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Comportement en temps long de
solutions d'EDP non linéaires : stabilité
des ondes progressives, dispersion,
intégrabilité, amortissement.

*Long-time behavior of solutions to nonlinear PDEs:
orbital stability of traveling waves, dispersion,
integrability, damping.*

Thèse de doctorat de l'université Paris-Saclay

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**Thèse présentée et soutenue à Paris-Saclay,
le 8 juin 2021, par**

Louise GASSOT

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Chapitre 1

Introduction

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Dans cette thèse, on s'intéresse à la façon dont le comportement en temps long de solutions d'équations aux dérivées partielles est influencé par les propriétés de dispersion de l'équation d'une part, celles d'intégrabilité d'autre part. Pour cela, on considère deux familles d'équations : l'équation de Schrödinger sur le groupe de Heisenberg, et celle de Benjamin-Ono sur le tore.

Dans un premier temps, on introduit l'équation de Schrödinger non linéaire cubique focalisante sur le groupe de Heisenberg dans le cas radial

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad t \in \mathbb{R}, \quad (x, y, s) \in \mathbb{H}^1. \quad (\text{NLS-}\mathbb{H}^1)$$

La notation $\Delta_{\mathbb{H}^1}$ fait référence au sous-laplacien sur le groupe de Heisenberg, qui s'écrit pour des fonctions radiales (i.e. qui ne dépendent que de t , $|x + iy|$ et s)

$$\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_{xx} + \partial_{yy}) + (x^2 + y^2)\partial_{ss}.$$

Le choix de la géométrie fait de (NLS- \mathbb{H}^1) une équation sans dispersion, ce qui conduit à des difficultés pour résoudre le problème de Cauchy. Par conséquent, on utilise des méthodes ne faisant pas appel à l'utilisation d'inégalités de Strichartz ni à d'autres arguments de dispersion pour construire une famille de solutions ondes progressives dans l'espace homogène d'énergie $\dot{H}^1(\mathbb{H}^1)$ puis établir des propriétés de stabilité orbitale pour

ces ondes progressives. Enfin, on étudie le problème de Cauchy randomisé dans le cas focalisant ou défocalisant pour l'équation simplifiée de Schrödinger-Grushin

$$i\partial_t u - \Delta_G u = |u|^2 u, \quad (\text{NLS-G})$$

avec $\Delta_G = \partial_{xx} + x^2 \partial_{yy}$, obtenue en remarquant que les variables x et y jouent le même rôle pour (NLS-H¹). Les données initiales aléatoires sont construites à partir de données initiales satisfaisant une certaine condition de décroissance, et on montre alors le caractère presque-sûrement localement bien posé du problème de Cauchy dans $H^k(\mathbb{R}^2)$ pour tout $k > 1$.

Dans un second temps, on s'intéresse à deux équations dérivées de l'équation de Benjamin-Ono sur le tore

$$\partial_t u = \partial_x (H \partial_x u - u^2). \quad (\text{BO})$$

Le symbole H désigne la transformée de Hilbert

$$Hf(x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} -i \operatorname{sgn}(n) \widehat{f}(n) e^{inx}, \quad f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}, \quad \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$$

L'équation de Benjamin-Ono est complètement intégrable, dans le sens où elle admet une application de Birkhoff qui ramène l'évolution temporelle des solutions à celle d'un système infini découplé de coordonnées évoluant à vitesse constante. On se sert de cette transformation pour décrire les solutions d'équations s'écrivant de façon proche de l'équation de Benjamin-Ono.

Ainsi, on introduit la troisième équation de la hiérarchie intégrable de Benjamin-Ono sur le tore

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right). \quad (\text{3-BO})$$

Pour cette équation, on détermine le seuil pour le caractère bien posé du problème de Cauchy, puis on classe les solutions ondes progressives et on étudie leurs propriétés de stabilité orbitale.

Enfin, on considère une équation de Benjamin-Ono avec amortissement sur le tore associée à un paramètre $\alpha > 0$

$$\partial_t u + \alpha(\langle u | \cos \rangle \cos + \langle u | \sin \rangle \sin) = \partial_x (H \partial_x u - u^2). \quad (\text{BO-}\alpha)$$

On établit l'existence globale et l'unicité des solutions dans l'espace $L^2_{r,0}(\mathbb{T})$ des fonctions de carré intégrable sur le tore à valeurs réelles et de moyenne nulle. Puis, on décrit les valeurs d'adhérence faibles des trajectoires en temps infini et on établit la relative compacité de ces trajectoires. Finalement, on montre que les normes de Sobolev d'ordre supérieur restent bornées en temps infini.

Dans les deux parties suivantes, on introduit ces deux familles d'équations.

1.1 Dispersion pour NLS sur le groupe de Heisenberg

Le phénomène de dispersion se lit sur la partie linéaire de l'équation aux dérivées partielles. Elle traduit le fait que pour une solution sous forme de superposition d'ondes planes progressives, deux fréquences différentes se propagent à des vitesses différentes, étalant

ainsi la solution. Sur \mathbb{R}^d , lorsque l'équation aux dérivées partielles linéaire est associée à un multiplicateur de Fourier ω , elle s'écrit en variable de Fourier

$$\partial_t \widehat{u}(t, \xi) + i\omega(\xi)\widehat{u}(t, \xi) = 0,$$

et on observe que chaque fréquence ξ se déplace avec la pulsation $\omega(\xi)$. Lorsque ω est différentiable, on est conduit à la définition suivante : l'équation est dite *non dispersive* si la vitesse $\nabla\omega$ est localement constante.

Une équation non dispersive typique est l'équation de transport $\partial_t u + c\partial_x u = 0$, pour une constante réelle c . On obtient alors $\omega(\xi) = c\xi$, de plus, les solutions de cette équation sont données par $u(t, x) = u(0, x - ct)$ et gardent leur amplitude au cours du temps. En revanche, l'équation de Schrödinger non linéaire focalisante sur \mathbb{R}^d

$$i\partial_t u - \Delta u = |u|^{p-1}u \quad (\text{NLS})$$

est dispersive puisque $\omega(\xi) = -\|\xi\|^2$. On peut également mentionner d'autres équations dispersives classiques, comme l'équation de Benjamin-Ono sur le tore (BO) qui donne $\omega(n) = -|n|n$, $n \in \mathbb{Z}$, ou l'équation de Korteweg-de Vries sur la droite

$$\partial_t u = -\partial_{xxx} u + 6u\partial_x u \quad (\text{KdV})$$

qui donne $\omega(\xi) = -\xi^3$. De même, l'équation de demi-onde cubique sur \mathbb{R}^d

$$i\partial_t u + |D|u = |u|^2 u, \quad \mathcal{F}(|D|u) := \|\xi\|\mathcal{F}(u), \quad (\text{HW})$$

est dispersive si et seulement si $d \geq 2$, puisque la relation de dispersion s'écrit $\omega(\xi) = -\|\xi\|$. L'équation de demi-onde permet de factoriser celle des ondes, en effet, on a l'identité $-\partial_{tt} + \Delta = (i\partial_t - |D|)(i\partial_t + |D|)$. L'étude du cas totalement non dispersif en dimension 1 pour l'équation de demi-onde et de ses liens avec l'équation (NLS-III¹) fera l'objet d'une attention particulière dans la partie 1.1.2.

Une façon de visualiser la dispersion consiste en quantifier la vitesse d'étalement du paquet d'ondes en le traduisant par une décroissance des normes L^p de la solution : on obtient alors des inégalités de Strichartz. En ce qui concerne l'équation de Schrödinger sur \mathbb{R}^d , suite aux travaux de Strichartz [Str77], Ginibre, Velo [GV95], Keel, Tao [KT98], etc., on a

$$\|e^{-it\Delta} u\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^d))} \lesssim \|u\|_{L^2(\mathbb{R}^d)}$$

pour toute paire $(p, q) \in [2, +\infty]^2$ vérifiant les conditions d'admissibilité $(p, q) \neq (2, +\infty)$ et $\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}$.

1.1.1 Équation de Schrödinger, dispersion et géométrie

Géométrie euclidienne Pour l'équation de Schrödinger non linéaire sur \mathbb{R}^d , la dispersion est en compétition avec la partie non linéaire qui peut concentrer la solution. Ainsi, on voit apparaître trois principaux régimes : des solutions globales avec dispersion linéaire vers zéro en temps infini, des solutions non linéaires qui développent des singularités, ou des objets intermédiaires comme les ondes progressives.

On peut construire des ondes progressives grâce à un argument variationnel qui remonte au travail de Weinstein [Wei83] dans le cas de l'équation de Schrödinger focalisante masse-critique

$$i\partial_t u - \Delta u = |u|^{\frac{4}{d}} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

Il existe une onde progressive $u(t, x) = e^{it}Q(x)$, où le profil $Q \in H^1(\mathbb{R}^d)$ est une solution minimisante positive de $\Delta Q - Q + Q^{1+\frac{4}{d}} = 0$. Ce profil est unique aux symétries près de l'équation [GNN79; Kwo89], et linéairement stable sur une variété de codimension $2d + 4$ liée à ces symétries [Wei85]. Par ailleurs, il y a existence globale et diffusion linéaire (ou « scattering ») dans $L^2(\mathbb{R}^d)$ des solutions dans $H^1(\mathbb{R}^d)$ dès que la masse de la donnée initiale est strictement inférieure à celle de Q [Dod15]. En revanche, au-delà de cette masse, les solutions peuvent exploser en temps fini. Ainsi, il existe une unique solution explosive de plus petite masse aux symétries près de l'équation [Mer93], obtenue en appliquant au minimiseur Q l'invariance par transformation conforme des solutions $u \rightsquigarrow (x, t) \mapsto \frac{1}{t^{d/2}} \exp\left(\frac{|x|^2}{4it}\right) u\left(\frac{x}{t}, -\frac{1}{t}\right)$.

Dans le cas du groupe de Heisenberg, on verra que l'on peut aussi construire des solutions ondes progressives à partir de profils minimisants qui s'apparentent plutôt à ceux associés à l'équation de Schrödinger focalisante énergie-critique sur \mathbb{R}^d , $d \geq 3$

$$i\partial_t u - \Delta u = |u|^{\frac{4}{d-2}} u. \quad (1.1)$$

Il existe une solution stationnaire minimisante donnée par

$$u(t, x) = W(x) = \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}}.$$

Cette solution est unique aux symétries près, c'est-à-dire que l'ensemble des solutions minimisantes est donné par l'orbite de W

$$\{x \mapsto e^{i\theta} \lambda^{\frac{d-2}{2}} W(\lambda(x + x_0)) \mid (\theta, x_0, \lambda) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+^*\}.$$

Étant donnée l'équation vérifiée par W , la multiplication d'un élément quelconque de cette orbite par une constante $C \in \mathbb{R}$ donne l'ensemble des minimiseurs pour l'injection de Sobolev $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$, dont l'étude remonte à Aubin et Talenti [Aub76; Tal76].

Par ailleurs, dans l'étude du comportement qualitatif des solutions de (1.1), l'énergie $E(W) = \frac{1}{2} \|W\|_{\dot{H}^1(\mathbb{R}^d)}^2 - \frac{1}{p+1} \|W\|_{L^{p+1}(\mathbb{R}^d)}^{p+1}$, avec $p = \frac{d+2}{d-2}$, et la norme homogène $\|W\|_{\dot{H}^1(\mathbb{R}^d)}$ jouent un rôle important. Kenig et Merle [KM06] montrent dans le cas radial que si $d \in \{3, 4, 5\}$ et si la donnée initiale $u_0 \in \dot{H}^1(\mathbb{R}^d)$ satisfait $E(u_0) < E(W)$ et $\|u_0\|_{\dot{H}^1(\mathbb{R}^d)} < \|W\|_{\dot{H}^1(\mathbb{R}^d)}$, alors la solution est globale et diffuse linéairement dans $\dot{H}^1(\mathbb{R}^d)$. En revanche, si $E(u_0) < E(W)$ et $\|u_0\|_{\dot{H}^1(\mathbb{R}^d)} > \|W\|_{\dot{H}^1(\mathbb{R}^d)}$, alors il y a explosion en temps fini.

Géométrie Riemannienne En changeant de géométrie, notamment en se plaçant dans des domaines bornés, la dispersion peut devenir moins importante et modifier le comportement qualitatif des solutions. Par exemple, Bourgain [Bou93b; Bou93a; Bou99] traite le cas du tore \mathbb{T}^d , sur lequel les inégalités de Strichartz ne suffisent pas à elles seules à résoudre le problème de Cauchy dans l'espace d'énergie en dimension $d \geq 3$ (voir [BGT04], partie 1.2, pour plus de détails).

Sur une variété Riemannienne (M, g) , on peut toujours écrire l'équation de Schrödinger en remplaçant le laplacien usuel par l'opérateur de Laplace-Beltrami Δ_g sur M . On fait référence à [Gér06] pour une bibliographie des résultats liés au problème de Cauchy dans ce cadre, et à [Bou12] pour des questions relatives à l'explosion des solutions. Sur une variété Riemannienne compacte sans bord quelconque, Burq, Gérard et Tzvetkov [BGT04] montrent l'existence d'inégalités de Strichartz avec perte de $\frac{1}{p}$ dérivées sur tout intervalle fini I

$$\|e^{-it\Delta_g} u_0\|_{L^p(I, L^q(M))} \lesssim_I \|u_0\|_{H^{\frac{1}{p}}(M)}, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad p \geq 2, \quad q < +\infty,$$

ce qui est la moitié de la perte de dérivées donnée par les injections de Sobolev $H^{2/p}(M) \hookrightarrow L^q(M)$. Les auteurs établissent également des inégalités de Strichartz améliorées sur $M = \mathbb{S}^d$, $d \geq 2$

$$\|e^{-it\Delta_g}u_0\|_{L^4(I \times \mathbb{S}^d)} \lesssim_I \|u_0\|_{H^{\frac{k}{2}}(\mathbb{S}^d)},$$

optimales dans le sens où elles sont vraies pour tout u_0 si et seulement si $k > \frac{d}{2} - 1$ pour $d \geq 3$ et $k > \frac{1}{4}$ pour $d = 2$. Sur une variété Riemannienne compacte avec bord ou une variété compacte sans bord associée à une métrique lipschitzienne, on dispose d'inégalités de Strichartz avec une perte de $\frac{4}{3p}$ dérivées [BSS08].

Par ailleurs, suivant la remarque 2.12 de [BGT05], et l'introduction de [GG10], l'existence d'un flot \mathcal{C}^3 au voisinage de l'origine dans l'espace de Sobolev $H^k(M)$ pour l'équation de Schrödinger cubique sur M

$$i\partial_t u - \Delta_g u = |u|^2 u, \quad (t, x) \in \mathbb{R} \times M$$

implique une inégalité de type Strichartz

$$\|e^{it\Delta_g}u_0\|_{L^4([0,1] \times M)} \leq C \|u_0\|_{H^{\frac{k}{2}}(M)}. \quad (1.2)$$

Ceci donne une indication de perte de dispersion dans \mathbb{S}^2 , puisque dans le cas euclidien, lorsque $d \leq 4$, l'inégalité (1.2) ci-dessus est vraie dès que $k = \max(0, \frac{d}{2} - 1)$. On peut noter que le problème de Cauchy n'est en fait pas uniformément bien posé dans $H^k(\mathbb{S}^2)$ lorsque $k < \frac{1}{4}$ [BGT02].

Géométrie sous-Riemannienne Pour aller plus loin, on peut se demander ce qu'il advient des inégalités de Strichartz lorsque l'on se place sur une variété sous-Riemannienne munie d'un sous-laplacien.

Bahouri, Gérard et Xu [BGX00] puis Gérard et Grellier [GG10] observent que la dispersion disparaît complètement pour le groupe de Heisenberg. En effet, l'existence d'un flot \mathcal{C}^3 sur $H^k(\mathbb{H}^1)$ pour l'équation de Schrödinger cubique implique comme pour les variétés Riemanniennes l'inégalité de Strichartz (1.2) en remplaçant M par \mathbb{H}^1 . Sur le groupe de Heisenberg, on a nécessairement $k \geq 2$ dans cette inégalité. On reprend les arguments de l'introduction de [GG10] dans le paragraphe suivant.

Comme les fonctions de Hermite $(h_n)_{n \geq 0}$ diagonalisent l'oscillateur harmonique $-\partial_{xx} + x^2$, l'espace $L^2(\mathbb{H}^1)$ s'écrit comme une somme directe orthogonale

$$L^2(\mathbb{H}^1) = \bigoplus_{(n, \pm) \in \mathbb{N} \times \{\pm\}} L^2(\mathbb{H}^1) \cap V_n^\pm, \quad (1.3)$$

où la projection orthogonale Π_n^\pm sur V_n^\pm est donnée après application de la transformation de Fourier en la dernière variable s par

$$\mathcal{F}(\Pi_n^\pm(u))(x, y, \sigma) = \sum_{n_1+n_2=n} f_{n_1, n_2}(\sigma) h_{n_1}(\sqrt{2|\sigma|x}) h_{n_2}(\sqrt{2|\sigma|y}) \mathbf{1}_{\pm\sigma \geq 0} \quad (1.4)$$

avec

$$f_{n_1, n_2}(\sigma) = |\sigma| \langle u | h_{n_1}(\sqrt{2|\sigma|x}) h_{n_2}(\sqrt{2|\sigma|y}) \rangle_{L^2_{x,y}}.$$

Dans cette décomposition, l'équation (NLS- \mathbb{H}^1) devient un système couplé d'équations de transport

$$i(\partial_t \mp (2n+1)\partial_s)\Pi_n^\pm(u) = \Pi_n^\pm(|u|^2 u),$$

avec comme données initiales $\Pi_n^\pm(u_0)$. En particulier, pour une donnée initiale $u_0 \in V_0^+$, le flot linéaire associé à (NLS- \mathbb{H}^1) est celui d'une équation de transport, donc on a

$$\|e^{it\Delta_{\mathbb{H}^1}} u_0\|_{L^4([0,1] \times \mathbb{H}^1)} = \|u_0\|_{L^4(\mathbb{H}^1)}.$$

Mais si u_0 est concentrée autour des modes de Fourier N^2 de plus en plus grands, par exemple

$$u_0(x, y, s) = N^2 F(Nx, Ny, N^2 s)$$

à partir d'un certain profil $F \in L^2(\mathbb{H}^1) \cap V_0^+$, on a $\|u_0\|_{\dot{H}^{\frac{k}{2}}(\mathbb{H}^1)} = N^{\frac{k}{2}} \|F\|_{\dot{H}^{\frac{k}{2}}(\mathbb{H}^1)}$ alors que $\|u_0\|_{L^4(\mathbb{H}^1)} = N \|F\|_{L^4(\mathbb{H}^1)}$, ce qui implique pour $N \rightarrow +\infty$ que nécessairement $k \geq 2$ dans (1.2).

L'absence de dispersion cause des difficultés pour résoudre le problème de Cauchy, notamment dans l'espace d'énergie correspondant à $k = 1$. Ainsi, pour l'équation (NLS- \mathbb{H}^1), l'existence globale de solution lisses et l'unicité des solutions faibles sont des problèmes ouverts. Dans le cas défocalisant, il serait possible de construire des solutions faibles globales par un argument de compacité, mais il faudrait des arguments supplémentaires pour garantir que ces solutions ont un comportement qualitatif pertinent.

Plus généralement, Del Hierro analyse les propriétés dispersives pour les groupes de type H dans [Del05], et montre des estimées de décroissance optimales pour l'équation de Schrödinger qui dépendent de la dimension du centre du groupe. Bahouri, Fermanian et Gallagher [BFG16] prouvent des inégalités de dispersion optimales pour les groupes de Lie stratifiés d'ordre 2 sous une hypothèse de rang maximal. Dans leur proposition 1.7, une autre classe de groupes ne satisfaisant pas l'hypothèse de rang maximal est exhibée, et pour ces groupes la dispersion disparaît complètement. Le groupe de Heisenberg en est alors un exemple.

Casarino et Peloso [CP15] observent que pour la sphère CR, liée au groupe de Heisenberg par la transformation de Cayley, le sous-laplacien est lui aussi peu dispersif, et montrent des estimations de Strichartz pour des espaces de type Sobolev anisotropes associés au sous-laplacien. Dans [BBG21], Bahouri, Barilari et Gallagher démontrent des inégalités de type Strichartz espace-temps anisotropes. Les auteurs adaptent la méthode de restriction de Fourier [Mül90] à la transformée de Fourier sur le groupe Heisenberg, en utilisant l'analyse de Fourier sur le groupe de Heisenberg [BCD18; BCD19]. Le choix des espaces est basé sur l'observation suivante : l'absence de dispersion sur le groupe de Heisenberg se manifeste par le fait que l'équation de Schrödinger linéaire se comporte comme une équation de transport dans une direction privilégiée, appelée direction verticale, mais ce n'est pas le cas dans les autres directions. Des estimations de Strichartz locales font également l'objet de [BG20a]. Une question qui se pose alors est de déterminer encore plus précisément pour quelles régularités on peut prouver des inégalités de type Strichartz sur le groupe de Heisenberg. Puis, il serait intéressant de comprendre si cela permettrait de résoudre le problème de Cauchy sur $H^k(\mathbb{H}^1)$ pour certains exposants k inférieurs ou égaux à 2, ou bien dans d'autres espaces adaptés.

1.1.2 Équation de demi-onde et équation de Szegő sur la droite

Un modèle pour l'équation de Schrödinger sur le groupe de Heisenberg On a vu plus haut que l'équation (NLS- \mathbb{H}^1) se décompose comme un système couplé d'équations de transport. Une telle décomposition existe également pour l'équation de Grushin

$$i\partial_t u - \Delta_G u = |u|^2 u, \tag{NLS-G}$$

pour laquelle on remplace le laplacien par l'opérateur de Grushin $\Delta_G = \partial_{xx} + x^2\partial_{yy}$ [GG10], comme on le verra dans la partie 1.1.4.

Une structure similaire plus simple apparaît pour l'équation de demi-onde sur la droite (HW)

$$i\partial_t u + |D|u = |u|^2 u,$$

qui est également non dispersive en dimension 1. En effet, notons $\Pi : L^2(\mathbb{R}) \rightarrow L^2_+(\mathbb{R})$ le projecteur de Szegő sur l'espace $L^2_+(\mathbb{R})$ des fonctions $u \in L^2(\mathbb{R})$ n'ayant que des fréquences de Fourier positives ou nulles. En écrivant $u_+ = \Pi(u)$ et $u_- = u - \Pi(u)$, on remarque que l'équation de demi-onde est équivalente au système couplé d'équations de transport

$$\begin{cases} i(\partial_t + \partial_x)u_+ = \Pi(|u|^2 u) \\ i(\partial_t - \partial_x)u_- = (\text{Id} - \Pi)(|u|^2 u) \\ u_+(0) = \Pi(u_0), \quad u_-(0) = (\text{Id} - \Pi)(u_0), \quad u = u_+ + u_- \end{cases}$$

Ceci conduit à s'intéresser à l'équation de Szegő sur la droite

$$i\partial_t u = \Pi(|u|^2 u), \tag{Sz}$$

qui constitue un système résonnant pour l'équation de demi-onde.

Un dernier exemple d'équation non dispersive est donné sur \mathbb{R}^2 par Gérard et Grellier dans [GG08], et s'écrit

$$i\partial_t u - (|D_{x_1}| + |D_{x_2}|)u = |u|^2 u.$$

En effet, un découpage entre fréquences de Fourier positives et négatives en x_1 et en x_2 ramène cette équation à un système couplé de quatre équations de transport.

Ondes progressives pour l'équation de demi-onde sur la droite L'équation de demi-onde et celle de Szegő ont d'abord été étudiées sur le tore par Gérard et Grellier [GG08; GG10; GG12a] comme modèle jouet d'équation totalement non dispersive. Sur la droite, on va voir que l'étude des ondes progressives pour ces équations constitue un modèle pour notre étude de (NLS- \mathbb{H}^1).

Le problème de Cauchy pour l'équation de demi-onde sur la droite (HW) est localement bien posé dans l'espace d'énergie $H^{1/2}(\mathbb{R})$ [GG12a; KLR13]. De plus, l'équation (HW) admet une solution onde progressive $u(t, x) = e^{it}Q_0(x)$, où Q_0 est le minimiseur pour l'inégalité de Gagliardo-Nirenberg

$$\|f\|_{L^4(\mathbb{R})}^4 \leq C \| |D|^{\frac{1}{2}} f \|_{L^2(\mathbb{R})}^2 \|f\|_{L^2(\mathbb{R})}^2.$$

D'après Frank, Lenzmann [FL13], le profil $Q_0 \in H^{1/2}(\mathbb{R})$ est l'unique solution minimisante positive de l'équation

$$|D|Q_0 + Q_0 - Q_0^3 = 0.$$

Si $u(0) \in H^{1/2}(\mathbb{R})$ et $\|u(0)\|_{L^2(\mathbb{R})} < \|Q_0\|_{L^2(\mathbb{R})}$, alors la solution u associée est globale. En revanche, il existe des solutions explosives en temps fini de même masse que Q_0 et d'énergie finie [KLR13].

On peut aussi construire une famille d'ondes progressives minimisantes pour (HW) de vitesse $\beta \in (-1, 1)$ (voir [KLR13])

$$u_\beta(t, x) = Q_\beta \left(\frac{x + \beta t}{1 - \beta} \right) e^{-it},$$

pour lesquelles la norme critique $\|u_\beta\|_{L^2(\mathbb{R})}$ tend vers 0 lorsque β tend vers 1. Concernant l'équation de Schrödinger L^2 critique sur \mathbb{R} , la symétrie par transformation de Galilée donne également une famille d'ondes progressives $u_\beta(t, x) = Q_\beta(x - \beta t)e^{i\gamma_\beta(t)}$ avec $Q_\beta(x) = Q(x)e^{i\beta x}$, mais une famille dont la masse tendrait vers 0 ne peut pas exister puisqu'il y a scattering pour les données initiales dont la masse est en-dessous de celle de Q [Dod15].

Pour l'équation (HW), lorsque $\beta \in (-1, 1)$, le profil Q_β est solution de l'équation

$$\frac{|D| - \beta D}{1 - \beta} Q_\beta + Q_\beta = |Q_\beta|^2 Q_\beta, \quad D = -i\partial_x,$$

de plus, on a $\lim_{\beta \rightarrow 0} \|Q_\beta - Q_0\|_{H^{1/2}(\mathbb{R})} = 0$ et $\|Q_\beta\|_{L^2(\mathbb{R})} < \|Q_0\|_{L^2(\mathbb{R})}$. L'existence des profils Q_β est une conséquence d'un argument variationnel standard, néanmoins, l'unicité est plus délicate à montrer. On peut procéder de la façon suivante lorsque la vitesse β est suffisamment proche de 1.

En considérant la limite photonique $\beta \rightarrow 1$, la suite $(Q_\beta)_\beta$ tend dans $H^{1/2}(\mathbb{R})$ vers une solution minimisante Q pour l'équation

$$DQ + Q = \Pi(|Q|^2 Q), \quad D = -i\partial_x.$$

On obtient en posant $u(t, x) = Q(x - t)e^{-it}$ une onde progressive pour l'équation de Szegő cubique

$$i\partial_t u = \Pi(|u|^2 u). \quad (\text{Sz})$$

Pocovnicu montre dans [Poc11b] que les solutions onde progressive pour (Sz), minimisantes ou non, sont uniques aux symétries près. En d'autres termes, au changement de fonction près $Q \rightsquigarrow e^{i\theta} Q(x + x_0)$ avec $\theta \in \mathbb{T}$ et $x_0 \in \mathbb{R}$, le profil Q est égal à

$$Q(x) = \frac{2}{2x + i}.$$

Un argument essentiel dans la preuve est l'existence d'une paire de Lax qui s'écrit de la façon suivante (voir [Poc11b], Proposition 1.4). Pour un potentiel $u \in H_+^{1/2}(\mathbb{R}) = \Pi(H^{1/2}(\mathbb{R}))$, notons

$$H_u : h \in L_+^2(\mathbb{R}) \mapsto \Pi(u\bar{h}) \in L_+^2(\mathbb{R})$$

l'opérateur de Hankel de symbole u , et pour $b \in L^\infty(\mathbb{R})$, notons

$$T_b : h \in L_+^2(\mathbb{R}) \mapsto \Pi(bh) \in L_+^2(\mathbb{R}) \quad (1.5)$$

l'opérateur de Toeplitz de symbole b . Alors pour toute solution de l'équation (Sz), on a

$$\frac{d}{dt} H_u = [B_u, H_u], \quad B_u = -iT_{|u|^2} + \frac{i}{2} H_u^2. \quad (1.6)$$

Stabilité orbitale des ondes progressives Dans [Poc12], Pocovnicu montre que le linéarisé autour du profil Q pour l'équation de Szegő est coercif. Ceci lui permet de déduire la stabilité orbitale en grand temps de l'onde progressive associée lorsque l'équation de Szegő est perturbée par un petit potentiel de Toeplitz $T_b(u) = \Pi(bu)$, $b \in H^1(\mathbb{R})$:

$$i\partial_t u = \Pi(|u|^2 u) + \varepsilon T_b(u), \quad 0 < \varepsilon \ll 1.$$

À partir de la coercivité du linéarisé, on peut également obtenir la stabilité orbitale pour l'onde progressive de l'équation de Szegő (voir [Poc11b], Théorème 1.3, pour une version qualitative).

Théorème 1.1.1 (Stabilité orbitale de Q pour l'équation de Szegő [GR16]). *Il existe $\varepsilon_0 > 0$ et $C > 0$ tels que pour toute solution u de l'équation de Szegő (Sz) associée à la donnée initiale $u_0 \in H_+^{1/2}(\mathbb{R})$, si*

$$\|u_0 - Q\|_{H^{1/2}(\mathbb{R})} \leq \varepsilon_0,$$

alors

$$\sup_{t \in \mathbb{R}} \inf_{(\gamma, x_0) \in \mathbb{T} \times \mathbb{R}} \|e^{-i\gamma} u(t, \cdot - x_0) - Q\|_{H^{1/2}(\mathbb{R})}^2 \leq C \|u_0 - Q\|_{H^{1/2}(\mathbb{R})}.$$

Enfin, Gérard, Lenzmann, Pocovnicu et Raphaël [Gér+18] déduisent de la coercivité du linéarisé autour de l'onde progressive associée à Q pour l'équation de Szegő que linéarisé autour de l'onde progressive associée à Q_β pour l'équation de demi-onde est inversible lorsque β est suffisamment proche de 1. Leurs estimées impliquent la stabilité orbitale de ces profils.

Théorème 1.1.2 (Stabilité orbitale de Q_β pour l'équation de demi-onde [GR16]). *Il existe $\beta_* \in (0, 1)$ tel que pour tout $\beta \in (\beta_*, 1)$, il existe $\varepsilon_0(\beta) > 0$ et $C(\beta) > 0$ tels que pour toute solution u de l'équation de demi-onde (HW) associée à la donnée initiale $u_0 \in H^{1/2}(\mathbb{R})$, si*

$$\|u_0 - Q_\beta\left(\frac{\cdot}{1-\beta}\right)\|_{H^{1/2}(\mathbb{R})} \leq \varepsilon_0(\beta),$$

alors

$$\sup_{t \in \mathbb{R}} \inf_{(\gamma, y) \in \mathbb{T} \times \mathbb{R}} \|e^{-i\gamma} u(t, \cdot - y) - Q_\beta\left(\frac{\cdot}{1-\beta}\right)\|_{H^{1/2}(\mathbb{R})}^2 \leq C \|u_0 - Q_\beta\left(\frac{\cdot}{1-\beta}\right)\|_{H^{1/2}(\mathbb{R})}.$$

Ces estimées sur l'inversibilité des linéarisés impliquent également l'unicité des Q_β aux symétries près pour β proche de 1, et on peut alors définir une courbe lisse $\beta \mapsto Q_\beta$ au voisinage de 1.

Ondes progressives pour l'équation de demi-onde en dimension supérieure En ce qui concerne les dimensions supérieures $d \geq 2$, les ondes progressives pour l'équation de demi-onde (dispersive) sur \mathbb{R}^d

$$i\partial_t u + \sqrt{-\Delta} u = |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d$$

sont également orbitalement stables dans le cas radial lorsque la non linéarité est L^2 sous-critique $1 < p < 1 + \frac{2}{d}$, mais orbitalement instables avec explosion en temps fini ou infini dans le régime L^2 sur-critique $1 + \frac{2}{d} < p < 1 + \frac{2}{d-1}$ [BGV18]. Enfin, dans les cas énergie-critique (resp. énergie sous-critique), Bellazzini, Georgiev, Lenzmann et Visiglia [Bel+19] montrent qu'il ne peut pas y avoir scattering dans l'espace d'énergie $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ (resp. $H^{1/2}(\mathbb{R}^d)$), même pour des petites données initiales, puisque l'on peut construire des ondes progressives non radiales d'énergie arbitrairement petite.

1.1.3 Résultats sur une famille d'ondes progressives pour l'équation de Schrödinger sur le groupe de Heisenberg

Dans cette partie, on reprend les résultats des articles [Gas20b; Gas21a].

Ondes progressives Sur le groupe de Heisenberg, il existe une famille d'ondes progressives de vitesse $\beta \in (-1, 1)$ sous la forme

$$u_\beta(t, x, y, s) = Q_\beta\left(\frac{x}{\sqrt{1-\beta}}, \frac{y}{\sqrt{1-\beta}}, \frac{s+\beta t}{1-\beta}\right). \quad (1.7)$$

De façon équivalente, en utilisant l'invariance par changement d'échelles qui pour $\alpha > 0$ s'écrit $u \rightsquigarrow \alpha u(\alpha^2 t, \alpha x, \alpha y, \alpha^2 s)$, on peut aussi définir Q_β par la formule

$$u_\beta(t, x, y, s) = \sqrt{1-\beta} Q_\beta(x, y, s + \beta t).$$

Le profil Q_β est solution de l'équation hypo-elliptique

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} Q_\beta = |Q_\beta|^2 Q_\beta. \quad (1.8)$$

On construit des solutions radiales pour cette dernière équation comme minimiseurs sur les fonctions radiales non nulles de $\dot{H}^1(\mathbb{H}^1)$ de l'inégalité de Gagliardo-Nirenberg

$$\|f\|_{L^4(\mathbb{H}^1)}^4 \leq C(-(\Delta_{\mathbb{H}^1} + \beta D_s)f, f)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}$$

qui provient de l'injection de Folland-Stein $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$ [FS74]. L'existence d'un minimiseur est garantie par un argument de concentration-compacité [CL82; Gér98] dont adaptation au groupe de Heisenberg figure dans les travaux de Benameur [Ben08].

Aux symétries près de l'équation $Q_\beta \rightsquigarrow e^{i\theta} \alpha Q_\beta(\alpha x, \alpha y, \alpha^2(s-s_0))$ pour $(s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{T}$, le même argument que pour l'équation de demi-onde permet de montrer que, lorsque β tend vers 1, alors Q_β converge dans $\dot{H}^1(\mathbb{H}^1)$ vers une solution minimisante Q pour l'équation

$$D_s Q = \Pi_0^+(|Q|^2 Q), \quad D_s = -i\partial_s. \quad (1.9)$$

On rappelle que l'opérateur Π_0^+ est la projection orthogonale de $L^2(\mathbb{H}^1)$ sur un sous-espace noté $L^2(\mathbb{H}^1) \cap V_0^+$ (voir (1.3)). En utilisant un lien entre l'espace $L^2(\mathbb{H}^1) \cap V_0^+$ et l'espace de Bergman $L^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ sur le demi-plan de Poincaré \mathbb{C}_+ donné par [Bék+12], on peut interpréter la projection Π_0^+ comme un projecteur de Bergman. Ce projecteur est en fait une simplification dans le cas radial du projecteur de Cauchy-Szegő sur le groupe de Heisenberg [SM93].

Remarquons qu'à cause de la convergence des profils Q_β vers Q , les ondes progressives correspondantes u_β ont une énergie qui tend vers 0 lorsque β tend vers 1

$$E(u_\beta(t)) = \frac{1}{2} \|u_\beta(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \frac{1}{4} \|u_\beta(t)\|_{L^4(\mathbb{H}^1)}^4 \sim (1-\beta) \frac{\pi^2}{2} \rightarrow 0,$$

ce qui implique l'existence de solutions qui ne diffusent pas en temps infini et d'énergie arbitrairement petite.

Unicité des ondes progressives Un calcul explicite implique que les solutions minimisantes Q de (1.9) sont uniques aux symétries près, égales à

$$Q(x, y, s) = \frac{\sqrt{2}i}{s + i(x^2 + y^2) + i}.$$

On montre que le linéarisé autour de Q est coercif hors des symétries de l'équation (voir le théorème 1.1.7 ci-dessous pour plus de détails). Par conséquent, en s'inspirant de la

stratégie employée pour l'équation de demi-onde, on déduit l'inversibilité des linéarisés autour des profils Q_β pour l'équation de Schrödinger lorsque β est proche de 1. Grâce à des propriétés additionnelles de régularité et de décroissance des profils Q_β , cela conduit à l'unicité des Q_β aux symétries près au voisinage de 1.

Théorème 1.1.3 (Unicité des ondes progressives au voisinage de 1 [Gas20b]). *Il existe $\beta_* \in (0, 1)$ tel que pour tout $\beta \in (\beta_*, 1)$ il existe un unique minimiseur Q_β aux symétries près pour l'équation (1.8)*

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta = |Q_\beta|^2 Q_\beta.$$

Pour $\beta \in (\beta_*, 1)$, on peut choisir Q_β de sorte que la suite $(Q_\beta)_\beta$ converge lorsque β tend vers 1 vers le profil Q , et de sorte que l'application $\beta \in (\beta_*, 1) \mapsto Q_\beta \in \dot{H}^1(\mathbb{H}^1)$ soit lisse. Dans ce cas, pour tout $k \in [1, +\infty)$, le profil Q_β appartient à l'espace de Sobolev homogène $\dot{H}^k(\mathbb{H}^1)$, et lorsque β tend vers 1, pour tout $\gamma \in (0, \frac{1}{4})$, on a

$$\|Q_\beta - Q\|_{\dot{H}^k(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^\gamma).$$

Stabilité orbitale des ondes progressives La coercivité du linéarisé autour de Q permet également de démontrer un résultat de stabilité orbitale de l'onde progressive associée à Q (donnée par $u(t, x, y, s) = Q(x, y, s + t)$) pour le système limite

$$i\partial_t u = \Pi_0^+(|u|^2 u). \quad (1.10)$$

En revenant à l'équation (NLS- \mathbb{H}^1), on en déduit un résultat de stabilité orbitale pour l'onde progressive associée à Q_β pour l'équation de Schrödinger lorsque β est proche de 1.

On définit la distance d'une fonction à l'orbite des ondes progressives concernées de la façon suivante.

Définition 1.1.4. Pour $u \in \dot{H}^1(\mathbb{H}^1)$ et $X = (s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$, on note $T_X u$ l'élément de $\dot{H}^1(\mathbb{H}^1)$ défini par

$$T_X u(x, y, s) := e^{i\theta} \alpha u(\alpha x, \alpha y, \alpha^2(s - s_0)), \quad (x, y, s) \in \mathbb{H}^1.$$

On note $\mathcal{M} = \{T_X Q \mid X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\}$ l'orbite de Q , et on définit la distance de $u \in \dot{H}^1(\mathbb{H}^1)$ à \mathcal{M} par

$$d(u, \mathcal{M}) = \inf_{X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|u - T_X Q\|_{\dot{H}^1(\mathbb{H}^1)}.$$

De même, la distance de u à l'orbite $\sqrt{1 - \beta} \mathcal{Q}_\beta$ de $\sqrt{1 - \beta} Q_\beta$, où $\mathcal{Q}_\beta = \{T_X Q_\beta \mid X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\}$, est donnée par

$$d(u, \sqrt{1 - \beta} \mathcal{Q}_\beta) = \inf_{X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|u - \sqrt{1 - \beta} T_X Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}.$$

On énonce maintenant les résultats de stabilité orbitale de Q pour le système limite et de Q_β pour l'équation de Schrödinger lorsque β est proche de 1.

Théorème 1.1.5 (Stabilité orbitale de Q pour le système limite [Gas21a]). *Il existe $c_0 > 0$ et $r_0 > 0$ tels que la propriété suivante soit vraie. Soit $r \leq r_0$ et $u_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ tels que*

$$\|u_0 - Q\|_{\dot{H}^1(\mathbb{H}^1)} < r^2. \quad (1.11)$$

Alors il existe une solution au sens faible de $u \in \mathcal{C}(\mathbb{R}, \dot{H}_w^1(\mathbb{H}^1))$ (l'indice w indique que $\dot{H}^1(\mathbb{H}^1)$ est équipé de la topologie faible) de l'équation (1.10) telle que $u(0) = u_0$ et

$$\sup_{t \in \mathbb{R}} d(u(t), \mathcal{M}) \leq c_0 r.$$

Théorème 1.1.6 (Stabilité orbitale de Q_β pour l'équation de Schrödinger [Gas21a]). *Il existe $c_0 > 0$ et $r_0 > 0$ tels que la propriété suivante soit vraie. Soit $r \in (0, r_0)$, alors il existe $\beta_*(r) \in (0, 1)$ tel que si $\beta \in (\beta_*(r), 1)$, et si $u_0 \in \dot{H}^1(\mathbb{H}^1)$ est une donnée initiale radiale satisfaisant*

$$(i) \text{ soit } u_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \text{ et } \|u_0 - \sqrt{1 - \beta} Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1 - \beta} r^2$$

$$(ii) \text{ soit } \|u_0 - \sqrt{1 - \beta} Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < (1 - \beta)r,$$

alors il existe une solution faible radiale $u \in \mathcal{C}(\mathbb{R}, \dot{H}_w^1(\mathbb{H}^1))$ (équipé de la topologie faible) de l'équation (NLS- \mathbb{H}^1) telle que $u(0) = u_0$ et

$$\sup_{t \in \mathbb{R}} d\left(u(t), \sqrt{1 - \beta} Q_\beta\right) \leq c_0 \sqrt{1 - \beta} r.$$

Ces deux théorèmes impliqueraient la stabilité orbitale des minimiseurs Q et Q_β dans le cas radial sous réserve d'un résultat d'unicité des solutions pour les équations (1.10) et (NLS- \mathbb{H}^1) dans $\dot{H}^1(\mathbb{H}^1)$.

L'hypothèse (ii) imposée sur une donnée initiale dans le cas général est plus forte que celle (i) dans le cas d'une donnée initiale dans V_0^+ . En effet, en supposant (ii), à r fixé, il faut dans la preuve choisir β_* suffisamment proche de 1 pour forcer $\Pi_0^+(u_0)$ à vérifier l'inégalité (i) à un facteur constant près. Par ailleurs, comme Q_β converge vers Q lorsque β tend vers 1 en $o(\sqrt{1 - \beta})$, l'hypothèse (i) est comparable à (1.11) au changement de fonction $u \rightsquigarrow \frac{1}{\sqrt{1 - \beta}} u(x, y, s - \beta t)$ près.

Les preuves de stabilité orbitale reposent sur des résultats quantitatifs d'inversibilité du linéarisé autour de Q pour l'équation (1.9), qui nous permettent dans les deux cas d'estimer la distance de u à l'orbite \mathcal{M} de Q en fonction de la différence de moment (défini par $\mathcal{P}(u) := \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2$) et d'énergie (définie par $\mathcal{E}(u) := \|u\|_{L^4(\mathbb{H}^1)}^4$). Plus précisément, en posant

$$\delta(u) := |\mathcal{P}(u) - \mathcal{P}(Q)| + |\mathcal{E}(u) - \mathcal{E}(Q)|,$$

il existe $\delta_0 > 0$ et $C > 0$ tels que pour tout $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, si $\delta(u) \leq \delta_0$, alors

$$d(u, \mathcal{M})^2 \leq C\delta(u).$$

On construit les solutions globales faibles comme limite de solutions lisses d'équations proches des équations (1.10) et (NLS- \mathbb{H}^1) obtenues en restreignant les fréquences, pour lesquelles le problème de Cauchy est bien posé. On contrôle la distance de ces solutions lisses à l'orbite de Q , puis on établit une borne sur les paramètres de modulation pour passer à la limite dans les estimations par un argument de compacité. Contrairement à l'équation de demi-onde, il n'est ici pas possible d'estimer directement la distance des solutions à l'orbite des profils Q_β uniquement au moyen de la différence de moments et d'énergie avec Q_β , ce qui nous force à revenir sur les propriétés de convergence des équations vérifiées par Q_β vers celle vérifiée par Q .

Coercivité du linéarisé autour du minimiseur pour le système limite On détaille maintenant la stratégie permettant d'établir la coercivité du linéarisé autour de Q pour l'équation limite (1.9) sur une variété de codimension finie. Le linéarisé s'écrit

$$\mathcal{L}h = D_s h - 2\Pi_0^+(|Q|^2 h) - \Pi_0^+(Q^2 \bar{h}), \quad h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+.$$

Théorème 1.1.7 (Coercivité du linéarisé [Gas20b]). *Il existe $c > 0$ telle que pour tout $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ orthogonal à $Q, iQ, \partial_s Q$ et $i\partial_s Q$ pour le produit scalaire du Hilbert $\dot{H}^1(\mathbb{H}^1)$,*

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq c \|h\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

Le résultat de coercivité implique que le linéarisé est non dégénéré, puisque son noyau n'est composé que des trois directions dues aux symétries de l'équation : $\text{Ker}(\mathcal{L}) = \text{Vect}_{\mathbb{R}}(\partial_s Q, iQ, Q + 2i\partial_s Q)$.

Contrairement au cas de l'équation de demi-onde, on ne dispose pas d'information sur l'existence ou non d'une paire de Lax pour (NLS- \mathbb{H}^1). Afin d'étudier le linéarisé, on se sert d'une invariance conforme entre le groupe de Heisenberg \mathbb{H}^1 et la sphère CR $\mathbb{S}^3 \subset \mathbb{C}^2$, appelée *transformation de Cayley*. Cette transformation relie l'étude du linéarisé \mathcal{L} avec celle du sous-Laplacien sur \mathbb{S}^3 , pour lequel le spectre et les fonctions propres ont été complètement déterminés par Stanton dans [Sta89]. Pour obtenir la coercivité de \mathcal{L} , on élimine les directions négatives potentielles au moyen des conditions d'orthogonalité.

En dimension d , la transformation de Cayley donne une équivalence entre le groupe de Heisenberg \mathbb{H}^d et la sphère CR \mathbb{S}^{2d+1} dans \mathbb{C}^d , de la même façon que la projection stéréographique fait le lien entre \mathbb{R}^d et la sphère euclidienne \mathbb{S}^d dans \mathbb{R}^{d+1} . Ces deux transformations ont déjà été utilisées pour étudier les minimiseurs pour des injections de Folland-Stein sur \mathbb{H}^d et de Sobolev sur \mathbb{R}^d .

Ainsi, sur \mathbb{R}^d , Lieb [Lie83] caractérise les minimiseurs pour les injections de Sobolev fractionnaires $\dot{H}^k(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $0 < k < \frac{d}{2}$, $p = \frac{2d}{d-2k}$, en utilisant la projection stéréographique : ce sont les fonctions qui s'écrivent aux symétries près

$$W(x) = \frac{1}{(1 + |x|^2)^{\frac{d-2k}{2}}}.$$

On retrouve les solutions de [Aub76; Tal76] dans le cas $k = 1$, et de [Ros71] lorsque $k = 2$ et $d = 3$. Grâce à la simplification du problème aux valeurs propres sur \mathbb{S}^d , les linéarisés pour les équations critiques vérifiées par les minimiseurs des injections de Sobolev sont non dégénérés [DDS13], et le reste pour l'inégalité de Hardy-Littlewood-Sobolev fractionnaire équivalente vérifie une estimée quadratique [CFW13].

De la même façon, sur le groupe de Heisenberg, Frank et Lieb se servent de la transformation de Cayley pour montrer que les minimiseurs pour les injections de Folland-Stein fractionnaires $\dot{H}^k(\mathbb{H}^d) \hookrightarrow L^p(\mathbb{H}^d)$, $0 < k < \frac{d'}{2}$, $p = \frac{2d'}{d'-2k}$, $d' = 2d + 2$, sont les fonctions qui s'écrivent à symétries près

$$H(u) = \frac{1}{((1 + \|w\|^2)^2 + \|s\|^2)^{\frac{d'-2k}{4}}}, \quad H \in \dot{H}^k(\mathbb{H}^d),$$

avec $u = (w, s)$ en identifiant \mathbb{H}^d à $\mathbb{C}^d \times \mathbb{R}^d$, et leur méthode ramenée à \mathbb{R}^n et à la projection stéréographique permet de retrouver les minimiseurs pour les injections de Sobolev [FL12]. Liu et Zhang [LZ15] en déduisent des estimées quadratiques pour le terme de reste sur \mathbb{H}^d . Lorsque $k = 1$, la formule pour le minimiseur remonte à [JL88]. On peut aussi remarquer que dans le cas de l'injection $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$, on a $H = |Q|$.

1.1.4 Résultats sur le problème de Cauchy randomisé associé à l'équation de Schrödinger-Grushin

Dans cette partie, on reprend les résultats de [GL21].

Étant données les difficultés évoquées plus haut pour aborder le problème de Cauchy en basse régularité pour l'équation (NLS- \mathbb{H}^1), on établit plutôt un résultat pour le caractère presque-sûrement bien posé de l'équation pour des familles de données initiales aléatoires. Pour simplifier le problème, comme les variables (x, y) jouent le même rôle dans l'expression du sous-laplacien radial $\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_{xx} + \partial_{yy}) + (x^2 + y^2)\partial_{ss}$, on remplace plutôt cet opérateur par l'opérateur de Grushin $\Delta_G = \partial_{xx} + x^2\partial_{yy}$, ce qui nous conduit à considérer l'équation de Schrödinger-Grushin

$$i\partial_t u - \Delta_G u = |u|^2 u. \quad (\text{NLS-G})$$

Pour cette équation, les espaces de Sobolev associés sont les espaces H_G^k sur \mathbb{R}^2 obtenus en remplaçant les puissances de l'opérateur usuel $\sqrt{-\Delta}$ par $\sqrt{-\Delta_G}$. De même que pour l'équation (NLS- \mathbb{H}^1), on sait que si un flot lisse est défini sur H_G^k , alors l'inégalité (1.2) est vérifiée, et on a alors nécessairement $k \geq \frac{3}{2}$, ce qui empêche la résolution du problème de Cauchy déterministe par des méthodes de point fixe.

L'approche probabiliste remonte à Bourgain [Bou94] pour l'équation de Schrödinger sur le tore. Dans ce cas, l'usage de mesures invariantes permet de construire des solutions globales. Cette approche a ensuite été étendue à d'autres équations dispersives, notamment dans [BT08a; BT08b]. La méthode des mesures invariantes impose de se placer dans des domaines compacts, mais dans le cas non compact, la méthode probabiliste employée dans [BT08a] s'adapte encore, comme on peut l'observer pour l'équation de Schrödinger sur \mathbb{R}^d [BOP15b; BOP15a] et pour l'équation des ondes sur \mathbb{R}^d [OP16; Poc17].

Pour l'équation (NLS-G), la méthode probabiliste nous conduit au résultat suivant. On considère une donnée initiale $u_0 \in H_G^k$, $k \in]1, \frac{3}{2}]$. On décompose u_0 de façon similaire à (1.3), c'est-à-dire qu'en variable de Fourier selon la dernière variable, $\mathcal{F}_{y \rightarrow \eta}(u_0)$ se décompose dans la base des fonctions de Hermite $(h_m(\sqrt{|\eta|x}))_{m \in \mathbb{N}}$, ce qui permet d'écrire

$$\mathcal{F}_{y \rightarrow \eta}(u_0)(x, \eta) = \sum_{m \in \mathbb{N}} f_m(\eta) h_m(\sqrt{|\eta|x}).$$

Puis on découpe encore ce potentiel en fonction de la valeur de la fréquence η en intervalles dyadiques $|\eta| \in [I, 2I]$:

$$u_0 = \sum_{(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}} u_{I, m}, \quad (1.12)$$

avec

$$\mathcal{F}(u_{I, m})(x, \eta) = f_m(\eta) h_m(\sqrt{|\eta|x}) \mathbf{1}_{|\eta| \in [I, 2I]}.$$

On suppose que u_0 vérifie, en plus de la régularité H^k , une condition supplémentaire de régularité en la variable y pour un certain $\rho \geq 0$, notée $u_0 \in \mathcal{X}_\rho^k$ avec

$$\|u_0\|_{\mathcal{X}_\rho^k}^2 := \sum_{(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m + 1)I)^k \langle I \rangle^\rho \|u_{I, m}\|_{L_G^2}^2 < +\infty,$$

où on a noté $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. La norme H^k est dans cette décomposition équivalente à

$$\|u_0\|_{H_G^k}^2 \sim \sum_{(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m + 1)I)^k \|u_{I, m}\|_{L_G^2}^2.$$

Fixons un univers de probabilité $(\Omega, \mathcal{A}, \mathbb{P})$. On s'intéresse aux données initiales $(u_0^\omega)_{\omega \in \Omega}$ obtenues en multipliant tous les termes de la décomposition (1.12) ci-dessus par des variables aléatoires $(X_{I,m})_{I,m}$ indépendantes et identiquement distribuées selon une loi gaussienne centrée :

$$u_0^\omega = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} X_{I,m}(\omega) u_{I,m}.$$

On montre alors que presque-sûrement, le problème de Cauchy associé à la donnée initiale u_0^ω est localement bien posé dans l'espace des fonctions qui s'écrivent $u(t) = z^\omega(t) + v(t)$ où $z^\omega(t) = e^{it\Delta_G} u_0^\omega$ suit le flot linéaire, et $v(t) \in H_G^\ell$, $\ell \in]\frac{3}{2}, k + \frac{1}{2}[$, est plus régulière.

Théorème 1.1.8 (Problème de Cauchy local randomisé [GL21]). *Soit $k \in]1, \frac{3}{2}[$, et $u_0 \in \mathcal{X}_1^k \subset H_G^k$.*

- (i) *Pour tout $\ell \in]\frac{3}{2}, k + \frac{1}{2}[$, pour presque-tout $\omega \in \Omega$, il existe $T > 0$ et une unique solution de (NLS-G) associée à la donnée initiale u_0^ω dans l'espace*

$$e^{it\Delta_G} u_0^\omega + \mathcal{C}([0, T], H_G^\ell) \subset \mathcal{C}([0, T], H_G^k).$$

Par ailleurs, il existe $c > 0$ telle que pour tout $R \geq 1$, en dehors d'un ensemble de probabilité au plus e^{-cR^2} , on puisse choisir $T \geq (R \|u_0\|_{\mathcal{X}_1^k})^{-2}$.

- (ii) *(Absence de régularisation par randomisation) Si $u_0 \in H_G^k \setminus \bigcup_{\varepsilon > 0} H_G^{k+\varepsilon}$, alors presque-sûrement, $u_0^\omega \in H_G^k \setminus \bigcup_{\varepsilon > 0} H_G^{k+\varepsilon}$.*

- (iii) *(Densité des mesures associées à des faibles régularités) Pour tout $\varepsilon > 0$ et $\rho \geq 0$, il existe $v_0 \in \mathcal{X}_\rho^k \setminus \bigcup_{\varepsilon' > 0} H_G^{k+\varepsilon'}$ telle que*

$$\mathbb{P}(v_0^\omega \in B_{H_G^k}(u_0, \varepsilon)) > 0.$$

Notre stratégie est inspirée des techniques de Burq, Thomann et Tzvetkov [BTT13] pour l'équation de Schrödinger cubique sur \mathbb{R}^d avec potentiel harmonique $-\Delta + x^2$, voir aussi les travaux [BT20; Lat20; Den12]. Afin d'établir le caractère localement bien posé, on écrit la formule de Duhamel pour l'équation satisfaite par v

$$v(t) = -i \int_0^t e^{i(t-\tau)\Delta_G} (|z^\omega + v|^2 (z^\omega + v))(\tau) d\tau,$$

et on cherche à appliquer un théorème de point fixe dans l'espace H_G^ℓ , qui est une algèbre puisque $\ell > \frac{3}{2}$. Pour cela, on doit gagner de la régularité pour tous les termes faisant intervenir z^ω , qui n'est que de régularité H^k , en développant le produit $|z^\omega + v|^2 (z^\omega + v)$. Cette étude fait l'objet des estimées bilinéaires et trinéaires suivantes.

Théorème 1.1.9 (Estimées bilinéaires et trinéaires [GL21]). *Soit $u_0 \in \mathcal{X}_1^k \subset H_G^k$, u_0^ω la donnée aléatoire associée et $z^\omega(t) = e^{it\Delta_G} u_0^\omega$. Soit $q \in [2, +\infty[$. Alors il existe $c > 0$ telle que pour tout $R \geq 1$, en dehors d'un ensemble de probabilité au plus e^{-cR^2} , on a*

$$\| (z^\omega)^2 \|_{L^q([0, T], H_G^{k+1/2})} + \| |z^\omega|^2 \|_{L^q([0, T], H_G^{k+1/2})} \leq R^2 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^2,$$

$$\| |z^\omega|^2 z^\omega \|_{L^q([0, T], H_G^{k+1/2})} \leq R^3 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^3.$$

On suppose de plus qu'il existe $\varepsilon_0 > 0$ tel que $u_0 \in \mathcal{X}_{1+\varepsilon_0}^k$. Alors pour tout $\ell \in]\frac{3}{2}, k + \frac{1}{2}[$, on peut choisir c en fonction de ℓ tel que pour tout $R \geq 1$, en dehors d'un ensemble de probabilité au plus e^{-cR^2} , pour tout $v, w \in L^\infty([0, T], H_G^\ell)$ (qui peuvent dépendre de ω),

$$\| z^\omega v w \|_{L^q([0, T], H_G^\ell)} \leq R T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon_0}^k} \|v\|_{L^\infty([0, T], H_G^\ell)} \|w\|_{L^\infty([0, T], H_G^\ell)}.$$

L'argument principal de la preuve est une estimée bilinéaire sur chaque bloc de la décomposition (1.12), qui s'écrit

$$\|u_{I,m}v_{J,n}\|_{L_G^2}^2 \lesssim \min\{I, J\} \min \left\{ \frac{1+I}{(1+(2m+1)I)^{\frac{1}{2}}}, \frac{1+J}{(1+(2n+1)J)^{\frac{1}{2}}} \right\} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2,$$

où $(1+(2m+1)I)$ et $(1+(2n+1)J)$ quantifient le gain de régularité. On montre cette inégalité grâce à une estimée bilinéaire sur les produits de fonctions de Hermite avec changement d'échelles $h_m(\alpha_1 \cdot)h_n(\alpha_2 \cdot)$, $\alpha_1, \alpha_2 > 0$, que l'on obtient à partir des estimations ponctuelles usuelles sur les fonctions de Hermite (voir par exemple [KT05], Lemme 5.1 et [KTZ07]).

Notons que notre méthode ne s'étend pas à l'équation de demi-onde (HW), puisque l'on utilise le fait que la décomposition (1.12) comporte une infinité de modes pour que l'hypothèse $u_0 \in \mathcal{X}_1^k$ n'augmente pas la régularité de la donnée initiale. On s'attend en revanche à ce que le résultat soit également vrai pour l'équation (NLS- \mathbb{H}^1) avec le même gain de régularité, voir la partie 1.3.

1.2 Intégrabilité pour l'équation de Benjamin-Ono

Dans cette partie, on emploie une autre approche pour étudier le comportement en temps long de solutions d'équations aux dérivées partielles non linéaires, qui consiste en utiliser l'intégrabilité éventuelle de l'équation en question. Comme pour la dimension finie, le principe est d'exhiber un système de coordonnées dans lequel un système hamiltonien donné a une forme simple.

Intégrabilité en dimension finie Rappelons le cadre. En dimension finie, on considère une variété M de dimension paire $\dim(M) = 2d$ munie d'une structure symplectique, c'est-à-dire d'une 2-forme différentielle ω fermée non dégénérée, et on considère un hamiltonien $\mathcal{H} : M \rightarrow \mathbb{R}$. En notant $X_{\mathcal{H}}$ le champ de vecteurs associé à \mathcal{H} , défini par $\iota_{X_{\mathcal{H}}}\omega = -d\mathcal{H}$, on s'intéresse à l'évolution hamiltonienne

$$\frac{d\gamma}{dt} = X_{\mathcal{H}}(\gamma).$$

Remarquons que le long d'une trajectoire γ , pour $F : M \rightarrow \mathbb{R}$, on a

$$\frac{dF(\gamma(t))}{dt} = \omega(X_{\mathcal{H}}(\gamma(t)), X_F(\gamma(t))).$$

Par conséquent, on dit que F est une loi de conservation pour le flot hamiltonien associé à \mathcal{H} si le crochet de Poisson entre \mathcal{H} et F , défini par $\{\mathcal{H}, F\} = \omega(X_{\mathcal{H}}, X_F)$, s'annule en tout point.

Grâce au théorème de Arnold-Liouville [Arn78; Lio55; Jos68; Min36], nous disposons d'un critère d'intégrabilité pour un système hamiltonien donné, basé sur l'existence de d lois de conservations. On dit que le système est *Liouville-intégrable* s'il existe d lois de conservation $F_1 = \mathcal{H}, \dots, F_d$ sur M lisses, linéairement indépendantes et en involution : en d'autres termes, on a $dF_1 \wedge \dots \wedge dF_d \neq 0$, et $\{F_i, F_j\} = 0$ en tout point pour tous i, j . Dans ce cas, on fixe un ensemble de valeurs $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ et on note $T(c) = F_1^{-1}(c_1) \cap \dots \cap F_d^{-1}(c_d)$ la courbe de niveau associée. On suppose que quitte à prendre une composante connexe, $T(c)$ est connexe, compacte, lisse et n'intersecte pas la

frontière de M . Alors le théorème d'Arnold-Liouville affirme que $T(c)$ est difféomorphe au tore \mathbb{T}^d . Plus précisément, il existe un symplectomorphisme lisse entre un voisinage $\mathcal{V}_{T(c)}$ de $T(c)$ dans M et le produit cartésien d'un ouvert $\mathcal{U} \subset \mathbb{R}^d$ avec \mathbb{T}^d

$$\Phi : m \in (\mathcal{V}_{T(c)}, \omega) \rightarrow (I_1, \dots, I_d, \theta_1, \dots, \theta_d) \in (\mathcal{U} \times \mathbb{T}^d, \sum_{i=1}^d dI_i \wedge d\theta_i),$$

De plus, sur ce voisinage, les valeurs de $F_1(m), \dots, F_d(m)$ ne dépendent que variables action I_1, \dots, I_d mais sont indépendantes des variables angle $\theta_1, \dots, \theta_d$. Dans ces nouvelles coordonnées action-angle, il existe une fonction des actions Ω telle que l'évolution hamiltonienne devient

$$\begin{cases} \frac{dI}{dt} = 0 \\ \frac{d\theta}{dt} = \Omega(I). \end{cases}$$

En résolvant ce système explicitement, on voit que I reste constant et que $\theta(t) = \theta(0) + t\Omega(I)$ est linéaire, ce qui implique que les trajectoires sont quasi-périodiques, c'est-à-dire périodiques avec un nombre fini de périodes temporelles.

Intégrabilité en dimension infinie En dimension infinie, on se place sur un Hilbert complexe $(H, \langle \cdot | \cdot \rangle)$ (par exemple $H = L^2(\mathbb{T})$ ou $H = L^2(\mathbb{R})$), et on considère une forme \mathbb{R} -bilineaire antisymétrique ω , non-dégénérée au sens où l'application

$$u \in H \mapsto \omega(u, \cdot) \in H^*$$

est bijective (par exemple, $\omega(u, v) = \text{Im}(\langle u | v \rangle)$ ou $\omega(u, v) = \text{Re}(\langle u | \partial_x v \rangle)$). On considère ensuite un hamiltonien $\mathcal{H} : D \subset H \rightarrow \mathbb{R}$ défini sur un sous-espace D dense dans H . Le champ hamiltonien associé à \mathcal{H} est défini à partir de la différentielle au sens de Gateaux $d\mathcal{H}$ par la formule

$$\text{Re}(\langle d\mathcal{H}(u) | h \rangle) = \omega(h, X_{\mathcal{H}}(u)),$$

et l'évolution temporelle pour le système hamiltonien associé est donnée par

$$\partial_t u = X_{\mathcal{H}}(u).$$

En dimension infinie, l'existence d'une infinité de lois de conservation, c'est-à-dire de fonctions $F : H \rightarrow \mathbb{R}$ qui vérifient $\{\mathcal{H}, F\} := \omega(X_{\mathcal{H}}, X_F) = 0$, n'est pas équivalente à l'existence d'un changement de coordonnées comme ci-dessus. Dans les parties qui suivent, on donne quelques exemples d'équations complètement intégrables au sens où elles admettent un changement de coordonnées appelé coordonnées de Birkhoff : l'équation de Benjamin-Ono, l'équation de Korteweg-de Vries, l'équation de Schrödinger cubique en dimension 1 et l'équation de Szegő cubique. Puis, on étudie plus précisément certaines équations de type Benjamin-Ono au travers de ces coordonnées.

1.2.1 L'équation de Benjamin-Ono

L'équation de Benjamin-Ono [Ben67; Ono75]

$$\partial_t u = \partial_x(|D|u - u^2) \tag{BO}$$

décrit un certain régime d'ondes internes longues à l'interface entre deux fluides soumis à la gravitation, la profondeur du fluide du bas étant infinie. Cette équation peut être

posée sur la droite ou sur le tore, la solution u est cherchée à valeurs réelles. C'est le cas périodique qui nous intéressera plus particulièrement dans la suite. On fait référence à Saut [Sau19] pour une description du modèle et une bibliographie détaillée, et à Gérard [Gér20] pour un récapitulatif des avancées récentes sur les propriétés d'intégrabilité de l'équation sur le tore.

Intégrabilité On sait depuis les travaux de Nakamura [Nak79] et Bock, Kruskal [BK79] que l'équation de Benjamin-Ono admet une paire de Lax. Suivant la formulation de [GK20], Appendice A, la paire de Lax s'écrit

$$\frac{d}{dt}L_u = [B_u, L_u],$$

avec

$$L_u = D - T_u, \quad B_u = iD^2 + 2iT_{D(\Pi u)} - 2iDT_u. \quad (1.13)$$

Ici, on note $D = -i\partial_x$, et T_u désigne l'opérateur de Toeplitz de symbole $u \in L^\infty(\mathbb{T})$ sur l'espace de Hardy $L_+^2(\mathbb{T}) = \{h \in L^2(\mathbb{T}) \mid \forall n < 0, \widehat{h}(n) = 0\}$, défini en (1.5) à partir du projecteur de Szegő $\Pi : L^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$ par la formule $T_u(h) = \Pi(uh)$, $h \in L_+^2(\mathbb{T})$. La paire de Lax conduit à l'existence d'une infinité de lois de conservation que l'on peut définir suivant [Nak79], [FF81] ou [KM98], voir [MX11] pour plus de détails.

Grâce aux travaux récents de Gérard, Kappeler et Topalov dans [GK20; GKT20b; GKT20a], on sait que l'équation de Benjamin-Ono admet des coordonnées de Birkhoff. Notons $H_{r,0}^s(\mathbb{T})$ l'espace des fonctions à valeurs réelles dans $H^s(\mathbb{T})$ de moyenne nulle et $h_+^{\frac{1}{2}+s}$ l'espace des suites

$$h_+^{\frac{1}{2}+s} = \{\zeta = (\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \|\zeta\|_{h_+^{\frac{1}{2}+s}}^2 = \sum_{n=1}^{+\infty} n^{1+2s} |\zeta_n|^2 < +\infty\}.$$

Théorème 1.2.1 (Coordonnées de Birkhoff pour (BO) [GK20; GKT20b]). *Il existe une transformation*

$$\Phi : u \in \bigcup_{s > -\frac{1}{2}} H_{r,0}^s(\mathbb{T}) \mapsto (\zeta_n(u))_{n \geq 1} \in \bigcup_{s > -\frac{1}{2}} h_+^{\frac{1}{2}+s} \quad (1.14)$$

telle que les propriétés suivantes soient vérifiées.

- Pour tout $s > -\frac{1}{2}$, $\Phi : H_{r,0}^s(\mathbb{T}) \rightarrow h_+^{\frac{1}{2}+s}$ est un homéomorphisme. De plus, cette application et sa réciproque échangent les bornés de ces deux espaces.
- Pour tout $u_0 \in H_{r,0}^s(\mathbb{T})$ avec s assez grand, si u est la solution de (BO) associée à la donnée initiale u_0 , pour tout $n \geq 1$ et pour tout $t \in \mathbb{R}$, on a

$$\zeta_n(u(t)) = e^{it\omega_n(u_0)} \zeta_n(u_0),$$

$$\omega_n(u_0) = n^2 - 2 \sum_{k \geq 1} \min(k, n) |\zeta_k(u_0)|^2.$$

- Pour tout $\tau \in \mathbb{T}$, $u \in \bigcup_{s > -\frac{1}{2}} H_{r,0}^s(\mathbb{T})$ et $n \geq 1$,

$$\zeta_n(u(\cdot + \tau)) = e^{in\tau} \zeta_n(u).$$

On peut interpréter les coordonnées de Birkhoff comme des coefficients de Fourier non linéaires adaptés à l'équation de Benjamin-Ono.

Seuil pour le caractère bien posé Grâce aux travaux de Saut [Sau79] et Abdelouhab, Bona, Felland et Saut [Abd+89], on sait que l'équation (BO) est globalement bien posée dans $H^s(\mathbb{T})$ (ou $H^s(\mathbb{R})$ sur la droite) pour $s > \frac{3}{2}$. Tao [Tao04] étend le caractère globalement bien posé de l'équation à $H^1(\mathbb{R})$ en utilisant une transformation appelée forme normale de Tao, et inspirée de la transformation de Cole-Hopf [Col51; Hop50]

$$w = \Pi_{>0}(e^{-i\partial_x^{-1}u}).$$

Le meilleur résultat connu sur le tore avant le résultat ci-dessus sur la complète intégrabilité de (BO) est le caractère globalement bien posé dans $L^2(\mathbb{T})$ [Mol08; MP12b]. Comme conséquence du théorème 1.2.1, Gérard, Kappeler et Topalov étendent en fait le flot de l'équation de Benjamin-Ono à $H_{r,0}^s(\mathbb{T})$ pour tout $s > -\frac{1}{2}$ [GKT20b], ce qui implique que l'équation de Benjamin-Ono est globalement bien posée au sens suivant.

Définition 1.2.2. *Soit $s \in \mathbb{R}$. Pour $u_0 \in H_{r,0}^s(\mathbb{T})$, on dit que $u \in \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ est une solution globale associée à la donnée initiale u_0 si pour toute suite de données initiales $u_0^{(k)}$ dans $H^\sigma(\mathbb{T})$, $\sigma > \frac{3}{2}$, convergeant vers u_0 dans $H_{r,0}^s(\mathbb{T})$, la suite de $u^{(k)}$ de solutions classiques de (BO) associées converge vers u dans $\mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$.*

L'équation (BO) est dite continûment globalement bien posée dans $H_{r,0}^s(\mathbb{T})$ si pour tout $u_0 \in H_{r,0}^s(\mathbb{T})$, il existe une solution globale u de (BO) associée à u_0 , et si de plus l'application solution $\mathcal{S} : H_{r,0}^s(\mathbb{T}) \rightarrow \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ est continue.

Ce nouveau seuil de $-\frac{1}{2}$ pour le problème de Cauchy est optimal dans le sens où il n'existe pas d'extension continue du flot à $H_{r,0}^s(\mathbb{T})$ pour $s \leq -\frac{1}{2}$ (voir le théorème 2 dans [GKT20a], voir aussi [PH10] pour le cas $s < -\frac{1}{2}$). Par ailleurs, le flot n'est nulle part localement uniformément continu pour $s < 0$ [GKT21a].

Nature des trajectoires Pour l'équation de Benjamin-Ono, les lois de conservation permettent d'établir que toutes les normes de Sobolev des trajectoires sont bornées. En fait, ces trajectoires sont même presque périodiques dans $H_{r,0}^s(\mathbb{T})$ pour tout $s > -\frac{1}{2}$, c'est-à-dire que l'orbite $\{u_\tau : x \mapsto u(x + \tau) \mid \tau \in \mathbb{R}\}$ est relativement compacte dans $\mathcal{C}_b(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ (voir [GK20], Théorème 1.3 et [GKT20a], Théorème 3 et Corollaire 8). Comme conséquence, les solutions de l'équation de Benjamin-Ono sont récurrentes : pour toute solution u associée à la donnée initiale $u_0 \in L_{r,0}^2(\mathbb{T})$, il existe une suite $t_n \rightarrow +\infty$ telle que $u(t_n)$ tend vers u_0 dans $L_{r,0}^2(\mathbb{T})$ (voir [GK20], Remarque 1.4 (ii)). Notons qu'une construction de solutions périodiques et quasi-périodiques non lisses fait l'objet de [GKT21b].

Une approche probabiliste à l'équation de Benjamin-Ono sur le tore, développée dans [Tzv10; TV13a; TV13b; TV15; Den15; DTV15] permettait déjà d'établir qu'avec probabilité 1 par rapport à la mesure concernée, les solutions de (BO) sont récurrentes (voir [TV13b], Corollaire 1.3 et [Sy18], Corollaire 1.2). Les auteurs construisent une suite $(\mu_n)_n$ de mesures gaussiennes sur $L^2(\mathbb{T})$, invariantes par le flot de (BO), et associée aux lois de conservation pour cette équation. Ainsi, chaque mesure μ_n se concentre sur $H^s(\mathbb{T})$ pour $s < n - \frac{1}{2}$, et satisfait $\mu_n(H^{n-\frac{1}{2}}(\mathbb{T})) = 0$. La mesure μ_n est formellement définie à partir de la n -ème loi de conservation \mathcal{H}_n comme une renormalisation de l'expression $d\mu_n = e^{-\mathcal{H}_n(u)} du$. Sy [Sy18] étend cette construction pour obtenir une mesure concentrée sur $\mathcal{C}^\infty(\mathbb{T})$. On peut se demander s'il est possible de construire des solutions globales pour des données initiales génériques très peu régulières, typiquement dans $H^{-\frac{1}{2}}(\mathbb{T})$, et pousser la compréhension de ces mesures à l'aide des coordonnées de Birkhoff.

Intéressons-nous maintenant à la classification des ondes progressives et à leurs propriétés de stabilité. Les ondes progressives sur la droite et sur le tore sont caractérisées depuis les travaux de Amick et Toland [AT91] : dans le cas périodique, ce sont les fonctions de la forme

$$u(t, x) = U(x - ct)$$

avec $c = \frac{1+r^2}{1-r^2}$ et

$$U(x) = N \frac{1 - r^2}{1 - 2r \cos(Nx + \alpha) + r^2} + a,$$

où $\alpha \in \mathbb{R}$, $a \in \mathbb{R}$ et $N \in \mathbb{N}^*$. En coordonnées de Birkhoff, ces solutions correspondent en fait aux solutions monomodes, c'est-à-dire ayant une seule coordonnée de Birkhoff non nulle (voir [GK20], Appendice B). De plus, la stabilité orbitale de ces ondes progressives, établie par Pava et Natali [PN08] dans $H^{1/2}$, s'étend à $H_{r,0}^s(\mathbb{T})$, $s > -\frac{1}{2}$ au moyen de ces coordonnées (voir [GKT20b], Théorème 4).

Équation de Benjamin-Ono sur la droite Sur la droite, on s'attend à ce que la dispersion change le comportement des solutions de l'équation de Benjamin-Ono. Ainsi, Ifrim et Tataru [IT19] montrent un résultat de décroissance dispersif sur des temps grands pour des données initiales petites et localisées. Ils conjecturent également qu'une solution associée à une donnée initiale petite est soit dispersive, soit se décompose comme la somme d'un soliton et d'une partie dispersive (conjecture de résolution en solitons).

L'intégrabilité au sens de Lax est toujours valide sur la droite, avec la même formulation (1.13) pour la paire de Lax, en remplaçant le projecteur de Szegő sur le tore par celui sur la droite. Néanmoins, la transformée de diffusion inverse (IST), écrite formellement par Ablowitz et Fokas dans [FA83], n'est complètement développée que pour des données initiales suffisamment petites et décroissantes (voir [CW90] puis [KM98]). Wu [Wu16] étudie le spectre de l'opérateur de Lax pour cette équation, puis résout le problème de diffusion direct pour des données suffisamment décroissantes [Wu17]. Récemment, Sun [Sun21] a construit la transformation de Birkhoff pour des multisolitons, ce qui constitue une première étape en direction de la conjecture de résolution en solitons.

Perturbations de l'équation de Benjamin-Ono Au moyen d'extensions de la théorie KAM, on peut s'intéresser au comportement en temps long de solutions pour des perturbations d'équations intégrables. En particulier, on se demande dans quelle mesure le caractère périodique, quasi-périodique ou presque-périodique des trajectoires est préservé par la perturbation. Des résultats dans ce sens remontent à Kuksin [Kuk87] et Wayne [Way90], on fait référence à Craig [Cra00] et Berti [Ber19] pour une bibliographie détaillée.

En ce qui concerne les perturbations de (BO) au voisinage de zéro, on peut citer les résultats suivants. Liu et Yuan [LY11] considèrent une famille de perturbations non bornées hamiltoniennes de l'équation de Benjamin-Ono sur le tore, pour lesquelles ils construisent des tores invariants, et en déduisent l'existence de trajectoires quasi-périodiques. Mi et Zhang [MZ14] étendent cette étude à des perturbations quasi-périodiques hamiltoniennes plus générales. Dans [Bal13], Baldi montre l'existence de solutions périodiques pour des perturbations non hamiltoniennes réversibles d'une équation de Benjamin-Ono avec non linéarité cubique

$$\partial_t u + H \partial_{xx} u + \partial_x (u^3) = 0.$$

Pour des équations Benjamin-Ono généralisées

$$\partial_t u + H \partial_{xx} u + \partial_x (f(u)) = 0,$$

Bernier et Grébert [BG20b] établissent une approximation sur des temps longs des trajectoires pour des données initiales génériques proches de l'origine.

Afin de pousser plus loin la théorie des perturbations hamiltoniennes de (BO) au moyen de la théorie KAM, Gérard, Kappeler et Topalov montrent que l'application moment $u \mapsto (|\zeta_n(u)|^2)_n$ est analytique réelle de $H_{r,0}^s(\mathbb{T})$ sur des espaces pondérés de suites ℓ^1 [GKT20a], puis que l'application de Birkhoff Φ s'étend en une application analytique complexe au voisinage de zéro dans $H_{r,0}^s(\mathbb{T})$ pour tout $s > \frac{1}{2}$ [GKT21a].

Amortissement pour l'équation de Benjamin-Ono Enfin, on peut également introduire plusieurs types d'amortissement physiques en fonction de la nature de la dissipation des ondes pour l'équation de Benjamin-Ono, de la même façon que pour l'équation de Korteweg-de Vries [OS70]. L'ajout de termes d'amortissement de type diffusif peut conduire à une décroissance des solutions et à l'existence d'attracteurs globaux. Un attracteur global est un ensemble compact, invariant par le flot de l'équation, qui attire toutes les trajectoires et ce de façon uniforme sur les ensembles bornés de données initiales.

Ainsi, Grimshaw, Smyth et Stepanyants (voir [GSS18] et les références associées) introduisent un modèle général d'équations dispersives de la forme

$$\partial_t u + \alpha u \partial_x u + \beta H \partial_{xx} u + \delta \mathcal{D}[u] = 0,$$

où $\mathcal{D}(u)$ est un multiplicateur de Fourier. Cet opérateur peut notamment correspondre à la dissipation de Rayleigh $\mathcal{D}u = u$, la dissipation de Reynolds $\mathcal{D}u = -\partial_{xx} u$ (on parle alors d'équation de Benjamin-Ono-Burgers), l'amortissement de Landau $\mathcal{D}u = H \partial_x u$, la dissipation des ondes internes sur un fond irrégulier $\mathcal{D}u = |u|u$, etc. Pour ces différents modèles, ils observent numériquement la vitesse de décroissance pour les solitons de l'équation de Benjamin-Ono.

La décroissance vers zéro ainsi qu'une description asymptotique des profils solutions lorsque $t \rightarrow +\infty$ figurent notamment dans [Dix91] pour la dissipation de Reynolds et [BL11] pour des équations de Benjamin-Ono-Burgers généralisées (voir aussi [Mat+07] pour l'étude des chocs dispersifs lorsque l'amortissement est petit). Pour une dissipation fractionnaire $\mathcal{D}u = |D|^q u$, $q > 0$, le comportement asymptotique et l'existence d'attracteurs globaux sont étudiés dans [Dix92; Ala93]. Guo et Huo [GH06] (voir aussi [BY95]) établissent l'existence d'un attracteur global dans $L^2(\mathbb{R})$ qui est compact dans $H^3(\mathbb{R})$ pour des équations de KdV-Benjamin-Ono généralisées faiblement amorties

$$\partial_t u + \alpha H \partial_{xx} u + \beta \partial_{xxx} u + \mu \partial_x u + \lambda u + u \partial_x u = f.$$

De la même façon, pour l'équation de Schrödinger cubique faiblement amortie avec forçage sur \mathbb{R} (resp. sur \mathbb{T})

$$\partial_t u + \gamma u + i \partial_{xx} u + i |u|^2 u = f,$$

Goubet et Molinet [GM09] (resp. [Mol09a]) prouvent l'existence d'un attracteur global dans L^2 , compact dans H^2 , voir aussi [Akr99] pour l'existence dans des espaces plus réguliers. On fait également référence à Burq, Raugel et Schlag [BRS17; BRS18] pour des équations de Klein-Gordon faiblement amorties.

1.2.2 Autres équations aux dérivées partielles intégrables

On mentionne maintenant des résultats de complète intégrabilité pour d'autres équations aux dérivées partielles.

Équation de Kortevveg-de Vries L'équation de Kortevveg-de Vries [KD95]

$$\partial_t u = -\partial_{xxx} u + 6u\partial_x u \quad (\text{KdV})$$

est un modèle de propagation de grandes ondes de surface dans un canal étroit et de faible profondeur. Pour une monographie détaillée sur cette équation et ses perturbations, on fait référence au livre de Kappeler et Pöschel [KP03].

La structure de Poisson pour (KdV) est donnée par le hamiltonien

$$\mathcal{H}(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 + \int_{\mathbb{T}} u^3$$

de domaine $D = H_0^{\frac{5}{2}}(\mathbb{T})$, et la forme $\omega(u, v) = \text{Re}(\langle u | \partial_x v \rangle)$, ce qui conduit au le crochet de Poisson $\{F, G\} = \text{Re}(\langle \nabla F | \partial_x \nabla G \rangle)$ pour des fonctions F et G suffisamment régulières.

Il existe une infinité de lois de conservations \mathcal{H}_k , $k \geq 0$ [Gar71; Miu68; MGK68], qui contrôlent les normes des trajectoires dans tous les espaces de Sobolev d'ordre supérieur, et que l'on peut retrouver au moyen de la paire de Lax [Lax68]

$$\frac{dL_u}{dt} = [B_u, L_u]$$

avec $L_u = -\frac{d^2}{dx^2} + u$ et $B_u = -4\frac{d^3}{dx^3} + 3u\frac{d}{dx} + 3\frac{d}{dx}u$. L'équation (KdV) est complètement intégrable dans le sens où elle admet des coordonnées de Birkhoff globales [Bät+95; BKM96]. En utilisant la formulation de [KM01], l'existence de variables action-angle globales s'écrit sous la forme suivante.

Théorème 1.2.3 (Coordonnées de Birkhoff globales pour (KdV) [Bät+95; BKM96; KM01]). *Pour $s \geq 0$, on note $h_r^s := \{x \in \ell^2(\mathbb{N}, \mathbb{R}) \mid \sum_{n \geq 1} n^{2s} |x_n|^2 < +\infty\}$. Il existe un difféomorphisme*

$$\Phi : u \in L_0^2(\mathbb{T}) \mapsto (x_n(u), y_n(u)) \in h_r^{\frac{1}{2}} \times h_r^{\frac{1}{2}}$$

tel que les propriétés suivantes soient vraies.

- L'application Φ est bijective, bi-analytique réelle, et préserve les crochets de Poisson.
- Pour tout entier $s \geq 0$, la restriction de Φ à $H_0^s(\mathbb{T})$ dans $h_r^{s+\frac{1}{2}} \times h_r^{s+\frac{1}{2}}$ est bijective et bi-analytique.
- Les variables $(x_n, y_n)_{n \geq 1}$ sont les coordonnées de Birkhoff pour l'équation de KdV dans $h_r^{\frac{3}{2}} \times h_r^{\frac{3}{2}}$, pour lesquelles les variables action-angle sont données par les coordonnées polaires $I_n = \frac{x_n^2 + y_n^2}{2}$ et $\theta_n = \arctan(\frac{y_n}{x_n})$.

Dans ces coordonnées, le Hamiltonien \mathcal{H} ne dépend que des actions $I = (I_n)_{n \geq 1}$, et l'équation devient

$$\begin{cases} \dot{x}_n = \omega_n(I) y_n \\ \dot{y}_n = -\omega_n(I) x_n \end{cases}$$

avec $\omega_n(I) = \frac{\partial \mathcal{H}}{\partial I_n}(I)$. Comme conséquence, les trajectoires sont presque-périodiques. C'est aussi le cas pour toutes les équations dans la hiérarchie de KdV, qui admettent les mêmes coordonnées de Birkhoff que l'équation de KdV, avec des régularités adaptées et des

fréquences adaptées $\omega_n^{(k)} = \frac{\partial \mathcal{H}_k}{\partial I_n}(I)$ correspondant aux hamiltoniens d'ordre supérieur \mathcal{H}_k . Notons que contrairement à l'équation de Benjamin-Ono, on ne dispose pas de formule explicite pour ces fréquences.

Comme la transformation de Birkhoff s'étend en une application bijective, bi-analytique $\Phi : H_0^s(\mathbb{T}) \rightarrow h_r^{s+\frac{1}{2}} \times h_r^{s+\frac{1}{2}}$ pour $s \geq -1$ [KMT05], Kappeler et Topalov déduisent que l'équation de KdV est globalement bien posée dans $H^{-1}(\mathbb{T})$ [KT06], de plus, les trajectoires sont relativement compactes dans $H^s(\mathbb{T})$ et presque-périodiques. Sur la droite, on peut mentionner Kilip et Visan [KV19] qui montrent que l'équation de KdV est globalement bien posée dans $H^{-1}(\mathbb{R})$, et que l'équation suivante dans la hiérarchie de KdV est globalement bien posée dans $L^2(\mathbb{R})$.

Équation de Schrödinger cubique Le formalisme hamiltonien est le suivant pour traiter les équations de Schrödinger focalisante (avec le signe $-$) et défocalisante (avec le signe $+$)

$$i\partial_t u \pm \partial_{xx} u = |u|^2 u.$$

On se ramène au système

$$\begin{cases} i\partial_t u_1 + \partial_{xx} u_1 = 2u_2 u_1^2 \\ i\partial_t u_2 - \partial_{xx} u_2 = -2u_1 u_2^2, \end{cases} \quad (1.15)$$

le cas défocalisant étant obtenu par restriction au sous-espace $\{u_2 = \overline{u_1}\}$, alors que le cas focalisant est obtenu par restriction à $\{u_2 = -\overline{u_1}\}$. Pour la structure

$$\omega((u_1, u_2), (v_1, v_2)) = -i \int_{\mathbb{T}} (u_1 v_2 - u_2 v_1) dx,$$

le crochet de Poisson de deux fonctions suffisamment régulières F, G sur $L^2(\mathbb{T})$ est défini par

$$\{F, G\} = -i \int_{\mathbb{T}} (\nabla_{u_1} F)(\nabla_{u_2} G) - (\nabla_{u_2} F)(\nabla_{u_1} G) dx,$$

ce qui permet d'écrire l'équation (1.15) sous la forme

$$\begin{cases} i\partial_t u_1 = \nabla_{u_2} \mathcal{H} \\ i\partial_t u_2 = -\nabla_{u_1} \mathcal{H} \end{cases}$$

avec

$$\mathcal{H} = \int_{\mathbb{T}} \partial_x u_1 \partial_x u_2 + u_1^2 u_2^2 dx.$$

La paire de Lax $\frac{dL_u}{dt} = [B_u, L_u]$ est donnée par Zakharov et Shabat [ZS74] par

$$L_u = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & u_1 \\ u_2 & 0 \end{pmatrix}, \quad B_u = \begin{pmatrix} 2i\partial_{xx} - iu_1 u_2 & u_1' + 2u_1 \partial_x \\ u_2' + 2u_2 \partial_x & -2i\partial_{xx} + iu_1 u_2 \end{pmatrix}.$$

Elle fournit une infinité de lois de conservation qui bornent les normes H^s pour tout $s \geq 0$.

On sait depuis Bourgain [Bou99] que le problème de Cauchy est globalement bien posé dans $L^2(\mathbb{T})$, de plus, ce seuil est optimal étant donné les résultats sur le caractère mal posé dans $H^s(\mathbb{T})$ lorsque $s < 0$ (voir [Mol09b] et les références associées).

Grébert et Kappeler [GK14] montrent que l'équation de Schrödinger cubique défocalisante sur le tore admet des coordonnées de Birkhoff globales dans $H^s(\mathbb{T})$ pour $s \in \mathbb{N}$.

Notons que l'on ne dispose pas de formule explicite en coordonnées de Birkhoff pour le hamiltonien \mathcal{H} , ni pour les fréquences associées $\omega_n = \frac{\partial \mathcal{H}}{\partial I_n}(I)$. Comme conséquence de la complète intégrabilité, les trajectoires pour l'équation de Schrödinger cubique défocalisante sont presque-périodiques dans $H^s(\mathbb{T})$ pour tout $s \geq 0$.

En revanche, l'équation de Schrödinger cubique focalisante sur le tore n'admet pas de coordonnées de Birkhoff locales autour de certains potentiels [KT19]. Kappeler et Topalov [KT20b] démontrent alors un théorème de type Arnold-Liouville en construisant des coordonnées action-angles autour des potentiels pour lesquels le spectre périodique de l'opérateur de Lax est simple, qui constituent un G_δ -dense de potentiels [KT20a].

Équation de Szegő L'équation de Szegő sur le tore admet également des variables action-angle globales dans $H_+^{1/2}(\mathbb{T})$ (voir [GG17], Théorème 2). Cette construction repose sur l'existence de deux paires de Lax [GG10; GG12b]. Pour un potentiel $u \in H_+^{1/2}(\mathbb{T})$, notons

$$H_u : h \in L_+^2(\mathbb{T}) \mapsto \Pi(u\bar{h}) \in L_+^2(\mathbb{T})$$

l'opérateur de Hankel de symbole u , et pour $b \in L^\infty(\mathbb{T})$, notons

$$T_b : h \in L_+^2(\mathbb{T}) \mapsto \Pi(bh) \in L_+^2(\mathbb{T})$$

l'opérateur de Toeplitz de symbole b . On définit aussi l'opérateur de décalage par $S : u \in L_+^2(\mathbb{T}) \mapsto e^{ix}u \in L_+^2(\mathbb{T})$, et l'opérateur de Hankel décalé par $K_u := S^*H_u = H_uS$. Alors pour toute solution de l'équation (Sz), on a

$$\frac{d}{dt}H_u = [B_u, H_u], \quad B_u = -iT_{|u|^2} + \frac{i}{2}H_u^2$$

et

$$\frac{d}{dt}K_u = [C_u, K_u], \quad C_u = -iT_{|u|^2} + \frac{i}{2}K_u^2.$$

Comme conséquence, les solutions sont presque périodiques dans $H_+^{1/2}(\mathbb{T})$ (voir [GG17], Théorème 1). En revanche, même si l'équation de Szegő admet une infinité de lois de conservation, celles-ci ne contrôlent pas les normes de Sobolev d'ordre supérieur à $\frac{1}{2}$. En fait, l'équation de Szegő présente de la turbulence faible (voir [GG17], Théorème 1) : il existe un G_δ -dense de données initiales dans $L_+^2 \cap \mathcal{C}^\infty$ telles que pour tout $s > \frac{1}{2}$, la solution u vérifie $\limsup_t \|u(t)\|_{H^s(\mathbb{T})} = +\infty$ et $\liminf_t \|u(t)\|_{H^s(\mathbb{T})} < +\infty$.

Sur la droite, on a vu que la première paire de Lax s'écrivait de la même façon (1.6) en prenant les opérateurs de Toeplitz et de Hankel sur la droite. Néanmoins, l'opérateur S disparaît, et doit être remplacé par un semi-groupe $(S(\eta))_{\eta \geq 0}$, où $S(\eta) : f \in L_+^2(\mathbb{R}) \mapsto e^{i\eta x}f \in L_+^2(\mathbb{R})$. Au moyen d'une formule explicite pour les solutions, Pocovnicu [Poc11a] montre que pour des données initiales génériques dans $H^s(\mathbb{R})$, $s \geq 0$, la conjecture de résolution en solitons est vraie et il existe des coordonnées action-angle généralisées dans ce cas. En revanche, il existe des données initiales non génériques pour lesquelles la norme de Sobolev H^s explose dès que $s > \frac{1}{2}$.

1.2.3 Résultats pour la hiérarchie de Benjamin-Ono sur le tore

Dans cette partie, on reprend les résultats de [Gas21b] autour de la troisième équation de la hiérarchie intégrable de Benjamin-Ono sur le tore

$$\partial_t u = \partial_x \left(-\partial_{xx}u - \frac{3}{2}uH\partial_x u - \frac{3}{2}H(u\partial_x u) + u^3 \right). \quad (3-BO)$$

On définit les équations de la hiérarchie pour l'équation de Benjamin-Ono par [Mat84]

$$\partial_t u = \partial_x (\nabla \mathcal{H}_{k+1}(u)), \quad (\text{k-BO})$$

où les lois de conservation \mathcal{H}_k sont obtenues à partir de l'opérateur de Lax L_u (1.13) par la formule

$$\mathcal{H}_{k+1}(u) = \langle L_u^{k+1} \mathbf{1} | \mathbf{1} \rangle.$$

En particulier, le hamiltonien pour la troisième équation dans la hiérarchie de Benjamin-Ono (3-BO) vaut

$$\mathcal{H}_4(u) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} (\partial_x u)^2 - \frac{3}{4} u^2 H \partial_x u + \frac{1}{4} u^4 \right) dx - \frac{1}{2} \mathcal{H}_2(u)^2, \quad (1.16)$$

avec $\mathcal{H}_2(u) = \frac{1}{2} \|u\|_{L^2(\mathbb{T})}^2$.

La transformation de Birkhoff pour l'équation de Benjamin-Ono est valable pour toutes équations de la hiérarchie. De fait, le hamiltonien \mathcal{H}_{k+1} ne dépend que des variables d'action $\gamma_p = |\zeta_p|^2$, de même que les fréquences

$$\omega_n^{(k+1)}(\zeta) = \frac{\partial(\mathcal{H}_{k+1} \circ \Phi^{-1})}{\partial |\zeta_n|^2}.$$

Ainsi, en coordonnées de Birkhoff, l'équation (k-BO) de la hiérarchie associée au hamiltonien \mathcal{H}_{k+1} devient

$$\frac{d}{dt} \zeta_n = i \omega_n^{(k+1)}(\zeta) \zeta_n, \quad n \geq 1.$$

Ce système d'équations différentielles a pour solution

$$\zeta_n(t) = \zeta_n(0) e^{i \omega_n^{(k+1)}(\zeta(0)) t}, \quad t \in \mathbb{R}, \quad n \geq 1. \quad (1.17)$$

Les fréquences sont bien définies et la formule pour l'évolution temporelle est valable dès que la suite de coordonnées de Birkhoff au temps initial $\zeta(0) = (\zeta_n(0))_{n \geq 1}$ est presque nulle (on dit alors que $u = \Phi^{-1}(\zeta(0))$ a un nombre fini de modes). Pour la troisième équation (3-BO) dans la hiérarchie de Benjamin-Ono (k-BO), on montre que les fréquences s'écrivent

$$\omega_n^{(4)}(\zeta) = n^3 + n \sum_{p \geq 1} p |\zeta_p|^2 - 3 \sum_{p \geq 1} \min(p, n)^2 |\zeta_p|^2 + 3 \sum_{p, q \geq 1} \min(p, q, n) |\zeta_p|^2 |\zeta_q|^2. \quad (1.18)$$

Caractère bien posé On détermine d'abord le seuil pour lequel l'équation (3-BO) est bien posée dans les espaces de Sobolev $H_{r,0}^s(\mathbb{T})$ de potentiels à valeurs réelles et de moyenne nulle sur le tore. Remarquons que pour tout $t \in \mathbb{R}$, le flot pour cette équation $\mathcal{S}^t : u_0 \mapsto u(t)$ est bien défini lorsque u_0 a un nombre fini de modes.

On montre que l'application solution s'étend globalement par continuité à $H_{r,0}^s(\mathbb{T})$ lorsque $s \geq 0$, ce qui est la manière naturelle d'étendre la notion de caractère globalement bien posé pour des solutions peu régulières, en accord avec la définition 1.2.2. En revanche, le flot n'admet pas d'extension par continuité, même pour des temps petits, dans $H_{r,0}^{-s}(\mathbb{T})$ lorsque $0 < s < \frac{1}{2}$.

Théorème 1.2.4 (Seuil pour le caractère bien posé [Gas21b]). *Pour tout $s \geq 0$, l'application solution pour (3-BO), définie pour les données initiales ayant un nombre fini de modes, s'étend de façon continue à $H_{r,0}^s(\mathbb{T})$, dans le sens où il existe une application solution $\mathcal{S} : H_{r,0}^s(\mathbb{T}) \rightarrow \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ telle que les propriétés suivantes soient vraies :*

- pour tout $t \in \mathbb{R}$, le flot $\mathcal{S}^t : H_{r,0}^s(\mathbb{T}) \rightarrow H_{r,0}^s(\mathbb{T})$ coïncide avec le flot défini pour les potentiels ayant un nombre fini de modes ;
- l'application solution $\mathcal{S} : H_{r,0}^s(\mathbb{T}) \rightarrow \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ est continue.

En revanche, pour tout $t > 0$, il n'existe pas d'extension continue du flot \mathcal{S}^t à $H_{r,0}^{-s}(\mathbb{T})$ lorsque $0 < s < \frac{1}{2}$.

De plus, pour tout $t \in \mathbb{R}$, l'application flot ainsi construite $\mathcal{S}^t : u_0 \mapsto u(t)$ est faiblement séquentiellement continue dans $H_{r,0}^s(\mathbb{T})$ lorsque $s > 0$, mais pas faiblement séquentiellement continue dans $L_{r,0}^2(\mathbb{T})$.

Lorsque la donnée initiale est dans $H_{r,0}^s(\mathbb{T})$ pour $s \geq \frac{1}{2}$, la solution donnée par le théorème donne une solution de l'équation (3-BO) au sens des distributions. En effet, on peut montrer que l'on obtient une solution de l'équation (3-BO) dans $H_{r,0}^{s-3}(\mathbb{T})$ au moyen des injections de Sobolev sur le tore, et des injections de Sobolev duales lorsque l'exposant de Sobolev est négatif. Par ailleurs, en approchant une donnée initiale par des données initiales régulières n'ayant pas nécessairement un nombre fini de modes, les solutions régulières correspondantes coïncident avec l'extension du théorème et convergent vers la solution donnée par le théorème.

On s'attend à ce que le seuil pour le caractère bien posé augmente d'une demi-dérivée pour chaque nouvelle équation de la hiérarchie. Ainsi, pour l'équation (k-BO) associée au $(k+1)$ -ème hamiltonien \mathcal{H}_{k+1} pour $k \geq 3$, le seuil devrait être au niveau de l'espace de Sobolev $H_{r,0}^{\frac{k+1}{2}-2}(\mathbb{T})$. Notons que pour toutes les équations de la hiérarchie, l'exposant de Sobolev critique est $-\frac{1}{2}$.

Mentionnons maintenant d'autres approches au problème de Cauchy pour des équations de Benjamin-Ono d'ordre supérieur. Tanaka [Tan19] montre le caractère localement bien posé dans $H^s(\mathbb{T})$ lorsque $s > \frac{5}{2}$ pour des équations du troisième ordre dans la hiérarchie plus générales, de la forme

$$\partial_t u = \partial_x(-\partial_{xx}u - c_1 u H \partial_x u - c_2 H(u \partial_x u) + u^3).$$

Il déduit le caractère globalement bien posé dans $H^s(\mathbb{T})$, $s \geq 3$, dans le cas intégrable $c_1 = c_2 = \frac{3}{2}$.

Sur la droite, Feng et Han [FH96] montrent le caractère localement bien posé dans $H^s(\mathbb{R})$ pour $s \geq 4$ de l'équation (3-BO). En ce qui concerne des équations du troisième ordre dans la hiérarchie plus générales, Linares, Pilod et Ponce [LPP11] montrent le caractère localement bien posé dans $H^s(\mathbb{R})$ pour $s \geq 2$, puis Molinet et Pilod [MP12a] montrent le caractère globalement bien posé dans $H^s(\mathbb{R})$ pour $s \geq 1$.

Pour une famille d'équations d'ordre 4 plus générales dans la hiérarchie de Benjamin-Ono sur le tore et sur la droite, Tanaka [Tan21] montre le caractère localement bien posé dans H^s pour $s > \frac{7}{2}$, et le caractère globalement bien posé dans H^s pour $s \geq 4$ dans le cas intégrable.

Nature des trajectoires Rappelons que pour l'équation de Benjamin-Ono, les trajectoires sont presque périodiques dans $H_{r,0}^s(\mathbb{T})$ pour tout $s > -\frac{1}{2}$. La preuve de presque périodicité (et comme conséquence la récurrence des solutions) dans les espaces de Sobolev adaptés s'étend en fait à toute la hiérarchie de Benjamin-Ono (k-BO). On peut également remarquer grâce au développement asymptotique $\omega_n^{(k)} \sim n^k$ lorsque $n \rightarrow +\infty$ que les trajectoires quasi-périodiques sont exactement les trajectoires ayant un nombre fini de modes.

On classe maintenant les ondes progressives pour l'équation (3-BO) dans $L^2_{r,0}(\mathbb{T})$, et on détermine leurs propriétés de stabilité orbitale. Les ondes progressives pour (3-BO) sont les solutions qui s'écrivent sous la forme $u(t, x) = u_0(x + ct)$ pour tout $t \in \mathbb{R}$ et $x \in \mathbb{T}$, où $u_0 \in L^2_{r,0}(\mathbb{T})$ et $c \in \mathbb{R}$.

Pour $N \geq 1$, un potentiel $u_0 \in L^2_{r,0}(\mathbb{T})$ est appelé N -mode si l'ensemble de ses modes non nuls $\{n \geq 1 \mid \zeta_n(u_0) \neq 0\}$, avec $\Phi(u_0) = (\zeta_n(u_0))_{n \geq 1}$, est fini et de cardinal N .

On rappelle également qu'une onde progressive associée à $u_0 \in L^2_{r,0}(\mathbb{T})$ est *orbitalement stable* si pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que pour toute solution v de l'équation (3-BO) associée à une donnée initiale $v_0 \in L^2_{r,0}(\mathbb{T})$ vérifiant $\|v_0 - u_0\|_{L^2(\mathbb{T})} \leq \delta$, alors

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{T}} \|v(t) - u_0(\cdot + \theta)\|_{L^2(\mathbb{T})} \leq \varepsilon.$$

Théorème 1.2.5 (Classification et stabilité orbitale des ondes progressives [Gas21b]).
Une donnée initiale $u_0 \in L^2_{r,0}(\mathbb{T})$ définit une onde progressive pour l'équation (3-BO) si et seulement si l'une des deux conditions suivantes est satisfaite.

- Soit u_0 est un 1-mode, dans ce cas, l'onde progressive est orbitalement stable.
- Soit u_0 est un 2-mode, tel que les indices des deux coordonnées de Birkhoff non nulles $p < q$ vérifient, en notant $\gamma_p = |\zeta_p|^2$ et $\gamma_q = |\zeta_q|^2$, l'inégalité

$$0 < \gamma_p < \frac{1}{2} \left(p + \sqrt{p^2 + 4q \frac{p+q}{3}} \right)$$

et l'égalité

$$\gamma_q = \frac{q \frac{p+q}{3} - \gamma_p^2 + p\gamma_p}{2\gamma_p + q}.$$

Dans ce cas, l'onde progressive est orbitalement instable.

1.2.4 Résultats pour une équation de Benjamin-Ono faiblement amortie

Dans cette partie, on reprend les résultats de [Gas20a].

On amortit l'équation de Benjamin-Ono sur le tore (BO) par les plus petits modes de Fourier cos et sin. Pour un paramètre fixé $\alpha > 0$, l'équation de Benjamin-Ono amortie s'écrit

$$\partial_t u + \alpha(\langle u | \cos \rangle \cos + \langle u | \sin \rangle \sin) = H \partial_{xx} u - \partial_x(u^2). \quad (\text{BO-}\alpha)$$

On s'intéresse à la façon dont l'amortissement influence le comportement en temps long des trajectoires. Notons que notre choix d'amortissement enlève à la fois la structure hamiltonienne et la structure intégrable propres à l'équation de Benjamin-Ono, par ailleurs, le terme d'amortissement ne peut pas être traité comme une petite perturbation de l'équation de Benjamin-Ono. Le cas d'un terme d'amortissement général vérifiant ces propriétés semble hors de portée des techniques usuelles.

Caractère bien posé Dans un premier temps, on montre que l'équation de Benjamin-Ono amortie (BO- α) est globalement bien posée dans $L^2_{r,0}(\mathbb{T})$, et que l'application solution est faiblement séquentiellement continue.

Théorème 1.2.6 (Caractère bien posé [Gas20a]). *Pour tout $T > 0$ et tout $u_0 \in L^2_{r,0}(\mathbb{T})$, il existe une unique solution de (BO- α) au sens des distributions de donnée initiale u_0 dans $\mathcal{C}([0, T], L^2_{r,0}(\mathbb{T}))$. De plus, l'application solution est continue et faiblement séquentiellement continue de $L^2_{r,0}(\mathbb{T})$ dans $\mathcal{C}([0, T], L^2_{r,0}(\mathbb{T}))$.*

Pour montrer que les solutions locales maximales sont globales, on borne la norme L^2 de la solution au moyen d'une fonctionnelle de Lyapunov

$$\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{T})}^2 + 2\alpha (|\langle u(t)|\cos \rangle|^2 + |\langle u(t)|\sin \rangle|^2) = 0. \quad (1.19)$$

Afin d'obtenir le caractère globalement bien posé de l'équation (BO- α) dans $L^2_{r,0}(\mathbb{T})$, on pourrait également utiliser des techniques employées pour l'équation de Benjamin-Ono par Molinet [Mol08] et Molinet, Pilod [MP12b]. Néanmoins, ces techniques ne conduisent pas à la continuité séquentielle faible du flot, qui est nécessaire à la description des valeurs d'adhérence faibles de la suite $(u(t))_t$ lorsque $t \rightarrow +\infty$.

Valeurs d'adhérence faibles et relative compacité des trajectoires Ensuite, on s'intéresse aux valeurs d'adhérence faibles des trajectoires. On montre qu'elles suivent le principe de LaSalle, puis que ce sont en fait des valeurs d'adhérence fortes, de sorte que les trajectoires sont relativement compactes dans $L^2_{r,0}(\mathbb{T})$.

Théorème 1.2.7 (Valeurs d'adhérence faibles des trajectoires [Gas20a]). *Soit u une solution globale de l'équation (BO- α) dans $L^2_{r,0}(\mathbb{T})$. Soit v_0 une valeur d'adhérence faible dans $L^2_{r,0}(\mathbb{T})$ de la suite $(u(t))_{t \geq 0}$ lorsque $t \rightarrow +\infty$. Alors les propriétés suivantes sont vraies.*

- (i) *La solution v de (BO- α) associée à la donnée initiale v_0 satisfait le principe de LaSalle :*

$$\forall t \in \mathbb{R}, \quad \langle v(t)|\cos \rangle = \langle v(t)|\sin \rangle = 0. \quad (1.20)$$

La condition (1.20) est équivalente au fait que la valeur d'adhérence faible v_0 n'a pas deux coordonnées de Birkhoff consécutives non nulles, c'est-à-dire avec la convention $\zeta_0(v_0) = 1$:

$$\forall n \in \mathbb{N}, \quad \zeta_n(v_0)\zeta_{n+1}(v_0) = 0.$$

- (ii) *Étant donnée une solution u , toutes les valeurs d'adhérence faibles associées à u en temps infini ont les mêmes actions : il existe une suite $(\gamma_n^\infty)_{n \geq 1}$ telle que pour toute valeur d'adhérence faible v_0 associée à u :*

$$\forall n \geq 1, \quad |\zeta_n(v_0)|^2 = \gamma_n^\infty.$$

- (iii) *Les trajectoires sont relativement compactes dans $L^2_{r,0}(\mathbb{T})$: si v_0 est une valeur d'adhérence faible associée à la suite $(u(t_k))_k$, alors*

$$\|u(t_k) - v_0\|_{L^2(\mathbb{T})} \xrightarrow{k \rightarrow +\infty} 0.$$

La caractérisation (1.20) implique que les trajectoires pour l'équation (BO- α) ne sont pas récurrentes à moins d'être également des trajectoires pour l'équation de Benjamin-Ono non amortie (BO), ce qui se produit lorsque les projections selon cos et sin restent nulles au cours du temps. Par conséquent, les trajectoires ne sont en général pas presque périodiques. Ceci est en contraste avec le comportement des trajectoires pour l'équation de Benjamin-Ono non amortie.

Borne sur les normes de Sobolev d'ordre supérieur Enfin, on montre que les normes de Sobolev H^s des solutions de (BO- α) restent bornées lorsque $0 \leq s < \frac{3}{2}$. Cette propriété est aussi vraie pour l'équation de Benjamin-Ono sans amortissement grâce à l'existence d'une infinité de lois de conservation qui contrôlent toutes les normes de Sobolev [Nak79; BK79].

Théorème 1.2.8 (Normes de Sobolev d'ordre supérieur [Gas20a]). *Soit $0 \leq s < \frac{3}{2}$. Soit u une solution de l'équation de Benjamin-Ono amortie (BO- α) dans $L^2_{r,0}(\mathbb{T})$ dont la donnée initiale u_0 appartient à $H^s(\mathbb{T})$. Alors il existe une constante $C_s(\|u_0\|_{H^s(\mathbb{T})}) > 0$ telle que pour tout $t \in \mathbb{R}_+$,*

$$\|u(t)\|_{H^s(\mathbb{T})} \leq C_s(\|u_0\|_{H^s(\mathbb{T})}).$$

Avec la même approche, on pourrait montrer que les normes de Sobolev H^s restent bornées pour tout $s \geq 0$. Néanmoins, la technicité augmente à mesure que l'exposant s augmente, c'est pourquoi nous ne donnons une preuve que pour des exposants $s < \frac{3}{2}$.

La stratégie de preuve consiste en l'utilisation de fonctionnelles de Lyapunov pour les normes de Sobolev d'ordre supérieur en coordonnées de Birkhoff. En effet, on sait grâce à [GKT20b], Proposition 5 en Appendice A, que l'application de Birkhoff Φ échange les bornés de $H^s_{r,0}(\mathbb{T})$ avec ceux de

$$h^{\frac{1}{2}+s}_+ = \left\{ \zeta = (\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \|\zeta\|_{h^{\frac{1}{2}+s}_+}^2 = \sum_n n^{1+2s} |\zeta_n|^2 < +\infty \right\}.$$

Les fonctionnelles de Lyapunov candidates s'écrivent alors $P_s(u) = \sum_{n \geq 1} w_n \gamma_n(u)$, où les poids w_n sont choisis de sorte que $w_n \approx n^{1+2s}$. La stratégie d'introduire des fonctionnelles de Lyapunov pour contrôler les normes de Sobolev d'ordre supérieur a déjà été employée dans [AB20; AMS20] pour plusieurs équations décrivant des dynamiques de fluides à surface libre, mais à notre connaissance, c'est la première fois que cette stratégie est mise en œuvre au travers des coordonnées de Birkhoff pour un système intégrable proche de l'équation étudiée.

Comparaison avec une équation de Szegő faiblement amortie Dans [GG20], Gérard et Grellier introduisent et étudient une équation avec le même amortissement, mais construite à partir de l'équation de Szegő cubique sur le tore. Pour un paramètre $\alpha > 0$, l'équation de Szegő amortie s'écrit

$$i\partial_t u + i\alpha \langle u | \mathbb{1} \rangle = \Pi(|u|^2 u), \tag{Sz- α }$$

où l'on rappelle que $\Pi : L^2(\mathbb{T}) \rightarrow L^2_+(\mathbb{T}) = \{u \in L^2(\mathbb{T}) \mid \forall n < 0, \widehat{u}(n) = 0\}$ est le projecteur de Szegő.

De la même façon que pour l'équation (BO- α), une valeur d'adhérence faible en temps infini v_0 dans l'espace naturel d'énergie $H^{1/2}_+(\mathbb{T}) = \Pi(H^{1/2}(\mathbb{T}))$ pour une solution de (Sz- α) suit le principe de LaSalle : la solution v de (Sz- α) associée à la donnée initiale v_0 satisfait $\langle v(t) | \mathbb{1} \rangle = 0$ pour tout $t \in \mathbb{R}$. En particulier, les trajectoires ne sont généralement pas presque-périodiques dans $H^{1/2}_+(\mathbb{T})$, alors que les trajectoires pour l'équation de Szegő le sont [GG17]. Ces propriétés sont similaires au cas de l'équation de Benjamin-Ono que nous venons de présenter.

Néanmoins, de nombreuses données initiales pour l'équation (Sz- α) donnent des trajectoires qui ne sont pas relativement compactes dans $H^{1/2}_+(\mathbb{T})$ (ce que l'on peut voir grâce

à la preuve du théorème 3.1 dans [GG20]), alors que l'on vient de voir que les trajectoires sont relativement compactes dans $L^2_{r,0}(\mathbb{T})$ pour (BO- α). Une autre différence avec l'étude de (BO- α) est l'utilisation par les auteurs d'une des deux paires de Lax pour l'équation de Szegő, qui est encore une paire de Lax pour l'équation amortie (Sz- α), alors que l'on ne connaît pas de paire de Lax pour (BO- α). Enfin, en ce qui concerne les normes de Sobolev d'ordre supérieur, Gérard et Grellier montrent dans [GG20] que pour l'équation de Szegő amortie (Sz- α), il existe un ouvert non vide de $H^{1/2}_+(\mathbb{T})$ qui intersecte $H^s(\mathbb{T})$ pour tout $s > \frac{1}{2}$, et tel que toute solution u dont la donnée initiale appartient à $H^s(\mathbb{T})$ a une trajectoire qui explose en temps infini dans $H^s(\mathbb{T})$:

$$\|u(t)\|_{H^s(\mathbb{T})} \xrightarrow{t \rightarrow +\infty} +\infty.$$

En comparaison, on a vu plus haut que l'équation de Szegő sans amortissement présente de la turbulence faible [GG17].

Stratégie de preuve La stratégie générale consiste en transformer l'équation (BO- α) en un système d'équations différentielles ordinaires en les coordonnées de Birkhoff

$$\frac{d}{dt}\zeta_n(u(t)) = i\omega_n(\zeta(u(t)))\zeta_n(u(t)) - \alpha (\langle u(t)|\cos \rangle d\zeta_n[u(t)].\cos + \langle u(t)|\sin \rangle d\zeta_n[u(t)].\sin). \quad (1.21)$$

On simplifie le second membre, et en particulier les termes $d\zeta_n[u(t)].\cos$ et $d\zeta_n[u(t)].\sin$, en mettant à part les fonctions des actions $\gamma = (|\zeta_n|^2)_{n \geq 1}$. Notons que si $u \in L^2_{r,0}(\mathbb{T})$, alors les actions γ sont dans l'espace

$$\ell^1_+ = \{z = (z_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \|z\|_{\ell^1_+} = \sum_{n \geq 1} |z_n| < +\infty\}.$$

On montre qu'il existe des fonctions des actions $p_n^*, q_n^*, A_{n,k}^*$ et $B_{n,k}^*$ de classe \mathcal{C}^1 telles que pour tout $u \in L^2_{r,0}(\mathbb{T})$,

$$d\zeta_n[u].\cos = p_n^*(\gamma)\zeta_{n-1} + q_n^*(\gamma)\zeta_{n+1} + \left(\sum_{k=0}^{+\infty} A_{n,k}^*(\gamma)\bar{\zeta}_k\zeta_{k+1} + B_{n,k}^*(\gamma)\zeta_k\bar{\zeta}_{k+1} \right) \zeta_n.$$

Les termes $p_n^*, q_n^*, A_{n,k}^*$ et $B_{n,k}^*$ sont uniformément bornés et Lipschitz sur les bornés de ℓ^1_+ : pour tout $\gamma \in \ell^1_+$ tel que $\|\gamma\|_{\ell^1_+} \leq R$ et tout $n, k \geq 0$, on a

$$|p_n^*(\gamma)| + |q_n^*(\gamma)| + |A_{n,k}^*(\gamma)| + |B_{n,k}^*(\gamma)| \leq C(R),$$

de plus, pour tout $h \in \ell^1_+$,

$$|dp_n^*(\gamma).h| + |dq_n^*(\gamma).h| + |dA_{n,k}^*(\gamma).h| + |dB_{n,k}^*(\gamma).h| \leq C(R)\|h\|_{\ell^1_+}.$$

Le terme $d\zeta_n[u].\sin$ vérifie une propriété similaire. De même, on sait grâce au Lemme 13 dans [GKT20b] qu'il existe des fonctions a_k^* des actions, uniformément bornées et Lipschitz sur les bornés de ℓ^1_+ telles que

$$\langle u|\cos \rangle - i\langle u|\sin \rangle = - \sum_{k \geq 0} a_k^*(\gamma)\bar{\zeta}_k\zeta_{k+1}. \quad (1.22)$$

Par conséquent, on a décomposé le champ de vecteurs en blocs de la forme $A(\zeta) := \sum_{k \geq 0} A_k^*(\gamma) \overline{\zeta_k} \zeta_{k+1}$, où les A_k^* ne dépendent que des actions et sont uniformément bornés et Lipschitz sur les bornés, de sorte que l'on peut appliquer le théorème de Cauchy-Lipschitz.

Pour étudier le comportement en temps long, on remarque ensuite que les actions γ évoluent lentement : on établit une borne de la forme

$$\sum_{n \geq 1} \left| \frac{d}{dt} \gamma_n(t) \right| \leq C(\|u_0\|_{L^2(\mathbb{T})}) |\langle u(t) | e^{ix} \rangle|,$$

sachant que la fonctionnelle de Lyapunov (1.19) implique que le membre de droite est de carré intégrable en temps. En revanche, les coordonnées de Birkhoff ζ sont très oscillantes à cause du facteur de phase $i\omega_n(\zeta)\zeta_n \approx in^2\zeta_n$ apparaissant dans la formule (1.21) pour la dérivée temporelle de ζ_n . Le caractère très oscillant reste vrai pour les interactions entre différents modes de la forme $\overline{\zeta_k \zeta_{k+1} \zeta_n \zeta_{n+1}}$, $k \neq n$. Par conséquent, on traite ces interactions par intégrations par parties en temps de la même façon que lorsque l'on étudie des intégrales oscillantes pour la transformée de Fourier usuelle.

En partant de la formule (1.22) dans la fonctionnelle de Lyapunov (1.19), on peut donc éliminer les interactions $\overline{\zeta_k \zeta_{k+1} \zeta_n \zeta_{n+1}}$, $k \neq n$, ce qui nous conduit à l'inégalité

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} \gamma_n(t) \gamma_{n+1}(t) dt < +\infty.$$

Par intégrations par parties temporelles successives, on déduit que pour des exposants s de plus en plus élevés, on a

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} n^{2s} \gamma_n(t) \gamma_{n+1}(t) dt < +\infty,$$

et ces intégrales permettent de contrôler l'évolution temporelle de nos fonctions de Lyapunov d'ordre supérieur $P_s(u) = \sum_{n \geq 1} w_n \gamma_n(u)$, avec $w_n \approx n^{1+2s}$.

1.3 Perspectives

Mentionnons maintenant quelques problèmes ouverts et perspectives de recherche.

Problème de Cauchy pour l'équation de Schrödinger sur le groupe de Heisenberg Dans la partie 1.1.4, on a montré le caractère localement bien posé de l'équation (NLS-G) pour des données aléatoires dans H_G^k pour tout $k \in]1, \frac{3}{2}]$, sous une hypothèse additionnelle de décroissance de modes de Fourier à l'infini. On s'attend à ce qu'un résultat similaire soit vrai pour l'équation (NLS- \mathbb{H}^1) pour des données aléatoires dans $H^k(\mathbb{H}^1)$, $k \in]\frac{3}{2}, 2]$ avec une hypothèse adaptée de décroissance. Dans le cas radial, les potentiels se décomposent plutôt selon les fonctions de Laguerre, dont l'asymptotique des normes L^p a par exemple été étudiée dans [Mar82]. Dans le cas non radial, il faudrait traiter un terme supplémentaire dans le sous-laplacien $\mathcal{L} = \Delta_{\mathbb{H}^1} + (y\partial_x - x\partial_y)\partial_s$, où on rappelle que $\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_{xx} + \partial_{yy}) + (x^2 + y^2)\partial_{ss}$.

Il est naturel de se demander si en changeant légèrement la randomisation, il est possible de considérer des potentiels de plus faible régularité, notamment des potentiels en dessous de l'espace d'énergie $H^1(\mathbb{H}^1)$. Dans ce cas, on souhaiterait ensuite observer si la

conservation de l'énergie permet d'obtenir le caractère globalement bien posé des solutions à donnée initiale aléatoire. Un obstacle majeur dans l'utilisation de la conservation de l'énergie est la forme des solutions locales construites $u(t) = z^\omega(t) + v(t)$, où $z^\omega(t) = e^{it\Delta_G} u_0^\omega \in H_G^k$ suit le flot linéaire associé à la donnée initiale aléatoire u_0^ω , et $v \in H_G^\ell$ pour $\ell \in]\frac{3}{2}, k + \frac{1}{2}[$. En effet, la conservation de l'énergie à elle seule s'avère insuffisante et on aurait besoin de contrôler la norme H_G^ℓ de v . Néanmoins, v suit une équation de la forme

$$i\partial_t v - \Delta_G v = |z^\omega + v|^2(z^\omega + v),$$

pour laquelle les normes de Sobolev ne sont a priori pas plus simples à étudier que pour l'équation de départ (NLS-G). Dans l'optique de construire des solutions globales, la construction de mesures invariantes pour l'équation (NLS-G) périodisée selon la dernière variable est une piste envisageable, néanmoins, la très faible régularité attendue de son support (dans les espaces de Sobolev d'exposant strictement négatif) laisse présager de nombreuses difficultés.

Le caractère globalement bien posé pour l'équation (NLS- \mathbb{H}^1) avec des données initiales déterministes reste à ce jour un problème ouvert. En effet, on ne dispose pas d'un moyen de contrôler la norme H^k pour des exposants $k > 1$. De plus, dans le cas focalisant, il n'y a pas à notre connaissance de résultat d'existence même locale de solutions pour des données initiales quelconques dans $H^k(\mathbb{H}^1)$ lorsque $k \leq 2$. Par des méthodes de compacité, on peut construire des solutions faibles globales dans l'espace d'énergie dans le cas défocalisant, mais rien ne garantit l'unicité de ces solutions. Dans cette optique, Bahouri, Barilari et Gallagher [BBG21; BG20a] montrent des estimations de Strichartz en absence de dispersion sur le groupe de Heisenberg, et on peut se demander dans quelle mesure ces estimations permettraient d'améliorer l'étude du problème de Cauchy.

Explosion en temps fini pour l'équation de Schrödinger sur le groupe de Heisenberg À cause de l'absence de dispersion sur le groupe de Heisenberg, l'équation (NLS- \mathbb{H}^1) peut être vue comme une équation de Schrödinger énergie sur-critique (voir l'introduction de [GG10]). On se demande alors si l'on peut construire des solutions de (NLS- \mathbb{H}^1) qui explosent en temps fini dans l'espace d'énergie. Une approche possible serait de chercher des solutions explosives proches d'une solution auto-similaire. On peut partir d'une onde progressive en faisant varier les paramètres de modulation liés aux symétries de l'équation, suivant la stratégie de Bourgain et Wang [BW97], et plus particulièrement s'inspirer des travaux sur l'équation de demi-onde [KLR13]. On pourrait ensuite comparer les vitesses possibles d'explosion avec celles du cas euclidien qui figurent notamment dans [MRR15].

Par ailleurs, les travaux récents de Merle, Raphaël, Rodnianski et Szeftel [Mer+19] mettent en évidence l'existence de solutions qui explosent en temps fini dans le cas défocalisant de l'équation de Schrödinger énergie sur-critique pour certaines dimensions $d \geq 5$. On mentionne également l'existence de critères d'explosion en temps fini des solutions pour la dimension 3 dans le cas défocalisant [Bul20]. Ces travaux nous invitent également à nous pencher sur l'existence de solutions explosives pour l'équation (NLS- \mathbb{H}^1) défocalisante.

Nature des trajectoires pour la hiérarchie de Benjamin-Ono Grâce à l'application de Birkhoff, on dispose d'une formule explicite pour l'évolution temporelle des solutions pour les équations de la hiérarchie de Benjamin-Ono (k-BO), donnée par (1.17)

$$\zeta_n(t) = \zeta_n(0) e^{i\omega_n^{(k)}(\zeta(0))t}, \quad t \in \mathbb{R}, \quad n \geq 1.$$

On sait également que les trajectoires sont presque périodiques, et que les trajectoires quasi-périodiques sont celles ayant un nombre fini de modes non nuls. Il reste encore à caractériser les données initiales qui donnent lieu à des trajectoires périodiques en temps, et à classifier les ondes progressives pour les équations successives de la hiérarchie.

Valeurs d'adhérence des trajectoires pour l'équation de Benjamin-Ono amortie Une approche plus poussée tant sur le plan numérique que sur le plan théorique est à envisager afin d'obtenir une meilleure compréhension des valeurs d'adhérence des trajectoires pour l'équation de Benjamin-Ono amortie (BO- α).

On sait en effet que l'ensemble des valeurs d'adhérence d'une trajectoire en temps infini est inclus dans un unique tore en coordonnées de Birkhoff. Il serait intéressant de pouvoir décrire plus précisément ce tore limite en fonction de la donnée initiale u_0 . Selon des simulations numériques effectuées par Christian Klein [Kle20], des données initiales même très simples en coordonnées de Birkhoff semblent conduire à des tores limites difficiles à caractériser. Ce phénomène est lié à la structure du champ de vecteur qui fait apparaître de nouveaux modes dès que $t > 0$, empêchant ainsi les potentiels avec un nombre fini de modes d'être invariants par le flot de l'équation.

Dans ce but, on peut s'intéresser à l'application

$$u_0 \in L^2_{r,0}(\mathbb{T}) \mapsto \lim_{t \rightarrow +\infty} \|u(t)\|_{L^2}^2 \in \mathbb{R}_+.$$

Cette application est bien définie et semi-continue supérieurement, de plus, toute valeur d'adhérence faible u_∞ de la suite $(u(t))_{t \geq 0}$ lorsque $t \rightarrow +\infty$ vérifie $\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2}^2 = \|u_\infty\|_{L^2}^2$: on aimerait avoir plus d'informations sur cette application. On peut par exemple se demander s'il est possible que le tore limite soit réduit à zéro, c'est-à-dire $\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2}^2 = 0$. Les simulations numériques de Christian Klein [Kle20] suggèrent que de nombreuses trajectoires ne tendent pas vers zéro en temps infini. C'est le cas par exemple pour une famille de données initiales monomodes pour lesquels l'unique mode non nul est ζ_1 , donnée pour un certain $r \in [0, 1)$ par (voir [GK20], Appendice B)

$$u_0(x) = \frac{1 - r^2}{1 - 2 \cos(x)r + r^2} + 1.$$

C'est aussi le cas pour des polynômes trigonométriques de la forme $u_0(x) = a \cos(x) + b \sin(x)$. Une question qui se pose également est la possibilité de quantifier le taux de convergence de $\|u(t)\|_{L^2}^2$ vers sa limite.

En ce qui concerne les normes de Sobolev d'ordre supérieur, les simulations numériques suggèrent la relative compacité des trajectoires dans les espaces associés aux lois de conservation. La question de dépendance des trajectoires en le paramètre d'amortissement α est ouverte, ainsi que celle de la croissance des normes de Sobolev lorsque $\alpha < 0$.

Équations proches de l'équation de Benjamin-Ono Il est naturel de se demander si la transformation de Birkhoff permet également d'étudier des équations qui s'écrivent de façon proche de l'équation de Benjamin-Ono ou d'une équation de sa hiérarchie. On a vu que cette stratégie était utile pour étudier l'équation de Benjamin-Ono faiblement amortie (BO- α), on peut observer si elle simplifierait ou non des équations dont les propriétés d'intégrabilité seraient inconnues, notamment des équations hamiltoniennes.

Un premier exemple est l'équation de Smith [Smi72]

$$\partial_t u = \partial_x ((1 + |D|^2)^{1/2} u - u^2),$$

qui diffère avec l'équation de Benjamin-Ono (BO) d'un terme égal à $\frac{1}{2}Hu$ au premier ordre. Un deuxième exemple est l'équation de Benjamin [Ben92], qui prend en compte les effets de capillarité

$$\partial_t u + u\partial_x u - H\partial_{xx}u - \delta\partial_{xxx}u = 0, \quad \delta > 0,$$

et que l'on peut voir comme une perturbation de l'équation du troisième ordre dans la hiérarchie (3-BO)

$$\partial_t u = \partial_x \left(-\partial_{xx}u - \frac{3}{2}uH\partial_x u - \frac{3}{2}H(u\partial_x u) + u^3 \right). \quad (3\text{-BO})$$

L'écriture d'une telle équation proche en coordonnées de Birkhoff, combinée à des résultats de régularité de l'application de Birkhoff, permet d'obtenir le caractère globalement bien posé du problème de Cauchy dans les espaces de Sobolev pertinents pour lesquels on dispose d'une loi de conservation. La question est de savoir si on peut également obtenir d'autres informations sur le comportement en temps long des trajectoires, comme la croissance des normes de Sobolev d'ordre supérieur, et la nature des trajectoires en temps infini.

Équation ILW On peut se demander si la complète intégrabilité au sens de l'existence d'une transformation de Birkhoff s'étend à l'équation ILW (« Intermediate Long Wave » en anglais). Cette équation décrit la propagation d'ondes internes faiblement non linéaires pour une interface entre deux fluides en profondeur finie δ [KKD78]. Elle est intermédiaire entre les équations (KdV) et (BO), en effet, si u_δ est solution de (ILW), alors $(x, t) \mapsto \frac{3}{\delta}u(x, \frac{3}{\delta}t)$ converge vers une solution de (KdV) lorsque $\delta \rightarrow 0$, alors que u_δ converge vers une solution de (BO) lorsque $\delta \rightarrow +\infty$ [Abd+89]. On fait référence à Saut [Sau19] pour une bibliographie détaillée des résultats concernant cette équation.

Dans le cas périodique, l'équation ILW s'écrit

$$\partial_t u + u\partial_x u + \frac{1}{\delta}\partial_x u + \mathcal{T}(\partial_{xx}u) = 0, \quad (ILW)$$

où $\delta > 0$ et

$$\mathcal{T}u(x) = \sum_{n \neq 0} -i \frac{1}{\tanh(n\delta)} \widehat{u}(n) e^{inx}.$$

L'équation (ILW) admet une paire de Lax [KSA81; KAS82] qui s'écrit de la façon suivante

$$\begin{cases} iW_x^+ + \zeta(W^+ - W^-) = -uW^+ \\ iW_t^\pm - 2i(\zeta - \frac{1}{2\delta})W_x^\pm + W_{xx}^\pm + (\pm iu_x - \mathcal{T}(u_x) + \rho)W^\pm = 0 \end{cases}$$

avec

$$\begin{aligned} W^\pm &= e^{ikx/2} e^{\mp i\delta/2} \frac{\text{Id} \mp i\mathcal{T}}{2} \psi, \\ \zeta &= \frac{1 + \coth(k\delta)}{2} k, \\ \rho &= -k(\zeta - \frac{1}{2\delta}) + \frac{k^2}{4} + \nu. \end{aligned}$$

Le formalisme pour la théorie IST (transformation de diffusion inverse) pour l'équation (ILW) sur le tore est décrit dans [Abl+82], voir aussi [Rad85].

Dans une autre direction, la construction de mesures invariantes pour l'équation (ILW) est ouverte.

Techniques non dispersives et intégrables combinées Pour conclure, remarquons que les techniques invoquées dans les deux parties 1.1 et 1.2 de cette introduction ne sont pas indépendantes, et il serait intéressant combiner autant que possible ces deux approches. Dans cette direction, on peut citer les travaux récents de Bahouri et Perelman [BP20] sur l'équation

$$i\partial_t u \pm \Delta u = \partial_x(|u|^2 u). \quad (\text{DNLS})$$

Les auteurs font usage à la fois de techniques dispersives comme le théorème de décomposition en profils mentionné en partie 1.1, et de propriétés d'intégrabilité liées à l'existence d'une infinité de lois de conservation liées à la partie 1.2, pour montrer le caractère globalement bien posé du problème de Cauchy dans $H^{1/2}(\mathbb{R})$. Remarquons que le flot est mal posé dans $H^s(\mathbb{R})$ pour $s < \frac{1}{2}$, au sens où il ne peut pas être uniformément continu par rapport à la donnée initiale [Tak01; BL01].

Partie I

Équation de Schrödinger sur le groupe de Heisenberg

Chapitre 2

Radially symmetric traveling waves for NLS on the Heisenberg group

Ce chapitre reprend les résultats de [Gas20b].

Résumé. Nous nous intéressons aux solutions radiales de l'équation de Schrödinger cubique sur le groupe de Heisenberg

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad \Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1.$$

Cette équation est un modèle d'équation d'évolution entièrement non dispersive. Nous montrons l'existence d'ondes progressives minimisantes de vitesse $\beta \in (-1, 1)$. Lorsque la vitesse β est suffisamment proche de 1, nous prouvons l'unicité de ces ondes progressives à symétrie près et leur régularité en fonction du paramètre β . L'ingrédient principal est l'émergence d'un système limite lorsque β tend vers 1, pour lequel nous établissons la stabilité linéaire de l'onde progressive minimisante.

Abstract. We consider radial solutions to the cubic Schrödinger equation on the Heisenberg group

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad \Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1.$$

This equation is a model for totally non-dispersive evolution equations. We show existence of ground state traveling waves with speed $\beta \in (-1, 1)$. When the speed β is sufficiently close to 1, we prove their uniqueness up to symmetries and their smoothness along the parameter β . The main ingredient is the emergence of a limiting system as β tends to 1, for which we establish linear stability of the ground state traveling wave.

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2.1 Introduction

2.1.1 Dispersion for nonlinear Schrödinger equations

We consider the cubic focusing Schrödinger equation on the Heisenberg group

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1, \quad (2.1)$$

where $\Delta_{\mathbb{H}^1}$ denotes the sub-Laplacian on the Heisenberg group. When the solution is radial, in the sense that it only depends on t , $|x + iy|$ and s , the sub-Laplacian becomes

$$\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2.$$

The Heisenberg group is a typical case of geometry where dispersive properties of the nonlinear Schrödinger equation disappear. Let us recall the motivation for this setting.

Fix a Riemannian manifold M , and denote by Δ the Laplace operator associated to the metric g on M . As observed by Burq, Gérard and Tzvetkov [BGT05], qualitative properties of the solutions to the nonlinear Schrödinger equation

$$i\partial_t u - \Delta u = |u|^2 u, \quad (t, x) \in \mathbb{R} \times M$$

are strongly influenced by the underlying geometry of the manifold M . When some loss of dispersion occurs, for example in the spherical geometry, a condition for well-posedness of the Cauchy problem in $H^s(M)$ is that s must be larger than a critical parameter.

To take it further, on sub-Riemannian manifolds, Bahouri, Gérard and Xu [BGX00] noticed that the dispersion properties totally disappear for the sub-Laplacian on the Heisenberg group, leaving the existence of smooth global in time solutions and the uniqueness of weak solutions as an open problem. In [Del05], Del Hierro analyzed the dispersion properties on H-type groups, proving sharp decay estimates for the Schrödinger equation depending on the dimension of the center of the group. More generally, Bahouri, Fermandian and Gallagher [BFG16] proved optimal dispersive estimates on stratified Lie groups of step 2 under some property of the canonical skew-symmetric form. In contrast, they

also give a class of groups without this property displaying total lack of dispersion, which includes the Heisenberg group.

In this spirit, Gérard and Grellier introduced the cubic Szegő equation on the torus [GG08; GG10] as a simpler model of a non-dispersive Hamiltonian equation in order to better understand the situation on the Heisenberg group. The cubic Szegő equation was then studied on the real line by Pocovnicu [Poc11b], where it has the form

$$i\partial_t u = \Pi(|u|^2 u), \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

$\Pi : L^2(\mathbb{R}) \rightarrow L^2_+(\mathbb{R})$ being the Szegő projector onto the space $L^2_+(\mathbb{R})$ of functions in $L^2(\mathbb{R})$ with nonnegative Fourier frequencies. The cubic Szegő equation displays a strong link with the mass-critical half-wave equation on the torus [GG12a] and on the real line [KLR13]. On the real line, the cubic focusing half-wave equation is written

$$i\partial_t u + |D|u = |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

where $\widehat{|D|f}(\xi) = |\xi|\widehat{f}(\xi)$. Some of the interactions between the Szegő equation and the half-wave equation will be discussed below, because they can be transferred to the setting of the Heisenberg group.

2.1.2 Traveling waves and limiting profiles

Constructing traveling wave solutions which are weak global solutions in the energy space can be obtained by a classical variational argument. For example, this technique was used to study the famous focusing mass-critical NLS problem

$$i\partial_t u - \Delta u = |u|^{\frac{4}{n}} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

From Weinstein's work [Wei83], the existence of a ground state positive solution $Q \in H^1(\mathbb{R}^n)$ to

$$\Delta Q - Q + Q^{1+\frac{4}{n}} = 0,$$

leads to a criterion for global existence of solutions in $H^1(\mathbb{R}^n)$. The uniqueness of this ground state (up to symmetries) holds [GNN79; Kwo89].

Concerning the half-wave equation, the Cauchy problem is locally well-posed in the energy space $H^{\frac{1}{2}}(\mathbb{R})$ [GG12a; KLR13]. Moreover, one also gets a global existence criterion, derived from the existence of a unique [FL13] ground state positive solution $Q \in H^{\frac{1}{2}}(\mathbb{R})$ to

$$|D|Q + Q - Q^3 = 0.$$

Contrary to the mass-critical Schrödinger equation on \mathbb{R}^n , the half-wave equation admits traveling waves with speed $\beta \in (-1, 1)$ (see Krieger, Lenzmann and Raphaël [KLR13])

$$u_\beta(t, x) = Q_\beta \left(\frac{x + \beta t}{1 - \beta} \right) e^{-it}.$$

The profile Q_β is a solution to

$$\frac{|D| - \beta D}{1 - \beta} Q_\beta + Q_\beta = |Q_\beta|^2 Q_\beta.$$

Moreover, it satisfies

$$\lim_{\beta \rightarrow 0} \|Q_\beta - Q\|_{H^{\frac{1}{2}}(\mathbb{R})} = 0 \quad \text{and} \quad \|Q_\beta\|_{L^2(\mathbb{R})} < \|Q\|_{L^2(\mathbb{R})}.$$

While the existence of the profiles Q_β follows from a standard variational argument, their uniqueness is more delicate to prove. This can be done through the study of the photonic limit $\beta \rightarrow 1$ as follows. It has been shown in [Gér+18] that the traveling waves converge as β tends to 1 to a solution of the cubic Szegő equation. More precisely, $(Q_\beta)_\beta$ converges in $H^{\frac{1}{2}}(\mathbb{R})$ to a profile Q_+ , which is a ground state solution to

$$DQ_+ + Q_+ = \Pi(|Q_+|^2 Q_+), \quad D = -i\partial_x.$$

From Q_+ , we recover a traveling wave solution to the cubic Szegő equation by setting

$$u(t, x) = Q_+(x - t) e^{-it}.$$

But Pocovnicu [Poc11b] showed that the traveling waves u are unique up to symmetries, and that Q_+ must have the form

$$Q_+(x) = \frac{2}{2x + i}.$$

Moreover, the linearized operator around Q_+ is coercive [Poc12], and in particular, the Szegő profile is orbitally stable. Gérard, Lenzmann, Pocovnicu and Raphaël [Gér+18] deduced the invertibility of the linearized operator for the half-wave equation around the profiles Q_β when β is close enough to 1 with quantitative estimates, leading to their uniqueness up to symmetries. This allowed them to define a smooth map of solutions $\beta \mapsto Q_\beta$ on a neighborhood of 1.

On the Heisenberg group, one can also construct a family of radial traveling waves with speed $\beta \in (-1, 1)$ under the form

$$u_\beta(t, x, y, s) = Q_\beta\left(\frac{x}{\sqrt{1-\beta}}, \frac{y}{\sqrt{1-\beta}}, \frac{s + \beta t}{1-\beta}\right). \quad (2.2)$$

The profile Q_β satisfies the following stationary hypoelliptic equation

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} Q_\beta = |Q_\beta|^2 Q_\beta. \quad (2.3)$$

There exist ground state solutions, constructed as optimizers for some Gagliardo-Nirenberg inequalities derived from the Folland-Stein embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$ [FS74]. The proof of existence relies on a concentration-compactness argument, which first appeared in the work of Cazenave and Lions [CL82] and was refined into a profile decomposition theorem on \mathbb{R}^n by Gérard [Gér98]. The profile decomposition theorem was then adapted to the Heisenberg group by Benameur [Ben08]. The family of traveling waves u_β is constructed in the radial setting for simplicity, but it seems realistic to establish the existence of non-radial traveling waves as minimizers for the same Gagliardo-Nirenberg inequalities restricted to some other set of functions.

Our purpose is to show the uniqueness of the profiles Q_β when the speed β is close to 1 up to some symmetries. Following the strategy deployed on the half-wave equation, we derive a limiting system in the photonic limit $\beta \rightarrow 1$. We then determine all ground state solutions to the limiting system and prove their linear stability. From the linear stability of the limiting ground states, we recover the uniqueness of the profiles Q_β up to symmetries when β is close to 1.

2.1.3 Main results

The Schrödinger equation on the Heisenberg group (2.1) enjoys the following symmetries: if u is a solution, then

- for all $s_0 \in \mathbb{R}$, $(t, x, y, s) \mapsto u(t, x, y, s + s_0)$ is a solution (translation in s);
- for all $\theta \in \mathbb{T}$, $(t, x, y, s) \mapsto e^{i\theta}u(t, x, y, s)$ is a solution (phase multiplication);
- for all $\lambda \in \mathbb{R}$, $(t, x, y, s) \mapsto \lambda u(\lambda^2 t, \lambda x, \lambda y, \lambda^2 s)$ is a solution (scaling).

Our main result is the uniqueness of the ground states Q_β when β is close to 1.

Theorem 2.1.1. *There exists $\beta_* \in (0, 1)$ such that the following holds. For all $\beta \in (\beta_*, 1)$, there is a unique ground state up to symmetries to (2.3)*

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta = |Q_\beta|^2 Q_\beta.$$

Denote by Q_β this ground state, then the set of all ground state solutions of the above equation can be described as

$$\{T_{s_0, \theta, \alpha} Q_\beta : (x, y, s) \in \mathbb{H}^1 \mapsto e^{i\theta} \alpha Q_\beta(\alpha x, \alpha y, \alpha^2(s + s_0)) \mid (s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\}.$$

For $\beta \in (\beta_*, 1)$, Q_β can be chosen such that as β tends to 1, it tends to the profile

$$Q_+ : (x, y, s) \in \mathbb{H}^1 \mapsto \frac{\sqrt{2}i}{s + i(x^2 + y^2) + i},$$

and so that the map $\beta \in (\beta_*, 1) \mapsto Q_\beta \in \dot{H}^1(\mathbb{H}^1)$ is smooth. In this case, for all $\gamma \in (0, \frac{1}{4})$ and for all $k \in [1, +\infty)$, Q_β lies in $\dot{H}^k(\mathbb{H}^1)$, and as β tends to 1,

$$\|Q_\beta - Q_+\|_{\dot{H}^k(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^\gamma).$$

We refer to Theorem 2.5.14 for a more precise statement.

Note that the energy of the traveling waves u_β defined in (2.2) vanishes as β goes to 1, indeed, $\|u_\beta\|_{\dot{H}^1(\mathbb{H}^1)} = \sqrt{1 - \beta} \|Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \rightarrow 0$. This is similar to the cubic half-wave equation, for which the critical norm $\|u_\beta\|_{L^2(\mathbb{R})}$ vanishes as β goes to 1 [KLR13].

We now briefly present the emergence of the profile Q_+ as a ground state solution to a limiting system, and the key ingredient for the proof of Theorem 2.1.1 which relies on the study of the limiting geometry.

We are interested in radial solutions with values in the homogeneous energy space $\dot{H}^1(\mathbb{H}^1)$, which is a Hilbert space endowed with the real scalar product

$$(u, v)_{\dot{H}^1(\mathbb{H}^1)} = \operatorname{Re} \left(\int_{\mathbb{H}^1} -\Delta_{\mathbb{H}^1} u(x, y, s) \overline{v(x, y, s)} \, dx \, dy \, ds \right).$$

For $u \in \dot{H}^{-1}(\mathbb{H}^1)$ and $v \in \dot{H}^1(\mathbb{H}^1)$, we will also make use of the duality product

$$(u, v)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \operatorname{Re} \left(\int_{\mathbb{H}^1} u(x, y, s) \overline{v(x, y, s)} \, dx \, dy \, ds \right).$$

Up to the three symmetries (translation, phase multiplication, scaling), one can show convergence as β tends to 1 of the profiles Q_β to some profile Q_+ in $\dot{H}^1(\mathbb{H}^1)$. Then, Q_+ is a ground state solution to

$$D_s Q_+ = \Pi_0^+(|Q_+|^2 Q_+), \quad D_s = -i\partial_s. \quad (2.4)$$

The operator Π_0^+ is an orthogonal projector from $L^2(\mathbb{H}^1)$ onto a subspace $L^2(\mathbb{H}^1) \cap V_0^+$, which will be defined in part 2.2.2. In order to study this projector and the space $L^2(\mathbb{H}^1) \cap V_0^+$, we introduce a link between the space $L^2(\mathbb{H}^1) \cap V_0^+$ and the Bergman space $L^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ on the complex upper half-plane [Bék+12]. The orthogonal projection Π_0^+ from $L^2(\mathbb{H}^1)$ onto $L^2(\mathbb{H}^1) \cap V_0^+$ then matches with a Bergman projector. This projector is a simplification of the usual Cauchy-Szegő projector for the Heisenberg group in the radial case.

A salutary fact is that the profile Q_+ can be determined explicitly, and is unique up to symmetry (see part 2.3.3):

$$Q_+(x, y, s) = \frac{\sqrt{2}i}{s + i(x^2 + y^2) + i}.$$

Our key result is the coercivity of the linearized operator \mathcal{L} around Q_+ on the orthogonal of a finite-dimensional manifold in some subspace $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$ of $\dot{H}^1(\mathbb{H}^1)$ (see part 2.2.2). On $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$, the linearized operator \mathcal{L} around Q_+ is defined by

$$\mathcal{L}h = D_s h - 2\Pi_0^+(|Q_+|^2 h) - \Pi_0^+(Q_+^2 \bar{h}).$$

Theorem 2.1.2. *For some constant $c > 0$, the following holds. Let $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, and suppose h is orthogonal to the directions $Q_+, iQ_+, \partial_s Q_+$ and $i\partial_s Q_+$ in the Hilbert space $\dot{H}^1(\mathbb{H}^1)$. Then*

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq c \|h\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

In particular, the linearized operator \mathcal{L} is nondegenerate in the sense that its kernel is composed only of three directions coming from the three symmetries of the equation

$$\text{Ker}(\mathcal{L}) = \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+).$$

Following the approach employed in the study of the half-wave equation [Gér+18], one can then prove the invertibility of the linearized operators \mathcal{L}_{Q_β} for the Schrödinger equation around the profiles Q_β for β close enough to 1. In order to do so, we need to combine the above coercivity result with some regularity estimates and decay properties for Q_β . This enables us to achieve our goal, which is the uniqueness of these profiles up to symmetries for β close to 1.

2.1.4 Stereographic projection and Cayley transform

Conclusive information on the linearized operator \mathcal{L} around Q_+ is not easy to obtain directly. Indeed, the operator \mathcal{L} would be self-adjoint acting on $L^2(\mathbb{H}^1)$, but the space we consider is the Hilbert space $\dot{H}^1(\mathbb{H}^1)$. In order to get a coercivity estimate, we rely on a conformal invariance between the Heisenberg group \mathbb{H}^1 and the CR sphere \mathbb{S}^3 in \mathbb{C}^2 called the Cayley transform

$$\begin{aligned} \mathcal{C} : \quad \mathbb{H}^1 &\rightarrow \mathbb{S}^3 \setminus (0, -1) \\ (w, s) &\mapsto \left(\frac{2w}{1+|w|^2+is}, \frac{1-|w|^2-is}{1+|w|^2+is} \right), \end{aligned}$$

where \mathbb{H}^1 is parametrized by the complex number $w = x + iy$ and by s .

This transformation links estimates for the linearized operator \mathcal{L} to the spectrum of the sub-Laplacian on the CR sphere, which is explicit [Sta89]. Potential negative eigenvalues are discarded by the orthogonality conditions from Theorem 2.1.2. This latter step follows from technical but direct calculations.

For the n -dimensional Heisenberg group \mathbb{H}^n , the Cayley transform gives an equivalence between \mathbb{H}^n and the CR sphere \mathbb{S}^{2n+1} in \mathbb{C}^n . This transform is the counterpart of stereographic projection, which links the space \mathbb{R}^n with the euclidean sphere \mathbb{S}^n in \mathbb{R}^{n+1} . Both transformations have been a useful tool in the study of fractional Folland-Stein inequalities on \mathbb{H}^n , resp. fractional Sobolev inequalities in \mathbb{R}^n , as we will now recall.

On the space \mathbb{R}^n , Lieb [Lie83] characterized all optimizers for the fractional Sobolev embeddings $\dot{H}^k(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, $0 < k < \frac{n}{2}$, $p = \frac{2n}{n-2k}$, as the set of functions which, up to translation, dilation and multiplication by a nonzero constant, coincide with

$$W(x) = \frac{1}{(1 + |x|^2)^{\frac{n-2k}{2}}}, \quad U \in \dot{H}^k(\mathbb{R}^n).$$

Stereographic projection appears in Lieb's paper in order to show that these functions are actually optimizers. The formula for W was first established with different methods for $k = 2$ and $n = 3$ by Rosen [Ros71], and then for $k = 1$ and arbitrary n by Aubin [Aub76] and Talenti [Tal76].

Stereographic projection also appears in the proofs of nondegeneracy of the optimizers for the critical Sobolev embeddings, indeed, this transformation provides a simpler form to the eigenvalue problem when transferred to the unit sphere \mathbb{S}^n . In this spirit, Dávila, Del Pino and Sire [DDS13] proved the nondegeneracy of the linearized operator for the critical equation corresponding to the fractional Sobolev embeddings. Chen, Frank and Weth [CFW13] showed a quadratic estimate for the remainder terms for the equivalent fractional Hardy-Littlewood-Sobolev inequalities.

On the Heisenberg group \mathbb{H}^n , Frank and Lieb [FL10] determined the optimizers for the fractional Folland-Stein embeddings $\dot{H}^k(\mathbb{H}^n) \hookrightarrow L^p(\mathbb{H}^n)$, $0 < k < \frac{Q}{2}$, $p = \frac{2Q}{Q-2k}$, $Q = 2n + 2$, with the use of the Cayley transform. These optimizers are the functions equal, up to translations, dilations and multiplication by a constant, to

$$H(u) = \frac{1}{((1 + \|w\|^2)^2 + \|s\|^2)^{\frac{Q-2k}{4}}}, \quad H \in \dot{H}^k(\mathbb{H}^n).$$

Here, the notation $u = (w, s)$ uses the identification of \mathbb{H}^n with $\mathbb{C}^n \times \mathbb{R}^n$. In [FL12], the same authors proved that a similar approach with stereographic projection on \mathbb{R}^n enables us to characterize the optimizers of the fractional Sobolev embeddings on \mathbb{R}^n . Liu and Zhang [LZ15] then carried the study of remainder term to the complex sphere \mathbb{S}^{2n+1} by using the Cayley transform. When $k = 1$, the optimizers were first determined by Jerison and Lee [JL88], who already made use of the Cayley transform. One can notice that fixing $n = k = 1$, $u = (x, y, s) \in \mathbb{H}^1$, we get

$$H(u) = \frac{1}{((1 + x^2 + y^2)^2 + s^2)^{\frac{1}{2}}}.$$

Therefore, up to multiplication by a constant, H coincides with $|Q_+|$, where Q_+ is the ground state we are interested in. In fact, the profile Q_+ is an optimizer for the Folland-Stein inequality $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$ restricted to the subspace $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$.

Plan of the paper In section 2.3, we prove the existence of the profiles Q_β and their convergence to a ground state solution to the limiting system (2.4). We then determine all the limiting profiles (part 2.3.3), in particular, we show that they are unique up to symmetries. In section 2.4, we focus on the linear stability of the limiting profile Q_+ . After recalling some results about orthogonal projections on Bergman spaces (part 2.4.1) and about the spectrum of the sub-Laplacian on the CR sphere (part 2.4.3), we prove the coercivity of the linearized operator around Q_+ . Finally, in section 2.5, we retrieve the uniqueness of the profiles Q_β up to symmetries for β close to 1. In order to do so, we first need to collect some regularity properties and decay estimates on the profiles Q_β , which come from the theory of elliptic and hypoelliptic equations (part 2.5.1).

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2.2 Notation

2.2.1 The Heisenberg group

We recall some facts about the Heisenberg group. We identify the Heisenberg group \mathbb{H}^1 with \mathbb{R}^3 . The group multiplication is given by

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2(x'y - xy')).$$

The Lie algebra of left-invariant vector fields on \mathbb{H}^1 is spanned by the vector fields $X = \partial_x + 2y\partial_s$, $Y = \partial_y - 2x\partial_s$ and $T = \partial_s = \frac{1}{4}[Y, X]$. The sub-Laplacian is defined as

$$\mathcal{L}_0 := \frac{1}{4}(X^2 + Y^2) = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2 + (y\partial_x - x\partial_y)\partial_s.$$

When u is a radial function, the sub-Laplacian coincides with the operator

$$\Delta_{\mathbb{H}^1} := \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2.$$

The space \mathbb{H}^1 is endowed with a smooth left invariant measure, the Haar measure, which in the coordinate system (x, y, s) is the Lebesgue measure $d\lambda_3(x, y, s)$. Sobolev spaces of positive order can then be constructed on \mathbb{H}^1 from powers of the operator $-\Delta_{\mathbb{H}^1}$, for example, $\dot{H}^1(\mathbb{H}^1)$ is the completion of the Schwarz space $\mathcal{S}(\mathbb{H}^1)$ for the norm

$$\|u\|_{\dot{H}^1(\mathbb{H}^1)} := \|(-\Delta_{\mathbb{H}^1})^{\frac{1}{2}}u\|_{L^2(\mathbb{H}^1)}.$$

The distance between two points (x, y, s) and (x', y', s') in \mathbb{H}^1 is defined as

$$d((x, y, s), (x', y', s')) := \left(((x - x')^2 + (y - y')^2)^2 + (s - s' + 2(x'y - xy'))^2 \right)^{\frac{1}{4}}.$$

For convenience, the distance to the origin is denoted by

$$\rho(x, y, s) := ((x^2 + y^2)^2 + s^2)^{\frac{1}{4}}.$$

2.2.2 Decomposition along the Hermite functions

In order to study radial functions valued on the Heisenberg group \mathbb{H}^1 , it is convenient to use their decomposition along Hermite-type functions (see for example [SM93], Chapters 12 and 13). The Hermite functions

$$h_m(x) = \frac{1}{\pi^{\frac{1}{4}} 2^{\frac{m}{2}} (m!)^{\frac{1}{2}}} (-1)^m e^{\frac{x^2}{2}} \partial_x^m (e^{-x^2}), \quad x \in \mathbb{R}, \quad m \in \mathbb{N},$$

form an orthonormal basis of $L^2(\mathbb{R})$. In $L^2(\mathbb{R}^2)$, the family of products of two Hermite functions $(h_m(x)h_p(y))_{m,p \in \mathbb{N}}$ diagonalizes the two-dimensional harmonic oscillator: for all $m, p \in \mathbb{N}$,

$$(-\Delta_{x,y} + x^2 + y^2)h_m(x)h_p(y) = 2(m+p+1)h_m(x)h_p(y).$$

Given $u \in \mathcal{S}(\mathbb{H}^1)$, we will denote by \widehat{u} its usual Fourier transform under the variable s , with corresponding variable σ :

$$\widehat{u}(x, y, \sigma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\sigma} u(x, y, s) ds.$$

For $m, p \in \mathbb{N}$, set $\widehat{h_{m,p}}(x, y, \sigma) := h_m(\sqrt{2|\sigma|x})h_p(\sqrt{2|\sigma|y})$. Then

$$\Delta_{\mathbb{H}^1} \widehat{h_{m,p}} = -(m+p+1)|\sigma| \widehat{h_{m,p}}.$$

Let $k \in \{-1, 0, 1\}$, and denote by $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ the subspace of functions in $\dot{H}^k(\mathbb{H}^1)$ spanned by $\{h_{m,p} \mid m, p \in \mathbb{N}, m+p=n\}$. A function $u_n^\pm \in \dot{H}^k(\mathbb{H}^1)$ belongs to $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ if there exist functions $f_{m,p}^\pm$ such that

$$\widehat{u_n^\pm}(x, y, \sigma) = \sum_{\substack{m,p \in \mathbb{N} \\ m+p=n}} f_{m,p}^\pm(\sigma) \widehat{h_{m,p}}(x, y, \sigma) \mathbb{1}_{\sigma \geq 0}.$$

For $u_n^\pm \in \dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$, the \dot{H}^k norm of u_n^\pm has the form

$$\begin{aligned} \|u_n^\pm\|_{\dot{H}^k(\mathbb{H}^1)}^2 &= \int_{\mathbb{R}_\pm} ((n+1)|\sigma|)^k \int_{\mathbb{R}^2} |\widehat{u_n^\pm}(x, y, \sigma)|^2 dx dy d\sigma \\ &= \sum_{\substack{m,p \in \mathbb{N} \\ m+p=n}} \int_{\mathbb{R}_\pm} ((n+1)|\sigma|)^k |f_{m,p}^\pm(\sigma)|^2 \frac{d\sigma}{2|\sigma|}. \end{aligned}$$

Any function $u \in \dot{H}^k(\mathbb{H}^1)$ admits a decomposition along the orthogonal sum of subspaces $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$: $u = \sum_\pm \sum_{n \in \mathbb{N}} u_n^\pm$ where $u_n^\pm \in \dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ for all (n, \pm) . Then

$$\|u\|_{\dot{H}^k(\mathbb{H}^1)}^2 = \sum_\pm \sum_{n \in \mathbb{N}} \|u_n^\pm\|_{\dot{H}^k(\mathbb{H}^1)}^2.$$

Note that rotations of the (x, y) variable commute with $-\Delta_{\mathbb{H}^1}$ so $u \in \dot{H}^k(\mathbb{H}^1)$ is radial if and only if for all (n, \pm) , u_n^\pm is radial. Moreover, $u \in \dot{H}^k(\mathbb{H}^1)$ belongs to $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ if and only if $-\Delta_{\mathbb{H}^1} u$ belongs to $\dot{H}^{k-2}(\mathbb{H}^1) \cap V_n^\pm$, and the same holds for $D_s u$.

For $k=0$, we get an orthogonal decomposition of the space $L^2(\mathbb{H}^1)$, and denote by Π_n^\pm the associated orthogonal projectors.

The particular space $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$ will be especially interesting in our discussion below. This space is spanned by a unique radial function h_0^+ , satisfying

$$\widehat{h_0^+}(x, y, \sigma) = \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)\sigma} \mathbf{1}_{\sigma \geq 0}.$$

Set $u \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$, then there exists f such that

$$\widehat{u}(x, y, s) = f(\sigma) \widehat{h_0^+}(x, y, \sigma),$$

and in this case

$$\|u\|_{\dot{H}^k(\mathbb{H}^1)}^2 = \int_{\mathbb{R}_+} |f(\sigma)|^2 \frac{d\sigma}{2\sigma^{1-k}}.$$

2.3 Existence of traveling waves and limiting profile

In this section, we prove the existence of ground states Q_β for equation (2.3) with speed $\beta \in (-1, 1)$ (part 2.3.1). Then, we show the convergence in $\dot{H}^1(\mathbb{H}^1)$ of the profiles Q_β to a limiting profile Q_+ as β tends to 1 (part 2.3.2). The profile Q_+ is a ground state solution of equation (2.4), which we will determine explicitly (part 2.3.3).

2.3.1 Existence of traveling waves with speed $\beta \in (-1, 1)$

A family of traveling wave solutions to the Schrödinger equation on the Heisenberg group (2.1) can be found under the form

$$u_\beta(t, x, y, s) = Q_\beta \left(\frac{x}{\sqrt{1-\beta}}, \frac{y}{\sqrt{1-\beta}}, \frac{s + \beta t}{1-\beta} \right),$$

with Q_β satisfying the equation

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} Q_\beta = |Q_\beta|^2 Q_\beta.$$

The Q_β are constructed as minimizers of some Gagliardo-Nirenberg inequalities. We will be adapting the proofs of Krieger, Lenzmann and Raphaël [KLR13] concerning the L^2 -critical half-wave equation on the real line. Our starting point is the Folland-Stein embedding.

Theorem 2.3.1 (Folland-Stein [FS74]). *Let $p \in (1, 4)$ and set $p^* = \frac{4p}{4-p}$. Then there exists $C_p > 0$ such that, for $u \in C_c^\infty(\mathbb{H}^1)$,*

$$\left(\int_{\mathbb{H}^1} |u(x, y, s)|^{p^*} dx dy ds \right)^{\frac{1}{p^*}} \leq C_p \left(\int_{\mathbb{H}^1} |(-\Delta_{\mathbb{H}^1})^{\frac{1}{2}} u(x, y, s)|^p dx dy ds \right)^{\frac{1}{p}}.$$

In particular, from the embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$, we deduce some Gagliardo-Nirenberg inequalities.

Proposition 2.3.2 (Gagliardo-Nirenberg). *Set $\beta \in (-1, 1)$. Then there exists some constant $C > 0$ such that for every $u \in \dot{H}^1(\mathbb{H}^1)$,*

$$\|u\|_{L^4(\mathbb{H}^1)}^4 \leq C (-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}.$$

Proof. Fix $u \in \dot{H}^1(\mathbb{H}^1)$, and decompose u along the spaces $V_n^+ \cup V_n^-$ as $u = \sum_{n \in \mathbb{N}} u_n$, where $u_n = u_n^+ + u_n^-$. Then

$$(-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} ((n+1)|\sigma| - \beta\sigma) |\widehat{u}_n(x, y, \sigma)|^2 dx dy d\sigma$$

and

$$\|u\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} (n+1)|\sigma| |\widehat{u}_n(x, y, \sigma)|^2 dx dy d\sigma.$$

We deduce the equivalence of norms

$$(1 - |\beta|) \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq (-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \leq (1 + |\beta|) \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2. \quad (2.5)$$

The result follows from the Folland-Stein embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$. \square

From the Gagliardo-Nirenberg inequalities, one knows that the infimum over nonzero radial functions $u \in \dot{H}^1(\mathbb{H}^1)$ of the functional

$$J_\beta(u) := \frac{(-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}}{\|u\|_{L^4(\mathbb{H}^1)}^4}$$

is positive. Let us denote by I_β the minimal value of J_β . We want to show that it is attained by some $Q_\beta \in \dot{H}^1(\mathbb{H}^1)$. We consider a minimizing sequence for J_β . Then this sequence converges to a minimizer for J_β thanks to the following profile decomposition theorem.

Definition 2.3.3. *The scaling-core pairs $((\tilde{h}_i)_{i \in \mathbb{N}}, (\tilde{s}_i)_{i \in \mathbb{N}})$ and $((h_i)_{i \in \mathbb{N}}, (s_i)_{i \in \mathbb{N}})$ of $(\mathbb{R}_+^*)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ are said to be strange if*

$$\left(\left| \log \frac{\tilde{h}_n}{h_n} \right| \xrightarrow{n \rightarrow +\infty} +\infty \right) \text{ or } \left((\tilde{h}_n)_n = (h_n)_n \text{ and } \frac{|\tilde{s}_n - s_n|}{h_n^2} \xrightarrow{n \rightarrow +\infty} +\infty \right).$$

Theorem 2.3.4 (Concentration-compactness [Ben08]). *Fix a bounded sequence $\underline{u} = (u_n)_{n \in \mathbb{N}}$ of radial functions in $\dot{H}^1(\mathbb{H}^1)$. Then there exist a subsequence $(u_{n_i})_{i \in \mathbb{N}}$, of \underline{u} , and sequences of cores $(s_{n_i}^{(j)})_{i, j \in \mathbb{N}} \subset \mathbb{R}$, scalings $(h_{n_i}^{(j)})_{i, j \in \mathbb{N}} \subset \mathbb{R}$, and radial functions $(U^{(j)})_{j \in \mathbb{N}} \subset \dot{H}^1(\mathbb{H}^1)$ satisfying the following conditions.*

1. *The pairs $((h_{n_i}^{(j)})_i, (s_{n_i}^{(j)})_i)$, $j \in \mathbb{N}$, are pairwise strange.*
2. *Let*

$$r_{n_i}^{(l)}(x, y, s) = u_{n_i}(x, y, s) - \sum_{j=1}^l \frac{1}{h_{n_i}^{(j)}} U^{(j)} \left(\frac{x}{h_{n_i}^{(j)}}, \frac{y}{h_{n_i}^{(j)}}, \frac{s - s_{n_i}^{(j)}}{(h_{n_i}^{(j)})^2} \right),$$

then

$$\lim_{l \rightarrow +\infty} \limsup_{i \rightarrow +\infty} \|r_{n_i}^{(l)}\|_{L^4(\mathbb{H}^1)} = 0.$$

Moreover, for all $l \geq 1$, one has the following orthogonality relations as i goes to $+\infty$:

$$\|u_{n_i}\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \sum_{j=1}^l \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \|r_{n_i}^{(l)}\|_{\dot{H}^1(\mathbb{H}^1)}^2 + o(1),$$

$$(D_s u_{n_i}, u_{n_i})_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \sum_{j=1}^l (D_s U^{(j)}, U^{(j)})_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} + (D_s r_{n_i}^{(l)}, r_{n_i}^{(l)})_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} + o(1),$$

and

$$\|u_{n_i}\|_{L^4(\mathbb{H}^1)}^4 \xrightarrow{i \rightarrow +\infty} \sum_{j=1}^{+\infty} \|U^{(j)}\|_{L^4(\mathbb{H}^1)}^4.$$

This result is an adaptation of a concentration-compactness argument due to Cazenave and Lions [CL82], which was refined into a profile decomposition theorem as above by Gérard [Gér98] for Sobolev spaces on \mathbb{R}^n . One can find a proof of this profile decomposition theorem for Sobolev spaces on the Heisenberg group in Benameur's work [Ben08], which is here restricted to the subspace of radial functions.

2.3.2 The limit $\beta \rightarrow 1^-$

In this part, we study the behavior of the traveling waves Q_β as β tends to the limit 1^- . We show that these traveling waves converge up to symmetries to a limiting profile. The strategy is similar to [Gér+18] for the half-wave equation.

For $\beta \in (-1, 1)$, let Q_β be a minimizer of J_β and $I_\beta = J_\beta(Q_\beta)$. Up to a change of functions $Q_\beta \rightsquigarrow \alpha Q_\beta$, one can choose Q_β such that

$$\frac{(-(\Delta_{\mathbb{H}^1} + \beta D_s)Q_\beta, Q_\beta)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}}{1 - \beta} = \|Q_\beta\|_{L^4(\mathbb{H}^1)}^4,$$

so that Q_β is a solution to equation (2.3).

Definition 2.3.5 (Minimizers in \mathcal{Q}_β). *For all $\beta \in (-1, 1)$, denote by \mathcal{Q}_β the set of minimizers Q_β of $J_\beta : u \mapsto \frac{(-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}}{\|u\|_{L^4}^4}$ which satisfy*

$$\frac{(-(\Delta_{\mathbb{H}^1} + \beta D_s)Q_\beta, Q_\beta)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}}{1 - \beta} = \|Q_\beta\|_{L^4}^4 = \frac{I_\beta}{(1 - \beta)^2}, \quad I_\beta = J_\beta(Q_\beta). \quad (2.6)$$

Note that equation (2.3) is satisfied for $Q_\beta \in \mathcal{Q}_\beta$

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta = |Q_\beta|^2 Q_\beta.$$

Definition 2.3.6 (Minimizers in \mathcal{Q}_+). *For all radial functions $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \setminus \{0\}$ whose Fourier transform have a nonzero component only along the Hermite-type function $\widehat{h_0^+}$, define*

$$J_+(u) := \frac{\|u\|_{\dot{H}^1(\mathbb{H}^1)}^4}{\|u\|_{L^4(\mathbb{H}^1)}^4}$$

(note that $-\Delta_{\mathbb{H}^1} = D_s$ on the space $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$). Denote by I_+ its infimum

$$I_+ := \inf \left\{ J_+(u) \mid u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \setminus \{0\} \right\}.$$

Let \mathcal{Q}_+ be the set of minimizers Q_+ of J_+ such that

$$\|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \|Q_+\|_{L^4(\mathbb{H}^1)}^4 = I_+, \quad I_+ = J_+(Q_+).$$

Then any $Q_+ \in \mathcal{Q}_+$ is a solution to equation (2.4)

$$D_s Q_+ = \Pi_0^+(|Q_+|^2 Q_+).$$

The minimum I_+ is attained and positive. The proof is similar to the one for the minimum I_β , all there is to do is to restrict the profile decomposition theorem to the closed subspace $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$ of $\dot{H}^1(\mathbb{H}^1)$.

The term $\Pi_0^+(|Q_+|^2 Q_+)$ may not seem suitable since $|Q_+|^2 Q_+$ belongs to $L^{\frac{4}{3}}(\mathbb{H}^1) \hookrightarrow \dot{H}^{-1}(\mathbb{H}^1)$ whereas Π_0^+ is a projector defined on $L^2(\mathbb{H}^1)$. Several arguments give sense to this term in later parts. On the one hand, we will see that $|Q_+|^2 Q_+ \in L^2(\mathbb{H}^1)$ (see part 2.3.3). On the other hand, the projector Π_0^+ extends to $L^p(\mathbb{H}^1)$ for all $p > 1$ (see Theorem 2.4.6).

We establish the following convergence result.

Theorem 2.3.7 (Convergence). *For all $\beta \in (-1, 1)$, fix $Q_\beta \in \mathcal{Q}_\beta$. Then, there exist a subsequence $\beta_n \rightarrow 1^-$, scalings $(\alpha_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, cores $(s_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and a function $Q_+ \in \mathcal{Q}_+$ such that*

$$\|\alpha_n Q_{\beta_n}(\alpha_n \cdot, \alpha_n \cdot, \alpha_n^2(\cdot + s_n)) - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0.$$

We first introduce the quantity $\delta(u)$, which quantifies the gap between the norms of a function u in $\dot{H}^1(\mathbb{H}^1)$ and those of the profiles $Q_+ \in \mathcal{Q}_+$. We prove that $\delta(Q_\beta)$ is small, and then show that $\delta(u)$ controls the distance up to symmetries from u to the profiles $Q_+ \in \mathcal{Q}_+$.

Definition 2.3.8. *For $u \in \dot{H}^1(\mathbb{H}^1)$, define*

$$\delta(u) = \left| \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2 - I_+ \right| + \left| \|u\|_{L^4(\mathbb{H}^1)}^4 - I_+ \right|.$$

Let us show a lemma about $\delta(Q_\beta)$, $Q_\beta \in \mathcal{Q}_\beta$.

Lemma 2.3.9. *There exist $C > 0$ and $\beta_* \in (0, 1)$ such that the following holds. For all $\beta \in (\beta_*, 1)$ fix $Q_\beta \in \mathcal{Q}_\beta$, and decompose Q_β along the Hermite-type functions from part 2.2.2*

$$Q_\beta = Q_\beta^+ + R_\beta,$$

where $Q_\beta^+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $R_\beta \in \dot{H}^1(\mathbb{H}^1) \cap \bigoplus_{(n,\pm) \neq (0,+)} V_n^\pm$. Then $\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \leq C(1-\beta)^{\frac{1}{2}}$, $\delta(Q_\beta^+) \leq C(1-\beta)^{\frac{1}{2}}$ and $\delta(Q_\beta) \leq C(1-\beta)^{\frac{1}{2}}$.

Proof. Fix $u \in \dot{H}^1(\mathbb{H}^1)$. Thanks to inequality (2.5)

$$(1 - |\beta|) \|u\|_{\dot{H}^1}^2 \leq (-\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \leq (1 + |\beta|) \|u\|_{\dot{H}^1}^2,$$

one knows that $I_\beta \geq (1 - \beta)^2 I_0$ when $\beta \in (0, 1)$.

Furthermore, let $Q_+ \in \mathcal{Q}_+$. Then, using the fact that $-\Delta_{\mathbb{H}^1} Q_+ = D_s Q_+$,

$$\begin{aligned} I_\beta &\leq J_\beta(Q_+) \\ &= \frac{(1 - \beta)^2 (D_s Q_+, Q_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}}{\|Q_+\|_{L^4}^4} \\ &= (1 - \beta)^2 I_+. \end{aligned}$$

Consequently, $(\frac{I_\beta}{(1-\beta)^2})_\beta$ is bounded above and below:

$$I_0 \leq \frac{I_\beta}{(1-\beta)^2} \leq I_+.$$

We will show that actually $\frac{I_\beta}{(1-\beta)^2} \rightarrow I_+$ as $\beta \rightarrow 1$.

Let us decompose a minimizer $Q_\beta \in \mathcal{Q}_\beta$ along the Hermite-type functions: $Q_\beta = Q_\beta^+ + R_\beta$, where $Q_\beta^+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, and $R_\beta \in \dot{H}^1(\mathbb{H}^1) \cap \bigoplus_{(n,\pm) \neq (0,+)} V_n^\pm$ is a remainder term which will go to zero.

Multiplying equation (2.3) by $\overline{R_\beta}$, we get that for all n ,

$$\left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} Q_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = (|Q_\beta|^2 Q_\beta, R_\beta)_{L^{\frac{4}{3}}(\mathbb{H}^1) \times L^4(\mathbb{H}^1)}.$$

Since the operators $\Delta_{\mathbb{H}^1}$ and D_s leave invariant the spaces V_n^\pm , we can replace Q_β by R_β in the left term of the equality

$$\left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} R_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = (|Q_\beta|^2 Q_\beta, R_\beta)_{L^{\frac{4}{3}}(\mathbb{H}^1) \times L^4(\mathbb{H}^1)}.$$

Applying Hölder's inequality, we deduce that

$$\left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} R_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \leq \|Q_\beta\|_{L^4(\mathbb{H}^1)}^3 \|R_\beta\|_{L^4(\mathbb{H}^1)}. \quad (2.7)$$

We now write more precisely the equivalence of norms (2.5) between the norms $\|u\|_{\dot{H}^1(\mathbb{H}^1)}$ and $(-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}^{\frac{1}{2}}$. The left inequality can be controlled with sharper constants which do not depend on β when we require the function $u \in \dot{H}^1(\mathbb{H}^1)$ to have a zero component u_0^+ . Indeed, remark that when $n \geq 1$,

$$n+1-\beta \geq n \geq (n+1)/2,$$

and when $n \geq 0$,

$$n+1+\beta \geq n+1 \geq (n+1)/2.$$

We deduce that for all $u \in \dot{H}^1(\mathbb{H}^1) \cap \bigoplus_{(n,\pm) \neq (0,+)} V_n^\pm$, decomposed as $u = \sum_{(n,\pm) \neq (0,+)} u_n^\pm$, $u_n^\pm \in \dot{H}^1(\mathbb{H}^1) \cap V_n^\pm$, then

$$\begin{aligned} (-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= \sum_{(n,\pm) \neq (0,+)} \int_{\mathbb{R}^3} ((n+1)|\sigma| - \beta\sigma) |\widehat{u}_n^\pm(x, y, \sigma)|^2 dx dy d\sigma \\ &\geq \frac{1}{2} \sum_{(n,\pm) \neq (0,+)} \int_{\mathbb{R}^3} (n+1)|\sigma| |\widehat{u}_n^\pm(x, y, \sigma)|^2 dx dy d\sigma. \end{aligned}$$

This implies the inequality

$$\|u\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq 2 (-(\Delta_{\mathbb{H}^1} + \beta D_s)u, u)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}, \quad u \in \dot{H}^1(\mathbb{H}^1) \cap \bigoplus_{(n,\pm) \neq (0,+)} V_n^\pm, \quad (2.8)$$

which we can use for $u = R_\beta$. Combining this inequality and the Folland-Stein inequality $\|u\|_{L^4(\mathbb{H}^1)} \leq C\|u\|_{\dot{H}^1(\mathbb{H}^1)}$ in (2.7), we get

$$\begin{aligned} & \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} R_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ & \leq C \|Q_\beta\|_{L^4(\mathbb{H}^1)}^3 \left(2(1 - \beta) \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} R_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \right)^{\frac{1}{2}}, \end{aligned}$$

so

$$\left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} R_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \leq 2C^2(1 - \beta) \|Q_\beta\|_{L^4(\mathbb{H}^1)}^6.$$

Since $(\|Q_\beta\|_{L^4(\mathbb{H}^1)})_\beta$ is bounded independently of β thanks to the norm conditions (2.6) and the boundedness of $(\frac{I_\beta}{(1-\beta)^2})_\beta$, we deduce that as β goes to 1,

$$\left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} R_\beta, R_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \mathcal{O}(1 - \beta).$$

This implies immediately that $\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \mathcal{O}(1 - \beta)$ and $\|R_\beta\|_{L^4(\mathbb{H}^1)}^2 = \mathcal{O}(1 - \beta)$. Using the orthogonal decomposition $Q_\beta = Q_\beta^+ + R_\beta$ in $\dot{H}^1(\mathbb{H}^1)$ and the fact that $-\Delta_{\mathbb{H}^1} = D_s$ on $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$, we get

$$\begin{aligned} \|Q_\beta^+\|_{\dot{H}^1(\mathbb{H}^1)}^2 &= \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta^+, Q_\beta^+ \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ &= \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta, Q_\beta \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} + \mathcal{O}(1 - \beta) \\ &= \frac{I_\beta}{(1 - \beta)^2} + \mathcal{O}(1 - \beta), \end{aligned}$$

and

$$\begin{aligned} \|Q_\beta^+\|_{L^4(\mathbb{H}^1)}^4 &= \|Q_\beta\|_{L^4(\mathbb{H}^1)}^4 + \mathcal{O}((1 - \beta)^{\frac{1}{2}}) \\ &= \frac{I_\beta}{(1 - \beta)^2} + \mathcal{O}((1 - \beta)^{\frac{1}{2}}). \end{aligned}$$

We can now prove that $\frac{I_\beta}{(1-\beta)^2} \xrightarrow{\beta \rightarrow 1^-} I_+$. From the definition of I_+ as a minimum on $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$,

$$\begin{aligned} I_+ &\leq \frac{\|Q_\beta^+\|_{\dot{H}^1(\mathbb{H}^1)}^4}{\|Q_\beta^+\|_{L^4(\mathbb{H}^1)}^4} \\ &= \frac{\left(\frac{I_\beta}{(1-\beta)^2} + \mathcal{O}(1 - \beta) \right)^2}{\frac{I_\beta}{(1-\beta)^2} + \mathcal{O}((1 - \beta)^{\frac{1}{2}})} \\ &= \frac{I_\beta}{(1 - \beta)^2} (1 + \mathcal{O}((1 - \beta)^{\frac{1}{2}})). \end{aligned}$$

We already know that $\frac{I_\beta}{(1-\beta)^2} \leq I_+$ for all β , so we conclude that

$$\frac{I_\beta}{(1-\beta)^2} \xrightarrow{\beta \rightarrow 1^-} I_+.$$

Moreover, the norms of Q_β^+ can be written as $\|Q_\beta^+\|_{\dot{H}^1(\mathbb{H}^1)}^2 = I_+ + \mathcal{O}((1-\beta)^{\frac{1}{2}})$ and $\|Q_\beta^+\|_{L^4(\mathbb{H}^1)}^4 = I_+ + \mathcal{O}((1-\beta)^{\frac{1}{2}})$. We conclude that

$$\delta(Q_\beta^+) = \mathcal{O}((1-\beta)^{\frac{1}{2}})$$

and

$$\delta(Q_\beta) = \delta(Q_\beta^+ + R_\beta) = \mathcal{O}((1-\beta)^{\frac{1}{2}}). \quad \square$$

The following stability result allows us to complete the proof of Theorem 2.3.7.

Proposition 2.3.10. *Fix a sequence $(u_n)_{n \in \mathbb{N}}$ of radial functions in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$. Suppose that $\delta(u_n) \xrightarrow{n \rightarrow +\infty} 0$. Then, up to a subsequence, there exist scalings $(\alpha_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, cores $(s_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and a ground state $Q_+ \in \mathcal{Q}_+$ optimizing*

$$I_+ = \inf \left\{ J_+(u) = \frac{\|u\|_{\dot{H}^1(\mathbb{H}^1)}^4}{\|u\|_{L^4(\mathbb{H}^1)}^4} \mid u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \setminus \{0\} \right\},$$

such that

$$\left\| \alpha_n u_n(\alpha_n \cdot, \alpha_n \cdot, \alpha_n^2(\cdot + s_n)) - Q_+ \right\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. Let $(u_n)_{n \in \mathbb{N}} \in (\dot{H}^1(\mathbb{H}^1) \cap V_0^+)^{\mathbb{N}}$ such that $\delta(u_n) \xrightarrow{n \rightarrow +\infty} 0$. Since $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$ is a closed subspace of $\dot{H}^1(\mathbb{H}^1)$, one can restrict the concentration-compactness theorem 2.3.4 to this subspace. In consequence, one can assume that the profiles $U^{(j)}$ from the theorem lie in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$. Therefore, up to a subsequence, there exist a core sequence $(s_n^{(j)})_{n,j \in \mathbb{N}} \subset \mathbb{R}$, a scaling sequence $(h_n^{(j)})_{n,j \in \mathbb{N}} \subset \mathbb{R}$, and radial functions $(U^{(j)})_{j \in \mathbb{N}} \subset \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ such that

- for all $j, k \in \mathbb{N}$, $j \neq k$, the pairs $((h_n^{(j)})_n, (s_n^{(j)})_n)$ are pairwise strange;
- let

$$r_n^{(l)}(x, y, s) = u_n(x, y, s) - \sum_{j=1}^l \frac{1}{h_n^{(j)}} U^{(j)} \left(\frac{x}{h_n^{(j)}}, \frac{y}{h_n^{(j)}}, \frac{s - s_n^{(j)}}{(h_n^{(j)})^2} \right),$$

then

$$\lim_{l \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|r_n^{(l)}\|_{L^4(\mathbb{H}^1)} = 0.$$

Moreover, for all l , as n goes to $+\infty$,

$$\|u_n\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \sum_{j=1}^l \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \|r_n\|_{\dot{H}^1(\mathbb{H}^1)}^2 + o(1) \quad (2.9)$$

and

$$\|u_n\|_{L^4(\mathbb{H}^1)}^4 \xrightarrow{n \rightarrow +\infty} \sum_{j=1}^{+\infty} \|U^{(j)}\|_{L^4(\mathbb{H}^1)}^4.$$

By construction, since $\delta(u_n)$ goes to 0, $\sum_{j=1}^{+\infty} \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^4 = I_+$, $\sum_{j=1}^{+\infty} \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq I_+$ and $\frac{\|u_n\|_{\dot{H}^1(\mathbb{H}^1)}^4}{\|u_n\|_{L^4(\mathbb{H}^1)}^4}$ tends to I_+ . But from the definition of I_+ as a minimum,

$$\begin{aligned} I_+^2 &\geq \left(\sum_{j=1}^{+\infty} \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^2 \right)^2 \geq \sum_{j=1}^{+\infty} \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^4 \\ &\geq I_+ \sum_{j=1}^{+\infty} \|U^{(j)}\|_{\dot{H}^1(\mathbb{H}^1)}^4 \geq I_+ \sum_{j=1}^{+\infty} \|U^{(j)}\|_{L^4(\mathbb{H}^1)}^4 = I_+^2. \end{aligned}$$

All the above inequalities must then be equalities.

In particular, because of the second inequality, only one of the profiles $U^{(j)}$ is allowed to be nonzero. We denote this profile by Q_+ , and by r_n, h_n and s_n the corresponding rests, scalings and cores. Then Q_+ must be a ground state of the functional J_+ , and

$$u_n(x, y, s) = \frac{1}{h_n} Q_+ \left(\frac{x}{h_n}, \frac{y}{h_n}, \frac{s - s_n}{h_n^2} \right) + r_n(x, y, s).$$

From relation (2.9), as n goes to $+\infty$,

$$\|u_n\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \|r_n\|_{\dot{H}^1(\mathbb{H}^1)}^2 + o(1).$$

Since $\|u_n\|_{\dot{H}^1(\mathbb{H}^1)}^2$ must converge to $\|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2$ because the inequalities turned to equalities, we get that $\|r_n\|_{\dot{H}^1(\mathbb{H}^1)}^2 \xrightarrow{n \rightarrow +\infty} 0$, and the sequence $h_n u_n(h_n \cdot, h_n \cdot, h_n^2(\cdot + s_n))$ converges to Q_+ in $\dot{H}^1(\mathbb{H}^1)$. \square

Proof of Theorem 2.3.7. Consider the sequence $(Q_\beta^+)_{\beta \in (-1, 1)}$ from Lemma 2.3.9. We know that $\delta(Q_\beta^+) = \mathcal{O}((1 - \beta)^{\frac{1}{2}})$.

Applying Proposition 2.3.10, there exist a subsequence $(Q_{\beta_n}^+)_{n \in \mathbb{N}}$ with $\beta_n \xrightarrow{n \rightarrow +\infty} 1^-$, a core sequence $(s_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, a scaling sequence $(\alpha_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, and a ground state $Q_+ \in \mathcal{Q}_+$ such that

$$\|\alpha_n Q_{\beta_n}^+(\alpha_n \cdot, \alpha_n \cdot, \alpha_n^2(\cdot + s_n)) - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0.$$

To conclude, since $R_{\beta_n} = Q_{\beta_n} - Q_{\beta_n}^+$ satisfies $\|R_{\beta_n}\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0$, and since the \dot{H}^1 norm is invariant by translation and scaling, we deduce that

$$\|\alpha_n Q_{\beta_n}(\alpha_n \cdot, \alpha_n \cdot, \alpha_n^2(\cdot + s_n)) - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0. \quad \square$$

2.3.3 Ground state solutions to the limiting equation

We now show that the optimizers for

$$I_+ := \inf \left\{ \frac{\|u\|_{\dot{H}^1(\mathbb{H}^1)}^4}{\|u\|_{L^4(\mathbb{H}^1)}^4} \mid u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \setminus \{0\} \right\}$$

are unique up to symmetries (translation, phase multiplication and scaling).

Proposition 2.3.11. *The minimum I_+ is equal to π^2 . Moreover,*

- *the set composed of all minimizing functions for I_+ is*

$$\left\{ (x, y, s) \in \mathbb{H}^1 \mapsto \frac{C}{s + s_0 + i(x^2 + y^2) + i\alpha} \mid (s_0, C, \alpha) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R}_+^* \right\};$$

- *the set \mathcal{Q}_+ composed of all minimizing functions for I_+ which satisfy*

$$\|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \|Q_+\|_{L^4(\mathbb{H}^1)}^4 = I_+$$

(so that Q_+ is a solution to equation (2.4)) is

$$\mathcal{Q}_+ = \left\{ (x, y, s) \in \mathbb{H}^1 \mapsto \frac{i e^{i\theta} \sqrt{2\alpha}}{s + s_0 + i(x^2 + y^2) + i\alpha} \mid (s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^* \right\}.$$

Proof. Let $U \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$. Let us transform the expression of the L^4 norm of U as follows

$$\|U\|_{L^4(\mathbb{H}^1)}^4 = \|U^2\|_{L^2(\mathbb{H}^1)}^2 = \|\widehat{U^2}\|_{L^2(\mathbb{H}^1)}^2 = \frac{1}{2\pi} \|\widehat{U} * \widehat{U}\|_{L^2(\mathbb{H}^1)}^2.$$

Let f be the function associated to U in the decomposition along \widehat{h}_0^+

$$\widehat{U}(x, y, s) = f(\sigma) \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)\sigma}.$$

Then

$$\|U\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \frac{1}{2} \int_0^{+\infty} |f(\sigma)|^2 d\sigma$$

and

$$\begin{aligned} \|U\|_{L^4(\mathbb{H}^1)}^4 &= \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_0^\sigma f(\sigma - \sigma') \frac{e^{-(x^2+y^2)(\sigma-\sigma')}}{\sqrt{\pi}} f(\sigma') \frac{e^{-(x^2+y^2)\sigma'}}{\sqrt{\pi}} d\sigma' \right|^2 dx dy d\sigma \\ &= \frac{1}{2\pi^2} \int_{\mathbb{R}_+} \left| \int_0^\sigma f(\sigma - \sigma') f(\sigma') d\sigma' \right|^2 \frac{d\sigma}{2\sigma}. \end{aligned}$$

Applying Cauchy-Schwarz's inequality,

$$\begin{aligned} \|U\|_{L^4(\mathbb{H}^1)}^4 &\leq \frac{1}{4\pi^2} \int_0^{+\infty} \int_0^\sigma |f(\sigma - \sigma') f(\sigma')|^2 d\sigma' \int_0^\sigma 1 d\sigma' \frac{d\sigma}{\sigma} \\ &= \frac{1}{4\pi^2} \int_0^{+\infty} \int_0^\sigma |f(\sigma - \sigma') f(\sigma')|^2 d\sigma' d\sigma \\ &= \frac{1}{4\pi^2} \|f\|_{L^2(\mathbb{R}_+)}^4 \\ &= \frac{1}{\pi^2} \|U\|_{\dot{H}^1(\mathbb{H}^1)}^4. \end{aligned}$$

Consequently, $I_+ \geq \pi^2$.

Equality holds if and only if there is equality in Cauchy-Schwarz's inequality, that is to say, for almost every $\sigma > 0$, and almost every $\sigma' \in]0, \sigma[$,

$$f(\sigma')f(\sigma - \sigma') = C(\sigma).$$

Fix an open interval I contained in $]0, \sigma[$ with positive length $|I|$. Then

$$\int_I f(\sigma')f(\sigma - \sigma') d\sigma' = |I|C(\sigma).$$

Therefore, C is continuous on \mathbb{R}_+^* as a convolution product of two L^2 functions. Since f is not identically zero, one can find an interval $J \subset \mathbb{R}_+^*$ such that

$$\int_J f(\zeta) d\zeta \neq 0.$$

Integrating the equality

$$f(\sigma)f(\zeta) = C(\sigma + \zeta), \quad (\sigma, \zeta) \in (\mathbb{R}_+^*)^2 \quad (2.10)$$

along the variable ζ , one gets that for all $\sigma \in \mathbb{R}_+^*$,

$$f(\sigma) \int_J f(\zeta) d\zeta = \int_J C(\sigma + \zeta) d\zeta = \int_{J+\sigma} C(\zeta) d\zeta.$$

Therefore, f has \mathcal{C}^1 regularity on \mathbb{R}_+^* , so C also has \mathcal{C}^1 regularity on \mathbb{R}_+^* . Fix $\zeta > 0$ such as $f(\zeta) \neq 0$. Letting $\sigma \rightarrow 0^+$ in (2.10), one knows that f admits a finite limit as $\sigma \rightarrow 0^+$ which is equal to

$$f(0^+) = \frac{C(\zeta)}{f(\zeta)}.$$

Likewise, computing the derivative along the variable σ of (2.10), there holds $f'(\sigma)f(\zeta) = C'(\sigma + \zeta)$, and one gets that f' admits a finite limit at 0^+ which is equal to

$$f'(0^+) = \frac{C'(\zeta)}{f(\zeta)}.$$

We deduce that f satisfies the differential equation

$$f'(\sigma)f(0^+) = f(\sigma)f'(0^+) = C'(\sigma), \quad \sigma \in \mathbb{R}_+^*.$$

Let us show that $f(0^+) \neq 0$. Supposing $f(0^+) = 0$, we would get that for all $\sigma > 0$, $C'(\sigma) = 0$. Then C would be a constant function, so f would be constant too since

$$f(\sigma) = \frac{C(\sigma + \zeta)}{f(\zeta)}.$$

As f is in $L^2(\mathbb{R}_+)$, this would imply that f is identically zero, a contradiction. Therefore, solving the differential equation, there exist some constants K and α such that, for all $\sigma \geq 0$,

$$f(\sigma) = K e^{-\alpha\sigma}.$$

The assumption $f \in L^2(\mathbb{R}_+)$ implies that $\text{Re}(\alpha) > 0$.

Computing the inverse Fourier transform leads to

$$\begin{aligned} U(x, y, s) &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{is\sigma} f(\sigma) \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)\sigma} d\sigma \\ &= \frac{K}{\pi\sqrt{2}} \int_0^{+\infty} e^{is\sigma - \alpha\sigma - (x^2+y^2)\sigma} d\sigma \end{aligned}$$

so

$$U(x, y, s) = \frac{K}{\pi\sqrt{2}} \frac{1}{x^2 + y^2 + \alpha - is}.$$

This is the first point of the proposition. Let us now prove the second point.

Since the equation and the result we want to show are both invariant under translation of the variable s , up to translating of a factor s_0 , we will assume from now on that α is a (positive) real number.

Now, there holds

$$\|U\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \frac{1}{2} |K|^2 \int_0^{+\infty} e^{-2\alpha\sigma} d\sigma = \frac{|K|^2}{4\alpha},$$

$$\begin{aligned} \|U\|_{L^4(\mathbb{H}^1)}^4 &= \frac{1}{4\pi^2} |K|^4 \int_0^{+\infty} \left| \int_0^\sigma e^{-\alpha(\sigma-\sigma')} e^{-\alpha\sigma'} d\sigma' \right|^2 \frac{d\sigma}{\sigma} \\ &= \frac{1}{4\pi^2} |K|^4 \int_0^{+\infty} \sigma e^{-2\alpha\sigma} d\sigma \\ &= \frac{1}{4\pi^2} \frac{|K|^4}{(2\alpha)^2}, \end{aligned}$$

so U satisfies $\|U\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \|U\|_{L^4(\mathbb{H}^1)}^4 = I_+$ if and only if $|K|^2 = 4\pi^2\alpha$. In this case, write $K = 2\pi\sqrt{\alpha} e^{i\theta}$ for some $\theta \in \mathbb{T}$, then,

$$U(x, y, s) = \frac{K}{\pi\sqrt{2}} \frac{1}{x^2 + y^2 + \alpha - is} = \frac{e^{i\theta}\sqrt{2\alpha}}{x^2 + y^2 + \alpha - is}. \quad \square$$

We proved that up to the symmetries of the equation, there is a unique minimizer Q_+ in \mathcal{Q}_+ , which is equal with the choice of parameters $(s_0, \theta, \alpha) = (0, 0, 1)$ to

$$Q_+(x, y, s) = \frac{i\sqrt{2}}{s + i(x^2 + y^2) + i},$$

with Fourier transform

$$\widehat{Q}_+(x, y, \sigma) = 2\pi e^{-\sigma} \widehat{h}_0^+(x, y, \sigma).$$

Note that the profile Q_+ has infinite mass.

2.4 The limiting problem

We now focus on the stability of Q_+ , which is the unique ground state solution up to symmetry to (2.4)

$$D_s Q_+ = \Pi_0^+(|Q_+|^2 Q_+).$$

Let us study the linearized operator \mathcal{L} close to Q_+

$$\mathcal{L}h = D_s h - 2\Pi_0^+(|Q_+|^2 h) - \Pi_0^+(Q_+^2 \bar{h}), \quad h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+.$$

We first consider the linearized operator on the real subspace spanned by the set of vectors $(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)$ with the help of the correspondence with Bergman spaces (parts 2.4.1 and 2.4.2). Then, on the orthogonal of this subspace in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$, we prove the coercivity of \mathcal{L} by using the spectral properties of the sub-Laplacian on the CR sphere via the Cayley transform (parts 2.4.3 and 2.4.4). We conclude this section with some estimates about the invertibility of the linearized operator \mathcal{L} (part 2.4.5).

2.4.1 Bergman spaces on the upper half plane

In order to better understand the spaces $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$, $k \in \{-1, 0, 1\}$, we introduce their link with Bergman spaces on the upper half-plane \mathbb{C}_+ . The space $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$ is the subspace of $\dot{H}^k(\mathbb{H}^1)$ spanned (after a Fourier transform under the variable s) by $\widehat{h}_0^+(x, y, \sigma) = \frac{1}{\sqrt{\pi}} \exp(-(x^2 + y^2)\sigma) \mathbf{1}_{\sigma \geq 0}$, so that $u \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$ if $u \in \dot{H}^k(\mathbb{H}^1)$ and

$$\widehat{u}(x, y, s) = f(\sigma) \widehat{h}_0^+(x, y, \sigma),$$

in this case,

$$\|u\|_{\dot{H}^k(\mathbb{H}^1)}^2 = \|(-\Delta_{\mathbb{H}^1})^{\frac{k}{2}} u\|_{L^2(\mathbb{H}^1)}^2 = \int_{\mathbb{R}_+} |f(\sigma)|^2 \frac{d\sigma}{2\sigma^{1-k}}.$$

Definition 2.4.1 (Weighted Bergman spaces). *Given $k < 1$ and $p \in [1, +\infty)$, the weighted Bergman space A_{1-k}^p is the subspace of $L_{1-k}^p := L^p(\mathbb{C}_+, \text{Im}(z)^{-k} d\lambda(z))$ composed of holomorphic functions of the complex upper half-plane \mathbb{C}_+ :*

$$A_{1-k}^p := \left\{ F \in \text{Hol}(\mathbb{C}_+) \mid \|F\|_{L_{1-k}^p}^p := \int_0^{+\infty} \int_{\mathbb{R}} |F(s+it)|^p ds \frac{dt}{t^k} < +\infty \right\}.$$

Thanks to the following Paley-Wiener theorem on weighted Bergman spaces, one can associate to each element of $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$ a function of the weighted Bergman space A_{1-k}^2 .

Theorem 2.4.2 (Paley-Wiener [Bék+12], Theorem 1.22). *Let $k < 1$. Then for all $f \in L^2(\mathbb{R}_+, \sigma^{k-1} d\sigma)$, the following integral is absolutely convergent on \mathbb{C}_+*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma, \quad (2.11)$$

and defines a function $F \in A_{1-k}^2$ which satisfies

$$\|F\|_{L_{1-k}^2}^2 = \frac{\Gamma(1-k)}{2^{1-k}} \int_0^{+\infty} |f(\sigma)|^2 \frac{d\sigma}{\sigma^{1-k}}. \quad (2.12)$$

Conversely, for every $F \in A_{1-k}^2$, there exists $f \in L^2(\mathbb{R}_+, \sigma^{k-1} d\sigma)$ such that (2.11) and (2.12) hold.

When dealing with functions from the space $\dot{H}^1(\mathbb{H}^1)$, we use the usual Paley-Wiener theorem.

Definition 2.4.3. *The Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ is the space of holomorphic functions of the upper half-plane \mathbb{C}_+ such that the following norm is finite:*

$$\|F\|_{\mathcal{H}^2(\mathbb{C}_+)}^2 := \sup_{t>0} \int_{\mathbb{R}} |F(s+it)|^2 ds < +\infty.$$

Theorem 2.4.4 (Paley-Wiener [Rud87]). *For every $f \in L^2(\mathbb{R}_+)$, the following integral is absolutely convergent on \mathbb{C}_+*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma, \quad (2.13)$$

and defines a function F in the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ which satisfies

$$\|F\|_{\mathcal{H}^2(\mathbb{C}_+)}^2 = \int_0^{+\infty} |f(\sigma)|^2 d\sigma. \quad (2.14)$$

Conversely, for every $F \in \mathcal{H}^2(\mathbb{C}_+)$, there exists $f \in L^2(\mathbb{R}_+)$ such that (2.13) and (2.14) hold.

Given any $h \in \dot{H}^k(\mathbb{H}^1)$ radial, one can define

$$F_h(s + i(x^2 + y^2)) := h(x, y, s).$$

If $h \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$, $k \in \{-1, 0, 1\}$, then F_h is holomorphic, since the holomorphic representation given by the suitable Paley-Wiener theorem is given by $\sqrt{\pi}F_h$. Note that

$$F_{D_s h} = -iF_{\partial_s h} = -iF'_h, \quad h \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$$

and

$$F_{gh} = F_g F_h, \quad g, h \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+.$$

Moreover, if $h \in L^2(\mathbb{H}^1)$,

$$\|h\|_{L^2(\mathbb{H}^1)}^2 = \pi \|F_h\|_{L^2(\mathbb{C}_+)}^2. \quad (2.15)$$

For example, the holomorphic representation in the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ of

$$Q_+(x, y, s) = \frac{i\sqrt{2}}{i(x^2 + y^2) + i + s}$$

is

$$F_{Q_+}(z) = \frac{i\sqrt{2}}{z + i}.$$

One can now identify the orthogonal projector Π_0^+ from the Hilbert space $L^2(\mathbb{H}^1)$ onto its closed subspace $L^2(\mathbb{H}^1) \cap V_0^+$ as a projector P_0 from $L^2(\mathbb{C}_+)$ onto $A_1^2 = L^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. More generally, for $k < 1$, the orthogonal projector from the Hilbert space $\dot{H}^k(\mathbb{H}^1)$ onto its closed subspace $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$ corresponds to the Bergman projector P_k from L_{1-k}^2 onto A_{1-k}^2 . For general $k < 1$, the Bergman projector P_k can be expressed as a convolution through a reproducing kernel called the Bergman kernel [Bék+12]. We are here interested in the case $k = 0$.

Proposition 2.4.5 (See [Bék+12], Proposition 1.32). *For all $F \in L^2(\mathbb{C}_+)$,*

$$P_0(F)(z) = -\frac{1}{\pi} \int_{\mathbb{C}_+} \frac{1}{(z - s + it)^2} F(s + it) ds dt.$$

For $h \in L^2(\mathbb{H}^1)$, the holomorphic function $F_{\Pi_0^+(h)}$ is the projection of F_h on the subspace A_1^2 of $L^2(\mathbb{C}_+)$:

$$F_{\Pi_0^+(h)}(z) = P_0(F_h)(z),$$

hence

$$F_{\Pi_0^+(h)}(z) = -\frac{1}{\pi} \int_{\mathbb{C}_+} \frac{1}{(z-s+it)^2} F_h(s+it) ds dt.$$

For $p \in (1, +\infty)$, the orthogonal projector P_0 can be extended as a bounded operator from the space $L^p(\mathbb{C}_+, d\lambda(z))$ onto the Bergman space A_1^p .

Theorem 2.4.6 (See [Bék+12], Theorem 1.34). *Let $p \in [1, +\infty)$. Then the Bergman projector P_0 is a bounded operator in $L^p(\mathbb{C}_+)$ if and only if $p > 1$.*

One has $\|h\|_{L^p(\mathbb{H}^1)}^p = \pi \|F_h\|_{L^p(\mathbb{C}_+)}^p$ when this quantity is finite. Therefore, if h_1, h_2, h_3 lie in $\dot{H}^1(\mathbb{H}^1)$ (which embeds in $L^4(\mathbb{H}^1)$), it makes sense to consider $\Pi_0^+(h_1 h_2 h_3)$.

2.4.2 Symmetries of the equation and orthogonality conditions

In this part, we decompose the linearized operator \mathcal{L} around Q_+

$$\mathcal{L}h = -\Delta_{\mathbb{H}^1} h - 2\Pi_0^+(|Q_+|^2 h) - \Pi_0^+(Q_+^2 \bar{h}), \quad h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$$

depending on the symmetries of the equation. This operator would be self-adjoint acting on $L^2(\mathbb{H}^1)$, but we are interested in elements of $\dot{H}^1(\mathbb{H}^1)$ endowed with its own scalar product. After studying the action of \mathcal{L} on the real subspace V spanned by $(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)$, we will try to find a new form for $(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}$ on the orthogonal of V in $\dot{H}^1(\mathbb{H}^1)$ which is more suitable for a spectral study.

Proposition 2.4.7. *In the real subspace V of $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$ spanned by the orthogonal basis of vectors $(\partial_s Q_+, iQ_+ - \partial_s Q_+, Q_+ + 2i\partial_s Q_+, Q_+)$, the linearized operator \mathcal{L} has the form*

$$\mathcal{L}|_V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.16)$$

Proof. We define

$$\tilde{\mathcal{L}}(F) := -iF' - 2P_0(|F_{Q_+}|^2 F) - P_0(F_{Q_+}^2 \bar{F}), \quad F \in \mathcal{H}^2(\mathbb{C}_+).$$

For $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, the holomorphic function $F_h \in \mathcal{H}^2(\mathbb{C}_+)$ satisfies

$$\tilde{\mathcal{L}}(F_h) = F_{\mathcal{L}h}.$$

We study $\tilde{\mathcal{L}}$ on $\mathcal{H}^2(\mathbb{C}_+)$. For $F \in \mathcal{H}^2(\mathbb{C}_+)$, define

$$\mathcal{F}(F) := -iF' - P_0(|F|^2 F).$$

Let U be a \mathcal{C}^1 function defined on a neighborhood of $t = 0$, valued in $\mathcal{H}^2(\mathbb{C}_+)$, satisfying $U(0) = F_{Q_+}$ and $U'(0) = F$. Then

$$\tilde{\mathcal{L}}F = \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(U(t)).$$

Thanks to the invariance under translation in the variable s , we consider $U : s_0 \in \mathbb{R} \mapsto F_{Q_+}(\cdot + s_0)$. For all $s_0 \in \mathbb{R}$, we have $\mathcal{F}(U(s_0)) = 0$, so

$$\tilde{\mathcal{L}}(F'_{Q_+}) = 0 = \mathcal{L}(\partial_s Q_+).$$

Following the same pattern, the invariance under phase multiplication gives, with $U : \theta \in \mathbb{R} \mapsto e^{i\theta} F_{Q_+}$, that $\mathcal{F}(U(\theta)) = 0$ for all θ , so

$$\tilde{\mathcal{L}}(iF_{Q_+}) = 0 = \mathcal{L}(iQ_+).$$

Finally, let $U : \lambda \in]-1, 1[\mapsto (1 + \lambda)F_{Q_+}((1 + \lambda)^2 \cdot)$, then $\mathcal{F}(U(\lambda)) = 0$ for all λ thanks to the scaling invariance, so

$$\tilde{\mathcal{L}}(F_{Q_+} + 2zF'_{Q_+}) = 0.$$

Remark that

$$zF'_{Q_+} = -\frac{i\sqrt{2}z}{(z+i)^2} = -F_{Q_+} - iF'_{Q_+}.$$

Consequently,

$$\mathcal{L}(Q_+ + 2i\partial_s Q_+) = 0.$$

In order to determine \mathcal{L} entirely on the subspace V , it is sufficient to calculate $\mathcal{L}(Q_+)$. Yet

$$\mathcal{L}(Q_+) = -i\partial_s Q_+ - 3\Pi_0^+(|Q_+|^2 Q_+) = 2i\partial_s Q_+.$$

We have proved that in the orthogonal basis $(\partial_s Q_+, iQ_+ - \partial_s Q_+, Q_+ + 2i\partial_s Q_+, Q_+)$ of V , \mathcal{L} admits the matrix representation (2.16). \square

We now work on the orthogonal of V , so we study the orthogonality conditions. For this part, it is more natural to work with the complex scalar product in $\dot{H}^1(\mathbb{H}^1)$

$$\begin{aligned} \langle h_1, h_2 \rangle_{\dot{H}^1(\mathbb{H}^1)} &= \int_{\mathbb{H}^1} (-\Delta_{\mathbb{H}^1} h_1) \overline{h_2} \, dx \, dy \, ds \\ &= \langle -\Delta_{\mathbb{H}^1} h_1, h_2 \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

We have

$$\langle h, Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1)} = (h, Q_+)_{\dot{H}^1(\mathbb{H}^1)} + i(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}.$$

Proposition 2.4.8. *Let $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $F_h(s + i(x^2 + y^2)) = h(x, y, s)$ its holomorphic counterpart. Then*

$$\langle h, Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1)} = \sqrt{2\pi^2} F_h(i).$$

Consequently,

- h is orthogonal to Q_+ and iQ_+ in $\dot{H}^1(\mathbb{H}^1)$ if and only if $F_h(i) = 0$;
- h is orthogonal to $\partial_s Q_+$ and $i\partial_s Q_+$ in $\dot{H}^1(\mathbb{H}^1)$ if and only if $F'_h(i) = 0$.

This proposition enables us to check easily that the basis $(\partial_s Q_+, iQ_+ - \partial_s Q_+, Q_+ + 2i\partial_s Q_+, Q_+)$ of V is orthogonal in $\dot{H}^1(\mathbb{H}^1)$.

Proof. We study the duality bracket in $\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)$ between $-\Delta_{\mathbb{H}^1}Q_+ = D_sQ_+$ and h , for which we use the holomorphic function F_h . Knowing that

$$F_{\partial_s Q_+}(z) = F'_{Q_+}(z) = -\frac{i\sqrt{2}}{(z+i)^2},$$

equality (2.15) $\|u\|_{L^2(\mathbb{H}^1)}^2 = \pi \|F_u\|_{L^2(\mathbb{C}_+)}^2$ for $u \in L^2(\mathbb{H}^1)$ leads to

$$\langle h, \partial_s Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)} = \pi \int_{\mathbb{C}_+} \frac{i\sqrt{2}}{(z+i)^2} F_h(z) d\lambda(z).$$

Let $t > 0$, and define $f_t : z \mapsto F_h(z+it)$ on $\{z \in \mathbb{C} \mid \text{Im}(z) > -t\}$. Applying the residue formula to $z \mapsto \frac{1}{(z-it-i)^2} f_t(z)$, which is holomorphic on $\{z \in \mathbb{C} \mid \text{Im}(z) > -t\} \setminus \{it+i\}$ with a simple pole at $it+i$, we get that on every rectangle $\mathcal{R} := [-a, a] + i[0, b]$ containing $it+i$,

$$\int_{\partial \mathcal{R}} \frac{1}{(z-it-i)^2} f_t(z) dz = 2i\pi f'_t(it+i) = 2i\pi F'_h(2it+i). \quad (2.17)$$

Since the integral of $z \mapsto \frac{1}{(z-it-i)^2} f_t(z)$ is absolutely convergent on $\{z \in \mathbb{C} \mid \text{Im}(z) > -t\}$, there are some sequences $(a_j)_{j \in \mathbb{N}}$ and $(b_j)_{j \in \mathbb{N}}$ of real numbers converging to $+\infty$ that satisfy

$$\begin{aligned} \int_{\mathbb{R}_+} \frac{1}{(-a_j + it' - it - i)^2} f_t(-a_j + it') dt' &\rightarrow 0, \\ \int_{\mathbb{R}_+} \frac{1}{(a_j + it' - it - i)^2} f_t(a_j + it') dt' &\rightarrow 0, \\ \int_{\mathbb{R}} \frac{1}{(s + ib_j - it - i)^2} f_t(s + ib_j) ds &\rightarrow 0. \end{aligned}$$

Applying formula (2.17) to the rectangles $[-a_j, a_j] \times [0, b_j]$ and passing to the limit $j \rightarrow +\infty$, one gets

$$\int_{\mathbb{R}} \frac{1}{(s-it-i)^2} f_t(s) ds = 2i\pi F'_h(2it+i).$$

Consequently

$$\begin{aligned} \langle h, \partial_s Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)} &= i\pi\sqrt{2}2i\pi \int_{\mathbb{R}_+} F'_h(2it+i) dt \\ &= -i\pi^2\sqrt{2}F_h(i), \end{aligned}$$

since $F_h(it)$ goes to 0 as t goes to $+\infty$. This latter fact can be established by using the function $f \in L^2(\mathbb{R}_+)$ associated to F_h , which satisfies for all $t \in \mathbb{R}_+^*$

$$F_h(it) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-t\sigma} f(\sigma) d\sigma.$$

Indeed, we have

$$|F_h(it)| \leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{+\infty} e^{-2t\sigma} d\sigma \right)^{\frac{1}{2}} \left(\int_0^{+\infty} |f(\sigma)|^2 d\sigma \right)^{\frac{1}{2}} = \frac{1}{2\sqrt{\pi t}} \|f\|_{L^2},$$

which goes to 0 as t goes to $+\infty$.

We have shown as wanted that

$$\langle h, Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1)} = \langle h, -i\partial_s Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)} = \sqrt{2}\pi^2 F_h(i).$$

In particular,

$$\langle h, \partial_s Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1)} = -\langle \partial_s h, -i\partial_s Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)} = -\sqrt{2}\pi^2 F'_h(i). \quad \square$$

We now check that $h \in \dot{H}^1(\mathbb{H}^1)$, $\mathcal{L}h$ decomposes in the Hilbert space $\dot{H}^{-1}(\mathbb{H}^1)$ as an orthogonal sum $\mathcal{L}h = \mathcal{L}|_V h + \mathcal{L}|_{V^\perp} h$, where V^\perp is the orthogonal of V in $\dot{H}^1(\mathbb{H}^1)$.

Corollary 2.4.9. *Let $h \in \dot{H}^1(\mathbb{H}^1)$ and decompose h as $h = h_0 + h_- + h_+$, where*

$$\begin{aligned} h_0 &\in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+), \\ h_- &\in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+), \\ h_+ &\in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap (Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^\perp, \dot{H}^1(\mathbb{H}^1). \end{aligned}$$

Then

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = (\mathcal{L}h_+, h_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} + (\mathcal{L}h_-, h_-)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}$$

and

$$\|\mathcal{L}h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 = \|\mathcal{L}h_+\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 + \|\mathcal{L}h_-\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2.$$

Proof. With this decomposition, we have $\mathcal{L}h = \mathcal{L}h_+ + \mathcal{L}h_-$.

Let us show that $\mathcal{L}h_+$ is orthogonal to Q_+ , iQ_+ , $\partial_s Q_+$ and $i\partial_s Q_+$ for the duality product space $\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)$. We treat separately each term of

$$\mathcal{L}h_+ = D_s h_+ - 2\Pi_0^+(|Q_+|^2 h_+) - \Pi_0^+(Q_+^2 \bar{h}_+).$$

By assumption on h_+ , $D_s h_+ = -\Delta_{\mathbb{H}^1} h_+$ so that

$$\langle D_s h_+, Q_+ \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \langle h_+, Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1)} = 0$$

and

$$\langle D_s h_+, \partial_s Q_+ \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \langle h_+, \partial_s Q_+ \rangle_{\dot{H}^1(\mathbb{H}^1)} = 0.$$

Moreover, using Proposition 2.4.8,

$$\begin{aligned} \langle \Pi_0^+(|Q_+|^2 h_+), Q_+ \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= \langle Q_+ h_+, Q_+^2 \rangle_{L^2(\mathbb{H}^1) \times L^2(\mathbb{H}^1)} \\ &= \langle Q_+ h_+, -i\sqrt{2}\partial_s Q_+ \rangle_{L^2(\mathbb{H}^1) \times L^2(\mathbb{H}^1)} \\ &= 2\pi^2 F_{Q_+ h_+}(i) \\ &= 0, \end{aligned}$$

since $F_{Q_+ h_+} = F_{Q_+} F_{h_+}$ and $F_{h_+}(i) = 0$. In the same way,

$$\langle \Pi_0^+(Q_+^2 \bar{h}_+), Q_+ \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \langle Q_+^2, Q_+ h_+ \rangle_{L^2(\mathbb{H}^1) \times L^2(\mathbb{H}^1)} = 0.$$

Finally,

$$\begin{aligned}
\langle \Pi_0^+(|Q_+|^2 h_+), \partial_s Q_+ \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= \frac{1}{2} \langle Q_+ h_+, \partial_s(Q_+^2) \rangle_{L^2(\mathbb{H}^1) \times L^2(\mathbb{H}^1)} \\
&= -\frac{1}{2} \langle \partial_s(Q_+ h_+), Q_+^2 \rangle_{\dot{H}^{-2}(\mathbb{H}^1) \times \dot{H}^2(\mathbb{H}^1)} \\
&= -\frac{1}{2} \langle \partial_s(Q_+ h_+), -i\sqrt{2}\partial_s Q_+ \rangle_{\dot{H}^{-2}(\mathbb{H}^1) \times \dot{H}^2(\mathbb{H}^1)} \\
&= -\pi^2 F'_{Q_+ h_+}(i) \\
&= 0,
\end{aligned}$$

and in the same way,

$$\begin{aligned}
\langle \Pi_0^+(Q_+^2 \overline{h_+}), \partial_s Q_+ \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= \langle Q_+^2, \partial_s(Q_+ h_+) \rangle_{L^2(\mathbb{H}^1) \times L^2(\mathbb{H}^1)} \\
&= 2\pi^2 \overline{F_{\partial_s(Q_+ h_+)}(i)} \\
&= 0.
\end{aligned}$$

Therefore, $\mathcal{L}h_+$ lies in $\dot{H}^{-1}(\mathbb{H}^1) \cap V_0^+ \cap (Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^{\perp, L^2(\mathbb{H}^1)}$, where the orthogonal is taken for the duality product $\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)$. In particular,

$$(\mathcal{L}h_+, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = (\mathcal{L}h_+, h_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}.$$

Now, since $\mathcal{L}h_-$ is in $\text{Vect}_{\mathbb{R}}(i\partial_s Q_+)$, write $\mathcal{L}h_- = \lambda i\partial_s Q_+ = \lambda \Delta_{\mathbb{H}^1} Q_+$ for some real number λ . One has

$$\begin{aligned}
(\mathcal{L}h_-, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= -\lambda(Q_+, h)_{\dot{H}^1(\mathbb{H}^1)} \\
&= -\lambda(Q_+, h_-)_{\dot{H}^1(\mathbb{H}^1)} \\
&= (\mathcal{L}h_-, h_-)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)},
\end{aligned}$$

which gives the first part of the proposition.

Finally,

$$(\mathcal{L}h_+, \mathcal{L}h_-)_{\dot{H}^{-1}(\mathbb{H}^1)} = (\mathcal{L}h_+, -\lambda Q_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = 0,$$

so we conclude that

$$\|\mathcal{L}h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 = \|\mathcal{L}h_+\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 + \|\mathcal{L}h_-\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2. \quad \square$$

We now give a simplified expression of $(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}$ when h is orthogonal to Q_+ and iQ_+ .

Proposition 2.4.10. *For $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$, the following identity is true*

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \langle -\Delta_{\mathbb{H}^1} h - 2|Q_+|^2 h, h \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}.$$

Note that it is more convenient to switch to a complex scalar product because contrary to \mathcal{L} , the operator $-\Delta_{\mathbb{H}^1} h - 2|Q_+|^2 h$ is a complex linear operator of the variable h .

Proof. We only have to show that $(\Pi_0^+(Q_+^2 \bar{h}), h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}$ is zero. We calculate

$$\begin{aligned} (\Pi_0^+(Q_+^2 \bar{h}), h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= (Q_+^2 \bar{h}, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ &= (Q_+^2, h^2)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ &= (-i\sqrt{2}\partial_s Q_+, h^2)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ &= 2\pi^2 \operatorname{Re}(F_{h^2}(i)). \end{aligned}$$

Now, $F_{h^2} = F_h^2$, therefore, $F_{h^2}(i) = 0$ as soon as $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \operatorname{Vect}_{\mathbb{R}}(Q_+, iQ_+)^\perp$. \square

2.4.3 Study of the limiting profile through the Cayley transform

We now study the spectrum of $-\Delta_{\mathbb{H}^1} - 2|Q_+|^2$, which is natural since we are searching for a coercivity estimate on \mathcal{L} and we just proved (Proposition 2.4.10) that

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \langle -\Delta_{\mathbb{H}^1} h - 2|Q_+|^2 h, h \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}.$$

This spectrum can be determined via the equivalence between the Heisenberg group \mathbb{H}^1 and the CR sphere \mathbb{S}^3 in \mathbb{C}^2 called the Cayley transform. We rely on [BFM13] in order to introduce this equivalence and its spectral consequences. In this part, we will denote by (w, s) the elements of the Heisenberg group, bearing in mind that $w = x + iy$ with the former notation. The Cayley transform is

$$\begin{aligned} \mathcal{C} : \quad \mathbb{H}^1 &\rightarrow \mathbb{S}^3 \setminus (0, -1) \\ (w, s) &\mapsto \left(\frac{2w}{1+|w|^2+is}, \frac{1-|w|^2-is}{1+|w|^2+is} \right). \end{aligned}$$

The inverse of \mathcal{C} is given by $\mathcal{C}^{-1}(\zeta_1, \zeta_2) = (\frac{\zeta_1}{1+\zeta_2}, \operatorname{Im}(\frac{1-\zeta_2}{1+\zeta_2}))$. The Jacobian of the Cayley transform is

$$|J_{\mathcal{C}}(w, s)| = \frac{8}{((1+|w|^2)^2 + s^2)^2}.$$

Notice that $|J_{\mathcal{C}}|$ is linked to Q_+ as follows

$$|J_{\mathcal{C}}(x + iy, s)| = 2|Q_+(x, y, s)|^4.$$

For any integrable function F on \mathbb{S}^3 , we have the relation

$$\int_{\mathbb{S}^3} F \, d\zeta = \int_{\mathbb{H}^1} (F \circ \mathcal{C}) |J_{\mathcal{C}}| \, d\lambda_3(w, s).$$

Here, $d\zeta$ denotes the standard euclidean volume element of \mathbb{S}^3 . We consider the complex scalar product on $L^2(\mathbb{S}^3)$

$$\langle F, G \rangle_{L^2(\mathbb{S}^3)} = \int_{\mathbb{S}^3} F \bar{G} \, d\zeta, \quad F, G \in L^2(\mathbb{S}^3).$$

One can note that

$$\int_{\mathbb{S}^3} |F|^2 \, d\zeta = \int_{\mathbb{H}^1} |F \circ \mathcal{C}|^2 |J_{\mathcal{C}}| \, d\lambda_3(w, s),$$

in particular, $|J_C| = 2|Q_+|^4$ is in $L^2(\mathbb{H}^1)$, so if a function F is such that $F \circ \mathcal{C}$ belongs to $L^4(\mathbb{H}^1)$ (for example if $F \circ \mathcal{C} \in \dot{H}^1(\mathbb{H}^1)$), then $|F \circ \mathcal{C}|^2$ belongs to $L^2(\mathbb{H}^1)$, and therefore F is in $L^2(\mathbb{S}^3)$.

On the standard sphere \mathbb{S}^3 , let us denote $\mathcal{R} = \zeta_1 \partial_{\zeta_1} + \zeta_2 \partial_{\zeta_2}$. Then the vector fields $T_i = \partial_{\zeta_i} - \bar{\zeta}_i \mathcal{R}$ for $i = 1, 2$ generate the holomorphic tangent space to \mathbb{S}^3 . The conformal sub-Laplacian is defined as

$$\mathcal{D} = -\frac{1}{2} \sum_{i=1}^2 (T_i \bar{T}_i + \bar{T}_i T_i) + \frac{1}{4},$$

where $\mathcal{D} - \frac{1}{4}$ is the sub-Laplacian. One can construct the Sobolev space

$$H^1(\mathbb{S}^3) := \{v \in L^2(\mathbb{S}^3) \mid \|v\|_{H^1(\mathbb{S}^3)} := \|\mathcal{D}^{\frac{1}{2}} v\|_{L^2(\mathbb{S}^3)} < +\infty\}.$$

The operator \mathcal{D} on the sphere has a direct link with the sub-Laplacian on the Heisenberg group via the Cayley transform: for any radial function $F \circ \mathcal{C}$ in $\dot{H}^1(\mathbb{H}^1)$,

$$-\Delta_{\mathbb{H}^1} \left((2|J_C|)^{\frac{1}{4}} (F \circ \mathcal{C}) \right) = (2|J_C|)^{\frac{3}{4}} (\mathcal{D}F) \circ \mathcal{C}.$$

Note that a function in $\dot{H}^1(\mathbb{H}^1)$ maps to a function in $H^1(\mathbb{S}^3)$ via the following transformation.

Proposition 2.4.11. *Let h be a function on \mathbb{H}^1 , and define a function v_h on \mathbb{S}^3 by*

$$h(x, y, s) = (2|J_C|)^{\frac{1}{4}} (v_h \circ \mathcal{C})(x + iy, s) = \sqrt{2}|Q_+| (v_h \circ \mathcal{C})(x + iy, s). \quad (2.18)$$

Then for radial h ,

$$\langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} = \frac{1}{2} \langle -\Delta_{\mathbb{H}^1} h, h \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}$$

and

$$\langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)} = \int_{\mathbb{H}^1} |h|^2 |Q_+|^2 d\lambda_3.$$

Therefore, v_h defines a function in $H^1(\mathbb{S}^3)$ if and only if h is in $\dot{H}^1(\mathbb{H}^1)$.

Proof. Fix a radial function h , and define v_h by (2.18). Then

$$\begin{aligned} (-\Delta_{\mathbb{H}^1} h) \cdot \bar{h} &= (2|J_C|)^{\frac{3}{4}} (\mathcal{D}v_h) \circ \mathcal{C} \cdot \overline{(2|J_C|)^{\frac{1}{4}} (v_h \circ \mathcal{C})} \\ &= 2|J_C| (\mathcal{D}v_h) \circ \mathcal{C} \cdot \overline{v_h \circ \mathcal{C}}, \end{aligned}$$

so

$$\langle -\Delta_{\mathbb{H}^1} h, h \rangle_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = 2 \langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)}.$$

Moreover, when $h \in L^4(\mathbb{H}^1)$, then $v_h \in L^2(\mathbb{S}^3)$ and

$$\begin{aligned} \langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)} &= \int_{\mathbb{S}^3} |v_h|^2 d\zeta = \int_{\mathbb{H}^1} |v_h \circ \mathcal{C}|^2 |J_C| d\lambda_3(w, s) \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{H}^1} |h|^2 |J_C|^{\frac{1}{2}} d\lambda_3(w, s) = \int_{\mathbb{H}^1} |h|^2 |Q_+|^2 d\lambda_3. \quad \square \end{aligned}$$

Propositions 2.4.10 and 2.4.11 combined imply the following corollary.

Corollary 2.4.12. *Let h in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap (Q_+, iQ_+)^\perp, \dot{H}^1(\mathbb{H}^1)$. Then*

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = 2\langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} - 2\langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)}.$$

The spectrum of the operator \mathcal{D} on $H^1(\mathbb{S}^3)$ is well known. Indeed, the space $L^2(\mathbb{S}^3)$ endowed with the inner product $\langle F, G \rangle_{L^2(\mathbb{S}^3)} = \int_{\mathbb{S}^3} F \bar{G} d\zeta$ admits the orthogonal decomposition

$$L^2(\mathbb{S}^3) = \bigoplus_{j,k \geq 0} \text{Ha}_{j,k},$$

where $\text{Ha}_{j,k}$ is the space of harmonic polynomials on \mathbb{C}^2 that are homogeneous of degree j in ζ_1, ζ_2 and k in $\bar{\zeta}_1, \bar{\zeta}_2$, restricted to the sphere \mathbb{S}^3 . Fix $j, k \geq 0$, then the dimension of $\text{Ha}_{j,k}$ is

$$m_{j,k} := \dim(\text{Ha}_{j,k}) = j + k + 1.$$

The spectrum of \mathcal{D} is as follows.

Proposition 2.4.13 (Spectrum of \mathcal{D} [Sta89]). *Let $\lambda_j = j + \frac{1}{2}$. Then for all $Y_{j,k} \in \text{Ha}_{j,k}$,*

$$\mathcal{D}Y_{j,k} = \lambda_j \lambda_k Y_{j,k}.$$

In particular, the smallest eigenvalue of $\mathcal{D} - \text{Id}$ is $\lambda_{0,0} - 1 = -\frac{3}{4}$, with multiplicity 1, for which the eigenvectors are the constant functions on \mathbb{S}^3 . The second one is also negative, equal to $\lambda_{1,0} - 1 = \lambda_{0,1} - 1 = -\frac{1}{4}$, with eigenvectors spanned by $\zeta_1, \zeta_2, \bar{\zeta}_1, \bar{\zeta}_2$. The third one is positive, equal to $\lambda_{2,0} - 1 = \lambda_{0,2} - 1 = \frac{1}{4}$.

Let us study the radial property on \mathbb{S}^3 . Let $h \in \dot{H}^1(\mathbb{H}^1)$ be a radial function and let v_h as in (2.18)

$$h(x, y, s) = (2|J_{\mathcal{C}}|)^{\frac{1}{4}} (v_h \circ \mathcal{C})(x + iy, s).$$

Since h and $|J_{\mathcal{C}}|$ only depend on $|x + iy|$ and s , so does $v_h \circ \mathcal{C}$, which means that v_h only depends on $|\zeta_1|, \zeta_2$ and $\bar{\zeta}_2$. This discards the eigenfunctions ζ_1 and $\bar{\zeta}_1$ in the orthogonal decomposition of v_h .

The last step is to treat the remaining eigenvectors with negative eigenvalues for the operator $\mathcal{D} - \text{Id}$, in order to find a lower bound in the quadratic form

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = 2\langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} - 2\langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)}$$

for $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^\perp, \dot{H}^1(\mathbb{H}^1)$. These eigenvectors are the constant function $e_1 = \mathbb{1}$ (with eigenvalue $-\frac{3}{4}$) and the harmonic polynomials $e_2 = \bar{\zeta}_2$ and $e_3 = \zeta_2$ (with eigenvalue $-\frac{1}{4}$). In order to do so, we reformulate the above spectral study back to the setting of holomorphic functions of the upper complex plane.

For fractional Sobolev embeddings on \mathbb{R}^n and fractional Folland-Stein embeddings on \mathbb{H}^n ([CFW13] and [LZ15]), the potential negative eigenvalues are naturally discarded by the orthogonality conditions, since they correspond to the tangent space to the manifold of functions equal, up to translation, dilation and multiplication by a nonzero constant, to the respective optimizers W and H :

$$\mathcal{M}(\mathbb{R}^n) = \left\{ cW \left(\frac{\cdot - x_0}{\varepsilon} \right) \mid c \in \mathbb{R}^*, x_0 \in \mathbb{R}^n, \varepsilon > 0 \right\}$$

resp.

$$\mathcal{M}(\mathbb{H}^1) = \{ cH(\delta(u)) \mid c \in \mathbb{R}^*, u \in \mathbb{H}^n, \delta > 0 \}.$$

2.4.4 Coercivity of the linearized operator

In this part, we use the spectrum of \mathcal{D} on the CR sphere in order to get a coercivity estimate on \mathcal{L} . The lowest eigenvalues of $\mathcal{D} - \text{Id}$ are, in increasing order,

$$\lambda_{0,0} - 1 = -\frac{3}{4} < \lambda_{0,1} - 1 = \lambda_{1,0} - 1 = -\frac{1}{4} < \lambda_{0,2} - 1 = \lambda_{2,0} - 1 = \frac{1}{4}.$$

The negative eigenfunctions are $e_1 = \mathbf{1}$ (for $\lambda_{0,0}$), $e_2 = \bar{\zeta}_2$ (for $\lambda_{0,1}$) and $e_3 = \zeta_2$ (for $\lambda_{1,0}$).

Let $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$ and let v as in (2.18)

$$h(x, y, s) = \sqrt{2}|Q_+|(v \circ \mathcal{C})(x + iy, s).$$

Decompose v as:

$$v = v_+ + \frac{\langle v, e_1 \rangle_{L^2(\mathbb{S}^3)}}{\langle e_1, e_1 \rangle_{L^2(\mathbb{S}^3)}} e_1 + \frac{\langle v, e_2 \rangle_{L^2(\mathbb{S}^3)}}{\langle e_2, e_2 \rangle_{L^2(\mathbb{S}^3)}} e_2 + \frac{\langle v, e_3 \rangle_{L^2(\mathbb{S}^3)}}{\langle e_3, e_3 \rangle_{L^2(\mathbb{S}^3)}} e_3, \quad v_+ \in \text{Vect}_{\mathbb{C}}(e_1, e_2, e_3)^{\perp}.$$

Since $e_1 \in \text{Ha}_{0,0}$, $e_2 \in \text{Ha}_{0,1}$ and $e_3 \in \text{Ha}_{1,0}$, these three vectors are pairwise orthogonal in $L^2(\mathbb{S}^3)$, and they are orthogonal to $\bigoplus_{(j,k) \notin \{(0,0), (0,1), (1,0)\}} \text{Ha}_{j,k}$. The knowledge of the eigenvalues of $\mathcal{D} - \text{Id}$ enables us to say that

$$\begin{aligned} \frac{1}{2}(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= \langle \mathcal{D}v, v \rangle_{L^2(\mathbb{S}^3)} - \langle v, v \rangle_{L^2(\mathbb{S}^3)} \\ &\geq \frac{1}{4} \|v_+\|_{L^2(\mathbb{S}^3)}^2 - \frac{1}{4} \frac{|\langle v, e_3 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_3, e_3 \rangle_{L^2(\mathbb{S}^3)}} - \frac{1}{4} \frac{|\langle v, e_2 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_2, e_2 \rangle_{L^2(\mathbb{S}^3)}} - \frac{3}{4} \frac{|\langle v, e_1 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_1, e_1 \rangle_{L^2(\mathbb{S}^3)}}. \end{aligned}$$

But

$$\|v\|_{L^2(\mathbb{S}^3)}^2 = \|v_+\|_{L^2(\mathbb{S}^3)}^2 + \frac{|\langle v, e_3 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_3, e_3 \rangle_{L^2(\mathbb{S}^3)}} + \frac{|\langle v, e_2 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_2, e_2 \rangle_{L^2(\mathbb{S}^3)}} + \frac{|\langle v, e_1 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_1, e_1 \rangle_{L^2(\mathbb{S}^3)}},$$

so

$$\frac{1}{2}(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq \frac{1}{4} \|v\|_{L^2(\mathbb{S}^3)}^2 - \frac{1}{2} \frac{|\langle v, e_3 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_3, e_3 \rangle_{L^2(\mathbb{S}^3)}} - \frac{1}{2} \frac{|\langle v, e_2 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_2, e_2 \rangle_{L^2(\mathbb{S}^3)}} - \frac{|\langle v, e_1 \rangle_{L^2(\mathbb{S}^3)}|^2}{\langle e_1, e_1 \rangle_{L^2(\mathbb{S}^3)}}.$$

Let us replace these latter terms by their expression on the Heisenberg group. We define

$$f_j = \sqrt{2}|Q_+|e_j \circ \mathcal{C}, \quad j = 1, 2, 3.$$

From the identity

$$\zeta_2 \circ \mathcal{C}(w, s) = \frac{1 - |w|^2 - is}{1 + |w|^2 + is} = \sqrt{2Q_+}(w, s) - \mathbf{1},$$

we get that

$$\begin{aligned} f_1 &= \sqrt{2}|Q_+|, \\ f_2 &= \sqrt{2}|Q_+|(\sqrt{2}Q_+ - 1), \\ f_3 &= \sqrt{2}|Q_+|(\sqrt{2Q_+} - 1). \end{aligned}$$

Thanks to Proposition 2.4.11, one knows that $\langle v, v \rangle_{L^2(\mathbb{S}^3)} = \langle hQ_+, hQ_+ \rangle_{L^2(\mathbb{H}^1)}$, so

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq \frac{1}{2} \|hQ_+\|_{L^2(\mathbb{H}^1)}^2 - \frac{|\langle hQ_+, f_3Q_+ \rangle_{L^2(\mathbb{H}^1)}|^2}{\|f_3Q_+\|_{L^2(\mathbb{H}^1)}^2} - \frac{|\langle hQ_+, f_2Q_+ \rangle_{L^2(\mathbb{H}^1)}|^2}{\|f_2Q_+\|_{L^2(\mathbb{H}^1)}^2} - 2 \frac{|\langle hQ_+, f_1Q_+ \rangle_{L^2(\mathbb{H}^1)}|^2}{\|f_1Q_+\|_{L^2(\mathbb{H}^1)}^2}.$$

For $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$, let us consider the space where F_{hQ_+} lies.

Since $h \in \dot{H}^1(\mathbb{H}^1)$, from the embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$, one knows that hQ_+ is in $L^2(\mathbb{H}^1)$ so F_{hQ_+} belongs to $L^2(\mathbb{C}_+)$.

From part 2.4.1, h being in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$, F_h (defined by $h(x, y, s) = F_h(s + i|x + iy|^2)$ for $(x, y, s) \in \mathbb{H}^1$) is a holomorphic function since F_h lies in the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$. This implies that the function $F_{hQ_+} = F_h F_{Q_+}$ is holomorphic too: we have shown that F_h is in the Bergman space $A_1^2 = L^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$.

Moreover, the fact that h is orthogonal to $\text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)$ in $\dot{H}^1(\mathbb{H}^1)$ is equivalent by Proposition 2.4.8 to $F_h(i) = F'_h(i) = 0$. But then, $F_{hQ_+} = F_h F_{Q_+}$ has a double zero at i . Proposition 2.4.8 again implies that

$$\langle hQ_+, \partial_s Q_+ \rangle_{L^2(\mathbb{H}^1)} = \langle hQ_+, \partial_s^2 Q_+ \rangle_{L^2(\mathbb{H}^1)} = 0,$$

which is equivalent to

$$\langle F_{hQ_+}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = \langle F_{hQ_+}, F''_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = 0.$$

Now, define $W := A_1^2 \cap \text{Vect}_{\mathbb{C}}(F'_{Q_+}, F''_{Q_+})^{\perp, L^2(\mathbb{C}_+)}$ and denote by P_W the orthogonal projection from $L^2(\mathbb{C}_+)$ onto W . We have shown that if $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ is orthogonal to $\text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)$ in $\dot{H}^1(\mathbb{H}^1)$, then $F_{hQ_+} \in W$. In particular, for $u \in L^2(\mathbb{H}^1)$,

$$\langle hQ_+, u \rangle_{L^2(\mathbb{H}^1)} = \pi \langle F_{hQ_+}, F_u \rangle_{L^2(\mathbb{C}_+)} = \pi \langle F_{hQ_+}, P_W(F_u) \rangle_{L^2(\mathbb{C}_+)}.$$

Back to the quadratic form, we deduce that

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq \pi \left(\frac{1}{2} \|F_{hQ_+}\|_{L^2(\mathbb{C}_+)}^2 - \frac{|\langle F_{hQ_+}, P_W(F_{f_3Q_+}) \rangle_{L^2(\mathbb{C}_+)}|^2}{\|F_{f_3Q_+}\|_{L^2(\mathbb{C}_+)}^2} - \frac{|\langle F_{hQ_+}, P_W(F_{f_2Q_+}) \rangle_{L^2(\mathbb{C}_+)}|^2}{\|F_{f_2Q_+}\|_{L^2(\mathbb{C}_+)}^2} - 2 \frac{|\langle F_{hQ_+}, P_W(F_{f_1Q_+}) \rangle_{L^2(\mathbb{C}_+)}|^2}{\|F_{f_1Q_+}\|_{L^2(\mathbb{C}_+)}^2} \right).$$

Let us denote

$$X_j = \frac{P_W(F_{f_jQ_+})}{\|F_{f_jQ_+}\|_{L^2(\mathbb{C}_+)}} = \frac{P_W(F_j)}{\|F_j\|_{L^2(\mathbb{C}_+)}} \quad j = 1, 2, 3,$$

with

$$F_1(z) = \frac{1}{|z+i|(z+i)},$$

$$F_2(z) = \frac{1}{|z+i|(z+i)} \left(\frac{2i}{z+i} - 1 \right),$$

$$F_3(z) = \frac{1}{|z+i|(z+i)} \left(\frac{-2i}{\bar{z}-i} - 1 \right).$$

We try to find an upper bound on the quadratic form on $L^2(\mathbb{C}_+)$

$$q(F) := 2 \left| \langle F, X_1 \rangle_{L^2(\mathbb{C}_+)} \right|^2 + \left| \langle F, X_2 \rangle_{L^2(\mathbb{C}_+)} \right|^2 + \left| \langle F, X_3 \rangle_{L^2(\mathbb{C}_+)} \right|^2, \quad F \in L^2(\mathbb{C}_+).$$

In particular, we want to show that this upper bound is strictly less than $\frac{1}{2}$.

Let us first write explicitly the orthogonal projector P_W from $L^2(\mathbb{C}_+)$ onto the subspace $W = A_1^2 \cap \text{Vect}_{\mathbb{C}}(F'_{Q_+}, F''_{Q_+})^{\perp, L^2(\mathbb{C}_+)}$. We start by finding an orthogonal basis of $\text{Vect}_{\mathbb{C}}(F'_{Q_+}, F''_{Q_+})$ for the scalar product on $L^2(\mathbb{C}_+)$. We know from Proposition 2.4.8 that

$$\langle u, \partial_s Q_+ \rangle_{L^2(\mathbb{H}^1)} = -i\sqrt{2}\pi^2 F_u(i), \quad u \in L^2(\mathbb{H}^1),$$

so

$$\langle F, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = -i\sqrt{2}\pi F(i), \quad F \in L^2(\mathbb{C}_+).$$

Using the equalities

$$F_Q(z) = \frac{i\sqrt{2}}{z+i}, \quad F'_Q(z) = \frac{-i\sqrt{2}}{(z+i)^2}, \quad F''_Q(z) = \frac{2i\sqrt{2}}{(z+i)^3}, \quad F'''_Q(z) = -\frac{6i\sqrt{2}}{(z+i)^4},$$

we obtain

$$F_{Q_+}(i) = \frac{1}{\sqrt{2}}, \quad F'_{Q_+}(i) = \frac{i}{2\sqrt{2}}, \quad F''_{Q_+}(i) = -\frac{1}{2\sqrt{2}}, \quad F'''_{Q_+}(i) = -\frac{3i}{4\sqrt{2}}.$$

Therefore,

$$\langle F''_{Q_+}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = -i\sqrt{2}\pi F''_{Q_+}(i) = i\frac{\pi}{2}.$$

In the same way,

$$\langle F'_{Q_+}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = -i\sqrt{2}\pi F'_{Q_+}(i) = \frac{\pi}{2},$$

so $\tilde{F} := F'_{Q_+} - \frac{\langle F'_{Q_+}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)}}{\langle F''_{Q_+}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)}} F''_{Q_+} = F'_{Q_+} + iF''_{Q_+}$ is orthogonal to F'_{Q_+} :

$$\langle \tilde{F}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = 0.$$

Moreover,

$$\begin{aligned} \langle \tilde{F}, \tilde{F} \rangle_{L^2(\mathbb{C}_+)} &= \langle \tilde{F}, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} + \langle \tilde{F}, iF''_{Q_+} \rangle_{L^2(\mathbb{C}_+)} \\ &= 0 + \langle i\tilde{F}', F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} \\ &= \sqrt{2}\pi \tilde{F}'(i). \end{aligned}$$

Since $\tilde{F}'(i) = F''_{Q_+}(i) + iF'''_{Q_+}(i) = \frac{1}{4\sqrt{2}}$, \tilde{F} is of norm

$$\langle \tilde{F}, \tilde{F} \rangle_{L^2(\mathbb{C}_+)} = \frac{\pi}{4}.$$

The orthogonal projection onto $\text{Vect}_{\mathbb{C}}(F'_{Q_+}, F''_{Q_+})^{\perp, L^2(\mathbb{C}_+)}$ in $L^2(\mathbb{C}_+)$ is then written

$$F \in L^2(\mathbb{C}_+) \mapsto F - \frac{2}{\pi} \langle F, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} F'_{Q_+} - \frac{4}{\pi} \langle F, F'_{Q_+} + iF''_{Q_+} \rangle_{L^2(\mathbb{C}_+)} (F'_{Q_+} + iF''_{Q_+}).$$

Besides, from Proposition 2.4.5, we know that the orthogonal projection P_0 from $L^2(\mathbb{C}_+)$ onto A_1^2 is given by

$$P_0 F(s+it) = -\frac{1}{\pi} \int_{\mathbb{C}_+} \frac{1}{(s-u+it+iv)^2} F(u+iv) du dv, \quad F \in L^2(\mathbb{C}_+).$$

Therefore, the orthogonal projection P_W on the space $W = A_1^2 \cap \text{Vect}_{\mathbb{C}}(F'_{Q_+}, F''_{Q_+})^{\perp, L^2(\mathbb{C}_+)}$, for $F \in L^2(\mathbb{C}_+)$, is written

$$\begin{aligned} P_W F(s+it) &= -\frac{1}{\pi} \int_{\mathbb{C}_+} \frac{1}{(s-u+it+iv)^2} F(u+iv) du dv - \frac{2}{\pi} \langle F, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} F'_{Q_+}(s+it) \\ &\quad - \frac{4}{\pi} \langle F, F'_{Q_+} + iF''_{Q_+} \rangle_{L^2(\mathbb{C}_+)} (F'_{Q_+} + iF''_{Q_+})(s+it). \end{aligned}$$

We use the following estimates of $\langle \pi P_0 F_j, F_j \rangle_{L^2(\mathbb{C}_+)}$, $j = 1, 2, 3$.

Lemma 2.4.14. *Set $\varepsilon = 10^{-10}$, then*

$$\begin{aligned} |\langle \pi P_0 F_1, F_1 \rangle_{L^2(\mathbb{C}_+)} - 2| &\leq \varepsilon, \\ |\langle \pi P_0 F_2, F_2 \rangle_{L^2(\mathbb{C}_+)} - \frac{10}{9}| &\leq \varepsilon, \\ |\langle \pi P_0 F_3, F_3 \rangle_{L^2(\mathbb{C}_+)} - 0.1303955989| &\leq \varepsilon. \end{aligned}$$

The proof of this lemma is rather technical and is postponed to Appendix 2.A. It involves simplifying the integrals defining $P_0 F_j$, $j = 1, 2, 3$: we determine explicitly the holomorphic function which coincides with $P_0 F_j$ on \mathbb{C}_+ thanks to a massive use of the residue formula. This part is necessary in order to compute numerically $\langle P_0 F_j, F_j \rangle_{L^2(\mathbb{C}_+)}$. Without this preliminary work, there is a four-dimensional numerical integration to perform and the error estimate is big with a naive approach.

A direct calculation gives

$$\begin{aligned} \langle F_1, F_1 \rangle_{L^2(\mathbb{C}_+)} &= \frac{\pi}{4}, \quad \langle F_2, F_2 \rangle_{L^2(\mathbb{C}_+)} = \langle F_3, F_3 \rangle_{L^2(\mathbb{C}_+)} = \frac{\pi}{8}, \\ \langle F_1, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} &= -\frac{2\sqrt{2}}{3}, \quad \langle F_2, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = -\frac{2\sqrt{2}}{9}, \quad \langle F_3, F'_{Q_+} \rangle_{L^2(\mathbb{C}_+)} = \frac{2\sqrt{2}}{15}, \\ \langle F_1, \tilde{F} \rangle_{L^2(\mathbb{C}_+)} &= -\frac{2\sqrt{2}}{15}, \quad \langle F_2, \tilde{F} \rangle_{L^2(\mathbb{C}_+)} = \frac{14\sqrt{2}}{45}, \quad \langle F_3, \tilde{F} \rangle_{L^2(\mathbb{C}_+)} = \frac{2\sqrt{2}}{35}. \end{aligned}$$

We deduce that

$$\left| 2 \frac{\langle P_W F_1, F_1 \rangle_{L^2(\mathbb{C}_+)}}{\langle F_1, F_1 \rangle_{L^2(\mathbb{C}_+)}} + \frac{\langle P_W F_2, F_2 \rangle_{L^2(\mathbb{C}_+)}}{\langle F_2, F_2 \rangle_{L^2(\mathbb{C}_+)}} + \frac{\langle P_W F_3, F_3 \rangle_{L^2(\mathbb{C}_+)}}{\langle F_3, F_3 \rangle_{L^2(\mathbb{C}_+)}} - 0.2046049976 \right| \leq 24\varepsilon.$$

This enables us to get a sufficiently precise estimate for the quadratic form. Indeed, we want to show that the norm of the following quadratic form is smaller than $\frac{1}{2}$

$$q(F) = 2 \left| \langle F, X_1 \rangle_{L^2(\mathbb{C}_+)} \right|^2 + \left| \langle F, X_2 \rangle_{L^2(\mathbb{C}_+)} \right|^2 + \left| \langle F, X_3 \rangle_{L^2(\mathbb{C}_+)} \right|^2, \quad F \in L^2(\mathbb{C}_+).$$

Applying Cauchy-Schwarz's inequality, for $F \in W$,

$$\begin{aligned} q(F) &= 2 \left| \frac{\langle F, F_1 \rangle_{L^2(\mathbb{C}_+)}}{\|F_1\|_{L^2(\mathbb{C}_+)}} \right|^2 + \left| \frac{\langle F, F_2 \rangle_{L^2(\mathbb{C}_+)}}{\|F_2\|_{L^2(\mathbb{C}_+)}} \right|^2 + \left| \frac{\langle F, F_3 \rangle_{L^2(\mathbb{C}_+)}}{\|F_3\|_{L^2(\mathbb{C}_+)}} \right|^2 \\ &\leq \|F\|_{L^2(\mathbb{C}_+)}^2 \left(2 \frac{\langle P_W F_1, F_1 \rangle_{L^2(\mathbb{C}_+)}}{\|F_1\|_{L^2(\mathbb{C}_+)}^2} + \frac{\langle P_W F_2, F_2 \rangle_{L^2(\mathbb{C}_+)}}{\|F_2\|_{L^2(\mathbb{C}_+)}^2} + \frac{\langle P_W F_3, F_3 \rangle_{L^2(\mathbb{C}_+)}}{\|F_3\|_{L^2(\mathbb{C}_+)}^2} \right). \end{aligned}$$

But we just estimated

$$C := 2 \frac{\langle P_W F_1, F_1 \rangle_{L^2(\mathbb{C}_+)}}{\|F_1\|_{L^2(\mathbb{C}_+)}^2} + \frac{\langle P_W F_2, F_2 \rangle_{L^2(\mathbb{C}_+)}}{\|F_2\|_{L^2(\mathbb{C}_+)}^2} + \frac{\langle P_W F_3, F_3 \rangle_{L^2(\mathbb{C}_+)}}{\|F_3\|_{L^2(\mathbb{C}_+)}^2}$$

as

$$C \approx 0.2046049976 < \frac{1}{2}.$$

Going back to h in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$,

$$\begin{aligned} (\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &\geq \pi \left(\frac{1}{2} \|F_{hQ_+}\|_{L^2(\mathbb{C}_+)}^2 - q(F_{hQ_+}) \right) \\ &\geq \pi \left(\frac{1}{2} \|F_{hQ_+}\|_{L^2(\mathbb{C}_+)}^2 - C \|F_{hQ_+}\|_{L^2(\mathbb{C}_+)}^2 \right) \\ &= \frac{1-2C}{2} \|hQ_+\|_{L^2(\mathbb{H}^1)}^2 \\ &= \frac{1-2C}{2} \|v_h\|_{L^2(\mathbb{S}^3)}^2. \end{aligned}$$

But

$$\frac{1}{2} (\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} - \langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)}$$

so

$$\langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} \geq \left(1 + \frac{1-2C}{4}\right) \langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)}$$

and

$$\langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} - \langle v_h, v_h \rangle_{L^2(\mathbb{S}^3)} \geq \left(1 - \frac{1}{1 + (1-2C)/4}\right) \langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)}.$$

Set $\delta = 2\left(1 - \frac{1}{1+(1-2C)/4}\right)$. Since $\langle \mathcal{D}v_h, v_h \rangle_{L^2(\mathbb{S}^3)} = \|h\|_{\dot{H}^1(\mathbb{H}^1)}^2$, the following theorem holds.

Theorem 2.4.15. *The linearized operator \mathcal{L} around Q_+*

$$\mathcal{L}h = D_s h - 2\Pi_0^+(|Q_+|^2 h) - \Pi_0^+(Q_+^2 \bar{h})$$

is coercive outside the finite-dimensional subspace spanned by Q_+ , iQ_+ , $\partial_s Q_+$ and $i\partial_s Q_+$: there exists $\delta > 0$ such that for all h in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap (Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$, then

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq \delta \|h\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

Note that for the Szegő equation, Pocovnicu proved in [Poc12] that the linearized operator is coercive in directions which are symplectically orthogonal to the manifold of solitons

$$\left\{ \frac{\alpha \mu e^{i\theta}}{\mu(x-a) + i} \mid \mu \in \mathbb{R}_+^*, \alpha \in \mathbb{R}_+^*, \theta \in \mathbb{T}, a \in \mathbb{R} \right\}.$$

The nondegeneracy follows from this theorem and the study of \mathcal{L} on the finite-dimensional subspace $V = \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)$ (part 2.4.2).

Corollary 2.4.16. *The linearized operator \mathcal{L} is non degenerate:*

$$\text{Ker}(\mathcal{L}) = \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+).$$

2.4.5 Invertibility of \mathcal{L}

The following corollaries of Theorem 2.4.15 make precise the invertibility of \mathcal{L} and the linear stability up to symmetries of the ground state Q_+ . These estimates will be useful in order to prove the invertibility of the linearized operators \mathcal{L}_{Q_β} around Q_β in section 2.5.

Corollary 2.4.17. *There exists $c > 0$ such that for all $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$,*

$$\|\mathcal{L}h\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \geq c\|h\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Proof. Let $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$. We decompose h into three orthogonal components $h = h_0 + h_- + h_+$, where $h_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)$, $h_- \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+)$ and $h_+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$. Then $\mathcal{L}h_0 = 0$, and $\mathcal{L}h_+$ satisfies the above coercivity estimate 2.4.15: for some $\delta > 0$,

$$\|\mathcal{L}h_+\|_{\dot{H}^{-1}(\mathbb{H}^1)} \geq \delta\|h_+\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Write $h_- = \lambda Q_+$ for some real number λ . Then $\mathcal{L}h_- = 2\lambda i\partial_s Q_+$, so

$$(\mathcal{L}h_-, h_-)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = 2\lambda^2 (i\partial_s Q_+, Q_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}.$$

But

$$\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2 = (-i\lambda\partial_s Q_+, \lambda Q_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)},$$

so $(\mathcal{L}h_-, h_-)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = -2\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2$. In particular, $\|\mathcal{L}h_-\|_{\dot{H}^{-1}(\mathbb{H}^1)} \geq 2\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}$.

Thanks to Corollary 2.4.9, we deduce that

$$\begin{aligned} \|\mathcal{L}h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 &= \|\mathcal{L}h_-\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 + \|\mathcal{L}h_+\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 \\ &\geq 4\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \delta^2\|h_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \\ &\geq (\min(2, \delta))^2 \|h_- + h_+\|_{\dot{H}^1(\mathbb{H}^1)}^2. \end{aligned}$$

Moreover, since h_0 is in the space spanned by $\partial_s Q_+$, iQ_+ and $Q_+ + 2iQ_+$, there exists some constant $0 < c \leq \min(2, \delta)$ such that

$$|(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \geq c\|h_0\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Therefore,

$$\|\mathcal{L}h\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \geq c\|h\|_{\dot{H}^1(\mathbb{H}^1)}. \quad \square$$

Let us recall that for $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, we have set in Definition 2.3.8

$$\delta(u) = \left| \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \right| + \left| \|u\|_{L^4(\mathbb{H}^1)}^4 - \|Q_+\|_{L^4(\mathbb{H}^1)}^4 \right|.$$

Corollary 2.4.18. *There exists $\varepsilon_0 > 0$, $c > 0$ such that for all $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, if $\|u - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \leq \varepsilon_0$, then*

$$\delta(u) + |(u, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(u, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(u, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \geq c\|u - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

Proof. Let $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and set $h = u - Q_+$. We decompose h as above in three orthogonal parts $h = h_0 + h_- + h_+$, where $h_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)$, $h_- \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap \text{Vect}_{\mathbb{R}}(Q_+)$ and $h_+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \cap (Q_+, iQ_+, \partial_s Q_+, i\partial_s Q_+)^\perp, \dot{H}^1(\mathbb{H}^1)$.

The link between $\delta(u)$ and the linearized operator \mathcal{L} appears through the functional

$$E(u) := \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \frac{1}{2}\|u\|_{L^4(\mathbb{H}^1)}^4.$$

Indeed,

$$|E(u) - E(Q_+)| \leq \delta(u),$$

but since Q_+ is a solution to $D_s Q_+ = \Pi_0^+(|Q_+|^2 Q_+)$ and h belongs to $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$, we have the Taylor expansion

$$E(u) - E(Q_+) = (\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} + \mathcal{O}(\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3).$$

Therefore,

$$\delta(u) \geq (\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} - \mathcal{O}(\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3).$$

From Corollary 2.4.9, we know that

$$(\mathcal{L}h, h)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = (\mathcal{L}h_+, h_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} + (\mathcal{L}h_-, h_-)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}.$$

Consequently, the coercivity estimate on \mathcal{L} implies that for some constants $c_1, C_1 > 0$,

$$\delta(u) \geq c_1\|h_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 - C_1(\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \|h\|_{\dot{H}^1(\mathbb{H}^1)}^3). \quad (2.19)$$

Let us focus on the term $\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2$. We use the fact that

$$\begin{aligned} \delta(u) &\geq |(Q_+ + h, Q_+ + h)_{\dot{H}^1(\mathbb{H}^1)} - (Q_+, Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \\ &\geq 2|(Q_+, h)_{\dot{H}^1(\mathbb{H}^1)}| - \|h\|_{\dot{H}^1(\mathbb{H}^1)}^2 \\ &= 2\|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}\|h_-\|_{\dot{H}^1(\mathbb{H}^1)} - \|h\|_{\dot{H}^1(\mathbb{H}^1)}^2 \end{aligned}$$

to deduce

$$\delta(u)^2 \geq 4\|Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \mathcal{O}(\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3).$$

We use this estimate to control $\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2$ in the lower bound (2.19) of $\delta(u)$. Up to decreasing ε_0 , one can absorb the term $\delta(u)^2$ into the term $\delta(u)$: there exist $c_2, C_2 > 0$ and $\varepsilon_0 > 0$ such that if $\|h\|_{\dot{H}^1(\mathbb{H}^1)} = \|u - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \leq \varepsilon_0$,

$$2\delta(u) \geq \delta(u) + C_2\delta(u)^2 \geq c_2\|h_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 + c_2\|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2 - C_2\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3.$$

We now control $\|h_0\|_{\dot{H}^1(\mathbb{H}^1)}^2$. If $\varepsilon_0 \leq 1$, we have an upper bound

$$\|h_0\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq \|h_0\|_{\dot{H}^1(\mathbb{H}^1)} \leq C(|(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}|).$$

In the end, there exist $c_3 > 0$ and $C_3 > 0$ such that for all $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$,

$$\begin{aligned} \delta(u) + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \\ \geq c_3(\|h_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \|h_-\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \|h_0\|_{\dot{H}^1(\mathbb{H}^1)}^2) - C_3\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3 \\ = c_3\|h\|_{\dot{H}^1(\mathbb{H}^1)}^2 - C_3\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3. \end{aligned}$$

Up to decreasing ε_0 again, we can absorb the term $\|h\|_{\dot{H}^1(\mathbb{H}^1)}^3$ into the term $\|h\|_{\dot{H}^1(\mathbb{H}^1)}^2$. Note that Q_+ is orthogonal in $\dot{H}^1(\mathbb{H}^1)$ to $\partial_s Q_+$, iQ_+ and $Q_+ + 2i\partial_s Q_+$, therefore $(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = (u, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}$, $(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)} = (u, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}$ and $(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = (u, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}$. \square

We now control the distance of a function $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ to the profile Q_+ up to symmetries by the difference of their norms $\delta(u)$.

Definition 2.4.19. Fix $h \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, $s_0 \in \mathbb{R}$, $\theta \in \mathbb{T}$ and $\alpha \in \mathbb{R}_+^*$. We denote by $T_{s_0, \theta, \alpha} h$ the function in $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$ defined by

$$T_{s_0, \theta, \alpha} h(x, y, s) := e^{i\theta} \alpha h(\alpha x, \alpha y, \alpha^2(s + s_0)), \quad (x, y, s) \in \mathbb{H}^1.$$

Corollary 2.4.20. There exist $\delta_0 > 0$ and $C > 0$ such that for all $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, if $\delta(u) \leq \delta_0$, then

$$\inf_{(s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|T_{s_0, \theta, \alpha} u - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq C\delta(u).$$

Proof. Assume by contradiction that there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ such that $\delta(u_n) \rightarrow 0$, but

$$\frac{1}{\delta(u_n)} \inf_{(s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|T_{s_0, \theta, \alpha} u_n - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \xrightarrow{n \rightarrow +\infty} +\infty.$$

According to the consequence of the profile decomposition theorem stated in Proposition 2.3.10, since $\delta(u_n) \rightarrow 0$, then, up to a subsequence, there exist cores $(s_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, an angle $\theta_0 \in \mathbb{T}$, and scalings $(\alpha_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that

$$\|T_{s_n, \theta_0, \alpha_n} u_n - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0.$$

We will make use of the implicit function theorem in order to apply Corollary 2.4.18 with some functions $T_{s'_n, \theta'_n, \alpha'_n} u_n$ orthogonal to $\partial_s Q_+$, iQ_+ and $Q_+ + 2i\partial_s Q_+$ and get a contradiction. Consider the maps

$$\begin{aligned} F : \dot{H}^1(\mathbb{H}^1) \cap V_0^+ &\rightarrow \mathbb{R}^3 \\ u &\mapsto ((u, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}, (u, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}, (u, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}), \end{aligned}$$

and

$$\begin{aligned} G : \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^* \times (\dot{H}^1(\mathbb{H}^1) \cap V_0^+) &\rightarrow \mathbb{R}^3 \\ (s, \theta, \alpha, u) &\mapsto F(T_{s, \theta, \alpha} u). \end{aligned}$$

Then $F(Q_+) = 0$ so $G(0, 0, 1, Q_+) = 0$. Moreover, G is smooth in (s, θ, α) and the Jacobian $d_{s, \theta, \alpha} G(0, 0, 1, Q_+)$ of this application along (s, θ, α) at $(s, \theta, \alpha, u) = (0, 0, 1, Q_+)$ is equal to

$$\begin{pmatrix} \|\partial_s Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 & (iQ_+, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} & (Q_+ + 2i\partial_s Q_+, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} \\ (\partial_s Q_+, iQ_+)_{\dot{H}^1(\mathbb{H}^1)} & \|iQ_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 & (Q_+ + 2i\partial_s Q_+, iQ_+)_{\dot{H}^1(\mathbb{H}^1)} \\ (\partial_s Q_+, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} & (iQ_+, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} & \|Q_+ + 2i\partial_s Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \end{pmatrix}.$$

Replacing all the terms by their values, we get

$$d_{s, \theta, \alpha} G(0, 0, 1, Q_+) = \begin{pmatrix} \frac{\pi^2}{2} & \frac{\pi^2}{2} & 0 \\ 0 & \frac{\pi^2}{2} & 0 \\ 0 & 0 & \pi^2 \end{pmatrix},$$

which is invertible. By the implicit function theorem, we get continuously differentiable functions $S_0(u)$, $\Theta(u)$ and $A(u)$, defined in a neighborhood \mathcal{V} of Q_+ and valued in a neighborhood of $(0, 0, 1)$: if $u \in \mathcal{V}$, then $\|T_{S_0(u), \Theta(u), A(u)} u - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \leq \varepsilon_0$ (where ε_0 is taken from Corollary 2.4.18). These functions satisfy $(S_0(Q_+), \Theta(Q_+), A(Q_+)) = (0, 0, 1)$ and

$$G(S_0(u), \Theta(u), A(u), u) = 0.$$

Now, since $\|T_{s_n, \theta_0, \alpha_n} u_n - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \xrightarrow{n \rightarrow +\infty} 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $T_{s_n, \theta_0, \alpha_n} u_n \in \mathcal{V}$. Therefore, defining $s'_n = s_n + S_0(T_{s_n, \theta_0, \alpha_n} u_n)$, $\theta'_n = \theta_0 + \Theta(T_{s_n, \theta_0, \alpha_n} u_n)$ and $\alpha'_n = \alpha_n + A(T_{s_n, \theta_0, \alpha_n} u_n)$, we get $\tilde{u}_n := T_{s'_n, \theta'_n, \alpha'_n} u_n \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ such that $\|\tilde{u}_n - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} \leq \varepsilon_0$ and

$$(\tilde{u}_n, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = (\tilde{u}_n, iQ_+)_{\dot{H}^1(\mathbb{H}^1)} = (\tilde{u}_n, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = 0.$$

Moreover, by invariance under symmetries, $\delta(\tilde{u}_n) = \delta(u_n)$, so applying Corollary 2.4.18 to $\tilde{u}_n = T_{s'_n, \theta'_n, \alpha'_n} u_n$, we get that for some constant $C > 0$,

$$\|T_{s'_n, \theta'_n, \alpha'_n} u_n - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq C\delta(u_n).$$

This is a contradiction with the assumption that

$$\frac{1}{\delta(u_n)} \inf_{(s, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|T_{s, \theta, \alpha} u_n - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2 \xrightarrow{n \rightarrow +\infty} +\infty. \quad \square$$

2.5 Uniqueness of traveling waves for the Schrödinger equation

In this section, we show that the study of the limiting profile Q_+ , and in particular the linear stability, enables us to prove some uniqueness results about the sequence of traveling waves Q_β with speed β sufficiently close to 1. The argument is similar to that of [Gér+18] for the half-wave equation: for β close to 1, Q_β is close to Q_+ so we can make a link between the respective linearized operators.

In order to do so, we first need to show some regularity properties and decay estimates on the profiles Q_β (part 2.5.1). For the half-wave equation, these estimates come from the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$, $2 \leq p < +\infty$, and the convergence in $H^{\frac{1}{2}}(\mathbb{R})$.

Recall that from Definition 2.3.5, \mathcal{Q}_β denotes the set of ground states Q_β satisfying (2.3). One can summarize the convergence of $(Q_\beta)_\beta$ from part 2.3.2 combined with the uniqueness result for Q_+ from section 2.3.3 as follows.

Proposition 2.5.1. *For all $\beta \in (-1, 1)$, fix a ground state $Q_\beta^0 \in \mathcal{Q}_\beta$ of speed β . Then there exist scalings $(\alpha_\beta)_\beta$ in \mathbb{R}_+^* , cores $(s_\beta)_\beta$ in \mathbb{R} , and an angle θ in \mathbb{T} such that after a change of functions $Q_\beta := e^{i\theta} \alpha_\beta Q_\beta(\alpha_\beta \cdot, \alpha_\beta \cdot, \alpha_\beta^2(\cdot + s_\beta))$, the sequence $(Q_\beta)_\beta$ of solutions to (2.3)*

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta = |Q_\beta|^2 Q_\beta$$

converges as $\beta \rightarrow 1$ in $\dot{H}^1(\mathbb{H}^1)$ to the unique (up to symmetries) ground state solution to (2.4), namely $D_s Q_+ = \Pi_0^+(|Q_+|^2 Q_+)$, which writes

$$Q_+(x, y, s) = \frac{i\sqrt{2}}{s + i(x^2 + y^2) + i}.$$

2.5.1 Regularity and decay of the traveling waves Q_β

In this part, we collect information on the regularity of the profiles Q_β . We show that after the transformations from Proposition 2.5.1, they are uniformly bounded in $L^p(\mathbb{H}^1)$ for all $p > 2$ when β is close to 1. We deduce a uniform bound in $L^\infty(\mathbb{H}^1)$, from which we estimate the decay of these profiles when the variable $(x, y, s) \in \mathbb{H}^1$ tends to infinity. Finally, we show that $(Q_\beta)_\beta$ is bounded in $\dot{H}^k(\mathbb{H}^1)$ for β close to 1 and fixed $k \geq 1$.

The operator $-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta}$ admits an explicit fundamental solution.

Theorem 2.5.2 (Fundamental solution [SM93]). *Let*

$$m_\beta(x, y, s) = -\frac{1 - \beta}{2\pi^2} \Gamma\left(\frac{1 - \beta}{2}\right) \Gamma\left(\frac{1 + \beta}{2}\right) \frac{1}{(x^2 + y^2 - is)^{\frac{1-\beta}{2}} (x^2 + y^2 + is)^{\frac{1+\beta}{2}}}.$$

Then m_β is a fundamental solution for $-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta}$: in the sense of distributions,

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} m_\beta = \delta_0.$$

The proof of regularity for the Q_β relies on the use of generalized Hölder's and Young's inequalities in weak Lebesgue spaces (see [Vet19] for the strategy). We define the Lorentz spaces as follows.

Definition 2.5.3 (Lorentz spaces). *Fix $p \in [1, \infty)$ and $q \in [1, \infty]$. The Lorentz space $L^{p,q}(\mathbb{H}^1)$ is the set of all functions $f : \mathbb{H}^1 \rightarrow \mathbb{C}$ with finite $L^{p,q}(\mathbb{H}^1)$ norm, where*

$$\|f\|_{L^{p,q}(\mathbb{H}^1)} := \begin{cases} \left(p \int_0^{+\infty} R^{q-1} \lambda_3(\{u \in \mathbb{H}^1 \mid |f(u)| \geq R\})^{\frac{q}{p}} dR \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{R>0} (R^p \lambda_3(\{u \in \mathbb{H}^1 \mid |f(u)| \geq R\}))^{\frac{1}{p}} & \text{if } q = \infty. \end{cases}$$

The usual $L^p(\mathbb{H}^1)$ spaces coincide with $L^{p,p}(\mathbb{H}^1)$ spaces. In general, $\|\cdot\|_{L^{p,q}(\mathbb{H}^1)}$ is not a norm since the Minkowski inequality may fail. The following inclusion relations are true.

Proposition 2.5.4 (Growth of $L^{p,q}$ spaces [SW71]). *Let $p \in [1, \infty)$ and $q_1, q_2 \in [1, \infty]$ such that $q_1 \leq q_2$. Then $L^{p,q_1}(\mathbb{H}^1) \subset L^{p,q_2}(\mathbb{H}^1)$.*

Note that the functions m_β , $\beta \in [0, 1)$, are uniformly bounded in $L^{2,\infty}$. Indeed, let $R > 0$, then

$$\lambda_3(\{(x, y, s) \in \mathbb{H}^1 \mid |x|^2 + |y|^2 + |s| \leq R^{-1}\}) = R^{-2} \lambda_3(\{(x', y', s') \in \mathbb{H}^1 \mid |x'|^2 + |y'|^2 + |s'| \leq 1\}),$$

moreover, the constants

$$c_\beta := -\frac{1-\beta}{2\pi^2} \Gamma\left(\frac{1-\beta}{2}\right) \Gamma\left(\frac{1+\beta}{2}\right)$$

are bounded for $\beta \in [0, 1)$.

Definition 2.5.5 (Convolution). *The convolution product of two functions f and g on \mathbb{H}^1 is defined by*

$$f \star g(u) = \int_{\mathbb{H}^1} f(v)g(v^{-1}u) d\lambda_3(v) = \int_{\mathbb{H}^1} f(uv^{-1})g(v) d\lambda_3(v).$$

Note that the convolution in \mathbb{H}^1 is not commutative, and that the relation

$$P(f \star g) = f \star Pg$$

holds for every left-invariant vector field P in \mathbb{H}^1 (for example, $P = -\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta}$), whereas in general $P(f \star g) \neq Pf \star g$.

Let us recall the generalizations of Hölder's and Young's inequalities for Lorentz spaces.

Lemma 2.5.6 (Hölder). *Let $p_1, p_2, p \in (0, \infty)$ and $q_1, q_2, q \in (0, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}$ with the convention $1/\infty = 0$. Then there exists $C = C(p_1, p_2, p, q_1, q_2, q)$ such that for any $f \in L^{p_1, q_1}(\mathbb{H}^1)$ and any $g \in L^{p_2, q_2}(\mathbb{H}^1)$, we have $fg \in L^{p, q}(\mathbb{H}^1)$ and*

$$\|fg\|_{L^{p, q}(\mathbb{H}^1)} \leq C \|f\|_{L^{p_1, q_1}(\mathbb{H}^1)} \|g\|_{L^{p_2, q_2}(\mathbb{H}^1)}.$$

Lemma 2.5.7 (Young). *Let $p_1, p_2, p \in (1, \infty)$ and $q_1, q_2, q \in (0, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}$ with the convention $1/\infty = 0$. Then there exists $C = C(p_1, p_2, p, q_1, q_2, q)$ such that for any $f \in L^{p_1, q_1}(\mathbb{H}^1)$ and any $g \in L^{p_2, q_2}(\mathbb{H}^1)$, we have $f \star g \in L^{p, q}(\mathbb{H}^1)$ and*

$$\|f \star g\|_{L^{p, q}(\mathbb{H}^1)} \leq C \|f\|_{L^{p_1, q_1}(\mathbb{H}^1)} \|g\|_{L^{p_2, q_2}(\mathbb{H}^1)}.$$

Theorem 2.5.2 implies the following formula for Q_β .

Corollary 2.5.8. *For all $\beta \in (-1, 1)$,*

$$Q_\beta = (|Q_\beta|^2 Q_\beta) \star m_\beta.$$

Let us now prove the boundedness of Q_β in $L^p(\mathbb{H}^1)$, $p > 2$.

Theorem 2.5.9. *For all $p > 2$, there exist $C_p > 0$ and $\beta_*(p) \in (0, 1)$ such that for all $\beta \in (\beta_*(p), 1)$, $\|Q_\beta\|_{L^p(\mathbb{H}^1)} \leq C_p$.*

Proof. We proceed by contradiction. Fix $p > 2$. Assume that there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ in $(0, 1)$ converging to 1 and such that $\|Q_{\beta_n}\|_{L^p(\mathbb{H}^1)} \in [n, +\infty]$ for all $n \in \mathbb{N}$. By

duality and density of $\mathcal{C}_c^\infty(\mathbb{H}^1)$ in $L^q(\mathbb{H}^1)$, $\frac{1}{p} + \frac{1}{q} = 1$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $L^q(\mathbb{H}^1) \cap L^{\frac{4}{3}}(\mathbb{H}^1)$ such that $\|\varphi_n\|_{L^q(\mathbb{H}^1)} \leq 1$ for all n and

$$\left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 \right| \xrightarrow{n \rightarrow +\infty} +\infty.$$

Let us define

$$K_n := \left\{ \varphi \in L^q(\mathbb{H}^1) \cap L^{\frac{4}{3}}(\mathbb{H}^1) \mid \|\varphi\|_{L^q(\mathbb{H}^1)} \leq \|\varphi_n\|_{L^q(\mathbb{H}^1)} \text{ and } \|\varphi\|_{L^{\frac{4}{3}}(\mathbb{H}^1)} \leq \|\varphi_n\|_{L^{\frac{4}{3}}(\mathbb{H}^1)} \right\}.$$

Since $Q_{\beta_n} \in L^4(\mathbb{H}^1)$, the supremum over functions $\varphi \in K_n$ of $\int_{\mathbb{H}^1} Q_{\beta_n} \varphi \, d\lambda_3$ is finite. Thus, if we change φ_n to another function φ from K_n where $\int_{\mathbb{H}^1} Q_{\beta_n} \varphi \, d\lambda_3$ is closer to this supremum, the K_n corresponding to φ and thus the new supremum will decrease. We therefore assume up to changing φ_n that

$$2 \left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 \right| \geq \sup_{\varphi \in K_n} \left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi \, d\lambda_3 \right|.$$

By density, let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{C}_c^\infty(\mathbb{H}^1)$ such that $\| |Q_+|^2 - f_k \|_{L^2(\mathbb{H}^1)} \xrightarrow{k \rightarrow +\infty} 0$. Denote, for $k, n \in \mathbb{N}$, $g_{n,k} := |Q_{\beta_n}|^2 - f_k$. We will use the fact that the functions $g_{n,k}$ have a small norm in $L^2(\mathbb{H}^1)$ when k and n are large enough thanks to Proposition 2.5.1. Let us cut

$$\begin{aligned} \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 &= \int_{\mathbb{H}^1} (|Q_{\beta_n}|^2 Q_{\beta_n}) \star m_{\beta_n} \varphi_n \, d\lambda_3 \\ &= \int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 + \int_{\mathbb{H}^1} ((g_{n,k} Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 \end{aligned}$$

in order to evaluate these terms separately.

Concerning the first term on the right-hand side, using Lemmas 2.5.6 and 2.5.7,

$$\begin{aligned} \left| \int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 \right| &\leq \|((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n\|_{L^{1,1}(\mathbb{H}^1)} \\ &\leq C_1(p) \|((f_k Q_{\beta_n}) \star m_{\beta_n})\|_{L^{p,p}(\mathbb{H}^1)} \|\varphi_n\|_{L^{q,q}(\mathbb{H}^1)} \\ &\leq C_2(p) \|f_k Q_{\beta_n}\|_{L^{\frac{2p}{2+p},p}(\mathbb{H}^1)} \|m_{\beta_n}\|_{L^{2,\infty}(\mathbb{H}^1)} \|\varphi_n\|_{L^q(\mathbb{H}^1)} \end{aligned}$$

(we used that $\frac{2p}{2+p} > 1$ since $p > 2$). Using again Lemma 2.5.6, choosing any $\tau \in (0, +\infty)$ such that $\frac{1}{\tau} \geq \frac{4-p}{4p}$ and $\sigma = \frac{4p}{4+p} > 1$, we get

$$\left| \int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 \right| \leq C_3(p) \|f_k\|_{L^{\sigma,\tau}(\mathbb{H}^1)} \|Q_{\beta_n}\|_{L^{4,4}(\mathbb{H}^1)} \|m_{\beta_n}\|_{L^{2,\infty}(\mathbb{H}^1)} \|\varphi_n\|_{L^q(\mathbb{H}^1)}.$$

We know that $\|\varphi_n\|_{L^q(\mathbb{H}^1)} \leq 1$ for all n , that $\|m_{\beta_n}\|_{L^{2,\infty}(\mathbb{H}^1)}$ is bounded independently of n and that $(Q_\beta)_{\beta \in [0,1]}$ is bounded in $L^4(\mathbb{H}^1)$, so there exists $C_4(p) > 0$ such that for all $k, n \in \mathbb{N}$,

$$\int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 \leq C_4(p) \|f_k\|_{L^{\sigma,\tau}(\mathbb{H}^1)}.$$

Applying Fubini's theorem to the second term in the right-hand side,

$$\begin{aligned} \int_{\mathbb{H}^1} ((g_{n,k} Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 &= \int_{\mathbb{H}^1} \int_{\mathbb{H}^1} (g_{n,k} Q_{\beta_n})(v) m_{\beta_n}(v^{-1}u) \varphi_n(u) \, d\lambda_3(u) \, d\lambda_3(v) \\ &= \int_{\mathbb{H}^1} \int_{\mathbb{H}^1} (g_{n,k} Q_{\beta_n})(v) \check{m}_{\beta_n}(u^{-1}v) \varphi_n(u) \, d\lambda_3(u) \, d\lambda_3(v) \\ &= \int_{\mathbb{H}^1} (g_{n,k} Q_{\beta_n})(v) (\varphi_n \star \check{m}_{\beta_n})(v) \, d\lambda_3(v), \end{aligned}$$

where

$$\begin{aligned} \check{m}_{\beta}(x, y, s) &= m_{\beta}((x, y, s)^{-1}) \\ &= -\frac{1-\beta}{2\pi^2} \Gamma\left(\frac{1-\beta}{2}\right) \Gamma\left(\frac{1+\beta}{2}\right) \frac{1}{(x^2 + y^2 + is)^{\frac{1-\beta}{2}} (x^2 + y^2 - is)^{\frac{1+\beta}{2}}} \end{aligned}$$

has the same bounds in $L^{2,\infty}(\mathbb{H}^1)$ as m_{β} .

But thanks to Lemmas 2.5.6 and 2.5.7,

$$\begin{aligned} \|g_{n,k}(\varphi_n \star \check{m}_{\beta_n})\|_{L^q(\mathbb{H}^1)} &\leq C'_1(p) \|g_{n,k}\|_{L^{2,\infty}(\mathbb{H}^1)} \|\varphi_n \star \check{m}_{\beta_n}\|_{L^{\frac{2p}{p-2},q}(\mathbb{H}^1)} \\ &\leq C'_2(p) \|g_{n,k}\|_{L^{2,\infty}(\mathbb{H}^1)} \|\varphi_n\|_{L^{q,q}(\mathbb{H}^1)} \|\check{m}_{\beta_n}\|_{L^{2,\infty}(\mathbb{H}^1)}. \end{aligned}$$

Note that the assumption $p > 2$ ensures that $\frac{2p}{p-2} \in (1, \infty)$.

Moreover, this last inequality still holds with the same reasoning when replacing p by 4 and its conjugate exponent q by $\frac{4}{3}$. Fix

$$C = \max(C'_2(p), C'_2(4)) \times \sup_{\beta \in [0,1]} \|\check{m}_{\beta}\|_{L^{2,\infty}(\mathbb{H}^1)}.$$

Then, when $g_{n,k}$ is nonzero in $L^2(\mathbb{H}^1)$, the function

$$\psi_{n,k} := \frac{1}{C \|g_{n,k}\|_{L^{2,\infty}(\mathbb{H}^1)}} g_{n,k}(\varphi_n \star \check{m}_{\beta_n})$$

belongs to K_n . Therefore by definition of φ_n , for all $k, n \in \mathbb{N}$,

$$\left| \int_{\mathbb{H}^1} Q_{\beta_n} g_{n,k}(\varphi_n \star \check{m}_{\beta_n}) \, d\lambda_3 \right| \leq 2C \|g_{n,k}\|_{L^{2,\infty}(\mathbb{H}^1)} \left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 \right|.$$

But

$$\begin{aligned} \|g_{n,k}\|_{L^{2,\infty}(\mathbb{H}^1)} &\leq \| |Q_{\beta_n}|^2 - f_k \|_{L^2(\mathbb{H}^1)} \\ &\leq \| |Q_{\beta_n}|^2 - |Q_+|^2 \|_{L^2(\mathbb{H}^1)} + \| |Q_+|^2 - f_k \|_{L^2(\mathbb{H}^1)}, \end{aligned}$$

and this quantity converges to 0 as $\min(n, k)$ goes to $+\infty$ thanks to Proposition 2.5.1 and the construction of $(f_k)_{k \in \mathbb{N}}$. Therefore, there exists n_0 such that, for all $k \geq n_0$ and $n \geq n_0$, $2C \|g_{n,k}\|_{L^{2,\infty}(\mathbb{H}^1)} \leq \frac{1}{2}$, or in other words,

$$\left| \int_{\mathbb{H}^1} Q_{\beta_n} g_{n,k}(\varphi_n \star \check{m}_{\beta_n}) \, d\lambda_3 \right| \leq \frac{1}{2} \left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 \right|.$$

Since

$$\int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 = \int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 + \int_{\mathbb{H}^1} Q_{\beta_n} g_{n,k}(\varphi_n \star \check{m}_{\beta_n}) \, d\lambda_3,$$

we get that for all $k \geq n_0$ and $n \geq n_0$,

$$\left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 \right| \leq 2 \left| \int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 \right|.$$

Fix $k \geq n_0$ and consider this inequality. There is a contradiction when n goes to $+\infty$, since the right-hand side $2 \left| \int_{\mathbb{H}^1} ((f_k Q_{\beta_n}) \star m_{\beta_n}) \varphi_n \, d\lambda_3 \right|$ remains bounded by $C_4(p) \|f_k\|_{L^{\sigma, \tau}(\mathbb{H}^1)}$, whereas the left-hand side $\left| \int_{\mathbb{H}^1} Q_{\beta_n} \varphi_n \, d\lambda_3 \right|$ tends to $+\infty$. \square

Corollary 2.5.10. *For all $p \in (2, \infty)$ and $q \in (1, \infty)$, there exist $C_{p,q} > 0$ and $\beta_*(p, q) \in (0, 1)$ such that for all $\beta \in (\beta_*(p, q), 1)$, $\|Q_\beta\|_{L^{p,q}(\mathbb{H}^1)} \leq C_{p,q}$.*

We now collect some estimates on the decay of Q_β when β is close to 1.

Theorem 2.5.11. *There exist $C > 0$ and $\beta_* \in (0, 1)$ such that, for all $\beta \in (\beta_*, 1)$ and all $(x, y, s) \in \mathbb{H}^1$,*

$$|Q_\beta(x, y, s)| \leq \frac{C}{\rho(x, y, s)^2 + 1},$$

where $\rho(x, y, s) = ((x^2 + y^2)^2 + s^2)^{\frac{1}{4}}$ is the distance from $(x, y, s) \in \mathbb{H}^1$ to the origin.

Proof. Let us first show that the Q_β are uniformly bounded in $L^\infty(\mathbb{H}^1)$ for $\beta \in (\beta_*, 1)$, where β_* is large enough. Let $u \in \mathbb{H}^1$. Applying Hölder's inequality 2.5.6 to the right-hand side,

$$\begin{aligned} |Q_\beta(u)| &= \left| \int_{v \in \mathbb{H}^1} |Q_\beta|^2 Q_\beta(v) m_\beta(v^{-1}u) \, d\lambda_3(v) \right| \\ &\leq \| |Q_\beta|^2 Q_\beta m_\beta(\cdot^{-1}u) \|_{L^1(\mathbb{H}^1)} \\ &\leq \| |Q_\beta|^2 Q_\beta \|_{L^{2,1}(\mathbb{H}^1)} \|m_\beta\|_{L^{2,\infty}(\mathbb{H}^1)} \\ &\leq \|Q_\beta\|_{L^{6,3}(\mathbb{H}^1)}^3 \|m_\beta\|_{L^{2,\infty}(\mathbb{H}^1)}. \end{aligned}$$

The conclusion follows from Corollary 2.5.10.

For every $R > 0$, we set $B_R = \{(x, y, s) \in \mathbb{H}^1 \mid \rho(x, y, s) \leq R\}$ and

$$M(R) = \sup_{(x,y,s) \in B_R^c} |Q_\beta(x, y, s)|.$$

Let $R > 0$, $u \in B_R^c$. We split the integral

$$\begin{aligned} |(|Q_\beta|^2 Q_\beta) \star m(u)| &\leq \left| \int_{v \in B_{R/2}} |Q_\beta|^2 Q_\beta(v) m_\beta(v^{-1}u) \, d\lambda_3(v) \right| \\ &\quad + \left| \int_{v \in B_{R/2}^c} |Q_\beta|^2 Q_\beta(v) m_\beta(v^{-1}u) \, d\lambda_3(v) \right|. \end{aligned}$$

For the first summand, if $v \in B_{R/2}$ then $uv^{-1} \in B_{R/2}^c$, so

$$\left| \int_{v \in B_{R/2}} |Q_\beta|^2 Q_\beta(v) m_\beta(v^{-1}u) \, d\lambda_3(v) \right| \leq \frac{|c_\beta|}{R^2} \|Q_\beta\|_{L^3(\mathbb{H}^1)}^3.$$

Thanks to Theorem 2.5.9, one knows that up to increasing β_* , there exists some constant C such that $|c_\beta| \|Q_\beta\|_{L^3(\mathbb{H}^1)}^3 \leq C$ for all $\beta \in (\beta_*, 1)$.

To estimate the second summand, we apply Hölder's inequality (Lemma 2.5.6)

$$\begin{aligned} \left| \int_{v \in B_{R/2}^c} |Q_\beta|^2 Q_\beta(v) m_\beta(v^{-1}u) \, d\lambda_3(v) \right| &\leq \| |Q_\beta|^2 m_\beta(\cdot^{-1}u) \|_{L^1(B_{R/2}^c)} M\left(\frac{R}{2}\right) \\ &\leq \| |Q_\beta|^2 \|_{L^{2,1}(B_{R/2}^c)} \| m_\beta(\cdot^{-1}u) \|_{L^{2,\infty}(B_{R/2}^c)} M\left(\frac{R}{2}\right) \\ &\leq \| Q_\beta \|_{L^{4,4}(B_{R/2}^c)} \| Q_\beta \|_{L^{4,4/3}(B_{R/2}^c)} \| m_\beta \|_{L^{2,\infty}(\mathbb{H}^1)} M\left(\frac{R}{2}\right). \end{aligned}$$

Thanks to the convergence of $(Q_\beta)_\beta$ to Q_+ in $\dot{H}^1(\mathbb{H}^1)$ as β tends to 1 and the Folland-Stein embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$, the sequence $(Q_\beta)_\beta$ converges to Q_+ in $L^4(\mathbb{H}^1)$ and therefore is tight in $L^4(\mathbb{H}^1)$. Moreover, the norms $\|Q_\beta\|_{L^{4,4/3}(\mathbb{H}^1)}$, for β close to 1, are bounded. Therefore, up to increasing β_* again, one can choose $R_0 > 0$ such that

$$\sup_{\beta \in (\beta_*, 1)} \left(\|Q_\beta\|_{L^{4,4}(B_{R_0/2}^c)} \|Q_\beta\|_{L^{4,4/3}(B_{R_0/2}^c)} \|m_\beta\|_{L^{2,\infty}(\mathbb{H}^1)} \right) \leq \frac{1}{8}.$$

Then, for every $R \geq R_0$,

$$\left| \int_{v \in B_{R/2}^c} |Q_\beta|^2 Q_\beta(v) m_\beta(v^{-1}u) \, d\lambda_3(v) \right| \leq \frac{1}{8} M\left(\frac{R}{2}\right).$$

Combining the two estimates and applying them to $R = 2^n$, $n \geq n_0$ so that $2^{n_0} \geq R_0$, we get

$$M(2^n) \leq \frac{C}{4^n} + \frac{1}{8} M(2^{n-1}).$$

Iterating, one knows that for all $n \geq n_0$,

$$\begin{aligned} M(2^n) &\leq C \sum_{k=0}^{n-n_0} \frac{1}{4^{n-k}} \frac{1}{8^k} + \frac{1}{8^{n-n_0+1}} M(2^{n_0-1}) \\ &\leq C 4^{-n} \sum_{k=0}^{n-n_0} 4^{-k} + 8^{n_0+1} M(2^{n_0-1}) 8^{-n} \\ &\leq (2C + 8^{n_0+1} M(2^{n_0-1})) 4^{-n}. \end{aligned}$$

Since $\rho(u) \sim 2^n$ for $2^n \leq \rho(u) \leq 2^{n+1}$, this completes the proof of the result. \square

Corollary 2.5.12. *For some $\beta_* \in (0, 1)$, for all $k \geq 1$, there exists $C_k > 0$ such that for all $\beta \in (\beta_*, 1)$,*

$$\|Q_\beta\|_{\dot{H}^k(\mathbb{H}^1)} \leq C_k.$$

Proof. It is enough to prove the first part of the claim for $k \in \mathbb{N}$. We proceed by induction on k . We already know that it is true for $k = 1$ because

$$\|Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq \frac{(-(\Delta_{\mathbb{H}^1} + \beta D_s)Q_\beta, Q_\beta)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}}{1 - \beta} = \frac{I_\beta}{(1 - \beta)^2},$$

and $(\frac{I_\beta}{(1-\beta)^2})_\beta$ is bounded (see part 2.3.2).

The following additional assumption will be useful in the induction step. Up to increasing β_* , we can assume that the Q_β are bounded in $L^6(\mathbb{H}^1)$ and in $L^\infty(\mathbb{H}^1)$ for $\beta \in (\beta_*, 1)$.

Suppose now that the Q_β are bounded in $\dot{H}^k(\mathbb{H}^1)$ for an integer $k \geq 1$. Then by Leibniz' rule, since for radial functions $\Delta_{\mathbb{H}^1} = \frac{1}{4}(X^2 + Y^2)$ with $X = \partial_x + 2y\partial_s$ and $Y = \partial_y - 2x\partial_s$, there exist some coefficients c_λ such that

$$\begin{aligned} -\Delta_{\mathbb{H}^1}^{k-1} \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta \right) &= -\Delta_{\mathbb{H}^1}^{k-1} (|Q_\beta|^2 Q_\beta) \\ &= \sum_{|\lambda_1| + |\lambda_2| + |\lambda_3| = 2k-2} c_\lambda \partial^{\lambda_1} (Q_\beta) \partial^{\lambda_2} (Q_\beta) \partial^{\lambda_3} (\overline{Q_\beta}). \end{aligned}$$

The notation is similar as in \mathbb{R}^N , λ_j being a finite sequence of letters X and Y of length $|\lambda_j|$, $\partial^X := X$, $\partial^Y := Y$. The following inequality can be easily proven via the Fourier transform:

$$\begin{aligned} (-\Delta_{\mathbb{H}^1}^{k+1} Q_\beta, Q_\beta)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= (-\Delta_{\mathbb{H}^1}^k Q_\beta, -\Delta_{\mathbb{H}^1} Q_\beta)_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)} \\ &\leq \left(-\Delta_{\mathbb{H}^1}^{k-1} \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta \right), -\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta \right)_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)} \\ &\leq \left(-\Delta_{\mathbb{H}^1}^{k-1} \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta \right), |Q_\beta|^2 Q_\beta \right)_{\dot{H}^1(\mathbb{H}^1) \times \dot{H}^{-1}(\mathbb{H}^1)}. \end{aligned}$$

We replace the term on the left by the above sum. By integration by parts and Leibniz' rule again, we can manage so that the following indexes of derivation μ_i all have length less or equal than $(k-1)$:

$$(-\Delta_{\mathbb{H}^1}^{k+1} Q_\beta, Q_\beta)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} = \sum_{\substack{|\mu_1| + \dots + |\mu_6| = 2k-2, \\ |\mu_1|, \dots, |\mu_6| \leq k-1}} c'_\mu \int_{\mathbb{H}^1} \partial^{\mu_1} (Q_\beta) \dots \partial^{\mu_4} (Q_\beta) \partial^{\mu_5} (\overline{Q_\beta}) \partial^{\mu_6} (\overline{Q_\beta}).$$

We now apply Hölder's inequality with exponents $p_1, \dots, p_6 \in (2, \infty)$ satisfying $\frac{1}{p_1} + \dots + \frac{1}{p_6} = 1$, to be chosen later. Then, denoting $m_j = |\mu_j|$,

$$\left| \int_{\mathbb{H}^1} \partial^{\mu_1} (Q_\beta) \dots \partial^{\mu_4} (Q_\beta) \partial^{\mu_5} (\overline{Q_\beta}) \partial^{\mu_6} (\overline{Q_\beta}) \right| \leq \|Q_\beta\|_{\dot{W}^{m_1, p_1}(\mathbb{H}^1)} \dots \|Q_\beta\|_{\dot{W}^{m_6, p_6}(\mathbb{H}^1)}.$$

Let us choose the p_i appropriately. The aim is to use complex interpolation, and in particular the following relation between homogeneous Sobolev spaces (see e.g. [BL76], Theorem 6.4.5, assertion (7))

$$(L^q(\mathbb{H}^1), \dot{H}^k(\mathbb{H}^1))_\theta = \dot{W}^{m, p}(\mathbb{H}^1)$$

where $p, q \in (2, \infty)$, $m = (1 - \theta)0 + \theta k$ and $\frac{1}{p} = \frac{1 - \theta}{q} + \frac{\theta}{2}$. For example, we choose $\theta_i = \frac{m_i}{k}$ and p_i such that $\frac{1}{p_i} = \frac{1}{6k} + \frac{m_i}{2k}$. Then

$$0 < \frac{1}{p_i} \leq \frac{1}{6k} + \frac{k-1}{2k} = \frac{1+3k-3}{6k} < \frac{1}{2}$$

so $p_i \in (2, \infty)$, and

$$\frac{1}{p_1} + \dots + \frac{1}{p_6} = \frac{1}{k} + \frac{2k-2}{2k} = 1.$$

Moreover, this choice leads to the exponents

$$q_i = \frac{6k}{1 - m_i/k}.$$

Since $0 \leq m_i \leq k-1$, $2 < 6k \leq q_i \leq 6k^2 < \infty$, we can therefore apply the interpolation result.

Since there is a finite number of terms in the sum, the boundedness of Q_β in $L^6(\mathbb{H}^1)$, in $L^\infty(\mathbb{H}^1)$ and in $\dot{H}^k(\mathbb{H}^1)$ for $\beta > \beta_*$ ensures that there exists $C_{k+1} > 0$ such that for $\beta > \beta_*$,

$$\|(-\Delta_{\mathbb{H}^1})^{\frac{k+1}{2}} Q_\beta\|_{L^2(\mathbb{H}^1)} \leq C_{k+1},$$

so the Q_β are bounded in $\dot{H}^{k+1}(\mathbb{H}^1)$. \square

2.5.2 Invertibility of \mathcal{L}_{Q_β}

For $\beta \in (-1, 1)$ the linearized operator around Q_β for the Schrödinger equation is

$$\mathcal{L}_{Q_\beta} h = -\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} h - 2|Q_\beta|^2 h - Q_\beta^2 \bar{h}, \quad h \in \dot{H}^1(\mathbb{H}^1).$$

We prove the invertibility of this operator on a space of finite codimension.

Proposition 2.5.13. *There exist a neighborhood \mathcal{V} of Q_+ , $\beta_* \in (0, 1)$ and some constant $c > 0$ such that for all $\beta \in (\beta_*, 1)$, for all $Q_\beta \in \mathcal{Q}_\beta \cap \mathcal{V}$, and for all $h \in \dot{H}^1(\mathbb{H}^1)$,*

$$\|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \geq c \|h\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Proof. Let $\beta \in (0, 1)$ and $Q_\beta \in \mathcal{Q}_\beta$. Let $h \in \dot{H}^1(\mathbb{H}^1)$. We decompose $h = h^+ + h_\perp$ where $h^+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $h_\perp = h - h^+ \in \dot{H}^1(\mathbb{H}^1) \cap \bigoplus_{(n,\pm) \neq (0,+)} V_n^\pm$.

We split $\mathcal{L}_{Q_\beta} h$ as

$$\mathcal{L}_{Q_\beta} h = \mathcal{L}h^+ - r_+ - r_- + \mathcal{L}_{Q_\beta}^- h,$$

where

$$\begin{aligned} \mathcal{L}h^+ &= -\Delta_{\mathbb{H}^1} h^+ - 2\Pi_0^+(|Q_+|^2 h^+) - \Pi_0^+(Q_+^2 \bar{h}^+), \\ r_+ &= 2\Pi_0^+((|Q_\beta|^2 - |Q_+|^2) h^+) + \Pi_0^+((Q_\beta^2 - Q_+^2) \bar{h}^+), \\ r_- &= 2\Pi_0^+(|Q_\beta|^2 h_\perp) + \Pi_0^+(Q_\beta^2 \bar{h}_\perp), \end{aligned}$$

and

$$\mathcal{L}_{Q_\beta}^- h = -\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} h_\perp - 2(\text{Id} - \Pi_0^+)(|Q_\beta|^2 h) - (\text{Id} - \Pi_0^+)(Q_\beta^2 \bar{h}).$$

We treat each term separately.

- Concerning $\mathcal{L}h^+$, thanks to Corollary 2.4.17,

$$\begin{aligned} \|\mathcal{L}h^+\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h^+, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h^+, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h^+, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \\ \geq c \|h^+\|_{\dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

Since $\partial_s Q_+$, iQ_+ and $(Q_+ + 2i\partial_s Q_+)$ are in V_0^+ , we have $(h^+, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = (h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}$, $(h^+, iQ_+)_{\dot{H}^1(\mathbb{H}^1)} = (h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}$ and $(h^+, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = (h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}$.

• Now consider r_+ and r_- . Let K be the constant in the Folland-Stein embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$

$$\|g\|_{L^4(\mathbb{H}^1)} \leq K \|g\|_{\dot{H}^1(\mathbb{H}^1)}, \quad g \in \dot{H}^1(\mathbb{H}^1).$$

Since the sequence $(\|Q_\beta\|_{L^4(\mathbb{H}^1)})_\beta$ is bounded by some constant C_1 ,

$$\|r_-\|_{L^{\frac{4}{3}}(\mathbb{H}^1)} \leq 3C_1^2 \|h_\perp\|_{L^4(\mathbb{H}^1)} \leq 3KC_1^2 \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}$$

and

$$\begin{aligned} \|r_+\|_{L^{\frac{4}{3}}(\mathbb{H}^1)} &\leq 3\|Q_\beta - Q_+\|_{L^4(\mathbb{H}^1)}(\|Q_\beta\|_{L^4(\mathbb{H}^1)} + \|Q_+\|_{L^4(\mathbb{H}^1)})\|h^+\|_{L^4(\mathbb{H}^1)} \\ &\leq 6C_1\|Q_\beta - Q_+\|_{L^4(\mathbb{H}^1)}\|h^+\|_{L^4(\mathbb{H}^1)} \\ &\leq 6KC_1\|Q_\beta - Q_+\|_{L^4(\mathbb{H}^1)}\|h^+\|_{\dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

Fix $\varepsilon > 0$ to be determined later. There exists $\beta_*(\varepsilon)$ such that for $\beta > \beta_*(\varepsilon)$,

$$\|Q_\beta - Q_+\|_{L^4(\mathbb{H}^1)} \leq \varepsilon.$$

We conclude by the dual embedding $L^{\frac{4}{3}}(\mathbb{H}^1) \hookrightarrow \dot{H}^{-1}(\mathbb{H}^1)$ that there exists a constant C_2 (independent of ε) such that for all $\beta \in (\beta_*(\varepsilon), 1)$,

$$\|r_+\|_{\dot{H}^{-1}(\mathbb{H}^1)} + \|r_-\|_{\dot{H}^{-1}(\mathbb{H}^1)} \leq C_2\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)} + C_2\varepsilon\|h^+\|_{\dot{H}^1(\mathbb{H}^1)}.$$

• Finally, we focus on

$$\mathcal{L}_{Q_\beta}^- h = -\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} h_\perp - 2(\text{Id} - \Pi_0^+)(|Q_\beta|^2 h) - (\text{Id} - \Pi_0^+)(Q_\beta^2 \bar{h}).$$

In order to bound the \dot{H}^{-1} norm of this term, we will use the fact that

$$\begin{aligned} \frac{1}{2}\|\mathcal{L}_{Q_\beta}^- h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 + \frac{1}{2}\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}^2 &\geq \|\mathcal{L}_{Q_\beta}^- h\|_{\dot{H}^{-1}(\mathbb{H}^1)}\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)} \\ &\geq (\mathcal{L}_{Q_\beta}^- h, h_\perp)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

On the one hand, by inequality (2.8),

$$\left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} h_\perp, h_\perp \right)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \geq \frac{1}{2} \frac{1}{1 - \beta} \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

On the other hand,

$$\begin{aligned} \left| (2(\text{Id} - \Pi_0^+)(|Q_\beta|^2 h) + (\text{Id} - \Pi_0^+)(Q_\beta^2 \bar{h}), h_\perp)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \right| &\leq 3C_1^2 \|h\|_{L^4(\mathbb{H}^1)} \|h_\perp\|_{L^4(\mathbb{H}^1)} \\ &\leq \varepsilon^2 \|h\|_{L^4(\mathbb{H}^1)}^2 + \frac{3C_1^2}{4\varepsilon^2} \|h_\perp\|_{L^4(\mathbb{H}^1)}^2. \end{aligned}$$

To summarize,

$$\frac{1}{2}\|\mathcal{L}_{Q_\beta}^- h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 + \frac{1}{2}\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}^2 \geq \frac{1}{2} \frac{1}{1 - \beta} \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \varepsilon^2 \|h\|_{L^4(\mathbb{H}^1)}^2 - \frac{3C_1^2}{4\varepsilon^2} \|h_\perp\|_{L^4(\mathbb{H}^1)}^2,$$

and by removing the squares appropriately,

$$\begin{aligned} \|\mathcal{L}_{Q_\beta}^- h\|_{\dot{H}^{-1}(\mathbb{H}^1)} &\geq \sqrt{\frac{\beta}{1 - \beta}} \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)} - \sqrt{2}\varepsilon \|h\|_{L^4(\mathbb{H}^1)} - \sqrt{\frac{3C_1^2}{2\varepsilon^2}} \|h_\perp\|_{L^4(\mathbb{H}^1)} \\ &\geq \sqrt{\frac{\beta}{1 - \beta}} \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)} - \sqrt{2}K\varepsilon \|h\|_{\dot{H}^1(\mathbb{H}^1)} - \sqrt{\frac{3C_1^2}{2\varepsilon^2}} K \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

• We conclude by combining all the estimates. Because of the orthogonality of the decomposition along the spaces $\dot{H}^{-1}(\mathbb{H}^1) \cap V_n^\pm$ in $\dot{H}^{-1}(\mathbb{H}^1)$,

$$\|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 = \|\mathcal{L}h^+ + r_+ + r_-\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2 + \|\mathcal{L}_{Q_\beta}^- h\|_{\dot{H}^{-1}(\mathbb{H}^1)}^2,$$

so we can add up the estimates to get

$$\begin{aligned} & \sqrt{2}\|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \\ & \geq c\|h^+\|_{\dot{H}^1(\mathbb{H}^1)} - C_2\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)} - C_2\varepsilon\|h^+\|_{\dot{H}^1(\mathbb{H}^1)} + \sqrt{\frac{\beta}{1-\beta}}\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)} \\ & \quad - \sqrt{2}K\varepsilon\|h\|_{\dot{H}^1(\mathbb{H}^1)} - \sqrt{\frac{3C_1^2}{2\varepsilon^2}}K\|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

The terms compensate as follows. Concerning $\|h^+\|_{\dot{H}^1(\mathbb{H}^1)}$, fix $\varepsilon > 0$ small enough in the sense that

$$(C_2 + \sqrt{2}K)\varepsilon < \frac{c}{2}.$$

Then for all $\beta > \beta_*(\varepsilon)$,

$$\begin{aligned} & \sqrt{2}\|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \\ & \geq \frac{c}{2}\|h^+\|_{\dot{H}^1(\mathbb{H}^1)} + \left(\sqrt{\frac{\beta}{1-\beta}} - \left(C_2 + \sqrt{2}K\varepsilon + \sqrt{\frac{3C_1^2}{2\varepsilon^2}}K \right) \right) \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

Now, let $\beta_* \in (0, 1)$ such that for all $\beta \in (\beta_*, 1)$,

$$\sqrt{\frac{\beta}{1-\beta}} \geq C_2 + \sqrt{2}K\varepsilon + \sqrt{\frac{3C_1^2}{2\varepsilon^2}} + \frac{c}{2}.$$

Then for all $\beta \in (\beta_*, 1)$,

$$\begin{aligned} & \sqrt{2}\|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)} + |(h, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, iQ_+)_{\dot{H}^1(\mathbb{H}^1)}| + |(h, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)}| \\ & \geq \frac{c}{2}(\|h^+\|_{\dot{H}^1(\mathbb{H}^1)} + \|h_\perp\|_{\dot{H}^1(\mathbb{H}^1)}) \geq \frac{c}{2}\|h\|_{\dot{H}^1(\mathbb{H}^1)}. \quad \square \end{aligned}$$

2.5.3 Uniqueness of the traveling waves for β close to 1^-

Theorem 2.5.14. *There exist $\beta_* \in (0, 1)$ and a neighborhood \mathcal{V} of Q_+ in $\dot{H}^1(\mathbb{H}^1)$ such that for all $\beta \in (\beta_*, 1)$, there is a unique $Q_\beta \in \mathcal{Q}_\beta \cap \mathcal{V} \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^\perp, \dot{H}^1(\mathbb{H}^1)$. Moreover,*

1. for all $\beta \in (\beta_*, 1)$,

$$\mathcal{Q}_\beta = \{T_{s_0, \theta, \alpha} Q_\beta : (x, y, s) \mapsto e^{i\theta} \alpha Q_\beta(\alpha x, \alpha y, \alpha^2(s + s_0)) \mid (s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\};$$

2. for all $\gamma \in (0, \frac{1}{4})$ and all $k \in [1, +\infty)$, $\|Q_\beta - Q_+\|_{\dot{H}^k(\mathbb{H}^1)} = \mathcal{O}((1-\beta)^\gamma)$;

3. the map $\beta \in (\beta_*, 1) \mapsto Q_\beta \in \dot{H}^1(\mathbb{H}^1)$ is smooth, tends to Q_+ as β tends to 1, and its derivative \dot{Q}_β is uniquely determined by

$$\begin{cases} \mathcal{L}_{Q_\beta}(\dot{Q}_\beta) = -\frac{\Delta_{\mathbb{H}^1} + D_s}{(1-\beta)^2} Q_\beta \\ \dot{Q}_\beta \in \dot{H}^1(\mathbb{H}^1) \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^\perp, \dot{H}^1(\mathbb{H}^1). \end{cases} \quad (2.20)$$

Proof. • Fix any neighborhood \mathcal{V} of Q_+ . We first prove the existence of a profile $Q_\beta \in \mathcal{Q}_\beta \cap \mathcal{V} \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$ for β close enough to 1. For $\beta \in (0, 1)$, we choose $Q_\beta \in \mathcal{Q}_\beta$ arbitrarily. By combining Corollary 2.4.20 with the fact that $\delta(Q_\beta) = \mathcal{O}((1 - \beta)^{\frac{1}{2}})$ from Lemma 2.3.9, we know that

$$\inf_{(s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|T_{s_0, \theta, \alpha} Q_\beta - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{\frac{1}{4}}).$$

The same argument as in the proof of Corollary 2.4.20, based on the implicit function theorem, enables us to state that for β close enough to 1, one can choose $(s_\beta, \theta_\beta, \alpha_\beta) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ such that $\tilde{Q}_\beta := T_{s_\beta, \theta_\beta, \alpha_\beta} Q_\beta \in \mathcal{V}$ and

$$(\tilde{Q}_\beta, \partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = (\tilde{Q}_\beta, iQ_+)_{\dot{H}^1(\mathbb{H}^1)} = (\tilde{Q}_\beta, Q_+ + 2i\partial_s Q_+)_{\dot{H}^1(\mathbb{H}^1)} = 0.$$

This gives the existence part of the result.

• We now prove uniqueness for some small neighborhood \mathcal{V} of Q_+ . We first set \mathcal{V} as the neighborhood of Q_+ from Proposition 2.5.13. Let $\beta \in (\beta_*, 1)$, and fix two profiles Q_β and \tilde{Q}_β in $\mathcal{Q}_\beta \cap \mathcal{V} \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$. We define

$$h := Q_\beta - \tilde{Q}_\beta \in \dot{H}^1(\mathbb{H}^1) \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}.$$

By subtracting the equations solved by Q_β and \tilde{Q}_β , h satisfies

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} h = 2\Pi_0^+(|Q_\beta|^2 h) + \Pi_0^+(Q_\beta^2 \bar{h}) + \mathcal{O}(\|h\|_{\dot{H}^1(\mathbb{H}^1)}^2),$$

so that

$$\mathcal{L}_{Q_\beta} h = \mathcal{O}(\|h\|_{\dot{H}^1(\mathbb{H}^1)}^2).$$

Since Q_β belongs to the neighborhood \mathcal{V} from Proposition 2.5.13, this means that for some constants $c > 0$ and $C > 0$,

$$C\|h\|_{\dot{H}^1(\mathbb{H}^1)}^2 \geq \|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)} \geq c\|h\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Up to reducing the neighborhood \mathcal{V} , one can chose it small enough such that h has to be the zero function.

• The description of the set \mathcal{Q}_β is then a direct consequence. Indeed, if $\beta \in (\beta_*, 1)$, fix $U_\beta \in \mathcal{Q}_\beta$. We know from the first point that β_* is sufficiently close to 1 to ensure the existence of $(s_\beta, \theta_\beta, \alpha_\beta) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ such that $T_{s_\beta, \theta_\beta, \alpha_\beta} U_\beta \in \mathcal{V} \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^{\perp, \dot{H}^1(\mathbb{H}^1)}$. By the uniqueness part, $T_{s_\beta, \theta_\beta, \alpha_\beta} U_\beta = Q_\beta$.

• We now show the convergence of $(Q_\beta)_\beta$ to Q_+ in $\dot{H}^k(\mathbb{H}^1)$ for all $k \geq 1$. Applying Corollary 2.4.18 to Q_β , we know that for β close to 1,

$$\delta(Q_\beta) \geq c\|Q_\beta - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

But $\delta(Q_\beta) = \mathcal{O}((1 - \beta)^{\frac{1}{2}})$ from Proposition 2.3.9, therefore $\|Q_\beta - Q_+\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{\frac{1}{4}})$.

One can now deduce that for all $0 < \gamma < \frac{1}{4}$, as β goes to 1,

$$\|Q_\beta - Q_+\|_{\dot{H}^k(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^\gamma).$$

Indeed, the interpolation formula [BL76] $(\dot{H}^m(\mathbb{H}^1), \dot{H}^1(\mathbb{H}^1))_{4\gamma} = \dot{H}^k(\mathbb{H}^1)$, with $m \in \mathbb{R}$ chosen so that $k = (1 - 4\gamma)m + 4\gamma$, leads to

$$\|Q_\beta - Q_+\|_{\dot{H}^k(\mathbb{H}^1)} \leq \|Q_\beta - Q_+\|_{\dot{H}^m(\mathbb{H}^1)}^{1-4\gamma} \|Q_\beta - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^{4\gamma},$$

and it only remains to use the fact that $(Q_\beta - Q_+)_\beta$ is bounded in $\dot{H}^m(\mathbb{H}^1)$ for β close to 1 (Corollary 2.5.12) and that $\|Q_\beta - Q_+\|_{\dot{H}^1(\mathbb{H}^1)}^{4\gamma} = \mathcal{O}((1-\beta)^\gamma)$ as β goes to 1.

• We now prove the last point of the theorem about the smoothness of the map $\beta \mapsto Q_\beta$. We first show that equation (2.20) uniquely determines a function \dot{Q}_β lying on the appropriate space

$$W_1 := \dot{H}^1(\mathbb{H}^1) \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^\perp, \dot{H}^1(\mathbb{H}^1).$$

Define

$$W_{-1} := \dot{H}^{-1}(\mathbb{H}^1) \cap (\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^\perp, L^2(\mathbb{H}^1),$$

and set

$$F : (\beta, U) \in (\beta_*, 1) \times W_1 \mapsto -\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} U - |U|^2 U \in \dot{H}^{-1}(\mathbb{H}^1).$$

Note that $\partial_\beta F$ takes values in the space W_{-1} . Indeed, the derivative $\partial_\beta F(\beta, U)$ is equal to

$$\partial_\beta F(\beta, U) = -\frac{\Delta_{\mathbb{H}^1} + D_s}{(1-\beta)^2} U.$$

In particular, since $Q_+, iQ_+, \partial_s Q_+$ and $i\partial_s Q_+$ belong to $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$, and since $-(\Delta_{\mathbb{H}^1} + D_s)$ vanishes on this space,

$$\begin{aligned} (\partial_\beta F(\beta, U), \partial_s Q_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} &= (\partial_\beta F(\beta, U), iQ_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ &= (\partial_\beta F(\beta, U), Q_+ + 2i\partial_s Q_+)_{\dot{H}^{-1}(\mathbb{H}^1) \times \dot{H}^1(\mathbb{H}^1)} \\ &= 0, \end{aligned}$$

or equivalently $\partial_\beta F(\beta, U) \in W_{-1}$.

Consider \mathcal{L}_{Q_β} as a self-adjoint operator on $L^2(\mathbb{H}^1)$. Then thanks to Proposition 2.5.13, we get that $\text{Ker}(\mathcal{L}_{Q_\beta}) \subset \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)$. Therefore,

$$\text{Im}(\mathcal{L}_{Q_\beta}) = \text{Ker}(\mathcal{L}_{Q_\beta})^\perp, L^2(\mathbb{H}^1) \supseteq \dot{H}^{-1}(\mathbb{H}^1) \cap \text{Vect}_{\mathbb{R}}(\partial_s Q_+, iQ_+, Q_+ + 2i\partial_s Q_+)^\perp, L^2(\mathbb{H}^1),$$

so $\text{Im}(\mathcal{L}_{Q_\beta}) = W_{-1}$. This implies that \mathcal{L}_{Q_β} is an isomorphism from W_1 to W_{-1} , with continuous inverse:

$$\|\mathcal{L}_{Q_\beta} h\|_{\dot{H}^{-1}(\mathbb{H}^1)} \geq c \|h\|_{\dot{H}^1(\mathbb{H}^1)}, \quad h \in W_1.$$

In particular, $\partial_\beta F(\beta, Q_\beta) \in W_{-1} = \text{Im}(\mathcal{L}_{Q_\beta})$, and by the invertibility of \mathcal{L}_{Q_β} from W_1 to W_{-1} , $\dot{Q}_\beta := (\mathcal{L}_{Q_\beta})^{-1}(\partial_\beta F(\beta, Q_\beta))$ is uniquely determined and satisfies (2.20).

We now show that \dot{Q}_β is a derivative of the map $\beta \in (\beta_*, 1) \mapsto Q_\beta \in \dot{H}^1(\mathbb{H}^1)$. Fix $\beta \in (\beta_*, 1)$. For $\varepsilon > 0$ small enough, $f_\varepsilon := \frac{Q_{\beta+\varepsilon} - Q_\beta}{\varepsilon} - \dot{Q}_\beta$ is well-defined. Moreover, since $(\beta + \varepsilon, Q_{\beta+\varepsilon})$ and (β, Q_β) are both solution to the equation $F(\alpha, U) = 0$, then

$$\begin{aligned} 0 &= F(\beta + \varepsilon, Q_{\beta+\varepsilon}) - F(\beta, Q_\beta) \\ &= F(\beta + \varepsilon, Q_{\beta+\varepsilon}) - F(\beta, Q_{\beta+\varepsilon}) + F(\beta, Q_{\beta+\varepsilon}) - F(\beta, Q_\beta) \\ &= \varepsilon \partial_\beta F(\beta + \varepsilon, Q_\beta) + \mathcal{L}_{Q_\beta}(Q_{\beta+\varepsilon} - Q_\beta) + \mathcal{O}(\varepsilon^2 + \|Q_{\beta+\varepsilon} - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2). \end{aligned}$$

Actually, since F is smooth in the β variable,

$$0 = \varepsilon \partial_\beta F(\beta, Q_\beta) + \mathcal{L}_{Q_\beta}(Q_{\beta+\varepsilon} - Q_\beta) + \mathcal{O}(\varepsilon^2 + \|Q_{\beta+\varepsilon} - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2).$$

Replacing $\partial_\beta F(\beta, Q_\beta)$ by $\mathcal{L}_{Q_\beta}(\dot{Q}_\beta)$, we get

$$\mathcal{L}_{Q_\beta}(f_\varepsilon) = \mathcal{O} \left(\varepsilon + \frac{\|Q_{\beta+\varepsilon} - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2}{\varepsilon} \right).$$

Since $f_\varepsilon \in W_1$, we know that $\|\mathcal{L}_{Q_\beta}(f_\varepsilon)\|_{\dot{H}^{-1}(\mathbb{H}^1)} \geq c\|f_\varepsilon\|_{\dot{H}^1(\mathbb{H}^1)}$. This implies that for some constant $C > 0$,

$$C \left(\varepsilon + \frac{\|Q_{\beta+\varepsilon} - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2}{\varepsilon} \right) \geq c\|f_\varepsilon\|_{\dot{H}^1(\mathbb{H}^1)}.$$

But $\|Q_{\beta+\varepsilon} - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2 = \varepsilon^2\|f_\varepsilon + \dot{Q}_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2$, so

$$C\varepsilon(1 + \|f_\varepsilon + \dot{Q}_\beta\|_{\dot{H}^1(\mathbb{H}^1)}) \geq c\|f_\varepsilon\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Letting $\varepsilon \rightarrow 0$, we get that $\|f_\varepsilon\|_{\dot{H}^1(\mathbb{H}^1)} \rightarrow 0$, so the map $\beta \mapsto Q_\beta$ is indeed \mathcal{C}^1 with derivative \dot{Q}_β . The smoothness follows from an implicit function theorem. Set

$$\Phi : (\beta, U, V) \in (\beta_*, 1) \times W_1 \times W_1 \mapsto \mathcal{L}_{Q_\beta}V - \partial_\beta F(\beta, U) \in W_{-1}.$$

If $\beta \mapsto Q_\beta$ has regularity \mathcal{C}^n for $\beta \in (\beta_*, 1)$, then the function Φ is also \mathcal{C}^n . For fixed $\beta \in (\beta_*, 1)$, $\Phi(\beta, Q_\beta, \dot{Q}_\beta) = 0$, and $\partial_V F(\beta, Q_\beta, \cdot) = \mathcal{L}_{Q_\beta}$, which is an isomorphism from W_1 to W_{-1} . Applying the implicit function theorem, there exists a \mathcal{C}^n map V defined on a neighborhood of (β, Q_β) in $(\beta_*, 1) \times W_1$ and valued in W_1 such that $V(\beta, Q_\beta) = \dot{Q}_\beta$ and that on this neighborhood, we have

$$F(\beta, U, V(\beta, U)) = 0.$$

In particular for β' close to β , $F(\beta', Q_{\beta'}, V(\beta', Q_{\beta'})) = 0$ and since $\dot{Q}_{\beta'}$ is uniquely determined by (2.20), $\dot{Q}_{\beta'} = V(\beta', Q_{\beta'})$. The function V being \mathcal{C}^n , supposing that $\beta \mapsto Q_\beta$ is \mathcal{C}^n for some integer n , then $\beta \mapsto \dot{Q}_\beta$ is \mathcal{C}^n , and therefore $\beta \mapsto Q_\beta$ is \mathcal{C}^{n+1} . \square

2.A Appendix: explicit formulas for some projections

We establish an explicit formula for the orthogonal projections P_0F_1 , P_0F_2 and P_0F_3 which are under integral form. Then, we estimate numerically $\langle P_0F_j, F_j \rangle_{L^2(\mathbb{C}_+)}$, $j = 1, 2, 3$, in order to get Lemma 2.4.14.

First term We know that

$$-\pi P_0(F_1)(s + it) = \int_{v \in \mathbb{R}_+} \int_{u \in \mathbb{R}} \frac{1}{(s - u + i(t + v))^2} \frac{1}{(u + i(v + 1))} \frac{1}{\sqrt{u^2 + (v + 1)^2}} du dv.$$

Using the change of variables $u = (v+1)\sinh(y)$, $du = (v+1)\cosh(y) dy = \sqrt{u^2 + (v+1)^2} dy$, then

$$-\pi P_0(F_1)(s + it) = \int_{v \in \mathbb{R}_+} \int_{y \in \mathbb{R}} \frac{1}{(s - (v+1)\sinh(y) + i(t + v))^2} \frac{1}{(\sinh(y) + i)(v+1)} dy dv.$$

We now apply the change of variables $x = \exp(y)$, $dx = \exp(y) dy$:

$$\begin{aligned} & -\pi P_0(F_1)(s + it) \\ &= \int_{v \in \mathbb{R}_+} \int_{y \in \mathbb{R}} \frac{8e^{3y}}{(2(s + i(t + v))e^y - (v + 1)e^{2y} + (v + 1))^2} \frac{1}{(e^{2y} - 1 + 2ie^y)(v + 1)} dy dv \\ &= \int_{v \in \mathbb{R}_+} \int_{x \in \mathbb{R}_+} \frac{8x^2}{(2(s + i(t + v))x - (v + 1)x^2 + (v + 1))^2} \frac{1}{(x^2 - 1 + 2ix)(v + 1)} dx dv. \end{aligned}$$

Thanks to Fubini's theorem, one can exchange the integral signs so that

$$\begin{aligned} & -\pi P_0(F_1)(s + it) \\ &= \int_{x \in \mathbb{R}_+} \frac{8x^2}{(x + i)^2} \int_{v \in \mathbb{R}_+} \frac{1}{(2(s + it)x - x^2 + 1 + v(-x^2 + 2ix + 1))^2} \frac{1}{(v + 1)} dx dv \\ &= \int_{x \in \mathbb{R}_+} \frac{8x^2}{(x + i)^2(x - i)^4} \int_{v \in \mathbb{R}_+} \frac{1}{\left(\frac{x^2 - 2(s + it)x - 1}{x^2 - 2ix - 1} + v\right)^2} \frac{1}{(v + 1)} dv dx. \end{aligned}$$

By the residue formula, for any rational function R such that $\int_{\mathbb{R}_+} R(v) dv$ is convergent, we have

$$\int_{\mathbb{R}_+} R(v) dv = - \sum_{w \in \mathbb{C}} \text{Res}_w(R(w) \log_0(w)),$$

where \log_0 is the positive determination of the logarithm. Here, we consider the rational function

$$R(v) = \frac{1}{\left(\frac{x^2 - 2zx - 1}{x^2 - 2ix - 1} + v\right)^2} \frac{1}{(v + 1)},$$

where $z = s + it$. We fix $\lambda = \frac{x^2 - 2zx - 1}{x^2 - 2ix - 1}$.

Assume that $z \neq i$, so $\lambda \neq 1$. The residues at the simple pole -1 and the double pole $-\lambda$ are

$$\text{Res}_{-1}(R(w) \log_0(w)) = \left(\frac{1}{(\lambda + w)^2} \log_0(w) \right) \Big|_{w=-1} = \frac{1}{(\lambda - 1)^2} i\pi,$$

$$\begin{aligned} \text{Res}_{-\lambda}(R(w) \log_0(w)) &= \frac{d}{dw} \left(\frac{1}{(w + 1)} \log_0(w) \right) \Big|_{w=-\lambda} \\ &= \left(\frac{1}{w(w + 1)} - \frac{\log_0(w)}{(w + 1)^2} \right) \Big|_{w=-\lambda} = \frac{1}{\lambda(\lambda - 1)} - \frac{\log_0(-\lambda)}{(\lambda - 1)^2}. \end{aligned}$$

Remark that

$$\lambda = 1 - 2(z - i) \frac{x}{(x - i)^2}, \quad \frac{1}{\lambda - 1} = -\frac{1}{2} \frac{(x - i)^2}{x} \frac{1}{z - i}, \quad \text{and} \quad \frac{1}{(\lambda - 1)^2} = \frac{1}{4} \frac{(x - i)^4}{x^2} \frac{1}{(z - i)^2}.$$

Therefore,

$$\begin{aligned} \text{Res}_{-1}(R(w) \log_0(w)) &= i\pi \frac{1}{4} \frac{(x - i)^4}{x^2} \frac{1}{(z - i)^2}, \\ \text{Res}_{-\lambda}(R(w) \log_0(w)) &= -\frac{(x - i)^2}{x^2 - 2zx - 1} \frac{(x - i)^2}{2x(z - i)} - \log_0 \left(-1 + \frac{2(z - i)x}{(x - i)^2} \right) \frac{(x - i)^4}{4x^2(z - i)^2}. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{8x^2}{(x+i)^2(x-i)^2} \int_{\mathbb{R}_+} R(v) \, dv &= \frac{-2i\pi}{(x+i)^2(z-i)^2} + \frac{4x}{(x^2-2zx-1)(x+i)^2(z-i)} \\ &\quad + 2 \log_0 \left(-1 + \frac{2(z-i)x}{(x-i)^2} \right) \frac{1}{(x+i)^2} \frac{1}{(z-i)^2}. \end{aligned}$$

We can integrate every term of the right-hand side. First,

$$\int_{x \in \mathbb{R}_+} \frac{-2i\pi}{(x+i)^2(z-i)^2} \, dx = \frac{-2\pi}{(z-i)^2}.$$

Then, an integration by parts leads to

$$\int_{x \in \mathbb{R}_+} \log_0 \left(-1 + \frac{2(z-i)x}{(x-i)^2} \right) \frac{1}{(x+i)^2} \, dx = \pi + 2(z-i) \int_{\mathbb{R}_+} \frac{1}{(x-i)(x^2-2zx-1)} \, dx.$$

We conclude that

$$\begin{aligned} -\pi P_0(F_1)(z) &= \frac{-2\pi}{(z-i)^2} + \frac{4}{z-i} \int_{x \in \mathbb{R}_+} \frac{1}{x^2-2zx-1} \frac{x}{(x+i)^2} \, dx \\ &\quad + \frac{2}{(z-i)^2} \left(\pi + 2(z-i) \int_{x \in \mathbb{R}_+} \frac{1}{(x-i)(x^2-2zx-1)} \, dx \right) \\ &= \frac{4}{z-i} \int_{x \in \mathbb{R}_+} \frac{1}{x^2-2zx-1} \frac{2x^2+ix-1}{(x+i)^2(x-i)} \, dx. \end{aligned}$$

We apply the residue formula to get an exact expression for $-\pi P_0(F_1)$. We consider the rational function

$$R(x) = \frac{1}{x^2-2zx-1} \frac{2x^2+ix-1}{(x+i)^2(x-i)}.$$

Fix $x_{\pm} := z \pm \sqrt{z^2+1}$. Since $z \neq i$, the rational function R admits three simple poles x_+ , x_- and i and one double pole $-i$. We calculate the residue

$$\text{Res}_{x_+}(R(w) \log_0(w)) = \frac{2x_+^2 + ix_+ - 1}{(x_+ - x_-)(x_+ + i)^2(x_+ - i)} \log_0(x_+).$$

The identities $x_+^2 = 2zx_+ + 1$, $(x_+ + i)^2 = 2(z+i)x_+$, $x_+x_- = -1$ and $(x_+ - i)(x_- - i) = -2i(z-i)$ enable simplification to

$$\text{Res}_{x_+}(R(w) \log_0(w)) = i \frac{(z+i)x_- - 2iz}{4(z^2+1)^{\frac{3}{2}}} \log_0(x_+).$$

The same arguments lead to

$$\begin{aligned} \text{Res}_{x_-}(R(w) \log_0(w)) &= \frac{2x_-^2 + ix_- - 1}{(x_- - x_+)(x_- + i)^2(x_- - i)} \log_0(x_-) \\ &= -i \frac{(z+i)x_+ - 2iz}{4(z^2+1)^{\frac{3}{2}}} \log_0(x_-). \end{aligned}$$

Moreover, the residue at the pole i is

$$\text{Res}_i(R(w) \log_0(w)) = \frac{1}{-1-2zi-1} \frac{-4i\pi}{-4} = -\frac{\pi}{4(z-i)}.$$

Finally, the residue at the double pole $-i$ is

$$\begin{aligned} \text{Res}_{-i}(R(w) \log_0(w)) &= \left[\frac{1}{x(x^2 - 2zx - 1)} \frac{2x^2 + ix - 1}{(x - i)} + \frac{4x + i}{(x^2 - 2zx - 1)(x - i)} \log_0(x) \right. \\ &\quad \left. - \frac{2x^2 + ix - 1}{(x^2 - 2zx - 1)(x - i)} \left(\frac{1}{x - x_+} + \frac{1}{x - x_-} + \frac{1}{x - i} \right) \log_0(x) \right]_{x=-i}, \end{aligned}$$

which simplifies to

$$\text{Res}_{-i}(R(w) \log_0(w)) = -\frac{i}{2(z + i)}.$$

We conclude that

$$\begin{aligned} &\int_{x \in \mathbb{R}_+} \frac{1}{x^2 - 2zx - 1} \frac{2x^2 + ix - 1}{(x + i)^2(x - i)} dx \\ &= -i \frac{(z + i)x_- - 2iz}{4(z^2 + 1)^{\frac{3}{2}}} \log_0(x_+) + i \frac{(z + i)x_+ - 2iz}{4(z^2 + 1)^{\frac{3}{2}}} \log_0(x_-) + \frac{\pi}{4(z - i)} + \frac{i}{2(z + i)}. \end{aligned}$$

Therefore, as soon as $z \neq i$,

$$-\pi P_0(F_1)(z) = -i \frac{(z + i)x_- - 2iz}{(z - i)(z^2 + 1)^{\frac{3}{2}}} \log_0(x_+) + i \frac{(z + i)x_+ - 2iz}{(z - i)(z^2 + 1)^{\frac{3}{2}}} \log_0(x_-) + \frac{\pi}{(z - i)^2} + \frac{2i}{z^2 + 1},$$

with $x_{\pm} = z \pm \sqrt{z^2 + 1}$. Note that $\log_0(x_{\pm})$ is well-defined because if $z \pm \sqrt{z^2 + 1}$ is real, then z should be real, which we exclude by assumption ($z \in \mathbb{C}_+$).

Second term We apply the same strategy for $(F_1 + F_2)(z) = \frac{2i}{(z+i)^2} \frac{1}{|z+i|}$. We have

$$-\frac{\pi}{2i} P_0(F_1 + F_2)(s + it) = \int_{v \in \mathbb{R}_+} \int_{u \in \mathbb{R}} \frac{1}{(s - u + i(t + v))^2} \frac{1}{(u + i(v + 1))^2} \frac{1}{\sqrt{u^2 + (v + 1)^2}} du dv.$$

With the change of variables $u = (v + 1) \sinh(y)$, $du = (v + 1) \cosh(y) dy = \sqrt{u^2 + (v + 1)^2} dy$, we get

$$\frac{i\pi}{2} P_0(F_1 + F_2(z)) = \int_{v \in \mathbb{R}_+} \int_{y \in \mathbb{R}} \frac{1}{(s - (v + 1) \sinh(y) + i(t + v))^2} \frac{1}{(\sinh(y) + i)^2 (v + 1)^2} dy dv.$$

Now apply the change of variables $x = \exp(y)$, $dx = \exp(y) dy$:

$$\begin{aligned} &\frac{i\pi}{2} P_0(F_1 + F_2(z)) \\ &= \int_{v \in \mathbb{R}_+} \int_{y \in \mathbb{R}} \frac{16e^{4y}}{(2(s + i(t + v))e^y - (v + 1)e^{2y} + (v + 1))^2} \frac{1}{(e^{2y} - 1 + 2ie^y)^2 (v + 1)^2} dy dv \\ &= \int_{v \in \mathbb{R}_+} \int_{x \in \mathbb{R}_+} \frac{16x^3}{(2(s + i(t + v))x - (v + 1)x^2 + (v + 1))^2} \frac{1}{(x^2 - 1 + 2ix)^2 (v + 1)^2} dx dv. \end{aligned}$$

Thanks to Fubini's theorem, one can exchange the integral signs so that

$$\begin{aligned} &\frac{i\pi}{2} P_0(F_1 + F_2(z)) \\ &= \int_{x \in \mathbb{R}_+} \frac{16x^3}{(x + i)^4} \int_{v \in \mathbb{R}_+} \frac{1}{(2(s + it)x - x^2 + 1 + v(-x^2 + 2ix + 1))^2} \frac{1}{(v + 1)^2} dx dv \\ &= \int_{x \in \mathbb{R}_+} \frac{16x^3}{(x + i)^4 (x - i)^4} \int_{v \in \mathbb{R}_+} \frac{1}{\left(\frac{x^2 - 2(s + it)x - 1}{x^2 - 2ix - 1} + v\right)^2} \frac{1}{(v + 1)^2} dv dx. \end{aligned}$$

We apply the consequence of the residue formula to

$$R(v) = \frac{1}{\left(\frac{x^2-2zx-1}{x^2-2ix-1} + v\right)^2} \frac{1}{(v+1)^2},$$

$z = s + it$. We fix $\lambda = \frac{x^2-2zx-1}{x^2-2ix-1}$ as in the first point.

Assume that $z \neq i$, therefore $\lambda \neq 1$. The residue at the double pole -1 is equal to

$$\begin{aligned} \text{Res}_{-1}(R(w) \log_0(w)) &= \frac{d}{dw} \left(\frac{1}{(\lambda + w)^2} \log_0(w) \right) \Big|_{w=-1} \\ &= \left(\frac{1}{w(w + \lambda)^2} - 2 \frac{\log_0(w)}{(w + \lambda)^3} \right) \Big|_{w=-1} \\ &= \frac{-1}{(-1 + \lambda)^2} - 2 \frac{1}{(\lambda - 1)^3} i\pi \\ &= -\frac{(x - i)^4}{4x^2(z - i)^2} + \frac{i\pi}{4} \frac{(x - i)^6}{x^3(z - i)^3}. \end{aligned}$$

The residue at the double pole $-\lambda$ is

$$\begin{aligned} \text{Res}_{-\lambda}(R(w) \log_0(w)) &= \frac{d}{dw} \left(\frac{1}{(w + 1)^2} \log_0(w) \right) \Big|_{w=-\lambda} \\ &= \left(\frac{1}{w(w + 1)^2} - 2 \frac{\log_0(w)}{(w + 1)^3} \right) \Big|_{w=-\lambda} \\ &= \frac{-1}{\lambda(\lambda - 1)^2} - 2 \frac{-\log_0(-\lambda)}{(\lambda - 1)^3} \\ &= -\frac{(x - i)^6}{x^2 - 2xz - 1} \frac{1}{4x^2(z - i)^2} - \frac{(x - i)^6}{4x^3(z - i)^3} \log_0 \left(-1 + \frac{2(z - i)x}{(x - i)^2} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{16x^3}{(x + i)^4(x - i)^4} \int_{\mathbb{R}_+} R(v) dv &= \frac{4x}{(x + i)^4(z - i)^2} - \frac{4i\pi(x - i)^2}{(x + i)^4(z - i)^3} \\ &+ \frac{4(x - i)^2x}{(x + i)^4(x^2 - 2xz - 1)(z - i)^2} + \frac{4(x - i)^2}{(x + i)^4(z - i)^3} \log_0 \left(-1 + \frac{2(z - i)x}{(x - i)^2} \right). \end{aligned}$$

We now integrate again in x to get that for all $z \neq i$,

$$\frac{i\pi}{2} P_0(F_1 + F_2(z)) = \frac{-2(z - 2i)}{3(z - i)(z + i)^2} - \frac{(1 + 2iz)(\log_0(z + \sqrt{z^2 + 1}) - \log_0(z - \sqrt{z^2 + 1}))}{3(z - i)(z + i)^2\sqrt{z^2 + 1}}.$$

Third term We do the last computation for $(F_1 + F_3)(z) = \frac{-2i}{(z+i)(\bar{z}-i)} \frac{1}{|z+i|} = \frac{-2i}{|z+i|^3}$. We have

$$-\frac{\pi}{-2i} P_0(F_1 + F_3)(s + it) = \int_{v \in \mathbb{R}_+} \int_{u \in \mathbb{R}} \frac{1}{(s - u + i(t + v))^2} \frac{1}{(u^2 + (v + 1)^2)^{3/2}} du dv.$$

Apply the change of variables $u = (v+1)\sinh(y)$, $du = (v+1)\cosh(y) dy = \sqrt{u^2 + (v+1)^2} dy$, then

$$-\frac{i\pi}{2} P_0(F_1 + F_3)(s + it) = \int_{v \in \mathbb{R}_+} \int_{y \in \mathbb{R}} \frac{1}{(s - (v + 1)\sinh(y) + i(t + v))^2} \frac{1}{\cosh(y)^2(v + 1)^2} dy dv.$$

We now put $x = \exp(y)$, $dx = \exp(y) dy$:

$$\begin{aligned} & -\frac{i\pi}{2}P_0(F_1 + F_3)(s + it) \\ &= \int_{v \in \mathbb{R}_+} \int_{y \in \mathbb{R}} \frac{16 e^{4y}}{(2(s + i(t + v)) e^y - (v + 1) e^{2y} + (v + 1))^2} \frac{1}{(e^{2y} + 1)^2 (v + 1)^2} dy dv \\ &= \int_{v \in \mathbb{R}_+} \int_{x \in \mathbb{R}_+} \frac{16x^3}{(2(s + i(t + v))x - (v + 1)x^2 + (v + 1))^2} \frac{1}{(x^2 + 1)^2 (v + 1)^2} dx dv. \end{aligned}$$

Thanks to Fubini's theorem, one can exchange the integral signs so that

$$\begin{aligned} & -\frac{i\pi}{2}P_0(F_1 + F_3)(s + it) \\ &= \int_{x \in \mathbb{R}_+} \frac{16x^3}{(x + i)^2(x - i)^2} \int_{v \in \mathbb{R}_+} \frac{1}{(2(s + it)x - x^2 + 1 + v(-x^2 + 2ix + 1))^2} \frac{1}{(v + 1)^2} dx dv \\ &= \int_{x \in \mathbb{R}_+} \frac{16x^3}{(x + i)^2(x - i)^6} \int_{v \in \mathbb{R}_+} \frac{1}{\left(\frac{x^2 - 2(s + it)x - 1}{x^2 - 2ix - 1} + v\right)^2} \frac{1}{(v + 1)^2} dv dx. \end{aligned}$$

We have already done the computation of the integral in the v variable in the latter point. We proved that putting

$$R(v) = \frac{1}{\left(\frac{x^2 - 2(s + it)x - 1}{x^2 - 2ix - 1} + v\right)^2} \frac{1}{(v + 1)^2},$$

we have

$$\begin{aligned} \frac{16x^3}{(x + i)^4(x - i)^4} \int_{\mathbb{R}_+} R(v) dv &= \frac{4x}{(x + i)^4(z - i)^2} - \frac{4i\pi(x - i)^2}{(x + i)^4(z - i)^3} \\ &+ \frac{4(x - i)^2x}{(x + i)^4(x^2 - 2xz - 1)(z - i)^2} + \frac{4(x - i)^2}{(x + i)^4(z - i)^3} \log_0 \left(-1 + \frac{2(z - i)x}{(x - i)^2} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{16x^3}{(x + i)^2(x - i)^6} \int_{v \in \mathbb{R}_+} R(v) dv &= \frac{4x}{(x + i)^2(x - i)^2(z - i)^2} - \frac{4i\pi}{(x + i)^2(z - i)^3} \\ &+ \frac{4x}{(x + i)^2(x^2 - 2xz - 1)(z - i)^2} \\ &+ \frac{4}{(x + i)^2(z - i)^3} \log_0 \left(-1 + \frac{2(z - i)x}{(x - i)^2} \right). \end{aligned}$$

We now integrate again in x to get that for all $z \neq i$,

$$-\frac{i\pi}{2}P_0(F_1 + F_3)(z) = \frac{2(z + 2i)}{(z - i)^2(z + i)} + \frac{(1 - 2iz)(\log_0(z + \sqrt{z^2 + 1}) - \log_0(z - \sqrt{z^2 + 1}))}{(z - i)^2(z + i)\sqrt{z^2 + 1}}$$

• Now we can compute numerically $\langle P_0F_j, F_j \rangle_{L^2(\mathbb{C}_+)}$, $j = 1, 2, 3$, the error estimate for every term can be chosen almost arbitrarily now that we know P_0F_j .

We set $\varepsilon = 10^{-10}$ and we deduce

$$\begin{aligned} |\langle \pi P_0F_1, F_1 \rangle_{L^2(\mathbb{C}_+)} - 2| &\leq \varepsilon, \\ |\langle \pi P_0F_2, F_2 \rangle_{L^2(\mathbb{C}_+)} - \frac{10}{9}| &\leq \varepsilon, \\ |\langle \pi P_0F_3, F_3 \rangle_{L^2(\mathbb{C}_+)} - 0.1303955989| &\leq \varepsilon. \end{aligned}$$

Chapitre 3

Orbital stability of a family of traveling waves for NLS on the Heisenberg group

Ce chapitre reprend les résultats de [Gas21a].

Résumé. On considère l'équation de Schrödinger cubique sur le groupe de Heisenberg dans le cas radial

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad \Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1,$$

qui est un modèle d'équation d'évolution non dispersive. Pour cette équation, l'existence de solutions globales régulières et l'unicité des solutions faibles dans l'espace d'énergie sont des problèmes ouverts. On s'intéresse à une famille d'ondes progressives de vitesse $\beta \in]-1, 1[$. On montre que les ondes progressives de vitesse proche de 1 présentent une forme de stabilité orbitale dans le sens suivant. Pour une donnée initiale radiale proche d'une onde progressive, il existe une solution faible qui reste proche de l'orbite de cette onde progressive en tout temps. Un résultat similaire est montré pour l'équation limite associée.

Abstract. We consider the focusing energy-critical Schrödinger equation on the Heisenberg group in the radial case

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad \Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1,$$

which is a model for non-dispersive evolution equations. For this equation, existence of global smooth solutions and uniqueness of weak solutions in the energy space are open problems. We are interested in a family of ground state traveling waves parameterized by their speed in $(-1, 1)$. We show that the traveling waves of speed close to 1 present some orbital stability in the following sense. If the initial data is radial and close enough to one traveling wave, then there exists a global weak solution which stays close to the orbit of this traveling wave for all times. A similar result is proven for the limiting system associated to this equation.

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3.1 Introduction

3.1.1 Motivation

We are interested in the Schrödinger equation on the Heisenberg group

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u \\ u(t=0) = u_0 \end{cases}, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1. \quad (3.1)$$

The operator $\Delta_{\mathbb{H}^1}$ denotes the sub-Laplacian on the Heisenberg group. When the solution is radial, in the sense that it only depends on t , $|x + iy|$ and s , the sub-Laplacian is written

$$\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2.$$

The Heisenberg group is a typical case of sub-Riemannian geometry where dispersive properties of the Schrödinger equation disappear (see Bahouri, Gérard and Xu [BGX00]). To take it further, Del Hierro [Del05] proves sharp decay estimates for the Schrödinger equation on H-type groups, depending on the dimension of the center of the group. More generally, Bahouri, Fermanian and Gallagher [BFG16] establish optimal dispersive estimates on stratified Lie groups of step 2 under some property of the canonical skew-symmetric form. In contrast, they also give a class of groups without this property displaying total lack of dispersion, which includes the Heisenberg group.

Dispersion impacts the way one can address the Cauchy problem for the Schrödinger equation. Indeed (see Burq, Gérard and Tzvetkov [BGT05], Remark 2.12), the existence of a smooth local in time flow map defined on some Sobolev space $H^k(M)$ for the Schrödinger equation on a Riemannian manifold M with Laplace-Beltrami operator Δ

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u \\ u(t=0) = u_0 \end{cases}$$

implies the following Strichartz estimate

$$\| e^{it\Delta} f \|_{L^4([0,1] \times M)} \leq C \| f \|_{H^{\frac{k}{2}}(M)}.$$

The argument also applies for the Heisenberg group, for which the inequality holds if and only if $k \geq 2$ [GG08]. In particular, without a conservation law controlling the H^2 -norm, there is no existence result of global smooth solutions. Moreover, existence and uniqueness of weak solutions in the energy space $H^1(\mathbb{H}^1)$ is an open problem, even if constructing global weak solutions to the Schrödinger equation on the Heisenberg group would still be possible in the defocusing case through a compactness argument. Note that for weak solutions, the energy of the solution is only bounded above by the initial energy. Therefore, the cancellation of the energy of the solution at some time may not imply that the solution is identically zero, and does not exclude the possibility of non-uniqueness of weak solutions as in the 2D incompressible Euler equation [DS09].

The aim of this paper is to construct some global weak solutions with a prescribed behaviour. More precisely, given an initial data close to some ground state traveling wave solution for the Schrödinger equation on the Heisenberg group, we want to construct a global weak solution which stays close to the orbit of the traveling wave at all times. Combined with an uniqueness result, this would lead to the orbital stability of this ground state traveling wave.

3.1.2 Main results

We consider a family of traveling waves with speed $\beta \in (-1, 1)$ under the form

$$u_\beta(t, x, y, s) = \sqrt{1 - \beta} Q_\beta(x, y, s + \beta t).$$

The profile Q_β satisfies the following stationary hypoelliptic equation (with $D_s = -i\partial_s$)

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta = |Q_\beta|^2 Q_\beta.$$

Because of the scaling invariance, it would have been equivalent in the rest of the study to define u_β as

$$u_\beta(t, x, y, s) = Q_\beta \left(\frac{x}{\sqrt{1 - \beta}}, \frac{y}{\sqrt{1 - \beta}}, \frac{s + \beta t}{1 - \beta} \right).$$

From [Gas20b], we know that as β tends to 1, the ground state solutions of speed β converge up to symmetries in $\dot{H}^1(\mathbb{H}^1)$ to some profile Q . Moreover, Q is solution to a limiting equation

$$D_s Q = \Pi_0^+(|Q|^2 Q), \tag{3.2}$$

for which the ground state solution is unique up to symmetries, equal to

$$Q(x, y, s) = \frac{i\sqrt{2}}{s + i(x^2 + y^2) + i}.$$

The operator Π_0^+ is an orthogonal projector onto a relevant space for our analysis denoted by V_0^+ . For more details, see the Notation part 3.2.

We first prove a mild form of orbital stability for the ground state Q in the limiting equation, and then focus on the orbital stability of the ground states Q_β in the Schrödinger equation on the Heisenberg group when β is close to 1.

Definition 3.1.1. For $u \in \dot{H}^1(\mathbb{H}^1)$ and $X = (s_0, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$, we denote by $T_X u$ the element of $\dot{H}^1(\mathbb{H}^1)$ satisfying

$$T_X u(x, y, s) := e^{i\theta} \alpha u(\alpha x, \alpha y, \alpha^2(s - s_0)), \quad (x, y, s) \in \mathbb{H}^1.$$

Let \mathcal{M} be the orbit of Q

$$\mathcal{M} = \{T_X Q \mid X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\},$$

then the distance of u to \mathcal{M} is defined as

$$d(u, \mathcal{M}) = \inf_{X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|u - T_X Q\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Similarly, denote by \mathcal{Q}_β the orbit of Q_β

$$\mathcal{Q}_\beta = \{T_X Q_\beta \mid X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\},$$

then the distance of u to $\sqrt{1-\beta}\mathcal{Q}_\beta$ is

$$d(u, \sqrt{1-\beta}\mathcal{Q}_\beta) = \inf_{X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|u - \sqrt{1-\beta}T_X Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}.$$

Our first result is an orbital stability result for the profile Q associated the evolution problem linked to the limiting equation

$$\begin{cases} i\partial_t u = \Pi_0^+(|u|^2 u) \\ u(t=0) = u_0. \end{cases} \quad (3.3)$$

Theorem 3.1.2 (Orbital stability for Q). *There exist $c_0 > 0$ and $r_0 > 0$ such that the following holds. Let $r \leq r_0$ and $u_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ such that*

$$\|u_0 - Q\|_{\dot{H}^1(\mathbb{H}^1)} < r^2. \quad (3.4)$$

Then there exists a weak solution $u \in \mathcal{C}(\mathbb{R}, \dot{H}^1(\mathbb{H}^1))$ (with the weak topology) to equation (3.3) such that for all $t \in \mathbb{R}$,

$$d(u(t), \mathcal{M}) \leq c_0 r.$$

Using the links between the limiting equation and the Schrödinger equation, we deduce our second result: an orbital stability result for the profiles Q_β for the Schrödinger equation when β is close to 1 in the radial case.

Theorem 3.1.3 (Orbital stability for Q_β). *There exist $c_0 > 0$ and $r_0 > 0$ such that the following holds. Let $r \in (0, r_0)$. Then there exists $\beta_* \in (0, 1)$ such that if $\beta \in (\beta_*, 1)$, and if $u_0 \in \dot{H}^1(\mathbb{H}^1)$ is radial and satisfies*

- if $u_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$:

$$\|u_0 - \sqrt{1-\beta}Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1-\beta}r^2 \quad (3.5)$$

- in the general case:

$$\|u_0 - \sqrt{1-\beta}Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < (1-\beta)r, \quad (3.6)$$

then there exists a weak radial solution $u \in \mathcal{C}(\mathbb{R}, \dot{H}^1(\mathbb{H}^1))$ (with the weak topology) to the Schrödinger equation on the Heisenberg group (3.1)

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u \\ u(t=0) = u_0 \end{cases}$$

such that for all $t \in \mathbb{R}$, $\frac{u(t)}{\sqrt{1-\beta}}$ is close to the orbit of Q_β :

$$d\left(u(t), \sqrt{1-\beta}\mathcal{Q}_\beta\right) \leq c_0 \sqrt{1-\beta}r.$$

Note that, unlike the weak solutions discussed in the first part 3.1.1, the energy of the weak solutions from Theorem 3.1.2 (resp. Theorem 3.1.3) is controlled, indeed, this energy is very close to the one of the ground state Q (resp. $\sqrt{1-\beta}Q_\beta$). Furthermore, these two theorems would imply the orbital stability of Q and Q_β in the radial case in both situations if we had a uniqueness result for the solutions.

The assumption required on a general initial condition for the Schrödinger equation (3.6) is stronger than the assumption on an initial data already in V_0^+ (3.5). Indeed, for r fixed, and general initial data u_0 , we will choose β_* sufficiently close to 1 so that the projection $\Pi_0^+(u_0)$ of u_0 onto V_0^+ satisfies $\|\Pi_0^+(u_0) - \sqrt{1-\beta}Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \leq C\sqrt{1-\beta}r^2$ (and so that the remainder term $(\text{Id} - \Pi_0^+)(u_0)$ is also bounded). Note that thanks to the convergence of Q_β to Q with a rate $o(\sqrt{1-\beta})$ (see Appendix 3.A), assumption (3.5) is comparable to assumption (3.4) up to a change of function $u \rightsquigarrow \frac{1}{\sqrt{1-\beta}}u(x, y, s - \beta t)$.

The key point in both proofs is the following local stability estimate for Q , which comes from the invertibility of the linearized operator around Q for the limiting equation (3.2) on a subspace of V_0^+ of finite co-dimension.

Definition 3.1.4. For $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, we denote by $\mathcal{P}(u)$ (resp. $\mathcal{E}(u)$) the momentum (resp. the energy) for the limiting equation (3.3)

$$\mathcal{P}(u) := \|u\|_{\dot{H}^1(\mathbb{H}^1)}^2$$

resp.

$$\mathcal{E}(u) := \|u\|_{L^4(\mathbb{H}^1)}^4,$$

then define

$$\delta(u) := |\mathcal{P}(u) - \mathcal{P}(Q)| + |\mathcal{E}(u) - \mathcal{E}(Q)|.$$

Proposition 3.1.5 (See [Gas20b], Corollary 4.20). *There exist $\delta_0 > 0$ and $C > 0$ such that for all $u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, if $\delta(u) \leq \delta_0$, then*

$$d(u, \mathcal{M})^2 \leq C\delta(u).$$

In order to prove Theorem 3.1.2, we construct the weak solution for the limiting initial value problem (3.3) as a limit of smooth functions. The approximating functions solve slightly modified equations where we have restricted frequencies, so that the Cauchy problem is globally well-posed. We show that we can control their distance to the orbit of the ground state Q using Proposition 3.1.5. Finally, we build modulation parameters which stay bounded on finite time intervals for the approximate solutions, and through a compactness argument, we control the distance of the weak solution to the orbit of Q when passing to the limit.

For Theorem 3.1.3, the idea for the construction is the same, however we only have at our disposal the information on the limiting equation from Proposition 3.1.5. Therefore, we need to take advantage of the fact that Q_β is close to Q when β is close to 1. In this spirit, in order to tackle Theorem 3.1.3 for the speed β , we first introduce Cauchy problems for the Schrödinger equation (3.1) with a parameter γ increasing from β to 1. We display some continuity between the Cauchy problems, therefore it is possible to show their convergence to a Cauchy problem for the limiting equation as γ tends to 1. In the proof, we combine this strategy with the above method: we approximate by smooth functions the weak solutions to the Cauchy problems with parameter γ by restricting frequencies. Finally, we are able to get back to the problem with speed β by continuity and conclude in the same way as the proof of Theorem 3.1.2, by constructing bounded modulation parameters for the approximate solutions.

3.1.3 Comparison with other equations

Concerning the focusing energy-critical Schrödinger equation on the euclidean plane \mathbb{R}^N

$$i\partial_t u - \Delta u = |u|^{p_c-1}u,$$

where $N \geq 3$ and $p_c = \frac{N+2}{N-2}$, there exists an explicit stationary solution

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}}.$$

The orbit $\{x \mapsto CW(\frac{x+x_0}{\lambda}) \mid (C, x_0, \lambda) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*\}$ of W is the set of minimizers for the Sobolev embedding $\dot{H}^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ (see the work of Talenti [Tal76] and Aubin [Aub76]). The energy $E(W) = \frac{1}{2}\|W\|_{\dot{H}^1(\mathbb{R}^N)}^2 - \frac{1}{p_c+1}\|W\|_{L^{p_c+1}(\mathbb{R}^N)}^{p_c+1}$ and the \dot{H}^1 norm $\|W\|_{\dot{H}^1(\mathbb{R}^N)}$ play an important role in the dynamical behaviour of the solutions. Kenig and Merle [KM06] proved in the radial case that if $N \in \{3, 4, 5\}$ and the initial condition $u_0 \in \dot{H}^1(\mathbb{R}^N)$ satisfies $E(u_0) < E(W)$ and $\|u_0\|_{\dot{H}^1(\mathbb{R}^N)} < \|W\|_{\dot{H}^1(\mathbb{R}^N)}$, then the solution is global and scatters in $\dot{H}^1(\mathbb{R}^N)$, whereas if $E(u_0) < E(W)$ and $\|u_0\|_{\dot{H}^1(\mathbb{R}^N)} > \|W\|_{\dot{H}^1(\mathbb{R}^N)}$, then the solution must blow up in finite time.

The situation is different for the Schrödinger equation on the Heisenberg group. Indeed, from the equation satisfied by Q_β , one can see that the traveling waves

$$u_\beta(t, x, y, s) = \sqrt{1 - \beta}Q_\beta(x, y, s + \beta t)$$

have a vanishing energy as β tends to 1:

$$E(u_\beta(t)) = \frac{1}{2}\|u_\beta(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \frac{1}{4}\|u_\beta(t)\|_{L^4(\mathbb{H}^1)}^4 \sim (1 - \beta)\frac{\pi^2}{2} \rightarrow 0,$$

therefore there exist solutions that do not scatter with arbitrary small energy.

A better parallel would be the mass-critical focusing half-wave equation on the real line

$$i\partial_t u + |D|u = |u|^2u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{3.7}$$

where $D = -i\partial_x$, $|\widehat{D}|f(\xi) = |\xi|\widehat{f}(\xi)$. The half-wave equation in one dimension also presents some lack of dispersion, and admits traveling waves with speed $\beta \in (-1, 1)$ (see Krieger, Lenzmann and Raphaël [KLR13])

$$u(t, x) = Q_\beta\left(\frac{x + \beta t}{1 - \beta}\right) e^{-it},$$

where the profile Q_β is a solution to

$$\frac{|D| - \beta D}{1 - \beta}Q_\beta + Q_\beta = |Q_\beta|^2Q_\beta.$$

The profiles Q_β in the half-wave equation converge [Gér+18] as β tends to 1 in $H^{\frac{1}{2}}(\mathbb{R})$ to a ground state solution Q to some limiting equation

$$DQ + Q = \Pi(|Q|^2Q),$$

where Π is the Szegő projector from $L^2(\mathbb{R})$ onto the space $L_+^2(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \text{Supp}(\widehat{u}) \subset \mathbb{R}_+\}$ of L^2 functions with nonnegative Fourier frequencies. From Q , we recover a traveling wave solution to the cubic Szegő equation

$$i\partial_t u = \Pi(|u|^2 u) \quad (3.8)$$

by setting $u(t, x) = Q(x - t)e^{-it}$. Moreover, the linearized operator around Q is coercive [Poc12]. One can deduce a similar estimate to Proposition 3.1.5, implying that the Szegő profile is orbitally stable in the relevant space for Q

$$H_+^{\frac{1}{2}}(\mathbb{R}) = \{u \in H^{\frac{1}{2}}(\mathbb{R}) \mid \text{Supp}(\widehat{u}) \subset \mathbb{R}_+\}.$$

The following theorem comes from a graduate course given by P. Gérard and F. Rousset [GR16], for which no online notes are available (see Pocovnicu [Poc11b], Theorem 1.3, for a non quantitative version, and [Poc12], Proposition 6.3 applied to a zero Toeplitz potential, for a similar result with finite times).

Theorem 3.1.6 (Orbital stability of Q for the Szegő equation). *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solution u of the Szegő equation (3.8) with initial condition $u_0 \in H_+^{\frac{1}{2}}(\mathbb{R})$, if*

$$\|u_0 - Q\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq \varepsilon_0,$$

then

$$\sup_{t \in \mathbb{R}} \inf_{(\gamma, y) \in \mathbb{T} \times \mathbb{R}} \|e^{-i\gamma} u(t, \cdot - y) - Q\|_{H^{\frac{1}{2}}(\mathbb{R})}^2 \leq C \|u_0 - Q\|_{H^{\frac{1}{2}}(\mathbb{R})}.$$

Gérard, Lenzmann, Pocovnicu and Raphaël [Gér+18] deduced the invertibility of the linearized operator for the half-wave equation around the profiles Q_β when β is close enough to 1. Their estimates imply the orbital stability of these profiles [GR16], indeed, one can prove a result similar to Proposition 3.1.5 for the profile Q_β with an adapted gap δ_β . However, this strategy does not work for the Schrödinger equation on the Heisenberg group, as we will see at the beginning of section 3.4.

Theorem 3.1.7 (Orbital stability of Q_β for the half-wave equation). *There exists $\beta_* \in (0, 1)$ such that the following holds. Let $\beta \in (\beta_*, 1)$. Then there exist $\varepsilon_0(\beta) > 0$ and $C(\beta) > 0$ such that for all solution u of the half-wave equation (3.7) with initial condition $u_0 \in H^{\frac{1}{2}}(\mathbb{R})$, if*

$$\|u_0 - Q_\beta(\frac{\cdot}{1-\beta})\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq \varepsilon_0(\beta),$$

then

$$\sup_{t \in \mathbb{R}} \inf_{(\gamma, y) \in \mathbb{T} \times \mathbb{R}} \|e^{-i\gamma} u(t, \cdot - y) - Q_\beta(\frac{\cdot}{1-\beta})\|_{H^{\frac{1}{2}}(\mathbb{R})}^2 \leq C \|u_0 - Q_\beta(\frac{\cdot}{1-\beta})\|_{H^{\frac{1}{2}}(\mathbb{R})}.$$

In higher dimensions $d \geq 2$, traveling waves for the half-wave equation on \mathbb{R}^d

$$i\partial_t u + \sqrt{-\Delta} u = |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

are also orbitally stable in the radial case for mass-subcritical nonlinearities $1 < p < 1 + \frac{2}{n}$, but orbitally unstable in the mass-supercritical regime $1 + \frac{2}{n} < p < 1 + \frac{2}{n-1}$ [BGV18]. Moreover, in the energy-critical and subcritical case, Bellazzini, Georgiev, Lenzmann and Visciglia [Bel+19] proved that there can be no small data scattering in the energy space because of the existence of traveling waves with arbitrary small energy.

As we will see in this paper, one cannot directly adapt the proofs for the half-wave equation because we lack information on the Cauchy problem. A second complication arising in comparison to the half-wave equation is the fact that only two conservation laws are available (energy and momentum), because the mass of the ground states may be infinite (this fact is easy to check for Q for instance). The method both for the Schrödinger equation on the Heisenberg group and for its limiting system is the construction of some weak solutions as a limit of smooth functions, and show that we can pass to the limit on their stability properties.

The paper is organized as follows. We first prove the orbital stability of Q for the limiting equation in Section 3.3. Then, we assess how close the solutions are to the limiting equation as β tends to 1, in order to study the orbital stability of Q_β for the Schrödinger equation in Section 3.4.

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3.2 Notation

3.2.1 The Heisenberg group

Let us now recall some facts about the Heisenberg group. We use coordinates and identify the Heisenberg group \mathbb{H}^1 with \mathbb{R}^3 . The group multiplication is given by

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2(x'y - xy')).$$

The Lie algebra of left-invariant vector fields on \mathbb{H}^1 is spanned by the vector fields $X = \partial_x + 2y\partial_s$, $Y = \partial_y - 2x\partial_s$ and $T = \partial_s = \frac{1}{4}[Y, X]$. The sub-Laplacian is defined as

$$\mathcal{L}_0 := \frac{1}{4}(X^2 + Y^2) = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2 + (y\partial_x - x\partial_y)\partial_s.$$

When the function is radial, the sub-Laplacian coincides with the operator

$$\Delta_{\mathbb{H}^1} := \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2 + y^2)\partial_s^2.$$

The space \mathbb{H}^1 is endowed with a smooth left invariant measure, the Haar measure, which in the coordinate system (x, y, s) is the Lebesgue measure $d\lambda_3(x, y, s)$. Sobolev spaces of positive order can then be constructed on \mathbb{H}^1 from powers of the operator $-\Delta_{\mathbb{H}^1}$, for example, $\dot{H}^1(\mathbb{H}^1)$ is the completion of the Schwarz space $\mathcal{S}(\mathbb{H}^1)$ for the norm

$$\|u\|_{\dot{H}^1(\mathbb{H}^1)} := \|(-\Delta_{\mathbb{H}^1})^{\frac{1}{2}}u\|_{L^2(\mathbb{H}^1)}.$$

3.2.2 Decomposition along the Hermite functions

In order to study radial functions defined on the Heisenberg group \mathbb{H}^1 , it is convenient to use their decomposition along Hermite-type functions (see for example [SM93], Chapters 12 and 13). The Hermite functions

$$h_m(x) = \frac{1}{\pi^{\frac{1}{4}} 2^{\frac{m}{2}} (m!)^{\frac{1}{2}}} (-1)^m e^{\frac{x^2}{2}} \partial_x^m (e^{-x^2}), \quad x \in \mathbb{R}, m \in \mathbb{N},$$

form an orthonormal basis of $L^2(\mathbb{R})$. In $L^2(\mathbb{R}^2)$, the family of products of two Hermite functions $(h_m(x)h_p(y))_{m,p \in \mathbb{N}}$ diagonalizes the two-dimensional harmonic oscillator: for all $m, p \in \mathbb{N}$,

$$(-\Delta_{x,y} + x^2 + y^2)h_m(x)h_p(y) = 2(m+p+1)h_m(x)h_p(y).$$

Given $u \in \mathcal{S}(\mathbb{H}^1)$, we will denote by \widehat{u} its usual Fourier transform under the s variable, with corresponding variable σ

$$\widehat{u}(x, y, \sigma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\sigma} u(x, y, s) ds.$$

For $m, p \in \mathbb{N}$, set $\widehat{h_{m,p}}(x, y, \sigma) := h_m(\sqrt{2|\sigma|x})h_p(\sqrt{2|\sigma|y})$. Then the family $(h_{m,p})_{m,p \in \mathbb{N}}$ diagonalizes the sub-Laplacian:

$$\Delta_{\mathbb{H}^1} \widehat{h_{m,p}} = -(m+p+1)|\sigma| \widehat{h_{m,p}}.$$

Let $k \in \{-1, 0, 1\}$, and denote by $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ the subspace of $\dot{H}^k(\mathbb{H}^1)$ spanned by $\{h_{m,p} \mid m, p \in \mathbb{N}, m+p=n\}$ in the following sense.

Definition 3.2.1. *Some $u_n^\pm \in \dot{H}^k(\mathbb{H}^1)$ belongs to $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ if there exists a family $(f_{m,p}^\pm)_{m+p=n}$ such that*

$$\widehat{u_n^\pm}(x, y, \sigma) = \sum_{\substack{m,p \in \mathbb{N}; \\ m+p=n}} f_{m,p}^\pm(\sigma) \widehat{h_{m,p}}(x, y, \sigma) \mathbb{1}_{\sigma \geq 0}.$$

For $u_n^\pm \in \dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$, the \dot{H}^k -norm of u_n^\pm writes

$$\begin{aligned} \|u_n^\pm\|_{\dot{H}^k(\mathbb{H}^1)}^2 &= \int_{\mathbb{R}_\pm} ((n+1)|\sigma|)^k \int_{\mathbb{R}^2} |\widehat{u_n^\pm}(x, y, \sigma)|^2 dx dy d\sigma \\ &= \sum_{\substack{m,p \in \mathbb{N}; \\ m+p=n}} \int_{\mathbb{R}_\pm} ((n+1)|\sigma|)^k |f_{m,p}^\pm(\sigma)|^2 \frac{d\sigma}{2|\sigma|}. \end{aligned}$$

Any function $u \in \dot{H}^k(\mathbb{H}^1)$ admits a decomposition along the orthogonal sum of the subspaces $\dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$. Let us write $u = \sum_\pm \sum_{n \in \mathbb{N}} u_n^\pm$ where $u_n^\pm \in \dot{H}^k(\mathbb{H}^1) \cap V_n^\pm$ for all (n, \pm) . Then

$$\|u\|_{\dot{H}^k(\mathbb{H}^1)}^2 = \sum_\pm \sum_{n \in \mathbb{N}} \|u_n^\pm\|_{\dot{H}^k(\mathbb{H}^1)}^2.$$

For $k=0$, we get an orthogonal decomposition of the space $L^2(\mathbb{H}^1)$, and denote by Π_n^\pm the associated orthogonal projectors.

The particular space $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$ is spanned by a unique radial function h_0^+ , satisfying

$$\widehat{h_0^+}(x, y, \sigma) = \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)\sigma} \mathbb{1}_{\sigma \geq 0}.$$

Set $u \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$, then there exists f such that

$$\widehat{u}(x, y, \sigma) = f(\sigma) \widehat{h_0^+}(x, y, \sigma),$$

and

$$\|u\|_{\dot{H}^k(\mathbb{H}^1)}^2 = \int_{\mathbb{R}_+} |f(\sigma)|^2 \frac{d\sigma}{2\sigma^{1-k}}.$$

3.3 Orbital stability for the ground state Q in the limiting equation

In this section, we prove Theorem 3.1.2 on the orbital stability for the ground state Q in the limiting equation (3.3)

$$\begin{cases} i\partial_t u = \Pi_0^+(|u|^2 u) \\ u(t=0) = u_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+ \end{cases}, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1.$$

For convenience, we replace in this part the elements $u \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$, $k \in \{-1, 0, 1\}$ with the corresponding function on the complex upper half-plane F_u defined as

$$F_u(s + i(x^2 + y^2)) := u(x, y, s).$$

Thanks to the equality of Sobolev norms (3.10) below, we will see that for $k \in \{-1, 0, 1\}$, u belongs to $\dot{H}^k(\mathbb{H}^1) \cap V_0^+$ if and only if F_u belongs to the space of holomorphic functions $\dot{H}^{\frac{k}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. Moreover, the Paley-Wiener theorem 3.3.2 enables us to identify the orthogonal projector Π_0^+ from $L^2(\mathbb{H}^1)$ onto its closed subspace $L^2(\mathbb{H}^1) \cap V_0^+$ with the orthogonal projector P_0 from $L^2(\mathbb{C}_+)$ onto the Bergman space $A_1^2 = L^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. The projector P_0 is a Bergman projector, which will be defined in equality (3.15). This change of functions then transforms the Cauchy problem for u in a Cauchy problem for F_u written as

$$\begin{cases} i\partial_t u = P_0(|u|^2 u) \\ u(t=0) = u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+) \end{cases}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+. \quad (3.9)$$

We establish the correspondence between u and F_u in part 3.3.1. Then we construct some smooth functions approximating a weak solution of equation (3.9) in part 3.3.2, prove their weak convergence in part 3.3.3, and deduce from their distance to the orbit of Q an upper bound on the distance of the weak limit to this orbit in part 3.3.4.

3.3.1 Weighted Bergman spaces

Let $u \in \dot{H}^k(\mathbb{H}^1) \cap V_0^+$, we first note that $F_u \in \dot{H}^{\frac{k}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. Indeed, the Fourier transform of u along the s variable corresponds to a function in $L^2(\mathbb{R}_+, \sigma^{k-1} d\sigma)$: there exists $f \in L^2(\mathbb{R}_+, \sigma^{k-1} d\sigma)$ such that $\widehat{u}(x, y, \sigma) = f(\sigma) \widehat{h_0^+}(x, y, \sigma)$. Therefore, we have

$$F_u(z) = \frac{1}{\pi\sqrt{2}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma,$$

so that F_u is holomorphic. Moreover, the Sobolev norms of u and F_u are linked by

$$\|u\|_{\dot{H}^k(\mathbb{H}^1)}^2 = \pi \|F_u\|_{\dot{H}^{\frac{k}{2}}(\mathbb{C}_+)}^2 = \pi \|(-i\partial_z)^{\frac{k}{2}} F_u\|_{L^2(\mathbb{C}_+)}^2 = \frac{1}{2} \int_0^{+\infty} |f(\sigma)|^2 \sigma^{k-1} d\sigma. \quad (3.10)$$

For $k < 1$, F_u belongs to the weighted Bergman space A_{1-k}^2 .

Definition 3.3.1 (Weighted Bergman spaces). *Given $k < 1$, the weighted Bergman space A_{1-k}^2 is the subspace of $L_{1-k}^2 := L^2(\mathbb{C}_+, \text{Im}(z)^{-k} d\lambda(z))$ composed of holomorphic functions of the complex upper half-plane \mathbb{C}_+ :*

$$A_{1-k}^2 := \left\{ F \in \text{Hol}(\mathbb{C}_+) \mid \|F\|_{L_{1-k}^2}^2 := \int_0^{+\infty} \int_{\mathbb{R}} |F(s + it)|^2 ds \frac{dt}{t^k} < +\infty \right\}.$$

Indeed, recall the Paley-Wiener theorem for Bergman spaces.

Theorem 3.3.2 (Paley-Wiener [Bék+12]). *Let $k < 1$. Then for every $f \in L^2(\mathbb{R}_+, \sigma^{k-1} d\sigma)$, the following integral is absolutely convergent on \mathbb{C}_+*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma, \quad (3.11)$$

and defines a function $F \in A_{1-k}^2$ which satisfies

$$\|F\|_{L_{1-k}^2}^2 = \frac{\Gamma(1-k)}{2^{1-k}} \int_0^{+\infty} |f(\sigma)|^2 \sigma^{k-1} d\sigma. \quad (3.12)$$

Conversely, for every $F \in A_{1-k}^2$, there exists $f \in L^2(\mathbb{R}_+, \sigma^{k-1} d\sigma)$ such that (3.11) and (3.12) hold.

For $k = 1$, F_u belongs to the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$.

Definition 3.3.3. *The Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ space of holomorphic functions of the upper half-plane \mathbb{C}_+ such that the following norm is finite:*

$$\|F\|_{\mathcal{H}^2(\mathbb{C}_+)}^2 := \sup_{t>0} \int_{\mathbb{R}} |F(s+it)|^2 ds < +\infty.$$

Theorem 3.3.4 (Paley-Wiener [Rud87]). *For every $f \in L^2(\mathbb{R}_+)$, the following integral is absolutely convergent on \mathbb{C}_+*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma, \quad (3.13)$$

and defines a function F in the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ which satisfies

$$\|F\|_{\mathcal{H}^2(\mathbb{C}_+)}^2 = \int_0^{+\infty} |f(\sigma)|^2 d\sigma. \quad (3.14)$$

Conversely, for every $F \in \mathcal{H}^2(\mathbb{C}_+)$, there exists $f \in L^2(\mathbb{R}_+)$ such that (3.13) and (3.14) hold.

In the following, we will work with the holomorphic representations, the solutions being valued in the Hardy space $\mathcal{H}^2(\mathbb{C}_+) = \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$.

3.3.2 Construction of approximate solutions

Given an initial data $u_0 \in \mathcal{H}^2(\mathbb{C}_+) = \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ close enough to the ground state

$$Q(z) = \frac{i\sqrt{2}}{z+i},$$

we want to construct a global solution to the Cauchy problem (3.9)

$$\begin{cases} i\partial_t u = P_0(|u|^2 u), & (t, z) \in \mathbb{R} \times \mathbb{C}_+ \\ u(t=0) = u_0 \end{cases}$$

which stays close to Q (up to symmetries) at all times. The Bergman projection P_0 from $L^2(\mathbb{C}_+)$ to A_1^2 writes (see eg [Bék+12])

$$P_0(u)(z) = -\frac{1}{\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{1}{(z-s+it)^2} u(s+it) ds dt, \quad z \in \mathbb{C}_+. \quad (3.15)$$

We approximate u by functions with higher regularity, satisfying equations for which we can use a classical global well-posedness result.

Construction of smoothing projectors $\tilde{P}_{\varepsilon, M}$: For $\varepsilon, M > 0$, we define the projector $\tilde{P}_{\varepsilon, M}$ as follows. Write $u \in \dot{H}^k(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$, $k \leq \frac{1}{2}$ (or $u \in H^k(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$, $k \geq 0$) as

$$u(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma,$$

then

$$\tilde{P}_{\varepsilon, M}(u)(z) := \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^M e^{iz\sigma} f(\sigma) d\sigma.$$

This projector removes the high and low frequencies of u , in order to add some regularity on the solutions. It defines a bounded projector from $\dot{H}^k(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ to itself for $k \leq \frac{1}{2}$, and from $H^k(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ to itself for $k \geq 0$.

Construction of a sequence of approximate solutions $(u_n)_n$: We consider $f \in L^2(\mathbb{R}_+)$ such that for all $z \in \mathbb{C}_+$,

$$u_0(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma,$$

which satisfies

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 = \frac{1}{2} \|f\|_{L^2(\mathbb{R}_+)}^2.$$

Let us fix a sequence of positive numbers $(\varepsilon_n)_n$ going to zero, and consider the following initial data belonging to $H^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$

$$u_0^n(z) := \tilde{P}_{\varepsilon_n, \frac{1}{\varepsilon_n}} u_0(z) = \frac{1}{\sqrt{2\pi}} \int_{\varepsilon_n}^{1/\varepsilon_n} e^{iz\sigma} f(\sigma) d\sigma.$$

We denote by $H_{\varepsilon}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ the space of functions $u \in H^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ satisfying $\tilde{P}_{\varepsilon, \frac{1}{\varepsilon}}(u) = u$. On this space, the \dot{H}^k -norms, $k \geq 0$, are equivalent:

$$\varepsilon^{2k} \|u\|_{L^2(\mathbb{C}_+)}^2 \leq \|u\|_{\dot{H}^k(\mathbb{C}_+)}^2 = \frac{1}{2} \int_{\varepsilon}^{1/\varepsilon} \sigma^{2k-1} |f(\sigma)|^2 d\sigma \leq \frac{1}{\varepsilon^{2k}} \|u\|_{L^2(\mathbb{C}_+)}^2.$$

Define the projection P_0^n as

$$P_0^n = \tilde{P}_{\varepsilon_n, \frac{1}{\varepsilon_n}} \circ P_0.$$

We consider the following Cauchy problem

$$\begin{cases} i\partial_t u_n = P_0^n(|u_n|^2 u_n) \\ u_n(t=0) = u_0^n, \end{cases} \quad (3.16)$$

which is globally well-posed in $H_{\varepsilon_n}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$.

Proposition 3.3.5. *Let $u_0^n \in H_{\varepsilon_n}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. Then there exists a unique solution $u_n \in \mathcal{C}^\infty(\mathbb{R}, H_{\varepsilon_n}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+))$ of (3.16) in the distribution sense.*

Proof. The local existence comes from the Cauchy-Lipschitz theory for ODEs. Indeed, P_0 defines a bounded projector from $H^2(\mathbb{C}_+)$ onto $H^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$, therefore P_0^n defines a bounded projector from $H^2(\mathbb{C}_+)$ onto $H_{\varepsilon_n}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. Let $r := \|u_0^n\|_{H^2(\mathbb{C}_+)}$ and denote

by $B(u_0, r)$ the ball centered at u_0 of radius r in $H_{\varepsilon_n}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. Since $H^2(\mathbb{C}_+)$ is an algebra, we deduce that there exists $T = T(r)$ such that the map

$$v \in \mathcal{C}([-T, T], B(u_0, r)) \mapsto \left(t \mapsto v_0 + \frac{1}{i} \int_0^t P_0^n(|v|^2 v)(\tau) \, d\tau \right) \in \mathcal{C}([-T, T], B(u_0, r))$$

defines a contraction mapping from $\mathcal{C}([-T, T], B(u_0, r))$ to itself. We conclude to the local well-posedness of equation (3.16). Moreover, the time of existence of the solution is bounded below by some constant which only depends on the norm of the initial data in $H^2(\mathbb{C}_+)$.

In order to prove that local solutions extend globally in time, we show that there is no blow-up of the H^2 norm in finite time. Thanks to the equivalence of the $\dot{H}^{\frac{1}{2}}$ norm and the H^2 norm in $H_{\varepsilon_n}^2(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$, it is enough to prove that equation (3.16) has conserved momentum

$$\mathcal{P}(u) := (u, -iu_z)_{L^2(\mathbb{C}_+)} = \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2.$$

But using the equation, a solution u satisfies

$$\begin{aligned} \frac{d}{dt} \mathcal{P}(u) &= 2\text{Re}(\partial_t u, -i\partial_z u)_{L^2(\mathbb{C}_+)} \\ &= 2\text{Re}(P_0^n(|u|^2 u), \partial_z u)_{L^2(\mathbb{C}_+)} \\ &= 2\text{Re}(|u|^2 u, -i\partial_z u)_{L^2(\mathbb{C}_+)}. \end{aligned}$$

By integration by parts, one knows that the complex scalar product $(|u|^2 u, -i\partial_z u)_{L^2(\mathbb{C}_+)}$ is imaginary, leading to the conservation of momentum. \square

Similarly, one can show that the energy $\mathcal{E}(u) = \|u\|_{L^4(\mathbb{C}_+)}^4$ is also conserved: using the equation,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(u) &= 2\text{Re}(\partial_t u, |u|^2 u)_{L^2(\mathbb{C}_+)} \\ &= 2\text{Re}(-iP_0^n(|u|^2 u), |u|^2 u)_{L^2(\mathbb{C}_+)} \\ &= 2\text{Re}(-iP_0^n(|u|^2 u), P_0^n(|u|^2 u))_{L^2(\mathbb{C}_+)} \\ &= 0. \end{aligned}$$

We now show that $u_n(t)$ is close to the orbit \mathcal{M} of the ground state Q . Thanks to Proposition 3.1.5, it is enough to focus on $\delta(u_n(t))$. But using the conservation laws, we know that for all $t \in \mathbb{R}$,

$$\delta(u_n(t)) = \delta(u_0^n).$$

Moreover, by construction of u_0^n , we know that $\|u_0^n - u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}$ tends to 0 as n tends to $+\infty$, therefore $\delta(u_0^n)$ tends to $\delta(u_0)$. Assume that $\delta(u_0) < \delta_0$, then $\delta(u_0^n) < \delta_0$ after some rank N . Thanks to Proposition 3.1.5, we deduce that for all $n \geq N$ and $t \in \mathbb{R}$,

$$d(u_n(t), \mathcal{M})^2 \leq C\delta(u_0^n). \quad (3.17)$$

3.3.3 Weak convergence

In this part, we show that the sequence $(u_n)_n$ has a weak limit u , which is a weak solution to equation (3.9). In order to do so, we first prove that $t \mapsto \partial_t u_n(t)$ is uniformly bounded in $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)$, then use Ascoli's theorem.

Because of the conservation of the momentum and the fact that $\mathcal{P}(u_0^n) \leq \mathcal{P}(u_0)$ for all $n \in \mathbb{N}$, we know that for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\|u_n(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq \|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}.$$

Using the equation satisfied by u_n , we also know that

$$\|\partial_t u_n(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \leq \|P_0^n(|u_n|^2 u_n)(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)}.$$

By the dual Sobolev embedding $L^{\frac{4}{3}}(\mathbb{C}_+) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)$ and the fact that P_0 extends to a bounded projector from $L^p(\mathbb{C}_+)$ to itself as soon as $1 < p < +\infty$ (see for instance [Bék+12], Theorem 1.34), we can estimate

$$\begin{aligned} \|P_0^n(|u_n|^2 u_n)(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} &\leq \|P_0(|u_n|^2 u_n)(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \\ &\leq C \|P_0(|u_n|^2 u_n)(t)\|_{L^{\frac{4}{3}}(\mathbb{C}_+)} \\ &\leq C' \| |u_n|^2 u_n(t) \|_{L^{\frac{4}{3}}(\mathbb{C}_+)} \\ &\leq C' \|u_n(t)\|_{L^4(\mathbb{C}_+)}^3. \end{aligned}$$

Since $u_n(t)$ is uniformly bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)$ and therefore in $L^4(\mathbb{C}_+)$, we conclude that the term $\|\partial_t u_n(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)}$ is also uniformly bounded.

We now prove that that up to a subsequence, $(u_n)_n$ converges in $\mathcal{C}([-T, T], \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+))$ (with the weak topology) to a function u for all $T > 0$.

We know that $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ is separable, since it is isometric to $L^2(\mathbb{R}_+)$. Moreover, by removing the high frequencies of the Fourier function f at infinity, one can see that $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ is dense in $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$. We can therefore consider a countable sequence $(\varphi_k)_k$ in $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ such that every function in $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ can be approximated by a subsequence of $(\varphi_k)_k$ for the $\dot{H}^{-\frac{1}{2}}$ -norm.

Fix $k \in \mathbb{N}$. Since $(t \mapsto \partial_t u_n(t))_n$ and $(t \mapsto u_n(t))_n$ are uniformly bounded in $\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)$ and in $\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)$ respectively, the sequence $\ell_n(\cdot, \varphi_k) : t \in [-T, T] \mapsto (u_n(t), \varphi_k)$ is equicontinuous and equibounded: for all n and t ,

$$|\partial_t \ell_n(t, \varphi_k)| = |(\partial_t u_n(t), \varphi_k)| \leq \|\partial_t u_n(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \|\varphi_k\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}$$

and

$$|\ell_n(t, \varphi_k)| = |(u_n(t), \varphi_k)| \leq \|u_n(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \|\varphi_k\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)}.$$

Applying Ascoli's theorem, for every $k \in \mathbb{N}$, there is a subsequence $(n_p)_p$ such that $(\ell_{n_p}(\cdot, \varphi_k))_p$ converges in $\mathcal{C}([-T, T], \mathbb{C})$ to some continuous function $\ell(\cdot, \varphi_k)$ as p tends to $+\infty$. By a diagonal argument, we can use the same subsequence for all $k \in \mathbb{N}$. Using a second diagonal argument on a sequence of times $(T_n)_n$ going to $+\infty$, we can assume that for all k , there exists $\ell(\cdot, \varphi_k) \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ such that for all $T > 0$, the sequence $(\ell_{n_p}(\cdot, \varphi_k))_p$ converges in $\mathcal{C}([-T, T], \mathbb{C})$ to $\ell(\cdot, \varphi_k)|_{[-T, T]}$.

By density, ℓ extends to a bounded linear map $\ell \in \mathcal{C}(\mathbb{R}, (\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+))^*)$ (with the weak topology). Now, by duality, ℓ can be represented by $u \in \mathcal{C}(\mathbb{R}, \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+))$: for all $\varphi \in \dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$,

$$\ell(t, \varphi) = (u(t), \varphi).$$

To conclude, by construction, for all $T > 0$, the sequence $(\ell_{n_p}|_{[-T, T]})_p$ converges weakly to $\ell|_{[-T, T]}$ in the space $\mathcal{C}([-T, T], (\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+))^*)$, therefore $(u_n)_n$ converges weakly to u in $\mathcal{C}([-T, T], \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+))$. Passing to the limit, we conclude that u is a global solution to the original equation (3.9) in the distribution sense.

We deduce that

$$\begin{aligned} d(u(t), \mathcal{M})^2 &= \inf_{X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|u(t) - T_X Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 \\ &\leq \inf_{X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \liminf_{n \rightarrow +\infty} \|u_n(t) - T_X Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2. \end{aligned}$$

However, since X is not compact, this inequality is not sufficient if we want to apply inequality (3.17) to estimate $d(u(t), \mathcal{M})$. In the following part, we construct a map $t \mapsto X_n(t)$, such that for all $t \in \mathbb{R}$, $u_n(t)$ is close to $T_{X_n(t)} Q$ and $(X_n(t))_{n \in \mathbb{N}}$ stays bounded, then use a compactness argument.

3.3.4 Modulation

Recall the notation from the introduction. Fix $u \in \mathcal{H}^2(\mathbb{C}_+) = \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \text{Hol}(\mathbb{C}_+)$ and $X = (s, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$, we denote by $T_X u$ the element of $\mathcal{H}^2(\mathbb{C}_+)$ satisfying

$$T_X u(z) := e^{i\theta} \alpha u(\alpha^2(z - s)), \quad z \in \mathbb{C}_+.$$

We write $X^{-1} = (-s, -\theta, \alpha^{-1})$ and

$$|X| := |s| + |\theta| + |\log(\alpha)|.$$

We have also defined the orbit of Q as

$$\mathcal{M} = \{T_X Q \mid X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\},$$

and the distance of u to \mathcal{M} as

$$d(u, \mathcal{M}) = \inf_{X=(s, \theta, \alpha) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*} \|T_X u - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}.$$

We choose $0 < r < 1$, and assume that $\|u_0 - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < r^2$. Given $K > 0$, for $n \geq N$ large enough, the regularized initial data u_0^n satisfies $\delta(u_0^n) < Kr^2$. Using the conservation of energy and momentum and Proposition 3.1.5, we deduce that there exist $c_0 > 0$ and $r_0 > 0$ such that if $0 < r < r_0$, then $d(u_n(t), \mathcal{M}) < c_0 r$ for all $t \in \mathbb{R}$.

We start from the observation that around time $t = 0$, one can choose $X_n(t) = (0, 0, 1)$ for all $n \geq N$ since $\|u_0^n - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < c_0 r$. By continuity, we know that $\|u_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1 + \varepsilon)c_0 r$ on some small time interval, which can be taken independently of n . Indeed, using the conservation of momentum, we have

$$\begin{aligned} \|u_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 &= \|u_n(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 + \|Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 - 2(u_n(t), Q)_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \\ &= \|u_0^n\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 + \|Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 - 2(u_n(t), -iQ_z)_{L^2(\mathbb{C}_+)}, \end{aligned}$$

therefore the derivative of $\|u_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2$ is bounded by

$$\begin{aligned} \left| \frac{d}{dt} \|u_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 \right| &= |2(\partial_t u_n(t), -iQ_z)_{L^2(\mathbb{C}_+)}| \\ &\leq 2\|\partial_t u_n(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \|-iQ_z\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}. \end{aligned}$$

But we have already seen that $\|\partial_t u_n(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)}$ is bounded independently of t and n , therefore there exists $K > 0$ such that for $n \geq N$ and $t \in \mathbb{R}$

$$\begin{aligned} \|u_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 &\leq \|u_n^0 - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 + K|t| \\ &\leq (c_0 r)^2 + K|t|. \end{aligned}$$

For fixed $\varepsilon > 0$, we conclude that the inequality $\|u_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1 + \varepsilon)c_0 r$ holds as long as $|t| \leq \frac{(1+\varepsilon)^2 - 1}{K}(c_0 r)^2$.

Set $\varepsilon > 0$ and $t_1 := \frac{(1+\varepsilon)^2 - 1}{K}(c_0 r)^2$. Assume that at time t_0 , there exists a bounded sequence $(X_n^0)_n$ in $\mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ such that for all n , $\|u_n(t_0) - T_{X_n^0} Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < c_0 r$. By the above method, one can show that $\|u_n(t) - T_{X_n^0} Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1 + \varepsilon)c_0 r$ on $[t_0 - t_1, t_0 + t_1]$. Indeed, let $v_n := T_{(X_n^0)^{-1}} u_n$. The equation satisfied by u_n is not invariant by scaling, but we can write down the equation satisfied by v_n . Recall that if

$$u(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma,$$

then

$$\tilde{P}_{\varepsilon, M} u(z) = \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^M e^{iz\sigma} f(\sigma) d\sigma.$$

Write $(X_n^0) =: (s_n^0, \theta_n^0, \alpha_n^0)$ and $\tilde{P}_0^n := \tilde{P}_{\frac{\varepsilon_n}{(\alpha_n^0)^2}, \frac{1}{\varepsilon_n(\alpha_n^0)^2}} \circ P_0$, then $v_n = T_{(X_n^0)^{-1}} u_n$ satisfies $\|v_n(t_0) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < c_0 r$ and

$$i\partial_t v_n = \tilde{P}_0^n (|v_n|^2 v_n).$$

Like equation (3.16), this equation conserves the energy $\|v_n(t)\|_{L^4(\mathbb{C}_+)}^4$ and the momentum $\|v_n(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2$. But for all ε and M , one can see from the expression (3.10) of the Sobolev norms that the projector $\tilde{P}_{\varepsilon, M}$ satisfies $\|\tilde{P}_{\varepsilon, M} u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \leq \|u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)}$. Therefore, we have the same inequalities as above

$$\begin{aligned} \|\partial_t v_n(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} &\leq \|\tilde{P}_0^n (|v_n|^2 v_n)(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \\ &\leq \|P_0 (|v_n|^2 v_n)(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{C}_+)} \\ &\leq C \|P_0 (|v_n|^2 v_n)(t)\|_{L^{\frac{4}{3}}(\mathbb{C}_+)} \\ &\leq C' \| |v_n|^2 v_n(t) \|_{L^{\frac{4}{3}}(\mathbb{C}_+)} \\ &\leq C' \|v_n(t)\|_{L^4(\mathbb{C}_+)}^3. \end{aligned}$$

Since $\|v_n(t)\|_{L^4(\mathbb{C}_+)} = \|u_n(t)\|_{L^4(\mathbb{C}_+)}$ is uniformly bounded by conservation of the L^4 -norm, we conclude that $\|v_n(t) - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} = \|u_n(t) - T_{X_n^0} Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1 + \varepsilon)c_0 r$ as long as $|t - t_0| \leq t_1$.

We construct X_n as a piecewise \mathcal{C}^1 functional on \mathbb{R} as follows. For $k \in \mathbb{Z}$, X_n is constant on $[kt_1, (k+1)t_1[$, equal to some $X_n^k \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ to be chosen. We first set $X_n^{-1} = X_n^0 = (0, 0, 1)$. Then, at time $t_k = kt_1$, $k \geq 1$, we use the fact that $d(u_n(t_k), \mathcal{M}) < r$ and choose X_n^k such that $\|u_n(t_k) - T_{X_n^k}Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < c_0r$. Then from the above paragraph, $\|u_n(t) - T_{X_n^k}Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1+\varepsilon)c_0r$ on $[t_k, t_k+t_1]$. We do a similar construction for negative times. The map X_n satisfies

$$\|u_n(t) - T_{X_n(t)}Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1+\varepsilon)c_0r, \quad t \in \mathbb{R}.$$

It remains to show that X_n is bounded independently of n on bounded intervals. In order to do so, it is enough to control the gap between X_n^{k-1} and X_n^k . By construction, at time t_k ,

$$\|u_n(t_k) - T_{X_n^{k-1}}Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1+\varepsilon)c_0r$$

and

$$\|u_n(t_k) - T_{X_n^k}Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < c_0r,$$

therefore

$$\|T_{X_n^{k-1}}Q - T_{X_n^k}Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (2+\varepsilon)c_0r.$$

Using the following Lemma, we conclude that if r is chosen small enough, then there exists a constant $c_1 > 0$ such that for all $n \geq N$ and $k \in \mathbb{Z}$,

$$|X_n^{k-1}(X_n^k)^{-1}| \leq c_1.$$

Lemma 3.3.6. *There exist $c_1 > 0$ and $r_1 > 0$ such that the following holds. Let $X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ such that*

$$\|T_XQ - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq r_1.$$

Then

$$|X| \leq c_1.$$

Proof. Thanks to the invariance of the $\dot{H}^{\frac{1}{2}}$ -norm by symmetries, one can assume that $X = (s, \theta, \alpha)$ with $\alpha \geq 1$ up to exchanging X and X^{-1} . We expand

$$\begin{aligned} \|T_XQ - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 &= \|T_XQ\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 + \|Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 - 2(T_XQ, Q)_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \\ &= 2\pi - 2(T_XQ, Q)_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}. \end{aligned}$$

Now, recall that

$$Q(z) = \frac{i\sqrt{2}}{z+i} = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{iz\sigma} f(\sigma) d\sigma$$

with

$$f(\sigma) = 2\sqrt{\pi}e^{-\sigma},$$

and

$$\|Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 = \frac{1}{2} \int_0^{+\infty} |f(\sigma)|^2 d\sigma = \pi.$$

With this notation, the function corresponding to $T_X Q$ is $g(\sigma) = 2\sqrt{\pi}e^{i\theta}e^{-is\sigma}e^{-\frac{\sigma}{\alpha^2}}\frac{1}{\alpha^2}$, therefore

$$\begin{aligned}\|T_X Q - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)}^2 &= 2\pi - 4\pi \operatorname{Re} \left(\int_0^{+\infty} e^{i\theta} e^{-is\sigma} e^{-\frac{\sigma}{\alpha^2}} \frac{1}{\alpha^2} e^{-\sigma} d\sigma \right) \\ &= 2\pi - 4\pi \operatorname{Re} \left(\frac{e^{i\theta}}{\alpha^2 \left(is + \frac{1}{\alpha^2} + 1 \right)} \right).\end{aligned}$$

Set $\alpha = 1 + \beta$ with $\beta \geq 0$. We want to bound s and β . By assumption,

$$\left| \operatorname{Re} \left(\frac{e^{i\theta}}{is \frac{(1+\beta)^2}{2} + 1 + \beta + \frac{\beta^2}{2}} \right) - 1 \right| \leq \frac{r_1^2}{2\pi} =: \delta_1.$$

Denote $z := \frac{e^{i\theta}}{is \frac{(1+\beta)^2}{2} + 1 + \beta + \frac{\beta^2}{2}}$. The fact $|\operatorname{Re}(z) - 1| \leq \delta_1$ implies that $|z| \geq \operatorname{Re}(z) \geq 1 - \delta_1$, and if $\delta_1 < 1$, that

$$\frac{1}{|z|} = \left| is \frac{(1+\beta)^2}{2} + 1 + \beta + \frac{\beta^2}{2} \right| \leq \frac{1}{1 - \delta_1}.$$

On the one hand, taking the real part,

$$1 + \beta + \frac{\beta^2}{2} \leq \frac{1}{1 - \delta_1}.$$

Since $\beta \in \mathbb{R}_+ \mapsto 1 + \beta + \frac{\beta^2}{2}$ is strictly increasing and going to $+\infty$ as β goes to $+\infty$, there exists some constant $c > 0$ such that $\beta \leq c$, or in other terms $0 \leq \log \alpha \leq \log(1 + c)$. On the other hand, since $\beta \geq 0$, the bound on the imaginary part implies that

$$|s| \leq \frac{2}{1 - \delta_1}. \quad \square$$

Using this lemma, assume that $3c_0 r < r_1$ and fix $t \in \mathbb{R}$. We now know that $(X_n(t))_n$ takes values in a compact set: up to extraction, one can assume that $(X_n(t))_n$ converges to some $X(t)$. Moreover, for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$, $\|u_n(t) - T_{X_n(t)} Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1 + \varepsilon)c_0 r$, therefore passing to the weak limit $n \rightarrow +\infty$ we conclude that $\|u(t) - T_{X(t)} Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} \leq (1 + \varepsilon)c_0 r$. Since $\varepsilon > 0$ can be taken arbitrarily small, we have proven the following reformulation of Theorem 3.1.2.

Theorem 3.3.7. *There exist $c_0 > 0$ and $r_0 > 0$ such that the following holds. Let $r \leq r_0$ and $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \operatorname{Hol}(\mathbb{C}_+)$ such that $\|u_0 - Q\|_{\dot{H}^{\frac{1}{2}}(\mathbb{C}_+)} < r^2$. Then there exists a weak solution $u \in \mathcal{C}(\mathbb{R}, \dot{H}^{\frac{1}{2}}(\mathbb{C}_+) \cap \operatorname{Hol}(\mathbb{C}_+))$ (with the weak topology) to equation (3.9)*

$$\begin{cases} i\partial_t u = P_0(|u|^2 u) \\ u(t=0) = u_0 \end{cases}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+$$

such that for all $t \in \mathbb{R}$,

$$d(u(t), \mathcal{M}) \leq c_0 r.$$

3.4 Orbital stability for the ground states Q_β in the Schrödinger equation

We now consider the Schrödinger equation on the Heisenberg group (3.1)

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u \\ u(t=0) = u_0 \end{cases}, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1.$$

For $\beta \in (\beta_*, 1)$, we are interested in solutions with initial data $u_0 \in \dot{H}^1(\mathbb{H}^1)$ satisfying

$$\|u_0 - \sqrt{1-\beta}Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < (1-\beta)r.$$

Let u be an eventual solution, and set

$$u(t, x, y, s) = \sqrt{1-\beta}U((1-\beta)t, x, y, s + \beta t),$$

so that U is a solution to

$$i\partial_t U - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} U = |U|^2 U, \quad (t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1. \quad (3.18)$$

The initial data U_0 satisfies

$$\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1-\beta}r.$$

There are two relevant conserved quantities for this equation: the energy

$$\mathcal{E}_\beta(V) := \frac{1}{2} \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} V, V \right)_{L^2(\mathbb{H}^1)} - \frac{1}{4} \|V\|_{L^4(\mathbb{H}^1)}^4,$$

and the momentum

$$\mathcal{P}(V) := (D_s V, V)_{L^2(\mathbb{H}^1)}, \quad V \in \dot{H}^1(\mathbb{H}^1).$$

Theorem 3.1.3 is equivalent to prove that if β is large, then one can construct a global weak solution U to equation (3.18) which stays close to the orbit of Q_β at all times, which leads to the following reformulation.

Theorem 3.4.1. *There exist some constants $c_0 > 0$ and $r_0 > 0$ such that for all $r \in (0, r_0)$, there exists a parameter $\beta^*(r) \in (0, 1)$ such that the following holds. Let $\beta \in (\beta^*(r), 1)$ and $U_0 \in \dot{H}^1(\mathbb{H}^1)$ satisfying*

- if $U_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$:

$$\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < r^2$$

- in the general case:

$$\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1-\beta}r.$$

Then there exists a global weak solution $U_\beta \in \mathcal{C}(\mathbb{R}, \dot{H}^1(\mathbb{H}^1))$ (with the weak topology) to equation (3.18)

$$\begin{cases} i\partial_t U_\beta - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} U_\beta = |U_\beta|^2 U_\beta \\ U_\beta(t=0) = U_0 \end{cases}$$

such that for all $t \in \mathbb{R}$, $U_\beta(t)$ is close to the orbit $\mathcal{Q}_\beta = \{T_X Q_\beta \mid X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*\}$ of Q_β :

$$d(U_\beta(t), \mathcal{Q}_\beta) \leq c_0 r.$$

Contrary to the strategy deployed for the half-wave equation [GR16], the gap

$$\delta_\beta(V) := |\mathcal{E}_\beta(V) - \mathcal{E}_\beta(Q_\beta)| + |\mathcal{P}(V) - \mathcal{P}(Q_\beta)|, \quad V \in \dot{H}^1(\mathbb{H}^1),$$

does not here directly control the distance of V to Q_β , so Proposition 3.1.5 does not hold for Q_β and δ_β . Indeed, even the fact that $\delta_\beta(V) = 0$ does not imply that V belongs to Q_β . This is due to the fact that we can only use two conservation laws (energy and momentum) here, whereas an additional conservation laws was available for the half-wave equation: the mass of the solution.

However, using that Q_β tends to Q as β tends to 1, one can instead show that the component of the solution along the space V_0^+ is close to Q and control the rest separately. More precisely, decompose

$$U(t) = U^+(t) + W(t),$$

where $U^+(t) \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $W(t) \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$. If we know that $W(t)$ is small enough, then $\delta_\beta(U(t)) \approx \delta(U^+(t))$. This enables us to estimate the distance $d(U^+(t), \mathcal{M})$ of $U^+(t)$ to the orbit of Q , and therefore the distance of $U(t)$ to the orbit of Q_β for β close to 1.

The plan of the proof is as follows. Fix $\beta \in (0, 1)$. We approximate the initial data and the equation by global smooth functions $(U_{\gamma,n})_{\gamma \in [\beta, 1], n \in \mathbb{N}}$ valued in $H^2(\mathbb{H}^1)$ in part 3.4.1. We then decompose

$$U_{\gamma,n}(t) = U_{\gamma,n}^+(t) + W_{\beta,n}(t),$$

where $U_{\gamma,n}^+(t) \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $W_{\beta,n}(t) \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$. In part 3.4.2, we fix $n \in \mathbb{N}$, and study the limit $\gamma \rightarrow 1$. We prove by using the conservation laws that $W_{\beta,n}(t)$ stays small and that the gap $\delta(U_{\gamma,n}^+(t))$ is controlled as $\delta(U_{\gamma,n}^+(t)) \lesssim r^2$ for $\gamma \geq \beta^*(n, t)$, which leads to an upper bound

$$d(U_{\gamma,n}(t), \mathcal{M}) < c_0 r, \quad t \in \mathbb{R}, \gamma \in [\max(\beta^*(n, t), \beta), 1). \quad (3.19)$$

Then, we show that the lower bound $\beta^*(n, t)$ can be taken independently of n and t . Finally, in part 3.4.3, we fix $\beta \geq \beta_*$ and use the same method as for the limiting equation to find an upper bound on the modulation parameters $(X_{\beta,n}(t))_{n \in \mathbb{N}}$ in order to pass to the limit $n \rightarrow +\infty$ in the above inequality (3.19).

3.4.1 Construction of approximate solutions

Construction of a sequence of smoothing projectors $\Pi^{(n)}$: We define a sequence of projectors $\Pi^{(n)}$ close to identity, mapping elements of $\dot{H}^s(\mathbb{H}^1)$ ($s = \pm 1$) to smoother functions, by removing the high and low Fourier frequencies and the high Hermite modes in the decomposition

$$\dot{H}^s(\mathbb{H}^1) = \bigoplus_{k \in \mathbb{N}} \bigoplus_{\pm} \dot{H}^s(\mathbb{H}^1) \cap V_k^\pm.$$

Using these projectors, we consider a sequence of equations approximating (3.18) for which the Cauchy problem is globally well-posed.

Let $u \in \dot{H}^s(\mathbb{H}^1)$, $s = \pm 1$, which we decompose as a series of elements of $\dot{H}^s(\mathbb{H}^1) \cap V_k^\pm$ for $(k, \pm) \in (\mathbb{N}, \pm)$. Write

$$u = \sum_{k=0}^{+\infty} \sum_{\pm} \Pi_{k,\pm}(u),$$

where for all $(k, \pm) \in (\mathbb{N}, \pm)$, $\Pi_{k,\pm}(u) \in \dot{H}^s(\mathbb{H}^1) \cap V_k^\pm$. Then

$$\|u\|_{\dot{H}^s(\mathbb{H}^1)}^2 = \sum_{k \in \mathbb{N}} \sum_{\pm} \int_{\mathbb{R}^\pm} ((k+1)|\sigma|)^s \int_{\mathbb{R}^2} |\widehat{\Pi_{k,\pm}(u)}(x, y, \sigma)|^2 dx dy d\sigma.$$

Let $n \in \mathbb{N}$, we define $\Pi^{(n)}(u)$ as follows. We take the n -th partial sum and remove the high and low frequencies $|\sigma| \rightarrow +\infty$ and $|\sigma| \rightarrow 0$:

$$\widehat{\Pi^{(n)}(u)}(x, y, \sigma) := \sum_{k=0}^n \sum_{\pm} \widehat{\Pi_{k,\pm}(u)}(x, y, \sigma) \mathbf{1}_{\frac{1}{n} \leq |\sigma| \leq n}. \quad (3.20)$$

Consequently,

$$\|\Pi^{(n)}(u)\|_{\dot{H}^s(\mathbb{H}^1)}^2 = \sum_{k=0}^n \sum_{\pm} \int_{\{\sigma \in \mathbb{R}^\pm\} \cap \{\frac{1}{n} \leq |\sigma| \leq n\}} ((k+1)|\sigma|)^s \int_{\mathbb{R}^2} |\widehat{\Pi_{k,\pm}(u)}(x, y, \sigma)|^2 dx dy d\sigma$$

converges to $\|u\|_{\dot{H}^s(\mathbb{H}^1)}^2$ as n goes to $+\infty$.

Moreover, if $u \in H^1(\mathbb{H}^1)$, then $\Pi^{(n)}(u)$ belongs to $H^3(\mathbb{H}^1)$. Indeed,

$$\|\Pi^{(n)}(u)\|_{H^3(\mathbb{H}^1)}^2 = \sum_{k=0}^n \sum_{\pm} \int_{\{\sigma \in \mathbb{R}^\pm\} \cap \{\frac{1}{n} \leq |\sigma| \leq n\}} (1 + (k+1)^3 |\sigma|^3) \int_{\mathbb{R}^2} |\widehat{\Pi_{k,\pm}(u)}(x, y, \sigma)|^2 dx dy d\sigma,$$

but on the set $\{\frac{1}{n} \leq |\sigma| \leq n\}$, and for $k \leq n$,

$$(1 + (k+1)^3 |\sigma|^3) \leq (n|\sigma| + (n+1)^2(k+1)n^2|\sigma|) \leq n(1 + n(n+1)^2)(k+1)|\sigma|,$$

therefore $\|\Pi^{(n)}(u)\|_{H^3(\mathbb{H}^1)}$ is finite.

Construction of a sequence of approximate solutions $(U_{\gamma,n})_{\gamma \in [\beta, 1], n \in \mathbb{N}}$: Fix $\beta \in (0, 1)$, $r > 0$ and $U_0 \in \dot{H}^1(\mathbb{H}^1)$ such that

- either $U_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < r^2$;
- either $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1 - \beta}r$.

We construct a global solution to (3.18)

$$\begin{cases} i\partial_t U_\beta - \frac{\Delta_{\mathbb{H}^1 + \beta D_s}}{1 - \beta} U_\beta = |U_\beta|^2 U_\beta \\ U_\beta(t=0) = U_0 \end{cases}$$

such that for all $t \in \mathbb{R}$,

$$d(U_\beta(t), \mathcal{Q}_\beta) \leq c_0 r.$$

By approximation, the idea would be to consider a sequence of equations

$$\begin{cases} i\partial_t U_{\beta,n} - \frac{\Delta_{\mathbb{H}^1 + \beta D_s}}{1 - \beta} U_{\beta,n} = \Pi^{(n)}(|U_{\beta,n}|^2 U_{\beta,n}) \\ U_{\beta,n}(t=0) = U_0^{\beta,n} = \Pi^{(n)}(U_0) \end{cases}, \quad n \in \mathbb{N}, \quad (3.21)$$

for which one can show that for all n large, there exists $\beta^*(n)$ such that if $\beta \geq \beta^*(n)$, then

$$d(U_{\beta,n}(t), \mathcal{Q}_\beta) \leq c_0 r, \quad t \in \mathbb{R}.$$

In order to get a lower bound β^* independent of n , we rather construct a set of initial data $(U_0^{\gamma,n})_{\gamma \in [\beta,1), n \in \mathbb{N}}$ and equations

$$\begin{cases} i\partial_t U_{\gamma,n} - \frac{\Delta_{\mathbb{H}^1} + \gamma D_s}{1-\gamma} U_{\gamma,n} = \Pi^{(n)}(|U_{\gamma,n}|^2 U_{\gamma,n}) \\ U_{\gamma,n}(t=0) = U_0^{\gamma,n} = \Pi^{(n)}(U_0^\gamma) \end{cases}, \quad n \in \mathbb{N}, \gamma \in [\beta, 1), \quad (3.22)$$

then use a continuity argument.

For $\gamma \in [\beta, 1)$, the initial data U_0^γ is defined as follows:

- if $U_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$, we choose U_0^γ constant equal to U_0 ;
- otherwise, we choose

$$U_0^\gamma := \frac{1-\gamma}{1-\beta} U_0 + \frac{\gamma-\beta}{1-\beta} Q.$$

We make this choice in the general case because we need the initial data U_0^γ to go to $\dot{H}^1(\mathbb{H}^1) \cap V_0^+$ as γ tends to 1.

Lemma 3.4.2. *Let $r > 0$ and $U_0 \in \dot{H}^1(\mathbb{H}^1)$. There exist $C_0 > 0$, $\beta_*(r) \in (0, 1)$ and $N(r, U_0) \in \mathbb{N}$ such that the following holds. Let $\beta \in (\beta_*(r), 1)$ and assume that U_0 satisfies*

- either $U_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < r^2$;
- either $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1-\beta}r$.

Then for all $n \geq N(r, U_0)$ and for all $\gamma \in [\beta, 1)$,

$$|\mathcal{E}_\gamma(U_0^{\gamma,n}) - \mathcal{E}(Q)| + |\mathcal{P}(U_0^{\gamma,n}) - \mathcal{P}(Q)| < C_0 r^2.$$

Proof. We use the following convergence rate of $(Q_\beta)_\beta$ to Q as β tends to 1 (proved in Appendix 3.A):

$$\|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)} = o(\sqrt{1-\beta}).$$

If $U_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < r^2$, we have chosen U_0^γ constant equal to U_0 and it is enough to use that $\|\Pi^{(n)}(U_0) - U_0\|_{\dot{H}^1(\mathbb{H}^1)} \rightarrow 0$ as $n \rightarrow +\infty$.

We now treat the case $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1-\beta}r$. By convergence of Q_β to Q , there exists $\beta_* = \beta_*(r) \in (0, 1)$ such that for all $\beta \in (\beta_*, 1)$,

$$\|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1-\beta}r.$$

We decompose

$$Q_\beta = Q_\beta^+ + R_\beta$$

and

$$U_0 = U_0^+ + W_0$$

where $Q_\beta^+, U_0^+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $R_\beta, W_0 \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$. In the same way, we decompose U_0^γ as

$$U_0^\gamma = (U_0^\gamma)^+ + W_0^\gamma$$

and $U_0^{\gamma,n} = \Pi^{(n)}(U_0^\gamma)$ as

$$U_0^{\gamma,n} = (U_0^{\gamma,n})^+ + W_0^{\gamma,n},$$

where $(U_0^\gamma)^+, (U_0^{\gamma,n})^+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $W_0^\gamma, W_0^{\gamma,n} \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$.

Since

$$\|W_0 - R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \leq \|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1 - \beta}r,$$

then W_0 satisfies

$$\begin{aligned} \|W_0\|_{\dot{H}^1(\mathbb{H}^1)} &\leq \|W_0 - R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} + \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \\ &\leq 2\sqrt{1 - \beta}r. \end{aligned}$$

Therefore, $W_0^\gamma = \frac{1-\gamma}{1-\beta}W_0$ satisfies

$$\|W_0^\gamma\|_{\dot{H}^1(\mathbb{H}^1)} \leq 2\frac{1-\gamma}{\sqrt{1-\beta}}r,$$

which implies that for all $n \in \mathbb{N}$,

$$\|W_0^{\gamma,n}\|_{\dot{H}^1(\mathbb{H}^1)} \leq 2\frac{1-\gamma}{\sqrt{1-\beta}}r.$$

In particular,

$$\begin{aligned} \left| \left(-\frac{\Delta_{\mathbb{H}^1} + \gamma D_s}{1-\gamma} W_0^{\gamma,n}, W_0^{\gamma,n} \right)_{L^2(\mathbb{H}^1)} \right| + |(D_s W_0^{\gamma,n}, W_0^{\gamma,n})_{L^2(\mathbb{H}^1)}| &\leq 8\frac{1-\gamma}{1-\beta}r^2 + 4\frac{(1-\gamma)^2}{1-\beta}r^2 \\ &\leq 12r^2 \end{aligned}$$

and

$$\|W_0^{\gamma,n}\|_{L^4(\mathbb{H}^1)} \leq C\|W_0^{\gamma,n}\|_{\dot{H}^1(\mathbb{H}^1)} \leq 2C\sqrt{1-\beta}r,$$

which is bounded by r^2 if $\beta \geq \beta_*(r)$ is large enough.

Given the form of the energy

$$\begin{aligned} \mathcal{E}_\gamma(U_0^{\gamma,n}) &= \frac{1}{2} \left(-\frac{\Delta_{\mathbb{H}^1} + \gamma D_s}{1-\gamma} W_0^{\gamma,n}, W_0^{\gamma,n} \right)_{L^2(\mathbb{H}^1)} + \frac{1}{2} (D_s(U_0^{\gamma,n})^+, (U_0^{\gamma,n})^+)_{L^2(\mathbb{H}^1)} \\ &\quad - \frac{1}{4} \|(U_0^{\gamma,n})^+ + W_0^{\gamma,n}\|_{L^4(\mathbb{H}^1)}^4, \end{aligned}$$

and given that all the terms involving $W_0^{\gamma,n}$ are bounded by some C_0r^2 , it is now enough to prove an estimate of the form $\|(U_0^{\gamma,n})^+ - Q\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_0r^2$. But

$$\begin{aligned} \|(U_0^{\gamma,n})^+ - (U_0^\gamma)^+\|_{\dot{H}^1(\mathbb{H}^1)} &\leq \frac{1-\gamma}{1-\beta} \|\Pi^{(n)}((U_0)^+) - U_0^+\|_{\dot{H}^1(\mathbb{H}^1)} + \frac{\gamma-\beta}{1-\beta} \|\Pi^{(n)}(Q) - Q\|_{\dot{H}^1(\mathbb{H}^1)} \\ &\leq \|\Pi^{(n)}((U_0)^+) - U_0^+\|_{\dot{H}^1(\mathbb{H}^1)} + \|\Pi^{(n)}(Q) - Q\|_{\dot{H}^1(\mathbb{H}^1)}, \end{aligned}$$

which converges to zero as n goes to $+\infty$ independently of γ and β . Moreover,

$$\begin{aligned} \|(U_0^\gamma)^+ - Q\|_{\dot{H}^1(\mathbb{H}^1)} &= \frac{1-\gamma}{1-\beta} \|U_0^+ - Q\|_{\dot{H}^1(\mathbb{H}^1)} \\ &\leq \frac{1-\gamma}{1-\beta} (\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} + \|W_0\|_{\dot{H}^1(\mathbb{H}^1)} + \|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)}) \\ &\leq \frac{1-\gamma}{1-\beta} 4\sqrt{1-\beta}r \\ &\leq 4r^2 \end{aligned}$$

for $\beta \geq \beta_*(r)$ large enough.

To conclude, there exist $C_0 > 0$, $r_0 > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $r \in (0, r_0)$, for all $\beta \in (\beta_*(r), 1)$ and $\gamma \in [\beta, 1)$,

$$|\mathcal{E}_\gamma(U_0^{\gamma,n}) - \mathcal{E}(Q)| + |\mathcal{P}(U_0^{\gamma,n}) - \mathcal{P}(Q)| < C_0 r^2. \quad \square$$

From now on, we assume that $\beta \geq \beta_*(r)$ and $n \geq N(r)$.

As in Proposition 3.3.5 for the limiting equation, equation (3.22) admits a unique global solution in $H_n^3(\mathbb{H}^1) := \Pi^{(n)}(H^3(\mathbb{H}^1))$.

Proposition 3.4.3. *Let $U_0^{\gamma,n} \in H_n^3(\mathbb{H}^1)$ and $\gamma \in [0, 1)$. Then there exists a unique $U_{\gamma,n} \in \mathcal{C}^\infty(\mathbb{R}, H_n^3(\mathbb{H}^1))$ such that (3.22) is satisfied in the distributional sense. Moreover, the solution map is continuous from $H_n^3(\mathbb{H}^1)$ to $\mathcal{C}^\infty(\mathbb{R}, H_n^3(\mathbb{H}^1))$.*

Proof. Local well-posedness comes from the Cauchy-Lipschitz theory from ODEs. Indeed, $H^3(\mathbb{H}^1)$ is an algebra, moreover, the Hermite modes k are restricted to $k \geq n$ and the frequencies σ are restricted to the set $\{\frac{1}{n} \leq |\sigma| \leq n\}$. Therefore, the map $V \mapsto \frac{\Delta_{\mathbb{H}^1} + \gamma D_s}{1-\gamma} V + \Pi^{(n)}(|V|^2 V)$ is well-defined and locally Lipschitz from the Banach space $H_n^3(\mathbb{H}^1)$ to itself.

In order to show that the local maximal solutions are global, we prove that there is no blow up in finite time thanks to the conservation of the momentum

$$\mathcal{P}(V) = (D_s V, V)_{L^2(\mathbb{H}^1)},$$

and the following inequality valid for $V \in H_n^3(\mathbb{H}^1)$:

$$(D_s V, V)_{L^2(\mathbb{H}^1)} \leq \|V\|_{H^3(\mathbb{H}^1)}^2 \leq (n + (n+1)^3 n^2) (D_s V, V)_{L^2(\mathbb{H}^1)}. \quad \square$$

3.4.2 Limit $\gamma \rightarrow 1$ for the n -th partial sum

In this part, we use the conservation of energy and momentum to recover an upper bound on $d(U_{\gamma,n}(t), \mathcal{M})$ for $\gamma \geq \beta^*(n, t)$ close to 1. Then, we prove that the lower bound for γ can be chosen independently of n and t .

For $t \in \mathbb{R}$, we decompose $U_{\gamma,n}(t)$ as

$$U_{\gamma,n}(t) = U_{\gamma,n}^+(t) + W_{\gamma,n}(t),$$

where

$$U_{\gamma,n}^+(t) = \Pi_0^+(U_{\gamma,n}(t)) \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+,$$

and therefore

$$W_{\gamma,n}(t) = (\text{Id} - \Pi_0^+)(U_{\gamma,n}(t)) \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$$

(see Definition 3.2.1 for the definition of the spaces in orthogonal sum V_k^\pm).

In what follows, we show that $U_{\gamma,n}^+(t)$ is the main part for which we control $\delta(U_{\gamma,n}^+(t))$, and $W_{\gamma,n}(t)$ is a remainder term which vanishes in the limit $\gamma \rightarrow 1$.

First, since $\mathcal{P}(U_{\gamma,n}(t)) = (D_s U_{\gamma,n}(t), U_{\gamma,n}(t))_{L^2(\mathbb{H}^1)}$ is conserved, bounded by $C_0 r^2 + \mathcal{P}(Q)$ for all $\gamma \in [\beta, 1)$ and equivalent to $\|U_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2$ in $H_n^2(\mathbb{H}^1)$, there exists some constant $C(n) > 0$ such that for all $t \in \mathbb{R}$ and $\gamma \in [\beta, 1)$,

$$\|U_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C(n). \quad (3.23)$$

But such a bound on $\|U_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)}$ and the conservation of energy imply that $W_{\gamma,n}(t)$ must vanish in $\dot{H}^1(\mathbb{H}^1)$ as γ tends to 1 thanks to the following lemma.

Lemma 3.4.4. *Let $r > 0$, $\beta \in (\beta_*(r), 1)$, $n \geq N(r)$ and assume that U_0 satisfies*

- *either $U_0 \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < r^2$;*
- *either $\|U_0 - Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} < \sqrt{1 - \beta}r$,*

There exists $C_1 > 0$ such that if there exists $C > 0$ (possibly depending on n), $t \in \mathbb{R}$ and $\gamma \in [\beta, 1)$ such that $\|U_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C$, then

$$\|W_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_1(1 + C^2)\sqrt{1 - \gamma}.$$

Proof. We use the conservation of energy

$$\begin{aligned} \mathcal{E}_\gamma(U_{\gamma,n}(t)) &= \frac{1}{2} \left(-\frac{\Delta_{\mathbb{H}^1} + \gamma D_s}{1 - \gamma} W_{\gamma,n}(t), W_{\gamma,n}(t) \right)_{L^2(\mathbb{H}^1)} + \frac{1}{2} (D_s U_{\gamma,n}^+(t), U_{\gamma,n}^+(t))_{L^2(\mathbb{H}^1)} \\ &\quad - \frac{1}{4} \|U_{\gamma,n}(t)\|_{L^4(\mathbb{H}^1)}^4, \end{aligned}$$

then apply Lemma 3.4.2 to get that

$$|\mathcal{E}_\gamma(U_{\gamma,n}(t)) - \mathcal{E}(Q)| < C_0 r^2.$$

Thanks to the embedding $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$, we know that

$$\|U_{\gamma,n}(t)\|_{L^4(\mathbb{H}^1)}^4 \leq K^4 \|U_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)}^4 = K^4 C^4.$$

Moreover, recall the equivalence of norms

$$\frac{1}{2} \|w\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq (-\Delta_{\mathbb{H}^1} + \gamma D_s)w, w)_{L^2(\mathbb{H}^1)} \leq 2 \|w\|_{\dot{H}^1(\mathbb{H}^1)}^2, \quad w \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm.$$

We conclude that

$$\frac{1}{4(1 - \gamma)} \|W_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq \mathcal{E}(Q) + C_0 r^2 + \frac{1}{4} K^4 C^4,$$

which implies the lemma. \square

Applying Lemma 3.4.4 and inequality (3.23), we know that for all $t \in \mathbb{R}$ and $\gamma \in [\beta, 1)$,

$$\|W_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_1(1 + C(n)^2)\sqrt{1 - \gamma},$$

which vanishes as γ tends to 1.

Fix $t \in \mathbb{R}$. We establish in Lemma 3.4.5 below that $\gamma \in [\beta, 1) \mapsto W_{\gamma,n}(t)$ is continuous, so that we can define $\beta_0(n, t) \geq \beta$ as the minimal element in $[\beta, 1)$ satisfying:

$$\forall \gamma \in [\beta_0(n, t), 1), \quad \|W_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq r^2.$$

Lemma 3.4.5. *For $t \in \mathbb{R}$, $\gamma \in [\beta, 1) \mapsto W_{\gamma,n}(t) \in \dot{H}^1(\mathbb{H}^1)$ is continuous.*

Proof. Fix $t \in \mathbb{R}$. One knows that the orthogonal projection $(\text{Id} - \Pi_0^+)$ onto the space $\bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$ is continuous since it satisfies the inequality $\|(\text{Id} - \Pi_0^+)(V)\|_{\dot{H}^1(\mathbb{H}^1)} \leq \|V\|_{\dot{H}^1(\mathbb{H}^1)}$ for all $V \in \dot{H}^1(\mathbb{H}^1)$. But by definition, $W_{\gamma,n}(t) = (\text{Id} - \Pi_0^+)(U_{\gamma,n}(t))$, therefore it is enough to show that the map $\gamma \in [\beta, 1) \mapsto U_{\gamma,n}(t) \in \dot{H}^1(\mathbb{H}^1)$ is continuous. Let $\gamma_1, \gamma_2 \in [\beta, 1)$ and set $R := U_{\gamma_1,n} - U_{\gamma_2,n}$. Then R is a solution to

$$\begin{aligned} i\partial_t R - \frac{\Delta_{\mathbb{H}^1} + \gamma_1 D_s}{1 - \gamma_1} R - \left(\frac{\Delta_{\mathbb{H}^1} + \gamma_1 D_s}{1 - \gamma_1} - \frac{\Delta_{\mathbb{H}^1} + \gamma_2 D_s}{1 - \gamma_2} \right) U_{\gamma_2,n} \\ = \Pi^{(n)}(|U_{\gamma_1,n}|^2 U_{\gamma_1,n}) - \Pi^{(n)}(|U_{\gamma_2,n}|^2 U_{\gamma_2,n}). \end{aligned}$$

We bound $\|\partial_t R(t)\|_{\dot{H}^1(\mathbb{H}^1)}$, which is equivalent to controlling $\|\partial_t R(t)\|_{\dot{H}^{-1}(\mathbb{H}^1)}$ since $\partial_t R(t) \in \Pi^{(n)}(\dot{H}^1(\mathbb{H}^1))$. We treat each term in the equation separately.

First,

$$\begin{aligned} \left\| -\frac{\Delta_{\mathbb{H}^1} + \gamma_1 D_s}{1 - \gamma_1} R(t) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} &\leq \frac{2}{1 - \gamma_1} \left\| -\Delta_{\mathbb{H}^1} R(t) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} \\ &\leq \frac{2}{1 - \gamma_1} \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)}. \end{aligned}$$

Then,

$$\begin{aligned} \left\| \left(\frac{\Delta_{\mathbb{H}^1} + \gamma_1 D_s}{1 - \gamma_1} - \frac{\Delta_{\mathbb{H}^1} + \gamma_2 D_s}{1 - \gamma_2} \right) U_{\gamma_2,n}(t) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} \\ \leq \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} \left(\left\| -\Delta_{\mathbb{H}^1} U_{\gamma_2,n}(t) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} + \left\| D_s U_{\gamma_2,n}(t) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} \right) \\ \leq \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} 2C(n). \end{aligned}$$

Finally, note that in the image $\dot{H}_n^{-1}(\mathbb{H}^1)$ of $\dot{H}^{-1}(\mathbb{H}^1)$ by $\Pi^{(n)}$, all the Sobolev norms are equivalent. Indeed, by definition of $\Pi^{(n)}$ (see equation (3.20)), the frequencies are restricted to the set $\{\frac{1}{n} \leq |\sigma| \leq n\}$ and the Hermite modes are bounded by n . Therefore, there exists $C'_1(n) > 0$ such that

$$\begin{aligned} \left\| \Pi^{(n)}(|U_{\gamma_1,n}|^2 U_{\gamma_1,n}(t)) - \Pi^{(n)}(|U_{\gamma_2,n}|^2 U_{\gamma_2,n}(t)) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} \\ \leq C'_1(n) \left\| \Pi^{(n)}(|U_{\gamma_1,n}|^2 U_{\gamma_1,n}(t)) - \Pi^{(n)}(|U_{\gamma_2,n}|^2 U_{\gamma_2,n}(t)) \right\|_{H^3(\mathbb{H}^1)}. \end{aligned}$$

Using the algebra property of $H^3(\mathbb{H}^1)$, we get

$$\left\| \Pi^{(n)}(|U_{\gamma_1,n}|^2 U_{\gamma_1,n}(t)) - \Pi^{(n)}(|U_{\gamma_2,n}|^2 U_{\gamma_2,n}(t)) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} \leq C'_2(n) \|R(t)\|_{H^3(\mathbb{H}^1)},$$

and using again the equivalence between \dot{H}^1 and H^3 norms, we deduce

$$\left\| \Pi^{(n)}(|U_{\gamma_1,n}|^2 U_{\gamma_1,n}(t)) - \Pi^{(n)}(|U_{\gamma_2,n}|^2 U_{\gamma_2,n}(t)) \right\|_{\dot{H}^{-1}(\mathbb{H}^1)} \leq C'_3(n) \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)}.$$

We now define $f(t) := \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2$ for $t \in \mathbb{R}$. Then there exists some constant $C''(n) > 0$ such that

$$\begin{aligned} f'(t) &\leq 2 \|\partial_t R(t)\|_{\dot{H}^1(\mathbb{H}^1)} \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)} \\ &\leq C''(n) \|\partial_t R(t)\|_{\dot{H}^{-1}(\mathbb{H}^1)} \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)} \\ &\leq C''(n) \left(\left(\frac{2}{1 - \gamma_1} + C'_3(n) \right) \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} 2C(n) \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)} \right) \\ &\leq C''(n) \left(\left(\frac{2}{1 - \gamma_1} + C'_3(n) + \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} C(n) \right) \|R(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 + \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} C(n) \right). \end{aligned}$$

Therefore, $f(t)$ satisfies a Gronwall-type inequality

$$f(t)' \leq K_1(n)f(t) + K_2(n) \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)}$$

with

$$K_1(n) = C''(n) \left(\frac{2}{1 - \gamma_1} + C_3'(n) + \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} C(n) \right),$$

and

$$K_2(n) = C''(n)C(n).$$

This inequality implies that for all $t \in \mathbb{R}$,

$$f(t) \leq f(0)e^{K_1(n)|t|} + \frac{K_2(n)}{K_1(n)} \frac{|\gamma_2 - \gamma_1|}{(1 - \gamma_1)(1 - \gamma_2)} (e^{K_1(n)|t|} - 1)$$

with

$$f(0) = \|\Pi^{(n)}(U_0^{\gamma_1} - U_0^{\gamma_2})\|_{\dot{H}^1(\mathbb{H}^1)}^2.$$

Fix $t \in \mathbb{R}$ and $\gamma_1 \in [\beta, 1)$, we see that if γ_2 tends to γ_1 , then $f(t)$ tends to 0. \square

Lemma 3.4.6. *Let $r > 0$, $\beta \in (\beta_*(r), 1)$ and $n \geq N(r)$. There exists some constant $\beta^*(r) \in (0, 1)$ such that if $\beta \geq \beta^*(r)$, then the solution $U_{\beta,n}$ to equation (3.21)*

$$\begin{cases} i\partial_t U_{\beta,n} - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} U_{\beta,n} = \Pi^{(n)}(|U_{\beta,n}|^2 U_{\beta,n}) \\ U_{\beta,n}(t = 0) = U_0^n = \Pi^{(n)}(U_0) \end{cases}$$

satisfies for all $t \in \mathbb{R}$

$$\|W_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq r^2$$

and

$$\|U_{\beta,n}^+(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq (\mathcal{P}(Q) + C_0 r^2)^{\frac{1}{2}}.$$

Proof. Fix $\beta \in (0, 1)$, recall that $\beta_0(n, t) \geq \beta$ is the minimal element in $[\beta, 1)$ satisfying:

$$\forall \gamma \in [\beta_0(n, t), 1), \quad \|W_{\gamma,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq r^2,$$

and assume that $\beta < \beta_0(n, t) =: \beta_0$. We find an upper bound for β_0 in $[\beta, 1)$ independent on n and t . The continuity of $\gamma \mapsto W_{\gamma,n}(t)$ implies that

$$\|W_{\beta_0,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} = r^2.$$

The projection of $U_{\beta_0,n}(t)$ on V_0^+ is bounded by

$$\begin{aligned} \|U_{\beta_0,n}^+(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 &\leq \mathcal{P}(U_{\beta_0,n}(t)) \\ &\leq \mathcal{P}(Q) + C_0 r^2, \end{aligned}$$

therefore

$$\|U_{\beta_0,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C,$$

where $C := r^2 + (\mathcal{P}(Q) + C_0 r^2)^{\frac{1}{2}}$ does not depend on n or t anymore. Lemma 3.4.4 now implies

$$\|W_{\beta_0,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_1(1 + C^2)\sqrt{1 - \beta_0}.$$

We conclude that

$$r^2 \leq C_1(1 + C^2)\sqrt{1 - \beta_0},$$

which means

$$\beta_0 \leq 1 - \left(\frac{r^2}{C_1(1 + C^2)} \right)^2 =: \beta^*(r),$$

therefore $\beta < \beta^*(r)$. Taking the converse, we have proven that if $\beta \geq \beta^*(r)$, then $\beta_0 = \beta$. \square

We now show that $U_{\beta,n}(t)$ is close to the orbit \mathcal{M} of Q for $t \in \mathbb{R}$ and $\beta \geq \beta^*(r)$.

Proposition 3.4.7. *There exist $r_0 > 0$ and $c_0 > 0$ such that if $r < r_0$, $\beta \in [\beta^*(r), 1)$ and $n \geq N(r)$, then for all $t \in \mathbb{R}$,*

$$d(U_{\beta,n}(t), \mathcal{M}) < c_0 r.$$

Proof. Fix $t \in \mathbb{R}$. It suffices to estimate $\delta(U_{\beta,n}^+(t))$ and apply Proposition 3.1.5.

On the one hand, since $(D_s W_{\beta,n}(t), W_{\beta,n}(t))_{L^2(\mathbb{H}^1)} \leq \|W_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq r^2$, the conservation of momentum and Lemma 3.4.2 lead to

$$|(D_s U_{\beta,n}^+(t), U_{\beta,n}^+(t))_{L^2(\mathbb{H}^1)} - (D_s Q, Q)_{L^2(\mathbb{H}^1)}| \leq (C_0 + 1)r^2. \quad (3.24)$$

On the other hand, we estimate $|\|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)}^4 - \|Q\|_{L^4(\mathbb{H}^1)}^4|$ via the conservation of energy. We know that

$$|\|U_{\beta,n}(t)\|_{L^4(\mathbb{H}^1)}^4 - \|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)}^4| \leq \|W_{\beta,n}(t)\|_{L^4(\mathbb{H}^1)} (\|U_{\beta,n}(t)\|_{L^4(\mathbb{H}^1)} + \|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)})^3.$$

Since $\|U_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)}$ is bounded thanks to Lemma 3.4.6, there exists $C_1 > 0$ such that

$$|\|U_{\beta,n}(t)\|_{L^4(\mathbb{H}^1)}^4 - \|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)}^4| \leq C_1 \|W_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_1 r^2. \quad (3.25)$$

Therefore, from (3.24) and (3.25), we get

$$\begin{aligned} \mathcal{E}_\beta(U_{\beta,n}(t)) &= \frac{1}{2} \left(-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} W_{\beta,n}(t), W_{\beta,n}(t) \right)_{L^2(\mathbb{H}^1)} + \frac{1}{2} (D_s U_{\beta,n}^+(t), U_{\beta,n}^+(t))_{L^2(\mathbb{H}^1)} - \frac{1}{4} \|U_{\beta,n}(t)\|_{L^4(\mathbb{H}^1)}^4 \\ &\geq \frac{1}{2} (D_s Q, Q)_{L^2(\mathbb{H}^1)} - \frac{1}{4} \|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)}^4 - \left(\frac{C_0 + 1}{2} + \frac{C_1}{4} \right) r^2. \end{aligned}$$

However, thanks to the conservation of energy and Lemma 3.4.2, we have

$$\mathcal{E}_\beta(U_{\beta,n}(t)) = \mathcal{E}_\beta(U_0^n) \leq \frac{1}{2} (D_s Q, Q)_{L^2(\mathbb{H}^1)} - \frac{1}{4} \|Q\|_{L^4(\mathbb{H}^1)}^4 + C_0 r^2,$$

therefore

$$\frac{1}{4} \|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)}^4 \geq \frac{1}{4} \|Q\|_{L^4(\mathbb{H}^1)}^4 - \left(\frac{3C_0 + 1}{2} + \frac{C_1}{4} \right) r^2.$$

For the reverse inequality, recall the link between Q and the best constant in the embedding $\dot{H}^1(\mathbb{H}^1) \cap V_0^+ \hookrightarrow L^4(\mathbb{H}^1)$: if

$$\inf_{u \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+} \frac{(D_s u, u)_{L^2(\mathbb{H}^1)}^2}{\|u\|_{L^4(\mathbb{H}^1)}^4} = I_+,$$

then

$$(D_s Q, Q)_{L^2(\mathbb{H}^1)} = \|Q\|_{L^4(\mathbb{H}^1)}^4 = I_+ = \pi^2.$$

This leads to

$$\begin{aligned} \|U_{\beta,n}^+(t)\|_{L^4(\mathbb{H}^1)}^4 &\leq \frac{1}{I_+} (D_s U_{\beta,n}^+(t), U_{\beta,n}^+(t))_{L^2(\mathbb{H}^1)}^2 \\ &\leq \frac{1}{I_+} ((D_s Q, Q)_{L^2(\mathbb{H}^1)} + (C_0 + 1)r^2)^2 \\ &\leq \frac{1}{I_+} (I_+ + (C_0 + 1)r^2)^2 \\ &\leq \|Q\|_{L^4(\mathbb{H}^1)}^4 + \frac{1}{I_+} (2I_+(C_0 + 1) + (C_0 + 1)^2 r^2) r^2. \end{aligned}$$

In the end, we have proven that if $r \leq 1$, then there exists $C_2 > 0$ such that

$$\delta(U_{\beta,n}^+(t)) \leq C_2 r^2,$$

and Proposition 3.1.5 immediately implies that for r small enough,

$$d(U_{\beta,n}^+(t), \mathcal{M})^2 \leq C C_2 r^2.$$

Since $\|W_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq r^2$, we get the Proposition. \square

3.4.3 Weak convergence

We now know that if $\beta \geq \beta_*(r)$, then for all $n \geq N$ and $t \in \mathbb{R}$,

$$d(U_{\beta,n}(t), \mathcal{M}) < c_0 r. \quad (3.26)$$

The aim is now to pass to the limit $n \rightarrow +\infty$ in equation (3.21)

$$\begin{cases} i\partial_t U_{\beta,n} - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} U_{\beta,n} = \Pi^{(n)}(|U_{\beta,n}|^2 U_{\beta,n}) \\ U_{\beta,n}(t=0) = U_0^n = \Pi^{(n)}(U_0) \end{cases}$$

and in inequality (3.26) in order to get a weak solution U_β to equation (3.18)

$$\begin{cases} i\partial_t U_\beta - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} U_\beta = |U_\beta|^2 U_\beta \\ U_\beta(t=0) = U_0 \end{cases}$$

which satisfies

$$d(U_\beta(t), \mathcal{M}) \leq c_0 r, \quad t \in \mathbb{R}.$$

The method is identical to parts 3.3.3 and 3.3.4 for the limiting equation: we use a uniform bound on $\|\partial_t U_{\beta,n}(t)\|_{\dot{H}^{-1}(\mathbb{H}^1)}$. Thanks to Ascoli's theorem, the sequence $(U_{\beta,n})_{n \in \mathbb{N}}$ admits a weak limit U_β which is a weak solution to (3.18). Then, we construct bounded modulation parameters $X_{\beta,n}(t)$ in order to control the distance between U_β and \mathcal{M} .

Lemma 3.4.8. *There exists $c_\beta > 0$ such that for all $n \geq N$, $t \in \mathbb{R}$ and $X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$,*

$$\|\partial_t(T_X U_{\beta,n})(t)\|_{\dot{H}^{-1}(\mathbb{H}^1)} \leq c_\beta.$$

Proof. We know from Lemma 3.4.6 that there exists some constant $C_1 > 0$ such that for all $n \geq N$ and $t \in \mathbb{R}$,

$$\|U_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_1.$$

Set $V_{\beta,n} := T_X U_{\beta,n}$. By symmetry invariance, $V_{\beta,n}$ satisfies that for all $n \geq N$ and $t \in \mathbb{R}$,

$$\|V_{\beta,n}(t)\|_{\dot{H}^1(\mathbb{H}^1)} \leq C_1.$$

Moreover, $V_{\beta,n}$ is a solution to some equation

$$\begin{cases} i\partial_t V_{\beta,n} - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} V_{\beta,n} = \tilde{\Pi}^{(n)}(|V_{\beta,n}|^2 V_{\beta,n}) \\ V_{\beta,n}(t=0) = V_0^n = \tilde{\Pi}^{(n)}(U_0). \end{cases}$$

The projector $\tilde{\Pi}^{(n)}$ is defined as follows. Write $X = (s, \theta, \alpha)$. For $u \in \dot{H}^{-1}(\mathbb{H}^1)$, we decompose

$$u = \sum_{k \in \mathbb{N}} \sum_{\pm} \Pi_{k,\pm}(u)$$

with $\Pi_{k,\pm}(u) \in \dot{H}^{-1}(\mathbb{H}^1) \cap V_k^\pm$ for $(k, \pm) \in \mathbb{N} \times \{\pm\}$. Then

$$\widehat{\tilde{\Pi}^{(n)}(u)}(x, y, \sigma) = \sum_{k=0}^n \sum_{\pm} \widehat{\Pi_{k,\pm}(u)}(x, y, \sigma) \mathbb{1}_{\frac{\alpha^2}{n} \leq |\sigma| \leq \alpha^2 n}.$$

Thanks to the fact that $\tilde{\Pi}^{(n)}$ is a projector and the embeddings $L^{\frac{4}{3}}(\mathbb{H}^1) \hookrightarrow \dot{H}^{-1}(\mathbb{H}^1)$ and $\dot{H}^1(\mathbb{H}^1) \hookrightarrow L^4(\mathbb{H}^1)$,

$$\begin{aligned} \|\partial_t V_{\beta,n}(t)\|_{\dot{H}^{-1}(\mathbb{H}^1)} &\leq \frac{1}{1-\beta} \|-(\Delta_{\mathbb{H}^1} + \beta D_s)V_{\beta,n}\|_{\dot{H}^{-1}(\mathbb{H}^1)} + \|\tilde{\Pi}^{(n)}(|V_{\beta,n}|^2 V_{\beta,n})\|_{\dot{H}^{-1}(\mathbb{H}^1)} \\ &\leq \frac{2}{1-\beta} \|-\Delta_{\mathbb{H}^1} V_{\beta,n}\|_{\dot{H}^{-1}(\mathbb{H}^1)} + \| |V_{\beta,n}|^2 V_{\beta,n} \|_{\dot{H}^{-1}(\mathbb{H}^1)} \\ &\leq \frac{2}{1-\beta} \|V_{\beta,n}\|_{\dot{H}^1(\mathbb{H}^1)} + K_1 \| |V_{\beta,n}|^2 V_{\beta,n} \|_{L^{\frac{4}{3}}(\mathbb{H}^1)} \\ &\leq \frac{2}{1-\beta} \|V_{\beta,n}\|_{\dot{H}^1(\mathbb{H}^1)} + K_2 \|V_{\beta,n}\|_{\dot{H}^1(\mathbb{H}^1)}^3 \\ &\leq \frac{2C_1}{1-\beta} + K_2 C_1^3. \end{aligned} \quad \square$$

We deduce the weak convergence of $(U_{\beta,n})_{n \in \mathbb{N}}$, for which the proof is identical to that in part 3.3.3 and is based on Ascoli's theorem.

Lemma 3.4.9. *Up to a subsequence, $(U_{\beta,n})_n$ converges weakly to a solution $U_\beta \in \mathcal{C}(\mathbb{R}, \dot{H}^1(\mathbb{H}^1))$ (with the weak topology) to (3.18)*

$$\begin{cases} i\partial_t U_\beta - \frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} U_\beta = |U_\beta|^2 U_\beta \\ U_\beta(t=0) = U_0. \end{cases}$$

Moreover, one can see that for all $X \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ and $t_0, t \in \mathbb{R}$, setting $V_{\beta,n} := T_{X^{-1}} U_{\beta,n}$,

$$\left| \frac{d}{dt} \|U_{\beta,n}(t) - T_X Q\|_{\dot{H}^1(\mathbb{H}^1)}^2 \right| = \left| \frac{d}{dt} \|V_{\beta,n}(t) - Q\|_{\dot{H}^1(\mathbb{H}^1)}^2 \right|.$$

Thanks to the conservation of the momentum for equation (3.18), we have

$$\begin{aligned} \left| \frac{d}{dt} \|U_{\beta,n}(t) - T_X Q\|_{\dot{H}^1(\mathbb{H}^1)}^2 \right| &= |2(\partial_t V_{\beta,n}(t), D_s Q)_{L^2(\mathbb{H}^1)}| \\ &\leq 2\|\partial_t V_{\beta,n}(t)\|_{\dot{H}^{-1}(\mathbb{H}^1)} \|D_s Q\|_{\dot{H}^1(\mathbb{H}^1)}, \end{aligned}$$

which implies that there exists $c_\beta > 0$ such that for all $t_0, t \in \mathbb{R}$,

$$\|U_{\beta,n}(t) - T_X Q\|_{\dot{H}^1(\mathbb{H}^1)}^2 \leq \|U_{\beta,n}(t_0) - T_X Q\|_{\dot{H}^1(\mathbb{H}^1)}^2 + c_\beta |t - t_0|.$$

Set $\varepsilon \in (0, 1)$ and define $t_1 := \frac{(1+\varepsilon)^2 - 1}{c_\beta} (c_0 r)^2$. Note that t_1 may depend on β , but this is not important because in this part the varying parameter is n whereas β is fixed. The construction of $X_{\beta,n}$ as a piecewise constant functional is now the same as for the limiting system. For $k \in \mathbb{Z}$, $X_{\beta,n}$ is constant on $[kt_1, (k+1)t_1[$, equal to some $X_{\beta,n}^k \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$ to be chosen. We first set $X_{\beta,n}^{-1} = X_{\beta,n}^0 = (0, 0, 1)$. Then, at time $t_k = kt_1$, $k \geq 1$, we use the fact that $d(U_{\beta,n}(t_k), \mathcal{M}) < c_0 r$ and choose $X_{\beta,n}^k$ such that $\|U_{\beta,n}(t_k) - T_{X_{\beta,n}^k} Q\|_{\dot{H}^1(\mathbb{H}^1)} < c_0 r$. By definition of t_1 , for all $k \geq 0$ and $t \in [t_k, t_k + t_1]$, $\|U_{\beta,n}(t) - T_{X_{\beta,n}^k} Q\|_{\dot{H}^1(\mathbb{H}^1)} \leq (1 + \varepsilon)c_0 r$. We do a similar construction for negative times. The map $X_{\beta,n}$ satisfies

$$\|U_{\beta,n}(t) - T_{X_{\beta,n}(t)} Q\|_{\dot{H}^1(\mathbb{H}^1)} \leq (1 + \varepsilon)c_0 r, \quad t \in \mathbb{R}. \quad (3.27)$$

It remains to show that $X_{\beta,n}$ is bounded independently of n on bounded intervals. In order to do so, it is enough to control the gap between $X_{\beta,n}^{k-1}$ and $X_{\beta,n}^k$. By construction, at time t_k ,

$$\|U_{\beta,n}(t_k) - T_{X_{\beta,n}^{k-1}} Q\|_{\dot{H}^1(\mathbb{H}^1)} \leq (1 + \varepsilon)c_0 r$$

and

$$\|U_{\beta,n}(t_k) - T_{X_{\beta,n}^k} Q\|_{\dot{H}^1(\mathbb{H}^1)} < c_0 r,$$

therefore

$$\|T_{X_{\beta,n}^{k-1}} Q - T_{X_{\beta,n}^k} Q\|_{\dot{H}^1(\mathbb{H}^1)} \leq (2 + \varepsilon)c_0 r.$$

Using Lemma 3.3.6, we conclude that if $r \leq r_0$ is small enough (for example if $3c_0 r_0 \leq \sqrt{\pi} r_1$), then

$$|X_n^{k-1} (X_n^k)^{-1}| \leq c_1.$$

Now, for fixed $t \in \mathbb{R}$, the sequence $(X_{\beta,n}(t))_{n \in \mathbb{N}}$ is bounded, therefore up to extraction, this sequence converges to some $X_\beta(t) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+^*$, and passing to the weak limit in (3.27),

$$\|U_\beta(t) - T_{X_\beta(t)} Q\|_{\dot{H}^1(\mathbb{H}^1)} \leq (1 + \varepsilon)c_0 r.$$

3.A Appendix: Rate of convergence of Q_β to Q

In order to conclude the proof of Theorem 3.4.1, it only remains to make precise the convergence rate of $(Q_\beta)_\beta$ to Q as β tends to 1. Decompose

$$Q_\beta = Q_\beta^+ + R_\beta,$$

where $Q_\beta^+ \in \dot{H}^1(\mathbb{H}^1) \cap V_0^+$ and $R_\beta \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$. We improve the bound from [Gas20b]

$$\delta(Q_\beta^+) + \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{\frac{1}{2}}).$$

Proposition 3.A.1. *Let $\varepsilon > 0$. Then as β tends to 1,*

$$\delta(Q_\beta^+) + \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{2-\varepsilon}),$$

which implies that

$$\|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{1-\frac{\varepsilon}{2}}).$$

Proof. Assume that we have proven that

$$\delta(Q_\beta^+) + \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^\gamma)$$

for some exponent $\gamma > 0$ (for example we already know that it is true for $\gamma = \frac{1}{2}$, see Lemma 3.9 in [Gas20b]), and therefore

$$\|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{\frac{\gamma}{2}}).$$

We increase the exponent γ by showing that actually

$$\delta(Q_\beta^+) + \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} = \mathcal{O}((1 - \beta)^{1+\frac{\gamma}{2}}).$$

Then, we conclude by iteration, since the sequence $\gamma_{n+1} = 1 + \frac{\gamma_n}{2}$ with $\gamma_0 = \frac{1}{2}$ is convergent to 2.

Since $R_\beta \in \bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$, the norms $\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)}$ and $\|-(\Delta_{\mathbb{H}^1} + \beta D_s)R_\beta\|_{\dot{H}^{-1}(\mathbb{H}^1)}$ are equivalent

$$\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \leq 2\|-(\Delta_{\mathbb{H}^1} + \beta D_s)R_\beta\|_{\dot{H}^{-1}(\mathbb{H}^1)}.$$

Projecting the equation satisfied by Q_β

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1 - \beta} Q_\beta = |Q_\beta|^2 Q_\beta$$

on $\bigoplus_{(k,\pm) \neq (0,+)} \dot{H}^1(\mathbb{H}^1) \cap V_k^\pm$, we deduce that

$$\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} \leq 2(1 - \beta)\|(\text{Id} - \Pi_{0,+})(|Q_\beta|^2 Q_\beta)\|_{\dot{H}^{-1}(\mathbb{H}^1)}.$$

Since $|Q|^2 Q = D_s Q \in \dot{H}^{-1}(\mathbb{H}^1) \cap V_0^+$, one can make this term appear inside the right-hand side term of the inequality:

$$\begin{aligned} \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} &\leq 2(1 - \beta)\|(\text{Id} - \Pi_{0,+})(|Q_\beta|^2 Q_\beta - |Q|^2 Q)\|_{\dot{H}^{-1}(\mathbb{H}^1)} \\ &\leq 2K(1 - \beta)\|(\text{Id} - \Pi_{0,+})(|Q_\beta|^2 Q_\beta - |Q|^2 Q)\|_{L^{\frac{4}{3}}(\mathbb{H}^1)}. \end{aligned}$$

Now, since $(\text{Id} - \Pi_{0,+})$ defines a bounded operator on $L^{\frac{4}{3}}(\mathbb{H}^1)$, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} &\leq C_1(1 - \beta)\| |Q_\beta|^2 Q_\beta - |Q|^2 Q \|_{L^{\frac{4}{3}}(\mathbb{H}^1)} \\ &\leq C_2(1 - \beta)\|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)}(\|Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)} + \|Q\|_{\dot{H}^1(\mathbb{H}^1)})^2. \end{aligned}$$

But since $(Q_\beta)_\beta$ is bounded in $\dot{H}^1(\mathbb{H}^1)$, we get that there exists $C_3 > 0$ such that

$$\begin{aligned} \|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} &\leq C_3(1 - \beta)\|Q_\beta - Q\|_{\dot{H}^1(\mathbb{H}^1)} \\ &= \mathcal{O}((1 - \beta)^{1+\frac{\gamma}{2}}). \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \|Q_\beta^+\|_{\dot{H}^1(\mathbb{H}^1)}^2 - \|Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}^2 \right| &\leq 2\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} (\|R_\beta\|_{\dot{H}^1(\mathbb{H}^1)} + \|Q_\beta\|_{\dot{H}^1(\mathbb{H}^1)}) \\ &= \mathcal{O}((1-\beta)^{1+\frac{\gamma}{2}}) \end{aligned}$$

and

$$\begin{aligned} \left| \|Q_\beta^+\|_{L^4(\mathbb{H}^1)}^4 - \|Q_\beta\|_{L^4(\mathbb{H}^1)}^4 \right| &\lesssim \|R_\beta\|_{L^4(\mathbb{H}^1)} (\|R_\beta\|_{L^4(\mathbb{H}^1)} + \|Q_\beta\|_{L^4(\mathbb{H}^1)})^3 \\ &= \mathcal{O}((1-\beta)^{1+\frac{\gamma}{2}}), \end{aligned}$$

which means that

$$\delta(Q_\beta^+) = \mathcal{O}((1-\beta)^{1+\frac{\gamma}{2}}).$$

It now remains to consider the sequence $\gamma_{n+1} = 1 + \frac{\gamma_n}{2}$, $\gamma_0 = \frac{1}{2}$, which is convergent to 2. \square

Chapitre 4

Probabilistic local well-posedness for NLS with the Grushin operator

Ce chapitre reprend les résultats de [GL21].

Résumé. On étudie le caractère localement bien posé de l'équation de Schrödinger associée à l'opérateur de Grushin avec donnée initiale aléatoire. À notre connaissance, on ne dispose d'aucun résultat sur le caractère bien posé pour cette équation dans les espaces de Sobolev H^k lorsque $k \leq \frac{3}{2}$. Dans cet article, on montre qu'il existe une grande famille de données initiales dans H^k , $k \in]1, \frac{3}{2}]$, pour lesquelles une randomisation appropriée conduit au caractère presque-sûrement localement bien posé du problème de Cauchy. La preuve repose sur des estimées bilinéaires et trinéaires.

Abstract. We study the local well-posedness of the nonlinear Schrödinger equation associated to the Grushin operator with random initial data. To the best of our knowledge, no well-posedness result is known in the Sobolev spaces H^k when $k \leq \frac{3}{2}$. In this article, we prove that there exists a large family of initial data such that, with respect to a suitable randomization in H^k , $k \in (1, \frac{3}{2}]$, almost-sure local well-posedness holds. The proof relies on bilinear and trilinear estimates.

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4.1 Introduction

4.1.1 The Schrödinger equation on the Heisenberg group and the Grushin equation

We consider the Grushin-Schrödinger equation

$$i\partial_t u - \Delta_G u = |u|^2 u, \quad (\text{NLS-G})$$

where $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ and $\Delta_G = \partial_{xx} + x^2 \partial_{yy}$. The natural associated Sobolev spaces in this case are the Grushin Sobolev spaces H_G^k on \mathbb{R}^2 , defined by replacing powers of the usual operator $\sqrt{-\Delta}$ by powers of $\sqrt{-\Delta_G}$.

This equation is a simplification of the semilinear Schrödinger equation on the Heisenberg group in the radial case

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad (\text{NLS-}\mathbb{H}^1)$$

where $(t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1$. In the radial case, the solution u only depends on $t, |x+iy|$ and s and the sub-Laplacian is written $\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_{xx} + \partial_{yy}) + (x^2 + y^2)\partial_{ss}$. Our simplification of this equation consists in removing one of the two variables x, y since they play the same role, leading to (NLS-G).

When $k > k_C$ where $k_C = 2$ (resp. $k_C = \frac{3}{2}$ for (NLS-G)), one can use the algebra property of the spaces $H^k(\mathbb{H}^1)$ (resp. H_G^k) and solve the Cauchy problem associated to (NLS- \mathbb{H}^1) (resp. (NLS-G)) locally in time, see Appendix 4.A.2 for details. However, the conservation of energy only controls the H^k norm when $k = 1$, and since no conservation law is known for $k > 1$, we have no information about global existence of maximal solutions in the range of Sobolev exponents $k > k_C$.

For Sobolev exponents below the critical exponent k_C , existence and uniqueness of general weak solutions is an open problem. To go further, the Schrödinger equation on the Heisenberg group displays a total lack of dispersion [BGX00], implying that the flow map for (NLS- \mathbb{H}^1) (resp. (NLS-G)) cannot be smooth in the Sobolev spaces H^k when $k < k_C$. We refer to the introduction of [GG10] and Remark 2.12 in [BGT04] for details.

4.1.2 Main results

In this subsection we only introduce the needed notations to state our main result, we refer to Section 4.2 for precise definitions.

Fix u_0 in some Sobolev space H_G^k for $k > 0$. Then u decomposes as a sum

$$u_0 = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} u_{I,m}, \quad (4.1)$$

where the $u_{I,m}$ will be defined by (4.8).

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider a sequence $(X_{I,m})_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}}$ of independent and identically distributed Gaussian random variables and define the measure μ_{u_0} as the image measure of \mathbb{P} under the *randomization map*

$$\omega \in \Omega \mapsto u_0^\omega := \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} X_{I,m}(\omega) u_{I,m}. \quad (4.2)$$

For $k \geq 0$ and $\rho \geq 0$, we introduce the subspace \mathcal{X}_ρ^k of H_G^k in which we will prove almost-sure local well-posedness. Denoting $\langle x \rangle = \sqrt{1 + x^2}$, the space \mathcal{X}_ρ^k corresponds to the norm

$$\|u_0\|_{\mathcal{X}_\rho^k}^2 := \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m + 1)I)^k \langle I \rangle^\rho \|u_{I,m}\|_{L_G^2}^2. \quad (4.3)$$

The powers $(1 + (2m + 1)I)^k$ refer to Sobolev regularity: for instance, when $\rho = 0$, then $\|u_0\|_{\mathcal{X}_0^k} \sim \|u_0\|_{H_G^k}$. However, the powers $\langle I \rangle^\rho$ only correspond to partial regularity with respect to the last variable, see the precise definition of decomposition (4.1) in Section 4.2 and Remark 4.2.4 for details.

Our main result is the following.

Theorem A (Local Cauchy Theory for (NLS-G)). *Let $k \in (1, \frac{3}{2}]$ and $u_0 \in \mathcal{X}_1^k \subset H_G^k$.*

(i) *For any $\ell \in (\frac{3}{2}, k + \frac{1}{2})$, for almost-every $\omega \in \Omega$, there exists $T > 0$ and a unique local solution with initial data u_0^ω to (NLS-G) in the space*

$$e^{it\Delta_G} u_0^\omega + \mathcal{C}^0([0, T], H_G^\ell) \subset \mathcal{C}^0([0, T], H_G^k).$$

More precisely, there exists $c > 0$ such that for all $R \geq 1$, outside a set of probability at most e^{-cR^2} , one can choose $T \geq (R\|u_0\|_{\mathcal{X}_1^k})^{-2}$.

(ii) *(Non-smoothing under randomization) If $u_0 \in H_G^k \setminus (\bigcup_{\varepsilon > 0} H_G^{k+\varepsilon})$, then*

$$\text{supp}(\mu_{u_0}) \subset H_G^k \setminus \left(\bigcup_{\varepsilon > 0} H_G^{k+\varepsilon} \right).$$

(iii) (Density of measures with rough potentials) Let $\varepsilon > 0$, then there exists $v_0 \in \mathcal{X}_1^k \setminus (\bigcup_{\varepsilon' > 0} H_G^{k+\varepsilon'})$ such that

$$\text{supp}(\mu_{v_0}) \cap B_{H_G^k}(u_0, \varepsilon) \neq \emptyset.$$

Remark 4.1.1 (Consequences of (i)). For every $k > 1 + 2\varepsilon > 1$ and $u_0 \in \mathcal{X}_{1+2\varepsilon+1-k}^k$, the continuous embedding $\mathcal{X}_{1+2\varepsilon+1-k}^k \hookrightarrow \mathcal{X}_1^{1+2\varepsilon}$ implies that for almost-every $\omega \in \Omega$, the initial data u_0^ω gives rise to a unique local solution

$$u \in e^{it\Delta_G} u_0^\omega + \mathcal{C}^0([0, T], H_G^{\frac{3}{2}+\varepsilon}),$$

where we check that $\ell = \frac{3}{2} + \varepsilon \in (\frac{3}{2}, 1 + 2\varepsilon + \frac{1}{2})$.

Therefore, in the case $k = \frac{3}{2}$, we observe that the limiting almost-sure well-posedness space is $\bigcap_{\varepsilon > 0} \mathcal{X}_{\frac{1}{2}+\varepsilon}^{\frac{3}{2}}$. We recall that for $k = \frac{3}{2} + \varepsilon$, local well-posedness is known to hold in $H_G^{\frac{3}{2}+\varepsilon} = \mathcal{X}_0^{\frac{3}{2}+\varepsilon}$. It is interesting to note that our approach loses an exponent $\rho = \frac{1}{2}$, since we do not recover the same limit space in the limit $k \rightarrow \frac{3}{2}$.

Remark 4.1.2 (Decomposition of the solution). In (i), we construct local solutions to (NLS-G) in the space $\mathcal{C}([0, T], H_G^k)$ for small values of k . However, uniqueness holds only on a smaller subset, as a consequence of an a priori decomposition of the solution as sum of the solution to the linear equation (LS-G) with initial data u_0^ω and a smoother part. This decomposition can be interpreted as a more nonlinear decomposition of the solution than seeking $v \in H_G^k$, as we seek for u in an affine space $e^{it\Delta_G} u_0^\omega + H_G^\ell$ instead of a vector space. Such a decomposition is the simplest nonlinear decomposition, akin to [BT08a], and is a key feature in most random data well-posedness works. More recently, even more nonlinear decompositions for the solutions to some dispersive equations have been exhibited in [OTW20] and in [DNY19] (see also [GIP15; Hai13; Hai14] in the context of stochastic equations).

Remark 4.1.3 (Regularity and density of the measures). Parts (ii) and (iii) give regularity properties of the measures μ_{u_0} . In fact, (ii) ensures that the measure μ_{u_0} does not charge solutions which are more regular than u_0 . Actually, the measure μ_{u_0} charges solutions that have regularity $W_G^{k+1/4,4}$ on L^p based Sobolev spaces, but no better regularity bound is expected to hold (see Proposition 4.3.1 and Remark 4.3.5), so this estimate alone is not enough to establish Theorem A.

Statement (iii) goes even further since we prove that

$$\bigcup \left\{ \text{supp}(\mu_{v_0}) \mid v_0 \in \mathcal{X}_1^k \setminus \left(\bigcup_{\varepsilon > 0} H_G^{k+\varepsilon} \right) \right\}$$

is dense in H_G^k . This result is related to the support density in the euclidean case. Indeed, for the nonlinear Schrödinger equation on the torus, probabilistic local well-posedness holds with respect to measures which are dense in Sobolev spaces, see for instance Appendix B of [BT08a] and [BT14].

Remark 4.1.4 (Admissible initial data). When u_0 only has a finite number of modes m , the assumption $u_0 \in \mathcal{X}_1^k$ is equivalent to the condition $u_0 \in H_G^{k+1}$, but since $k+1 > k_C = \frac{3}{2}$, the result is void. For this reason, our result does not extend to the nonlinear half-wave equation $\partial_t u \pm \sqrt{-\Delta} u = |u|^2 u$, which also admits a similar decomposition to (4.1) but only with a finite number of modes m . However, we will see in Remark 4.2.4 that the condition $u_0 \in \mathcal{X}_1^k$ still allows a general set of low regularity initial data in our context.

Remark 4.1.5 (Defocusing case). *One can replace equation (NLS-G) by its defocusing variant and get the exact same local well-posedness theory. Indeed, we only address local well-posedness, which mainly depends on the order of magnitude of the nonlinearity and not its sign.*

Remark 4.1.6 (Randomization). *The measures μ_{u_0} are defined in (4.2) with Gaussian random variables. However, most of the results in this article are stated for more general subgaussian random variables (see Definition 4.2.11), except when using the Wiener chaos estimates from Corollary 4.2.15, which is stated only for Gaussian random variables.*

Note that the randomization on a unit scale in the variable m is quite classical, as this is the variable along which we establish our bilinear estimates. However, the variable I plays a different role which allows us to only take a randomization on a dyadic scale. One could compare this choice with the construction of adapted dilated cubes in [BOP15b].

4.1.3 Further work

We expect that for the Schrödinger equation (NLS- \mathbb{H}^1) on the Heisenberg group, the local well-posedness theory for the randomized Cauchy problem holds with a same gain of almost $\frac{1}{2}$ derivative compared to the critical exponent $k_C = 2$. More precisely, for $u_0 \in H^k(\mathbb{H}^1)$, $k \in (\frac{3}{2}, 2]$, there holds a decomposition similar to (4.1). Assuming that u_0 belongs to a space similar to (4.3), we conjecture that there exists a unique local solution with random initial data u_0^ω in the space $e^{it\Delta_{\mathbb{H}^1}} u_0^\omega + \mathcal{C}^0([0, T], H^\ell(\mathbb{H}^1)) \subset \mathcal{C}^0([0, T], H^k(\mathbb{H}^1))$ with $\ell \in (2, k + \frac{1}{2})$. This will be the object of a subsequent work. In the non radial case, we would have to tackle the additional terms in the expression of the sub-Laplacian on the Heisenberg group $\mathcal{L} = \Delta_{\mathbb{H}^1} + (y\partial_x - x\partial_y)\partial_s$.

4.1.4 Deterministic and probabilistic Cauchy theory for (NLS- \mathbb{H}^1)

As mentioned at the beginning of this introduction, the nonlinear Schrödinger equation on the Heisenberg group lacks dispersion, therefore the dispersive paradigm cannot be applied for lowering the critical well-posedness exponent below $k_C = 2$ (resp. $k_C = \frac{3}{2}$ for (NLS-G)) given by the Sobolev embedding. More precisely, the lack of dispersion precludes the usual way in which Strichartz estimates are proven, that is a combination of a dispersive estimate and a duality TT^* argument. The result in [BGX00] goes even further, as the non smoothness of the flow map for (NLS- \mathbb{H}^1) in H^k for $k < k_C$ makes it impossible to implement a fixed point argument.

The lack of dispersion and the lack of Strichartz estimates for the Schrödinger equation on the Heisenberg group have been recently investigated in [BG20a] and [BBG21]. In these works, the authors prove that there exist anisotropic Strichartz estimates [BBG21], local in space versions of the dispersive estimates (Theorem 1 in [BG20a]) and local version of Strichartz estimates (Theorem 3 in [BG20a]). These results follow the general strategy of Fourier restriction methods for proving Strichartz estimates, dating back to Strichartz [Str77], and use the Fourier analysis on the Heisenberg group [BCD18; BCD19]. We also refer to [Mül90] for restriction theorems on the Heisenberg group.

Probabilistic methods have proven to be very useful to break the scaling barrier in the context of dispersive equations. Such a study has been pioneered in [Bou94]: the purpose is to construct global solutions for nonlinear Schrödinger equations posed on the torus, using invariant measures and a probabilistic local Cauchy theory. In [BT08a; BT08b],

the authors extend these results to other dispersive equations, opening the way to a very active area of research and leading to an immense body of results.

Invariant measure methods mostly reduce their scope to compact spaces, the setting of the torus being used on many works. For non-compact spaces, the probabilistic method of [BT08a] remains largely adaptable through the use of Gaussian random initial data. We refer for example to [BOP15b; BOP15a] where probabilistic local well-posedness is obtained for the nonlinear Schrödinger equations on \mathbb{R}^d , and to [OP16; Poc17] for similar results with the wave equation.

Several works go beyond the euclidean Laplacian. For instance, in [BTT13] the authors replace the standard Laplacian $-\Delta$ with a harmonic oscillator $-\Delta + x^2$ and study the local Cauchy theory for the associated nonlinear Schrödinger equation. Our work is partly inspired from this work, and also [Den12; BT20; Lat20]. Indeed, in our case, rescaled harmonic oscillators parameterized by one of the variables appear when one considers a partial Fourier transform of the equation.

We point out that although no progress had been obtained in the direction of local well-posedness up to now, traveling waves for the Schrödinger equation on the Heisenberg group and their stability have been studied in [Gas21a; Gas20b].

4.1.5 Outline of the proof and main arguments

In this section, we briefly review the main ideas leading to the proof of Theorem A.

General strategy for almost-sure local well-posedness

We follow the probabilistic approach to the local well-posedness problem from [BT08a] in order to study (NLS-G). We fix $u_0 \in H_G^k$, where $0 < k < k_C = \frac{3}{2}$, and consider the randomization u_0^ω defined in (4.2). We seek for solutions to (NLS-G) under the form

$$u(t) = e^{it\Delta_G} u_0^\omega + v(t)$$

where $v(t)$ belongs to some space $H_G^{\frac{3}{2}+\varepsilon}$, $\varepsilon > 0$, on which a deterministic local well-posedness theory is known to hold. Plugging this ansatz in the Duhamel representation of (NLS-G) leads to

$$u(t) = e^{it\Delta_G} u_0^\omega - i \int_0^t e^{i(t-t')\Delta_G} \left(|e^{it'\Delta_G} u_0^\omega + v(t')|^2 (e^{it'\Delta_G} u_0^\omega + v(t')) \right) dt',$$

so that with the notation $z^\omega(t) = e^{it\Delta_G} u_0^\omega$, we expect that $v(t) = \Phi v(t) \in H_G^{\frac{3}{2}+\varepsilon}$, where

$$\Phi v(t) = -i \int_0^t e^{i(t-t')\Delta_G} (|z^\omega(t') + v(t')|^2 (z^\omega(t') + v(t'))) dt'.$$

In view of the lack of Strichartz estimates, the best known bounds on Φv are the trivial estimates (we take $t \leq T$ and forget time estimates, as we only give heuristic arguments)

$$\|\Phi v\|_{H_G^{\frac{3}{2}+\varepsilon}} \lesssim \|(v + z^\omega)^3\|_{H_G^{\frac{3}{2}+\varepsilon}} \lesssim \|v^3\|_{H_G^{\frac{3}{2}+\varepsilon}} + \|(z^\omega)^3\|_{H_G^{\frac{3}{2}+\varepsilon}} + \|z^\omega v^2\|_{H_G^{\frac{3}{2}+\varepsilon}} + \|(z^\omega)^2 v\|_{H_G^{\frac{3}{2}+\varepsilon}}.$$

The term v^3 is handled using the algebra property of the space $H_G^{\frac{3}{2}+\varepsilon}$, since v has high regularity. The terms involving z^ω are more difficult to handle because $z^\omega \in H_G^k \setminus H_G^{\frac{3}{2}+\varepsilon}$

only has H^k regularity since the randomization does not gain derivatives, as stated in Theorem A (ii) and proven in Section 4.3. A first approach would be to estimate

$$\|(z^\omega)^3\|_{H_G^{\frac{3}{2}+\varepsilon}} \sim \| \langle -\Delta_G \rangle^{\frac{3}{4}+\frac{\varepsilon}{2}} z^\omega (z^\omega)^2 \|_{L_G^2} \lesssim \|z^\omega\|_{W_G^{\frac{3}{2},\infty}} \|z^\omega\|_{L_G^4}^2,$$

thus it is important to study the effect of the randomization on u_0 in terms of regularity in L^p based Sobolev spaces. As proven in Proposition 4.3.1, linear estimates in $W_G^{k,p}$ spaces gain up to $\frac{1}{4}$ derivatives. However, the linear estimates alone are not sufficient to gain the $\frac{1}{2}$ derivatives in regularity and therefore deal with low values of k in Theorem A.

In order to improve our estimates, we establish bilinear estimates: we prove that given the random solutions z^ω with initial data in H_G^k to the linear Schrödinger equation

$$i\partial_t z - \Delta_G z = 0, \quad (\text{LS-G})$$

then almost-surely we have $(z^\omega)^2 \in H_G^{k+\frac{1}{2}}$. In this article, we prove the following bilinear and trilinear estimates in the spaces \mathcal{X}_ρ^k , which could be of independent interest.

Theorem B (Bilinear and trilinear estimates for random solutions). *Let $k \in (1, \frac{3}{2}]$ and $u_0 \in \mathcal{X}_1^k \subset H_G^k$ (see (4.3)). Let u_0^ω as in (4.2), and let us denote by $z^\omega = e^{it\Delta_G} u_0^\omega \in C^0(\mathbb{R}, H_G^k)$ the solution to (LS-G) associated to u_0^ω . Fix $q \in [2, \infty)$. For $T > 0$, denote $L_T^q := L^q([0, T])$.*

(i) *For all $R \geq 1$ and $T > 0$, there exists $c > 0$ such that outside a set of probability at most e^{-cR^2} , one has*

$$\|(z^\omega)^2\|_{L_T^q H_G^{k+\frac{1}{2}}} + \| |z^\omega|^2 \|_{L_T^q H_G^{k+\frac{1}{2}}} \leq R^2 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^2, \quad (4.4)$$

$$\| |z^\omega|^2 z^\omega \|_{L_T^q H_G^{k+\frac{1}{2}}} \leq R^3 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^3. \quad (4.5)$$

(ii) *We further require that $u_0 \in \mathcal{X}_{1+\varepsilon_0}^k$ for some $\varepsilon_0 > 0$. Let $\ell < k + \frac{1}{2}$. Then there exists $c = c(\ell) > 0$ such that for all $R \geq 1$ and $T > 0$, there exists a set $E_{R,T}$ of probability at least $1 - e^{-cR^2}$ such that the following holds. Fix $\omega \in E_{R,T}$ and $v, w \in L_T^\infty H_G^\ell$, then*

$$\|z^\omega v w\|_{L_T^q H_G^\ell} \leq R T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon_0}^k} \|v\|_{L_T^\infty H_G^\ell} \|w\|_{L_T^\infty H_G^\ell}. \quad (4.6)$$

Note that v and w may depend on ω .

Remark 4.1.7 (Time dependence). *The time variable does not play an important role. Indeed, in the course of the proof, we establish deterministic pointwise estimates in the time variable, and the L_T^q norm instead of L_T^∞ only appears in order to apply the Khinchine inequality and Wiener chaos estimates from part 4.2.4. In comparison, bilinear smoothing estimates for the nonlinear Schrödinger equation on \mathbb{R}^2 crucially exploit the time variable, as the smoothing occurs in time averages.*

The heuristic explained above for proving Theorem A by using Theorem B is implemented rigorously in Section 4.8.

Multilinear random estimates

The bulk of this paper aims at establishing Theorem B, thus we briefly outline the main aspects of the proof.

First, because of the random nature of the z^ω , we use random decoupling in order to reduce the estimates to “building block estimates”, that is estimating products $\|u_a v_b w_c\|_{H_G^{\frac{3}{2}+\varepsilon}}$ for u_a , v_b and w_c obtained by restricting the Sobolev frequencies of u , v and w around the values a , b and c . This reduction is a consequence of Corollary 4.2.15.

In the euclidean setting, the Bernstein estimates and the Littlewood-Paley decomposition would justify the heuristics $\nabla(u_a v_b w_c) \simeq \nabla(u_a) v_b w_c + u_a \nabla(v_b) w_c + u_a v_b \nabla(w_c)$ and thus reduce the analysis of $\|u_a v_b w_c\|_{H_G^{\frac{3}{2}+\varepsilon}}$ to that of $\|u_a v_b w_c\|_{L_G^2}$. In our case, this result still holds, but rigorous justification is more intricate and is the content of Section 4.4.

The purpose of Section 4.5 is to prove “building-block” estimates of the form

$$\|u_a v_b\|_{L_G^2} \leq \frac{C}{\max\{a, b\}^{\frac{1}{2}}} \|u_a\|_{L_G^2} \|v_b\|_{L_G^2}.$$

To give the main idea of proof, it is instructing to see that in partial Fourier transform along the last variable, we can think of u_a and v_b as

$$\mathcal{F}_{y \rightarrow \eta}(u_a)(x, \eta) = f(\eta) h_m(|\eta|^{\frac{1}{2}} x) \text{ and } \mathcal{F}_{y \rightarrow \eta}(v_b)(x, \eta) = g(\eta) h_n(|\eta|^{\frac{1}{2}} x),$$

where h_m, h_n are Hermite functions. Thus we can see that estimating $u_a v_b$ involves estimating convolution products of rescaled Hermite functions. The key fact is now that Hermite functions, due to their localization and normalization, enjoy bilinear estimates that are better than trivial Hölder bounds, see [BTT13], relying on pointwise estimates from [KT05], see also [KTZ07].

Bilinear random-deterministic estimates

It turns out to be more difficult to prove a multilinear estimate on probabilistic-deterministic products such as $u^\omega v w$, where v and w are deterministic. This is the content of (4.6). In this case, one should ensure that the required set of ω constructed in Theorem B (ii) does not depend on v and w . This precludes a direct use of decoupling offered by Corollary 4.2.15 as exploited in the proof of Theorem B (i). To understand the difference between the treatment of $|z^\omega|^2 z^\omega$ and $z^\omega v w$, remark that for example in (4.27) the set of ω which is removed depends on the $z_{I,m}$, that is on z , which is fixed in Theorem B. In the case of $z^\omega v^2$ this would remove a set of ω depending on v and w .

In order to circumvent such a difficulty, the idea is to apply probabilistic decoupling only on terms involving the random part z^ω . The implementation of this strategy is carried out in Section 4.7 and relies on a preparatory step introduced in Section 4.7.1 aiming at splitting the analysis of $z^\omega v w$ into a deterministic part $\mathbf{K} := \mathbf{K}(v, w)$ and a probabilistic part $\mathbf{J}^\omega := \mathbf{J}(z^\omega)$, which are treated in Section 4.7.2 and Section 4.7.3.

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4.2 Notation and preliminary estimates

The purpose of this section is twofold. First we introduce decomposition (4.1), which is due to the structure of the Grushin operator. Then, we recall some useful estimates, such as Sobolev embeddings, product laws, eigenfunction estimates and probabilistic decoupling estimates.

We will use the notation $f \lesssim g$ to denote that there exists $C > 0$ such that $f \leq Cg$.

4.2.1 Decomposition along Hermite functions for the Grushin operator

In this subsection we give an explicit description of the Grushin operator $\Delta_G = \partial_x^2 + x^2 \partial_y^2$, acting on $L^2(\mathbb{R}^2)$.

Let us consider the orthonormal basis of $L^2(\mathbb{R})$ given by the Hermite functions $(h_m)_{m \geq 0}$. By definition, the Hermite functions are eigenfunctions of the harmonic oscillator: for all $m \geq 0$, we have

$$(-\partial_x^2 + x^2)h_m = (2m + 1)h_m.$$

Taking the Fourier transform in y , with Fourier variable η , we observe that for all $\eta \in \mathbb{R}$, we have

$$(-\partial_x^2 + x^2 \eta^2)h_m(|\eta|^{\frac{1}{2}}x) = (2m + 1)|\eta|h_m(|\eta|^{\frac{1}{2}}x).$$

Therefore, one can decompose the Fourier transform $\mathcal{F}_{y \rightarrow \eta}(u)(\cdot, \eta)$ of $u \in H_G^k$ along the basis $(h_m(|\eta|^{\frac{1}{2}}\cdot))_{m \geq 0}$, so u becomes a sum

$$\mathcal{F}_{y \rightarrow \eta}(u)(x, \eta) = \sum_{m \geq 0} f_m(\eta)h_m(|\eta|^{\frac{1}{2}}x).$$

Moreover, this decomposition is invariant by the action of the Grushin operator $-\Delta_G$, so that we can explicitly write the H_G^k norm as

$$\|u\|_{H_G^k}^2 := \|(\text{Id} - \Delta_G)^{\frac{k}{2}}u\|_{L_G^2}^2 = \sum_{m \geq 0} \int_{\mathbb{R}} (1 + (2m + 1)|\eta|)^k |f_m(\eta)|^2 |\eta|^{-\frac{1}{2}} d\eta. \quad (4.7)$$

Remark 4.2.1. *The quantity $(2m + 1)|\eta|$ plays the role of “taking two” derivatives, that is a similar role as the Fourier variable $|\xi|^2$ in the context of the euclidean Sobolev spaces. Keep in mind that however this quantity mixes the Hermite modes m with the partial Fourier variable η .*

Remark 4.2.2. *In (4.7), the extra factor $|\eta|^{-\frac{1}{2}}$ should be understood as a normalization factor. Indeed, the L_x^2 norm of the function $x \mapsto h_m(|\eta|^{\frac{1}{2}}x)$ is $|\eta|^{-\frac{1}{4}}$.*

In order to deal with the Sobolev norms, we further decompose u according to its regularity in the y variable as (4.1)

$$u = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} u_{I,m}.$$

The definition for the $u_{I,m}$ is the following. Taking the Fourier transform along the y variable, the support of $\mathcal{F}_{y \rightarrow \eta}(u_{I,m})(x, \eta)$ satisfies the condition $|\eta| \in [I, 2I]$ for some dyadic relative integer $I \in 2^{\mathbb{Z}}$:

$$\mathcal{F}_{y \rightarrow \eta}(u_{I,m})(x, \eta) = f_m(\eta)h_m(|\eta|^{\frac{1}{2}}x)\mathbf{1}_{|\eta| \in [I, 2I]}. \quad (4.8)$$

When using the bilinear estimates, it will be useful to regroup the global frequencies $1+(2m+1)|\eta|$ in dyadic blocs $1+(2m+1)I \in [A, 2A]$ where $A \in 2^{\mathbb{N}}$ is a dyadic integer. For the shortness of notation, we will write $(m+1)I \sim A$ instead of $1+(2m+1)I \in [A, 2A]$. Therefore, we denote

$$u_A := \sum_{\substack{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m+1)I \sim A}} u_{I,m} \quad (4.9)$$

so that

$$u = \sum_{A \in 2^{\mathbb{N}}} u_A.$$

It is useful to note that, writing $\langle I \rangle = \sqrt{1+I^2}$, we have $\frac{1}{2m+1} \lesssim \frac{\langle I \rangle}{1+(2m+1)I}$.

Because of the orthogonality of the functions h_m , we have the following useful identities, which we will refer to as using *orthogonality*.

Lemma 4.2.3 (Orthogonality). *For all $k \in \mathbb{R}$, there holds*

$$\|u\|_{H_G^k}^2 = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I,m}\|_{H_G^k}^2 = \sum_{A \in 2^{\mathbb{N}}} \|u_A\|_{H_G^k}^2.$$

Observe that on the support of $u_{I,m}$, we have

$$1+(2m+1)|\eta| \in [1+(2m+1)I, 1+(2m+1)2I] \subset [A, 4A],$$

so that

$$\|u_{I,m}\|_{H_G^k} \sim (1+(2m+1)I)^{\frac{k}{2}} \|u_{I,m}\|_{L_G^2} \sim A^{\frac{k}{2}} \|u_{I,m}\|_{L_G^2}.$$

Using orthogonality, one also has: for any $A \in 2^{\mathbb{N}}$,

$$\|u_A\|_{H_G^k} \sim A^{\frac{k}{2}} \|u_A\|_{L_G^2}.$$

Remark 4.2.4. *With the above notation, one can interpret the norm (4.3) in \mathcal{X}_ρ^k*

$$\|u\|_{\mathcal{X}_\rho^k}^2 := \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1+(2m+1)I)^k \langle I \rangle^\rho \|u_{I,m}\|_{L_G^2}^2$$

as

$$\|u\|_{\mathcal{X}_\rho^k}^2 \sim \sum_{m \geq 0} \int_{\mathbb{R}} (1+(2m+1)|\eta|)^k (1+|\eta|)^\rho |f_m(\eta)|^2 |\eta|^{-\frac{1}{2}} d\eta.$$

In particular, every function $u \in H_G^k$ with additional partial regularity of $\frac{\rho}{2}$ in the y variable belongs to \mathcal{X}_ρ^k .

4.2.2 Hermite functions

Let us first recall some pointwise bounds for the Hermite functions $(h_m)_{m \geq 0}$. We denote by $\lambda_m = \sqrt{2m+1}$ the square root of the m -th eigenvalue for the harmonic oscillator.

Theorem 4.2.5 (Pointwise estimates for Hermite functions [KT05], Lemma 5.1). *For any $m \geq 0$ and $x \in \mathbb{R}$, there holds*

$$|h_m(x)| \lesssim \begin{cases} \frac{1}{|\lambda_m^2 - x^2|^{1/4}} & \text{if } |x| < \lambda_m - \lambda_m^{-1/3} \\ \lambda_m^{-1/6} & \text{if } ||x| - \lambda_m| \leq \lambda_m^{-1/3} \\ \frac{e^{-sm(x)}}{|\lambda_m^2 - x^2|^{1/4}} & \text{if } |x| > \lambda_m + \lambda_m^{-1/3}, \end{cases}$$

where

$$s_m(x) = \int_{\lambda_m}^x \sqrt{t^2 - \lambda_m^2} dt.$$

Remark 4.2.6. *In order to understand how bilinear estimates on $h_m h_n$ are proven, one may roughly picture h_m to be concentrated on $[-\sqrt{2m+1}, \sqrt{2m+1}]$, and work with models of the form*

$$h_m(x) \sim \frac{1}{\sqrt{2}} (2m+1)^{-\frac{1}{4}} \mathbf{1}_{[-\sqrt{2m+1}, \sqrt{2m+1}]}(x),$$

as long as one does not take pointwise estimates, and as long as one does not consider L^p norms for p too big (see [BTT13]).

The pointwise estimates imply the following lemma on the L^p norm of the Hermite functions.

Lemma 4.2.7 (L^p norms for Hermite functions [KT05], Corollary 3.2). *For any $p \geq 2$ there holds uniformly in m*

$$\|h_m\|_{L^p(\mathbb{R})} \lesssim \frac{1}{\lambda_m^{\zeta(p)}},$$

where $\lambda_m = \sqrt{2m+1}$ and

$$\zeta(p) = \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if } 2 \leq p \leq 4 \\ \frac{1}{6} + \frac{1}{3p} & \text{if } 4 < p \leq +\infty. \end{cases} \quad (4.10)$$

However, these L^p norm estimates will not be sufficient for our purpose, and we will rather make use of the following simplified pointwise estimates.

Corollary 4.2.8 (Rough pointwise estimates for Hermite functions). *There exists $c > 0$ such that for any $m \geq 0$ and $x \in \mathbb{R}$,*

$$|h_m(x)| \lesssim \begin{cases} \lambda_m^{-\frac{1}{2}} & \text{if } |x| \leq \frac{\lambda_m}{2} \\ \left(\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2| \right)^{-\frac{1}{4}} & \text{if } \frac{\lambda_m}{2} \leq |x| \leq 2\lambda_m \\ e^{-\frac{1}{8}x^2} & \text{if } |x| \geq 2\lambda_m. \end{cases}$$

For a proof of Corollary 4.2.8 based on Theorem 4.2.5, see Appendix 4.A.1.

4.2.3 Sobolev spaces

We will use on several occasions Sobolev embeddings for the Grushin operator, which correspond to the Folland-Stein embedding for the Sobolev spaces on the Heisenberg group [FS74].

Theorem 4.2.9 (Folland-Stein embedding). *Let $p \in [2, \infty)$.*

(i) *For $k > \frac{3}{2}$, then $H_G^k \hookrightarrow L_G^\infty$.*

(ii) *For $k \leq \frac{3}{2}$, if $\frac{1}{p} \geq \frac{1}{2} - \frac{k}{3}$, one has $H_G^k \hookrightarrow L_G^p$.*

Proposition 4.2.10. *Let $k > 0$ and $u, v \in H_G^k$. Then*

(i) *(Product rule) $\|uv\|_{H_G^k} \lesssim \|u\|_{H_G^k} \|v\|_{L_G^\infty} + \|u\|_{L_G^\infty} \|v\|_{H_G^k}$;*

(ii) *(Algebra property) if $k > \frac{3}{2}$, $\|uv\|_{H_G^k} \lesssim \|u\|_{H_G^k} \|v\|_{H_G^k}$;*

(iii) *(Chain rule) if $k > \frac{3}{2}$, for every $p \in \mathbb{N}^*$, $\|u^p\|_{H_G^k} \lesssim \|u\|_{H_G^k}^p$.*

More details about the proof of Proposition 4.2.10 can be found in Appendix 4.A.2.

4.2.4 Probabilistic preliminaries

Only basic probability notions will be used in this article. Recall that we have fixed once and for all a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and denote $\omega \in \Omega$.

Our main probabilistic tool is the Khinchine inequality for subgaussian random variables, and a is a multilinear version for Gaussian random variables.

Definition 4.2.11 (Subgaussian random variables). *We say that the family of independent and identically distributed complex-valued random variables $(X_{I,m})_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}}$ is subgaussian if there exists $c > 0$ such that for all $\gamma > 0$,*

$$\mathbb{E} \left[e^{\gamma X_{I,m}} \right] \leq e^{c\gamma^2}.$$

Theorem 4.2.12 (Khinchine inequality / Kolmogorov-Paley-Zygmund [BT08a], Lemma 3.1). *Let \mathcal{I} be a countable set, and let $(X_n)_{n \in \mathcal{I}}$ be a sequence of independent and identically distributed complex-valued subgaussian random variables. Then there exists $C > 0$ such that for every complex-valued sequence $(\Psi_n)_{n \in \mathcal{I}} \in \ell^2(\mathcal{I})$ and all $r \in [2, \infty)$, one has:*

$$\left\| \sum_{n \in \mathcal{I}} \Psi_n X_n \right\|_{L^r_\Omega} \leq C \sqrt{r} \left(\sum_{n \in \mathcal{I}} |\Psi_n|^2 \right)^{\frac{1}{2}}.$$

Corollary 4.2.13 (Probabilistic decoupling). *Let \mathcal{I} be a countable set, and let $(X_n)_{n \in \mathcal{I}}$ be a sequence of independent and identically distributed complex-valued subgaussian random variables. Then there exists $c > 0$ such that the following holds.*

Fix a sequence $(\Psi_n)_{n \in \mathcal{I}}$ of functions of the variable $\psi \in \mathbb{R}^d$ in L^p_ψ , $p = (p_1, \dots, p_d) \in [2, \infty)^d$. Fix a countable set \mathcal{P} and a partition of \mathcal{I} denoted $(\mathcal{I}_k)_{k \in \mathcal{P}}$. Then there exists $R_0(p)$ large enough such that for $R \geq R_0(p)$, outside a set of probability less than e^{-cR^2} , there holds:

$$\sum_{k \in \mathcal{P}} \left\| \sum_{n \in \mathcal{I}_k} \Psi_n X_n(\omega) \right\|_{L^p_\psi}^2 \leq R^2 \sum_{n \in \mathcal{I}} \|\Psi_n\|_{L^p_\psi}^2.$$

Proof. First, from the triangle inequality, we have

$$\left\| \sum_{k \in \mathcal{P}} \left\| \sum_{n \in \mathcal{I}_k} \Psi_n X_n(\omega) \right\|_{L^p_\psi}^2 \right\|_{L^r_\Omega} \leq \sum_{k \in \mathcal{P}} \left\| \sum_{n \in \mathcal{I}_k} \Psi_n X_n(\omega) \right\|_{L^{2r} L^p_\psi}^2.$$

Now, by the Minkowski inequality and Theorem 4.2.12 we have for $2r \geq \max\{p_1, \dots, p_d\}$, for all $k \in \mathcal{P}$,

$$\left\| \sum_{n \in \mathcal{I}_k} \Psi_n X_n \right\|_{L^{2r} L^p_\psi} \leq C \sqrt{r} \left\| \left(\sum_{n \in \mathcal{I}_k} |\Psi_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p_\psi},$$

and applying the Minkowski inequality again,

$$\left\| \sum_{n \in \mathcal{I}_k} \Psi_n X_n \right\|_{L^{2r} L^p_\psi} \leq C \sqrt{r} \left(\sum_{n \in \mathcal{I}_k} \|\Psi_n\|_{L^p_\psi}^2 \right)^{\frac{1}{2}}.$$

Thus by the Markov inequality, there exists $C > 0$ such that

$$\mathbb{P} \left(\sum_{k \in \mathcal{P}} \left\| \sum_{n \in \mathcal{I}_k} \Psi_n X_n \right\|_{L_\psi^p}^2 > R^2 \sum_{n \in \mathcal{I}} \|\Psi_n\|_{L_\psi^p}^2 \right) \leq \left(\frac{C\sqrt{r}}{R} \right)^r,$$

and the conclusion follows by optimizing in r , which leads to the choice $r = \frac{R^2}{4C}$. \square

We have the following multilinear version for Gaussian variables.

Theorem 4.2.14 (Wiener Chaos estimates [OT18], Lemma 2.6 and [Sim74], Lemma I.18-I.22). *Let \mathcal{I} be a countable set, $\ell \geq 1$ an integer, and let $\Psi : \mathcal{I}^\ell \rightarrow \mathbb{R}$. Let $(g_n)_{n \in \mathcal{I}}$ be independent and identically distributed standard real-valued Gaussian variables. Then there exists $C(\ell)$ such that the following holds. Let*

$$F^\omega := \sum_{(n_1, \dots, n_\ell) \in \mathcal{I}^\ell} \Psi_{n_1, \dots, n_\ell} g_1^\omega \cdots g_\ell^\omega,$$

and assume that $F^\omega \in L_\Omega^2$. Then one has that for any $r \geq 2$,

$$\|F^\omega\|_{L_\Omega^r} \leq C(\ell) r^{\frac{\ell}{2}} \|F^\omega\|_{L_\Omega^2}.$$

We now state the main consequences of this theorem and of the Markov inequality that we will use in this article.

Corollary 4.2.15 (Probabilistic decoupling). *Let \mathcal{I} be a countable set. Let $X_n = g_n + ih_n$ complex Gaussian random variables, where the $\{g_n, h_n\}_{n \in \mathcal{I}}$ are independent and identically distributed real-valued Gaussian variables. There exists $c > 0$ such that the following holds. Let $p = (p_1, \dots, p_d) \in [2, \infty)^d$ and $(\Psi_{n,n'})_{n,n' \in \mathcal{I}}$ and $(\Psi_{n,n',n''})_{n,n',n'' \in \mathcal{I}}$ be functions of the variable $\psi \in \mathbb{R}^d$ belonging to $L_\psi^p = L_{\psi_1}^{p_1} \dots L_{\psi_d}^{p_d}$. Then for $R \geq R_0(p)$ large enough the following holds.*

(i) *Outside a set of probability at most e^{-cR^2} ,*

$$\left\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'} \right\|_{L_\psi^p}^2 \leq R^4 \sum_{n,n' \in \mathcal{I}} \|\Psi_{n,n'}\|_{L_\psi^p}^2 + R^4 \left(\sum_{n \in \mathcal{I}} \|\Psi_{n,n}\|_{L_\psi^p} \right)^2$$

$$\left\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'} X_n X_{n'} \right\|_{L_\psi^p}^2 \leq R^4 \sum_{n,n' \in \mathcal{I}} \|\Psi_{n,n'}\|_{L_\psi^p}^2.$$

(ii) *We assume that $\Psi_{n,n',n''} = \Psi_{n',n,n''}$ for every $n, n', n'' \in \mathcal{I}$. Outside a set of probability at most e^{-cR^2} ,*

$$\left\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''} \right\|_{L_\psi^p}^2 \leq R^6 \sum_{n,n',n'' \in \mathcal{I}} \|\Psi_{n,n',n''}\|_{L_\psi^p}^2 + R^6 \sum_{n \in \mathcal{I}} \left(\sum_{n' \in \mathcal{I}} \|\Psi_{n,n,n'}\|_{L_\psi^p} \right)^2.$$

(iii) We assume that $\psi = (\psi_-, \psi_+)$, $p = (p_-, p_+)$, and we relax the assumption on p as $p_- = (p_1, \dots, p_{d_-}) \in [1, \infty)^{d_-}$ and $p_+ = (p_{d_+}, \dots, p_d) \in [2, \infty)^{d_+}$. Then outside a set of probability at most e^{-cR^2} ,

$$\left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'} X_n \bar{X}_{n'} \right\|_{L_\psi^p} \leq R^2 \left\| \left(\sum_{n, n' \in \mathcal{I}} \|\Psi_{n, n'}\|_{L_{\psi_+}^{p_+}}^2 \right)^{1/2} + \sum_{n \in \mathcal{I}} \|\Psi_{n, n}\|_{L_{\psi_+}^{p_+}} \right\|_{L_{\psi_-}^{p_-}}.$$

Proof. The proof follows from Theorem 4.2.14 by expansion, writing that $\Psi_{n, n'} = b_{n, n'} + ic_{n, n'}$ and using the independence of g_n from h_n . We fix $r \geq \max\{p_1, \dots, p_d\}$.

(i) Applying the Minkowski inequality, Theorem 4.2.14 and the Markov inequality, we get that outside a set of probability at most e^{-cR^2} there holds

$$\left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'} X_n \bar{X}_{n'} \right\|_{L_\psi^p} \leq R^2 \left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'} X_n \bar{X}_{n'} \right\|_{L_\psi^p L_\Omega^2}.$$

But for every $x \in \mathbb{R}^d$, we expand

$$\begin{aligned} \left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'}(\psi) X_n \bar{X}_{n'} \right\|_{L_\Omega^2}^2 &= \mathbb{E} \left[\left| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'}(\psi) X_n \bar{X}_{n'} \right|^2 \right] \\ &= \sum_{n_1, n_2, n'_1, n'_2 \in \mathcal{I}} \mathbb{E}[X_{n_1} \bar{X}_{n'_1} \bar{X}_{n_2} X_{n'_2}] \Psi_{n_1, n'_1}(\psi) \overline{\Psi_{n_2, n'_2}(\psi)}. \end{aligned}$$

The crucial observation is then the following, $\mathbb{E}[X_{n_1} \bar{X}_{n'_1} \bar{X}_{n_2} X_{n'_2}] = 0$ unless $\{n_1, n'_2\} = \{n'_1, n_2\}$. Indeed, since $X := X_{I, m}$ is a complex gaussian, we have $0 = \mathbb{E}[X] = \mathbb{E}[X^2]$, and the same holds for \bar{X} . Therefore we are left with:

$$\left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'}(\psi) X_n \bar{X}_{n'} \right\|_{L_\Omega^2}^2 = C \sum_{n, n' \in \mathcal{I}} |\Psi_{n, n'}(\psi)|^2 + C' \left(\sum_{n \in \mathcal{I}} |\Psi_{n, n}(\psi)| \right)^2.$$

Taking the norm L_ψ^p of the square root of this term and using the Minkowski inequality lead to the first inequality. For the second inequality, the proof is the same except for the remark that $\mathbb{E}[X_{n_1} X_{n'_1} \bar{X}_{n_2} \bar{X}_{n'_2}] = 0$ unless $\{n_1, n'_1\} = \{n_2, n'_2\}$.

(ii) Applying the Minkowski inequality, Theorem 4.2.14 and the Markov inequality, we get that outside a set of probability at most e^{-cR^2} there holds

$$\left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''} X_n X_{n'} \bar{X}_{n''} \right\|_{L_\psi^p} \leq R^3 \left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''} X_n X_{n'} \bar{X}_{n''} \right\|_{L_\psi^p L_\Omega^2}.$$

We first fix $x \in \mathbb{R}^d$ and estimate $\left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''}(\psi) X_n X_{n'} \bar{X}_{n''} \right\|_{L_\Omega^2}$ by expanding

$$\begin{aligned} \left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''}(\psi) X_n X_{n'} \bar{X}_{n''} \right\|_{L_\Omega^2}^2 &= \sum_{n_1, n'_1, n''_1, n_2, n'_2, n''_2 \in \mathcal{I}} \mathbb{E}[X_{n_1} X_{n'_1} \bar{X}_{n''_1} \bar{X}_{n_2} \bar{X}_{n'_2} X_{n''_2}] \Psi_{n_1, n'_1, n''_1}(\psi) \overline{\Psi_{n_2, n'_2, n''_2}(\psi)}. \end{aligned}$$

Since $X := X_n$ is a complex gaussian, we have $0 = \mathbb{E}[X] = \mathbb{E}[X^2] = \mathbb{E}[X^3]$, and the same holds for \bar{X} . Therefore, if

$$\mathbb{E}[X_{n_1} X_{n'_1} \bar{X}_{n''_1} \bar{X}_{n_2} \bar{X}_{n'_2} X_{n''_2}] \neq 0,$$

then one can see by double inclusion that

$$\{n_1, n'_1, n''_1\} = \{n''_1, n_2, n'_2\}.$$

More precisely, using that $\mathbb{E}[X^2 \bar{X}] = 0$, we even have that the indices n_1, n'_1, n''_1 are in bijection with the indices n''_1, n_2, n'_2 . This implies that up to permutation of the pairs of indices playing the same role, namely the pair of indices n_1 and n'_1 and the pair of indices n_2 and n'_2 , we are in one of the following two cases:

- $n_1 = n_2, n'_1 = n'_2, n''_1 = n''_2$;
- $n_1 = n''_1, n'_1 = n'_2$ and $n_2 = n''_2$.

Therefore, outside a set of probability at most e^{-cR^2} , there holds

$$\left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''} X_n X_{n'} \bar{X}_{n''} \right\|_{L_\psi^p}^2 \leq R^6 \left\| \left(\sum_{n, n', n'' \in \mathcal{I}} |\Psi_{n, n', n''}|^2 \right)^{1/2} \right\|_{L_\psi^p}^2 + R^6 \left\| \left(\sum_{n' \in \mathcal{I}} \left(\sum_{n \in \mathcal{I}} |\Psi_{n, n', n}| \right)^2 \right)^{1/2} \right\|_{L_\psi^p}^2,$$

so that from the Minkowski inequality on the sum over n, n', n'' resp. n' ,

$$\left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''} X_n X_{n'} \bar{X}_{n''} \right\|_{L_\psi^p}^2 \leq R^6 \sum_{n, n', n'' \in \mathcal{I}} \|\Psi_{n, n', n''}\|_{L_\psi^p}^2 + R^6 \sum_{n' \in \mathcal{I}} \left\| \sum_{n \in \mathcal{I}} |\Psi_{n, n', n}| \right\|_{L_\psi^p}^2,$$

and from the triangle inequality on the sum over n ,

$$\left\| \sum_{n, n', n'' \in \mathcal{I}} \Psi_{n, n', n''} X_n X_{n'} \bar{X}_{n''} \right\|_{L_\psi^p}^2 \leq R^6 \sum_{n, n', n'' \in \mathcal{I}} \|\Psi_{n, n', n''}\|_{L_\psi^p}^2 + R^6 \sum_{n' \in \mathcal{I}} \left(\sum_{n \in \mathcal{I}} \|\Psi_{n, n', n}\|_{L_\psi^p} \right)^2.$$

(iii) Applying the Minkowski inequality, Theorem 4.2.14 and the Markow inequality, we have that outside a set of probability at most e^{-cR^2} there holds

$$\left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'} X_n \bar{X}_{n'} \right\|_{L_\psi^p} \leq R^2 \left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'} X_n \bar{X}_{n'} \right\|_{L_\psi^p L_\Omega^2}.$$

For $\psi \in \mathbb{R}^d$, we expand

$$\begin{aligned} \left\| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'}(\psi) X_n \bar{X}_{n'} \right\|_{L_\Omega^2}^2 &= \mathbb{E} \left[\left| \sum_{n, n' \in \mathcal{I}} \Psi_{n, n'}(\psi) X_n \bar{X}_{n'} \right|^2 \right] \\ &= \sum_{n_1, n_2, n'_1, n'_2 \in \mathcal{I}} \mathbb{E}[X_{n_1} \bar{X}_{n'_1} \bar{X}_{n_2} X_{n'_2}] \Psi_{n_1, n'_1}(\psi) \overline{\Psi_{n_2, n'_2}(\psi)}. \end{aligned}$$

Since $X := X_n$ is a complex gaussian, we have $0 = \mathbb{E}[X] = \mathbb{E}[X^2]$, and the same holds for \bar{X} . Therefore, if $\mathbb{E}[X_{n_1} \bar{X}_{n'_1} \bar{X}_{n_2} X_{n'_2}] \neq 0$, then one can see that either $(n_1 = n'_1$ and $n_2 = n'_2)$ or $(n_1 = n_2$ and $n'_1 = n'_2)$. Therefore, we have

$$\left\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'} \right\|_{L^p_\psi} \leq R^2 \left\| \left(\sum_{n,n' \in \mathcal{I}} |\Psi_{n,n'}|^2 \right)^{1/2} + \sum_{n \in \mathcal{I}} |\Psi_{n,n}| \right\|_{L^p_\psi}.$$

Moreover, applying the Minkowski inequality for the admissible exponents $p_+ \in [2, \infty)^{d-d_-}$, we obtain that outside a set of probability at most e^{-cR^2} there holds

$$\left\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'} \right\|_{L^p_\psi} \leq R^2 \left\| \left(\sum_{n,n' \in \mathcal{I}} \|\Psi_{n,n'}\|_{L^{p_+}}^2 \right)^{1/2} + \sum_{n \in \mathcal{I}} \|\Psi_{n,n}\|_{L^{p_+}} \right\|_{L^{p_-}}. \quad \square$$

4.3 Random data and linear random estimates

In this section, we study in detail how the multiplication of each mode by independent normal distributions improves the integrability of the potential in the L^p spaces, without changing the Sobolev regularity.

Let us recall the construction of the random linear solutions associated to some fixed initial data $u_0 \in \mathcal{X}_\rho^k \subset H_G^k$. From (4.3), the $u_{I,m}$ defined by (4.8) satisfy

$$\|u_0\|_{\mathcal{X}_\rho^k}^2 = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m+1)I)^k \langle I \rangle^\rho \|u_{I,m}\|_{L_G^2}^2.$$

Given a family of Gaussian independent and identically distributed random variables denoted $(X_{I,m})_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}}$, the probability measure μ_{u_0} is the push-forward of \mathbb{P} under the map (4.2)

$$\omega \mapsto u_0^\omega := \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} X_{I,m}(\omega) u_{I,m}.$$

We denote $z^\omega(t) = e^{it\Delta_G} u_0^\omega$ the solution to the linear equation (LS-G) associated to the initial data u_0^ω . In particular, we have

$$z^\omega = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} X_{I,m}(\omega) z_{I,m}, \quad (4.11)$$

with $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$.

4.3.1 Probabilistic integrability and smoothing estimates

In this part, we prove that the randomized potential z^ω belongs to the space $L_T^q L_G^p$ for every $p, q \in [2, \infty)$, and even to the space $L_T^q W_G^{k+\zeta(p),p}$, $\zeta(p) \leq \frac{1}{4}$ being defined in (4.10).

Proposition 4.3.1 (Integrability improvement). *There exists $c > 0$ such that the following holds. Let $k \in \mathbb{R}$. Fix $u_0 \in \mathcal{X}_{\zeta(p)+\frac{1}{2}-\frac{1}{p}}^k$, denote u_0^ω its randomization from (4.2), and*

write $z^\omega = e^{it\Delta_G} u_0^\omega$. Let $p, q \in [2, \infty)$ and $\varepsilon > 0$. Then outside a set of probability at most e^{-cR^2} , there holds

$$\|z^\omega\|_{L_T^q W_G^{k+\zeta(p), p}} \leq RT^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}^k}, \quad (4.12)$$

$$\sum_{A \in 2^{\mathbb{N}}} A^{k+\zeta(p)-\varepsilon} \|z_A^\omega\|_{L_T^q W_G^{\varepsilon, p}}^2 \leq R^2 T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}^k}^2. \quad (4.13)$$

Note that we have $\zeta(p) + \frac{3}{2} - \frac{3}{p} = 2 - \frac{4}{p}$ when $p \leq 4$ and $\zeta(p) + \frac{3}{2} - \frac{3}{p} = \frac{5}{3} - \frac{8}{3p}$ when $p > 4$. The derivative gain is maximal when $\zeta(p)$ is, that is when $p = 4$ leading to a $\frac{1}{4}$ gain of derivatives.

Remark 4.3.2 (Case $p = \infty$). Since $\frac{4}{p} \times p = 4 > 3$ one can use the embedding $W_G^{\frac{4}{p}, p} \hookrightarrow L_G^\infty$ and obtain that outside a set of probability at most e^{-cR^2} , there holds

$$\|z^\omega\|_{L_T^q W_G^{\varepsilon, \infty}} \lesssim_\varepsilon RT^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}^{\frac{4}{p}-\zeta(p)+\varepsilon}},$$

which is obtained by taking $k = \frac{4}{p} - \zeta(p) + \varepsilon$ in (4.12). Observe that $\mathcal{X}_1^1 \hookrightarrow \mathcal{X}_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}^{\frac{4}{p}-\zeta(p)+\varepsilon}$ when $\varepsilon + \frac{1}{2} + \frac{1}{p} \leq 1$, satisfied as soon as $\varepsilon < \frac{1}{2}$ by choosing p large enough. Therefore, for $\varepsilon \in [0, \frac{1}{2})$, outside a set of probability at most e^{-cR^2} , there holds

$$\|z^\omega\|_{L_T^q W_G^{\varepsilon, \infty}} \lesssim_\varepsilon RT^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^1}.$$

Similarly, one obtains that for $\varepsilon \in (0, \frac{1}{2})$, outside a set of probability at most e^{-cR^2} , there holds

$$\sum_{A \in 2^{\mathbb{N}}} A^\varepsilon \|z_A^\omega\|_{L_T^q L_G^\infty}^2 \lesssim_\varepsilon R^2 T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_1^1}^2. \quad (4.14)$$

The proof of Proposition 4.3.1 relies on the following deterministic estimates.

Lemma 4.3.3. *With the notation from Proposition 4.3.1, let $p, q \in [2, \infty)$. Then for all $T > 0$, writing $L_T^q = L^q([0, T])$,*

$$\sum_{(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m + 1)I)^{k+\zeta(p)} \|z_{I, m}\|_{L_T^q L_G^p}^2 \lesssim T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}^k}^2. \quad (4.15)$$

Proof of Lemma 4.3.3. Let $p \in [2, \infty)$. We apply the Hausdorff-Young inequality in the y variable, so that if we denote p' the conjugate exponent of p , we obtain that for all $t \in [0, T]$,

$$\begin{aligned} \|z_{I, m}(t)\|_{L_G^p} &\lesssim \|e^{it(2m+1)|\eta|} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \mathbf{1}_{|\eta| \in [I, 2I]}\|_{L_x^p L_\eta^{p'}} \\ &= \|f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \mathbf{1}_{|\eta| \in [I, 2I]}\|_{L_x^p L_\eta^{p'}}. \end{aligned}$$

Moreover, using the Minkowski inequality, since $p \geq p'$, we have

$$\|z_{I, m}(t)\|_{L_G^p} \lesssim \|f_m(\eta) h_m(|\eta|^{1/2} x) \mathbf{1}_{|\eta| \in [I, 2I]}\|_{L_\eta^{p'} L_x^p}.$$

Since $z_{I, m}(0) = u_{I, m}$, this implies

$$\|z_{I, m}(t)\|_{L_G^p} \lesssim \|u_{I, m}\|_{L_\eta^{p'} L_x^p}.$$

Therefore, estimate (4.15) is a consequence of the inequality

$$\sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m + 1)I)^{k + \zeta(p)} \|u_{I,m}\|_{L_{\eta}^{p'} L_x^p}^2 \lesssim \|u_0\|_{\mathcal{X}^k_{\zeta(p) + \frac{3}{2} - \frac{3}{p}}}^2, \quad (4.16)$$

which we will now establish.

Using the upper bounds in L^p on the Hermite functions h_m from Lemma 4.2.7, and the notation (4.10) for the exponent $\zeta(p)$, we have

$$\begin{aligned} \|u_{I,m}\|_{L_{\eta}^{p'} L_x^p} &= \|f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \mathbf{1}_{|\eta| \in [I, 2I]}\|_{L_{\eta}^{p'} L_x^p} \\ &\lesssim (2m + 1)^{-\frac{\zeta(p)}{2}} \left\| f_m(\eta) \eta^{-\frac{1}{4}} \mathbf{1}_{|\eta| \in [I, 2I]} \eta^{\frac{1}{4} - \frac{1}{2p}} \right\|_{L_{\eta}^{p'}} \\ &\lesssim (2m + 1)^{-\frac{\zeta(p)}{2}} I^{\frac{1}{4} - \frac{1}{2p}} \|f_m(\eta) \eta^{-\frac{1}{4}} \mathbf{1}_{|\eta| \in [I, 2I]}\|_{L_{\eta}^{p'}}. \end{aligned}$$

From Hölder's inequality in η (and because the interval $[I, 2I]$ has length I), we get

$$\begin{aligned} \|u_{I,m}\|_{L_G^p} &\lesssim (2m + 1)^{-\frac{\zeta(p)}{2}} I^{\frac{1}{4} - \frac{1}{2p} - \frac{1}{2} + \frac{1}{p'}} \|f_m(\eta) \eta^{-\frac{1}{4}} \mathbf{1}_{|\eta| \in [I, 2I]}\|_{L_{\eta}^2} \\ &\lesssim (1 + (2m + 1)I)^{-\frac{\zeta(p)}{2}} \langle I \rangle^{\frac{\zeta(p)}{2} + \frac{1}{4} - \frac{1}{2p} - \frac{1}{2} + \frac{1}{p'}} \|u_{I,m}\|_{L_G^2}, \end{aligned}$$

which leads to (4.16) with $\zeta(p) + \frac{1}{2} - \frac{1}{p} - 1 + \frac{2}{p'} = \zeta(p) + \frac{3}{2} - \frac{3}{p}$. \square

Proof of Proposition 4.3.1. Let us first establish (4.12). We start with an application of the probabilistic decoupling from Corollary 4.2.13: outside a set of probability at most e^{-cR^2} , we have

$$\|z^{\omega}\|_{L_T^q W_G^{k + \zeta(p), p}}^2 \lesssim R^2 \sum_{I,m} \|z_{I,m}\|_{L_T^q W_G^{k + \zeta(p), p}}^2.$$

Since for all $t \in \mathbb{R}$ and all $k' \in \mathbb{R}$, we have $\|(-\Delta_G)^{k'/2} z_{I,m}(t)\|_{L_G^p}^2 \lesssim (1 + (2m + 1)I)^{k'/2} \|z_{I,m}(t)\|_{L_G^p}^2$, we deduce that

$$\|z^{\omega}\|_{L_T^q W_G^{k + \zeta(p), p}} \lesssim R \left(\sum_{I,m} (1 + (2m + 1)I)^{k + \zeta(p)} \|z_{I,m}\|_{L_T^q L_G^p}^2 \right)^{\frac{1}{2}}.$$

Inequality (4.12) is now a consequence of (4.15).

Similarly, fix $\varepsilon > 0$. For $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$, we denote by A the dyadic integer such that $(m + 1)I \sim A$. Then from Corollary 4.2.13 applied to the partition given by the $\{(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N} \mid (2m + 1)I \sim A\}$ for $A \in 2^{\mathbb{N}}$. Therefore, outside a set of probability at most e^{-cR^2} , we have

$$\begin{aligned} \sum_{A \in 2^{\mathbb{N}}} A^{k + \zeta(p) - \varepsilon} \|z_A^{\omega}\|_{L_T^q W_G^{\varepsilon, p}}^2 &\lesssim R^2 \sum_{I,m} A^{k + \zeta(p) - \varepsilon} \|z_{I,m}\|_{L_T^q W_G^{\varepsilon, p}}^2 \\ &\lesssim R^2 \sum_{I,m} (1 + (2m + 1)I)^{k + \zeta(p)} \|z_{I,m}\|_{L_T^q L_G^p}^2, \end{aligned}$$

enabling us to conclude thanks to (4.15) again. \square

4.3.2 Non-smoothing properties of the randomization

We now prove Theorem A (ii), stating that the randomization does not improve the Sobolev regularity in the H^k spaces, in a similar spirit as in [BT08a].

Proposition 4.3.4 (Non-smoothing for random initial data). *We assume that $X_{I,m}$ is not almost-surely equal to 0. Let $u_0 \in H_G^k \setminus (\bigcup_{\varepsilon>0} H_G^{k+\varepsilon})$ and let u_0^ω defined by (4.2). Then, almost surely, we have $u_0^\omega \in H_G^k \setminus (\bigcup_{\varepsilon>0} H_G^{k+\varepsilon})$.*

Proof. Without loss of generality, we assume that $k = 0$. First, let us remark that $\mathbb{E}[u_0^\omega] = 0$. Then, since by orthogonality one has

$$\|u_0^\omega\|_{L_G^2}^2 = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} |X_{I,m}(\omega)|^2 \|u_{I,m}\|_{L_G^2}^2,$$

and since the $X_{I,m}$ have a finite variance, we conclude that

$$\mathbb{E} \left[\|u_0^\omega\|_{L_G^2}^2 \right] \lesssim \|u_0\|_{L_G^2}^2 < \infty.$$

Let $\varepsilon > 0$, it remains to prove that almost surely we have $u_0^\omega \notin H_G^\varepsilon$. In fact, one only needs to show that

$$\mathbb{E} \left[e^{-\|u_0^\omega\|_{H_G^\varepsilon}^2} \right] = 0. \quad (4.17)$$

In order to establish (4.17), we expand decomposition (4.2) using the independence of the random variables $X_{I,m}$:

$$\mathbb{E} \left[e^{-\|u_0^\omega\|_{H_G^\varepsilon}^2} \right] = \prod_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \mathbb{E} \left[e^{-|X_{I,m}|^2 \|u_{I,m}\|_{H_G^\varepsilon}^2} \right].$$

Since for all I, m , there holds $\|u_{I,m}\|_{H_G^\varepsilon}^2 \geq (1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2$, we get

$$\mathbb{E} \left[e^{-\|u_0^\omega\|_{H_G^\varepsilon}^2} \right] \leq \prod_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \mathbb{E} \left[e^{-|X_{I,m}|^2 (1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2} \right].$$

Then we have the following alternative.

First case: Assume that the terms $(1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2$ do not go to zero as $(I, m) \rightarrow \infty$. Then there exists $\gamma > 0$ and an infinite set S of indices $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$ satisfying $(1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2 \geq \gamma$. This leads to the bound

$$\mathbb{E} \left[e^{-\|u_0^\omega\|_{H_G^\varepsilon}^2} \right] \leq \prod_{(I,m) \in S} \mathbb{E} \left[e^{-|X_{I,m}|^2 (1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2} \right] \leq \prod_{(I,m) \in S} \mathbb{E} \left[e^{-|X_{I,m}|^2 \gamma} \right] = 0,$$

where in the last step we used that S is infinite, the fact that the $X_{I,m}$ are identically distributed and the assumption $\mathbb{P}(X_{I,m} = 0) < 1$.

Second case: Assume that $(1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2 \rightarrow 0$ as $(I, m) \rightarrow \infty$. In this case, we fix $R > 0$ such that $\delta_R := \mathbb{P}(|X_{I,m}| > R) > 0$, which is possible thanks to the assumption $\mathbb{P}(X_{I,m} = 0) < 1$ on the $X_{I,m}$. Moreover, δ_R does not depend on (I, m) since the $X_{I,m}$ are identically distributed.

For every I, m , we have

$$\mathbb{E} \left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2} \right] \leq (1 - \delta_R) + \delta_R e^{-R^2(1+(2m+1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2}.$$

Since the sequence $\left((1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2 \right)_{I,m}$ is convergent to zero as $(I, m) \rightarrow \infty$, this sequence is bounded by some constant $C > 0$. But on the interval $[0, R^2C]$, there holds the inequality $e^{-x} \leq 1 - c_R x$ for some $c_R > 0$, so that

$$\mathbb{E} \left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2} \right] \lesssim 1 - \delta_R c_R R^2 (1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2,$$

and the upper bound is positive. Moreover, since

$$\sum_{I,m} \delta_R c_R R^2 (1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2 \gtrsim \delta_R c_R R^2 \|u_0\|_{H_G^\varepsilon}^2 = \infty,$$

we conclude that the infinite product with general term $(1 - \delta_R c_R R^2 (1 + (2m + 1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2)$ is zero, so that

$$\prod_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \mathbb{E} \left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon \|u_{I,m}\|_{L_G^2}^2} \right] = 0.$$

In any case the above discussion proves (4.17). \square

Remark 4.3.5. Note that using the fact that the decay rate for the L^p norms of h_m is optimal in Lemma 4.2.7 (see Lemma 5.1 in [KT05]) and the probabilistic decoupling argument from Corollary 4.2.15, it seems highly unlikely that u_0^ω belongs to $W_G^{\zeta(p),p}$ for $\varepsilon > 0$ (see also [IRT16]).

4.3.3 Density of the measure μ_{u_0}

In this part, we establish Theorem A (iii). Before we turn to the density properties of the measures μ_{u_0} associated to rough potentials u_0 , we briefly justify that we can construct functions $u_0 \in \mathcal{X}_1^k$ such that $u_0 \in H_G^k \setminus (\bigcup_{\varepsilon > 0} H_G^{k+\varepsilon})$.

Lemma 4.3.6 (Existence of rough potentials). *Let $k \geq 0$ and $\rho \geq 0$. There exists a function $u_0 \in \mathcal{X}_\rho^k \subset H_G^k$ such that for all $\varepsilon > 0$, $u_0 \notin H_G^{k+\varepsilon}$.*

Remark 4.3.7. In fact, this lemma implies that there exists uncountably many such functions. Indeed, we can apply another randomization argument to the potential u_0 from the lemma. Take $v_0^\omega = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \varepsilon_{I,m}(\omega) u_{I,m}$, where $\varepsilon_{I,m}$ are independent random signs, then the functions v_0^ω almost-surely satisfy the requirements.

Proof. Let $k, \rho \geq 0$. We consider $u_0 := \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} u_{I,m}$ defined in Fourier variable by $u_{I,m} = 0$ if $I < 1$, and if $I \geq 1$, we take $u_{I,m}$ "constant" by parts

$$\mathcal{F}_{y \rightarrow \eta}(u_{I,m})(x, \eta) = \|u_{I,m}\|_{L_G^2} |\eta|^{\frac{1}{4}} h_m \left(|\eta|^{\frac{1}{2}} x \right).$$

where for any $(I, m) \in 2^{\mathbb{N}} \times \mathbb{N}$, we choose

$$\|u_{I,m}\|_{L_G^2}^2 = \frac{1}{(1 + (2m + 1)I)^k \langle I \rangle^\rho \log(1 + I)^2 (m + 1) \log(m + 2)^2}.$$

First, we observe that for $\varepsilon \in \mathbb{R}$,

$$\|u_0\|_{H_G^{k+\varepsilon}}^2 \sim \sum_{I \in 2^{\mathbb{N}}} \frac{1}{\langle I \rangle^\rho \log(1+I)^2} \sum_{m \in \mathbb{N}} \frac{(1+(2m+1)I)^\varepsilon}{(m+1) \log(m+2)^2}.$$

This series with positive general term is divergent when $\varepsilon > 0$. For $\varepsilon \leq 0$, this series is bounded by $C \sum_{I \in 2^{\mathbb{N}}} \frac{1}{\langle I \rangle^\rho \log(1+I)^2}$ which is convergent.

It remains to prove that $u_0 \in \mathcal{X}_\rho^k$. In order to do so, we compute:

$$\|u_0\|_{\mathcal{X}_\rho^k}^2 = \sum_{(I,m) \in 2^{\mathbb{N}} \times \mathbb{N}} \frac{1}{\log(1+I)^2 (m+1) \log(m+2)^2},$$

which is indeed finite. \square

We now establish density properties of the support of the measures with rough potentials (Theorem A (iii)).

Proposition 4.3.8 (Density of measures with rough potentials). *Let $k \geq 0$. We assume that for any $r > 0$, $\mathbb{P}(|X_{I,m} - 1| < r) > 0$. Let $u_0 \in H_G^k$ and $\varepsilon > 0$. Then there exists $v_0 \in \mathcal{X}_1^k \setminus (\bigcup_{\varepsilon' > 0} H_G^{k+\varepsilon'})$ such that*

$$\mu_{v_0} \left(B_{H_G^k}(u_0, \varepsilon) \right) > 0.$$

Proof. Without loss of generality, we assume that $k = 0$, and we only deal with the Grushin case. Fix $\varepsilon > 0$ and $u_0 \in L_G^2$. We first construct $v_0 \in \mathcal{X}_2^0 \setminus (\bigcup_{\varepsilon' > 0} H_G^{\varepsilon'})$ satisfying the condition $\|u_0 - v_0\|_{L_G^2} \leq \varepsilon$, then we prove that v_0 meets the requirements from the proposition.

To construct v_0 , let $K_0 > 0$ be such that

$$\sum_{|(I,m)| > K_0} \|u_{I,m}\|_{L_G^2}^2 \leq \varepsilon^2,$$

where $|(I,m)| = \max\{|\log(1+I)|, m\}$. Then, let $\delta > 0$ to be determined later, and denote by

$$\tilde{u}_0 = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \tilde{u}_{I,m} \in \mathcal{X}_1^0 \setminus \left(\bigcup_{\varepsilon' > 0} H_G^{\varepsilon'} \right)$$

the potential constructed in Lemma 4.3.6 for $k = 0$ and $\rho = 1$. Set

$$v_{I,m} := \begin{cases} u_{I,m} & \text{if } |(I,m)| \leq K_0 \\ \delta \tilde{u}_{I,m} & \text{if } |(I,m)| > K_0. \end{cases}$$

Since the coefficients $v_{I,m}$ for $|(I,m)| > K_0$ are the ones from Lemma 4.3.6, we know that v belongs to $\mathcal{X}_1^0 \setminus (\bigcup_{\varepsilon' > 0} H_G^{\varepsilon'})$. Then we compute that when δ is small enough,

$$\|u_0 - v_0\|_{L_G^2}^2 \leq \sum_{|(I,m)| > K_0} \|u_{I,m}\|_{L_G^2}^2 + \delta^2 \sum_{|(I,m)| > K_0} \|\tilde{u}_{I,m}\|_{L_G^2}^2 \leq 2\varepsilon^2.$$

In the end of this proof, we establish that

$$\mu_{v_0} \left(w \in L_G^2, \|w - v_0\|_{L_G^2} \leq \varepsilon \right) > 0,$$

as this would imply

$$\mu_{v_0} \left(w \in L_G^2, \|w - u_0\|_{L_G^2} \leq 2\varepsilon \right) > 0.$$

For future occurrences, for $K > 0$ and $w = \sum_{I,m} w_{I,m} \in L_G^2$, let us denote

$$\Pi_K(w) := \sum_{|(I,m)| \leq K} w_{I,m}.$$

Because of the inclusion

$$\begin{aligned} \left\{ w \in L_G^2 \mid \|\Pi_K(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right\} \cap \left\{ w \in L_G^2 \mid \|(\text{Id} - \Pi_K)(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right\} \\ \subset \left\{ w \in L_G^2 \mid \|w - v_0\|_{L_G^2} \leq \varepsilon \right\}, \end{aligned}$$

we know by independence that

$$\begin{aligned} \mu_{v_0} \left(w \in L_G^2, \|w - v_0\|_{L_G^2} \leq \varepsilon \right) &\geq \mu_{v_0} \left(w \in L_G^2, \|\Pi_K(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right) \\ &\quad \times \mu_{v_0} \left(w \in L_G^2, \|(\text{Id} - \Pi_K)(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right). \end{aligned}$$

To handle the second term in the right-hand side, since $\Pi_K v_0$ tends to v_0 in L_G^2 as $K \rightarrow \infty$, one has that for large enough $K > 0$,

$$\mu_{v_0} \left(w \in L_G^2, \|(\text{Id} - \Pi_K)(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right) \geq \mu \left(w \in L_G^2, \|(\text{Id} - \Pi_K)w\|_{L_G^2} \leq \frac{\varepsilon}{4} \right).$$

But we have

$$\mu_{v_0} \left(w \in L_G^2, \|(\text{Id} - \Pi_K)w\|_{L_G^2} > \frac{\varepsilon}{4} \right) \xrightarrow{K \rightarrow \infty} 0,$$

since from the Markov inequality,

$$\begin{aligned} \mu_{v_0} \left(w \in L_G^2, \|(\text{Id} - \Pi_K)w\|_{L_G^2} > \frac{\varepsilon}{4} \right) &= \mathbb{P} \left(\|(\text{Id} - \Pi_K)v^\omega\|_{L_G^2} > \frac{\varepsilon}{4} \right) \\ &\lesssim \varepsilon^{-2} \mathbb{E} \left[\|(\text{Id} - \Pi_K)v^\omega\|_{L_G^2}^2 \right] \lesssim \varepsilon^{-2} \sum_{|(I,m)| > K} \|v_{I,m}\|_{L_G^2}^2, \end{aligned}$$

which goes to zero as K goes to infinity. We therefore fix $K > 0$ large enough so that

$$\mu_{v_0} \left(w \in L_G^2, \|(\text{Id} - \Pi_K)(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right) > 0.$$

It only remains to handle the first term in the right-hand side and prove that

$$\mu_{v_0} \left(w \in L_G^2, \|\Pi_K(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right) > 0. \quad (4.18)$$

Let $c > 0$ small enough so that the following inclusion holds:

$$\begin{aligned} \bigcap_{|(I,m)| < K} \left\{ w \in L_G^2 \mid \exists \tilde{X}_{I,m} \in \mathbb{R}, w_{I,m} = \tilde{X}_{I,m} v_{I,m} \text{ and } \|(\tilde{X}_{I,m} - 1)v_{I,m}\|_{L_G^2} \leq c\varepsilon \right\} \\ \subset \left\{ w \in L_G^2 \mid \|\Pi_K(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right\}. \end{aligned}$$

This implies by independence of the random variables $X_{I,m}$ that

$$\mu_{v_0} \left(w \in L_G^2, \|\Pi_K(w - v_0)\|_{L_G^2} \leq \frac{\varepsilon}{2} \right) \geq \prod_{|(I,m)| \leq K} \mathbb{P} \left(\|(X_{I,m}(\omega) - 1)v_{I,m}\|_{L_G^2} \leq c\varepsilon \right),$$

which is positive thanks to the assumption $\mathbb{P}(|X_{I,m} - 1| < r) > 0$ for all $r > 0$, thus (4.18) holds. \square

4.4 Action of the Laplace operator

In the remaining of this article, our general strategy is to group the decompositions (4.1) of u and v in dyadic packets (4.9)

$$u = \sum_{A \in 2^{\mathbb{N}}} u_A \quad \text{and} \quad v = \sum_{B \in 2^{\mathbb{N}}} v_B,$$

and write the estimates for the H^k norm of the product uv in terms of the L^2 norms of products $u_A v_B$ for dyadic A and B .

In this section, we consider two elements $u, v \in H_G^k$ and provide useful estimates for the H^k norm of the products $u_A v_B$ and $u_A v$ in terms of the L^2 norms of the products $u_A v_B$ for dyadic A and B .

4.4.1 The action of $-\Delta_G$ on a product of functions

In this part, we prove that for $u, v \in H_G^k$, the term $-\Delta_G(uv)$ can be expressed thanks to shifted versions of u and v , defined as follows.

Definition 4.4.1 (δ -shifted functions). *Let and $u \in H_G^k$, decomposed as (4.1) $u = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} u_{I,m}$, where $u_{I,m}$ is defined in (4.8) as*

$$\mathcal{F}_{y \rightarrow \eta}(u_{I,m})(x, \eta) = f_{I,m}(\eta) h_m(|\eta|^{\frac{1}{2}} x),$$

and $f_{I,m}(\eta) = f_m(\eta) \mathbf{1}_{|\eta| \in [I, 2I]}$.

For $\delta \in D_1 := \{-1, 0, 1\} \times \{+, -\}$ we write $\delta = (\delta_0, \pm)$ and for $m \in \mathbb{N}$, we write $m + \delta$ as a shortcut for $m + \delta := m + \delta_0$. We introduce the shifted function u^δ from its decomposition (4.1) $u^\delta = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} u_{I,m}^\delta$ as follows: for all $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$,

$$\mathcal{F}_{y \rightarrow \eta}(u_{I,m}^\delta)(x, \eta) = F_{I,m}^\delta(\eta) f_{I,m}(\eta) h_{m+\delta}(|\eta|^{\frac{1}{2}} x),$$

and if $A \in 2^{\mathbb{N}}$ is the dyadic integer such that $(m+1)I \sim A$,

$$F_{I,m}^\delta(\eta) := \begin{cases} \left(\frac{(2m+1)|\eta|}{4A} \right)^{\frac{1}{2}} & \text{if } \delta \in \{(-1, +), (1, +)\} \\ \left(\frac{(2m+1)}{4A} \right)^{\frac{1}{2}} \frac{\eta}{\sqrt{|\eta|}} & \text{if } \delta \in \{(-1, -), (1, -)\} \\ \frac{(2m+1)|\eta|}{4A} & \text{if } \delta = (0, +) \\ 1 & \text{if } \delta = (0, -). \end{cases} \quad (4.19)$$

Note that by definition, for all $k \geq 0$ and $u \in H_G^k$, we have $\|u^\delta\|_{H_G^k} \leq \|u\|_{H_G^k}$.

Lemma 4.4.2 (Action of Δ_G). *Let $A, B \in 2^{\mathbb{N}}$, $I, J \in 2^{\mathbb{Z}}$ and $m, n \in \mathbb{N}$ such that $(m+1)I \sim A$ and $(n+1)J \sim B$. We denote $D_2 := D_1 \times D_1$. Then there holds:*

$$(\text{Id} - \Delta_G)(u_{I,m} v_{J,n}) = \sum_{(\delta_1, \delta_2) \in D_2} C_{A,B}(\delta_1, \delta_2) u_{I,m}^{\delta_1} v_{J,n}^{\delta_2},$$

where for some explicit numerical constants $c(\delta_1, \delta_2)$,

$$C_{A,B}(\delta_1, \delta_2) = \begin{cases} 4A & \text{if } (\delta_1, \delta_2) = ((0, -), (0, +)) \\ 4B & \text{if } (\delta_1, \delta_2) = ((0, +), (0, -)) \\ c(\delta_1, \delta_2) \sqrt{AB} & \text{if } (\delta_1, \delta_2) \in (\{-1, 1\} \times \{+, -\})^2 \\ 1 & \text{if } (\delta_1, \delta_2) = ((0, -), (0, -)) \\ 0 & \text{if } (\delta_1, \delta_2) = ((0, +), (0, +)), . \end{cases}$$

Proof. Taking the Fourier transform of $u_{I,m}v_{J,n}$ in y , we transform the product into a convolution product

$$\mathcal{F}_{y \rightarrow \eta}(u_{I,m}v_{J,n})(x, \eta) = \int f_{I,m}(\eta_1)h_m(|\eta_1|^{\frac{1}{2}}x)g_{J,n}(\eta - \eta_1)h_n(|\eta - \eta_1|^{\frac{1}{2}}x)d\eta_1.$$

In Fourier variable, the Grushin operator acts as $\partial_{xx} - x^2|\eta|^2$, therefore we have

$$\mathcal{F}(\Delta_G(u_{I,m}v_{J,n}))(x, \eta) = \int f_{I,m}(\eta_1)g_{J,n}(\eta - \eta_1)(\partial_{xx} - x^2|\eta|^2) \left(h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x) \right) d\eta_1.$$

We now use the fact that the Hermite functions are eigenvectors of the Harmonic oscillator:

$$\frac{d^2}{dx^2}h_m(x) = -(2m + 1)h_m + x^2h_m$$

to deduce the formula

$$\begin{aligned} & (\partial_{xx} - x^2|\eta|^2)(h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x)) \\ &= -((2m + 1)|\eta_1| + (2n + 1)|\eta - \eta_1|)h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x) \\ & \quad + x^2(|\eta_1|^2 + |\eta - \eta_1|^2 - |\eta|^2)h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x) + 2\partial_x(h_m(|\eta_1|^{\frac{1}{2}}x))\partial_x(h_n(|\eta - \eta_1|^{\frac{1}{2}}x)). \end{aligned}$$

Note that if η_1 and $\eta - \eta_1$ have the same sign, then $|\eta_1|^2 + |\eta - \eta_1|^2 - |\eta|^2 = -2\eta_1(\eta - \eta_1)$, and if η_1 and $\eta - \eta_1$ are of opposite sign, $|\eta_1|^2 + |\eta - \eta_1|^2 - |\eta|^2 = 2|\eta_1||\eta - \eta_1| = -2\eta_1(\eta - \eta_1)$.

We now use the identities

$$xh_m(x) = \sqrt{\frac{m}{2}}h_{m-1} + \sqrt{\frac{m+1}{2}}h_{m+1},$$

$$\frac{d}{dx}h_m(x) = \sqrt{\frac{m}{2}}h_{m-1} - \sqrt{\frac{m+1}{2}}h_{m+1},$$

to remove the weight x^2 and simplify the products $\partial_x(h_m(|\eta_1|^{\frac{1}{2}}x))\partial_x(h_n(|\eta - \eta_1|^{\frac{1}{2}}x))$. We deduce that

$$\begin{aligned} & (\partial_{xx} - x^2|\eta|^2)(h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x)) \\ &= -((2m + 1)|\eta_1| + (2n + 1)|\eta - \eta_1|)h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x) \\ & \quad - 2\frac{\eta_1(\eta - \eta_1)}{\sqrt{|\eta_1||\eta - \eta_1|}} \left(\sqrt{\frac{m}{2}}h_{m-1}(|\eta_1|^{\frac{1}{2}}x) + \sqrt{\frac{m+1}{2}}h_{m+1}(|\eta_1|^{\frac{1}{2}}x) \right) \\ & \quad \times \left(\sqrt{\frac{n}{2}}h_{n-1}(|\eta - \eta_1|^{\frac{1}{2}}x) + \sqrt{\frac{n+1}{2}}h_{n+1}(|\eta - \eta_1|^{\frac{1}{2}}x) \right) \\ & \quad + 2\sqrt{|\eta_1||\eta - \eta_1|} \left(\sqrt{\frac{m}{2}}h_{m-1}(|\eta_1|^{\frac{1}{2}}x) - \sqrt{\frac{m+1}{2}}h_{m+1}(|\eta_1|^{\frac{1}{2}}x) \right) \\ & \quad \times \left(\sqrt{\frac{n}{2}}h_{n-1}(|\eta - \eta_1|^{\frac{1}{2}}x) - \sqrt{\frac{n+1}{2}}h_{n+1}(|\eta - \eta_1|^{\frac{1}{2}}x) \right). \end{aligned}$$

Now, we observe that by definition of the support of $\mathcal{F}_{y \rightarrow \eta}(u_{I,m})(x, \eta)$ and $\mathcal{F}_{y \rightarrow \eta}(v_{J,n})(x, \eta)$, we have $1 + (2m + 1)|\eta_1| \in [A, 4A]$ and $1 + (2n + 1)|\eta - \eta_1| \in [B, 4B]$. Therefore, for some

numerical constants $c(\delta_1, \delta_2)$, using the notations (4.19) for $F_{I,m}^\delta$, one can write

$$\begin{aligned} & (1 - \partial_{xx} + x^2|\eta|^2)(h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x)) \\ &= \left(1 + 4AF_{I,m}^{(0,+)}(\eta_1) + 4BF_{J,n}^{(0,+)}(\eta - \eta_1)\right) h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta - \eta_1|^{\frac{1}{2}}x) \\ &+ \sum_{(\delta_1, \delta_2) \in \{-1, 1\} \times \{+, -\}^2} c(\delta_1, \delta_2) \sqrt{AB} F_{I,m}^{\delta_1}(\eta_1) F_{J,n}^{\delta_2}(\eta - \eta_1) h_{m+\delta_1}(|\eta_1|^{\frac{1}{2}}x) h_{n+\delta_2}(|\eta - \eta_1|^{\frac{1}{2}}x). \end{aligned}$$

We denote $D_2 = D_1^2$. Then we define $C_{A,B}((0, -), (0, -)) = 1$, $C_{A,B}((0, +), (0, +)) = 0$, $C_{A,B}((0, +), (0, -)) = 2A$, $C_{A,B}((0, -), (0, +)) = 2B$ and $C_{A,B}(\delta_1, \delta_2) = c(\delta_1, \delta_2) \sqrt{AB}$ if $(\delta_1, \delta_2) \in (\{-1, 1\} \times \{+, -\})^2$. With the notation from Definition 4.4.1 (and a similar notation for the $v_{J,n}^{\delta_2}$), we conclude that

$$\begin{aligned} & \mathcal{F}((\text{Id} - \Delta_G)(u_{I,m} v_{J,n}))(x, \eta) \\ &= \sum_{(\delta_1, \delta_2) \in D_2} C_{A,B}(\delta_1, \delta_2) \int f_{I,m}^{\delta_1}(\eta_1) g_{J,n}^{\delta_2}(\eta - \eta_1) h_{m+\delta_1}(|\eta_1|^{\frac{1}{2}}x) h_{n+\delta_2}(|\eta - \eta_1|^{\frac{1}{2}}x) d\eta_1, \end{aligned}$$

so that

$$(\text{Id} - \Delta_G)(u_{I,m} v_{J,n}) = \sum_{(\delta_1, \delta_2) \in D_2} C_{A,B}(\delta_1, \delta_2) u_{I,m}^{\delta_1} v_{J,n}^{\delta_2}. \quad \square$$

Lemma 4.4.3 (Action of Δ_G for the product of three terms). *There exists a finite set $D_3 \subset D_1 \times D_1 \times D_1$ such that the following holds. Let $u^{(1)}, u^{(2)}, u^{(3)} \in L_G^2$, $A_1, A_2, A_3 \in 2^{\mathbb{N}}$, $I_1, I_2, I_3 \in 2^{\mathbb{Z}}$ and $m_1, m_2, m_3 \in \mathbb{N}$ such that $(m_i + 1)I_i \sim A_i$ for $i = 1, 2, 3$. Then*

$$(\text{Id} - \Delta_G)(u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}) = \sum_{\delta = (\delta_1, \delta_2, \delta_3) \in D_3} C_{A_1, A_2, A_3}(\delta) (u_{I_1, m_1}^{(1)})^{\delta_1} (u_{I_2, m_2}^{(2)})^{\delta_2} (u_{I_3, m_3}^{(3)})^{\delta_3},$$

where the shifted functions $(u_{I_i, m_i}^{(i)})^{\delta_i}$ are defined in Definition 4.4.1 and for all $\delta \in D_3$,

$$|C_{A_1, A_2, A_3}(\delta)| \lesssim \max\{A_1, A_2, A_3\}.$$

Proof. Taking the Fourier transform in y , we transform the product $u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}$ into a convolution product

$$\begin{aligned} \mathcal{F}_{y \rightarrow \eta}(u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)})(x, \eta) &= \int f_{I_1, m_1}^{(1)}(\eta_1) f_{I_2, m_2}^{(2)}(\eta_2) f_{I_3, m_3}^{(3)}(\eta - \eta_1 - \eta_2) \\ & h_{m_1}(|\eta_1|^{\frac{1}{2}}x) h_{m_2}(|\eta_2|^{\frac{1}{2}}x) h_{m_3}(|\eta - \eta_1 - \eta_2|^{\frac{1}{2}}x) d\eta_1 d\eta_2. \end{aligned}$$

Then, we write $h_{m_1, m_2, m_3}(\eta_1, \eta_2, \eta_3) := h_{m_1}(|\eta_1|^{\frac{1}{2}}x) h_{m_2}(|\eta_2|^{\frac{1}{2}}x) h_{m_3}(|\eta_3|^{\frac{1}{2}}x)$ and expand

$$\begin{aligned} & (\partial_{xx} - x^2|\eta_1 + \eta_2 + \eta_3|^2)(h_{m_1, m_2, m_3}(\eta_1, \eta_2, \eta_3)) \\ &= -((2m_1 + 1)|\eta_1| + (2m_2 + 1)|\eta_2| + (2m_3 + 1)|\eta_3|) h_{m_1, m_2, m_3}(\eta_1, \eta_2, \eta_3) \\ &+ x^2(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 - |\eta_1 + \eta_2 + \eta_3|^2) h_{m_1, m_2, m_3}(\eta_1, \eta_2, \eta_3) \\ &+ 2(\mathbb{I}_{1,2} + \mathbb{I}_{1,3} + \mathbb{I}_{2,3}), \end{aligned}$$

where

$$\begin{aligned} \mathbb{I}_{i,j} &= \partial_x(h_{m_i}(|\eta_i|^{\frac{1}{2}}x)) \partial_x(h_{m_j}(|\eta_j|^{\frac{1}{2}}x)) \\ &= \sqrt{|\eta_i||\eta_j|} \left(\sqrt{\frac{m_i}{2}} h_{m_i-1}(|\eta_i|^{\frac{1}{2}}x) - \sqrt{\frac{m_i+1}{2}} h_{m_i+1}(|\eta_i|^{\frac{1}{2}}x) \right) \\ &\quad \left(\sqrt{\frac{m_j}{2}} h_{m_j-1}(|\eta_j|^{\frac{1}{2}}x) - \sqrt{\frac{m_j+1}{2}} h_{m_j+1}(|\eta_j|^{\frac{1}{2}}x) \right). \end{aligned}$$

In particular, expanding $\mathbb{I}_{i,j}$, this term writes as a combination of terms

$$C_{A_1, A_2, A_3}(\delta) F_{I_1, m_1}^{\delta_1}(\eta_1) F_{I_2, m_2}^{\delta_2}(\eta_2) F_{I_3, m_3}^{\delta_3}(\eta_3) h_{m_1 + \delta_1, m_2 + \delta_2, m_3 + \delta_3}(\eta_1, \eta_2, \eta_3), \quad (4.20)$$

where $|C_{A_1, A_2, A_3}(\delta)| \lesssim \max\{A_1, A_2, A_3\}$ for $\delta = (\delta_1, \delta_2, \delta_3)$ in some finite set $D_3 \subset D_1^3$.

By comparing the signs of η_1, η_2 and η_3 , one can see that $|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 - |\eta_1 + \eta_2 + \eta_3|^2$ is a linear combination of terms $|\eta_i \eta_j|$ and $\eta_i \eta_j$ for $i \neq j$. We now use the identity

$$x h_m(x) = \sqrt{\frac{m}{2}} h_{m-1} + \sqrt{\frac{m+1}{2}} h_{m+1}$$

to remove the weight x^2 and write the term

$$x^2 (|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 - |\eta_1 + \eta_2 + \eta_3|^2) h_{m_1, m_2, m_3}(\eta_1, \eta_2, \eta_3)$$

as a linear combination of terms as (4.20) above. Finally, we conclude that

$$\begin{aligned} & \mathcal{F}((\text{Id} - \Delta_G)(u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}))(x, \eta) \\ &= \sum_{\delta \in D_3} C_{A_1, A_2, A_3}(\delta) \int (f_{I_1, m_1}^{(1)})^{\delta_1}(\eta_1) (f_{I_2, m_2}^{(2)})^{\delta_2}(\eta_2) (f_{I_3, m_3}^{(3)})^{\delta_3}(\eta - \eta_1 - \eta_2) \\ & \quad h_{m_1 + \delta_1}(|\eta_1|^{\frac{1}{2}} x) h_{m_2 + \delta_2}(|\eta_2|^{\frac{1}{2}} x) h_{m_3 + \delta_3}(|\eta - \eta_1 - \eta_2|^{\frac{1}{2}} x) d\eta_1 d\eta_2 \end{aligned}$$

for some $|C_{A_1, A_2, A_3}(\delta)| \lesssim \max\{A_1, A_2, A_3\}$, leading to the lemma. \square

4.4.2 A bound on the Sobolev norm of a high-low product

We are now able to estimate the H^ℓ norm of a product $u_A v_B$ by its L^2 norm as follows.

Corollary 4.4.4. *Let $\ell \in [0, 2]$. Then there exists $C > 0$ such that for all $u, v \in H_G^\ell$, for all $A, B \in 2^{\mathbb{N}}$, there holds*

$$\|u_A v_B\|_{H_G^\ell} \leq C \max\{A, B\}^{\frac{\ell}{2}} \sum_{(\delta_1, \delta_2) \in D_2} \|(u_A)^{\delta_1} (v_B)^{\delta_2}\|_{L_G^2}.$$

For general $\ell \geq 0$, the same result holds up to taking bigger finite sets D_2 (depending on ℓ) and applying the shift several times for each function.

Proof. We proceed by interpolation.

In the case $\ell = 0$, there is nothing to do. In the case $\ell = 2$, we use Lemma 4.4.2 to write that for any $(m+1)I \sim A$ and $(n+1)J \sim B$,

$$(\text{Id} - \Delta_G)(u_{I, m} v_{J, n}) = \sum_{(\delta_1, \delta_2) \in D_2} C_{A, B}(\delta_1, \delta_2) u_{I, m}^{\delta_1} v_{J, n}^{\delta_2},$$

and thus by summation and taking the L_G^2 norm, we obtain

$$\|(\text{Id} - \Delta_G)(u_A v_B)\|_{L_G^2}^2 \lesssim \left\| \sum_{(\delta_1, \delta_2) \in D_2} C_{A, B}(\delta_1, \delta_2) \sum_{\substack{(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m+1)I \sim A}} \sum_{\substack{(J, n) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (n+1)J \sim B}} u_{I, m}^{\delta_1} v_{J, n}^{\delta_2} \right\|_{L_G^2}^2.$$

We use the triangle inequality on the sum over (δ_1, δ_2) and the fact that $|C_{A,B}(\delta_1, \delta_2)| \lesssim \max(A, B)$ to deduce that

$$\begin{aligned} \|(\text{Id} - \Delta_G)(u_A v_B)\|_{L_G^2}^2 &\lesssim A^2 \sum_{(\delta_1, \delta_2) \in D_2} \left\| \sum_{\substack{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m+1)I \sim A}} \sum_{\substack{(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (n+1)J \sim B}} u_{I,m}^{\delta_1} v_{J,n}^{\delta_2} \right\|_{L_G^2}^2 \\ &\lesssim A^2 \sum_{(\delta_1, \delta_2) \in D_2} \|(u_A)^{\delta_1} (v_B)^{\delta_2}\|_{L_G^2}^2. \end{aligned}$$

To get the result when $\ell \in (0, 2)$, it only remains to interpolate thanks to the inequality

$$\|u_A v_B\|_{H_G^\ell} \leq \|u_A v_B\|_{L_G^2}^{1-\frac{\ell}{2}} \|u_A v_B\|_{H_G^2}^{\frac{\ell}{2}}$$

when $u_A, v_B \in H_G^2$, and conclude by density.

Similarly, for even integer ℓ , we apply Lemma 4.4.2 successively $\frac{\ell}{2}$ times to get the estimate, and conclude by interpolation for exponents between ℓ and $\ell + 2$. \square

Using Lemma 4.4.3 instead of 4.4.2, we get the following adaptation of Corollary 4.4.4 for the product of three terms.

Corollary 4.4.5. *Let $\ell \in [0, 2]$ and $u, v, w \in H_G^\ell$. Then for all $A, B, C \in 2^{\mathbb{N}}$, there holds*

$$\|u_A v_B w_C\|_{H_G^\ell} \lesssim \max\{A, B, C\}^{\frac{\ell}{2}} \sum_{(\delta_1, \delta_2, \delta_3) \in D_3} \|(u_A)^{\delta_1} (v_B)^{\delta_2} (w_C)^{\delta_3}\|_{L_G^2}.$$

For general $\ell \geq 0$, the same result holds up to taking bigger finite sets D_3 (depending on ℓ) and applying the shift several times for each function.

Proof. By interpolation over ℓ , it is sufficient to establish the following inequality for even integer ℓ : for all $B, C \leq A$,

$$\|(\text{Id} - \Delta_G)^{\ell/2} (u_A v_B w_C)\|_{L_G^2}^2 \lesssim \sum_{(\delta_1, \delta_2, \delta_3) \in D_3} A^\ell \|(u_A)^{\delta_1} (v_B)^{\delta_2} (w_C)^{\delta_3}\|_{L_G^2}^2,$$

then apply this result to $u, P_{\leq A} v$ and $P_{\leq A} w$.

Since in the case $\ell = 0$ there is nothing to do, let us assume $\ell = 2$. Using Lemma 4.4.3, we can write

$$(\text{Id} - \Delta_G)(u_{I_1, m_1} v_{I_2, m_2} w_{I_3, m_3}) = \sum_{(\delta_1, \delta_2, \delta_3) \in D_3} C_{A,B,C}(\delta_1, \delta_2, \delta_3) u_{I_1, m_1}^{\delta_1} v_{I_2, m_2}^{\delta_2} w_{I_3, m_3}^{\delta_3},$$

and therefore

$$\|(\text{Id} - \Delta_G)(u_A v_B w_C)\|_{L_G^2}^2 \lesssim \left\| \sum_{(\delta_1, \delta_2, \delta_3) \in D_3} C_{A,B,C}(\delta_1, \delta_2, \delta_3) \sum_{m_1, m_2, m_3 \in \mathbb{N}} \sum_{\substack{(m_1+1)I_1 \sim A \\ (m_2+1)I_2 \sim B \\ (m_3+1)I_3 \sim C}} u_{I_1, m_1}^{\delta_1} v_{I_2, m_2}^{\delta_2} w_{I_3, m_3}^{\delta_3} \right\|_{L_G^2}^2.$$

We use the triangle inequality on the sum over $(\delta_1, \delta_2, \delta_3)$ and the fact that $|C_{A,B,C}(\delta_1, \delta_2, \delta_3)| \lesssim \max(A, B, C) \leq A$ to deduce that

$$\|(\text{Id} - \Delta_G)(u_A v_B w_C)\|_{L_G^2}^2 \lesssim A^2 \sum_{(\delta_1, \delta_2, \delta_3) \in D_3} \|(u_A)^{\delta_1} (v_B)^{\delta_2} (w_C)^{\delta_3}\|_{L_G^2}^2.$$

In the case of a general even integer ℓ , successive applications of Lemma 4.4.3 lead to a similar result. \square

4.4.3 A refined Sobolev estimate of a product

Finally, we estimate the H^ℓ norm of a product $u_A v$ by the L^2 norm of products $u_A v_B$ and the H^ℓ norm of v as follows.

In rough terms, we should have the following. Let $\ell \in [0, 2]$, $u, v \in L_G^2$ and $A \in 2^\mathbb{N}$. Then for all $\varepsilon > 0$, we have

$$\|uv\|_{H_G^\ell}^2 \lesssim \sum_{(\delta_1, \delta_2) \in D_2} \sum_{A, B: B \leq A} A^{\ell+\varepsilon} \|(u_A)^{\delta_1} (v_B)^{\delta_2}\|_{L_G^2}^2 + \sum_{\delta \in D_1} \sum_A A^\varepsilon \|(u_A)^{\delta_1}\|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2,$$

where we recall that D_1 is some finite set and $D_2 = D_1 \times D_1$.

However, in order to get nice mapping properties in L^p spaces for $p \neq 2$ which are necessary during the course of this proof, see Appendix 4.A.3, we introduce a cutoff function $\chi \in C_c^\infty[0, 1]$ such that $\chi \equiv 1$ on $[0, \frac{1}{2}]$. Then, for $A \in 2^\mathbb{N}$, we define the projection $P_{\leq A}$ as the Fourier multiplier

$$P_{\leq A} = \chi \left(\frac{\text{Id} - \Delta_G}{A} \right),$$

which is a smooth counterpart for the projection on the Sobolev modes $1 + (2m+1)|\eta| \leq A$, acting on the decomposition (4.1) as

$$\mathcal{F}_{y \rightarrow \eta}(P_{\leq A} u)(x, \eta) = \sum_{(I, m) \in 2^{\mathbb{Z} \times \mathbb{N}}} \chi \left(\frac{1 + (2m+1)|\eta|}{A} \right) \widehat{u_{I, m}}(x, \eta).$$

Note that since on the support of $\widehat{u_{I, m}}$, we have $|\eta| \in [I, 2I]$, the sum can be restricted to the indices $1 + (2m+1)I \leq A$. The projection $P_{\leq A}$ commutes with the block decomposition (4.9) and the Grushin operator: for $B \in 2^\mathbb{N}$, we have

$$P_{\leq A}(v_B) = (P_{\leq A} v)_B$$

and for $k \in \mathbb{R}$, we have

$$P_{\leq A}((-\Delta_G)^{k/2} v) = (-\Delta_G)^{k/2} (P_{\leq A} v).$$

Moreover, for all $u \in L_G^2$, we have $\|P_{\leq A} u\|_{L_G^2} \leq \|u\|_{L_G^2}$. We also denote $P_{> A} = \text{Id} - P_{\leq A}$.

Lemma 4.4.6 (Sobolev norm for the product of two terms). *Let $\ell \in [0, 2]$, $u, v \in L_G^2$ and $A \in 2^\mathbb{N}$. Then for all $\varepsilon > 0$, we have*

$$\|u_A v\|_{H_G^\ell}^2 \lesssim \sum_{(\delta_1, \delta_2) \in D_2} \sum_{B: B \leq A} A^\ell B^\varepsilon \|(u_A)^{\delta_1} (P_{\leq A} v_B)^{\delta_2}\|_{L_G^2}^2 + \sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2,$$

where we recall that $D_1 = \{-1, 0, 1, \emptyset\}$ and $D_2 \subset D_1 \times D_1$.

As a consequence, we have

$$\|uv\|_{H_G^\ell}^2 \lesssim \sum_{(\delta_1, \delta_2) \in D_2} \sum_{A, B: B \leq A} A^{\ell+\varepsilon} \|(u_A)^{\delta_1} (P_{\leq A} v_B)^{\delta_2}\|_{L_G^2}^2 + \sum_{\delta \in D_1} \sum_A A^\varepsilon \|(u_A)^{\delta_1}\|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2.$$

For general $\ell \geq 0$, the same result holds up to taking bigger finite sets D_1, D_2 (depending on ℓ) and applying the shift several times for each function.

Proof. The consequence is a simple application of the Cauchy-Schwarz' inequality.

We decompose $v = P_{\leq A}v + P_{> A}v$ and $v = \sum_{B \in 2^{\mathbb{N}}} v_B$. Since $(P_{\leq A}v_B) = v_B$ for $4B \leq A$ and $(P_{\leq A}v_B) = 0$ for $B > A$, we have

$$\|u_A v\|_{H_G^\ell}^2 \lesssim \left\| \sum_{B \leq A} u_A(P_{\leq A}v_B) \right\|_{H_G^\ell}^2 + \left\| \sum_{4B > A} u_A(P_{> A}v_B) \right\|_{H_G^\ell}^2.$$

We treat the two parts separately, and in each case we proceed by interpolation between successive even integers ℓ .

Step 1: upper bound for $\left\| \sum_{B \leq A} u_A(P_{\leq A}v_B) \right\|_{H_G^\ell}$. Concerning the first term, thanks to the Cauchy-Schwarz inequality, we have

$$\left\| \sum_{B \leq A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A}v_B)) \right\|_{L_G^2}^2 \lesssim \left(\sum_{B \leq A} B^{-\varepsilon} \right) \left(\sum_{B \leq A} B^\varepsilon \left\| (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A}v_B)) \right\|_{L_G^2}^2 \right),$$

and as $\sum_{B \in 2^{\mathbb{N}}} B^{-\varepsilon} \lesssim 1$, we infer that

$$\left\| \sum_{B \leq A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A}v_B)) \right\|_{L_G^2}^2 \lesssim \sum_{B \leq A} B^\varepsilon \left\| (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A}v_B)) \right\|_{L_G^2}^2.$$

But using Corollary 4.4.4 to $(P_{\leq A}v)$ instead of v , we have

$$\left\| (\text{Id} - \Delta_G)^{\ell/2} (u_A v_B) \right\|_{L_G^2}^2 \lesssim A^\ell \sum_{(\delta_1, \delta_2) \in D_2} \left\| (u_A)^{\delta_1} (P_{\leq A}v_B)^{\delta_2} \right\|_{L_G^2}^2.$$

Step 2: upper bound for $\left\| \sum_{4B > A} u_A(P_{> A}v_B) \right\|_{H_G^\ell}$. For the second term, we establish the following estimate by interpolation:

$$\left\| \sum_{4B > A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{> A}v_B)) \right\|_{L_G^2}^2 \lesssim \sum_{\delta \in D_1} \left\| (u_A)^{\delta_1} \right\|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2. \quad (4.21)$$

In the case $\ell = 0$, we only need to note that from the orthogonality of the different modes v_B , we have

$$\left\| u_A \sum_{4B > A} (P_{> A}v_B) \right\|_{L_G^2}^2 \lesssim \|u_A\|_{L_G^\infty}^2 \left\| \sum_{4B > A} (P_{> A}v_B) \right\|_{L_G^2}^2 \lesssim \|u_A\|_{L_G^\infty}^2 \|v\|_{L_G^2}^2.$$

In the case $\ell = 2$, we use Lemma 4.4.2 to get that for any $m, n \in \mathbb{N}$ and $I, J \in 2^{\mathbb{Z}}$ such that $(m+1)I \sim A$ and $(n+1)J \sim B$,

$$(\text{Id} - \Delta_G)(u_{I,m}(P_{> A}v)_{J,n}) = \sum_{(\delta_1, \delta_2) \in D_2} C_{A,B}(\delta_1, \delta_2) u_{I,m}^{\delta_1} (P_{> A}v)_{J,n}^{\delta_2},$$

and therefore obtain

$$\left\| \sum_{4B > A} (\text{Id} - \Delta_G) u_A(P_{> A}v_B) \right\|_{L_G^2}^2 \lesssim \left\| \sum_{(\delta_1, \delta_2) \in D_2} \sum_{4B > A} C_{A,B}(\delta_1, \delta_2) \sum_{m, n \in \mathbb{N}} \sum_{\substack{(m+1)I \sim A \\ (n+1)J \sim B}} u_{I,m}^{\delta_1} (P_{> A}v)_{J,n}^{\delta_2} \right\|_{L_G^2}^2.$$

We use the triangle inequality on the sum over (δ_1, δ_2) to get that for fixed A ,

$$\begin{aligned} \left\| \sum_{4B>A} (\text{Id} - \Delta_G) u_A(P_{>A} v_B) \right\|_{L_G^2}^2 &\lesssim \sum_{(\delta_1, \delta_2) \in D_2} \left\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) \sum_{m,n \in \mathbb{N}} \sum_{\substack{(m+1)I \sim A \\ (n+1)J \sim B}} u_{I,m}^{\delta_1}(P_{>A} v)^{\delta_2}_{J,n} \right\|_{L_G^2}^2 \\ &= \sum_{(\delta_1, \delta_2) \in D_2} \left\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) u_A^{\delta_1}(P_{>A} v_B)^{\delta_2} \right\|_{L_G^2}^2 \\ &\lesssim \sum_{(\delta_1, \delta_2) \in D_2} \|u_A^{\delta_1}\|_{L_G^\infty}^2 \left\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) (P_{>A} v_B)^{\delta_2} \right\|_{L_G^2}^2. \end{aligned}$$

Now, by orthogonality and the fact that $|C_{A,B}(\delta_1, \delta_2)| \lesssim B$ for $4B > A$, we have

$$\begin{aligned} \left\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) (P_{>A} v_B)^{\delta_2} \right\|_{L_G^2}^2 &= \sum_{4B>A} |C_{A,B}(\delta_1, \delta_2)|^2 \| (P_{>A} v_B)^{\delta_2} \|_{L_G^2}^2 \\ &\lesssim \sum_{4B>A} B^2 \| (P_{>A} v_B) \|_{L_G^2}^2 \lesssim \|v\|_{H_G^2}^2, \end{aligned}$$

which concludes the proof in the case $\ell = 2$.

Estimate (4.21) is now a consequence of an interpolation result, stated and proven in Lemma 4.A.6, based on the Stein's interpolation theorem. Instead of using this lemma, one could invoke the interpolation result between Sobolev spaces $[H^0, H^\ell]_\theta = H^{\theta\ell}$ for $\theta \in [0, 1]$, that holds for Sobolev spaces on \mathbb{R}^d (see for instance [BL76]) and could be extended to the setting of the Grushin operator. For general ℓ , successive uses of Lemma 4.4.2 lead to the result. \square

In what follows, we will actually use the trilinear version of Lemma 4.4.6.

Corollary 4.4.7 (Sobolev norm for the product of three terms). *Let $\ell \in [0, 2]$ and $u, v, w \in L_G^2$. Then for all $\varepsilon > 0$ there holds*

$$\|uvw\|_{H_G^\ell}^2 \lesssim \sum_{\substack{(\delta_1, \delta_2, \delta_3) \in D_3 \\ A, B, C: B, C \leq A}} A^{\ell+\varepsilon} \| (u_A)^{\delta_1} (P_{\leq A} v_B)^{\delta_2} (P_{\leq A} w_C)^{\delta_3} \|_{L_G^2}^2 + \sum_{\delta_1 \in D_1} \sum_A A^\varepsilon \| (u_A)^{\delta_1} \|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2 \|w\|_{H_G^\ell}^2.$$

For $\ell > 2$, a similar result holds up to taking bigger finite sets D_3 and taking successive shifted functions.

Proof. We mimic the proof of Lemma 4.4.6. It is enough to establish that for every A , we have

$$\|u_A v w\|_{H_G^\ell}^2 \lesssim \sum_{\substack{(\delta_1, \delta_2, \delta_3) \in D_3 \\ B, C: B, C \leq A}} A^\ell (BC)^\varepsilon \| (u_A)^{\delta_1} (P_{\leq A} v_B)^{\delta_2} (P_{\leq A} w_C)^{\delta_3} \|_{L_G^2}^2 + \sum_{\delta_1 \in D_1} \| (u_A)^{\delta_1} \|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2 \|w\|_{H_G^\ell}^2.$$

We start by writing that

$$\|u_A v w\|_{H_G^\ell}^2 \lesssim \left\| \sum_{B, C \leq A} u_A (P_{\leq A} v_B) (P_{\leq A} w_C) \right\|_{H_G^\ell}^2 + \left\| \sum_{B, C} u_A (v_B w_C - (P_{\leq A} v_B) (P_{\leq A} w_C)) \right\|_{H_G^\ell}^2.$$

Step 1: upper bound for $\left\| \sum_{B,C \leq A} u_A(P_{\leq A} v_B)(P_{\leq A} w_C) \right\|_{H_G^\ell}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left\| \sum_{B,C \leq A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A} v_B)(P_{\leq A} w_C)) \right\|_{L_G^2}^2 \\ & \lesssim \left(\sum_{B,C \leq A} (BC)^{-\varepsilon} \right) \left(\sum_{B,C \leq A} (BC)^\varepsilon \left\| (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A} v_B)(P_{\leq A} w_C)) \right\|_{L_G^2}^2 \right) \\ & \lesssim \sum_{B,C \leq A} (BC)^\varepsilon \left\| (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A} v_B)(P_{\leq A} w_C)) \right\|_{L_G^2}^2. \end{aligned}$$

We then conclude thanks to Corollary 4.4.5 that

$$\left\| \sum_{B,C \leq A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{\leq A} v_B)(P_{\leq A} w_C)) \right\|_{L_G^2}^2 \lesssim \sum_{\substack{(\delta_1, \delta_2, \delta_3) \in D_3 \\ B, C: B, C \leq A}} A^\ell (BC)^\varepsilon \left\| (u_A)^{\delta_1} (P_{\leq A} v_B)^{\delta_2} (P_{\leq A} w_C)^{\delta_3} \right\|_{L_G^2}^2.$$

Step 2: upper bound for $\left\| \sum_{B,C} u_A(v_B w_C - (P_{\leq A} v_B)(P_{\leq A} w_C)) \right\|_{H_G^\ell}$. We split

$$v_B w_C - (P_{\leq A} v_B)(P_{\leq A} w_C) = (P_{> A} v_B) w_C + (P_{\leq A} v_B)(P_{> A} w_C).$$

For the first term, using the algebra property of H_G^ℓ for $\ell > \frac{3}{2}$ we write

$$\begin{aligned} \left\| \sum_{B,C} u_A(P_{> A} v_B) w_C \right\|_{H_G^\ell}^2 & \lesssim \left\| \sum_B u_A(P_{> A} v_B) \right\|_{H_G^\ell}^2 \left\| \sum_{C: C \leq A} w_C \right\|_{H_G^\ell}^2 \\ & \lesssim \left\| \sum_{B: 4B > A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{> A} v_B)) \right\|_{L_G^2}^2 \|w\|_{H_G^\ell}^2, \end{aligned}$$

hence inequality (4.21) in the Step 2 of Lemma 4.4.6 applies and gives the expected bound. A similar treatment can be applied to the second term. \square

4.5 Deterministic bilinear estimates

In this section, we first establish deterministic bilinear and trilinear estimates for the product of two Hermite functions with rescaling, of the form $h_m(\alpha_1 \cdot) h_n(\alpha_2 \cdot)$ and $h_{m_1}(\alpha_1 \cdot) h_{m_2}(\alpha_2 \cdot) h_{m_3}(\alpha_3 \cdot)$. We then deduce deterministic bilinear estimates for the products $u_{I,m} v_{J,n}$.

4.5.1 Bilinear estimates for rescaled Hermite functions

We start by establishing bilinear estimates for the rescaled Hermite functions, which will be at the core of our future bilinear and trilinear smoothing estimates. Let us recall that $\lambda_m = \sqrt{2m+1}$.

Lemma 4.5.1. *Let $\alpha > 0$, $m, n \in \mathbb{N}$, and assume that $\lambda_n \leq \frac{\alpha}{4} \lambda_m$. Then there holds*

$$\|h_m h_n(\alpha \cdot)\|_{L^2}^2 \lesssim \frac{1}{\alpha \lambda_m}.$$

Proof. We cut the Hermite functions between the space regions delimited by the pointwise estimates of Corollary 4.2.8.

(1) In the region $R_1 := \{|x| \leq \frac{1}{2}\lambda_m\}$, we use the bound $|h_m(x)| \lesssim \lambda_m^{-\frac{1}{2}}$. Therefore there holds

$$\|h_m h_n(\alpha \cdot)\|_{L^2(R_1)}^2 \lesssim \lambda_m^{-1} \|h_n(\alpha \cdot)\|_{L^2(R_1)}^2 \lesssim \lambda_m^{-1} \alpha^{-1}.$$

(2) In the region $R_2 := \{\frac{1}{2}\lambda_m \leq |x| \leq 2\lambda_m\}$, we know by assumption that $2\lambda_n \leq \frac{\alpha}{2}\lambda_m \leq \alpha|x|$. We therefore use the bounds $|h_m(x)| \lesssim \left(\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|\right)^{-\frac{1}{4}}$ and $|h_n(\alpha x)| \lesssim e^{-c(\alpha x)^2}$ with $c = \frac{1}{8}$, and get

$$\|h_m h_n(\alpha \cdot)\|_{L^2(R_2)}^2 \lesssim \int_{\frac{1}{2}\lambda_m}^{2\lambda_m} \frac{e^{-2c\alpha^2 x^2}}{\sqrt{\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|}} dx.$$

We fix the limits of the integral by setting $y = \frac{x}{\lambda_m}$, which leads to the bound:

$$\begin{aligned} \int_{\frac{1}{2}\lambda_m}^{2\lambda_m} \frac{e^{-2c\alpha^2 x^2}}{\sqrt{\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|}} dx &= \int_{\frac{1}{2}}^2 \frac{e^{-2c(\alpha\lambda_m)^2 y^2}}{\sqrt{\lambda_m^{-\frac{4}{3}} + |y^2 - 1|}} dy \\ &\lesssim \int_{\frac{1}{2}}^2 \frac{e^{-2c(\alpha\lambda_m)^2 y^2}}{\sqrt{|y^2 - 1|}} dy \\ &\lesssim e^{-\frac{c}{2}(\alpha\lambda_m)^2} \int_{\frac{1}{2}}^2 \frac{dy}{\sqrt{|y^2 - 1|}}. \end{aligned}$$

Now we remark that $e^{-\frac{c}{2}(\alpha\lambda_m)^2} \lesssim \frac{1}{\alpha\lambda_m}$, giving the required estimate.

(3) In the last region $R_3 := \{|x| \geq 2\lambda_m\}$ we use the exponential bounds for the two terms $|h_m(x)| \leq e^{-cx^2}$ and $|h_n(\alpha x)| \leq e^{-c(\alpha x)^2}$ with $c = \frac{1}{8}$. This leads to

$$\|h_m h_n(\alpha \cdot)\|_{L^2(R_3)}^2 \lesssim \int_{|x| \geq 2\lambda_m} e^{-2cx^2(1+\alpha^2)} dx.$$

Then using that for $C = 2c(1 + \alpha^2)$ and $X = 2\lambda_m$, we have

$$\int_X^\infty e^{-Cx^2} dx \leq \frac{1}{2CX} \int_X^\infty 2Cxe^{-Cx^2} dx \leq \frac{1}{2CX},$$

this gives the result.

All the previous bounds put together imply the lemma. \square

In what follows, we will use the bilinear estimate from Lemma 4.5.1 under the following form.

Corollary 4.5.2 (Bilinear estimates for rescaled Hermite functions). *Let $m, n \in \mathbb{N}$ and $\alpha_1, \alpha_2 > 0$. Then*

$$\|h_m(\alpha_1 \cdot) h_n(\alpha_2 \cdot)\|_{L^2}^2 \lesssim \min \left\{ \frac{1}{\alpha_1^2(2n+1)}, \frac{1}{\alpha_2^2(2m+1)} \right\}^{\frac{1}{2}}.$$

Proof. In the first scenario, assume that $\alpha_1\sqrt{2n+1} \leq \frac{1}{4}\alpha_2\sqrt{2m+1}$. We start with a change of variable:

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 = \frac{1}{\alpha_1} \left\| h_m h_n \left(\frac{\alpha_2\cdot}{\alpha_1} \right) \right\|_{L^2}^2.$$

Denote $\alpha := \frac{\alpha_2}{\alpha_1}$. Since $\alpha_1\sqrt{2n+1} \leq \frac{1}{4}\alpha_2\sqrt{2m+1}$, we have $\lambda_n \leq \frac{\alpha}{4}\lambda_m$. Then Lemma 4.5.1 implies

$$\left\| h_m h_n \left(\frac{\alpha_2\cdot}{\alpha_1} \right) \right\|_{L^2}^2 \lesssim \frac{1}{\alpha\sqrt{2m+1}} = \sqrt{\frac{\alpha_1^2}{\alpha_2^2(2m+1)}}.$$

Consequently,

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \sqrt{\frac{1}{\alpha_2^2(2m+1)}}.$$

This is enough to conclude since by assumption, we have $\sqrt{\frac{1}{\alpha_2^2(2m+1)}} \leq \sqrt{\frac{1}{\alpha_1^2(2n+1)}}$.

In the second scenario, assume that $\alpha_2\sqrt{2m+1} \leq \frac{1}{4}\alpha_1\sqrt{2n+1}$. We exchange the roles of m and n and the roles of α_1 and α_2 to get

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \sqrt{\frac{1}{\alpha_1^2(2n+1)}}.$$

In the last scenario, there exists $C_0 > 0$ such that $\frac{1}{4}\alpha_2\sqrt{2m+1} \leq \alpha_1\sqrt{2n+1} \leq 4\alpha_2\sqrt{2m+1}$. Then from Hölder's inequality, one obtains the bound

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \|h_m(\alpha_1\cdot)\|_{L^4}^2 \|h_n(\alpha_2\cdot)\|_{L^4}^2.$$

From the L^4 norm estimate $\|h_m\|_{L^4} \leq \frac{C}{(2m+1)^{\frac{1}{8}}}$, $m \in \mathbb{N}$ (see Lemma 4.2.7), we deduce

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \frac{1}{(\alpha_1^2(2m+1))^{\frac{1}{4}}(\alpha_2^2(2n+1))^{\frac{1}{4}}}.$$

Since $\alpha_1\sqrt{2n+1}$ and $\alpha_2\sqrt{2m+1}$ differ from a factor at most 4, in this situation, one can bound the right-hand side by either $\sqrt{\frac{1}{\alpha_2^2(2m+1)}}$ or $\sqrt{\frac{1}{\alpha_1^2(2n+1)}}$. \square

Corollary 4.5.3 (Trilinear estimates for rescaled Hermite functions). *Let $m_1, m_2, m_3 \in \mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_3 > 0$. Then*

$$\|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim \frac{1}{\alpha_1\alpha_2\alpha_3} \min_{i \neq j \in \{1,2,3\}} \left\{ \frac{\alpha_i\alpha_j}{(2m_i+1)^{\frac{1}{8}}(2m_j+1)^{\frac{1}{2}}} \right\}.$$

As a consequence, if $A \in 2^{\mathbb{N}}$, $I_1, I_2, I_3 \in 2^{\mathbb{Z}}$ are such that $(m_1+1)I_1 \sim A$, for $\alpha_1^2 \in [I_1, 2I_1]$, $\alpha_2^2 \in [I_2, 2I_2]$ and $\alpha_3^2 \in [I_3, 2I_3]$, we have

$$\alpha_1\alpha_2\alpha_3 \|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim C_{\{I_i, m_i\}}^2,$$

where one can choose

$$C_{\{I_i, m_i\}}^2 = \frac{\langle I_1 \rangle (I_2 I_3)^{\frac{1}{4}}}{A^{\frac{1}{2}} (2m_2+1)^{\frac{1}{12}} (2m_3+1)^{\frac{1}{12}}}.$$

Note that the gain with power $\frac{1}{12}$ induced by the L^∞ norm on the Hermite functions is not necessary in the sequel, but we keep this exponent for the sake of security.

Proof. We start by breaking the symmetry of roles of the three terms

$$\|h_{m_1}(\alpha_1 \cdot) h_{m_2}(\alpha_2 \cdot) h_{m_3}(\alpha_3 \cdot)\|_{L^2}^2 \leq \|h_{m_1}(\alpha_1 \cdot)\|_{L^\infty}^2 \|h_{m_2}(\alpha_2 \cdot) h_{m_3}(\alpha_3 \cdot)\|_{L^2}^2.$$

Then we use Lemma 4.2.7 and Corollary 4.5.2 to bound

$$\|h_{m_1}(\alpha_1 \cdot) h_{m_2}(\alpha_2 \cdot) h_{m_3}(\alpha_3 \cdot)\|_{L^2}^2 \lesssim \frac{1}{(2m_1 + 1)^{\frac{1}{6}}} \min \left\{ \frac{1}{\alpha_2 (2m_3 + 1)^{\frac{1}{2}}}, \frac{1}{\alpha_3 (2m_2 + 1)^{\frac{1}{2}}} \right\}.$$

By symmetry between the roles of m_1, m_2, m_3 , this implies

$$\|h_{m_1}(\alpha_1 \cdot) h_{m_2}(\alpha_2 \cdot) h_{m_3}(\alpha_3 \cdot)\|_{L^2}^2 \lesssim \frac{1}{\alpha_1 \alpha_2 \alpha_3} \min \left\{ \frac{\alpha_i \alpha_j}{(2m_i + 1)^{\frac{1}{6}} (2m_j + 1)^{\frac{1}{2}}} \right\}_{i \neq j \in \{1,2,3\}}.$$

In order to establish the second bound, for every $j = 1, 2, 3$, we have $\alpha_j \leq (2I_j)^{\frac{1}{2}}$, therefore

$$\min \left\{ \frac{\alpha_i \alpha_j}{(2m_i + 1)^{\frac{1}{6}} (2m_j + 1)^{\frac{1}{2}}} \right\}_{i \neq j \in \{1,2,3\}} \lesssim \min \left\{ \frac{I_i^{\frac{1}{2}} I_j^{\frac{1}{2}}}{(2m_i + 1)^{\frac{1}{6}} (2m_j + 1)^{\frac{1}{2}}} \right\}_{i \neq j \in \{1,2,3\}}.$$

Now, let us remark that

$$\min \left\{ \frac{\alpha_i \alpha_j}{(2m_i + 1)^{\frac{1}{6}} (2m_j + 1)^{\frac{1}{2}}} \right\}_{i \neq j \in \{1,2,3\}} \leq \frac{I_2^{\frac{1}{2}} I_1^{\frac{1}{2}}}{(2m_2 + 1)^{\frac{1}{6}} (2m_1 + 1)^{\frac{1}{2}}},$$

and since the same bound holds when exchanging the roles of (I_2, m_2) and (I_3, m_3) , we obtain by interpolation

$$\min \left\{ \frac{\alpha_i \alpha_j}{(2m_i + 1)^{\frac{1}{6}} (2m_j + 1)^{\frac{1}{2}}} \right\}_{i \neq j \in \{1,2,3\}} \leq \frac{(I_2 I_3)^{\frac{1}{4}} \langle I_1 \rangle}{((2m_2 + 1)(2m_3 + 1))^{\frac{1}{12}} A^{\frac{1}{2}}}. \quad \square$$

4.5.2 Bilinear and trilinear estimates for the product of unimodal blocks

In this part, we provide a bilinear estimate for the “building blocks” $u_{I,m} v_{J,n}$, where $u_{I,m}$ and $v_{J,n}$ are frequency localized and unimodal.

Proposition 4.5.4 (Bilinear block estimate). *Let $m, n \geq 0$ and $I, J \in 2^{\mathbb{Z}}$. Let $u_{I,m}$ (resp. $v_{J,n}$) in L_G^2 defined by (4.8)*

$$\mathcal{F}_{y \rightarrow \eta}(u_{I,m})(x, \eta) = f_{I,m}(\eta) h_m(|\eta|^{\frac{1}{2}} x)$$

resp.

$$\mathcal{F}_{y \rightarrow \eta}(v_{J,n})(x, \eta) = g_{J,n}(\eta) h_n(|\eta|^{\frac{1}{2}} x),$$

the function $f_{I,m}$ (resp. $g_{J,n}$) being supported on the set $\{|\eta| \in [I, 2I]\}$ (resp. on the set $\{|\eta| \in [J, 2J]\}$). Then, one has

$$\|u_{I,m}v_{J,n}\|_{L_G^2}^2 \lesssim \min\{I, J\} \min\left\{\frac{I}{2m+1}, \frac{J}{2n+1}\right\}^{\frac{1}{2}} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2.$$

As a consequence,

$$\|u_{I,m}v_{J,n}\|_{L_G^2}^2 \lesssim \min\left\{\frac{J\langle I \rangle}{(1+(2m+1)I)^{\frac{1}{2}}}, \frac{I\langle J \rangle}{(1+(2n+1)J)^{\frac{1}{2}}}\right\} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2.$$

Proof. From the Parseval formula, we have $\|u_{I,m}v_{J,n}\|_{L_G^2}^2 = \|\hat{u}_{I,m} * \hat{v}_{J,n}\|_{L_{x,\eta}^2}^2$, where the convolution is the classical convolution product in the variable η . We expand the norm of this convolution product and obtain

$$\|u_{I,m}v_{J,n}\|_{L_G^2}^2 = \int_{\mathbb{R}^3} f_{I,m}(\eta_1)f_{I,m}(\eta_2)g_{J,n}(\eta - \eta_1)g_{J,n}(\eta - \eta_2)\mathbb{I}_{m,n}(\eta_1, \eta_2, \eta - \eta_1, \eta - \eta_2) d\eta_1 d\eta_2 d\eta,$$

with

$$\mathbb{I}_{m,n}(\eta_1, \eta_2, \eta'_1, \eta'_2) := \int_{\mathbb{R}} h_m(|\eta_1|^{\frac{1}{2}}x)h_m(|\eta_2|^{\frac{1}{2}}x)h_n(|\eta'_1|^{\frac{1}{2}}x)h_n(|\eta'_2|^{\frac{1}{2}}x) dx.$$

By Cauchy-Schwarz' inequality, one has

$$|\mathbb{I}_{m,n}(\eta_1, \eta_2, \eta'_1, \eta'_2)| \leq \|h_m(|\eta_1|^{\frac{1}{2}}\cdot)h_n(|\eta'_1|^{\frac{1}{2}}\cdot)\|_{L^2} \|h_m(|\eta_2|^{\frac{1}{2}}\cdot)h_n(|\eta'_2|^{\frac{1}{2}}\cdot)\|_{L^2}.$$

Using Corollary 4.5.2, we deduce the estimate

$$|\mathbb{I}_{m,n}(\eta_1, \eta_2, \eta'_1, \eta'_2)| \lesssim \min\left\{\frac{1}{|\eta_1|(2n+1)}, \frac{1}{|\eta'_1|(2m+1)}\right\}^{\frac{1}{4}} \min\left\{\frac{1}{|\eta_2|(2n+1)}, \frac{1}{|\eta'_2|(2m+1)}\right\}^{\frac{1}{4}}.$$

Going back to the blocks $u_{I,m}$ and $v_{J,n}$, this estimate implies

$$\|u_{I,m}v_{J,n}\|_{L_G^2}^2 \lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \min\left\{\frac{1}{|\eta_1|(2n+1)}, \frac{1}{|\eta - \eta_1|(2m+1)}\right\}^{\frac{1}{4}} |f_{I,m}(\eta_1)g_{J,n}(\eta - \eta_1)| d\eta_1 \right)^2 d\eta.$$

By extracting the minimum, we conclude

$$\|u_{I,m}v_{J,n}\|_{L_G^2}^2 \lesssim \min\left\{\frac{1}{\sqrt{2n+1}} \left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\right\|_{L^2}^2, \frac{1}{\sqrt{2m+1}} \left\| |f_{I,m}| * \frac{|g_{J,n}(\cdot)|}{|\cdot|^{1/4}} \right\|_{L^2}^2\right\}.$$

Using the symmetry of the roles of u and v , we estimate for instance $\left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\right\|_{L^2}^2$. We apply Young's inequality

$$\left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\right\|_{L^2}^2 \leq \left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} \right\|_{L^1}^2 \|g_{J,n}\|_{L^2}^2.$$

Thanks to Cauchy-Schwarz' inequality, since the interval $[I, 2I]$ has length I , we get

$$\left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} \right\|_{L^1}^2 \lesssim I \left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} \right\|_{L^2}^2.$$

Moreover, we know that $\|g_{J,n}\|_{L^2}^2 \lesssim \sqrt{J} \|v_{J,n}\|_{L^2}^2$, therefore

$$\left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}| \right\|_{L^2}^2 \leq I \sqrt{J} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2.$$

But we also have from Young's inequality

$$\left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}| \right\|_{L^2}^2 \leq \left\| \frac{f_{I,m}(\cdot)}{|\cdot|^{1/4}} \right\|_{L^2}^2 \|g_{J,n}\|_{L^1}^2,$$

where from Cauchy-Schwarz' inequality, $\|g_{J,n}\|_{L^1}^2 \leq J \|g_{J,n}\|_{L^2}^2 \lesssim J^{\frac{3}{2}} \|v_{J,n}\|_{L_G^2}^2$, so that actually

$$\left\| \frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}| \right\|_{L^2}^2 \leq \min\{I, J\} \sqrt{J} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2.$$

To conclude, using the symmetry of the roles of u and v , we have proven that

$$\|u_{I,m} v_{J,n}\|_{L_G^2}^2 \lesssim \min\{I, J\} \min\left\{ \frac{I}{2m+1}, \frac{J}{2n+1} \right\}^{\frac{1}{2}} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2. \quad \square$$

4.6 Probabilistic bilinear and trilinear estimates

In this section we provide the proof of Theorem B (i). The main idea of proof is to use the deterministic bilinear estimate for the products $u_{I,m} v_{J,n}$ given by Proposition 4.5.4 in order to prove a smoothing effect on quadratic and cubic expressions of a random function.

4.6.1 Bilinear estimate for random interactions

To deal with purely random interactions of the form $(u_0^\omega)^2$ and $|u_0^\omega|^2$, we heavily rely on the frequency decoupling offered by the randomization. This is akin to the bilinear estimates obtained in [BTT13].

Let us recall some notation. We fix a sequence of independent identically distributed subgaussian random variables $(X_{I,m})_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}}$ and $u_0 \in H_G^k$. We use decomposition (4.2)

$$u_0^\omega = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} X_{I,m}(\omega) u_{I,m}.$$

For all $t \in \mathbb{R}$, we consider the time evolution under the linear flow (LS-G) with initial data $u_{I,m}$ denoted $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$, $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$. Then, we define the random counterpart z^ω of u as in (4.11)

$$z^\omega(t) = e^{it\Delta_G} u_0^\omega = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} X_{I,m}(\omega) z_{I,m}(t).$$

Proof of (4.4) in Theorem B (i). We establish estimate (4.4), that is, we bound the norms of the products $\|(z^\omega)^2\|_{L_T^q H_G^\ell}$ and $\| |z^\omega|^2 \|_{L_T^q H_G^\ell}$ for $\ell = k + \frac{3}{2}$.

Step 1: reduction of inequality (4.4) to deterministic estimates. We start with $\|(z^\omega)^2\|_{L_T^q H_G^\ell}$. Fix $t \in [0, T]$ and $(x, y) \in \mathbb{R}^2$. Applying Corollary 4.2.15 (i) with the norm $L_T^q L_{x,y}^2$ and

$\Psi_{(I,m),(J,n)} = (\text{Id} - \Delta_G)^{\ell/2}(z_{I,m}z_{J,n})$, we obtain that outside a set of probability at most e^{-cR^2} there holds

$$\|(z^\omega)^2\|_{L_T^q H_G^\ell}^2 \leq R^4 \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \sum_{(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|z_{I,m}z_{J,n}\|_{L_T^q H_G^\ell}^2.$$

It remains to prove that for every $u_0, v_0 \in \mathcal{X}_1^k$, denoting $z_{I,m}(t) = e^{it\Delta_G}u_{I,m}$ and $\tilde{z}_{J,n}(t) = e^{it\Delta_G}v_{J,n}$ for $t \in \mathbb{R}$, we have

$$\sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \sum_{(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|z_{I,m}\tilde{z}_{J,n}\|_{L_T^q H_G^\ell}^2 \lesssim T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_1^k}^2 \|v_0\|_{\mathcal{X}_1^k}^2. \quad (4.22)$$

Indeed we will conclude by taking $u_0 = v_0$.

Similarly for $\|z^\omega\|_{L_T^q H_G^\ell}^2$, applying Corollary 4.2.15 (i) with the norm $L_T^q L_{x,y}^2$ and $\Psi_{(I,m),(J,n)} = (\text{Id} - \Delta_G)^{\ell/2}(z_{I,m}\overline{z_{J,n}})$, we obtain that outside a set of probability at most e^{-cR^2} there holds

$$\|z^\omega\|_{L_T^q H_G^\ell}^2 \leq R^4 \sum_{(I,m),(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|z_{I,m}\overline{z_{J,n}}\|_{L_T^q H_G^\ell}^2 + R^4 \left(\sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|z_{I,m}\|_{L_T^q H_G^\ell}^2 \right)^2.$$

The upper bound is handled using inequality (4.22) and establishing

$$\sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|z_{I,m}\tilde{z}_{I,m}\|_{L_T^q H_G^\ell} \lesssim T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k} \|v_0\|_{\mathcal{X}_1^k}, \quad (4.23)$$

which we apply to u and $v = \bar{u}$.

Step 2: Proof of (4.22). We claim that (4.22) is a consequence of the time independent inequality

$$\sum_{(I,m),(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I,m}v_{J,n}\|_{H_G^\ell}^2 \lesssim \|u_0\|_{\mathcal{X}_1^k}^2 \|v_0\|_{\mathcal{X}_1^k}^2. \quad (4.24)$$

Indeed we apply (4.24) to $z_{I,m}(t) = e^{it\Delta_G}u_{I,m}$ and $\tilde{z}_{J,n}(t) = e^{it\Delta_G}v_{J,n}$ for $t \in \mathbb{R}$ instead of $u_{I,m}$ and $v_{J,n}$, then we use \mathcal{X}_1^k isometry property of the linear flow, where we recall that we have defined the norm (4.3)

$$\|u_0\|_{\mathcal{X}_1^k}^2 = \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} (1 + (2m+1)I)^k \langle I \rangle \|u_{I,m}\|_{L^2}^2.$$

Finally we integrate in time using the Hölder inequality, yielding to (4.22).

In order to prove (4.24), fix $I, J \in 2^{\mathbb{Z}}$ and $m, n \in \mathbb{N}$. Let $A, B \in 2^{\mathbb{N}}$ such that $(m+1)I \sim A$ and $(n+1)J \sim B$. We apply Corollary 4.4.4 and get

$$\|u_{I,m}v_{J,n}\|_{H_G^\ell}^2 \lesssim \max\{A, B\}^\ell \|u_{I,m}v_{J,n}\|_{L_G^2}^2.$$

Then, from the unit block bound of Proposition 4.5.4, it follows that

$$\|u_{I,m}v_{J,n}\|_{H_G^\ell}^2 \lesssim \max\{A, B\}^\ell \min\left\{\frac{J\langle I \rangle}{A^{\frac{1}{2}}}, \frac{I\langle J \rangle}{B^{\frac{1}{2}}}\right\} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2. \quad (4.25)$$

Separating the cases $A \leq B$ and $A > B$, we obtain the bounds

$$\begin{aligned} \sum_{(I,m),(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I,m} v_{J,n}\|_{H_G^\ell}^2 &\lesssim \sum_{\substack{(I,m),(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ A \leq B}} B^{\ell - \frac{1}{2}} I \langle J \rangle \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2 \\ &+ \sum_{\substack{(I,m),(J,n) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ A > B}} A^{\ell - \frac{1}{2}} J \langle I \rangle \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2, \end{aligned}$$

implying (4.24) since $\ell - \frac{1}{2} = k$.

Step 3: Proof of (4.23). We claim that (4.23) is a consequence of the time independent inequality

$$\sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I,m} v_{I,m}\|_{H_G^\ell} \lesssim \|u_0\|_{\mathcal{X}_1^k} \|v_0\|_{\mathcal{X}_1^k}. \quad (4.26)$$

Indeed we apply (4.26) to $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$ and $\tilde{z}_{I,m}(t) = e^{it\Delta_G} v_{I,m}$ for $t \in \mathbb{R}$ instead of $u_{I,m}$, then we use \mathcal{X}_1^k isometry property of the linear flow and integrate in time using the Hölder inequality. This yields (4.23).

In order to prove (4.26), for $I \in 2^{\mathbb{Z}}$ and $m \in \mathbb{N}$, we denote $A \in 2^{\mathbb{N}}$ such that $(m+1)I \sim A$. From (4.25) applied to $I = J$, $m = n$ and $A = B$, we therefore write

$$\sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I,m} v_{I,m}\|_{H_G^\ell} \lesssim \sum_{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}} A^{\frac{\ell}{2} - \frac{1}{4}} \langle I \rangle \|u_{I,m}\|_{L_G^2} \|v_{I,m}\|_{L_G^2}.$$

An application of Cauchy-Schwarz' inequality implies (4.26) since $\ell - \frac{1}{2} = k$. \square

4.6.2 Trilinear estimate for random interactions

Proof of (4.5) in Theorem B (i). We now estimate $\| |z^\omega|^2 z^\omega \|_{L_T^q H_G^\ell}$.

Step 1: reduction of inequality (4.5) to deterministic estimates. Using Corollary 4.2.15 (ii), we know that outside a set of probability at most e^{-cR^2} , there holds

$$\begin{aligned} \| |z^\omega|^2 z^\omega \|_{L_T^q H_G^\ell}^2 &\leq R^6 \sum_{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|z_{I_1, m_1} z_{I_2, m_2} \overline{z_{I_3, m_3}}\|_{L_T^q H_G^\ell}^2 \\ &+ R^6 \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \| |z_{I_1, m_1}|^2 z_{I_2, m_2} \|_{L_T^q H_G^\ell} \right)^2. \quad (4.27) \end{aligned}$$

As in the two subsections above, using Hölder's inequality in time, inequality (4.5) is now a consequence of the time independent inequalities

$$\sum_{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I_1, m_1} u_{I_2, m_2} \overline{u_{I_3, m_3}}\|_{H_G^\ell}^2 \lesssim \|u_0\|_{\mathcal{X}_1^k}^6, \quad (4.28)$$

$$\sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \| |u_{I_1, m_1}|^2 u_{I_2, m_2} \|_{H_G^\ell} \right)^2 \lesssim \|u_0\|_{\mathcal{X}_1^k}^6. \quad (4.29)$$

Step 2: Proof of (4.28). We rather prove inequality

$$\sum_{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \lesssim \|u^{(1)}\|_{\mathcal{X}_1^k}^2 \|u^{(2)}\|_{\mathcal{X}_1^k}^2 \|u^{(3)}\|_{\mathcal{X}_1^k}^2, \quad (4.30)$$

for fixed $u^{(1)}, u^{(2)}, u^{(3)} \in L_G^2$ decomposed as in (4.1), as this implies (4.28) with $u^{(1)} = u^{(2)} = u_0$ and $u^{(3)} = \bar{u}_0$. Let A_1, A_2, A_3 be the dyadic integers such that $(m_1 + 1)I_1 \sim A_1$, $(m_2 + 1)I_2 \sim A_2$ and $(m_3 + 1)I_3 \sim A_3$.

We apply Corollary 4.4.5 and get

$$\|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \lesssim \max\{A_1, A_2, A_3\}^\ell \|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{L_G^2}^2.$$

Assuming up to permutation that $\max\{A_1, A_2, A_3\} = A_1$, we deduce

$$\|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \lesssim A_1^\ell \|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)}\|_{L_G^2}^2 \|u_{I_3, m_3}^{(3)}\|_{L_G^\infty}^2.$$

Then, from the unit block bound of Proposition 4.5.4, it follows that if $\max\{A_1, A_2, A_3\} = A_1$, then

$$\|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \lesssim A_1^{\ell - \frac{1}{2}} \langle I_1 \rangle \langle I_2 \rangle \|u_{I_1, m_1}^{(1)}\|_{L_G^2}^2 \|u_{I_2, m_2}^{(2)}\|_{L_G^2}^2 \|u_{I_3, m_3}^{(3)}\|_{L_G^\infty}^2. \quad (4.31)$$

This implies that

$$\begin{aligned} & \sum_{\substack{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ \max\{A_1, A_2, A_3\} = A_1}} \|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \\ & \lesssim \sum_{\substack{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ \max\{A_1, A_2, A_3\} = A_1}} A_1^\ell \frac{\langle I_1 \rangle \langle I_2 \rangle}{A_1^{\frac{1}{2}}} \|u_{I_1, m_1}^{(1)}\|_{L_G^2}^2 \|u_{I_2, m_2}^{(2)}\|_{L_G^2}^2 \|u_{I_3, m_3}^{(3)}\|_{L_G^\infty}^2. \end{aligned}$$

By definition of \mathcal{X}_ρ^k , we get that for $k = \ell - \frac{1}{2}$,

$$\begin{aligned} & \sum_{\substack{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ \max\{A_1, A_2, A_3\} = A_1}} \|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \\ & \lesssim \|u^{(1)}\|_{\mathcal{X}_1^k}^2 \|u^{(2)}\|_{\mathcal{X}_1^0}^2 \sum_{(I_3, m_3) \in \mathbb{N}^* \times \mathbb{N}} \|u_{I_3, m_3}^{(3)}\|_{L_G^\infty}^2. \end{aligned}$$

It only remains to use estimate (4.15) from Lemma 4.3.3 and the embedding $W_G^{\varepsilon, p} \hookrightarrow L_G^\infty$ for arbitrary small $\varepsilon > 0$ and $p > \frac{3}{\varepsilon}$ to deduce that one has

$$\sum_{(I_3, m_3) \in \mathbb{N}^* \times \mathbb{N}} \|u_{I_3, m_3}^{(3)}\|_{L_G^\infty}^2 \lesssim \|u^{(3)}\|_{\mathcal{X}_{\zeta(p) + \frac{3}{2} - \frac{3}{p}}^{-\zeta(p) + \varepsilon}}^2.$$

Remark that for large p we have $\mathcal{X}_1^k \hookrightarrow \mathcal{X}_{\zeta(p) + \frac{3}{2} - \frac{3}{p}}^{-\zeta(p) + \varepsilon}$. This is indeed the case because $\varepsilon + \frac{1}{2} - \frac{3}{p} \leq k$ for small ε since $k > \frac{1}{2}$. We conclude that

$$\sum_{\substack{(I_1, m_1), (I_2, m_2), (I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ \max\{A_1, A_2, A_3\} = A_1}} \|u_{I_1, m_1}^{(1)} u_{I_2, m_2}^{(2)} u_{I_3, m_3}^{(3)}\|_{H_G^\ell}^2 \lesssim \|u^{(1)}\|_{\mathcal{X}_1^k}^2 \|u^{(2)}\|_{\mathcal{X}_1^k}^2 \|u^{(3)}\|_{\mathcal{X}_1^k}^2.$$

By symmetry, this inequality is also valid when $\max\{A_1, A_2, A_3\} = A_2$ or A_3 . This implies (4.30) and therefore (4.28).

Step 3: Proof of (4.29). In order to prove (4.29), fix $I_1, I_2 \in 2^{\mathbb{Z}}$ and $m_1, m_2 \in \mathbb{N}$.

If $\max\{A_1, A_2\} = A_1$, we use inequality (4.31) with $u^{(1)} = u^{(2)} = u_0$ and $u^{(3)} = \bar{u}_0$:

$$\|u_{I_1, m_1} u_{I_2, m_2} \bar{u}_{I_1, m_1}\|_{H_G^\ell}^2 \lesssim A_1^\ell \frac{\langle I_1 \rangle \langle I_2 \rangle}{A_1^{\frac{1}{2}}} \|u_{I_1, m_1}\|_{L_G^2}^2 \|u_{I_2, m_2}\|_{L_G^2}^2 \|u_{I_1, m_1}\|_{L_G^\infty}^2.$$

Otherwise, we have $\max\{A_1, A_2\} = A_2$. In this case, we still use inequality (4.31) with $u^{(1)} = u^{(2)} = u_0$ and $u^{(3)} = \bar{u}_0$:

$$\|u_{I_2, m_2} u_{I_1, m_1} \bar{u}_{I_1, m_1}\|_{H_G^\ell}^2 \lesssim A_2^\ell \frac{\langle I_1 \rangle \langle I_2 \rangle}{A_2^{\frac{1}{2}}} \|u_{I_2, m_2}\|_{L_G^2}^2 \|u_{I_1, m_1}\|_{L_G^2}^2 \|u_{I_1, m_1}\|_{L_G^\infty}^2.$$

We deduce by summation that

$$\begin{aligned} & \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \| |u_{I_1, m_1}|^2 u_{I_2, m_2} \|_{H_G^\ell} \right)^2 \\ & \lesssim \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} A_1^{\frac{\ell}{2} - \frac{1}{4}} (\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}} \|u_{I_1, m_1}\|_{L_G^2} \|u_{I_2, m_2}\|_{L_G^2} \|u_{I_1, m_1}\|_{L_G^\infty} \right)^2 \\ & \quad + \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I_1, m_1}\|_{L_G^2} A_2^{\frac{\ell}{2} - \frac{1}{4}} (\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}} \|u_{I_2, m_2}\|_{L_G^2} \|u_{I_1, m_1}\|_{L_G^\infty} \right)^2. \end{aligned}$$

Now we apply Cauchy-Schwarz' inequality and get

$$\begin{aligned} & \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \| |u_{I_1, m_1}|^2 u_{I_2, m_2} \|_{H_G^\ell} \right)^2 \\ & \lesssim \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} A_1^{\frac{\ell}{2} - \frac{1}{2}} \langle I_1 \rangle \|u_{I_1, m_1}\|_{L_G^2}^2 \right) \langle I_2 \rangle \|u_{I_2, m_2}\|_{L_G^2}^2 \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I_1, m_1}\|_{L_G^\infty}^2 \right) \\ & \quad + \sum_{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N}} \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \langle I_1 \rangle \|u_{I_1, m_1}\|_{L_G^2}^2 \right) A_2^{\frac{\ell}{2} - \frac{1}{2}} \langle I_2 \rangle \|u_{I_2, m_2}\|_{L_G^2}^2 \left(\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I_1, m_1}\|_{L_G^\infty}^2 \right). \end{aligned}$$

In only remains to use the definition (4.3) of \mathcal{X}_1^k and the estimate proven in the above step $\sum_{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N}} \|u_{I_1, m_1}\|_{L_G^\infty}^2 \lesssim \|u_0\|_{\mathcal{X}_1^k}^2$ to get (4.29). \square

4.7 Deterministic-probabilistic trilinear estimate

In this section, we establish Theorem B (ii). Let $\varepsilon_0 > 0$ such that $u_0 \in \mathcal{X}_{1+\varepsilon_0}^k$ and $v, w \in L_T^\infty H_G^\ell$. We recall that z, v and w have a decomposition

$$z = \sum_{A \in 2^{\mathbb{N}}} z_A = \sum_{A \in 2^{\mathbb{N}}} \sum_{\substack{(I_1, m_1) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m_1+1)I_1 \sim A}} z_{I_1, m_1},$$

$$v = \sum_{B \in 2^{\mathbb{N}}} v_B = \sum_{B \in 2^{\mathbb{N}}} \sum_{\substack{(I_2, m_2) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m_2+1)I_2 \sim B}} v_{I_2, m_2}$$

and

$$w = \sum_{C \in 2^{\mathbb{N}}} w_C = \sum_{C \in 2^{\mathbb{N}}} \sum_{\substack{(I_3, m_3) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m_3+1)I_3 \sim C}} w_{I_3, m_3}.$$

Let $\varepsilon > 0$ to be chosen later. Using Corollary 4.4.7, we get that for all t , there holds

$$\begin{aligned} \|z^\omega v w(t)\|_{H_G^\ell}^2 &\lesssim \sum_{\substack{(\delta_1, \delta_2, \delta_3) \in D_3 \\ A, B, C: B, C \leq A}} A^{\ell+\varepsilon} \|(z_A^\omega)^{\delta_1} (P_{\leq A} v_B)^{\delta_2} (P_{\leq A} w_C)^{\delta_3}(t)\|_{L_G^2}^2 \\ &\quad + \sum_{\substack{\delta_1 \in D_1 \\ A \in 2^{\mathbb{N}}}} A^\varepsilon \|(z_A^\omega)^{\delta_1}(t)\|_{L_G^\infty}^2 \|v(t)\|_{H_G^\ell}^2 \|w(t)\|_{H_G^\ell}^2. \end{aligned} \quad (4.32)$$

Using estimate (4.14) for the second term in the right hand side, we infer that outside of a set of probability e^{-cR^2} , there holds

$$\sum_{A \in 2^{\mathbb{N}}} A^\varepsilon \|z_A^\omega\|_{L_T^q L_G^\infty}^2 \lesssim_\varepsilon R^2 T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_1^k}^2.$$

Moreover, since applying the shifts $\delta_1 \in D_1$ to every mode z_A give equivalent estimates for the L^p norms, this leads similarly to

$$\left\| \left(\sum_{\substack{\delta_1 \in D_1 \\ A \in 2^{\mathbb{N}}}} A^\varepsilon \|(z_A^\omega)^{\delta_1}\|_{L_G^\infty}^2 \right)^{1/2} \|v\|_{H_G^\ell} \|w\|_{H_G^\ell} \right\|_{L_T^q} \lesssim R T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k} \|v\|_{L_T^\infty H_G^\ell} \|w\|_{L_T^\infty H_G^\ell}.$$

For the sake of simplicity, we note that the shifted function u^δ is nothing but a shift of the indices $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$ of order at most one and a multiplication of every mode (I, m) by a function of modulus at most 1 in Fourier variable. Similarly, the projection $P_{\leq A}$ is a multiplication of every mode (I, m) by a function of modulus at most 1 in Fourier variable. Therefore, we assume without loss of generality that in the right-hand side of (4.32) there is no shift ($\delta_1 = \delta_2 = \delta_3 = \emptyset$) and no projection $P_{\leq A}$, up to applying the proof to $(P_{\leq A} v_B)^{\delta_1}$ instead of v_B and doing a similar transformation for w . In order to estimate $\|z^\omega v w\|_{L_T^q H_G^\ell}$, we therefore write

$$\left\| \left(\sum_{A, B, C: B, C \leq A} A^{\ell+\varepsilon} \|z_A^\omega v_B w_C\|_{L_G^2}^2 \right)^{1/2} \right\|_{L_T^q} = \left\| \sum_{A, B, C: B, C \leq A} A^{\ell+\varepsilon} \|z_A^\omega v_B w_C\|_{L_G^2}^2 \right\|_{L_T^{q/2}}^{1/2},$$

and our aim is to prove that for our choice of q and ε , then with probability greater than $1 - e^{-cR^2}$, we have

$$\left\| \sum_{A, B, C: B, C \leq A} A^{\ell+\varepsilon} \|z_A^\omega v_B w_C\|_{L_G^2}^2 \right\|_{L_T^{q/2}} \lesssim_\varepsilon R^2 T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon}^k}^2 \|v\|_{L_T^\infty H_G^\ell}^2 \|w\|_{L_T^\infty H_G^\ell}^2.$$

By homogeneity, it is enough to prove that there exists C_ε such that for every $R > 0$, with probability greater than $1 - e^{-cR^2}$, for every $v, w \in L_T^\infty H_G^\ell$ satisfying $\|v\|_{L_T^\infty H_G^\ell} \leq 1$ and $\|w\|_{L_T^\infty H_G^\ell} \leq 1$,

$$\left\| \sum_{A,B,C:B,C \leq A} A^{\ell+\varepsilon} \|z_A^\omega v_B w_C(t)\|_{L_G^2}^2 \right\|_{L_T^{q/2}} \leq C_\varepsilon R^2 T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon}^k}^2. \quad (4.33)$$

4.7.1 Step 1: Decoupling z^ω from vw in (4.33).

We fix $t \in \mathbb{R}$ and $A, B, C \in 2^{\mathbb{N}}$ such that $B, C \leq A$. Then we use the Plancherel formula to get

$$\begin{aligned} \|z_A^\omega v_B w_C(t)\|_{L_G^2}^2 &= \int dy \int dx (z_A^\omega \overline{z_A^\omega})(x, y) (v_B \overline{v_B} w_C \overline{w_C})(x, y) \\ &= \int d\eta \int dx (\widehat{z_A^\omega} * \widehat{\overline{z_A^\omega}})(x, \eta) (\widehat{v_B} * \widehat{\overline{v_B}} * \widehat{w_C} * \widehat{\overline{w_C}})(x, \eta). \end{aligned}$$

We use decomposition (4.8). For $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$, we denote

$$\mathcal{F}_{y \rightarrow \eta}(z_{I,m}^\omega)(t, x, \eta) = f_{I,m}^\omega(t, \eta) h_m(\sqrt{|\eta|x}),$$

where $f_{I,m}^\omega(t, \eta) = X_{I,m}(\omega) e^{-t(2m+1)|\eta|} f_{I,m}(\eta)$ and $f_{I,m}(\eta) = f_m(\eta) \mathbf{1}_{|\eta| \in [I, 2I]}$. Similarly for v , we write

$$\mathcal{F}_{y \rightarrow \eta}(v_{I,m})(t, x, \eta) = g_{I,m}^\omega(t, \eta) h_m(\sqrt{|\eta|x}),$$

where the dependence of $g_{I,m}^\omega$ along the variables t and ω is not explicit. For w we do the same by making use of functions $\widetilde{g}_{I,m}^\omega$. Then we expand everything:

$$\begin{aligned} \|z_A^\omega v_B w_C(t)\|_{L_G^2}^2 &= \int d\eta \int dx \int d\eta_1 \int d\eta_2 \int d\eta_3 \int d\eta_4 \\ &\quad \sum_{\substack{m_1, m'_1 \in \mathbb{N} \\ (m_1+1)I_1, (m'_1+1)I'_1 \sim A}} f_{I_1, m_1}^\omega(t, \eta_1) \overline{f_{I'_1, m'_1}^\omega(t, \eta - \eta_1)} h_{m_1}(\sqrt{|\eta_1|x}) h_{m'_1}(\sqrt{|\eta - \eta_1|x}) \\ &\quad \sum_{\substack{m_2, m'_2 \in \mathbb{N} \\ (m_2+1)I_2, (m'_2+1)I'_2 \sim B}} g_{I_2, m_2}^\omega(t, \eta_2) \overline{g_{I'_2, m'_2}^\omega(t, \eta_3)} h_{m_2}(\sqrt{|\eta_2|x}) h_{m'_2}(\sqrt{|\eta_3|x}) \\ &\quad \sum_{\substack{m_3, m'_3 \in \mathbb{N} \\ (m_3+1)I_3, (m'_3+1)I'_3 \sim C}} \widetilde{g}_{I_3, m_3}^\omega(t, \eta_4) \overline{\widetilde{g}_{I'_3, m'_3}^\omega(t, \eta - \eta_2 - \eta_3 - \eta_4)} h_{m_3}(\sqrt{|\eta_4|x}) h_{m'_3}(\sqrt{|\eta - \eta_2 - \eta_3 - \eta_4|x}). \end{aligned}$$

Our aim is to apply the probabilistic decoupling to the series involving products $f_{I_1, m_1}^\omega \overline{f_{I'_1, m'_1}^\omega}$. However, since v and w may depend on ω , we first isolate the terms involving g_{I_2, m_2}^ω , $g_{I'_2, m'_2}^\omega$, $\widetilde{g}_{I_3, m_3}^\omega$ and $\widetilde{g}_{I'_3, m'_3}^\omega$.

The expanded formula is rather long, we reorganize it as follows. We define

$$\begin{aligned} \mathbb{J}_{\{m_i, m'_i\}}(\eta, \eta_1, \eta_2, \eta_3, \eta_4) &:= \int dx h_{m_1}(\sqrt{|\eta_1|x}) h_{m'_1}(\sqrt{|\eta - \eta_1|x}) h_{m_2}(\sqrt{|\eta_2|x}) h_{m'_2}(\sqrt{|\eta_3|x}) \\ &\quad h_{m_3}(\sqrt{|\eta_4|x}) h_{m'_3}(\sqrt{|\eta - \eta_2 - \eta_3 - \eta_4|x}). \end{aligned} \quad (4.34)$$

We define the “random” part

$$\begin{aligned} \mathbf{J}_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^\omega(t, \eta, \eta_2, \eta_3, \eta_4) &:= \int d\eta_1 \sum_{\substack{(I_1, m_1), (I'_1, m'_1) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m_1+1)I_1, (m'_1+1)I'_1 \sim A}} f_{I_1, m_1}^\omega(t, \eta_1) \overline{f_{I'_1, m'_1}^\omega(t, \eta - \eta_1)} \\ &\mathbb{J}_{\{m_i, m'_i\}}(\eta, \eta_1, \eta_2, \eta_3, \eta_4) |\eta_2 \eta_3 \eta_4 (\eta - \eta_2 - \eta_3 - \eta_4)|^{1/4} \\ &\mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I'_2, 2I'_2]} \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \mathbf{1}_{|\eta - \eta_2 - \eta_3 - \eta_4| \in [I'_3, 2I'_3]}, \quad (4.35) \end{aligned}$$

whereas the “deterministic” part is written

$$\mathbf{K}_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^\omega(t, \eta, \eta_2, \eta_3, \eta_4) = \frac{g_{I_2, m_2}^\omega(t, \eta_2)}{|\eta_2|^{1/4}} \frac{\overline{g_{I'_2, m'_2}^\omega(t, \eta_3)}}{|\eta_3|^{1/4}} \frac{\widetilde{g}_{I_3, m_3}^\omega(t, \eta_4)}{|\eta_4|^{1/4}} \frac{\overline{\widetilde{g}_{I'_3, m'_3}^\omega(t, \eta - \eta_2 - \eta_3 - \eta_4)}}{|\eta - \eta_2 - \eta_3 - \eta_4|^{1/4}}.$$

Then the expanded formula becomes

$$\begin{aligned} \|z_A^\omega v_B w_C(t)\|_{L_G^2}^2 &= \int d\eta \int d\eta_2 \int d\eta_3 \int d\eta_4 \sum_{\substack{(I_2, m_2), (I'_2, m'_2) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m_2+1)I_2, (m'_2+1)I'_2 \sim B}} \sum_{\substack{(I_3, m_3), (I'_3, m'_3) \in 2^{\mathbb{Z}} \times \mathbb{N} \\ (m_3+1)I_3, (m'_3+1)I'_3 \sim C}} \\ &\mathbf{J}_{A, I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^\omega(t, \eta, \eta_2, \eta_3, \eta_4) \\ &\mathbf{K}_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^\omega(t, \eta, \eta_2, \eta_3, \eta_4). \end{aligned}$$

Taking the absolute value, this leads by summation to a condensed expression

$$\left\| \sum_{A, B, C: B, C \leq A} A^{\ell+\varepsilon} \|z_A^\omega v_B w_C(t)\|_{L_G^2}^2 \right\|_{L_T^{q/2}} \leq \|A^{\ell+\varepsilon} \mathbf{J}^\omega \mathbf{K}^\omega\|_{L_\psi^p},$$

where, ψ stands for a huge tuple

$$\psi = (T, A, B, C, I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3, \eta, \eta_2, \eta_3, \eta_4).$$

Moreover, we take $L_\psi^p = L_{\psi_1}^{p_1} \dots L_{\psi_d}^{p_d}$ norm with the exponents

$$L_\psi^p = L_T^{q/2} L_A^1 L_{B, C, I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3, \eta, \eta_2, \eta_3, \eta_4}^1,$$

where the L^{p_j} norm implicitly denotes the ℓ^{p_j} norm when we consider sequence spaces. It is now possible to take successive Hölder inequalities, as one easily checks that for $\frac{1}{q_1} + \frac{1}{q_1} = \frac{1}{p_1}$ and $\frac{1}{q_2} + \frac{1}{q_2} = \frac{1}{p_2}$, we have

$$\|fg\|_{L_x^{p_1} L_y^{p_2}} \leq \left\| \|f\|_{L_y^{q_2}} \|g\|_{L_y^{q'_2}} \right\|_{L_x^{p_1}} \leq \|f\|_{L_x^{q_1} L_y^{q_2}} \|g\|_{L_x^{q'_1} L_y^{q'_2}}.$$

Our purpose is to isolate the $L_T^\infty H_G^\ell$ norm of vw by using Hölder inequalities. For this, we note that it better to split between $(BC)^\ell \mathbf{K}^\omega$ and $A^{\ell+\varepsilon} \mathbf{J}^\omega (BC)^{-\ell}$. As we will see below, our actual splitting also involves some extra other terms which aim at balancing the two terms and make every series converge.

We fix p_1 and p_2 to be very large exponents in $[2, \infty)$ to be chosen later, with respective conjugate exponents denoted p'_1 and p'_2 : $\frac{1}{p_1} + \frac{1}{p'_1} = 1$ and $\frac{1}{p_2} + \frac{1}{p'_2} = 1$. For the term involving \mathbf{J}^ω , we choose the exponents

$$L_\psi^{p(J)} = L_T^{q/2} L_A^1 L_{B, C}^{p_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta, \eta_2, \eta_3, \eta_4}^{p_1}$$

and for the term involving \mathbf{K}^ω , we choose the exponents

$$L_\psi^{p(K)} = L_T^\infty L_A^\infty L_{B,C}^{p'_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}.$$

The reason why we introduce the exponents p_1 and p_2 instead of taking them directly equal to ∞ is that we need finite exponents in order to apply the probabilistic decoupling to \mathbf{J}^ω .

Our choice of exponents is compatible with the assumptions for applying successive Hölder inequalities, so that

$$\begin{aligned} & \left\| \sum_{A, B, C: B, C \leq A} A^{\ell+\varepsilon} \|z_A^\omega v_B w_C\|_{L_G^2}^2 \right\|_{L_T^{q/2}} \\ & \leq \|A^{\ell+\varepsilon} (BC)^{-\ell+\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2-\varepsilon_1} \mathbf{J}^\omega\|_{L_\psi^{p(J)}} \|(BC)^{\ell-\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{-1/2+\varepsilon_1} \mathbf{K}^\omega\|_{L_\psi^{p(K)}}. \end{aligned} \quad (4.36)$$

The exponents $\varepsilon_1, \varepsilon_2 > 0$ will be chosen later depending on the following step.

4.7.2 Step 2: Evaluating the deterministic part \mathbf{K}^ω in (4.36).

We first estimate the term

$$\|(BC)^{\ell-\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{-1/2+\varepsilon_1} \mathbf{K}^\omega\|_{L_\psi^{p(K)}}$$

with

$$L_\psi^{p(K)} = L_A^\infty L_T^\infty L_{B,C}^{p'_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}.$$

For the intuition, we should keep in mind that $p'_1, p'_2 \approx 1$. By definition, of \mathbf{K}^ω , we have

$$\begin{aligned} |\mathbf{K}_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^\omega(t, \eta, \eta_2, \eta_3, \eta_4)| &= \frac{|g_{I_2, m_2}^\omega(t, \eta_2)|}{|\eta_2|^{1/4}} \frac{|g_{I'_2, m'_2}^\omega(t, \eta_3)|}{|\eta_3|^{1/4}} \\ & \quad \frac{|\tilde{g}_{I_3, m_3}^\omega(t, \eta_4)|}{|\eta_4|^{1/4}} \frac{|\tilde{g}_{I'_3, m'_3}^\omega(t, \eta - \eta_2 - \eta_3 - \eta_4)|}{|\eta - \eta_2 - \eta_3 - \eta_4|^{1/4}}, \end{aligned}$$

therefore

$$\|\mathbf{K}^\omega\|_{L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}} = \left\| \left(\frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|\tilde{g}_{I_3, m_3}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|\tilde{g}_{I'_3, m'_3}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1}.$$

We apply Young's inequality

$$\|\mathbf{K}^\omega\|_{L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}} \leq \left\| \left(\frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1} \left\| \left(\frac{|\tilde{g}_{I_3, m_3}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|\tilde{g}_{I'_3, m'_3}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1}.$$

Applying Young's inequality again, we get

$$\left\| \left(\frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1} \leq \left\| \left(\frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1} \left\| \left(\frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1}.$$

But since

$$\left\| \left(\frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1} = \left\| \frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} (t, \eta) \right\|_{L_\eta^{p'_1}},$$

the Hölder inequality with the choice of exponents $\frac{1}{p'_1} = \frac{1}{2} + \frac{p_1-2}{2p_1}$ leads to

$$\begin{aligned} \left\| \left(\frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1} &\leq \left\| \frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} (t, \eta) \right\|_{L_\eta^{2p_1}} \|\mathbf{1}_{|\eta| \in [I_2, 2I_2]}\|_{L_\eta^{2p_1/(p_1-2)}} \\ &\leq \|v_{I_2, m_2}(t)\|_{L_G^2} (I_2)^{(p_1-2)/(2p_1)}. \end{aligned}$$

By symmetry of the roles of I_2, m_2 and I'_2, m'_2 , we deduce

$$\left\| \left(\frac{|g_{I_2, m_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} * \left(\frac{|g_{I'_2, m'_2}^\omega|}{|\cdot|^{1/4}} \right)^{p'_1} (t, \eta) \right\|_{L_\eta^1}^{1/p'_1} \leq \|v_{I_2, m_2}(t)\|_{L_G^2} \|v_{I'_2, m'_2}(t)\|_{L_G^2} (I_2 I'_2)^{(p_1-2)/(2p_1)}.$$

By symmetry of the roles of u and v , we conclude

$$\|\mathbf{K}^\omega\|_{L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}} \leq \|v_{I_2, m_2}(t)\|_{L_G^2} \|v_{I'_2, m'_2}(t)\|_{L_G^2} \|w_{I_3, m_3}(t)\|_{L_G^2} \|w_{I'_3, m'_3}(t)\|_{L_G^2} (I_2 I'_2 I_3 I'_3)^{(p_1-2)/(2p_1)}.$$

Note that $\frac{p_1-2}{2p_1} = \frac{1}{2} - \frac{1}{p_1} = \frac{1}{2} - \varepsilon_1$ where $\varepsilon_1 = \frac{1}{p_1}$ is very small. This leads to

$$\begin{aligned} &\|(I_2 I'_2 I_3 I'_3)^{-1/2+\varepsilon_1} \mathbf{K}^\omega\|_{L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}} \\ &\leq \left(\sum_{\substack{m_2, m'_2, m_3, m'_3, I_2, I'_2, I_3, I'_3 \\ (m_2+1)I_2, (m'_2+1)I'_2 \sim B \\ (m_3+1)I_3, (m'_3+1)I'_3 \sim C}} \|v_{I_2, m_2}(t)\|_{L_G^2}^2 \|v_{I'_2, m'_2}(t)\|_{L_G^2}^2 \|w_{I_3, m_3}(t)\|_{L_G^2}^2 \|w_{I'_3, m'_3}(t)\|_{L_G^2}^2 \right)^{1/2} \\ &= \|v_B(t)\|_{L_G^2}^2 \|w_C(t)\|_{L_G^2}^2. \end{aligned}$$

Finally we get

$$\begin{aligned} &\|(BC)^{\ell-\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{-1/2+\varepsilon_1} \mathbf{K}^\omega\|_{L_{B, C}^{p'_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}} \\ &\leq \left(\sum_{B, C} (BC)^{(\ell-\varepsilon_2)p'_2} \|v_B(t)\|_{L_G^2}^{2p'_2} \|w_C(t)\|_{L_G^2}^{2p'_2} \right)^{1/p'_2}. \end{aligned}$$

The exponent p'_2 is greater than 1, which could be troublesome, but we recall that we have chosen v and w satisfying $\|v\|_{L_T^\infty H_G^\ell} \leq 1$ and $\|w\|_{L_T^\infty H_G^\ell} \leq 1$ in the assumption of the desired estimate (4.33). Therefore, for every t, B, C , we have $\|v_B(t)\|_{L_G^2} \leq 1$ and $\|w_C(t)\|_{L_G^2} \leq 1$, leading to

$$\begin{aligned} &\|(BC)^{\ell-\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{-1/2+\varepsilon_1} \mathbf{K}^\omega\|_{L_{B, C}^{p'_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta, \eta_2, \eta_3, \eta_4}^{p'_1}} \\ &\leq \left(\sum_{B, C} (BC)^{(\ell-\varepsilon_2)p'_2} \|v_B(t)\|_{L_G^2}^2 \|w_C(t)\|_{L_G^2}^2 \right)^{1/p'_2}. \end{aligned}$$

We choose ε_2 such that $(\ell - \varepsilon_2)p'_2 = \ell$, i.e. $\varepsilon_2 = \frac{\ell}{p_2}$. Then the upper bound is bounded by

$$\begin{aligned} & \| (BC)^{\ell - \varepsilon_2} (I_2 I'_2 I_3 I'_3)^{-1/2 + \varepsilon_1} \mathbf{K}^\omega \|_{L_{B,C}^{p'_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta_1, \eta_2, \eta_3, \eta_4}^{p'_1}} \\ & \leq \left(\sum_{B,C} \|v_B(t)\|_{H_G^\ell}^2 \|w_B(t)\|_{H_G^\ell}^2 \right)^{1/p'_2} \\ & \leq (\|v(t)\|_{H_G^\ell} \|w(t)\|_{H_G^\ell})^{2/p'_2} \\ & \leq 1. \end{aligned}$$

Taking the $L_T^\infty L_A^\infty$ norm, we conclude that

$$\| (BC)^{\ell - \varepsilon_2} (I_2 I'_2 I_3 I'_3)^{-1/2 + \varepsilon_1} \mathbf{K}^\omega \|_{L_\psi^{p(K)}} \leq 1.$$

4.7.3 Step 3: Evaluating the random part \mathbf{J}^ω in (4.36).

We now estimate the term

$$\| A^{\ell + \varepsilon} (BC)^{-\ell + \varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2 - \varepsilon_1} \mathbf{J}^\omega \|_{L_\psi^{p(J)}}$$

with $\varepsilon_1 = \frac{1}{p_1}$, $\varepsilon_2 = \frac{\ell}{p_2}$ and

$$L_\psi^{p(J)} = L_T^{q/2} L_A^1 L_{B,C}^{p_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2 L_{\eta_1, \eta_2, \eta_3, \eta_4}^{p_1}.$$

Step 3.1: probabilistic decoupling. We first apply the probabilistic decoupling: let us check that the assumptions in order to apply Corollary 4.2.15 (iii) are met. It is now more convenient to write

$$A^{\ell + \varepsilon} (BC)^{-\ell + \varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2 - \varepsilon_1} \mathbf{J}^\omega = \sum_{\substack{m_1, m'_1, I_1, I'_1 \\ (m_1+1)I_1, (m'_1+1)I'_1 \sim A}} X_{I_1, m_1}(\omega) \overline{X_{I'_1, m'_1}(\omega)} \Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi),$$

where from the definition (4.35) of \mathbf{J}^ω , we have

$$\begin{aligned} \Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi) &= A^{\ell + \varepsilon} (BC)^{-\ell + \varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2 - \varepsilon_1} \int d\eta_1 f_{I_1, m_1}(t, \eta_1) \overline{f_{I'_1, m'_1}(t, \eta - \eta_1)} \\ & \quad \mathbb{J}_{\{m_i, m'_i\}}(\eta, \eta_1, \eta_2, \eta_3, \eta_4) |\eta_2 \eta_3 \eta_4 (\eta - \eta_2 - \eta_3 - \eta_4)|^{1/4} \\ & \quad \mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I'_2, 2I'_2]} \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \mathbf{1}_{|\eta - \eta_2 - \eta_3 - \eta_4| \in [I'_3, 2I'_3]}, \end{aligned} \quad (4.37)$$

and we recall that \mathbb{J} is defined in (4.34).

Applying Corollary 4.2.15 (iii) with

$$\psi_- = (T, A, B, C, I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3), \quad \psi_+ = (\eta, \eta_2, \eta_3, \eta_4),$$

$$L_{\psi_-}^{p_-(J)} = L_T^{q/2} L_A^1 L_{B,C}^{p_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2, \quad L_{\psi_+}^{p_+(J)} = L_{\eta, \eta_2, \eta_3, \eta_4}^{p_1},$$

since

$$\begin{aligned} & \| A^{\ell + \varepsilon} (BC)^{-\ell + \varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2 - \varepsilon_1} \mathbf{J}^\omega \|_{L_\psi^{p(J)}} \\ &= \left\| \sum_{\substack{m_1, m'_1, I_1, I'_1 \\ (m_1+1)I_1, (m'_1+1)I'_1 \sim A}} X_{I_1, m_1}(\omega) \overline{X_{I'_1, m'_1}(\omega)} \Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi) \right\|_{L_\psi^{p(J)}}, \end{aligned}$$

we get that outside of a set of probability e^{-cR^2} , there holds

$$\begin{aligned} & \|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L_{\psi}^{p(J)}} \\ & \leq R^2 \left\| \left(\sum_{\substack{m_1, m_1', I_1, I_1' \\ (m_1+1)I_1, (m_1'+1)I_1' \sim A}} \|\Psi_{(I_1, m_1), (I_1', m_1')}(\psi)\|_{L_{\psi_+}^{p_+(J)}}^2 \right)^{1/2} \right\|_{L_{\psi_-}^{p_-(J)}} \\ & \quad + R^2 \left\| \sum_{\substack{m_1, I_1 \\ (m_1+1)I_1 \sim A}} \|\Psi_{(I_1, m_1), (I_1, m_1)}(\psi)\|_{L_{\psi_+}^{p_+(J)}} \right\|_{L_{\psi_-}^{p_-(J)}}. \end{aligned} \quad (4.38)$$

Step 3.2: deterministic trilinear estimates. Now, we apply the deterministic trilinear estimates from Corollary 4.5.3 in order to gain $\frac{1}{2}$ additional derivatives, *i.e.* $\frac{1}{2}$ powers of A .

Let us fix I_1, I_1', m_1, m_1' and ψ . We estimate $\Psi_{(I_1, m_1), (I_1', m_1')}(\psi)$ defined in (4.37). We start with \mathbb{J} defined in (4.34) as

$$\begin{aligned} \mathbb{J}_{\{m_i, m_i'\}}(\eta, \eta_1, \eta_2, \eta_3, \eta_4) &= \int dx h_{m_1}(\sqrt{|\eta_1|x}) h_{m_1'}(\sqrt{|\eta - \eta_1|x}) h_{m_2}(\sqrt{|\eta_2|x}) h_{m_2'}(\sqrt{|\eta_3|x}) \\ & \quad h_{m_3}(\sqrt{|\eta_4|x}) h_{m_3'}(\sqrt{|\eta - \eta_2 - \eta_3 - \eta_4|x}). \end{aligned}$$

We apply Cauchy-Schwarz' inequality

$$\begin{aligned} |\mathbb{J}_{\{m_i, m_i'\}}(\eta, \eta_1, \eta_2, \eta_3, \eta_4)| &\leq \|h_{m_1}(\sqrt{|\eta_1|x}) h_{m_2}(\sqrt{|\eta_2|x}) h_{m_3}(\sqrt{|\eta_4|x})\|_{L_x^2} \\ & \quad \|h_{m_1'}(\sqrt{|\eta - \eta_1|x}) h_{m_2'}(\sqrt{|\eta_3|x}) h_{m_3'}(\sqrt{|\eta - \eta_2 - \eta_3 - \eta_4|x})\|_{L_x^2}. \end{aligned}$$

We apply Corollary 4.5.3 for $\alpha_1^2 = |\eta_1| \in [I_1, 2I_1]$, $\alpha_2^2 = |\eta_2| \in [I_2, 2I_2]$ and $\alpha_3^2 = |\eta_4| \in [I_3, 2I_3]$:

$$\alpha_1 \alpha_2 \alpha_3 \|h_{m_1}(\alpha_1 \cdot) h_{m_2}(\alpha_2 \cdot) h_{m_3}(\alpha_3 \cdot)\|_{L^2}^2 \lesssim C_{\{I_i, m_i\}}^2,$$

where

$$C_{\{I_i, m_i\}}^2 = \frac{\langle I_1 \rangle (I_2 I_3)^{1/4}}{A^{1/2} ((2m_2 + 1)(2m_3 + 1))^{1/12}}.$$

We also apply Corollary 4.5.3 for $\alpha_1^2 = |\eta - \eta_1| \in [I_1', 2I_1']$, $\alpha_2^2 = |\eta_3| \in [I_2', 2I_2']$ and $\alpha_3^2 = |\eta - \eta_2 - \eta_3 - \eta_4| \in [I_3', 2I_3']$. We conclude that

$$|\eta_1(\eta - \eta_1)\eta_2\eta_3\eta_4(\eta - \eta_2 - \eta_3 - \eta_4)|^{1/4} |\mathbb{J}_{\{m_i, m_i'\}}(\eta, \eta_1, \eta_2, \eta_3, \eta_4)| \lesssim C_{\{I_i, m_i\}} C_{\{I_i', m_i'\}}.$$

Moreover, since

$$\int d\eta_1 \left| \frac{f_{I_1, m_1}(t, \eta_1)}{|\eta_1|^{1/4}} \frac{\overline{f_{I_1', m_1'}(t, \eta - \eta_1)}}{|\eta - \eta_1|^{1/4}} \right| = \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) * \left(\frac{|f_{I_1', m_1'}|}{|\cdot|^{1/4}} \right) (t, \eta),$$

we get the estimate

$$\begin{aligned} |\Psi_{(I_1, m_1), (I_1', m_1')}(\psi)| &\leq A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1} \\ & \quad C_{\{I_i, m_i\}} C_{\{I_i', m_i'\}} \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) * \left(\frac{|f_{I_1', m_1'}|}{|\cdot|^{1/4}} \right) (t, \eta) \\ & \quad \mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I_2', 2I_2']} \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \mathbf{1}_{|\eta - \eta_2 - \eta_3 - \eta_4| \in [I_3', 2I_3']}. \end{aligned}$$

Step 3.3: Integration bounds. It is now time to evaluate $\|\Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi)\|_{L_{\psi_+}^{p_+(J)}}$ with

$$L_{\psi_+}^{p_+(J)} = L_{\eta, \eta_2, \eta_3, \eta_4}^{p_1}.$$

First, as indicators functions are bounded by 1, we have

$$\begin{aligned} & \left\| \mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I'_2, 2I'_2]} \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \mathbf{1}_{|\eta - \eta_2 - \eta_3 - \eta_4| \in [I'_3, 2I'_3]} \right\|_{L_{\eta_2, \eta_3, \eta_4}^{p_1}} \\ & \leq \left\| \mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I'_2, 2I'_2]} \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \right\|_{L_{\eta_2, \eta_3, \eta_4}^{p_1}}^{1/p_1} \leq (I_2 I'_2 I_3)^{1/p_1}. \end{aligned}$$

By symmetry of roles, recalling that $\varepsilon_1 = \frac{1}{p_1}$, we write that

$$\left\| \mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I'_2, 2I'_2]} \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \mathbf{1}_{|\eta - \eta_2 - \eta_3 - \eta_4| \in [I'_3, 2I'_3]} \right\|_{L_{\eta_2, \eta_3, \eta_4}^{p_1}} \leq (I_2 I'_2 I_3 I'_3)^{3\varepsilon_1/4}.$$

Hence we have discarded the integration over η_1, η_2, η_3 and the term $\mathbf{1}_{|\eta_2| \in [I_2, 2I_2]} \mathbf{1}_{|\eta_3| \in [I'_2, 2I'_2]} \times \mathbf{1}_{|\eta_4| \in [I_3, 2I_3]} \mathbf{1}_{|\eta - \eta_2 - \eta_3 - \eta_4| \in [I'_3, 2I'_3]}$:

$$\begin{aligned} \|\Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi)\|_{L_{\eta_2, \eta_3, \eta_4}^{p_1}} & \leq A^{\ell+\varepsilon} (BC)^{-\ell+\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2-\varepsilon_1/4} \\ & C_{\{I_i, m_i\}} C_{\{I'_i, m'_i\}} \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) * \left(\frac{|f_{I'_1, m'_1}|}{|\cdot|^{1/4}} \right) (\eta). \end{aligned}$$

Then, we integrate in η . Fixing I_1, I'_1, m_1, m'_1 , we have thanks to Young's inequality with $1 + \frac{1}{p_1} = 2\frac{p_1+1}{2p_1}$,

$$\left\| \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) * \left(\frac{|f_{I'_1, m'_1}|}{|\cdot|^{1/4}} \right) (\eta) \right\|_{L_\eta^{p_1}} \leq \left\| \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) \right\|_{L_\eta^{2p_1/(p_1+1)}} \left\| \left(\frac{|f_{I'_1, m'_1}|}{|\cdot|^{1/4}} \right) \right\|_{L_\eta^{2p_1/(p_1+1)}}$$

and from Hölder's inequality with $\frac{p_1+1}{2p_1} = \frac{1}{2} + \frac{1}{2p_1}$,

$$\begin{aligned} \left\| \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) * \left(\frac{|f_{I'_1, m'_1}|}{|\cdot|^{1/4}} \right) (t, \eta) \right\|_{L_\eta^{p_1}} & \leq (I_1 I'_1)^{1/(2p_1)} \left\| \left(\frac{|f_{I_1, m_1}|}{|\cdot|^{1/4}} \right) (t, \eta) \right\|_{L_\eta^2} \left\| \left(\frac{|f_{I'_1, m'_1}|}{|\cdot|^{1/4}} \right) (t, \eta) \right\|_{L_\eta^2} \\ & = (I_1 I'_1)^{1/(2p_1)} \|z_{I_1, m_1}(t)\|_{L_G^2} \|z_{I'_1, m'_1}(t)\|_{L_G^2} \\ & = (I_1 I'_1)^{1/(2p_1)} \|u_{I_1, m_1}\|_{L_G^2} \|u_{I'_1, m'_1}\|_{L_G^2}, \end{aligned}$$

where in the latter equality we have used the L^2 isometry property of the linear flow. We have proven that

$$\begin{aligned} \|\Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi)\|_{L_{\psi_+}^{p_+(J)}} & \leq A^{\ell+\varepsilon} (BC)^{-\ell+\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2-\varepsilon_1/4} \\ & C_{\{I_i, m_i\}} C_{\{I'_i, m'_i\}} (I_1 I'_1)^{1/(2p_1)} \|u_{I_1, m_1}\|_{L_G^2} \|u_{I'_1, m'_1}\|_{L_G^2}. \end{aligned}$$

Recalling that $\varepsilon_1 = \frac{1}{p_1}$ and replacing $C_{\{I_i, m_i\}}^2 = \frac{\langle I_1 \rangle \langle I_2 I_3 \rangle^{1/4}}{A^{1/2} ((2m_2+1)(2m_3+1))^{1/12}}$ by its value, this leads to

$$\begin{aligned} \|\Psi_{(I_1, m_1), (I'_1, m'_1)}(\psi)\|_{L_{\psi_+}^{p_+(J)}} & \leq A^{\ell+\varepsilon} (BC)^{-\ell+\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2-\varepsilon_1/4} \\ & \frac{(\langle I_1 \rangle \langle I'_1 \rangle)^{1/2} (I_2 I'_2 I_3 I'_3)^{1/8}}{A^{1/2} ((2m_2+1)(2m'_2+1)(2m_3+1)(2m'_3+1))^{1/24}} (I_1 I'_1)^{\varepsilon_1/2} \|u_{I_1, m_1}\|_{L_G^2} \|u_{I'_1, m'_1}\|_{L_G^2}. \end{aligned}$$

Step 3.4: decoupled summation. We now consider the sums over I_1, I'_1, m_1, m'_1 in (4.38).

For the second term in the right-hand side of (4.38), we take $I'_1 = I_1, m'_1 = m_1$ and incorporate the term $\langle I_1 \rangle^{1+\varepsilon_1}$, which is the only term still dependent of I_1, m_1 in the upper bound. By summation, this leads to

$$\sum_{\substack{m_1, I_1 \\ (m_1+1)I_1 \sim A}} \langle I_1 \rangle^{1+\varepsilon_1} \|u_{I_1, m_1}\|_{L_G^2}^2 = \|u_A\|_{\mathcal{X}_{1+\varepsilon_1}^0}^2.$$

For the first term in the right-hand side of (4.38), we also have

$$\left(\sum_{\substack{m_1, m'_1, I_1, I'_1 \\ (m_1+1)I_1, (m'_1+1)I'_1 \sim A}} \langle I_1 \rangle^{1+\varepsilon_1} \langle I'_1 \rangle^{1+\varepsilon_1} \|u_{I_1, m_1}\|_{L_G^2}^2 \|u_{I'_1, m'_1}\|_{L_G^2}^2 \right)^{1/2} = \|u_A\|_{\mathcal{X}_{1+\varepsilon_1}^0}^2.$$

Therefore (4.38) becomes

$$\begin{aligned} & \|A^{\ell+\varepsilon} (BC)^{-\ell+\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2-\varepsilon_1} \mathbf{J}^\omega\|_{L_\psi^{p(J)}} \\ & \lesssim R^2 \left\| A^{\ell-\frac{1}{2}+\varepsilon} (BC)^{-\ell+\varepsilon_2} \frac{(I_2 I'_2 I_3 I'_3)^{1/2+1/8-\varepsilon_1/4}}{((2m_2+1)(2m'_2+1)(2m_3+1)(2m'_3+1))^{1/24}} \|u_A\|_{\mathcal{X}_{1+\varepsilon_1}^0}^2 \right\|_{L_{\psi_-}^{p_-(J)}}, \end{aligned}$$

where

$$L_{\psi_-}^{p_-(J)} = L_T^{q/2} L_A^1 L_{B,C}^{p_2} L_{I_2, I'_2, I_3, I'_3, m_2, m'_2, m_3, m'_3}^2.$$

Step 3.5: Hölder bounds. Let us now compute the norm L_{I_2, m_2}^2 . Thanks to the condition $1 + (2m_2 + 1)I_2 \in [B, 2B]$ implying $m_2 \leq \frac{B}{I_2}$ and $I_2 \leq B$, we have

$$\begin{aligned} \sum_{\substack{m_2, I_2 \\ (m_2+1)I_2 \sim B}} \frac{I_2^{1+1/4-\varepsilon_1/2}}{(m_2+1)^{1/12}} & \leq \sum_{I_2 \in 2^{\mathbb{Z}}: I_2 \leq B} I_2^{1+1/4-\varepsilon_1/2} \sum_{m_2 \leq \frac{B}{I_2}} \frac{1}{(m_2+1)^{1/12}} \\ & \leq \sum_{I_2 \in 2^{\mathbb{Z}}: I_2 \leq B} I_2^{1+1/4-\varepsilon_1/2} \left(\frac{B}{I_2}\right)^{1-1/12} \\ & \lesssim B^{1-1/12} \sum_{I_2 \in 2^{\mathbb{Z}}: I_2 \leq B} I_2^{1/4+1/12-\varepsilon_1/2}. \end{aligned}$$

The series over $I_2 \leq 1$ is convergent since for $\varepsilon_1 > 0$ sufficiently small, the exponent $1/4 + 1/12 - \varepsilon_1/2$ is positive. Moreover, one can see that the gain of $1/12$ is actually useless here in the argument. For the series over $I_2 \geq 1$, we get a bound $B^{1/4+1/12-\varepsilon_1/2}$, so that

$$\sum_{\substack{m_2, I_2 \\ (m_2+1)I_2 \sim B}} \frac{I_2^{1+1/4-\varepsilon_1/2}}{(m_2+1)^{1/12}} \lesssim B^{1+1/4-\varepsilon_1/2} \lesssim B^{5/4}.$$

We do the same for all the other indices $I'_2, m'_2, I_3, m_3, I'_3, m'_3$ and get

$$\|A^{\ell+\varepsilon} (BC)^{-\ell+\varepsilon_2} (I_2 I'_2 I_3 I'_3)^{1/2-\varepsilon_1} \mathbf{J}^\omega\|_{L_\psi^{p(J)}} \lesssim R^2 \left\| A^{\ell-1/2+\varepsilon} (BC)^{-\ell+\varepsilon_2+5/4} \|u_A\|_{\mathcal{X}_{1+\varepsilon_1}^0}^2 \right\|_{L_T^{q/2} L_A^1 L_{B,C}^{p_2}}.$$

We now take the L^{p_2} norm over B, C . Since $\ell > \frac{3}{2}$, the exponent in front of the term (BC) is $-\ell + \varepsilon_2 + \frac{5}{4} < -\frac{3}{2} + \varepsilon_2 + \frac{5}{4}$. When $\varepsilon_2 < \frac{1}{4}$ is small, we see that this exponent is negative, so that the L^{p_2} norm over B and C is convergent and bounded by some constant C_0 . This leads to

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L_\psi^{p(J)}} \lesssim R^2 \left\| A^{\ell-\frac{1}{2}+\varepsilon} \|u_A\|_{\mathcal{X}_{1+\varepsilon_1}^0}^2 \right\|_{L_T^{q/2} L_A^1}.$$

We finally conclude that

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L_\psi^{p(J)}} \lesssim R^2 \| \|u_0\|_{\mathcal{X}_{1+\varepsilon_1}^{\ell-\frac{1}{2}+\varepsilon}}^2 \|_{L_T^{q/2}}.$$

It only remains to take the $L_T^{q/2}$ norm, but one can see that our upper bound does not depend on T anymore, so this only adds a $T^{2/q}$ factor:

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L_\psi^{p(J)}} \lesssim R^2 T^{2/q} \|u_0\|_{\mathcal{X}_{1+\varepsilon_1}^{\ell-\frac{1}{2}+\varepsilon}}^2.$$

When $\ell + \varepsilon < k + \frac{1}{2}$ (this is equivalent to taking ε small enough), we conclude that

$$\|z^\omega v w\|_{L_T^q H_G^\ell}^2 \lesssim_\varepsilon R^2 T^{\frac{2}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon_1}^k}^2,$$

where $\varepsilon_1 = \frac{1}{p_1}$ can be chosen arbitrarily small and in particular no greater than ε_0 . Using the homogeneity, we remove the assumption $\|v\|_{L_T^\infty H_G^\ell} \leq 1$, $\|w\|_{L_T^\infty H_G^\ell} \leq 1$ and deduce that up to removing an extra set of probability not larger than e^{-cR^2} , inequality (4.6) holds. This concludes the proof of Theorem B (ii).

4.8 Local well-posedness

This section is devoted to the proof of Theorem A .

We fix $k \in (1, \frac{3}{2})$ and $u_0 \in H_G^k$. We assume that there exists $\varepsilon_0 > 0$ such that $u_0 \in \mathcal{X}_{1+\varepsilon}^k$. We denote by u_0^ω its associate randomization, defined by (4.2), and recall that we write $z^\omega(t) := e^{it\Delta_G} u_0^\omega$ for the solution to the linear flow (LS-G) associated to the initial data u_0^ω . We seek for a solution u to (NLS-G) of the form

$$u(t) = z^\omega(t) + v(t),$$

where $v(0) = 0$ and $v(t) \in H_G^\ell$ with $\ell \in (\frac{3}{2}, k + \frac{1}{2})$. We will prove local well-posedness for $v \in \mathcal{C}^0([0, T], H_G^\ell)$ solving

$$\begin{cases} i\partial_t v - \Delta_G v = |z^\omega + v|^2(z^\omega + v) \\ v(0) = 0, \end{cases} \quad (4.39)$$

thanks to a fixed point argument. We consider the map $\Phi : v \in \mathcal{C}^0([0, T], H_G^\ell) \mapsto \Phi(v) \in \mathcal{C}^0([0, T], H_G^\ell)$ defined by

$$\Phi(v) : t \in [0, T] \mapsto -i \int_0^t e^{i(t-t')\Delta_G} (|z^\omega + v|^2(z^\omega + v)) dt'. \quad (4.40)$$

Observe that by the Duhamel formula, v solves (4.39) if and only if $\Phi(v) = v$. Note that v may depend on ω .

We introduce the following set of initial data:

$$E_{R,T} = \{\omega \in \Omega \mid (4.41), (4.42), (4.43) \text{ and } (4.44) \text{ hold}\},$$

where

$$\|(z^\omega)^2\|_{L_T^1 H_G^\ell} + \| |z^\omega|^2 \|_{L_T^1 H_G^\ell} \leq TR^2 \|u_0\|_{\mathcal{X}_1^k}^2, \quad (4.41)$$

$$\| |z^\omega|^2 z^\omega \|_{L_T^1 H_G^\ell} \leq TR^3 \|u_0\|_{\mathcal{X}_1^k}^3, \quad (4.42)$$

$$\|z^\omega v w\|_{L_T^1 H_G^\ell} \leq TR \|u_0\|_{\mathcal{X}_1^k} \|v\|_{L_T^\infty H_G^\ell} \|w\|_{L_T^\infty H_G^\ell} \text{ for all } v, w \in L_T^\infty H_G^\ell, \quad (4.43)$$

$$\|z^\omega\|_{L_T^2 L_G^\infty}^2 \leq TR^2 \|u_0\|_{\mathcal{X}_1^k}^2. \quad (4.44)$$

We have the following estimate of $E_{R,T}$.

Lemma 4.8.1. *Let $u_0 \in H_G^k$. Then there exists a constant $c > 0$ which depends on the basis function u_0 of the randomization, such that for all $R, T > 0$, $\mathbb{P}(\Omega \setminus E_{R,T}) \leq e^{-cR^2}$.*

Proof. Outside a set of probability at most e^{-cR^2} the bounds (4.41) and (4.42) follow from Theorem B (i). Similarly, (4.43) follows from Theorem B (ii) with $u_0 \in \mathcal{X}_1^k \subset \mathcal{X}_{1+\varepsilon}^{k-\varepsilon}$ for ε chosen small enough so that $\ell < k - \varepsilon + \frac{1}{2}$. Moreover, (4.44) follows from Proposition 4.3.1. \square

The key estimate in proving Theorem A is the following.

Proposition 4.8.2 (A priori estimate). *Let $u_0^\omega \in E_{R,T}$ and $\ell \in (\frac{3}{2}, k + \frac{1}{2})$. Then for any $v \in \mathcal{C}^0([0, T], H_G^\ell)$ there holds*

$$\|\Phi(v)\|_{L_T^\infty H_G^\ell} \lesssim T \left(\|v\|_{L_T^\infty H_G^\ell}^3 + (R \|u_0\|_{\mathcal{X}_1^k})^3 \right). \quad (4.45)$$

Similarly, for any $v_1, v_2 \in \mathcal{C}^0([0, T], H_G^\ell)$ there holds

$$\|\Phi(v_2) - \Phi(v_1)\|_{L_T^\infty H_G^\ell} \lesssim T \|v_2 - v_1\|_{L_T^\infty H_G^\ell} \left((R \|u_0\|_{\mathcal{X}_1^k})^2 + \|v_1\|_{L_T^\infty H_G^\ell}^2 + \|v_2\|_{L_T^\infty H_G^\ell}^2 \right). \quad (4.46)$$

Proof. Let $\omega \in E_{R,T}$. Estimates (4.45) and (4.46) reduce to multilinear estimates, as a consequence of the triangle inequality in the Duhamel formula (4.40) and the fact that $e^{it\Delta_G}$ is an isometry in H_G^ℓ . Indeed, for $t \in [0, T]$, there holds

$$\begin{aligned} \|\Phi(v)(t)\|_{H_G^\ell} &\leq \int_0^t \|e^{i(t-t')\Delta_G} |z^\omega + v|^2 (z^\omega + v)(t')\|_{H_G^\ell} dt' \\ &\lesssim \int_0^t \| |z^\omega + v|^2 (z^\omega + v)(t') \|_{H_G^\ell} dt' \\ &\lesssim \| |z^\omega + v|^2 (z^\omega + v) \|_{L_T^1 H_G^\ell}. \end{aligned}$$

We expand the cubic term and bound

$$\begin{aligned} \|\Phi(v)\|_{L_T^\infty H_G^\ell} &\lesssim \| |z^\omega|^2 z^\omega \|_{L_T^1 H_G^\ell} + \| (z^\omega)^2 \bar{v} \|_{L_T^1 H_G^\ell} + \| |z^\omega|^2 v \|_{L_T^1 H_G^\ell} \\ &\quad + \| z^\omega |v|^2 \|_{L_T^1 H_G^\ell} + \| z^\omega \bar{v}^2 \|_{L_T^1 H_G^\ell} + T \| |v|^2 v \|_{L_T^\infty H_G^\ell}. \end{aligned} \quad (4.47)$$

Similarly, let $v_1, v_2 \in H_G^\ell$, then we have

$$\|\Phi(v_2) - \Phi(v_1)\|_{L_T^\infty H_G^\ell} \lesssim \| |z^\omega + v_2|^2(z^\omega + v_2) - |z^\omega + v_1|^2(z^\omega + v_1) \|_{L_T^1 H_G^\ell},$$

so that

$$\begin{aligned} \|\Phi(v_2) - \Phi(v_1)\|_{L_T^\infty H_G^\ell} &\lesssim \|(z^\omega)^2(\overline{v_2 - v_1})\|_{L_T^1 H_G^\ell} + \| |z^\omega|^2(v_2 - v_1) \|_{L_T^1 H_G^\ell} \\ &\quad + \|z^\omega(|v_2|^2 - |v_1|^2)\|_{L_T^1 H_G^\ell} + \|z^\omega(\bar{v}_2^2 - \bar{v}_1^2)\|_{L_T^1 H_G^\ell} + T\| |v_2|^2 v_2 - |v_1|^2 v_1 \|_{L_T^\infty H_G^\ell}. \end{aligned} \quad (4.48)$$

We now provide upper bounds for all the terms in (4.47) and (4.48).

- We begin with $(z^\omega)^2 \bar{v}$, $(z^\omega)^2(\overline{v_2 - v_1})$, $|z^\omega|^2 v$ and $|z^\omega|^2(v_2 - v_1)$. Let $w = v$ or $w = v_2 - v_1$. By the product law in H_G^k of Proposition 4.2.10, we have

$$\|(z^\omega)^2 \bar{w}(t)\|_{H_G^\ell} \lesssim \left(\|(z^\omega)^2(t)\|_{H_G^\ell} + \|z^\omega(t)\|_{L_G^\infty}^2 \right) \|w(t)\|_{H_G^\ell}.$$

Using (4.41) and (4.44) from the assumption $\omega \in E_{R,T}$, this gives:

$$\|(z^\omega)^2 \bar{w}\|_{L_T^1 H_G^\ell} \lesssim \left(\|(z^\omega)^2\|_{L_T^1 H_G^\ell} + \|z^\omega\|_{L_T^2 L_G^\infty}^2 \right) \|w\|_{L_T^\infty H_G^\ell} \lesssim TR^2 \|u_0\|_{\mathcal{X}_1^k}^2 \|w\|_{L_T^\infty H_G^\ell}.$$

Similarly one has

$$\| |z^\omega|^2 w \|_{L_T^1 H_G^\ell} \lesssim TR^2 \|u_0\|_{\mathcal{X}_1^k}^2 \|w\|_{L_T^\infty H_G^\ell}.$$

We have both proven

$$\|(z^\omega)^2 \bar{v}\|_{L_T^1 H_G^\ell} + \| |z^\omega|^2 v \|_{L_T^1 H_G^\ell} \lesssim TR^2 \|u_0\|_{\mathcal{X}_1^k}^2 \|v\|_{L_T^\infty H_G^\ell}$$

and

$$\|(z^\omega)^2(\overline{v_2 - v_1})\|_{L_T^1 H_G^\ell} + \| |z^\omega|^2(v_2 - v_1) \|_{L_T^1 H_G^\ell} \lesssim TR^2 \|u_0\|_{\mathcal{X}_1^k}^2 \|v_2 - v_1\|_{L_T^\infty H_G^\ell}.$$

- Let us estimate $z^\omega |v|^2$, $z^\omega(|v_2|^2 - |v_1|^2)$, $z^\omega \bar{v}^2$ and $z^\omega(\bar{v}_2^2 - \bar{v}_1^2)$. Using (4.43) from the assumption that $\omega \in E_{R,T}$, we infer:

$$\|z^\omega |v|^2\|_{L_T^1 H_G^\ell} + \|z^\omega \bar{v}^2\|_{L_T^1 H_G^\ell} \lesssim TR \|u_0\|_{\mathcal{X}_1^k} \|v\|_{L_T^\infty H_G^\ell}^2$$

and

$$\begin{aligned} \|z^\omega(|v_2|^2 - |v_1|^2)\|_{L_T^1 H_G^\ell} + \|z^\omega(\bar{v}_2^2 - \bar{v}_1^2)\|_{L_T^1 H_G^\ell} \\ \lesssim TR \|u_0\|_{\mathcal{X}_1^k} (\|v_2 + v_1\|_{L_T^\infty H_G^\ell}^2 + \|v_2 - v_1\|_{L_T^\infty H_G^\ell}^2) \|v_2 - v_1\|_{L_T^\infty H_G^\ell}^2 \\ \lesssim TR \|u_0\|_{\mathcal{X}_1^k} (\|v_2\|_{L_T^\infty H_G^\ell}^2 + \|v_1\|_{L_T^\infty H_G^\ell}^2) \|v_2 - v_1\|_{L_T^\infty H_G^\ell}^2. \end{aligned}$$

- Observe that thanks to the algebra property of H_G^ℓ (since $\ell > \frac{3}{2}$) of Lemma 4.2.10, we have

$$\| |v|^2 v \|_{L_T^\infty H_G^\ell} \lesssim \|v\|_{L_T^\infty H_G^\ell}^3$$

and

$$\| |v_2|^2 v_2 - |v_1|^2 v_1 \|_{L_T^\infty H_G^\ell} \lesssim (\|v_2\|_{L_T^\infty H_G^\ell} + \|v_1\|_{L_T^\infty H_G^\ell})^2 \|v_2 - v_1\|_{L_T^\infty H_G^\ell}.$$

All these bounds combined together with assumption (4.42) in estimates (4.47) and (4.48) imply (4.45) and (4.46). \square

Proof of Theorem A. Let $\omega \in E_{R,T}$. Thanks Proposition 4.8.2, we know that there exists $C > 0$ such that the map $\Phi : \mathcal{C}^0([0, T], H_G^\ell) \rightarrow \mathcal{C}^0([0, T], H_G^\ell)$ is bounded Lipschitz on finite balls: if $\|v\|_{L_T^\infty H_G^\ell} \leq R\|u_0\|_{\mathcal{X}_1^k}$, we have

$$\|\Phi(v)\|_{L_T^\infty H_G^\ell} \leq CT(R\|u_0\|_{\mathcal{X}_1^k})^3,$$

and for $\|v_1\|_{L_T^\infty H_G^\ell} \leq R$ and $\|v_2\|_{L_T^\infty H_G^\ell} \leq R$, we have

$$\|\Phi(v_2) - \Phi(v_1)\|_{L_T^\infty H_G^\ell} \leq CT(R\|u_0\|_{\mathcal{X}_1^k})^2 \|v_2 - v_1\|_{L_T^\infty H_G^\ell}.$$

Thus, taking $T = \frac{1}{2C(R\|u_0\|_{\mathcal{X}_1^k})^2}$, we see that Φ stabilizes the ball $B(0, R\|u_0\|_{\mathcal{X}_1^k})$ in $\mathcal{C}^0([0, T], H_G^\ell)$, moreover, Φ is a contraction on the ball $B(0, R\|u_0\|_{\mathcal{X}_1^k})$. The existence and uniqueness of v solving (4.39) then follows from standard contraction mapping arguments.

We have obtained that for any $\omega \in E_{R,T}$ (and $T = \frac{1}{2C(R\|u_0\|_{\mathcal{X}_1^k})^2}$), there exists a unique solution to (NLS-G) in the space

$$e^{it\Delta_G} u_0^\omega + \mathcal{C}^0([0, T], H_G^\ell) \subset \mathcal{C}^0([0, T], H_G^k).$$

Then the set

$$E := \bigcup_{k \geq 1} \bigcap_{n \geq k} E_{n, \frac{1}{2n^2}}$$

satisfies the requirements of Theorem A (i). Indeed, it remains to see that $\mathbb{P}(\Omega \setminus E) = 0$. Since the sequence of sets $\bigcup_{n \geq k} E_{n, \frac{1}{2n^2}}$ is non-increasing, we have

$$\mathbb{P}(\Omega \setminus E) \leq \limsup_{k \rightarrow \infty} \mathbb{P} \left(\bigcup_{n \geq k} E_{n, \frac{1}{2n^2}} \right) \leq \limsup_{k \rightarrow \infty} \sum_{n \geq k} e^{-cn^2},$$

which is 0 since $\sum e^{-cn^2}$ converges. \square

4.A Appendices

4.A.1 Pointwise estimates on Hermite functions

The purpose of this appendix is to explain how to prove the estimates of Corollary 4.2.8 as a consequence of the pointwise estimates on the Hermite functions from Theorem 4.2.5.

Proof of Corollary 4.2.8. We study the bounds on distinct regions of space. Let us fix $m \in \mathbb{N}$.

(1) For $|x| \leq \frac{1}{2}\lambda_m$, there holds $|x^2 - \lambda_m^2| \geq \frac{3}{4}\lambda_m^2$ thus

$$|h_m(x)| \lesssim |x^2 - \lambda_m^2|^{-\frac{1}{4}} \lesssim \lambda_m^{-\frac{1}{2}}.$$

(2) For $\frac{1}{2}\lambda_m \leq |x| \leq \lambda_m - \lambda_m^{-\frac{1}{3}}$, we have $x^2 \leq \lambda_m^2 - 2\lambda_m^{\frac{2}{3}} + \lambda_m^{-\frac{2}{3}} \leq \lambda_m^2 - \lambda_m^{\frac{2}{3}}$ thus $|\lambda_m^2 - x^2| = (\lambda_m^2 - x^2) \geq \lambda_m^{\frac{2}{3}}$, which implies that:

$$\lambda_m^{\frac{2}{3}} + |\lambda_m^2 - x^2| \leq 2|\lambda_m^2 - x^2|.$$

Finally, we get

$$|h_m(x)| \lesssim |\lambda_m^2 - x^2|^{-\frac{1}{4}} \lesssim \left(\lambda_m^{\frac{2}{3}} + |\lambda_m^2 - x^2| \right)^{-\frac{1}{4}}.$$

(3) For $||x| - \lambda_m| \leq \lambda_m^{-\frac{1}{3}}$, we have $|x^2 - \lambda_m^2| \leq ||x| - \lambda_m| \cdot ||x| + \lambda_m| \lesssim \lambda_m^{\frac{2}{3}}$, so that

$$|h_m(x)| \lesssim \lambda_m^{-\frac{1}{6}} = (\lambda_m^{\frac{2}{3}})^{-\frac{1}{4}} \lesssim \left(\lambda_m^{\frac{2}{3}} + |\lambda_m^2 - x^2| \right)^{-\frac{1}{4}}.$$

(4) For $\lambda_m + \lambda_m^{-\frac{1}{3}} \leq |x| \leq 2\lambda_m$ there holds $|x^2 - \lambda_m^2| \gtrsim \lambda_m^{\frac{2}{3}}$, thus the crude bound $e^{-s_m(x)} \leq 1$ gives

$$|h_m(x)| \leq \frac{e^{-s_m(x)}}{|x^2 - \lambda_m^2|^{\frac{1}{4}}} \lesssim \left(\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2| \right)^{-\frac{1}{4}}.$$

(5) Let $|x| \geq 2\lambda_m$. Observe that by change of variable $t = \lambda_m y$ we have

$$s_m(x) = \lambda_m^2 \int_1^{\frac{x}{\lambda_m}} \sqrt{y^2 - 1} \, dy \geq \lambda_m^2 \int_1^{\frac{x}{\lambda_m}} (y - 1) \, dy = \frac{(x - \lambda_m)^2}{2},$$

where we used that for $y \geq 1$, $\sqrt{y^2 - 1} \geq y - 1$. Then, observe that $x - \lambda_m \geq \frac{x}{2}$ by definition of x . This implies $s_m(x) \geq \frac{x^2}{8}$ and finally, since $|x^2 - \lambda_m^2| \geq \lambda_m^2 \geq 1$, we conclude:

$$|h_m(x)| \lesssim \frac{e^{-s_m(x)}}{|x^2 - \lambda_m^2|^{\frac{1}{4}}} \lesssim e^{-\frac{1}{8}x^2}. \quad \square$$

4.A.2 Algebra property, product laws and local Cauchy theory

Proof of the functional inequalities

In the Grushin case, the proof of Proposition 4.2.10 is a consequence of the following results. In the context of the Heisenberg sub-Laplacian, the proof of Proposition 4.2.10 about the algebra property of the Sobolev spaces H^k can be found in [BG01] and relies on representation theoretic formulæ.

Lemma 4.A.1 (See the proof of Lemma 3.6 in [BFG16]). *Let $H = \partial_{xx} + x^2$ denote the Harmonic oscillator. For all $k \geq 0$, there exists $C(k) > 0$ such that for all $m \in \mathbb{N}$,*

$$\frac{1}{C(k)} \|H^{k/2} h_m\|_{L_x^2} \leq \|\partial_x^k h_m\|_{L_x^2} + \|x^k h_m\|_{L_x^2} \leq C(k) \|H^{k/2} h_m\|_{L_x^2}.$$

Corollary 4.A.2. *For all $k \geq 0$, there exists $C(k) > 0$ such that for all $u \in H_G^k$, there holds*

$$\frac{1}{C(k)} \|(\text{Id} - \Delta_G)^{k/2} u\|_{L_G^2} \leq \|\langle \partial_x \rangle^k u\|_{L_G^2} + \|\langle x \partial_y \rangle^k u\|_{L_G^2} \leq C(k) \|(\text{Id} - \Delta_G)^{k/2} u\|_{L_G^2}.$$

Proof. We decompose u as

$$\mathcal{F}_{y \rightarrow \eta} u(x, \eta) = \sum_m f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x).$$

Then

$$\|(\text{Id} - \Delta_G)^{k/2} u\|_{L_x^2}^2 = \sum_m \int |f_m(\eta)|^2 d\eta \int (1 + (2m + 1)|\eta|)^k h_m(|\eta|^{\frac{1}{2}} x)^2 dx.$$

Hence we see that $\|(\text{Id} - \Delta_G)^{k/2} u\|_{L_G^2} \sim_k \|u\|_{L_x^2} + \|(-\Delta_G)^{k/2} u\|_{L_G^2}$, and

$$\begin{aligned} \|(-\Delta_G)^{k/2} u\|_{L_G^2}^2 &\sim_k \sum_m \int |f_m(\eta)|^2 d\eta \int ((2m+1)|\eta|)^k h_m(|\eta|^{\frac{1}{2}} x)^2 dx \\ &= \sum_m \int |f_m(\eta)|^2 d\eta \int |\eta|^k (H^{k/2} h_m)(|\eta|^{\frac{1}{2}} x)^2 dx. \end{aligned}$$

Now we use a change of variables to get

$$\|(-\Delta_G)^{k/2} u\|_{L_G^2}^2 \sim_k \sum_m \int |f_m(\eta)|^2 d\eta |\eta|^{k-1/2} \int (H^k h_m)(x)^2 dx.$$

Then we use that $\|H^{k/2} h_m\|_{L_x^2} \sim_k \|\partial_x^k h_m\|_{L_x^2} + \|x^k h_m\|_{L_x^2}$ and get

$$\begin{aligned} \|(-\Delta_G)^{k/2} u\|_{L_G^2}^2 &\sim_k \sum_m \int |f_m(\eta)|^2 d\eta |\eta|^{k-1/2} \int ((\partial_x^k + x^k) h_m)(x)^2 dx \\ &= \sum_m \int |f_m(\eta)|^2 d\eta \int |\eta|^k ((\partial_x^k + x^k) h_m)(|\eta|^{\frac{1}{2}} x)^2 dx \\ &= \sum_m \int |f_m(\eta)|^2 d\eta \int (\partial_x^k + (|\eta|x)^k) h_m(|\eta|^{\frac{1}{2}} x)^2 dx \\ &= \|\partial_x^k u\|_{L_G^2}^2 + \|(x\partial_y)^k u\|_{L_G^2}^2. \end{aligned} \quad \square$$

We are now ready to give a proof of the product laws.

Proof of Proposition 4.2.10. (i) We start by using the above Corollary 4.A.2 and get

$$\|uv\|_{H_G^k} \lesssim \|\langle \partial_x \rangle^k (uv)\|_{L_{x,y}^2} + \|\langle x\partial_y \rangle^k (uv)\|_{L_{x,y}^2}.$$

Now, the classical product rule in Sobolev spaces applied in x (resp. in y) implies

$$\|\langle \partial_x \rangle^k (uv)\|_{L_x^2} \lesssim \|\langle \partial_x \rangle^k u\|_{L_x^2} \|v\|_{L_x^\infty} + \|u\|_{L_x^\infty} \|\langle \partial_x \rangle^k v\|_{L_x^2},$$

resp.

$$\|\langle x\partial_y \rangle^k (uv)\|_{L_y^2} \lesssim \|\langle x\partial_y \rangle^k u\|_{L_y^2} \|v\|_{L_y^\infty} + \|u\|_{L_y^\infty} \|\langle x\partial_y \rangle^k v\|_{L_y^2},$$

which combined with Hölder estimates in y (resp. x) yields

$$\begin{aligned} \|uv\|_{H_G^k} &\lesssim \|\langle \partial_x \rangle^k u\|_{L_{x,y}^2} \|v\|_{L_{x,y}^\infty} + \|u\|_{L_{x,y}^\infty} \|\langle \partial_x \rangle^k v\|_{L_{x,y}^2} \\ &\quad + \|\langle x\partial_y \rangle^k u\|_{L_{x,y}^2} \|v\|_{L_{x,y}^\infty} + \|u\|_{L_{x,y}^\infty} \|\langle x\partial_y \rangle^k v\|_{L_{x,y}^2}. \end{aligned}$$

Proposition 4.2.10 (i) follows by an other application of Corollary 4.A.2.

(ii) is a direct consequence of (i) and the Sobolev embedding $H_G^k \hookrightarrow L_G^\infty$ when $k > \frac{3}{2}$.

(iii) When p is an integer, the result follows from (ii) by iteration. \square

Lemma 4.A.3 (Limit Sobolev embedding). *The following statements hold.*

(i) *There exists $C > 0$ such that for every $p > 2$ and $u \in H_G^{\frac{3}{2}}$, there holds*

$$\|u\|_{L_G^p} \leq C \sqrt{p} \|u\|_{H_G^{\frac{3}{2}}}. \quad (4.49)$$

(ii) (Brezis-Gallouët) For any $k > \frac{3}{2}$, there exists $C_k > 0$ such that there holds

$$\|u\|_{L_G^\infty} \leq C_k \|u\|_{H_G^{\frac{3}{2}}} \log^{\frac{1}{2}} \left(1 + \frac{\|u\|_{H_G^k}}{\|u\|_{H_G^{\frac{3}{2}}}} \right). \quad (4.50)$$

Proof. (i) Let $p > 2$ and $u \in L_G^p$. By the triangle inequality we have

$$\|u\|_{L_G^p} \leq \sum_{A \in 2^{\mathbb{N}}} \|u_A\|_{L_G^p}.$$

Then, the Sobolev embedding yields

$$\|u\|_{L_G^p} \lesssim \sum_{A \in 2^{\mathbb{N}}} \|u_A\|_{H_G^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}} \lesssim \sum_{A \in 2^{\mathbb{N}}} A^{-\frac{3}{2p}} \|u_A\|_{H_G^{\frac{3}{2}}}.$$

An application of the Cauchy-Schwarz inequality and orthogonality provide us with

$$\|u\|_{L_G^p} \lesssim \left(\sum_{A \in 2^{\mathbb{N}}} A^{-\frac{3}{p}} \right)^{\frac{1}{2}} \|u\|_{H_G^{\frac{3}{2}}},$$

which gives the conclusion since $\sum_{A \in 2^{\mathbb{N}}} A^{-\frac{3}{p}} \sim Cp$ as p goes to infinity.

(ii) Let $A_0 \geq 1$ be a dyadic integer. Start by writing $u = \sum_{A \leq A_0} u_A + \sum_{A > A_0} u_A$, and by Cauchy-Schwarz

$$\|u\|_{L^\infty} \leq \log^{\frac{1}{2}}(A_0) \left(\sum_{A \leq A_0} \|u_A\|_{L^\infty}^2 \right)^{\frac{1}{2}} + \sum_{A > A_0} \|u_A\|_{L^\infty}^2.$$

Observe that for $p \geq 2$, $\|u_A\|_{L_G^p} \lesssim A^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})} \|u_A\|_{L_G^2}$, which gives when letting $p \rightarrow \infty$

$$\|u_A\|_{L_G^\infty} \lesssim \|u_A\|_{H_G^{\frac{3}{2}}}, \quad (4.51)$$

and this latter inequality also implies

$$\|u_A\|_{L_G^\infty} \lesssim A^{-\frac{1}{2}(k-\frac{3}{2})} \|u_A\|_{H_G^k}. \quad (4.52)$$

The bound (4.51) when $A \leq A_0$ and (4.52) when $A > A_0$ imply

$$\|u\|_{L_G^\infty} \lesssim \|u\|_{H_G^{\frac{3}{2}}} \log^{\frac{1}{2}} A_0 + A_0^{-\frac{1}{2}(k-\frac{3}{2})} \|u\|_{H_G^k},$$

which gives the result after optimisation in A_0 . \square

Deterministic local Cauchy theory

We finish this appendix with a summary of well-posedness results for (NLS- \mathbb{H}^1) (resp. (NLS-G)) for $k > 2$ (resp. $k > \frac{3}{2}$). To the best of our knowledge, the best well-posedness result for (NLS- \mathbb{H}^1) is the following.

Proposition 4.A.4 (Well-posedness for (NLS- \mathbb{H}^1), see [BG01]). *For $k > 2$, the Cauchy problem for (NLS- \mathbb{H}^1) is locally well-posed in $\mathcal{C}^0([0, T^*), H^k(\mathbb{H}^1))$ and $T^* = T^*(\|u_0\|_{H^k(\mathbb{H}^1)}) \gtrsim \|u_0\|_{H^k(\mathbb{H}^1)}^{-2}$.*

Similarly, we recall the best local theory for (NLS-G).

Proposition 4.A.5 (Well-posedness theory for (NLS-G)). *The following well-posedness statements hold:*

(i) For $k > \frac{3}{2}$, the Cauchy problem for (NLS-G) is locally well-posed in $\mathcal{C}([0, T^*), H_G^k)$. Moreover, for $u_0 \in H_G^k$, the maximal time T^* satisfies $T^* \gtrsim \|u_0\|_{L_G^\infty}^{-2}$.

(ii) The blow-up criterion can be refined:

$$T^* < \infty \implies \|u(t)\|_{H_G^{\frac{3}{2}} t \rightarrow T^*} \longrightarrow \infty.$$

Sketch of proof for Proposition 4.A.5. We only give formal arguments, which are easily converted into fully rigorous proofs by standard means.

(i) Let $k > \frac{3}{2}$. Applying the operator $(-\Delta_G)^{\frac{k}{2}}$ to (NLS-G), then multiplying by $(-\Delta_G)^{\frac{k}{2}}u$ and integrating by parts in space, we compute that:

$$\frac{d}{dt} \|u(t)\|_{H_G^k}^2 \lesssim \|(-\Delta_G)^{\frac{k}{2}}(|u|^2u)(-\Delta_G)^{\frac{k}{2}}\bar{u}\|_{L_G^1}.$$

Applying the Hölder inequality, the algebra property of Lemma 4.2.10 and the Hölder inequality again, we finally arrive at the estimate

$$\frac{d}{dt} \|u(t)\|_{H_G^k}^2 \lesssim \|u\|_{L_G^\infty}^2 \|u\|_{H_G^k}^2,$$

which by the Sobolev embedding and the Grönwall inequality gives an *a priori* estimate in H^k and implies the local theory.

(ii) This follows from the energy estimate and inequality (4.50), which give

$$\frac{d}{dt} \|u(t)\|_{H_G^k}^2 \lesssim \|u\|_{L_G^\infty}^2 \|u\|_{H_G^k}^2 \lesssim \|u\|_{H^{\frac{3}{2}}}^2 \|u\|_{H_G^k}^2 \log \left(1 + \frac{\|u\|_{H_G^k}}{\|u\|_{H^{\frac{3}{2}}}} \right),$$

which implies the result after an application of Grönwall's inequality. \square

Note that a similar argument to that of (ii), which relies on the inequality (4.49) can be used to prove that, if solutions exists in $H_G^{\frac{3}{2}}$, they are unique. This argument goes back to Yudovich [Yud63], then has been used by Vladimirov [Vla84] in the context of Schrödinger equations. Namely, if u_1, u_2 are two $H_G^{\frac{3}{2}}$ solutions in $L^\infty([0, T], H_G^{\frac{3}{2}})$, introduce $\phi(t) = \|u_1(t) - u_2(t)\|_{L_G^2}^2$ and fix $T_1 < T$, we prove that $\phi(t) = 0$ for all $t \in [0, T_1]$. Denote by $w(t) = u_1(t) - u_2(t)$. Then

$$\phi'(t) = 2 \int_{\mathbb{R}^2} w'(t) \bar{w}(t) \, dx = 2i \int_{\mathbb{R}^2} \Delta w(t) \bar{w}(t) \, dx - 2i \int_{\mathbb{R}^2} (|u_1|^2 u_1 - |u_2|^2 u_2) \bar{w}(t) \, dx$$

Since the first terms equals $-2i\|\nabla w\|_{L^2}^2 \in i\mathbb{R}$ and ϕ is real-valued, we have

$$\phi'(t) \leq 2 \left| \int_{\mathbb{R}^2} (|u_1|^2 u_1 - |u_2|^2 u_2) \bar{w}(t) \, dx \right| \lesssim \int_{\mathbb{R}^2} |w(t)|^2 (|u_1(t)|^2 + |u_2(t)|^2) \, dx.$$

Then for all $t \leq T_1$, we have

$$\begin{aligned} \phi'(t) &\lesssim \int_{\mathbb{R}^2} |(u_1 - u_2)(t)|^{2(1-1/p)} (|u_1(t)|^{2(1+1/p)} + |u_2(t)|^{2(1+1/p)}) \, dx \\ &\lesssim \sqrt{p} \phi(t)^{1-\frac{1}{p}} \left(\|u_1\|_{H_G^{\frac{3}{2}}}^{2(1+1/p)} + \|u_2\|_{H_G^{\frac{3}{2}}}^{2(1+1/p)} \right), \end{aligned}$$

where we use (4.49) in the last step. Since $\|u_1(t)\|_{L^\infty([0, T_1], H^{\frac{3}{2}})} = C(T_1)$ and $2(1 + \frac{1}{p}) \leq 3$, we obtain

$$\phi'(t) \lesssim \sqrt{p} \phi(t)^{1-\frac{1}{p}}$$

The, we integrate on $[0, t]$ to get

$$\phi(t) \leq \left(\frac{t}{\sqrt{p}} \right)^p,$$

which goes to 0 as $p \rightarrow \infty$, hence $\phi = 0$.

4.A.3 An interpolation lemma

The aim of this appendix is to prove that if inequality (4.21)

$$\left\| \sum_{2B>A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{>A}v_B)) \right\|_{L_G^2}^2 \lesssim \sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|_{L_G^\infty}^2 \|v\|_{H_G^\ell}^2$$

holds for $\ell = 0$ and $\ell = 2$, then this inequality holds for all $\ell \in [0, 2]$ by interpolation. Writing $w = (\text{Id} - \Delta_G)^{\ell/2} v$, we have

$$\left\| \sum_{B>A} (\text{Id} - \Delta_G)^{\ell/2} (u_A(P_{>A}v_B)) \right\|_{L_G^2} = \left\| (\text{Id} - \Delta_G)^{\ell/2} \left(u_A \sum_{2B>A} (\text{Id} - \Delta_G)^{-\ell/2} (P_{>A}w)_B \right) \right\|_{L_G^2}.$$

Lemma 4.A.6 (Interpolation lemma). *Let $\ell \in [0, 1]$, then for any $w \in L_G^2$, there holds*

$$\|(\text{Id} - \Delta_G)^\ell (u_{A \times > A} (\text{Id} - \Delta_G)^{-\ell} w)\|_{L_G^2} \lesssim \left(\sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|_{L_G^\infty}^2 \right)^{1/2} \|w\|_{L_G^2}.$$

The proof of this lemma is an application of Stein's interpolation theorem in the case when the operators are bounded by the same constant.

Theorem 4.A.7 (Stein interpolation theorem [Ste56], Theorem 1). *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 0, 1$ be two measured spaces, $p_i, q_i \in [1, \infty]$, $S := \{z \in \mathbb{C} \mid 0 < \text{Re}(z) < 1\}$ and $(T_z)_{z \in \bar{S}}$ a family of operators from simple functions in $L^1(\mu_1)$ to μ_2 -measurable functions. Assume that there exists $c < \pi$ such that the following holds.*

- (i) *For any fixed simple functions f and g on Ω_0 and Ω_1 respectively, $z \in \bar{S} \mapsto \int_{\Omega_1} T_z(f)g \, d\mu_1$ is continuous on \bar{S} and holomorphic in S , and satisfies*

$$\sup_{z \in S} e^{-c|\text{Im}(z)|} \log \left| \int_{\Omega_1} T_z(f)g \, d\mu_1 \right| < \infty.$$

(ii) The operator $T_z : L^{p_0}(\Omega_0) \rightarrow L^{q_0}(\Omega_1)$ is continuous whenever $\operatorname{Re}(z) = 0$: there exists C_0 such that for all $f \in L^{p_0}(\Omega_0)$ and $r \in \mathbb{R}$,

$$\|T_{0+ir}f\|_{L^{q_0}} \leq C_0(r)\|f\|_{L^{p_0}},$$

and similarly, whenever $\operatorname{Re}(z) = 1$, the operator $T_z : L^{p_1}(\Omega_0) \rightarrow L^{q_1}(\Omega_1)$ is continuous: there exists C_1 such that for all $f \in L^{p_1}(\Omega_0)$ and $r \in \mathbb{R}$,

$$\|T_{1+ir}f\|_{L^{q_1}} \leq C_1(r)\|f\|_{L^{p_1}}.$$

Moreover, for $i \in \{0, 1\}$,

$$\sup_{r \in \mathbb{R}} e^{-c|r|} \log|C_i(r)| < \infty.$$

Then for $\theta \in [0, 1]$, setting $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, the operator $T_\theta : L^{p_\theta}(\Omega_0) \rightarrow L^{q_\theta}(\Omega_1)$ is bounded. More precisely, there exists $C_\theta = C(\theta, C_0, C_1)$ such that

$$\|T_\theta f\|_{L^{q_\theta}} \leq C_\theta \|f\|_{L^{p_\theta}}, \quad f \in L^{p_\theta}(\Omega_0), \quad r \in \mathbb{R}.$$

Proof of Lemma 4.A.6. We apply Theorem 4.A.7 to the operators $\left(\sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|_{L^\infty}^2\right)^{-1/2} T_z$, with

$$T_z f := (\operatorname{Id} - \Delta_G)^z (u_A P_{>A} (\operatorname{Id} - \Delta_G)^{-z} f),$$

and $(\operatorname{Id} - \Delta_G)^z = \exp(z \log(\operatorname{Id} - \Delta_G))$. We make the choice $\Omega_0 = \Omega_1 = \mathbb{R}^2$, $\mu_0 = \mu_1$ is the Lebesgue measure λ , and $p_0 = p_1 = p_\theta = q_\theta = q_0 = q_1 = 2$.

Observe that for any real r , the operator $(\operatorname{Id} - \Delta_G)^{ir}$ acts by rotation of the Fourier coefficients in L_G^2 thus it is bounded in L^2 with norm 1. Indeed, the decomposition $\mathcal{F}(w)(x, \eta) = \sum_{m \in \mathbb{N}} f_m(\eta) h_m(|\eta|^{1/2} x)$ leads to

$$\mathcal{F}((\operatorname{Id} - \Delta_G)^{ir} w)(x, \eta) = \sum_{m \in \mathbb{N}} (1 + (2m + 1)|\eta|)^{ir} f_m(\eta) h_m(|\eta|^{1/2} x),$$

so that we can conclude by orthogonality of the decomposition. This implies that for all $w \in L^2(\mathbb{R}^2)$ and $z \in \bar{S}$, we have

$$\|T_z f\|_{L^2(\mathbb{R}^2)} = \|T_{\operatorname{Re}(z)}((\operatorname{Id} - \Delta_G)^{-ir} f)\|_{L^2(\mathbb{R}^2)},$$

with $\|(\operatorname{Id} - \Delta_G)^{-ir} w\|_{L^2(\mathbb{R}^2)} = \|w\|_{L^2(\mathbb{R}^2)}$. Thanks to the fact that inequality (4.21) holds in the case $\ell = 0$ and $\ell = 2$, assumption (ii) from Theorem 4.A.7 then holds with some constant functions $C_0 = C_1$ independent of $r = \operatorname{Im}(z)$.

We now establish assumption (i). Fix two simple functions f, g on \mathbb{R}^2 . Then the map $z \in \bar{S} \mapsto \int_{\mathbb{R}^2} T_z(f)g \, d\lambda$ is continuous and holomorphic. Moreover, for all $z \in S$, we have

$$\left| \int_{\mathbb{R}^2} T_z(f)g \, d\lambda \right| \leq \|T_z f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)}.$$

We write $z = \ell + ir$ with $(\ell, r) \in [0, 1] \times \mathbb{R}$. When $\ell = 0$, we simply write

$$\|T_z f\|_{L_G^2} \lesssim \|u_A\|_{L^\infty} \|P_{>A} f\|_{L_G^2},$$

and observe that $\|u_A\|_{L_G^\infty}^2 \lesssim \|u_A\|_{H_G^{\frac{3}{2}+\varepsilon}}^2 \lesssim A^{\frac{3}{2}+\varepsilon} \|u_A\|_{L_G^2}^2 < +\infty$. Otherwise, if $\ell > 0$, start by an application of the product law in Proposition 4.2.10, which gives

$$\begin{aligned} \|T_z f\|_{L_G^2} &= \|u_A P_{>A} (\text{Id} - \Delta_G)^{-z} f\|_{H_G^\ell} \\ &\lesssim \|u_A\|_{L_G^\infty} \|P_{>A} (\text{Id} - \Delta_G)^{-z} f\|_{H_G^\ell} + \|u_A\|_{H_G^h} \|P_{>A} (\text{Id} - \Delta_G)^{-z} f\|_{L_G^\infty} \\ &\lesssim \|u_A\|_{L_G^\infty} \|f\|_{L_G^2} + \|u_A\|_{H_G^\ell} \|P_{>A} (\text{Id} - \Delta_G)^{-ir} f\|_{W_G^{-\ell, \infty}}. \end{aligned}$$

Then we observe that $\|u_A\|_{H_G^\ell} \lesssim A^{\frac{\ell}{2}} \|u_A\|_{L_G^2} < \infty$, $\|u_A\|_{L_G^\infty} < \infty$, as well as $\|f\|_{L_G^2} < \infty$. It remains to study $\|P_{>A} (\text{Id} - \Delta_G)^{-ir} f\|_{W_G^{-\ell, \infty}}$. We first use the dual Sobolev embedding $L_G^p \hookrightarrow W_G^{-\ell, \infty}$ where $\frac{1}{p} - \frac{\ell}{3} = 0$ (so that $3 \leq p < \infty$):

$$\|P_{>A} (\text{Id} - \Delta_G)^{-ir} f\|_{W_G^{-\ell, \infty}} \lesssim \|P_{>A} (\text{Id} - \Delta_G)^{-ir} f\|_{L_G^p}.$$

Now, we conclude by using the continuity of $P_{>A} (\text{Id} - \Delta_G)^{-ir}$ on L_G^p .

Indeed, $P_{>A} (\text{Id} - \Delta_G)^{-ir} = F(-\Delta_G)$, where

$$F(\lambda) = \left(1 - \chi\left(\frac{1+\lambda}{A}\right)\right) (1+\lambda)^{ir}.$$

Since $\chi \in \mathcal{C}_c^\infty[0, 1)$, one knows that $F \in W^{2, \infty}(\mathbb{R})$. Now, Theorem 1 in [MS12] implies that for all $p \in (1, \infty)$, $F(-\Delta_G)$ is bounded in $\mathcal{L}(L^p(\mathbb{R}^2))$. \square

Partie II

Équations de type Benjamin-Ono

Chapitre 5

The third order Benjamin-Ono equation on the torus : well-posedness, traveling waves and stability

Ce chapitre reprend les résultats de [Gas21b].

Résumé. Nous nous intéressons à la troisième équation dans la hiérarchie de Benjamin-Ono sur le tore

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right).$$

Nous montrons que pour tout $t \in \mathbb{R}$, le flot s'étend continûment à $H_{r,0}^s(\mathbb{T})$ si $s \geq 0$, mais n'admet pas d'extension continue à $H_{r,0}^{-s}(\mathbb{T})$ si $0 < s < \frac{1}{2}$. Par ailleurs, nous prouvons que l'extension du flot n'est pas faiblement séquentiellement continue dans $L_{r,0}^2(\mathbb{T})$. Enfin, nous classifions les solutions onde progressive pour la troisième équation dans la hiérarchie de Benjamin-Ono dans $L_{r,0}^2(\mathbb{T})$, et nous étudions leur stabilité orbitale.

Abstract. We consider the third order Benjamin-Ono equation on the torus

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right).$$

We prove that for any $t \in \mathbb{R}$, the flow map continuously extends to $H_{r,0}^s(\mathbb{T})$ if $s \geq 0$, but does not admit a continuous extension to $H_{r,0}^{-s}(\mathbb{T})$ if $0 < s < \frac{1}{2}$. Moreover, we show that the extension is weakly sequentially continuous in $H_{r,0}^s(\mathbb{T})$ if $s > 0$, but is not weakly sequentially continuous in $L_{r,0}^2(\mathbb{T})$. We then classify the traveling wave solutions for the third order Benjamin-Ono equation in $L_{r,0}^2(\mathbb{T})$ and study their orbital stability.

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5.1 Introduction

We are interested in the third equation of the integrable Benjamin-Ono hierarchy on the torus

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right). \quad (5.1)$$

The operator H is the Hilbert transform, defined as

$$Hf(x) = \sum_{n \in \mathbb{Z}} -i \operatorname{sgn}(n) \widehat{f}(n) e^{inx}, \quad f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}, \quad \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx,$$

with the convention that $\operatorname{sgn}(\pm n) = \pm 1$ if $n \geq 1$, and $\operatorname{sgn}(0) = 0$.

5.1.1 Benjamin-Ono equations and integrability

The Benjamin-Ono equation on the torus

$$\partial_t u = H \partial_{xx} u - \partial_x(u^2)$$

was introduced by Benjamin [Ben67] and Ono [Ono75] in order to describe long internal waves in a two-layer fluid of great depth. This equation admits an infinite number of conserved quantities $\mathcal{H}_k, k \geq 1$ (see Nakamura [Nak79] for a proof on the real line). The evolution equations associated to the conservation laws

$$\partial_t u = \partial_x(\nabla \mathcal{H}_k(u)) \quad (5.2)$$

are the equations for the Benjamin-Ono hierarchy [Mat84].

Since the work of Nakamura [Nak79] and Bock, Kruskal [BK79], we know that the Benjamin-Ono equation admits a Lax pair. In what follows, we will adopt the Lax pair formulation from [GK20], Appendix A

$$\frac{d}{dt} L_u = [B_u, L_u],$$

$$L_u = D - T_u, \quad B_u = iD^2 + 2iT_D(\Pi u) - 2iDT_u,$$

where $D = -i\partial_x$ and T_u is the Toeplitz operator on the Hardy space

$$L_+^2(\mathbb{T}) = \{h \in L^2(\mathbb{T}) \mid \forall n < 0, \widehat{h}(n) = 0\}$$

defined from the Szegő projector $\Pi : L^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$ as

$$T_u : h \in L_+^2(\mathbb{T}) \mapsto \Pi(uh) \in L_+^2(\mathbb{T}).$$

The Hamiltonians \mathcal{H}_k are defined from the Lax operator L_u as

$$\mathcal{H}_k(u) = \langle L_u^k \mathbf{1} | \mathbf{1} \rangle. \quad (5.3)$$

In particular, the Hamiltonian for equation (5.1) is

$$\mathcal{H}_4(u) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} (\partial_x u)^2 - \frac{3}{4} u^2 H \partial_x u + \frac{1}{4} u^4 \right) dx - \frac{1}{2} \mathcal{H}_2(u)^2, \quad (5.4)$$

with

$$\mathcal{H}_2(u) = \frac{1}{2} \|u\|_{L^2(\mathbb{T})}^2.$$

In [GK20], Gérard and Kappeler constructed global Birkhoff coordinates for the Benjamin-Ono equation on the torus. In these coordinates, the evolution equations for the Benjamin-Ono hierarchy are easier to understand. Indeed, denote by Φ the Birkhoff map

$$\Phi : u \in L_{r,0}^2(\mathbb{T}) \mapsto (\zeta_n(u))_{n \geq 1} \in h_+^{\frac{1}{2}},$$

where $L_{r,0}^2(\mathbb{T})$ is the subspace of real valued functions in $L^2(\mathbb{T})$ with zero mean, and

$$h_+^{\frac{1}{2}} = \left\{ (\zeta_n)_{n \geq 1} \mid \sum_{n \geq 1} n |\zeta_n|^2 < +\infty \right\}.$$

Then in Birkhoff coordinates, the equation (5.2) of the hierarchy associated to \mathcal{H}_k becomes

$$\partial_t \zeta_n = i \omega_n^{(k)} \zeta_n, \quad n \geq 1$$

when the frequencies

$$\omega_n^{(k)} = \frac{\partial(\mathcal{H}_k \circ \Phi^{-1})}{\partial |\zeta_n|^2}$$

are well-defined. For instance, this formula is valid if the sequence $\zeta(0) = (\zeta_n(0))_{n \geq 1}$ only has a finite number of nonzero terms, or in other words, if $\Phi^{-1}(\zeta(0))$ is a finite gap potential. In this case, the frequencies $\omega_n^{(k)}$ only depend on the actions $|\zeta_p|^2$, and the evolution simply reads

$$\zeta_n(t) = \zeta_n(0) e^{i \omega_n^{(k)}(\zeta(0)) t}, \quad t \in \mathbb{R}, \quad n \geq 1.$$

For (5.1), the third equation of the hierarchy (5.2), the frequencies are written

$$\omega_n^{(4)}(\zeta) = n^3 + n \sum_{p \geq 1} p |\zeta_p|^2 - 3 \sum_{p \geq 1} \min(p, n)^2 |\zeta_p|^2 + 3 \sum_{p, q \geq 1} \min(p, q, n) |\zeta_p|^2 |\zeta_q|^2. \quad (5.5)$$

More details about the frequencies $\omega_n^{(k)}$ and formula (5.5) can be found in Appendix 5.A.1.

We refer to Saut [Sau19] for a detailed survey of the Benjamin-Ono equation and of its hierarchy.

5.1.2 Main results

Our first main result is the determination of the well-posedness threshold for the third order Benjamin-Ono equation. For $s \in \mathbb{R}$, we use the notation

$$H_{r,0}^{-s}(\mathbb{T}) = \{u \in H^s(\mathbb{T}, \mathbb{R}) \mid \langle u | \mathbf{1} \rangle = 0\}.$$

As we will see at the beginning of section 5.2, for all $t \in \mathbb{R}$, the flow map for equation (5.1) $\mathcal{S}^t : u_0 \mapsto u(t)$ is well-defined for finite gap potentials. We prove that the flow map is globally \mathcal{C}^0 -well-posed in $H_{r,0}^s(\mathbb{T})$ when $s \geq 0$ (in the sense of Definitions 1 and 2 from [GKT20b]), but is not globally \mathcal{C}^0 -well-posed in $H_{r,0}^{-s}(\mathbb{T})$ when $0 < s < \frac{1}{2}$.

Theorem 5.1.1 (Global well-posedness threshold). *For all $s \geq 0$, the flow map defined for finite gap potentials admits a unique continuous extension to $H_{r,0}^s(\mathbb{T})$ in following sense. There exists a unique solution map $\mathcal{S} : H_{r,0}^s(\mathbb{T}) \rightarrow \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ satisfying*

- for all $t \in \mathbb{R}$, the flow map $\mathcal{S}^t : H_{r,0}^s(\mathbb{T}) \rightarrow H_{r,0}^s(\mathbb{T})$ coincides with the flow map for finite gap potentials;
- the solution map $\mathcal{S} : H_{r,0}^s(\mathbb{T}) \rightarrow \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ is continuous.

Moreover, there exists no continuous extension of the flow map to $H_{r,0}^{-s}(\mathbb{T})$ for $0 < s < \frac{1}{2}$.

In particular, by approximating an initial data by smooth functions (not necessarily finite gaps), we get smooth solutions which coincide with the extension from the theorem and converge to the solution obtained in the theorem.

Remark 5.1.2. *Note that if $s \geq \frac{1}{2}$, the maps $t \in \mathbb{R} \mapsto u(t)$ constructed in this way are solutions to equation (5.1) in the distribution sense. Indeed, one can see that in this case, equation (5.1) is satisfied in $H_{r,0}^{s-3}(\mathbb{T})$ by using the Sobolev embeddings on the torus, and the dual embeddings for negative Sobolev exponents.*

We explain for instance how to get that $\partial_x(uH\partial_x u) \in H^{s-3}(\mathbb{T})$ if $u \in H^s(\mathbb{T})$ and $s \geq \frac{1}{2}$. We know that $H\partial_x u \in H^{s-1}(\mathbb{T})$, we want to show that $uH\partial_x u \in H^{s-2}(\mathbb{T})$.

- If $s > \frac{3}{2}$, the space $H^{s-1}(\mathbb{T})$ is an algebra, therefore $uH\partial_x u \in H^{s-1}(\mathbb{T})$.
- If $\frac{1}{2} \leq s < \frac{3}{2}$, the space $H^{s-1}(\mathbb{T})$ is included in $H^{-s}(\mathbb{T})$ thanks to the condition $s \geq \frac{1}{2}$, therefore $uH\partial_x u \in L^1(\mathbb{T})$. It is now enough to note the dual Sobolev embedding $L^1(\mathbb{T}) \hookrightarrow H^{s-2}(\mathbb{T})$ since $s - 2 < -\frac{1}{2}$.
- If $s = \frac{3}{2}$, one can use the Sobolev embeddings for $H^{\frac{1}{2}}(\mathbb{T})$ to see that $uH\partial_x u \in L^2(\mathbb{T})$.

We also investigate the question of the sequential weak continuity for the flow map.

Theorem 5.1.3 (Sequential weak continuity for the flow map). *For all $t \in \mathbb{R}$, the extension of flow map for equation (5.1) $\mathcal{S}^t : u_0 \mapsto u(t)$ is weakly sequentially continuous in $H_{r,0}^s(\mathbb{T})$ for $s > 0$, but is not weakly sequentially continuous in $L_{r,0}^2(\mathbb{T})$.*

In [GKT20b], Gérard, Kappeler and Topalov proved that the flow map for the Benjamin-Ono equation is globally \mathcal{C}^0 -well-posed in $H_{r,0}^s(\mathbb{T})$ for $s > -\frac{1}{2}$, whereas from [PH10] there is no continuous extension of the flow map to $H_{r,0}^s(\mathbb{T})$ when $s \leq -\frac{1}{2}$. We expect that the well-posedness threshold on the torus increases by $\frac{1}{2}$ for each new equation in the hierarchy : for the equation corresponding to the k -th Hamiltonian \mathcal{H}_k , $k \geq 4$, the threshold

should be $H_{r,0}^{\frac{k}{2}-2}(\mathbb{T})$ (see Remarks 5.2.6 and 5.A.1). Note that all the equations for the Benjamin-Ono hierarchy have critical Sobolev exponent $-\frac{1}{2}$.

Let us mention former approaches to the Cauchy problem for higher order Benjamin-Ono equations. Tanaka [Tan19] considered more general third order type Benjamin-Ono equations on the torus

$$\partial_t u = \partial_x(-\partial_{xx}u - c_1 u H \partial_x u - c_2 H(u \partial_x u) + u^3),$$

and proved local well-posedness in $H^s(\mathbb{T})$ for $s > \frac{5}{2}$. He deduced global well-posedness in $H^s(\mathbb{T})$, $s \geq 3$ for the integrable case $c_1 = c_2 = \frac{3}{2}$.

On the real line, Feng and Han [FH96] proved local well-posedness in $H^s(\mathbb{R})$, $s \geq 4$ for the third equation of the Benjamin-Ono hierarchy (5.1). Considering more general third order type Benjamin-Ono equations under the form

$$\partial_t u - bH\partial_{xx}u - a\partial_{xxx}u = cv\partial_x v - d\partial_x(vH\partial_x v + H(v\partial_x v)),$$

Linares, Pilod and Ponce [LPP11] established local well-posedness in $H^s(\mathbb{R})$, $s \geq 2$, then Molinet and Pilod [MP12a] proved global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$.

Concerning Benjamin-Ono equations of fourth order on the torus and on the real line, Tanaka [Tan21] proved local well-posedness in H^s , $s > \frac{7}{2}$ for a more general family of fourth order type Benjamin-Ono equations, and deduced global well-posedness in H^s , $s \geq 4$ in the integrable case.

Our second main result is the classification of the traveling waves for the third order Benjamin-Ono equation in $L_{r,0}^2(\mathbb{T})$, i.e. the solutions to (5.1) under the form $u(t, x) = u_0(x + ct)$, $t \in \mathbb{R}$, $x \in \mathbb{T}$, $u_0 \in L_{r,0}^2(\mathbb{T})$, for some $c \in \mathbb{R}$.

Definition 5.1.4. For $N \geq 1$, we say that $u \in L_{r,0}^2(\mathbb{T})$ is a N gap potential if the set $\{n \geq 1 \mid \zeta_n(u) \neq 0\}$, where $\Phi(u) = (\zeta_n(u))_{n \geq 1}$, is finite and of cardinality N .

Theorem 5.1.5 (Classification of traveling waves). A potential $u_0 \in L_{r,0}^2(\mathbb{T})$ defines a traveling wave for equation (5.1) if and only if

- either u_0 is a one gap potential;
- either u_0 is a two gap potential, and the two nonzero indexes $p < q$ satisfy, with $\gamma_p = |\zeta_p|^2$ and $\gamma_q = |\zeta_q|^2$,

$$0 < \gamma_p < \frac{1}{2} \left(p + \sqrt{p^2 + 4q \frac{p+q}{3}} \right)$$

and

$$\gamma_q = \frac{q \frac{p+q}{3} - \gamma_p^2 + p\gamma_p}{2\gamma_p + q}.$$

Note that from [GK20], Appendix B, the one gap potentials are the only traveling wave solutions to the Benjamin-Ono equation; they have been characterized by Amick and Toland [AT91].

Our last main result answers the question of orbital stability for these two types of traveling waves.

Definition 5.1.6. Let $u_0 \in L^2_{r,0}(\mathbb{T})$ be a one gap traveling wave. We say the u_0 is orbitally stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If v is a solution to (5.1) with initial condition $v_0 \in L^2_{r,0}(\mathbb{T})$ such that $\|v_0 - u_0\|_{L^2(\mathbb{T})} \leq \delta$, then

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{T}} \|v(t) - u_0(\cdot + \theta)\|_{L^2(\mathbb{T})} \leq \varepsilon.$$

Theorem 5.1.7 (Orbital stability of the traveling waves). *The one gap traveling waves are orbitally stable, whereas the two gap traveling waves are orbitally unstable.*

For the Benjamin-Ono equation, Pava and Natali [PN08] proved the orbital stability of the traveling wave solutions in $H^{\frac{1}{2}}_{r,0}(\mathbb{T})$. In [GKT20b], Theorem 4, Gérard, Kappeler and Topalov improved the orbital stability of these solutions to $H^s_{r,0}(\mathbb{T})$, $s > -\frac{1}{2}$.

Plan of the paper The paper is organized as follows. We first prove the well-posedness threshold for the third order Benjamin-Ono equation (5.1) in Section 5.2. Then, in Section 5.3, we classify the traveling wave solutions and study their orbital stability properties.

In Appendix 5.A.1, we describe how to compute the Hamiltonians \mathcal{H}_k and frequencies $\omega_n^{(k)} = \frac{\partial \mathcal{H}_k \circ \Phi^{-1}}{\partial |\zeta_n|^2}$ in terms of the action variables $|\zeta_p|^2$. In Appendix 5.A.2, we retrieve the Hamiltonian and frequencies of the third order Benjamin-Ono equation (see formulas (5.4) and (5.5)) by starting from the definition (5.3) of the higher order Hamiltonians. In Appendix 5.A.3, we provide an alternative proof of a result from [TV13b] about the structure of the higher order Hamiltonians by using formula (5.3), which may be of independent interest.

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5.2 Well-posedness threshold for the fourth Hamiltonian

For $N \in \mathbb{N}$, let \mathcal{U}_N be the set

$$\mathcal{U}_N = \{u \in L^2_{r,0}(\mathbb{T}) \mid \zeta_N(u) \neq 0, \quad \zeta_j(u) = 0 \quad \forall j > N\}.$$

We know from [GK20], Theorem 7.1, that the restriction of the Birkhoff map Φ to \mathcal{U}_N is a real analytic diffeomorphism onto some euclidean space. In Birkhoff coordinates, the evolution along the flow of equation (5.1) for an initial data $u_0 \in \mathcal{U}_N$ writes

$$\begin{cases} \partial_t \zeta_n = i\omega_n^{(4)}(u_0)\zeta_n \\ \zeta_n(0) = \zeta_n(u_0) \end{cases}, \quad n \geq 1,$$

where for all $n \geq 1$, the frequencies $\omega_n^{(4)}(u_0)$ are given by (5.5)

$$\begin{aligned} \omega_n^{(4)}(u_0) = n^3 + n \sum_{p \geq 1} p |\zeta_p(u_0)|^2 - 3 \sum_{p \geq 1} \min(p, n)^2 |\zeta_p(u_0)|^2 \\ + 3 \sum_{p, q \geq 1} \min(p, q, n) |\zeta_p(u_0)|^2 |\zeta_q(u_0)|^2. \end{aligned}$$

This implies that

$$\zeta_n(u(t)) = \zeta_n(u_0) e^{i\omega_n^{(4)}(u_0)t}, \quad t \in \mathbb{R}, \quad n \geq 1.$$

Therefore, for any finite gap initial data u_0 belonging to some of the sets \mathcal{U}_N , the flow map $\mathcal{S}^t : u_0 \in \mathcal{U}_N \mapsto u(t) \in \mathcal{U}_N$ is well-defined.

In part 5.2.1, we prove that for all $t \in \mathbb{R}$, this flow map extends by continuity to $H_{r,0}^s(\mathbb{T})$ for $s \geq 0$. We also show that the extension is sequentially weakly continuous in $H_{r,0}^s(\mathbb{T})$ for $s > 0$, but not in $L_{r,0}^2(\mathbb{T})$. In part 5.2.2, we prove that the flow map does not extend by continuity to $H_{r,0}^{-s}(\mathbb{T})$ for $s > 0$. This gives a threshold for the global \mathcal{C}^0 -well-posedness of the third order Benjamin-Ono equation in the sense of Definitions 1 and 2 from [GKT20b].

5.2.1 Well-posedness in $H_{r,0}^s(\mathbb{T})$, $s \geq 0$

Proposition 5.2.1. *Let $s \geq 0$. For any $u_0 \in H_{r,0}^s(\mathbb{T})$, there exists a continuous map $t \in \mathbb{R} \mapsto \mathcal{S}^t(u_0) = u(t) \in H_{r,0}^s(\mathbb{T})$ with $u(0) = u_0$ such that the following holds.*

For any finite gap sequence $(u_0^k)_k$ converging to u_0 in $H_{r,0}^s(\mathbb{T})$, for any $t \in \mathbb{R}$, $u_k(t) = \mathcal{S}^t(u_0^k)$ converges to $u(t)$ in $H_{r,0}^s(\mathbb{T})$ as k goes to infinity.

Moreover, the extension of the flow map $\mathcal{S} : u_0 \in H_{r,0}^s(\mathbb{T}) \mapsto (t \mapsto u(t)) \in \mathcal{C}(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$ is continuous.

We use the following result from Proposition 5 of Appendix A in [GKT20b]. For $s \geq 0$, the Birkhoff map Φ defines a homeomorphism between $H_{r,0}^s(\mathbb{T})$ and the space

$$h_+^{\frac{1}{2}+s} = \left\{ (\zeta_n)_{n \geq 1} \mid \sum_{n \geq 1} n^{1+2s} |\zeta_n|^2 < +\infty \right\}.$$

The proof of Proposition 5.2.1 therefore relies on the following sequential convergence result obtained after applying the Birkhoff map.

Lemma 5.2.2. *Fix $s \geq 0$. Let $\zeta^k = (\zeta_n^k)_{n \geq 1}$, $k \in \mathbb{N}$, and ζ be elements of $h_+^{\frac{1}{2}+s}$ such that $\|\zeta^k - \zeta\|_{h_+^{\frac{1}{2}+s}} \xrightarrow{k \rightarrow +\infty} 0$. Then for all $t \in \mathbb{R}$,*

$$\|(\zeta_n^k e^{i\omega_n^{(4)}(\zeta^k)t})_n - (\zeta_n e^{i\omega_n^{(4)}(\zeta)t})_n\|_{h_+^{\frac{1}{2}+s}} \xrightarrow{k \rightarrow +\infty} 0,$$

where the convergence is uniform on bounded time intervals.

Proof. Note that since $(\zeta^k)_k$ converges to ζ in $h_+^{\frac{1}{2}+s}$, then for all $n \in \mathbb{N}$, formula (5.5) for $\omega_n^{(4)}(\zeta^k)$ implies that $\omega_n^{(4)}(\zeta^k)$ converges to $\omega_n^{(4)}(\zeta)$ as k goes to infinity.

Let $\varepsilon > 0$. Fix $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\|\zeta^k - \zeta\|_{h_+^{\frac{1}{2}+s}} \leq \varepsilon.$$

Using that $\zeta \in h_+^{\frac{1}{2}+s}$, fix $N \in \mathbb{N}$ such that

$$\left(\sum_{n \geq N} n^{1+2s} |\zeta_n|^2 \right)^{\frac{1}{2}} \leq \varepsilon.$$

Now, if $k \geq K$,

$$\begin{aligned} & \|(\zeta_n^k e^{i\omega_n^{(4)}(\zeta^k)t})_n - (\zeta_n e^{i\omega_n^{(4)}(\zeta)t})_n\|_{h_+^{\frac{1}{2}+s}} \\ & \leq \|(\zeta_n^k)_n - (\zeta_n)_n\|_{h_+^{\frac{1}{2}+s}} + \|(\zeta_n(e^{i\omega_n^{(4)}(\zeta^k)t} - e^{i\omega_n^{(4)}(\zeta)t}))_n\|_{h_+^{\frac{1}{2}+s}} \\ & \leq 3\varepsilon + \left(\sum_{n=0}^{N-1} n^{1+2s} |\zeta_n(e^{i\omega_n^{(4)}(\zeta^k)t} - e^{i\omega_n^{(4)}(\zeta)t})|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which is less than 4ε for k large enough by convergence term by term of the elements in the sum. Moreover, this convergence is uniform on bounded time intervals. \square

Proof of Proposition 5.2.1. Let $s \geq 0$ and $u_0 \in H_{r,0}^s(\mathbb{T})$. Fix $t \in \mathbb{R}$, and a sequence of finite gap initial data $(u_0^k)_k$ converging to u_0 in $H_{r,0}^s(\mathbb{T})$.

We first establish that for all $t \in \mathbb{R}$, $(u_k(t))_k$ has a limit in $H_{r,0}^s(\mathbb{T})$ as k goes to $+\infty$. By assumption, $\Phi(u_0^k)$ converges to $\Phi(u_0)$ in $h_+^{\frac{1}{2}+s}$. Define the sequence $\zeta(t)$ by

$$\zeta_n(t) := \zeta_n(u_0) e^{i\omega_n^{(4)}(u_0)t}, \quad n \in \mathbb{N}.$$

Lemma 5.2.2 immediately implies that the sequence $(\Phi(u_k(t)))_k$ converges to $\zeta(t)$ in $h_+^{\frac{1}{2}+s}$. Since Φ^{-1} defines a continuous application from $h_+^{\frac{1}{2}+s}$ to $H_{r,0}^s(\mathbb{T})$, we deduce that $u_k(t)$ converges in $H_{r,0}^s(\mathbb{T})$ to $u(t) := \Phi^{-1}(\zeta(t))$. Fix $T > 0$. Because the convergence of $\Phi(u_k(t))$ to $\zeta(t)$ in $h_+^{\frac{1}{2}+s}$ is uniform on bounded time intervals, the union of $\Phi(u_k(t))$ and $\zeta(t)$ for $k \in \mathbb{N}$ and $t \in [0, T]$ is a compact. By continuity and therefore uniform continuity of Φ^{-1} on this compact, we deduce that u_k converges to u in $\mathcal{C}([0, T], H_{r,0}^s(\mathbb{T}))$.

We now prove the continuity of the flow map \mathcal{S}^t . Let $u_0^k \in H_{r,0}^s(\mathbb{T})$, $k \in \mathbb{N}$, be a sequence of initial data converging to some u_0 in $H_{r,0}^s(\mathbb{T})$. Then $\Phi(u_0^k)$ converges to $\Phi(u_0)$ in $h_+^{\frac{1}{2}+s}$, and the above Lemma 5.2.2 again implies that $\Phi(u_k(t))$ converges to $\Phi(u(t))$ in $h_+^{\frac{1}{2}+s}$. In other terms, $u_k(t)$ converges to $u(t)$ in $H_{r,0}^s(\mathbb{T})$. With the same argument as in the latter paragraph, this convergence is uniform on bounded intervals. \square

Corollary 5.2.3. *For all $s > 0$ and all $t \in \mathbb{R}$, the extension of the flow map restricted to $H_{r,0}^s(\mathbb{T}) : u_0 \in H_{r,0}^s(\mathbb{T}) \mapsto u(t) \in H_{r,0}^s(\mathbb{T})$ is sequentially weakly continuous.*

Proof. Let $u_0^k \in H_{r,0}^s(\mathbb{T})$, $k \in \mathbb{N}$, be a sequence weakly converging in $H_{r,0}^s(\mathbb{T})$ to $u_0 \in H_{r,0}^s(\mathbb{T})$. Since the embedding $H_{r,0}^s(\mathbb{T}) \hookrightarrow L_{r,0}^2(\mathbb{T})$ is compact, $(u_0^k)_k$ is strongly convergent to u_0 in $L_{r,0}^2(\mathbb{T})$. By continuity of the flow map \mathcal{S}^t , one deduces that $(u_k(t))_k$ converges strongly to $u(t)$ in $L_{r,0}^2(\mathbb{T})$. This implies that $(u_k(t))_k$ converges weakly to $u(t)$ in $H_{r,0}^s(\mathbb{T})$. \square

Proposition 5.2.4. *For all $t \in \mathbb{R}^*$, the extension of the flow map restricted to $L_{r,0}^2(\mathbb{T}) : u_0 \in L_{r,0}^2(\mathbb{T}) \mapsto u(t) \in L_{r,0}^2(\mathbb{T})$ is not sequentially weakly continuous.*

Proof. Fix $t \in \mathbb{R}^*$ and $u_0 \in L_{r,0}^2(\mathbb{T}) \setminus \{0\}$. We construct a sequence $(u_0^k)_k$ in $L_{r,0}^2(\mathbb{T})$ weakly convergent to u_0 in $L_{r,0}^2(\mathbb{T})$ but such that $u_k(t) = \mathcal{S}^t(u_0^k)$ is not weakly convergent to $u(t) = \mathcal{S}^t(u_0)$ in $L_{r,0}^2(\mathbb{T})$.

Let $\alpha > 0$ to be chosen later. For $k \in \mathbb{N}$, we choose $(\zeta_p(u_0^k))_p$ converging weakly to $(\zeta_p(u_0))_p$ in $h_+^{\frac{1}{2}}$ (so that u_0^k converges weakly to u_0 in $L_{r,0}^2(\mathbb{T})$ by weak sequential continuity of Φ^{-1} , see [GKT20b], Remark 6 (iii)) and such that

$$|\zeta_p(u_0^k)|^2 = |\zeta_p(u_0)|^2 + \frac{\alpha}{p} \delta_{k,p}, \quad p \geq 1.$$

For instance, for $p \neq k$ we make the choice $\zeta_p(u_0^k) = \zeta_p(u_0)$, and for $p = k$, we make the choice $\zeta_k(u_0^k) = \sqrt{|\zeta_k(u_0)|^2 + \frac{\alpha}{k} \frac{\zeta_k(u_0)}{|\zeta_k(u_0)|}}$ if $\zeta_k(u_0) \neq 0$ and $\zeta_k(u_0^k) = \frac{\alpha}{k}$ if $\zeta_k(u_0) = 0$.

Fix $t \neq 0$. If $u_k(t)$ was weakly convergent to u_0 in $L_{r,0}^2(\mathbb{T})$, then $(\zeta_p(u_k(t)))_p$ would converge weakly to $(\zeta_p(u(t)))_p$ in $h_+^{\frac{1}{2}}$, and therefore component by component :

$$\zeta_p(u_0^k) e^{i\omega_p^{(4)}(u_0^k)t} \xrightarrow[k \rightarrow +\infty]{} \zeta_p(u_0) e^{i\omega_p^{(4)}(u_0)t}, \quad p \geq 1.$$

In particular, let $p \geq 1$ such that $\zeta_p(u_0) \neq 0$. Then there exists a sequence $(n_k)_k$ of integers such that

$$\omega_p^{(4)}(u_0^k)t + 2\pi n_k \xrightarrow[k \rightarrow +\infty]{} \omega_p^{(4)}(u_0)t.$$

From the expression (5.5) of $\omega_p^{(4)}(u_0^k)$ and the strong convergence of $(\zeta_p(u_0^k))_p$ to $(\zeta_p(u_0))_p$ in $\ell_+^2 = \{(\zeta_p)_{p \geq 1} \mid \sum_{p \geq 1} |\zeta_p|^2 < +\infty\}$ by compactness of the embedding $h_+^{\frac{1}{2}} \hookrightarrow \ell_+^2$, we deduce

$$\sum_{p=1}^{+\infty} p |\zeta_p(u_0)|^2 + \alpha + \frac{2\pi n_k}{t} = \sum_{p=1}^{+\infty} p |\zeta_p(u_0^k)|^2 + \frac{2\pi n_k}{t} \xrightarrow[k \rightarrow +\infty]{} \sum_{p=1}^{+\infty} p |\zeta_p(u_0)|^2.$$

We get a contradiction by choosing $\alpha \notin \frac{2\pi}{t}\mathbb{Z}$. □

5.2.2 Ill-posedness in $H_{r,0}^{-s}(\mathbb{T})$, $s > 0$

Proposition 5.2.5. *For all $t > 0$, there exists no continuous local extension of the flow map \mathcal{S}^t to $H_{r,0}^{-s}(\mathbb{T})$ when $0 < s < \frac{1}{2}$, not even when $H_{r,0}^{-s}(\mathbb{T})$ is equipped with the weak topology.*

Proof. Let us fix $0 < s < \frac{1}{2}$ and an initial data $u_0 \in H_{r,0}^{-s}(\mathbb{T}) \setminus L_{r,0}^2(\mathbb{T})$. Let u_0^k , $k \in \mathbb{N}$, be a sequence finite gap initial data, to be chosen later, such that u_0^k converges in $H_{r,0}^{-s}(\mathbb{T})$ to u_0 .

From [GKT20b], Theorem 6, the Birkhoff map extends by continuity as an homeomorphism

$$\Phi : u \in H_{r,0}^{-s} \mapsto \Phi(u) = (\zeta_n(u))_{n \geq 1} \in h_+^{\frac{1}{2}-s}$$

where

$$h_+^{\frac{1}{2}-s} = \left\{ (\zeta_n)_{n \geq 1} \mid \sum_{n \geq 1} n^{1-2s} |\zeta_n|^2 < +\infty \right\}.$$

Therefore, $(\zeta_n(u_0))_{n \geq 1} := \Phi(u_0) \in h_+^{\frac{1}{2}-s}$ is well-defined. For $k \in \mathbb{N}$, write

$$\Phi(u_0^k) = (\zeta_n(u_0^k) \mathbb{1}_{n \leq N_k})_n.$$

Since u_0^k is a finite gap potential, it belongs to $L_{r,0}^2(\mathbb{T})$. Recall that

$$\omega_n^{(4)}(u_0^k) - n \sum_{p=1}^{N_k} p |\zeta_p(u_0^k)|^2 = \widetilde{\omega}_n(u_0^k)$$

where

$$\widetilde{\omega}_n(u_0^k) = n^3 - 3 \sum_{p=1}^{N_k} \min(p, n)^2 |\zeta_p(u_0^k)|^2 + 3 \sum_{p=1}^{N_k} \sum_{q=1}^{N_k} \min(p, q, n) |\zeta_p(u_0^k)|^2 |\zeta_q(u_0^k)|^2.$$

Since u_0^k converges to u_0 in $H_{r,0}^{-s}(\mathbb{T})$, the series $\sum_{p \geq 1} |\zeta_p(u_0)|^2$ is convergent, and

$$\sum_{p \geq 1} |\zeta_p(u_0^k)|^2 \xrightarrow{k \rightarrow +\infty} \sum_{p \geq 1} |\zeta_p(u_0)|^2.$$

In particular, the term $\widetilde{\omega}_n(u_0^k)$ converges as k goes to infinity to

$$\widetilde{\omega}_n(u_0) = n^3 - 3 \sum_{p=1}^{+\infty} \min(p, n)^2 |\zeta_p(u_0)|^2 + 3 \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} \min(p, q, n) |\zeta_p(u_0)|^2 |\zeta_q(u_0)|^2.$$

For $k \in \mathbb{N}$, let

$$\tau_k := \sum_{p=1}^{N_k} p |\zeta_p(u_0^k)|^2 = \frac{1}{2} \|u_0^k\|_{L^2(\mathbb{T})}^2$$

and

$$v_k(t, \cdot) := u_k(t, \cdot - \tau_k t).$$

We use the following identity from the proof of Proposition B.1. in [GK20] :

$$\zeta_n(u(\cdot + \tau)) = \zeta_n(u) e^{i\tau n}, \quad \tau \in \mathbb{R}, \quad u \in L_{r,0}^2(\mathbb{T}),$$

to deduce that for $n \in \mathbb{N}$,

$$\begin{aligned} \zeta_n(v_k(t)) &= \zeta_n(u_k(t)) e^{-in\tau_k t} \\ &= \zeta_n(u_0^k) e^{i(\omega_n^{(4)}(u_0^k) - n\tau_k)t}. \end{aligned}$$

Since

$$\omega_n^{(4)}(u_0^k) - n\tau_k \xrightarrow{k \rightarrow +\infty} \widetilde{\omega}_n(u_0),$$

the sequence $(\zeta_n(v_k(t)))_k$ is convergent :

$$\zeta_n(v_k(t)) \xrightarrow{k \rightarrow +\infty} \zeta_n(u_0) e^{i\widetilde{\omega}_n(u_0)t}. \quad (5.6)$$

Let $t > 0$. If there was a local extension of the flow map \mathcal{S}^t in $H_{r,0}^{-s}(\mathbb{T})$ (equipped with the weak topology), then $u_k(t)$ would be weakly convergent to $u(t)$ in $H_{r,0}^{-s}(\mathbb{T})$. Applying the Birkhoff map, which is weakly sequentially continuous (see [GKT20b], Remark 6 (iii)), $\Phi(u_k(t))$ would converge weakly to $\Phi(u(t))$ in $h_+^{\frac{1}{2}-s}$. In particular, for all n ,

$$\zeta_n(v_k(t)) e^{i\tau_k n t} = \zeta_n(u_k(t)) \xrightarrow{k \rightarrow +\infty} \zeta_n(u(t)). \quad (5.7)$$

We deduce from (5.6) and (5.7) that if $\zeta_n(u_0) \neq 0$, then

$$e^{i\tau_k nt} \xrightarrow[k \rightarrow +\infty]{} \frac{\zeta_n(u(t))}{\zeta_n(u_0)} e^{-i\widetilde{\omega}_n(u_0)t}. \quad (5.8)$$

We construct the sequence $(u_0^k)_k$ in order to contradict this latter point. Let $n \in \mathbb{N}$ such that $\zeta_n(u_0) \neq 0$. Fix $k \in \mathbb{N}$. From the fact that u_0 does not belong to $L^2_{r,0}(\mathbb{T})$,

$$\sum_{p>k} p|\zeta_p(u_0)|^2 = +\infty,$$

therefore one can choose $N_k \geq k + 1$ such that

$$\sum_{p=k+1}^{N_k} p|\zeta_p(u_0)|^2 \geq \frac{2\pi}{nt}.$$

Let $0 < \alpha_k < 1$ such that there exists an integer m_k such that

$$\sum_{p \leq k} p|\zeta_p(u_0)|^2 + \alpha_k \sum_{p=k+1}^{N_k} p|\zeta_p(u_0)|^2 = \frac{1}{nt}(k\pi + 2\pi m_k).$$

We define u_0^k by

$$\zeta_p(u_0^k) = \begin{cases} \zeta_p(u_0) & \text{if } p \leq k \\ \sqrt{\alpha_k} \zeta_p(u_0) & \text{if } k < p \leq N_k \\ 0 & \text{if } N_k < p \end{cases}, \quad p \in \mathbb{N}.$$

By construction, u_0^k is finite gap and converges to u_0 in $H^{-s}_{r,0}(\mathbb{T})$. However,

$$\begin{aligned} \tau_k &= \sum_{p \leq k} p|\zeta_p(u_0)|^2 + \alpha_k \sum_{p=k+1}^{N_k} p|\zeta_p(u_0)|^2 \\ &= \frac{1}{nt}(k\pi + 2\pi m_k), \end{aligned}$$

which implies that

$$e^{i\tau_k nt} = (-1)^k.$$

In particular, the sequence $(e^{i\tau_k nt})_k$ is not convergent, and we get a contradiction with (5.8). \square

Remark 5.2.6. *We expect that with a similar argument, one can prove the following fact. For the higher equations of the hierarchy, the well-posedness threshold increases by $\frac{1}{2}$ for each equation (see Remark 5.A.1).*

5.3 Traveling waves for the fourth Hamiltonian

In this part, we classify all traveling wave solutions to equation (5.1)

$$\partial_x(-cu - \partial_{xx}u - \frac{3}{2}u^2 H \partial_x u - \frac{3}{2}H(u \partial_x u) + u^3) = 0.$$

A traveling wave of speed $c \in \mathbb{R}$ is a solution to (5.1) of the form $u(t, x) = u_0(x + ct)$.

The argument of [GK20], Proposition B.1., applies for the higher equations of the Benjamin-Ono hierarchy, implying that a potential $u_0 \in L^2_{r,0}(\mathbb{T})$ is a traveling wave solution to (5.1) of speed $c \in \mathbb{R}$ if and only if for all $t \in \mathbb{R}$ and $n \geq 1$,

$$e^{icnt} \zeta_n(u_0) = e^{i\omega_n^{(4)}(u_0)t} \zeta_n(u_0),$$

or in other words :

$$\text{for all } n \geq 1, \text{ if } \zeta_n(u_0) \neq 0, \text{ then } cn = \omega_n^{(4)}(u_0). \quad (5.9)$$

In particular, all one gap potentials are traveling wave solutions. Note that these potentials are the only traveling wave solutions to the Benjamin-Ono equation and write (see [GK20], Appendix B)

$$u_0(x) = \frac{pw e^{ipx}}{1 - w e^{ipx}} + \frac{p\bar{w} e^{-ipx}}{1 - \bar{w} e^{-ipx}}$$

with the nonzero gap being at index $p \geq 1$ and $w = \frac{\zeta_p(u_0)}{\sqrt{p + \gamma_p(u_0)}}$. As we will see in this section, the one gap potentials are not the only traveling wave solutions for equation (5.1).

In part 5.3.1, we first show that the traveling waves in $L^2_{r,0}(\mathbb{T})$ are necessarily one gap and two gap potentials, then provide a classification of the two gap traveling waves in term of their actions. In part 5.3.2, we prove that the one gap traveling waves are orbitally stable whereas the two gap traveling waves are orbitally unstable.

5.3.1 Classification of traveling wave solutions

In the following, it will be more convenient to work with the actions $\gamma_p = |\zeta_p(u_0)|^2$. Formula (5.5) for $\omega_n^{(4)}(u_0)$

$$\omega_n^{(4)}(u_0) = n^3 + n \sum_{p \geq 1} p\gamma_p - 3 \sum_{p \geq 1} \min(p, n)^2 \gamma_p + 3 \sum_{p, q \geq 1} \min(p, q, n) \gamma_p \gamma_q$$

shows that $\frac{\omega_n^{(4)}(u_0)}{n}$ is equivalent to n^2 as n goes to infinity, therefore, from condition (5.9), the traveling waves for the third equation of the hierarchy (5.1) are necessarily finite gap solutions.

Proposition 5.3.1. *Let u_0 be a two gap potential, and $p < q$ be the indices of the two nonzero gaps with gaps $\gamma_p > 0$ and $\gamma_q > 0$. Then u_0 is a traveling wave for equation (5.1) if and only if*

$$0 < \gamma_p < \frac{1}{2} \left(p + \sqrt{p^2 + 4q \frac{p+q}{3}} \right)$$

and

$$\gamma_q = \frac{q \frac{p+q}{3} + p\gamma_p - \gamma_p^2}{2\gamma_p + q}.$$

Proof. Let u_0 be such a two gap potential. Then u_0 is a traveling wave if and only if $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$, where

$$\frac{\omega_p^{(4)}(u_0)}{p} = p^2 + (p\gamma_p + q\gamma_q) - 3p(\gamma_p + \gamma_q) + 3(\gamma_p^2 + 2\gamma_p\gamma_q + \gamma_q^2)$$

and

$$\frac{\omega_q^{(4)}(u_0)}{q} = q^2 + (p\gamma_p + q\gamma_q) - 3\left(\frac{p^2}{q}\gamma_p + q\gamma_q\right) + 3\left(\frac{p}{q}\gamma_p^2 + 2\frac{p}{q}\gamma_p\gamma_q + \gamma_q^2\right).$$

Taking the difference of the two terms, $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$, if and only if

$$0 = q^2 - p^2 + 3\left(-p\left(\frac{p}{q} - 1\right)\gamma_p - (q - p)\gamma_q + \left(\frac{p}{q} - 1\right)\gamma_p^2 + 2\left(\frac{p}{q} - 1\right)\gamma_p\gamma_q\right).$$

Dividing by $3(1 - \frac{p}{q})$, this necessary and sufficient condition becomes

$$0 = q\frac{p+q}{3} + p\gamma_p - q\gamma_q - \gamma_p^2 - 2\gamma_p\gamma_q,$$

i.e.

$$(2\gamma_p + q)\gamma_q = q\frac{p+q}{3} + p\gamma_p - \gamma_p^2. \quad (5.10)$$

Fix $\gamma_p > 0$ and γ_q satisfying this latter equality. We get that $\gamma_q > 0$ if and only if the right-hand side of the equality is positive, i.e.

$$0 < \gamma_p < \frac{1}{2}\left(p + \sqrt{p^2 + 4q\frac{p+q}{3}}\right). \quad (5.11)$$

Conversely, any two gap solution u_0 satisfying (5.10) and (5.11) verifies $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$, therefore is a traveling wave solution. \square

Let us give an idea of the form of a two gap potential u_0 with gaps at indices $p < q$. By Theorem 7.1 in [GK20], the extension of Πu_0 as an holomorphic function on the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ satisfies

$$\Pi u_0(z) = -z \frac{Q'(z)}{Q(z)}$$

where $Q(z) = \det(\text{Id} - zM)$ and $M = (M_{nm})_{0 \leq n, m \leq q-1}$ is a $q \times q$ matrix defined by

$$M_{nm} = \begin{cases} \delta_{m, n+1} & \text{if } \zeta_{n+1} = 0 \\ \sqrt{\mu_{n+1}} \sqrt{\frac{\kappa_m}{\kappa_{n+1}} \frac{\zeta_m(u_0) \overline{\zeta_{n+1}(u_0)}}{(\lambda_m - \lambda_{n-1})}} & \text{if } \zeta_{n+1} \neq 0 \end{cases}.$$

A precise definition of μ_n , κ_n and λ_n can be found in [GK20]. Therefore, Q and Q' respectively write

$$Q(z) = 1 - z^p M_{p-1,0} - z^q M_{q-1,0} M_{p-1,p}$$

and

$$Q'(z) = -pz^{p-1} M_{p-1,0} - qz^{q-1} M_{q-1,0} M_{p-1,p}$$

This leads to

$$u_0(x) = \frac{p e^{ipx} M_{p-1,0} + q e^{iqx} M_{q-1,0} M_{p-1,p}}{1 - e^{ipx} M_{p-1,0} - e^{iqx} M_{q-1,0} M_{p-1,p}} + \frac{p e^{-ipx} \overline{M_{p-1,0}} + q e^{-iqx} \overline{M_{q-1,0} M_{p-1,p}}}{1 - e^{-ipx} \overline{M_{p-1,0}} - e^{-iqx} \overline{M_{q-1,0} M_{p-1,p}}}.$$

Proposition 5.3.2. *There are no three gap traveling waves.*

Proof. Let $p < q < r$ the indices for the nonzero gaps for a three gap potential u_0 . The speeds for each mode write

$$\begin{aligned} \frac{\omega_p^{(4)}(u_0)}{p} &= p^2 + (p\gamma_p + q\gamma_q + r\gamma_r) - 3p(\gamma_p + \gamma_q + \gamma_r) \\ &\quad + 3(\gamma_p^2 + 2\gamma_p(\gamma_q + \gamma_r) + \gamma_q^2 + 2\gamma_q\gamma_r + \gamma_r^2), \end{aligned}$$

$$\begin{aligned} \frac{\omega_q^{(4)}(u_0)}{q} &= q^2 + (p\gamma_p + q\gamma_q + r\gamma_r) - 3\left(\frac{p^2}{q}\gamma_p + q(\gamma_q + \gamma_r)\right) \\ &\quad + 3\left(\frac{p}{q}\gamma_p^2 + 2\frac{p}{q}\gamma_p(\gamma_q + \gamma_r) + \gamma_q^2 + 2\gamma_q\gamma_r + \gamma_r^2\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\omega_r^{(4)}(u_0)}{r} &= r^2 + (p\gamma_p + q\gamma_q + r\gamma_r) - 3\left(\frac{p^2}{r}\gamma_p + \frac{q^2}{r}\gamma_q + r\gamma_r\right) \\ &\quad + 3\left(\frac{p}{r}\gamma_p^2 + 2\frac{p}{r}\gamma_p(\gamma_q + \gamma_r) + \frac{q}{r}\gamma_q^2 + 2\frac{q}{r}\gamma_q\gamma_r + \gamma_r^2\right). \end{aligned}$$

Let us now subtract the equalities.

$$\begin{aligned} \frac{\omega_q^{(4)}(u_0)}{q} - \frac{\omega_p^{(4)}(u_0)}{p} &= q^2 - p^2 - 3\left(\left(\frac{p}{q} - 1\right)p\gamma_p + (q - p)(\gamma_q + \gamma_r)\right) \\ &\quad + 3\left(\left(\frac{p}{q} - 1\right)\gamma_p^2 + 2\left(\frac{p}{q} - 1\right)\gamma_p(\gamma_q + \gamma_r)\right). \end{aligned}$$

Dividing by $3(1 - \frac{p}{q})$, we get that if $\frac{\omega_q^{(4)}(u_0)}{q} = \frac{\omega_p^{(4)}(u_0)}{p}$, then

$$0 = q\frac{p+q}{3} + p\gamma_p - q(\gamma_q + \gamma_r) - \gamma_p^2 - 2\gamma_p(\gamma_q + \gamma_r),$$

or equivalently

$$(q + 2\gamma_p)(\gamma_q + \gamma_r) = q\frac{p+q}{3} + p\gamma_p - \gamma_p^2. \quad (5.12)$$

Doing the same for indices p and r ,

$$\begin{aligned} \frac{\omega_r^{(4)}(u_0)}{r} - \frac{\omega_p^{(4)}(u_0)}{p} &= r^2 - p^2 - 3\left(\left(\frac{p}{r} - 1\right)p\gamma_p + \left(\frac{q^2}{r} - p\right)\gamma_q + (r - p)\gamma_r\right) \\ &\quad + 3\left(\left(\frac{p}{r} - 1\right)\gamma_p^2 + 2\left(\frac{p}{r} - 1\right)\gamma_p(\gamma_q + \gamma_r) + \left(\frac{q}{r} - 1\right)\gamma_q^2 + 2\left(\frac{q}{r} - 1\right)\gamma_q\gamma_r\right). \end{aligned}$$

Dividing by $3(1 - \frac{p}{r})$, if $\frac{\omega_r^{(4)}(u_0)}{r} = \frac{\omega_p^{(4)}(u_0)}{p}$, then

$$0 = r\frac{r+p}{3} + p\gamma_p + \frac{pr - q^2}{r-p}\gamma_q - r\gamma_r - \gamma_p^2 - 2\gamma_p(\gamma_q + \gamma_r) - \frac{r-q}{r-p}\gamma_q^2 - 2\frac{r-q}{r-p}\gamma_q\gamma_r. \quad (5.13)$$

Subtracting (5.13) and (5.12), if $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q} = \frac{\omega_r^{(4)}(u_0)}{r}$, then

$$(r - q + 2\frac{r-q}{r-p}\gamma_q)\gamma_r = \frac{r^2 - q^2 + p(r-q)}{3} + \left(\frac{pr - q^2}{r-p} + q\right)\gamma_q - \frac{r-q}{r-p}\gamma_q^2.$$

Since

$$\frac{pr - q^2}{r - p} + q = \frac{pr - q^2 + qr - pq}{r - p} = (p + q)\frac{r - q}{r - p},$$

by multiplication by $\frac{r-p}{r-q}$, we get

$$(r - p + 2\gamma_q)\gamma_r = (r - p)\frac{r + q + p}{3} + (p + q)\gamma_q - \gamma_q^2. \quad (5.14)$$

Now, if u_0 is a traveling wave, the first equality (5.12) implies

$$\begin{aligned} \gamma_q + \gamma_r &= \frac{q\frac{p+q}{3} + p\gamma_p - \gamma_p^2}{q + 2\gamma_p} \\ &= \frac{p + q}{3} + \frac{-2\gamma_p\frac{p+q}{3} + p\gamma_p - \gamma_p^2}{q + 2\gamma_p} \\ &= \frac{p + q}{3} + \frac{\gamma_p}{3(q + 2\gamma_p)}(p - 2q - 3\gamma_p), \end{aligned}$$

in particular, since $p < q$ and $\gamma_p > 0$, necessarily

$$\gamma_q + \gamma_r < \frac{p + q}{3}. \quad (5.15)$$

However, the second equality (5.14) implies that

$$\begin{aligned} \gamma_r &= \frac{(r - p)\frac{r+q+p}{3} + (p + q)\gamma_q - \gamma_q^2}{r - p + 2\gamma_q} \\ &= \frac{(r - p)\frac{q+p}{3} + (p + q)\gamma_q - \gamma_q^2}{r - p + 2\gamma_q} + \frac{(r - p)\frac{r}{3}}{r - p + 2\gamma_q} \\ &= \frac{p + q}{3} + \frac{-2\gamma_q\frac{p+q}{3} + (p + q)\gamma_q - \gamma_q^2}{r - p + 2\gamma_q} + \frac{(r - p)\frac{r}{3}}{r - p + 2\gamma_q} \\ &= \frac{p + q}{3} + \gamma_q\frac{\frac{p+q}{3} - \gamma_q}{r - p + 2\gamma_q} + \frac{(r - p)r}{3(r - p + 2\gamma_q)}. \end{aligned}$$

Since from (5.15),

$$\gamma_q < \frac{p + q}{3},$$

we get from this latter equality for γ_r that

$$\gamma_r > \frac{p + q}{3},$$

but this is a contradiction with (5.15). \square

Corollary 5.3.3. *There are no N gap traveling waves for $N \geq 3$.*

Proof. The proof is the same as for the three gap traveling waves case, but with some additional terms which might hinder understanding for a first reading. We explain here how to adapt the proof.

Let u_0 be a N gap potential, $N \geq 3$, and $p < q < r < r_4 < \dots < r_N$ be the indices for the nonzero gaps. Let

$$\Gamma_r := \gamma_r + \sum_{k=4}^N \gamma_{r_k}.$$

The speeds for the three smallest modes at indices p, q and r write

$$\begin{aligned} \frac{\omega_p^{(4)}(u_0)}{p} &= p^2 + (p\gamma_p + q\gamma_q + r\gamma_r + \sum_{k=4}^N r_k \gamma_{r_k}) - 3p(\gamma_p + \gamma_q + \Gamma_r) \\ &\quad + 3(\gamma_p^2 + 2\gamma_p(\gamma_q + \Gamma_r) + \gamma_q^2 + 2\gamma_q\Gamma_r + \Gamma_r^2), \end{aligned}$$

$$\begin{aligned} \frac{\omega_q^{(4)}(u_0)}{q} &= q^2 + (p\gamma_p + q\gamma_q + r\gamma_r + \sum_{k=4}^N r_k \gamma_{r_k}) - 3\left(\frac{p^2}{q}\gamma_p + q(\gamma_q + \Gamma_r)\right) \\ &\quad + 3\left(\frac{p}{q}\gamma_p^2 + 2\frac{p}{q}\gamma_p(\gamma_q + \Gamma_r) + \gamma_q^2 + 2\gamma_q\Gamma_r + \Gamma_r^2\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\omega_r^{(4)}(u_0)}{r} &= r^2 + (p\gamma_p + q\gamma_q + r\gamma_r + \sum_{k=4}^N r_k \gamma_{r_k}) - 3\left(\frac{p^2}{r}\gamma_p + \frac{q^2}{r}\gamma_q + r\Gamma_r\right) \\ &\quad + 3\left(\frac{p}{r}\gamma_p^2 + 2\frac{p}{r}\gamma_p(\gamma_q + \Gamma_r) + \frac{q}{r}\gamma_q^2 + 2\frac{q}{r}\gamma_q\Gamma_r + \Gamma_r^2\right). \end{aligned}$$

The rest of the proof is identical up to replacing γ_r by Γ_r everywhere from this point on. \square

5.3.2 Orbital stability

Proposition 5.3.4. *The one gap traveling waves in $L_{r,0}^2(\mathbb{T})$ are orbitally stable.*

Proof. Let u_0 be a one gap traveling wave, u_0^k a sequence of initial data converging to u_0 in $L_{r,0}^2(\mathbb{T})$, and t_k a sequence of times. We prove that up to some subsequence,

$$\inf_{\theta \in \mathbb{T}} \|u_k(t_k) - u_0(\cdot + \theta)\|_{L_{r,0}^2(\mathbb{T})} \xrightarrow{k \rightarrow +\infty} 0.$$

It is enough to show that there exists $\theta \in \mathbb{T}$ such that in $L_{r,0}^2(\mathbb{T})$,

$$u_k(t_k) \xrightarrow{k \rightarrow +\infty} u_0(\cdot + \theta),$$

i.e. such that in $h_+^{\frac{1}{2}}$,

$$\Phi(u_k(t_k)) \xrightarrow{k \rightarrow +\infty} \Phi(u_0(\cdot + \theta)).$$

Recall that

$$\Phi(u_k(t_k)) = (\zeta_n(u_0^k) e^{i\omega_n^{(4)}(u_0^k)t_k})_n.$$

Let $p \geq 1$ be the index for which $\zeta_p(u_0) \neq 0$. Up to some subsequence, there exists $\theta \in \mathbb{T}$ such that $e^{i\omega_p^{(4)}(u_0^k)t_k} \xrightarrow{k \rightarrow +\infty} e^{i\theta}$. Moreover, since u_0^k converges to u_0 in $L_{r,0}^2(\mathbb{T})$,

$$p|\zeta_p(u_0^k) - \zeta_p(u_0)|^2 + \sum_{n \neq p} n |\zeta_n(u_0^k)|^2 \xrightarrow{k \rightarrow +\infty} 0.$$

We deduce that

$$\begin{aligned} \|\Phi(u_k(t_k)) - \Phi(u_0(\cdot + \theta))\|_{h_+^{\frac{1}{2}}}^2 &= p|\zeta_p(u_0^k) e^{i\omega_p^{(4)}(u_0^k)t_k} - \zeta_p(u_0) e^{i\theta}|^2 + \sum_{n \neq p} n|\zeta_n(u_0^k)|^2 \\ &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

□

Proposition 5.3.5. *The two gap traveling waves in $L_{r,0}^2(\mathbb{T})$ are orbitally unstable.*

Proof. Let u_0 be a two gap traveling wave such that the nonzero terms of the sequence $\Phi(u_0) = (\zeta_n(u_0))_{n \geq 1}$ are $\zeta_p(u_0)$ and $\zeta_q(u_0)$. We define the sequence u_0^k of two gap initial data by their nonzero gaps at indices p and q , denoted $\zeta_p(u_0^k)$ and $\zeta_q(u_0^k)$, as follows. We fix $\zeta_p(u_0^k) := \zeta_p(u_0)$ and choose any sequence of nonzero complex numbers $(\zeta_q(u_0^k))_k$ such that $\zeta_q(u_0^k) \xrightarrow{k \rightarrow +\infty} \zeta_q(u_0)$ but for all $k \in \mathbb{N}$, $\varepsilon_k := |\zeta_q(u_0^k)|^2 - |\zeta_q(u_0)|^2 \neq 0$. Then we construct $t_k \in \mathbb{R}$ in order to negate the orbital stability of u_0 .

Assume by contradiction that

$$\inf_{\theta \in \mathbb{T}} \|u_k(t_k) - u_0(\cdot + \theta)\|_{L_{r,0}^2(\mathbb{T})} \xrightarrow{k \rightarrow +\infty} 0.$$

Then there exists a sequence $\theta_k \in \mathbb{T}$, $k \in \mathbb{N}$, such that in $L_{r,0}^2(\mathbb{T})$,

$$\|u_k(t_k, \cdot - \theta_k) - u_0\|_{L_{r,0}^2(\mathbb{T})} \xrightarrow{k \rightarrow +\infty} 0.$$

Applying the Birkhoff map, which is continuous on $L_{r,0}^2(\mathbb{T})$,

$$\zeta_p(u_0^k) e^{i\omega_p^{(4)}(u_0^k)t_k - ip\theta_k} = \zeta_p(u_k(t_k)) e^{-ip\theta_k} \xrightarrow{k \rightarrow +\infty} \zeta_p(u_0)$$

and

$$\zeta_q(u_0^k) e^{i\omega_q^{(4)}(u_0^k)t_k - iq\theta_k} = \zeta_q(u_k(t_k)) e^{-iq\theta_k} \xrightarrow{k \rightarrow +\infty} \zeta_q(u_0).$$

This implies by taking the arguments that for some integers $n_{p,k}$ and $n_{q,k}$,

$$\omega_p^{(4)}(u_0^k)t_k - p\theta_k + 2\pi n_{p,k} \xrightarrow{k \rightarrow +\infty} 0$$

and

$$\omega_q^{(4)}(u_0^k)t_k - q\theta_k + 2\pi n_{q,k} \xrightarrow{k \rightarrow +\infty} 0,$$

therefore

$$pqt_k \left(\frac{\omega_q^{(4)}(u_0^k)}{q} - \frac{\omega_p^{(4)}(u_0^k)}{p} \right) + 2\pi(pn_{q,k} - qn_{p,k}) \xrightarrow{k \rightarrow +\infty} 0. \quad (5.16)$$

However, writing $\varepsilon_k = |\zeta_q(u_0^k)|^2 - |\zeta_q(u_0)|^2 = \gamma_q(u_0^k) - \gamma_q(u_0)$, we get that the speeds of the two modes p and q for the initial data u_0^k are given by

$$\frac{\omega_p^{(4)}(u_0^k)}{p} = \frac{\omega_p^{(4)}(u_0)}{p} + q\varepsilon_k - 3p\varepsilon_k + 6\varepsilon_k(\gamma_p(u_0) + \gamma_q(u_0)) + 3\varepsilon_k^2$$

and

$$\frac{\omega_q^{(4)}(u_0^k)}{q} = \frac{\omega_q^{(4)}(u_0)}{q} + q\varepsilon_k - 3q\varepsilon_k + 6\varepsilon_k\left(\frac{p}{q}\gamma_p(u_0) + \gamma_q(u_0)\right) + 3\varepsilon_k^2.$$

Since u_0 is a traveling wave, $\frac{\omega_p^{(4)}(u_0)}{p} = \frac{\omega_q^{(4)}(u_0)}{q}$ and therefore

$$\frac{\omega_q^{(4)}(u_0^k)}{q} - \frac{\omega_p^{(4)}(u_0^k)}{p} = -3(q - p + 2(1 - \frac{p}{q})\gamma_p(u_0))\varepsilon_k.$$

Since $\varepsilon_k \neq 0$, then $\frac{\omega_q^{(4)}(u_0^k)}{q} \neq \frac{\omega_p^{(4)}(u_0^k)}{p}$. It is therefore possible to choose a sequence t_k such that the limit (5.16) does not hold and get a contradiction. For instance, we can choose t_k such that

$$-3pqt_k(q - p + 2(1 - \frac{p}{q})\gamma_p)\varepsilon_k = \pi.$$

□

5.A Appendices

5.A.1 About the hierarchy

The aim of this Appendix is to provide a way to compute the Hamiltonians \mathcal{H}_k and frequencies $\omega_n^{(k)}$ for the higher order Benjamin-Ono equations in terms of the actions $\gamma_p = |\zeta_p(u_0)|^2$. In particular, we establish formula (5.5)

$$\omega_n^{(4)}(u_0) = n^3 + n \sum_{p \geq 1} p\gamma_p - 3 \sum_{p \geq 1} \min(p, n)^2 \gamma_p + 3 \sum_{p, q \geq 1} \min(p, q, n) \gamma_p \gamma_q$$

for finite gap potentials u_0 .

We first recall some notation. Given $u \in L_{r,0}^2(\mathbb{T})$, we consider its Lax operator $L_u = -i\partial_x - T_u$ acting on $L_+^2(\mathbb{T})$, with domain $H_+^1(\mathbb{T}) = H^1(\mathbb{T}) \cap L_+^2(\mathbb{T})$. The spectrum of L_u is discrete, with eigenvalues

$$\lambda_0(u) < \lambda_1(u) < \dots < \lambda_n(u) < \dots.$$

Moreover (see [GK20], Proposition 2.2, Corollary 3.4 and equality (4.1)), for all $n \geq 1$,

$$\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1$$

is nonnegative and satisfies $\gamma_n(u) = |\zeta_n(u)|^2$. We also define $f_n(u) \in H_+^1(\mathbb{T})$ as the L^2 -normalized eigenfunction for L_u associated to the eigenvalue $\lambda_n(u)$.

Let u_0 be a finite gap potential and use the notation $\lambda_n = \lambda_n(u_0)$, $f_n = f_n(u_0)$. From [GK20], equality (3.8), a variant of the generating function, denoted by $\widetilde{\mathcal{H}}_\varepsilon$, is defined as

$$\widetilde{\mathcal{H}}_\varepsilon = \sum_{n=0}^{+\infty} \frac{|\langle \mathbb{1} | f_n \rangle|^2}{1 + \varepsilon \lambda_n}.$$

From the decomposition $\mathbb{1} = \sum_{n=1}^{+\infty} \langle \mathbb{1} | f_n \rangle f_n$, we know using formula (5.3) that

$$\begin{aligned} \mathcal{H}_k(u_0) &= \sum_{n=0}^{+\infty} |\langle \mathbb{1} | f_n \rangle|^2 \lambda_n^k \\ &= \frac{(-1)^k}{k!} \frac{d^k}{d\varepsilon^k} \Big|_{\varepsilon=0} \widetilde{\mathcal{H}}_\varepsilon. \end{aligned}$$

We now make use of the generating function to derive a recurrence formula for the \mathcal{H}_k . Set

$$g_\varepsilon := -\frac{d}{d\varepsilon} \log \widetilde{\mathcal{H}}_\varepsilon = -\frac{1}{\widetilde{\mathcal{H}}_\varepsilon} \frac{d}{d\varepsilon} \widetilde{\mathcal{H}}_\varepsilon,$$

then from [GK20], equality (3.11), g_ε writes

$$g_\varepsilon = \frac{\lambda_0}{1 + \varepsilon\lambda_0} + \sum_{n=1}^{+\infty} \frac{\gamma_n}{(1 + \varepsilon(\lambda_{n-1} + 1))(1 + \varepsilon\lambda_n)}.$$

Using the identity

$$\begin{aligned} -\frac{d^{k+1}}{d\varepsilon^{k+1}} \widetilde{\mathcal{H}}_\varepsilon &= \frac{d^k}{d\varepsilon^k} (g_\varepsilon \widetilde{\mathcal{H}}_\varepsilon) \\ &= \sum_{l=0}^k \binom{k}{l} \end{aligned}$$

and defining

$$P_l := \frac{(-1)^l}{l!} \frac{d^l}{d\varepsilon^l} \Big|_{\varepsilon=0} (g_\varepsilon),$$

we get the recurrence relation

$$\mathcal{H}_{k+1} = \frac{1}{k+1} \sum_{l=0}^k P_l \mathcal{H}_{k-l}. \quad (5.17)$$

Moreover, the frequencies $\omega_n^{(k)} = \frac{\partial \mathcal{H}_k}{\partial \gamma_n}$, satisfy the recurrence formula

$$\omega_n^{(k+1)} = \frac{1}{k+1} \sum_{l=0}^k \frac{\partial P_l}{\partial \gamma_n} \mathcal{H}_{k-l} + P_l \omega_n^{(k-l)}. \quad (5.18)$$

We now simplify P_l and $\frac{\partial P_l}{\partial \gamma_n}$:

$$P_l = \lambda_0^{l+1} + \sum_{n \geq 1} \gamma_n \sum_{m=0}^l (\lambda_{n-1} + 1)^m \lambda_n^{l-m},$$

and since $\lambda_{n-1} + 1 = \lambda_n - \gamma_n$,

$$P_l = \lambda_0^{l+1} + \sum_{n \geq 1} \lambda_n^{l+1} - (\lambda_n - \gamma_n)^{l+1}.$$

From (3.13), $\lambda_n = n - s_{n+1}$ where $s_n = \sum_{k=n}^{\infty} \gamma_k$ for $n \geq 1$, therefore

$$P_l = (-1)^{l+1} s_1^{l+1} + \sum_{n \geq 1} (n - s_{n+1})^{l+1} - (n - s_n)^{l+1}. \quad (5.19)$$

We deduce

$$\begin{aligned}
\frac{1}{l+1} \frac{\partial P_l}{\partial \gamma_n} &= (-1)^{l+1} s_1^l + (n - s_n)^l - \sum_{p < n} (p - s_{p+1})^l - (p - s_p)^l \\
&= \sum_{p=1}^n (p - s_p)^l - (p - 1 - s_p)^l \\
&= n^l + \sum_{p=1}^n \sum_{m=1}^{l-1} \binom{l}{m} (p^m - (p-1)^m) (-1)^{l-m} s_p^{l-m} \\
&= n^l + \sum_{m=1}^{l-1} \binom{l}{m} (-1)^{l-m} \sum_{\substack{p_1, p_1, \dots, p_{l-m} \\ 1 \leq p \leq \min(n, p_1, \dots, p_{l-m})}} (p^m - (p-1)^m) \gamma_{p_1} \dots \gamma_{p_{l-m}},
\end{aligned}$$

therefore

$$\frac{1}{l+1} \frac{\partial P_l}{\partial \gamma_n} = n^l + \sum_{m=1}^{l-1} \binom{l}{m} (-1)^{l-m} \sum_{p_1, \dots, p_{l-m}} \min(n, p_1, \dots, p_{l-m})^m \gamma_{p_1} \dots \gamma_{p_{l-m}}.$$

Let us compute the first small terms by using (5.17) and (5.18). The Hamiltonian with index 0 is constant $\mathcal{H}_0 = 1$, leading to $\omega_n^{(0)} = 0$, $P_0 = 0$ and $\frac{\partial P_0}{\partial \gamma_n} = 0$.

Concerning index 1, $\mathcal{H}_1 = -\langle u | \mathbf{1} \rangle = 0$ because we assumed that $u \in L_{r,0}^2(\mathbb{T})$ is of average zero. This leads to $\omega_n^{(1)} = 0$, $P_1 = 2 \sum_{p \geq 1} p \gamma_p$ and $\frac{\partial P_1}{\partial \gamma_n} = 2n$. Because of cancellations for several of these small terms, the recurrence relations (5.17) and (5.18) write, for $k \geq 2$,

$$\mathcal{H}_k = \frac{1}{k} (P_1 \mathcal{H}_{k-2} + P_2 \mathcal{H}_{k-3} + \dots + P_{k-3} \mathcal{H}_2 + P_{k-1}) \quad (5.20)$$

and

$$\begin{aligned}
\omega_n^{(k)} &= \frac{1}{k} \left(\frac{\partial P_1}{\partial \gamma_n} \mathcal{H}_{k-2} + \frac{\partial P_2}{\partial \gamma_n} \mathcal{H}_{k-3} + \dots + \frac{\partial P_{k-3}}{\partial \gamma_n} \mathcal{H}_2 \right. \\
&\quad \left. + P_1 \omega_n^{(k-2)} + P_2 \omega_n^{(k-3)} + \dots + P_{k-3} \omega_n^{(2)} + \frac{\partial P_{k-1}}{\partial \gamma_n} \right). \quad (5.21)
\end{aligned}$$

The second index leads to the conservation of the mass

$$\begin{aligned}
\mathcal{H}_2 &= \frac{\|u\|^2}{2} = \frac{P_1}{2}, \\
\omega_n^{(2)} &= n.
\end{aligned}$$

Moreover,

$$P_2 = -s_1^3 + \sum_{p=1}^{\infty} (p - s_{p+1})^3 - (p - s_p)^3 = 3 \sum_{p \geq 1} p^2 \gamma_p - 3 \sum_{p \geq 1} s_p^2$$

and

$$\frac{\partial P_2}{\partial \gamma_n} = 3(n^2 - 2 \sum_{p=1}^{\infty} \min(p, n) \gamma_p).$$

For the third index, we retrieve identity (8.6) from [GK20] for the Hamiltonian

$$\mathcal{H}_3 = \frac{P_2}{3} = \sum_{p \geq 1} p^2 \gamma_p - \sum_{p \geq 1} s_p^2$$

and formula (8.4) from [GK20] for the frequencies

$$\omega_n^{(3)} = n^2 - 2 \sum_{p=1}^{+\infty} \min(p, n) \gamma_p.$$

We now use that

$$P_3 = s_1^4 + \sum_{p=1}^{+\infty} (p - s_{p+1})^4 - (p - s_p)^4$$

and

$$\frac{\partial P_3}{\partial \gamma_n} = 4(n^3 - 3 \sum_p \min(p, n)^2 \gamma_p + 3 \sum_{p, q} \min(p, q, n) \gamma_p \gamma_q)$$

to get the formula for the Hamiltonian of index 4

$$\mathcal{H}_4 = \frac{1}{4}(P_3 + P_1 \mathcal{H}_2) = \frac{1}{4}P_3 + \frac{1}{2}\mathcal{H}_2^2$$

and the frequencies

$$\begin{aligned} \omega_n^{(4)} &= \frac{1}{4} \frac{\partial P_3}{\partial \gamma_n} + n \mathcal{H}_2 \\ &= n^3 + n \sum_{p \geq 1} p \gamma_p - 3 \sum_{p \geq 1} \min(p, n)^2 \gamma_p + 3 \sum_{p, q \geq 1} \min(p, q, n) \gamma_p \gamma_q. \end{aligned}$$

In the same way, using that

$$P_4 = -s_1^5 + \sum_{p=1}^{+\infty} (p - s_{p+1})^5 - (p - s_p)^5$$

and

$$\frac{1}{5} \frac{\partial P_4}{\partial \gamma_n} = n^4 - 4 \sum_{p \geq 1} \min(p, n)^3 \gamma_p + 6 \sum_{p, q \geq 1} \min(p, q, n)^2 \gamma_p \gamma_q - 4 \sum_{p, q, r \geq 1} \min(p, q, r, n) \gamma_p \gamma_q \gamma_r,$$

we can get a formula for the Hamiltonian with index 5

$$\mathcal{H}_5 = \frac{1}{5}(P_4 + P_2 \mathcal{H}_2 + P_1 \mathcal{H}_3)$$

and the frequencies

$$\begin{aligned} \omega_n^{(5)} &= \frac{1}{5} \left(\frac{\partial P_1}{\partial \gamma_n} \mathcal{H}_3 + P_1 \omega_n^{(3)} + \frac{\partial P_2}{\partial \gamma_n} \mathcal{H}_2 + P_2 \omega_n^{(2)} + \frac{\partial P_4}{\partial \gamma_n} \right) \\ &= \frac{2}{5} n \left(\sum_{p \geq 1} p^2 \gamma_p - \sum_{p \geq 1} s_p^2 \right) + \frac{2}{5} \left(\sum_{p \geq 1} p \gamma_p \right) (n^2 - 2 \sum_{p \geq 1} \min(p, n) \gamma_p) \\ &\quad + \frac{3}{5} (n^2 - 2 \sum_{p \geq 1} \min(p, n) \gamma_p) \left(\sum_{p \geq 1} p \gamma_p \right) + \frac{3}{5} \left(\sum_{p \geq 1} p^2 \gamma_p - \sum_{p \geq 1} s_p^2 \right) n + \frac{1}{5} \frac{\partial P_4}{\partial \gamma_n}, \end{aligned}$$

leading to

$$\begin{aligned} \omega_n^{(5)} &= n \left(\sum_{p \geq 1} p^2 \gamma_p - \sum_{p \geq 1} s_p^2 \right) + \left(\sum_{p \geq 1} p \gamma_p \right) (n^2 - 2 \sum_{p \geq 1} \min(p, n) \gamma_p) \\ &+ n^4 - 4 \sum_{p \geq 1} \min(p, n)^3 \gamma_p + 6 \sum_{p, q \geq 1} \min(p, q, n)^2 \gamma_p \gamma_q - 4 \sum_{p, q, r \geq 1} \min(p, q, r, n) \gamma_p \gamma_q \gamma_r. \end{aligned} \quad (5.22)$$

Remark 5.A.1. Note that formulas (5.5) and (5.22) for $\omega_n^{(4)}(u_0)$ and $\omega_n^{(5)}(u_0)$, which have been established for finite gap potentials u_0 , still make sense for $\omega_n^{(4)}(u_0)$ if $u_0 \in L_{r,0}^2(\mathbb{T})$ and for $\omega_n^{(5)}(u_0)$ if $u_0 \in H_{r,0}^{\frac{1}{2}}(\mathbb{T})$, but diverge if $u_0 \in H_{r,0}^s(\mathbb{T})$ for s right below these respective exponents ($s < 0$ and $s < \frac{1}{2}$).

One can actually show by induction the following facts. In the formula (5.19) for P_k , there is one term $c_1 \sum_{p=1}^{+\infty} p^k \gamma_p$, the other terms being convergent if $\sum_{p=1}^{+\infty} p^{k-1} \gamma_p < +\infty$. This implies that in the formula (5.20) for \mathcal{H}_k appears one term $c_2 \sum_{p=1}^{+\infty} p^{k-1} \gamma_p$, the other terms being convergent if $\sum_{p=1}^{+\infty} p^{k-2} \gamma_p < +\infty$. Consequently, in formula (5.21) for $\omega_n^{(k)}$, $k \geq 4$, appears one term $c_3 \sum_{p=1}^{+\infty} p^{k-3} \gamma_p$, the other terms being convergent if $\sum_{p=1}^{+\infty} p^{k-4} \gamma_p < +\infty$.

From these facts, one can see that the formula for $\omega_n^{(k)}$ can be extended by continuity to potentials in $H_{r,0}^{s_k}(\mathbb{T})$ where $s_k = \frac{k}{2} - 2$, however there is no continuous extension to $H_{r,0}^{s_k}(\mathbb{T})$ when $s_k - \frac{1}{2} < s < s_k$. This explains why the well-posedness threshold for the equation associated to the Hamiltonian \mathcal{H}_k in the hierarchy should be $H_{r,0}^{s_k}(\mathbb{T})$.

5.A.2 Equation for the fourth Hamiltonian

From formula (5.3) and the decomposition (2.14) in [GK20] : $\Pi u = -\sum_{n=0}^{+\infty} \lambda_n \langle \mathbb{1} | f_n \rangle f_n$, we see that for $k \geq 2$,

$$\mathcal{H}_k(u) = \langle L_u^{k-2} \Pi u | \Pi u \rangle, \quad (5.23)$$

where

$$L_u(h) = Dh - \Pi(uh), \quad D = -i\partial_x, \quad h \in H_+^1(\mathbb{T}).$$

For instance,

$$\mathcal{H}_3(u) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} u H \partial_x u - \frac{1}{3} u^3 \, dx$$

leads to the Benjamin-Ono equation

$$\partial_t u = H \partial_x^2 u - \partial_x(u^2).$$

Proposition 5.A.2. The Hamiltonian for the third order equation of the Benjamin-Ono hierarchy (5.1) is

$$\mathcal{H}_4(u) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} (\partial_x u)^2 - \frac{3}{4} u^2 H \partial_x u + \frac{1}{4} u^4 \right) - \frac{1}{8} \|u\|_{L^2(\mathbb{T})}^4,$$

therefore the third order equation of the Benjamin-Ono hierarchy writes

$$\partial_t u = \partial_x \left(-\partial_{xx} u - \frac{3}{2} u H \partial_x u - \frac{3}{2} H(u \partial_x u) + u^3 \right)$$

Proof. Let $u_0 \in L^2_{r,0}(\mathbb{T})$. We develop

$$\mathcal{H}_4(u) = \|D\Pi u\|_{L^2(\mathbb{T})}^2 - 2\operatorname{Re}\langle D\Pi u | \Pi(u\Pi u) \rangle + \|\Pi(u\Pi u)\|_{L^2(\mathbb{T})}^2,$$

and study each term separately.

First, since u is real, $\widehat{u}(-n) = \overline{\widehat{u}(n)}$, therefore

$$\|D\Pi u\|_{L^2(\mathbb{T})}^2 = \sum_{n \geq 0} |n|^2 |\widehat{u}(n)|^2 = \frac{1}{2} \sum_{n \in \mathbb{Z}} |n|^2 |\widehat{u}(n)|^2 = \frac{1}{2} \|\partial_x u\|_{L^2(\mathbb{T})}^2.$$

Then, u being with average zero, $u = \Pi u + \overline{\Pi u}$, leading to

$$\begin{aligned} \langle D\Pi u | \Pi(u\Pi u) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} D(\Pi u) u \overline{\Pi u} \, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} D(u) u \overline{\Pi u} \, dx - \frac{1}{2\pi} \int_0^{2\pi} D(\overline{\Pi u}) u \overline{\Pi u} \, dx \end{aligned}$$

so that

$$\begin{aligned} 2\langle D\Pi u | \Pi(u\Pi u) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} D(u^2) \overline{\Pi u} \, dx - \frac{1}{2\pi} \int_0^{2\pi} D(\overline{\Pi u}^2) u \, dx \\ &= -\frac{1}{2\pi} \int_0^{2\pi} u^2 D(\overline{\Pi u}) \, dx + \frac{1}{2\pi} \int_0^{2\pi} \overline{\Pi u}^2 D u \, dx. \end{aligned}$$

Taking the real part,

$$\begin{aligned} 4\operatorname{Re}\langle D\Pi u | \Pi(u\Pi u) \rangle &= 2 \left(\langle D\Pi u | \Pi(u\Pi u) \rangle + \overline{\langle D\Pi u | \Pi(u\Pi u) \rangle} \right) \\ &= -\frac{1}{2\pi} \int_0^{2\pi} u^2 (D(\overline{\Pi u}) + \overline{D\Pi u}) \, dx + \frac{1}{2\pi} \int_0^{2\pi} \overline{\Pi u}^2 D u + (\Pi u)^2 \overline{D u} \, dx. \end{aligned}$$

Using that $\overline{Df} = -D\overline{f}$,

$$\begin{aligned} 4\operatorname{Re}\langle D\Pi u | \Pi(u\Pi u) \rangle &= -\frac{1}{2\pi} \int_0^{2\pi} u^2 (D(\overline{\Pi u}) - D(\Pi u)) \, dx + \frac{1}{2\pi} \int_0^{2\pi} (\overline{\Pi u}^2 - (\Pi u)^2) D u \, dx \\ &= -\frac{1}{2\pi} \int_0^{2\pi} u^2 (D(\overline{\Pi u}) - D(\Pi u)) \, dx + \frac{1}{2\pi} \int_0^{2\pi} (\overline{\Pi u} - \Pi u) u D u \, dx \\ &= -\frac{1}{2\pi} \int_0^{2\pi} u^2 (D(\overline{\Pi u}) - D(\Pi u)) \, dx - \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} D(\overline{\Pi u} - \Pi u) u^2 \, dx \\ &= -\frac{3}{2} \frac{1}{2\pi} \int_0^{2\pi} u^2 (D(\overline{\Pi u}) - D(\Pi u)) \, dx. \end{aligned}$$

It now remains to remark that $D(\overline{\Pi u}) - D(\Pi u) = -H\partial_x u$ in order to conclude the identity

$$2\operatorname{Re}\langle D\Pi u | \Pi(u\Pi u) \rangle = \frac{3}{4} \langle H\partial_x u | u^2 \rangle.$$

Finally, we treat the last term $\|\Pi(u\Pi u)\|_{L^2(\mathbb{T})}^2$. Note that by decomposing $u = \Pi u + \overline{\Pi u}$,

$$\Pi(u\Pi u) = (\Pi u)^2 + \Pi(\overline{\Pi u}\Pi u),$$

therefore

$$|\Pi(u\Pi u)|^2 = |\Pi u|^4 + (\Pi u)^2 \overline{\Pi(\overline{\Pi u \Pi u})} + \overline{\Pi u}^2 \Pi(\overline{\Pi u \Pi u}) + |\Pi(\overline{\Pi u \Pi u})|^2.$$

By removing the useless projections,

$$\begin{aligned} \|\Pi(u\Pi u)\|_{L^2(\mathbb{T})}^2 &= \frac{1}{2\pi} \int_0^{2\pi} |\Pi u|^4 + (\Pi u)^2 \overline{\Pi u \Pi u} + \overline{\Pi u}^2 \overline{\Pi u \Pi u} + |\Pi(\overline{\Pi u \Pi u})|^2 dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\Pi u)^2 \overline{\Pi u}^2 + (\Pi u)^3 \overline{\Pi u} + \overline{\Pi u}^3 \Pi u + |\Pi(\overline{\Pi u \Pi u})|^2 dx. \end{aligned}$$

But if we take the fourth power of the identity $u = \Pi u + \overline{\Pi u}$

$$u^4 = (\Pi u)^4 + \overline{\Pi u}^4 + 4(\Pi u)^3 \overline{\Pi u} + 4\overline{\Pi u}^3 \Pi u + 6\overline{\Pi u}^2 (\Pi u)^2,$$

and make use of the fact that the mean of u is zero, we get

$$\|u\|_{L^4(\mathbb{T})}^4 = \frac{1}{2\pi} \int_0^{2\pi} 4(\Pi u)^3 \overline{\Pi u} + 4\overline{\Pi u}^3 \Pi u + 6(\Pi u)^2 \overline{\Pi u}^2 dx.$$

By subtraction, the following cancellations happen :

$$\|\Pi(u\Pi u)\|_{L^2(\mathbb{T})}^2 - \frac{1}{4} \|u\|_{L^4(\mathbb{T})}^4 = \frac{1}{2\pi} \int_0^{2\pi} |\Pi(\overline{\Pi u \Pi u})|^2 dx - \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} (\Pi u)^2 \overline{\Pi u}^2 dx.$$

To conclude, since $\overline{\Pi u \Pi u}$ is real, we can use the identity $\|f\|^2 + |\langle f | \mathbf{1} \rangle|^2 = 2\|\Pi f\|^2$ for real valued functions $f \in L^2(\mathbb{T})$ to get

$$\frac{1}{2\pi} \int_0^{2\pi} |\Pi(\overline{\Pi u \Pi u})|^2 dx = \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} |\overline{\Pi u \Pi u}|^2 dx - \frac{1}{2} |\langle \overline{\Pi u \Pi u}, \mathbf{1} \rangle|^2,$$

leading to

$$\begin{aligned} \|\Pi(u\Pi u)\|_{L^2(\mathbb{T})}^2 - \frac{1}{4} \|u\|_{L^4(\mathbb{T})}^4 &= -\frac{1}{2} |\langle \overline{\Pi u \Pi u}, \mathbf{1} \rangle|^2 \\ &= -\frac{1}{2} \|\Pi u\|_{L^2(\mathbb{T})}^4 \\ &= -\frac{1}{8} \|u\|_{L^2(\mathbb{T})}^4. \end{aligned}$$

□

5.A.3 Structure of the higher order Hamiltonians

The aim of this Appendix is to give an alternative proof of Proposition 2.2 in [TV13b].

We first recall the notation introduced in [TV13b] for the sake of completeness. For a smooth function $u \in \mathcal{C}^\infty(\mathbb{T})$, define by induction the sets $\mathcal{P}_n(u)$ as

$$\mathcal{P}_1(u) = \{H^{\varepsilon_1} \partial_x^{\alpha_1} u \mid \varepsilon_1 \in \{0, 1\}, \quad \alpha_1 \in \mathbb{N}\},$$

$$\mathcal{P}_2(u) = \{(H^{\varepsilon_1} \partial_x^{\alpha_1} u)(H^{\varepsilon_2} \partial_x^{\alpha_2} u) \mid \varepsilon_1, \varepsilon_2 \in \{0, 1\}, \quad \alpha_1, \alpha_2 \in \mathbb{N}\}$$

and for $n \geq 2$,

$$\mathcal{P}_n(u) = \left\{ \prod_{l=1}^k H^{\varepsilon_l} p_{j_l}(u) \mid k \in \llbracket 2, n \rrbracket, \quad \varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}, \quad \sum_{l=1}^k j_l = n, \quad p_{j_l}(u) \in \mathcal{P}_{j_l}(u) \right\}.$$

Moreover, for $p_n(u) \in \mathcal{P}_n(u)$, the term $\tilde{p}_n(u)$ is uniquely defined from $p_n(u)$ by removing all the symbols H in the expression of $p_n(u)$ and only keeping the symbols $\partial_x^{\alpha_i} u$. In this case, if

$$\tilde{p}_n(u) = \prod_{i=1}^n \partial_x^{\alpha_i} u,$$

the maximal order of derivative involved and the sum of these orders are respectively denoted

$$|p_n(u)| = \sup_{i \in \llbracket 1, n \rrbracket} \alpha_i$$

and

$$\|p_n(u)\| = \sum_{i=1}^n \alpha_i.$$

We now retrieve a proof of the following result (Proposition 2.2 in [TV13b]).

Proposition 5.A.3. *Let $k = 2(m + 1)$ be an even integer. Then there exists $c \in \mathbb{R}$ such that the $(k + 2)$ -th Hamiltonian \mathcal{H}_{k+2} writes, for all $u \in \mathcal{C}^\infty(\mathbb{T})$,*

$$\mathcal{H}_{k+2}(u) = \frac{1}{2} \|u\|_{\dot{H}^{m+1}(\mathbb{T})}^2 + c \int_0^{2\pi} u(H\partial_x^m u)(\partial_x^{m+1} u) \, dx + R,$$

where for some real numbers $c(p)$,

$$R = \sum_{j=3}^{2m+4} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\| = 2m+4-j \\ |p(u)| \leq m}} c(p) \int_0^{2\pi} p(u) \, dx.$$

Note that $\mathcal{H}_{k+2}(u) = \frac{1}{2} E_{k/2}(u)$ with the notation from [TV13b].

Proof. Recall formula (5.23)

$$\begin{aligned} \mathcal{H}_{k+2}(u) &= \langle L_u^k \Pi u | \Pi u \rangle \\ &= \langle L_u^{m+1} \Pi u | L_u^{m+1} \Pi u \rangle \\ &= \langle (D - T_u)^{m+1} \Pi u | (D - T_u)^{m+1} \Pi u \rangle, \end{aligned}$$

where $D = -i\partial_x$ and $T_u : h \in L_+^2(\mathbb{T}) \mapsto \Pi(uh)$.

We expand $\mathcal{H}_{k+2}(u)$ as a sum of terms depending on whether we applied the operator D or the operator T_u when applying L_u .

It is possible to decompose $\mathcal{H}_{k+2}(u)$ as follows :

$$\mathcal{H}_{k+2}(u) = \frac{1}{2} \|u\|_{\dot{H}^{m+1}(\mathbb{T})}^2 + A + B,$$

where $\|\Pi u\|_{\dot{H}^{m+1}(\mathbb{T})}^2 = \frac{1}{2}\|u\|_{\dot{H}^{m+1}(\mathbb{T})}^2$ is obtained when one only applies operator D , A is obtained when one applies only once the operator T_u and $(2m+1)$ times the operator D

$$\begin{aligned} A &= -2\operatorname{Re} \left(\sum_{j=0}^m \langle D^{m-j} \Pi(uD^j \Pi u) | D^{m+1} \Pi u \rangle \right) \\ &= -2\operatorname{Re} \left(\sum_{j=0}^m \langle D^{m-j} (uD^j \Pi u) | D^{m+1} \Pi u \rangle \right), \end{aligned}$$

and B is obtained when we apply at least twice in total the operator T_u .

- We first prove that one can decompose A as

$$A = c \int_0^{2\pi} u(H\partial_x^m u)(\partial_x^{m+1} u) dx + \tilde{A}$$

where for some real numbers $c(p)$,

$$\tilde{A} = \sum_{\substack{p(u) \in \mathcal{P}_3(u) \\ \|p(u)\| = 2m+1 \\ |p(u)| \leq m}} c(p) \int_0^{2\pi} p(u) dx. \quad (5.24)$$

Let $j \in \llbracket 0, m \rrbracket$. By integration by parts and Leibniz' formula,

$$\begin{aligned} \overline{\langle D^{m-j} (uD^j \Pi u) | D^{m+1} \Pi u \rangle} &= \langle D^{m+1-j} (uD^j \Pi u) | D^m \Pi u \rangle \\ &= \langle u D^{m+1} (\Pi u) | D^m \Pi u \rangle + \langle D^{m+1-j} (u) D^j (\Pi u) | D^m \Pi u \rangle \\ &\quad + \sum_{k=1}^{m-j} \binom{m+1-j}{k} \langle D^k (u) D^{m+1-k} (\Pi u) | D^m \Pi u \rangle. \end{aligned}$$

We take the real part and sum over the indices j . When distinguishing the cases $j = 0$ and $j \geq 1$, we see that for some suitable \tilde{A} as in (5.24), A decomposes as

$$A = -2(m+1)\operatorname{Re}(\langle u D^{m+1} (\Pi u) | D^m \Pi u \rangle) - 2\operatorname{Re}(\langle D^{m+1} (u) \Pi u | D^m \Pi u \rangle) + \tilde{A}.$$

Write $\Pi u = \frac{u+iHu}{2}$, then there exists some real constants $c(\varepsilon_1, \varepsilon_2)$ such that

$$A = \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} c(\varepsilon_1, \varepsilon_2) \int_0^{2\pi} u \partial_x^m (H^{\varepsilon_1} u) \partial_x^{m+1} (H^{\varepsilon_2} u) dx - \operatorname{Re}(\langle D^{m+1} (u) iHu | D^m (\Pi u) \rangle) + \tilde{A}.$$

On the one hand, the terms in the sum are simplified as follows (see the remark from Tzvetkov and Visciglia [TV13b]). When $\varepsilon_1 = \varepsilon_2$,

$$\begin{aligned} \int_0^{2\pi} u \partial_x^m (H^{\varepsilon_1} u) \partial_x^{m+1} (H^{\varepsilon_1} u) dx &= \frac{1}{2} \int_0^{2\pi} u \partial_x ((\partial_x^m (H^{\varepsilon_1} u))^2) dx \\ &= -\frac{1}{2} \int_0^{2\pi} \partial_x (u) (\partial_x^m (H^{\varepsilon_1} u))^2 dx \end{aligned}$$

so this term is a remainder term to be added to \tilde{A} . Moreover, by integration by parts,

$$\int_0^{2\pi} u \partial_x^m (u) \partial_x^{m+1} (Hu) dx = - \int_0^{2\pi} \partial_x (u) \partial_x^m (u) \partial_x^m (Hu) dx - \int_0^{2\pi} u \partial_x^{m+1} (u) \partial_x^m (Hu) dx.$$

Therefore, the sum is a linear combination of the term $\int_0^{2\pi} u(H\partial_x^m u)\partial_x^{m+1}u \, dx$ and other terms that can be added to the remainder \tilde{A} .

On the other hand,

$$\begin{aligned} \operatorname{Re}(\langle D^{m+1}(u)iHu|D^m(\Pi u)\rangle) &= \operatorname{Re}(\langle \partial_x^{m+1}(u)Hu|\partial_x^m(\Pi u)\rangle) \\ &= \frac{1}{2} (\operatorname{Re}(\langle \partial_x^{m+1}(u)Hu|\partial_x^m u\rangle) + \operatorname{Re}(\langle \partial_x^{m+1}(u)Hu|i\partial_x^m(Hu)\rangle)) \end{aligned}$$

Since u is real valued, so is Hu , therefore

$$\operatorname{Re}(\langle \partial_x^{m+1}(u)Hu|i\partial_x^m(Hu)\rangle) = 0.$$

By integration by parts, we then write

$$\begin{aligned} \operatorname{Re}(\langle D^{m+1}(u)iHu|D^m(\Pi u)\rangle) &= \frac{1}{4\pi} \int_0^{2\pi} \partial_x^{m+1}(u)Hu\partial_x^m(u) \, dx \\ &= -\frac{1}{8\pi} \int_0^{2\pi} \partial_x(Hu)(\partial_x^m(u))^2 \, dx \end{aligned}$$

as a remainder term to be added to \tilde{A} .

• We now tackle term B , for which we have applied T_u at least twice. We show that it can be written for some real numbers $c(p)$ as a sum

$$B = \sum_{j=4}^{2m+4} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=2m+4-j \\ |p(u)| \leq m}} c(p) \int_0^{2\pi} p(u) \, dx.$$

Let \tilde{B} be one of the terms in B obtained by applying T_u ($j - 1$) times on the left side and ($k - 1$) times on the right side.

Assume that we have applied T_u at least once in each side of the brackets, i.e. $j - 1 \in \llbracket 1, m + 1 \rrbracket$ and $k - 1 \in \llbracket 1, m + 1 \rrbracket$. Then we can apply Leibniz' rule and decompose the left side as a complex linear combination of terms of the form $p(u)$ where $p(u) \in \mathcal{P}_j(u)$, $\|p(u)\| = m + 2 - j$ and $|p(u)| \leq m$ (for the right side we just replace j by k). The term \tilde{B} is therefore a complex linear combination of terms $\int_0^{2\pi} p(u) \, dx$, where $p(u) \in \mathcal{P}_l(u)$ for some $l = j + k \in \llbracket 4, 2m + 4 \rrbracket$, $\|p(u)\| = 2m + 4 - l$ and $|p(u)| \leq m$.

Otherwise, we have applied T_u at least twice in the same side of the brackets, let us say the left, and we only have applied the operator D on the other side : $j - 1 \in \llbracket 2, m + 1 \rrbracket$ and $k - 1 = 0$. Again by Leibniz' rule, \tilde{B} decomposes as a sum

$$\tilde{B} = \sum_{j=3}^{m+2} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=m+2-j \\ |p(u)| \leq m-1}} c(p) \langle p(u)|D^{m+1}\Pi u \rangle.$$

But then by integration by parts and Leibniz' rule again,

$$\begin{aligned}
 \tilde{B} &= \sum_{j=3}^{m+2} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=m+2-j \\ |p(u)| \leq m-1}} c(p) \overline{\langle Dp(u) | D^m \Pi u \rangle} \\
 &= \sum_{j=3}^{m+2} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=m+3-j \\ |p(u)| \leq m}} c'(p) \overline{\langle p(u) | D^m \Pi u \rangle} \\
 &= \sum_{j=4}^{m+3} \sum_{\substack{p(u) \in \mathcal{P}_j(u) \\ \|p(u)\|=2m+4-j \\ |p(u)| \leq m}} c''(p) \int_0^{2\pi} p(u) dx,
 \end{aligned}$$

which is of the desired form. □

Chapitre 6

Long time behavior of solutions for a damped Benjamin-Ono equation

Ce chapitre reprend les résultats de [Gas20a].

Résumé. On considère une équation de Benjamin-Ono sur le tore amortie par les petits modes de Fourier (cos et sin). On montre d'abord le caractère bien posé de cette équation dans $L^2_{r,0}(\mathbb{T})$. Ensuite, on décrit les limites faibles des trajectoires dans $L^2_{r,0}(\mathbb{T})$ lorsque le temps tend vers l'infini, et on prouve que ces limites faibles sont des limites fortes. Enfin, on montre que les normes de Sobolev d'ordre supérieur restent bornées pour cette équation. Notre outil principal est la transformation de Birkhoff pour l'équation de Benjamin-Ono, que nous utilisons comme une transformée de Fourier non linéaire.

Abstract. We consider the Benjamin-Ono equation on the torus with an additional damping term on the smallest Fourier modes (cos and sin). We first prove global well-posedness of this equation in $L^2_{r,0}(\mathbb{T})$. Then, we describe the weak limit points of the trajectories in $L^2_{r,0}(\mathbb{T})$ when time goes to infinity, and show that these weak limit points are strong limit points. Finally, we prove the boundedness of higher-order Sobolev norms for this equation. Our key tool is the Birkhoff map for the Benjamin-Ono equation, that we use as an adapted nonlinear Fourier transform.

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6.1 Introduction

In this paper, we consider the Benjamin-Ono equation on the torus [Ben67], [Ono75]

$$\partial_t u = H\partial_{xx}u - \partial_x(u^2) \quad (\text{BO})$$

with an additional weak damping term on the smallest Fourier modes defined as follows. Fix a parameter $\alpha > 0$, the damped Benjamin-Ono equation writes

$$\partial_t u + \alpha(\langle u|\cos\rangle \cos + \langle u|\sin\rangle \sin) = H\partial_{xx}u - \partial_x(u^2). \quad (\text{BO-}\alpha)$$

The operator H is the Hilbert transform

$$Hf(x) = \sum_{n \in \mathbb{Z}} -i \operatorname{sgn}(n) \widehat{f}(n) e^{inx}, \quad f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}, \quad \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx,$$

with the convention that $\operatorname{sgn}(\pm n) = \pm 1$ if $n \geq 1$, and $\operatorname{sgn}(0) = 0$.

Our aim is to investigate how this damping can affect the long-time behavior of trajectories. Note that our choice of damping breaks both the Hamiltonian and the integrable structures of the Benjamin-Ono equation, moreover, the damping term is not a small perturbation of the Benjamin-Ono equation. For a general damping term with such properties, the problem seems out of the scope of usual techniques.

Our main tool is the construction and study of a Birkhoff map for the Benjamin-Ono equation by Gérard, Kappeler and Topalov in [GK20; GKT20b; GKT20a] (see Gérard [Gér20] for a recent survey of these results), implying that the Benjamin-Ono equation is integrable in the strongest possible sense. This transformation should be seen as a nonlinear Fourier transform adapted to the Benjamin-Ono equation. The classical Fourier transform has proven to be a powerful tool for studying nonlinear partial differential equations, for instance with the development of pseudodifferential and paradifferential calculus. In the same spirit, we will see that the Birkhoff map transforms the damped equation (BO- α) into a system of ODEs on the nonlinear Fourier coefficients $(\zeta_n(u))_{n \geq 1}$, which enables us to study the qualitative properties of the solutions in infinite time.

The Birkhoff map writes

$$\Phi : u \in L^2_{r,0}(\mathbb{T}) \mapsto (\zeta_n(u))_{n \geq 1} \in h^{\frac{1}{2}}_+, \quad (6.1)$$

where $L^2_{r,0}(\mathbb{T})$ is the space of real-valued functions in $L^2(\mathbb{T})$ with zero mean, and

$$h^{\frac{1}{2}}_+ = \left\{ \zeta = (\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \|\zeta\|_{h^{\frac{1}{2}}_+}^2 = \sum_{n=1}^{+\infty} n |\zeta_n|^2 < +\infty \right\}.$$

This map provides a system of coordinates in which the Benjamin-Ono equation can be solved by quadrature. Indeed, if u is a solution to the Benjamin-Ono equation with initial data u_0 , then there exist frequencies $\omega_n(u_0) = n^2 - 2 \sum_{k \geq 1} \min(k, n) |\zeta_k(u_0)|^2$, $n \geq 1$, such that for $t \in \mathbb{R}$ and $n \geq 1$,

$$\zeta_n(u(t)) = \zeta_n(u_0) e^{i\omega_n(u_0)t}.$$

6.1.1 Main results

First, we establish the global well-posedness of the damped Benjamin-Ono equation (BO- α) in $L^2_{r,0}(\mathbb{T})$, and the weak sequential continuity of the solution map.

Theorem 6.1.1 (Global well-posedness). *For all $T > 0$ and $u_0 \in L^2_{r,0}(\mathbb{T})$, there exists a unique solution of equation (BO- α) in the distribution sense in $\mathcal{C}([0, T], L^2_{r,0}(\mathbb{T}))$ with initial data u_0 .*

Moreover, the solution map is continuous and weakly sequentially continuous from $L^2_{r,0}(\mathbb{T})$ to $\mathcal{C}([0, T], L^2_{r,0}(\mathbb{T}))$.

The solutions are global thanks a Lyapunov functional controlling the L^2 norm

$$\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{T})}^2 + 2\alpha |\langle u(t) | e^{ix} \rangle|^2 = 0. \quad (6.2)$$

In order to establish global well-posedness of equation (BO- α) in $L^2_{r,0}(\mathbb{T})$, one could also apply the techniques used for the Benjamin-Ono equation from Molinet [Mol08] and Molinet, Pilod [MP12b]. However, their techniques do not lead to the weak sequential continuity of the flow map, which is needed for describing the weak limit points of $u(t)$ as $t \rightarrow +\infty$.

Then, we show that the weak limit points of trajectories as time goes to infinity follow the LaSalle principle. Moreover, we prove that the trajectories are relatively compact in $L^2_{r,0}(\mathbb{T})$, in other words, all weak limit points are strong limit points.

Theorem 6.1.2 (Weak limit points). *Let u be a solution to the damped Benjamin-Ono equation (BO- α) in $L^2_{r,0}(\mathbb{T})$. Let v_0 be a limit point of the sequence $(u(t))_{t \geq 0}$ for the weak topology in $L^2_{r,0}(\mathbb{T})$ as $t \rightarrow +\infty$. Then we have the following results.*

1. *The solution v to the Benjamin-Ono equation (BO) with initial data v_0 satisfies:*

$$\forall t \in \mathbb{R}, \quad \langle v(t) | e^{ix} \rangle = 0. \quad (6.3)$$

Moreover, condition (6.3) is equivalent to the fact that the weak limit point v_0 does not have two consecutive nonzero Birkhoff coordinates:

$$\forall n \in \mathbb{N}, \quad \zeta_n(v_0) \zeta_{n+1}(v_0) = 0,$$

with the convention $\zeta_0(v_0) = 1$.

2. *Given a solution u , all the limit points for u have the same actions: there exists a sequence $(\gamma_n^\infty)_{n \geq 1}$ such that for all limit point v_0 associated to u , for all $n \geq 1$,*

$$|\zeta_n(v_0)|^2 = \gamma_n^\infty.$$

3. The convergence is strong in $L^2_{r,0}(\mathbb{T})$: if v_0 is a weak limit point associated to the sequence $(u(t_k))_k$, then

$$\|u(t_k) - v_0\|_{L^2(\mathbb{T})} \xrightarrow{k \rightarrow +\infty} 0.$$

Gérard, Kappeler and Topalov proved that the trajectories of the undamped Benjamin-Ono equation (BO) are almost-periodic in $H^s_{r,0}(\mathbb{T})$ for all $s > -\frac{1}{2}$ (see [GK20], Theorem 1.3 and [GKT20a], Theorem 3 and Corollary 8). As a consequence, the solutions are recurrent (see [GK20], Remark 1.4 (ii)): for all solutions u with initial data $u_0 \in L^2_{r,0}(\mathbb{T})$, there exists a sequence $t_n \rightarrow +\infty$ such that $u(t_n)$ tends to u_0 in $L^2_{r,0}(\mathbb{T})$. Both properties do not hold for the damped equation (BO- α) because of characterization (6.3).

In [Tzv10; TV13a; TV13b; TV15; Den15; DTV15], the authors build a sequence $(\mu_n)_n$ of gaussian measures on $L^2(\mathbb{T})$, invariant by the flow of the periodic Benjamin-Ono equation (BO), and associated to the conservation laws for this equation. Each measure μ_n is concentrated on $H^s(\mathbb{T})$ for $s < n - \frac{1}{2}$, and satisfies $\mu_n(H^{n-\frac{1}{2}}(\mathbb{T})) = 0$. Formally, it is defined from the n -th conservation law \mathcal{H}_n as a renormalization of the formula $d\mu_n = e^{-\mathcal{H}_n(u)} du$. Sy [Sy18] extends this construction to get a measure concentrated on $\mathcal{C}^\infty(\mathbb{T})$. With this approach, it was already possible to prove that with probability 1 with respect to the corresponding measure, the solutions of (BO) display the recurrence property mentioned above (see [TV13b], Corollary 1.3 and [Sy18], Corollary 1.2).

An equation similar to (BO- α) has been introduced and studied by Gérard and Grellier in [GG20] starting from the Szegő equation. For $\alpha > 0$, the damped Szegő equation on the torus writes

$$i\partial_t u + i\alpha \langle u | \mathbb{1} \rangle = \Pi(|u|^2 u), \quad (\text{Sz-}\alpha)$$

where Π is the Szegő projector from $L^2(\mathbb{T})$ onto the space

$$L^2_+(\mathbb{T}) = \{u \in L^2(\mathbb{T}) \mid \forall n < 0, \hat{u}(n) = 0\}.$$

Characterization (6.3) is then similar for (BO- α) and (Sz- α). Indeed, let v_0 be a weak limit point of a solution $(u(t))_{t \in \mathbb{R}}$ of (Sz- α) for the weak topology of the natural energy space $H^{\frac{1}{2}}_+(\mathbb{T}) := \Pi(H^{\frac{1}{2}}(\mathbb{T}))$, then the solution v to the cubic Szegő equation with initial data v_0 satisfies $\langle v(t) | \mathbb{1} \rangle = 0$ for all times. As a consequence, the trajectories of (Sz- α) cannot be almost-periodic in $H^{\frac{1}{2}}_+(\mathbb{T})$, whereas almost-periodicity holds for the Szegő equation [GG17], similarly as the Benjamin-Ono case.

However, our result implies that the trajectories for (BO- α) are relatively compact in $L^2_{r,0}(\mathbb{T})$, whereas many initial data for (Sz- α) lead to trajectories which are not relatively compact in the natural energy space $H^{\frac{1}{2}}_+(\mathbb{T})$ (this fact is a consequence of the proof of Theorem 3.1 in [GG20]). One can also note that in the study of (Sz- α), the authors strongly use the fact that one of the two known Lax pairs for the Szegő equation still holds, whereas there is no known Lax pair for (BO- α).

The last part of our paper is devoted to the boundedness for higher-order Sobolev norms. For the Benjamin-Ono equation on the torus (BO), the infinite number conservation laws from Nakamura [Nak79] and Bock, Kruskal [BK79] provide bounds on all the Sobolev norms H^s , $s \geq 0$. We prove that the H^s Sobolev norms also stay bounded when $0 \leq s < \frac{3}{2}$ for the damped Benjamin-Ono equation (BO- α).

Theorem 6.1.3 (Higher-order Sobolev norms). *Fix $0 \leq s < \frac{3}{2}$. Let u be a solution to the damped equation (BO- α) in $L^2_{r,0}(\mathbb{T})$ such that the initial data u_0 belongs to $H^s(\mathbb{T})$. Then there exists some constant $C_s(\|u_0\|_{H^s(\mathbb{T})}) > 0$ such that for all $t \in \mathbb{R}_+$,*

$$\|u(t)\|_{H^s(\mathbb{T})} \leq C_s(\|u_0\|_{H^s(\mathbb{T})}).$$

The proof of Theorem 6.1.3 could be adapted to show that there is no growth of Sobolev norms in $H^s_{r,0}(\mathbb{T})$ for any $s \geq 0$. However, our method becomes more and more technical as the exponent s increases, this is why we chose to stop at $s = \frac{3}{2}$. See Remark 6.5.6 for more details.

The higher-order Sobolev norms for the Benjamin-Ono equation with and without damping have the same behavior, which is in contrast with the Szegő equation. Indeed, Gérard and Grellier proved in [GG20] that for the damped Szegő equation (Sz- α), there exists a non empty open subset of $H^{\frac{1}{2}}_+(\mathbb{T})$ for which every solution u with initial data in $H^s(\mathbb{T})$, $s > \frac{1}{2}$, has an exploding orbit in $H^s(\mathbb{T})$:

$$\|u(t)\|_{H^s(\mathbb{T})} \xrightarrow[t \rightarrow +\infty]{} +\infty.$$

However, the undamped Szegő equation exhibits weak turbulence [GG17], in the sense that there exists a dense G_δ subset of initial data in $L^2_+ \cap \mathcal{C}^\infty$ such that for every $s > \frac{1}{2}$, the solution u satisfies $\limsup_t \|u(t)\|_{H^s(\mathbb{T})} = +\infty$, whereas $\liminf_t \|u(t)\|_{H^s(\mathbb{T})} < +\infty$.

In order to prove Theorem 6.1.3, we introduce Lyapunov functionals controlling the higher-order Sobolev norms in Birkhoff coordinates. Note that from [GKT20b], Proposition 5 in Appendix A, the Birkhoff map Φ and its inverse map bounded subsets of the Sobolev space $H^s_{r,0}(\mathbb{T})$ to bounded subsets of sequences in

$$h^{\frac{1}{2}+s}_+ = \left\{ \zeta = (\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \|\zeta\|_{h^{\frac{1}{2}+s}_+}^2 = \sum_n n^{1+2s} |\zeta_n|^2 < +\infty \right\}.$$

This leads us to consider higher-order Lyapunov functionals in Birkhoff coordinates of the form

$$P_s(u) = \sum_{n \geq 1} w_n \gamma_n(u),$$

where we choose weights w_n satisfying $w_n \approx n^{1+2s}$. The strategy of introducing Lyapunov functionals controlling higher-order Sobolev norms has been implemented for instance in [AB20] and [AMS20] for various equations describing free surface flows in fluid dynamics. In our paper, this strategy is implemented for the first time using the Birkhoff coordinates of a close integrable system.

6.1.2 Long-time behavior for equations of Benjamin-Ono type

As a comparison, let us now recall how the long-time dynamics for the Benjamin-Ono equation would be affected by the change of geometry, the introduction of a perturbation or the addition of a diffusive damping term. We refer to Saut [Sau19] for a general survey on the Benjamin-Ono equation.

Benjamin-Ono equation on the real line On the real line, one expects the dispersion to change the behavior of Benjamin-Ono solutions. Ifrim and Tataru [IT19] proved that for small localized initial data, there holds linear dispersive decay for large time scales. It is also conjectured that a solution associated to small initial data either is dispersive, or decomposes into the sum of a soliton and a dispersive part (soliton resolution conjecture).

The Benjamin-Ono equation on the line is also integrable. The Lax pair formulation is the same as on the torus, however, the inverse spectral theory, formally described in Ablowitz, Fokas [FA83], is only performed for sufficiently small and decaying initial data [CW90]. Wu [Wu16] studied the spectrum of the Lax operator for the Benjamin-Ono equation, then solved the direct scattering problem for arbitrary sufficiently decaying initial data [Wu17]. The recent construction of the Birkhoff map for multisolitons by Sun [Sun21] constitutes a first step towards the soliton resolution conjecture.

Perturbations of the Benjamin-Ono equation on the torus Thanks to extensions of the KAM theory, one can investigate if the periodic, quasi periodic or almost periodic nature of trajectories is preserved by perturbation of an integrable equation. This kind of studies goes back to Kuksin [Kuk87] and Wayne [Way90], we refer to Craig [Cra00] and Berti [Ber19] for a detailed survey.

Concerning perturbations of the Benjamin-Ono equation in the neighborhood of the origin, we mention the following results. In [LY11], Liu and Yuan consider a class of unbounded Hamiltonian perturbations of the periodic Benjamin-Ono equation (BO), construct KAM tori and deduce that there exist quasi-periodic trajectories. Mi and Zhang [MZ14] extend this result to more general unbounded quasi-periodic Hamiltonian perturbations. In [Bal13], Baldi proves that there exist periodic solutions for some non Hamiltonian reversible perturbations of a nonlinear cubic Benjamin-Ono equation

$$\partial_t u + H\partial_{xx}u + \partial_x(u^3) = 0.$$

For generalized Benjamin-Ono equations of the form

$$\partial_t u + H\partial_{xx}u + \partial_x(f(u)) = 0,$$

Bernier and Grébert [BG20b] establish approximations of trajectories on large time scales for generic initial data close to the origin.

In order to investigate further Hamiltonian perturbations of the Benjamin-Ono equation through the KAM theory and tackle trajectories which are not necessarily close to the origin, it would be useful to prove that the Birkhoff map Φ is real analytic. As a first step, Gérard, Kappeler et Topalov prove in [GKT20a] that the action map $u \mapsto (|\zeta_n(u)|^2)_n$ is real analytic from Sobolev spaces $H_{r,0}^s(\mathbb{T})$ onto weighted ℓ^1 spaces.

Damped Benjamin-Ono equations Adding a damping of diffusive type can lead to a decay of solutions or the existence of a global attractor. A global attractor is a compact set, invariant by the flow of the equation, and attracting all trajectories uniformly on bounded sets of initial data.

Several physical damping terms can be chosen depending on the nature of dissipation, in the same way as for the Korteweg-de Vries equation, see Ott and Sudan [OS70]. Grimshaw, Smyth and Stepanyants (see [GSS18] and references inside) introduce and study numerically a general model of dispersive equations under the form

$$\partial_t u + \alpha u \partial_x u + \beta H\partial_{xx}u + \delta \mathcal{D}[u] = 0,$$

where $\mathcal{D}(u)$ is a Fourier multiplier. For instance, the operator \mathcal{D} can describe the Rayleigh dissipation $\mathcal{D}u = u$, the Reynolds dissipation $\mathcal{D}u = -\partial_{xx}u$ leading to the Benjamin-Ono-Burgers equation, the Landau damping $\mathcal{D}u = H\partial_x u$, the dissipation of internal solitary waves over a rough bottom $\mathcal{D}u = |u|u$ [Gri03], etc.

The decay to zero and an asymptotic profile of solutions as $t \rightarrow +\infty$ can be found in Dix [Dix91] for Reynolds dissipation, and Bona, Luo [BL11] for generalized Benjamin-Ono-Burgers equations (see also [Mat+07] for dispersive shockwaves when the damping is small). For a fractional dissipation $\mathcal{D}u = |D|^q u$, $q > 0$, the asymptotic behavior and the existence of global attractors is investigated in Dix [Dix92] and Alarcon [Ala93]. Guo and Huo [GH06] prove the existence of a global attractor in $L^2(\mathbb{R})$, compact in $H^3(\mathbb{R})$, for generalized weakly-damped KdV-Benjamin-Ono equations, under the form

$$\partial_t u + \alpha H \partial_{xx} u + \beta \partial_{xxx} u + \mu \partial_x u + \lambda u + u \partial_x u = f.$$

The existence of a global attractor in $H^1(\mathbb{R})$ for the weakly-damped Benjamin-Ono equation had already been proven in [BY95].

We also mention the works of Burq, Raugel and Schlag [BRS17; BRS18] concerning weakly-damped Klein-Gordon equations.

Note that in equation (BO- α), the damping term is weaker than the ones mentioned above. Indeed, there are many trajectories which do not go to zero as time goes to infinity, for instance, this is the case for the solutions satisfying condition (6.3) since the L^2 norm of these solutions is preserved. Numerical simulations implemented by Klein [Kle20] tend to suggest that there are many other trajectories which do not go to zero, and that a global attractor may not exist (see related open questions in part 6.1.4).

6.1.3 Equation in Birkhoff coordinates

In the study of the damped Benjamin-Ono equation (BO- α), our general strategy consists in applying the Birkhoff map Φ and transform this equation into a system of ODEs on the Birkhoff coordinates

$$\frac{d}{dt} \zeta_n(u) = i\omega_n(\zeta(u))\zeta_n(u) - \alpha (\langle u | \cos \rangle d\zeta_n[u] \cdot \cos + \langle u | \sin \rangle d\zeta_n[u] \cdot \sin), \quad (6.4)$$

where we recall that $\omega_n(\zeta) = n^2 - 2 \sum_{k \geq 1} \min(k, n) |\zeta_k|^2$ for $n \geq 1$.

Our key result is a simplification of the right-hand side of equation (6.4), in particular the terms $d\zeta_n[u(t)] \cdot \cos$ and $d\zeta_n[u(t)] \cdot \sin$, as a function of the Birkhoff coordinates $\zeta = (\zeta_n)_{n \geq 1}$, and of their actions $\gamma = (|\zeta_n|^2)_{n \geq 1}$. Before stating the theorem, we introduce some notation for the spaces at stake. If $u \in L^2_{r,0}(\mathbb{T})$, then the Birkhoff coordinates ζ belong to $h^{\frac{1}{2}}_+ \subset \ell^2_+$, where ℓ^p_+ denotes the space of complex sequences

$$\ell^p_+ := \left\{ z = (z_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \|z\|_{\ell^p_+} = \sum_{n \geq 1} |z_n|^p < +\infty \right\}.$$

In particular, the actions γ belong to the space ℓ^1_+ .

Theorem 6.1.4 (Structure of the vector field). *There exist functions $p_n^*, q_n^*, A_{n,k}^*$ and $B_{n,k}^*$ such that for all $u \in L^2_{r,0}(\mathbb{T})$, if $\zeta = \Phi(u)$, then*

$$d\zeta_n[u] \cdot \cos = p_n^*(\gamma)\zeta_{n-1} + q_n^*(\gamma)\zeta_{n+1} + \left(\sum_{k=0}^{+\infty} A_{n,k}^*(\gamma)\bar{\zeta}_k \zeta_{k+1} + B_{n,k}^*(\gamma)\zeta_k \bar{\zeta}_{k+1} \right) \zeta_n,$$

with the convention $\zeta_0 = 1$ and the notation $\gamma = (|\zeta_n|^2)_{n \geq 1}$. The terms p_n^* , q_n^* , $A_{n,k}^*$ and $B_{n,k}^*$ are \mathcal{C}^1 functions of the actions $\gamma \in \ell_+^1$, they are uniformly bounded and Lipschitz on finite balls: for all $\gamma \in \ell_+^1$ satisfying $\|\gamma\|_{\ell_+^1} \leq R$, for all $n, k \geq 0$,

$$|p_n^*(\gamma)| + |q_n^*(\gamma)| + |A_{n,k}^*(\gamma)| + |B_{n,k}^*(\gamma)| \leq C(R),$$

moreover, for all $h \in \ell_+^1$,

$$|dp_n^*(\gamma).h| + |dq_n^*(\gamma).h| + |dA_{n,k}^*(\gamma).h| + |dB_{n,k}^*(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}.$$

The same holds for $d\zeta_n[u].\sin$, which also writes under the form

$$d\zeta_n[u].\sin = p_n^*(\gamma)\zeta_{n-1} + q_n^*(\gamma)\zeta_{n+1} + \left(\sum_{k=0}^{+\infty} A_{n,k}^*(\gamma)\bar{\zeta}_k\zeta_{k+1} + B_{n,k}^*(\gamma)\zeta_k\overline{\zeta_{k+1}} \right) \zeta_n,$$

with similar estimates for the new terms p_n^* , q_n^* , $A_{n,k}^*$ and $B_{n,k}^*$.

Precise formulas for p_n^* and q_n^* in the case of $d\zeta_n[u].\cos$ are given by equalities (6.12) and (6.13) below.

Similarly, as proven in [GKT20b], Lemma 13, there exist functions a_k^* of the actions, which are uniformly bounded and Lipschitz on finite balls of ℓ_+^1 , such that

$$\langle u|\cos \rangle - i\langle u|\sin \rangle = - \sum_{k \geq 0} a_k^*(\gamma)\bar{\zeta}_k\zeta_{k+1}. \quad (6.5)$$

The decomposition of the vector field into blocks of the form

$$A(\zeta) := \sum_{k \geq 0} A_k^*(\gamma)\bar{\zeta}_k\zeta_{k+1}, \quad (6.6)$$

where A_k^* are functions of the actions only, enables us to get bounds and Lipschitz estimates on finite balls, so that one can apply the Cauchy-Lipschitz theorem for ODEs. Thanks to this, we avoid going back and forth from u to ζ in the equation, and we do not need to use analicity results on the Birkhoff map (see [GKT20a; GKT21a] for results in this direction).

We now explain the main idea in the study of the long time asymptotics. Functions of the actions γ are slowly varying, indeed, we will establish in inequality (6.24) below that if $\|u_0\|_{L^2(\mathbb{T})} \leq R$, then for all $t \geq 0$,

$$\sum_{n \geq 1} \left| \frac{d}{dt} \gamma_n(t) \right| \leq C(R) |\langle u(t) | e^{ix} \rangle|,$$

where the right-hand side is square integrable over time thanks to the Lyapunov functional (6.2). However, the Birkhoff coordinates ζ are highly oscillating over time, because of the phase factor $i\omega_n(\zeta)\zeta_n \approx in^2\zeta_n$ appearing in the formula (6.4) of its time derivative. Moreover, interactions of different modes under the form $\bar{\zeta}_k\zeta_{k+1}\zeta_n\overline{\zeta_{n+1}}$, $k \neq n$, are also highly oscillating over time, again because of the phase factor

$$i(-\omega_k(\zeta) + \omega_{k+1}(\zeta) + \omega_n(\zeta) - \omega_{n+1}(\zeta))\bar{\zeta}_k\zeta_{k+1}\zeta_n\overline{\zeta_{n+1}} \approx 2i(k-n)\bar{\zeta}_k\zeta_{k+1}\zeta_n\overline{\zeta_{n+1}}$$

in the time derivative. By analogy with usual Fourier coefficients, such highly oscillating terms have a small time integral. We retrieve this fact here by performing an integration by parts.

Using these remarks, we start from the observation that $|\langle u|e^{ix}\rangle|^2$ is integrable over time, and expand the square modulus of formula (6.5). By discarding the interactions of different modes $\overline{\zeta_k}\zeta_{k+1}\zeta_n\overline{\zeta_{n+1}}$, $k \neq n$, we deduce that

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} \gamma_n(t)\gamma_{n+1}(t) dt < +\infty$$

(see Proposition 6.4.3 for more details). Successive integration by parts allow us to go even further and prove that for higher and higher exponents s ,

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} n^{2s} \gamma_n(t)\gamma_{n+1}(t) dt < +\infty.$$

Those latter integrals enable us to control the time evolution of our candidate higher-order Lyapunov functionals.

6.1.4 Open questions

Let us now mention some open questions about equation (BO- α).

First, thanks to the Lyapunov functional (6.2) and Theorem 6.1.2, one knows that the map

$$u_0 \in L^2_{r,0}(\mathbb{T}) \mapsto \lim_{t \rightarrow +\infty} \|u(t)\|_{L^2}^2 \in \mathbb{R}_+$$

is well-defined, upper semi-continuous, and that for every weak limit point u_∞ of the sequence $(u(t))_{t \geq 0}$ as $t \rightarrow +\infty$, one has $\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2}^2 = \|u_\infty\|_{L^2}^2$. It would be interesting to have more information about this map, for instance, one could ask whether it is possible to have $u_\infty = 0$ for some nonzero initial data u_0 . According to numerical simulations implemented by Klein [Kle20], there exist families of initial data which do not decay to zero as time goes to infinity. For instance, this seems to be the case for a family of potentials with only one nonzero gap γ_1 at index 1, given for some $r \in [0, 1[$ by (see [GK20], Appendix B)

$$u_0(x) = \frac{1 - r^2}{1 - 2 \cos(x)r + r^2} - 1.$$

Finally, one knows from Theorem 6.1.2, point 2, that given an initial data u_0 , all weak limit points u_∞ have the same actions $(\gamma_n^\infty)_n$, so that these limit points belong to the same Benjamin-Ono torus. One could ask if for some initial data u_0 , it is possible to provide a precise description of the limiting torus. Again, the numerical simulations from Klein [Kle20] suggest that even for potentials with only one nonzero gap γ_1 at index 1, the weak limits could have more than one gap. This is linked to the structure of the equation in Theorem 6.1.4, since the terms p_n^* and q_n^* given by formulas (6.12) and (6.13) are nonzero and lead to the emergence of new modes in the time evolution. As a consequence, finite gap manifolds are not invariant by the flow of equation (BO- α), whereas as a comparison, there exist finite dimensional invariant manifolds for equation (Sz- α) (see [GG20]).

6.1.5 Plan of the paper

First, in section 6.2, we write the damped Benjamin-Ono equation (BO- α) in Birkhoff coordinates as a system of ODEs, in particular, we prove Theorem 6.1.4 about the decomposition of the terms $d\zeta_n[u].\cos$ and $d\zeta_n[u].\sin$.

Then, in section 6.3, we establish global well-posedness of the damped Benjamin-Ono equation (BO- α) and weak sequential continuity of the solution map (Theorem 6.1.1).

Section 6.4 is devoted to the characterization of the weak limit points for the trajectories according to the LaSalle principle, and the fact that the convergence is actually strong in $L^2_{r,0}(\mathbb{T})$ (Theorem 6.1.2).

Finally, we bound the higher-order Sobolev norms in section 6.5 by studying corresponding Lyapunov functionals in Birkhoff coordinates (Theorem 6.1.3).

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6.2 Structure of the equation

Let us consider the flow map for the damped Benjamin-Ono equation (BO- α) in Birkhoff coordinates. Fix $n \geq 1$, then the time derivative of the n -th Birkhoff coordinate ζ_n is

$$\frac{d}{dt}\zeta_n(u(t)) = d\zeta_n[u(t)].\partial_t u(t).$$

Using the partial differential equation (BO- α) satisfied by u , this time derivative becomes (6.4)

$$\frac{d}{dt}\zeta_n(u) = i\omega_n(\zeta(u))\zeta_n(u) - \alpha (\langle u|\cos\rangle d\zeta_n[u].\cos + \langle u|\sin\rangle d\zeta_n[u].\sin),$$

with $\omega_n(\zeta) = n^2 - 2\sum_{k \geq 1} \min(k, n)|\zeta_k|^2$.

In this section, we simplify the different terms of the right-hand side of this equation, then study their Lipschitz properties. Part 6.2.1 is devoted to the study of the terms which only depend on the actions $\gamma_n = |\zeta_n|^2$, $n \geq 1$, for which we establish bounds and Lipschitz properties on balls of finite radius in ℓ^1_+ . In part 6.2.2, we observe that blocks under the form (6.6)

$$A(\zeta) = \sum_{k \geq 0} A_k^*(\gamma)\bar{\zeta}_k\zeta_{k+1},$$

where A_k^* is a function of the actions γ only, are also bounded and Lipschitz on finite balls as soon as the A_k^* are. This is for instance the case for $\langle u|\cos\rangle - i\langle u|\sin\rangle$, which can be expressed under this form. In part 6.2.3, we establish a formula for the differential $d\zeta_n[u].h$ of the n -th Birkhoff coordinate at $u \in L^2_{r,0}(\mathbb{T})$ applied to $h \in L^2_{r,0}(\mathbb{T})$. Finally, in part 6.2.4, we simplify this formula when $h = \cos, \sin$ and prove Theorem 6.1.4. Since $d\zeta_n[u].\cos$ and $d\zeta_n[u].\sin$ have an expression based on blocks of the above type (6.6), we deduce bounds and Lipschitz properties on finite balls for these terms.

6.2.1 Functions of the actions

In this part, we consider some spectral parameters which will appear all along our study of the damped Benjamin-Ono equation (BO- α). These parameters are functions of the actions, defined for $n \geq 1$ as $\gamma_n(u) = |\zeta_n(u)|^2$.

First, let us recall some links between the Lax operator for the Benjamin-Ono equation (see [Nak79], [BK79]) and the actions. With the notation from [GK20], Appendix A, the Lax operator L_u writes $L_u = D - T_u$, where $D = -i\partial_x$, and T_u is the Toeplitz operator

$$T_u : h \in L_+^2(\mathbb{T}) \mapsto \Pi(uh) \in L_+^2(\mathbb{T}).$$

defined from the Szegő projector $\Pi : L^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$, where

$$L_+^2(\mathbb{T}) = \{h \in L^2(\mathbb{T}) \mid \forall n < 0, \widehat{h}(n) = 0\}.$$

The eigenvalues $(\lambda_n)_{n \geq 0}$ of L_u satisfy (see identity (3.13) in [GK20])

$$\lambda_n(u) = n - \sum_{k=n+1}^{+\infty} \gamma_k(u), \quad n \geq 0.$$

Therefore, for all $n, p \in \mathbb{N}$, we have the inequalities

$$|p - n| \leq |\lambda_p - \lambda_n| \leq |p - n| + \sum_{k \geq 1} \gamma_k.$$

We now define the spectral parameters κ_n and μ_n as in Corollary 3.4 and formula (4.9) from [GK20].

Definition 6.2.1. *Let $u \in L_{r,0}^2(\mathbb{T})$. We define*

$$\kappa_0(u) = \prod_{p=1}^{+\infty} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_0}\right)$$

and for $n \geq 1$,

$$\kappa_n(u) = \frac{1}{\lambda_n - \lambda_0} \prod_{\substack{p=1 \\ p \neq n}}^{+\infty} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right).$$

For $n \geq 1$, we also set

$$\mu_n(u) = \left(1 - \frac{\gamma_n}{\lambda_n - \lambda_0}\right) \prod_{\substack{p=1 \\ p \neq n}}^{+\infty} \frac{\left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right)}{\left(1 - \frac{\gamma_p}{\lambda_p - \lambda_{n-1} - 1}\right)}.$$

The parameters κ_n and μ_n are bounded above and below.

Lemma 6.2.2. *For all $R > 0$, there exists $C(R) > 0$ such that for all $u \in L_{r,0}^2(\mathbb{T})$ satisfying $\|\gamma(u)\|_{\ell_+^1} \leq R$ the following holds: for all $n \in \mathbb{N}$,*

$$\frac{1}{C(R)(n+1)} \leq \kappa_n(u) \leq \frac{C(R)}{n+1},$$

moreover, for all $n \geq 1$,

$$\frac{1}{C(R)} \leq \mu_n(u) \leq C(R).$$

Proof. • As a first step, we prove that for all $n \in \mathbb{N}$,

$$\left| \sum_{\substack{p=1 \\ p \neq n}}^{+\infty} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) \right| \leq C(R).$$

Let us first show that the sum of logarithms has a lower bound. For $p < n$, since $\lambda_p < \lambda_n$, we have $-\ln(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}) \leq 0$. Consequently, we discard the negative terms in the following sum:

$$-\sum_{p \neq n} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) \leq -\sum_{p \geq n+1} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right).$$

For $p \geq n + 1$, since by assumption $\gamma_p \in [0, R]$,

$$0 \leq \frac{\gamma_p}{\lambda_p - \lambda_n} \leq \frac{\gamma_p}{1 + \gamma_p} \leq \frac{R}{1 + R},$$

therefore one can use that \ln is $C(R)$ -Lipschitz on $[\frac{1}{1+R}, 1]$ to get

$$\begin{aligned} -\sum_{p \geq n+1} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) &\leq C(R) \sum_{p \geq n+1} \frac{\gamma_p}{\lambda_p - \lambda_n} \\ &\leq C(R)R. \end{aligned}$$

Similarly, we find an upper bound for the sum of logarithms. We first discard the negative terms in the sum

$$\sum_{p \neq n} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) \leq \sum_{p \leq n-1} \ln \left(1 + \frac{\gamma_p}{\lambda_n - \lambda_p} \right).$$

Then, since $0 \leq \frac{\gamma_p}{\lambda_n - \lambda_p} \leq R$ for $p \leq n - 1$, and since \ln is $C(R)$ -Lipschitz on $[1, R + 1]$, we get

$$\begin{aligned} \sum_{p \leq n-1} \ln \left(1 + \frac{\gamma_p}{\lambda_n - \lambda_p} \right) &\leq C(R) \sum_{p \leq n-1} \frac{\gamma_p}{\lambda_n - \lambda_p} \\ &\leq C(R)R. \end{aligned}$$

• Now, we prove the inequalities for κ_n , $n \in \mathbb{N}$. For $n \geq 1$, we have

$$\begin{aligned} |\ln(\lambda_n - \lambda_0) + \ln(\kappa_n)| &= \left| \sum_{p \neq n} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) \right| \\ &\leq C(R), \end{aligned}$$

where $|\lambda_n - \lambda_0 - n| \leq C(R)$. Similarly,

$$\begin{aligned} |\ln(\kappa_0)| &= \left| \sum_{p \geq 1} \ln \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_0} \right) \right| \\ &\leq C(R). \end{aligned}$$

The proof is the same for μ_n , $n \geq 1$. □

Since the formulas for κ_n and μ_n only depend on the sequence of actions $\gamma(u) = (|\zeta_n(u)|^2)_n$, one can also consider these parameters as functions of the variable $\gamma \in \ell_+^1$ only. In order to avoid confusion, we will denote $\kappa_n^*(\gamma)$ and $\mu_n^*(\gamma)$ in this case. For all $u \in L_{r,0}^2(\mathbb{T})$, we therefore have the link

$$\kappa_n^*(\gamma(u)) = \kappa_n(u) \quad \text{and} \quad \mu_n^*(\gamma(u)) = \mu_n(u).$$

We now estimate the differential of the spectral parameters κ_n^* and μ_n^* with respect to the action variables.

Lemma 6.2.3. *Let $R > 0$. Then there exists $C(R) > 0$ such that for all $\gamma \in \ell_+^1$ satisfying $\|\gamma\|_{\ell_+^1} \leq R$ and for all $h \in \ell_+^1$,*

$$|\mathrm{d} \ln(\kappa_n^*)[\gamma].h| \leq C(R) \|h\|_{\ell_+^1}, \quad n \geq 0$$

and

$$|\mathrm{d} \ln(\mu_n^*)[\gamma].h| \leq C(R) \|h\|_{\ell_+^1}, \quad n \geq 1.$$

Proof. For instance in the case $n \geq 1$, the differential of $\ln(\kappa_n^*)$ at γ is given by

$$\begin{aligned} \mathrm{d} \ln(\kappa_n^*)[\gamma].h &= -\frac{\mathrm{d}\lambda_n^*[\gamma].h - \mathrm{d}\lambda_0^*[\gamma].h}{\lambda_n - \lambda_0} \\ &\quad + \sum_{p \neq n} \frac{1}{1 - \frac{\gamma_p}{\lambda_p - \lambda_n}} \left(\frac{\mathrm{d}\lambda_p^*[\gamma].h - \mathrm{d}\lambda_n^*[\gamma].h}{(\lambda_p - \lambda_n)^2} \gamma_p - \frac{h_p}{\lambda_p - \lambda_n} \right), \end{aligned}$$

where $h = (h_p)_{p \geq 1} \in \ell_+^1$. We write $\mathrm{d}\lambda_n^*$ in order to specify that we differentiate the eigenvalue $\lambda_n = n - \sum_{k \geq n+1} \gamma_k$ as a function of γ . In particular, for all $\gamma, h \in \ell_+^1$, we have

$$|\mathrm{d}\lambda_n^*[\gamma].h| \leq \|h\|_{\ell_+^1}.$$

The conclusion follows. We proceed similarly for $\ln(\kappa_0^*)$ and for $\ln(\mu_n^*)$, $n \geq 1$. \square

6.2.2 Lipschitz properties of blocks

In this part, we start from bounds and Lipschitz properties on finite balls for general blocks of the form (6.6)

$$A(\zeta) = \sum_{k \geq 0} A_k^*(\gamma) \overline{\zeta_k} \zeta_{k+1}.$$

We then prove that the terms $\langle u | \cos \rangle$ and $\langle u | \sin \rangle$ can be written under this form, then deduce bounds and Lipschitz estimates on finite balls.

Proposition 6.2.4. *Let A_k^* , $k \geq 0$, be uniformly bounded and Lipschitz functions on finite balls of ℓ_+^1 : for all $R > 0$, there exists $C_0(R) > 0$ such that for all $\gamma^{(1)}, \gamma^{(2)} \in \ell_+^1$ satisfying $\|\gamma^{(1)}\|_{\ell_+^1} \leq R$ and $\|\gamma^{(2)}\|_{\ell_+^1} \leq R$, then for all $k \geq 0$,*

$$|A_k^*(\gamma^{(1)})| + |A_k^*(\gamma^{(2)})| \leq C_0(R)$$

and

$$|A_k^*(\gamma^{(1)}) - A_k^*(\gamma^{(2)})| \leq C_0(R) \|\gamma^{(1)} - \gamma^{(2)}\|_{\ell_+^1}.$$

For $\zeta \in \ell_+^2$ we write $\gamma = (|\zeta_n|^2)_{n \geq 1}$ and adopt the convention $\zeta_0 = 1$. Let us consider

$$A(\zeta) := \sum_{k \geq 0} A_k^*(\gamma) \bar{\zeta}_k \zeta_{k+1}.$$

Then A is a bounded Lipschitz function of ζ on balls of finite radius in ℓ_+^2 : there exists $C(R) > 0$ such that for all $\zeta^{(1)}, \zeta^{(2)} \in \ell_+^2$ satisfying $\|\zeta^{(1)}\|_{\ell_+^2}^2 \leq R$ and $\|\zeta^{(2)}\|_{\ell_+^2}^2 \leq R$, there holds

$$|A(\zeta^{(1)})| + |A(\zeta^{(2)})| \leq C(R)$$

and

$$|A(\zeta^{(1)}) - A(\zeta^{(2)})| \leq C(R) \|\zeta^{(1)} - \zeta^{(2)}\|_{\ell_+^2}.$$

The proof is quite direct and omitted here.

We now establish a formula of the above form (6.6) for $\langle u | \cos \rangle$ and $\langle u | \sin \rangle$.

Let $(f_p)_{p \geq 0}$ be the orthonormal basis of eigenfunctions for the Lax operator L_u associated to the Benjamin-Ono equation in $L_{r,0}^2(\mathbb{T})$ uniquely determined by the additional conditions $\langle \mathbf{1} | f_0 \rangle > 0$ and $\langle f_p | S f_{p-1} \rangle > 0$ for $p \geq 1$ (see Definition 2.8 in [GK20]). The formula for $\langle u | \cos \rangle$ and $\langle u | \sin \rangle$ depends on the matrix M of the adjoint operator $S^* = T_{e^{-ix}}$ of the shift operator $S : h \in L_+^2(\mathbb{T}) \mapsto e^{ix} h \in L_+^2(\mathbb{T})$.

Definition 6.2.5. For all $n, p \geq 0$, let

$$\begin{aligned} M_{n,p} &= \langle f_p | S f_n \rangle \\ &= \begin{cases} \frac{\sqrt{\mu_{n+1}}}{\kappa_{n+1}} \langle f_p | \mathbf{1} \rangle \overline{\langle f_{n+1} | \mathbf{1} \rangle} \frac{1}{\lambda_p - \lambda_{n-1}} & \text{if } \zeta_{n+1} \neq 0 \\ \delta_{p,n+1} & \text{if } \zeta_{n+1} = 0 \end{cases}. \end{aligned}$$

In other words, since $\zeta_n = \frac{\langle \mathbf{1} | f_n \rangle}{\sqrt{\kappa_n}}$, for $n \geq 0$, (see equality (4.1) in [GK20]), we have

$$M_{n,p} = \begin{cases} \sqrt{\mu_{n+1}} \frac{\sqrt{\kappa_p}}{\sqrt{\kappa_{n+1}}} \bar{\zeta}_p \zeta_{n+1} \frac{1}{\lambda_p - \lambda_{n-1}} & \text{if } p \neq n+1 \\ \sqrt{\mu_{n+1}} & \text{if } p = n+1 \end{cases}.$$

Lemma 6.2.6. Let $u \in L_{r,0}^2(\mathbb{T})$. Then

$$\langle u | e^{ix} \rangle = - \sum_{n \geq 0} \bar{\zeta}_n \zeta_{n+1} \sqrt{\frac{\kappa_n}{\kappa_{n+1}}} \sqrt{\mu_{n+1}}$$

with the convention $\zeta_0 = 1$.

This formula for $\langle u | \cos \rangle$ and $\langle u | \sin \rangle$ was already proven in [GKT20b], Lemma 13, in the special case when $\gamma_n(u) \neq 0$ for all n . We justify here how the proof stays valid in the general case.

Proof. We note that $\langle u | e^{ix} \rangle = \langle \Pi u | e^{ix} \rangle$, then decompose Πu and e^{ix} in the orthonormal basis $(f_p)_{p \geq 0}$

$$\langle u | e^{ix} \rangle = \sum_{p \geq 0} \langle \Pi u | f_p \rangle \langle f_p | e^{ix} \rangle.$$

We now remark that $\langle f_p | e^{ix} \rangle = \langle S^* f_p | \mathbb{1} \rangle$, and decompose $S^* f_p$ and $\mathbb{1}$ along the orthonormal basis $(f_n)_{n \geq 0}$

$$\begin{aligned} \langle f_p | e^{ix} \rangle &= \sum_{n \geq 0} \langle S^* f_p | f_n \rangle \langle f_n | \mathbb{1} \rangle \\ &= \sum_{n \geq 0} M_{n,p} \langle f_n | \mathbb{1} \rangle \\ &= \sum_{\substack{n \geq 0 \\ \gamma_{n+1} \neq 0}} \frac{\sqrt{\mu_{n+1}}}{\kappa_{n+1}} \langle f_p | \mathbb{1} \rangle \overline{\langle f_{n+1} | \mathbb{1} \rangle} \frac{1}{\lambda_p - \lambda_n - 1} \langle f_n | \mathbb{1} \rangle. \end{aligned}$$

From the identity $\langle \Pi u | f_p \rangle = -\lambda_p \langle \mathbb{1} | f_p \rangle$, we get

$$\begin{aligned} \langle u | e^{ix} \rangle &= \sum_{p \geq 0} -\lambda_p \langle \mathbb{1} | f_p \rangle \sum_{\substack{n \geq 0 \\ \gamma_{n+1} \neq 0}} \frac{\sqrt{\mu_{n+1}}}{\kappa_{n+1}} \langle f_p | \mathbb{1} \rangle \overline{\langle f_{n+1} | \mathbb{1} \rangle} \frac{1}{\lambda_p - \lambda_n - 1} \langle f_n | \mathbb{1} \rangle \\ &= - \sum_{\substack{n \geq 0 \\ \gamma_{n+1} \neq 0}} \frac{\sqrt{\mu_{n+1}}}{\kappa_{n+1}} \langle f_n | \mathbb{1} \rangle \overline{\langle f_{n+1} | \mathbb{1} \rangle} \sum_{p \geq 0} \frac{\lambda_p |\langle f_p | \mathbb{1} \rangle|^2}{\lambda_p - \lambda_n - 1}. \end{aligned}$$

Recall the definition of the generating function \mathcal{H}_λ for $\lambda \in \mathbb{C}$ (formula (3.2) in [GK20])

$$\mathcal{H}_\lambda(u) = \sum_{p \geq 0} \frac{|\langle \mathbb{1} | f_p \rangle|^2}{\lambda_p + \lambda},$$

then if $\gamma_{n+1} \neq 0$, we have the identity

$$\begin{aligned} \sum_{p \geq 0} \frac{\lambda_p |\langle f_p | \mathbb{1} \rangle|^2}{\lambda_p - \lambda_n - 1} &= (\lambda_n + 1) \sum_{p \geq 0} \frac{|\langle f_p | \mathbb{1} \rangle|^2}{\lambda_p - \lambda_n - 1} + \sum_{p \geq 0} |\langle f_p | \mathbb{1} \rangle|^2 \\ &= (\lambda_n + 1) \mathcal{H}_{-\lambda_n - 1}(u) + 1. \end{aligned}$$

Moreover, the formula from [GK20] (see Proposition 3.1 (i))

$$\mathcal{H}_{-\lambda_n - 1}(u) = -\frac{1}{\lambda_n + 1 - \lambda_0} \prod_{p=1}^{+\infty} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n - 1} \right)$$

implies that $\mathcal{H}_{-\lambda_n - 1}(u) = 0$ because of the term appearing in the product when $p = n + 1$. Using that $\zeta_n = \frac{\langle \mathbb{1} | f_n \rangle}{\sqrt{\kappa_n}}$, we conclude the proof of the lemma. \square

Let us write

$$\langle u | e^{ix} \rangle = \sum_{n \geq 0} -a_n^*(\gamma) \overline{\zeta_n} \zeta_{n+1},$$

where

$$a_n^*(\gamma) = \sqrt{\mu_{n+1}^*(\gamma)} \frac{\sqrt{\kappa_n^*(\gamma)}}{\sqrt{\kappa_{n+1}^*(\gamma)}}. \quad (6.7)$$

Thanks to part 6.2.1, we know that if $\|\gamma\|_{\ell_+^1} \leq R$, then there exists $C(R) > 0$ such that for all $n \geq 0$,

$$\frac{1}{C(R)} \leq a_n^*(\gamma) \leq C(R) \quad (6.8)$$

and for $h \in \ell_+^1$,

$$|da_n^*[\gamma].h| \leq C(R) \|h\|_{\ell_+^1}. \quad (6.9)$$

Proposition 6.2.4 implies that the maps $\zeta \in h_+^{\frac{1}{2}} \mapsto \langle \Phi^{-1}(\zeta) | \cos \rangle$ and $\zeta \in h_+^{\frac{1}{2}} \mapsto \langle \Phi^{-1}(\zeta) | \sin \rangle$ are bounded and Lipschitz on finite balls of ℓ_+^2 .

Corollary 6.2.7. *Fix $R > 0$. Then there exists $C(R) > 0$ such that for all $\zeta^{(1)}, \zeta^{(2)} \in h_+^{\frac{1}{2}}$ satisfying $\|\zeta^{(1)}\|_{\ell_+^2} \leq R$ and $\|\zeta^{(2)}\|_{\ell_+^2} \leq R$, writing $u^{(1)} = \Phi^{-1}(\zeta^{(1)})$ and $u^{(2)} = \Phi^{-1}(\zeta^{(2)})$, we have*

$$|\langle u^{(1)} | \cos \rangle| + |\langle u^{(1)} | \sin \rangle| + |\langle u^{(2)} | \cos \rangle| + |\langle u^{(2)} | \sin \rangle| \leq C(R)$$

and

$$|\langle u^{(1)} | \cos \rangle - \langle u^{(2)} | \cos \rangle| + |\langle u^{(1)} | \sin \rangle - \langle u^{(2)} | \sin \rangle| \leq C(R) \|\zeta^{(1)} - \zeta^{(2)}\|_{\ell_+^2}.$$

6.2.3 Formula for the differential of the Birkhoff map

In this part, we establish a formula for the differential of the normalized Birkhoff coordinates ζ_n (with $\zeta_0 = 1$), which are linked to the unnormalized Birkhoff coordinates $\langle \mathbf{1} | f_n \rangle$ through the formulas (see equality (4.1) in [GK20])

$$\zeta_n = \frac{\langle \mathbf{1} | f_n \rangle}{\sqrt{\kappa_n}}, \quad n \geq 0.$$

These coordinates are known to admit a differential thanks to Lemma 3.4 in [GK20]. The differential of ζ_n at $u \in L_{r,0}^2(\mathbb{T})$ therefore satisfies: for all $h \in L_{r,0}^2(\mathbb{T})$,

$$d\zeta_n[u].h = d\langle \mathbf{1} | f_n \rangle[u].h \frac{1}{\sqrt{\kappa_n}} - \frac{1}{2} \zeta_n d \ln(\kappa_n)[u].h. \quad (6.10)$$

We prove the following decomposition of the differential of the unnormalized Birkhoff map.

Proposition 6.2.8 (Differential of the unnormalized Birkhoff map). *We fix u and h in $L_{r,0}^2(\mathbb{T})$, then for all $n \geq 0$, the term $d\langle \mathbf{1} | f_n \rangle[u].h$ is written*

$$d\langle \mathbf{1} | f_n \rangle[u].h = \delta_{\parallel} \langle \mathbf{1} | f_n \rangle + \delta_{\perp} \langle \mathbf{1} | f_n \rangle,$$

where

$$\begin{aligned} \delta_{\parallel} \langle \mathbf{1} | f_n \rangle &= -i \operatorname{Im}(\langle \tilde{\xi}_n | h \rangle) \langle \mathbf{1} | f_n \rangle, \\ \tilde{\xi}_n &= \frac{1}{\langle f_0 | \mathbf{1} \rangle} \sum_{p=1}^{+\infty} \frac{\langle \mathbf{1} | f_p \rangle}{\lambda_p - \lambda_0} \overline{f_0} f_p - \sum_{k=1}^n \frac{1}{\langle f_k | S f_{k-1} \rangle} \psi_k, \\ \psi_n &= \sum_{\substack{p \geq 0 \\ p \neq n}} \frac{\langle f_p | S f_{n-1} \rangle}{\lambda_p - \lambda_n} f_n \overline{f_p} - \sum_{\substack{p \geq 0 \\ p \neq n-1}} \frac{\langle S f_p | f_n \rangle}{\lambda_p - \lambda_{n-1}} f_{n-1} \overline{f_p} \end{aligned}$$

and

$$\delta_{\perp} \langle \mathbf{1} | f_n \rangle = \sum_{\substack{p \geq 0 \\ p \neq n}} \frac{\langle f_p \overline{f_n} | h \rangle}{\lambda_p - \lambda_n} \langle \mathbf{1} | f_p \rangle.$$

Proof. For $n \geq 0$, since $d\langle \mathbf{1}|f_n\rangle[u].h = \langle \mathbf{1}|df_n[u].h\rangle$, we decompose the two terms inside the brackets along the orthonormal basis $(f_p)_{p \geq 0}$ and get

$$d\langle \mathbf{1}|f_n\rangle[u].h = \sum_{p \geq 0} \langle \mathbf{1}|f_p\rangle \langle f_p|df_n[u].h\rangle.$$

We now compute $\langle df_n[u].h|f_p\rangle$ for $n, p \geq 0$.

For $n \geq 0$, recall that f_n is the L^2 -normalized eigenfunction of $L_u = D - T_u$ associated to the eigenvalue λ_n , and that the family $(f_n)_n$ is uniquely determined by the additional conditions $\langle \mathbf{1}|f_0\rangle > 0$ and $\langle f_n|Sf_{n-1}\rangle > 0$ for $n \geq 1$. Therefore, we get by differentiating

$$(L_u - \lambda_n)df_n[u].h = d\lambda_n[u].hf_n + T_h f_n.$$

For $p \neq n$, f_p is orthogonal to f_n and therefore

$$\langle df_n[u].h|f_p\rangle = \frac{\langle f_n \bar{f}_p|h\rangle}{\lambda_p - \lambda_n}.$$

This gives the formula for $\delta_\perp \langle \mathbf{1}|f_n\rangle$.

For $p = n$, we see that $\langle df_n[u].h|f_n\rangle$ is purely imaginary because $\|f_n\|_{L^2(\mathbb{T})}^2 = 1$. Moreover, the conditions $\langle f_0|\mathbf{1}\rangle > 0$ and $\langle f_n|Sf_{n-1}\rangle > 0$ for $n \geq 1$ imply that

$$\text{Im}(\langle df_0[u].h|\mathbf{1}\rangle) = 0$$

and

$$\text{Im}(\langle df_n[u].h|Sf_{n-1}\rangle + \langle f_n|Sdf_{n-1}[u].h\rangle) = 0, \quad n \geq 1.$$

Decomposing $\mathbf{1}$ in the orthonormal basis $(f_p)_{p \geq 0}$, we get for $n = 0$

$$\begin{aligned} 0 &= \text{Im} \left(\langle df_0[u].h|f_0\rangle \langle f_0|\mathbf{1}\rangle + \sum_{p=1}^{+\infty} \langle df_0[u].h|f_p\rangle \langle f_p|\mathbf{1}\rangle \right) \\ &= \text{Im} \left(\langle df_0[u].h|f_0\rangle \langle f_0|\mathbf{1}\rangle + \sum_{p=1}^{+\infty} \frac{\langle f_0 \bar{f}_p|h\rangle}{\lambda_p - \lambda_0} \langle f_p|\mathbf{1}\rangle \right), \end{aligned}$$

or

$$\begin{aligned} \langle df_0[u].h|f_0\rangle &= -i \text{Im} \left(\frac{1}{\langle f_0|\mathbf{1}\rangle} \sum_{p=1}^{+\infty} \frac{\langle f_0 \bar{f}_p|h\rangle}{\lambda_p - \lambda_0} \langle f_p|\mathbf{1}\rangle \right) \\ &= i \text{Im}(\langle \tilde{\xi}_0|h\rangle). \end{aligned}$$

Similarly, for $n \geq 1$, decomposing Sf_{n-1} and S^*f_n in the orthonormal basis $(f_p)_{p \geq 0}$, we have

$$0 = \text{Im} \left(\sum_{p \geq 0} \langle df_n[u].h|f_p\rangle \langle f_p|Sf_{n-1}\rangle + \langle S^*f_n|f_p\rangle \langle f_p|df_{n-1}[u].h\rangle \right),$$

and by isolating the index $p = n$ in the first term and $p = n - 1$ in the second term, we get the formula

$$\begin{aligned} &\langle f_n|Sf_{n-1}\rangle \text{Im}(\langle df_n[u].h|f_n\rangle - \langle df_{n-1}[u].h|f_{n-1}\rangle) \\ &= -\text{Im} \left(\sum_{p \neq n} \langle df_n[u].h|f_p\rangle \langle f_p|Sf_{n-1}\rangle - \sum_{p \neq n-1} \overline{\langle f_n|Sf_p\rangle} \langle df_{n-1}[u].h|f_p\rangle \right), \end{aligned}$$

leading to

$$\begin{aligned} & \langle f_n | S f_{n-1} \rangle \operatorname{Im}(\langle d f_n[u].h | f_n \rangle - \langle d f_{n-1}[u].h | f_{n-1} \rangle) \\ &= -\operatorname{Im} \left(\sum_{p \neq n} \frac{\langle f_n \bar{f}_p | h \rangle}{\lambda_p - \lambda_n} \langle f_p | S f_{n-1} \rangle - \sum_{p \neq n-1} \langle S f_p | f_n \rangle \frac{\langle f_{n-1} \bar{f}_p | h \rangle}{\lambda_p - \lambda_{n-1}} \right). \end{aligned}$$

We retrieve the expression of $\delta_{\parallel} \langle \mathbb{1} | f_n \rangle$. □

6.2.4 Decomposition for $d\zeta_n[u].\cos$ and $d\zeta_n[u].\sin$

In this part, we establish Theorem 6.1.4: we decompose $d\zeta_n[u].\cos$ and $d\zeta_n[u].\sin$ using blocks of the form (6.6) $\sum_{k \geq 0} A_k^*(\gamma) \bar{\zeta}_k \zeta_{k+1}$, where A_k^* are functions of the actions only.

Let us now give the organization of this part. Recall that since $\zeta_n = \frac{\langle \mathbb{1} | f_n \rangle}{\sqrt{\kappa_n}}$, we have equality (6.10)

$$d\zeta_n[u].h = d\langle \mathbb{1} | f_n \rangle[u].h \frac{1}{\sqrt{\kappa_n}} - \frac{1}{2} \zeta_n d \ln(\kappa_n)[u].h,$$

where $d\langle \mathbb{1} | f_n \rangle[u].h$ decomposes from Proposition 6.2.8 as

$$d\langle \mathbb{1} | f_n \rangle[u].h = \delta_{\parallel} \langle \mathbb{1} | f_n \rangle + \delta_{\perp} \langle \mathbb{1} | f_n \rangle.$$

This leads us to study successively each part of the decomposition: $d \ln(\kappa_n)[u].h$, $\delta_{\parallel} \langle \mathbb{1} | f_n \rangle$ and $\delta_{\perp} \langle \mathbb{1} | f_n \rangle$, in the particular cases $h = \cos$ and $h = \sin$. Then, in subpart 6.2.4, we use part 6.2.2 in order to deduce that $d\zeta_n[u].\cos$ and $d\zeta_n[u].\sin$ are bounded and Lipschitz on balls of finite radius.

The case of $d \ln(\kappa_n)[u].h$

For $\zeta \in h_+^{\frac{1}{2}}$, we denote $u = \Phi^{-1}(\zeta)$ and define

$$\delta \kappa_n(\zeta) := d \ln(\kappa_n)[u].\cos - i d \ln(\kappa_n)[u].\sin, \quad n \in \mathbb{N}.$$

To avoid confusion, we shall precise that $\ln(\kappa_n)$ is a function of $u \in L_{r,0}^2(\mathbb{T})$, and that $d \ln(\kappa_n)[u].\cos$ denotes its differential with respect to the variable u applied to \cos . However, $\delta \kappa_n$ is a function of the Birkhoff coordinates ζ , since $u = \Phi^{-1}(\zeta)$ in the definition.

Lemma 6.2.9. *There exist functions $A_{n,k}^* \in \mathcal{C}^1(\ell_+^1, \mathbb{R})$, $n, k \in \mathbb{N}$, such that for all $n \geq 0$, for all $\zeta \in h_+^{\frac{1}{2}}$,*

$$\delta \kappa_n(\zeta) = \sum_{k \geq 0} A_{n,k}^*(\gamma) \bar{\zeta}_k \zeta_{k+1}.$$

Moreover, for all $R > 0$, there exists $C(R) > 0$ such that for all $\gamma \in \ell_+^1$ satisfying $\|\gamma\|_{\ell_+^1} \leq R$, for all $n, k \geq 0$,

$$|A_{n,k}^*(\gamma)| \leq C(R)$$

and for all $h \in \ell_+^1$,

$$|dA_{n,k}^*(\gamma).h| \leq C(R) \|h\|_{\ell_+^1}.$$

Proof. We first compute the differential of the eigenvalues λ_n , $n \in \mathbb{N}$, with respect to the variable u and applied to $h = \cos$ and $h = \sin$. From [GK20], Corollary 5.3, we know that

$$d\lambda_n[u].h = -\langle |f_n|^2 |h \rangle, \quad h \in L_{r,0}^2(\mathbb{T}),$$

which leads to a formula for the differential of the moments $\gamma_n = \lambda_n - \lambda_{n-1} - 1$ for $n \geq 1$

$$d\gamma_n[u].h = \langle |f_{n-1}|^2 - |f_n|^2 |h \rangle, \quad h \in L_{r,0}^2(\mathbb{T}). \quad (6.11)$$

We now simplify

$$\begin{aligned} d\gamma_n[u].\cos - i d\gamma_n[u].\sin &= \langle |f_{n-1}|^2 - |f_n|^2 |e^{ix} \rangle \\ &= M_{n-1,n-1} - M_{n,n}. \end{aligned}$$

We denote

$$m_n = M_{n,n} = -a_n^*(\gamma) \overline{\zeta_n} \zeta_{n+1},$$

where we recall that a_n^* is a function of γ defined as

$$a_n^* = \sqrt{\mu_{n+1}^*} \frac{\sqrt{\kappa_n^*}}{\sqrt{\kappa_{n+1}^*}}.$$

In the rest of the proof, we drop the star exponent for a_n^* , λ_n^* , κ_n^* and μ_n^* in order to avoid heaviness. With this notation, we have

$$d\gamma_n[u].\cos - i d\gamma_n[u].\sin = m_{n-1} - m_n$$

and

$$d\lambda_n[u].\cos - i d\lambda_n[u].\sin = -m_n.$$

We are now ready to study the differential of $\ln(\kappa_n)$. Recall that for $n \geq 1$,

$$\kappa_n(u) = \frac{1}{\lambda_n - \lambda_0} \prod_{\substack{p=1 \\ p \neq n}}^{+\infty} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right)$$

and

$$\kappa_0(u) = \prod_{p=1}^{+\infty} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_0} \right).$$

We get that for $n \geq 1$,

$$\delta\kappa_n = \frac{m_n - m_0}{\lambda_n - \lambda_0} + \sum_{\substack{p \geq 1 \\ p \neq n}} \frac{1}{1 - \frac{\gamma_p}{\lambda_p - \lambda_n}} \left(\frac{m_n - m_p}{(\lambda_p - \lambda_n)^2} \gamma_p - \frac{m_{p-1} - m_p}{\lambda_p - \lambda_n} \right)$$

and similarly,

$$\delta\kappa_0 = \sum_{p \geq 1} \frac{1}{1 - \frac{\gamma_p}{\lambda_p - \lambda_0}} \left(\frac{m_0 - m_p}{(\lambda_p - \lambda_0)^2} \gamma_p - \frac{m_{p-1} - m_p}{\lambda_p - \lambda_0} \right).$$

For $n \geq 1$, one can therefore write $\delta\kappa_n(\zeta) = \sum_k A_{n,k}^*(\gamma) \overline{\zeta_k} \zeta_{k+1}$, where for $k \notin \{0; n\}$,

$$A_{n,k}^*(\gamma) = \frac{1}{1 - \frac{\gamma_k}{\lambda_k - \lambda_n}} \left(\frac{a_k}{(\lambda_k - \lambda_n)^2} \gamma_k - \frac{a_k}{\lambda_k - \lambda_n} \right) + \frac{1}{1 - \frac{\gamma_{k+1}}{\lambda_{k+1} - \lambda_n}} \frac{a_k}{\lambda_{k+1} - \lambda_n},$$

for $k = n$,

$$A_{n,n}^*(\gamma) = -\frac{a_n}{\lambda_n - \lambda_0} + \left(\sum_{k \neq n} \frac{1}{1 - \frac{\gamma_k}{\lambda_k - \lambda_n}} \frac{-a_n}{(\lambda_k - \lambda_n)^2 \gamma_k} \right) + \frac{1}{1 - \frac{\gamma_{n+1}}{\lambda_{n+1} - \lambda_n}} \frac{a_n}{\lambda_{n+1} - \lambda_n},$$

and for $k = 0$,

$$A_{n,0}^*(\gamma) = \frac{a_0}{\lambda_n - \lambda_0} + \frac{1}{1 - \frac{\gamma_1}{\lambda_1 - \lambda_n}} \frac{a_0}{\lambda_1 - \lambda_n}.$$

In the case $n = 0$, we have for $k \geq 1$,

$$A_{0,k}^*(\gamma) = \frac{1}{1 - \frac{\gamma_k}{\lambda_k - \lambda_0}} \left(\frac{a_k}{(\lambda_k - \lambda_0)^2 \gamma_k} - \frac{a_k}{\lambda_k - \lambda_0} \right) + \frac{1}{1 - \frac{\gamma_{k+1}}{\lambda_{k+1} - \lambda_0}} \frac{a_k}{\lambda_{k+1} - \lambda_0}$$

and for $k = 0$,

$$A_{0,0}^*(\gamma) = \left(\sum_{k \geq 1} \frac{1}{1 - \frac{\gamma_k}{\lambda_k - \lambda_0}} \frac{-a_0}{(\lambda_k - \lambda_0)^2 \gamma_k} \right) + \frac{1}{1 - \frac{\gamma_1}{\lambda_1 - \lambda_0}} \frac{a_0}{\lambda_1 - \lambda_0}.$$

We have seen in (6.8) and (6.9) that if $\|\gamma\|_{\ell_+^1} \leq R$, then there exists $C(R) > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C(R)} \leq a_n(\gamma) \leq C(R)$$

and for all $h \in \ell_+^1$,

$$|da_n(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}.$$

Moreover, the estimate $|d\lambda_n^*(\gamma).h| \leq \|h\|_{\ell_+^1}$ holds for the differential of λ_n^* with respect to γ . We deduce that for all $n, k \in \mathbb{N}$,

$$|A_{n,k}^*(\gamma)| \leq C(R)$$

and for all $h \in \ell_+^1$,

$$|dA_{n,k}^*(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}.$$

□

The case of $\delta_{\parallel} \langle \mathbf{1} | f_n \rangle$

We now study, for $h = \cos$ and $h = \sin$, the term

$$\delta_{\parallel} \langle \mathbf{1} | f_n \rangle = -i \operatorname{Im}(\langle \tilde{\xi}_n | h \rangle) \langle \mathbf{1} | f_n \rangle,$$

where we recall that

$$\tilde{\xi}_n = \frac{1}{\langle f_0 | \mathbf{1} \rangle} \sum_{p=1}^{+\infty} \frac{\langle \mathbf{1} | f_p \rangle}{\lambda_p - \lambda_0} \overline{f_0} f_p - \sum_{k=1}^n \frac{1}{\langle f_k | S f_{k-1} \rangle} \psi_k$$

and

$$\psi_k = \sum_{\substack{p \geq 0 \\ p \neq k}} \frac{\langle f_p | S f_{k-1} \rangle}{\lambda_p - \lambda_k} f_k \overline{f_p} - \sum_{\substack{p \geq 0 \\ p \neq k-1}} \frac{\langle S f_p | f_k \rangle}{\lambda_p - \lambda_{k-1}} f_{k-1} \overline{f_p}.$$

Lemma 6.2.10. For all $n \geq 0$, denote

$$c_n^\pm := \langle \tilde{\xi}_n | e^{\pm ix} \rangle,$$

then for $n, k \geq 0$, there exists $A_{n,k}^\pm \in C^1(\ell_+^1, \mathbb{R})$, such that for all $n \geq 0$, for all $\zeta \in h_+^{\frac{1}{2}}$,

$$c_n^+(\zeta) = \sum_{k=0}^{+\infty} A_{n,k}^+(\gamma) \overline{\zeta_k} \zeta_{k+1} \quad \text{and} \quad c_n^-(\zeta) = \sum_{k=0}^{+\infty} A_{n,k}^-(\gamma) \zeta_k \overline{\zeta_{k+1}}.$$

Moreover, for all $R > 0$, there exists $C(R) > 0$ such that for all $\gamma \in \ell_+^1$ satisfying $\|\gamma\|_{\ell_+^1} \leq R$,

$$|A_{n,k}^\pm(\gamma)| \leq C(R)$$

and for all $h \in \ell_+^1$,

$$|dA_{n,k}^\pm(\gamma).h| \leq C(R) \|h\|_{\ell_+^1}.$$

Proof. • By definition of $M_{n,p} = \langle f_p | S f_n \rangle$, we have

$$c_n^+ = \frac{1}{\sqrt{\kappa_0}} \sum_{p=1}^{+\infty} \frac{\sqrt{\kappa_p} \zeta_p}{\lambda_p - \lambda_0} M_{0,p} - \sum_{k=1}^n \frac{1}{\sqrt{\mu_k}} \langle \psi_k | e^{ix} \rangle$$

and

$$\langle \psi_k | e^{ix} \rangle = \sum_{p \neq k} \frac{M_{k-1,p}}{\lambda_p - \lambda_k} M_{p,k} - \sum_{p \neq k-1} \frac{\overline{M_{p,k}}}{\lambda_p - \lambda_{k-1}} M_{p,k-1}.$$

We replace the terms $M_{n,p}$ by their expressions (see Definition 6.2.5), and make use of colors to emphasize the oscillating terms (of the form $\overline{\zeta_k} \zeta_{k+1}$ or $\zeta_k \overline{\zeta_{k+1}}$). Since $\zeta_0 = 1$, we have

$$c_n^+ = \sqrt{\frac{\mu_1 \kappa_1}{\kappa_0}} \frac{\overline{\zeta_0} \zeta_1}{1 + \gamma_1} + \sqrt{\frac{\mu_1}{\kappa_0 \kappa_1}} \overline{\zeta_0} \zeta_1 \sum_{p=2}^{+\infty} \frac{\kappa_p \overline{\zeta_p} \zeta_p}{(\lambda_p - \lambda_0)(\lambda_p - \lambda_0 - 1)} - \sum_{k=1}^n \frac{1}{\sqrt{\mu_k}} \langle \psi_k | e^{ix} \rangle.$$

Moreover, by isolating the indices $p = k - 1$ and $p = k - 2$ in the first and second sum respectively in the formula for $\langle \psi_k | e^{ix} \rangle$, we get

$$\begin{aligned} \langle \psi_k | e^{ix} \rangle &= \frac{\mu_k}{1 + \gamma_k} \sqrt{\frac{\kappa_{k-1}}{\kappa_k}} \overline{\zeta_{k-1}} \zeta_k \\ &+ \overline{\zeta_k} \zeta_k \sum_{p \neq k-1, k} \sqrt{\mu_k \mu_{p+1}} \sqrt{\frac{\kappa_p}{\kappa_{p+1}}} \frac{\overline{\zeta_p} \zeta_{p+1}}{(\lambda_p - \lambda_k)(\lambda_p - \lambda_{k-1} - 1)(\lambda_k - \lambda_p - 1)} \\ &+ \frac{\mu_{k-1}}{(1 + \gamma_{k-1})(1 + \gamma_k + \gamma_{k-1})} \sqrt{\frac{\kappa_k}{\kappa_{k-1}}} \overline{\zeta_k} \zeta_{k-1} \\ &- \overline{\zeta_k} \overline{\zeta_{k-1}} \sum_{p \neq k-2, k-1} \mu_{p+1} \frac{\sqrt{\kappa_k \kappa_{k-1}}}{\kappa_{p+1}(\lambda_p - \lambda_{k-1})} \frac{\zeta_{p+1} \overline{\zeta_{p+1}}}{(\lambda_k - \lambda_p - 1)(\lambda_{k-1} - \lambda_p - 1)}. \end{aligned}$$

One can therefore write $c_n^+ = \sum_{k=1}^{+\infty} A_{n,k-1}^+ \overline{\zeta_{k-1}} \zeta_k$, with

$$\begin{aligned} A_{n,k-1}^+ = & \mathbb{1}_{k=1} \left(\sqrt{\frac{\mu_1 \kappa_1}{\kappa_0}} \frac{1}{1 + \gamma_1} + \sqrt{\frac{\mu_1}{\kappa_0 \kappa_1}} \sum_{p=2}^{+\infty} \frac{\kappa_p \gamma_p}{(\lambda_p - \lambda_0)(\lambda_p - \lambda_0 - 1)} \right) \\ & + \mathbb{1}_{1 \leq k \leq n} \left(\frac{\sqrt{\mu_k}}{1 + \gamma_k} \sqrt{\frac{\kappa_{k-1}}{\kappa_k}} \right. \\ & + \frac{\mu_{k-1}}{\sqrt{\mu_k}(1 + \gamma_{k-1})(1 + \gamma_k + \gamma_{k-1})} \sqrt{\frac{\kappa_k}{\kappa_{k-1}}} \\ & - \sum_{p \neq k-2, k-1} \frac{\mu_{p+1}}{\sqrt{\mu_k}} \frac{\sqrt{\kappa_k \kappa_{k-1}}}{\kappa_{p+1}(\lambda_p - \lambda_{k-1})} \frac{\gamma_{p+1}}{(\lambda_k - \lambda_p - 1)(\lambda_{k-1} - \lambda_p - 1)} \Big) \\ & + \sum_{\substack{1 \leq l \leq n \\ k-1 \neq l-1, l}} \sqrt{\mu_k} \sqrt{\frac{\kappa_{k-1}}{\kappa_k}} \frac{\gamma_l}{(\lambda_{k-1} - \lambda_l)(\lambda_{k-1} - \lambda_{l-1} - 1)(\lambda_l - \lambda_{k-1} - 1)}. \end{aligned}$$

• We now prove bounds and Lipschitz estimates on finite balls of ℓ_+^1 .

We know that for $p \neq k$, $\frac{1}{|\lambda_p - \lambda_k|} \leq \frac{1}{|p-k|}$, and for $p \neq k+1$, $\frac{1}{|\lambda_p - \lambda_{k-1}|} \leq 1$. Moreover, $\mu_p \leq C(R)$ and $\kappa_p \leq \frac{C(R)}{p+1}$.

The first term is therefore bounded as

$$\begin{aligned} \sqrt{\frac{\mu_1 \kappa_1}{\kappa_0}} \frac{1}{1 + \gamma_1} + \sqrt{\frac{\mu_1}{\kappa_0 \kappa_1}} \sum_{p=2}^{+\infty} \frac{\kappa_p \gamma_p}{(\lambda_p - \lambda_0)(\lambda_p - \lambda_0 - 1)} & \leq C(R) \left(1 + \sum_{p=2}^{+\infty} \kappa_p \gamma_p \right) \\ & \leq C'(R). \end{aligned}$$

Then, for $1 \leq k \leq n$, we have

$$\frac{\sqrt{\mu_k}}{1 + \gamma_k} \sqrt{\frac{\kappa_{k-1}}{\kappa_k}} + \frac{\mu_{k-1}}{\sqrt{\mu_k}(1 + \gamma_{k-1})(1 + \gamma_k + \gamma_{k-1})} \sqrt{\frac{\kappa_k}{\kappa_{k-1}}} \leq C(R)$$

and since the following estimate can be obtained by comparing $(k-1)$ to $\frac{p}{2}$

$$\frac{\sqrt{\kappa_k \kappa_{k-1}}}{\kappa_{p+1}(\lambda_p - \lambda_{k-1})} \leq C(R) \frac{p}{k|p - (k-1)|} \leq C'(R),$$

we get

$$\begin{aligned} \left| \sum_{p \neq k-2, k-1} \frac{\mu_{p+1}}{\sqrt{\mu_k}} \frac{\sqrt{\kappa_k \kappa_{k-1}}}{\kappa_{p+1}(\lambda_p - \lambda_{k-1})} \frac{\gamma_{p+1}}{(\lambda_k - \lambda_p - 1)(\lambda_{k-1} - \lambda_p - 1)} \right| \\ \leq C(R) \sum_{p \neq k-2, k-1} \gamma_{p+1} \leq C'(R). \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{\substack{1 \leq l \leq n \\ k-1 \neq l-1, l}} \sqrt{\mu_k} \sqrt{\frac{\kappa_{k-1}}{\kappa_k}} \frac{\gamma_l}{(\lambda_{k-1} - \lambda_l)(\lambda_{k-1} - \lambda_{l-1} - 1)(\lambda_l - \lambda_{k-1} - 1)} \\ \leq C(R) \sum_{\substack{1 \leq l \leq n \\ k-1 \neq l-1, l}} \gamma_l \leq C'(R). \end{aligned}$$

We conclude that

$$|A_{n,p}^+| \leq C(R).$$

We finally check that for all $h \in \ell_+^1$,

$$|dA_{n,p}^+(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}.$$

This is a consequence of the formulas, and the fact that for all $n \in \mathbb{N}$, the spectral parameters, seen as functions of the actions γ , satisfy $|d\lambda_n^*(\gamma).h| \leq \|h\|_{\ell_+^1}$, $|d\mu_n^*(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}$ and $|(n+1)d\kappa_n^*(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}$ if $\|\gamma\|_{\ell_+^1} \leq R$.

• Similarly, we write the formula for c_n^-

$$c_n^- = \frac{1}{\sqrt{\kappa_0}} \sum_{p=1}^{+\infty} \frac{\sqrt{\kappa_p} \zeta_p}{\lambda_p - \lambda_0} \overline{M_{p,0}} - \sum_{k=1}^n \frac{1}{\sqrt{\mu_k}} \langle \psi_k | e^{-ix} \rangle,$$

where

$$\langle \psi_k | e^{-ix} \rangle = \sum_{p \neq k} \frac{M_{k-1,p}}{\lambda_p - \lambda_k} \overline{M_{k,p}} - \sum_{p \neq k-1} \frac{\overline{M_{p,k}}}{\lambda_p - \lambda_{k-1}} \overline{M_{k-1,p}}.$$

Using the expression of $M_{n,p}$, we get

$$c_n^- = \sum_{p=1}^{+\infty} \sqrt{\mu_{p+1}} \sqrt{\frac{\kappa_p}{\kappa_{p+1}}} \frac{\zeta_p \overline{\zeta_{p+1}}}{(\lambda_p - \lambda_0)(\lambda_0 - \lambda_p - 1)} - \sum_{k=1}^n \frac{1}{\sqrt{\mu_k}} \langle \psi_k | e^{-ix} \rangle,$$

where we isolate the terms $p = k + 1$ and $p = k$ in the first and second term respectively in the formula for $\langle \psi_k | e^{-ix} \rangle$

$$\begin{aligned} \langle \psi_k | e^{-ix} \rangle &= \frac{\sqrt{\mu_k \mu_{k+1}}}{(1 + \gamma_{k+1} + \gamma_k)(1 + \gamma_{k+1})} \sqrt{\frac{\kappa_{k+1}}{\kappa_k}} \overline{\zeta_{k+1}} \zeta_k \\ &+ \overline{\zeta_k \zeta_{k+1}} \sum_{p \neq k, k+1} \sqrt{\mu_k \mu_{k+1}} \frac{\kappa_p}{\sqrt{\kappa_k \kappa_{k+1}} (\lambda_p - \lambda_k)} \frac{\zeta_p \overline{\zeta_p}}{(\lambda_p - \lambda_{k-1} - 1)(\lambda_p - \lambda_k - 1)} \\ &\quad + \frac{\sqrt{\mu_{k+1} \mu_k}}{1 + \gamma_k} \sqrt{\frac{\kappa_k}{\kappa_{k+1}}} \zeta_k \overline{\zeta_{k+1}} \\ &- \overline{\zeta_k \zeta_k} \sum_{p \neq k-1, k} \sqrt{\mu_{p+1} \mu_k} \sqrt{\frac{\kappa_p}{\kappa_{p+1}}} \frac{\overline{\zeta_{p+1}} \zeta_p}{(\lambda_p - \lambda_{k-1})(\lambda_k - \lambda_p - 1)(\lambda_p - \lambda_{k-1} - 1)}. \end{aligned}$$

One can therefore write $c_n^- = \sum_k A_{n,k}^- \zeta_k \overline{\zeta_{k+1}}$, with

$$\begin{aligned} A_{n,k}^- &= \mathbf{1}_{k \neq 0} \sqrt{\mu_{k+1}} \sqrt{\frac{\kappa_k}{\kappa_{k+1}}} \frac{1}{(\lambda_k - \lambda_0)(\lambda_0 - \lambda_k - 1)} \\ &+ \mathbf{1}_{1 \leq k \leq n} \left(\frac{\sqrt{\mu_{k+1}}}{(1 + \gamma_{k+1} + \gamma_k)(1 + \gamma_{k+1})} \sqrt{\frac{\kappa_{k+1}}{\kappa_k}} \right. \\ &+ \sum_{p \neq k, k+1} \sqrt{\mu_{k+1}} \frac{\kappa_p}{\sqrt{\kappa_k \kappa_{k+1}} (\lambda_p - \lambda_k)} \frac{\gamma_p}{(\lambda_p - \lambda_{k-1} - 1)(\lambda_p - \lambda_k - 1)} \\ &\left. + \frac{\sqrt{\mu_{k+1}}}{1 + \gamma_k} \sqrt{\frac{\kappa_k}{\kappa_{k+1}}} \right) \\ &- \sum_{\substack{1 \leq l \leq n \\ k \neq l-1, l}} \sqrt{\mu_{k+1}} \sqrt{\frac{\kappa_k}{\kappa_{k+1}}} \frac{\gamma_l}{(\lambda_k - \lambda_{l-1})(\lambda_l - \lambda_k - 1)(\lambda_k - \lambda_{l-1} - 1)}. \end{aligned}$$

The proof of the estimates

$$|A_{n,p}^-| \leq C(R)$$

and

$$|dA_{n,p}^-(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}$$

are similar to the terms $A_{n,p}^+$. □

The case of $\delta_\perp \langle \mathbf{1} | f_n \rangle$

Finally, we focus on the term

$$\delta_\perp \langle \mathbf{1} | f_n \rangle = \sum_{p \neq n} \frac{\langle f_p \overline{f_n} | h \rangle}{\lambda_p - \lambda_n} \langle \mathbf{1} | f_p \rangle$$

when $h = \cos$ and $h = \sin$.

Lemma 6.2.11. *For $n \geq 0$, let us write*

$$b_n^\pm := \sum_{p \neq n} \frac{\langle f_p \overline{f_n} | e^{\pm ix} \rangle}{\lambda_p - \lambda_n} \langle \mathbf{1} | f_p \rangle.$$

Then there exist $p_n^, B_{n,k}^*, q_n^* \in \mathcal{C}^1(\ell_+^1, \mathbb{R})$, $n, k \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, for all $\zeta \in h_+^{\frac{1}{2}}$,*

$$\frac{b_n^+}{\sqrt{\kappa_n}} = q_n^*(\gamma) \zeta_{n+1}$$

and

$$\frac{b_n^-}{\sqrt{\kappa_n}} = p_n^*(\gamma) \zeta_{n-1} + \left(\sum_{k=0}^{+\infty} B_{n,k}^*(\gamma) \zeta_k \overline{\zeta_{k+1}} \right) \zeta_n.$$

The terms p_n^ , $B_{n,k}^*$ and q_n^* are uniformly bounded and Lipschitz on finite balls: for all $R > 0$, there exists $C(R) > 0$ such that for all $\gamma \in \ell_+^1$ satisfying $\|\gamma\|_{\ell_+^1} \leq R$, for all $n, k \geq 0$, we have*

$$|p_n^*(\gamma)| + |B_{n,k}^*(\gamma)| + |q_n^*(\gamma)| \leq C(R)$$

and for all $h \in \ell_+^1$,

$$|dp_n^*(\gamma).h| + |dB_{n,k}^*(\gamma).h| + |dq_n^*(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}.$$

Proof. We first simplify the formulas for b_n^+ and b_n^- : on the one hand,

$$\begin{aligned} b_n^+ &= \sum_{p \neq n} \frac{M_{n,p}}{\lambda_p - \lambda_n} \sqrt{\kappa_p} \zeta_p \\ &= \sqrt{\mu_{n+1}} \zeta_{n+1} \left(\frac{\sqrt{\kappa_{n+1}}}{1 + \gamma_{n+1}} + \frac{1}{\sqrt{\kappa_{n+1}}} \sum_{p \neq n, n+1} \frac{\kappa_p}{(\lambda_p - \lambda_n)(\lambda_p - \lambda_n - 1)} \zeta_p \overline{\zeta_p} \right), \end{aligned}$$

and on the other hand,

$$\begin{aligned} b_n^- &= \sum_{p \neq n} \frac{\overline{M_{p,n}}}{\lambda_p - \lambda_n} \sqrt{\kappa_p} \zeta_p \\ &= -\frac{\sqrt{\mu_n} \sqrt{\kappa_{n-1}}}{1 + \gamma_n} \zeta_{n-1} + \sqrt{\kappa_n} \zeta_n \sum_{p \neq n, n-1} \frac{\sqrt{\mu_{p+1}}}{(\lambda_p - \lambda_n)(\lambda_n - \lambda_p - 1)} \sqrt{\frac{\kappa_p}{\kappa_{p+1}}} \zeta_p \overline{\zeta_{p+1}}. \end{aligned}$$

We now define

$$p_n^*(\gamma) = -\frac{\sqrt{\mu_n}\sqrt{\kappa_{n-1}}}{\sqrt{\kappa_n}(1+\gamma_n)}, \quad (6.12)$$

$$B_{n,k}^*(\gamma) = \mathbb{1}_{k \neq n, n-1} \frac{\sqrt{\mu_{k+1}}}{(\lambda_k - \lambda_n)(\lambda_n - \lambda_k - 1)} \sqrt{\frac{\kappa_k}{\kappa_{k+1}}}$$

and

$$q_n^*(\gamma) = \frac{\sqrt{\mu_{n+1}}}{\sqrt{\kappa_n}} \left(\frac{\sqrt{\kappa_{n+1}}}{1+\gamma_{n+1}} + \frac{1}{\sqrt{\kappa_{n+1}}} \sum_{\substack{p \geq 1 \\ p \neq n, n+1}} \frac{\kappa_p \gamma_p}{(\lambda_p - \lambda_n)(\lambda_p - \lambda_n - 1)} \right). \quad (6.13)$$

The estimates on p_n^* and $B_{n,k}^*$ follow from their definitions. Note that in order to bound the sum on the right in the expression of q_n^* , we use the inequality $|\lambda_p - \lambda_n| \geq |p - n|$ and the estimate

$$\sum_{\substack{p \geq 1 \\ p \neq n}} \frac{1}{p|p-n|} \leq \frac{C}{n+1},$$

which can be obtained when comparing p to $\frac{n}{2}$. \square

Proof of Theorem 6.1.4. Using the expression (6.10) of $d\zeta_n[u] \cdot \cos$ and $d\zeta_n[u] \cdot \sin$, we have the formulas

$$d\zeta_n[u] \cdot \cos = \zeta_n \left(-i \operatorname{Im}(\langle \tilde{\xi}_n | \cos \rangle) - \frac{d \ln(\kappa_n)[u] \cdot \cos}{2} \right) + \frac{b_n^+ + b_n^-}{2\sqrt{\kappa_n}}$$

and

$$d\zeta_n[u] \cdot \sin = \zeta_n \left(-i \operatorname{Im}(\langle \tilde{\xi}_n | \sin \rangle) - \frac{d \ln(\kappa_n)[u] \cdot \sin}{2} \right) - \frac{b_n^+ - b_n^-}{2i\sqrt{\kappa_n}}.$$

The proof is now a direct consequence of Lemmas 6.2.9, 6.2.10 and 6.2.11. \square

Lipschitz properties

We now deduce from Theorem 6.1.4 that the terms $d\zeta_n[u] \cdot \cos$ and $d\zeta_n[u] \cdot \sin$ are bounded and Lipschitz functions of ζ on finite balls for the norms

$$h_+^{\frac{1}{2}-s} = \{(\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{+\infty} n^{1-2s} |\zeta_n|^2 < +\infty\}, \quad 0 \leq s < \frac{1}{2}.$$

Corollary 6.2.12 (Bounds and Lipschitz properties). *Fix $R > 0$. There exists $C(R) > 0$ such that the following holds.*

Let $s < -\frac{1}{2}$. If $\zeta \in h_+^{\frac{1}{2}}$ and $\|\zeta\|_{\ell^2} \leq R$, then writing $u = \Phi^{-1}(\zeta)$,

$$\|(d\zeta_n[u] \cdot \cos)_n\|_{h_+^{\frac{1}{2}+s}} + \|(d\zeta_n[u] \cdot \sin)_n\|_{h_+^{\frac{1}{2}+s}} \leq C(R) \|\zeta\|_{h_+^{\frac{1}{2}+s}}.$$

Moreover, for all $\zeta^{(1)}, \zeta^{(2)} \in h_+^{\frac{1}{2}}$ satisfying $\|\zeta^{(1)}\|_{h_+^{\frac{1}{2}+s}}, \|\zeta^{(2)}\|_{h_+^{\frac{1}{2}+s}} \leq R$, if $u^{(1)} = \Phi^{-1}(\zeta^{(1)})$

and $u^{(2)} = \Phi^{-1}(\zeta^{(2)})$, then

$$\|(d\zeta_n[u^{(1)}] \cdot \cos)_n - (d\zeta_n[u^{(2)}] \cdot \cos)_n\|_{h_+^{\frac{1}{2}+s}} \leq C(R) \|\zeta^{(1)} - \zeta^{(2)}\|_{h_+^{\frac{1}{2}+s}}$$

and

$$\|(d\zeta_n[u^{(1)}] \cdot \sin)_n - (d\zeta_n[u^{(2)}] \cdot \sin)_n\|_{h_+^{\frac{1}{2}+s}} \leq C(R) \|\zeta^{(1)} - \zeta^{(2)}\|_{h_+^{\frac{1}{2}+s}}.$$

Proof. Let $\zeta \in h_+^{\frac{1}{2}}$ such that $\|\zeta\|_{\ell_+^2} \leq R$. We write

$$d\zeta_n[u].\cos = p_n^*(\gamma)\zeta_{n-1} + \left(\sum_{k=0}^{+\infty} A_{n,k}^*(\gamma)\overline{\zeta_k}\zeta_{k+1} + B_{n,k}^*(\gamma)\zeta_k\overline{\zeta_{k+1}} \right) \zeta_n + q_n^*(\gamma)\zeta_{n+1},$$

where the terms p_n^* , q_n^* , $A_{n,k}^*$ and $B_{n,k}^*$ are bounded and Lipschitz on balls of finite radius in ℓ_+^1 , and $\gamma = (|\zeta_n|^2)_n \in \ell_+^1$. We have $\|\gamma\|_{\ell_+^1} \leq R^2$, therefore there exists $C(R) > 0$ such that

$$|p_n^*(\gamma)| + |q_n^*(\gamma)| + |A_{n,k}^*(\gamma)| + |B_{n,k}^*(\gamma)| \leq C(R)$$

and for all $h \in \ell_+^1$,

$$|dp_n^*(\gamma).h| + |dq_n^*(\gamma).h| + |dA_{n,k}^*(\gamma).h| + |dB_{n,k}^*(\gamma).h| \leq C(R)\|h\|_{\ell_+^1}.$$

We deduce that for all $n \geq 1$,

$$|d\zeta_n[u].\cos| \leq C(R)(|\zeta_{n-1}| + |\zeta_n| + |\zeta_{n+1}|).$$

Let $s > -\frac{1}{2}$ and assume $\|\zeta\|_{\ell_+^2} \leq R$, a summation leads to

$$\sum_{n=1}^{+\infty} n^{1+2s} |d\zeta_n[u].\cos|^2 \leq C(R) \sum_{n=1}^{+\infty} n^{1+2s} |\zeta_n|^2 = C(R) \|\zeta\|_{h_+^{\frac{1}{2}+s}}^2.$$

Now, define, for $\zeta \in h_+^{\frac{1}{2}}$,

$$S_n(\zeta) := d\zeta_n[\Phi^{-1}(\zeta)].\cos.$$

We prove that the differential of S_n is bounded on balls of finite radius of $h_+^{\frac{1}{2}+s}$ for $s > -\frac{1}{2}$. Fix $h \in \ell_+^2$ and denote $H = (H_n)_{n \geq 1}$ the differential of the map $\zeta \in \ell_+^2 \mapsto (|\zeta_n|^2)_{n \geq 1} \in \ell_+^1$ at ζ applied to h . For all $n \geq 1$, we have $H_n = \zeta_n \overline{h_n} + \overline{\zeta_n} h_n$, therefore H is in ℓ_+^1 and

$$\|H\|_{\ell_+^1} \leq 2\|\zeta\|_{\ell_+^2} \|h\|_{\ell_+^2}.$$

The differential of S_n at ζ applied to h now writes

$$\begin{aligned} dS_n(\zeta).h &= dp_n^*(\gamma).H\zeta_{n-1} + dq_n^*(\gamma).H\zeta_{n+1} + p_n^*(\gamma)h_{n-1} + q_n^*(\gamma)h_{n+1} \\ &\quad + \left(\sum_{k=0}^{+\infty} dA_{n,k}^*(\gamma).H\overline{\zeta_k}\zeta_{k+1} + dB_{n,k}^*(\gamma).H\zeta_k\overline{\zeta_{k+1}} \right) \zeta_n \\ &\quad + \left(\sum_{k=0}^{+\infty} A_{n,k}^*(\gamma)(\overline{h_k}\zeta_{k+1} + \overline{\zeta_k}h_{k+1}) + B_{n,k}^*(\gamma)(h_k\overline{\zeta_{k+1}} + \zeta_k\overline{h_{k+1}}) \right) \zeta_n \\ &\quad + \left(\sum_{k=0}^{+\infty} A_{n,k}^*(\gamma)\overline{\zeta_k}\zeta_{k+1} + B_{n,k}^*(\gamma)\zeta_k\overline{\zeta_{k+1}} \right) h_n. \end{aligned}$$

When $\|\zeta\|_{h_+^{\frac{1}{2}+s}} \leq R$, a summation leads to

$$\begin{aligned} \sum_{n=1}^{+\infty} n^{1-2s} |dS_n(\zeta).h|^2 &\leq C(R)\|H\|_{\ell_+^1}^2 + C(R) \sum_{n=1}^{+\infty} n^{1+2s} |h_n|^2 \\ &\leq C'(R)\|h\|_{h_+^{\frac{1}{2}+s}}^2. \end{aligned}$$

These arguments are also valid in the case of $d\zeta_n[u].\sin$. □

6.3 Flow map

In this section, we prove global well-posedness for equation (BO- α) and establish the weak sequential continuity of the flow map.

Thanks to Theorem 6.1.4, we have simplified the system of ODEs satisfied by the Birkhoff coordinates ζ . In order for the vector field to be Lipschitz along the second variable, we need to remove the oscillatory part in the frequencies $\omega_n = n^2 - 2 \sum_{k \geq 1} \min(k, n) |\zeta_k|^2$. Therefore, we introduce a change of functions $z_n(t) = e^{-in^2 t} \zeta_n(t)$, $n \geq 1$: the new coordinates $z = (z_n)_{n \geq 1}$ are solution to an ODE

$$z'(t) = F(t, z(t)). \quad (6.14)$$

The vector field $F = (F_n)_{n \geq 1}$ is defined as

$$F_n(t, z) = e^{-in^2 t} \left(i \tilde{\omega}_n(z) e^{in^2 t} z_n - \alpha \tilde{Z}_n(t, z) \right), \quad (6.15)$$

where $\tilde{\omega}_n$ are the new frequencies

$$\tilde{\omega}_n(z) = -2 \sum_{k=1}^{+\infty} \min(k, n) |z_k|^2$$

and

$$\tilde{Z}_n(t, z) = Z_n(\zeta(t, z)) := \langle u | \cos \rangle d\zeta_n[u] \cdot \cos + \langle u | \sin \rangle d\zeta_n[u] \cdot \sin, \quad (6.16)$$

with

$$\zeta_n(t, z) = e^{in^2 t} z_n, \quad n \geq 1$$

and

$$u(t, z) = \Phi^{-1}(\zeta(t, z)).$$

We apply the Cauchy-Lipschitz theorem for ordinary differential equations in part 6.3.1 to get local well-posedness for equation (BO- α). In part 6.3.2, global well-posedness follows from the existence of a Lyapunov functional controlling the L^2 norm of the solution and a compactness argument. Finally, we prove the weak sequential continuity of the flow map in part 6.3.3.

6.3.1 Local well-posedness

In this part, we prove that the vector field F is locally Lipschitz with respect to the second variable and apply the Cauchy-Lipschitz theorem for ODEs.

Theorem 6.3.1. *The map F is bounded and Lipschitz on finite balls in the following sense. Let $R > 0$. Then there exists $C(R) > 0$ such that if $\|z^{(1)}\|_{h^{\frac{1}{2}}_+} \leq R$ and $\|z^{(2)}\|_{h^{\frac{1}{2}}_+} \leq R$, then for all $t \in \mathbb{R}$,*

$$\|F(t, z^{(1)})\|_{h^{\frac{1}{2}}_+} + \|F(t, z^{(2)})\|_{h^{\frac{1}{2}}_+} \leq C(R),$$

and

$$\|F(t, z^{(2)}) - F(t, z^{(1)})\|_{h^{\frac{1}{2}}_+} \leq C(R) \|z^{(2)} - z^{(1)}\|_{h^{\frac{1}{2}}_+}.$$

Moreover, F is weakly sequentially continuous with respect to the second variable.

Proof. We start from the definition of F from (6.15):

$$F_n(t, z) = e^{-in^2t} \left(i\tilde{\omega}_n(z)e^{in^2t}z_n - \alpha\tilde{Z}_n(t, z) \right).$$

We consider the terms $i\tilde{\omega}_n(z)z_n$ and \tilde{Z}_n separately.

• First, the part $(t, z) \in \mathbb{R} \times h_+^{\frac{1}{2}} \mapsto (i\tilde{\omega}_n(z)z_n)_{n \geq 1} \in h_+^{\frac{1}{2}}$ is bounded and Lipschitz on finite balls with respect to the variable z . Indeed, fix $z^{(1)}, z^{(2)} \in h_+^{\frac{1}{2}}$ bounded by R in $h_+^{\frac{1}{2}}$. We have

$$|\tilde{\omega}_n(z^{(1)})| = 2 \sum_{k=1}^{+\infty} \min(k, n) |z_k^{(1)}|^2 \leq 2R^2,$$

so that

$$\|(\tilde{\omega}_n(z^{(1)})z_n^{(1)})_{n \geq 1}\|_{h_+^{\frac{1}{2}}} \leq 2R^3.$$

Moreover, thanks to Cauchy-Schwarz' inequality,

$$\begin{aligned} |\tilde{\omega}_n(z^{(2)}) - \tilde{\omega}_n(z^{(1)})| &= 2 \left| \sum_{k=1}^{+\infty} \min(k, n) \left(|z_k^{(2)}|^2 - |z_k^{(1)}|^2 \right) \right| \\ &\leq C(R) \|z^{(2)} - z^{(1)}\|_{h_+^{\frac{1}{2}}}, \end{aligned}$$

so that

$$\|(\tilde{\omega}_n(z^{(2)})z_n^{(2)})_{n \geq 1} - (\tilde{\omega}_n(z^{(1)})z_n^{(1)})_{n \geq 1}\|_{h_+^{\frac{1}{2}}} \leq C'(R) \|z^{(2)} - z^{(1)}\|_{h_+^{\frac{1}{2}}}.$$

Let us now establish the weak sequential continuity. Let $z^{(k)}$, $k \geq 1$, be a sequence of elements of $h_+^{\frac{1}{2}}$ weakly convergent to some $z \in h_+^{\frac{1}{2}}$. By compactness, this sequence is strongly convergent in the space ℓ_+^2 . Therefore, for all $n \geq 1$,

$$|\tilde{\omega}_n(z^{(k)}) - \tilde{\omega}_n(z)| \leq 2n \sum_{p=1}^{+\infty} \left| |z_p^{(k)}|^2 - |z_p|^2 \right|$$

converges to zero as k goes to infinity. We conclude that $(\tilde{\omega}_n(z^{(k)})z_n^{(k)})_{n \geq 1}$ is weakly convergent to $(\tilde{\omega}_n(z)z_n)_{n \geq 1}$ in $h_+^{\frac{1}{2}}$ as k goes to infinity.

• We now prove that $\tilde{Z} = (\tilde{Z}_n)_n$ defines a bounded and Lipschitz map on finite balls from $h_+^{\frac{1}{2}-s}$ to itself for all $s \in [0, \frac{1}{2})$. The introduction of a variable Sobolev space $h_+^{\frac{1}{2}-s}$ in the Lipschitz properties will be useful when considering the weak sequential continuity.

From Corollary 6.2.7, we know that $\zeta \in h_+^{\frac{1}{2}} \mapsto \langle u | \cos \rangle \in \mathbb{R}$ and $\zeta \in h_+^{\frac{1}{2}} \mapsto \langle u | \sin \rangle \in \mathbb{R}$, where $u = \Phi^{-1}(\zeta)$, define real-valued maps which are bounded by $C(R)$, and Lipschitz with respect to the ℓ_+^2 norm: if $\|\zeta^{(1)}\|_{\ell_+^2} \leq R$ and $\|\zeta^{(2)}\|_{\ell_+^2} \leq R$, then

$$|\langle u^{(1)} | \cos \rangle| + |\langle u^{(1)} | \sin \rangle| + |\langle u^{(2)} | \cos \rangle| + |\langle u^{(2)} | \sin \rangle| \leq C(R)$$

and

$$|\langle u^{(1)} | \cos \rangle - \langle u^{(2)} | \cos \rangle| + |\langle u^{(1)} | \sin \rangle - \langle u^{(2)} | \sin \rangle| \leq C(R) \|\zeta^{(1)} - \zeta^{(2)}\|_{\ell_+^2}.$$

From Corollary 6.2.12, we also know that the terms $d\zeta_n[u].\cos$ and $d\zeta_n[u].\sin$, when restricted to a ball of radius R , are bounded by $C(R)$ and Lipschitz with respect to the $h_+^{\frac{1}{2}-s}$ norm for all $s \in [0, \frac{1}{2})$: if $\|\zeta^{(1)}\|_{h_+^{\frac{1}{2}-s}} \leq R$ and $\|\zeta^{(2)}\|_{h_+^{\frac{1}{2}-s}} \leq R$, then

$$\|(d\zeta_n[u^{(1)}].\cos)_n\|_{h_+^{\frac{1}{2}-s}} + \|(d\zeta_n[u^{(1)}].\sin)_n\|_{h_+^{\frac{1}{2}-s}} \leq C(R),$$

$$\|(\mathrm{d}\zeta_n[u^{(2)}] \cdot \cos)_n\|_{h_+^{\frac{1}{2}-s}} + \|(\mathrm{d}\zeta_n[u^{(2)}] \cdot \sin)_n\|_{h_+^{\frac{1}{2}-s}} \leq C(R),$$

$$\|(\mathrm{d}\zeta_n[u^{(1)}] \cdot \cos)_n - (\mathrm{d}\zeta_n[u^{(2)}] \cdot \cos)_n\|_{h_+^{\frac{1}{2}-s}} \leq C(R)\|\zeta^{(1)} - \zeta^{(2)}\|_{h_+^{\frac{1}{2}-s}}$$

and

$$\|(\mathrm{d}\zeta_n[u^{(1)}] \cdot \sin)_n - (\mathrm{d}\zeta_n[u^{(2)}] \cdot \sin)_n\|_{h_+^{\frac{1}{2}-s}} \leq C(R)\|\zeta^{(1)} - \zeta^{(2)}\|_{h_+^{\frac{1}{2}-s}}.$$

We deduce that $Z = (Z_n)_n$ is bounded and Lipschitz with respect to the $h_+^{\frac{1}{2}-s}$ norm when $s \in [0, \frac{1}{2})$: if $\|\zeta^{(1)}\|_{h_+^{\frac{1}{2}}} \leq R$ and $\|\zeta^{(2)}\|_{h_+^{\frac{1}{2}}} \leq R$, then

$$\|Z(\zeta^{(1)})\|_{h_+^{\frac{1}{2}-s}} + \|Z(\zeta^{(2)})\|_{h_+^{\frac{1}{2}-s}} \leq C(R)$$

and

$$\|Z(\zeta^{(1)}) - Z(\zeta^{(2)})\|_{h_+^{\frac{1}{2}-s}} \leq C(R)\|\zeta^{(1)} - \zeta^{(2)}\|_{h_+^{\frac{1}{2}-s}}.$$

Now, fix $t \in \mathbb{R}$ and $z^{(1)}, z^{(2)} \in h_+^{\frac{1}{2}}$ such that $\|z^{(1)}\|_{h_+^{\frac{1}{2}}} \leq R$ and $\|z^{(2)}\|_{h_+^{\frac{1}{2}}} \leq R$. Then by definition of $\zeta^{(1)}$ and $\zeta^{(2)}$ as $\zeta_n^{(k)}(t, z) = e^{in^2 t} z_n^{(k)}$, for $n \geq 1$ and $k = 1, 2$, we see that $\|\zeta^{(k)}\|_{h_+^{\frac{1}{2}}} = \|z^{(k)}\|_{h_+^{\frac{1}{2}}}$, moreover, for $s \in [0, \frac{1}{2})$, $\|\zeta^{(1)} - \zeta^{(2)}\|_{h_+^{\frac{1}{2}-s}} = \|z^{(1)} - z^{(2)}\|_{h_+^{\frac{1}{2}-s}}$. We have therefore proven that

$$\|\tilde{Z}(z^{(1)})\|_{h_+^{\frac{1}{2}-s}} + \|\tilde{Z}(z^{(2)})\|_{h_+^{\frac{1}{2}-s}} \leq C(R)$$

and

$$\|\tilde{Z}(z^{(1)}) - \tilde{Z}(z^{(2)})\|_{h_+^{\frac{1}{2}-s}} \leq C(R)\|z^{(1)} - z^{(2)}\|_{h_+^{\frac{1}{2}-s}}. \quad (6.17)$$

We now prove that \tilde{Z} is weakly sequentially continuous with respect to z . Indeed, let $z^{(k)}$, $k \geq 1$, be a sequence of elements of $h_+^{\frac{1}{2}}$ weakly convergent to some $z \in h_+^{\frac{1}{2}}$. We fix $s \in (0, \frac{1}{2})$ and use the fact that the embedding $h_+^{\frac{1}{2}} \hookrightarrow h_+^{\frac{1}{2}-s}$ is compact: from the Rellich theorem, $z^{(k)}$ is strongly convergent to z in $h_+^{\frac{1}{2}-s}$. But inequality (6.17) now implies that $\tilde{Z}(z^{(k)})$ is strongly convergent to $\tilde{Z}(z)$ in $h_+^{\frac{1}{2}-s}$, and therefore weakly convergent in $h_+^{\frac{1}{2}}$. \square

We deduce from the Cauchy-Lipschitz theorem for ODEs that the initial value problem for the damped Benjamin-Ono equation in Birkhoff coordinates (6.14) is locally well-posed in $h_+^{\frac{1}{2}}$.

Corollary 6.3.2. *For all $z(0) \in h_+^{\frac{1}{2}}$, there exists $T > 0$ such that equation (6.14)*

$$z'(t) = F(t, z(t))$$

with initial data $z(0)$ admits a unique solution $z \in \mathcal{C}^1([0, T], h_+^{\frac{1}{2}})$. Moreover, the solution map is continuous from $h_+^{\frac{1}{2}}$ to $\mathcal{C}^1([0, T], h_+^{\frac{1}{2}})$.

Applying the inverse of the Birkhoff map, we get that the damped Benjamin-Ono equation (BO- α) is locally well-posed $L_{r,0}^2(\mathbb{T})$: for every $u_0 \in L_{r,0}^2(\mathbb{T})$, there exists $T > 0$ such that (BO- α) admits a unique solution $u \in \mathcal{C}([0, T], L_{r,0}^2(\mathbb{T}))$ with initial data u_0 in the distribution sense; moreover, the solution map is continuous. Indeed, thanks to (4.5) in Lemma 4.2 in [GK20], one can prove that the inverse Birkhoff map admits a differential from $h_+^{\frac{1}{2}}$ to $L_{r,0}^2(\mathbb{T})$.

6.3.2 Global well-posedness

Thanks to a Lyapunov functional controlling the L^2 norm, we now establish global well-posedness for the damped Benjamin-Ono equation (BO- α) in $L^2_{r,0}(\mathbb{T})$.

Proposition 6.3.3 (Lyapunov functional). *Let $u_0 \in L^2_{r,0}(\mathbb{T})$, and $([0, T^*), u)$ be the corresponding maximal solution. Then for all $t \in [0, T^*)$,*

$$\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{T})}^2 + 2\alpha |\langle u(t) | e^{ix} \rangle|^2 = 0.$$

Moreover,

$$2\alpha \int_0^{T^*} |\langle u(t) | e^{ix} \rangle|^2 dt \leq \|u_0\|_{L^2(\mathbb{T})}^2$$

and if $T^* = +\infty$, then $|\langle u(t) | e^{ix} \rangle|$ tends to 0 as t goes to $+\infty$.

Proof. Using the equation,

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{T})}^2 &= 2\operatorname{Re} [\langle \partial_t u | u(t) \rangle] \\ &= -2\alpha (|\langle u(t) | \cos \rangle|^2 + |\langle u(t) | \sin \rangle|^2) + 2\operatorname{Re} [\langle \partial_x (H\partial_x u - u^2) | u(t) \rangle] \\ &= -2\alpha |\langle u(t) | e^{ix} \rangle|^2. \end{aligned}$$

It remains to show that $|\langle u(t) | e^{ix} \rangle|$ tends to 0 as t goes to $+\infty$. Taking the derivative,

$$\begin{aligned} \frac{d}{dt} |\langle u(t) | \cos \rangle|^2 &= 2\operatorname{Re} [\langle \partial_t u | \cos \rangle \langle \cos | u(t) \rangle] \\ &= -2\alpha \operatorname{Re} [(\langle u(t) | \cos \rangle \langle \cos | \cos \rangle + \langle u(t) | \sin \rangle \langle \sin | \cos \rangle) \langle \cos | u(t) \rangle] \\ &\quad + 2\operatorname{Re} [\langle H\partial_{xx} u - \partial_x(u^2) | \cos \rangle \langle \cos | u(t) \rangle] \\ &= -\alpha |\langle u(t) | \cos \rangle|^2 + 2\operatorname{Re} [(-\langle u(t) | H\partial_{xx} \cos \rangle + \langle u^2(t) | \partial_x \cos \rangle) \langle \cos | u(t) \rangle]. \end{aligned}$$

Since $H \cos = \sin$, we deduce that

$$\frac{d}{dt} |\langle u(t) | \cos \rangle|^2 = -\alpha |\langle u(t) | \cos \rangle|^2 + 2\operatorname{Re} [(\langle u(t) | \sin \rangle - \langle u^2(t) | \sin \rangle) \langle \cos | u(t) \rangle].$$

But

$$|\langle u(t) | \cos \rangle| \leq \|u\|_{L^2(\mathbb{T})} \leq \|u_0\|_{L^2(\mathbb{T})},$$

$$|\langle u(t) | \sin \rangle| \leq \|u_0\|_{L^2(\mathbb{T})}$$

and

$$|\langle u^2(t) | \sin \rangle| \leq \|u\|_{L^2(\mathbb{T})}^2 \leq \|u_0\|_{L^2(\mathbb{T})}^2,$$

therefore $\frac{d}{dt} |\langle u(t) | \cos \rangle|^2$ is bounded, and the same can be proven for $\frac{d}{dt} |\langle u(t) | \sin \rangle|^2$. This observation combined with the fact that $|\langle u(t) | e^{ix} \rangle|$ is square integrable implies that $|\langle u(t) | e^{ix} \rangle|$ tends to zero as time goes to infinity if $T^* = +\infty$. \square

Proposition 6.3.4. *For all $z(0) \in h^{\frac{1}{2}}_+$, the maximal solution $t \mapsto z(t) \in h^{\frac{1}{2}}_+$ is global.*

The proof of this proposition relies on the fact that, given an initial condition $z(0) \in h_+^{\frac{1}{2}}$, it is possible to construct a weighted space $\ell(w)$ which contains $z(0)$ and which compactly embeds into $h_+^{\frac{1}{2}}$. With a Gronwall type argument, we then prove that the maximal solution $t \mapsto z(t)$ is bounded in $\ell(w)$. Because of the compactness of the embedding $\ell(w) \hookrightarrow h_+^{\frac{1}{2}}$ and the local well-posedness of the Cauchy problem, this ensures that the maximal solution is global.

Lemma 6.3.5. *Let $(x_n)_{n \geq 1} \in (\mathbb{R}_+)^{\mathbb{N}}$ be a sequence of nonnegative real numbers such that the series with general term x_n is convergent: $\sum_{n \geq 1} x_n < +\infty$. Then there exist positive weights $(w_n)_{n \geq 1}$ such that $w_n \rightarrow +\infty$ and*

$$\sum_{n \geq 1} w_n x_n < +\infty.$$

Moreover, one can assume that for all $n \geq 1$, $1 \leq \frac{w_{n+1}}{w_n} \leq 2$.

Proof. Let $r_n := \sum_{p > n} x_p$ and define

$$w_n := \frac{1}{2^{-n} + \sqrt{r_n} + \sqrt{r_{n-1}}}.$$

Then $w_n \rightarrow +\infty$, and

$$\begin{aligned} w_n x_n &= \frac{r_{n-1} - r_n}{2^{-n} + \sqrt{r_n} + \sqrt{r_{n-1}}} \\ &\leq \sqrt{r_{n-1}} - \sqrt{r_n}. \end{aligned}$$

The general term of the upper bound defines a telescopic sum, so that the series $\sum_{n \geq 1} w_n x_n$ is convergent.

We now need to ensure that for all $n \geq 1$, $1 \leq \frac{w_{n+1}}{w_n} \leq 2$. Let $\tilde{w}_0 := w_0$ and define by induction

$$\tilde{w}_n := \min(w_n, 2\tilde{w}_{n-1}), \quad n \geq 1.$$

We check that \tilde{w}_n satisfies the required assumptions.

- Since $\tilde{w}_n \leq w_n$, we know that $\sum_{n \geq 1} \tilde{w}_n x_n < +\infty$.
- By definition, the sequence $(w_n)_n$ is increasing, therefore

$$\tilde{w}_n \geq \min(w_{n-1}, 2\tilde{w}_{n-1}) \geq \tilde{w}_{n-1}.$$

Moreover, the other side of the inequality is immediate $\tilde{w}_n \leq 2\tilde{w}_{n-1}$: we have proven that $1 \leq \frac{\tilde{w}_n}{\tilde{w}_{n-1}} \leq 2$.

- We now prove that $(\tilde{w}_n)_n$ tends to infinity. Let $(n_k)_{k \geq 1}$ be the increasing sequence of indexes n for which $\tilde{w}_n = w_n$ (this sequence might be finite).

If $(n_k)_{k \geq 1}$ is finite, then there exists N such that for all $n \geq N$,

$$\tilde{w}_n := 2\tilde{w}_{n-1}.$$

The weights being positive, this implies that $(\tilde{w}_n)_n$ tends to infinity.

Otherwise, since the sequence $(w_n)_n$ goes to infinity, so does the sequence $(w_{n_k})_k$ as k goes to infinity. The sequence $(\tilde{w}_n)_n$ being increasing, we deduce that $(\tilde{w}_n)_n$ also goes to infinity.

□

We now fix a sequence of positive weights $w = (w_n)_{n \geq 1}$ and define the weighted space

$$\ell(w) := \left\{ z \in h_+^{\frac{1}{2}} \mid \|z\|_{\ell(w)}^2 = \sum_{n \geq 1} n w_n |z_n|^2 < +\infty \right\}.$$

Lemma 6.3.6. *Assume that $w_n \rightarrow +\infty$ and $1 \leq \frac{w_{n+1}}{w_n} \leq 2$ for all $n \geq 1$. Fix $z(0) \in \ell(w)$. Then the maximal solution $t \mapsto z(t) \in h_+^{\frac{1}{2}}$ is global.*

Proof. Let I be the maximal time interval on which the solution $t \mapsto z(t)$ is defined. On this interval, we consider the functional $f : t \mapsto \|z(t)\|_{\ell(w)}^2$.

We prove that f stays bounded on the interval I . Indeed,

$$\frac{d}{dt} f(t) = \sum_{n \geq 1} n w_n \frac{d}{dt} \gamma_n.$$

For $n \geq 1$, the time derivative of γ_n is

$$\begin{aligned} \frac{d}{dt} \gamma_n(u(t)) &= d\gamma_n[u(t)] \cdot \partial_t u \\ &= -\alpha \langle u(t) | \cos \rangle d\gamma_n[u(t)] \cdot \cos - \alpha \langle u(t) | \sin \rangle d\gamma_n[u(t)] \cdot \sin. \end{aligned}$$

From the expression (6.11) of the differential of γ_n , we have

$$d\gamma_n[u(t)] \cdot \cos = \operatorname{Re}(\langle |f_{n-1}|^2 - |f_n|^2 | e^{ix} \rangle)$$

and

$$d\gamma_n[u(t)] \cdot \sin = -\operatorname{Im}(\langle |f_{n-1}|^2 - |f_n|^2 | e^{ix} \rangle),$$

where

$$\langle |f_{n-1}|^2 - |f_n|^2 | e^{ix} \rangle = -\sqrt{\mu_n} \frac{\sqrt{\kappa_{n-1}}}{\sqrt{\kappa_n}} \bar{\zeta}_{n-1} \zeta_n + \sqrt{\mu_{n+1}} \frac{\sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}} \bar{\zeta}_n \zeta_{n+1}.$$

Since μ_n and $\frac{\kappa_n}{\kappa_{n+1}}$ are bounded by $C(R)$, we see that actually

$$\left| \frac{d}{dt} \gamma_n \right| \leq C(R) |\langle u(t) | e^{ix} \rangle| (|\zeta_{n-1} \zeta_n| + |\zeta_n \zeta_{n+1}|) \quad (6.18)$$

so that

$$\left| \frac{d}{dt} f(t) \right| \leq C(R) |\langle u(t) | e^{ix} \rangle| \sum_{n \geq 1} n w_n (|\zeta_{n-1} \zeta_n| + |\zeta_n \zeta_{n+1}|).$$

Using Cauchy-Schwarz' inequality, we deduce that

$$\begin{aligned} \left| \frac{d}{dt} f(t) \right| &\leq C(R) |\langle u(t) | e^{ix} \rangle| \left(\sum_{n \geq 1} n w_n |\zeta_n|^2 \right)^{\frac{1}{2}} \\ &\quad \left(\left(\sum_{n \geq 1} n w_n |\zeta_{n-1}|^2 \right)^{\frac{1}{2}} + \left(\sum_{n \geq 1} n w_n |\zeta_{n+1}|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

For all $n \geq 1$, because $1 \leq \frac{w_{n+1}}{w_n} \leq 2$, we have $1 \leq \frac{(n+1)w_{n+1}}{nw_n} \leq 4$. Hence there exists $C(R) > 0$ such that for all $t \in I$,

$$\left| \frac{d}{dt} f(t) \right| \leq C(R) |\langle u(t) | e^{ix} \rangle| f(t).$$

Now, Gronwall's lemma implies that for all $t \in I$,

$$|f(t)| \leq |f(0)| e^{C(R) \int_0^t |\langle u(t) | e^{ix} \rangle| dt}.$$

Applying Proposition 6.3.3, we deduce that

$$|f(t)| \leq |f(0)| e^{C'(R)\sqrt{t}}.$$

To conclude, for all $T > 0$, f stays on a bounded subset of $\ell(w)$ on the time interval $([0, T^*) \cap [0, T])$. Since the embedding $\ell(w) \hookrightarrow h_+^{\frac{1}{2}}$ is compact because of the condition $w_n \rightarrow +\infty$, the solution $t \mapsto z(t)$ stays in a compact set of $h_+^{\frac{1}{2}}$. This implies that there cannot be finite time blowup: $[0, T) \subset [0, T^*)$, therefore the solution is global. \square

Proof of Proposition 6.3.4. Let $z(0) \in h_+^{\frac{1}{2}}$. Thanks to Lemma 6.3.5, we construct a sequence $(w_n)_{n \geq 1}$ of positive weights such that $z(0) \in \ell(w)$, $w_n \rightarrow +\infty$ and $1 \leq \frac{w_{n+1}}{w_n} \leq 2$ for all $n \geq 1$. Now Lemma 6.3.6 with the weights $(w_n)_n$ ensures that the maximal solution $t \mapsto z(t) \in h_+^{\frac{1}{2}}$ is global. \square

6.3.3 Weak sequential continuity of the flow map

In this part, we prove that the flow map for the damped Benjamin-Ono equation (BO- α) is weakly sequentially continuous. Note that in view of the weak sequential continuity of the Birkhoff map and its inverse, it is equivalent to establish the weak sequential continuity of the flow map for the damped Benjamin-Ono equation in Birkhoff coordinates (6.14).

Proposition 6.3.7. *The flow map for the damped Benjamin-Ono equation (BO- α) is weakly sequentially continuous.*

More precisely, let $u^{(k)}(0) \rightharpoonup u(0)$ in $L_{r,0}^2(\mathbb{T})$. Then for all $T > 0$, the sequence of solutions $u^{(k)}$ associated to the initial data $u^{(k)}(0)$ converges to the solution u associated to the initial data u_0 in $\mathcal{C}_w([0, T], L_{r,0}^2(\mathbb{T}))$ (with the weak topology).

Proof. We consider a sequence $u^{(k)}(0)$ weakly convergent to $u(0)$ in $L_{r,0}^2(\mathbb{T})$.

For $k \in \mathbb{N}$, denote $z^{(k)}(0) := \Phi(u^{(k)}(0))$. By weak sequential continuity of the Birkhoff map Φ (see [GKT20b], Remark 6 (iii)), we have $z^{(k)}(0) \rightharpoonup z(0) := \Phi(u(0))$ in $h_+^{\frac{1}{2}}$, therefore this sequence is bounded in $h_+^{\frac{1}{2}}$ by some $R > 0$. But the solutions of the damped Benjamin-Ono equation have a decreasing norm in $L_{r,0}^2(\mathbb{T})$ (Proposition 6.3.3) and from the Parseval formula $\|u\|_{L^2(\mathbb{T})}^2 = 2 \sum_{n \geq 1} n |\zeta_n|^2$ (Remark 1.2 (i) in [GK20]), their Birkhoff coordinates have a decreasing norm in $h_+^{\frac{1}{2}}$. Therefore for all $t \geq 0$, the sequence $(z^{(k)}(t))_k$ is also bounded in $h_+^{\frac{1}{2}}$ by R .

Recall from Theorem 6.3.1 that there exists $C(R) > 0$ such that for all $t \geq 0$,

$$\left\| \frac{dz^{(k)}(t)}{dt} \right\|_{h_+^{\frac{1}{2}}} = \|F(t, z^{(k)}(t))\|_{h_+^{\frac{1}{2}}} \leq C(R).$$

We conclude that the sequence $\left\| \frac{dz^{(k)}(t)}{dt} \right\|_{h_+^{\frac{1}{2}}}$, $k \in \mathbb{N}$, is bounded in $h_+^{\frac{1}{2}}$ uniformly in time.

Fix $T > 0$. From Ascoli's theorem, we know that up to a subsequence, $z^{(k)}$ converges in $\mathcal{C}_w([0, T], h_+^{\frac{1}{2}})$ (with the weak topology) to a function \tilde{z} . In particular, since Φ^{-1} is weakly sequentially continuous, for all $t \in [0, T]$, the sequence $u^{(k)}(t) = \Phi^{-1}(z^{(k)}(t))$ is weakly convergent to $\tilde{u}(t) := \Phi^{-1}(\tilde{z}(t))$. But F is sequentially weakly continuous with respect to the second variable (see Theorem 6.3.1), therefore $F(t, z^{(k)}(t))$ is weakly convergent to $F(t, \tilde{z}(t))$.

Passing to the limit in the equation, we conclude that \tilde{z} is a solution on $[0, T]$ to the original equation (6.14) in the distribution sense with initial data $z(0)$. By uniqueness of such solutions (see Corollary 6.3.2), we deduce that $\tilde{z} = z$. \square

6.4 Long time asymptotics

In this section, we describe the weak limit points for the flow map (point 1 of Theorem 6.1.2) in part 6.4.1, and prove that the convergence to these weak limit points is actually strong in $L_{r,0}^2(\mathbb{T})$ (points 2 and 3 of Theorem 6.1.2) in part 6.4.3. In order to get the strong convergence, we show that the integral $\int_0^{+\infty} \sum_{n=0}^{+\infty} \gamma_n(t) \gamma_{n+1}(t) dt$ is finite in part 6.4.2.

6.4.1 Weak limit points of trajectories as $t \rightarrow +\infty$

Using the LaSalle principle, we study the limit points of $(u(t))_{t \in \mathbb{R}}$ for the weak topology in $L_{r,0}^2(\mathbb{T})$ as t goes to $+\infty$ and prove Theorem 6.1.2.

Proposition 6.4.1. *Let u be a solution of the damped equation (BO- α) with initial data $u_0 \in L_{r,0}^2(\mathbb{T})$. Then any weak limit v_0 of the sequence $(u(t))_{t \in \mathbb{R}}$ in $L_{r,0}^2(\mathbb{T})$ as t goes to $+\infty$ defines a solution v to the Benjamin-Ono equation (BO) such that for all $t \geq 0$,*

$$\langle v(t) | e^{ix} \rangle = 0.$$

Proof. Let v_0 be a weak limit of some sequence $(u(t_k))_{k \in \mathbb{N}}$ in $L_{r,0}^2(\mathbb{T})$, where $t_k \rightarrow +\infty$, and let v be the solution to the damped Benjamin-Ono equation (BO- α) with initial data v_0 . By weak sequential continuity of the flow in $L_{r,0}^2(\mathbb{T})$ (Proposition 6.3.7), for all $t \in \mathbb{R}$, we have

$$u(t + t_k) \xrightarrow[k \rightarrow +\infty]{} v(t)$$

and in particular

$$\langle u(t + t_k) | e^{ix} \rangle \xrightarrow[k \rightarrow +\infty]{} \langle v(t) | e^{ix} \rangle.$$

However, from Proposition 6.3.3, $\langle u(t + t_k) | e^{ix} \rangle$ tends to 0 as k goes to $+\infty$. We deduce that $\langle v(t) | e^{ix} \rangle = 0$ for all $t \in \mathbb{R}$. \square

Proposition 6.4.2. *An initial data $v_0 \in L_{r,0}^2(\mathbb{T})$ defines a solution v to the Benjamin-Ono equation such that for all $t \in \mathbb{R}$,*

$$\langle v(t) | e^{ix} \rangle = 0$$

if and only if v_0 does not have two consecutive nonzero gaps:

$$\forall n \in \mathbb{N}, \quad \zeta_n(v_0) \zeta_{n+1}(v_0) = 0.$$

Proof. Assume that v is a solution to the Benjamin-Ono equation with initial data v_0 . Then the Birkhoff coordinates evolve as

$$\zeta_n(v(t)) = e^{i\omega_n(v_0)t} \zeta_n(v_0),$$

where

$$\omega_n(v_0) = n^2 - 2 \sum_{p=1}^{+\infty} \min(p, n) \gamma_p.$$

Applying Lemma 6.2.6, we have

$$\langle v(t) | e^{ix} \rangle = \sum_{n \geq 0} m_n e^{i(\omega_n(v_0) - \omega_{n+1}(v_0))t},$$

where m_n is constant along the flow of the Benjamin-Ono equation:

$$m_n = M_{n,n} = -\overline{\zeta_n(v_0)} \zeta_{n+1}(v_0) \sqrt{\frac{\kappa_n(v_0)}{\kappa_{n+1}(v_0)}} \sqrt{\mu_{n+1}(v_0)}.$$

One can see that if the condition $\overline{\zeta_n(v_0)} \zeta_{n+1}(v_0) = 0$ is satisfied for all $n \in \mathbb{N}$, then all the coefficients m_n , $n \in \mathbb{N}$, vanish, therefore $\langle v(t) | e^{ix} \rangle = 0$ for all $t \in \mathbb{R}$.

Conversely, assume that $\langle v(t) | e^{ix} \rangle = 0$ for all $t \in \mathbb{R}$. Since

$$\omega_{n+1}(v_0) - \omega_n(v_0) = 2n + 1 - 2 \sum_{p=n+1}^{+\infty} \gamma_p,$$

the sequence $(\omega_{n+1}(v_0) - \omega_n(v_0))_{n \in \mathbb{N}}$ is strictly increasing. Fix $n \in \mathbb{N}$, and choose $T > 0$. Then

$$\begin{aligned} 0 &= \frac{1}{T} \int_0^T e^{i(\omega_{n+1}(v_0) - \omega_n(v_0))t} \langle v(t) | e^{ix} \rangle dt \\ &= m_n + \frac{1}{T} \sum_{p \neq n} m_p \frac{e^{i(\omega_{n+1}(v_0) - \omega_n(v_0) + \omega_p(v_0) - \omega_{p+1}(v_0))T} - 1}{i(\omega_{n+1}(v_0) - \omega_n(v_0) + \omega_p(v_0) - \omega_{p+1}(v_0))}. \end{aligned}$$

Taking $T \rightarrow +\infty$ in this equality, we deduce that $m_n = 0$. Note that $\kappa_n > 0$, $\kappa_{n+1} > 0$ and $\mu_{n+1} > 0$, therefore, we have $\overline{\zeta_n(v_0)} \zeta_{n+1}(v_0) = 0$. \square

6.4.2 Time integrability for products of two consecutive modes

Let $u \in \mathcal{C}(\mathbb{R}_+, L^2_{r,0}(\mathbb{T}))$ be a solution to the damped Benjamin-Ono equation (BO- α). We denote $\gamma_n(t) := \gamma_n(u(t))$ for all $n \geq 1$ and $t \in \mathbb{R}_+$ (with the convention $\gamma_0(t) = 1$).

Proposition 6.4.3. *Let $R := \|u_0\|_{L^2(\mathbb{T})}$. Then there exists $C(R) > 0$ such that*

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} \gamma_n(t) \gamma_{n+1}(t) dt \leq C(R).$$

Moreover, there exists a map ε_u such that $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$ and the following holds. For all $n \in \mathbb{N}$, fix $a_n \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{C})$ satisfying:

$$\forall t \geq 0, \quad |a_n(t)| \leq 1 \quad \text{and} \quad |a'_n(t)| \leq |\langle u(t) | e^{ix} \rangle|. \quad (6.19)$$

Then for all $T \geq 0$, we have

$$\int_T^{+\infty} \left| \sum_{n=0}^{+\infty} a_n(t) \zeta_n(t) \overline{\zeta_{n+1}(t)} \right|^2 dt \leq \varepsilon_u(T).$$

Remark 6.4.4. Note that by homogeneity, if the maps $a_n \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{C})$ satisfy:

$$\forall t \geq 0, \quad |a_n(t)| \leq \frac{K}{n+1} \quad \text{and} \quad |\dot{a}_n(t)| \leq K |\langle u(t) | e^{ix} \rangle|,$$

then for all $T > 0$,

$$\int_T^{+\infty} \left| \sum_{n=0}^{+\infty} a_n(t) \zeta_n(t) \overline{\zeta_{n+1}(t)} \right|^2 dt \leq K^2 \varepsilon_u(T).$$

Let $0 \leq T < T' < +\infty$. We denote

$$I(T, T') := \int_T^{T'} \sum_{n=0}^{+\infty} \gamma_n(t) \gamma_{n+1}(t) dt.$$

For a family $a = (a_n)_n$ of maps satisfying assumption (6.19) of Proposition 6.4.3, we also define

$$J_a(T, T') = \int_T^{T'} \sum_{\substack{n,p \\ n \neq p}} a_n(t) a_p(t) \overline{\zeta_n(t)} \zeta_{n+1}(t) \zeta_p(t) \overline{\zeta_{p+1}(t)} dt.$$

The integrals $I(T, T')$ and $J_a(T, T')$ are well-defined since for all $t \in \mathbb{R}_+$, $\sum_n n \gamma_n(t) \leq R^2/2$ thanks to the Lyapunov functional (see Proposition 6.3.3) and the Parseval formula (see [GK20], Remark 1.2(i)).

The key step in the proof of Proposition 6.4.3 is the following estimation of $J_a(T, T')$ depending on $I(T, T')$.

Lemma 6.4.5. Let $R = \|u_0\|_{L^2(\mathbb{T})}$. There exist $C(R) > 0$ and a map ε_u with $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$ such that the following holds. For all $0 \leq T < T' < +\infty$ and for all family $a = (a_n)_n$ of maps satisfying assumption (6.19) of Proposition 6.4.3,

$$|J_a(T, T')| \leq \varepsilon_u(T) (1 + \sqrt{I(T, T')}).$$

Moreover, for $T = 0$, one has $|\varepsilon_u(0)| \leq C(R)$.

Given Lemma 6.4.5, the strategy of proof for Proposition 6.4.3 will be the following. We observe that for the choice of family $a^* = (a_n^*)_n$ given by (6.7)

$$a_n^*(t) = \sqrt{\mu_{n+1}^*(\gamma(t))} \sqrt{\frac{\kappa_n^*(\gamma(t))}{\kappa_{n+1}^*(\gamma(t))}},$$

we have thanks to the Lyapunov functional (Proposition 6.3.3)

$$I(0, T') \leq C(R) + C(R) |J_{a^*}(0, T')|.$$

We deduce from Lemma 6.4.5 that $I(0, T')$ is bounded independently of T' . As a consequence, a new application of Lemma 6.4.5 implies that for all family $a = (a_n)_n$ satisfying (6.19), $J_a(T, T')$ is bounded by some $\varepsilon_u(T)$, where this upper bound is independent of T' and of the choice of a .

Proof of Lemma 6.4.5. For $n, p \geq 0$, $n \neq p$, we denote $\eta_{n,p} := \overline{\zeta_n \zeta_{n+1} \zeta_p \zeta_{p+1}}$.

Fix a family $a = (a_n)_n$ of elements of $\mathcal{C}^1(\mathbb{R}_+, \mathbb{C})$ satisfying assumption (6.19). Since for all $t \in \mathbb{R}_+$, $\sum_n n \gamma_n(t) \leq R^2/2$, there exists $C(R) > 0$ such that for all $t \in \mathbb{R}_+$,

$$\sum_{\substack{n,p, \\ n \neq p}} |a_n(t) a_p(t) \eta_{n,p}(t)| \leq C(R).$$

Therefore, one can exchange the summation sign with the time integral:

$$J_a(T, T') = \sum_{\substack{n,p, \\ n \neq p}} \int_T^{T'} a_n(t) a_p(t) \eta_{n,p}(t) dt. \quad (6.20)$$

For each term in the series over the indexes n and p , we perform an integration by parts by using the differential equation satisfied by $\eta_{n,p}$, which we will now establish.

Recall from (6.4) that the time derivative of ζ_n is

$$\frac{d}{dt} \zeta_n(t) = i\omega_n(t) \zeta_n(t) - \alpha Z_n(t)$$

with

$$\omega_n(t) = n^2 - 2 \sum_{k=1}^{+\infty} \min(k, n) \gamma_k(t)$$

and Z_n defined in (6.16) as

$$Z_n(t) = \langle u(t) | \cos \rangle d\zeta_n[u(t)]. \cos + \langle u(t) | \sin \rangle d\zeta_n[u(t)]. \sin.$$

Therefore, the time derivative of $\eta_{n,p} = \overline{\zeta_n \zeta_{n+1} \zeta_p \zeta_{p+1}}$ writes

$$\frac{d}{dt} \eta_{n,p}(t) = i\Omega_{n,p}(t) \eta_{n,p}(t) - \alpha F_{n,p}(t) \quad (6.21)$$

where

$$\Omega_{n,p} = -\omega_n + \omega_{n+1} + \omega_p - \omega_{p+1}$$

and

$$F_{n,p} = (\overline{Z_n \zeta_{n+1}} + \overline{\zeta_n Z_{n+1}}) \zeta_p \overline{\zeta_{p+1}} + (Z_p \overline{\zeta_{p+1}} + \zeta_p \overline{Z_{p+1}}) \overline{\zeta_n} \zeta_{n+1}. \quad (6.22)$$

Note that

$$\Omega_{n,p} = 2 \left((n-p) - \sum_{k \geq n+1} \gamma_k + \sum_{k \geq p+1} \gamma_k \right),$$

hence for $n \neq p$,

$$|\Omega_{n,p}| \geq 2|n-p| > 0.$$

When $n \neq p$, we can therefore divide the differential equation satisfied by $\eta_{n,p}$ by the factor $i\Omega_{n,p}$ and get

$$\int_T^{T'} a_n(t) a_p(t) \eta_{n,p}(t) dt = \int_T^{T'} \frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \left(\frac{d}{dt} \eta_{n,p}(t) + \alpha F_{n,p}(t) \right) dt.$$

An integration by parts now leads to

$$\int_T^{T'} a_n(t)a_p(t)\eta_{n,p}(t) dt = \left[\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)}\eta_{n,p}(t) \right]_T^{T'} - \int_T^{T'} \frac{d}{dt} \left(\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) dt + \alpha \int_T^{T'} \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} F_{n,p}(t) dt. \quad (6.23)$$

In order to determine an upper bound for $\sum_{n,p,n \neq p} |\int_T^{T'} a_n(t)a_p(t)\eta_{n,p}(t) dt|$, we consider the three terms in the right-hand side of this equality separately.

1. For all $t \geq 0$, we have

$$\sum_{\substack{n,p, \\ n \neq p}} \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)}\eta_{n,p}(t) \right| \leq \left(\sum_{n=0}^{+\infty} \gamma_n(t) \right)^2 \leq R^4,$$

therefore the series with general term $\sum_{n,p,n \neq p} \left[\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)}\eta_{n,p}(t) \right]_T^{T'}$ is absolutely convergent and bounded by some constant $C(R) > 0$ independent of T and T' .

Moreover, for $N \in \mathbb{N}$, one can cut the sum between the indexes $n \leq N$ and the indexes $n > N$ to deduce

$$\begin{aligned} & \sum_{\substack{n,p, \\ n \neq p}} \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)}\eta_{n,p}(t) \right| \\ & \leq \sum_{\substack{n,p,n \neq p \\ n \leq N, p \leq N}} \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)}\eta_{n,p}(t) \right| + \sum_{\substack{n,p,n \neq p \\ n > N \text{ or } p > N}} \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)}\eta_{n,p}(t) \right| \\ & \leq \left(\sum_{n=0}^N \sqrt{\gamma_n(t)\gamma_{n+1}(t)} \right)^2 + 2 \left(\sum_{n=N+1}^{+\infty} \gamma_n(t) \right) \left(\sum_{p=0}^{+\infty} \gamma_p(t) \right). \end{aligned}$$

Let $\varepsilon > 0$. Since

$$\left(\sum_{n=N+1}^{+\infty} \gamma_n(t) \right) \left(\sum_{p=0}^{+\infty} \gamma_p(t) \right) \leq \frac{1}{N+1} \left(\sum_{n=N+1}^{+\infty} n\gamma_n(t) \right) \left(\sum_{p=0}^{+\infty} \gamma_p(t) \right) \leq \frac{C(R)}{N+1},$$

we know that there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $t \in \mathbb{R}_+$,

$$\left(\sum_{n=N+1}^{+\infty} \gamma_n(t) \right) \left(\sum_{p=0}^{+\infty} \gamma_p(t) \right) \leq \varepsilon.$$

Besides, for all n , we have $\gamma_n(t)\gamma_{n+1}(t) \rightarrow 0$ as $t \rightarrow +\infty$ by weak sequential continuity of the Birkhoff map and the description of the weak limit points in Birkhoff coordinates (see the first point of Theorem 6.1.2). Therefore, there exists $T_0 = T_0(\varepsilon)$ such that for all $t \geq T_0$,

$$\left(\sum_{n=0}^N \sqrt{\gamma_n(t)\gamma_{n+1}(t)} \right)^2 \leq \varepsilon.$$

We conclude that there exists ε_u such that $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$ and for all $T < T'$,

$$\sum_{\substack{n,p \\ n \neq p}} \left| \left[\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \eta_{n,p}(t) \right]_T^{T'} \right| \leq \varepsilon_u(T),$$

moreover ε_u is independent of the choice of family a .

2. Next, we develop the time derivative

$$\frac{d}{dt} \left(\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right) = \frac{\dot{a}_n(t)a_p(t) + a_n(t)\dot{a}_p(t)}{i\Omega_{n,p}(t)} - \frac{a_n(t)a_p(t)\dot{\Omega}_{n,p}(t)}{i\Omega_{n,p}(t)^2}.$$

From the definition of $\Omega_{n,p}$, we know that

$$|\dot{\Omega}_{n,p}(t)| \leq 2\|\dot{\gamma}(t)\|_{\ell_+^1}.$$

However, the time derivative of γ_n is bounded from inequality (6.18) as

$$|\dot{\gamma}_n(t)| \leq C(R)|\langle u(t)|e^{ix}\rangle|(|\zeta_{n-1}\zeta_n| + |\zeta_n\zeta_{n+1}|),$$

so that

$$\|\dot{\gamma}(t)\|_{\ell_+^1} \leq C(R)|\langle u(t)|e^{ix}\rangle|, \quad (6.24)$$

and therefore

$$|\dot{\Omega}_{n,p}(t)| \leq 2C(R)|\langle u(t)|e^{ix}\rangle|.$$

Using the assumptions (6.19) on a_n , this implies that there exists $C(R) > 0$ such that

$$\int_T^{T'} \left| \frac{d}{dt} \left(\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) \right| dt \leq C(R) \int_T^{T'} |\langle u(t)|e^{ix}\rangle| \frac{|\eta_{n,p}(t)|}{|n-p|} dt. \quad (6.25)$$

Applying the Cauchy-Schwarz' inequality, we deduce

$$\begin{aligned} & \sum_{\substack{n,p \\ n \neq p}} \int_T^{T'} \left| \frac{d}{dt} \left(\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) \right| dt \\ & \leq C(R) \left(\sum_{\substack{n,p \\ n \neq p}} \int_T^{T'} |\langle u(t)|e^{ix}\rangle|^2 \frac{\gamma_n(t)}{|n-p|^2} dt \right)^{\frac{1}{2}} \\ & \quad \left(\sum_{\substack{n,p \\ n \neq p}} \int_T^{T'} \gamma_{n+1}(t)\gamma_p(t)\gamma_{p+1}(t) dt \right)^{\frac{1}{2}}. \end{aligned}$$

But there exists $C > 0$ such that for all n , $\sum_{p,p \neq n} \frac{1}{|n-p|^2} \leq C$, moreover, for all $t \geq 0$, we have $\sum_n \gamma_n(t) \leq R^2/2$. We obtain

$$\sum_{\substack{n,p \\ n \neq p}} \int_T^{T'} \left| \frac{d}{dt} \left(\frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) \right| dt \leq C(R) \left(\int_T^{+\infty} |\langle u(t)|e^{ix}\rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I(T, T')}.$$

3. Finally, we prove that

$$\sum_{\substack{n,p \\ n \neq p}} \int_T^{T'} \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} F_{n,p}(t) \right| dt \leq C(R) \left(\int_T^{+\infty} |\langle u(t)|e^{ix}\rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I(T, T')}.$$

Note that from the definition $Z_n = \langle u|\cos\rangle d\zeta_n[u].\cos + \langle u|\sin\rangle d\zeta_n[u].\sin$ in formula (6.16), and from Corollary 6.2.12 about the bounds on $d\zeta_n[u(t)].\cos$ and $d\zeta_n[u(t)].\sin$, we have

$$\sum_{n \geq 0} n|Z_n(t)|^2 \leq C(R)|\langle u(t)|e^{ix}\rangle|^2.$$

One can apply Cauchy-Schwarz' inequality to deduce

$$\begin{aligned} \sum_{\substack{n,p \\ n \neq p}} \int_T^{T'} \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right| |Z_n(t)\zeta_{n+1}(t)\zeta_p(t)\overline{\zeta_{p+1}(t)}| dt \\ \leq \left(\int_T^{T'} \sum_{\substack{n,p \\ n \neq p}} \frac{|Z_n(t)|^2}{|n-p|^2} dt \right)^{\frac{1}{2}} \left(\int_T^{T'} \sum_{\substack{n,p \\ n \neq p}} \gamma_{n+1}\gamma_p\gamma_{p+1} dt \right)^{\frac{1}{2}} \\ \leq C(R) \left(\int_T^{+\infty} |\langle u(t)|e^{ix}\rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I(T, T')}. \end{aligned}$$

Using the same strategy to the three other terms composing $F_{n,p}$ (up to exchanging the roles of n and $n + 1$ and the roles of n and p), we get the desired inequality.

To conclude, we use the square integrability in time of $|\langle u(t)|e^{ix}\rangle|$ (Proposition 6.3.3) to get

$$\int_0^{+\infty} |\langle u(t)|e^{ix}\rangle|^2 dt \leq R^2.$$

We have therefore proven that there exists ε_u satisfying $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$, and such that for all family $a = (a_n)_n$ satisfying the assumptions in (6.19),

$$\sum_{\substack{n,p \\ n \neq p}} \left| \int_T^{T'} a_n(t)a_p(t)\eta_{n,p}(t) dt \right| \leq \varepsilon_u(T)(1 + \sqrt{I(T, T')}),$$

moreover, one has $|\varepsilon_u(0)| \leq C(R)$. Plugging this into equality (6.20), we deduce that

$$|J_a(T, T')| \leq \varepsilon_u(T)(1 + \sqrt{I(T, T')}).$$

□

Proof of Proposition 6.4.3. We first establish a bound of $I(0, T')$. We start from the formula for $\langle u(t)|e^{ix}\rangle$ from Lemma 6.2.6:

$$\langle u(t)|e^{ix}\rangle = - \sum_{n=0}^{+\infty} a_n^*(t)\overline{\zeta_n(t)}\zeta_{n+1}(t),$$

where

$$a_n^*(t) = \sqrt{\mu_{n+1}(t)} \sqrt{\frac{\kappa_n(t)}{\kappa_{n+1}(t)}}.$$

Now, we expand

$$|\langle u(t) | e^{ix} \rangle|^2 = \sum_{n=0}^{+\infty} a_n^*(t)^2 \gamma_n(t) \gamma_{n+1}(t) + \operatorname{Re} \left(\sum_{\substack{n,p \\ n \neq p}} a_n^*(t) a_p^*(t) \overline{\zeta_n(t)} \zeta_{n+1}(t) \zeta_p(t) \overline{\zeta_{p+1}(t)} \right)$$

and since $a_n^* \geq \frac{1}{C(R)}$ (see inequality (6.8)), we deduce that

$$\frac{1}{C(R)^2} \sum_{n=0}^{+\infty} \gamma_n(t) \gamma_{n+1}(t) \leq |\langle u(t) | e^{ix} \rangle|^2 - \operatorname{Re} \left(\sum_{\substack{n,p \\ n \neq p}} a_n^*(t) a_p^*(t) \eta_{n,p}(t) \right).$$

In particular, if we denote $a^* = (a_n^*)_n$, an integration in time leads to the inequality

$$I(0, T') \leq \int_0^{T'} |\langle u(t) | e^{ix} \rangle|^2 dt + |J_{a^*}(0, T')| \leq C(R) + |J_{a^*}(0, T')|. \quad (6.26)$$

We now use Lemma 6.4.5 applied to the family a^* . It only remains to check that up to division by some constant $C(R)$, this family satisfies assumption (6.19) of Proposition 6.4.3. Using inequalities (6.8) and (6.9), we have

$$|a_n^*(t)| \leq C(R)$$

and

$$|\dot{a}_n^*(t)| \leq C(R) \|\dot{\gamma}(t)\|_{\ell_1^+},$$

so that from inequality (6.24), there exists $C(R) > 0$ such that

$$|\dot{a}_n^*(t)| \leq C(R) |\langle u(t) | e^{ix} \rangle|.$$

Now, Remark 6.4.4 following Lemma 6.4.5 implies that

$$|J_{a^*}(0, T')| \leq C_1(R) (1 + \sqrt{I(0, T')}).$$

Therefore, one can use the inequality $2xy \leq x^2 + y^2$ on the second term of the right-hand side to get

$$|J_{a^*}(0, T')| \leq C_2(R) + \frac{1}{2} I(0, T').$$

Plugging this into inequality (6.26), we deduce an inequality of the form

$$I(0, T') \leq C_3(R) + \frac{1}{2} I(0, T').$$

We conclude that $I(0, T')$ is bounded by some constant $2C_3(R)$ independent of T' . In particular, one can pass to the limit $T' \rightarrow +\infty$ and deduce that for some $C(R) > 0$,

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} \gamma_n(t) \gamma_{n+1}(t) dt \leq C(R).$$

To conclude, fix family a satisfying assumption (6.19). From Lemma 6.4.5 and the bound $I(T, T') \leq C(R)$, we deduce that there exists a map ε_u independent of a such that $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$ and for all $0 \leq T < T' < +\infty$,

$$|J_a(T, T')| \leq \varepsilon_u(T).$$

Finally, expanding

$$\begin{aligned} \int_T^{T'} \left| \sum_{n=0}^{+\infty} a_n(t) \zeta_n(t) \overline{\zeta_{n+1}(t)} \right|^2 dt &\leq \int_T^{T'} \sum_{n=0}^{+\infty} |a_n(t)|^2 \gamma_n(t) \gamma_{n+1}(t) dt + |\operatorname{Re}(J_a(T, T'))| \\ &\leq I(T, T') + |J_a(T, T')| \\ &\leq \varepsilon_u(T), \end{aligned}$$

we see that this integral is bounded independently of T' and a , and the bound $\varepsilon_u(T)$ goes to 0 as $T \rightarrow +\infty$. We deduce the second part of the proposition. \square

6.4.3 Strong relative compactness of trajectories as $t \rightarrow +\infty$

In this part, we prove points 2 and 3 of Theorem 6.1.2: let u be a solution to the damped Benjamin-Ono equation (BO- α), then there exists a sequence $(\gamma_n^\infty)_{n \geq 1} \in \ell_+^1$ such that for every weak limit u_∞ in $L_{r,0}^2(\mathbb{T})$ of $(u(t))_{t \geq 0}$ as $t \rightarrow +\infty$, and for all $n \geq 1$, we have $\gamma_n(u_\infty) = \gamma_n^\infty$; moreover, if u_∞ is a weak limit associated to a subsequence $(u(t_k))_{k \in \mathbb{N}}$ with $t_k \rightarrow +\infty$, then the convergence of $(u(t_k))_{k \in \mathbb{N}}$ to u_∞ is strong in $L_{r,0}^2(\mathbb{T})$.

We start with the proof of point 2. Fix $n \geq 1$ and recall inequality (6.18)

$$|\dot{\gamma}_n(t)| \leq C(R) |\langle u(t) | e^{ix} \rangle| (|\zeta_{n-1} \zeta_n| + |\zeta_n \zeta_{n+1}|).$$

Since $|\langle u(t) | e^{ix} \rangle| \in L^2(\mathbb{R}_+)$ (see Proposition 6.3.3) and $|\zeta_{n-1} \zeta_n| \in L^2(\mathbb{R}_+)$ (see Proposition 6.4.3), we get that $\dot{\gamma}_n \in L^1(\mathbb{R}_+)$. Therefore, there exists γ_n^∞ such that $\gamma_n(t) \rightarrow \gamma_n^\infty$ as $t \rightarrow +\infty$. Let now u_∞ be a weak limit in $L_{r,0}^2(\mathbb{T})$ of a subsequence $(u(t_k))_{k \in \mathbb{N}}$ with $t_k \rightarrow +\infty$. By weak sequential continuity of the Birkhoff map, we get that for all n , $\gamma_n(u(t_k)) \rightarrow \gamma_n(u_\infty)$, and therefore $\gamma_n(u_\infty) = \gamma_n^\infty$.

We now prove that $\|u(t)\|_{L^2(\mathbb{T})} \rightarrow \|u_\infty\|_{L^2(\mathbb{T})}$ as $t \rightarrow +\infty$, so that the convergence of $(u(t_k))_{k \in \mathbb{N}}$ to u_∞ is strong in $L_{r,0}^2(\mathbb{T})$. In order to do so, we consider the generating function

$$\mathcal{H}_\mu(u) = \sum_{n=0}^{+\infty} \frac{\kappa_n \gamma_n}{\lambda_n + \mu}.$$

The strategy is as follows. Let $R = \|\zeta(u_0)\|_{h^{\frac{1}{2}}}$. Using the differential equation satisfied by $t \mapsto \mathcal{H}_\mu(u(t))$, we prove that there exists a map ε_u such that $\varepsilon_u(t) \rightarrow 0$ as $t \rightarrow +\infty$ and for any weak limit u_∞ , for all $\mu \geq R^2 + 1$,

$$|\mathcal{H}_\mu(u(t)) - \mathcal{H}_\mu(u_\infty)| \leq C(R) \frac{\varepsilon_u(t)}{\mu^3}.$$

Then, for fixed $t \in \mathbb{R}_+$, we compare the asymptotic expansion of $\mathcal{H}_\mu(u(t))$ as $\mu \rightarrow +\infty$ with the norm $\|u(t)\|_{L^2(\mathbb{T})}$. Finally, we combine these two steps to get the convergence of $\|u(t)\|_{L^2(\mathbb{T})}$ to $\|u_\infty\|_{L^2(\mathbb{T})}$ when $t \rightarrow +\infty$.

Fix $\mu \geq R^2 + 1$. For all $t \geq 0$, $\|\zeta(t)\|_{h_+^{\frac{1}{2}}} \leq R$, we have $\mu + \lambda_0(t) \geq \mu - R^2 > 0$, therefore $\mathcal{H}_\mu(u(t))$ is well-defined. Moreover, by weak sequential continuity of the generating function $v \in L_{r,0}^2(\mathbb{T}) \mapsto \mathcal{H}_\mu(v) \in \mathbb{R}$ (see [GKT20b], Lemma 7), we have $\mathcal{H}_\mu(u(t_k)) \rightarrow \mathcal{H}_\mu(u_\infty)$ as $k \rightarrow +\infty$. We now quantify the rate of convergence of $\mathcal{H}_\mu(u(t_k))$ to $\mathcal{H}_\mu(u_\infty)$ by estimating the time derivative of $\mathcal{H}_\mu(u(t))$.

Lemma 6.4.6. *Let $R = \|\zeta(u_0)\|_{h_+^{\frac{1}{2}}}$. Then there exists a map ε_u with $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$, such that the following holds. For all $T > 0$ and for all $\mu \geq R^2 + 1$, we have*

$$\int_T^{+\infty} \left| \frac{d}{dt} \mathcal{H}_\mu(u(t)) \right| dt \leq \frac{\varepsilon_u(T)}{(\mu - R^2)^3}.$$

Proof. Using the expression $\mathcal{H}_\mu(u) = \langle (L_u + \mu \text{Id})^{-1} \mathbb{1} | \mathbb{1} \rangle$, the time derivative of $t \mapsto \mathcal{H}_\mu(u(t))$ writes

$$\frac{d}{dt} \mathcal{H}_\mu(u(t)) = \langle T_{\partial_t u}(w_\mu^{u(t)}) | w_\mu^{u(t)} \rangle,$$

where we denote $w_\mu^{u(t)} = (L_{u(t)} + \mu \text{Id})^{-1} \mathbb{1}$. Since $\mathbb{1} \in L_+^2$, one can remove the projector Π in the expression of $T_{\partial_t u}$, so that

$$\frac{d}{dt} \mathcal{H}_\mu(u(t)) = \langle \partial_t u \cdot w_\mu^{u(t)} | w_\mu^{u(t)} \rangle.$$

Write $\partial_t u = \partial_x \nabla \mathcal{H}(u) - \alpha P_1(u)$, where $\mathcal{H}(u)$ is the Hamiltonian for the Benjamin-Ono equation without damping (BO) and

$$P_1(u) = \langle u | \cos \rangle \cos + \langle u | \sin \rangle \sin.$$

Then

$$\langle \partial_x \nabla \mathcal{H}(u(t)) \cdot w_\mu^{u(t)} | w_\mu^{u(t)} \rangle = \frac{d}{dt} \mathcal{H}_\mu(v(t)),$$

where v is the global solution to (BO) satisfying $v(t) = u(t)$ at time t . Since the map $t' \mapsto \mathcal{H}_\mu(v(t'))$ is constant, we deduce $\langle \partial_x \nabla \mathcal{H}(u(t)) \cdot w_\mu^{u(t)} | w_\mu^{u(t)} \rangle = 0$. Therefore,

$$\frac{d}{dt} \mathcal{H}_\mu(u(t)) = -\alpha \langle P_1(u(t)) w_\mu^{u(t)} | w_\mu^{u(t)} \rangle.$$

We factor $\langle P_1(u) w_\mu^u | w_\mu^u \rangle$ by using complex numbers

$$\begin{aligned} \langle P_1(u) w_\mu^u | w_\mu^u \rangle &= \langle u | \cos \rangle \langle \cos w_\mu^u | w_\mu^u \rangle + \langle u | \sin \rangle \langle \sin w_\mu^u | w_\mu^u \rangle \\ &= \text{Re}(\langle u | e^{ix} \rangle \langle e^{ix} w_\mu^u | w_\mu^u \rangle). \end{aligned}$$

In order to complete the proof, it is now enough to show that $\langle e^{ix} w_\mu^u | w_\mu^u \rangle \in L^2(\mathbb{R}_+)$, and that there holds an estimate of the form

$$\left(\int_T^{+\infty} |\langle e^{ix} w_\mu^{u(t)} | w_\mu^{u(t)} \rangle|^2 dt \right)^{\frac{1}{2}} \leq \frac{\varepsilon_u(T)}{(\mu - R^2)^3}, \tag{6.27}$$

with $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$.

First, note that for all $t \geq 0$, since $w_\mu^{u(t)} = (L_{u(t)} + \mu \text{Id})^{-1} \mathbb{1} \in L_+^2$, we have the cancellation $\langle e^{ix} w_\mu^{u(t)} | \mathbb{1} \rangle = 0$. Therefore one can write $\langle e^{ix} w_\mu^u | w_\mu^u \rangle = \langle e^{ix} w_\mu^u | w_\mu^u - \mathbb{1}/\mu \rangle$.

Then, let us decompose w_μ^u and $w_\mu^u - \mathbf{1}/\mu$ along the basis of eigenfunctions $(f_n)_n$ of L_u to get

$$\begin{aligned}\langle e^{ix} w_\mu^u | w_\mu^u \rangle &= \langle e^{ix} w_\mu^u | w_\mu^u - \mathbf{1}/\mu \rangle \\ &= \sum_{n,p \in \mathbb{N}} \langle w_\mu^u | f_n \rangle \overline{\langle w_\mu^u - \mathbf{1}/\mu | f_p \rangle} \langle e^{ix} f_n | f_p \rangle.\end{aligned}$$

By definition of w_μ^u as $w_\mu^u = (L_u + \mu \text{Id})^{-1} \mathbf{1}$, we have

$$\langle w_\mu^u | f_n \rangle = \frac{\langle \mathbf{1} | f_n \rangle}{\lambda_n + \mu} = \frac{\sqrt{\kappa_n} \zeta_n}{\lambda_n + \mu},$$

and

$$\langle w_\mu^u - \mathbf{1}/\mu | f_p \rangle = -\frac{\langle \mathbf{1} | f_p \rangle \lambda_p}{\mu(\lambda_p + \mu)} = -\frac{\sqrt{\kappa_p} \zeta_p \lambda_p}{\mu(\lambda_p + \mu)}.$$

From the formula $\langle e^{ix} f_n | f_p \rangle = \overline{M_{n,p}}$ and Definition 6.2.5 of $M_{n,p}$, we conclude

$$\begin{aligned}-\mu \langle e^{ix} w_\mu^u | w_\mu^u \rangle &= \sum_{\substack{n,p \\ p \neq n+1}} \frac{\kappa_p \lambda_p \sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}} \frac{\sqrt{\mu_{n+1}} \zeta_n \overline{\zeta_p} \overline{\zeta_{n+1}}}{(\lambda_p - \lambda_n - 1)(\lambda_n + \mu)(\lambda_p + \mu)} \\ &\quad + \sum_n \frac{\sqrt{\kappa_n \kappa_{n+1}} \lambda_{n+1} \sqrt{\mu_{n+1}} \zeta_n \overline{\zeta_{n+1}}}{(\lambda_n + \mu)(\lambda_{n+1} + \mu)}.\end{aligned}$$

Let us consider the two sums separately.

On the one hand, the Cauchy-Schwarz' inequality on the sums over indexes n implies

$$\begin{aligned}&\left| \sum_{n,p,p \neq n+1} \frac{\kappa_p \lambda_p \sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}} \frac{\sqrt{\mu_{n+1}} \zeta_n \overline{\zeta_p} \overline{\zeta_{n+1}}}{(\lambda_p - \lambda_n - 1)(\lambda_n + \mu)(\lambda_p + \mu)} \right| \\ &\leq C(R) \sum_p \frac{\kappa_p \gamma_p \lambda_p}{\lambda_p + \mu} \left(\sum_n \gamma_n \gamma_{n+1} \right)^{\frac{1}{2}} \left(\sum_{n,n \neq p-1} \frac{1}{(\lambda_p - \lambda_n - 1)^2 (\lambda_n + \mu)^2} \right)^{\frac{1}{2}}.\end{aligned}$$

For all $p \in \mathbb{N}$, we have $\sum_{n,n \neq p-1} \frac{1}{(\lambda_p - \lambda_n - 1)^2} \leq C$, moreover, for all $n \in \mathbb{N}$,

$$\frac{1}{\lambda_n + \mu} \leq \frac{1}{\mu - R^2}$$

and finally,

$$\sum_p \frac{\kappa_p \gamma_p \lambda_p}{\lambda_p + \mu} \leq \frac{C(R)}{\mu - R^2}.$$

Therefore,

$$\left| \sum_{n,p,p \neq n+1} \frac{\kappa_p \lambda_p \sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}} \frac{\sqrt{\mu_{n+1}} \zeta_n \overline{\zeta_p} \overline{\zeta_{n+1}}}{(\lambda_p - \lambda_n - 1)(\lambda_n + \mu)(\lambda_p + \mu)} \right| \leq \frac{C(R)}{(\mu - R^2)^2} \left(\sum_n \gamma_n \gamma_{n+1} \right)^{\frac{1}{2}},$$

so that

$$\begin{aligned}\int_T^{+\infty} \left| \sum_{n,p,p \neq n+1} \frac{\kappa_p \lambda_p \sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}} \frac{\sqrt{\mu_{n+1}} \zeta_n \overline{\zeta_p} \overline{\zeta_{n+1}}}{(\lambda_p - \lambda_n - 1)(\lambda_n + \mu)(\lambda_p + \mu)} \right|^2 dt \\ \leq \frac{C(R)}{(\mu - R^2)^4} \int_T^{+\infty} \sum_n \gamma_n(t) \gamma_{n+1}(t) dt.\end{aligned}$$

On the other hand, let us denote

$$b_n := \frac{\sqrt{\kappa_n \kappa_{n+1}} \lambda_{n+1} \sqrt{\mu_{n+1}}}{(\lambda_n + \mu)(\lambda_{n+1} + \mu)}.$$

Then, there exists $C(R) > 0$ such that for all $n \in \mathbb{N}$,

$$|b_n| \leq \frac{C(R)}{(\mu - R^2)^2}.$$

and

$$|\dot{b}_n| \leq \frac{C(R)}{(\mu - R^2)^2} |\langle u | e^{ix} \rangle|.$$

As a consequence, Remark 6.4.4 following Proposition 6.4.3, applied to the family $b = (b_n)_n$, implies

$$\int_T^{+\infty} \left| \sum_n \frac{\sqrt{\kappa_n \kappa_{n+1}} \lambda_{n+1} \sqrt{\mu_{n+1}} \zeta_n \bar{\zeta}_{n+1}}{(\lambda_n + \mu)(\lambda_{n+1} + \mu)} \right|^2 dt \leq \frac{\varepsilon_u(T)}{(\mu - R^2)^4}.$$

To conclude, we have proven the existence of $\varepsilon_u(T) \rightarrow 0$ as $T \rightarrow +\infty$ such that for all $\mu \geq R^2 + 1$,

$$\int_T^{+\infty} |\mu \langle e^{ix} w_\mu^{u(t)} | w_\mu^{u(t)} \rangle|^2 dt \leq \frac{\varepsilon_u(T)}{(\mu - R^2)^4}.$$

This completes the proof of inequality (6.27), and therefore of the lemma. \square

We now study the asymptotic expansion of $\mathcal{H}_\mu(u)$ when $\mu \rightarrow +\infty$.

Lemma 6.4.7. *Let $u \in L^2_{r,0}(\mathbb{T})$. Then we have the following asymptotic expansion at order 3 of the generating function when $\mu \rightarrow +\infty$:*

$$\mathcal{H}_\mu(u) = \frac{1}{\mu} + \frac{1}{2\mu^3} \|u\|_{L^2(\mathbb{T})}^2 + o\left(\frac{1}{\mu^3}\right).$$

Proof. By using the identity $\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x}$, we have

$$\mathcal{H}_\mu(u) = \frac{1}{\mu} \sum_{n=0}^{+\infty} \frac{\kappa_n \gamma_n}{\lambda_n / \mu + 1} = \frac{1}{\mu} \sum_{n=0}^{+\infty} \kappa_n \gamma_n - \frac{1}{\mu^2} \sum_{n=0}^{+\infty} \lambda_n \kappa_n \gamma_n + \frac{1}{\mu^3} \sum_{n=0}^{+\infty} \frac{\lambda_n^2}{\lambda_n / \mu + 1} \kappa_n \gamma_n.$$

Remark that

$$\sum_{n=0}^{+\infty} \kappa_n \gamma_n = \sum_{n=0}^{+\infty} |\langle \mathbb{1} | f_n \rangle|^2 = 1$$

and

$$\sum_{n=0}^{+\infty} \lambda_n \kappa_n \gamma_n = \sum_{n=0}^{+\infty} \lambda_n |\langle \mathbb{1} | f_n \rangle|^2 = -\langle \Pi u | \mathbb{1} \rangle = 0.$$

Consequently, the asymptotic development simplifies as

$$\mathcal{H}_\mu(u) = \frac{1}{\mu} + \frac{1}{\mu^3} \sum_{n=0}^{+\infty} \frac{\lambda_n^2}{\lambda_n / \mu + 1} \kappa_n \gamma_n.$$

For $\mu \geq -2\lambda_0(u)$, we have $\lambda_0/\mu + 1 \geq \frac{1}{2}$, so that for all n , $\frac{\lambda_n^2}{\lambda_n/\mu+1} \leq 2\lambda_n^2$. Moreover,

$$\sum_{n=0}^{+\infty} \lambda_n^2 \kappa_n \gamma_n = \langle \Pi u | \Pi u \rangle = \frac{1}{2} \|u\|_{L^2(\mathbb{T})}^2.$$

Therefore, one can pass to the limit $\mu \rightarrow +\infty$ in the summation sum and deduce

$$\mathcal{H}_\mu(u) = \frac{1}{\mu} + \frac{1}{2\mu^3} \|u\|_{L^2(\mathbb{T})}^2 + o\left(\frac{1}{\mu^3}\right).$$

□

We apply this lemma to

$$\mathcal{H}_\mu(u(T)) = \frac{1}{\mu} + \frac{\|u(T)\|_{L^2(\mathbb{T})}^2}{2\mu^3} + o_T\left(\frac{1}{\mu^3}\right)$$

and

$$\mathcal{H}_\mu(u_\infty) = \frac{1}{\mu} + \frac{\|u_\infty\|_{L^2(\mathbb{T})}^2}{2\mu^3} + o\left(\frac{1}{\mu^3}\right).$$

Note that all the possible weak limits u_∞ have the same norm in $L^2_{r,0}(\mathbb{T})$, indeed, we know from point 2 in Theorem 6.1.2 that for all $n \geq 1$, $\gamma_n(u_\infty) = \gamma_n^\infty$.

In light of Lemma 6.4.6, we deduce that for fixed $T > 0$,

$$|\mathcal{H}_\mu(u(T)) - \mathcal{H}_\mu(u_\infty)| = \left| \frac{\|u(T)\|_{L^2(\mathbb{T})}^2 - \|u_\infty\|_{L^2(\mathbb{T})}^2}{2\mu^3} + o_T\left(\frac{1}{\mu^3}\right) \right| \leq \frac{\varepsilon_u(T)}{(\mu - R^2)^3}.$$

We now consider the limit $\mu \rightarrow +\infty$ and get

$$\left| \|u(T)\|_{L^2(\mathbb{T})}^2 - \|u_\infty\|_{L^2(\mathbb{T})}^2 \right| \leq 2\varepsilon_u(T).$$

Taking the limit $T \rightarrow +\infty$ in this inequality, we conclude that $\|u(T)\|_{L^2(\mathbb{T})} \rightarrow \|u_\infty\|_{L^2(\mathbb{T})}$. We have proven that for any weak limit u_∞ of a subsequence $(u(t_k))_k$ in $L^2_{r,0}(\mathbb{T})$, the convergence is strong in $L^2_{r,0}(\mathbb{T})$.

6.5 Higher-order Sobolev norms

In this section, we consider a solution with higher regularity: we assume that the initial data u_0 belongs to $H^s_{r,0}(\mathbb{T})$ for some exponent $s \geq 0$. The Birkhoff transformation and its inverse transformation map bounded subsets of $H^s_{r,0}(\mathbb{T})$ to bounded subsets of $h^{\frac{1}{2}+s}_+$ (see [GKT20a], Proposition 5 in Appendix A). Therefore, it is equivalent to study the image of the H^s norm by this transformation and consider $\sum_{n \geq 0} n^{1+2s} \gamma_n$. This leads us to introduce a Lyapunov functional of the form $P_s = \sum_n w_n \gamma_n$, where $w_n \approx n^{1+2s}$, in order to prove Theorem 6.1.3.

Note that the Cauchy problem for (BO- α) is locally well-posed in $H^s_{r,0}(\mathbb{T})$, $s \geq 0$, by a simple adaptation of the proof in part 6.3.1 and by using Corollary 6.2.12.

6.5.1 Formula for the derivative of Sobolev norms

Proposition 6.5.1. Fix $s \geq 0$, and let u be a solution to the damped Benjamin-Ono equation (BO- α) with initial data $u_0 \in H_{r,0}^s(\mathbb{T})$. Define $c_n := n^{2s}$ with $c_0 = 0$, $w_n := \sum_{k=1}^{n-1} c_k$, so that we have $\frac{n^{1+2s}}{C} \leq w_n \leq Cn^{1+2s}$, and denote

$$P_s(t) := \sum_{n \geq 1} w_n \gamma_n(t).$$

Then

$$\frac{d}{dt} P_s(t) = -\alpha \sum_{n=0}^{+\infty} c_n a_n^*(t)^2 \gamma_n(t) \gamma_{n+1}(t) - \frac{\alpha}{2} \sum_{\substack{n,p \geq 0 \\ n \neq p}} (c_n + c_p) a_n^*(t) a_p^*(t) \eta_{n,p}(t),$$

with the notation $a_n^* = \sqrt{\mu_{n+1}} \frac{\sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}}$ and $\eta_{n,p} = \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}}$.

Proof. We have already seen in the proof of Lemma 6.3.6 that the time derivative of γ_n is

$$\frac{d}{dt} \gamma_n(u(t)) = -\alpha (\langle u(t) | \cos \rangle d\gamma_n[u(t)] \cdot \cos + \langle u(t) | \sin \rangle d\gamma_n[u(t)] \cdot \sin),$$

where

$$d\gamma_n[u(t)] \cdot \cos - i d\gamma_n[u(t)] \cdot \sin = m_{n-1} - m_n$$

and

$$m_n = -\sqrt{\mu_{n+1}} \frac{\sqrt{\kappa_n}}{\sqrt{\kappa_{n+1}}} \overline{\zeta_n} \zeta_{n+1}.$$

Moreover, from Lemma 6.2.6, the following formula holds

$$\langle u(t) | \cos \rangle - i \langle u(t) | \sin \rangle = \sum_{p \geq 0} m_p.$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt} \gamma_n(u(t)) &= -\alpha \left(\operatorname{Re}(m_{n-1} - m_n) \operatorname{Re} \left(\sum_{p \geq 0} m_p \right) + \operatorname{Im}(m_{n-1} - m_n) \operatorname{Im} \left(\sum_{p \geq 0} m_p \right) \right) \\ &= -\alpha \operatorname{Re} \left((m_{n-1} - m_n) \overline{\sum_{p \geq 0} m_p} \right). \end{aligned}$$

In particular, a summation leads to

$$\frac{d}{dt} P_s(t) = -\alpha \operatorname{Re} \left(\sum_{n \geq 1} w_n (m_{n-1} - m_n) \overline{\sum_{p \geq 0} m_p} \right).$$

We rewrite

$$\sum_{n \geq 1} w_n (m_{n-1} - m_n) = \sum_{n \geq 0} c_n m_n$$

with

$$c_n = w_{n+1} - w_n = n^{2s}.$$

Then

$$\frac{d}{dt} P_s(t) = -\alpha \operatorname{Re} \left(\sum_{n,p \geq 0} c_n m_n \overline{m_p} \right),$$

and taking the real part leads to the formula. \square

We now consider the decomposition

$$\frac{d}{dt}P_s(t) = -\alpha \sum_{n=0}^{+\infty} c_n a_n^*(t)^2 \gamma_n(t) \gamma_{n+1}(t) - \frac{\alpha}{2} \sum_{\substack{n,p \geq 0 \\ n \neq p}} (c_n + c_p) a_n^*(t) a_p^*(t) \eta_{n,p}(t).$$

We establish an improvement of Lemma 6.4.5 so as to see the second term in the right-hand side as a remainder term compared to the first term in the right-hand side.

In what follows, we drop the star exponent to the term a_n^* because we are only going to consider the only family $a_n = \sqrt{\mu_{n+1}} \sqrt{\frac{\kappa_n}{\kappa_{n+1}}}$. For $T \geq 0$, let us define

$$J^s(T) := \int_0^T \sum_{\substack{n,p \\ n \neq p}} (c_n + c_p) a_n(t) a_p(t) \eta_{n,p}(t) dt.$$

Assume that for all $t \in \mathbb{R}_+$, we have $\sum_p c_p \gamma_p(t) \leq C(R)$ (in particular, we already know that this condition is always satisfied if $s \leq \frac{1}{2}$). Then one can see that $J^s(T)$ is well-defined for all $T \geq 0$.

Our aim is to estimate $J^s(T)$ depending on

$$I^s(T) := \int_0^T \sum_{n=0}^{+\infty} c_n \gamma_n(t) \gamma_{n+1}(t) dt.$$

In the following lemma, we show that provided some upper bound on $J^s(T)$ is satisfied (assumption (6.28) below), then the higher-order Lyapunov functional P_s is bounded.

Lemma 6.5.2. *Fix $s \geq 0$. Assume that for some $R > 0$, and for all $t \geq 0$, the Birkhoff coordinates of the solution are bounded in h_+^s :*

$$\sum_n c_n \gamma_n(t) \leq R$$

(the constant R depends on the choice of s since $c_n = n^{2s}$). Also assume that for all $\varepsilon > 0$, there exists $C_s(R, \varepsilon) > 0$ such that for all $t \geq 0$,

$$J^s(t) \leq C_s(R, \varepsilon) + C_s(R, \varepsilon) \sqrt{I^s(t)} + \varepsilon I^s(t). \quad (6.28)$$

If the initial data belongs to $h_+^{\frac{1}{2}+s}$, i.e. if $P_s(0)$ is finite, then P_s is bounded: there exists $C'_s(R)$ such that for all $t \geq 0$,

$$P_s(t) \leq C'_s(R) + P_s(0),$$

moreover,

$$\int_0^{+\infty} \sum_{n=0}^{+\infty} c_n \gamma_n(t) \gamma_{n+1}(t) dt \leq C'_s(R) + P_s(0).$$

Proof. We use the formula for the time derivative of P_s

$$\frac{d}{dt}P_s(t) = -\alpha \sum_{n=0}^{+\infty} c_n a_n(t)^2 \gamma_n(t) \gamma_{n+1}(t) - \frac{\alpha}{2} \sum_{\substack{n,p \geq 0 \\ n \neq p}} (c_n + c_p) a_n(t) a_p(t) \eta_{n,p}(t).$$

Since for all n , $a_n(t) \geq \frac{1}{C(R)}$, an integration in time leads to

$$P_s(T) - P_s(0) \leq -\frac{\alpha}{C(R)^2} I^s(T) + \frac{\alpha}{2} |J^s(T)|.$$

We fix $\varepsilon := \frac{1}{2C(R)^2}$. Using that by assumption,

$$\begin{aligned} |J^s(T)| &\leq C_s(R, \varepsilon) + C_s(R, \varepsilon) \sqrt{I^s(T)} + \frac{1}{2C(R)^2} I^s(T) \\ &\leq C'_s(R, \varepsilon) + \frac{1}{C(R)^2} I^s(T), \end{aligned}$$

we deduce that

$$P_s(T) + \frac{\alpha}{2C(R)^2} I^s(T) \leq P_s(0) + \frac{\alpha}{2} C'_s(R, \varepsilon).$$

□

6.5.2 The case $s < \frac{1}{2}$

In the parts that follow, we prove that assumption (6.28) on $J^s(T)$ is satisfied. In this purpose, we first remove the part of $J^s(T)$ that can be bounded with the same strategy as the proof of Lemma 6.4.5. In the case $s < \frac{1}{2}$, we obtain a bound on $J^s(T)$ itself, and we deduce that if $u_0 \in H_{r,0}^s(\mathbb{T})$, then the solution u stays bounded in $H_{r,0}^s(\mathbb{T})$.

Lemma 6.5.3. *Let $s \geq 0$. We assume that for all $t \geq 0$, $\sum_{n \geq 0} c_n \gamma_n(t) \leq R$ (where $c_n = n^{2s}$). Then there exists $C(R) > 0$ such that the following holds. For all $T > 0$,*

$$|J^s(T) - \alpha K^s(T)| \leq C(R),$$

where

$$\begin{aligned} K^s(T) = \sum_{\substack{n,p \\ n \neq p}} \int_0^T \frac{a_n(t) a_p(t)}{i \Omega_{n,p}(t)} &\left(c_p \left(\overline{Z_n(t)} \zeta_{n+1}(t) + \overline{\zeta_n(t)} Z_{n+1}(t) \right) \zeta_p(t) \overline{\zeta_{p+1}(t)} \right. \\ &\left. + c_n \left(Z_p(t) \overline{\zeta_{p+1}(t)} + \zeta_p(t) \overline{Z_{p+1}(t)} \right) \overline{\zeta_n(t)} \zeta_{n+1}(t) \right) dt \quad (6.29) \end{aligned}$$

and from (6.16),

$$Z_n = \langle u | \cos \rangle d\zeta_n[u] \cdot \cos + \langle u | \sin \rangle d\zeta_n[u] \cdot \sin.$$

Moreover, if $s < \frac{1}{2}$, there exists $C_s(R) > 0$ such that

$$|J^s(T)| \leq C_s(R) + C_s(R) \sqrt{I^s(T)}.$$

Proof. We follow the strategy of proof of Lemma 6.4.5. For $n, p \geq 0$, $n \neq p$, recall the notation $\eta_{n,p} = \overline{\zeta_n} \zeta_{n+1} \overline{\zeta_p} \zeta_{p+1}$. Since for all $t \geq 0$, $\sum_n c_n \gamma_n(t) \leq R$, there exists $C(R) > 0$ such that for all $t \in \mathbb{R}$,

$$\sum_{\substack{n,p \\ n \neq p}} |(c_n + c_p) a_n(t) a_p(t) \eta_{n,p}(t)| \leq C(R).$$

Therefore, one can exchange the summation sign with the time integral:

$$J^s(T) = \sum_{\substack{n,p \\ n \neq p}} \int_0^T (c_n + c_p) a_n(t) a_p(t) \eta_{n,p}(t) dt.$$

For each term in the series over the indexes n and p , we use equality (6.23) from the proof of Lemma 6.4.5, which came from the differential equation (6.21) satisfied by $\eta_{n,p}$:

$$\begin{aligned} \int_0^T a_n(t) a_p(t) \eta_{n,p}(t) dt &= \left[\frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \eta_{n,p}(t) \right]_0^T - \int_0^T \frac{d}{dt} \left(\frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) dt \\ &\quad + \alpha \int_0^T \frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} F_{n,p}(t) dt, \end{aligned}$$

with

$$F_{n,p} = (\overline{Z_n} \zeta_{n+1} + \overline{\zeta_n} Z_{n+1}) \zeta_p \overline{\zeta_{p+1}} + (Z_p \overline{\zeta_{p+1}} + \zeta_p \overline{Z_{p+1}}) \overline{\zeta_n} \zeta_{n+1}.$$

We now consider the three terms in the right-hand side separately.

1. For all $t \geq 0$, we have

$$\sum_{\substack{n,p \\ n \neq p}} c_n \left| \frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \eta_{n,p}(t) \right| \leq C(R) \left(\sum_{n=0}^{+\infty} c_n \gamma_n(t) \right) \left(\sum_{p=0}^{+\infty} \gamma_p(t) \right) \leq C'(R).$$

The upper bound is independent of t by assumption. Since n and p play symmetric roles, we also have the estimate $\sum_{n,p, n \neq p} c_p \left| \frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \eta_{n,p}(t) \right| \leq C'(R)$.

Therefore, the series with general term $\sum_{n,p, n \neq p} (c_n + c_p) \left[\frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \eta_{n,p}(t) \right]_0^T$ is absolutely convergent and bounded by some constant $C''(R)$.

2. Next, recall from inequality (6.25) that

$$\int_0^T \left| \frac{d}{dt} \left(\frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) \right| dt \leq C(R) \int_0^T |\langle u(t) | e^{ix} \rangle| \frac{|\eta_{n,p}(t)|}{|n-p|} dt.$$

Since n and p play symmetric roles in the above upper bound, we only estimate

$$\begin{aligned} &\sum_{\substack{n,p \\ n \neq p}} c_n \int_0^T \left| \frac{d}{dt} \left(\frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) \right| dt \\ &\leq C(R) \left(\sum_{\substack{n,p \\ n \neq p}} \int_0^T |\langle u(t) | e^{ix} \rangle|^2 \frac{c_n \gamma_n(t)}{|n-p|^2} dt \right)^{\frac{1}{2}} \left(\sum_{\substack{n,p \\ n \neq p}} \int_0^T c_n \gamma_{n+1}(t) \gamma_p(t) \gamma_{p+1}(t) dt \right)^{\frac{1}{2}}. \end{aligned}$$

But there exists $C > 0$ such that for all n , $\sum_{p, p \neq n} \frac{1}{|n-p|^2} \leq C$ and moreover, by assumption, $\sum_n c_n \gamma_n(t) \leq R$. We conclude that

$$\sum_{\substack{n,p \\ n \neq p}} c_n \int_0^T \left| \frac{d}{dt} \left(\frac{a_n(t) a_p(t)}{i\Omega_{n,p}(t)} \right) \eta_{n,p}(t) \right| dt \leq C(R) \left(\int_0^{+\infty} |\langle u(t) | e^{ix} \rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I^0(T)}.$$

3. Finally, we prove that

$$\begin{aligned} \sum_{\substack{n,p \\ n \neq p}} c_n \int_0^T \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \left(\overline{Z_n(t)}\zeta_{n+1}(t) + \overline{\zeta_n(t)}Z_{n+1}(t) \right) \zeta_p(t)\overline{\zeta_{p+1}(t)} \right| dt \\ \leq C_s(R) \left(\int_0^{+\infty} |\langle u(t)|e^{ix} \rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I^0(T)}. \end{aligned}$$

By symmetry, we would also get

$$\begin{aligned} \sum_{\substack{n,p \\ n \neq p}} c_p \int_0^T \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \left(Z_p(t)\overline{\zeta_{p+1}(t)} + \zeta_p(t)\overline{Z_{p+1}(t)} \right) \overline{\zeta_n(t)}\zeta_{n+1}(t) \right| dt \\ \leq C_s(R) \left(\int_0^{+\infty} |\langle u(t)|e^{ix} \rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I^0(T)}. \end{aligned}$$

In this purpose, we use Corollary 6.2.12: if for all $t \geq 0$, we have $\sum_n c_n \gamma_n(t) \leq R$ (recall that $c_n = n^{2s}$), then for all $t \geq 0$,

$$\|d\zeta_n[u].\cos\|_{h_+^s} + \|d\zeta_n[u].\sin\|_{h_+^s} \leq C(R).$$

We deduce that $Z_n = \langle u|\cos \rangle d\zeta_n[u].\cos + \langle u|\sin \rangle d\zeta_n[u].\sin$ satisfies

$$\sum_{n \geq 0} c_n |Z_n(t)|^2 \leq C(R) |\langle u(t)|e^{ix} \rangle|^2. \quad (6.30)$$

Applying Cauchy-Schwarz' inequality, we conclude

$$\begin{aligned} \sum_{\substack{n,p \\ n \neq p}} c_n \int_0^T \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \overline{Z_n(t)}\zeta_{n+1}(t)\zeta_p(t)\overline{\zeta_{p+1}(t)} \right| dt \\ \leq C(R) \left(\int_0^T \sum_{\substack{n,p \\ n \neq p}} \frac{c_n |Z_n(t)|^2}{|n-p|^2} dt \right)^{\frac{1}{2}} \left(\int_0^T \sum_{\substack{n,p \\ n \neq p}} c_n \gamma_{n+1} \gamma_p \gamma_{p+1} dt \right)^{\frac{1}{2}} \\ \leq C'(R) \left(\int_0^{+\infty} |\langle u(t)|e^{ix} \rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I^0(T)}. \end{aligned}$$

One can apply the same strategy to the other term (involving $\overline{\zeta_n}Z_{n+1}\zeta_p\overline{\zeta_{p+1}}$) and get the desired inequality.

To conclude, since $\int_0^{+\infty} |\langle u(t)|e^{ix} \rangle|^2 dt \leq R^2$, we have proven that there exists $C(R) > 0$ such that for all $T \geq 0$,

$$|J^s(T) - \alpha K^s(T)| \leq C(R) + C(R)\sqrt{I^0(T)} \leq C'(R).$$

For $s < \frac{1}{2}$, we copy the proof of point 3. and show that actually

$$\sum_{n,p, n \neq p} (c_n + c_p) \int_0^T \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} F_{n,p}(t) \right| dt \leq C_s(R) \left(\int_0^{+\infty} |\langle u(t)|e^{ix} \rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I^s(T)}.$$

Indeed, applying Cauchy-Schwarz' inequality, we have

$$\begin{aligned} & \sum_{n,p,n \neq p} (c_n + c_p) \int_0^T \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right| \left| \overline{Z_n(t)}\zeta_{n+1}(t)\zeta_p(t)\overline{\zeta_{p+1}(t)} \right| dt \\ & \leq \left(\int_0^T \sum_{n,p,n \neq p} \frac{(c_n + c_p)|Z_n(t)|^2}{|n-p|^2} dt \right)^{\frac{1}{2}} \left(\int_0^T \sum_{n,p,n \neq p} (c_n + c_p)\gamma_{n+1}\gamma_p\gamma_{p+1} dt \right)^{\frac{1}{2}}. \end{aligned}$$

Since $s < \frac{1}{2}$, we have an estimate of the form $\sum_{p,p \neq n} \frac{c_p}{|n-p|^2} \leq C_s c_n$ by comparing n to $\frac{p}{2}$. This leads to

$$\begin{aligned} & \sum_{n,p,n \neq p} (c_n + c_p) \int_0^T \left| \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \right| \left| \overline{Z_n(t)}\zeta_{n+1}(t)\zeta_p(t)\overline{\zeta_{p+1}(t)} \right| dt \\ & \leq C_s(R) \left(\int_0^{+\infty} |\langle u(t)|e^{ix}\rangle|^2 dt \right)^{\frac{1}{2}} \sqrt{I^s(T)}. \end{aligned}$$

One can apply the same strategy to the three other terms composing $F_{n,p}$ (see the definition (6.22) of $F_{n,p}$) and get the desired result. \square

6.5.3 The case $\frac{1}{2} \leq s < \frac{3}{2}$

We now consider the part left to study when $s \geq \frac{1}{2}$, which has been defined in (6.29):

$$\begin{aligned} K^s(T) = & \sum_{\substack{n,p \\ n \neq p}} \int_0^T \frac{a_n(t)a_p(t)}{i\Omega_{n,p}(t)} \left(c_p \left(\overline{Z_n(t)}\zeta_{n+1}(t) + \overline{\zeta_n(t)}Z_{n+1}(t) \right) \zeta_p(t)\overline{\zeta_{p+1}(t)} \right. \\ & \left. + c_n \left(Z_p(t)\overline{\zeta_{p+1}(t)} + \zeta_p(t)\overline{Z_{p+1}(t)} \right) \overline{\zeta_n(t)}\zeta_{n+1}(t) \right) dt. \end{aligned}$$

In order to find an upper bound for $K^s(T)$, we apply Theorem 6.1.4 and get a decomposition for the term $Z_n = \langle u|\cos\rangle d\zeta_n[u].\cos + \langle u|\sin\rangle d\zeta_n[u].\sin$. We can therefore write $K^s(T)$ as a sum of terms of the form

$$\sum_{n,p} \int_0^T c_p \frac{a_n a_p}{i\Omega_{n,p}} G_{n,p} dt,$$

where up to exchanging the roles of n and p , $G_{n,p}$ is in one of the following two cases.

- In the first case, we take the terms corresponding to p_n^* or q_n^* from Theorem 6.1.4

$$G_{n,p} = \langle u|h_0\rangle r_n \overline{\zeta_{n+\sigma}} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}},$$

where $\sigma \in \{-1, 1\}$, $h_0 = e^{ix}$ or e^{-ix} , and r_n satisfies the estimates

$$|r_n| \leq C(R) \quad \text{and} \quad |\dot{r}_n| \leq C(R) |\langle u|e^{ix}\rangle|,$$

up to exchanging the roles of the indexes n , $n+1$, p and $p+1$: we can also have

$$G_{n,p} = \langle u|h_0\rangle r_n \overline{\zeta_n} \zeta_{n+1+\sigma} \zeta_p \overline{\zeta_{p+1}}.$$

- In the second case, we consider the terms corresponding to $A_{n,k}^*$ or $B_{n,k}^*$ from Theorem 6.1.4

$$G_{n,p} = \langle u|h_0 \rangle \sum_k A_{n,k} \zeta_k \overline{\zeta_{k+1}} \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}},$$

or

$$G_{n,p} = \langle u|h_0 \rangle \sum_k A_{n,k} \overline{\zeta_k} \zeta_{k+1} \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}},$$

where $h_0 = e^{ix}$ or e^{-ix} , and $A_{n,k}$ satisfies the estimates

$$|A_{n,k}| \leq C(R) \quad \text{and} \quad |\dot{A}_{n,k}| \leq C(R) |\langle u|e^{ix} \rangle|,$$

up to replacing $A_{n,k}$ by $A_{n+1,k}$, $B_{n,k}$ or $B_{n+1,k}$, which satisfy the same estimates.

For the sake of simplicity, we only treat the first expression of $G_{n,p}$ for each situation and assume that $h_0 = e^{ix}$:

- either

$$G_{n,p} = \langle u|e^{ix} \rangle r_n \overline{\zeta_{n+\sigma}} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}}$$

- or

$$G_{n,p} = \langle u|e^{ix} \rangle \sum_k A_{n,k} \zeta_k \overline{\zeta_{k+1}} \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}},$$

but the other cases are similar.

In both cases, we decompose $\langle u|h_0 \rangle$ thanks to the formula from Lemma 6.2.6

$$\langle u|e^{ix} \rangle = - \sum_q a_q \overline{\zeta_q} \zeta_{q+1}$$

(for treating the case $h_0 = e^{-ix}$, we would need to take the conjugate of this expression). Then the strategy follows the proof of Lemma 6.4.5, where $\eta_{n,p}$ is replaced by a longer expression, typically a product $\eta_{n,p,q}^\sigma$ of three or four of terms of the form $\overline{\zeta_q} \zeta_{q+1} \overline{\zeta_{n+\sigma}} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}}$ or $\overline{\zeta_q} \zeta_{q+1} \zeta_k \overline{\zeta_{k+1}} \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}}$. In particular, we use the differential equation satisfied by the oscillating part and integrate by parts.

Border terms

Lemma 6.5.4. *We consider a term in the first situation*

$$G_{n,p} = \sum_q a_q r_n \overline{\zeta_q} \zeta_{q+1} \overline{\zeta_{n+\sigma}} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}}$$

such that $\sigma \in \{-1, 1\}$, and for all $t \geq 0$, we have $|r_n(t)| \leq C(R)$ and $|r'_n(t)| \leq C(R) |\langle u(t)|e^{ix} \rangle|$. Assume that $\frac{1}{2} \leq s < \frac{3}{2}$ and for all $t \geq 0$, $\sum_n c_n \gamma_n(t) \leq R$ (where $c_n = n^{2s}$). Then for all $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that for all $T \geq 0$,

$$\left| \sum_{\substack{n,p \\ n \neq p}} c_p \int_0^T \frac{a_n a_p}{i\Omega_{n,p}} G_{n,p} dt \right| \leq C_s(R, \varepsilon) + C_s(R, \varepsilon) \sqrt{I^s(T)} + \varepsilon I^s(T).$$

Proof. The idea is to decompose $G_{n,p}$ between a part which has a small time derivative, and a part which oscillates rapidly for which we establish a differential equation. We have

$$\frac{a_n a_p}{i\Omega_{n,p}} G_{n,p} = \sum_q \frac{a_n a_p a_q r_n}{i\Omega_{n,p}} \eta_{n,p,q}^\sigma,$$

where

$$\eta_{n,p,q}^\sigma = \overline{\zeta_q \zeta_{q+1}} \overline{\zeta_{n+\sigma} \zeta_{n+1}} \zeta_p \overline{\zeta_{p+1}}$$

(for the other possible forms for $G_{n,p}$, we would have a similar formula up to placing the σ elsewhere in the product or taking the conjugate of $\overline{\zeta_q \zeta_{q+1}}$).

The oscillating part $\eta_{n,p,q}^\sigma$ satisfies the differential equation

$$\frac{d}{dt} \eta_{n,p,q}^\sigma = i\Omega_{n,p,q}^\sigma \eta_{n,p,q}^\sigma - \alpha H_{n,p,q}^\sigma, \quad (6.31)$$

where

$$H_{n,p,q} = (\overline{Z_q \zeta_{q+1}} + \overline{\zeta_q} Z_{q+1}) \overline{\zeta_{n+\sigma} \zeta_{n+1}} \zeta_p \overline{\zeta_{p+1}} + \overline{\zeta_q \zeta_{q+1}} (\overline{Z_{n+\sigma} \zeta_{n+1}} + \overline{\zeta_{n+\sigma}} Z_{n+1}) \zeta_p \overline{\zeta_{p+1}} \\ + \overline{\zeta_q \zeta_{q+1}} \overline{\zeta_{n+\sigma} \zeta_{n+1}} (Z_p \overline{\zeta_{p+1}} + \zeta_p \overline{Z_{p+1}}),$$

and

$$\Omega_{n,p,q}^\sigma = -\omega_q + \omega_{q+1} - \omega_{n+\sigma} + \omega_{n+1} + \omega_p - \omega_{p+1} \\ = 2((1-\sigma)n + q - p) + 1 - \sigma^2 - 2 \sum_{k \geq q+1} \gamma_k \\ + 2 \sum_{k \geq n+\sigma+1} \min(k, n+\sigma) \gamma_k - 2 \sum_{k \geq n+2} \min(k, n+1) \gamma_k + 2 \sum_{k \geq p+1} \gamma_k.$$

We write

$$\Omega_{n,p,q}^\sigma = 2 \left((1-\sigma)n + q - p + \tilde{\Omega}_{n,p,q}^\sigma \right),$$

with $|\tilde{\Omega}_{n,p,q}^\sigma| \leq C_0(R)$ and $|\frac{d}{dt} \tilde{\Omega}_{n,p,q}^\sigma| \leq C_0(R) |u| e^{ix}$. For all n, p, q such that $|(1-\sigma)n + q - p| > C_0(R) + 1$, one has $\Omega_{n,p,q}^\sigma \neq 0$, so that one can divide by $\Omega_{n,p,q}^\sigma$ in the differential equation satisfied by $\eta_{n,p,q}^\sigma$.

We split

$$\sum_{n,p, n \neq p} c_p \int_0^T \frac{a_n a_p}{i\Omega_{n,p}} G_{n,p} dt = \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q r_n}{i\Omega_{n,p}} \eta_{n,p,q}^\sigma dt \\ + \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q r_n}{i\Omega_{n,p}} \eta_{n,p,q}^\sigma dt.$$

• We first show that for all $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that the part with indexes n, p, q satisfying $|(1-\sigma)n + q - p| \leq C_0(R) + 1$ is bounded by $\varepsilon I^s(T) + C_s(R, \varepsilon) \sqrt{I^s(T)}$.

Indeed, since $|a_n a_p a_q r_n| \leq C(R)$, we have from the Cauchy-Schwarz' inequality

$$\begin{aligned} & \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \left| \int_0^T \frac{a_n a_p a_q r_n}{i\Omega_{n,p}} \eta_{n,p,q}^\sigma dt \right| \\ & \leq C(R) \left(\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \gamma_q \gamma_{q+1} \gamma_{n+\sigma} dt \right)^{\frac{1}{2}} \\ & \quad \left(\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} dt \right)^{\frac{1}{2}}. \end{aligned}$$

For fixed n and q , the possible indexes p in the sum belong to an interval of length $C(R)$. Moreover, since $p \leq C(R) + 4n + q$, we have $c_p \leq C(R) \max(c_n, c_q) \leq C(R) c_n c_q$, so that

$$\begin{aligned} \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \gamma_q \gamma_{q+1} \gamma_{n+\sigma} & \leq C(R) \sum_n c_n \gamma_{n+\sigma} \sum_q c_q \gamma_q \gamma_{q+1} \\ & \leq C'(R) \sum_q c_q \gamma_q \gamma_{q+1}, \end{aligned}$$

and

$$\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \gamma_q \gamma_{q+1} \gamma_{n+\sigma} dt \leq C'(R) I^s(T).$$

Now, fix $\varepsilon > 0$. Then there exists $N_0 = N_0(\varepsilon)$ such that for all $t \geq 0$, $\sum_{n \geq N_0} \gamma_{n+1}(t) \leq \varepsilon$ (recall that by assumption, for all $t \geq 0$, $\sum_n n^{2s} \gamma_n(t) \leq R$). Moreover, up to increasing N_0 , we also have

$$\sum_{n, |n-p| \geq N_0} \frac{\gamma_{n+1}}{|n-p|^2} \leq C(R) \sum_{n, |n-p| \geq N_0} \frac{1}{|n-p|^2} \leq \varepsilon.$$

In the two cases $n \geq N_0$ or $|n-p| \geq N_0$, we deduce

$$\begin{aligned} \sum_{\substack{n,p,q, n \neq p \\ n \geq N_0 \text{ or } |n-p| \geq N_0 \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} & \leq C(R) \sum_{\substack{n,p, n \neq p \\ n \geq N_0 \text{ or } |n-p| \geq N_0}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} \\ & \leq \varepsilon C'(R) \sum_p c_p \gamma_p \gamma_{p+1}. \end{aligned}$$

Otherwise, we have $n \leq N_0$ and $|n-p| \leq N_0$, therefore

$$\begin{aligned} \sum_{\substack{n,p,q, n \neq p \\ n \leq N_0 \text{ and } |n-p| \leq N_0 \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} \\ \leq C(R) \sum_{\substack{n,p, n \neq p \\ n \leq N_0 \text{ and } |n-p| \leq N_0}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} & \leq C'(R) N_0^{2s} \sum_p \gamma_p \gamma_{p+1}. \end{aligned}$$

Since $\int_0^T \sum_p \gamma_p \gamma_{p+1} dt \leq C(R)$ (see Proposition 6.4.3), we deduce that for any $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that

$$\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} dt \leq \varepsilon C(R) I^s(T) + C_s(R, \varepsilon).$$

Therefore we conclude that

$$\begin{aligned} \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \left| \int_0^T \frac{a_n a_p a_q r_n}{i \Omega_{n,p}} \eta_{n,p,q}^\sigma dt \right| \\ \leq C(R) \sqrt{I^s(T)} \sqrt{\varepsilon C(R) I^s(T) + C_s(R, \varepsilon)}. \end{aligned}$$

In other words, for all $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that

$$\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| \leq C_0(R)+1}} c_p \left| \int_0^T \frac{a_n a_p a_q r_n}{i \Omega_{n,p}} \eta_{n,p,q}^\sigma dt \right| \leq \varepsilon I^s(T) + C_s(R, \varepsilon) \sqrt{I^s(T)}.$$

• We now tackle the indexes n, p and q such that $n \neq p$ and $|(1-\sigma)n+q-p| > C_0(R)+1$ (so that $\Omega_{n,p,q}^\sigma \neq 0$). We use the differential equation (6.31) satisfied by $\eta_{n,p,q}^\sigma$, and get that

$$\int_0^T \frac{a_n a_p}{i \Omega_{n,p}} G_{n,p} dt = - \int_0^T \frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \left(\frac{d}{dt} \eta_{n,p,q}^\sigma + \alpha H_{n,p,q}^\sigma \right) dt.$$

We perform an integration by parts for the first term in the right-hand side of this equality:

$$\begin{aligned} \int_0^T \frac{a_n a_p}{i \Omega_{n,p}} G_{n,p} dt = \left[- \frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \eta_{n,p,q}^\sigma \right]_0^T + \int_0^T \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \eta_{n,p,q}^\sigma dt \\ - \alpha \int_0^T \frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} H_{n,p,q}^\sigma dt. \end{aligned}$$

It remains to study the summability properties for each of those three terms.

1. First, we see that since $\sum_p c_p \gamma_p(t) \leq R$ for all $t \geq 0$, then

$$\begin{aligned} \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \left| \left[- \frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \eta_{n,p,q}^\sigma \right]_0^T \right| \\ \leq C(R) \sum_{n,p,q} c_p \sqrt{\gamma_q \gamma_{q+1} \gamma_{n+\sigma} \gamma_{n+1} \gamma_p \gamma_{p+1}} \leq C'(R). \end{aligned}$$

2. Then, we apply Cauchy-Schwarz' inequality to the second term

$$\begin{aligned} & \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \int_0^T \left| \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \eta_{n,p,q}^\sigma \right| dt \\ & \leq C(R) \left(\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \gamma_p \gamma_{n+\sigma} \left| \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \right|^2 dt \right)^{\frac{1}{2}} \\ & \quad \left(\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \gamma_{p+1} \gamma_{n+1} \gamma_q \gamma_{q+1} dt \right)^{\frac{1}{2}}. \end{aligned}$$

On the one hand, we have

$$\int_0^T \sum_{n,p,q} c_p \gamma_{p+1} \gamma_{n+1} \gamma_q \gamma_{q+1} dt \leq C(R) I^0(T).$$

On the other hand, we establish a bound for $\left| \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \right|$. We have already seen that

$$\left| \frac{d}{dt} \left(\frac{a_n a_p a_q}{i \Omega_{n,p}} \right) \right| \leq C(R) \frac{|\langle u | e^{ix} \rangle|}{|\Omega_{n,p}|}.$$

Since we also have

$$\left| \frac{d}{dt} \Omega_{n,p,q}^\sigma \right| \leq C(R) |\langle u | e^{ix} \rangle|$$

and by assumption,

$$\left| \frac{d}{dt} r_n \right| \leq C(R) |\langle u | e^{ix} \rangle|,$$

we deduce

$$\int_0^T c_p \gamma_p \gamma_{n+\sigma} \left| \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \right|^2 dt \leq C(R) \int_0^T \frac{c_p \gamma_p \gamma_{n+\sigma}}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} |\langle u | e^{ix} \rangle|^2 dt.$$

But since $|\Omega_{n,p,q}^\sigma| \geq 2(|(1-\sigma)n+q-p| - C_0(R)) \geq 1$, we can use that

$$\sum_{\substack{q \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \frac{1}{|\Omega_{n,p,q}^\sigma|^2} \leq C(R),$$

and we get that the following series is convergent and bounded by $C(R)$:

$$\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \frac{c_p \gamma_p \gamma_{n+\sigma}}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} \leq C(R) \sum_{\substack{n,p \\ n \neq p}} \frac{c_p \gamma_p \gamma_{n+\sigma}}{|n-p|^2} \leq C'(R).$$

We deduce

$$\int_0^T \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \gamma_p \gamma_{n+\sigma} \left| \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \right|^2 dt \leq C(R).$$

To conclude, we have proven that

$$\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \int_0^T \left| \frac{d}{dt} \left(\frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \right) \eta_{n,p,q}^\sigma \right| dt \leq C(R) \sqrt{I^0(T)}.$$

3. Finally, recall that

$$H_{n,p,q}^\sigma = (\overline{Z_q \zeta_{q+1}} + \overline{\zeta_q Z_{q+1}}) \overline{\zeta_{n+\sigma} \zeta_{n+1} \zeta_p \zeta_{p+1}} + \overline{\zeta_q \zeta_{q+1}} (\overline{Z_{n+\sigma} \zeta_{n+1}} + \overline{\zeta_{n+\sigma} Z_{n+1}}) \zeta_p \overline{\zeta_{p+1}} \\ + \overline{\zeta_q \zeta_{q+1} \zeta_{n+\sigma} \zeta_{n+1}} (\overline{Z_p \zeta_{p+1}} + \overline{\zeta_p Z_{p+1}}).$$

We estimate for instance the term

$$\left| \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \overline{Z_q \zeta_{q+1} \zeta_{n+\sigma} \zeta_{n+1} \zeta_p \zeta_{p+1}} dt \right| \\ \leq C(R) \left(\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \int_0^T c_p \gamma_p \gamma_{p+1} \gamma_{q+1} \gamma_{n+\sigma} dt \right)^{\frac{1}{2}} \\ \left(\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \int_0^T \frac{c_p \gamma_{n+1}}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} |Z_q|^2 dt \right)^{\frac{1}{2}}.$$

On the one hand,

$$\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \int_0^T c_p \gamma_p \gamma_{p+1} \gamma_{q+1} \gamma_{n+\sigma} dt \leq C(R) I^s(T).$$

On the other hand, we first estimate the sum over indexes p :

$$\sum_{\substack{p, p \neq n \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \frac{c_p}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} \\ \leq C(R) \sum_{\substack{p, p \neq n \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \frac{p^{2s}}{|n-p|^2 (1 + |(1-\sigma)n+q-p|)^2}.$$

When $p \geq 2n$ and $p \geq 2((1-\sigma)n+q)$, the general term of the series is bounded by $C_s \frac{p^{2s}}{p^4}$, and this defines a convergent series since $s < \frac{3}{2}$. Otherwise, $p^{2s} \leq C_s \max(n, q)^{2s}$ and the series $\sum_{p, p \neq n} \frac{1}{|n-p|^2}$ is convergent. We deduce that

$$\sum_{\substack{p, p \neq n \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \frac{c_p}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} \leq C_s \max(n, q)^{2s},$$

so that

$$\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \frac{c_p \gamma_{n+1}}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} |Z_q|^2 \leq C_s \sum_{n,q} \max(n, q)^{2s} \gamma_{n+1} |Z_q|^2.$$

But by assumption, $\sum_n n^{2s} \gamma_n \leq R$, moreover, we have seen in (6.30) that

$$\sum_k c_k |Z_k|^2 \leq C(R) |\langle u | e^{ix} \rangle|^2.$$

Therefore, we get an estimate for this second term

$$\sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} \int_0^T \frac{c_p \gamma_{n+1}}{|\Omega_{n,p} \Omega_{n,p,q}^\sigma|^2} |Z_q|^2 dt \leq C_s(R).$$

To conclude, have proven that

$$\left| \sum_{\substack{n,p,q, n \neq p \\ |(1-\sigma)n+q-p| > C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q r_n}{\Omega_{n,p} \Omega_{n,p,q}^\sigma} \overline{Z_q} \zeta_{q+1} \overline{\zeta_{n+\sigma} \zeta_{n+1} \zeta_p \zeta_{p+1}} dt \right| \leq C_s(R) \sqrt{I^s(T)}.$$

This proof also works when exchanging the roles of n and q and with a small variant when exchanging the roles of n and p . □

Central terms

Lemma 6.5.5. *Let us consider a term in the second situation*

$$G_{n,p} = \sum_{q,k} a_q A_{n,k} \eta_{n,p,q,k},$$

with

$$\eta_{n,p,q,k} = \overline{\zeta_q \zeta_{q+1} \zeta_k \zeta_{k+1} \zeta_n \zeta_{n+1} \zeta_p \zeta_{p+1}},$$

and such that for all $t \geq 0$, $|A_{n,k}(t)| \leq C(R)$ and $|\dot{A}_{n,k}(t)| \leq C(R) |\langle u | e^{ix} \rangle|$. Fix $\frac{1}{2} \leq s < \frac{3}{2}$, and assume that for all $t \geq 0$, $\sum_n c_n \gamma_n(t) \leq R$ (where $c_n = n^{2s}$). Then for all $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that for all $T \geq 0$,

$$\left| \sum_{\substack{n,p \\ n \neq p}} c_p \int_0^T \frac{a_n a_p}{i \Omega_{n,p}} G_{n,p} dt \right| \leq C_s(R, \varepsilon) + C_s(R, \varepsilon) \sqrt{I^s(T)} + \varepsilon I^s(T).$$

Proof. The proof follows the proof in the above part 6.5.3. We first compute the differential equation satisfied by $\eta_{n,p,q,k}$:

$$\frac{d}{dt} \eta_{n,p,q,k} = i \Omega_{n,p,q,k} \eta_{n,p,q,k} - \alpha H_{n,p,q,k}, \tag{6.32}$$

where

$$\begin{aligned}\Omega_{n,p,q,k} &= -\omega_q + \omega_{q+1} + \omega_k - \omega_{k+1} - \omega_n + \omega_{n+1} + \omega_p - \omega_{p+1} \\ &= 2(q - k + n - p) - 2 \sum_{l \geq q+1} \gamma_l + 2 \sum_{l \geq k+1} \gamma_l - 2 \sum_{l \geq n+1} \gamma_l + 2 \sum_{l \geq p+1} \gamma_l \\ &= 2((q - k + n - p) + \tilde{\Omega}_{n,p,q,k}),\end{aligned}$$

where $|\tilde{\Omega}_{n,p,q,k}| \leq C_0(R)$, $|\frac{d}{dt}\tilde{\Omega}_{n,p,q,k}| \leq C_0(R)|\langle u|e^{ix}\rangle|$, and

$$\begin{aligned}H_{n,p,q,k} &= (\overline{Z}_q \zeta_{q+1} + \overline{\zeta}_q Z_{q+1}) \zeta_k \overline{\zeta_{k+1}} \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}} \\ &\quad + \overline{\zeta}_q \zeta_{q+1} (\overline{Z}_k \zeta_{k+1} + \overline{\zeta}_k Z_{k+1}) \overline{\zeta_n} \zeta_{n+1} \zeta_p \overline{\zeta_{p+1}} + \overline{\zeta}_q \zeta_{q+1} \zeta_k \overline{\zeta_{k+1}} (\overline{Z}_n \zeta_{n+1} + \overline{\zeta}_n Z_{n+1}) \zeta_p \overline{\zeta_{p+1}} \\ &\quad + \overline{\zeta}_q \zeta_{q+1} \zeta_k \overline{\zeta_{k+1}} \overline{\zeta_n} \zeta_{n+1} (\overline{Z}_p \zeta_{p+1} + \overline{\zeta}_p Z_{p+1}).\end{aligned}$$

For all n, p, q, k such that $|q - k + n - p| - C_0(R) \geq 1$, one has $\Omega_{n,p,q,k} \neq 0$, so that one can divide by $\Omega_{n,p,q,k}$ in the differential equation satisfied by $\eta_{n,p,q,k}$.

We split

$$\begin{aligned}\sum_{\substack{n,p \\ n \neq p}} c_p \int_0^T \frac{a_n a_p a_q}{i\Omega_{n,p}} G_{n,p} dt &= \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q A_{n,k}}{i\Omega_{n,p}} \eta_{n,p,q,k} dt \\ &\quad + \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q A_{n,k}}{i\Omega_{n,p}} \eta_{n,p,q,k} dt.\end{aligned}$$

• We first show that the part with indexes n, p, q and k such that $n \neq p$ and $|q - k + n - p| \leq C_0(R) + 1$ is bounded by $\varepsilon I^s(T) + C_s(R, \varepsilon) \sqrt{I^s(T)}$. Indeed, we have

$$\begin{aligned}\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \left| \int_0^T \frac{a_n a_p a_q A_{n,k}}{i\Omega_{n,p}} \eta_{n,p,q,k} dt \right| \\ \leq C(R) \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|} \int_0^T \sqrt{\gamma_q \gamma_{q+1} \gamma_k \gamma_{k+1} \gamma_n \gamma_{n+1} \gamma_p \gamma_{p+1}} dt,\end{aligned}$$

so that from the Cauchy-Schwarz' inequality,

$$\begin{aligned}\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \left| \int_0^T \frac{a_n a_p a_q A_{n,k}}{i\Omega_{n,p}} \eta_{n,p,q,k} dt \right| \\ \leq C(R) \left(\int_0^T \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \gamma_q \gamma_{q+1} \gamma_k \gamma_n dt \right)^{\frac{1}{2}} \\ \left(\int_0^T \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{k+1} \gamma_{n+1} \gamma_p \gamma_{p+1} dt \right)^{\frac{1}{2}}.\end{aligned}$$

For fixed n, q and k , the possible indexes p lie in an interval of length $C(R)$. Moreover, since $p \leq C(R) + q + k + n$, we have $c_p \leq C(R) \max(c_n, c_q, c_k) \leq C(R)c_n c_q c_k$, so that

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \gamma_q \gamma_{q+1} \gamma_k \gamma_n \leq C(R) \sum_k c_k \gamma_k \sum_n c_n \gamma_n \sum_q c_q \gamma_q \gamma_{q+1}$$

and

$$\int_0^T \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \gamma_q \gamma_{q+1} \gamma_k \gamma_n \leq C'(R) I^s(T).$$

Moreover, fix $\varepsilon > 0$. Then there exists $N_0 = N_0(\varepsilon)$ such that for all $t \geq 0$,

$$\sum_{n \geq N_0} \gamma_{n+1}(t) \leq \varepsilon$$

and

$$\sum_{n, |n-p| \geq N_0} \frac{\gamma_{n+1}}{|n-p|^2} \leq C(R) \sum_{n, |n-p| \geq N_0} \frac{1}{|n-p|^2} \leq \varepsilon.$$

In the two cases $n \geq N_0$ or $|n-p| \geq N_0$, we deduce

$$\begin{aligned} \sum_{\substack{n,p,q,k, n \neq p \\ n \geq N_0 \text{ or } |n-p| \geq N_0 \\ |q-k+n-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_{k+1} \gamma_p \gamma_{p+1} \\ \leq C(R) \sum_{\substack{n,p, n \neq p \\ n \geq N_0 \text{ or } |n-p| \geq N_0}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} \leq \varepsilon C(R) \sum_p c_p \gamma_p \gamma_{p+1}. \end{aligned}$$

Otherwise,

$$\begin{aligned} \sum_{\substack{n,p,q,k, n \neq p \\ n \leq N_0 \text{ and } |n-p| \leq N_0 \\ |q-k+n-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_{k+1} \gamma_p \gamma_{p+1} \\ \leq C(R) \sum_{\substack{n,p, n \neq p \\ n \leq N_0 \text{ and } |n-p| \leq N_0}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_p \gamma_{p+1} \leq C'(R) N_0^{2s} \sum_p \gamma_p \gamma_{p+1}. \end{aligned}$$

Since $\int_0^T \sum_p \gamma_p \gamma_{p+1} dt \leq C(R)$, we deduce that for any $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that

$$\int_0^T \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} \frac{c_p}{|n-p|^2} \gamma_{n+1} \gamma_{k+1} \gamma_p \gamma_{p+1} dt \leq \varepsilon C(R) I^s(T) + C_s(R, \varepsilon).$$

We conclude that for all $\varepsilon > 0$, there exists $C_s(R, \varepsilon)$ such that for all $T \geq 0$,

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| \leq C_0(R)+1}} c_p \left| \int_0^T \frac{a_n a_p a_q A_{n,k}}{i \Omega_{n,p}} \eta_{n,p,q,k} dt \right| \leq \varepsilon I^s(T) + C_s(R, \varepsilon) \sqrt{I^s(T)}.$$

• We now tackle the indexes n, p, q and k such that $|q - k + n - p| > C_0(R) + 1$, so that $\Omega_{n,p,q,k} \neq 0$. We use the differential equation (6.32) satisfied by $\eta_{n,p,q,k}$, and get that if $|q - k + n - p| > C_0(R) + 1$, then

$$\int_0^T \frac{a_n a_p}{i\Omega_{n,p}} G_{n,p} dt = - \int_0^T \frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \left(\frac{d}{dt} \eta_{n,p,q,k} + \alpha H_{n,p,q,k} \right) dt.$$

We perform an integration by parts for the first term in the right-hand side of this equality:

$$\begin{aligned} \int_0^T \frac{a_n a_p}{i\Omega_{n,p}} G_{n,p} dt &= \left[-\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \eta_{n,p,q,k} \right]_0^T + \int_0^T \frac{d}{dt} \left(\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \right) \eta_{n,p,q,k} dt \\ &\quad - \alpha \int_0^T \frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} H_{n,p,q,k} dt. \end{aligned}$$

It remains to study the summability properties for each of those three terms.

1. First, we see that thanks to the assumption $\sum_p c_p \gamma_p \leq C(R)$, we have

$$\begin{aligned} \sum_{\substack{n,p,q,k \\ |q-k+n-p| > C_0(R)+1}} c_p \left| \left[-\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \eta_{n,p,q,k} \right]_0^T \right| \\ \leq C(R) \sum_{n,p,q,k} c_p \sqrt{\gamma_q \gamma_{q+1} \gamma_k \gamma_{k+1} \gamma_n \gamma_{n+1} \gamma_p \gamma_{p+1}} \leq C'(R). \end{aligned}$$

2. Then, we apply Cauchy-Schwarz' inequality to the second term:

$$\begin{aligned} \sum_{\substack{n,p,q,k \\ |q-k+n-p| > C_0(R)+1}} c_p \int_0^T \left| \frac{d}{dt} \left(\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \right) \eta_{n,p,q,k} \right| dt \\ \leq C(R) \left(\int_0^T \sum_{\substack{n,p,q,k \\ |q-k+n-p| > C_0(R)+1}} c_p \gamma_p \gamma_k \gamma_n \left| \frac{d}{dt} \left(\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \right) \right|^2 dt \right)^{\frac{1}{2}} \\ \left(\int_0^T \sum_{\substack{n,p,q,k \\ |q-k+n-p| > C_0(R)+1}} c_p \gamma_{p+1} \gamma_{k+1} \gamma_{n+1} \gamma_q \gamma_{q+1} dt \right)^{\frac{1}{2}}. \end{aligned}$$

On the one hand, we have

$$\int_0^T \sum_{n,p,q} c_p \gamma_{p+1} \gamma_{k+1} \gamma_{n+1} \gamma_q \gamma_{q+1} dt \leq C(R) I^0(T).$$

On the other hand, we know that

$$\left| \frac{d}{dt} \left(\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \right) \right| \leq C(R) \frac{|\langle u | e^{ix} \rangle|}{|\Omega_{n,p} \Omega_{n,p,q,k}|}.$$

But using that $\sum_{q, |q-k+n-p| > C_0(R)+1} \frac{1}{|\Omega_{n,p,q,k}|^2} \leq C(R)$, we get that the following series is convergent and bounded by $C(R)$:

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} \frac{c_p \gamma_p \gamma_k \gamma_n}{|\Omega_{n,p} \Omega_{n,p,q,k}|^2} \leq C(R) \sum_{\substack{n,p,k \\ n \neq p}} \frac{c_p \gamma_p \gamma_k \gamma_n}{|n-p|^2} \leq C'(R).$$

We deduce

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} c_p \int_0^T \left| \frac{d}{dt} \left(\frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \right) \eta_{n,p,q,k} \right| dt \leq C(R) \sqrt{I^0(T)}.$$

3. Finally, recall that

$$\begin{aligned} H_{n,p,q,k} = & (\overline{Z_q} \zeta_{q+1} + \overline{\zeta_q} Z_{q+1}) \zeta_k \overline{\zeta_{k+1}} \zeta_n \overline{\zeta_{n+1}} \zeta_p \overline{\zeta_{p+1}} \\ & + \overline{\zeta_q} \zeta_{q+1} (\overline{Z_k} \zeta_{k+1} + \overline{\zeta_k} Z_{k+1}) \zeta_n \overline{\zeta_{n+1}} \zeta_p \overline{\zeta_{p+1}} \\ & + \overline{\zeta_q} \zeta_{q+1} \zeta_k \overline{\zeta_{k+1}} (\overline{Z_n} \zeta_{n+1} + \overline{\zeta_n} Z_{n+1}) \zeta_p \overline{\zeta_{p+1}} \\ & + \overline{\zeta_q} \zeta_{q+1} \zeta_k \overline{\zeta_{k+1}} \zeta_n \overline{\zeta_{n+1}} (\overline{Z_p} \zeta_{p+1} + \overline{\zeta_p} Z_{p+1}). \end{aligned}$$

We estimate for instance the term

$$\begin{aligned} & \left| \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p} \Omega_{n,p,q,k}} \overline{Z_q} \zeta_{q+1} \zeta_k \overline{\zeta_{k+1}} \zeta_n \overline{\zeta_{n+1}} \zeta_p \overline{\zeta_{p+1}} dt \right| \\ & \leq C(R) \left(\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} \int_0^T c_p \gamma_p \gamma_{p+1} \gamma_{q+1} \gamma_{k+1} \gamma_n dt \right)^{\frac{1}{2}} \\ & \quad \left(\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} \int_0^T \frac{c_p \gamma_{n+1} \gamma_k}{|\Omega_{n,p} \Omega_{n,p,q,k}|^2} |Z_q|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

On the one hand,

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} \int_0^T c_p \gamma_p \gamma_{p+1} \gamma_{q+1} \gamma_{k+1} \gamma_n dt \leq C(R) I^s(T).$$

On the other hand, we first estimate the sum over indexes p :

$$\begin{aligned} & \sum_{\substack{p, p \neq n \\ |q-k+n-p| > C_0(R)+1}} \frac{c_p}{|\Omega_{n,p} \Omega_{n,p,q,k}|^2} \\ & \leq C(R) \sum_{\substack{p, p \neq n \\ |q-k+n-p| > C_0(R)+1}} \frac{p^{2s}}{|n-p|^2 (1 + |q-k+n-p|)^2}. \end{aligned}$$

When $p \geq 2n$ and $p \geq 2(q - k + n)$, the general term inside the summation term is bounded by $C_s \frac{p^{2s}}{p^4}$, and this defines a convergent series since $s < \frac{3}{2}$. Otherwise, we have $p^{2s} \leq \max(n, q, k)^{2s}$ and the series $\sum_{p,p \neq n} \frac{1}{|n-p|^2}$ is convergent. We deduce that

$$\sum_{\substack{p, p \neq n \\ |q-k+n-p| > C_0(R)+1}} \frac{c_p}{|\Omega_{n,p}\Omega_{n,p,q,k}|^2} \leq C_s \max(n, q, k)^{2s},$$

so that

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} \frac{c_p \gamma_{n+1} \gamma_k}{|\Omega_{n,p}\Omega_{n,p,q,k}|^2} |Z_q|^2 \leq C_s \sum_{n,q,k} \max(n, q, k)^{2s} \gamma_{n+1} \gamma_k |Z_q|^2.$$

Since $\sum_l l^{2s} \gamma_l \leq C(R)$, and since from inequality (6.30) we have

$$\sum_l l^{2s} |Z_l|^2 \leq C(R) |\langle u | e^{ix} \rangle|^2,$$

we get the bound

$$\sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} \int_0^T \frac{c_p \gamma_{n+1} \gamma_k}{|\Omega_{n,p}\Omega_{n,p,q,k}|^2} |Z_q|^2 dt \leq C_s(R).$$

To conclude, we have proven that

$$\left| \sum_{\substack{n,p,q,k, n \neq p \\ |q-k+n-p| > C_0(R)+1}} c_p \int_0^T \frac{a_n a_p a_q A_{n,k}}{\Omega_{n,p}\Omega_{n,p,q,k}} \overline{Z_q} \zeta_{q+1} \overline{\zeta_k} \zeta_{k+1} \overline{\zeta_n} \zeta_{n+1} \overline{\zeta_p} \zeta_{p+1} dt \right| \leq C_s(R) \sqrt{I^s(T)}.$$

This proof also works when exchanging the roles of n and q , and with a small variant when exchanging the roles of n and p .

□

Remark 6.5.6. We have proven that $\sum_n n^{1+2s} \gamma_n$ is bounded for exponents s satisfying $s < \frac{3}{2}$. To increase the range of exponents s , we would need to further decompose the term Z_q appearing in point 3 of subparts 6.5.3 and 6.5.3. This would lead us to consider finite products of the following form. Let $h_0 = \cos$ or $h_0 = \sin$, $N = (n_1, \dots, n_k)$ be a finite set of indexes, and $\Sigma = (\sigma_1, \dots, \sigma_k)$ be a finite set of indexes bounded by some finite constant $c(k)$. Then we would have to study

$$G_{N,\Sigma} = \langle u | h_0 \rangle A_{N,\Sigma} \overline{\zeta_{n_1}} \zeta_{n_1+\sigma_1} \cdots \zeta_{n_k} \overline{\zeta_{n_k+\sigma_k}},$$

provided the uniform estimates

$$|A_{N,\Sigma}| \leq C(R) \quad \text{and} \quad |\dot{A}_{N,\Sigma}| \leq C(R) |\langle u | e^{ix} \rangle|.$$

In particular, we should use the differential equation satisfied by this term and integrate by parts in the same way as before, following the idea of proof from Lemma 6.4.5.

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Titre: Comportement en temps long de solutions d'EDP non linéaires : stabilité des ondes progressives, dispersion, intégrabilité, amortissement

Mots clés: Équation de Schrödinger non linéaire, équation de Benjamin-Ono, ondes progressives, dispersion, systèmes intégrables, comportement en temps long

Résumé: Cette thèse est consacrée à l'étude du comportement en temps long de solutions pour deux familles d'EDP non linéaires.

D'une part, on s'intéresse à l'équation de Schrödinger non linéaire cubique sur le groupe de Heisenberg, qui constitue une équation sans dispersion. On sait qu'il existe des solutions ondes progressives minimisantes radiales pour toute vitesse dans $]-1,1[$, et on montre des propriétés de stabilité orbitale de ces ondes progressives à partir de l'étude d'un système limite obtenu en faisant tendre la vitesse vers 1. Ce système limite joue le même rôle que l'équation de Szegő cubique envers l'équation de demi-onde cubique. Puis, on établit en basse régularité le caractère presque-sûrement localement bien

posé du problème de Cauchy randomisé associé à une équation de Schrödinger-Grushin proche pour une grande famille de données initiales.

D'autre part, on analyse deux équations apparentées à l'équation de Benjamin-Ono en utilisant les propriétés d'intégrabilité de l'équation de Benjamin-Ono. Pour l'équation du troisième ordre dans la hiérarchie de Benjamin-Ono, on étudie le problème de Cauchy, on caractérise les solutions ondes progressives, puis on détermine leurs propriétés de stabilité orbitale. Pour une équation de Benjamin-Ono amortie par les petits modes de Fourier cos et sin, on décrit les limites faibles des trajectoires en temps infini, puis on montre l'absence de croissance des normes de Sobolev d'ordre supérieur en temps infini.

Title: Long-time behavior of solutions to nonlinear PDEs: orbital stability of traveling waves, dispersion, integrability, damping

Keywords: Nonlinear Schrödinger equation, Benjamin-Ono equation, traveling waves, dispersion, integrable systems, long-time behavior

Abstract: This thesis is devoted to the study of the long-time behavior of solutions to two families of nonlinear PDEs.

First, we are interested in the nonlinear cubic Schrödinger equation on the Heisenberg group, which displays total lack of dispersion. We know that there exist ground state radial traveling waves solutions with arbitrary speed in $(-1,1)$, and we establish orbital stability properties for these traveling waves, starting from a limiting system obtained when the speeds tends to 1. This limiting system plays the same role as the Szegő equation towards the cubic half-wave equation. Then, we establish a low regularity almost-sure local well-posedness theorem for the randomized Cauchy problem associated

to a close Schrödinger-Grushin equation for a large family of initial data.

Finally, we analyze the solutions to two equations related to the Benjamin-Ono equation, by using the integrability properties of the Benjamin-Ono equation. For the third order equation in the Benjamin-Ono hierarchy, we study the Cauchy problem, characterize the traveling wave solutions and determine their orbital stability properties. For a Benjamin-Ono equation with a damping term on the smallest Fourier modes cos and sin, we describe the weak limits points of the trajectories in infinite time, then prove the boundedness of higher order Sobolev norms in infinite time.