_j\eta_{N+1}) - \int_{S(0)} \pi(\xi)\nabla(\eta_j - c_j\eta_{N+1}) \\ &= \int_{\partial S(t)} np(t, x)(\eta_i(X^{-1}(t, x)) - c_i(t)\eta_{N+1}(X^{-1}(t, x)))dS. \end{aligned}$$

Then the equations for the solid are given by

$$\begin{aligned} & \sum_j \int_{S(0)} \ddot{\alpha}_i(\eta_i - c_i\eta_{N+1})(\eta_j - c_j\eta_{N+1}) - \sum_j \int_{S(0)} \dot{\alpha}_i\dot{c}_i\eta_{N+1}(\eta_j - c_j\eta_{N+1}) \\ & \quad - \alpha_j\lambda_j - \lambda_{N+1}\delta c_j \\ &= \int_{\partial S(t)} np(t, x)(\eta_i(X^{-1}(t, x)) - c_i(t)\eta_{N+1}(X^{-1}(t, x)))dS. \end{aligned}$$

Due to the fact that the pressure depends in part on the acceleration of the fluid which in turn depends in part on the acceleration of the solid thanks to the impermeability condition, we wish to extract from the last term the part that depends on  $\ddot{\alpha}$ . In order to do that we will make some definitions. We introduce the Kirchhoff potentials  $\Phi_i$  defined as the solutions to

$$\begin{aligned} \Delta\Phi_i &= z(x) \int_{\partial S(t)} (\eta_i(X^{-1}(t, x)) - c_i(t)\eta_{N+1}(X^{-1}(t, x))) \cdot n(t, x) && \text{in } \mathcal{F}(t), \\ \frac{\partial\Phi_i}{\partial n} &= (\eta_i(X^{-1}(t, x)) - c_i(t)\eta_{N+1}(X^{-1}(t, x))) \cdot n(t, x) && \text{in } \partial\mathcal{F}(t) \cap \partial\mathcal{S}(t), \\ \frac{\partial\Phi_i}{\partial n} &= 0 && \text{in } \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t). \end{aligned} \tag{4.3}$$

where  $z : \Omega \rightarrow \mathbb{R}$  is a fixed function such that

$$\int_{\Omega} z dx = 1,$$

and such that the support of  $z$  never encounters the solid  $\mathcal{S}(t)$ . The purpose of this function is to satisfy the compatibility conditions of the Neumann problem. When we consider the superposition of the Kirchhoff potentials,

the influence of  $z$  will disappear thanks to the volume preserving condition. Integrating by parts the boundary terms:

$$\int_{\partial\mathcal{S}(t)} np(t, x)(\eta_i(X^{-1}(t, x)) - c_i(t)\eta_{N+1}(X^{-1}(t, x)))dS = \int_{\mathcal{F}(t)} \nabla p \nabla \Phi_i(t, x) dS,$$

Using Euler's equation to remove the pressure we obtain:

$$\begin{aligned} (\mathcal{M})\ddot{\alpha} - \Lambda\alpha + \Gamma(t)\dot{\alpha} - \lambda_{N+1}\delta c = & - \int_{\mathcal{F}(t)} u_t \nabla \Phi_i(t, x) dS \\ & - \int_{\mathcal{F}(t)} (u \cdot \nabla) u \nabla \Phi_i(t, x) dS, \end{aligned} \quad (4.4)$$

where we introduced the proper mass matrix

$$\mathcal{M}(t)_{ij} = \int_{\mathcal{S}(0)} (\eta_i - c_i\eta_{N+1})(\eta_j - c_j\eta_{N+1}),$$

the diagonal matrix of eigenvalues of the Lamé operator  $\Lambda$  given by

$$\Lambda_{ij} = \lambda_i \delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker delta, and finally the auxiliary matrix

$$\Gamma(t) = \sum_j \int_{\mathcal{S}(0)} \dot{c}_j \eta_{N+1} (\eta_j - c_j \eta_{N+1}).$$

In order to extract the term  $\ddot{\alpha}$  from  $\partial_t u$ , we will do the following decomposition of the velocity noticing that it solves the linear system (provided that we know  $\omega$ ):

$$\begin{aligned} \operatorname{div} u &= 0 && \text{in } \mathcal{F}(t) \\ \operatorname{curl} u &= \omega && \text{in } \mathcal{F}(t) \\ u \cdot n &= 0 && \text{in } \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t) \\ u \cdot n &= \sum_i \dot{\alpha}_i (\eta_i((X^{-1}) - c_i \eta_{N+1}(X^{-1})) \cdot n && \text{in } \partial\mathcal{F}(t) \cap \partial\mathcal{S}(t), \partial\mathcal{S}(t) \\ \int_{\gamma_i} \tau \cdot u dl &= \int_{\gamma_i} \tau \cdot u_0 dl && i \in \{1, \dots, m\} \end{aligned} \quad (4.5)$$

which solution we can write as

$$u = \sum_i^N \dot{\alpha}_i(t) \nabla \Phi_i(t, x) + K[\omega] + \sum_i^m \mu_i h_i(t) \nabla^\perp \hat{\Phi}_i(t, x), \quad (4.6)$$

where in addition to the Kirchhoff potentials, we consider the Biot-Savart operator given by the solution to the system:

$$\begin{aligned}
\operatorname{div} K[\omega] &= 0 \quad , \text{ in } \Omega, \\
\operatorname{curl} K[\omega] &= \omega \quad \text{ in } \Omega, \\
K[\omega] \cdot n &= 0 \quad \text{ in } \partial\Omega. \\
\int_{\gamma_i} \tau \cdot K[\omega] dl &= 0 \quad .
\end{aligned} \tag{4.7}$$

As well as functions  $\hat{\Phi}_i$  satisfying

$$\begin{aligned}
\Delta \hat{\Phi}_i &= 0 \quad \text{ in } \mathcal{F}(t), \\
\hat{\Phi}_i &= 1 \quad \text{ in } \gamma_i, \\
\hat{\Phi}_i &= 0 \quad \text{ in } \partial\mathcal{F}(t) \setminus \gamma_i.
\end{aligned} \tag{4.8}$$

and the functions  $h_i$  are

$$h_i(t) = \frac{1}{\int_{\gamma_i} \frac{\partial \hat{\Phi}}{\partial n} dx}.$$

which is well defined thanks to the maximum principle and the Hopf's lemma.

Then the acceleration is given by

$$\partial_t u = \sum_i^N (\ddot{\alpha}_i(t) \nabla \Phi_i(t, x) + \alpha_i(t) \partial_t \nabla \Phi_i(t, x)) + \partial_t K[\omega] + \sum_i^m \mu_i h_i(t) \partial_t \nabla^\perp \hat{\Phi}_i(t, x) + \dot{h}_i(t) \nabla^\perp \hat{\Phi}_i(t, x), \tag{4.9}$$

With this we can rewrite the equation as

$$\begin{aligned}
(\mathcal{M})\ddot{\alpha} - \Lambda\alpha + \Gamma(t)\dot{\alpha} - \lambda_{N+1}\delta c &= - \int_{\mathcal{F}(t)} \partial_t K[\omega] \nabla \Phi_i(t, x) \\
&\quad - \sum_{j=1}^N \int_{\mathcal{F}(t)} (\dot{\alpha}_j \partial_t \nabla \Phi_j + \ddot{\alpha}_j \nabla \Phi_j) \nabla \Phi_i(t, x) \\
&\quad - \sum_{j=1}^m \mu_j h_j(t) \int_{\mathcal{F}(t)} \partial_t \nabla^\perp \hat{\Phi}_i(t, x) \cdot \nabla \Phi_i(t, x) dS \\
&\quad - \sum_{j=1}^m \mu_j \dot{h}_j(t) \int_{\mathcal{F}(t)} \nabla^\perp \hat{\Phi}_i(t, x) \cdot \nabla \Phi_i(t, x) dS \\
&\quad - \int_{\mathcal{F}(t)} (u \cdot \nabla) u \nabla \Phi_i(t, x) dS, \tag{4.10}
\end{aligned}$$



So we will define a Yudovich solution of system (4.1) as functions  $\alpha \in W^2([0, T])$ ,  $u \in L^\infty(\mathcal{L}\mathcal{L}(\mathcal{F}(t)))$ ,  $\omega \in L^\infty(L^\infty)$ ,  $\Theta \in C^{0,\alpha}([0, T] \times \mathcal{F}(t))$  such that

$$\left\{ \begin{array}{ll} (\mathcal{M})\ddot{\alpha} - \Lambda\alpha + \Gamma(t)\dot{\alpha} - \lambda_{N+1}\delta c = \tilde{f}(\alpha, \dot{\alpha}, \ddot{\alpha}, u) & t \in [0, T] \\ \dot{\delta}(t) = -\sum_i^N c_i(\alpha_j(t), \delta(t))\dot{\alpha}_i(t), & t \in [0, T] \\ \operatorname{div} u = 0 & (t, x) \in [0, T] \times \mathcal{F}(t) \\ \operatorname{curl} u = \omega & (t, x) \in [0, T] \times \mathcal{F}(t) \\ \\ \int_\gamma \tau \cdot u dl = \int_\gamma \tau \cdot u_0 dl & t \in [0, T] \\ \Theta = x + \int_0^t u(t, \Theta(t, x)) dx & (t, x) \in [0, T] \times \mathcal{F}(t) \\ \omega = \omega_0 \circ \Theta & (t, x) \in [0, T] \times \mathcal{F}(t) \\ u \cdot n = \partial_t X \cdot n & (t, x) \in [0, T] \times \partial\mathcal{S}(t) \\ u \cdot n = 0 & (t, x) \in [0, T] \times \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t) \end{array} \right. \quad (4.11)$$

where the components of  $f$  are given by

$$\begin{aligned} \tilde{f}_i(\alpha, \dot{\alpha}, \ddot{\alpha}, u) = & - \int_{\mathcal{F}(t)} \partial_t K[\omega] \cdot \nabla \Phi_i(t, x) \\ & - \sum_{j=1}^N \int_{\mathcal{F}(t)} (\dot{\alpha}_j \partial_t \nabla \Phi_j + \ddot{\alpha}_j \nabla \Phi_j) \nabla \Phi_i(t, x) \\ & - \sum_{j=1}^m \mu_j h_j(t) \int_{\mathcal{F}(t)} \partial_t \nabla^\perp \hat{\Phi}_i(t, x) \cdot \nabla \Phi_i(t, x) dS \\ & - \sum_{j=1}^m \mu_j \dot{h}_j(t) \int_{\mathcal{F}(t)} \nabla^\perp \hat{\Phi}_i(t, x) \cdot \nabla \Phi_i(t, x) dS \\ & - \int_{\mathcal{F}(t)} (u \cdot \nabla) u \nabla \Phi_i(t, x) dS. \end{aligned} \quad (4.12)$$

### 4.3 Main result

The main result of this article is the following:

**Theorem 4.3.1.** *If the boundary of  $\mathcal{F}(0)$  is of class  $C^{2,\alpha}$  and  $\omega_0 \in L^\infty(\mathcal{F}(0))$ ,  $u_0 \in \mathcal{L}\mathcal{L}(\mathcal{F}(0))$  and  $\int_{\partial\mathcal{S}(0)} \eta_{N+1}(x) \cdot \hat{n} dS \neq 0$ , then there is  $T > 0$  such that there*

is a solution  $(\alpha, u, \omega, \Theta) \in W^{2,\infty}([0, T]) \times L^\infty([0, T], \mathcal{L}\mathcal{L}(\mathcal{F}(t))) \times L^\infty([0, T] \times \infty(\Omega)) \times C^{0,\alpha}([0, T] \times \mathcal{F}(t))$  to the system (4.11) and moreover  $T$  satisfies

$$T > T_0(\|\alpha_0\|, \|X^{-1}(0, \cdot)\|, \int_{\partial S(0)} \eta_{N+1}(x) \cdot \hat{n} dS)$$

that is, the minimal time only depends uniformly on  $\alpha_0, X^{-1}(0, \cdot)$  and  $\int_{\partial S(0)} \eta_{N+1}(x) \cdot \hat{n} dS$ .

## 4.4 Added mass matrix

In order to remove the second time derivative from the right hand side of (4.10) we introduce the added mass matrix

$$\mathcal{A}(t)_{ij} = \int_{\mathcal{F}(t)} \nabla \Phi_i \nabla \Phi_j dx,$$

which is semi-definite positive as it is a Gramian matrix, in effect

$$\xi^t \mathcal{A}(t) \xi = \int_{\mathcal{F}(t)} \left| \sum_i \xi_i \nabla \Phi_i \right|^2 dx \geq 0$$

With this we can write the term involving the second time derivate in (4.10) as

$$\sum_i \ddot{\alpha}_i \int_{\mathcal{F}(t)} \nabla \Phi_i \nabla \Phi_j dx = \mathcal{A}(t) \ddot{\alpha},$$

and system (4.10) reads as:

$$(\mathcal{M} + \mathcal{A}(t)) \ddot{\alpha} + \Gamma(t) \dot{\alpha} + \Lambda \alpha - \lambda_{N+1} \delta c = f(\alpha, \dot{\alpha}, u) \quad (4.13)$$

where the components of  $f$  are

$$\begin{aligned}
f_i(\alpha, \dot{\alpha}, \ddot{\alpha}, u) = & - \int_{\mathcal{F}(t)} (\partial_t K[\omega] \nabla \Phi_i(t, x) \\
& - \sum_{j=1}^N \int_{\mathcal{F}(t)} \dot{\alpha}_j \partial_t \nabla \Phi_j \nabla \Phi_i(t, x) \\
& - \sum_{j=1}^m \mu_j h_j(t) \int_{\mathcal{F}(t)} \partial_t \nabla^\perp \hat{\Phi}_i(t, x) \cdot \nabla \Phi_i(t, x) dS \\
& - \sum_{j=1}^m \mu_j \dot{h}_j(t) \int_{\mathcal{F}(t)} \nabla^\perp \hat{\Phi}_i(t, x) \cdot \nabla \Phi_i(t, x) dS \\
& - \int_{\mathcal{F}(t)} (u \cdot \nabla) u \nabla \Phi_i(t, x) dS, \quad (4.14)
\end{aligned}$$

where the main difference is that we put the second order time derivatives in the left hand side. This will be important when we consider the fixed point argument to have the appropriate regularity of the fixed point map.

In order for this equation to be well defined, we need that the "complete" mass matrix (proper mass + added mass)

$$\mathcal{M}(t) + \mathcal{A}(t),$$

to be invertible and, in order to have good estimates, that the inverse is bounded. This can be obtained from the fact that  $\mathcal{M}$  is definite positive as well because the functions  $\xi_i$  are an orthonormal.

## 4.5 Fixed Point Argument

In this section we will provide the construction of the map whose fixed points would correspond to solutions to system (4.11). Later we will prove, using Schauder's fixed point theorem, that such fixed point exists indeed. Let us consider the set

$$\begin{aligned}
\mathcal{C}_R = & \{(\alpha, \omega) \in C^1([0, T], \mathbb{R}^N) \times L^\infty([0, T], L^\infty(\Omega)) \mid \\
& \|\alpha - \alpha(0)\|_{C^1} \leq R, \|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}\},
\end{aligned}$$

with the topology of  $C^1([0, T], \mathbb{R}^{N+1}) \times L^\infty([0, T], L^2(\Omega))$ .

Let us take  $\alpha_i \in C^1([0, T])$  and  $\omega \in L^\infty(\Omega)$ . Then we associate the displacement

$$\xi(t, x) = \sum_i^N \alpha_i(t) \eta_i(x) + \delta(t) \eta_{N+1}(x),$$

where  $\delta$  is the solution to the ODE

$$\dot{\delta}(t) = - \sum_i^N c_i(\alpha_j(t), \delta(t)) \dot{\alpha}_i(t),$$

and the function  $c$  is given by

$$c_i(\alpha, \delta) = \frac{\int_{\partial S_{(\alpha, \delta)}} \eta_i \cdot \hat{n}_{(\alpha, \delta)} dS}{\int_{\partial S_{(\alpha, \delta)}} \eta_{N+1} \cdot \hat{n}_{(\alpha, \delta)} dS},$$

and  $S_{(\alpha, \delta)} = (Id + \sum_i^N \alpha_i \eta_i + \delta \eta_{N+1}) \mathcal{S}(0)$ . With an abuse of notation we will write

$$c_i(t) = c_i(\alpha(t), \delta(t)).$$

The flow of the solid is given by

$$X(t, x) = x + \xi(t, x),$$

in particular

$$X \in C^1([0, T], H^3(\mathcal{S}(0))) \subset C^1([0, T], C^1(\mathcal{S}(0))),$$

and for small enough times the inverse flow is defined and satisfies

$$X^{-1}(t) \in C^1([0, T], H^3(\mathcal{S}(0))).$$

With this we construct a solution to the Euler system

$$u = \sum_i^N \alpha_i(t) \nabla \Phi_i(t, x) + K[\omega] + \sum_i^m \mu_i \nabla^\perp \hat{\Phi}_i(t, x), \quad (4.15)$$

as discussed earlier.

Provided that  $u$  is log-Lipschitz, we can define the flow  $\Theta(t, x)$ . With log-Lipschitz we refer to bounded functions that satisfy the inequality

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|(1 + \ln^- |x - y|)} < \infty.$$

We denote the space of such functions by  $\mathcal{LL}(\mathcal{F}(t))$  and we endow it with then norm

$$\|u\|_{\mathcal{LL}(\mathcal{F}(t))} = \|u\|_{L^\infty(\mathcal{F}(t))} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|(1 + \ln^- |x - y|)}.$$

Then we can define a new vorticity

$$\tilde{\omega}(t, x) = \omega_0(\Theta(t, x)),$$

and finally we solve the elasticity equations with the added mass term

$$(\mathcal{M} + \mathcal{A}(t))\ddot{\tilde{\alpha}} + \Gamma(t)\dot{\tilde{\alpha}} - \Lambda\tilde{\alpha} - \lambda_{N+1}\delta c = f(\alpha, \dot{\alpha}, u) \quad (4.16)$$

with this we define the map

$$\begin{aligned} \mathcal{G} : \mathcal{C}_R &\rightarrow \mathcal{C}_R \\ (\alpha, \omega) &\rightarrow (\tilde{\alpha}, \tilde{\omega}). \end{aligned}$$

In the next sections, our aim is to proof that this map has a fixed point, and that such fixed point provides a solution to (4.11).

## 4.6 Estimates

In this section we provide all the estimates necessary to prove that the fixed point map is well defined, continuous and that maps the set  $\mathcal{C}_R$  into itself. In what follows, we assume that  $T$  is bounded by a fixed  $T^* > 0$ , so when we say that a constant does not depend on  $T$ , we mean that it does not blow up on  $[0, T^*]$ .

### 4.6.1 Estimates on the Kirchhoff potentials

**Lemma 4.6.1.** *Let  $\alpha, \delta \in \mathcal{C}_R$  and the Kirchhoff potential  $\Phi_i$  defined as in (4.3), then  $\Phi_i$  is of class  $C^{2,\alpha}(\mathcal{F}(0))$  and we have the uniform estimate*

$$\|\Phi_i\|_{C^{2,\alpha}(\mathcal{F}(0))} \leq C_R,$$

with  $C_R$  independent of  $T$ .

*Proof.* In order to obtain uniform estimates on the Kirchhoff potentials let us consider a smooth function  $\Psi(t, x)$  such that

$$\Psi(t, \cdot)\mathcal{F}(0) = \mathcal{F}(t),$$

and

$$\begin{aligned} \Psi(t, x) &= X(t, x), x \in \partial\mathcal{S}(t), \\ \Psi(t, x) &= x, x \in \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t), \end{aligned}$$

we can achieve this by considering solutions to the equations

$$\Delta\theta_i = 0,$$

$$\theta_i = \partial_t X_i,$$

$$\theta_i = 0,$$

and then define  $\Psi$  as the flow of  $\theta_i$ . Let  $\Phi$  the Kirchhoff potentials satisfying

$$\Delta\Phi(t, x) = f(t, x),$$

$$n(t, x) \cdot \nabla\Phi = g(t, Y(t, x))n(t, x),$$

then if we define

$$\tilde{\Phi}(t, x) = \Phi(t, \Psi(t, x)),$$

we have that  $\tilde{\Phi}$  satisfies the elliptic equation

$$-\sum_{k,l} a_{kl} \partial_{kl} \tilde{\Phi} + \sum_k \Delta(\Psi^{-1}) \partial_k \tilde{\Phi} = f(t, \Psi(t, x)),$$

$$n(t, \Psi(t, x)) D\Psi^{-1} \cdot \nabla \tilde{\Phi} = n(t, \Psi(t, x)) \cdot g(t, x),$$

where

$$a_{kl} = \sum_i \partial_i(\Psi_k^{-1}) \partial_i(\Psi_l^{-1}),$$

Using Hölder regularity estimates (see for instance [28] page 127) we obtain the estimates

$$\|\tilde{\Phi}\|_{C^{2,\alpha}(\mathcal{F}(0))} \leq C(\|\tilde{\Phi}\|_{C^0(\mathcal{F}(0))} + \|f\|_{C^{0,\alpha}(\mathcal{F}(0))} + \|g(t, \cdot) n(t, \Psi(t, \cdot))\|_{C^{1,\alpha}(\mathcal{F}(0))}),$$

where the constant  $C$  depends uniformly on

$$\|\Psi^{-1}\|_{C^{2,\alpha}},$$

and

$$\|n(t, \Psi(t, \cdot)) D\Psi^{-1}\|_{C^{1,\alpha}}.$$

□

### 4.6.2 Estimates on the fluid velocity

For the velocity we are interested in log-Lipschitz estimates in order to be able to define the flow and to have holder estimates on the flow.

**Lemma 4.6.2.** *Let  $u$  be as in (4.9), then, there is  $C_R$  independent of time such that,*

$$\|u\|_{\mathcal{L}\mathcal{L}} \leq C\|\omega_0\| + C_R.$$

*Proof.* By estimates each term separately we obtain:

$$\|u\|_{\mathcal{L}\mathcal{L}} \leq C\|\omega\|_{L^\infty} + \|\alpha\|_{C^1} \max_i \|\nabla\Phi_i\|_{\mathcal{L}\mathcal{L}} + \max_j \|\nabla^\perp\phi_j\|,$$

where we used the fact that (see for instance [48])

$$\|K[\omega]\|_{\mathcal{L}\mathcal{L}} \leq C\|\omega\|_{L^\infty}.$$

so we obtain

$$\|u\|_{\mathcal{L}\mathcal{L}} \leq C\|\omega\|_{L^\infty} + \|\alpha\|_{C^1} \max_i \|\Phi_i\|_{C^{2,\varepsilon}} + \max_j \|\nabla^\perp\phi_j\|,$$

and finally

$$\leq C\|\omega\|_{L^\infty} + \|\alpha\|_{C^1} \max_i \|(\eta_i - c_i(t)\eta_{N+1}) \cdot n(t, x)\|_{C^{1,\varepsilon}} + \max_j \|\nabla^\perp\phi_j\|.$$

□

### 4.6.3 Spatial regularity of the flow and its inverse

If we have

$$\|X(t, x) - x\|_{C^1(\mathcal{S}(0))} \leq \ell < 1, \forall t \in [0, T]$$

then we know, as a consequence of Banach fixed point argument that  $X$  is invertible and its inverse, denoted  $Y$ , is  $C^1$ . We now study the Sobolev norm of the inverse of the flow.

**Lemma 4.6.3.** *If  $X \in H^3(\mathcal{S}(0))$  and the inverse flow is defined in  $[0, T]$ , we have the estimate*

$$\|Y\|_{H^3(\mathcal{S}(t))} \leq \frac{C}{\inf |\det \nabla X|^{\frac{7}{2}}} \|X\|_{H^3(\mathcal{S}(0))} (1 + \|Y\|_{C^1(\mathcal{S}(t))}^3). \forall t \in [0, T]$$

*Proof.* Let us denote

$$J = \det(DX).$$

From the inverse function theorem we have

$$DY(x) = (DX \circ Y)^{-1} = \frac{1}{J \circ Y} \text{Adj}(DX \circ Y)$$

where *adj* is the adjugate matrix (which is a linear operator in dimension 2).

$$\partial_i DY = \sum_j \frac{-1}{J^2 \circ Y} (\partial_j J \circ Y) (\partial_i Y_j) \text{Adj}(DX \circ Y) + \frac{1}{J \circ Y} \text{Adj}(\partial_j DX \circ Y) (\partial_i Y_j),$$

Henceforth

$$|\partial_i DY| \leq \sum_j \frac{1}{\inf |J|^2} |\partial_j J \circ Y| |\partial_i Y_j| |DX \circ Y| + \frac{1}{\inf |J|} |\partial_j DX \circ Y| |\partial_i Y_j|,$$

so for the  $L^2$  norm we obtain

$$\begin{aligned} \|\partial_i DY\|_{L^2(\mathcal{S}(t))} &\leq \|Y\|_{C^1} \sum_j \frac{1}{\inf |J|^2} \|\partial_j J \circ Y\|_{L^2(\mathcal{S}(t))} \|X\|_{C^1} \\ &\quad + \frac{1}{\inf |J|} \|\partial_j DX \circ Y\|_{L^2(\mathcal{S}(t))}, \end{aligned} \quad (4.17)$$

and using the fact that  $\|f \circ Y\|_{L^2(\mathcal{S}(t))} \leq \frac{1}{\inf J^{\frac{1}{2}}} \|f\|_{L^2(\mathcal{S}(0))}$

$$\|\partial_i DY\|_{L^2(\mathcal{S}(t))} \leq \|Y\|_{C^1} \sum_j \frac{1}{\inf |J|^{\frac{5}{2}}} \|\partial_j J\|_{L^2(\mathcal{S}(0))} + \frac{1}{\inf |J|^{\frac{3}{2}}} \|\partial_j DX\|_{L^2(\mathcal{S}(0))}.$$

Similarly if we compute the second derivatives of  $DY$ , we obtain

$$\begin{aligned} \partial_{im} DY &= \sum_{j,k} \frac{2}{J^3 \circ Y} (\partial_j J \circ Y) (\partial_m J \circ Y) (\partial_i Y_j) (\partial_k Y_m) \text{Adj}(DX \circ Y) \\ &\quad + \sum_{j,k} \frac{-1}{J^2 \circ Y} (\partial_{jm} J \circ Y) (\partial_k Y_m) (\partial_i Y_j) \text{Adj}(DX \circ Y) \\ &\quad + \sum_j \frac{-1}{J^2 \circ Y} (\partial_j J \circ Y) (\partial_{ik} Y_j) \text{Adj}(DX \circ Y) \\ &\quad + 2 \sum_{j,m} \frac{-1}{J^2 \circ Y} (\partial_j J \circ Y) (\partial_i Y_j) \text{Adj}(\partial_m DX \circ Y) \partial_k Y_m \end{aligned}$$



$$\begin{aligned}
& + \sum_{j,m} \frac{1}{J \circ Y} \text{Adj}(\partial_{jm} DX \circ Y)(\partial_i Y_j)(\partial_k Y_m) \\
& + \sum_j \frac{1}{J \circ Y} \text{Adj}(\partial_j DX \circ Y)(\partial_{ik} Y_j),
\end{aligned}$$

Due to the fact that  $H^2$  is an algebra in dimension 2, and  $J$  is sum of product of derivatives of  $X$ , we have

$$\|J\|_{H^2} \leq C \|X\|_{H^3}^2.$$

□

#### 4.6.4 Regularity of the flow $X_{\alpha,\delta}$ and $Y_{\alpha,\delta}$

Now we are concerned with the regularity with respect to  $\alpha, \delta$ . For  $X_{\alpha,\delta}$  we have

$$\begin{aligned}
\partial_\delta X_{\alpha,\delta}(t, x) &= \eta_{N+1}(x), \\
\partial_{\alpha_i} X_{\alpha,\delta}(t, x) &= \eta_i(x).
\end{aligned}$$

In the case of the inverse  $Y_{\alpha,\delta}$  we have the following result

**Lemma 4.6.4.** *Given  $\bar{\alpha}, \bar{\delta} \in \mathbb{R}^{N+1}$  there is  $\varepsilon > 0$  such that  $Y_{\alpha,\delta}$  is of class  $C^1$  in an  $\varepsilon$ -neighbourhood of  $\bar{\alpha}, \bar{\delta}$  and*

$$\|Y_i\|_{C^1(B_\varepsilon(\bar{\alpha}, \bar{\delta}))} \leq C.$$

*Proof.* From the implicit function theorem we obtain that  $Y_{\alpha,\delta}$  is  $C^1$  with respect to  $\alpha$  and  $\delta$  and moreover,

$$\begin{aligned}
\partial_\delta Y_{\alpha,\delta}(t, x) &= -DY_{\alpha,\delta}(t, x) \partial_\delta X_{\alpha,\delta}(t, Y(t, x)) = -DY_{\alpha,\delta}(t, x) \eta_{N+1}(Y(t, x)), \\
\partial_{\alpha_i} Y_{\alpha,\delta}(t, x) &= -DY_{\alpha,\delta}(t, x) \partial_{\alpha_i} X_{\alpha,\delta}(t, Y(t, x)) = -DY_{\alpha,\delta}(t, x) \eta_i(Y(t, x)).
\end{aligned}$$

Indeed, if we consider the function for fixed  $t$  and  $x$ ,

$$F : \mathbb{R}^{N+1} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$F(\alpha, \delta, y) = X(\alpha, \delta, y) - x$$

then its Fréchet derivative with respect to  $y$  is given by  $DX_{\alpha,\delta}$  which is invertible in  $\mathcal{C}_R$ , so applying the implicit function theorem to  $F$ , around a fixed  $\hat{\alpha}, \hat{\delta}$ , we obtain that the solution to

$$F(\alpha, \delta, y) = 0 = X(\alpha, \delta, y) - x,$$

is  $C^1$  in  $\alpha, \delta$  but by definition  $Y_{\alpha,\delta}$  satisfies this equation, so we obtain the desired regularity to justify the computations. □

### 4.6.5 Regularity of $c_i(\alpha, \delta)$

In this section we study the functions

$$c_i(\alpha, \delta) = \frac{\int_{\partial S_{(\alpha, \delta)}} \eta_i \cdot \hat{n}_{(\alpha, \delta)} dS}{\int_{\partial S_{(\alpha, \delta)}} \eta_{N+1} \cdot \hat{n}_{(\alpha, \delta)} dS},$$

where  $S_{(\alpha, \delta)} = (Id + \sum_i^N \alpha_i \eta_i + \delta \eta_{N+1})\mathcal{S}(0)$ .

**Lemma 4.6.5.** *Given  $\bar{\alpha}, \bar{\delta} \in \mathbb{R}^{N+1}$  there is  $\varepsilon > 0$  such that  $c_i$  is of class  $C^1$  in an  $\varepsilon$ -neighborhood of  $\bar{\alpha}, \bar{\delta}$  and*

$$\|c_i\|_{C^1(B_\varepsilon(\bar{\alpha}, \bar{\delta}))} \leq C.$$

*Proof.* Let us define

$$\begin{aligned} f_i(\alpha, \delta) &= \int_{\partial S_{(\alpha, \delta)}} \eta_i(X_{(\alpha, \delta)}^{-1}(t, x)) \cdot \hat{n}_{(\alpha, \delta)} dS = \int_{S_{(\alpha, \delta)}} \operatorname{dive}(\eta_i(X_{(\alpha, \delta)}^{-1}(t, x))) dx \\ &= \int_{S_{(\alpha, \delta)}} D\eta_i(Y_{(\alpha, \delta)}(t, x)) : DY_{(\alpha, \delta)}^t(t, x) dx \\ &= \int_{S(0)} D\eta_i(x) : DY_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \det(DX_{\alpha, \delta}) dx \end{aligned}$$

So for the partial derivatives

$$\begin{aligned} \partial_\delta f_i(\alpha, \delta) &= \int_{S(0)} D\eta_i(x) : \partial_\delta DY_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \det(DX_{\alpha, \delta}) dx \\ &+ \int_{S(0)} D\eta_i(x) : (D^2 Y_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \partial_\delta X_{(\alpha, \delta)}(t, x)) \det(DX_{\alpha, \delta}) dx \\ &+ \int_{S(0)} D\eta_i(x) : DY_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \partial_\delta \det(DX_{\alpha, \delta}) dx \end{aligned}$$

Similarly for the derivative with respect to  $\alpha_i$

$$\begin{aligned} \partial_{\alpha_i} f_i(\alpha, \delta) &= \int_{S(0)} D\eta_i(x) : \partial_{\alpha_i} DY_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \det(DX_{\alpha, \delta}) dx \\ &+ \int_{S(0)} D\eta_i(x) : (D^2 Y_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \partial_{\alpha_i} X_{(\alpha, \delta)}(t, x)) \det(DX_{\alpha, \delta}) dx \\ &+ \int_{S(0)} D\eta_i(x) : DY_{(\alpha, \delta)}^t(t, X_{(\alpha, \delta)}(t, x)) \partial_{\alpha_i} \det(DX_{\alpha, \delta}) dx \end{aligned}$$

Going back to the functions  $c_i$  we have

$$\partial_\delta c_i(\alpha, \delta) = \frac{\partial_\delta f_i}{f_{N+1}} - \frac{f_i \partial_\delta f_{N+1}}{f_{N+1}^2}$$

$$\partial_\delta c_i(\alpha, \delta) = \frac{\partial_\delta f_i}{f_{N+1}} - \frac{f_i \partial_\delta f_{N+1}}{f_{N+1}^2}$$

Thanks to the continuity of  $f_{N+1}$  and the fact that

$$f_{N+1}(\alpha(0), \delta(0)) \neq 0$$

we have that there is  $\epsilon_R$  such that if  $|\alpha - \alpha(0)| + |\delta - \delta(0)| \leq \epsilon_R$

$$|f_{N+1}(\alpha, \delta)| \geq \delta_R$$

so we obtain that

$$\|c_i\|_{C^1} \leq C_R$$

in an neighborhood of  $\alpha(0), \delta(0)$ . □

#### 4.6.6 Time derivative of the Kirchhoff potentials

Now we consider the terms involving the time derivative of the Kirchhoff potentials. We have the following lemma:

**Lemma 4.6.6.** *Let  $\Phi[w]$  defined as in (4.3) and assuming that  $(\alpha, \omega) \in \mathcal{C}_R$ , then we have:*

$$\left| \sum_{i=1}^N \int_{\mathcal{F}(t)} \nabla \partial_t \Phi_i \cdot \nabla \Phi_j dx \right| \leq C_R,$$

where  $C_R$  is independent of  $T$ .

*Proof.* In this section we study the Kirchhoff potentials satisfying

$$\Delta \Phi_i = f_i \quad \text{in } \mathcal{F}(t)$$

$$n \cdot \nabla \Phi_i = n \cdot g_i(t, Y(t, x)) \quad \text{in } \partial \mathcal{S}(t)$$

where  $f_i$  and  $g_i$  satisfy the compatibility condition

$$\int_{\mathcal{F}(t)} f_i(t, x) dx = \int_{\partial \mathcal{S}(t)} n \cdot g_i(t, Y(t, x)) dS(x).$$

Let  $\Psi(t, x)$  be such that

$$\Psi(t, \cdot) \mathcal{F}(0) = \mathcal{F}(t),$$

and

$$\Psi(t, x) = X(t, x),$$

for  $x \in \partial\mathcal{S}(t)$ . Then

$$\Delta\Phi_i(t, \Psi(t, x)) = f(t, \Psi(t, x)) \quad \text{in } \mathcal{F}(0),$$

$$n(t, \Psi(t, x)) \cdot \nabla\Phi_i(t, \Psi(t, x)) = n(t, \Psi(t, x)) \cdot g(t, \Psi(t, x)) \quad \text{in } \partial\mathcal{S}(0)$$

Differentiating with respect to  $t$

$$\partial_t\Delta\Phi_i(t, \Psi(t, x)) = \partial_t f(t, \Psi(t, x)) \quad \text{in } \mathcal{F}(0),$$

$$n(t, \Psi(t, x)) \cdot \nabla\partial_t\Phi_i(t, \Psi(t, x)) = -nD^2\Phi_i(t, \Psi(t, x))\partial_t X,$$

$$+\partial_t(n(t, \Psi(t, x)))(g(t, \Psi(t, x)) - \nabla\Phi_i(t, \Psi(t, x))) + n(t, \Psi(t, x))\partial_t g(t, x) \quad \text{in } \partial\mathcal{S}(0)$$

In order to compute the derivative of the normal, we consider the tangent to the boundary of  $\mathcal{S}(t)$

$$\tau(t, x) = \frac{DX(t, Y(t, x))\tau_0(Y(t, x))}{\|DX(t, Y(t, x))\tau_0(Y(t, x))\|},$$

and then

$$n(t, x) = \tau(t, x)^\perp,$$

$$n(t, \Psi(t, x)) = \frac{(DX(t, x)\tau_0(x))^\perp}{\|DX(t, x)\tau_0(x)\|}.$$

If  $F = DX(t, x)\tau_0(x)$  then

$$\partial_t\tau(t, \Psi(t, x)) = \left(I - \frac{FF^t}{\|F\|^2}\right) \frac{\partial_t F}{\|F\|}.$$

Henceforth

$$\|\partial_t n(t, \Psi(t, x))\| \leq \frac{2\|\partial_t DX(t, x)\tau_0\|}{\|F\|},$$

and we notice that, because  $DX$  is invertible and  $\tau_0$  is unitary

$$\|F\| \geq \|DY\|^{-1},$$

so

$$\|\partial_t n(t, \Psi(t, x))\| \leq 2\|DY\| \|\partial_t DX(t, x)\|.$$

In particular

$$\begin{aligned} \int_{\mathcal{F}(t)} \nabla\partial_t\Phi_i \cdot \nabla\Phi_j dx &= - \int_{\mathcal{F}(t)} \partial_t f_i \Phi_j + \int_{\partial\mathcal{S}(t)} nD^2\Phi_i \partial_t X \Phi_j \\ &\quad + \partial_t n(g_i - \nabla\Phi_i) \Phi_j + n(\partial_t g_i(t, x)). \end{aligned} \quad (4.18)$$

Due to the condition  $\sum_i f_i = 0$ , we have

$$\sum_{i=1}^N \int_{\mathcal{F}(t)} \nabla \partial_t \Phi_i \cdot \nabla \Phi_j dx = \sum_{i=1}^N \int_{\partial \mathcal{S}(t)} n D^2 \Phi_i \partial_t X \Phi_j + \partial_t n (g_i - \nabla \Phi_i) \Phi_j + n (\partial_t g_i(t, x)),$$

so we obtain

$$\left| \sum_{i=1}^N \int_{\mathcal{F}(t)} \nabla \partial_t \Phi_i \cdot \nabla \Phi_j dx \right| \leq C_R,$$

where  $C_R$  is independent of  $T$ .  $\square$

#### 4.6.7 Holder estimates on the flow of the velocity

Given that the velocity field is Log-Lipschitz, we have the following Holder estimates on its flow:

**Lemma 4.6.7.** *If  $u \in L^\infty([0, T], \mathcal{LL}(\mathcal{F}(t)))$  then its flow  $\Theta$  is of class  $C^{0,\alpha}$  and moreover there is  $C > 0$  independent of  $T$  such that*

$$\|\Theta\|_{C^\alpha([0, T], \Omega)} \leq C \exp(1 - T\|u\|_{L^\infty(\mathcal{LL})}) + CT^{1-\alpha}\|u\|_{L^\infty},$$

where  $\alpha = \exp(-T\|u\|_{\mathcal{LL}})$ .

*For a proof of this classical result, see for instance [36]*

#### 4.6.8 Time derivative of the Biot-Savart Operator

Similarly as for the Kirchhoff potentials, we have the following uniform estimate on the time derivative of the Biot-Savart Operator:

**Lemma 4.6.8.** *Let  $K[w]$  defined as in (4.7) and assuming that  $(\alpha, \omega) \in \mathcal{C}_R$ , then we have:*

$$\left| \int_{\mathcal{F}(t)} \partial_t K[\omega] \cdot \nabla \Phi_j \right| \leq C_R$$

where  $C_R$  is independent of  $T$

*Proof.* For the Biot-savart operator we do similar computations to the Kirchhoff potentials:

$$\begin{aligned} (\operatorname{div} K)(t, \Psi(t, x)) &= 0 \quad \text{in } \mathcal{F}(t) \\ (\operatorname{curl} K)(t, \Psi(t, x)) &= w(t, \Psi(t, x)) \quad \text{in } \mathcal{F}(t) \\ n(t, \Psi(t, x)) \cdot K(t, \Psi(t, x)) &= 0 \quad \text{in } \partial \mathcal{S}(0) \end{aligned}$$

then

$$\begin{aligned}\operatorname{div} \partial_t K &= 0 \quad \text{in } \mathcal{F}(t) \\ \operatorname{curl} \partial_t K &= \partial_t \omega = -u \cdot \nabla \omega \quad \text{in } \mathcal{F}(t) \\ n \cdot \partial_t K &= -n^t (DK) \partial_t X - (\partial_t n) K \quad \text{in } \partial \mathcal{S}(0)\end{aligned}$$

In particular

$$\int_{\mathcal{F}(t)} \partial_t K[\omega] \cdot \nabla \Phi_j = - \int_{\partial \mathcal{S}(t)} n^t (DK[\omega]) \partial_t X \Phi_j dS - \int_{\partial \mathcal{S}(t)} \partial_t n K[\omega] \Phi_j dS$$

so we obtain

$$\left| \int_{\mathcal{F}(t)} \partial_t K[\omega] \cdot \nabla \Phi_j \right| \leq C_R$$

where  $C_R$  is independent of  $T$ . □

#### 4.6.9 Time derivative of $\nabla^\perp \hat{\Phi}_i$

Similarly as for the Kirchhoff potentials, we have the following uniform estimate on the time derivative of the Biot-Savart Operator:

**Lemma 4.6.9.** *Let  $\hat{\Phi}_i$  defined as in (4.8) and assuming that  $(\alpha, \omega) \in \mathcal{C}_R$ , the we have:*

$$\left| \int_{\mathcal{F}(t)} h_i(t) \partial_t \nabla^\perp \hat{\Phi}_i \cdot \nabla \Phi_j \right| + \left| \int_{\mathcal{F}(t)} \dot{h}_i(t) \nabla^\perp \hat{\Phi}_i \cdot \nabla \Phi_j \right| \leq C_R$$

where  $C_R$  is independent of  $T$

*Proof.* By considering the shape derivative of the equation for  $\hat{\Phi}_i$  as we did for the other equations before, we obtain

$$\Delta(\partial_t \hat{\Phi}_i) = 0, \quad \text{in } \mathcal{F}(t)$$

$$\partial_t \hat{\Phi}_i = -\nabla \hat{\Phi} \partial_t X, \quad \text{in } \partial \mathcal{F}(t)$$

using classical estimates for the equations for  $\hat{\Phi}_i$  and  $\partial_t \hat{\Phi}_i$ , we can obtain the inequality. □

### 4.6.10 Estimates on the solid displacement

Thanks to the fact that  $(\mathcal{M} + \mathcal{A}(t))^{-1}$  is uniformly bounded, we have the following estimates

**Lemma 4.6.10.** *Let  $\alpha$  as in (4.16), then we have the following estimate*

$$\|\alpha\|_{W^{2,\infty}([0,T])} \leq |\alpha_0| + TC_R,$$

where  $C_R$  is uniform in  $\mathcal{C}_R$ .

*Proof.* Using the differential equation for  $\tilde{\alpha}$ , that is equation (4.16),

$$\begin{aligned} |\ddot{\tilde{\alpha}}| &\leq C\|(u \cdot \nabla)u \nabla \Phi_i(t, \cdot)\|_{L^1(\Omega)} \\ &\quad + C \sum_j \max_i \|\alpha\|_{C^1} \|\partial_t \nabla \Phi_j \nabla \Phi_i(t, \cdot)\|_{L^1(\Omega)} \\ &\quad + C \max_i \|\partial_t K[\omega] \nabla \Phi_i(t, \cdot)\|_{L^1(\Omega)} \\ &+ \sum_j \mu_j \max_i \|(h_i \partial_t \nabla^\perp \hat{\Phi}_i(t, x) + \dot{h}_i \nabla^\perp \hat{\Phi}_i(t, x)) \nabla \Phi_i(t, \cdot)\|_{L^1(\Omega)} + |\lambda_{N+1} \delta c|, \end{aligned} \tag{4.19}$$

so we have

$$\begin{aligned} \|\tilde{\alpha}\|_{W^{2,\infty}([0,T])} &\leq |\alpha_0| + T\|(u \cdot \nabla)u \nabla \Phi_i(t, x)\|_{L^\infty([0,T], L^1)} \\ &\quad + CT\|\alpha\|_{C^1} \|\partial_t \nabla \Phi_j \nabla \Phi_i(t, x)\|_{L^\infty([0,T], L^1)} \\ &\quad + CT\|(\partial_t K[\omega] \nabla \Phi_i(t, x) \nabla \Phi_i(t, x))\|_{L^\infty([0,T], L^1)}, \\ &+ CT \sum_j \mu_j \max_i \|(h_i \partial_t \nabla^\perp \hat{\Phi}_i(t, x) + \dot{h}_i \nabla^\perp \hat{\Phi}_i(t, x)) \nabla \Phi_i(t, \cdot)\|_{L^1(\Omega)} CT\|\lambda_{N+1} \delta c\|_{L^\infty} \end{aligned}$$

henceforth

$$\begin{aligned} \|\tilde{\alpha}\|_{W^{2,\infty}([0,T])} &\leq |\alpha_0| + CT\|(u \cdot \nabla)u\|_{L^\infty([0,T], L^2)} \|\nabla \Phi_i(t, x)\|_{L^\infty([0,T], L^2)} \\ &\quad + CT\|\alpha\|_{C^1} \|\partial_t \nabla \Phi_j \nabla \Phi_i(t, x)\|_{L^\infty([0,T], L^1)} \\ &\quad + CT\|(\partial_t K[\omega] \nabla \Phi_i(t, x) \nabla \Phi_i(t, x))\|_{L^\infty([0,T], L^1)} \\ &+ \sum_j \mu_j \max_i \|(h_i \partial_t \nabla^\perp \hat{\Phi}_i(t, x) + \dot{h}_i \nabla^\perp \hat{\Phi}_i(t, x)) \nabla \Phi_i(t, \cdot)\|_{L^1(\Omega)} + CT\|\lambda_{N+1} \delta c\|_{L^\infty} \end{aligned} \tag{4.20}$$

finally, using lemmas 4.6.3, 4.6.6, 4.6.8 and 4.6.9, we obtain

$$\|\tilde{\alpha}\|_{W^{2,\infty}([0,T])} \leq |\alpha_0| + TC_R.$$

□

### 4.6.11 Compactness

In order to proof compactness of the set  $A(\mathcal{C}_R)$  we use the fact that

$$W^{2,\infty}([0, T]) \subset\subset C^1([0, T]).$$

To obtain compactness for the vorticity we use the following lemma

**Lemma 4.6.11.** *Let  $C > 0$  and  $\omega_0 \in L^\infty(\Omega)$ . Then the set*

$A(\omega_0) = \{\omega_0 \circ \Theta \text{ for } \Theta \in C^{0,\alpha} \text{ such that } \|\Theta\|_{C^{0,\alpha}} \leq C \text{ and measure preserving}\}$   
*is relatively compact in  $L^\infty([0, T], L^2(\Omega))$ .*

*Proof.* This follows from Arzelà-Ascoli's theorem after we replace  $\omega_0$  with a continuous  $L^2$  approximation, that is,  $\hat{\omega} \in C^0(\Omega)$  such that  $\|\omega - \hat{\omega}\|_{L^2} \leq \varepsilon$ , then we know that we can find an  $\varepsilon$  cover in the continuous norm (as Holder function are compact in the space of continuous functions), from which we can construct a  $2\varepsilon$  cover in the  $L^\infty([0, T], L^2(\Omega))$  metric and thus is compact as  $\varepsilon$  is arbitrary.  $\square$

With this lemma is clear that

$$\mathcal{G}(\mathcal{C}_R) \subseteq B_{W^{2,\infty}([0,T])} \times A(\omega_0) \subset\subset \mathcal{C}_R.$$

proving that the fixed point map fixes the set  $\mathcal{C}_R$ . In the following part we will proof that it is continuous as well.

### 4.6.12 Continuity of the fixed point map $\mathcal{G}$

Now we prove the continuity of the map  $\mathcal{G}$ .

**Lemma 4.6.12.** *The map  $\mathcal{G} : \mathcal{C}_R \longrightarrow C^2(\mathbb{R}^{N+1}) \times L^\infty((0, T), L^\infty(\Omega))$  is continuous in the  $C^2(\mathbb{R}^{N+1}) \times L^\infty((0, T), L^2(\Omega))$  topology.*

*Proof.* Let  $(\alpha^n, \omega^n) \in \mathcal{C}_R$  converging to  $(\alpha, \omega)$  in the  $C^1([0, T], \mathbb{R}^N) \times L^\infty([0, T], L^\infty(\Omega) \cap L^q(\Omega))$  topology. Evidently

$$\sum_i^N \alpha_i^n(t) \eta_i(x) \longrightarrow_{n \rightarrow \infty} \sum_i^N \alpha_i(t) \eta_i(x), \text{ in } C^1(H^3(\mathcal{S}(0)))$$

In order to proof the convergence of

$$\delta^n(t) \longrightarrow \delta(t) \text{ in } C^1([0, T]),$$



we consider

$$\begin{aligned} |\delta^n - \delta^m| &\leq \sum_i \int_0^T |c_i(\alpha^n, \delta^n) \dot{\alpha}^n - c_i(\alpha^m, \delta^m) \dot{\alpha}^m| dt \\ &\leq C_R \int_0^1 |\delta^n - \delta^m| + |\dot{\alpha}^n - \dot{\alpha}^m| + |\alpha^n - \alpha^m| dt \quad (4.21) \end{aligned}$$

So applying Gronwall's lemma

$$|\delta^n - \delta^m| \leq C \exp(Ct) \|\alpha^n - \alpha^m\|_{C^1}.$$

Henceforth

$$\xi_i^n \rightarrow \xi_i, \text{ in } C^1([0, T], H^3(\mathcal{S}(0))),$$

in particular, because of the embedding  $H^3(\mathcal{S}(0)) \subseteq C^1(\mathcal{S}(0))$ , we have

$$\xi_i^n \rightarrow \xi_i, \text{ in } C^1([0, T] \times \mathcal{S}(0)),$$

so the flow satisfies

$$X^n \rightarrow X, \text{ in } C^1([0, T] \times \mathcal{S}(0)).$$

If  $T$  is small enough,

$$(X^n)^{-1} \rightarrow X^{-1} \text{ in } C^1([0, T] \times \mathcal{S}(0)) \cap C^1([0, T], H^3(\mathcal{S}(t)))$$

from this we obtain convergence of the Kirchhoff potentials

$$\Phi_i^n \rightarrow \Phi_i \text{ in } C^{2,\alpha}(\Omega),$$

which implies, together with the fact that the Biot-Savart operator is continuous, that

$$u^n \rightarrow u,$$

in the log-Lipschitz norm. With this, all the right hand side terms in (4.16) converge in  $L^\infty[0, T]$ , which implies that  $\hat{\alpha}^n$  converges in  $C^1([0, T])$

□

## 4.7 Proof of the main theorem

With the results of all the previous sections we have all the elements to perform the proof to the main theorem: Lets consider the fixed point map  $\mathcal{G}$ . From the previous sections, we know that there is a time  $T^*$  such that  $\mathcal{G}$  is well defined for  $T < T^*$ , and it is continuous and compact from  $\mathcal{C}_R$  to

$C^1([0, T], \mathbb{R}^N) \times L^\infty([0, T], L^2(\Omega))$ . In order to apply Schauder theorem it only remains to proof that

$$\mathcal{G}(\mathcal{C}_R) \subseteq \mathcal{C}_R.$$

Due to the fact that  $\Theta$  is measure preserving, we have that

$$\|\tilde{\omega}\|_{L^\infty(L^2)} \leq \|\omega_0\|_{L^2},$$

from the estimates for the solid displacement, we have

$$\|\alpha(t)\|_{C^1} \leq \|\alpha(0)\| + TC_R,$$

where  $C_R$  is independent of  $T$ . Taking  $T$  small enough we obtain

$$\|\alpha(t)\|_{C^1} \leq R.$$

So the map  $\mathcal{G}$  has a fixed point. By construction, this fixed point will satisfy the equations

$$\begin{aligned} (\mathcal{M} + \mathcal{A}(t))\ddot{\alpha} + \Gamma(t)\dot{\alpha} - \Lambda\alpha - \lambda_{N+1}\delta c &= f(\alpha, \dot{\alpha}, u), \\ u(t, x) &= \sum_i^N \dot{\alpha}_i(t) \nabla \Phi_i(t, x) + K[\omega](t, x) + \sum_i^m \mu_i h_i(t) \nabla^\perp \hat{\Phi}_i(t, x), \\ \Theta(t, x) &= x + \int_0^t u(t, \Theta(t, x)) dx, \\ \omega(t, x) &= \omega_0(\Theta(t, \theta)). \end{aligned}$$

By design,  $u$  satisfies

$$\begin{aligned} \operatorname{div} u &= 0 && \text{in } \mathcal{F}(t) \\ \operatorname{curl} u &= \omega && \text{in } \mathcal{F}(t) \\ \int_{\gamma_i} \tau \cdot u dl &= \int_{\gamma_i} \tau \cdot u_0 dl && i \in \{1, \dots, m\} \\ u \cdot n &= \partial_t X \cdot n && \text{in } \partial\mathcal{S}(t) \\ u \cdot n &= 0 && \text{in } \partial F(t) \setminus \partial\mathcal{S}(t) \end{aligned} \tag{4.22}$$

by removing the added mass term, we obtain existence of a solution for the system (4.11).

## 4.8 Conclusions

We manage to prove existence for the finite modes approximation of the problem for small time. The more obvious open problems now are to extend the proof to the general case, by obtaining uniform estimates on the finite modes solutions, independent on the number of modes. Another possible approach would be to work directly with a general solution, but our main difficulty in this case was to prove the invertibility of the added mass operator in this case.

Another open problem is to extend the local time result to a "global" in time, that is, until a collision happens (either the solid hits the boundary of the domain or the solid hits itself), or the flow of the solid becomes non-invertible. In particular would be necessary to get rid of the condition of the nonvanishing of the  $N + 1$  component of the flux through the boundary, namely

$$\int_{\partial S(t)} n \cdot \eta_{N+1} dx \neq 0.$$

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## RÉSUMÉ

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Dans cette thèse, on étudie des problèmes de contrôle et des problèmes de Cauchy dans la mécanique des milieux continus, en particulier sur la mécanique de fluides. On montre un résultat de contrôlabilité pour la equation de Navier-Stokes non isentropique, où la pression dépend de la température et de la densité, un résultat de stabilisation avec un contrôle en boucle fermée dans la densité. Finalement, on montre un résultat d'existence pour le problème de Cauchy pour un solide élastique plongé dans un fluide Eulerien.

## MOTS CLÉS

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Contrôlabilité, Stabilisation, Dynamique des fluides, Interaction fluide-structure.

## ABSTRACT

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In this thesis we study control related problems and Cauchy problems that appear in continuum mechanics, with an emphasis in fluids. We present a local null controllability result for the non-isentropic Navier-Stokes equations where the pressure depends on the temperature as well as the density, a local stabilization with state feedback law on the density for the isentropic case of Navier-Stokes, and finally, we present an existence result for the Cauchy problem of a linear elastic solid submerged on an Eulerian fluid in the case of a finite number of modes approximation.

## KEYWORDS

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Controllability, Stabilization, Fluids-dynamics, fluid-structure interaction.

