Integrable systems and gauge/string dualities

Mikhail Alfimov

THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à l'École Normale Supérieure Paris

Systèmes integrablés et dualités jauge/corde
Integrable systems and gauge/string dualities

École doctorale n°564

PHYSIQUE EN ÎLE-DE-FRANCE

Spécialité PHYSIQUE

COMPOSITION DU JURY :

Emery Sokatchev
Université Savoie Mont Blanc, Président

Vladimir Kazakov
ENS Paris, Directeur de these

Gregory Korchemsky
IPhT CEA Saclay, CoDirecteur de these

João Penedones
EPFL, Examinateur

Simon Caron-Huot
McGill University, Rapporteur

Roberto Tateo
Turin University, Rapporteur

Soutenue par **Mikhail Alfimov**
le 29.10.2018

Dirigée par **Vladimir Kazakov et Gregory Korchemsky**

ENS | PSL RESEARCH UNIVERSITY PARIS

Acknowledgements

First of all I wish to express all my respect and gratitude to my PhD adviser Vladimir Kazakov. At the time of our acquaintance I was thinking about integrable structure in the BFKL limit of QCD. Then I discovered Volodya's article on the solution of AdS/CFT Y-system, the topic I was interested in. Alexander Belavin introduced me to him and since then our fruitful joint work started. Volodya showed me the fascinating and charming world of integrable quantum field theory. After I became his PhD student he introduced me to the theoretical physics community from which I benefited a lot in my scientific efforts. In addition I would like to point out Volodya's confidence in the success of our projects, which in my opinion allowed us to reach the goals we set.

In the period my presence in Saclay I have to mention about the great scientific atmosphere during the discussions with my co-adviser Gregory Korchemsky. The outstanding clearness and wideness of Grisha's knowledge of the BFKL physics allowed me to explore a lot of new facets in my domain of research.

Also I am particularly grateful to Nikolay Gromov, who taught me a lot in integrability theory and computational methods. His help allowed me to reach a new level of ability to calculate in Mathematica. Our joint works lie in the basis of this thesis.

I wish to thank for inspiring and stimulating discussions my colleagues Fedor Levkovich-Maslyuk, Alexey Litvinov, Benjamin Basso, Mikhail Bershtein, Alexander Belavin, Sergei Lukyanov, Evgeny Sobko, Konstantin Zarembo, Ivan Kostov, Didina Serban, Zoltan Bajnok and Vincent Pasquier.

Last but not least, I would like to express gratitude to my friends and family, especially my mother Tatiana and my grandmother Rimma, whose constant and kind support helped me to finish the work on this thesis.

Contents

Chapter 1

Introduction

$\mathcal{N} = 4$ Super-Yang-Mills theory has been playing an important role in our understanding of Quantum Field Theories, especially in an AdS/CFT context. Due to the Kotikov-Lipatov maximal transcendentality principle [1, 2] some of the results obtained in this theory can be directly exported to more realistic planar QCD. In this work we describe how to efficiently perform calculations in this theory for one of the key QCD observables – BFKL spectrum, using integrability at any value of the 't Hooft coupling constant λ, which was discovered initially by L.N. Lipatov in the LO BFKL spectrum [3], and developed far beyond the perturbative regime in the $\mathcal{N} = 4$ SYM in recent years. Lev Nikolaevich was one of the main driving forces behind this progress and it is deeply saddening for us to know that he left us in September 2017.

In the beginning we are going to briefly describe the meaning of the quantities studied in the present work in the context of high energy scattering. The total cross-section $\sigma(s)$ for the high-energy scattering of two colorless particles A and B can be written as [4]

$$\sigma(s) = \int \frac{d^2q d^2q'}{(2\pi)^2 q^2 q'^2} \Phi_A(q)\Phi_B(q') \int\limits_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(q, q') \,, \tag{1.1}$$

where $\Phi_i(q_i)$ are the impact factors, $G_\omega(q, q')$ is the t-channel partial wave for the gluon-gluon scattering, $s_0 = |q||q'|$ and depends on the transverse momenta and $s = 2p_A p_B$, where p_A and p_B are the 4-momenta of the particles A and B respectively. For the t-channel partial wave the following Bethe-Salpeter equation holds

$$\omega G_\omega(q, q_1) = \delta^{D-2}(q - q_1) + \int d^{D-2}q_2 K(q, q_2) G_\omega(q_2, q_1) \,. \tag{1.2}$$

It appears to be possible to classify the eigenvalues ω of the BFKL kernel K using two quantum numbers: integer n (conformal spin) and real ν

$$\omega = \omega(n, \nu) \,. \tag{1.3}$$

The function $\omega(n, \nu)$ is called the Pomeron eigenvalue of the BFKL kernel or just the BFKL Pomeron eigenvalue and its values for different n and ν constitute the BFKL spectrum. For the phenomenological applications of the BFKL kernel eigenvalues with

non-zero conformal spin see [5]. The object $\omega(n,\nu)$[1] in the planar $\mathcal{N}=4$ SYM will be studied in this work by means of integrability.

Before turning to the consideration of the integrable structure of the 4d gauge theories let us briefly remind the action of $\mathcal{N}=4$ SYM and how the states in this theory are characterized. The action calculated in [6] is given by

$$S=\int d^4x\left\{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\frac{1}{2}D_\mu\phi^a_{ij}D^\mu\phi^{aij}+i\bar{\chi}^a_i\gamma^\mu D_\mu L\chi^{ai}-\right.$$
$$\left.-\frac{i}{2}g_{\rm YM}f_{abc}\left(\bar{\tilde{\chi}}^{ai}L\chi^{jb}\phi^c_{ij}-\bar{\chi}^a_iR\tilde{\chi}^b_j\phi^{ijc}\right)-\frac{1}{4}g^2_{\rm YM}\left(f_{abc}\phi^b_{ij}\phi^c_{kl}\right)\left(f_{ade}\phi^{ijd}\phi^{kle}\right)\right\}\ . \quad (1.4)$$

All the fields in the action (1.4) are decorated with the index of the adjoint representation of the gauge group $SU(N)$, which runs from 1 to $(N^2-1)/2$. In what follows we designate the $SU(N)$ structure constant by f_{abc}. The field content of the theory includes the gauge field A^a_μ, for which $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g_{YM}f_{abc}A^b_\mu A^c_\nu$. Then, there are the scalar fields, described by an antisymmetric matrix of complex fields ϕ^a_{ij} with $i,j=1,\dots,4$ and i and j indices being rotated by the R-symmetry group $SU(4)$ and subject to the conditions

$$\phi^{jka}=\frac{1}{2}\epsilon^{jklm}\phi^a_{lm}=\left(\phi^a_{jk}\right)^*\ . \quad (1.5)$$

Imposition of the conditions (1.5) leads to only 3 independent components in ϕ^a_{ij}, which are usually denoted as $X=\phi^a_{12}T^a$, $Y=\phi^a_{13}T^a$ and $Z=\phi^a_{14}T^a$, where T^a are the $SU(N)$ generators. D_μ is a covariant derivative in the adjoint representation and acts on the field as $D_\mu\mathcal{F}^a=\partial_\mu\mathcal{F}^a-g_{\rm YM}f_{abc}A^b_\mu\mathcal{F}^c$. Also, there are four 4-component spinors χ^i, $i=1,\dots,4$ with the projectors $L=(1+\gamma_5)/2$ and $R=(1-\gamma_5)/2$, which transform in the fundamental representation of the R-symmetry group $SU(4)$. In addition, the spinor $\tilde{\chi}_i=C(\bar{\chi}^i)^t$ transforming in the antifundamental of $SU(4)$, where $C=-i\gamma^2\gamma^0$ is the charge conjugation matrix. The t'Hooft coupling constant is connected with the Yang-Mills coupling constant as $\lambda=g^2_{\rm YM}N$. In what follows we are going to utilize the coupling constant $g=\sqrt{\lambda}/(4\pi)$.

The symmetry group of the action (1.4) is the superconformal group $PSU(2,2|4)$. The states in $\mathcal{N}=4$ SYM are labelled by the quantum numbers which are the values of the Cartan charges of the corresponding Lie superalgebra $\mathfrak{psu}(2,2|4)$. They are usually written as $(\Delta,S_1,S_2|J_1,J_2,J_3)$, where Δ is the dimension, S_1 and S_2 are called spins and J_i, $i=1,2,3$ correspond to the R-symmetry subgroup $SU(4)$ of the whole superconformal group.

The study of integrable structures in 4d gauge theory has long and interesting history of development. Integrability in QCD and supersymmetric Yang-Mills theories appeared in two contexts. First, in the gauge theory, namely QCD, the Bartels-Kwiecinski-Praszalowicz (BKP) equation [7,8] for multi-reggeon states was reformulated by L.N. Lipatov [3] as the model with holomorphic and antiholomorphic hamiltonians, which has a set of mutually commuting operators originating from the monodromy matrix satisfying the Yang-Baxter equation. After that L.D. Faddeev and G.P. Korchemsky in [9] proved this model to be completely integrable and equivalent to the spectral problem for $SL(2,\mathbb{C})$ XXX Heisenberg spin chain. Then, in the context of high-energy scattering

[1] In [4] the function ω is used with the different argument $\gamma=1/2+i\nu$.

there was considered a certain class of light-cone operators in QCD and supersymmetric Yang-Mills theories and in [10–13] the problem of finding the anomalous dimensions of the light-cone operators was formulated in terms of $SL(2,\mathbb{R})$ Heisenberg spin chain.

The other achievement was that the maximally supersymmetric $\mathcal{N}=4$ Yang-Mills theory in 4 dimensions, which is dual to $\mathrm{AdS}_5\times\mathrm{S}^5$ type IIB superstring theory was shown to be integrable [14, 15]. The study of the integrability structure of the latter theory allowed to explore its spectrum in the non-perturbative regime. The solution to the spectral problem was formulated in terms of the Quantum Spectral Curve (QSC) [16, 17] (for the recent reviews see [18] and [19]), following a long and successful study of this problem during the last decade and a half [15]. The QSC approach has already a history of a number of non-trivial tests and applications. In the weak coupling limit, the one-loop dimensions for twist-L operators of the type tr $ZD_+^SZ^{L-1}$ of the $\mathfrak{sl}(2)$ sector (these operators are identified with the states with the Cartan charges $S_1=S$, $S_2=0$, $J_1=L$ and $J_2=J_3=0$) were reproduced [16] and then the method was applied for calculating the dimension of Konishi operator at 10 loops [20]. For the small S expansion of anomalous dimension of twist-2 operator $\gamma=f_1(\lambda)S+f_2(\lambda)S^2+\mathcal{O}(S^3)$, the slope function f_1 [21], exact at any 't Hooft coupling λ, was reproduced from the QSC and the slope-to-slope function f_2 was then found in [22]. The results for the cusp anomalous dimension at small angle of the cusp, known from localization [23] and Thermodynamic Bethe Ansatz [22, 24, 25], was reproduced in an elementary way from QSC in [16]. The QSC method was generalized to the case of ABJM theory [26, 27] which allowed the efficient calculation of the ABJM slope function and helped to identify the mysterious interpolating function fixing the dependence of dispersion relation on the 't Hooft coupling λ [28][2] and gave the last missing element in the solution of the spectral problem for this model.

Nevertheless, until recently it was not known how to build the bridge between the integrability in the BFKL limit and integrability found in the AdS/CFT framework. In [30] the 4-loop Asymptotic Bethe Ansatz (ABA) contribution to the anomalous dimension of the twist-2 $\mathfrak{sl}(2)$ operators was analytically continued to the non-integer spins and compared with the corresponding prediction from the BFKL Pomeron eigenvalues. This analytic continuation to non-integer spins was incorporated into the QSC formalism in [22] for twist-2 operators from the $\mathfrak{sl}(2)$ sector and in this work we explain how to derive the Faddeev-Korchemsky Baxter equation [9] for Lipatov $SL(2,\mathbb{C})$ spin chain correctly reproducing the leading order (LO) BFKL Pomeron eigenvalue. In addition, QSC allowed to calculate analytically [31] the previously unknown next-to-next-to-leading order (NNLO) BFKL eigenvalue in the $\mathcal{N}=4$ supersymmetric Yang-Mills theory. At the same time, a very efficient numerical algorithm was constructed in [32], which allows to study not only the BFKL limit of the spectrum of the theory, but the whole anomalous dimension of a given operator for arbitrary values of the charges.

Let us consider the twist-2 $\mathfrak{sl}(2)$ operators of the form

$$\mathcal{O}=\mathrm{tr}ZD_+^SZ+(\text{permutations}) \tag{1.6}$$

and remember that from the perturbative calculations in the gauge theory for the case of even integer S we know the dimension of these operators Δ as a function of S up to several loops order. In the QSC framework the solution of the Baxter equation for

[2]Tested by a heroic strong coupling two loop calculation in [29].

the spectrum of such operators in the case of zero conformal spin n and integer even spins S was obtained in [16]. Then in [33,34] there was found the solution of this Baxter equation valid for arbitrary spin S, which leads to the anomalous dimension of the twist-2 $\mathfrak{sl}(2)$ operators analytically continued for non-integer spin S. After making this analytic continuation in the BFKL regime we are able to exchange the roles of Δ and S obtaining $S+1=\omega(n=0,\nu)$, where $\nu=-i\Delta/2$ and Δ is the dimension of the operator in question.

Here we demonstrate another application of the QSC to an important problem – the calculation of conformal dimensions Δ of the operators of a type (1.6) belonging to the $\mathfrak{sl}(2)$ sector in the BFKL limit, corresponding to a double scaling regime of small 't Hooft constant $g\equiv\sqrt{\lambda}/(4\pi)\to 0$ and the Lorentz spin S approaching to -1, whereas the ratio $\Lambda\equiv g^2/(S+1)$ is kept fixed. We will reproduce the famous formula for this dimension, obtained in [2,35,36] from the direct re-summation of Feynman graphs

$$\frac{1}{4\Lambda}=-\psi\left(\frac{1}{2}-\frac{\Delta}{2}\right)-\psi\left(\frac{1}{2}+\frac{\Delta}{2}\right)+2\psi(1)+\mathcal{O}(g^2)\,, \tag{1.7}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$. Remarkably, this result is also known to be valid for the pure Yang-Mills theory in the planar limit since only the gluons appear inside the Feynman diagrams of $\mathcal{N}=4$ SYM at LO! In [37] the conformal invariance of the BFKL kernel with the characteristic function (1.7) was shown. This formula was a result of a long and remarkable history of applications of the BFKL method to the study of Regge limit of high energy scattering amplitudes and correlators in QCD [9,38–41] and in the $\mathcal{N}=4$ SYM theory [2,30,42–48]. The effective action for the high-energy processes in nonabelian gauge theories was derived in [49]. Not a long time ago certain scattering amplitudes describing the adjoint sector (single reggeized gluon) were computed by means of all loop integrability in the BFKL limit [50] in the integrable polygonal Wilson loop formalism [51].

To recover the formula (1.7) from the QSC approach, we will have to compute certain quantities not only in the LO, but also in the NLO. In particular, we extract from the analytic Q-system describing QSC the Baxter-Faddeev-Korchemsky equation for the Pomeron wave function [3,9,52–55] in the LO and generalize it to the NLO. Some other ingredients of the QSC, entering the underlying so-called $\mathbf{P}\mu$- and $\mathbf{Q}\omega$-equations, will be determined in the LO or even up to NLO. These calculations lay out a good basis for the construction of a systematic BFKL expansion of this anomalous dimension in planar $\mathcal{N}=4$ SYM, now known up to NNLO correction to (1.7) from the direct computation of [2,4,56] and from the QSC [31].

Our method, designed here for the case of Pomeron singularity (a bound state of two reggeized gluons) should be applicable to the study of a bound state of L reggeized gluons as well.

Let us stress that one of the main results of this work – the correct reproduction of the formula (1.7) from the QSC – is a very non-trivial test of the QSC as well as of the whole integrability approach to planar AdS/CFT spectrum. It sums up an infinite number of the so-called wrapping corrections infinitely many loops for a highly non-trivial non-BPS quantity.

In the present work we consider the generalization allowing for an arbitrary value of the conformal spin. Namely, we consider the operators

$$\mathcal{O}=\mathrm{tr}ZD_+^{S_1}\partial_\perp^{S_2}Z+(\text{permutations})\,. \tag{1.8}$$

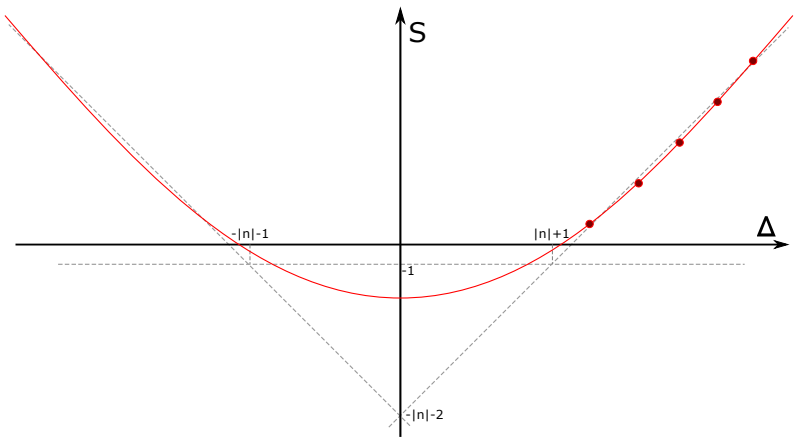

Figure 1.1: Trajectory of the length-2 operator for conformal spin $n = S_2$ as a function of the full dimension Δ. The dots correspond to local operators $\mathrm{tr} Z D_+^S \partial_\perp^n Z$. For the local operators $S + n$ is restricted to be even.

For the operators (1.8) we follow the same strategy as for the case of zero conformal spin. Analogously to that case we build the analytic continuation in the spins S_1 and S_2, which are identified with the spin S and conformal spin n respectively. Let us illustrate this analytic continuation with the Figure 1.1. The physical operators, for which the sum of non-negative integer $S = S_1$ and $n = S_2$ is even, are depicted with the dots. Then, flipping the roles of the dimension Δ and $S = S_1$ we can reach the BFKL regime described by the quantity $\omega(n = S_2, \nu) = S_1 + 1$, where $\nu = -i\Delta/2$.

The way to proceed with the problem in question is to first generalize the QSC approach to non-integer values of S_1 (as was already done in [22]) and then also to non-integer values of S_2. We describe the technical details of this procedure in the Section 4.1. This allows to treat $\omega(n, \nu)$ as an analytic function of both its parameters, which simplifies both analytical and numerical considerations. This gives a universal framework for studying the BFKL spectrum in full generality for all values of the parameters on equal footing within the extended QSC formalism.

Having formulated the problem as an extension of the initial QSC, a number of methods, initially developed for the local operators, became available for the BFKL problem. In particular we are enabled to employ a very powerful numerical algorithm [32] after some modifications. As we take the spins S_1 to be continuous variable we can consider instead of the function $\Delta(S_1, S_2)$ the function $S_1(\Delta, S_2)$. Then, using the algorithm we build the operator trajectories for different values of conformal spin S_2 and the dependencies of the spin S_1 on the coupling constant g for different values of conformal spin S_2 and dimension Δ (including a particular interesting intercept function corresponding to $\Delta = 0$). Having the numerical results for the operator trajectories we were able to fit the numerical values of the BFKL kernel eigenvalues[3], which were confirmed in [58] using a different method.

Another method available within the QSC formalism is an efficient perturbative expansion developed in [20, 31, 59–61]. We applied this method to find the value of the

[3]See M.Alfimov's presentation at the GATIS Training Event at DESY [57].

Pomeron intercept for an arbitrary value of conformal spin up to 3 loops. Our result is in full agreement with [58] at the NNLO level, but we also give a prediction for the next (NNNLO) order.

Then, we found and studied in detail a particularly interesting point in the space of parameters of the BFKL Pomeron. This is the "BPS" point $\Delta = 0$ and $n = 1$. As we have confirmed both numerically and analytically, the operator trajectory goes through the point $S = -1$, $n = 1$ and $\Delta = 0$ for any value of the coupling constant g. Studying the vicinity of this point we were able to find two non-perturbative quantities: "slope-to-intercept function" and "curvature function". The first function is the first derivative of $S(\Delta, n)$ with respect to n at the point $\Delta = 0$, $n = 1$ and the second function is the second derivative of $S(\Delta, n)$ with respect to Δ at the same point. We used the methods developed in [22] to compute analytically these quantities non-perturbatively to all orders in g.

Finally, we were able to identify the intercept function in the strong coupling expansion up to the 4th order. To obtain it we utilized the dependencies of the intercept on the coupling constant calculated by the QSC numerical method. By conducting the numerical fit of these dependencies for different values of conformal spin n we predict the formula for the intercept strong coupling expansion up to the 4th order for arbitrary conformal spin.

Let us present a brief summary of the quantities we calculated. They include the NNLO intercept function (4.182) and the non-rational part of the NNNLO intercept function (4.184). The other quantities we computed exactly to all orders in the 't Hooft coupling constant are the slope-to-intercept (4.227) and the curvature (4.315) functions with the strong coupling expansions of these functions given by (4.233) and (4.320) respectively. In addition, there was written the strong coupling expansion (4.322) of the intercept function for arbitrary conformal spin n. We also implemented the numerical method for finding the eigenvalues at arbitrary values of the parameters in `Mathematica`, the corresponding files `code_for_arxiv.nb` and `BFKLdata.mx` can be found in the attachments to the arXiv submission [62]. See `description.txt` file for the description.

This work is organized as follows. In the Chapter 2 we report a brief description of the Quantum Spectral Curve method. It includes the Section 2.1 on the Y- and T-systems, a sketchy derivation of the $\mathbf{P}\mu$-system in the Section 2.2, motivation from the weak coupling in 2.3 and the QSC generalities in 2.4. The Chapter 3 is devoted to the application of the QSC to the twist-2 operators with zero conformal spin and consists of the Section 3.1 on the weak coupling limit and 3.2 on the BFKL limit. In the Chapter 4 we turn to a more general class of the length-2 operators with non-zero conformal spin and address several issues there. In the Section 4.1 we give the general introduction into the QSC approach, extending it to the situation when both spins are non-integer. The Section 4.2 describes our numerical results. The Section 4.3 contains the weak coupling analysis. In the Section 4.4 we analyze the expansion near the BPS point to find the non-perturbative quantities such as the slope-to-intercept and the curvature functions. In the Section 4.5 we analyze the Pomeron intercept at strong coupling.

Chapter 2

Quantum Spectral Curve for $\mathcal{N} = 4$ SYM

In the present Chapter we are going to cover several topics. First of all, in the two initial Sections we will explain the derivation of the QSC method, which we want to utilize to find the spectrum of our operators of interest in $\mathcal{N} = 4$ SYM. Second, we motivate the QSC method from considering $\mathfrak{sl}(2)$ Heisenberg spin chain. And thirdly we explain the formulation of the QSC suitable for analysis of the spectrum of $\mathcal{N} = 4$ SYM for a general state.

Before starting to explain the QSC method we would like to introduce several convenient notations. The shifts of the functions are denoted as

$$f\left(u + \frac{in}{2}\right) = f^{[n]}(u) \ . \tag{2.1}$$

For the case $n = \pm 1$ we introduce the following short notation

$$f\left(u \pm \frac{i}{2}\right) = f^{\pm}(u) \ . \tag{2.2}$$

In addition, analytic continuation of the functions under the cut on the real axis is denoted by tilde $\tilde{f}(u)$. Thus, as the cuts we consider are quadratic we have

$$\tilde{\tilde{f}}(u) = f(u) \ . \tag{2.3}$$

Besides, we are going to use LHS and RHS for left- and right-hand side respectively.

2.1 Y- and T-systems for $\mathcal{N} = 4$ SYM

First we would like to shortly remind the history of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ integrability. Since the discovery of the AdS/CFT correspondence [63–65] the integrable structures were found both on the side of the gauge theory, which is $\mathcal{N} = 4$ SYM, and on the side of the string theory, which is the $\mathrm{AdS}_5 \times \mathrm{S}^5$ IIB superstring. In $\mathcal{N} = 4$ SYM the dilatation operator was shown [14] to be integrable. On the other side, the classical integrable structure [66] and the S-matrix [66–68] of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. Using this S-matrix it is possible to find all-loop asymptotic Bethe ansatz equations [69] and

Lüscher corrections [70–73] from the TBA equations, from which the Y-system for the AdS_5/CFT_4 case was first conjectured in [74]

$$Y^+_{a,s}Y^-_{a,s} = \frac{(1+Y_{a,s+1})\,(1+Y_{a,s-1})}{\left(1+\frac{1}{Y_{a+1,s}}\right)\left(1+\frac{1}{Y_{a-1,s}}\right)}\,, \tag{2.4}$$

where the Y-functions are depicted by the circles on the T-hook (see the Figure 2.1). For

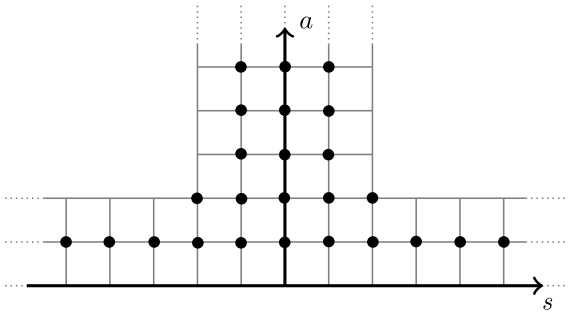

Figure 2.1: T-hook for AdS_5/CFT_4 Y- and T-systems.

the values of a and s outside the circular nodes the Y-functions are equal to the following (see [74])

$$Y_{0,s} = \infty\ ,\ s \in \mathbb{Z}\ ,\quad Y_{2,s} = \infty\ ,\ |s| \geq 3\ ,\quad Y_{a,\pm 2} = 0\ ,\ a \geq 3\ . \tag{2.5}$$

The Y-functions are defined on the Riemann sheet with the long cuts $(-\infty, -2g + ik/2] \cup [2g + ik/2, +\infty)$ for some k. The equations (2.4) have to be supplemented with the three discontinuity relations, first of which is

$$\operatorname{disc}\log\left[Y^{[2n]}_{1,1}Y^{[2n]}_{2,2}\prod_{a=1}^{n}\left(1+Y^{[2n-a]}_{a,0}\right)\right] = 0\ ,\quad n \geq 1\ , \tag{2.6}$$

and the other two together with (2.6) can be found in [75, 76].

The Y-functions in the AdS/CFT can be parametrized in terms of the Hirota T-functions in the following way

$$Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}\,, \tag{2.7}$$

where the indices of the T-function go over all the nodes of the T-hook and the indices of the Y-function go over the circular nodes of the same T-hook. After the substitution of (2.7), the Hirota equation looks as follows

$$T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}\ . \tag{2.8}$$

The statement above means that if the set of the T-functions solves the equations (2.8), then the Y-functions given by (2.7) represent a solution of the (2.4).

It should be noted that the Hirota equation (2.8) possesses the following gauge symmetry [77]

$$T_{a,s} \to g_1^{[a+s]}g_2^{[a-s]}g_3^{[-a-s]}g_4^{[-a+s]}T_{a,s}\ . \tag{2.9}$$

From [77] we know that there exists such a gauge for the T-functions, which satisfies the first discontinuity relation (2.6) and for which the T-functions $\mathbf{T}_{a-1,0}$, $\mathbf{T}_{a,\pm1}$ and $\mathbf{T}_{a+1,\pm2}$ are analytic inside the strip $-a/2 < \mathrm{Im}\, u < a/2$ for $a \geq 1$. In addition, for all $n \geq 3$ $\mathbf{T}_{2,n} = \mathbf{T}_{n,2}$, $\mathbf{T}_{2,-n} = \mathbf{T}_{n,-2}$, $\mathbf{T}_{0,0}$ is i-periodic and $\mathbf{T}_{0,s} = \mathbf{T}_{0,0}^{[+s]}$. The other two discontinuity relations mentioned together with (2.6) are naturally encoded in the other gauge

$$\mathbb{T}_{a,s} \equiv (-1)^{as} \mathbf{T}_{a,s} \left(\mathbf{T}_{0,0}^{[a+s]}\right)^{\frac{a-2}{2}} \ . \tag{2.10}$$

Namely, they lead to $\mathbb{T}_{0,s} = 1$ and on the Riemann sheet with the short cuts $\hat{\mathbb{T}}_{1,s}$ has only two of them for $|s| > 1$ and $\hat{\mathbb{T}}_{2,s} = \hat{\mathbb{T}}_{1,1}^{[+s]} \hat{\mathbb{T}}_{1,1}^{[-s]}$ (hat and check denote the function on the Riemann sheet with the short and long cuts respectively).

Its solution for the right band of the T-hook is parametrized by two functions $\mathbf{P}_1$ and $\mathbf{P}_2$ for $s \geq 1$

$$\mathbb{T}_{1,s} = \mathbf{P}_1^{[+s]} \mathbf{P}_2^{[-s]} - \mathbf{P}_2^{[+s]} \mathbf{P}_1^{[-s]} = \begin{vmatrix} \mathbf{P}_1^{[+s]} & \mathbf{P}_1^{[-s]} \\ \mathbf{P}_2^{[+s]} & \mathbf{P}_2^{[-s]} \end{vmatrix} \ , \tag{2.11}$$

where these $\mathbf{P}$-functions have only one short cut from $+2g$ to $-2g$ on their defining sheet (here we omitted hat for brevity). We are also able to write down a similar solution for the left band of the T-hook, which is parametrized by the functions $\mathbf{P}^3$ and $\mathbf{P}^4$ for $s \leq -1$

$$\mathbb{T}_{1,s} = \mathbf{P}^{4[+s]} \mathbf{P}^{3[-s]} - \mathbf{P}^{3[+s]} \mathbf{P}^{4[-s]} = \begin{vmatrix} \mathbf{P}^{4[+s]} & \mathbf{P}^{4[-s]} \\ \mathbf{P}^{3[+s]} & \mathbf{P}^{3[-s]} \end{vmatrix} \ . \tag{2.12}$$

Our aim is to reveal the whole analytic structure of these functions. In [77] it was shown that if we also use the function $\mu_{12} = (\mathbf{T}_{0,1})^{\frac{1}{2}}$, which is i-periodic on the Riemann sheet with the long cuts, together with (2.11) and (2.12) we are able to express all the T-functions in terms of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}^3$, $\mathbf{P}^4$ and μ_{12}. However, we have to derive the set of restrictions on $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}^3$, $\mathbf{P}^4$ and μ_{12} to resolve the analyticity requirements for the $\mathbf{T}$-functions mentioned in this Section in the paragraph before (2.10) in the upper band of the T-hook on the Fig. 2.1. To achieve this and simplify our reasonings, we turn to the specific left-right symmetric states.

2.2 $\mathbf{P}\mu$-system in the left-right symmetric case

As we are interested in the case of twist-2 operators with the charges $S_2 = 0$, $J_1 = 2$ and $J_2 = J_3 = 0$, this results in the left-right symmetry of the T-system, namely

$$\mathbb{T}_{a,s}(u) = \mathbb{T}_{a,-s}(u) \ . \tag{2.13}$$

This symmetry (2.13) can be motivated from the strong coupling limit. In this limit the T-functions are equal to the super-characters of $SU(2,2|4)$ group [78] and (2.13) is fulfilled. The symmetry (2.13) is satisfied by the identification $\mathbf{P}^4 = \mathbf{P}_1$ and $\mathbf{P}^3 = -\mathbf{P}_2$.

Let us introduce two additional functions $\mathbf{P}_3$ and $\mathbf{P}_4$, which also have only one short cut $[-2g, 2g]$ on their defining sheet and an infinite tower of short cuts if we analytically continue under this cut on the real axis. We assume that if we go under the cut the result $\tilde{\mathbf{P}}_a$ can be expressed as a linear combination of initial $\mathbf{P}$-functions

$$\tilde{\mathbf{P}}_a = \mu_{ab} \chi^{bc} \mathbf{P}_c \ , \tag{2.14}$$

where $\mu_{ab}(u)$ are subject to the conditions $\mu_{12}\mu_{34}-\mu_{13}\mu_{24}+\mu_{14}^2=1$ and $\mu_{23}=\mu_{14}$ and χ^{bc} are antisymmetric matrices and $\mu_{ab}(u)$ has an infinite ladder of cuts $[-2g+in,2g+in]$, $n\in\mathbb{Z}$, while $\chi^{14}=-\chi^{23}=\chi^{32}=-\chi^{41}=-1$ with the other entries being zeroes. Therefore, the analytic continuation $\tilde{\mathbf{P}}_a$ also has the same infinite ladder of cuts on its sheet. As μ_{ab} is i-periodic on the sheet with the long cuts the analytic continuation of μ_{ab} on the sheet with the short cuts is given by the relation

$$\tilde{\mu}_{ab}(u)=\mu_{ab}(u+i)\ . \tag{2.15}$$

Therefore $\tilde{\mu}_{ab}(u)$ has the same cut structure as $\mu_{ab}(u)$.

Because the branch points are quadratic, going under the cut twice gives the same result

$$\mathbf{P}_a=\tilde{\tilde{\mathbf{P}}}_a=\tilde{\mu}_{ab}\chi^{bc}\tilde{\mathbf{P}}_c\ . \tag{2.16}$$

After this, using the relation $\mu_{ab}\chi^{bc}\mu_{cd}\chi^{de}=\tilde{\mu}_{ab}\chi^{bc}\tilde{\mu}_{cd}\chi^{de}=\delta_a^e$, we get two equations

$$\begin{aligned}\tilde{\mathbf{P}}_a&=\mu_{ab}\chi^{bc}\mathbf{P}_c\ ,\\ \tilde{\mathbf{P}}_a&=\tilde{\mu}_{ab}\chi^{bc}\mathbf{P}_c\ ,\end{aligned} \tag{2.17}$$

from which we immediately obtain

$$(\tilde{\mu}_{ab}-\mu_{ab})\chi^{bc}\mathbf{P}_c=0\ . \tag{2.18}$$

Or, in other words, we can formulate the following system of equations

$$\begin{cases}\tilde{\mu}_{ab}\chi^{bc}\mathbf{P}_c=\mu_{ab}\chi^{bc}\mathbf{P}_c\ ,\\ \tilde{\mu}_{ab}\chi^{bc}\tilde{\mathbf{P}}_c=\mu_{ab}\chi^{bc}\tilde{\mathbf{P}}_c\ ,\\ \tilde{\mu}_{ab}=-\tilde{\mu}_{ba}\ ,\\ \tilde{\mu}_{ab}\chi^{bc}\tilde{\mu}_{cd}\chi^{de}=\delta_a^e\ .\end{cases} \tag{2.19}$$

Note, it is possible that not all of the equations (2.19) are independent. If we assume, that the matrix of the system of the first two equations

$$\begin{pmatrix}-\mathbf{P}_4 & \mathbf{P}_3 & -\mathbf{P}_2 & \mathbf{P}_1\\ -\tilde{\mathbf{P}}_4 & \tilde{\mathbf{P}}_3 & -\tilde{\mathbf{P}}_2 & \tilde{\mathbf{P}}_1\end{pmatrix} \tag{2.20}$$

has rank 2, or, equivalently, vectors $\mathbf{P}_a$ and $\tilde{\mathbf{P}}_a$ are linear independent, then, the solution to the first two equations from (2.19) is given by

$$\tilde{\mu}_{ab}=\mu_{ab}+c_{1a}\mathbf{P}_b+c_{2a}\tilde{\mathbf{P}}_b\ . \tag{2.21}$$

From the requirement $\tilde{\mu}_{aa}=0$ for all a we get

$$c_{2a}=-\frac{\mathbf{P}_a}{\tilde{\mathbf{P}}_a}c_{1a} \tag{2.22}$$

and

$$\tilde{\mu}_{ab}=\mu_{ab}+\frac{c_{1a}}{\tilde{\mathbf{P}}_a}(\tilde{\mathbf{P}}_a\mathbf{P}_b-\mathbf{P}_a\tilde{\mathbf{P}}_b)\ . \tag{2.23}$$

Using the equation $\tilde{\mu}_{ab} = -\tilde{\mu}_{ba}$ for $a \neq b$ we obtain

$$\frac{c_{1a}}{\tilde{\mathbf{P}}_a} = \frac{c_{1b}}{\tilde{\mathbf{P}}_b} \,. \tag{2.24}$$

Therefore, the result is

$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u) + e(u)(\mathbf{P}_a(u)\tilde{\mathbf{P}}_b(u) - \mathbf{P}_b(u)\tilde{\mathbf{P}}_a(u)) \,. \tag{2.25}$$

Then, using the fact that $\mathbf{P}_a\chi^{ab}\mathbf{P}_b = \tilde{\mathbf{P}}_a\chi^{ab}\mathbf{P}_b = \tilde{\mathbf{P}}_a\chi^{ab}\tilde{\mathbf{P}}_b = 0$ it is easy to check, that the fourth equation from (2.19) is also satisfied by the result (2.25).

Now it remains to fix the coefficient $e(u)$. First, let us calculate the product of $Y_{1,1}Y_{2,2}$

$$Y_{1,1}Y_{2,2} = \frac{\check{\mathbb{T}}_{1,0}(u)}{\check{\mu}_{12}(u)} \,. \tag{2.26}$$

We are going to calculate this product for $-i/2 < \operatorname{Im} u < i/2$. Let's consider the domain $0 < \operatorname{Im} u < i/2$. After some calculations we find

$$\check{\mathbb{T}}_{1,0} = \check{\mu}_{1,2} + \frac{\check{\mathbb{T}}^{+}_{1,1}\check{\mathbb{T}}^{-}_{1,1}}{\check{\mathbb{T}}_{1,2}} - \frac{\check{\mathbb{T}}^{+}_{2,2}\check{\mathbb{T}}^{-}_{2,2}}{\check{\mathbb{T}}_{1,2}\check{\mathbb{T}}_{2,3}} \,. \tag{2.27}$$

The subtlety here is that for $0 < \operatorname{Im} u < i/2$ we have $\check{\mathbb{T}}^{+}_{1,1} \neq \mathbf{P}_1^{++}\mathbf{P}_2 - \mathbf{P}_2^{++}\mathbf{P}_1$ and $\check{\mathbb{T}}^{+}_{2,2} \neq \check{\mathbb{T}}^{[+3]}_{1,1}\check{\mathbb{T}}^{[-1]}_{1,1}$, because we are out of the analyticity strip and have to take an analytical continuation. The result is

$$\begin{aligned} \check{\mathbb{T}}^{+}_{1,1} &= \mathbf{P}_1^{++}\tilde{\mathbf{P}}_2 - \mathbf{P}_2^{++}\tilde{\mathbf{P}}_1 \,, \\ \check{\mathbb{T}}^{+}_{2,2} &= (\tilde{\mathbf{P}}_1\mathbf{P}_2^{--} - \tilde{\mathbf{P}}_2\mathbf{P}_1^{--})\mathbb{T}^{[+3]}_{1,1} \,. \end{aligned} \tag{2.28}$$

Then, calculating with Mathematica we obtain the result for $0 < \operatorname{Im} u < i/2$

$$Y_{1,1}Y_{2,2} = 1 + \frac{\mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1}{\mu_{12}} = 1 - \frac{1}{e(u)} + \frac{\tilde{\mu}_{12}}{e(u)\mu_{12}} \,. \tag{2.29}$$

Analogously for the case $-i/2 < \operatorname{Im} u < 0$ we have

$$\begin{aligned} \check{\mathbb{T}}^{-}_{1,1} &= \tilde{\mathbf{P}}_1\mathbf{P}_2^{--} - \tilde{\mathbf{P}}_2\mathbf{P}_1^{--} \,, \\ \check{\mathbb{T}}^{-}_{2,2} &= (\mathbf{P}_1^{++}\tilde{\mathbf{P}}_2 - \mathbf{P}_2^{++}\tilde{\mathbf{P}}_1)\hat{\mathbb{T}}^{[-3]}_{1,1} \,. \end{aligned} \tag{2.30}$$

And again, calculation with Mathematica gives for $-i/2 < \operatorname{Im} u < 0$

$$Y_{1,1}Y_{2,2} = 1 + \frac{\tilde{\mathbf{P}}_1\mathbf{P}_2 - \tilde{\mathbf{P}}_2\mathbf{P}_1}{\tilde{\mu}_{12}} = 1 - \frac{1}{e(u)} + \frac{\mu_{12}}{e(u)\tilde{\mu}_{12}} \,. \tag{2.31}$$

Then, using the equation from [77], which is valid on the long cut on the real axis,

$$Y_{1,1}Y_{2,2}(u + i0) = \frac{1}{Y_{1,1}Y_{2,2}(u - i0)} \tag{2.32}$$

we obtain

$$\tilde{\mu}_{12} - \mu_{12} = \mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1 \ . \tag{2.33}$$

Therefore

$$Y_{1,1}Y_{2,2}(u) = \frac{\tilde{\mu}_{12}(u)}{\mu_{12}(u)} = \frac{\mu_{12}(u+i)}{\mu_{12}(u)} \ , \tag{2.34}$$

which is consistent with (2.32). Therefore, we can write down

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \ . \tag{2.35}$$

In the work [77] it was shown that the equations (2.35) and (2.19) (remembering that $\mu_{23} = \mu_{14}$) together with the analytic structure of $\mathbf{P}_3$ and $\mathbf{P}_4$ ensure that the analytic conditions for the upper band T-functions $\mathbf{T}_{a-1,0}$, $\mathbf{T}_{a,\pm1}$ and $\mathbf{T}_{a+1,\pm2}$ are satisfied.

Therefore in the left-right symmetric case we can express all the T-functions in terms of the three functions $\mathbf{P}_1$, $\mathbf{P}_2$ and μ_{12} and then, using additional functions $\mathbf{P}_3$ and $\mathbf{P}_4$ and μ_{ab} besides μ_{12} with certain analytic properties and the equations (2.19) and (2.35), all the analytic properties of the T-functions are guaranteed. Now we can turn to the symmetries of the $\mathbf{P}\mu$-system.

2.2.1 Symmetries of the Pμ-system

Let us write down once more the equations of the $\mathbf{P}\mu$-system

$$\begin{aligned} &\tilde{\mathbf{P}}_a = \mu_{ab}\chi^{bc}\mathbf{P}_c \ , \\ &\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \ . \end{aligned} \tag{2.36}$$

The transformation law of the μ-functions, which follows from $\mathbf{P}_a \to (H_B)_a^b\,\mathbf{P}_b$ and the second equation of (2.36), is

$$\mu_{ab} \to (H_B)_a^c\,\mu_{cd}\,(H_B)_b^d \ . \tag{2.37}$$

Note, that to keep the equation $\mu_{ab}\chi^{bc}\mu_{cd}\chi^{de} = \delta_a^e$ satisfied it is sufficient to impose the condition

$$(H_B)_b^a\chi^{bc}(H_B)_c^d = \chi^{ad} \ , \tag{2.38}$$

which means that the matrix H_B is symplectic and $H_B \in \mathrm{Sp}(4,\mathbb{C})$. Therefore, we are interested in the symplectic transformations, which preserve the asymptotics of the $\mathbf{P}$-functions. Without ruining the asymptotics and maintaining the parity of the $\mathbf{P}$-functions, which takes place for the states with $J_1 = 2$ and $J_2 = J_3 = 0$ and is dictated by them, the matrix H_B is

$$H_B = \begin{pmatrix} (H_B)_1^1 & 0 & 0 & 0 \\ 0 & (H_B)_2^2 & 0 & 0 \\ (H_B)_3^1 & 0 & (H_B)_3^3 & 0 \\ 0 & (H_B)_4^2 & 0 & (H_B)_4^4 \end{pmatrix} \tag{2.39}$$

The fact that H_B is symplectic imposes $(H_B)^i_i = 1/(H_B)^{4-i}_{4-i}$ for $i = 3, 4$ and $(H_B)^1_3 = -(H_B)^2_4(H_B)^1_1/(H_B)^2_2$. Thus

$$H_B = \begin{pmatrix} (H_B)^1_1 & 0 & 0 & 0 \\ 0 & (H_B)^2_2 & 0 & 0 \\ -(H_B)^2_4\frac{(H_B)^1_1}{(H_B)^2_2} & 0 & \frac{1}{(H_B)^2_2} & 0 \\ 0 & (H_B)^2_4 & 0 & \frac{1}{(H_B)^1_1} \end{pmatrix} = \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{(H_B)^2_4}{(H_B)^2_2} & 0 & 1 & 0 \\ 0 & (H_B)^2_4 & 0 & 1 \end{pmatrix} \begin{pmatrix} (H_B)^1_1 & 0 & 0 & 0 \\ 0 & (H_B)^2_2 & 0 & 0 \\ 0 & 0 & \frac{1}{(H_B)^2_2} & 0 \\ 0 & 0 & 0 & \frac{1}{(H_B)^1_1} \end{pmatrix} . \quad (2.40)$$

The factorized transformation matrix (2.40) shows that the H-rotation is a superposition of a rescaling of the **P**-functions and a rotation of them, which conserves the asymptotics.

As the **P**-functions have only one short cut on their defining sheet, we are able to introduce the following parametrization of them

$$\mathbf{P}_a = x^{-\tilde{M}_a}\left(g^{-\tilde{M}_a}A_a\left(1+\frac{\delta_{a,4}}{x^2}\right)+\sum_{k=1}^{+\infty}\frac{c_{a,k}}{x^{2k}}\right) , \quad (2.41)$$

which takes into account the parity of the **P**-functions manifesting itself in the even powers in the series in the Zhukovsky variable $x(u)$.

2.3 Motivation from weak coupling

To get an idea of the QSC it is instructive to start from the example of a non-compact, $\mathfrak{sl}(2)$ Heisenberg spin chain with negative values of spins $-s$ (usually denoted as XXX_{-s}), where s is not necessarily integer or half-integer. In particular, the Heisenberg spin chain with $s = 1/2$ describes the twist-2 operators at weak coupling at one loop. Furthermore, we will see that it is also responsible for the BFKL regime of these operators. Let us consider the case with two particles (two nodes of the spin chain) for simplicity. For this integrable model the problem of finding of its spectrum reduces to the Baxter equation

$$T(u)Q(u) + (u - is)^2Q(u - i) + (u + is)^2Q(u + i) = 0 , \quad (2.42)$$

where $T(u)$ is some 2^{nd} order polynomial which encodes the total spin S of the state via $T(u) = -2u^2 + S^2 - S + 4sS + 2s^2$ for zero momentum states. For the case of the one-loop spectrum of the twist-2 operators S corresponds to the operator with S covariant derivatives.

The equation (2.42) is in many respects similar to the usual Schrödinger equation, where $Q(u)$ plays the role of a wave function and $T(u)$ is an external potential. When S is integer one can find a polynomial solution of (2.42) of degree S, which we denote as $Q_1(u)$. The energy of the state is then given by

$$\Delta = 2 + S + 2ig^2\,\partial_u \log\frac{Q_1(u+is)}{Q_1(u-is)}\Big|_{u=0} + \mathcal{O}(g^4) . \quad (2.43)$$

For $s=1/2$ this polynomial can be found explicitly (see, for example, [79]) and it gives for the energy

$$\Delta(S) = 2 + S - 8g^2 \sum_{n=1}^{S} \frac{1}{n} + \mathcal{O}(g^4) \,. \tag{2.44}$$

An important point is that since (2.42) is a second degree finite difference equation there must be another solution to it. It is easy to show, by plugging u^α into the equation and taking $u \to \infty$, that the second linearly independent solution, denoted as $Q_2(u)$ has the asymptotics u^{1-4s-S}. It is clear from here that for $s > 0$ both solutions cannot be polynomial. Unavoidably, $Q_2(u)$ should have infinitely many zeros and poles. To see the positions of these poles we build the Wronskian $Q_{12}(u)$ out of these two solutions

$$Q_{12} = Q_1^+ Q_2^- - Q_1^- Q_2^+ \,. \tag{2.45}$$

As a consequence of (2.42) it satisfies $Q_{12}^+/Q_{12}^- = (u-is)^2/(u+is)^2$ which can be solved to give

$$Q_{12} = \left(\frac{\Gamma(-iu+1/2-s)}{\Gamma(-iu+1/2+s)}\right)^2 \,, \tag{2.46}$$

from where we see explicitly that Q_{12} has second order poles at $u = i(n-s+1/2)$, $n \in \mathbb{Z}_{\geq 0}$ and is analytic in the upper half plane. We also see that Q_2 should have double poles at $u = -i(n-s+1)$.

The normalization in (2.43) is chosen so that for $s=1/2$ it gives precisely the 1-loop dimensions of twist-2 operators with S covariant derivatives and two scalars. To pass to the BFKL regime and take the limit $S \to -1$ we have to analytically continue away from even integer S. The analytic continuation of the energy itself $\Delta(S)$ is naturally given by the following rewriting of (2.43)

$$\Delta(S) = 2 + S - 8g^2 \sum_{n=1}^{\infty} \left(\frac{1}{n+S} - \frac{1}{n}\right) + \mathcal{O}(g^4) \,. \tag{2.47}$$

In this form the sum is meaningful for non-integer S and we also clearly see a pole in this energy at $w = S+1 \to 0$

$$\Delta(-1+w) = 1 + w - g^2 \left(\frac{8}{w} + \mathcal{O}(w^0)\right) + \mathcal{O}(g^4) \,, \tag{2.48}$$

which reproduces the BFKL prediction (1.7) at one loop (which can be found from (1.7) by inverting the series for the expansion $\Delta \to 1$). However, at the level of Q-functions it is not immediately clear how to make the analytic continuation. Indeed, Q_1 as a solution with u^S asymptotics could no longer be polynomial and must also have poles. Requiring the power-like asymptotics the best possible thing to achieve is to cancel the second order pole and build a unique, up to a constant multiplier, Q_1 so that it has only simple poles. The singularities of the both "big" solution Q_1 and "small" solution Q_2 are located at $u = -in - i/2$ for all positive integers n as we discussed before. These poles result in infinities in the expression for the energy (2.43). The way to avoid such infinities is to form a regular combination, still solving the Baxter equation,

$$Q_1(u) + \cosh(2\pi u) Q_2(u) \,, \tag{2.49}$$

where all poles are cancelled, having however an exponential asymptotics. One can show that there is a unique, up to an overall normalization, combination of this form which is regular everywhere in the whole complex plane [33,34]. It automatically gives the correct analytic continuation for the dimension (2.47). In other words, one should find a regular solution of (2.42) with the large positive u asymptotics $u^S + \text{const}\ e^{2\pi|u|}u^{-1-S}$ and plug it into (2.43) to get the correct analytic continuation to non-integer S. We will see how a similar prescription allows to define the QSC for non-physical operators for any S.

2.4 Quantum Spectral Curve – Generalities

The QSC gives a generalization of the above construction to all loops. When we go away from weak coupling regime we start exploring all other degrees of freedom of the dual super-string in 10D. Thus the full symmetry group PSU(2, 2|4) emerges, which simply means that we should consider generalized Baxter equations with $2+2+4 = 8$ Q-functions with one index, which we denote as $\mathbf{P}_a,\ a = 1, \dots, 4$ (S^5 part) and $\mathbf{Q}_i,\ i = 1, \dots, 4$ (AdS_5 part). Out of them, we can form Wronskians like in (2.45) - which give another $8*7/2 = 28$ Q-functions with two indices, then we can iterate the process several times. In total we get 2^8 various Q-functions.

Another effect which happens at finite coupling is that the poles of Q-functions in the lower half-plane, described above, resolve into cuts $[-2g, 2g]$ (where $g = \sqrt{\lambda}/4\pi$).

Finally, we have to introduce new objects – the monodromies μ_{ab} and ω_{ij} corresponding to the analytic continuation of the functions $\mathbf{P}_a$ and $\mathbf{Q}_j$ under these cuts. They will be given by the equations (2.53) and (2.60).

Below we describe in more details this construction following [17]. We also derive some new relation important for the BFKL applications.

2.4.1 Algebraic properties

The AdS/CFT Q-system is formed by 2^8 Q-functions which we denote as $\mathcal{Q}_{A|J}(u)$ where $A, J \subset \{1, 2, 3, 4\}$ are two ordered subsets of indices. They satisfy the QQ-relations[1], generalizing (2.45)

$$\mathcal{Q}_{A|I}\mathcal{Q}_{Aab|I} = \mathcal{Q}^+_{Aa|I}\mathcal{Q}^-_{Ab|I} - \mathcal{Q}^-_{Aa|I}\mathcal{Q}^+_{Ab|I}\ , \tag{2.50a}$$

$$\mathcal{Q}_{A|I}\mathcal{Q}_{A|Iij} = \mathcal{Q}^+_{A|Ii}\mathcal{Q}^-_{A|Ij} - \mathcal{Q}^-_{A|Ii}\mathcal{Q}^+_{A|Ij}\ , \tag{2.50b}$$

$$\mathcal{Q}_{Aa|I}\mathcal{Q}_{A|Ii} = \mathcal{Q}^+_{Aa|Ii}\mathcal{Q}^-_{A|I} - \mathcal{Q}^+_{A|I}\mathcal{Q}^-_{Aa|Ii} \tag{2.50c}$$

and reshuffling a pair of individual indices (small letters a, b, i, j) we can express all Q-functions through 8 basic ones. The QQ-relations (2.50) can be nicely illustrated by the so-called Hasse diagram on the Fig. 2.2. Namely, the left picture on the Fig. 2.2 depicts a facet of the Hasse diagram corresponding to the bosonic QQ-relation (2.50a) with $A = \emptyset$. The right cubic Hasse diagram on the Fig. 2.2 designates other bosonic QQ-relations from (2.50a). The whole $AdS_5 \times S^5$ Q-system is encoded by the $\mathfrak{gl}(4|4)$ Hasse diagram. Comprehensive description of the Hasse diagrams including ones connected to the other

[1] As usually, we will denote the shifts w.r.t. the spectral parameter as $f^{\pm} \equiv f(u \pm i/2)$ and $f^{[n]} \equiv f(u + in/2)$

QQ-relations (2.50b) and (2.50c) can be found in the articles [17, 19, 80]. In addition we also impose the constraints[2] $\mathcal{Q}_{\emptyset|\emptyset} = \mathcal{Q}_{1234|1234} = 1$, the first one being a normalization and the second can be interpreted as a consequence of the unimodularity of the symmetry group [81]. The Hodge dual of this Q-system, built out of the Q-functions defined through the old ones as $\mathcal{Q}^{A|J} \equiv (-1)^{|A|\,|J|}\mathcal{Q}_{\bar{A}|\bar{J}}$ (notations are the same as in [17]) satisfies the same QQ-relations. Here the bar over a subset means the subset complementary w.r.t. the full set $\{1, 2, 3, 4\}$ and $|X|$ denotes the number of indexes in X. We use special notations for 16 most important Q-functions mentioned before: $\mathbf{P}_a \equiv Q_{a|\emptyset}$, $\mathbf{P}^a \equiv Q^{a|\emptyset}$, $\mathbf{Q}_j \equiv Q_{\emptyset|j}$ and $\mathbf{Q}^j \equiv Q^{\emptyset|j}$, where $a, j = 1, 2, 3, 4$.

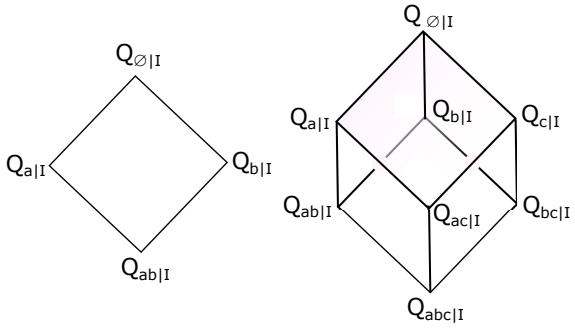

Figure 2.2: Facet and a cubic Hasse diagram.

One can think of $\mathbf{P}_a$ (and $\mathbf{P}^a$) as of quantum counterparts of the classical quasimomenta describing the S^5 part of the string motion, whereas $\mathbf{Q}_i$ (and $\mathbf{Q}^i$) correspond to the AdS_5 part. A nice feature of the Q-system is that any Q-function can be expressed in terms of $\mathbf{P}_a$ and $\mathbf{P}^a$ or, alternatively, in terms of $\mathbf{Q}_i$ and $\mathbf{Q}^i$. Furthermore, the discontinuity relations for $\mathbf{P}$'s decouple from the rest of the system and form a closed system of equations, called $\mathbf{P}\mu$-system, which carries complete information about the spectrum of the whole $AdS_5 \times S^5$ worldsheet sigma model. Alternatively, one can decouple $\mathbf{Q}_i$ and $\mathbf{Q}^i$ from the rest of the system getting a description more natural for the AdS type of excitations, called $\mathbf{Q}\omega$-system. In different situations one or another description could be more convenient, or even a completely new set of basic Q-functions could be chosen to form a closed set of equations. At the same time one can always pass from one description to another.

Here we present our new result which allows for the direct transition between these two equivalent systems. We show in the Appendix A.1 that, as a consequence of the QQ-relations, $\mathbf{P}$'s and $\mathbf{Q}$'s are related through the following 4th order finite-difference equation [82]

$$\mathbf{Q}^{[+4]} - \mathbf{Q}^{[+2]}\left[D_1 - \mathbf{P}_a^{[+2]}\mathbf{P}^{a[+4]}D_0\right] + \mathbf{Q}\left[D_2 - \mathbf{P}_a\mathbf{P}^{a[+2]}D_1 + \mathbf{P}_a\mathbf{P}^{a[+4]}D_0\right] - \\ - \mathbf{Q}^{[-2]}\left[\bar{D}_1 + \mathbf{P}_a^{[-2]}\mathbf{P}^{a[-4]}\bar{D}_0\right] + \mathbf{Q}^{[-4]} = 0\,, \quad (2.51)$$

[2] By $\emptyset$ we denote the empty set.

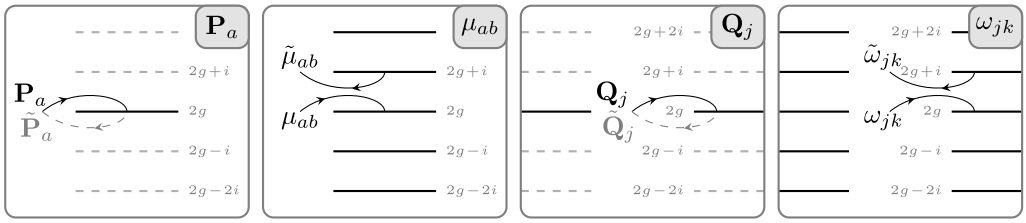

Figure 2.3: Cut structure of $\mathbf{P}$ and μ, $\mathbf{Q}$ and ω and their analytic continuations $\tilde{\mathbf{P}}$ and $\tilde{\mu}$, $\tilde{\mathbf{Q}}$ and $\tilde{\omega}$ [16, 17].

where

$$D_0 = \det \begin{pmatrix} \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad D_1 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix},$$

$$D_2 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad \bar{D}_1 = \det \begin{pmatrix} \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \end{pmatrix},$$

$$\bar{D}_0 = \det \begin{pmatrix} \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \end{pmatrix}. \tag{2.52}$$

The four solutions of this equation give four functions $\mathbf{Q}_j$. This relation will be useful for us since, whether as $\mathbf{P}\mu$-system is simpler at weak coupling, $\mathbf{Q}\omega$-system a priori is more suitable for the $\mathfrak{sl}(2)$ sector to which the twist-2 operators belong.[3]

2.4.2 Analytic properties, $\mathbf{P}\mu$- and $\mathbf{Q}\omega$-systems

The Q-system is a generic Grassmanian algebraic construction, based entirely on the symmetry group. To apply it to our particular model we have to complete it by analyticity properties. An important analytic feature of the AdS/CFT Q-functions is that they are multi-valued functions of u, with infinitely many Riemann sheets connected by cuts, parallel to $\mathbb{R}$, with fixed quadratic branch-points at $u \in \pm 2g + i\mathbb{Z}$ or $u \in \pm 2g + i(\mathbb{Z} + 1/2)$. According to the arguments of [17] there are no other singularities on the whole Riemann surface of any Q-function. The basic 16 Q-functions $\mathbf{Q}$ and $\mathbf{P}$ have particularly nice properties: $\mathbf{P}_a$ and $\mathbf{P}^a$ have only one "short" cut $u \in [-2g, 2g]$ on their main, defining sheet of its Riemann surface, whether $\mathbf{Q}_j$ and $\mathbf{Q}^j$ have only one "long" cut $u \in (-\infty, -2g] \cup [2g, \infty)$ on their main sheet. The rest of the Q-functions can be expressed in terms of either 8 $\mathbf{P}$'s or 8 $\mathbf{Q}$'s using QQ-relations. Depending on this choice we have two equivalent systems of equations described below.

As we explained above, we can focus on a much smaller closed subsystem constituted of 8 functions $\mathbf{P}_a$ and $\mathbf{P}^a$, having only one short cut on the real axis on their defining sheet. To close the system we have to describe their analytic continuation under this

[3] We also note that in a similar way one can derive similar relation (2.51) with $\mathbf{P}$ and $\mathbf{Q}$ exchanged.

cut, to the next sheet, as shown in Fig. 2.3. Denoting this continuation by $\tilde{\mathbf{P}}$ we simply have [17]

$$\tilde{\mathbf{P}}_a = \mu_{ab}(u)\mathbf{P}^b \ , \quad \tilde{\mathbf{P}}^a = \mu^{ab}(u)\mathbf{P}_b \ , \tag{2.53}$$

where $\mu_{ab}(u)$ is an antisymmetric matrix with unit Pfaffian, having infinitely many branch points at $u \in \pm 2g + i\mathbb{Z}$ and $\mu^{ab} = -\frac{1}{2}\epsilon^{abcd}\mu_{cd}$ is its inverse. To distinguish between short cut/long cut version of the same function we add hat/check over the symbol. Then for $\check{\mu}_{ab}$ we have the i-periodicity condition

$$\check{\mu}_{ab}(u+i) = \check{\mu}_{ab}(u) \ . \tag{2.54}$$

This means that all cuts are exact copies of each other with the distance i between them. The analytic continuation under these cuts is again very simple and is given by [17]

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \ . \tag{2.55}$$

Note that if we decide to consider $\hat{\mu}_{ab}$ instead of the periodic $\check{\mu}_{ab}$, we can combine the two above relations into a linear finite difference equation for $\hat{\mu}_{ab}$

$$\hat{\mu}^{++}_{ab} = \hat{\mu}_{ab} + \mathbf{P}_a\hat{\mu}_{bc}\mathbf{P}^c - \mathbf{P}_b\hat{\mu}_{ac}\mathbf{P}^c \ . \tag{2.56}$$

To see this we can take u to be slightly below the real axis, then $\check{\mu}_{ab}(u+i) = \tilde{\hat{\mu}}_{ab}(u)$, as shown in Fig. 2.3.

Finally $\mathbf{P}$'s satisfy the orthogonality relations $\mathbf{P}_a\mathbf{P}^a = 0$ and at large u they should behave as

$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \simeq \begin{pmatrix} A_1\, u^{\frac{-J_1-J_2+J_3-2}{2}} \\ A_2\, u^{\frac{-J_1+J_2-J_3}{2}} \\ A_3\, u^{\frac{+J_1-J_2-J_3-2}{2}} \\ A_4\, u^{\frac{+J_1+J_2+J_3}{2}} \end{pmatrix} , \quad \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \\ \mathbf{P}^4 \end{pmatrix} \simeq \begin{pmatrix} A^1\, u^{\frac{+J_1+J_2-J_3}{2}} \\ A^2\, u^{\frac{+J_1-J_2+J_3-2}{2}} \\ A^3\, u^{\frac{-J_1+J_2+J_3}{2}} \\ A^4\, u^{\frac{-J_1-J_2-J_3-2}{2}} \end{pmatrix} . \tag{2.57}$$

We note, that the coefficients A_a and A^a could be also determined solely in terms of the global symmetry Cartan charges $(\Delta, S_1, S_2|J_1, J_2, J_3)$ of the state, including the energy Δ. We will briefly discuss this below.

It may seem that the description in terms of $\mathbf{P}\mu$-system breaks the symmetry between AdS_5 and S^5 parts of the string background. It is possible however to pass to an alternative, equivalent description where the roles of these parts are interchanged. We will see that we also have to interchange short and long cuts. To construct this alternative system we can use (2.51), which, for a given $\mathbf{P}_a$, gives us 4 linear independent solutions $\mathbf{Q}_i$ (similarly we construct $\mathbf{Q}^i$). Knowing $\mathbf{P}_a$ and $\mathbf{Q}_i$ we construct $\mathcal{Q}_{a|i}$ using (A.1) which allows us to define ω_{ij}

$$\omega_{ij} = \mathcal{Q}^-_{a|i}\mathcal{Q}^-_{b|j}\mu^{ab} \ . \tag{2.58}$$

One can show that $\mathbf{Q}_a$ defined in this way will have one long cut. Also $\hat{\omega}_{ij}$, with short cuts, happens to be periodic $\hat{\omega}^+_{ij} = \hat{\omega}^-_{ij}$, similarly to its counterpart with long cuts $\check{\mu}_{ab}$! Finally, their discontinuities are given by

$$\tilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i\tilde{\mathbf{Q}}_j - \mathbf{Q}_j\tilde{\mathbf{Q}}_i \ , \tag{2.59}$$

$$\tilde{\mathbf{Q}}_i = \omega_{ij}\mathbf{Q}^j \ . \tag{2.60}$$

Similarly to (2.57), we have the large u asymptotics originating from the classical limit of the superstring sigma model

$$\begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{pmatrix} \simeq \begin{pmatrix} B_1\, u^{\frac{+\Delta-S_1-S_2}{2}} \\ B_2\, u^{\frac{+\Delta+S_1+S_2-2}{2}} \\ B_3\, u^{\frac{-\Delta-S_1+S_2}{2}} \\ B_4\, u^{\frac{-\Delta+S_1-S_2-2}{2}} \end{pmatrix} , \quad \begin{pmatrix} \mathbf{Q}^1 \\ \mathbf{Q}^2 \\ \mathbf{Q}^3 \\ \mathbf{Q}^4 \end{pmatrix} \simeq \begin{pmatrix} B^1\, u^{\frac{-\Delta+S_1+S_2-2}{2}} \\ B^2\, u^{\frac{-\Delta-S_1-S_2}{2}} \\ B^3\, u^{\frac{+\Delta+S_1-S_2-2}{2}} \\ B^4\, u^{\frac{+\Delta-S_1+S_2}{2}} \end{pmatrix} . \tag{2.61}$$

Now we are able to analyze the asymptotics of $\mu_{ab}(u)$ in the limit of big u. Let us assume that $\mu_{12}(u) \sim u^{\Delta-L}$ if $u \to \infty$. In other words, we consider the state with $S_1 = S$, $S_2 = 0$, $J_1 = L$ and $J_2 = J_3 = 0$. Then, suggesting that all the terms in the RHS of the formula for $\tilde{\mathbf{P}}_a$ have the same asymptotics, we obtain

$$(\mu_{34}, \mu_{13}, \mu_{14}, \mu_{24}, \mu_{12}) \simeq \left(B_{34}u^{\Delta+L}, B_{13}u^{\Delta+1}, B_{14}u^{\Delta}, B_{24}u^{\Delta-1}, B_{12}u^{\Delta-L}\right) . \tag{2.62}$$

For large u the system for the difference $\tilde{\mu}_{ab} - \mu_{ab}$ converts to

$$i\mu'_{ab}(u) \simeq -\mathbf{P}_a\mathbf{P}_e\chi^{ec}\mu_{cb} + \mathbf{P}_b\mathbf{P}_e\chi^{ec}\mu_{ca} , \tag{2.63}$$

which contains five non-equivalent equations. Putting the asymptotics of $\mu_{ab}(u)$ in this system of five first order differential equations gives us the linear system of five equations on the variables B_{ab}. This system has to have a non-trivial solution, which means that its determinant is equal to zero. This gives the 5-th order equation for Δ, which has the solutions $(\pm\alpha, \pm\beta, 0)$ with

$$\begin{aligned} \alpha &= i^{\frac{3}{2}} \frac{\sqrt{i + iL^2 + 2(1+L)A_2A_3 + 2(1-L)A_1A_4 + \sqrt{-(L^2-1)(-4i(L-1)A_2A_3 + (L+1)(L-1+4iA_1A_4))}}}{\sqrt{2}} , \\ \beta &= i^{\frac{1}{2}} \frac{\sqrt{-i - iL^2 - 2(1+L)A_2A_3 - 2(1-L)A_1A_4 + \sqrt{-(L^2-1)(-4i(L-1)A_2A_3 + (L+1)(L-1+4iA_1A_4))}}}{\sqrt{2}} . \end{aligned} \tag{2.64}$$

For the operators from the $\mathfrak{sl}(2)$ sector, identifying one of the roots with Δ and the other with $S-1$, where S is the Lorentz spin of the operator, we obtain

$$A_2A_3 = -i\frac{((L+S)^2-\Delta^2)((L-S+2)^2-\Delta^2)}{16L(L+1)}, \tag{2.65}$$

$$A_1A_4 = -i\frac{((L-S)^2-\Delta^2)((L+S-2)^2-\Delta^2)}{16L(L+1)}. \tag{2.66}$$

We note that since $\mathbf{P}$'s and $\mathbf{Q}$'s are not independent due to (2.51) there is a non-trivial compatibility condition for their asymptotics (2.57) and (2.61), which, in particular, fixes [17] the AA products for the most general state characterized by the $\mathfrak{psu}(2,2|4)$ Cartan charges $(\Delta, S_1, S_2|J_1, J_2, J_3)$

$$
\begin{aligned}
A_1A^1 &= \frac{\left((J_1+J_2-J_3-S_2+1)^2-(\Delta+S_1-1)^2\right)\left((J_1+J_2-J_3+S_2+1)^2-(\Delta-S_1+1)^2\right)}{-16i\,(J_1+J_2+1)\,(J_1-J_3)\,(J_2-J_3+1)}\,,\\
A_2A^2 &= \frac{\left((J_1-J_2+J_3-S_2-1)^2-(\Delta+S_1-1)^2\right)\left((J_1-J_2+J_3+S_2-1)^2-(\Delta-S_1+1)^2\right)}{+16i\,(J_1-J_2-1)\,(J_1+J_3)\,(J_2-J_3+1)}\,,\\
A_3A^3 &= \frac{\left((J_1-J_2-J_3+S_2-1)^2-(\Delta+S_1-1)^2\right)\left((J_1-J_2-J_3-S_2-1)^2-(\Delta-S_1+1)^2\right)}{-16i\,(J_1-J_2-1)\,(J_1-J_3)\,(J_2+J_3+1)}\,,\\
A_4A^4 &= \frac{\left((J_1+J_2+J_3-S_2+1)^2-(\Delta-S_1+1)^2\right)\left((J_1+J_2+J_3+S_2+1)^2-(\Delta+S_1-1)^2\right)}{+16i\,(J_1+J_2+1)\,(J_1+J_3)\,(J_2+J_3+1)}\,.
\end{aligned}
\tag{2.67}
$$

Now we will apply these general formulas, true for any local operator, to our current problems – the spectrum of twist-2 operators in the different limits.

Chapter 3

Twist-2 operators with zero conformal spin

In this Chapter we consider the class of the twist-2 operators (1.6), which are particularly important in the context of the BFKL limit of the high-energy scattering. For this we are going to utilize the Quantum Spectral Curve method, including the $\mathbf{P}\mu$- and $\mathbf{Q}\omega$-systems, described in the Chapter 2. It is logically divided into two Sections. In the first one we study the weak coupling limit of the $\mathbf{P}\mu$-system for the twist-2 operators (1.6) first for the integer spin S and then analytically continue in S. In the second Section using the aforementioned analytic continuation we find the QSC solution in the BFKL limit restoring the seminal LO BFKL kernel eigenvalue. The second Section of this Chapter is based on the article [82]. The main author's contribution to this part is the conjecture of the ansatz for the solution of the $\mathbf{P}\mu$-system in the BFKL limit.

3.1 Weak coupling limit

This Section is devoted to the consideration of the weak coupling limit of the QSC. Namely, we start from solving it at 1-loop order for the integer spin S, obtaining the second order Baxter equation. Then we find the analytic continuation of the solution of this equation in S, which correctly reproduces the analytically continued in S 1-loop anomalous dimension of the twist-2 operators. This allows us to reconstruct the whole $\mathbf{P}\mu$-system at 1-loop order and partially at 2-loop order.

3.1.1 QSC solution for non-negative even $S_1 = S$

Before proceeding with solving the system of equations we have to describe their scaling in the limit of interest, i.e. small g and non-negative even $S_1 = S$ and $S_2 = n = 0$. In this limit $\Delta = 2 + S + \mathcal{O}(g^2)$ and therefore $A_1 A_4 = \mathbf{A}_4 = -\mathbf{A}_1 = \mathcal{O}(g^2)$ and $A_2 A_3 = \mathbf{A}_2 = -\mathbf{A}_3 = \mathcal{O}(g^0)$. Using the rescaling symmetry, we can set $A_1 = \mathcal{O}(g^2)$ and $A_4 = \mathcal{O}(g^0)$ along with $A_2 = \mathcal{O}(g^0)$ and $A_3 = \mathcal{O}(g^0)$. Or, in other words,

$$A_a = \left(g^2, 1, -\mathbf{A}_3, \frac{\mathbf{A}_4}{g^2}\right) , \quad A^a = \left(\frac{\mathbf{A}_1}{g^2}, \mathbf{A}_2, -1, g^2\right) . \tag{3.1}$$

Application of the rotation symmetry from (2.40), taking into account the number of free parameters in (2.40), also allows us to set the coefficient $c_{4,1} = 0$ in all orders in the coupling constant.

Assuming that the scaling of each $\mathbf{P}$-function is determined by its coefficient in front of the leading asymptotic and introducing the notation

$$\mathbf{P}_a = \sum_{k=0}^{+\infty} \mathbf{P}_a^{(k)} g^{2k} \,, \tag{3.2}$$

we conclude that

$$\mathbf{P}_1^{(0)} = 0 \tag{3.3}$$

in the weak coupling limit. The scaling (3.2) and (3.3) imposes the following restrictions on the coefficients in (2.41)

$$c_{1,k} = \mathcal{O}(g^{-2k}) \quad \text{and} \quad c_{a,k} = \mathcal{O}(g^{-2k-\tilde{M}_a}) \,, \quad a = 2,3,4 \,. \tag{3.4}$$

To understand the scaling of the μ-functions we can do the following. Let us consider

$$\tilde{\mathbf{P}}_1 = \frac{A_1}{g^2} x^2 + \sum_{k=1}^{+\infty} c_{1,k} x^{2k+2} \,. \tag{3.5}$$

The leading term of the RHS of (3.5) scales as g^{-2}. We assume that the scaling of $\tilde{\mathbf{P}}_1$ is determined by its leading term and $\tilde{\mathbf{P}}_1 \sim g^{-2}$. Thus, from the RHS of the $\mathbf{P}\mu$-equations for we conclude that at least some of the μ-functions have to scale as g^{-2}. As the μ-functions are present also in the RHS of the $\mathbf{P}\mu$-equations for $\tilde{\mathbf{P}}_a$ for $a = 2,3,4$, we see that the scaling g^{-2} is transferred to the other components of $\tilde{\mathbf{P}}_a$. Therefore a natural hypothesis for the scaling of the μ-functions would be

$$\mu_{ab} = \sum_{k=0} \mu_{ab}^{(k)} g^{2k-2} \,. \tag{3.6}$$

The scaling of the $\mathbf{P}$-functions $\tilde{\mathbf{P}}_a = \mathcal{O}(g^{-2})$, taking into account the expansion (3.2), implies the scaling $c_{a,k} = \mathcal{O}(g^{\tilde{M}_a+2(k-1)})$ and the following weak coupling expansion of these coefficients

$$c_{a,k} = \sum_{m=0}^{+\infty} c_{a,k}^{(m)} g^{2(m+k-1)+\tilde{M}_a} \,. \tag{3.7}$$

Having now (3.2),(3.6) and (3.7) we are able to proceed with solving the $\mathbf{P}\mu$-equations in the LO.

The system of equations for μ_{ab} in the LO in g taking into account that $\mathbf{P}_1^{(0)} = 0$

$$\begin{pmatrix} \tilde{\mu}_{12}^{(0)} - \mu_{12}^{(0)} \\ \tilde{\mu}_{13}^{(0)} - \mu_{13}^{(0)} \\ \tilde{\mu}_{14}^{(0)} - \mu_{14}^{(0)} \\ \tilde{\mu}_{24}^{(0)} - \mu_{24}^{(0)} \\ \tilde{\mu}_{34}^{(0)} - \mu_{34}^{(0)} \end{pmatrix} = \begin{pmatrix} -\mathbf{P}_2^{(0)}\mathbf{P}_3^{(0)} & \left(\mathbf{P}_2^{(0)}\right)^2 & 0 & 0 & 0 \\ -\left(\mathbf{P}_3^{(0)}\right)^2 & \mathbf{P}_2^{(0)}\mathbf{P}_3^{(0)} & 0 & 0 & 0 \\ -\mathbf{P}_3^{(0)}\mathbf{P}_4^{(0)} & \mathbf{P}_2^{(0)}\mathbf{P}_4^{(0)} & 0 & 0 & 0 \\ -\left(\mathbf{P}_4^{(0)}\right)^2 & 0 & 2\mathbf{P}_2^{(0)}\mathbf{P}_4^{(0)} & -\mathbf{P}_2^{(0)}\mathbf{P}_3^{(0)} & \left(\mathbf{P}_2^{(0)}\right)^2 \\ 0 & -\left(\mathbf{P}_4^{(0)}\right)^2 & 2\mathbf{P}_3^{(0)}\mathbf{P}_4^{(0)} & -\left(\mathbf{P}_3^{(0)}\right)^2 & \mathbf{P}_2^{(0)}\mathbf{P}_3^{(0)} \end{pmatrix} \begin{pmatrix} \mu_{12}^{(0)} \\ \mu_{13}^{(0)} \\ \mu_{14}^{(0)} \\ \mu_{24}^{(0)} \\ \mu_{34}^{(0)} \end{pmatrix} . \tag{3.8}$$

On can see from (3.8) that the equations for $\mu_{12}^{(0)}$ and $\mu_{13}^{(0)}$ decouple from the system. Excluding $\mu_{13}^{(0)}$ from them with the formula

$$\mu_{13}^{(0)} = \frac{\mu_{12}^{(0)++}}{\left(\mathbf{P}_2^{(0)}\right)^2} + \left(\frac{\mathbf{P}_3^{(0)}}{\mathbf{P}_2^{(0)}} - \frac{1}{\left(\mathbf{P}_2^{(0)}\right)^2}\right)\mu_{12}^{(0)} \tag{3.9}$$

we get the equation

$$\frac{\mu_{12}^{(0)[+3]}}{\left(\mathbf{P}_2^{(0)+}\right)^2} + \left(\frac{\mathbf{P}_3^{(0)+}}{\mathbf{P}_2^{(0)+}} - \frac{\mathbf{P}_3^{(0)-}}{\mathbf{P}_2^{(0)-}} - \frac{1}{\left(\mathbf{P}_2^{(0)+}\right)^2} - \frac{1}{\left(\mathbf{P}_2^{(0)-}\right)^2}\right)\mu_{12}^{(0)+} + \frac{\mu_{12}^{(0)-}}{\left(\mathbf{P}_2^{(0)-}\right)^2} = 0 \tag{3.10}$$

and the equation for $\mu_{13}^{(0)}$

$$\frac{\mu_{13}^{(0)[+3]}}{\left(\mathbf{P}_3^{(0)+}\right)^2} - \left(\frac{\mathbf{P}_2^{(0)+}}{\mathbf{P}_3^{(0)+}} - \frac{\mathbf{P}_2^{(0)-}}{\mathbf{P}_3^{(0)-}} + \frac{1}{\left(\mathbf{P}_3^{(0)+}\right)^2} + \frac{1}{\left(\mathbf{P}_3^{(0)-}\right)^2}\right)\mu_{13}^{(0)+} + \frac{\mu_{13}^{(0)-}}{\left(\mathbf{P}_4^{(0)-}\right)^2} = 0\,. \tag{3.11}$$

Introducing a new notation $\mu_{12}^{(0)+} = \alpha Q$, where α is the normalization coefficient, we rewrite the equation (3.10)

$$TQ + \frac{1}{\left(\mathbf{P}_2^{(0)-}\right)^2}Q^{--} + \frac{1}{\left(\mathbf{P}_2^{(0)+}\right)^2}Q^{++} = 0\,, \tag{3.12}$$

where

$$T = \frac{\mathbf{P}_3^{(0)+}}{\mathbf{P}_2^{(0)+}} - \frac{\mathbf{P}_3^{(0)-}}{\mathbf{P}_2^{(0)-}} - \frac{1}{\left(\mathbf{P}_2^{(0)-}\right)^2} - \frac{1}{\left(\mathbf{P}_2^{(0)+}\right)^2}\,. \tag{3.13}$$

From (3.2) and (3.7) together with the normalization $A_1 = A_2 = 1$ we derive that

$$\mathbf{P}_1^{(0)} = \frac{1}{u^2}\,, \quad \mathbf{P}_2^{(0)} = \frac{1}{u}\,, \quad \mathbf{P}_3^{(0)} = A_3^{(0)}\,, \quad \mathbf{P}_4^{(0)} = A_4^{(0)}u\,, \tag{3.14}$$

then T is known. Therefore all the coefficients in (3.12) are polynomials and (3.12) takes the form

$$\left(u + \frac{i}{2}\right)^2 Q^{++} + \left(u - \frac{i}{2}\right)^2 Q^{--} - \left(2u^2 - S(S+1) - \frac{1}{2}\right)Q = 0\,. \tag{3.15}$$

Now we are going to analyze the analyticity properties of μ_{ab}. Let us represent μ_{ab} in the following form

$$\mu_{ab} = \left(\frac{\mu_{ab} + \mu_{ab}^{++}}{2}\right) + \sqrt{u^2 - 4g^2}\left(\frac{\mu_{ab} - \mu_{ab}^{++}}{2\sqrt{u^2 - 4g^2}}\right)\,. \tag{3.16}$$

Thanks to the continuity $\mu_{ab}(u \pm i0) = \tilde{\mu}_{12}(u \mp i0)$ the expressions in the brackets have no jump on the cut. This means that in the LO the function $\mu_{12}^{(0)}$ is analytic in the whole complex plane. For the twist-2 physical states from the $\mathfrak{sl}(2)$ sector of the function

$\mu_{12} \sim u^{\Delta-2}$ and we see that $\mu_{12}^{(0)} = Q^- \sim u^S$ has the polynomial asymptotic as S is non-negative integer in our consideration. Thus from the Liouville's theorem about entire functions we conclude that Q is a polynomial of degree S.

Now we are to find the solution of (3.12) for the case of non-negative even S. A polynomial solution to (3.12) can be expressed in terms of the hypergeometric function

$$Q(u;S) = {}_3F_2\left(-S, 1+S, \frac{1}{2} - iu; 1, 1; 1\right) , \tag{3.17}$$

which is a polynomial of the order S if S is a non-negative integer.

Up to the normalization in the LO $\mu_{12}^{(0)} = \alpha Q^-(u;S)$, where α is the normalization constant to be determined. To do this, let us use the equation

$$\mu_{12}^{(0)++} - \mu_{12}^{(0)} = -\tilde{\mathbf{P}}_1^{(0)}\mathbf{P}_2^{(0)} . \tag{3.18}$$

In the LO in g we get

$$\tilde{\mathbf{P}}_1^{(0)} = u^2 + \sum_{k=1}^{+\infty} c_{1,k}^{(0)} u^{2k+2} , \tag{3.19}$$

which leads us to

$$\alpha\left(Q^+(u;S) - Q^-(u;S)\right) = -u - \sum_{k=1}^{+\infty} c_{1,k}^{(0)} u^{2k+1} . \tag{3.20}$$

Solving (3.20) with respect to α gives us

$$\alpha = -\frac{1}{(\partial_u Q^+(u;S) - \partial_u Q^-(u;S))|_{u=0}} \tag{3.21}$$

Thus,

$$\mu_{12}^{(0)} = -\frac{iQ^-(u;S)}{8\cos^2\frac{\pi S}{2} S_1(S)} , \tag{3.22}$$

where $S_1(S)$ is the value of the harmonic sum at even non-negative S.

Finding $\tilde{\mathbf{P}}_2$ and 1-loop anomalous dimension for integer S

To proceed further, let us write down two of the $\mathbf{P}\mu$-system equations in the LO

$$\begin{aligned} \tilde{\mathbf{P}}_2^{(0)} &= \mu_{12}^{(0)}\mathbf{P}_4^{(0)} - \mu_{14}^{(0)}\mathbf{P}_2^{(0)} , \\ \tilde{\mathbf{P}}_2^{(0)} &= \mu_{12}^{(0)++}\mathbf{P}_4^{(0)} - \mu_{14}^{(0)++}\mathbf{P}_2^{(0)} . \end{aligned} \tag{3.23}$$

The important observation is that if we express from the RHS of (3.23)

$$\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}} = \frac{\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}}\mu_{12}^{(0)} - \mu_{14}^{(0)} \tag{3.24}$$

taking into account the **P**-functions in the LO (3.14) and the fact that all $\mu_{ab}^{(0)}$ are polynomials for integer S, we conclude that $\tilde{\mathbf{P}}_2^{(0)}/\mathbf{P}_2^{(0)}$ is a polynomial. Then, combining and shifting the equations (3.23), we obtain

$$\left(\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}}\right)^+ - \left(\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}}\right)^- = \alpha Q(u;S)\left(\left(\frac{\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}}\right)^+ - \left(\frac{\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}}\right)^-\right) . \tag{3.25}$$

In the LO in g according to (3.14) we know the **P**-functions(3.7), thus, the RHS of (3.25) reduces to

$$\left(\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}}\right)^+ - \left(\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}}\right)^- = 2i\alpha Q(u;S)A_4^{(0)}u . \tag{3.26}$$

The polynomial solution of (3.26) taken from the article [20] supplemented by the solution of the homogeneous version of (3.26) is

$$\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}} = u^2K\left((iu+\delta)\left(Q^+ - Q^-\right) + \frac{1}{2}\left(Q^+ + Q^-\right)\right) + a , \tag{3.27}$$

where K, δ and a are the constants to be fixed. They can be fixed from the conjectured form of $\tilde{\mathbf{P}}_2$ in the leading order in g and small u. So, we have

$$\begin{aligned}
\tilde{\mathbf{P}}_2^{(0)} &= u + \sum_{k=1}^{+\infty} c_{2,k}^{(0)} u^{2k+1} , \\
\tilde{\mathbf{P}}_2^{(0)} &= \frac{a}{u} + KQ\left(\frac{i}{2};S\right)u + K\delta\left(\partial_u Q^+ - \partial_u Q^-\right)\Big|_{u=0} u^2 + \mathcal{O}\left(u^3\right) .
\end{aligned} \tag{3.28}$$

Thus, comparing the first and second equations of (3.28), we obtain $K = 1/Q(i/2;S)$ and $\delta = a = 0$. The solution takes the form

$$\tilde{\mathbf{P}}_2^{(0)} = \frac{u}{Q\left(\frac{i}{2};S\right)}\left(iu(Q^+ - Q^-) + \frac{1}{2}\left(Q^+ + Q^-\right)\right) . \tag{3.29}$$

Plugging this result (3.29) together with (3.14) into the equation (3.26), we are able to express $A_4^{(0)}$

$$A_4^{(0)} = -\frac{(S+2)(S-1)}{2\alpha Q\left(\frac{i}{2};S\right)} . \tag{3.30}$$

On the other hand from (3.1) we know

$$A_4^{(0)} = \mathbf{A}_4^{(1)} = -\frac{i(S+2)(S-1)}{4}\gamma^{(1)} , \tag{3.31}$$

where $\gamma^{(1)}$ is the 1-loop anomalous dimension of the considered twist-2 $\mathfrak{sl}(2)$ operator. Therefore, comparing (3.30) and (3.31) and remembering (3.21), we obtain

$$\gamma^{(1)} = 2i\partial_u \log\frac{Q^+(u;S)}{Q^-(u;S)}\bigg|_{u=0} . \tag{3.32}$$

To sum up, using (3.32), we are able to write down a particularly simple formula for the 1-loop anomalous dimension of the twist-2 operators of the $\mathfrak{sl}(2)$ sector

$$\Delta = 2 + S + 2ig^2\partial_u \log \frac{Q^+(u;S)}{Q^-(u;S)}\bigg|_{u=0} + \mathcal{O}(g^4) \,. \tag{3.33}$$

Substituting (3.17) into (3.33) we derive the formula

$$\Delta = 2 + S + 8g^2 \sum_{k=1}^{S} \frac{1}{k} + \mathcal{O}(g^4) \,. \tag{3.34}$$

However, the presented solution allows to find the solution of the $\mathbf{P}\mu$-system and 1-loop anomalous dimension only in the case of the physical operators, i.e. when S is a non-negative even integer. Our aim now is to generalize the solution above to the case of non-integer S.

3.1.2 Analytic continuation to non-integer spin S

The aim of the present Subsection is to generalize the solution presented in the previous Subsection for integer S for the case of non-integer S. Let us start with writing down the analytic continuation of the 1-loop anomalous dimension

$$\Delta = 2 + S + 8g^2(\psi(S+1) - \psi(1)) + \mathcal{O}(g^4) \,, \tag{3.35}$$

which has the infinite series of poles in the points $S = -1, -2, -3, \ldots$ due to the presence of the digamma function. Also, the 1-loop anomalous dimension can be rewritten in terms of the harmonic sums $\gamma^{(1)} = 8S_1(S)$. Now we are going to build the solution of the $\mathbf{P}\mu$, which produces (3.35) for non-integer S.

We consider now the situation when S is not integer anymore and it is clear that the $\mathbf{P}$-functions are the same in the LO, i.e. given by the formula (3.14). In the weak coupling limit $\mathbf{P}_1 = \mathcal{O}(g^2)$, which allowed us to decouple the equations for $\mu_{12}^{(0)}$ and $\mu_{13}^{(0)}$ decouple and we obtained the Baxter equation (3.15) for $\mu_{12}^{(0)+} = \alpha Q$

$$\left(2u^2 - S(S+1) - \frac{1}{2}\right) Q = \left(u + \frac{i}{2}\right)^2 Q^{++} + \left(u - \frac{i}{2}\right)^2 Q^{--} \,. \tag{3.36}$$

For further convenience let us consider a more general equation with an additional parameter j

$$\Big(2u^2 - S(S+1) + 2j(j-1)\Big) Q = (u+ij)^2 Q^{++} + (u-ij)^2 Q^{--} \,. \tag{3.37}$$

For $j = 1/2$ the equation (3.37) turns into (3.15). It is notable that if $Q(u;S,j)$ is a solution of (3.37), then due to the symmetries of this Baxter equation $Q(u;-S-1,j)$ and $Q(-u;-S-1,j)$ are also solutions. The solution to this equation (3.37) has the form

$$Q_0(u;S,j) = \frac{1}{\Gamma^2(2j)}\, {}_3F_2(2j-1-S, 2j+S, j-iu; 2j, 2j; 1) \tag{3.38}$$

and it is true that $Q_0(u;S,1/2)=Q(u;S)$ from (3.17). This solution (3.38) is symmetric under the flip $S\to -S-1$. Its asymptotic expansion at $u\to\infty$ is (it can be obtained by the method described, for example, in [34])

$$Q_0(u;S,j)\simeq \frac{\Gamma(2S+1)}{\Gamma^2(S+1)\Gamma(2j+S)}ie^{i\pi\left(\frac{S}{2}-j\right)}u^{-2j+S+1}+ \\ +\frac{\Gamma(-2S-1)}{\Gamma^2(-S)\Gamma(2j-S-1)}e^{-i\pi\left(\frac{S}{2}+j\right)}u^{-2j-S}\ . \quad (3.39)$$

This function (3.38) is meromorphic in the complex plane, i.e. it has the infinite series of poles in the points $u=i(1-j)+ik$, $k=0,1,2,\ldots$. The residue in the pole $u=i(1-j)$ is[1]

$$Q_0(u;S,j)=-\frac{i}{\Gamma(2j-1-S)\Gamma(2j+S)}\frac{1}{u-i(1-j)}+\mathcal{O}\left((u-i(1-j))^0\right)\ . \quad (3.40)$$

If we use the Baxter equation (3.37), we can find the residues in all points $u=i(1-j)+ik$, where $k=1,2,3,\ldots$ From the formula (3.40) we see that $Q_0(u;S,1/2)$ does not reproduce the 1-loop anomalous dimension (3.35), therefore we have to look for the other solution.

We can obtain the second linear independent solution to the Baxter equation (3.37) by taking

$$q(u;S,j)=\left(\frac{\Gamma(1-j+iu)}{\Gamma(j+iu)}\right)^2 Q_0(u;S,1-j)\ . \quad (3.41)$$

The second solution $q(u;S,j)$ has the following asymptotic expansion at $u\to\infty$

$$q(u;S,j)\simeq \frac{\Gamma(2S+1)}{\Gamma^2(S+1)\Gamma(2-2j+S)}ie^{i\pi\left(\frac{S}{2}-j\right)}u^{-2j+S+1}+ \\ +\frac{\Gamma(-2S-1)}{\Gamma^2(-S)\Gamma(-2j-S+1)}e^{-i\pi\left(\frac{S}{2}+j\right)}u^{-2j-S}\ . \quad (3.42)$$

Also due to the factor consisting of gamma functions, $q(u;S,j)$ has the second order poles at the points $u=i(1-j)+ik$, where $k=0,1,2,\ldots$ and first order zeroes at the points $u=ij+ik$, where $k=0,1,2,\ldots$

Let us now restrict ourselves to the case $j=1/2$. First, we would like to build the decaying at infinity solution for $S>-1/2$, i.e. with the asymptotic u^{-1-S}, from these two: (3.38) and (3.41). This can be achieved by

$$Q_{1'}(u;S)=\lim_{j\to\frac{1}{2}}\frac{1}{j-\frac{1}{2}}\left(q(u;S,j)-\frac{\Gamma(2j+S)}{\Gamma(2-2j+S)}Q_0(u;S,j)\right)\ . \quad (3.43)$$

It can be easily found that

$$Q_{1'}(u;S)=-4\psi\left(\frac{1}{2}+iu\right)Q_0\left(u;S,\frac{1}{2}\right)- \\ -4\psi(S+1)Q_0\left(u;S,\frac{1}{2}\right)-2\left(\partial_j Q_0\left(u;S,j\right)\right)\big|_{j=\frac{1}{2}}\ . \quad (3.44)$$

[1]Note that the residue in this pole can be obtained by taking the asymptotic expansion of the terms of the hypergeometric series at large n. The sum of the obtained series gives $\zeta(2-j+iu)/(\Gamma(2j-1-S)\Gamma(2j+S)\Gamma(j-iu))$.

Note that $Q_{1'}(u;S)$ behaves as u^{-1-S} at $u\to\infty$. Taking into account the expansion

$$Q_0\left(u;S,\frac{1}{2}\right)=\frac{i\sin(\pi S)}{\pi\left(u-\frac{i}{2}\right)}+\cos(\pi S)+\frac{2}{\pi}\sin(\pi S)\left(\psi(S+1)+\gamma\right)+\mathcal{O}\left(u-\frac{i}{2}\right)\ , \quad (3.45)$$

in the vicinity of the point $u=i/2$ we have

$$Q_{1'}(u;S)=\frac{2\sin(\pi S)}{\pi\left(u-\frac{i}{2}\right)^2}-\frac{4i\sin(\pi S)\left(\psi(S+1)+\gamma\right)}{\pi\left(u-\frac{i}{2}\right)}+\mathcal{O}\left(\left(u-\frac{i}{2}\right)^0\right)\ . \quad (3.46)$$

By definition (3.43) the solution $Q_{1'}(u;S)$ is analytic for Im $u<1/2$ and for $j=1/2$ has an infinite series of second order poles at the points $u=i/2+ik$, $k=0,1,2,\dots$ and the expansion at these points can be understood from the Baxter equation (3.37).

The solution with the same asymptotics and analytic properties as $Q_{1'}(u;S)$ can be constructed by an equivalent method by requiring from this solution to be decaying for $S>-1/2$ at $u\to\infty$ (u^{-S-1}) and to have the second order poles at the points $u=i/2+ik$, $k=0,1,2,\dots$ determined by (3.46) and no singularities in the LHP. Taking into account that $Q_0(-u;S;1/2)$ is also a solution, these requirements lead us to the formula

$$Q_1(u;S)=2\pi\left(\cot(\pi S)Q_0\left(u;S,\frac{1}{2}\right)-\right.$$
$$\left.-\frac{1}{\sin(\pi S)}Q_0\left(-u;S,\frac{1}{2}\right)-i\tanh(\pi u)Q_0\left(u;S,\frac{1}{2}\right)\right)\ . \quad (3.47)$$

First, one can check that $Q_1(u,S)\sim u^{-S-1}$ if $u\to\infty$. Then, the poles at $u=-\frac{i}{2}-ik$, $k=0,1,2,...$ originating from $Q_0(-u,S;1/2)$ and $\tanh(\pi u)$ should be cancelled. We checked the cancellation of the pole at $u=-i/2$. The second order pole in $u=i/2$ is given by

$$Q_1(u;S)=\frac{2\sin(\pi S)}{\pi\left(u-\frac{i}{2}\right)^2}-\frac{4i\sin(\pi S)\left(\psi(S+1)+\gamma\right)}{\pi\left(u-\frac{i}{2}\right)}+\mathcal{O}\left(\left(u-\frac{i}{2}\right)^0\right)\ . \quad (3.48)$$

Because $Q_0(u;S,1/2)$ and $Q_1(u;S)$ have the poles at the same points, we can try to build a combination of these solutions, which has no singularities in the whole complex plane

$$Q_2(u;S)=Q_0\left(u;S,\frac{1}{2}\right)+\frac{i}{4\pi}Q_1(u;S)\sinh(2\pi u)\ . \quad (3.49)$$

At least, in $u=i/2$ poles annihilate each other and $Q_2(u,S)=\mathcal{O}((u-i/2)^0)$. Thus, as (3.49) is a solution of (3.37) with $j=1/2$ absence of the poles in the points $u=-i/2$ and $u=i/2$ allows us to conclude that in the points $u=i/2+ik$, $k=1,2,3,\dots$ there are also no poles.

One can obtain the linear independent solution by taking $Q_2(-u;S)$. Because, the solution to the Baxter equation we are looking for possesses the symmetry $Q(i/2,S)=Q(-i/2,S)$ (due to the symmetry of the μ-function $\mu_{12}^{(0)}(i)=\mu_{12}^{(0)}(0)$, which follows from $\mu_{12}^{+}(-u)=\mu_{12}^{+}(u)$), it is natural to take the combination

$$Q(u;S)=Q_2(u;S)+Q_2(-u;S)\ . \quad (3.50)$$

After the substitution of (3.47) and (3.49), we get[2]

$$Q(u;S) = \left(Q_0\left(u;S,\frac{1}{2}\right) + Q_0\left(-u;S,\frac{1}{2}\right)\right)\cosh^2(\pi u) + \\ + \frac{i}{2}\left(Q_0\left(u;S,\frac{1}{2}\right) - Q_0\left(-u;S,\frac{1}{2}\right)\right)\cot\left(\frac{\pi S}{2}\right)\sinh(2\pi u)\ . \quad (3.51)$$

First of all, using (3.45) we can calculate from (3.51)

$$Q\left(\frac{i}{2},S\right) = 2\cos^2\left(\frac{\pi S}{2}\right)\ . \quad (3.52)$$

For the derivatives with respect to u

$$\partial_u Q(u;S) = \partial_u\left(Q_0\left(u;S,\frac{1}{2}\right) + Q_0\left(-u;S,\frac{1}{2}\right)\right)\cosh^2(\pi u) + \\ + \left(Q_0\left(u;S,\frac{1}{2}\right) + Q_0\left(-u;S,\frac{1}{2}\right)\right)\pi\sinh(2\pi u) + \\ + \frac{i}{2}\partial_u\left(Q_0\left(u;S,\frac{1}{2}\right) - Q_0\left(-u;S,\frac{1}{2}\right)\right)\cot\left(\frac{\pi S}{2}\right)\sinh(2\pi u) + \\ + \pi i\left(Q_0\left(u,S,\frac{1}{2}\right) - Q_0\left(-u,S,\frac{1}{2}\right)\right)\cot\left(\frac{\pi S}{2}\right)\cosh(2\pi u)\ . \quad (3.53)$$

When we go to the point $u = i/2$, the double poles from the third and fourth terms of (3.53) annihilate each other and we have only the contributions from the first, second and fourth terms

$$\partial_u Q(u;S)|_{u=\frac{i}{2}} = -4i\cos^2\left(\frac{\pi S}{2}\right)(\psi(S+1) - \psi(1))\ . \quad (3.54)$$

$Q(u;S)$ is an even function, therefore $\partial_u Q(u;S)|_{u=-i/2} = -\partial_u Q(u;S)|_{u=i/2}$. Thus, the formula for the dimension (3.33) simplifies to

$$\Delta = 2 + S + 4ig^2\left.\frac{\partial_u Q^+(u;S)}{Q^+(u;S)}\right|_{u=0} + \mathcal{O}(g^4)\ . \quad (3.55)$$

Substituting the obtained results (3.52) and (3.54) into (3.55), we have

$$\Delta = 2 + S + 8g^2(\psi(S+1) - \psi(1)) + \mathcal{O}(g^4) = 2 + S + 8g^2 S_1(S) + \mathcal{O}(g^4)\ . \quad (3.56)$$

[2]Let us cite the result of [34]

$$Q_1(u) = \frac{i\sinh(2\pi u)}{2\sin(\pi M)}\left(\left(1 - i\tan\left(\frac{\pi M}{2}\right)\coth(\pi u)\right)\tilde{Q}(u) - \right. \\ \left. - \left(1 + i\tan\left(\frac{\pi M}{2}\right)\coth(\pi u)\right)\tilde{Q}(-u)\right)\ .$$

With the identification $\tilde{Q}(u) = Q_0(u;S,1/2)$ and $M = S$ we get for our solution $Q(u;S)$

$$Q(u,S) = 2\cos^2\left(\frac{\pi S}{2}\right)Q_1(u)\ ,$$

which shows the consistency of $Q(u;S)$ with the solution in [34].

which is equal to (3.35). Therefore, we managed to build the solution of the Baxter equation (3.15), which reproduces the 1-loop anomalous dimension (3.35) for non-integer S. However, the asymptotic of the solution to this Baxter equation is no longer u^S, but is a mixture of the component and the solution with the asymptotic u^{-S-1} enhanced by the exponentials $e^{2\pi u}$ and $e^{-2\pi u}$. Let us now proceed with the finding of the other μ-functions in the LO in the coupling constant g.

3.1.3 Reconstructing other P and μ. Leading order in g^2.

Up to the normalization in the leading order $\mu_{12}^{(0)} = \alpha Q^-(u;S)$, where α is the normalization constant to be determined. To do this, let us use the equation

$$\mu_{12}^{++} - \mu_{12} = \mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1 \tag{3.57}$$

expanded in the LO

$$\mu_{12}^{(0)++} - \mu_{12}^{(0)} = -\mathbf{P}_2^{(0)}\tilde{\mathbf{P}}_1^{(0)} \ . \tag{3.58}$$

Exactly as in [20], but for non-integer S, we get

$$\alpha = -\frac{1}{(\partial_u Q^+(u;S) - \partial_u Q^-(u;S))|_{u=0}} = -\frac{i}{8\cos^2\frac{\pi S}{2} S_1(S)} \ , \tag{3.59}$$

where S_1 is a harmonic sum. One should note that also due to the formula (3.9) and taking into account that $A_3^{(0)} = -iS(S+1)$ we can immediately write $\mu_{13}^{(0)}$

$$\mu_{13}^{(0)} = \alpha u^2\left(Q^+ - Q^-\right) - \alpha i S(S+1) u Q^- \ . \tag{3.60}$$

Note, that as we introduce two expansion parameters $w = S+1$ and $\Lambda = g^2/w$, it means that we have μ_{12} and μ_{13} in the order Λ^{-1} and in all orders in w.

In addition, we are able now to update $\mathbf{P}_1$. Let us start from writing $\tilde{\mathbf{P}}_1$

$$\tilde{\mathbf{P}}_1 = A_1\frac{x^2}{g^2} + \sum_{k=1}^{+\infty} c_{1,k} x^{2k+2} = g^{-2}\left(u + \sum_{k=1}^{+\infty} c_{1,k}^{(0)} u^{2k+1}\right) + \mathcal{O}(g^0) \ . \tag{3.61}$$

Thus, we obtain

$$\mu_{12}^{(0)++} - \mu_{12}^{(0)} = \alpha(Q^+(u;S) - Q^-(u;S)) = u + c_{1,1}^{(0)} u^3 + \mathcal{O}(u^5) \ . \tag{3.62}$$

We know that $Q(u;S)$ is an even function, then $Q^+(u;S) - Q^-(u;S)$ is an odd function and therefore its expansion near $u = 0$ does not contain the term u^2, which is consistent with the RHS of (3.62). In addition, we derive from (3.62) that

$$c_{1,1}^{(0)} = \frac{\alpha}{3} Q'''\left(\frac{i}{2}; S\right) \ . \tag{3.63}$$

Finding $\tilde{\mathbf{P}}_2^{(0)}$ and $\mathbf{P}_4^{(0)}$

As in the case of integer S, to proceed further, let us write down two of the $\mathbf{P}\mu$-system equations in the LO

$$\begin{aligned}\tilde{\mathbf{P}}_2^{(0)} &= \mu_{12}^{(0)}\mathbf{P}_4^{(0)} - \mu_{14}^{(0)}\mathbf{P}_2^{(0)} \,, \\ \tilde{\mathbf{P}}_2^{(0)} &= \mu_{12}^{(0)++}\mathbf{P}_4^{(0)} - \mu_{14}^{(0)++}\mathbf{P}_2^{(0)} \,.\end{aligned} \tag{3.64}$$

Then, combining and shifting the equations (3.64), we obtain

$$\left(\frac{\tilde{\mathbf{P}}_2}{\mathbf{P}_2}\right)^{(0)+} - \left(\frac{\tilde{\mathbf{P}}_2}{\mathbf{P}_2}\right)^{(0)-} = \mu_{12}^{(0)+}\left(\left(\frac{\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}}\right)^{+} - \left(\frac{\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}}\right)^{-}\right) \,, \tag{3.65}$$

whose right part in the leading g^2 order reduces to

$$\left(\frac{\tilde{\mathbf{P}}_2}{\mathbf{P}_2}\right)^{(0)+} - \left(\frac{\tilde{\mathbf{P}}_2}{\mathbf{P}_2}\right)^{(0)-} = 2i\alpha Q(u;S)A_4^{(0)}u \,. \tag{3.66}$$

The solution from [20] considered for non-integer S gives us the solution of (3.66)

$$\frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}} = u^2K((iu+\delta)(Q^+ - Q^-) + \frac{1}{2}(Q^+ + Q^-)) + A(u) \,, \tag{3.67}$$

where K and δ are the constants to be fixed, $A(u)$ is an i-periodic function. In the case of integer S the constant a was shown to be equal to 0, here we can also choose $A(u) = 0$ and this will also be consistent with the integer S case. They can be fixed from the conjectured form of $\tilde{\mathbf{P}}_2$ in the leading order in g and small u. So, we have

$$\begin{aligned}\tilde{\mathbf{P}}_2^{(0)} &= u + \sum_{k=1}^{+\infty} c_{2,k}u^{2k+1} \,, \\ \tilde{\mathbf{P}}_2^{(0)} &= KQ\left(\frac{i}{2};S\right)u + K\delta\left(\partial_u Q^+ - \partial_u Q^-\right)\Big|_{u=0}u^2 + \mathcal{O}\left(u^4\right) \,.\end{aligned} \tag{3.68}$$

Thus, from (3.68) we see that $K = 1/Q(i/2;S)$ and $\delta = 0$. The solution takes the form

$$\tilde{\mathbf{P}}_2 = \frac{u\left(iu(Q^+ - Q^-) + \frac{1}{2}(Q^+ + Q^-)\right)}{Q\left(\frac{i}{2};S\right)} \,. \tag{3.69}$$

Plugging this result into the equation (3.66), we reproduce $A_4^{(0)}$

$$A_4^{(0)} = \frac{(S+2)(S-1)}{2}\frac{(\partial_u Q^+(u;S) - \partial_u Q^-(u;S))|_{u=0}}{Q(\frac{i}{2};S)} = -2i(S+2)(S-1)S_1(S) \,. \tag{3.70}$$

It should be noted, that actually one can add a periodic function to $\tilde{\mathbf{P}}_2$. Also, let's find $c_{2,1}^{(0)}$ from (3.69)

$$c_{2,1}^{(0)} = \frac{2iQ'\left(\frac{i}{2};S\right) + \frac{1}{2}Q''\left(\frac{i}{2};S\right)}{Q\left(\frac{i}{2};S\right)} \,. \tag{3.71}$$

Now we are able to proceed with finding the other elements of the $\mathbf{P}\mu$-system.

Finding $\mu_{14}^{(0)}$

Using one of the equations of the $\mathbf{P}\mu$-system, we find

$$\tilde{\mathbf{P}}_2^{(0)} = \mu_{12}^{(0)}\mathbf{P}_4^{(0)} - \mu_{14}^{(0)}\mathbf{P}_2^{(0)} \,. \tag{3.72}$$

From this equation we are able to find $\mu_{14}^{(0)}$ in the leading order

$$\mu_{14}^{(0)} = \mu_{12}^{(0)}\frac{\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}} - \frac{\tilde{\mathbf{P}}_2^{(0)}}{\mathbf{P}_2^{(0)}} \,. \tag{3.73}$$

After the easy calculations we find

$$\mu_{14}^{(0)} = -\frac{u^2}{Q\left(\frac{i}{2};S\right)}\left[\left(iu+\frac{1}{2}\right)Q^{+} + \left(-iu+\frac{S^2+S-1}{2}\right)Q^{-}\right] \,. \tag{3.74}$$

Using the fact that $Q(u;S)$ is a solution of the Baxter equation (3.15), we can rewrite μ_3 in a very nice form

$$\mu_{14}^{(0)+} = \frac{\left(u^2+\frac{1}{4}\right)^2}{2Q\left(\frac{i}{2};S\right)}(Q^{++} - 2Q + Q^{--}) \,. \tag{3.75}$$

It should be noted, that $\mu_{14}^{(0)+}(-u) = \mu_{14}^{(0)+}(u)$.

Finding $\mu_{24}^{(0)}$ and $\mu_{34}^{(0)}$

To find the other μ-functions we have to use the other $\mathbf{P}\mu$-system equations. The $\mathbf{P}\mu$-equations for $\mu_{24}^{(0)}$ and $\mu_{34}^{(0)}$ are

$$\begin{aligned}
\mu_{24}^{(0)++} - \mu_{24}^{(0)} &= -\left(\mathbf{P}_4^{(0)}\right)^2\mu_{12}^{(0)} + 2\mathbf{P}_2^{(0)}\mathbf{P}_4^{(0)}\mu_{14}^{(0)} - \mathbf{P}_2^{(0)}\mathbf{P}_3^{(0)}\mu_{24}^{(0)} + \left(\mathbf{P}_2^{(0)}\right)^2\mu_{34}^{(0)} \,, \\
\mu_{34}^{(0)++} - \mu_{34}^{(0)} &= -\left(\mathbf{P}_4^{(0)}\right)^2\mu_{13}^{(0)} + 2\mathbf{P}_3^{(0)}\mathbf{P}_4^{(0)}\mu_{14}^{(0)} - \left(\mathbf{P}_3^{(0)}\right)^2\mu_{24}^{(0)} + \mathbf{P}_2^{(0)}\mathbf{P}_3^{(0)}\mu_{34}^{(0)} \,.
\end{aligned} \tag{3.76}$$

Excluding $\mu_{34}^{(0)}$ with the help of the formula

$$\mu_{34}^{(0)} = \frac{\left(\mathbf{P}_4^{(0)}\right)^2}{\left(\mathbf{P}_2^{(0)}\right)^2}\mu_{12}^{(0)} - \frac{2\mathbf{P}_4^{(0)}}{\mathbf{P}_2^{(0)}}\mu_{14}^{(0)} + \frac{1}{\left(\mathbf{P}_2^{(0)}\right)^2}\mu_{24}^{(0)++} + \left(\frac{\mathbf{P}_3^{(0)}}{\mathbf{P}_2^{(0)}} - \frac{1}{\left(\mathbf{P}_2^{(0)}\right)^2}\right)\mu_{24}^{(0)} \,, \tag{3.77}$$

we obtain the Baxter equation with source, which after shifting by $-i/2$ takes the form

$$\begin{aligned}
&\frac{1}{\left(\mathbf{P}_2^{(0)+}\right)^2}\mu_{24}^{(0)[+3]} + \frac{1}{\left(\mathbf{P}_2^{(0)-}\right)^2}\mu_{24}^{(0)-} + \left(\frac{\mathbf{P}_3^{(0)+}}{\mathbf{P}_2^{(0)+}} - \frac{\mathbf{P}_3^{(0)-}}{\mathbf{P}_2^{(0)-}} - \frac{1}{\left(\mathbf{P}_2^{(0)+}\right)^2} - \frac{1}{\left(\mathbf{P}_2^{(0)-}\right)^2}\right)\mu_{24}^{(0)+} = \\
&\qquad = -\left(\frac{\mathbf{P}_4^{(0)+}}{\mathbf{P}_2^{(0)+}}\right)^2\mu_{12}^{(0)+} + \left(\frac{\mathbf{P}_3^{(0)-}}{\mathbf{P}_2^{(0)-}} + \frac{1}{\left(\mathbf{P}_2^{(0)-}\right)^2}\right)\left(\mathbf{P}_4^{(0)-}\right)^2\mu_{12}^{(0)-} - \\
&\qquad\qquad - \left(\mathbf{P}_4^{(0)-}\right)^2\mu_{13}^{(0)-} + 2\frac{\mathbf{P}_4^{(0)+}}{\mathbf{P}_2^{(0)+}}\mu_{14}^{(0)+} - 2\frac{\mathbf{P}_4^{(0)-}}{\mathbf{P}_2^{(0)-}}\mu_{14}^{(0)-} \,.
\end{aligned} \tag{3.78}$$

Substituting **P**-functions from (3.14) and remembering that $c^{(0)}_{4,1}=0$, we obtain

$$\left(u+\frac{i}{2}\right)^2\mu^{(0)[+3]}_{24}+\left(u-\frac{i}{2}\right)^2\mu^{(0)-}_{24}-\left(2u^2-S(S+1)-\frac{1}{2}\right)\mu^{(0)+}_{24}=-\frac{(S-1)(S+2)}{2\alpha Q^2\left(\frac{i}{2};S\right)}\times$$
$$\times\, u\left[4u\left(u-\frac{i}{2}\right)^2Q^{--}+Q\left(2u\left(S(S+1)+\frac{1}{2}-2u^2\right)+i(8u^2-1)\right)\right]\ . \quad (3.79)$$

Or, utilizing the Baxter equation (3.36), we can write down (3.79) in a more symmetric form

$$\left(u+\frac{i}{2}\right)^2\mu^{(0)[+3]}_{24}+\left(u-\frac{i}{2}\right)^2\mu^{(0)-}_{24}-\left(2u^2-S(S+1)-\frac{1}{2}\right)\mu^{(0)+}_{24}=$$
$$-\frac{(S-1)(S+2)}{2\alpha Q^2(i/2)}u\left[2u\left(u-\frac{i}{2}\right)^2Q^{--}-2u\left(u+\frac{i}{2}\right)^2Q^{++}+i(8u^2-1)Q\right]\ . \quad (3.80)$$

This equation (3.80) is of the second order and is hard to solve. So our strategy is as follows: we find some solution of the inhomogeneous Baxter equation (3.80) by some other means, add to it the solution of the homogeneous equation, which is $Q(u;S)$ multiplied by a constant, and then fix this constant. Let us use the constraint

$$\mu_{12}\mu_{34}-\mu_{13}\mu_{24}+\mu^2_{14}=1\ , \quad (3.81)$$

from which it follows that in the leading order in g

$$\mu^{(0)}_{34}=\mu^{(0)}_{13}\frac{\mu^{(0)}_{24}}{\mu^{(0)}_{12}}-\frac{\mu^{(0)2}_{14}}{\mu^{(0)}_{12}}\ . \quad (3.82)$$

Then, putting (3.82) into the equation (3.77), we get the simpler equation for $\mu^{(0)}_{24}$

$$\mu^{(0)}_{13}\frac{\mu^{(0)}_{24}}{\mu^{(0)}_{12}}-\frac{\mu^{(0)2}_{14}}{\mu^{(0)}_{12}}=$$
$$=\frac{\left(\mathbf{P}^{(0)}_4\right)^2}{\left(\mathbf{P}^{(0)}_2\right)^2}\mu^{(0)}_{12}-\frac{2\mathbf{P}^{(0)}_4}{\mathbf{P}^{(0)}_2}\mu^{(0)}_{14}+\frac{\mu^{(0)++}_{24}}{\left(\mathbf{P}^{(0)}_2\right)^2}+\left(\frac{\mathbf{P}^{(0)}_3}{\mathbf{P}^{(0)}_2}-\frac{1}{\left(\mathbf{P}^{(0)}_2\right)^2}\right)\mu^{(0)}_{24}\ , \quad (3.83)$$

which is not the second order Baxter equation. Because we know all the quantities there, after some simplifications we get

$$\frac{\mu^{(0)++}_{24}}{\left(\mathbf{P}^{(0)}_2\right)^2}+\left(\frac{\mathbf{P}^{(0)}_3}{\mathbf{P}^{(0)}_2}-\frac{1}{\left(\mathbf{P}^{(0)}_2\right)^2}-\frac{\mu^{(0)}_{13}}{\mu^{(0)}_{12}}\right)\mu^{(0)}_{24}=-\mu^{(0)}_{12}\left(\frac{\mathbf{P}^{(0)}_4}{\mathbf{P}^{(0)}_2}-\frac{\mu^{(0)}_{14}}{\mu^{(0)}_{12}}\right)^2\ . \quad (3.84)$$

Substituting all **P**-functions from (3.14) and μ-functions from (3.60) and (3.74) and dividing by u^2Q^+, we obtain the following equation

$$\left(\frac{\mu^{(0)}_{24}}{Q^-}\right)^{++}-\left(\frac{\mu^{(0)}_{24}}{Q^-}\right)=\frac{u^2}{\alpha Q^2\left(\frac{i}{2};S\right)}\left(\left(u-\frac{i}{2}\right)^2\frac{Q^+}{Q^-}+\left(u+\frac{i}{2}\right)^2\frac{Q^-}{Q^+}-2u^2-\frac{1}{2}\right)\ . \quad (3.85)$$

Due to the fact that

$$\left(u^2\left(u-\frac{i}{2}\right)^2\frac{Q^+}{Q^-}\right)^{++} = \\ = -u^2\left(u+\frac{i}{2}\right)^2\frac{Q^-}{Q^+}+\left(u+\frac{i}{2}\right)^2\left(2\left(u+\frac{i}{2}\right)^2-S^2-S-\frac{1}{2}\right)\ , \quad (3.86)$$

we are able to find the solution of the equation (3.85). Making the substitution

$$\frac{\mu_{24}^{(0)}}{Q^-}=\frac{F(u)}{\alpha Q^2\left(\frac{i}{2};S\right)}-\frac{u^2}{\alpha Q^2\left(\frac{i}{2};S\right)}\left(u-\frac{i}{2}\right)^2\frac{Q^+}{Q^-}\ , \quad (3.87)$$

the equation for $F(u)$ is

$$F\left(u+\frac{i}{2}\right)-F\left(u-\frac{i}{2}\right)=-4iu^3+(S-1)(S+2)u^2+\frac{iu}{2} \quad (3.88)$$

and it has the solution

$$F(u)=\frac{u}{3}\left(u^2+\frac{1}{4}\right)(3u+i(S-1)(S+2))+\beta\alpha Q^2\left(\frac{i}{2};S\right)\ , \quad (3.89)$$

where β is a constant. Thus, substituting (3.89) into (3.87), we get

$$\mu_{24}^{(0)}=\frac{u\left(u-\frac{i}{2}\right)}{\alpha Q^2\left(\frac{i}{2};S\right)}\left(\frac{1}{3}\left(u+\frac{i}{2}\right)(3u+i(S-1)(S+2))Q^- - u\left(u-\frac{i}{2}\right)Q^+\right)+\beta Q^-\ . \quad (3.90)$$

However, it is possible to have the last expression in a more symmetric form. Let us first shift $u\to u+i/2$, then

$$\mu_{24}^{(0)+}=\frac{u\left(u+\frac{i}{2}\right)}{\alpha Q^2(i/2)}\left(\frac{1}{3}(u+i)\left(3u+i\left(S^2+S-\frac{1}{2}\right)\right)Q+u\left(u+\frac{i}{2}\right)Q^{++}\right)+\beta Q\ . \quad (3.91)$$

Using the Baxter equation (3.36) for Q, we derive $\mu_{24}^{(0)+}$ in a more symmetric form

$$\mu_{24}^{(0)+}=-\frac{iu(u^2+1/4)}{3\alpha Q^2\left(\frac{i}{2};S\right)}\left(\left(u+\frac{i}{2}\right)(u-i)Q^{++}-2(u-i)(u+i)Q+ \right. \\ \left. +\left(u-\frac{i}{2}\right)(u+i)Q^{--}\right)+\beta Q\ . \quad (3.92)$$

One can now check that (3.92) really solves the second order Baxter equation (3.80). However we still have one non-fixed parameter β, which we are going to find further.

Fixing the coefficient β

We can fix the coefficient β from the equation

$$\mu_{24}^{(0)++}-\mu_{24}^{(0)}=\mathbf{P}_2^{(0)}\tilde{\mathbf{P}}_4^{(0)}-\mathbf{P}_4^{(0)}\tilde{\mathbf{P}}_2^{(0)}\ . \quad (3.93)$$

From the fact that $c^{(0)}_{4,1} = 0$ we conclude that $\tilde{\mathbf{P}}^{(0)}_4 = 0$ and in the leading order in g^2 and close to $u = 0$ we obtain

$$\mu^{(0)++}_{24}(u) - \mu^{(0)}_{24}(u) = \mathcal{O}(u^2) \,, \tag{3.94}$$

where we also used (3.14) and the first formula from (3.26). On the other hand, one can get for the obtained solution (3.92) for $\mu_{24}(u)$

$$\mu^{(0)++}_{24} - \mu^{(0)}_{24} = -8i\beta \cos^2\left(\frac{\pi S}{2}\right) S_1(S)u + \mathcal{O}(u^2) \,, \tag{3.95}$$

which immediately gives $\beta = 0$. Now we are able to find the last remaining μ-function from the LO $\mathbf{P}\mu$-system.

Calculation of $\mu^{(0)}_{34}$

Because we know all the quantities of the LO $\mathbf{P}\mu$-system except for $\mu^{(0)}_{34}$, we can just use (3.82), which gives us

$$\mu^{(0)}_{34} = \frac{u^2(u-i)}{12\alpha g^4 Q^2(i/2)} \Big((-4i(S^2+S+1)u^2 + (S(S+1)(S^2+S+2) - 2)u + \\ + i(S-1)S(S+1)(S+2))Q^- + 4i(S^2+S+1)u\left(u - \frac{i}{2}\right)Q^+\Big) \,. \tag{3.96}$$

However, it can be rewritten in a simpler form using the Baxter equation (3.36) for Q

$$\mu^{(0)+}_{34} = \frac{\left(u^2+\frac{1}{4}\right)^2}{12\alpha Q^2\left(\frac{i}{2}\right)} \Big[2i(S^2+S+1)\left(\left(u+\frac{i}{2}\right)Q^{++} - \left(u-\frac{i}{2}\right)Q^{--}\right) + \\ + (S^4+2S^3+7S^2+6S+2)Q\Big] \,. \tag{3.97}$$

Having found now all the elements of the $\mathbf{P}\mu$-system we can compactly write them down.

3.1.4 Summary of the leading order P- and μ-functions

Let us start from writing the values of the $\mathbf{P}$-functions in the LO in g^2

$$\mathbf{P}^{(0)}_1 = \frac{1}{u^2} \,, \quad \mathbf{P}^{(0)}_2 = \frac{1}{u} \,, \quad \mathbf{P}^{(0)}_3 = -iS(S+1) \,, \quad \mathbf{P}^{(0)}_4 = -\frac{(S-1)(S+2)}{2\alpha Q\left(\frac{i}{2};S\right)}u \,, \tag{3.98}$$

where α is given by (3.59). The list of the LO μ-functions looks as follows

$$\begin{aligned}
&\mu_{12}^{(0)} = \alpha Q^- \ , \quad \mu_{13}^{(0)} = \alpha u^2 \left(Q^+ - Q^-\right) - \alpha i S(S+1) u Q^- \ , \\
&\mu_{14}^{(0)} = -\frac{u^2}{Q\left(\frac{i}{2};S\right)} \left[\left(iu + \frac{1}{2}\right) Q^+ + \left(-iu + \frac{S^2+S-1}{2}\right) Q^-\right] \ , \\
&\mu_{24}^{(0)} = \frac{u\left(u-\frac{i}{2}\right)}{\alpha Q^2\left(\frac{i}{2};S\right)} \left(\frac{1}{3}\left(u+\frac{i}{2}\right)(3u + i(S-1)(S+2))Q^- - u\left(u-\frac{i}{2}\right)Q^+\right) \\
&\mu_{34}^{(0)} = \frac{u^2(u-i)}{12\alpha g^4 Q^2(i/2)} \Big((-4i(S^2+S+1)u^2 + (S(S+1)(S^2+S+2)-2)u + \\
&+ i(S-1)S(S+1)(S+2))Q^- + 4i(S^2+S+1)u\left(u-\frac{i}{2}\right)Q^+\Big) \ .
\end{aligned} \tag{3.99}$$

Then, we are able to present the list of μ-functions in a different form with the usage of (3.36), in which their parity is more manifest

$$\begin{aligned}
&\mu_{12}^{(0)+} = \alpha Q \ , \quad \mu_{13}^{(0)+} = i\alpha\left(u^2+\frac{1}{4}\right)\left(\left(u+\frac{i}{2}\right)Q^{++} - 2uQ + \left(u-\frac{i}{2}\right)Q^{--}\right) \ , \\
&\mu_{14}^{(0)+} = \frac{\left(u^2+\frac{1}{4}\right)^2}{2Q\left(\frac{i}{2};S\right)}(Q^{++} - 2Q + Q^{--}) \ , \\
&\mu_{24}^{(0)+} = -\frac{iu(u^2+1/4)}{3\alpha Q^2\left(\frac{i}{2};S\right)}\left(\left(u+\frac{i}{2}\right)(u-i)Q^{++} - 2(u-i)(u+i)Q + \right. \\
&\left. + \left(u-\frac{i}{2}\right)(u+i)Q^{--}\right) \ , \\
&\mu_{34}^{(0)+} = \frac{\left(u^2+\frac{1}{4}\right)^2}{12\alpha Q^2\left(\frac{i}{2}\right)}\Big[2i(S^2+S+1)\left(\left(u+\frac{i}{2}\right)Q^{++} - \left(u-\frac{i}{2}\right)Q^{--}\right) + \\
&+ (S^4+2S^3+7S^2+6S+2)Q\Big] \ .
\end{aligned} \tag{3.100}$$

Now, having the results (3.98), (3.99) and (3.100) we can study their expansion in the vicinity of the point $S=-1$ as this point is particularly interesting for the study of the BFKL limit.

3.1.5 Expansion around the point $S=-1$

The most important ingredient to find the expansion of the obtained LO solution (3.98), (3.99) and (3.100) of the $\mathbf{P}\mu$-system is to calculate the function $Q(u;S)$ given by (3.51) in the vicinity of the point $S=-1$. The main obstacle is the generalized hypergeometric function, which we treat by using the series representation

$$Q_0\left(u;S,\frac{1}{2}\right) = 1 + \sum_{k=1}^{+\infty} \frac{(-S)_k\,(1+S)_k\left(\frac{1}{2}-iu\right)_k}{(k!)^3} \ , \tag{3.101}$$

where $(x)_k$ is the Pochhammer symbol. We derive the following from (3.101)

$$Q_0\left(u;-1+w,\frac{1}{2}\right)=1-w\left(\psi^{(0)}\left(\frac{1}{2}+iu\right)+\gamma_E\right)+ \\ +w^2\left(-\frac{1}{2}\psi^{(1)}\left(\frac{1}{2}+iu\right)+\frac{1}{2}\left(\psi^{(0)}\left(\frac{1}{2}+iu\right)+\gamma_E\right)^2+\frac{\pi^2}{12}\right)+\mathcal{O}(w^3)\ . \quad (3.102)$$

Expanding the solution (3.100) in the parameter $w=S+1$

$$\begin{aligned}
\mu_{12}^{(0)+}&=\frac{i\cosh^2(\pi u)}{\pi^2 w}\left(1-\frac{w}{2}\Psi(u)\right)+\mathcal{O}(w)\ , \quad (3.103)\\
\mu_{13}^{(0)+}&=\frac{wu\cosh^2(\pi u)}{\pi^2}+\mathcal{O}(w^2)\ ,\\
\mu_{14}^{(0)+}&=-\frac{(4u^2+1)\cosh^2(\pi u)}{2\pi^2 w}\left(1-\frac{w}{2}\left(\Psi(u)-2\right)\right)+\mathcal{O}(w)\ ,\\
\mu_{24}^{(0)+}&=\frac{4u(4u^2+1)\cosh^2(\pi u)}{3\pi^2 w^3}\left(1-\frac{w}{2}\left(\Psi(u)-4\right)\right)+\mathcal{O}(w^{-1})\ ,\\
\mu_{34}^{(0)+}&=-\frac{i(16u^4-8u^2-3)\cosh^2(\pi u)}{12\pi^2 w}+\mathcal{O}(w^0)\ ,
\end{aligned}$$

where

$$\Psi(u)=\psi^{(0)}\left(\frac{1}{2}-iu\right)+\psi^{(0)}\left(\frac{1}{2}+iu\right)-2\psi^{(0)}(1)\ . \quad (3.104)$$

Utilizing the rescaling symmetry of the $\mathbf{P}\mu$-system with the transformation matrix

$$H_B=\begin{pmatrix} w & 0 & 0 & 0\\ 0 & \frac{1}{w^2} & 0 & 0\\ 0 & 0 & w^2 & 0\\ 0 & 0 & 0 & \frac{1}{w}\end{pmatrix} \quad (3.105)$$

we are able to make all the μ-functions scale in the same way as w approaches 0. This fact will be very important in our consideration of the BFKL limit in the subsequent Sections. Now let us turn to the calculation of the $\mathbf{P}$- and μ-functions in the NLO in g.

3.1.6 Next-to-leading order

Let us start from the brief summary of $\mathbf{P}$-functions in the NLO order in g

$$\mathbf{P}_1=\frac{g^2}{u^2}+\mathcal{O}(g^4)\ ,\quad \mathbf{P}_2=\frac{1}{u}+\frac{g^2}{u^3}+\mathcal{O}(g^4)\ , \quad (3.106)$$
$$\mathbf{P}_3=A_3^{(0)}+g^2\left(A_3^{(1)}+\frac{c_{3,1}^{(0)}}{u^2}\right)+\mathcal{O}(g^4)\ ,\quad \mathbf{P}_4=A_4^{(0)}u+g^2\left(A_4^{(1)}u+\frac{c_{4,1}^{(0)}}{u}\right)+\mathcal{O}(g^4)\ .$$

Updating $\mathbf{P}_3$

We can update $\mathbf{P}_3$ from using the asymptotic conditions for the coefficient A_3

$$A_3=-\mathbf{A}_3=-\frac{[S^2-\Delta^2][(2-S)^2-\Delta^2]}{32i}\ , \quad (3.107)$$

from which $A_3^{(1)}$ can be found just putting $\Delta = 2 + S + 8g^2 S_1(S) + \mathcal{O}(g^4)$

$$A_3^{(1)} = -2i(S+2)(3S+1)S_1(S) \; . \tag{3.108}$$

To fix the coefficient $c_{3,1}^{(0)}$ we use the following trick. The principal value of μ_{ab}, i.e. $\not{\mu}_{ab} = (\mu_{ab} + \tilde{\mu}_{ab})/2 = (\mu_{ab} + \mu_{ab}^{++})/2$ is free of any singularities in the complex plane, thus taking the half-sum of the $\mathbf{P}\mu$-equations, we obtain

$$\tilde{\mathbf{P}}_a = \not{\mu}_{ab} \chi^{bc} \mathbf{P}_c \; . \tag{3.109}$$

Now let us turn to the $\mathbf{P}\mu$-equation for $\tilde{\mathbf{P}}_1$

$$\tilde{\mathbf{P}}_1 = \not{\mu}_{12}\mathbf{P}_3 - \not{\mu}_{13}\mathbf{P}_2 + \not{\mu}_{14}\mathbf{P}_1 \; . \tag{3.110}$$

Note, that to find $\mathbf{P}_3$ in the order $\mathcal{O}(g^2)$ at the first sight we need $\not{\mu}_{12,13,14}$ in the order $\mathcal{O}(g^0)$. In fact, that's not true, because we are looking for a singular part of $\mathbf{P}_3$. First, consider $\tilde{\mathbf{P}}_1$

$$\begin{aligned} \tilde{\mathbf{P}}_1 = x^2 + \sum_{k=1}^{+\infty} c_{1,k} x^{2(k+1)} = \\ = \frac{u^2}{g^2} - 2 + \sum_{k=1}^{+\infty} \frac{u^{2(k+1)}}{g^2} \left(c_{1,k}^{(0)} + g^2 \left(c_{1,k}^{(1)} - \frac{2(k+1)}{u^2} c_{1,k}^{(0)} \right) \right) + \mathcal{O}(g^2) \; , \end{aligned} \tag{3.111}$$

or, briefly,

$$\tilde{\mathbf{P}}_1 = \frac{1}{g^2}\left(u^2 + c_{1,1}^{(0)} u^4 + \mathcal{O}(u^6)\right) - 2\left(1 + c_{1,1}^{(0)} u^2 + \mathcal{O}(u^4)\right) + \mathcal{O}(g^2) \; , \tag{3.112}$$

which means, that $\tilde{\mathbf{P}}_1$ does not contain any singularities in the orders $\mathcal{O}(g^{-2})$ and $\mathcal{O}(g^0)$ in the point $u = 0$. To proceed let us consider the equation (3.110) in the NLO

$$\tilde{\mathbf{P}}_1^{(1)} = \not{\mu}_{12}^{(0)}\mathbf{P}_3^{(1)} + \not{\mu}_{12}^{(1)}\mathbf{P}_3^{(0)} - \not{\mu}_{13}^{(0)}\mathbf{P}_2^{(1)} - \not{\mu}_{13}^{(1)}\mathbf{P}_2^{(0)} + \not{\mu}_{14}^{(0)}\mathbf{P}_1^{(1)} \; . \tag{3.113}$$

The term $\not{\mu}_{12}^{(1)}\mathbf{P}_3^{(0)} = A_3^{(0)}\not{\mu}_{12}^{(1)}$ has no singularities as well because the functions $\not{\mu}_{ab}$ are analytic in the complex plane. We know the expression

$$\not{\mu}_{13}^{(0)}\mathbf{P}_2^{(1)} = -\frac{i\alpha S(S+1)Q\left(\frac{i}{2}; S\right)}{u^2} + \mathcal{O}(u^0) \; . \tag{3.114}$$

Remembering that μ_{13}^{+} is an odd function, we see that the Taylor expansion of $\not{\mu}_{13}(u)$ in the vicinity of $u = 0$ goes over the odd powers of u. This means that $\not{\mu}_{13}^{(1)}\mathbf{P}_2^{(0)} = \not{\mu}_{13}^{(1)}/u$ is analytic when u approaches 0. Let us analyse $\not{\mu}_{14}^{(0)}\mathbf{P}_1^{(1)} = \not{\mu}_{14}^{(0)}/u^2$, the latter one. We have

$$\not{\mu}_{14}^{(0)} = -\frac{u^2}{4Q\left(\frac{i}{2}; S\right)}\left((S^2 + S + 4iu)Q^{+} + (S^2 + S - 4iu)Q^{-}\right) + \mathcal{O}(g^2) \; , \tag{3.115}$$

which possesses the following expansion at $u = 0$

$$\not{\mu}_{14}^{(0)} = -\frac{S(S+1)}{2}u^2 + \mathcal{O}(u^4) \,. \tag{3.116}$$

Then the term $\not{\mu}_{14}^{(0)}\mathbf{P}_1^{(1)}$ has no no singularities at the origin. Finally, let us consider

$$\not{\mu}_{12}^{(0)}\mathbf{P}_3^{(1)} = \frac{\alpha c_{3,1}^{(0)} Q\left(\frac{i}{2}; S\right)}{u^2} + \mathcal{O}(u^0) \,. \tag{3.117}$$

Recalling that the only singular terms in the equation (3.113), which are (3.114) and (3.117), the discussion above leads us to the conclusion that $c_{3,1}^{(0)} = -iS(S+1)$. Let us now turn to the determination of the function μ_{12} in the NLO as now we have all $\mathbf{P}$- and μ-functions in the LO and all $\mathbf{P}$-functions in the NLO up to one coefficient $A_4^{(1)}$, which contains the 2-loop anomalous dimension $\gamma^{(2)}$, which was not yet fixed.

Updating $\mathbf{P}_4$

Using the energy in the NLO, we can completely reconstruct $\mathbf{P}$'s in the next to leading order. Let us first write the result for the energy

$$\begin{aligned}
&\Delta = 2 + S + g^2\gamma^{(1)} + g^4\gamma^{(2)} + \mathcal{O}(g^6) \,, \\
&\gamma^{(1)} = 8S_1(S) = -\frac{8}{w} + \frac{4\pi^2 w}{3} + \mathcal{O}(w^2) \,, \\
&\gamma^{(2)} = -16(S_3(S) + S_{-3}(S) - 2S_{-2,1}(S) + 2S_1(S)(S_2(S) + S_2(S))) = \mathcal{O}(w^0) \,.
\end{aligned} \tag{3.118}$$

From the above formulas for A_3 we easily obtain

$$A_3^{(0)} = -iS(S+1) \,, \quad A_3^{(1)} = -\frac{i}{4}(S+2)(3S+1)\gamma^{(1)} \,. \tag{3.119}$$

And for A_4

$$A_4^{(0)} = -\frac{i}{4}(S-1)(S+2)\gamma^{(1)} \,, \quad A_4^{(1)} = -\frac{i}{24}((S^2+7S+1)(\gamma^{(1)})^2 + 6(S^2+S-2)\gamma^{(2)}) \,. \tag{3.120}$$

To find the coefficient $c_{4,1}^{(0)}$ we can use one of the $\mathbf{P}\mu$-equations $\tilde{\mathbf{P}}_4 = \mu_{14}\mathbf{P}_1 - \mu_{24}\mathbf{P}_3 + \mu_{34}\mathbf{P}_2$ in the LO, for which from one side

$$\tilde{\mathbf{P}}_4^{(0)} = c_{4,1}^{(0)} u + \mathcal{O}(u^3) \tag{3.121}$$

and from the other side we can calculate the RHS in the LO using the result (3.99)

$$\begin{aligned}
\tilde{\mathbf{P}}_4^{(0)} = \frac{iu^2}{12\alpha Q^2\left(\frac{i}{2}; S\right)} \Big(&2\left(u + \frac{i}{2}\right)\left(4\left(u - \frac{i}{2}\right) + S(S+1)(u-i)\right)Q^{--} - \\
&-2\left(u - \frac{i}{2}\right)\left(4\left(u + \frac{i}{2}\right) + S(S+1)(u+i)\right)Q^{++}\Big) \,.
\end{aligned} \tag{3.122}$$

As the expansion of (3.122) is $\tilde{\mathbf{P}}_4^{(0)} = \mathcal{O}(u^3)$, we arrive to the following

$$c_{4,1}^{(0)} = -\frac{i}{4}(S-1)(S+2)\gamma^{(1)} \ . \tag{3.123}$$

The list of the $\mathbf{P}$'s in the NLO

$$\begin{aligned}
\mathbf{P}_1 &= \frac{g^2}{u^2} + \mathcal{O}(g^4) \ , \\
\mathbf{P}_2 &= \frac{1}{u} + \frac{g^2}{u^3} + \mathcal{O}(g^4) \ , \\
\mathbf{P}_3 &= -iS(S+1) + g^2\left(-\frac{i}{4}(S+2)(3S+1)\gamma^{(1)} - iS(S+1)\frac{1}{u^2}\right) + \mathcal{O}(g^4) \ , \\
\mathbf{P}_4 &= -\frac{i}{4}(S-1)(S+2)\gamma^{(1)}u + \\
&+ g^2\left(-\frac{i}{24}((S^2+7S+1)(\gamma^{(1)})^2 + 6(S-1)(S+2)\gamma^{(2)}\right)u + \mathcal{O}(g^4) \ .
\end{aligned} \tag{3.124}$$

In the vicinity of the point $S = -1 + w$

$$\begin{aligned}
\mathbf{P}_1 &= \frac{g^2}{u^2} + \mathcal{O}(g^4) \ , \\
\mathbf{P}_2 &= \frac{1}{u} + \frac{g^2}{u^3} + \mathcal{O}(g^4) \ , \\
\mathbf{P}_3 &= iw + \mathcal{O}(w^2) + g^2\left(-\frac{4i}{w} + 2i + \mathcal{O}(w) + (iw + \mathcal{O}(w^2))\frac{1}{u^2}\right) + \mathcal{O}(g^4) \ , \\
\mathbf{P}_4 &= \left(-\frac{4i}{w} - 2i + \mathcal{O}(w)\right)u + g^2\left(\frac{40i}{3w^2} - \frac{40i}{3w} + \mathcal{O}(w^0)\right)u + \mathcal{O}(g^4) \ .
\end{aligned} \tag{3.125}$$

Let us proceed now with the calculation of the μ_{12} in the NLO.

Finding μ_{12} in the NLO

To derive the equation for μ_{12} in the NLO, we are going to use the all-loop equation for μ_1, which can be derived by excluding $\tilde{\mathbf{P}}_1$ and μ_{13} from

$$\begin{aligned}
\tilde{\mathbf{P}}_1 &= \mu_{12}\mathbf{P}_3 - \mu_{13}\mathbf{P}_2 + \mu_{14}\mathbf{P}_1 \ , \\
\tilde{\mathbf{P}}_1 &= \mu_{12}^{++}\mathbf{P}_3 - \mu_{13}^{++}\mathbf{P}_2 + \mu_{14}^{++}\mathbf{P}_1 \ .
\end{aligned} \tag{3.126}$$

Such an equation takes the form

$$\begin{aligned}
\frac{\mu_{12}^{[+3]}}{(\mathbf{P}_2^+)^2} + \left(\frac{\mathbf{P}_3^+}{\mathbf{P}_2^+} - \frac{\mathbf{P}_3^-}{\mathbf{P}_2^-} - \frac{1}{(\mathbf{P}_2^+)^2} - \frac{1}{(\mathbf{P}_2^-)^2}\right)\mu_{12}^+ + \frac{\mu_{12}^-}{(\mathbf{P}_2^-)^2} = \\
= \frac{\mathbf{P}_1^+\tilde{\mathbf{P}}_2^+}{(\mathbf{P}_2^+)^2} - \frac{\mathbf{P}_1^-\tilde{\mathbf{P}}_2^-}{(\mathbf{P}_2^-)^2} + \left(\frac{\mathbf{P}_1^-}{\mathbf{P}_2^-} - \frac{\mathbf{P}_1^+}{\mathbf{P}_2^+}\right)\mu_{14}^+ \ .
\end{aligned} \tag{3.127}$$

Let us first write $\mu_{12} = g^{-2}\alpha Q^- + \mu_{12}^{(1)}$, thus we have to extract the coefficient in the equation (3.127) above in the order g^0. We start from the RHS of (3.127) in the NLO

$$\begin{aligned}\frac{\mathbf{P}_1^{(1)+}\tilde{\mathbf{P}}_2^{(0)+}}{(\mathbf{P}_2^{(0)+})^2} - \frac{\mathbf{P}_1^{(1)-}\tilde{\mathbf{P}}_2^{(0)-}}{(\mathbf{P}_2^{(0)-})^2} + \left(\frac{\mathbf{P}_1^{(1)-}}{\mathbf{P}_2^{(0)-}} - \frac{\mathbf{P}_1^{(1)+}}{\mathbf{P}_2^{(0)+}}\right)\mu_{14}^{(0)+} = \\ = \frac{1}{Q\left(\frac{i}{2};S\right)}\left(\left(u+\frac{i}{2}\right)Q^{++} - \left(u-\frac{i}{2}\right)Q^{--} - \frac{3}{2}iS(S+1)Q\right) = \\ = 4i\alpha S_1(S)\left(\left(u+\frac{i}{2}\right)Q^{++} - \left(u-\frac{i}{2}\right)Q^{--} - \frac{3}{2}iS(S+1)S_1(S)Q\right)\ , \quad (3.128)\end{aligned}$$

where Q is the function from (3.51). The LHS of (3.127) in the LO contains besides the LO Baxter equation for μ_{12} the part of the order $\mathcal{O}(g^0)$. It is given by

$$\begin{aligned}\frac{\mu_{12}^{[+3]}}{(\mathbf{P}_2^+)^2} + \left(\frac{\mathbf{P}_3^+}{\mathbf{P}_2^+} - \frac{\mathbf{P}_3^-}{\mathbf{P}_2^-} - \frac{1}{(\mathbf{P}_2^+)^2} - \frac{1}{(\mathbf{P}_2^-)^2}\right)\mu_{12}^+ + \frac{\mu_1^-}{(\mathbf{P}_2^-)^2} = \\ = \left(u+\frac{i}{2}\right)^2\mu_{12}^{(1)[+3]} + \left(u-\frac{i}{2}\right)^2\mu_{12}^{(1)-} - \left(2u^2 - S(S+1) - \frac{1}{2}\right)\mu_{12}^{(1)+} - \\ - 2\alpha Q^{++} - 2\alpha Q^{--} + (4 + 2(3S^2+7S+2)S_1(S))\alpha Q + \mathcal{O}(g^2)\ . \quad (3.129)\end{aligned}$$

Therefore, we get the equation

$$\begin{aligned}\left(u+\frac{i}{2}\right)^2\mu_{12}^{(1)[+3]} + \left(u-\frac{i}{2}\right)^2\mu_{12}^{(1)-} - \left(2u^2 - S(S+1) - \frac{1}{2}\right)\mu_{12}^{(1)+} = \\ = \left(2 + 4iS_1(S)\left(u+\frac{i}{2}\right)\right)\alpha Q^{++} + \left(2 - 4iS_1(S)\left(u-\frac{i}{2}\right)\right)\alpha Q^{--} - \\ - (4 + 4S_1(S)(2S+1))\,\alpha Q\ . \quad (3.130)\end{aligned}$$

Note, that the equation (3.130) is a little bit different from the result [34], which looks as

$$\begin{aligned}\left(u+\frac{i}{2}\right)^2 Q_2^{++} + \left(u-\frac{i}{2}\right)^2 Q_2^{--} - \left(2u^2 - S(S+1) - \frac{1}{2}\right)Q_2 = \\ = \left(2 - 4iS_1(S)\left(u+\frac{i}{2}\right)\right)Q_1^{++} + \left(2 + 4iS_1(S)\left(u-\frac{i}{2}\right)\right)Q_1^{--} - \\ - (4 + 4S_1(S)(2S+1))Q_1\ . \quad (3.131)\end{aligned}$$

We have to underline that here the functions Q_1 and Q_2 are different from the ones in the Subsection 3.1.2. The functions Q and Q_1 only differ by some numerical constant. Let us remember the asymptotics of these functions from the Subsection 3.1.2

$$Q \simeq au^S + bu^{-S-1}e^{2\pi|u|} \quad (3.132)$$

Thus, in our equation (3.130) the right hand side has the asymptotic behaviour $a_1u^S + b_1u^{-S-3}e^{2\pi|u|}$, in (3.131) $a_2u^{S-2} + b_2u^{-S-1}e^{2\pi|u|}$. Let us try to find the solution of the equations (3.130) and (3.131) in the forms u^α and $u^\beta \log u$.

For our equation (3.130) we obtain $\alpha = -S-1$ and $\beta = S$. The solution with the asymptotic $u^{-S-1}\sinh 2\pi u$ is consistent with the leading asymptotic of the RHS

$u^{-S-3}e^{2\pi|u|}$ and the solution with the asymptotic $u^S \log u$ is consistent with the subleading asymptotic of the RHS u^S. So, the resulting asymptotic of the solution is

$$\mu_{12}^{(1)+} \simeq Au^S \log u + Bu^{-S-1}e^{2\pi|u|} \,, \tag{3.133}$$

which agrees with $\mu_{12} \simeq Au^{\Delta-2} + Bu^{-S-1}e^{2\pi|u|}$, if we expand Δ in g^2.

For the equation (3.131) we obtain $\alpha = S$ and $\beta = S$. The solution with the asymptotic $u^{-S-1}e^{2\pi|u|}\log u$ is consistent with the leading asymptotic of the RHS $u^{-S-1}e^{2\pi|u|}$, and the solution with the asymptotic u^S is consistent with the subleading asymptotic of the RHS u^{S-2}. The asymptotic behaviour of the solution of (3.131)

$$Q_2 \simeq \tilde{A}u^S + \tilde{B}u^{-S-1}e^{2\pi|u|}\log u \,. \tag{3.134}$$

Now let us proceed with the solving the equation (3.130). Remembering the notation $\gamma^{(1)} = 8S_1(S)$ for 1-loop anomalous dimension and $\mu_{12}^{(1)} = \alpha g^2 Q_2^-$, we rewrite the NLO Baxter equation in the following way

$$\begin{aligned}&\left(u+\frac{i}{2}\right)^2 Q_2^{++} + \left(u-\frac{i}{2}\right)^2 Q_2^{--} - \left(2u^2 - S(S+1) - \frac{1}{2}\right) Q_2 = \\ &= \left(2 + i\frac{\gamma^{(1)}}{2}\left(u+\frac{i}{2}\right)\right) Q^{++} + \left(2 - i\frac{\gamma^{(1)}}{2}\left(u-\frac{i}{2}\right)\right) Q^{--} - \left(4 + \frac{\gamma^{(1)}}{2}(2S+1)\right) Q \,,\end{aligned} \tag{3.135}$$

where Q in (3.135) is understood as $Q(u;S)$ from (3.51). Due to the fact that the equation (3.135) is linear, we can divide its solution into several stages. First, let us solve the equation

$$\begin{aligned}&\left(u+\frac{i}{2}\right)^2 Q_0^{A'++} + \left(u-\frac{i}{2}\right)^2 Q_0^{A'--} - \left(2u^2 - S(S+1) - \frac{1}{2}\right) Q_0^{A'} = \\ &\qquad = i\left(u+\frac{i}{2}\right) Q_0^{++} - i\left(u-\frac{i}{2}\right) Q_0^{--} - (2S+1)Q_0 \,.\end{aligned} \tag{3.136}$$

where Q_0 is $Q_0(u;S,1/2)$ from (3.38). To find the solution, let us use (3.7) from [83] for

$$Q_\delta^A(u) = {}_3F_2\left(-S-2\delta, S+1, \frac{1}{2} - iu; 1-\delta, 1; 1\right) \,, \tag{3.137}$$

which gives (note, that $Q_0^A = Q_0$)

$$\begin{aligned}&\left(u+\frac{i}{2}\right)^2 Q_0^{A'++} + \left(u-\frac{i}{2}\right)^2 Q_0^{A'--} - \left(2u^2 - S(S+1) - \frac{1}{2}\right) Q_0^{A'} = \\ &\qquad = i\left(u+\frac{i}{2}\right) Q_0^{++} - i\left(u-\frac{i}{2}\right) Q_0^{--} - (2S+1)Q_0 \,.\end{aligned} \tag{3.138}$$

Thus, $Q_0^{A'} \equiv \partial Q_\delta^A/\partial\delta|_{\delta=0}$ is a part of the solution. Analogously, the functions

$$\begin{aligned}Q_\delta^B(u) &= {}_3F_2\left(-S, S+1, \frac{1}{2} - iu; 1+\delta, 1-\delta; 1\right) \,, \\ Q_\delta^C(u) &= {}_3F_2\left(-S, S+1, \frac{1}{2} - iu + \delta; 1+\delta, 1+\delta; 1\right) \,.\end{aligned} \tag{3.139}$$

produce the solutions to the equations

$$\left(u+\frac{i}{2}\right)^2 Q_0^{B,C''++}+\left(u-\frac{i}{2}\right)^2 Q_0^{B,C''--}-\left(2u^2-S(S+1)-\frac{1}{2}\right)Q_0^{B,C''}= \\ =2Q_0(u)-2Q_0(u\mp i)\,. \quad (3.140)$$

Therefore, the solution to the equation (3.130) is given by

$$\tilde{Q}_2(u)=\frac{\gamma^{(1)}}{2}\left.\frac{\partial Q_\delta^A(u)}{\partial\delta}\right|_{\delta=0}-\left.\frac{\partial^2 Q_\delta^B(u)}{\partial\delta^2}\right|_{\delta=0}-\left.\frac{\partial^2 Q_\delta^C(u)}{\partial\delta^2}\right|_{\delta=0}\,, \quad (3.141)$$

where $Q_\delta^A(u)$, $Q_\delta^B(u)$ and $Q_\delta^C(u)$ are given by (3.137) and (3.139). After this we can easily derive the bare solution to the equation (3.130)

$$Q_2^{\text{bare}}(u)=\left[\tilde{Q}_2(u)+\tilde{Q}_2(-u)\right]\cosh^2(\pi u)+ \\ +\frac{i}{2}\left[\tilde{Q}_2(u)-\tilde{Q}_2(-u)\right]\cot\left(\frac{\pi S}{2}\right)\sinh(2\pi u)\,. \quad (3.142)$$

This is one of the solutions of (3.130). Unfortunately, it has some poles. From the $\mathbf{P}\mu$-system we have the requirement, that near the points $\pm i/2$ the solution $Q_2(u)$ has to behave in the following way

$$Q_2(u)=\pm\frac{8i\cos^2\left(\frac{\pi S}{2}\right)S_1(S)}{u\mp\frac{i}{2}}+\mathcal{O}\left(\left(u\mp\frac{i}{2}\right)^0\right)\,. \quad (3.143)$$

The analysis of the poles of the bare solution gives

$$Q_2^{\text{bare}}(u)= \\ =\frac{4\cos^2\left(\frac{\pi S}{2}\right)}{\left(u\mp\frac{i}{2}\right)^2}\mp\frac{4i\cos^2\left(\frac{\pi S}{2}\right)\left(-2S_1(S)+\pi\tan\left(\frac{\pi S}{2}\right)\right)}{u\mp\frac{i}{2}}+\mathcal{O}\left(\left(u\mp\frac{i}{2}\right)^0\right)\,. \quad (3.144)$$

To cancel the second order poles let us add the term

$$-2\pi^2\tanh^2(\pi u)Q(u)=-\frac{4\cos^2\left(\frac{\pi S}{2}\right)}{\left(u\mp\frac{i}{2}\right)^2}\mp\frac{-8i\cos^2\left(\frac{\pi S}{2}\right)S_1(S)}{u\mp\frac{i}{2}}+\mathcal{O}\left(\left(u\mp\frac{i}{2}\right)^0\right)\,. \quad (3.145)$$

Therefore, we have

$$Q_2^{\text{bare}}(u)-2\pi^2\tanh^2(\pi u)Q(u)= \\ =\mp\frac{4i\cos^2\left(\frac{\pi S}{2}\right)\left(-4S_1(S)+\pi\tan\left(\frac{\pi S}{2}\right)\right)}{u\mp\frac{i}{2}}+\mathcal{O}\left(\left(u\mp\frac{i}{2}\right)^0\right)\,. \quad (3.146)$$

It can be easily seen, that to obtain the right combination with the desired poles we have to add the following term

$$Q_2(u)=Q_2^{\text{bare}}(u)-2\pi^2\tanh^2(\pi u)Q(u;S)+ \\ +2\pi\left(\pi-2S_1(S)\cot\left(\frac{\pi S}{2}\right)\right)\left(Q_0\left(u;S,\frac{1}{2}\right)+Q_0\left(-u;S,\frac{1}{2}\right)\right)+\tilde{\beta}Q(u;S)\,. \quad (3.147)$$

The residues of this solution in the points $u = \pm i/2$ are

$$Q_2(u) = \pm \frac{8i \cos^2\left(\frac{\pi S}{2}\right) S_1(S)}{u \mp \frac{i}{2}} + \mathcal{O}\left(\left(u \mp \frac{i}{2}\right)^0\right) . \tag{3.148}$$

Returning to $\mu_{12}^{(1)}$ we can write

$$\mu_{12}^{(1)} = \frac{1}{u-i} + \mathcal{O}((u-i)^0) , \quad \mu_{12}^{(1)} = -\frac{1}{u} + \mathcal{O}(u^0) , \tag{3.149}$$

which perfectly agrees with the equation of the $\mathbf{P}\mu$-system, expanded in the vicinity of the origin

$$\mu_{12}^{++} - \mu_{12} = g^{-2}\left(-u + \mathcal{O}(u^2)\right) + \left(\frac{2}{u} + \mathcal{O}(u^0)\right) + \mathcal{O}(g^2) . \tag{3.150}$$

Remembering that $\mu_{12}^{(1)}(u) = \alpha Q_2(u)$ with α given by (3.59) and substituting (3.148) into (3.149) and (3.150) we find that our result is consistent.

Now it remains to fix the constant $\tilde{\beta}$. To do this let us again use the equation for μ_{12}

$$\mu_{12}^{++} - \mu_{12} = \mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1 , \tag{3.151}$$

which looks in the NLO as

$$\mu_{12}^{(1)++} - \mu_{12}^{(1)} = \mathbf{P}_1^{(1)}\tilde{\mathbf{P}}_2^{(0)} - \mathbf{P}_2^{(0)}\tilde{\mathbf{P}}_1^{(1)} - \mathbf{P}_2^{(1)}\tilde{\mathbf{P}}_1^{(0)} . \tag{3.152}$$

Then, using (3.112) that

$$\begin{aligned} \tilde{\mathbf{P}}_1^{(0)} &= u^2 + c_{1,1}^{(0)} u^4 + \mathcal{O}(u^6) , \\ \tilde{\mathbf{P}}_1^{(1)} &= -2 - 2c_{1,1}^{(0)} u^2 + \mathcal{O}(u^4) , \\ \tilde{\mathbf{P}}_2^{(0)} &= u + c_{2,1}^{(0)} u^3 + \mathcal{O}(u^5) , \end{aligned} \tag{3.153}$$

we obtain after combining with (3.106)

$$\mu_{12}^{(1)++} - \mu_{12}^{(1)} = \frac{2}{u} + \left(c_{1,1}^{(0)} + c_{2,1}^{(0)}\right) u + \mathcal{O}(u^3) . \tag{3.154}$$

We can find $\tilde{\beta}$ from the condition that

$$\alpha(Q_2^+ - Q_2^-) + \tilde{\beta}(Q^+ - Q^-) = \frac{2}{u} + \left(c_{1,1}^{(0)} + c_{1,2}^{(0)}\right) u + \mathcal{O}(u^3) , \tag{3.155}$$

where $c_{1,1}^{(0)}$ and $c_{2,1}^{(0)}$ are given by (3.63) and (3.71) respectively It remains to find the expansion of $Q_2(u)$ in the vicinity of the pole $u = i/2$ up to the linear term. We start with $-2\pi^2 \tanh^2(\pi u) Q(u)$

$$\begin{aligned} &-2\pi^2 \tanh^2\left(\pi\left(u + \frac{i}{2}\right)\right)\left(Q^+(u;S) - Q^-(u;S)\right) = \\ &\qquad = -\frac{4Q'\left(\frac{i}{2};S\right)}{u} - \frac{2}{3}\left(Q'''\left(\frac{i}{2};S\right) + 4\pi^2 Q'\left(\frac{i}{2};S\right)\right) u + \mathcal{O}(u^3) . \end{aligned} \tag{3.156}$$

By combining the the previous results (3.155) and (3.156) one can find the coefficient $\tilde{\beta}$, whose explicit form we leave for the future research.

In the present Section we built the weak coupling solution of the QSC for twist-2 operators in the LO and partially NLO. These results can be used for the inspiration in considering the BFKL limit of the QSC for the same operators. The next Section is devoted to this issue.

3.2 BFKL limit

For the twist-2 operators in question, the charges are fixed to $J_2 = J_3 = S_2 = 0$ and $J_1 = 2$, and we will use the notation $S_1 \equiv S \equiv -1 + w$. These operators belong to the so called left-right symmetric sector for which we have the following reduction [17]

$$\mathbf{P}^a = \chi^{ac}\mathbf{P}_c \,, \quad \mathbf{Q}^i = \chi^{ij}\mathbf{Q}_j \,, \tag{3.157}$$

where χ is the antisymmetric constant 4×4 matrix with the only non-zero entries $\chi^{23} = \chi^{41} = -\chi^{14} = -\chi^{32} = 1$. From (3.157), (2.56), (2.59) and (2.60) we see that $\mu_{23} = \mu_{14}$ and $\omega_{23} = \omega_{14}$, i.e. we have only 5 linearly independent components in each of these antisymmetric matrices in addition to the non-linear condition of unit Pfaffian. The asymptotics (2.57) and (2.61) are simplified to

$$\mathbf{P}_a \simeq (A_1 u^{-2}, A_2 u^{-1}, A_3, A_4 u) \,, \tag{3.158}$$

$$\mathbf{Q}_j \simeq (B_1 u^{\frac{\Delta+1-w}{2}}, B_2 u^{\frac{\Delta-3+w}{2}}, B_3 u^{\frac{-\Delta+1-w}{2}}, B_4 u^{\frac{-\Delta-3+w}{2}}) \tag{3.159}$$

and (2.67) reduces to

$$A_1 A_4 = -A_1 A^1 = \frac{1}{96i}((5-w)^2 - \Delta^2)((1+w)^2 - \Delta^2) \,, \tag{3.160}$$

$$A_2 A_3 = +A_2 A^2 = \frac{1}{32i}((1-w)^2 - \Delta^2)((3-w)^2 - \Delta^2) \,. \tag{3.161}$$

Note that one can always make a suitable rescaling to set $A_1 = A_2 = 1$, then A_3 and A_4 are fixed uniquely by (3.160). This is the normalization we use below in this Section.

Prescription for analytic continuation in S

Before finding the solution for QSC with the asymptotics (3.158) and (3.159), we should specify the prescription for analytic continuation in S at the level of QSC. In the Section 3.1.2 we explained how the continuation works at weak coupling, in one-loop approximation. We have to translate this prescription into the QSC language. The role of the Q-function (2.42) in the QSC construction is played by μ_{12} [16]. To make a direct link with the prescription (2.49) we consider the 1^{st} order equation (2.56) for 5 independent components of μ_{ab}. For fixed $\mathbf{P}_a$ it has 5 independent solutions whose asymptotics follow from the asymptotics of $\mathbf{P}$'s (3.158). One finds that for μ_{12} one could have one of the following 5 asymptotics $(u^{-1-S}, u^{+\Delta-2}, u^{-2}, u^{-\Delta-2}, u^{+S-3})$, where we ordered the possible asymptotics according to their magnitude in the BFKL regime, i.e. when $S \to -1$ and $0 < \Delta < 1$ (see Fig. 3.1). Note that for the usual perturbative regime considered in the introduction we have $\Delta = 2 + S + \mathcal{O}(g^2)$ and thus we can recognize in

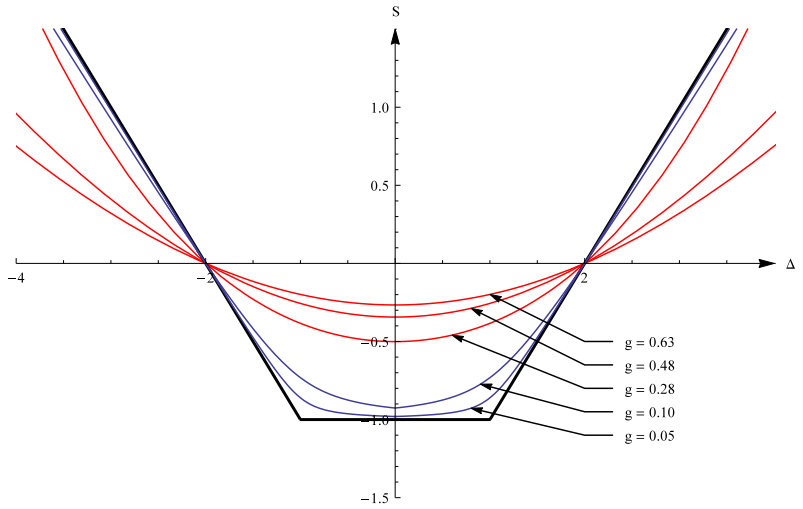

Figure 3.1: Operator trajectories $S(\Delta)$ corresponding to the twist-2 operator tr $Z(D_+)^S Z$ and different values of g.

the first two solutions Q_2 and Q_1 correspondingly! This motivates the prediction which was put forward and tested by [22], stating that in order to analytically continue the QSC to non-physical domain of non-integer S one should relax the power-like behavior of μ_{ab} (required for all physical states) allowing for the following leading and subleading terms in the asymptotics

$$\mu_{12} \sim \text{const}\ u^{+\Delta-2} + e^{2\pi u}\text{const}\ u^{-1-S} + \dots\ . \tag{3.162}$$

This is the generalization of (2.49) to a finite coupling. In [22] it was proven that with this prescription there is a unique solution for any coupling, at least in the vicinity of $S = 0$ up to the order S^2 inclusively. We also know that at weak coupling there exists the unique solution for these asymptotics. We consider this to be a strong indication towards uniqueness of such solution to an arbitrary number of loops, which would be however very interesting to prove rigorously. As we will also see below, such a solution is also unique in the BFKL regime. As this asymptotic is also consistent with the asymptotics for the physical states it should thus provide an analytic continuation of the physical solutions to an arbitrary non-integer S.

3.2.1 Leading order solution for Pμ-system

The logic of this Section is the following: we begin by arguing a certain scaling in the small parameter $w \equiv S + 1$ for various quantities and then write an ansatz for $\mathbf{P}_a$ and μ_{ab}. First, we assume that $\mathbf{P}_a \sim w^0$, in accordance with its large u asymptotics (3.160). Second, we keep in mind that the BFKL regime is still a regime of weak coupling, even though it re-sums all singularities of Δ of the type $(g^2/w)^n$. This means that all cuts are collapsed to a point and, in a generic situation, all functions should be regular on the entire complex plane. However, there could be some special cases where this rule is violated. Namely, consider a function $f(u) = 1/(gx)^2$, where $x(u) = (u+\sqrt{u^2-4g^2})/(2g)$

is the Zhukovsky variable. In the BFKL regime $g \to 0$

$$f(u) \simeq \frac{1}{u^2} , \quad \tilde{f}(u) = \frac{x^2}{g^2} \simeq \frac{u^2}{g^4} \simeq \frac{u^2}{w^2\Lambda^2} . \tag{3.163}$$

This shows that in principle even in the limit when the cut totally disappears some functions still can develop a singularity by the cost of being very large (but regular) on the next sheet. Note, that this exception is clearly not applicable to $\mu(u)$ since $\tilde{\mu}(u)$ is the same as shifted $\mu(u+i)$, so that both μ and $\tilde{\mu}$ should be of the same order and thus are regular at the leading order in w.

Nevertheless, $\mathbf{P}_1(u)$ must be exactly a function of this type. Indeed, at large u it behaves as $1/u^2$ and has no other singularities except for the cut. From that we conclude that we must have a double pole at zero or even a stronger singularity. The residue at the double pole is uniquely fixed at this order by (3.158), i.e. $\mathbf{P}_1(u) = 1/u^2$ whereas we will see that the stronger singularities could indeed appear at the next order in w. This $1/u^2$ singularity at zero implies that at the next sheet the function scales with our expansion parameter as $\tilde{\mathbf{P}} \sim 1/w^2$, which is only possible if at least some components of μ_{ab} scale as $\mu_{ab} \sim 1/w^2$, as we can see from (2.53). Consequently, $1/w^2$ will propagate via (2.53) into all components of $\mathbf{P}_a$. To summarize, we have to find a solution with the following scaling in w

$$\mathbf{P}_a \sim w^0 , \quad \mu_{ab} \sim w^{-2} , \quad \tilde{\mathbf{P}}_a \sim w^{-2} . \tag{3.164}$$

This scaling will lead us to an ansatz which we then plug into the $\mathbf{P}\mu$-system to fix the remaining freedom. We start from the $\mathbf{P}$-functions which have the simplest analytic structure: only one short cut on the main sheet, and integer powers in asymptotics. We can thus uniformize them by Zhukovsky map $u = g(x + 1/x)$, with the inverse $x(u)$ introduced above, such that $\tilde{x} = 1/x$, and expand $\mathbf{P}_a$ into the Laurent series around $x = 0$ [20]

$$\mathbf{P}_a = \sum_{n=-1}^{\infty} \frac{c_{a,n}}{x^n}. \tag{3.165}$$

It is guaranteed to converge for $|x| > |1/x(2g+i)|$ which allows to cover the whole upper sheet and even a finite part of the next sheet and leads to the corresponding ansatz for $\tilde{\mathbf{P}}_a$:

$$\tilde{\mathbf{P}}_a = \sum_{n=-1}^{\infty} c_{a,n} x^n . \tag{3.166}$$

To reduce the number of coefficients we note that for our observable there must be a parity symmetry $u \to -u$ (or equivalently $x \to -x$). Of course all $\mathbf{P}\mu-$system equations are invariant w.r.t. this transformation, which means that this symmetry in general maps one solution to another. As we know that our state with these quantum numbers is unique we conclude of course that the parity transformation should map our solution to an equivalent solution. Using the arguments similar to [17] in the Section 4.4.2 of that article it is possible to show that we can fix the remaining freedom in the construction and choose solution where $\mathbf{P}_a$ are mapped to themselves. From the asymptotics we see that $\mathbf{P}_1$ is even, whereas $\mathbf{P}_2$ is odd etc. To summarize, we impose

$$\mathbf{P}_1, \mathbf{P}_3 \ - \ \text{even} \quad , \quad \mathbf{P}_2, \mathbf{P}_4 \ - \ \text{odd} . \tag{3.167}$$

Similarly μ's should be covariant under the parity transformation. As the parity transformation is sensitive to the choice of cuts we should also take $\hat{\mu}_{ab}$ with short cuts to be covariant under the parity transformation which implies that $\hat{\mu}_{12}$ is even, however as a consequence of this $\check{\mu}_{12}$ should transforms nontrivially $\check{\mu}_{12}(-u) = \tilde{\check{\mu}}_{12}(u) = \check{\mu}_{12}(u+i)$ which by itself implies, by changing $u \to u - i/2$, that μ^+_{12} is even. To summarize, we have

$$\mu^+_{12},\ \mu^+_{14},\ \mu^+_{34}\ -\ \text{even}\quad ,\quad \mu^+_{13},\ \mu^+_{24}\ -\ \text{odd}\,. \tag{3.168}$$

This conditions allows us to drop each other coefficient in our ansatz $c_{2,2n} = c_{1,2n-1} = c_{4,2n} = c_{3,2n-1} \equiv 0$ at any n. It also follows from (3.158) that the coefficients $c_{2,-1} = c_{1,0} \equiv 0$. After that we still have one-parametric scalar freedom in our construction [22]:

$$\begin{aligned} &\mathbf{P}_3 \to \mathbf{P}_3 + \gamma \mathbf{P}_1\,, \qquad \mathbf{P}_4 \to \mathbf{P}_4 - \gamma \mathbf{P}_2\,, \\ &\mu_{14} \to \mu_{14} - \gamma \mu_{12}\,, \qquad \mu_{34} \to \mu_{34} + 2\gamma\mu_{14} - \gamma^2 \mu_{12}\,, \end{aligned} \tag{3.169}$$

preserving the leading $u \to \infty$ asymptotics but modifying the subleading ones. This allows to fix in addition $c_{3,2} = 0$. Finally, as we use the normalization with $A_1 = A_2 = 1$ we have to fix $c_{2,1} = \frac{1}{\sqrt{\Lambda w}}$ and $c_{1,2} = \frac{1}{\Lambda w}$.

Let us now restrict the possible scaling of the coefficients $c_{a,n}$ in the BFKL limit. In this limit $g^2 \sim w \to 0$ the Zhukovsky cut shrinks into a point and the $x(u)$ becomes

$$x(u) = \frac{u}{g} - \frac{g}{u} - \frac{g^3}{u^3} + \dots\,. \tag{3.170}$$

To satisfy the scaling $\tilde{\mathbf{P}}_a \sim 1/w^2 \sim 1/g^4$ the coefficients $c_{a,k}$ should become smaller and smaller with k rising and in general they scale as $c_{a,n} \sim g^{n-4}$. We thus denote

$$c_{m,n} = g^{n-4} \sum_{k=0}^{+\infty} c^{(k)}_{m,n} w^k. \tag{3.171}$$

where $c^{(k)}_{m,n}$ are already ~ 1. To the leading order in w we thus simply get

$$\mathbf{P}^{(0)}_1 = \frac{1}{u^2}\,,\quad \mathbf{P}^{(0)}_2 = \frac{1}{u}\,,\quad \mathbf{P}_3 = A^{(0)}_3\,,\quad \mathbf{P}^{(0)}_4 = A^{(0)}_4 u + \frac{c^{(1)}_{4,1}}{\Lambda u}\,. \tag{3.172}$$

where from (3.160) we have $A^{(0)}_4 = -i(\Delta^2-1)(\Delta^2-25)/96$, $A^{(0)}_3 = -i(\Delta^2-1)(\Delta^2-9)/32$ and the only coefficient to fix is $c^{(1)}_{4,1}$. Now, when $\mathbf{P}$-functions are essentially fixed, we can use the 5^{th} order equation (2.56) to find the μ-functions. Note that $\mathbf{P}_1$ and $\mathbf{P}_2$ are singular, whereas $\tilde{\mathbf{P}}_a$ should be regular at $u = 0$. This is only possible if μ_{ab} are regular and have a sufficient amount of zeros at $u = 0$. This observation singles out one solution out of 5 possible ones, with $\mu_{12} \sim u^{-S-1}$ for which all components of μ_{ab} have a polynomial asymptotic for large u: $(\mu_{12}, \mu_{13}, \mu_{14}, \mu_{24}, \mu_{34})^{(0)} \sim (u^0, u^1, u^2, u^3, u^4)$. To find this solution we plug a polynomial ansatz into (2.56) and also remember the expressions

(3.103), to find

$$\begin{aligned}
\mu_{12}^{(0)+} &= +\mathcal{P}, \\
\mu_{13}^{(0)+} &= -\frac{\mathcal{P}}{16} iu \left(\Delta^2-1\right)^2 , \\
\mu_{14}^{(0)+} &= -\frac{\mathcal{P}}{128} i \left(4u^2+1\right)\left(\Delta^2-1\right)^2 , \\
\mu_{24}^{(0)+} &= -\frac{\mathcal{P}}{192} iu \left(4u^2+1\right)\left(\Delta^2-1\right)^2 , \\
\mu_{34}^{(0)+} &= -\frac{\mathcal{P}}{49152}\left(16u^4-8u^2-3\right)\left(\Delta^2-1\right)^4 ,
\end{aligned} \tag{3.173}$$

and in addition (2.56) also to fix $c_{4,1}^{(1)} = -i(\Delta^2-1)^2\Lambda/96$. Thus we fix μ_{ab} up to a common multiplier $\mathcal{P}$. As we deal with a finite difference equation this multiplier could be only an i-periodic function, which has to be chosen in accordance with the prescription (3.162) and which respects the parity (3.168). The most general choice is

$$\mathcal{P} = C_1 + C_2 \cosh^2(\pi u) \tag{3.174}$$

for some constants C_1 and C_2. Thus we have only two constants to fix and still several nontrivial conditions to satisfy, namely (2.55) and (2.53). From (2.53) we find

$$\begin{gathered}
\tilde{\mathbf{P}}_1^{(0)} = -\frac{i\left(\Delta^2-1\right)\mathcal{P}^-}{4} , \quad \tilde{\mathbf{P}}_2^{(0)} = -\frac{i\left(\Delta^2-1\right)u\mathcal{P}^-}{4} , \\
\tilde{\mathbf{P}}_3^{(0)} = -\frac{\left(\Delta^2-1\right)^3 u^2\mathcal{P}^-}{128} , \quad \tilde{\mathbf{P}}_4^{(0)} = -\frac{\left(\Delta^2-1\right)^3\left(u^3+u\right)\mathcal{P}^-}{384}
\end{gathered} \tag{3.175}$$

To fix C_1 and C_2 we note that from the ansatz for $\tilde{\mathbf{P}}_1$ (3.166) we should have

$$\tilde{\mathbf{P}}_1 = \frac{u^2+\mathcal{O}(u^4)}{w^2\Lambda^2} + \mathcal{O}(w^{-1}) , \tag{3.176}$$

which is also clear from our basic discussion (3.163). Comparing with (3.175) we fix

$$C_1 = 0 , \quad C_2 = \frac{4i}{\pi^2\Lambda^2 w^2(\Delta^2-1)} . \tag{3.177}$$

We found a consistent solution with no free parameters left. We also might expect that we could get a relation between the energy Δ and the coupling Λ, which are so far completely independent. However, it is not the case at this order. The reason for this is that we are not able to use efficiently the remaining condition $\mathrm{Pf}\,\mu_{ab} = 1$, because the LHS is of the order $1/w^4$ and with our precision we cannot distinguish the 1 in the RHS from any other finite number. We would have to continue the procedure to the next 4 orders in w until we get this condition to work. As we will see, a much more efficient way to overcome this difficulty is to pass to $\mathbf{Q}\omega$-system.

3.2.2 Next-to-leading order solution

We can also extend the consideration of the previous Section to the next order in w. Using the ansatz (3.165) and (3.171) we get, up to w^1 terms

$$\begin{aligned}
\mathbf{P}_1 &= \frac{1}{u^2} + \frac{2\Lambda w}{u^4} + \mathcal{O}(w^2) \,, \\
\mathbf{P}_2 &= \frac{1}{u} + \frac{2\Lambda w}{u^3} + \mathcal{O}(w^2) \,, \\
\mathbf{P}_3 &= A_3^{(0)} + A_3^{(1)} w + \mathcal{O}(w^2) \,, \\
\mathbf{P}_4 &= A_4^{(0)} u - \frac{i(\Delta^2-1)^2}{96u} + \left(A_4^{(1)} u + \frac{c_{4,1}^{(2)}}{u\Lambda} - \frac{i(\Delta^2-1)^2\Lambda}{48u^3} \right) w + \mathcal{O}(w^2) \,.
\end{aligned} \tag{3.178}$$

where from (3.160) $A_3^{(1)} = -i(\Delta^2-3)/4$, $A_4^{(1)} = -i(\Delta^2+5)/12$. Again there is only one missing constant $c_{4,1}^{(2)}$. To fix it we have to proceed further to find μ in the NLO. At this order the solution cannot be just a polynomial as the asymptotic u^{-w} of μ_{12} suggests that the ansatz is more complicated. We discuss details of this calculation in Appendix A.2 where we find that the missing constant is

$$c_{4,1}^{(2)} = -\frac{i\Lambda}{24}(\Delta^2-1)\left[2(\Delta^2-1)\Lambda - 1\right] \,. \tag{3.179}$$

We will use this result to find the NLO for $\mathbf{Q}$-functions and also the LO result for the BFKL dimension.

3.2.3 Passing to Qω-system

An important step in our calculation is to switch now to the $\mathbf{Q}\omega$-system. It is especially easy having at hand the equation (2.51). We simply plug the already known $\mathbf{P}$'s (3.172) into (2.51) an get a 4-th order linear finite difference equation on $\mathbf{Q}_j$ with explicit polynomial coefficients. As a good sign that we are on the right track, the finite difference operator of this 4-th order equation can be nicely factorized as follows

$$\left[(u+2i)^2 D + (u-2i)^2 D^{-1} - 2u^2 - \frac{17-\Delta^2}{4}\right]\left[D + D^{-1} - 2 - \frac{1-\Delta^2}{4u^2}\right]\mathbf{Q}^{(0)} = 0 \,, \tag{3.180}$$

where $D = e^{i\partial_u}$ is the shift operator. This implies that two out of four $\mathbf{Q}$-functions satisfy the 2-nd order equation

$$\mathbf{Q}_j^{(0)} \frac{\Delta^2 - 1 - 8u^2}{4u^2} + \mathbf{Q}_j^{(0)--} + \mathbf{Q}_j^{(0)++} = 0 \,. \tag{3.181}$$

Even before solving this equation, it is easy to check that there are two independent solutions with the large u asymptotics $u^{\frac{+\Delta+1}{2}}$ and $u^{\frac{-\Delta+1}{2}}$, which indicates, together with (3.159), that they can be identified as $\mathbf{Q}_1^{(0)}$ and $\mathbf{Q}_3^{(0)}$.

Notice that (3.181) after redefinition $Q = \mathbf{Q}_j^{(0)}/u^2$ is precisely the famous $\mathfrak{sl}(2,\mathbb{C})$ Baxter equation defining, through Sklyanin's separation of variables method, the Pomeron

LO BFKL wave function [3, 9, 52–55]! It can be solved (for example, by using Mellin transform method), taking into account the asymptotics and the UHP analyticity, in terms of a hypergeometric function

$$\mathbf{Q}_1^{(0)} = 2iu\,{}_3F_2\left(iu+1, \frac{1}{2}-\frac{\Delta}{2}, \frac{1}{2}+\frac{\Delta}{2}; 1, 2; 1\right) \tag{3.182}$$
$$\mathbf{Q}_3^{(0)}(u) = \mathbf{Q}_1^{(0)}(-u)\sec\frac{\pi\Delta}{2} + \mathbf{Q}_1^{(0)}(u)\left[-i\coth(\pi u) + \tan\frac{\pi\Delta}{2}\right] .$$

where $\mathbf{Q}_\mathbf{1}^{(\mathbf{0})}$ is a solution chosen to be analytic in the upper half plane, and it can be read off from the papers [3, 9, 52–55]. At the same time, it serves as a building block for the second solution $\mathbf{Q}_3^{(0)}$, which has a smaller (for $\Delta > 0$) asymptotic. To build $\mathbf{Q}_3^{(0)}$ we use that $\mathbf{Q}_1^{(0)}(-u)$ is also an independent solution of the same Baxter equation, which, however, has poles in the upper half plane. We cancel these poles by adding $\mathbf{Q}_1^{(0)}(u)$ we a suitable periodic coefficient, and finally extract const $\times\, \mathbf{Q}_1^{(0)}(u)$ to ensure the right large u asymptotics. The choice appears to be unique.

Although for the rest of the paper we will not need to determine the other two $\mathbf{Q}$-functions, $\mathbf{Q}_2^{(0)}$ and $\mathbf{Q}_4^{(0)}$ we give here for completeness the inhomogeneous equation following from (3.180) expressing them through $\mathbf{Q}_1^{(0)}$ and $\mathbf{Q}_3^{(0)}$

$$\mathbf{Q}_j^{(0)++} + \mathbf{Q}_j^{(0)--} - \left(2 + \frac{1-\Delta^2}{4u^2}\right)\mathbf{Q}_j^{(0)} = \\ = \frac{\pi^2(\Delta^2-1)^2}{16\cos\frac{\pi\Delta}{2}}\left(\frac{\mathbf{Q}_{j-1}^{(0)++}}{(u+i)^2} + \frac{\mathbf{Q}_{j-1}^{(0)--}}{(u-i)^2} - 2\frac{\mathbf{Q}_{j-1}^{(0)}}{u^2}\right) , \quad j = 2, 4 . \tag{3.183}$$

The $\mathbf{Q}_2$ and $\mathbf{Q}_4$ may become useful for the calculation of dimension in NLO and NNLO of the BFKL approximation.

Similarly we can use our knowledge of the NLO $\mathbf{P}$'s (3.178) to construct the NLO Baxter equation which takes into account the $\mathcal{O}(w)$ terms in (2.51). For that we have to plug there $\mathbf{P}_a$ given by (3.178), (3.179). Again, this 4-th order finite difference equation appears to be factorizable and as a result we get the following 2-nd order equation for $\mathbf{Q}_1$ and $\mathbf{Q}_3$, generalizing (3.181)

$$\mathbf{Q}_j\left(\frac{\Delta^2-1-8u^2}{4u^2} + w\frac{(\Delta^2-1)\,\Lambda - u^2}{2u^4}\right) + \\ + \mathbf{Q}_j^{--}\left(1 - \frac{iw}{2(u-i)}\right) + \mathbf{Q}_j^{++}\left(1 + \frac{iw}{2(u+i)}\right) = 0 , \quad j = 1, 3 . \tag{3.184}$$

Its solution can be found using Mellin transformation method of [9] and is given in Appendix A.3. For our present goal – the calculation of BFKL dimension, we only need a simple fact about the NLO $\mathbf{Q}$. Namely, we want to know its behavior around $u = 0$. This information is easier to extract directly from the Baxter equation (3.184) by shifting $u \to u + i$. Recalling that $\mathbf{Q}_j$ must be regular in the upper half plane we obtain from the second term

$$\left(1 - \frac{iw}{2u}\right)\mathbf{Q}_j(u) = \text{regular at } u \sim 0 , \tag{3.185}$$

which gives the relation between the behaviour at the origin of the leading in w order $\mathbf{Q}_j^{(0)}$ and the subleading order $\mathbf{Q}_j^{(1)}$

$$\frac{\mathbf{Q}_j^{(1)}(u)}{\mathbf{Q}_j^{(0)}(u)} = +\frac{iw}{2u} + \mathcal{O}(u^0) \ , \quad j = 1, 3 \ . \tag{3.186}$$

The strategy is now to compute this ratio independently, using the $\mathbf{Q}\omega$-system. Matching these two results we will recover the BFKL Pomeron eigenvalue.

Going to the next sheet

So far we mostly recycled the information from the $\mathbf{P}\mu$-system into $\mathbf{Q}$'s. To get something new we have to work a bit harder and reconstruct ω's. This will allow us to compute, for example, $\tilde{\mathbf{Q}}_3(0)$ from which we will instantly determine the pole in the ratio (3.186). Let us remind the relation between μ_{ab} and ω_{ij}. They are related to each other by a Q-function with $2+2$ indices

$$\mu_{ab} = \mathcal{Q}_{ab|ij-}\mathcal{Q}_{b|j-}\omega^{ij} \ , \tag{3.187}$$

where $\mathcal{Q}_{ab|ij}$ can be decomposed in terms of $\mathbf{P}_a$ and $\mathbf{Q}_i$ and as a result should be $\sim w^0$. This implies that at least one of the components of ω^{ij} should scale as $1/w^2$. As we know, at the leading order in w, up to a periodic function, $\mu_{12} \sim u^{-S-1}$ and we can precisely identify it with $\omega^{13} = \omega_{24}$ which should be $\sim 1/w^2$ whereas other components should be smaller. We will see that the consistent scaling is $\omega_{12} \sim \omega_{14} \sim \omega_{34} \sim w^0$ and $\omega_{24} \sim w^{-2}$ and $\omega_{13} \sim w^2$. The reason why ω_{13} appears to be w^2 is due to the fact that this component multiplies ω_{24} in the Pfaffian, which is set to 1.

With this insight coming from μ's we can see that only two terms survive in the relation for $\tilde{\mathbf{Q}}_1$ and $\tilde{\mathbf{Q}}_3$ since the terms with ω_{13} are too small

$$\begin{aligned} \tilde{\mathbf{Q}}_1^{(0)}(u) &= +\omega_{14}^{(0)}(u)\mathbf{Q}_1^{(0)}(u) + \omega_{12}^{(0)}(u)\mathbf{Q}_3^{(0)}(u) \ , \\ \tilde{\mathbf{Q}}_3^{(0)}(u) &= +\omega_{34}^{(0)}(u)\mathbf{Q}_1^{(0)}(u) - \omega_{14}^{(0)}(u)\mathbf{Q}_3^{(0)}(u) \ . \end{aligned} \tag{3.188}$$

Note that since these components are suppressed compared to ω_{24} no explicit information about their form can be extracted from the $\mathbf{P}\mu$-system at the given order in w. At the same time we can say that $\omega_{24}^{(0)} = B\sinh^2(\pi u)$ just because the Q-function in (3.187) has a power-like asymptotics and all the exponents can only originate from the factor $\omega_{24}^{(0)}$. Another thing to notice, is that $\mathbf{Q}_1^{(0)}$ and $\mathbf{Q}_3^{(0)}$ decouple from the rest of the $\mathbf{Q}\omega$-system. This explains in particular the mysterious factorization of the 4th order equation (3.180).

We will now fix ω's appearing in (3.188) using some elementary properties of $\mathbf{Q}_1^{(0)}$ and $\mathbf{Q}_3^{(0)}$ found explicitly in (3.182). We already pointed out that $\mathbf{Q}_1^{(0)}(-u)$ and $\mathbf{Q}_3^{(0)}(-u)$ would also be solutions of the same finite difference equation and thus they can be re-expanded in terms of the basis $\mathbf{Q}_1^{(0)}(u)$, $\mathbf{Q}_3^{(0)}(u)$. Right from (3.182) we have

$$\begin{aligned} \mathbf{Q}_1^{(0)}(-u) &= +\mathbf{Q}_1^{(0)}(u)\frac{i\cosh\left(\pi\left(u+\frac{i\Delta}{2}\right)\right)}{\sinh(\pi u)} + \mathbf{Q}_3^{(0)}(u)\cos\left(\frac{\pi\Delta}{2}\right) \ , \\ \mathbf{Q}_3^{(0)}(-u) &= -\mathbf{Q}_1^{(0)}(u)\frac{\cos\left(\frac{\pi\Delta}{2}\right)}{\sinh^2(\pi u)} + \mathbf{Q}_3^{(0)}(u)\frac{i\cosh\left(\pi\left(u-\frac{i\Delta}{2}\right)\right)}{\sinh(\pi u)} \ . \end{aligned} \tag{3.189}$$

this equation in many respects is similar to the equation we want to recover, (3.188). Indeed, both $\tilde{\mathbf{Q}}^{(0)}(u)$ and $\mathbf{Q}^{(0)}(-u)$ are analytic below the real axis, and the coefficients in the RHS are periodic functions of u as ω's should be. We have to find a relation between $\tilde{\mathbf{Q}}^{(0)}(u)$ and $\mathbf{Q}^{(0)}(-u)$, which we may already expect to be simple. Combining (3.189) and (3.188) we can write

$$\tilde{\mathbf{Q}}_1^{(0)}(u) = a_{11}\mathbf{Q}_1^{(0)}(-u) + a_{13}\mathbf{Q}_3^{(0)}(-u) \,, \tag{3.190}$$

$$\tilde{\mathbf{Q}}_3^{(0)}(u) = a_{31}\mathbf{Q}_1^{(0)}(-u) + a_{33}\mathbf{Q}_3^{(0)}(-u) \,. \tag{3.191}$$

A priori a_{ij} are some periodic functions of u. Let us show that they must be constants. Firstly, they should have no poles. That is because both $\tilde{\mathbf{Q}}^{(0)}(u)$ and $\mathbf{Q}^{(0)}(-u)$ could not have any poles below the real axis, from the explicit form of $\mathbf{Q}_1^{(0)}$ and $\mathbf{Q}_3^{(0)}$ we can verify they do not vanish at $u = -in$ and cannot cancel the poles themselves, furthermore the cancellation of the poles between the two terms in the RHS is impossible as $\mathbf{Q}_3^{(0)}$ decays faster and soon becomes negligible comparing to the term with $\mathbf{Q}_1^{(0)}$, when we go down in the complex plane. Secondly, a's cannot grow exponentially at infinity as $\mathbf{Q}^{(0)}(-u)$'s and $\tilde{\mathbf{Q}}^{(0)}(u)$'s behave power-like in the lower half-plane. Therefore, according to the Liouville theorem these coefficients are constants. Thus our problem of finding ω's is already simplified enormously and reduced to the problem of finding a few constants.

Next, we have to remember that $\tilde{\mathbf{Q}}_i$ is an analytic continuation of $\mathbf{Q}_i$ and so they should match at $u = 0$ in the LO as well

$$\tilde{\mathbf{Q}}_1{}^{(0)}(0) = \mathbf{Q}_1^{(0)}(0) = 0 \,, \quad \tilde{\mathbf{Q}}_3^{(0)}(0) = \mathbf{Q}_3^{(0)}(0) \neq 0 \,, \tag{3.192}$$

which fixes $a_{13} = 0$ and $a_{33} = 1$. Now we can combine this information with (3.189) to see what it implies for ω's. We notice that in (3.188) there are only 3 different coefficients, which gives a nontrivial constraint on a's

$$ia_{11}\frac{\cosh\left(\pi u + \frac{i\pi\Delta}{2}\right)}{\sinh(\pi u)} = -a_{31}\cos\frac{\pi\Delta}{2} - i\frac{\cosh\left(\pi u - \frac{i\pi\Delta}{2}\right)}{\sinh(\pi u)} \,. \tag{3.193}$$

On the first sight, it seems to be impossible to satisfy with constant a_{11} and a_{31} for any u. Luckily, it is solved at once for $a_{11} = -1$ and $a_{31} = -2\tan(\pi\Delta/2)$! From where we obtain in particular

$$\begin{aligned} \tilde{\mathbf{Q}}_1^{(0)}(u) &= -\mathbf{Q}_1^{(0)}(-u) \,, \\ \tilde{\mathbf{Q}}_3^{(0)}(u) &= +\mathbf{Q}_3^{(0)}(-u) - 2\tan\left(\frac{\pi\Delta}{2}\right)\mathbf{Q}_1^{(0)}(-u) \,. \end{aligned} \tag{3.194}$$

One can also read ω's from this expression, which is done in Appendix A.4.

3.2.4 LO BFKL dimension

We are just one step away from the main result – BFKL dimension. For that we notice that the knowledge of, say, $\mathbf{Q}_3$ and $\tilde{\mathbf{Q}}_3$ in the $u \sim 1$ scaling gives an access to the leading singularity in $\mathbf{Q}_3$ at $u = 0$ to all orders in w. Indeed, the combination $\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3$ changes the sign when we go under the Zhukovsky cut and thus is proportional to $\sqrt{u^2 - 4g^2}$. In

other words $(\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3)/\sqrt{u^2-4g^2}$ does not have the cut $[-2g, 2g]$ anymore and thus it is regular in $g \ll u \sim 1$ scaling. Same is true about the even combination $\mathbf{Q}_3 + \tilde{\mathbf{Q}}_3$. Thus we can rewrite

$$\mathbf{Q}_3 = \frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{2\sqrt{u^2-4g^2}}\sqrt{u^2-4g^2} + \frac{\mathbf{Q}_3 + \tilde{\mathbf{Q}}_3}{2} = \\ = \left[\frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{\sqrt{u^2-4g^2}}\right]\left(-\frac{\Lambda w}{u} - \frac{\Lambda^2 w^2}{u^3} + \dots\right) + \text{regular} \quad (3.195)$$

and note that we know explicitly the expression in the square brackets at the leading order in w from (3.194) and (3.182). Its small u expansion gives

$$\frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{\sqrt{u^2-4g^2}} = 2i\mathbf{Q}_3^{(0)}(0)\Psi(\Delta) + \mathcal{O}(w) + \mathcal{O}(u)\ , \quad (3.196)$$

where

$$\Psi(\Delta) \equiv \psi\left(\frac{1}{2} - \frac{\Delta}{2}\right) + \psi\left(\frac{1}{2} + \frac{\Delta}{2}\right) - 2\psi(1)\ . \quad (3.197)$$

From that we can immediately find that the pole in u at the first order in w should be

$$\mathbf{Q}_3(u) = -\frac{2i\mathbf{Q}_3^{(0)}(0)\Psi(\Delta)\Lambda w}{u} + \text{regular} + \mathcal{O}(w^2)\ , \quad (3.198)$$

which compared with (3.186) leads to

$$-4\Psi(\Delta)\Lambda = 1\ . \quad (3.199)$$

This is precisely the formula (1.7) for the eigenvalue of the QCD BFKL kernel, or, equivalently, for the dimension of twist-2 operator in BFKL approximation at the leading Regge singularity!

Chapter 4

Length-2 operators with non-zero conformal spin

The present Chapter is devoted to the study of a more general class of operators of $\mathcal{N} = 4$ SYM – length-2 operators with non-zero conformal spin. To achieve this aim we first consider a QSC-based framework particularly useful to analyze such class of operators. Then by adopting the QSC numerical algorithm [32] for the states without left-right symmetry we manage to calculate the operator trajectories for non-zero conformal spin. After this we present the weak coupling expansion, reproducing the known Pomeron eigenvalue of the BFKL integral kernel for non-zero n. Also, applying the iterative procedure [31] in our case we find the intercept function up to the NNLO and partially NNNLO in the coupling constant. In addition, the QSC framework we use allowed us to derive two non-perturbative quantities – slope-to-intercept and curvature functions together with their weak and strong coupling expansions. Finally, from fitting of our numerical results we guess the formula for the strong coupling expansion of the intercept function for arbitrary conformal spin. We used the material of the article [62] to write this Chapter. The main author's contribution are the Sections devoted to the QSC based framework, near-BPS all-loop expansion and intercept function at strong coupling. In the Sections on the weak coupling solution the author's role was to find the starting point for both numerical algorithm and iterative procedure, which is the weak coupling QSC solution for non-zero conformal spin in the BFKL limit. In the numerical results the author computed the dependencies of the intercept $S(0, n)$ on the coupling constant g for different values of conformal spin n.

4.1 Description of the QSC based framework

In this Section we are going to present the framework which we use to solve the QSC [16, 17] and whose derivation is based on the analytic and asymptotic properties of the Q-functions. First, we reformulate the QSC in terms of gluing matrix. Namely, we start from the several axioms concerning the analytic structure of the Q-system and the symmetries which preserve the QQ-relations and derive from them the so-called gluing conditions. These gluing conditions already appeared in [17, 31] but our approach presented below does not utilize the notion of μ- and ω-functions to obtain the gluing matrix. Second,

using the connection between the asymptotics of the certain subset of Q-functions and the global charges together with their analytic properties, the system of constraints for the gluing matrix is derived. It appears to be possible to solve these equations in some physically interesting cases. Namely, we find the gluing matrix for the case when both AdS spins S_1 and S_2 are integers of the same parity and its form appears to be very simple and in complete agreement with the result of [17]. Then we consider a more general case of non-integer AdS spins S_1 and S_2, which is particularly interesting for the exploration of the BFKL regime. For this case we have not found the general solution for the gluing matrix, however we found the certain subclass of solutions and it appears to be applicable to our quantities of interest. We mostly follow the original paper [17], but the discussion of the gluing matrix and the extension to the non-integer quantum numbers is new. The reader familiar with the QSC formalism could skip to Subsection 4.1.5.

4.1.1 Algebraic part of the construction

QSC consists of a set of Q-functions of the complex spectral parameter u and relations between them. We will restrict ourselves to the most essential parts of the construction but still keeping the discussion self-contained. For more detailed description of the QSC see [16, 22] and for the pedagogical introduction see [17].

In total there are 256 Q-functions $Q_{a_1,\dots,a_n|i_1,\dots,i_m}(u)$ totally antisymmetric in the two groups of "bosonic" (a's) and "fermionic" (i's) indices with $1 \le n, m \le 4$, however not all of them are independent. The main building blocks of the QSC construction are the $4+4$ "elementary" Q-functions: $Q_{a|\emptyset}(u)$, where $a = 1, \dots, 4$, and $Q_{\emptyset|i}(u)$, where $i = 1, \dots, 4$. Setting the normalization $Q_{\emptyset|\emptyset} = 1$ and starting from these 8 Q-functions, one can recover the whole Q-system applying the QQ-relations written in [17]. In particular the QQ-relation for the Q-function with one "bosonic" and one "fermionic" index looks as follows

$$Q_{a|i}\left(u + \frac{i}{2}\right) - Q_{a|i}\left(u - \frac{i}{2}\right) = Q_{a|\emptyset}(u) Q_{\emptyset|i}(u) \,. \tag{4.1}$$

and $Q_{a|i}(u)$ is a solution of (4.1). From now on we are going to use the shorthand notation for the shift in the variable u: $f(u + ik/2) = f^{[k]}(u)$. In a similar way one can build all 256 Q-functions out of the basic 8 mentioned above. One should also impose the quantum unimodularity condition

$$Q_{1234|1234} = 1. \tag{4.2}$$

An important symmetry of the QSC is the Hodge-duality, which exchanges

$$\begin{aligned} Q_{a_1,\dots,a_n|i_1,\dots,i_m} \leftrightarrow Q^{a_1,\dots,a_n|i_1,\dots,i_m} \equiv \\ \equiv (-1)^{(4-n)m} \epsilon^{b_{n+1}\dots b_4 a_1 \dots a_n} \epsilon^{j_{m+1}\dots j_4 i_1 \dots i_m} Q_{b_{n+1},\dots,b_4|j_{m+1},\dots,j_4} \,, \end{aligned} \tag{4.3}$$

where in the right-hand side of (4.3) there is no summation over the repeated indices. The Hodge-dual Q-functions (4.3) with the upper indices also satisfy the same QQ-relations as the Q-functions with the lower indices.

Due to (4.2) we are able to obtain the relations which allow to get fast to the Hodge-dual Q's

$$Q^{a|i} Q_{a|j} = -\delta^i_j \,, \quad Q^{a|i} Q_{b|i} = -\delta^a_b \,. \tag{4.4}$$

The Q-function $Q^{a|i}$ allows to write the Q-functions with one upper index in a concise form

$$Q^{a|\emptyset} = (Q^{a|i})^{+} Q_{\emptyset|i} \tag{4.5}$$

and

$$Q^{\emptyset|i} = (Q^{a|i})^{+} Q_{a|\emptyset} \, . \tag{4.6}$$

From the condition (4.2) it can be shown that

$$Q_{a|\emptyset} Q^{a|\emptyset} = 0 \, , \quad Q_{\emptyset|i} Q^{\emptyset|i} = 0 \, . \tag{4.7}$$

In addition, the Q-system has a symmetry, which is called the H-symmetry [17] and which leaves the QQ-relations intact. It corresponds to the transformations of Q-functions by i-periodic matrices that rotate the "bosonic" and "fermionic" indices separately. Their form for all Q-functions can be found in [17], but in this Section we need the explicit form of them only for the Q-functions with one index. They are

$$Q_{a|\emptyset} \to (H_B)^c_a Q_{c|\emptyset} \, , \quad Q^{a|\emptyset} \to (H_B^{-1})^a_c Q^{c|\emptyset} \, , \quad Q_{\emptyset|i} \to (H_F)^j_i Q_{\emptyset|j} \, , \quad Q^{\emptyset|i} \to (H_F^{-1})^i_j Q^{\emptyset|j}, \tag{4.8}$$

where $H_B(u)$ and $H_F(u)$ are i-periodic 4×4 matrices. The determinants of these matrices have to satisfy

$$\det H_B(u) \det H_F(u) = 1 \tag{4.9}$$

for the quantum unimodularity condition (4.2) not to change under such H-rotations. The important particular case of this symmetry is the rescaling of the Q-functions with one index. It acts as follows

$$Q_{a|\emptyset} \to \alpha_a Q_{a|\emptyset} \, , \quad Q_{\emptyset|i} \to \beta_i Q_{\emptyset|i} \, , Q^{a|\emptyset} \to \frac{1}{\alpha_a} Q^{a|\emptyset} \, , \quad Q^{\emptyset|i} \to \frac{1}{\beta_i} Q^{\emptyset|i}. \tag{4.10}$$

The equation (4.1) allows to obtain a 4th order Baxter equation for the functions $Q_{\emptyset|i}(u)$, $i =, 1 \dots, 4$. In [82] this equation was derived and looks as follows

$$\begin{aligned} Q^{[+4]}_{\emptyset|i} D_0 - Q^{[+2]}_{\emptyset|i} \left[D_1 - Q^{[+2]}_{a|\emptyset} Q^{a|\emptyset|[+4]} D_0 \right] + \\ + Q_{\emptyset|i} \left[D_2 - Q_{a|\emptyset} Q^{a|\emptyset[+2]} D_1 + Q_{a|\emptyset} Q^{a|\emptyset[+4]} D_0 \right] - \\ - Q^{[-2]}_{\emptyset|i} \left[\bar{D}_1 + Q^{[-2]}_{a|\emptyset} Q^{a|\emptyset[-4]} \bar{D}_0 \right] + Q^{[-4]}_{\emptyset|i} \bar{D}_0 = 0 \, , \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} D_k &= \det_{1 \le i,j \le 4} Q^{\emptyset|j[4-2i+2\delta_{i,k}]} \, , \; k = 0, 1 \, , \\ D_2 &= \det_{1 \le i,j \le 4} Q^{\emptyset|j[4-2i+2\delta_{i,1}+2\delta_{i,2}]} \, , \\ \bar{D}_k &= \det_{1 \le i,j \le 4} Q^{\emptyset|j[-4+2i-2\delta_{i,k}]} \, , \; k = 0, 1 \, . \end{aligned} \tag{4.12}$$

It is also possible to show from the same equation as (4.1) for the Q-functions with upper indices that the functions $Q^{\emptyset|i}(u)$, $i = 1, \dots, 4$ satisfy the 4th order Baxter equation, which looks as (4.11) but with $Q_{a|\emptyset}$ exchanged with $Q^{a|\emptyset}$. For the sake of conciseness we do not write this Baxter equation explicitly.

After finishing the description of the algebraic structure of the QSC essential for the formulation of the QSC equations in the next Subsection we describe the analyticity properties of the Q-functions, which constitute the crucial part of our QSC framework.

4.1.2 Analytic part of the construction

To describe the analytic structure of the Q-system we have to first define the analyticity properties of the basic set of Q-functions: $Q_{a|\emptyset}(u)$, $a = 1, \dots, 4$ and $Q_{\emptyset|i}(u)$, $i = 1, \dots, 4$. The only singularities of these functions are quadratic branch points which come in pairs at the positions $\pm 2g + ik$, where $k \in \mathbb{Z}$. For each pair of branch points we can choose either short cut on the interval $[-2g + ik, 2g + ik]$ or a long cut $(-\infty + ik, -2g + ik] \cup [2g + ik, ik + \infty)$, where k is some integer. In what follows the sheet of the Q-function with only the short cuts is called physical and the function on this sheet is denoted by $\hat{Q}(u)$, while on the sheet, where all the cuts are long, is called mirror and the function is designated by $\check{Q}(u)$ on it. The continuation of any function $f(u)$ under the cut on the real axis is denoted by $\tilde{f}(u)$. The branch points of all the functions we will consider are quadratic, i.e. $\tilde{\tilde{f}}(u) = f(u)$.

In what follows we will denote the functions $Q_{a|\emptyset}(u)$, $Q^{a|\emptyset}(u)$, $Q_{\emptyset|i}(u)$ and $Q^{\emptyset|i}(u)$, with prescribed analytical properties, as $\mathbf{P}_a(u)$, $\mathbf{P}^a(u)$, $\mathbf{Q}_i(u)$ and $\mathbf{Q}^i(u)$ respectively. To proceed let us write the asymptotics of the Q-functions with one index. We know that all the Q-functions including $\mathbf{P}_a$, $\mathbf{P}^a$, $\mathbf{Q}_i$ and $\mathbf{Q}^i$ have the power-like asymptotics at large u, which for the basic 8 Q-functions can be taken from [17]

$$\mathbf{P}_a \simeq A_a u^{-\tilde{M}_a} , \quad \mathbf{P}^a \simeq A^a u^{\tilde{M}_a - 1} , \quad \mathbf{Q}_i \simeq B_i u^{\hat{M}_i - 1} , \quad \mathbf{Q}^i \simeq B^i u^{-\hat{M}_i} , \tag{4.13}$$

where $\tilde{M}_a$, $a = 1, \dots, 4$ and $\hat{M}_i$, $i = 1, \dots, 4$ are functions of the values of the 6 Cartan generators of the $\mathfrak{psu}(2,2|4)$ symmetry algebra of the $\mathcal{N} = 4$ SYM: integer J_1, J_2, J_3 ($\tilde{M}_a$) and Δ, S_1, S_2 ($\hat{M}_i$), which are specified below

$$\begin{aligned}
\tilde{M}_a &= \left\{ \frac{1}{2}\left(J_{1+2-3} + 2\right) , \frac{1}{2} J_{1-2+3} , \frac{1}{2}\left(-J_{1-2-3} + 2\right) , -\frac{1}{2} - J_{1+2+3} \right\} , \\
\hat{M}_i &= \left\{ \frac{1}{2}\left(\Delta - S_{1+2} + 2\right), \frac{1}{2}\left(\Delta + S_{1+2}\right), \frac{1}{2}\left(-\Delta - S_{1-2} + 2\right), \frac{1}{2}\left(-\Delta + S_{1-2}\right) \right\} ,
\end{aligned} \tag{4.14}$$

where $J_{i+j} \equiv J_i + J_j$ and $S_{i+j} \equiv S_i + S_j$.

As we know from the classical integrability of the dual superstring σ-model (see, for example, [18]), the **P**- and **Q**-functions at least have the quadratic branch points at $u = \pm 2g$. From the asymptotics (4.13) and (4.14) we can expect the cuts of **P**-functions to be short[1]. The minimal choice for the functions $\mathbf{P}_a(u)$ and $\mathbf{P}^a(u)$, $a = 1, \dots, 4$ is to have only one short cut on the real axis. From the asymptotics of the **Q**-functions we can see that they have a nontrivial monodromy around infinity, thus we have to assume the cuts of these functions to be long. So again the minimal choice for $\mathbf{Q}_i(u)$ and $\mathbf{Q}^i(u)$, $i = 1, \dots, 4$ would be to have only one long cut on the real axis. The analytic structure of **P**- and **Q**-functions is illustrated on the Figure 4.1. Notice, that because the functions $\mathbf{Q}_i$ and $\mathbf{Q}^i$ have the long cuts in the complex plane, their asymptotics prescribed from the large u limit of the superstring σ-model hold in the upper half-plane and in the lower half-plane they can be different, therefore the third and fourth formulas from (4.13) are valid for $\mathrm{Im}\, u > 0$[2].

[1] For some values of the Cartan charges J_1, J_2 and J_3 of the $\mathcal{N} = 4$ SYM symmetry algebra $\mathfrak{psu}(2,2|4)$ there could appear a quadratic branch cut going to infinity. However, the **P**-functions usually come in

Figure 4.1: Analytic structure of the **P**- and **Q**-functions on their defining sheet.

It is convenient for us to introduce some short-hand notations as in [17]: UHP – upper half-plane, LHP – lower half-plane, UHPA – upper half-plane analytic and LHPA – lower half-plane analytic.

As we have introduced the analytic structure of the basic set of Q-functions, let us proceed with the consideration of the other ones. We define the function $\mathcal{Q}_{a|i}$ as an UHPA solution of the equation (4.1)

$$\mathcal{Q}^+_{a|i} - \mathcal{Q}^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i \,, \quad \text{Im } u > 0 \tag{4.15}$$

with the asymptotic

$$\mathcal{Q}_{a|i} \simeq -i \frac{A_a B_i}{-\tilde{M}_a + \hat{M}_i} u^{-\tilde{M}_a + \hat{M}_i}. \tag{4.16}$$

In what follows we will denote the UHPA Q-functions of the Q-system obtained from $\mathbf{P}_a$, $\mathbf{Q}_i$ and $\mathcal{Q}_{a|i}$ by the application of the QQ-relations by curly $\mathcal{Q}$ (as it was done in [17] to underline that these Q-functions have certain analytic properties and where the corresponding Q-system was called fundamental). For the Hodge-dual Q-functions, which are UHPA as well and satisfy the same QQ-relations, we will also use curly $\mathcal{Q}$ to depict its Q-functions.

Substitution of (4.13) and (4.16) into $\mathbf{Q}_i = -\mathcal{Q}^+_{a|i}\mathbf{P}^a$ and $\mathbf{P}_a = -\mathcal{Q}^+_{a|i}\mathbf{Q}^i$ expressed from (4.5) and (4.6) themselves leads us to the systems of equations for $A_a A^a$ and $B_i B^i$ respectively, which can be solved [17, 18] and give the result

$$A_{a_0}A^{a_0} = i \frac{\prod\limits_{j=1}^{4} \left(\tilde{M}_{a_0} - \hat{M}_j\right)}{\prod\limits_{\substack{b=1 \\ b\neq a_0}} \left(\tilde{M}_{a_0} - \tilde{M}_b\right)} \,, \quad B_{i_0}B^{i_0} = -i \frac{\prod\limits_{a=1}^{4} \left(\hat{M}_{i_0} - \tilde{M}_a\right)}{\prod\limits_{\substack{j=1 \\ j\neq i_0}}^{4} \left(\hat{M}_{i_0} - \hat{M}_j\right)} \,, \quad a_0, i_0 = 1, \ldots, 4 \,, \tag{4.17}$$

where there is no summation over the indices a_0 and i_0 implied. For further convenience we introduce the shorthand notations $A_{a_0}A^{a_0} = \mathbf{A}_{i_0}$ and $B_{i_0}B^{i_0} = \mathbf{B}_{i_0}$.

Each function $\mathcal{Q}_{a|i}(u)$ is analytic for Im $u > -1/2$ due to the fact that both $\mathbf{P}_a(u)$ and $\mathbf{Q}_i(u)$ are UHPA. As it was mentioned in the Subsection 4.1.1 with the usage of the QQ-relations we can restore the remaining Q-functions thus building the UHPA Q-system. The Hodge-duality (4.3) does not change the analytic properties, therefore the Hodge-dual Q-system with the upper indices has the same analytic properties, i.e. UHPA.

Now we are going to turn to the analytic structure of the functions $\mathbf{Q}_i(u)$, $i = 1, \ldots, 4$. Let us remember the formulas (4.5). One can notice that the functions $\mathbf{Q}_i$ and $\mathbf{Q}^i$ with

bilinear combinations and these cuts cancel each other in these combinations.

[2] It should be noted that in the case, when at least one of the spins S_1 and S_2 is non-integer, the asymptotics of the **Q**-functions in the lower half-plane on the sheet with the long cuts can be not power-like but instead become some power times an exponential factor.

$-\mathcal{Q}^+_{a|i}\mathbf{P}^a$ and $\mathcal{Q}^{a|i+}\mathbf{P}_a$ respectively coincide only in the UHP, because their analytic structure in the LHP is different. Indeed, we can see that if we rewrite the QQ-relations for $\mathcal{Q}_{a|i}$ using (4.5)

$$\mathcal{Q}^-_{a|i} = \left(1 + \mathbf{P}_a\mathbf{P}^b\right)\mathcal{Q}^+_{b|i}\,, \tag{4.18}$$

it is possible to find the values of $\mathcal{Q}_{a|i}$ for $\text{Im}\ u < -1/2$. In the strip $-k-1 < \text{Im}\ u < -k$ for $k = 0, 1, 2, \ldots$ the functions $\mathcal{Q}^-_{a|i}$ are given by

$$\mathcal{Q}^-_{a|i} = \prod_{l=0}^{k}\left(1 + \mathbf{P}_a\mathbf{P}^b\right)^{[2l]}\mathcal{Q}^{[2k+1]}_{a|i}\,. \tag{4.19}$$

From (4.19) we can see that the functions $\mathcal{Q}_{a|i}$ have the infinite number of short cuts at the horizontal lines with $\text{Im}\ u = -k - 1/2$ for $k = 0, 1, 2, \ldots$ and due to (4.4) $\mathcal{Q}^{a|i}$ has the same structure of cuts in the complex plane. Therefore, the functions $-\mathcal{Q}^+_{a|i}\mathbf{P}^a$ and $\mathcal{Q}^{a|i+}\mathbf{P}_a$ have also the infinite ladder of cuts at $\text{Im}\ u = -k$ for $k = 0, 1, 2, \ldots$, which clearly do not coincide with the analytic structure of $\mathbf{Q}_i$ and $\mathbf{Q}^i$ in the LHP, who are LHPA.

However, we can resolve this difficulty by interpreting the analytic continuation of $-\mathcal{Q}^+_{a|i}\mathbf{P}^a$ and $\mathcal{Q}^{a|i+}\mathbf{P}_a$ under their short cut on the real axis from above as $\mathbf{Q}_i$ and $\mathbf{Q}^i$ respectively. To formulate this clearly let us use the hats and checks introduced in the beginning of the present Subsection, which denote the values of the $\mathbf{P}$- and $\mathbf{Q}$-functions on the different sheets. In these notations first of all the equations (4.15) and (4.19) determine $\hat{\mathcal{Q}}_{a|i}$ on the physical sheet with the short cuts. Then, the values of the $\hat{\mathbf{Q}}_i$ and $\hat{\mathbf{Q}}^i$ on their physical sheet with the short cuts are given by

$$\hat{\mathbf{Q}}_i = -\hat{\mathcal{Q}}^+_{a|i}\hat{\mathbf{P}}^a\,, \quad \hat{\mathbf{Q}}^i = \hat{\mathcal{Q}}^{a|i+}\hat{\mathbf{P}}_a \tag{4.20}$$

and coincide with the $\mathbf{Q}$-functions on the mirror sheet with the long cut on the real axis in the UHP

$$\hat{\mathbf{Q}}_i = \check{\mathbf{Q}}_i\,, \quad \hat{\mathbf{Q}}^i = \check{\mathbf{Q}}^i\,, \quad \text{Im}\ u > 0\,. \tag{4.21}$$

Whereas in the LHP we interpret the analytic continuation of $\hat{\mathbf{Q}}_i$ and $\hat{\mathbf{Q}}^i$ under their cut on the real axis as $\check{\mathbf{Q}}_i$ and $\check{\mathbf{Q}}^i$ in the LHP

$$\tilde{\hat{\mathbf{Q}}}_i = \check{\mathbf{Q}}_i\,, \quad \tilde{\hat{\mathbf{Q}}}^i = \check{\mathbf{Q}}^i\,, \quad \text{Im}\ u < 0\,, \tag{4.22}$$

where tilde denotes the analytic continuation under the cut on the real axis. We illustrated the equalities (4.21) and (4.22) describing the analytic structure of the $\mathbf{Q}$-functions of the Fig. 4.2.

Looking at the obtained picture from above, we conclude that there is no fundamental reason to choose the generated Q-system to be UHPA. Indeed, there exists a transformation of complex conjugation, which preserves the QQ-relations but interchanges UHPA with LHPA. Its explicit form is written in [17]

$$\mathcal{Q}_{a_1,\ldots,a_n|i_1,\ldots,i_m}(u) \to (-1)^{\frac{(m+n)(m+n-1)}{2}}\bar{\mathcal{Q}}_{a_1,\ldots,a_n|i_1,\ldots,i_m}(u)\,, \tag{4.23}$$

where the complex conjugated Q-functions are defined as $\bar{Q}(u) = (Q(u^*))^*$.

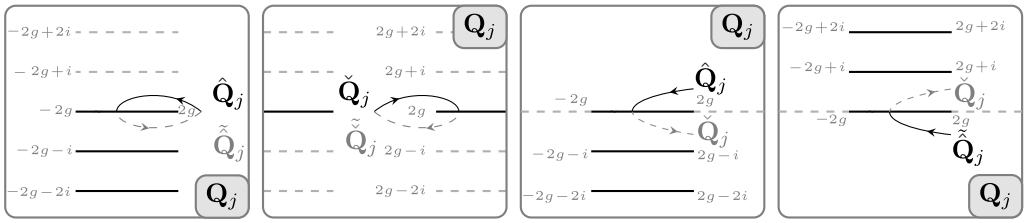

Figure 4.2: Analytic structure of the **Q**-functions on the different sheets of the Riemann surface.

The transformation (4.23) generates the Q-system which is LHPA and satisfies the same QQ-relations as the initial UHPA Q-system. It should be noted that the Hodge-dual Q-system also admits such a transformation

$$\mathcal{Q}^{a_1,\dots,a_n|i_1,\dots,i_m}(u) \to (-1)^{\frac{(m+n)(m+n-1)}{2}} \bar{\mathcal{Q}}^{a_1,\dots,a_n|i_1,\dots,i_m}(u) \ . \tag{4.24}$$

Let us now remember the analyticity properties of the function $\mathbf{Q}^i$ from (4.21) and (4.22). We see that the functions $\bar{\tilde{\hat{\mathbf{Q}}}}^i$ are LHPA and therefore have the same analyticity properties as the functions $\bar{\hat{\mathbf{Q}}}_i$ and $\bar{\hat{\mathbf{Q}}}^i$. From the strong coupling limit of the superstring σ-model and its classical integrability (see the pedagogical explanation of this in [18]) we know that for the case of integer spins S_1 and S_2 each function $\bar{\tilde{\hat{\mathbf{Q}}}}^i$, $i = 1, \dots, 4$ coincides with the certain function from the set $\bar{\hat{\mathbf{Q}}}_j$, $j = 1, \dots, 4$ in this limit. Thus, summarizing all this, we impose the equality of the LHPA functions $\bar{\tilde{\hat{\mathbf{Q}}}}^i$ and $\bar{\hat{\mathbf{Q}}}_j$ up to some matrix $M^{ij}(u)$

$$\bar{\tilde{\hat{\mathbf{Q}}}}^i(u) = M^{ij}(u)\bar{\hat{\mathbf{Q}}}_j(u) \ . \tag{4.25}$$

From now on we will call (4.25) the gluing condition and $M^{ij}(u)$ the gluing matrix, whose properties we will analyze below. We formulated QSC in the form (4.25) because this form is convenient to analytically continue the QSC solution for the case of non-integer spins S_1 and S_2. The transformation (4.23) generates the Q-system which is LHPA and satisfies the same QQ-relations as the initial UHPA Q-system. As there is no principal difference between UHPA and LHPA Q-systems and they describe the same spectral problem and due to the unitarity of the $\mathcal{N} = 4$ SYM theory, the UHPA and LHPA Q-systems have to be related by the symmetries of the Q-system, namely, the combination of the Hodge duality and H-symmetry[3]. Thus, we can interpret the gluing matrix M^{ij} as an i-periodic matrix of the H-transformation, which, in particular, relates the **Q**-functions with one lower and one upper "fermionic" index on the mirror sheet

$$\check{\mathbf{Q}}^i = M^{ij}\bar{\check{\mathbf{Q}}}_j \ , \quad \check{\mathbf{Q}}_i = \left(M^{-t}\right)_{ij} \bar{\check{\mathbf{Q}}}^j \ , \tag{4.26}$$

where $-t$ means that the inverse matrix is transposed.

Using the analyticity properties of the functions $\check{\mathbf{Q}}^i$ and $\bar{\check{\mathbf{Q}}}_j$ we are able to establish some properties of the matrix M^{ij}. Utilizing the i-periodicity of the matrix M^{ij}, it

[3]The presence of the Hodge duality can explained from the consideration of the classical limit of the superstring σ-model, which says that the analytic continuation of the Q-functions with the lower indices $\tilde{\hat{\mathbf{Q}}}_i(u)$ are related to the **Q**-functions with the upper indices $\hat{\mathbf{Q}}^i$, not the lower ones.

is possible to express its elements in terms of $\check{\mathbf{Q}}^i$ and $\bar{\check{\mathbf{Q}}}_j$. From the i-periodicity of $M^{ij}(u)$, (4.26) and remembering the QQ-relation for Q-function with 4 "fermionic" indices $Q_{\emptyset|1234} = \det_{1\leq k,l\leq 4} Q^{[5-2l]}_{\emptyset|k}$ one can show that

$$M^{ij}\bar{\check{\mathcal{Q}}}_{\emptyset|1234} = \det \begin{pmatrix} \bar{\check{\mathbf{Q}}}_1^{[+3]} & \bar{\check{\mathbf{Q}}}_1^{[+1]} & \bar{\check{\mathbf{Q}}}_1^{[-1]} & \bar{\check{\mathbf{Q}}}_1^{[-3]} \\ \cdots & & & \\ \check{\mathbf{Q}}^{i[+3]} & \check{\mathbf{Q}}^{i[+1]} & \check{\mathbf{Q}}^{i[-1]} & \check{\mathbf{Q}}^{i[-3]} \\ \cdots & & & \\ \bar{\check{\mathbf{Q}}}_4^{[+3]} & \bar{\check{\mathbf{Q}}}_4^{[+1]} & \bar{\check{\mathbf{Q}}}_4^{[-1]} & \bar{\check{\mathbf{Q}}}_4^{[-3]} \end{pmatrix} \leftarrow j\text{-th row} . \tag{4.27}$$

First of all let us show that the matrix elements M^{ij} do not have any branch points. To see this let us notice that for Im $u < -3/2$ the Q-function on the LHS and the determinant on the RHS of (4.27) are analytic and do not have branch points. Therefore in the same region and due to the i-periodicity M^{ij} has to be free from the branch points in the whole complex plane.

Second, in principle, M^{ij} can have poles. As M^{ij} is i-periodic, the existence of a pole, for example, in the point u_0 automatically leads to the infinite number of poles in the points $u_0 + ik$, where $k \in \mathbb{Z}$. However, at least in the region Im $u < -3/2$ the RHS of (4.27) is analytic, thus the poles of M^{ij} have to be compensated by the zeroes of $\bar{\check{\mathcal{Q}}}_{\emptyset|1234}$ in the same points or, in other words, there exists k_0 such that $\bar{\check{\mathcal{Q}}}_{\emptyset|1234}(u_0 + ik) = 0$ for $k \leq k_0$. In its turn this means that the number of zeroes of $\bar{\check{\mathcal{Q}}}_{\emptyset|1234}$ is infinite and these zeroes accumulate at infinity. We know that $\bar{\check{\mathcal{Q}}}_{\emptyset|1234}$ has power-like asymptotic, then there exist such a and b that

$$\bar{\check{\mathcal{Q}}}_{\emptyset|1234}(u) - au^b = \mathcal{O}\left(u^{b-1}\right) , \quad u \to \infty . \tag{4.28}$$

However, evaluating the LHS of (4.28) at $u = u_0 + ik$ for $k \leq k_0$ leads to a contradiction with the RHS of (4.28). From this contradiction we conclude that M^{ij} cannot have any poles.

Summarizing what was said we see that the gluing matrix M^{ij} is analytic in the whole complex plane. For the physical state, which means that the spins S_1 and S_2 are integer and the asymptotics of the functions $\tilde{\mathbf{Q}}^i$ are the power-like, analyticity and i-periodicity of the gluing matrix M^{ij} leads us to the conclusion that it is constant in this case.

Now let us return back to the equations (4.26). As we know that the matrix M^{ij} is free of any singularities, then analytically continuing both sides of the equations (4.26) to the sheet with the short cuts we obtain

$$\hat{\tilde{\mathbf{Q}}}^i = M^{ij}\bar{\hat{\mathbf{Q}}}_j , \quad \bar{\hat{\tilde{\mathbf{Q}}}}_i = \left(M^{-t}\right)_{ij} \hat{\mathbf{Q}}^j . \tag{4.29}$$

Analytic properties of the $\mathbf{Q}$-functions allow us to establish one important property of the gluing matrix. Analytically continuing both sides of the first equation from (4.29), using the fact that due to the quadratic nature of the branch points analytic continuation and complex conjugation commute with each other, and then applying the second equation from (4.29) we derive

$$\hat{\mathbf{Q}}^i = M^{ij}\left(\bar{M}^{-t}\right)_{jk} \hat{\mathbf{Q}}^k . \tag{4.30}$$

Noticing that (4.30) is true for any point u and applying the same trick as above and using the analyticity properties of the gluing matrix, we arrive to the conclusion that

$$M^{ij}\left(\bar{M}^{-t}\right)_{jk}=\delta^i_k \tag{4.31}$$

and the gluing matrix is hermitian

$$\bar{M}^{ij}(u)=M^{ji}(u)\ . \tag{4.32}$$

In what follows we are going mainly to deal with the Q-functions on the physical sheet therefore from now on we omit the hats and checks above the designations of the Q-functions implying that all the Q-functions are considered on the sheet with the short cuts if the opposite is not mentioned specifically.

Now we are ready to find the other constraints on the gluing matrix which follow from the conjugation and parity symmetries of the Q-system. To do this we will need the 4th order Baxter equation for the functions $\mathbf{Q}_i(u)$ to see if the certain properties of the $\mathbf{P}_a(u)$ and $\mathbf{P}^a(u)$ can allow us to relate $\mathbf{Q}_i(u)$, $\bar{\mathbf{Q}}_i(u)$ and $\mathbf{Q}_i(-u)$. In the two subsequent Subsections we analyze the implications of the conjugation and parity properties respectively.

4.1.3 Complex conjugation symmetry

Let us now concentrate on the conjugation properties of the **P**- and **Q**-functions assuming the charges Δ, S_1 and S_2 to be real. In [17] from the reality of the energy, Y- and T-functions and the fact that the complex conjugation supplemented by the certain sign factor is the symmetry of the Q-system it is shown that complex conjugation is equivalent to some H-symmetry transformation already mentioned in the Subsection 4.1.1. The matrix h_B of this transformation was proven in [17] to be constant due to analytic properties and power-like asymptotic of the **P**-functions. Then there was found a transformation which allows to make all **P**-functions with lower indices purely real and thus the **P**-functions with upper indices pure imaginary. However, in our calculations we use the different normalization and make the other H-rotation by multiplying $\mathbf{P}_3$ and $\mathbf{P}_4$ by i and thus $\mathbf{P}^1$ and $\mathbf{P}^2$ also by i and obtain

$$\bar{\mathbf{P}}_{1,2}=\mathbf{P}_{1,2}\ ,\quad \bar{\mathbf{P}}_{3,4}=-\mathbf{P}_{3,4}\ ,\quad \bar{\mathbf{P}}^{1,2}=-\mathbf{P}^{1,2}\ ,\quad \bar{\mathbf{P}}^{3,4}=\mathbf{P}^{3,4}$$

or, in other words

$$\bar{\mathbf{P}}_a=C^b_a\mathbf{P}_b\ ,\quad \bar{\mathbf{P}}^a=-C^a_b\mathbf{P}^b\ ,\quad C=\mathrm{diag}\{1,1,-1,-1\}\ . \tag{4.33}$$

Given the conjugation properties (4.33), we see that the Baxter equation (4.11), written for the Q-functions with prescribed analytic properties

$$\mathbf{Q}_i^{[+4]}-\mathbf{Q}_i^{[+2]}\left[D_1-\mathbf{P}_a^{[+2]}\mathbf{P}^{a[+4]}D_0\right]+\mathbf{Q}_i\left[D_2-\mathbf{P}_a\mathbf{P}^{a[+2]}D_1+\mathbf{P}_a\mathbf{P}^{a[+4]}D_0\right]-$$
$$-\mathbf{Q}_i^{[-2]}\left[\bar{D}_1+\mathbf{P}_a^{[-2]}\mathbf{P}^{a[-4]}\bar{D}_0\right]+\mathbf{Q}_i^{[-4]}=0 \tag{4.34}$$

with D_k and $\bar{D}_k$ given by (4.12) with $Q^{a|\emptyset}=\mathbf{P}^a$, remains the same, but for $\bar{\mathbf{Q}}_i(u)$ now. Thus, the functions $\bar{\mathbf{Q}}_i$ satisfy the equation (4.34) too. As the functions $\bar{\mathbf{Q}}_i(u)$ constitute

a basis in the space of solutions of the Baxter equation (4.34) this means that there has to exist an i-periodic matrix $\Omega_i^j(u)$ such that

$$\bar{\mathbf{Q}}_i(u) = \Omega_i^j(u)\mathbf{Q}_j(u). \tag{4.35}$$

As in [84] from (4.33) and (4.5) this matrix can be found to be

$$\Omega_i^j = \bar{\mathcal{Q}}^-_{a|i} C_b^a \mathcal{Q}^{b|j-} \ . \tag{4.36}$$

It is i-periodic $\Omega_i^{j++} = \Omega_i^j$ (see Appendix A.5 for the proof) and using this it is not hard to show that

$$\Omega_i^j \bar{\Omega}_j^k = \bar{\mathcal{Q}}^-_{a|i} C_b^a \mathcal{Q}^{b|j-} \mathcal{Q}^+_{c|j} C_d^c \bar{\mathcal{Q}}^{d|k+} = \bar{\mathcal{Q}}^-_{a|i} C_b^a \mathcal{Q}^{b|j-} \mathcal{Q}^-_{c|j} C_d^c \bar{\mathcal{Q}}^{d|k-} = \delta_i^k \ , \tag{4.37}$$

which means that $\Omega^{-1} = \bar{\Omega}$ and

$$\mathbf{Q}_i(u) = \bar{\Omega}_i^j(u)\bar{\mathbf{Q}}_j(u) \ . \tag{4.38}$$

The matrix $\bar{\Omega}_i^j$ also relates $\bar{\mathbf{Q}}^j$ and $\mathbf{Q}^i$

$$\bar{\mathbf{Q}}^j = -\bar{\Omega}_i^j \mathbf{Q}^i \tag{4.39}$$

and vice versa

$$\mathbf{Q}^j = -\Omega_i^j \bar{\mathbf{Q}}^i \ . \tag{4.40}$$

To determine the consequences of the conjugation symmetry for the gluing matrix we substitute (4.35) into the first gluing condition from (4.29) and obtain

$$\tilde{\mathbf{Q}}^i = M^{ij}\Omega_j^k \mathbf{Q}_k \ . \tag{4.41}$$

Let us analyze the matrix $M^{ij}\Omega_j^k$ more closely. As it is a product of two i-periodic matrices it has also to be i-periodic. We remember that according to its definition (4.36) the matrix $\Omega_j^k(u)$ has an infinite ladder of short cuts. Using the result of [84] we get the discard of $\Omega_j^k(u)$

$$\tilde{\Omega}_j^k - \Omega_j^k = -\bar{\tilde{\mathbf{Q}}}_j \tilde{\mathbf{Q}}^k + \bar{\mathbf{Q}}_j \mathbf{Q}^k \ . \tag{4.42}$$

Multiplying both sides of (4.42) by M^{ij} and utilizing the first gluing condition from (4.29) we derive the equation[4]

$$M^{ij}\tilde{\Omega}_j^k - M^{ij}\Omega_j^k = -\mathbf{Q}^i\tilde{\mathbf{Q}}^k + \mathbf{Q}^k\tilde{\mathbf{Q}}^i \ . \tag{4.43}$$

We note that the RHS of (4.43) is antisymmetric in the indices i and k, thus we conclude that the function $M^{ij}\Omega_j^k + M^{kj}\Omega_j^i$ has no cuts on the real axis. As this function is i-periodic it follows that $M^{ij}\Omega_j^k + M^{kj}\Omega_j^i$ is analytic in the whole complex plane.

Let us introduce a new notation

$$\omega^{ik} \equiv M^{ij}\Omega_j^k \ , \tag{4.44}$$

[4]Notice that the RHS of the equation (4.43) coincides with the RHS of the equation of the $\mathbf{Q}\omega$-system $\tilde{\omega}^{ij} - \omega^{ij} = -\mathbf{Q}^i\tilde{\mathbf{Q}}^j + \mathbf{Q}^j\tilde{\mathbf{Q}}^i$.

where ω^{ik} are the i-periodic functions on the sheet with the short cuts. Remembering (4.41) and applying (4.44) on the sheet with the short cuts we obtain the equation

$$\tilde{\hat{\omega}}^{ik} - \hat{\omega}^{ik} = \left(\delta^i_j \hat{\mathbf{Q}}^k \hat{\mathbf{Q}}_l - \delta^k_j \hat{\mathbf{Q}}^i \hat{\mathbf{Q}}_l\right) \hat{\omega}^{jl} \ . \tag{4.45}$$

On the other hand, the functions ω^{ik} are i-periodic on the sheet with the short cuts, thus on the sheet with the long cuts their analytic continuation under the cut on the real axis is given by the simple formula

$$\tilde{\check{\omega}}^{ik} = \left(\check{\omega}^{ik}\right)^{++} \ . \tag{4.46}$$

Rewriting (4.45) on the sheet with the long cuts gives us

$$\left(\check{\omega}^{ik}\right)^{++} - \check{\omega}^{ik} = \left(\delta^i_j \check{\mathbf{Q}}^k \check{\mathbf{Q}}_l - \delta^k_j \check{\mathbf{Q}}^i \check{\mathbf{Q}}_l\right) \check{\omega}^{jl} \ . \tag{4.47}$$

If we return several steps back, we can derive from the QQ-relations the equation for the function $\mathcal{Q}^{ab|ij}$, which looks almost exactly like (4.47)

$$\check{\mathcal{Q}}^{ab|ij+} - \check{\mathcal{Q}}^{ab|ij-} = \left(\delta^i_k \check{\mathbf{Q}}^j \check{\mathbf{Q}}_l - \delta^j_k \check{\mathbf{Q}}^i \check{\mathbf{Q}}_l\right) \check{\mathcal{Q}}^{ab|kl-} \ . \tag{4.48}$$

Recalling the notion of μ-functions introduced in the QSC framework in [16, 17], which are i-periodic on the sheet with the long cuts, we multiply both sides by $\check{\mu}_{ab}$, which leads us to

$$\left(\check{\mu}_{ab}\check{\mathcal{Q}}^{ab|ij-}\right)^{++} - \left(\check{\mu}_{ab}\check{\mathcal{Q}}^{ab|ij-}\right) = \left(\delta^i_k \check{\mathbf{Q}}^j \check{\mathbf{Q}}_l - \delta^j_k \check{\mathbf{Q}}^i \check{\mathbf{Q}}_l\right) \left(\check{\mu}_{ab}\check{\mathcal{Q}}^{ab|kl-}\right) \ . \tag{4.49}$$

Therefore, the functions $\check{\omega}^{ij}$ and $\check{\mu}_{ab}\check{\mathcal{Q}}^{ab|ij-}$ satisfy the same equation. As the functions $\check{\mu}_{ab}\check{\mathcal{Q}}^{ab|ij-}$ are antisymmetric in i and j due to the antisymmetry of the Q-function $\mathcal{Q}^{ab|ij}$, it is natural to impose the constraint that ω^{ik} is also antisymmetric and

$$\omega^{ik} = M^{ij}\Omega^k_j = -M^{kj}\Omega^i_j = -\omega^{ki} \ . \tag{4.50}$$

In the following Subsections we are going to exploit (4.50) to constrain the gluing matrix for different spins S_1 and S_2.

4.1.4 Parity symmetry

Now we are going to describe the parity properties of the Q-system. For a large class of states the **P**-functions possess the certain parity. Such states include the states with the charges $J_1 = 2$, $J_2 = J_3 = 0$, which we consider in the rest of the paper and also the ground state with the charges $J_1 = 3$ and $J_2 = J_3 = 0$, which is relevant for the BFKL Odderon eigenvalue (see [85, 86]). Thus, for the case $J_1 = 2$, $J_2 = J_3 = 0$ we have

$$\begin{aligned} &\tilde{M}_a = \{2, 1, 0, -1\} \, , \\ &\hat{M}_i = \left\{ \frac{1}{2}\left(\Delta - S_{1+2} + 2\right) \, , \frac{1}{2}\left(\Delta + S_{1+2}\right) \, , \frac{1}{2}\left(-\Delta - S_{1-2} + 2\right) \, , \frac{1}{2}\left(-\Delta + S_{1-2}\right) \right\} \ . \end{aligned} \tag{4.51}$$

As we understood the analytic structure of **P**- and **Q**-functions, taking into account the asymptotics of these functions expressed in terms of the charges (4.51), it is natural

to assume the existence of the certain symmetry between the Q-functions with lower and upper indices, which also changes sign of Δ. This symmetry takes a particularly simple form for the following choice of the normalization of the **P**-

$$A_a = (1, 1, -\mathbf{A}_3, \mathbf{A}_4) \ , \quad A^a = (\mathbf{A}_1, \mathbf{A}_2, -1, 1) \ , \tag{4.52}$$

and **Q**-functions

$$B_i = (-\mathbf{B}_1, 1, -\mathbf{B}_3, -1) \ , \quad B^i = (-1, \mathbf{B}_2, -1, -\mathbf{B}_4) \ , \tag{4.53}$$

which can be set with the usage of the rescaling symmetry (4.10). We obtain[5]

$$\mathbf{P}^a(\Delta, u) = \chi^{ab}\mathbf{P}_b(-\Delta, u) \ , \quad \mathbf{Q}^i(\Delta, u) = \eta^{ij}\mathbf{Q}_j(-\Delta, u) \ , \tag{4.54}$$

where

$$\chi^{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \quad \eta^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.55}$$

and χ^{ab} is the same matrix for the left-right symmetric states as in [16, 17].

For the operators we examine ($J_1 = 2$ and $J_2 = J_3 = 0$) in the following Sections the **P**-functions have the certain parity. Their parity is dictated by the asymptotics of the **P**-functions (4.13)

$$\mathbf{P}_a(-u) = (-1)^{a+1}\mathbf{P}_a(u) \ , \quad \mathbf{P}^a(-u) = (-1)^a\mathbf{P}^a(u) \ . \tag{4.56}$$

The symmetry (4.56) is a symmetry of the Baxter equation (4.34), thus $\mathbf{Q}_i(-u)$ is also a solution of (4.34). Utilizing the same logic as in the case of the complex conjugation, we conclude that there exists an i-periodic matrix $\Theta_i^j(u)$ (see the proof in Appendix A.5) such that

$$\mathbf{Q}_i(-u) = \Theta_i^j(u)\mathbf{Q}_j(u) \ . \tag{4.57}$$

It is possible also to find the matrix with such a property. Utilizing again (4.5), we obtain

$$\Theta_i^j(u) = (-1)^{a+1}\mathcal{Q}^-_{a|i}(-u)\mathcal{Q}^{a|j-}(u) \ , \tag{4.58}$$

where the summation over a is implied. The matrix Θ has the property $\Theta_i^j(u)\Theta_j^k(-u) = \delta_i^k$ and thus $\Theta^{-1}(u) = \Theta(-u)$. We can write

$$\mathbf{Q}_i(u) = \Theta_i^j(-u)\mathbf{Q}_j(-u). \tag{4.59}$$

The matrix $\Theta_i^j(-u)$ also relates $\mathbf{Q}^j(-u)$ and $\mathbf{Q}^i(u)$

$$\mathbf{Q}^j(-u) = \Theta_i^j(-u)\mathbf{Q}^i(u) \tag{4.60}$$

[5]The change of the sign $\Delta \to -\Delta$ is a symmetry of the equation and it should map one solution to another solution. One can check that $\mathbf{P}'^a = \chi^{ab}\mathbf{P}_b$, $\mathbf{P}'_a = \left(\chi^{-1}\right)_{ab}\mathbf{P}^b$, $\mathbf{Q}'^i = \eta^{ij}\mathbf{Q}_j$, $\mathbf{Q}'_i = \left(\eta^{-1}\right)_{ij}\mathbf{Q}^j$ and $M'^{ij} = \eta^{ik}\left(M^{-t}\right)_{kl}\eta^{lj}$ is also a solution to the QQ-relations and the gluing conditions but with Δ flipped to $-\Delta$. As it can be seen explicitly in the Appendix 4.3.1 in the weak coupling limit these two solutions coincide therefore our solution is mapped onto itself. Given the starting point the recursive procedure described in the Section 4.3 is non-ambiguous we conclude that this property holds to all orders in the coupling constant.

and vice versa

$$\mathbf{Q}^j(u) = \Theta_i^j(u)\mathbf{Q}^i(-u). \tag{4.61}$$

Analogously to the consideration of parity symmetry, we can find the discard of Θ_j^k on the cut which is situated on the real axis

$$\tilde{\Theta}_j^k(u) - \Theta_j^k(u) = -\tilde{\mathbf{Q}}_j(-u)\tilde{\mathbf{Q}}^k(u) + \mathbf{Q}_j(-u)\mathbf{Q}^k(u)\ . \tag{4.62}$$

Using the matrix $\Theta_i^j(u)$ from (4.58), the gluing conditions (4.41) and (4.59) we are able to introduce another gluing matrix

$$\tilde{\mathbf{Q}}^i(u) = L^{ij}(u)\mathbf{Q}_j(-u)\ , \quad \tilde{\mathbf{Q}}_i = \left(L^{-t}\right)_{ij}(u)\mathbf{Q}^j(-u)\ , \tag{4.63}$$

where

$$L^{il}(u) = M^{ij}(u)\Omega_j^k(u)\Theta_k^l(-u)\ , \tag{4.64}$$

and as the $\mathbf{Q}$-functions on the both sides of the gluing conditions are LHPA, then by the same arguments as for $M^{ij}(u)$ the matrix $L^{ij}(u)$ is also analytic in the whole complex plane. Analogously to the case of the gluing matrix $M^{ij}(u)$ by going under the cut twice we derive the property

$$L^{ji}(u) = L^{ij}(-u)\ . \tag{4.65}$$

Together with (4.32), (4.50) and (4.64) condition (4.65) constitutes the set of equations which are used to calculate the gluing matrix for different values of the spins S_1 and S_2.

Since for the states in question the $\mathbf{P}$-functions have the certain parity, this has some consequences for the asymptotic expansion of the $\mathbf{Q}$-functions. As it is explained in detail in the Section 4.2 with the description of the numerical algorithm the certain parity of the $\mathbf{P}$-functions leads to the form (4.103) of the asymptotic expansion of $\mathcal{Q}_{a|i}$

$$\mathcal{Q}_{a|i}(u) \simeq u^{-\tilde{M}_a+\hat{M}_i}\sum_{l=0}^{+\infty}\frac{B_{a|i,2l}}{u^{2l}}\ . \tag{4.66}$$

By applying analogous arguments to the QQ-relation for the Hodge dual function $\mathcal{Q}^{a|i}$, we conclude that the asymptotic expansion of $\mathcal{Q}^{a|i}$ is also given by

$$\mathcal{Q}^{a|i}(u) \simeq u^{\tilde{M}_a-\hat{M}_i}\sum_{l=0}^{+\infty}\frac{B^{a|i,2l}}{u^{2l}}\ . \tag{4.67}$$

Then we remember that $\mathbf{Q}_i = -\mathcal{Q}^+_{a|i}\mathbf{P}^a$ and $\mathbf{Q}^i = \mathcal{Q}^{a|i+}\mathbf{P}_a$, which after the substitution of (4.98), (4.66) and (4.67) lead us to the asymptotic expansions at infinity of the $\mathbf{Q}$-functions

$$\mathbf{Q}_i(u) \simeq u^{\hat{M}_i-1}\left(B_i + \sum_{l=1}^{+\infty}\frac{B_{i,2l}}{u^{2l}}\right)\ , \quad \mathbf{Q}^i(u) \simeq u^{-\hat{M}_i}\left(B^i + \sum_{l=1}^{+\infty}\frac{B^{i,2l}}{u^{2l}}\right)\ . \tag{4.68}$$

In what follows we will regard to the $\mathbf{Q}$-functions with the asymptotic expansions (4.68) as having the "pure" asymptotic, as these expansions contain the powers of u, which differ only by an integer number. The asymptotic expansions (4.68) will be important in determining the structure of the matrices $\Omega_i^j(u)$ and $\Theta_i^j(u)$, which are analyzed below.

4.1.5 Constraining the gluing matrix

In the present Subsection we are going to derive the set of equations for the elements of the gluing matrix originating from the conditions found in the previous Subsections. To remind briefly the QSC framework we are using let us recall the constraints on the gluing matrices known by now. The non-degenerate matrices $M^{ij}(u)$ and $L^{ij}(u)$ satisfy the following set of constraints

$$\begin{aligned} &\bar{M}^{ji}(u) = M^{ij}(u)\ , \quad M^{ij}(u)\Omega^k_j(u) = -M^{kj}(u)\Omega^i_j(u)\ , \quad \left(\Omega^{-1}\right)^j_i(u) = \bar{\Omega}^j_i(u)\ , \\ &L^{il}(u) = M^{ij}(u)\Omega^k_j(u)\Theta^l_k(-u)\ , \quad L^{li}(-u) = L^{il}(u)\ , \quad \left(\Theta^{-1}\right)^k_j(u) = \Theta^k_j(-u)\ . \end{aligned} \tag{4.69}$$

Now we are able to consider the gluing matrix for the case of different AdS spins S_1 and S_2 solving the set of constraints (4.69).

Integer S_1 and S_2

Let us start our study from the situation when all the charges except for the dimension Δ are integer. More precisely, in the present Subsection we address the case when the spins S_1 and S_2 have the same parity. This is motivated by the fact that for $S_2 = 0$ the physical states have even non-negative S_1[6]. To analyze the constraints (4.69) more closely we need to find the properties of the matrices $\Omega^j_i(u)$ and $\Theta^j_i(u)$.

In what follows we will need the asymptotics of the matrices Ω^j_i and $\Theta^j_i(u)$. To analyze them let us remember (4.35) and (4.57). As the asymptotics of the **Q**-functions on the both sides of (4.35) and (4.57) are power-like and the Ω-matrix consists of the i-periodic functions, the series expansions of $\Omega^j_i(u)$ and $\Theta^j_i(u)$ for $|\mathrm{Re}\ u| \gg 1$ are given by

$$\Omega^j_i(u) = \sum_{k=0}^{+\infty} \left(\Omega^{(k)}_{\pm}\right)^j_i e^{\mp 2\pi k u}\ , \quad \Theta^j_i(u) = \sum_{k=0}^{+\infty} \left(\Theta^{(k)}_{\pm}\right)^j_i e^{\mp 2\pi k u}\ . \tag{4.70}$$

where the signs correspond to expansion at $+\infty$ and $-\infty$ respectively. It should be noted that (4.70) does not have any growing terms on the RHS, because this would violate the power-like asymptotic of the **Q**-functions.

The asymptotics of the **Q**-functions are pure and the asymptotic expansion is given by (4.68). Looking at the values of $\hat{M}_i$, $i = 1, \dots, 4$ one may think that because $\hat{M}_1 - \hat{M}_2$ and $\hat{M}_3 - \hat{M}_4$ are integers, there could potentially appear a mixing of $\mathbf{Q}_1(u)$ with $\mathbf{Q}_2(u)$ and $\mathbf{Q}_3(u)$ with $\mathbf{Q}_4(u)$ as this does not violate the purity of the asymptotics. However as the spins S_1 and S_2 have the same parity and

$$\hat{M}_1 - \hat{M}_2 = 1 \mod 2\ , \quad \hat{M}_3 - \hat{M}_4 = 1 \mod 2\ , \tag{4.71}$$

the functions $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$, $\mathbf{Q}_4$ cannot mix, because their asymptotic expansions (4.68) contain only even powers of u in the round brackets. Therefore the matrices $(\Omega^{(0)}_{\pm})^j_i$ and $(\Theta^{(0)}_{\pm})^j_i$ have to be diagonal. Let us now find them.

[6]This is due to the cyclicity constraint on the states of the $\mathfrak{sl}(2)$ Heisenberg spin chain, which is only consistent with the symmetric distribution of roots leading to S_1 even.

First, we consider the matrix $\Omega_i^j(u)$ and remember (4.35). If Re u tends to $+\infty$, then, as $\bar{\mathbf{Q}}_i \simeq \bar{B}_i u^{\hat{M}_i-1}$, the diagonal element $(\Omega_+^{(0)})_i^i$ is equal to $\bar{B}_i/B_i$. But if Re u tends to $-\infty$ the situation is a little more subtle. The functions $\mathbf{Q}_i(u)$ have an infinite ladder of short cuts going down from the real axis, while the functions $\bar{\mathbf{Q}}_i(u)$ have the same ladder of cuts going up. Then taking the limit of Re u to $-\infty$ we have to go to $-\infty$ along the semicircle in the UHP for $\mathbf{Q}_i(u)$, i.e. $\mathbf{Q}_i(u) \simeq B_i e^{i\pi(\hat{M}_i-1)}(-u)^{\hat{M}_i-1}$ and along the semicircle in the LHP for $\bar{\mathbf{Q}}_i(u)$, i.e. $\bar{\mathbf{Q}}_i(u) \simeq \bar{B}_i e^{-i\pi(\hat{M}_i-1)}(-u)^{\hat{M}_i-1}$, therefore we see that the diagonal element of $(\Omega_-^{(0)})_i^i$ is equal to $\bar{B}_i/B_i e^{-2i\pi\hat{M}_i}$. To sum up, we obtain

$$\Omega_i^j(u) = \begin{cases} \delta_i^j e^{2i\phi_{B_j}} + (\Omega_+^{(1)})_i^j e^{-2\pi u} + \mathcal{O}\left(e^{-4\pi u}\right), & \text{Re } u \gg 1\,, \\ \delta_i^j e^{2i\phi_{B_j} - 2i\pi\hat{M}_j} + (\Omega_-^{(1)})_i^j e^{2\pi u} + \mathcal{O}\left(e^{4\pi u}\right), & \text{Re } u \ll -1\,, \end{cases} \tag{4.72}$$

where $e^{2i\phi_{B_i}} = \bar{B}_i/B_i$.

Second, analyzing the matrix $\Theta_i^j(u)$ from (4.57) is analogous. Thus, applying the arguments from the previous paragraph, we see that at Re u tending to $+\infty$ we have to go around the semicircle in the UHP and $\mathbf{Q}_i(-u) \simeq B_i e^{i\pi(\hat{M}_i-1)} u^{\hat{M}_i-1}$, while at Re u tending to $-\infty$ we have $\mathbf{Q}_i(-u) \simeq B_i(-u)^{\hat{M}_i-1}$. Then we obtain

$$\Theta_i^j(u) = \begin{cases} -\delta_i^j e^{i\pi\hat{M}_j} + (\Theta_+^{(1)})_i^j e^{-2\pi u} + \mathcal{O}\left(e^{-4\pi u}\right), & \text{Re } u \gg 1\,, \\ -\delta_i^j e^{-i\pi\hat{M}_j} + (\Theta_-^{(1)})_i^j e^{2\pi u} + \mathcal{O}\left(e^{4\pi u}\right), & \text{Re } u \ll -1\,. \end{cases} \tag{4.73}$$

As it was explained for example in [18] in the strong coupling limit the asymptotics of the functions $\tilde{\mathbf{Q}}^i(u)$ are some powers of u, then the only possible ansatz for the gluing matrix $M^{ij}(u)$ is to assume it to be a constant matrix. Thus for the case in question we obtain from (4.50) and (4.72) rather simple conditions

$$M^{ji} = -M^{ij} e^{2i(\phi_{B_j} - \phi_{B_i})}, \tag{4.74}$$

$$M^{ji} = -M^{ij} e^{2i(\phi_{B_j} - \phi_{B_i}) + 2i\pi(\hat{M}_i - \hat{M}_j)}. \tag{4.75}$$

Combining the two conditions (4.74) and (4.75) we obtain the following

$$M^{ij}\left(e^{2i\pi\left(\hat{M}_i - \hat{M}_j\right)} - 1\right) = 0. \tag{4.76}$$

Let us see now which additional restrictions do we have in the case $J_2 = J_3 = 0$ and $J_1 = 2$. First of all from our assumptions about the asymptotics of the functions $\mathbf{Q}^i(u)$ we understand that the gluing matrix L^{ij} given by (4.64) has to be constant, i.e. $L^{ij}(u) = L^{ij}$ is a symmetric matrix. Therefore, from the (4.64), (4.72) and (4.73) we immediately find

$$L^{ij} = -M^{ij} e^{2i\phi_{B_j} - i\pi\hat{M}_j}\,. \tag{4.77}$$

Then, using (4.74) and the symmetry of L^{ij}, we derive

$$M^{ij}\left(e^{i\pi\left(\hat{M}_i - \hat{M}_j\right)} + 1\right) = 0\,. \tag{4.78}$$

It is easy to see that if (4.78) is true then (4.76) is also true. We have to calculate the differences between the charges $\hat{M}_i$ to determine which elements of the matrix M^{ij} are

non-vanishing. It appears that only $\hat{M}_1 - \hat{M}_2$ and $\hat{M}_3 - \hat{M}_4$ are integers

$$\begin{aligned} \hat{M}_1 - \hat{M}_2 &= -S_1 - S_2 + 1 \,, \\ \hat{M}_3 - \hat{M}_4 &= -S_1 + S_2 + 1 \,. \end{aligned} \tag{4.79}$$

Thus for the case of integer spins S_1 and S_2 we are left with the spins S_1 and S_2 with the same parity, which is consistent with our initial setup. Therefore, only the matrix elements $M^{12} = \bar{M}^{21}$ and $M^{34} = \bar{M}^{43}$ are non-zero. Then, in the case of integer spins S_1 and S_2 of the same parity we obtain the following gluing matrix

$$M^{ij} = \begin{pmatrix} 0 & M^{12} & 0 & 0 \\ \bar{M}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & M^{34} \\ 0 & 0 & \bar{M}^{34} & 0 \end{pmatrix} . \tag{4.80}$$

Using also (4.74) and (4.75), which are equivalent for S_1 of S_2 of the same parity as $\hat{M}_1 - \hat{M}_2$ and $\hat{M}_3 - \hat{M}_4$ equal 1 modulo 2, we are able to fix the phases of the non-zero matrix elements of (4.80)

$$M^{12} = \left|M^{12}\right| e^{i\left(\pm\frac{\pi}{2} + \phi_{B_1} - \phi_{B_2}\right)} \,, \quad M^{34} = \left|M^{34}\right| e^{i\left(\pm\frac{\pi}{2} + \phi_{B_3} - \phi_{B_4}\right)} \,. \tag{4.81}$$

Now let us start the consideration of the case when at least one of the spins is not integer as this is particularly interesting for the BFKL limit.

Non-integer S_1 and S_2

First of all, from the asymptotics (4.51) we immediately see that if at least one of the charges S_1 or S_2 is non-integer, then not to violate the purity of the asymptotic expansions (4.68) the matrices $(\Omega^{(0)}_{\pm})^j_i$ and $(\Theta^{(0)}_{\pm})^j_i$ cannot mix different **Q**-functions and have to be diagonal. Therefore, these matrices are given by (4.70). This means, that in the case of at least one non-integer spin under the assumption that the gluing matrix is constant we obtain the same constraint (4.78). However, as S_1 or S_2 or both spins are non-integer, all the differences $\hat{M}_i - \hat{M}_j$ are non-integer in general, therefore we conclude that $M^{ij} = 0$. Then we have to modify the ansatz for $M^{ij}(u)$.

The matrix $M^{ij}(u)$ is analytic and i-periodic, so the minimal choice would be to add the terms proportional to $e^{2\pi u}$ and $e^{-2\pi u}$

$$M^{ij}(u) = M^{ij}_1 + M^{ij}_2 e^{2\pi u} + M^{ij}_3 e^{-2\pi u} \tag{4.82}$$

and this is consistent with what we know from the consideration of the BFKL limit for which S_1 approaches -1 and $S_2 = 0$ (see [82]). From the previous conditions (4.32) it follows that the matrices $M^{ij}_{1,2,3}$ are hermitian.

Substituting (4.82) into (4.50) we obtain the following conditions for the matrix $M^{ij}(u)$

$$\begin{aligned} M^{ji}_2 &= -M^{ij}_2 e^{2i\left(\phi_{B_j} - \phi_{B_i}\right)} \,, \\ M^{ji}_3 &= -M^{ij}_3 e^{2i\left(\phi_{B_j} - \phi_{B_i}\right) + 2i\pi\left(\hat{M}_i - \hat{M}_j\right)} \,, \end{aligned} \tag{4.83}$$

where the summation over the repeated indices is not implied. For $i = j$ we immediately see from (4.83) that

$$M_2^{ii} = M_3^{ii} = 0 \, . \tag{4.84}$$

Let us remember that for the case in question $J_2 = J_3 = 0$ and $J_1 = 2$. The matrix $L^{ij}(u)$ is given by the formula (4.64). Taking the limits $u \to \pm\infty$ and remembering the expansions (4.72) and (4.73) we have to assume the existence of the exponential contributions to $L^{ij}(u)$

$$L^{ij}(u) = L_1^{ij} + L_2^{ij} e^{2\pi u} + L_3^{ij} e^{-2\pi u} \, , \tag{4.85}$$

where the matrix L_1^{ij} is symmetric and $L_2^{ji} = L_3^{ij}$ due to (4.65) and the latter two of them are given by

$$L_2^{il} = M_2^{ij} (\Omega_+^{(0)})_j^k (\Theta_-^{(0)})_k^l = -M_2^{il} e^{2i\phi_{B_l} - i\pi\hat{M}_l} \, , \tag{4.86}$$

$$L_3^{il} = M_3^{ij} (\Omega_-^{(0)})_j^k (\Theta_+^{(0)})_k^l = -M_3^{il} e^{2i\phi_{B_l} - i\pi\hat{M}_l} \, . \tag{4.87}$$

Exploiting the symmetry $L_3^{ji} = L_2^{ij}$ and the relation from (4.83) we derive

$$M_3^{ij} = -M_2^{ij} e^{i\pi(\hat{M}_j - \hat{M}_i)} \, . \tag{4.88}$$

As we observed in the case of integer spins S_1 and S_2 the determinant of the gluing matrix is constant. According to (4.82) in the case of at least one non-integer spin this determinant is not guaranteed to be integer. However, if we assume for a moment that the determinant of (4.82) contains exponents, the form of the second gluing condition from (4.29) will contain exponents in the denominator. But as there is no preference to upper and lower indices, which get exchanged under $\Delta \to -\Delta$ symmetry, we have to assume that both gluing conditions (4.29) include the exponents $e^{2\pi u}$ only in the numerator of $M^{ij}(u)$, therefore we impose a new constraint

$$\det_{1 \le i,j \le 4} M^{ij}(u) = \text{const} \, . \tag{4.89}$$

Let us now show that staring from some simple ansatz for the gluing matrix $M^{ij}(u)$ we are able to solve the constraints (4.88) and (4.89). We saw from the implementation of the numerical algorithm described in the Section 4.2 that in the case when both spins S_1 and S_2 are non-integer it is sufficient for convergence of the numerical procedure to allow presence of exponents in $M^{13}(u)$ and $M^{14}(u)$ only. Thus, we have the following ansatz for the hermitian matrices $M_{2,3}^{ij}$

$$M_{2,3}^{ij} = \begin{pmatrix} 0 & 0 & M_{2,3}^{13} & M_{2,3}^{14} \\ 0 & 0 & 0 & 0 \\ M_{2,3}^{31} & 0 & 0 & 0 \\ M_{2,3}^{41} & 0 & 0 & 0 \end{pmatrix} \, , \tag{4.90}$$

where the non-zero matrix elements are subject to the relations (4.83) and (4.88).

From the constraint (4.50) we find the equation

$$e^{2i\phi_{B_2}} M_1^{22} = e^{2i\phi_{B_2} - 2i\pi\hat{M}_2} M_1^{22} = 0 \, , \tag{4.91}$$

from which we have

$$M_1^{22} = 0 \,. \tag{4.92}$$

Application of the constraint (4.89), i.e. the demand of the absence of the powers of $e^{2\pi u}$ in the determinant of the gluing matrix leads us to the following equations

$$\begin{aligned} &M_1^{23} M_2^{14} = M_1^{24} M_2^{13} \,, \\ &M_1^{24} M_2^{13} \left(e^{i\pi(S_2 - S_1)} - 1 \right) = 0 \,. \end{aligned} \tag{4.93}$$

As the spins S_1 and S_2 are non-integer and we assume their difference to be non-integer too and from (4.93) we obtain

$$M_1^{23} M_2^{14} = M_1^{24} M_2^{13} = 0 \,. \tag{4.94}$$

In the case in question M_2^{13} and M_2^{14} are not equal to zero, therefore we are left with the equality

$$M_1^{23} = M_1^{24} = 0 \,. \tag{4.95}$$

Summarizing the results (4.92) and (4.95) we obtain[7]

$$\begin{aligned} M = &\begin{pmatrix} M_1^{11} & M_1^{12} & M_1^{13} & M_1^{14} \\ \bar{M}_1^{12} & 0 & 0 & 0 \\ \bar{M}_1^{13} & 0 & M_1^{33} & M_1^{34} \\ \bar{M}_1^{14} & 0 & \bar{M}_1^{34} & M_1^{44} \end{pmatrix} + \\ &+ \begin{pmatrix} 0 & 0 & M_2^{13} & M_2^{14} \\ 0 & 0 & 0 & 0 \\ \bar{M}_2^{13} & 0 & 0 & 0 \\ \bar{M}_2^{14} & 0 & 0 & 0 \end{pmatrix} e^{2\pi u} + \begin{pmatrix} 0 & 0 & M_3^{13} & M_3^{14} \\ 0 & 0 & 0 & 0 \\ \bar{M}_3^{13} & 0 & 0 & 0 \\ \bar{M}_3^{14} & 0 & 0 & 0 \end{pmatrix} e^{-2\pi u} \,, \end{aligned} \tag{4.96}$$

where the elements of the matrices $M_{2,3}^{ij}$ are subject to (4.83) and (4.88). As we have the relation (4.88) it is sufficient to write only the phases of the non-zero matrix elements of the matrix M_2^{ij} extracted from (4.83)

$$M_2^{13} = \left| M_2^{13} \right| e^{i\left(\pm \frac{\pi}{2} + \phi_{B_1} - \phi_{B_3} \right)} \,, \quad M_2^{14} = \left| M_2^{14} \right| e^{i\left(\pm \frac{\pi}{2} + \phi_{B_1} - \phi_{B_4} \right)} \,. \tag{4.97}$$

Let us point out that the construction presented above will provide an analytic continuation to all values of S_2 from the integer values $S_2 \geq 0$. However, this analytic continuation breaks down the symmetry $S_2 \to -S_2$, which is naively present in the QSC, as one can see from the asymptotic (4.14). The analytic continuation, which describes perfectly positive integer S_2 will produce poles at negative integer S_2. This could look a bit puzzling, but the resolution of this paradox is in the existence of the second solution for the mixing matrix which is obtained by relabeling indices in accordance with $S_2 \to -S_2$. In practice result must be even in S_2 and it is enough to consider $S_2 \geq 0$ so it is sufficient to use the mixing matrix presented above.

[7] In the case when the spin S_2 is integer, the gluing matrix simplifies to (4.107) and we have $M^{14}(u) = M^{44}(u) = 0$ as we will see it in the Section 4.2 describing the applied QSC numerical algorithm.

To sum up the contents of the present Section, we have to point out several things. First, we formulated the algebraic structure of the Q-system by writing down the QQ-relations and 4th order Baxter equation originating from them. Second, the analytic structure of the Q-system was motivated from the solution of the classically integrable dual superstring σ-model and the QQ-relations. The symmetries of the Q-system allowed us to introduce the gluing conditions for which we managed to impose several constraints. These constraints were partially solved for different values of the spins S_1 and S_2. We examined the case when both spins are integer and non-integer. In the next Section we are going to appreciate an importance of the derived gluing conditions and see how they appear in the QSC numerical algorithm.

4.2 Numerical solution

The equations of QSC are especially well-suited for numerical analysis: simple analytical properties of the **P**-functions allow to parametrize them in terms of a truncated Laurent series and then constrain these coefficients by the gluing condition. Numerical algorithms for solving QSC equations were developed and applied in [32, 84, 87, 88]. In a non-symmetric case, such as BFKL with $S_2 = n \neq 0$, the procedure has to be modified in a way which we will describe here. We attached a `Mathematica` notebook named `code_for_arxiv.nb` implementing the algorithm, which we used to obtain the results described in this Section.

Let us start by briefly reminding the main steps of the numerical algorithm. A comprehensive description of the algorithm for the left-right symmetric case can be found in [18, 32]. Here we will point out the main features we have to take into account in the case without left-right symmetry. As in the left-right symmetric case, for the **P**-functions there is a sheet with only one cut in the complex plane where the following parametrisation is valid

$$\begin{aligned}\mathbf{P}_a(u) &= x^{-\tilde{M}_a}\left(g^{-\tilde{M}_a}A_a\left(1+\frac{\delta_{a,4}}{x^2}\right)+\sum_{k=1}^{+\infty}\frac{c_{a,k}}{x^{2k}}\right)\,, \\ \mathbf{P}^a(u) &= x^{\tilde{M}^a-1}\left(g^{\tilde{M}_a-1}A^a\left(1+\frac{\delta_{a,1}}{x^2}\right)+\sum_{k=1}^{+\infty}\frac{c^{a,k}}{x^{2k}}\right)\,,\end{aligned} \tag{4.98}$$

where $x(u) = \frac{u+\sqrt{u-2g}\sqrt{u+2g}}{2g}$. The expansions (4.98) contain only the even powers of Zhukovsky variable $x(u)$ because for the state in question the **P**-functions possess the certain parity determined by their asymptotics from (4.13) and (4.51). However, the coefficients $c_{a,k}$ and $c^{a,k}$ are not independent and are subject to the conditions following from

$$\mathbf{P}_a\mathbf{P}^a = 0\,. \tag{4.99}$$

In the left-right symmetric case the condition (4.99) was satisfied automatically.

Since $\mathcal{Q}_{a|i}$ is analytic in the UHP and has a power-like behaviour at $u \to \infty$, its asymptotic expansion for sufficiently large Im u in the UHP can be written as

$$\mathcal{Q}_{a|i}(u) \simeq u^{-\tilde{M}_a+\hat{M}_i}\sum_{k=0}^{+\infty}\frac{B_{a|i,k}}{u^k}\,, \tag{4.100}$$

where

$$B_{a|i,0} = -i\frac{A_a B_i}{-\tilde{M}_a + \hat{M}_i} \,. \tag{4.101}$$

Plugging the ansatz (4.98) and the expansion of $\mathcal{Q}_{a|i}$ into the equation

$$\mathcal{Q}^+_{a|i} - \mathcal{Q}^-_{a|i} = -\mathbf{P}_a \mathbf{P}^b \mathcal{Q}^+_{b|i}, \tag{4.102}$$

we are able to fix the coefficients $B_{a|i,k}$ in terms of the operator charges and the coefficients $c_{a,k}$ and $c^{a,k}$. The fact that the $\mathbf{P}$-functions have the certain parity and they are given by (4.98) leads to the disappearance of the odd coefficients $B_{a|i,2l+1} = 0$ for $l = 0, 1, 2, \ldots$ in (4.100) and we obtain

$$\mathcal{Q}_{a|i}(u) \simeq u^{-\tilde{M}_a + \hat{M}_i} \sum_{l=0}^{+\infty} \frac{B_{a|i,2l}}{u^{2l}} \,. \tag{4.103}$$

After doing this, using the same finite difference equation in the form

$$\mathcal{Q}^-_{a|i} = \left(\delta^b_a + \mathbf{P}_a \mathbf{P}^b\right) \mathcal{Q}^+_{a|i} \tag{4.104}$$

we find the numerical value of $\mathcal{Q}_{a|i}$ in the vicinity of the real axis.

One remaining ingredient of the iterative numerical procedure is the loss function — a function which is zero for the exact solution and which should decrease as each iteration brings us closer to the exact solution. We have the following loss function

$$S = \sum_{i,j} |F^i(u_j)|^2 \,, \tag{4.105}$$

which is zero when the gluing condition is satisfied. Here

$$F^i(u) = \mathcal{Q}^{a|i+}(u)\tilde{\mathbf{P}}_a(u) + M^{ij}(u)\bar{\mathcal{Q}}^-_{b|i}(u)\bar{\mathbf{P}}^b(u) \tag{4.106}$$

and $\{u_i\}$ is a set of points on the interval $[-2g; 2g]$. Every function $F^i(u)$ depends on the charges S, Δ, n, the coefficients $c_{a,k}$ and $c^{a,k}$ and the coefficients of the gluing matrix. As a starting point for the numerical algorithm one can use the weak coupling data from Appendix 4.3.1.

In the present work we are interested in the case of non-integer spin $S_1 = S$. As it was already shown in the Section 4.1 in this situation we cannot keep all the gluing conditions (4.80). However, we found that only two gluing conditions F^2 and F^4 are sufficient to constrain all the coefficients c and are still valid even for non-integer S_1 providing thus a natural way to analytically continue to non-integer spins. In terms of (4.105) this means that the sum in that formula goes only over $i = 2, 4$. After the loss function and all the constraints are formulated, the algorithm searches for the parameters which minimize the loss function subject to the constraints using a numerical optimisation procedure (Levenberg-Marquardt algorithm). Then, using the obtained numerical values of the $\mathbf{Q}$-functions, we are able to restore the ansatz for the gluing matrix, which tells us which elements of the gluing matrix contain the exponential terms and which of them are equal to zero. This allows to verify the modification proposed in the Section 4.1 for the gluing matrix (4.107) (in agreement with [31]). Thus for integer $S_2 = n$ this leads to the gluing matrix for non-integer S_1 given by (4.96) with

$$M^{14}(u) = M^{44}(u) = 0 \,. \tag{4.107}$$

In the situation when $S_2 = n$ is not integer the gluing matrix (4.107) needs further modification. To achieve this we use the fact that relaxing the conditions (4.107) is sufficient to make the numerical procedure convergent. Restoring again the gluing matrix, for general real value of the spin S_2 (or conformal spin n in high-energy scattering terminology) we are left with the gluing matrix which coincides with (4.96).

Using the proposed numerical algorithm, we managed to calculate several numerical quantities for the cases when n is non-zero and even non-integer. On the Figure 4.3 one can find the length-2 operator trajectory for $n = 1$.

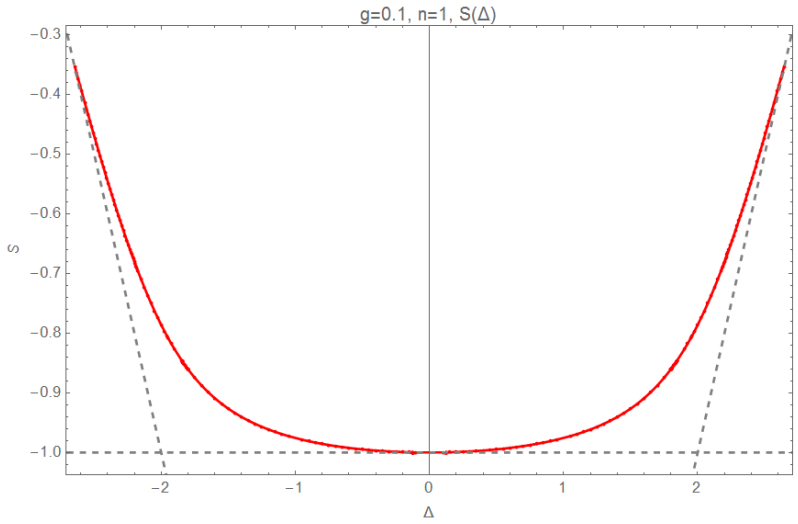

Figure 4.3: Trajectory of the length-2 operator for conformal spin $n = 1$ and coupling constant $g = 0.1$.

It is also possible to numerically calculate the dependence of the spin S on the coupling constant g for the fixed dimension Δ. On the Figure 4.4 you can see the dependence $S(g)$ for $\Delta = 0.45$ and $n = 1$ in comparison with the same result calculated perturbatively as the sum of LO and NLO BFKL eigenvalues.

Additionally, this numerical scheme allows us to compare the numerical values of BFKL kernel eigenvalues with the known perturbative eigenvalues at LO and NLO orders. In the Table 4.1 the numerical values of the BFKL kernel eigenvalue fitted from the plots of the Figure 4.4 are written in the first four orders together with perturbative results in the first two orders calculated for $n = 1$ and $\Delta = 0.45$. In the LO, NLO and NNLO order we observe the agreement with 22, 20 and 16 digits precision respectively.

From now on let us concentrate on the numerical calculation of the intercept function. On the Figure 4.5 one can find the dependencies of the intercept on the coupling constant g for the different values of conformal spin n. The dashed lines are plotted according to the intercept function from Section 4.3 calculated in the small coupling regime. The continuous lines correspond to the strong coupling expansion of the intercept function from Section 4.5, which was fitted from numerical data obtained in the present Section.

In the next Section we are going to analyze the weak coupling expansion of the intercept function. To achieve this we apply the iterative method first applied in [31].

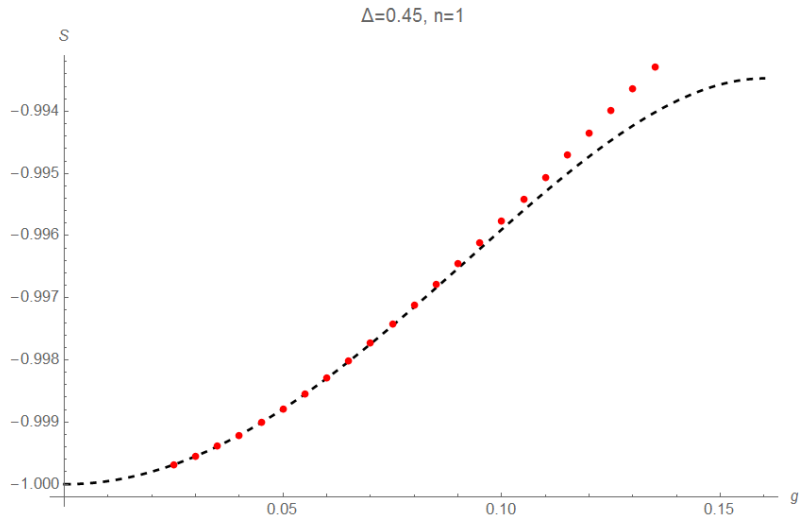

Figure 4.4: Dependence of S on the coupling constant for $\Delta = 0.45$ and $n = 1$. Red dots depict the numerical result and dashed line depicts the sum of analytical perturbative results at LO and NLO orders.

	Numerical fit	Exact perturbative
LO	0.5091953983611833709185 9	0.5091953983611833706918 60
NLO	-9.9263626361061612225	-9.9363626361061612225
NNLO	151.9290181554014	151.9290181554014
NNNLO	-2136.77907308	?

Table 4.1: BFKL kernel eigenvalues calculated numerically up to NNNLO and perturbatively up to NNLO for the conformal spin $n = 1$ and dimension $\Delta = 0.45$.

4.3 Weak coupling expansion

In this Section we explore the function $S(\Delta, n)$ perturbatively at weak coupling for arbitrary integer conformal spin n. In particular, we are interested in the BFKL intercept $j(n) = S(0, n) + 1$. The calculation of this quantity consists of two steps.

First, we apply the QSC iterative procedure introduced in [31] to the calculation of the intercept function for some integer n[8]. To do this we adopt this procedure to the case without the left-right symmetry. We repeat the main points of the iterative algorithm introduced in [31] and describe the functions which are used in it for the case $n \neq 0$.

In the second part we formulate an ansatz for the weak coupling expansion of the intercept function for arbitrary value of conformal spin in terms of binomial harmonic sums and fix the coefficients of this ansatz using the values of the intercept at several integer n, which we calculate solving the QSC iteratively order by order. This approach appears to be successful in the NNLO order in the coupling constant allowing us to find

[8]The reason we have to take specific n in our analytic calculations is that for arbitrary n the leading order solution is already quite complicated hypergeometric function. It would be really great to extend the method of [31] to be able to deal with this class of functions iteratively. This would allow one to derive the result for arbitrary n at once.

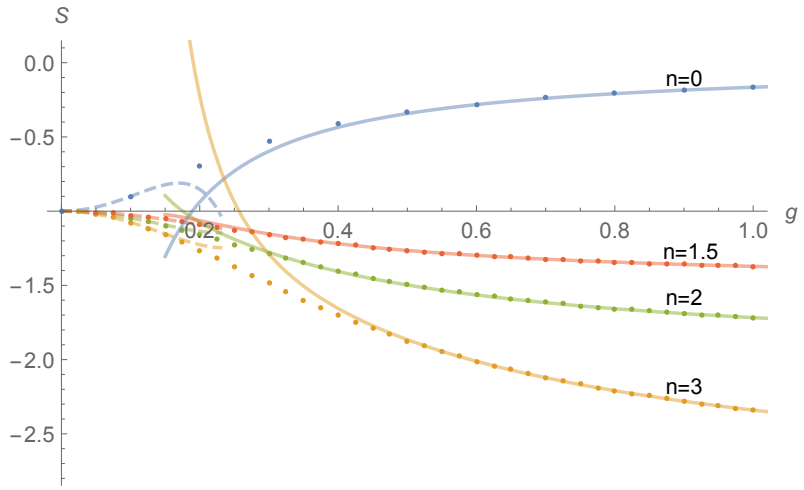

Figure 4.5: Intercept $S(0,n)$ as the function of the coupling constant g for conformal spins $n=0$, $n=3/2$, $n=2$ and $n=3$ (dots), weak coupling expansion of the intercept (dashed lines) and strong coupling expansion (continuous lines).

the intercept function at this order, but at NNNLO order we were not able to fix the rational part of the result for arbitrary n (see the details in the Subsection 4.3.3). This is due to the lack of the generalized-"reciprocity" at the NNNLO order. It would be very interesting to understand why and how the reciprocity in n is violated, which would allow to obtain the rational part with a greatly reduced basis of functions. We postpone this question to the future investigation.

Let us also mention that the method we explain here should be also applicable for non-zero Δ when $\Delta+n$ takes odd integer values. The case of $n=0$ was considered in [31], and it was sufficient to take a few values of Δ in order to fix the NNLO dimension. This would be also interesting to investigate in the future.

4.3.1 BFKL limit with nonzero $S_2 = n$

In this Subsection we consider BFKL limit of the QSC with non-zero conformal spin. Let us first of all briefly remind what BFKL limit is. We are going to study the regime when at the same time the coupling constant $g \to 0$ and one of the spins $S_1 = S \to -1$ while keeping the ratio $g^2/(S+1)$ finite. LO BFKL in this limit corresponds to resumming all the contributions of the form $(g^2/(S+1))^k$, NLO BFKL – to the contributions of the form $(S+1)(g^2/(S+1))^k$ and so on. However, in the present Subsection we take the second spin (conformal spin) $S_2 = n \neq 0$ and there appear some differences from the BFKL regime with zero conformal spin.

In order to find the BFKL kernel eigenvalue we are going to utilize the old fashioned method of $\mathbf{P}\mu$-system. The $\mathbf{P}\mu$-system consists of the functions $\mathbf{P}_a(u)$, $\mathbf{P}^a(u)$, which we introduced before and of an antisymmetric matrix $\mu_{ab}(u)$ (see [16, 17] for the detailed

description). They satisfy the following equations

$$\begin{aligned}
&\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \ , \quad \tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b \ , \\
&\tilde{\mu}^{ab} - \mu^{ab} = \mathbf{P}^a\tilde{\mathbf{P}}^b - \mathbf{P}^b\tilde{\mathbf{P}}^a \ , \quad \tilde{\mathbf{P}}^a = \mu^{ab}\mathbf{P}_b \ , \\
&\mathbf{P}_a\mathbf{P}^a = 0 \ , \quad \mu_{ab}\mu^{bc} = \delta_a^c \ , \quad \tilde{\mu}_{ab} = \mu_{ab}^{++} \ , \quad \tilde{\mu}^{ab} = \mu^{ab++} \ .
\end{aligned} \tag{4.108}$$

Before proceeding we are going to introduce a couple of new notations. Our notation for the BFKL scaling parameter is $w = S + 1$. It is also convenient to introduce the notation $\Lambda = g^2/w$.

To start solving the $\mathbf{P}\mu$-system in the BFKL regime we have to determine the scaling of the $\mathbf{P}$- and μ-functions in the limit $w \to 0$. In what follows we are going to use the arguments from [82], where the left-right symmetric case with zero conformal spin was considered. First, we assume that the scaling of the $\mathbf{P}$-functions coincides with the scaling of their leading coefficients in the large u asymptotics. Thus as from (4.17) for the length-2 state in question in the BFKL limit $A_aA^a = \mathcal{O}(w^0)$ for $a = 1, \dots, 4$ these functions can be chosen to scale as w^0

$$\mathbf{P}_a = \sum_{k=0}^{+\infty} \mathbf{P}_a^{(k)} w^k \ , \quad \mathbf{P}^a = \sum_{k=0}^{+\infty} \mathbf{P}^{(k)a} w^k \ . \tag{4.109}$$

Second, because the asymptotic of the function $\mathbf{P}_1(u)$ is the same as in the left-right symmetric case, for $S_2 = n \neq 0$ the argument from [82] about the scaling of the μ-functions is applicable and they scale as w^{-2}

$$\mu_{ab} = \sum_{k=0}^{+\infty} \mu_{ab}^{(k)} w^{k-2} \ , \quad \mu^{ab} = \sum_{k=0}^{+\infty} \mu^{(k)ab} w^{k-2} \ . \tag{4.110}$$

Additionally, as all the $\mathbf{P}$-functions for the length-2 states being considered possess the certain parity from the $\mathbf{P}\mu$-system equations (4.108) we can conclude that the functions $\mu_{ab}^+(u)$ have the certain parity, which will be specified below.

Finally, we are to determine the asymptotics of the μ-functions for the case of non-integer $S_1 = S$ for non-zero $S_2 = n$. Let us restrict ourselves from now on in this Section to the case of integer conformal spin $S_2 = n$. The μ-functions with the lower indices are given by the formula from [17] combined with (4.44) from the Section 4.1

$$\mu_{ab} = \frac{1}{2}\mathcal{Q}^-_{ab|ij}\omega^{ij} = \frac{1}{2}\mathcal{Q}^-_{ab|ij}M^{ik}\Omega^j_k \ . \tag{4.111}$$

Substituting the matrices (4.107) and (4.72), which are applicable to the case of integer conformal spin, into (4.111), we are able to determine the asymptotics of the ω-functions. Taking into account also the antisymmetry of $\omega^{ij} = M^{ik}\Omega^k_j$ from (4.50) we find that they are

$$\omega^{ij} \sim \left(\text{const}, e^{2\pi|u|}, e^{-2\pi|u|}, e^{-2\pi|u|}, e^{-2\pi|u|}, \text{const}\right) \ . \tag{4.112}$$

Using (4.112) we see that the leading asymptotics of the μ-functions with the lower indices at $u \to \pm\infty$ are

$$\mu_{ab} \sim (u^{-S-1}, u^{-S}, u^{-S+1}, u^{-S+1}, u^{-S+2}, u^{-S+3})e^{2\pi|u|} \ , \tag{4.113}$$

while the μ-functions with the upper indices have the same asymptotics but in the reverse order.

In addition, we know that the $\mathbf{P}$-functions have only one cut on one of the sheets, therefore they can be written as a series in the Zhukovsky variable. The parametrization of the $\mathbf{P}$-functions for the case $J_1 = 2$ and $J_2 = J_3 = 0$ is already known to us and can be taken from the formula (4.98). Utilizing the $\mathbf{P}$-functions rescaling (4.10) we can set the coefficients $A_1 = 1$, $A_2 = 1$, $A^3 = -1$ and $A^4 = 1$. Then, we are also allowed to apply the certain H-transformation of the $\mathbf{P}$-functions, which do not alter their asymptotics and parity

$$\mathbf{P}_a \to (h_B)_a^b \, \mathbf{P}_b \,, \quad \mathbf{P}^a \to \left(h_B^{-t}\right)_b^a \mathbf{P}^b \,, \tag{4.114}$$

where

$$(h_B)_a^b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_1 & 0 & 1 & 0 \\ 0 & \alpha_2 & 0 & 1 \end{pmatrix} \,. \tag{4.115}$$

As (4.115) has two parameters then, applying this transformation, we can set the coefficients $c_{3,1}$ and $c^{2,1}$ to zero. Thus, we arrive to the formulas

$$\begin{aligned}
\mathbf{P}_1 &= \frac{1}{\Lambda w x^2} + \sum_{k=1}^{+\infty} \frac{c_{1,k}}{x^{2k+2}} \,, \quad \mathbf{P}^1 = A^1 \sqrt{\Lambda w} \left(x + \frac{1}{x}\right) + \sum_{k=1}^{+\infty} \frac{c^{1,k}}{x^{2k-1}} \,, \\
\mathbf{P}_2 &= \frac{1}{\sqrt{\Lambda w} x} + \sum_{k=2}^{+\infty} \frac{c_{2,k}}{x^{2k+1}} \,, \quad \mathbf{P}^2 = A^2 + \sum_{k=2}^{+\infty} \frac{c^{2,k}}{x^{2k}} \,, \\
\mathbf{P}_3 &= A_3 + \sum_{k=2}^{+\infty} \frac{c_{3,k}}{x^{2k}} \,, \quad \mathbf{P}^3 = -\frac{1}{\sqrt{\Lambda w} x} + \sum_{k=1}^{+\infty} \frac{c^{3,k}}{x^{2k+1}} \,, \\
\mathbf{P}_4 &= A_4 \sqrt{\Lambda w} \left(x + \frac{1}{x}\right) + \sum_{k=1}^{+\infty} \frac{c_{4,k}}{x^{2k-1}} \,, \quad \mathbf{P}^4 = \frac{1}{\Lambda w x^2} + \sum_{k=1}^{+\infty} \frac{c^{4,k}}{x^{2k+2}} \,.
\end{aligned} \tag{4.116}$$

The scaling of $\mathbf{P}$-functions (4.109) suggests that the coefficients should have the expansions

$$c_{m,n} = \left(\sqrt{\Lambda w}\right)^{2n-m-1} \sum_{k=0}^{+\infty} c_{n,m}^{(k)} w^k \,, \quad c^{m,n} = \left(\sqrt{\Lambda w}\right)^{2n+m-6} \sum_{k=0}^{+\infty} c^{n,m(k)} w^k \,. \tag{4.117}$$

In what follows for the sake of convenience we change the numeration of the μ-functions to $\mu_{12} = \mu_1$, $\mu_{13} = \mu_2$, $\mu_{14} = \mu_3$, $\mu_{23} = \mu_4$, $\mu_{24} = \mu_5$ and $\mu_{34} = \mu_6$ and the same for μ with the upper indices. Having set the scaling and the expansions of $\mathbf{P}$- and μ-functions in the scaling parameter together with their asymptotics we are able to proceed with the solution of the $\mathbf{P}\mu$-system order by order in the scaling parameter.

LO solution

In the present part we will find the LO solution of the $\mathbf{P}\mu$-system in the BFKL regime. Taking into account that $\tilde{\mathbf{P}}$ scale as $1/w^2$, in the LO in w we obtain for the $\mathbf{P}$-functions

from (4.116) and (4.117)

$$\mathbf{P}_1^{(0)} = \frac{1}{u^2}\,, \quad \mathbf{P}_2^{(0)} = \frac{1}{u}\,, \quad \mathbf{P}_3^{(0)} = A_3^{(0)}\,, \quad \mathbf{P}_4^{(0)} = A_4^{(0)}u + \frac{c_{4,1}^{(1)}}{\Lambda u}\,, \tag{4.118}$$
$$\mathbf{P}^{(0)1} = A^{1(0)}u + \frac{c^{1,1(1)}}{\Lambda u}\,, \quad \mathbf{P}^{(0)2} = A^{2(0)}\,, \quad \mathbf{P}^{(0)3} = -\frac{1}{u}\,, \quad \mathbf{P}^{(0)4} = \frac{1}{u^2}\,,$$

where we implied $c_{4,1} = \frac{1}{(\Lambda w)^{3/2}}\left(c_{4,1}^{(1)}w + \mathcal{O}(w^2)\right)$, $c^{1,1} = \frac{1}{(\Lambda w)^{3/2}}\left(c^{1,1(1)}w + \mathcal{O}(w^2)\right)$ and

$$A_3^{(0)} = -\frac{i\left((\Delta-n)^2-1\right)\left((\Delta+n)^2-9\right)}{32}\,,\ A_4^{(0)} = -\frac{i\left((\Delta+n)^2-1\right)\left((\Delta-n)^2-25\right)}{96}\,. \tag{4.119}$$

Let us now examine the μ-functions in the vicinity of the point $S=-1$. In the LO, when $S=-1$, μ-functions with the lower indices at $u\to\infty$ have the asymptotics

$$(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6) \sim (u^0,u^1,u^2,u^2,u^3,u^4)e^{2\pi|u|}\,. \tag{4.120}$$

To find the μ-functions with the lower indices in the LO analogously to the left-right symmetric case we notice that $\mathbf{P}_{1,2}^{(0)}$ and $\mathbf{P}^{(0)3,4}$ are singular, while $\tilde{\mathbf{P}}_{1,2}$ and $\tilde{\mathbf{P}}^{3,4}$ are regular at $u=0$. This can be guaranteed if $\mu_{ab}^{(0)}$ are regular at $u=0$ and have a zero of sufficient order at this point. Therefore, as in the left-right symmetric case we can assume that in the LO the only solution which contributes in $\mu_{ab}^{(0)}$ is the one with the asymptotics (4.120). Consequently, we conjecture $\mu_{ab}^{(0)}$ to possess the form of a polynomial times some i-periodic factor. As for the length-2 operators the functions μ_{ab}^{+} have the certain parity, we are led to the ansatz

$$\mu_1^{(0)+} = \mathcal{P}(u)b_{1,1}\,, \quad \mu_2^{(0)+} = \mathcal{P}(u)b_{2,1}u\,, \tag{4.121}$$
$$\mu_3^{(0)+} = \mathcal{P}(u)(b_{3,1}u^2+b_{3,2})\,, \quad \mu_4^{(0)+} = \mathcal{P}(u)(b_{4,1}u^2+b_{4,2})\,,$$
$$\mu_5^{(0)+} = \mathcal{P}(u)(b_{5,1}u^3+b_{5,2}u)\,, \quad \mu_6^{(0)+} = \mathcal{P}(u)(b_{6,1}u^4+b_{6,2}u^2+b_{6,3})\,.$$

where $\mathcal{P}(u)$ is an i-periodic function. Let us first determine only the polynomial part of the solution. Plugging the ansatz (4.121) into the equations $\mu_{ab}^{++} = \mu_{ab} + \mathbf{P}_a\mu_{bc}\tilde{\mathbf{P}}^c - \mathbf{P}_b\mu_{ac}\tilde{\mathbf{P}}^c$ we fix all the coefficients besides $b_{1,1}$ and $c^{1,1(1)} = -c_{4,1}^{(1)}$. The latter equality automatically leads to the validity of the requirement

$$\mathbf{P}_a^{(0)}\mathbf{P}^{(0)a} = 0 \tag{4.122}$$

in the LO.

Now let us return to the general solution. Because in the leading order the equations for μ's are homogeneous, the solution is determined up to some i-periodic function. This periodic function grows not faster than $e^{2\pi|u|}$, then the most general ansatz for it is

$$B_{1,1}e^{2\pi u} + B_{1,2}e^{-2\pi u} + B_{1,3}\,. \tag{4.123}$$

Also we have to remember about the requirements of analyticity, i.e. that the following expressions have no cuts

$$\mathbf{P}+\tilde{\mathbf{P}}\,, \quad \frac{\mathbf{P}-\tilde{\mathbf{P}}}{\sqrt{u^2-4\Lambda w}}\,, \quad \mu+\tilde{\mu}\,, \quad \frac{\mu-\tilde{\mu}}{\sqrt{u^2-4\Lambda w}}\,. \tag{4.124}$$

The constraints (4.124) lead us to the results

$$B_{1,2} = B_{1,1}\ ,\ B_{1,3} = -2B_{1,1}\ ,\ c^{(1)}_{4,1} = -\frac{i\Lambda}{96}\left(\left(\Delta^2-1\right)^2 - 2\left(\Delta^2+1\right)n^2 + n^4\right)\ . \quad (4.125)$$

With these conditions (4.125) the parity requirement for μ^+ is automatically satisfied.

Summarizing the obtained results and setting $B_{1,1} = -B_1/4$, all the requirements give us the unique solution (up to a multiplicative constant) given by

$$\begin{aligned}
\mu^{(0)+}_1 &= B_1\cosh^2(\pi u)\ , \quad (4.126)\\
\mu^{(0)+}_2 &= -B_1\frac{i}{16}\left(\left(\Delta^2-1\right)^2 - 2\left(\Delta^2+1\right)n^2 + n^4\right)u\cosh^2(\pi u)\ ,\\
\mu^{(0)+}_3 &= \mu^{(0)+}_4 = -B_1\frac{i}{128}\left(\left(\Delta^2-1\right)^2 - 2\left(\Delta^2+1\right)n^2 + n^4\right)\left(4u^2+1\right)\cosh^2(\pi u)\ ,\\
\mu^{(0)+}_5 &= -B_1\frac{i}{192}\left(\left(\Delta^2-1\right)^2 - 2\left(\Delta^2+1\right)n^2 + n^4\right)u\left(4u^2+1\right)\cosh^2(\pi u)\ ,\\
\mu^{(0)+}_6 &= -B_1\frac{1}{49152}\left(\left(\Delta^2-1\right)^2 - 2\left(\Delta^2+1\right)n^2 + n^4\right)^2\left(4u^2-3\right)\left(4u^2+1\right)\cosh^2(\pi u)\ .
\end{aligned}$$

Having obtained the LO solution given by (4.118), (4.126) and (4.119) we are able to proceed with finding the NLO solution of the $\mathbf{P}\mu$-system together with the coefficient B_1, which will be done in the next Subsection.

NLO solution

Now it remains to fix the coefficient B_1 and find the $\mathbf{P}$-functions in the next order in w. To do this we need to calculate $\tilde{\mathbf{P}}$. First, we have to understand, which coefficients it is necessary to find in order to calculate $\mathbf{P}$ in the NLO order w^1 using the scaling or the $\mathbf{P}$- (4.109) and $\tilde{\mathbf{P}}$-functions (w^{-2})

$$\begin{aligned}
\mathbf{P}^{(1)}_1 &= \frac{2\Lambda}{u^4}\ ,\quad \mathbf{P}^{(1)1} = A^{1(1)}u + \frac{c^{1,1(2)}}{\Lambda u} + \frac{c^{1,1(1)} + \Lambda c^{1,2(0)}}{u^3}\ , \quad (4.127)\\
\mathbf{P}^{(1)}_2 &= \frac{1}{u} + \frac{\left(1+c^{(0)}_{2,2}\right)\Lambda w}{u^3} + \mathcal{O}(w^2)\ ,\quad \mathbf{P}^{(1)2} = A^{2(0)} + A^{2(1)}w + \mathcal{O}(w^2)\ ,\\
\mathbf{P}^{(1)}_3 &= A^{(1)}_3\ ,\quad \mathbf{P}^{(1)3} = \frac{\left(-1+c^{3,2(0)}\right)\Lambda}{u^3}\ ,\\
\mathbf{P}^{(1)}_4 &= A^{(1)}_4 u + \frac{c^{(2)}_{4,1}}{\Lambda u} + \frac{c^{(1)}_{4,1} + \Lambda c^{(0)}_{4,2}}{u^3}\ ,\quad \mathbf{P}^{(1)4} = \frac{2\Lambda}{u^4}\ .
\end{aligned}$$

Notice, that we already know $c^{(1)}_{4,1} = -c^{1,1(1)}$ given by (4.125) and $A^{(1)}_3$, $A^{(1)}_4$, $A^{1(1)}$ and $A^{2(1)}$

$$\begin{aligned}
A^{(1)}_3 &= -\frac{i}{4}\left(\Delta^2 + \Delta n + n^2 - 3\right)\ ,\quad A^{(1)}_4 = -\frac{i}{12}\left(\Delta^2 + 3\Delta n + n^2 + 5\right)\ , \quad (4.128)\\
A^{1(1)} &= \frac{i}{12}\left(\Delta^2 - 3\Delta n + n^2 + 5\right)\ ,\quad A^{2(1)} = -\frac{i}{4}\left(\Delta^2 - \Delta n + n^2 - 3\right)\ .
\end{aligned}$$

Thus, we have to find $c^{(0)}_{2,2}$, $c^{(2)}_{4,1}$, $c^{(0)}_{4,2}$, $c^{1,1(2)}$, $c^{1,2(0)}$ and $c^{3,2(0)}$. Comparing the $\tilde{\mathbf{P}}$-functions calculated from the LO result and the ones found with the equation $\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$ we can fix some unknown coefficients. On the one hand we get

$$\begin{aligned}
\tilde{\mathbf{P}}_1 &= \frac{1}{\Lambda^2 w^2}\left(u^2 + \mathcal{O}\left(u^4\right)\right) + \mathcal{O}\left(w^{-1}\right) , \\
\tilde{\mathbf{P}}_2 &= \frac{1}{\Lambda^2 w^2}\left(c^{(0)}_{2,3}u^3 + \mathcal{O}\left(u^5\right)\right) + \mathcal{O}\left(w^{-1}\right) , \\
\tilde{\mathbf{P}}_3 &= \frac{1}{\Lambda^2 w^2}\left(c^{(0)}_{3,4}u^4 + \mathcal{O}\left(u^6\right)\right) + \mathcal{O}\left(w^{-1}\right) , \\
\tilde{\mathbf{P}}_4 &= \frac{1}{\Lambda^2 w^2}\left(c^{(0)}_{4,3}u^5 + \mathcal{O}\left(u^7\right)\right) + \mathcal{O}\left(w^{-1}\right) .
\end{aligned} \tag{4.129}$$

On the other hand

$$\begin{aligned}
\tilde{\mathbf{P}}_1 &= -\frac{B_1}{w^2}\frac{i}{4}\left((\Delta-n)^2-1\right)\sinh^2(\pi u) + \mathcal{O}\left(w^{-1}\right) , \\
\tilde{\mathbf{P}}_2 &= -\frac{B_1}{w^2}\frac{i}{4}\left((\Delta-n)^2-1\right)u\sinh^2(\pi u) + \mathcal{O}\left(w^{-1}\right) , \\
\tilde{\mathbf{P}}_3 &= -\frac{B_1}{w^2}\frac{1}{128}\left((\Delta-n)^2-1\right)^2\left((\Delta+n)^2-1\right)u^2\sinh^2(\pi u) + \mathcal{O}\left(w^{-1}\right) , \\
\tilde{\mathbf{P}}_4 &= -\frac{B_1}{w^2}\frac{1}{384}\left((\Delta-n)^2-1\right)^2\left((\Delta+n)^2-1\right)u(u^2+1)\sinh^2(\pi u) + \mathcal{O}\left(w^{-1}\right) .
\end{aligned} \tag{4.130}$$

The result is the following

$$\begin{aligned}
&B_1 = \frac{4i}{\pi^2\left((\Delta-n)^2-1\right)\Lambda^2} , \quad c^{(0)}_{2,2} = 1 , \\
&c^{(0)}_{4,1} = 0 , \quad c^{(0)}_{4,2} = -\frac{i}{96}\left(\left(\Delta^2-1\right)^2 - 2\left(\Delta^2+1\right)n^2 + n^4\right) .
\end{aligned} \tag{4.131}$$

Thus, the final answer for the LO μ-functions is

$$\begin{aligned}
\mu^{(0)+}_1 &= \frac{4i}{\pi^2\Lambda^2}\frac{\cosh^2(\pi u)}{(\Delta-n)^2-1} , \\
\mu^{(0)+}_2 &= \frac{1}{4\pi^2\Lambda^2}\left((\Delta+n)^2-1\right)u\cosh^2(\pi u) , \\
\mu^{(0)+}_3 &= \mu^{(0)+}_4 = \frac{1}{32\pi^2\Lambda^2}\left((\Delta+n)^2-1\right)\left(4u^2+1\right)\cosh^2(\pi u) , \\
\mu^{(0)+}_5 &= \frac{1}{48\pi^2\Lambda^2}\left((\Delta+n)^2-1\right)u\left(4u^2+1\right)\cosh^2(\pi u) , \\
\mu^{(0)+}_6 &= -\frac{i}{12288\pi^2\Lambda^2}\left((\Delta-n)^2-1\right)\left((\Delta+n)^2-1\right)^2\left(4u^2-3\right)\left(4u^2+1\right)\cosh^2(\pi u) .
\end{aligned} \tag{4.132}$$

Using the equation

$$\mu_{ab}\mu^{bc} = \delta^c_a , \tag{4.133}$$

we are able to determine the μ-functions with the upper indices in the LO up to a multiplication constant. Then, inserting them into $\mathbf{P}\mu$-system equations, we fix this

constant obtaining

$$\mu^{(0)1+} = \frac{i}{12288\pi^2\Lambda^2}\left((\Delta+n)^2-1\right)\left((\Delta-n)^2-1\right)^2\left(4u^2-3\right)\left(4u^2+1\right)\cosh^2(\pi u)\ , \tag{4.134}$$

$$\mu^{(0)2+} = \frac{1}{48\pi^2\Lambda^2}\left((\Delta-n)^2-1\right)u\left(4u^2+1\right)\cosh^2(\pi u)\ ,$$

$$\mu^{(0)3+} = \mu^{(0)4+} = -\frac{1}{32\pi^2\Lambda^2}\left((\Delta-n)^2-1\right)\left(4u^2+1\right)\cosh^2(\pi u)\ ,$$

$$\mu^{(0)5+} = \frac{1}{4\pi^2\Lambda^2}\left((\Delta-n)^2-1\right)u\cosh^2(\pi u)\ ,$$

$$\mu^{(0)6+} = -\frac{4i}{\pi^2\Lambda^2}\frac{\cosh^2(\pi u)}{(\Delta+n)^2-1}$$

and some other coefficients in the **P**-functions with the upper indices

$$c^{1,2(0)} = \frac{i}{96}\left(\left(\Delta^2-n^2\right)^2-2\left(\Delta^2+n^2\right)+1\right)\ ,\quad c^{3,2(0)} = -1\ . \tag{4.135}$$

We are able even to go further and calculate completely the μ-functions in the NLO and partly fix the expansion coefficients in the **P**-functions in the NNLO. Let us now fix the other coefficients in the NLO in w. To find the unknown coefficient $c^{(2)}_{4,1}$, we have to build an ansatz for μ in the NLO. The asymptotics of the NLO μ are $\{u^0, u^3, u^2, u^1, u^5\}\log u$. First we introduce a new function

$$\Psi_0(u) = \psi\left(\frac{1}{2}+iu\right)+\psi\left(\frac{1}{2}-iu\right)-2\psi(1)\ . \tag{4.136}$$

This function has the infinite series of poles in the points $\frac{i}{2}+i\mathbb{Z}$. As the case of general integer conformal spin n has to be consistent with the known left-right symmetric case $n=0$, it is natural to assume that the ansatz for the NLO μ-functions is essentially the same as for $n=0$, but relaxing the requirement that μ_3 is equal to μ_4. So, we use the following ansatz

$$\begin{aligned}
\mu^{(1)+}_1 &= (B_1\Psi_0(u)+b_1)\cosh^2(\pi u)+K_1\ , \\
\mu^{(1)+}_4 &= (B_2u\Psi_0(u)+b_{21}u)\cosh^2(\pi u)+K_2u\ , \\
\mu^{(1)+}_3 &= (B_3(4u^2+1)\Psi_0(u)+b_{31}u^2+b_{32})\cosh^2(\pi u)+K_3(4u^2+1)\ , \\
\mu^{(1)+}_4 &= (B_4(4u^2+1)\Psi_0(u)+b_{41}u^2+b_{42})\cosh^2(\pi u)+K_4(4u^2+1)\ , \\
\mu^{(1)+}_5 &= (B_5u(4u^2+1)\Psi_0(u)+b_{51}u^3+b_{52}u)\cosh^2(\pi u)+K_5u(4u^2+1)\ , \\
\mu^{(1)+}_6 &= (B_6(4u^2-3)(4u^2+1)\Psi_0(u)+b_{61}u^4+b_{62}u^2+b_{63})\cosh^2(\pi u)+ \\
&+K_5(4u^2-3)(4u^2+1)\ .
\end{aligned} \tag{4.137}$$

Substituting this ansatz (4.121) into the **P**μ-system equations (4.108), first, we get the unknown coefficient

$$c^{(2)}_{4,1} = -\frac{i\Lambda}{24}\left(\Delta^2-1\right)\left(2\left(\Delta^2-1\right)\Lambda-1\right) \tag{4.138}$$

and obtain μ with the lower indices in the NLO

$$\mu_1^{(1)+} = \frac{1}{(\Delta-n)^2-1}\left(\left(-\frac{2i\Psi(u)}{\pi^2\Lambda^2}-\frac{16i\left(\pi^2\Lambda\left((\Delta-n)^2-1\right)+\frac{3}{2}\right)}{3\pi^2\Lambda^2}\right)\cosh^2(\pi u)+\frac{8i}{\Lambda}\right) ,$$

$$\mu_2^{(1)+} = ((\Delta+n)^2-1)\left(\left(-\frac{u\Psi(u)}{8\pi^2\Lambda^2}-\frac{u}{3\Lambda}+\frac{u}{2\pi^2\Lambda^2((\Delta+n)^2-1)}\right)\cosh^2(\pi u)+\frac{u}{2\Lambda}\right) ,$$

$$\mu_3^{(1)+} = ((\Delta+n)^2-1)\left(\left(-\frac{\left(4u^2+1\right)\Psi(u)}{64\pi^2\Lambda^2}-\frac{\pi^2\left(4u^2+1\right)-6}{24\pi^2\Lambda}-\right. \tag{4.139}$$

$$\left.-\frac{4u^2+1}{16\pi^2\Lambda^2((\Delta+n)^2-1)}\right)\cosh^2(\pi u)+\frac{4u^2+1}{16\Lambda}\right) ,$$

$$\mu_4^{(1)+} = ((\Delta+n)^2-1)\left(\left(-\frac{\left(4u^2+1\right)\Psi(u)}{64\pi^2\Lambda^2}-\frac{\pi^2\left(4u^2+1\right)-6}{24\pi^2\Lambda}-\frac{4u^2+1}{16\pi^2\Lambda^2((\Delta+n)^2-1)}\right.\right.$$

$$\left.-\frac{\left(4u^2+1\right)\Delta n}{2\pi^2\Lambda^2\left((\Delta-n)^2-1\right)\left((\Delta+n)^2-1\right)}\right)\cosh^2(\pi u)+\frac{4u^2+1}{16\Lambda}\Bigg) ,$$

$$\mu_5^{(1)+} = ((\Delta+n)^2-1)\left(\left(-\frac{u\left(4u^2+1\right)\Psi(u)}{96\pi^2\Lambda^2}-\frac{u\left(4u^2+1\right)}{36\Lambda}-\frac{u\left(4u^2+1\right)}{8\pi^2\Lambda^2\left((\Delta-n)^2-1\right)}\right.\right.$$

$$\left.+\frac{u\left(4u^2+1\right)\Delta n}{6\pi^2\Lambda^2((\Delta+n)^2-1)\left((\Delta-n)^2-1\right)}\right)\cosh^2(\pi u)+\frac{u\left(4u^2+1\right)}{24\Lambda}\Bigg) ,$$

$$\mu_6^{(1)+} = ((\Delta+n)^2-1)^2\left((\Delta-n)^2-1\right)\left(\left(\frac{i\left(4u^2-3\right)\left(4u^2+1\right)\Psi(u)}{24576\pi^2\Lambda^2}+\right.\right.$$

$$+\frac{i\left(\pi^2\left(4u^2-3\right)+36\right)\left(4u^2+1\right)}{9216\pi^2\Lambda}+\frac{i\left(4u^2-3\right)\left(4u^2+1\right)}{6144\pi^2\Lambda^2\left((\Delta-n)^2-1\right)}\Bigg)\cosh^2(\pi u)-$$

$$\left.-\frac{i\left(4u^2-3\right)\left(4u^2+1\right)}{6144\Lambda}\right) .$$

The μ-functions with the upper indices in the NLO are given by the formula

$$\mu^{(1)ab}(n,u) = \chi^{ac}\mu^{(1)}_{cd}(-n,u)\chi^{db} . \tag{4.140}$$

where the matrix χ is given by (4.55). Usage of the equation $\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$ allows to

partially fix the **P**-functions in the NNLO. They are written in the formulas below

$$\begin{aligned}
&\mathbf{P}_1^{(2)} = \frac{\pi^2\Lambda^2}{3u^4} + \frac{5\Lambda^2}{u^6}\ , \quad \mathbf{P}^{(2)1} = -\frac{i\left(\Delta^2+n^2-3\right)}{48}u + \frac{c^{1,1(3)}}{\Lambda u} - \\
&- i\Lambda\left((\Delta+n)^2-1\right)\left(\frac{24-\left(51+\pi^2\right)\left((\Delta-n)^2-1\right)}{288} - \frac{\Delta n}{6\left((\Delta+n)^2-1\right)}\right)\frac{1}{u^3} + \\
&+ \frac{5i\Lambda^2\left((\Delta+n)^2-1\right)\left((\Delta-n)^2-1\right)}{96u^5}\ , \\
&\mathbf{P}_2^{(2)} = \left(\frac{\pi^2\Lambda^2}{3} - \frac{4\Lambda}{(\Delta-n)^2-1}\right)\frac{1}{u^3} + \frac{5\Lambda^2}{u^5}\ , \\
&\mathbf{P}^{(2)2} = \frac{i\left(\Delta^2+n^2-11\right)}{16} - \frac{i\Lambda^2\left((\Delta+n)^2-1\right)\left((\Delta-n)^2+1\right)}{32u^4}\ , \\
&\mathbf{P}_3^{(2)} = \frac{i\left(\Delta^2+n^2-11\right)}{16} - \frac{i\Lambda^2\left((\Delta+n)^2-1\right)\left((\Delta-n)^2-1\right)}{32u^4}\ , \\
&\mathbf{P}^{(2)3} = \left(-\frac{\pi^2\Lambda^2}{3} + \frac{4\Lambda}{((\Delta+n)^2-1)}\right)\frac{1}{u^3} - \frac{5\Lambda^2}{u^5}\ , \\
&\mathbf{P}_4^{(2)} = \frac{i\left(\Delta^2+n^2-3\right)}{48}u + \frac{c_{4,1}^{(3)}}{\Lambda u} + \\
&+ i\Lambda\left((\Delta-n)^2-1\right)\left(\frac{24-\left(51+\pi^2\right)\left((\Delta+n)^2-1\right)}{288} + \frac{\Delta n}{6\left((\Delta-n)^2-1\right)}\right)\frac{1}{u^3} - \\
&- \frac{5i\Lambda^2\left((\Delta+n)^2-1\right)\left((\Delta-n)^2-1\right)}{96u^5}\ , \\
&\mathbf{P}^{(2)4} = \frac{\pi^2\Lambda^2}{3u^4} + \frac{5\Lambda^2}{u^6}\ .
\end{aligned} \tag{4.141}$$

Also, from the obtained results one can notice that the **P**- and μ-functions possess the following symmetry

$$\mathbf{P}^a(n,u) = \chi^{ab}\mathbf{P}_b(-n,u)\ , \quad \mu^{ab}(n,u) = \chi^{ac}\mu_{cd}(-n,u)\chi^{db}\ . \tag{4.142}$$

where the matrix χ is given by (4.55). In what follows we will assume that this symmetry is present in all orders of the perturbative expansion.

Passing to Qω-system

To proceed we need to use the system which is dual to **P**μ-system – **Q**ω-system. The equations of this system look as follows [16, 17]

$$\begin{aligned}
&\tilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i\tilde{\mathbf{Q}}_j - \mathbf{Q}_j\tilde{\mathbf{Q}}_i\ , \quad \tilde{\mathbf{Q}}_i = \omega_{ij}\mathbf{Q}^j\ , \\
&\tilde{\omega}^{ij} - \omega^{ij} = \mathbf{Q}^i\tilde{\mathbf{Q}}^j - \mathbf{Q}^j\tilde{\mathbf{Q}}^i\ , \quad \tilde{\mathbf{Q}}^i = \omega^{ij}\mathbf{Q}_j\ , \\
&\mathbf{Q}_i\mathbf{Q}^i = 0\ , \quad \omega_{ij}\omega^{jk} = \delta_i^k\ , \quad \omega_{ij}^{++} = \omega_{ij}\ .
\end{aligned} \tag{4.143}$$

We assume the scaling of the ω-functions is the same as in the case $n = 0$ considered in [82], which is motivated by the fact that the case of general n has to be consistent with the known data for $n = 0$. Thus, the function ω^{13} scales as w^{-2}, ω^{12}, ω^{14}, ω^{23} and ω^{34} scale as w^0 and ω^{24} scales as w^2. For the lower indices we have ω_{24} as w^{-2}, ω_{12}, ω_{14}, ω_{23} and ω_{34} as w^0 and ω_{13} as w^2. Let us remind the connection between μ- and ω-functions

$$\mu^{+}_{ab} = \frac{1}{2}\mathcal{Q}_{ab|ij}\omega^{ij+} \ , \quad \mu^{ab+} = \frac{1}{2}\mathcal{Q}^{ab|ij}\omega^{+}_{ij} \ . \tag{4.144}$$

With the obtained $\mathbf{P}$- and μ-functions in the LO (4.118), (4.132) and NLO (4.127), (4.139) we can extract the functions $\mathcal{Q}_{ab|13}$ and $\mathcal{Q}^{ab|24}$ in the LO and NLO as well. Let us proceed with further calculations taking $\mathcal{Q}_{ab|13}$ only as the actions with $\mathcal{Q}^{ab|24}$ are completely the same but with the exchange $n \to -n$.

One of the QQ-relations says

$$\mathcal{Q}_{a|1}\mathcal{Q}_{b|3} - \mathcal{Q}_{a|3}\mathcal{Q}_{b|1} = \mathcal{Q}_{ab|13} \tag{4.145}$$

and it allows to express $\mathcal{Q}_{3|1}$ and $\mathcal{Q}_{4|1}$ in terms of $\mathcal{Q}_{1|1}$, $\mathcal{Q}_{2|1}$ and $\mathcal{Q}_{ab|13}$. Using this fact and the equation

$$\mathcal{Q}^{+}_{a|i} - \mathcal{Q}^{-}_{a|i} = -\mathbf{P}_a\mathbf{P}^b\mathcal{Q}^{+}_{a|i} \tag{4.146}$$

we are able to derive the following second order Baxter equation for $\mathcal{Q}_{1|1}$ in the LO

$$\left(u^2 + \frac{1}{4}\right)\mathcal{Q}^{(0)++}_{1|1} + \left(u^2 + \frac{1}{4}\right)\mathcal{Q}^{(0)--}_{1|1} + \left(-2u^2 + \frac{(\Delta - n)^2 - 3}{4}\right)\mathcal{Q}^{(0)}_{1|1} = 0 \ . \tag{4.147}$$

Then, utilizing

$$\mathcal{Q}^{+}_{1|1} - \mathcal{Q}^{-}_{1|1} = \mathbf{P}_1\mathbf{Q}_1 \tag{4.148}$$

we obtain the second order Baxter equation

$$\mathbf{Q}^{(0)++}_{1,3} + \mathbf{Q}^{(0)--}_{1,3} + \left(-2 + \frac{(\Delta - n)^2 - 1}{4u^2}\right)\mathbf{Q}^{(0)}_{1,3} = 0 \ , \tag{4.149}$$

where an additional index 3 appears because the Baxter equation for $\mathbf{Q}_3$ is the same.

Repeating analogous calculations with $\mathcal{Q}^{ab|24}$ we get the similar equation for $\mathbf{Q}^2$ and $\mathbf{Q}^4$ in the LO

$$\mathbf{Q}^{(0)2,4++} + \mathbf{Q}^{(0)2,4--} + \left(-2 + \frac{(\Delta + n)^2 - 1}{4u^2}\right)\mathbf{Q}^{(0)2,4} = 0 \ . \tag{4.150}$$

As we know the $\mathbf{P}$-functions completely in the LO and NLO orders the equations (4.149) and (4.150) are known to us up to NLO order and will be used below.

It was said in the Section 4.1 that we should find the solutions to the Baxter equation with pure asymptotics. Let us consider $n = 0$ in (4.149) and (4.150) for simplicity (we always can restore them by shifting $\Delta \to \Delta - n$ for the lower indices or $\Delta \to \Delta + n$ for the upper indices). The Baxter equation

$$\mathbf{q}^{++} + \mathbf{q}^{--} + \left(-2 + \frac{\Delta^2 - 1}{4u^2}\right)\mathbf{q} = 0 \tag{4.151}$$

is known to have two independent solutions, which are UHPA

$$\mathbf{q}_I(\Delta,u) = 2iu\,{}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},1+iu;1,2;1\right)\,, \tag{4.152}$$
$$\mathbf{q}_{II}(\Delta,u) = -i\coth(\pi u)\mathbf{q}_I(\Delta,u) + \frac{\mathbf{q}_I(\Delta,-u)}{\cos\frac{\pi\Delta}{2}}\,.$$

We found two solutions $\mathbf{q}_1$ and $\mathbf{q}_2$ with the pure asymptotics $u^{\frac{\Delta+1}{2}}$ and $u^{-\frac{\Delta-1}{2}}$ respectively

$$\mathbf{q}_1(\Delta,u) = -\tan\frac{\pi\Delta}{2}\mathbf{q}_I(\Delta,u) + \mathbf{q}_{II}(\Delta,u) = \left(-i\coth(\pi u) - \tan\frac{\pi\Delta}{2}\right)\mathbf{q}_I(\Delta,u) +$$
$$+\frac{\mathbf{q}_I(\Delta,-u)}{\cos\frac{\pi\Delta}{2}}\,,$$
$$\mathbf{q}_2(\Delta,u) = \tan\frac{\pi\Delta}{2}\mathbf{q}_I(\Delta,u) + \mathbf{q}_{II}(\Delta,u) = \left(-i\coth(\pi u) + \tan\frac{\pi\Delta}{2}\right)\mathbf{q}_I(\Delta,u) +$$
$$+\frac{\mathbf{q}_I(\Delta,-u)}{\cos\frac{\pi\Delta}{2}}\,. \tag{4.153}$$

Then, the solutions of (4.149) and (4.150) with the pure asumptotics can be expressed as follows

$$\mathbf{Q}^{(0)}_{1,3}(u) = \mathbf{q}_{1,2}(\Delta-n,u)\,, \tag{4.154}$$
$$\mathbf{Q}^{(0)2,4}(u) = \mathbf{q}_{2,1}(\Delta+n,u)\,.$$

Let us now turn back to the gluing conditions for the integer conformal spin (4.107) and write down two of them containing only the $\mathbf{Q}$-functions from (4.154) in the LO in the scaling parameter w

$$\tilde{\mathbf{Q}}^2 = \bar{M}_1^{(0)12}\bar{\mathbf{Q}}_1\,, \tag{4.155}$$
$$\tilde{\mathbf{Q}}^4 = \bar{M}_1^{(0)34}\bar{\mathbf{Q}}_3\,.$$

To find $M_1^{(0)12}$ and $M_1^{(0)34}$ we can use the continuity of the functions $\mathbf{Q}^2$ and $\mathbf{Q}^4$ on the cut $\tilde{\mathbf{Q}}^2(0) = \mathbf{Q}^2(0)$ and $\tilde{\mathbf{Q}}^4(0) = \mathbf{Q}^4(0)$. The result is

$$M_1^{(0)12} = M_1^{(0)34} = \frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1}\,. \tag{4.156}$$

In terms of the $\mathbf{Q}$-functions with the pure asymptotics, we derive the following gluing conditions in the LO

$$\tilde{\mathbf{Q}}^{(0)2,4}(u) = \frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1}\bar{\mathbf{Q}}^{(0)}_{1,3}(u)\,. \tag{4.157}$$

LO BFKL eigenvalue

Here we will obtain the LO BFKL eigenvalue in a way similar to [82]. To begin with, let us write down the Baxter equation for $\mathbf{Q}^{2,4}$ in the NLO

$$\mathbf{Q}^{(1)2,4++} + \mathbf{Q}^{(1)2,4--} + \left(-2 + \frac{(\Delta+n)^2-1}{4u^2}\right)\mathbf{Q}^{(1)2,4} = \\ = -\frac{i}{2(u+i)}\mathbf{Q}^{(0)2,4++} + \frac{i}{2(u-i)}\mathbf{Q}^{(0)2,4--} + \frac{u^2-\Lambda(\Delta+n)^2-1)}{2u^4}\mathbf{Q}^{(0)2,4} . \quad (4.158)$$

From one side from the Baxter equation (4.158) it follows that

$$\frac{\mathbf{Q}^{(1)j}(u)}{\mathbf{Q}^{(0)j}(u)} = \frac{i}{2u} + \mathcal{O}(u^0) , \quad j = 2, 4 . \quad (4.159)$$

On the other side, we can apply the following trick to find the singular part of $\mathbf{Q}^j$ in the NLO

$$\mathbf{Q}^j = \frac{\mathbf{Q}^j + \tilde{\mathbf{Q}}^j}{2} + \frac{\mathbf{Q}^j - \tilde{\mathbf{Q}}^j}{2\sqrt{u^2-4\Lambda w}}\sqrt{u^2-4\Lambda w} . \quad (4.160)$$

For $\mathbf{Q}^2$ we obtain

$$\frac{\mathbf{Q}^2 - \tilde{\mathbf{Q}}^2}{2\sqrt{u^2-4\Lambda w}} = \frac{1}{2u}\left(\mathbf{Q}^{(0)2}(u) - \frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1}\bar{\mathbf{Q}}_1^{(0)}(u)\right) + \mathcal{O}(w) . \quad (4.161)$$

Combining (4.161) and the previously obtained results (4.154), we get

$$\mathbf{Q}^{(1)2}(u) = \left(-\frac{i\mathbf{Q}^{(0)2}(0)(\Psi(\Delta+n)+\Psi(\Delta-n))\Lambda}{u} + \mathcal{O}(u^0)\right) w + \mathcal{O}(w^2) , \quad (4.162)$$

where

$$\Psi(\Delta) \equiv \psi\left(\frac{1}{2} - \frac{\Delta}{2}\right) + \psi\left(\frac{1}{2} + \frac{\Delta}{2}\right) - 2\psi(1) . \quad (4.163)$$

Thus, comparing two independent results (4.159) and (4.162), we have the relation

$$-2(\Psi(\Delta+n)+\Psi(\Delta-n))\Lambda = 1 . \quad (4.164)$$

After some calculations, we obtain for the integer n

$$\frac{1}{4\Lambda} = -\frac{1}{2}\left(\Psi(\Delta+n)+\Psi(\Delta-n)\right) + \mathcal{O}(w) = \\ = -\psi\left(\frac{1+n-\Delta}{2}\right) - \psi\left(\frac{1+n+\Delta}{2}\right) + 2\psi(1) + \mathcal{O}(w) . \quad (4.165)$$

Rewriting the result in terms of the usual expansion parameters, we obtain the well-known LO BFKL kernel eigenvalue for nonzero integer conformal spin n [4]

$$S = -1 - 4g^2\left(\psi\left(\frac{1+n-\Delta}{2}\right) + \psi\left(\frac{1+n+\Delta}{2}\right) - 2\psi(1)\right) + \mathcal{O}(g^4) . \quad (4.166)$$

Having finished with reproducing the LO BFKL kernel eigenvalue with non-zero conformal spin from the QSC, we can now turn to the iterative procedure to calculate the intercept function.

4.3.2 Description of the iterative procedure

The iterative procedure increasing the number of orders is essentially the same as described in [31], but modified to account for non left-right symmetric case. The procedure is based on applying a version of variation of parameters method applied to the equation

$$\mathcal{Q}^{+}_{a|i} - \mathcal{Q}^{-}_{a|i} = -\mathbf{P}_a \mathbf{P}^b \mathcal{Q}^{-}_{b|i} \,, \tag{4.167}$$

which is a simple consequence of (4.1) and (4.5). Indeed, suppose the function $\mathcal{Q}^{(0)}_{a|i}$ solves the equation (4.167) up to a discrepancy $dS_{a|i}$

$$\mathcal{Q}^{(0)+}_{a|i} - \mathcal{Q}^{(0)-}_{a|i} + \mathbf{P}_a \mathbf{P}^b \mathcal{Q}^{(0)-}_{b|i} = dS_{a|i} \,. \tag{4.168}$$

The exact solution can be represented as the zero-order solution plus a correction. We expand the correction in the basis over the components of the zero-order solution

$$\mathcal{Q}_{a|i} = \mathcal{Q}^{(0)}_{a|i} + b^{j+}_i \mathcal{Q}^{(0)}_{a|j} \,. \tag{4.169}$$

If the discrepancy is of the order of some small parameter ϵ to the power m, i.e. $dS_{a|i} = \mathcal{O}(\epsilon^m)$, then we can obtain the equation for the coefficients b^k_i with the doubled precision

$$b^{k++}_i - b^k_i = (dS_{a|i} + b^j_i dS_{a|j}) \mathcal{Q}^{(0)a|k+} = dS_{a|i} \mathcal{Q}^{(0)a|k+} + \mathcal{O}(\epsilon^{2m}) \,. \tag{4.170}$$

The discrepancy thus becomes two orders smaller with each iteration step.

For the problem in question we restrict ourselves to the situation when $\Delta = 0$, n is an integer number and we perform the expansion in the parameter $\epsilon = g^2$. To start solving the finite difference equation (4.170), we have to find the zero-order in g solution $\mathcal{Q}^{(0)}_{a|i}$. The way to find these functions is to consider the 4-th order Baxter equation (4.34) with the $\mathbf{P}$-functions in the coefficients of it in the LO given by (4.118), (4.119), (4.125) and (4.142) with $\Delta = 0$. For example, the solution of this equation with the pure asymptotics (4.68) in the LO for $n = 7$ is

$$\begin{aligned}
&\mathbf{Q}^{(0)}_1 = \frac{3}{175}\left(-iu^4\eta_2 + \frac{i}{5}u^2\eta_2 + u^3 + \frac{i}{2}u^2 - \frac{11}{30}u - \frac{i}{10}\right) \,,\ \mathbf{Q}^{(0)}_2 = u^2 \,,\ \mathbf{Q}^{(0)}_3 = u^4 - \frac{u^2}{5} \,, \\
&\mathbf{Q}^{(0)}_4 = -\frac{9}{1715}\left(\left(-iu^4 + \frac{8iu^2}{49}\right)\eta_2 + \frac{9i}{49}\left(u^4 - \frac{u^2}{5}\right)\eta_4 + \right. \\
&\qquad\qquad \left. + u^3 + \frac{iu^2}{2} - \frac{115}{294}u - \frac{17i}{98} + \frac{33}{490u} + \frac{9i}{490u^2}\right) \,,
\end{aligned} \tag{4.171}$$

where $\eta_s(u)$ are examples of the so-called η-functions, whose definition is given below. As we checked by solving the 4th order Baxter equation (4.34) for different values of the conformal spin n, for odd values of n, as for the case $n = 0$ and odd Δ described in [31], the $\mathbf{Q}$-functions in the LO in g at $\Delta = 0$ can be expressed as linear combinations of the η-functions with the coefficients being Laurent polynomials[9] in u plus a Laurent polynomial in u without η-function multiplying it as in (4.171). The η-functions were

[9]Laurent polynomial is a polynomial with both positive and negative powers of the variable plus a constant term.

introduced in [89] and then used in [20, 31, 87] with their generalized version in [84] and for ABJM theory in [90] with an application in [91] as

$$\eta_{s_1,\dots,s_k}(u) = \sum_{n_1>n_2>\dots>n_k\geq 0} \frac{1}{(u+in_1)^{s_1}\dots(u+in_k)^{s_k}} \,, \tag{4.172}$$

where $s_i \geq 1$, $i = 1,\dots,k$. For non-zero even n and $\Delta = 0$ the asymptotics of the $\mathbf{Q}$-functions are half-integer, therefore in this case the $\mathbf{Q}$-functions cannot be described as linear combinations of the η-functions with the Laurent polynomial coefficients plus a Laurent polynomial, as the latter have only integer asymptotics. Unfortunately, we were not able to determine the class of functions to which $\mathbf{Q}_i^{(0)}$ belong for non-zero even n, therefore from this moment we restrict ourselves to the odd values of n. Then, applying the equation

$$\mathcal{Q}_{a|i}^{(0)+} - \mathcal{Q}_{a|i}^{(0)-} = \mathbf{P}_a^{(0)}\mathbf{Q}_i^{(0)} \,, \tag{4.173}$$

where $\mathbf{P}_a^{(0)}$ are given by (4.118) with $\Delta = 0$, we can find the functions $\mathcal{Q}_{a|i}$ in the leading order in the coupling constant.

To describe the solution to (4.173) we have to explain some properties of η-functions. This class of functions is particulary convenient because it is closed under all relevant for us operations. First, a product of two η-functions can be expressed as a linear combination of η-functions using the so-called "stuffle" relations [92]. Second, the solution of the equation of the form

$$f(u+i) - f(u) = u^n \eta_{s_1,\dots,s_k} \tag{4.174}$$

can be expressed as a linear combination of η-functions with the coefficients being Laurent polynomials in u. These two properties make η-functions very useful when solving the QSC perturbatively at weak coupling [20, 31, 89]. The described properties of the η-functions lead us to the conclusion that at least $\mathcal{Q}_{a|i}^{(0)}$ are also expressed as linear combinations of the η-functions with the coefficients being Laurent polynomials in u plus a Laurent polynomial in u. Then, recalling that the $\mathbf{P}$-functions in the NLO in the coupling constant are Laurent polynomials in u as well (see, for example, the formulas (4.127) and (4.141)), we see that the discrepancy $dS_{a|i}$ and the product in the RHS of (4.170) are also of the form of a linear combination of the η-functions with the Laurent polynomial coefficients plus a Laurent polynomial.

We call the operation inverting the linear operator in the LHS of (4.174) "periodization" (for a more precise definition see 4.4.2). It is easy to see that the "periodization" operation solves the equation (4.170) if the zero-order approximation entering the RHS is expressed as a linear combination of η-functions with the coefficients being Laurent polynomials in u plus a Laurent polynomial in u. We were able to find such representation for odd values of n, but not for even ones. After the zero-order solution is found, we iterate it as described above, applying the operation of periodization to the RHS of (4.170) in order to find the coefficients b_i^k. Because of the two properties of η-functions mentioned above, at each iteration the solution is again obtained in the form of a linear combination of η-functions with Laurent polynomial coefficients plus a Laurent polynomial.

After finding the corrected $\mathcal{Q}_{a|i}$ (4.169) at the given iteration step we still have some unfixed coefficients in it including the quantity of our interest $S(0,n)$. To find them we calculate the $\mathbf{Q}$-functions from (4.6) and (4.5) and apply the gluing conditions for the case

of integer conformal spin n, i.e. (4.96) with (4.107) satisfied. As it was explained in the Section 4.1 in these gluing conditions the **Q**-functions has to possess the pure asymptotics (4.68). Therefore we find the combinations of the **Q**-functions with the pure asymptotics. It appears to be sufficient to use only 2 of 4 gluing conditions, namely

$$\begin{aligned} \tilde{\mathbf{Q}}^2 &= \bar{M}_1^{12}\bar{\mathbf{Q}}_1 \,, \\ \tilde{\mathbf{Q}}^4 &= \bar{M}_1^{34}\bar{\mathbf{Q}}_3 \,, \end{aligned} \tag{4.175}$$

taking the **Q**-functions on the cut on the real axis. This procedure allows to fix the remaining unknown coefficients including the function $S(0,n)$. The described method allowed us to find the values of the intercept functions for odd conformal spins in the range from $n=1$ to $n=91$ up to NNNLO order in the coupling constant. These data will be used in the next Subsection, where we put forward the ansatz for the structure of the intercept function for arbitrary value of conformal spin.

4.3.3 Multiloop expansion of the intercept function for arbitrary conformal spin

Using the procedure described in the previous Subsection 4.3.2 we have calculated the expansion of the BFKL eigenvalue intercept for odd n up to $n=91$ in the weak coupling limit up to the order g^8 (NNNLO). These data are valuable by themselves, as they can serve as a test for future higher-order or non-perturbative calculations. What is more important, however, is that it allowed us to find NNLO and partially NNNLO BFKL eigenvalue intercept as a function of the conformal spin n.

We start by noticing that LO and NLO BFKL intercept can be represented as a linear combination of nested harmonic sums of uniform transcendentality. Indeed, the LO and NLO BFKL Pomeron eigenvalues themselves can be expressed (see, for example, [2] and Appendix A.6, where this calculation is explained in details) through the nested harmonic sums described, for example, in [48]

$$S_{a_1,a_2,\ldots,a_n}(x) = \sum_{y=1}^{x} \frac{\mathrm{sign}(a_1)^y}{y^{|a_1|}} S_{a_2,\ldots,a_n}(y) \,, \quad S_\emptyset(x) = 1 \,, \tag{4.176}$$

where x is a positive integer. The indices a_i are non-zero integer numbers and the transcendentality of the given nested harmonic sum is defined as the sum $\sum\limits_{i=1}^{n} |a_i|$. Note that if one of the indices a_i is negative, the formula (4.176) holds only for even integer x. In the literature [93–97] there was described the analytic continuation of the harmonic sums in question from the positive integer even x. To work with such a continuation we utilize the Mathematica package `Supppackage` applied in [31]. One can take $\Delta = 0$ in these eigenvalues, which after some simple algebra gives

$$\begin{aligned} j_{LO}(n) &= 8S_1\left(\frac{n-1}{2}\right) \,, \\ j_{NLO}(n) &= 4S_3\left(\frac{n-1}{2}\right) + 4S_{-3}\left(\frac{n-1}{2}\right) - 8S_{-2,1}\left(\frac{n-1}{2}\right) + \frac{2\pi^2}{3}S_1\left(\frac{n-1}{2}\right) \,. \end{aligned} \tag{4.177}$$

Here and below the transcendentality is computed as follows: the transcendentality of a product is assumed to be equal to the sum of transcendentalities of the factors and

transcendentality of a rational number is 0. Transcendentality of $\log 2$ is 1 and transcendentality of ζ_k is k. Since ζ_k for even k is proportional to π^2, it is easy to see that transcendentality of π is 1.

To conduct the calculations with harmonic sums one can use the `HarmonicSums` package for Mathematica [97–103] or the `Supppackage` utilized by the authors of [31]. It should be noted, that in the present work we utilize the same conventions for the analytic continuation of harmonic sums as in the latter work [31]. As we see, the argument of all the harmonic sums in (4.177) is $(n-1)/2$. This leads one to an idea of trying to find NNLO and NNNLO intercepts as analogous linear combinations of harmonic sums with transcendental coefficients of uniform transcendentality. The coefficients of the linear combination can be constrained using the data generated by the iterative procedure. But the number of harmonic sums of certain transcendentality grows fast as transcendentality increases. Fortunately, one can drastically reduce the number of harmonic sums in the ansatz by conjecturing a certain property of the result we call reciprocity.

The property in question [104–107] is parallel to the Gribov-Lipatov reciprocity [108, 109] and was observed in the weak coupling expansion of the scaling dimensions of the twist operators. Let us remind the statement of the reciprocity: if one defines an auxiliary function $\mathcal{P}$ [104–106] such that the anomalous dimension γ of the operator with the spin M satisfies in all orders in the coupling constant

$$\gamma(M) = \mathcal{P}\left(M + \frac{\gamma(M)}{2}\right) \tag{4.178}$$

then the inverse Mellin transform of $\mathcal{P}$ has the property

$$\{\mathcal{M}^{-1}\mathcal{P}\}(x) = -x\{\mathcal{M}^{-1}\mathcal{P}\}\left(\frac{1}{x}\right) . \tag{4.179}$$

The asymptotic expansion of the function $\mathcal{P}(M)$ for large M then acquires a nice property: it consists only of the powers and possibly logarithms of $M(M+1)$ thus possessing the symmetry $M \to -1 - M$.

Thus the function $\mathcal{P}(M)$ is much more convenient to work with than $\gamma(M)$ itself: the function $\gamma(M)$ can be expressed through the nested harmonic sums, while $\mathcal{P}$, on the other hand, can be expressed through a much smaller class of functions which satisfy the property (4.179). Such functions were identified and used in [105,110–112] as reciprocity-respecting harmonic sums. However, in our calculations we use another basis of the functions satisfying (4.179), which was applied in the works [59,113–118]. These functions are called the binomial harmonic sums and for integer M they are defined as (see [103])

$$\mathbb{S}_{i_1,\ldots,i_k}(M) = (-1)^M \sum_{j=1}^{M} (-1)^j \binom{M}{j} \binom{M+j}{j} S_{i_1,\ldots,i_k}(j) . \tag{4.180}$$

Note that we consider only the positive indices i_l, $l = 1,\ldots,k$ in the definition (4.180). Those are exactly the sums whose asymptotic expansion is even at infinity after the argument is shifted by 1/2.

All this is directly applicable to our case and we are able to formulate an ansatz for the NNLO intercept function. From the LO and NLO expressions (4.177) we see that their asymptotic expansions at large n are even in n. Since we are using the harmonic

sums of the argument $M = (n-1)/2$, we need to keep only the harmonic sums invariant under the transformation $M \to -1 - M$ or $n \to -n$ in our notations. Those are exactly the binomial sums (4.180). The expressions (4.177) for LO and NLO intercepts can be easily expressed through them

$$j_{LO} = 4\mathbb{S}_1 \,, \tag{4.181}$$
$$j_{NLO} = 8\,(\mathbb{S}_{2,1} + \mathbb{S}_3) + \frac{4\pi^2}{3}\mathbb{S}_1 \,,$$

where the arguments of the sums are again $(n-1)/2$.

In order to find the NNLO intercept we make an ansatz in a form of a linear combination of binomial harmonic sums with transcendental coefficients. The maximal transcendentality principle, formulated by L.N. Lipatov and A.V. Kotikov [1, 2], holds for the intercept as well: every term in the sum should be of the total transcenedentality 5. The terms of the sum can of course be multiplied by arbitrary rational coefficients which do not affect the transcendentality. Having constructed the ansatz in this way, we can now constrain its coefficients by the iterative data: we evaluate the ansatz (a linear combination of binomial nested harmonic sums) at several integer values of n and match the result to the data obtained from the numerical procedure for the corresponding n. Equating the coefficients in front of each unique product of transcendental constants in these two expressions, we get a linear system for the rational coefficients of the ansatz. Solving it we obtain a surprisingly simple expression

$$j_{NNLO} = 32\,(\mathbb{S}_{1,4} - \mathbb{S}_{3,2} - \mathbb{S}_{1,2,2} - \mathbb{S}_{2,2,1} - 2\mathbb{S}_{2,3}) - \frac{16\pi^2}{3}\mathbb{S}_3 - \frac{32\pi^4}{45}\mathbb{S}_1 \,. \tag{4.182}$$

The result (4.182) for the intercept function for arbitrary n can be compared with the other known quantities. First of them is the NNLO BFKL Pomeron eigenvalue for the conformal spin $n = 0$ calculated in [31]. Taking in this eigenvalue $\Delta = 0$ and comparing it with (4.182) for $n = 0$ we see perfect agreement. Second, for non-zero conformal spins the formulas for the Pomeron trajectories were found in [58], from which we can extract the intercept for given n. We also checked that the result of that work coincides with our result (4.182) for several first non-negative conformal spins n, thus representing an independent confirmation of the correctness of our calculation.

The same procedure can be repeated in the NNNLO. The values of the NNNLO intercept for several first odd values of the conformal spin n are given in the Appendix A.7. Again, as for NNLO, an ansatz in a form of a linear combination of binomial harmonic sums with transcendental coefficients of uniform transcendentality 7 can be constructed and we attempted to fit it to the iterative data. However, we found that the basis of binomial harmonic sums is insufficient to fit the data. This signals that reciprocity understood as parity under n to $-n$ seems to be broken down in this case. The reasons for this are unclear and will be the subject of further work. A possible way to proceed is to try to use an ansatz consisting of linear combinations of nested harmonic sums (4.176) with transcendental coefficients of uniform transcendentality 7. But this basis is much larger than the one with binomial harmonic sums and we do not have enough data to fix all the coefficients in it. However, we managed to fit the certain part of the NNNLO data.

For each odd n we took for the calculation (see Appendix A.7 for several first conformal spins n), the value of the NNNLO intercept is a linear combination of the transcendental

constants consisting of π, ζ_3, ζ_5 and ζ_7 with rational coefficients and a rational number (see the file `intercept_values_Nodd.mx` with the data for the odd conformal spins from $n = 1$ to $n = 91$ in the arXiv submission of [62]). Let us restrict ourselves to the values of the conformal spin in our data equal to $n = 4k + 1$ with $k = 0, 1, \ldots, 22$. The reason for this is that for such $n = 4k+1$ the argument of each harmonic sum is $(n-1)/2 = 2k$, i.e. an even integer number, and at these point harmomic sums take rational values. In these points the values of the intercept functions are given by

$$j_{NNNLO}(4k+1) = \pi^2 j^{\pi^2}_{NNNLO} + \pi^4 j^{\pi^4}_{NNNLO} + \pi^6 j^{\pi^6}_{NNNLO} + \pi^2\zeta_3 j^{\pi^2\zeta_3}_{NNNLO} + \\ + \zeta_3 j^{\zeta_3}_{NNNLO} + \zeta_5 j^{\zeta_5}_{NNNLO} + j^{\text{rat.}}_{NNNLO} \,, \quad (4.183)$$

where all coefficients in front of the transcendental constants on the RHS of (4.183) are rational functions of k . Each coefficient in the RHS of (4.183) is conjectured to be a linear combination with rational coefficients of the binomial harmonic sums with the transcendentality, supplementing the transcendentality of the corresponding coefficient to 7. We were able to fit all the contributions except for $j^{\zeta_3}_{NNNLO}$ and $j^{\text{rat.}}_{NNNLO}$, which, as other harmonic sums, take rational values at the points $n = 4k + 1$ for integer $k \geq 0$. However, we found that the term $j^{\zeta_3}_{NNNLO}$ cannot be fitted with the ansatz consisting of the binomial harmonic sums (4.180). This motivated us to try to fit this contribution with the nested harmonic sums (4.176). This appeared to be really the case and we managed to fit this part with the ordinary harmonic sums, which means that the reciprocity, i.e. the symmetry $n \to -n$ in the asymptotic expansion, is violated. For the last, rational contribution $j^{\text{rat.}}_{NNNLO}$, we also found that it is not described by the binomial harmonic sums. Unfortunately, fitting this contribution with the ordinary harmonic sums did not lead us to completely fixing this contribution due to the lack of data. Therefore combining the obtained results we write down the non-rational part of the answer for the points $n = 4k + 1$, which is the sum of the terms in the RHS of (4.183) except for $j^{\text{rat.}}_{NNNLO}$

$$j^{\text{non-rat.}}_{NNNLO}(4k+1) = -\frac{32\pi^2}{3}\left(3\mathbb{S}_{1,4} - 3\mathbb{S}_{2,3} - \mathbb{S}_{3,2} + \mathbb{S}_{1,1,3} - 2\mathbb{S}_{1,2,2} + \mathbb{S}_{2,2,1} - \mathbb{S}_{3,1,1}\right) + \\ + \frac{16\pi^4}{15}\left(4\mathbb{S}_3 - \mathbb{S}_{2,1}\right) + \frac{56\pi^6}{135}\mathbb{S}_1 + \frac{32\pi^2\zeta_3}{3}\mathbb{S}_{1,1} + 224\zeta_5\mathbb{S}_{1,1} - 128\zeta_3\left(S_{-3,1} + 2S_{-2,2} - 5S_{1,-3} - \right. \\ - 15S_{1,3} - 4S_{2,-2} - 12S_{2,2} - 15S_{3,1} - 4S_{-2,1,1} + 2S_{1,-2,1} + 8S_{1,1,-2} + 12S_{1,1,2} + \\ \left. + 12S_{1,2,1} + 12S_{2,1,1} + S_{-4} + 9S_4\right) \,. \quad (4.184)$$

One can find the full values of the NNNLO intercept function including the rational terms for the conformal spins $n = 4k + 1$ from 1 to 89 in the arXiv submission of this paper in the file named `intercept_values_Nodd.mx`. As the part of the harmonic sum consisting of the nested harmonic sums of the transcendentality 7, which constitutes the rational part at the points $n = 4k + 1$, was not fitted we are unable to write an expression for the NNNLO intercept working for all conformal spins leaving this task for future studies.

Let us briefly summarize the results of the present Section. In the first Subsection we reproduce the LO BFKL kernel eigenvalue by the application of the QSC. In the second Subsection applying the QSC iterative algorithm we found the values of the intercept up to NNNLO order in the coupling constant in the certain range of odd values of the conformal spin. In the third Subsection we saw that the LO and NLO intercept functions satisfy the

reciprocity symmetry, which allowed us to rewrite them in terms of the binomial harmonic sums and using the ansatz in terms of these sums in the NNLO order, fix the answer for the NNLO intercept for arbitrary conformal spin. In the NNNLO order the reciprocity breaks down, nevertheless, for the conformal spins $n = 4k + 1$ we managed to describe the non-rational part of the NNNLO intercept function in terms of binomial and ordinary nested harmonic sums.

4.4 Near-BPS all loop expansion

In this Section we are going to analyze the QSC equations near the BPS point $\Delta = 0$, $S_1 = -1$ and $S_2 = 1$. It appears that it is possible to calculate two non-perturbative quatities in this point by the methods of QSC. Another BPS-point $S_2 = 0,\ S_1 = 0$ was analyzed in detail in [21,22,119]. In this Section we follow closely the Near-BPS expansion method by [22].

4.4.1 Slope of the intercept near the BPS point

A particularly important role in BFKL computations is played by the intercept function $j(n) = S(0, n) + 1$, where $S = S_1$ and $n = S_2$. As we mentioned above, the point $\Delta = 0,\ n = 1$ is BPS, by which we mean that it is fixed for any 't Hooft coupling. The group-theoretical argument explaining this phenomenon should be based on the shortening condition. From the QSC perspective the BPS points are the points where $A^a A_a = 0$ simultaneously for all $a = 1, \dots, 4$. In this Subsection we study small deviations from this BPS point and calculate the slope of $j(n)$ with respect to n in the point $n = 1$ in all orders in the coupling constant g.

LO solution

The fact that at the BPS point $A^a A_a = 0$ usually leads to more powerful condition $\mathbf{P}_a = \mathbf{P}^a = 0$, which is known to lead to considerable simplifications [22] (see also [25] for a similar simplification in the TBA equations). Based on that we also expect that in our situation the Q-system simplifies a lot near the BPS point. Let us show that, indeed, $\mathcal{Q}_{a|i}$ takes a very simple form for $S_2 = 1, \Delta = 0$.

Recall that the functions $\mathcal{Q}_{a|i}$ satisfy the equation

$$\mathcal{Q}^+_{a|i} - \mathcal{Q}^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i \ . \tag{4.185}$$

Let us look at how the right and left hand sides of the equation (4.185) behave as S_2 approaches 1. Scaling of $\mathbf{P}_a$ and $\mathbf{Q}_i$ can be deduced from their leading coefficients A_a and B_i. For the convenience of the calculations we perform the rescaling of the $\mathbf{P}$-functions. The H-symmetry (4.8) and its particular case rescaling symmetry (4.10) explained in [17] allow us to set some of the coefficients from (4.98) to some fixed values. To describe them we introduce a scaling parameter $\nu = \sqrt{S_2 - 1}$, which is real for $S_2 \geq 1$. For the coefficients of the $\mathbf{P}$-functions we obtain

$$A_a = \left(\nu, \nu, -\frac{\mathbf{A}_3}{\nu}, \frac{\mathbf{A}_4}{\nu}\right) \ , \quad c_{3,1} = 0 \ , \quad A^a = \left(\frac{\mathbf{A}_1}{\nu}, \frac{\mathbf{A}_2}{\nu}, -\nu, \nu\right) \ , \quad c^{2,1} = 0 \ . \tag{4.186}$$

For the **Q**-functions in their turn

$$B_i = \left(-\frac{\mathbf{B}_1}{\nu}, \nu, -\frac{\mathbf{B}_3}{\nu}, -\nu\right) \ , \quad B^i = \left(-\nu, \frac{\mathbf{B}_2}{\nu}, -\nu, -\frac{\mathbf{B}_4}{\nu}\right) \ . \tag{4.187}$$

As follows from the AA-, BB-relations (4.17), (4.186) and (4.187) in the small ν limit

$$A_a = \left(\nu, \nu, \frac{1+\theta}{2}i\nu, \frac{1-\theta}{2}i\nu\right) + \mathcal{O}\left(\nu^3\right) \ , \quad B_i = \left(-\frac{1+\theta}{2}i\nu, \nu, \frac{1-\theta}{2}i\nu, -\nu\right) + \mathcal{O}\left(\nu^3\right) \ . \tag{4.188}$$

Notice that then from (4.17), (4.186) and (4.187) we derive that A^a and B^i all scale as ν and are given by the formulas

$$A^a = \left(-\frac{1-\theta}{2}i\nu, \frac{1+\theta}{2}i\nu, -\nu, \nu\right) + \mathcal{O}\left(\nu^3\right) \ , \quad B^i = \left(-\nu, -\frac{1+\theta}{2}i\nu, -\nu, -\frac{1-\theta}{2}i\nu\right) + \mathcal{O}\left(\nu^3\right) \ , \tag{4.189}$$

where

$$\theta = \left.\frac{\partial S_1}{\partial S_2}\right|_{\substack{\Delta=0 \\ S_2=1}} = \left.\frac{\partial j}{\partial n}\right|_{n=1} \tag{4.190}$$

is the slope-to-intercept function, which is the quantity of interest in the present Subsection. This means that the right-hand side in (4.234) is small and the functions $\mathcal{Q}_{a|i}$ are i-periodic in the LO. Since they are also analytic in the upper half plane, they are analytic everywhere. Recall that the asymptotics of $\mathcal{Q}_{a|i}$ are given by

$$\mathcal{Q}_{a|j} \sim -i\frac{A_a B_j}{-\tilde{M}_a + \hat{M}_j} u^{-\tilde{M}_a + \hat{M}_j} \ , \quad u \to \infty \ . \tag{4.191}$$

After plugging the global charges into the expressions (4.51) for $\tilde{M}_a$ and $\hat{M}_i$ taken at the BPS point $\Delta = 0$, $S_1 = -1$ and $S_2 = 0$ we see that all components of $\mathcal{Q}_{a|i}$ are either zero or scale like constants at infinity in the LO. Constant at infinity entire function is constant everywhere, so in the LO

$$\mathcal{Q}_{a|i} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\nu) \ . \tag{4.192}$$

When $\Delta = 0$ and $S_2 = n$ is arbitrary, the left-right symmetry is restored, which simplifies the solution a lot. Despite the normalizations (4.186) and (4.187) differ from (4.52) and (4.53) by a rescaling symmetry one can see that the left-right symmetry is restored (which is also confirmed by the weak coupling data for arbitrary Δ and S_2 from Appendix 4.3.1). At this point the symmetry (4.54) takes the form

$$\mathbf{P}^a = \chi^{ab}\mathbf{P}_b, \quad \mathbf{Q}^i = \eta^{ij}\mathbf{Q}_j, \tag{4.193}$$

where χ^{ab} and η^{ij} are given by (4.55) and χ^{ab} is the same matrix as for the left-right symmetric states as in [16, 17]. As we consider the case when both spins S_1 and S_2 are

not integer, let us recall the gluing matrix for this case (4.96) taking into account the hermiticity of the gluing matrix

$$\begin{aligned}
\tilde{\mathbf{Q}}^1 &= M_1^{11}\bar{\mathbf{Q}}_1 + M_1^{12}\bar{\mathbf{Q}}_2 + \left(M_1^{13} + M_2^{13}e^{2\pi u} + M_3^{13}e^{-2\pi u}\right)\bar{\mathbf{Q}}_3 + \left(M_1^{14} + M_2^{14}e^{2\pi u} + M_3^{14}e^{-2\pi u}\right)\bar{\mathbf{Q}}_4 \,, \\
\tilde{\mathbf{Q}}^2 &= \bar{M}_1^{12}\bar{\mathbf{Q}}_1 \,, \\
\tilde{\mathbf{Q}}^3 &= \left(\bar{M}_1^{13} + \bar{M}_2^{13}e^{2\pi u} + \bar{M}_3^{13}e^{-2\pi u}\right)\bar{\mathbf{Q}}_1 + M_1^{33}\bar{\mathbf{Q}}_3 + M_1^{34}\bar{\mathbf{Q}}_4 \,, \\
\tilde{\mathbf{Q}}^4 &= \left(\bar{M}_1^{14} + \bar{M}_2^{14}e^{2\pi u} + \bar{M}_3^{14}e^{-2\pi u}\right)\bar{\mathbf{Q}}_1 + \bar{M}_1^{34}\bar{\mathbf{Q}}_3 + M_1^{44}\bar{\mathbf{Q}}_4
\end{aligned} \tag{4.194}$$

and keeping in mind that M_1^{11}, M_1^{33} and M_1^{44} are real.

As now we have the matrix η from (4.55), which relates the $\mathbf{Q}$-functions with lower and upper indices, it is possible to obtain an additional constraint on the gluing matrix. Plugging this relation into the first gluing condition (4.29) we derive

$$\tilde{\mathbf{Q}}_i = \eta_{ik}M^{kl}\eta_{lj}\bar{\mathbf{Q}}^j \,, \tag{4.195}$$

which after comparison with the second gluing condition from (4.29) leads us to $(M^{-t})_{ij} = \eta_{ik}M^{kl}\eta_{lj}$, and after multiplication of the both sides by $(M^t)^{mj}$ we obtain an additional constraint for the gluing matrix

$$\eta_{ik}M^{kl}\eta_{lm}(M^t)^{mj} = \delta_i^j \,. \tag{4.196}$$

Substitution of (4.82) into (4.196) leads us to the following equations

$$\eta_{ik}M_{2,3}^{kl}\eta_{lm}(M_{2,3}^t)^{mj} = 0 \,, \tag{4.197}$$

$$\eta_{ik}M_1^{kl}\eta_{lm}(M_{2,3}^t)^{mj} + \eta_{ik}M_{2,3}^{kl}\eta_{lm}(M_1^t)^{mj} = 0 \,, \tag{4.198}$$

$$\eta_{ik}M_1^{kl}\eta_{lm}(M_1^t)^{mj} + \eta_{ik}M_2^{kl}\eta_{lm}(M_3^t)^{mj} + \eta_{ik}M_3^{kl}\eta_{lm}(M_2^t)^{mj} = \delta_i^j \,. \tag{4.199}$$

It should be noted that the first equation (4.197) is satisfied for the gluing matrix (4.96) (the same as in (4.194)).

To start solving the constraint (4.196) order by order in ν we use the following expansion of the gluing matrix, which is motivated by the scaling of the $\mathbf{Q}$-functions (4.188) and (4.189)

$$M^{ij}(u) = \sum_{k=0}^{+\infty} M^{(k)ij}(u)\nu^{2k} \,. \tag{4.200}$$

where

$$M^{(k)ij}(u) = M_1^{(k)ij} + M_2^{(k)ij}e^{2\pi u} + M_3^{(k)ij}e^{-2\pi u} \,. \tag{4.201}$$

In the LO in ν the constraints (4.83), if we take into account (4.188) and $S = -1 + \theta(n-1) + \mathcal{O}((n-1)^2)$, lead us to

$$\begin{aligned}
M_2^{(0)31} &= -M_2^{(0)13} \,, \quad M_2^{(0)41} = M_2^{(0)14} \,, \\
M_3^{(0)31} &= -M_3^{(0)13} \,, \quad M_3^{(0)41} = M_3^{(0)14} \,.
\end{aligned} \tag{4.202}$$

Together with the hermiticity of the gluing matrix (4.202) means that $M_{2,3}^{(0)13}$ are pure imaginary, while $M_{2,3}^{(0)14}$ are real. In addition, from (4.88) in the LO in ν we obtain

$$M_3^{(0)13} = M_2^{(0)13} \,, \quad M_3^{(0)14} = -M_2^{(0)14} \,. \tag{4.203}$$

Now we have to apply the constraint (4.196) in the LO in ν. Substitution of (4.202) and (4.203) into (4.198) allows us to fix

$$M_1^{(0)33} = M_1^{(0)44} = 0 \,, \quad M_1^{(0)12} = M_1^{(0)34} = M_1^{(0)43} \,, \tag{4.204}$$

given that $M_2^{(0)13}$ and $M_2^{(0)14}$ are non-zero. Combination of (4.199), (4.202), (4.203) and (4.204) makes it possible for us to derive the solution

$$M_1^{(0)12} = M_1^{(0)34} = -1 \,, \quad M_1^{(0)31} = -M_1^{(0)13} \,, \quad M_1^{(0)41} = M_1^{(0)14} \,. \tag{4.205}$$

Also there exists a solution with the opposite sign of $M_1^{(0)12}$ and $M_1^{(0)34}$, but as we will see below, it is not relevant for us. Summarizing (4.202), (4.203), (4.204) and (4.205), we are able to write down the solution

$$\begin{aligned} M^{(0)ij} = &\begin{pmatrix} M_1^{(0)11} & -1 & M_1^{(0)13} & M_1^{(0)14} \\ -1 & 0 & 0 & 0 \\ -M_1^{(0)13} & 0 & 0 & -1 \\ M_1^{(0)14} & 0 & -1 & 0 \end{pmatrix} + \\ &+ 2M_2^{(0)13} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cosh(2\pi u) + 2M_2^{(0)14} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sinh(2\pi u) \end{aligned} \tag{4.206}$$

where $M_1^{(0)11}$ and $M_{1,2}^{(0)14}$ are real and $M_{1,2}^{(0)13}$ are pure imaginary.

Let us start solving these equations in the LO. We have already found $\mathcal{Q}_{a|i}$ and thus

$$\mathcal{Q}^{a|i} = -(\mathcal{Q}_{a|i})^{-t} \,. \tag{4.207}$$

As the scaling of the $\mathbf{P}$- and $\mathbf{Q}$-functions is determined by the scaling (4.188) and (4.189) of the leading coefficient at large u, we are left with the following expansions of these quantities in the small ν limit

$$\begin{aligned} \mathbf{P}_a = \nu \sum_{k=0}^{+\infty} \mathbf{P}_a^{(k)} \nu^{2k} \,, \quad \mathbf{P}^a = \nu \sum_{k=0}^{+\infty} \mathbf{P}^{(k)a} \nu^{2k} \,, \\ \mathbf{Q}_i = \nu \sum_{k=0}^{+\infty} \mathbf{Q}_i^{(k)} \nu^{2k} \,, \quad \mathbf{Q}^i = \nu \sum_{k=0}^{+\infty} \mathbf{Q}^{(k)i} \nu^{2k} \,. \end{aligned} \tag{4.208}$$

Then, the correspondence between $\mathbf{Q}$- and $\mathbf{P}$-functions (4.5) and (4.6) in the LO taking into account (4.192) looks as follows

$$\begin{aligned} \mathbf{Q}_1^{(0)} &= -\mathbf{P}^{(0)2} = -\mathbf{P}_3^{(0)} = \mathbf{Q}^{(0)2} \,, \\ \mathbf{Q}_2^{(0)} &= -\mathbf{P}^{(0)3} = \mathbf{P}_2^{(0)} = -\mathbf{Q}^{(0)1} \,, \\ \mathbf{Q}_3^{(0)} &= -\mathbf{P}^{(0)1} = \mathbf{P}_4^{(0)} = -\mathbf{Q}^{(0)4} \,, \\ \mathbf{Q}_4^{(0)} &= -\mathbf{P}^{(0)4} = -\mathbf{P}_1^{(0)} = \mathbf{Q}^{(0)3} \,. \end{aligned} \tag{4.209}$$

First of all let us substitute (4.209) into the gluing conditions (4.194) written in the LO with the gluing matrix (4.206) and use the conjugacy properties of the **P**-functions (4.33)

$$\begin{aligned}
&\tilde{\mathbf{P}}_1^{(0)} = -\mathbf{P}_1^{(0)} + \left(M_1^{(0)13} + 2M_2^{(0)13}\cosh(2\pi u)\right)\mathbf{P}_3^{(0)} \,, \qquad (4.210)\\
&\tilde{\mathbf{P}}_2^{(0)} = \left(M_1^{(0)14} + 2M_2^{(0)14}\sinh(2\pi u)\right)\mathbf{P}_1^{(0)} + \mathbf{P}_2^{(0)} - M_1^{(0)11}\mathbf{P}_3^{(0)} + \left(M_1^{(0)13} + 2M_2^{(0)13}\cosh(2\pi u)\right)\mathbf{P}_4^{(0)} \,,\\
&\tilde{\mathbf{P}}_3^{(0)} = \mathbf{P}_3^{(0)} \,,\\
&\tilde{\mathbf{P}}_4^{(0)} = -\left(M_1^{(0)14} + 2M_2^{(0)14}\sinh(2\pi u)\right)\mathbf{P}_3^{(0)} - \mathbf{P}_4^{(0)} \,.
\end{aligned}$$

According to (4.98) we see that in the LO $\mathbf{P}_3^{(0)}$ is simply a constant. Furthermore the constant is fixed by the leading coefficient (4.188) in the large u asymptotics

$$\mathbf{P}_3^{(0)} = \frac{i}{2}(1+\theta) \,. \qquad (4.211)$$

To find the solutions of the other equations from (4.210) we have to introduce the following notations for the expansion of the hyperbolic functions

$$\cosh(2\pi u) = \cosh_+^u + I_0 + \cosh_-^u \,, \quad \sinh(2\pi u) = \sinh_+^u + \sinh_-^u \,, \qquad (4.212)$$

where

$$\cosh_\pm^u = \sum_{k=1}^{+\infty} I_{2k}(4\pi g)x^{\pm 2k}(u) \,, \quad \sinh_\pm^u = \sum_{k=1} I_{2k-1}(4\pi g)x^{\pm(2k-1)}(u) \,. \qquad (4.213)$$

In what follows we will usually omit the superscript with the variable u if the context does not imply the usage of $\cosh_\pm$ and $\sinh_\pm$ with different arguments and the expression $4\pi g$ in the argument of the generalized Bessel function for the sake of conciseness.

Substituting $\mathbf{P}_3$ from (4.211) into the first equation of (4.210), we have

$$\tilde{\mathbf{P}}_1^{(0)} + \mathbf{P}_1^{(0)} = \left(M_1^{(0)13} + 2M_2^{(0)13}I_0 + 2M_2^{(0)13}(\cosh_+ + \cosh_-)\right)\frac{i}{2}(1+\theta) \,. \qquad (4.214)$$

We need to find the solution for $\mathbf{P}_1^{(0)}$ as power series in x with the leading asymptotic $A_1/(gx)^2$, where A_1 is given by (4.188). Expanding the right-hand side as power series in x it is easy to see that the unique solution is

$$M_1^{(0)13} = -2M_2^{(0)13}I_0 \,, \quad M_2^{(0)13} = -\frac{i}{(1+\theta)g^2 I_2} \qquad (4.215)$$

and

$$\mathbf{P}_1^{(0)} = \frac{\cosh_-}{g^2 I_2} \,. \qquad (4.216)$$

Substituting again $\mathbf{P}_3$ from (4.211) and $\mathbf{P}_1$ from (4.216) into the fourth equation of (4.210), we have

$$\tilde{\mathbf{P}}_4^{(0)} + \mathbf{P}_4^{(0)} = -\left(M_1^{(0)14} + M_2^{(0)14}(\sinh_+ + \sinh_-)\right)i(1+\theta) \,, \qquad (4.217)$$

from which and (4.188) together with the oddity of $\mathbf{P}_4$ it follows that $M_1^{(0)14} = 0$ and the solution for $\mathbf{P}_4^{(0)}$ is

$$\mathbf{P}_4^{(0)} = \frac{1-\theta}{2}ig\left(x - \frac{1}{x}\right) - (1+\theta)iM_2^{(0)14}\sinh_- \,. \qquad (4.218)$$

Now let us substitute (4.211), (4.216), (4.218), and (4.215) into the second equation of (4.210). It gives the following equation

$$\tilde{\mathbf{P}}_2^{(0)} - \mathbf{P}_2^{(0)} = -\frac{1+\theta}{2} i M_1^{(0)11} + \frac{1-\theta}{1+\theta} g \left(x - \frac{1}{x}\right) (\cosh_+ + \cosh_-) + \\ + 2M_2^{(0)14} (\sinh_+ \cosh_- - \sinh_- \cosh_+) \ . \quad (4.219)$$

As the LHS of (4.219) does not contain the even powers of x for the RHS we have

$$M_1^{(0)11} = 0 \ . \quad (4.220)$$

The equation (4.219) takes the form

$$\tilde{\mathbf{P}}_2^{(0)} - \mathbf{P}_2^{(0)} = \frac{1-\theta}{1+\theta} \frac{1}{gI_2} \left(x - \frac{1}{x}\right) (\cosh_+ + \cosh_-) + \frac{2M_2^{(0)14}}{g^2 I_2} (\sinh_+ \cosh_- - \cosh_+ \sinh_-) \ , \quad (4.221)$$

which still depends on the unknown coefficient $M_2^{(0)14}$, which we will fix in the next Section, but going to the next order.

So far, starting from the gluing conditions for non-integer conformal spin n (4.194) we managed to solve the constraints on it in the LO getting (4.206) and then, using the connection between the **Q**- and **P**-functions in the LO formulated the system of equations (4.210) for the **P**-functions in the LO. Then from this system we found all the **P**-functions in the LO except for $\mathbf{P}_2$ for which we derived the equation (4.221), still containing one unknown constant. In the next Subsection using this equation we are going to show how to find the slope-to-intercept function.

Result for the slope-to-intercept function

From now on let us consider the equations (4.1) for the $\mathcal{Q}_{a|i}$ functions in the NLO. We start with $\mathcal{Q}_{3|3}$

$$\mathcal{Q}_{3|3}^{(1)+} - \mathcal{Q}_{3|3}^{(1)-} = \mathbf{P}_3^{(0)} \mathbf{Q}_3^{(0)} = \\ = \mathbf{P}_3^{(0)} \mathbf{P}_4^{(0)} = \frac{1+\theta}{2} i \left(\frac{1-\theta}{2} i \left(u - \frac{2g}{x} \right) - (1+\theta) i M_2^{(0)14} \sinh_- \right) \ . \quad (4.222)$$

As $\mathcal{Q}_{3|3}^{(0)} = 0$, then $\mathcal{Q}_{3|3}^{(1)}$ should not contain $\log u$ in its large u asymptotic, which can only appear from the expansion of non-integer powers $Au^\lambda \simeq A + A\lambda \log u + \ldots$. Thus, in the large u expansion of the RHS of (4.222) the coefficient in front of $1/u$ (which would produce $\log u$ in $\mathcal{Q}_{3|3}^{(1)}$) has to be equal to 0, which is guaranteed by

$$M_2^{(0)14} = -\frac{1-\theta}{1+\theta} \frac{g}{I_1} \ . \quad (4.223)$$

The substitution of (4.223) into (4.221) gives

$$\tilde{\mathbf{P}}_2^{(0)} - \mathbf{P}_2^{(0)} = \frac{1-\theta}{1+\theta} \left(\frac{1}{gI_2} \left(x - \frac{1}{x}\right) (\cosh_+ + \cosh_-) - \right. \\ \left. - \frac{2}{gI_1 I_2} (\sinh_+ \cosh_- - \cosh_+ \sinh_-) \right) \ . \quad (4.224)$$

We do not need to solve completely (4.224), because only the first coefficient of the expansion of $\mathbf{P}_2$ in the powers of x is sufficient to find the slope-to-intercept function

$$\mathbf{P}_2^{(0)} = -\frac{1-\theta}{1+\theta}\left(1+\frac{2}{I_1 I_2}\sum_{k=1}^{+\infty}(-1)^k I_k I_{k+1}\right)\frac{1}{gx}+\dots\ . \tag{4.225}$$

Remembering the leading coefficient of the large u asymptotic of $\mathbf{P}_2$ from (4.188) we obtain the equation

$$-\frac{1-\theta}{1+\theta}\left(1+\frac{2}{I_1 I_2}\sum_{k=1}^{+\infty}(-1)^k I_k I_{k+1}\right)=1\ . \tag{4.226}$$

The previous equation (4.226) fixes θ, which is now equal to

$$\boxed{\theta(g)=1+\frac{I_1 I_2}{\sum\limits_{k=1}^{+\infty}(-1)^k I_k I_{k+1}}}\ , \tag{4.227}$$

and constitutes our result for the slope-to-intercept function.

The weak coupling expansion of the obtained result (4.227) is given by

$$\theta(g)=-\frac{2\pi^2}{3}g^2+\frac{4\pi^4}{9}g^4-\frac{28\pi^6}{135}g^6+\frac{8\pi^8}{405}g^8+\mathcal{O}\left(g^{10}\right)\ . \tag{4.228}$$

Since the slope-to-intercept function by definition is the derivative of the intercept function with respect to the conformal spin n at $n=1$ we can immediately compare the first few coefficients in the weak coupling expansion (4.227) with the derivative of (4.177) and (4.182). This will also show that these expressions provide the formulas compatible with our analytic continuation in n away from the integer values. To find the derivatives of the binomial harmonic sums with respect to the argument we apply the `SuppPackage` used in [31], which expresses the nested harmonic sums (4.176) in terms of the η-functions (4.172) and allows to find the derivatives of these sums. The derivatives of the intercept function (4.177) and (4.182) can be calculated and we find that they are in full agreement with (4.228)

$$\left.\frac{dj}{dn}\right|_{n=1}=-\frac{2\pi^2}{3}g^2+\frac{4\pi^4}{9}g^4-\frac{28\pi^6}{135}g^6+\mathcal{O}\left(g^8\right)\ , \tag{4.229}$$

confirming our result (4.227).

In the next Section we compute the strong coupling expansion of our result for the slope-to-intercept function. As we will see the calculation is less straightforward than at weak coupling, even thought the result is still quite simple.

Strong coupling expansion of the slope-to-intercept function

To obtain the strong coupling expansion of the slope-to-intercept function (4.227) first we calculate the following expansion at strong coupling

$$\begin{aligned}\frac{I_k(4\pi g)I_{k+1}(4\pi g)}{I_1(4\pi g)I_2(4\pi g)} &= 1-\frac{1}{\lambda^{1/2}}\left(k^2+k-2\right)+\frac{1}{2\lambda}\left(k^4+2k^3-4k^2-5k+6\right)-\\ &-\frac{1}{24\lambda^{3/2}}\left(4k^6+12k^5-26k^4-72k^3+67k^2+105k-90\right)+\\ &+\frac{1}{24\lambda^2}\left(k^8+4k^7-10k^6-44k^5+35k^4+148k^3-71k^2-153k+90\right)-\\ &-\frac{1}{1920\lambda^{5/2}}\left(16k^{10}+80k^9-240k^8-1440k^7+1128k^6+\right.\\ &\left.+8760k^5-2140k^4-20720k^3+3261k^2+15345k-4050\right)+\mathcal{O}\left(\frac{1}{\lambda^3}\right), \quad (4.230)\end{aligned}$$

where

$$\lambda=(4\pi g)^2\,. \quad (4.231)$$

If we just sum the series (4.230) multiplying it by $(-1)^k$, it appears to be divergent. However, simple ζ-regularization gives the right result, as we verified numerically with high precision. Namely, multiplying this expression by k^δ and understanding the result as the limit $\delta\to 0$, we get the following answer

$$\begin{aligned}&\sum_{k=1}^{+\infty}\frac{(-1)^k I_k(4\pi g)I_{k+1}(4\pi g)}{I_1(4\pi g)I_2(4\pi g)}=\\ &=-\frac{1}{2}-\frac{3}{4(4\pi g)}-\frac{3}{4(4\pi g)^2}-\frac{9}{32(4\pi g)^3}+\frac{9}{8(4\pi g)^4}+\frac{2331}{512(4\pi g)^5}+\mathcal{O}\left(\frac{1}{g^6}\right)\,. \quad (4.232)\end{aligned}$$

Then, substituting (4.232) into the expression for the slope-to-intercept function (4.227), we obtain the strong coupling expansion for it

$$\theta=-1+\frac{3}{\lambda^{1/2}}-\frac{3}{2\lambda}-\frac{9}{8\lambda^{3/2}}-\frac{9}{4\lambda^2}-\frac{711}{128\lambda^{5/2}}+\mathcal{O}\left(\frac{1}{\lambda^3}\right)\,. \quad (4.233)$$

This expansion (4.233) will be useful for us in the Section 4.5 when we are able to compare it with the derivative of our formula for the strong coupling expansion of the intercept function for arbitrary conformal spin n taken at the point $n=1$, which is based on intensive numerical analysis.

4.4.2 Curvature function near the BPS point

As it was mentioned above in the Subsection 4.4.1 we are considering the expansion in the vicinity of the BPS point $\Delta=0$, $S_1=-1$ and $S_2=1$. In the previous Subsection we expanded in the powers of S_2-1, however, to find the curvature we keep $S_2=1$ and expand in the powers of Δ. The scheme of solving the QSC equations in this case is similar but with one difference. Since the function $S(\Delta,1)$ is an even function of Δ we have to expand to the NLO order in Δ to get a non-trivial result.

As in the previous Section we will utilize a simplification in the $\mathcal{Q}_{a|i}$ function to solve the Q-system explicitly.

LO solution

Recall that the functions $\mathcal{Q}_{a|i}$ satisfy the equation

$$\mathcal{Q}^{+}_{a|i} - \mathcal{Q}^{-}_{a|i} = \mathbf{P}_a \mathbf{Q}_i \ . \tag{4.234}$$

Let us look at how the right and left hand sides of the equation behave as Δ approaches 0. Behaviour of $\mathbf{P}_a$ and $\mathbf{Q}_i$ can be deduced from their leading coefficients A_a and B_i. Analogously to the case of the slope-to-intercept function the H-symmetry (4.8) and its particular case rescaling symmetry (4.10) explained in [17] allow us to set some of the coefficients from (4.98) to some fixed values. To describe them we introduce a scaling parameter $\epsilon = \sqrt{\Delta}$, which is real for $\Delta \geq 0$. For the coefficients of the $\mathbf{P}$-functions we obtain

$$A_a = \left(\epsilon, \epsilon, -\frac{\mathbf{A}_3}{\epsilon}, \frac{\mathbf{A}_4}{\epsilon}\right) \ , \quad c_{3,1} = 0 \ , \quad A^a = \left(\frac{\mathbf{A}_1}{\epsilon}, \frac{\mathbf{A}_2}{\epsilon}, -\epsilon, \epsilon\right) \ , \quad c^{2,1} = 0 \ . \tag{4.235}$$

For the $\mathbf{Q}$-functions in their turn

$$B_i = \left(-\frac{\mathbf{B}_1}{\epsilon}, \epsilon, -\frac{\mathbf{B}_3}{\epsilon}, -\epsilon\right) \ , \quad B^i = \left(-\epsilon, \frac{\mathbf{B}_2}{\epsilon}, -\epsilon, -\frac{\mathbf{B}_4}{\epsilon}\right) \ . \tag{4.236}$$

As follows from the AA-, BB-relations (4.17), (4.235) and (4.236) in the small ϵ limit

$$A_a = \left(\epsilon, \epsilon, \frac{1+\alpha}{2} i\epsilon, \frac{1-\alpha}{2} i\epsilon\right) + \mathcal{O}\left(\epsilon^3\right) \ , \ B_i = \left(\frac{1-\alpha}{2} i\epsilon, \epsilon, -\frac{1+\alpha}{2} i\epsilon, -\epsilon\right) + \mathcal{O}\left(\epsilon^3\right) \ , \tag{4.237}$$

where

$$\alpha = \left.\frac{\partial S_1}{\partial \Delta}\right|_{\substack{\Delta=0 \\ S_2=1}} \ . \tag{4.238}$$

Note that due to the parity symmetry $\Delta \to -\Delta$ we expect $\alpha = 0$, which will be the consistency check of our calculation. We only get a non-trivial result in the NLO order.

Then from (4.17), (4.235) and (4.236) we derive that A^a and B^i all scale as ϵ and are given by the formulas

$$A^a = \left(\frac{1+\alpha}{2} i\epsilon, -\frac{1-\alpha}{2} i\epsilon, -\epsilon, \epsilon\right) + \mathcal{O}\left(\epsilon^3\right) \ , \quad B^i = \left(-\epsilon, -\frac{1+\alpha}{2} i\epsilon, -\epsilon, -\frac{1-\alpha}{2} i\epsilon\right) + \mathcal{O}\left(\epsilon^3\right) \ . \tag{4.239}$$

This means that the right-hand side in (4.185) is small and the functions $\mathcal{Q}_{a|i}$ are i-periodic in the LO in ϵ. Since they are also analytic in the upper half plane, they are analytic everywhere. Recall that the asymptotics of $\mathcal{Q}_{a|i}$ are given by

$$\mathcal{Q}_{a|j} \sim -i \frac{A_a B_j}{-\tilde{M}_a + \hat{M}_j} u^{-\tilde{M}_a + \hat{M}_j} \ , \quad u \to \infty \ . \tag{4.240}$$

After plugging the global charges into the expressions (4.51) for $\tilde{M}$ and $\hat{M}$ we see that all components of $\mathcal{Q}_{a|i}$ are either zero in the LO or scale like constants at infinity. Constant at infinity entire function is constant everywhere, so again in the LO

$$\mathcal{Q}_{a|i} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon) \tag{4.241}$$

the result is the same as in the case of the slope-to-intercept function.

As in the case of the slope-to-intercept function we can exploit the symmetry between the **P**- and **Q**-functions with lower and upper indices. Since Δ is different from 0 we remember the symmetry (4.54). Despite the normalizations (4.235) and (4.236) differ from (4.52) and (4.53) by a rescaling one can see that the symmetry (4.54) takes the form (which is also confirmed by the weak coupling data for arbitrary Δ and S_2 from Appendix 4.3.1)

$$\mathbf{P}^a(\Delta,u)=\chi_c^{ab}\mathbf{P}_b(-\Delta,u)\ ,\quad \mathbf{Q}^i(\Delta,u)=\eta_c^{ij}\mathbf{Q}_j(-\Delta,u)\ , \tag{4.242}$$

where we set $\epsilon(-\Delta)=\sqrt{-\Delta}=i\sqrt{\Delta}=i\epsilon$ choosing the branch of the square root with the cut going from 0 to $+\infty$ and

$$\chi_c^{ab}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\ ,\quad \eta_c^{ij}=\begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}\ . \tag{4.243}$$

Usage of (4.242) and the substitution of (4.237) and (4.239) leads us to the equality

$$\alpha=0 \tag{4.244}$$

as it should be. Substituting α from (4.244) into (4.237) and (4.239) we obtain A's and B's in the LO

$$\begin{aligned}A_a&=\left(\epsilon,\epsilon,\frac{i\epsilon}{2},\frac{i\epsilon}{2}\right)+\mathcal{O}\left(\epsilon^3\right),\quad A^a=\left(\frac{i\epsilon}{2},-\frac{i\epsilon}{2},-\epsilon,\epsilon\right)+\mathcal{O}\left(\epsilon^3\right),\\ B_i&=\left(\frac{i\epsilon}{2},\epsilon,-\frac{i\epsilon}{2},-\epsilon\right)+\mathcal{O}\left(\epsilon^3\right),\quad B^i=\left(-\epsilon,-\frac{i\epsilon}{2},-\epsilon,-\frac{i\epsilon}{2}\right)+\mathcal{O}\left(\epsilon^3\right).\end{aligned} \tag{4.245}$$

As we consider the case when the spin S_1 is not integer and the spin S_2 is integer, we can use the gluing conditions (4.107) from the Section 4.2, which were also mentioned in the Section 4.1, thus

$$\begin{aligned}\tilde{\mathbf{Q}}^1&=M_1^{11}\bar{\mathbf{Q}}_1+M_1^{12}\bar{\mathbf{Q}}_2+\left(M_1^{13}+M_2^{13}e^{2\pi u}+M_3^{13}e^{-2\pi u}\right)\bar{\mathbf{Q}}_3\ ,\\ \tilde{\mathbf{Q}}^2&=\bar{M}_1^{12}\bar{\mathbf{Q}}_1\ ,\\ \tilde{\mathbf{Q}}^3&=\left(\bar{M}_1^{13}+\bar{M}_2^{13}e^{2\pi u}+\bar{M}_3^{13}e^{-2\pi u}\right)\bar{\mathbf{Q}}_1+M_1^{33}\bar{\mathbf{Q}}_3+M_1^{34}\bar{\mathbf{Q}}_4\ ,\\ \tilde{\mathbf{Q}}^4&=\bar{M}_1^{34}\bar{\mathbf{Q}}_3\ ,\end{aligned} \tag{4.246}$$

keeping in mind that M_1^{11} and M_1^{33} are real.

As now we have the matrix η_c from (4.55), which relates the **Q**-functions with lower and upper indices with Δ replaced by $-\Delta$, it is possible to obtain an additional constraint on the gluing matrix. Plugging the relation (4.242) into the first gluing condition (4.29) we derive

$$\tilde{\mathbf{Q}}_i(-\Delta,u)=(\eta_c)_{ik}M^{kl}(\Delta,u)(\bar{\eta}_c)_{lj}\bar{\mathbf{Q}}^j(-\Delta,u)\ , \tag{4.247}$$

which after comparison with the second gluing condition from (4.29) leads us to $(M^{-t})_{ij}(-\Delta,u)=(\eta_c)_{ik}M^{kl}(\Delta,u)(\eta_c)_{lj}$ and after multiplication of the both sides by $(M^t)^{mj}(-\Delta,u)$ we obtain an additional constraint for the gluing matrix

$$(\eta_c)_{ik}M^{kl}(\Delta,u)(\bar{\eta}_c)_{lm}(M^t)^{mj}(-\Delta,u)=\delta_i^j\ . \tag{4.248}$$

Substitution of (4.82) into (4.248) leads us to the following equations

$$(\eta_c)_{ik}M^{kl}_{2,3}(\Delta)(\bar{\eta}_c)_{lm}(M^t_{2,3})^{mj}(-\Delta) = 0 \,, \tag{4.249}$$

$$(\eta_c)_{ik}M^{kl}_{1}(\Delta)(\bar{\eta}_c)_{lm}(M^t_{2,3})^{mj}(-\Delta) + (\eta_c)_{ik}M^{kl}_{2,3}(\Delta)(\bar{\eta}_c)_{lm}(M^t_{1})^{mj}(-\Delta) = 0 \,, \tag{4.250}$$

$$(\eta_c)_{ik}M^{kl}_{1}(\Delta)(\bar{\eta}_c)_{lm}(M^t_{1})^{mj}(-\Delta) + (\eta_c)_{ik}M^{kl}_{2}(\Delta)(\bar{\eta}_c)_{lm}(M^t_{3})^{mj}(-\Delta) + \tag{4.251}$$
$$+ (\eta_c)_{ik}M^{kl}_{3}(\Delta)(\bar{\eta}_c)_{lm}(M^t_{2})^{mj}(-\Delta) = \delta^j_i \,.$$

It should be noted that the first equation (4.249) is satisfied for the gluing matrix (4.246) (the same as in (4.194)).

To start solving the constraint (4.248) order by order in ϵ analogously to the case of the slope-to-intercept function we use the following expansion of the gluing matrix, which is motivated by the scaling of the $\mathbf{Q}$-functions (4.237) and (4.239)

$$M^{ij}(u) = \sum_{k=0}^{+\infty} M^{(k)ij}(u)\epsilon^{2k} \,, \tag{4.252}$$

where

$$M^{(k)ij}(u) = M^{(k)ij}_1 + M^{(k)ij}_2 e^{2\pi u} + M^{(k)ij}_3 e^{-2\pi u} \,. \tag{4.253}$$

In the LO in ϵ the constraints (4.83), if we take into account (4.237) and $S(\Delta, 1) = -1 + \gamma(g)\Delta^2 + \mathcal{O}(\Delta^4)$, where $\gamma(g)$ is the curvature function, lead us to

$$\begin{aligned} M^{(0)31}_2 &= -M^{(0)13}_2 \,, \\ M^{(0)31}_3 &= -M^{(0)13}_3 \,, \end{aligned} \tag{4.254}$$

which together with the hermiticity of the gluing matrix (4.32) means that $M^{(0)13}_{2,3}$ are pure imaginary. In addition, from (4.88) in the LO in ϵ we obtain

$$M^{(0)13}_3 = M^{(0)13}_2 \,. \tag{4.255}$$

Now we have to apply the constraint (4.248) in the LO in ϵ. Substitution of (4.254) and (4.255) into (4.250) allows us to fix

$$M^{(0)12}_1 = -M^{(0)34}_1 \,, \tag{4.256}$$

given that $M^{(0)13}_2$ is non-zero. Combination of (4.199), (4.254), (4.255) and (4.256) allows us to fix the following elements of the gluing matrix on the LO

$$\left|M^{(0)12}_1\right| = 1 \,, \quad M^{(0)11}_1 = M^{(0)33}_1 = 0 \,, \quad M^{(0)31}_1 = -M^{(0)13}_1 \,. \tag{4.257}$$

In the LO after some calculations we obtain under the assumptions that $M^{(0)13}_2$ is not zero the solution which is

$$M^{(0)ij} = \begin{pmatrix} 0 & M^{(0)12}_1 & M^{(0)13}_1 + 2M^{(0)13}_2\cosh(2\pi u) & 0 \\ \bar{M}^{(0)12}_1 & 0 & 0 & 0 \\ -M^{(0)13}_1 - 2M^{(0)13}_2\cosh(2\pi u) & 0 & 0 & -M^{(0)12}_1 \\ 0 & 0 & -\bar{M}^{(0)12}_1 & 0 \end{pmatrix} , \tag{4.258}$$

where $M_{1,2}^{(0)13}$ are pure imaginary.

Let us start solving these equations in the LO. We have already found $\mathcal{Q}_{a|i}$ and thus

$$\mathcal{Q}^{a|i} = -\left(\mathcal{Q}_{a|i}\right)^{-t} . \tag{4.259}$$

As the scaling of the **P**- and **Q**-functions is determined by the scaling (4.237) and (4.239) of the leading coefficient at large u, we are left with the following expansions of these quantities in the small ϵ limit

$$\mathbf{P}_a = \epsilon \sum_{k=0}^{+\infty} \mathbf{P}_a^{(k)} \epsilon^{2k} , \quad \mathbf{P}^a = \epsilon \sum_{k=0}^{+\infty} \mathbf{P}^{(k)a} \epsilon^{2k} , \quad \mathbf{Q}_i = \epsilon \sum_{k=0}^{+\infty} \mathbf{Q}_i^{(k)} \epsilon^{2k} , \quad \mathbf{Q}^i = \epsilon \sum_{k=0}^{+\infty} \mathbf{Q}^{(k)i} \epsilon^{2k} . \tag{4.260}$$

Then, the correspondence between **Q**- and **P**-functions (4.5) and (4.6) in the LO taking into account (4.241) looks as follows

$$\begin{aligned} \mathbf{Q}_1^{(0)} &= -\mathbf{P}^{(0)2} = \mathbf{P}_3^{(0)} = -\mathbf{Q}^{(0)2} , \\ \mathbf{Q}_2^{(0)} &= -\mathbf{P}^{(0)3} = \mathbf{P}_2^{(0)} = -\mathbf{Q}^{(0)1} , \\ \mathbf{Q}_3^{(0)} &= -\mathbf{P}^{(0)1} = -\mathbf{P}_4^{(0)} = \mathbf{Q}^{(0)4} , \\ \mathbf{Q}_4^{(0)} &= -\mathbf{P}^{(0)4} = -\mathbf{P}_1^{(0)} = \mathbf{Q}^{(0)3} . \end{aligned} \tag{4.261}$$

First of all we substitute (4.261) to the gluing conditions (4.246) written in the LO with the gluing matrix (4.258) and take into account the conjugacy properties of the **P**-functions (4.33)

$$\begin{aligned} \tilde{\mathbf{P}}_1^{(0)} &= -M_1^{(0)12} \mathbf{P}_1^{(0)} - \left(M_1^{(0)13} + 2M_2^{(0)13} \cosh(2\pi u)\right) \mathbf{P}_3^{(0)} , \\ \tilde{\mathbf{P}}_2^{(0)} &= -M_1^{(0)12} \mathbf{P}_2^{(0)} - \left(M_1^{(0)13} + 2M_2^{(0)13} \cosh(2\pi u)\right) \mathbf{P}_4^{(0)} , \\ \tilde{\mathbf{P}}_3^{(0)} &= \bar{M}_1^{(0)12} \mathbf{P}_3^{(0)} , \\ \tilde{\mathbf{P}}_4^{(0)} &= \bar{M}_1^{(0)12} \mathbf{P}_4^{(0)} . \end{aligned} \tag{4.262}$$

Let us consider the third equation from (4.262). According to (4.98) the LHS contains only the non-negative powers of Zhukovsky variable $x(u)$, whereas the RHS has only the non-positive powers of $x(u)$. Therefore $\mathbf{P}_3^{(0)}$ is given only by the constant term from its expansion in the powers of $x(u)$ and taking into account the scaling of the **P**-functions (4.245) we see that in the LO

$$M_1^{(0)12} = 1 \tag{4.263}$$

and

$$\mathbf{P}_3^{(0)} = \frac{i}{2} . \tag{4.264}$$

Substituting $\mathbf{P}_3$ from (4.264) and $M_1^{(0)12}$ from (4.263) into the first equation of (4.262), we have

$$\tilde{\mathbf{P}}_1^{(0)} + \mathbf{P}_1^{(0)} = -\left(M_1^{(0)13} + 2M_2^{(0)13} I_0 + 2M_2^{(0)13} \left(\cosh_+ + \cosh_-\right)\right) \frac{i}{2} . \tag{4.265}$$

After inserting the x-expansion of $\mathbf{P}_1^{(0)}$ from (4.98), we see that the LHS of (4.265) contains all even powers of $x(u)$ except for x^0, thus the same term in the x-expansion of the RHS of (4.265) has to vanish

$$M_1^{(0)13} = -2M_2^{(0)13} I_0 \tag{4.266}$$

and then we obtain

$$\tilde{\mathbf{P}}_1^{(0)} + \mathbf{P}_1^{(0)} = -M_2^{(0)13} \left(\cosh_+ + \cosh_-\right) i \ . \tag{4.267}$$

Of course, the solution of (4.265) is defined modulo the solution of the homogeneous equation ((4.265) with zero RHS), but as the solution to this equation has to contain the positive powers of $x(u)$ it cannot contribute to $\mathbf{P}_1^{(0)}$. Therefore recalling the expansion (4.98) it follows that the solution for $\mathbf{P}_1^{(0)}$ is

$$\mathbf{P}_1^{(0)} = -iM_2^{(0)13} \cosh_- \ . \tag{4.268}$$

As we know the leading coefficient of $\mathbf{P}_1^{(0)}$ then, remembering (4.245), we obtain

$$M_2^{(0)13} = \frac{i}{g^2 I_2} \tag{4.269}$$

and write down the solution

$$\mathbf{P}_1^{(0)} = \frac{\cosh_-}{g^2 I_2} \ . \tag{4.270}$$

Considering the fourth equation from (4.262) with the substituted (4.263)

$$\tilde{\mathbf{P}}_4^{(0)} = \mathbf{P}_4^{(0)} \tag{4.271}$$

we see that in accordance with the expansion (4.98) the RHS has the odd powers of $x(u)$ less or equal to 1 and the LHS has the odd powers of $x(u)$ greater or equal to -1. This means that $\mathbf{P}_4^{(0)}$ contains only the terms with $x(u)$ and $1/x(u)$ and the only combination satisfying (4.271) is

$$\mathbf{P}_4^{(0)} = \frac{i}{2} g \left(x + \frac{1}{x}\right) \ . \tag{4.272}$$

Now let us substitute (4.264), (4.270) and (4.272) into the second equation of (4.262). It gives the following expression

$$\tilde{\mathbf{P}}_2^{(0)} + \mathbf{P}_2^{(0)} = -\left(M_1^{(0)13} + 2M_2^{(0)13} I_0 + 2M_2^{(0)13} \left(\cosh_+ + \cosh_-\right)\right) \frac{i}{2} g \left(x + \frac{1}{x}\right) \ . \tag{4.273}$$

Remembering (4.266) and (4.269) we obtain

$$\tilde{\mathbf{P}}_2^{(0)} + \mathbf{P}_2^{(0)} = \frac{1}{g I_2} \left(x + \frac{1}{x}\right) \left(\cosh_+ + \cosh_-\right) \ . \tag{4.274}$$

Analogously to the equation (4.265) for $\mathbf{P}_1$, the solution to (4.274) for $\mathbf{P}_2^{(0)}$ may potentially include the solution of the homogeneous equation. However, this solution of (4.274) with

the zero RHS inevitably has positive powers of $x(u)$ in it, thus it cannot contribute to $\mathbf{P}_2^{(0)}$. Then, from (4.274) we can extract the LO solution for $\mathbf{P}_2^{(0)}$

$$\mathbf{P}_2^{(0)} = \frac{1}{gI_2}\left(x + \frac{1}{x}\right)\cosh_- \ . \tag{4.275}$$

To summarize this part, we started from the gluing conditions for integer conformal spin n (4.246) and we managed to solve the constraints on it in the LO getting (4.258) and then, using the connection between the **Q**- and **P**-functions in the LO formulated the system of equations (4.262) for the **P**-functions in the LO. Then from this system we found all the **P**-functions. In what follows using these functions we are going to show how to find the NLO solution.

NLO solution

Since the function $S(\Delta, n)$ is even, we have to consider the next-to-leading order. Our strategy is to find the **P**-functions in the NLO order and extract the quantity of interest – the curvature function – from the leading coefficients of these functions. Indeed, from (4.235) and (4.17) we derive

$$\begin{aligned} A_a &= \left(\epsilon, \epsilon, \frac{i}{2}\epsilon + i\left(\frac{\gamma}{2} + \frac{3}{8}\right)\epsilon^3, \frac{i}{2}\epsilon - i\left(\frac{\gamma}{2} - \frac{7}{12}\right)\epsilon^3\right) + \mathcal{O}\left(\epsilon^5\right) \ , \\ A^a &= \left(\frac{i}{2}\epsilon + i\left(\frac{\gamma}{2} - \frac{7}{12}\right)\epsilon^3, -\frac{i}{2}\epsilon + i\left(\frac{\gamma}{2} + \frac{3}{8}\right)\epsilon^3, -\epsilon, \epsilon\right) + \mathcal{O}\left(\epsilon^3\right) \ , \end{aligned} \tag{4.276}$$

where

$$\gamma = \left.\frac{\partial^2 S_1}{\partial \Delta^2}\right|_{\substack{\Delta=0 \\ S_2=1}} \tag{4.277}$$

is the curvature function.

We begin by finding the correction to $\mathcal{Q}_{a|i}^{(0)}$. The equation (4.234), taking into account (4.5) and the scaling (4.260) and expanding it in the NLO, takes the form

$$\mathcal{Q}_{a|i}^{(1)+} - \mathcal{Q}_{a|i}^{(1)-} = -i\mathbf{P}_a^{(0)}\mathcal{Q}_{b|i}^{(0)}\chi_c^{bd}\mathbf{P}_d^{(0)} \ . \tag{4.278}$$

As the functions $\mathbf{P}_a^{(0)}$ were completely fixed in the previous calculations we are able to determine all $\mathcal{Q}_{a|i}^{(1)}$ up to a constant. In solving (4.278) we are going to act in a way similar to the one presented in [22]. We have to find an UHPA solution to the equation of the form

$$f^{++}(u) - f(u) = h(u) \ , \tag{4.279}$$

where the RHS has one cut on the real axis and it can be represented as a series in the Zhukovsky variable $x(u)$, whose powers are bounded from above. To build the solution of such an equation let us rewrite the RHS in the following form

$$h(u) = h_{\rm pol}(u) + h_-(u) \ , \tag{4.280}$$

where $h_{\rm pol}(u)$ is a polynomial in u and $h_-(u)$ is a series in the Zhukovsky variable $x(u)$ starting from the power not greater than $x^{-1}(u)$. Since $h(u)$ is a series in $x(u)$ bounded from above, we can always rewrite $x^a(u)$ with $a>0$ as

$$x^a = \left(x^a + \frac{1}{x^a}\right) - \frac{1}{x^a}\,, \tag{4.281}$$

where $x^a + 1/x^a$ is a polynomial in u. Thus, using (4.281) we are able to replace any positive power of $x(u)$ in $h(u)$ with the difference of polynomial in u and the negative power of $x(u)$ which justifies the form (4.280).

The UHPA solution of (4.279) is given by

$$f(u) = \Sigma\left(h_{\rm pol}\right)(u) + \left(\Gamma_U \cdot h_-\right)(u-i) + c\,, \tag{4.282}$$

where $\Sigma\left(h_{\rm pol}\right)(u)$ is the solution of

$$f^{++}_{\rm pol}(u) - f_{\rm pol}(u) = h_{\rm pol}(u) \tag{4.283}$$

subject to the condition

$$\Sigma\left(h_{\rm pol}\right)\left(\frac{i}{2}\right) = 0 \tag{4.284}$$

and $\left(\Gamma_U \cdot h_-\right)(u)$ is the solution of

$$f^{++}_U(u) - f_U(u) = h_-(u) \tag{4.285}$$

and c is a constant. The operator Γ_U is determined by

$$(\Gamma_U \cdot h)(u) = \oint\limits_{-2g}^{2g} \frac{dv}{2\pi i} i\psi^{(0)}(-i(u-v)+1)h(v)\,, \tag{4.286}$$

where the integration contour around the cut goes clockwise. Also we determine the operator Γ_D which we will use to obtain the LHPA solution

$$(\Gamma_D \cdot h)(u) = \oint\limits_{-2g}^{2g} \frac{dv}{2\pi i} i\psi^{(0)}(i(u-v)+1)h(v)\,. \tag{4.287}$$

It is not hard to check that

$$\begin{aligned}
&(\Gamma_U \cdot h_-)(u+i) - (\Gamma_U \cdot h_-)(u) = h_-(u+i)\,, \\
&(\Gamma_D \cdot h_-)(u+i) - (\Gamma_D \cdot h_-)(u) = h_-(u)\,.
\end{aligned} \tag{4.288}$$

Rewriting the RHS of (4.278) in the form (4.280) we can find the solution of it

$$\mathcal{Q}^{(1)+}_{a|i} = \left(\mathcal{Q}^{(1)+}_{a|i}\right)_{\rm pol} + \left(\mathcal{Q}^{(1)+}_{a|i}\right)_U + c_{a|i}\,, \tag{4.289}$$

where $(\mathcal{Q}^{(1)+}_{a|i})_{\rm pol}$ and $(\mathcal{Q}^{(1)+}_{a|i})_U$ are given in Appendix A.8 by the formulas (A.51) and (A.52) respectively. Using the equation (4.102), it is possible to fix the coefficients $c_{a|i}$ in

terms of the curvature function γ (4.277) and the NLO coefficient $c^{(1)}_{4,1}$ of the x-series of the function $\mathbf{P}_4$ (see the formula (A.53) in the Appendix A.8).

In what follows we will also need the complex conjugated function $\bar{\mathcal{Q}}^{(1)}_{a|i}$, for which we have the equation

$$\bar{\mathcal{Q}}^{(1)+}_{a|i} - \bar{\mathcal{Q}}^{(1)-}_{a|i} = -i\bar{\mathbf{P}}^{(0)}_a \mathcal{Q}^{(0)}_{b|i} \bar{\chi}^{bd}_c \bar{\mathbf{P}}^{(0)}_d \ . \tag{4.290}$$

Conjugating the solution (4.289) and noting that this operation transforms the UHPA part of it with the kernel Γ_U into the LHPA one with the kernel Γ_D we get the solution of the previous equation (4.290)

$$\bar{\mathcal{Q}}^{(1)-}_{a|i} = \left(\bar{\mathcal{Q}}^{(1)-}_{a|i}\right)_{\rm pol} + \left(\bar{\mathcal{Q}}^{(1)-}_{a|i}\right)_D + c^*_{a|i} \ , \tag{4.291}$$

where the functions $(\bar{\mathcal{Q}}^{(1)-}_{a|i})_{\rm pol}$ and $(\bar{\mathcal{Q}}^{(1)-}_{a|i})_D$ are given by the formulas (A.55) and (A.56) respectively (see Appendix A.8).

To proceed we are to solve the equations for the gluing matrix in the NLO in Δ. The constraints (4.83) in the NLO in ϵ lead us to the following equalities

$$M^{(1)31}_2 = -M^{(1)13}_2 \ , \quad M^{(1)31}_3 = -M^{(1)13}_3 - 2i\pi M^{(0)13}_3 \ , \tag{4.292}$$

whereas the constraint (4.88) gives

$$M^{(1)13}_3 = M^{(1)13}_2 - i\pi M^{(0)13}_2 \ , \tag{4.293}$$

where the elements of the LO gluing matrix $M^{(0)ij}(u)$ are known to us. After this it remains to solve the constraints (4.250) and (4.251) in the NLO order. The solution can be compactly described by the formula

$$M^{(1)13}_2 = \frac{i\left(M^{(1)12}_1 - M^{(1)34}_1\right)}{2g^2 I_2} \ , \tag{4.294}$$

while the other matrix elements of $M^{(0)ij}_{2,3}$ are determined by the relations (4.292) and (4.293) holds and $M^{(1)11}_1$, $M^{(1)12}_1$, $M^{(1)33}_1$ together with $M^{(1)34}_1$ are real.

To proceed let us slightly rewrite the gluing conditions expressing the $\mathbf{Q}$-functions in terms of the $\mathbf{P}$-functions as

$$\tilde{\mathbf{P}}_a = \mathcal{Q}^+_{a|i} M^{ij} \bar{\mathcal{Q}}^-_{b|j} \bar{\mathbf{P}}^b \ . \tag{4.295}$$

The same equation (4.295) written in the NLO

$$\begin{aligned}\tilde{\mathbf{P}}^{(1)}_a = &-\mathcal{Q}^{(0)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} i\chi^{be}_c \bar{\mathbf{P}}^{(1)}_e + \\ &+ \left(\mathcal{Q}^{(1)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} + \mathcal{Q}^{(0)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(1)-}_{b|j}\right) i\chi^{be}_c \bar{\mathbf{P}}^{(0)}_e + \mathcal{Q}^{(0)+}_{a|i} M^{(1)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} i\chi^{be}_c \bar{\mathbf{P}}^{(0)}_e \ .\end{aligned} \tag{4.296}$$

Recalling the LO gluing matrix (4.258) we are able to rewrite (4.296) in the following way

$$\begin{aligned}\tilde{\mathbf{P}}^{(1)}_{1,2} - \mathbf{P}^{(1)}_{1,2} &= \frac{2i}{g^2 I_2} \left(\cosh_+ + \cosh_-\right) \mathbf{P}^{(1)}_{3,4} + R_{1,2} \ , \\ \tilde{\mathbf{P}}^{(1)}_{3,4} + \mathbf{P}^{(1)}_{3,4} &= R_{3,4} \ ,\end{aligned} \tag{4.297}$$

where

$$R_a = \left(\mathcal{Q}^{(1)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} + \mathcal{Q}^{(0)+}_{a|i} M^{(0)ij} \bar{\mathcal{Q}}^{(1)-}_{b|j}\right) i\chi_c^{be} \bar{\mathbf{P}}^{(0)}_e + \mathcal{Q}^{(0)+}_{a|i} M^{(1)ij} \bar{\mathcal{Q}}^{(0)-}_{b|j} i\chi_c^{be} \bar{\mathbf{P}}^{(0)}_e \ . \tag{4.298}$$

In finding the solution of (4.297) we follow the same method as in [22]. Let us briefly sketch its main points. One can notice that the equations for the **P**-functions in the NLO (4.297) have one of the two forms

$$\tilde{F}(u) + F(u) = G(u) \quad \text{and} \quad \tilde{F}(u) - F(u) = G(u) \ , \tag{4.299}$$

where $F(u)$ is a power series in Zhukovsky variable $x(u)$ and the function $G(u)$ can be represented as Laurent series in Zhukovsky variable $x(u)$ in the vicinity of the point $x(u) = 0$. As it can be seen, the equations (4.299) are self-consistent only if, respectively, the conditions

$$\tilde{G}(u) - G(u) = 0 \quad \text{and} \quad \tilde{G}(u) + G(u) = 0 \tag{4.300}$$

are satisfied. As it was explained in [17], the unique solutions to the equations (4.299) on the classes of functions non-growing and decaying at infinity are respectively

$$F(u) = (H \cdot G)(u) \equiv \oint\limits_{-2g}^{2g} dv H(u,v) G(v) \text{ and } F(u) = (K \cdot G)(u) \equiv \oint\limits_{-2g}^{2g} dv K(u,v) G(v) \ , \tag{4.301}$$

where the integral kernels are given by

$$H(u,v) = -\frac{1}{4\pi i} \frac{\sqrt{u^2-4g^2}}{\sqrt{v^2-4g^2}} \frac{1}{u-v} \ , \quad K(u,v) = \frac{1}{4\pi i} \frac{1}{u-v} \ . \tag{4.302}$$

It should be noted that if the asymptotic of $F(u)$ is not specified to be non-growing of decaying respectively the solutions of (4.299) may include the zero-modes, i.e. the solutions of the homogeneous equations (4.299) with zero RHS.

With the usage of the integral kernels $H(u,v)$ defined in (4.302) first the 3rd and 4th equations of (4.297) are solved, then the obtained $\mathbf{P}^{(1)}_3$ and $\mathbf{P}^{(1)}_4$ are substituted into the 1st and 2nd equations of (4.297), which are solved by utilizing the integral kernel $K(u,v)$ and we derive the answer for $\mathbf{P}^{(1)}_1$ and $\mathbf{P}^{(1)}_2$. But before starting to solve (4.297) we are to check the validity of the conditions (4.300) in the case in question. First of all, from the form of the 3rd and 4th equations in (4.297) it follows that the equations

$$\tilde{R}_{3,4} - R_{3,4} = 0 \tag{4.303}$$

are satisfied exactly and do not fix any constants in the RHS of (4.297). But from the 1st and 2nd equations of (4.297) we obtain

$$\tilde{R}_{1,2} + R_{1,2} + \frac{2i}{g^2 I_2} \left(\cosh_+ + \cosh_-\right) R_{3,4} = 0 \ , \tag{4.304}$$

which is solved fixing $M^{(1)13}_1$ in terms of $M^{(1)12}_1$ and $M^{(1)34}_1$ together with $M^{(1)11}_1$ and $M^{(1)33}_1$. One can find the details of this calculation and the values of these constants in

Appendix A.9. Therefore we checked that the RHS of (4.297) satisfy (4.300) and we can apply (4.301).

As $\mathbf{P}_3^{(1)}$ has the constant asymptotic at infinity, the unique solution for $\mathbf{P}_3^{(1)}$ is given by

$$\mathbf{P}_3^{(1)} = H \cdot R_3 \tag{4.305}$$

and contains the constants $M_1^{(1)12}$, $c_{4,1}^{(1)}$ and the curvature function $\gamma(g)$ defined in (4.277), which we have to fix. With the usage of (4.235) and (4.276) we can express $M_1^{(1)12}$ and $c_{4,1}^{(1)}$ in terms of γ

$$\begin{aligned} M_1^{(1)12} &= 2\gamma - \frac{1}{2} + \frac{iH_0\left(u\left(\Gamma\cdot\cosh_-^v\right)(u)\right)}{g^2 I_2} - \frac{iH_0\left(\left(\Gamma\cdot v\cosh_-^v\right)(u)\right)}{g^2 I_2}\,, \\ c_{4,1}^{(1)} &= \left(\frac{I_4}{6I_2}+\frac{1}{3}\right) ig + \frac{i}{24g} + \frac{iH_2\left(u\left(\Gamma\cdot\cosh_-^v\right)(u)\right)}{g^3 I_2} - \frac{iH_2\left(\left(\Gamma\cdot v\cosh_-^v\right)(u)\right)}{g^3 I_2}\,, \end{aligned} \tag{4.306}$$

where the integral kernel Γ is defined as

$$\left(\Gamma\cdot h(v)\right)(u) = \left(\Gamma_U\cdot h(v)\right)(u) + \left(\Gamma_D\cdot h(v)\right)(u) \tag{4.307}$$

and H_k is the k-th coefficient in the large u expansion of the convolution of the integral kernel $H(u,v)$ with the function on which it acts.

The asymptotic of $\mathbf{P}_4^{(1)}(u)$ is growing as u, then the solution (4.301) with the kernel $H(u,v)$ is not unique and we have to add the solution of the homogeneous equation (zero-mode) to it. To find this zero-mode we first notice that according to the asymptotic of $\mathbf{P}_4^{(1)}(u)$ it can include only the powers $x(u)$ and $1/x(u)$ and thus it is proportional to $x(u)-1/x(u)$. The coefficient in front of this zero-mode is determined by the coefficients (4.276), then $\mathbf{P}_4^{(1)}$ is equal to

$$\mathbf{P}_4^{(1)} = \left(\frac{7}{24}-\frac{\gamma}{2}\right) ig\left(x-\frac{1}{x}\right) + H\cdot R_4 \tag{4.308}$$

and contains the constants $M_1^{(1)34}$ and γ, which are not fixed yet.

To proceed with the solution of the 3rd and 4th equations of (4.297) we take each equation and subtract the same equation conjugated

$$\begin{aligned} \tilde{\mathbf{P}}_1^{(1)} - \mathbf{P}_1^{(1)} &= -\frac{i}{g^2 I_2}\left(\cosh_+ + \cosh_-\right)\left(R_3 - 2\mathbf{P}_3^{(1)}\right) + \frac{R_1-\tilde{R}_1}{2}\,, \\ \tilde{\mathbf{P}}_2^{(1)} - \mathbf{P}_2^{(1)} &= -\frac{i}{g^2 I_2}\left(\cosh_+ + \cosh_-\right)\left(R_4 - 2\mathbf{P}_4^{(1)}\right) + \frac{R_2-\tilde{R}_2}{2}\,. \end{aligned} \tag{4.309}$$

As both the functions $\mathbf{P}_1^{(1)}(u)$ and $\mathbf{P}_2^{(1)}(u)$ have the decaying at infinity asymptotics, the solutions of (4.309) are uniquely determined by the formula (4.301) with the kernel $K(u,v)$. Calculating R_3 with the usage of (4.305) and (4.306), we find the solution for $\mathbf{P}_1^{(1)}$

$$\mathbf{P}_1^{(1)} = -\frac{i}{g^2 I_2} K\cdot\left(\left(\cosh_+ + \cosh_-\right)\left(R_3 - 2\mathbf{P}_3^{(1)}\right)\right) + K\cdot\frac{R_1-\tilde{R}_1}{2}\,. \tag{4.310}$$

Doing the same for R_4 with the usage of (4.308), we find the solution for $\mathbf{P}_2^{(1)}$

$$\mathbf{P}_2^{(1)} = -\frac{i}{g^2 I_2} K \cdot \left((\cosh_+ + \cosh_-) \left(R_4 - 2\mathbf{P}_4^{(1)} \right) \right) + K \cdot \frac{R_2 - \tilde{R}_2}{2} \,. \tag{4.311}$$

From (4.276) we conclude that

$$A_1^{(1)} = A_2^{(1)} = 0 \,, \tag{4.312}$$

which leads to fixing the following constants

$$M_1^{(1)34} = \frac{7}{6} + \frac{iH_0 \left(\left(\Gamma \cdot v \cosh_-^v \right)(u) - u \left(\Gamma \cdot \cosh_-^v \right)(u) \right)}{g^2 I_2} + \\ + \frac{2iK_2 \left(\cosh_-^u \left(\Gamma \cdot v \cosh_-^v \right)(u) - u \cosh_-^u \left(\Gamma \cdot \cosh_-^v \right)(u) \right)}{g^4 I_2} \,, \tag{4.313}$$

where K_k is the k-th coefficient in the large u expansion of the convolution of the kernel $K(u,v)$ with the function on which it acts.

To sum up, we managed to solve the constraints for the gluing matrix in the NLO, which allowed us to write down the system of equations for the **P**-functions in the NLO (4.296). In the NLO some Q-functions contain infinite series of cuts, which led us to the usage of the integral kernels (4.286), (4.287) and (4.302). After finding the **P**-functions in the NLO by solving the system for them we are ready to fix the curvature function by utilizing the values of the coefficients found in the previous calculations.

Result for the curvature function

To obtain the curvature function, we have to remember that $c_{4,1}^{(1)}$ from (4.306) determines the coefficient in front of $1/x$ in $\mathbf{P}_4^{(1)}$. Comparing this coefficient found from (4.308) with the usage of (4.313) and the one from (4.306) we find the answer

$$\gamma(g) = \frac{iH_0 \left(\left(\Gamma \cdot v \cosh_-^v \right)(u) - u \left(\Gamma \cdot \cosh_-^v \right)(u) \right)}{2g^2 I_2} + \frac{iH_1 \left(u \left(\Gamma \cdot v \cosh_-^v \right)(u) - \left(\Gamma \cdot v^2 \cosh_-^v \right)(u) \right)}{4g^2 I_2} + \\ + \frac{iH_2 \left(\left(\Gamma \cdot v \cosh_-^v \right)(u) - u \left(\Gamma \cdot \cosh_-^v \right)(u) \right)}{4g^2 I_2} + \frac{iK_2 \left(\cosh_-^u \left(\left(\Gamma \cdot v \cosh_-^v \right)(u) - u \left(\Gamma \cdot \cosh_-^v \right)(u) \right) \right)}{g^4 I_2} \,. \tag{4.314}$$

By expanding the integral kernels (4.302) at large u we can rewrite the curvature function in a more concise form

$$\boxed{\begin{aligned} \gamma(g) = {} & \frac{1}{4\pi g^4 I_2^2} \oint_{-2g}^{2g} dv (v \cosh_-^v \left(\Gamma \cdot u \cosh_-^u \right)(v) - v^2 \cosh_-^v \left(\Gamma \cdot \cosh_-^u \right)(v)) + \\ & + \frac{1}{16\pi g^5 I_2} \oint_{-2g}^{2g} dv \left(\frac{v^3 \left(\Gamma \cdot \cosh_-^u \right)(v) - 2v^2 \left(\Gamma \cdot u \cosh_-^u \right)(v) + v \left(\Gamma \cdot u^2 \cosh_-^u \right)}{x_v - \frac{1}{x_v}} \right) \,, \end{aligned}} \tag{4.315}$$

where

$$\left(\Gamma \cdot h(v) \right)(u) = \oint_{-2g}^{2g} \frac{dv}{2\pi i} \partial_u \log \frac{\Gamma\left[i(u-v) + 1 \right]}{\Gamma\left[-i(u-v) + 1 \right]} h(v) \,. \tag{4.316}$$

After obtaining (4.315) we can compare it with the other known results. In the first two orders we know the BFKL Pomeron eigenvalues for arbitrary conformal spin including $n = 1$, therefore we are able to calculate the curvature from these eigenvalues in these two first orders. Comparing with the weak coupling expansion of the formula (4.315) for the curvature function

$$\gamma(g) = 2\zeta_3 g^2 + \left(-\frac{2\pi^2}{3}\zeta_3 - 35\zeta_5\right) g^4 + \left(\frac{16\pi^4}{45}\zeta_3 + \frac{22\pi^2}{3}\zeta_5 + 504\zeta_7\right) g^6 + \left(-\frac{28\pi^6}{135}\zeta_3 - \frac{8\pi^4}{3}\zeta_5 - \right.$$
$$\left. -56\pi^2\zeta_7 - 6930\zeta_9\right) g^8 + \left(\frac{136\pi^8}{2835}\zeta_3 + \frac{668\pi^6}{189}\zeta_5 - \frac{112\pi^4}{3}\zeta_7 + 508\pi^2\zeta_9 + 93720\zeta_{11}\right) g^{10} +$$
$$+ \left(\frac{754\pi^{10}}{42525}\zeta_3 - \frac{1402\pi^8}{567}\zeta_5 - \frac{73\pi^6}{45}\zeta_7 + \frac{4618\pi^4}{3}\zeta_9 - 12969\pi^2\zeta_{11} - 1234233\zeta_{13}\right) g^{12} + O\left(g^{14}\right) \tag{4.317}$$

we find that the first two terms of the expansion (4.317) coincide with the values obtained from the BFKL Pomeron eigenvalues, which represents a check of our result. Having computed the weak coupling expansion (4.317), in the next part we analyze the other interesting limit, i.e. the strong coupling expansion, which is a separate computational task in the case of the curvature function.

Strong coupling expansion of the curvature function

For the calculation of the strong coupling expansion of the curvature function (4.315) we utilize the same method as the one used in [22]. Let us briefly sketch the scheme of this calculation. The curvature function can be represented in the following form

$$\gamma(g) = \oint\limits_{-2g}^{2g} du \oint\limits_{-2g}^{2g} dv F(x_u, x_v)\, \partial_u \log \frac{\Gamma\left(i(u-v)+1\right)}{\Gamma\left(-i(u-v)+1\right)} \,. \tag{4.318}$$

Integrating by parts (4.318) and changing the integration variables to x_u and x_v respectively, we obtain

$$\gamma(g) = \oint dx_u \oint dx_v G(x_u, x_v) \log \frac{\Gamma\left(i(u-v)+1\right)}{\Gamma\left(-i(u-v)+1\right)} \,, \tag{4.319}$$

where the integrals go over the unit circle and $G(x_u, x_v)$ is a polynomial in the variables x_u, x_v, $1/x_u$, $1/x_v$, $\cosh^u_-$, $\cosh^v_-$ and $\sinh^u_-$. Then, expanding $\cosh_-$ and $\sinh_-$ in a series in the Zhukovsky variable x, we can represent (4.319) as an infinite series in the coefficients of the BES dressing phase [69, 120–122], which admits the large g expansion in the form of asymptotic series [69].

Using the computed series we calculated the numerical value of the curvature function for a number of points in the range $5 \leq g \leq 40$ and obtained the first coefficients of the strong coupling expansion of the curvature function with high precision by fitting the data with inverse powers of g. The first several coefficients appear to be simple rational numbers. In the 6th coefficient it was expected to have ζ-function, which we managed to fit utilizing the `EZFace` [123] webpage. As in the case of the slope-to-slope function [22], usage of the exact value fit in the order $1/g^k$ increases accuracy of the fit in the next order $1/g^{k+1}$, which represents a non-trivial check of the validity of the used method.

Application of the numerical method described above yields us the large g expansion of the curvature function

$$\gamma = \frac{1}{2\lambda^{1/2}} - \frac{1}{4\lambda} - \frac{33}{16\lambda^{3/2}} - \frac{81}{16\lambda^2} - \\ - \frac{2265}{256\lambda^{5/2}} + \left(\frac{1440\zeta_5}{64} - \frac{765}{64}\right)\frac{1}{\lambda^3} + \left(\frac{207360\zeta_5}{2048} - \frac{22545}{2048}\right)\frac{1}{\lambda^{7/2}} + \mathcal{O}\left(\frac{1}{\lambda^4}\right), \quad (4.320)$$

where λ is given by (4.231). Together with the curvature function (4.315) and its weak coupling expansion (4.317) the formula (4.320), containing the strong coupling expansion of this function, concludes the list of the results of the present Section.

4.5 Intercept function at strong coupling

The other interesting limit of the intercept function besides the small coupling limit considered in the Section 4.3 is the strong coupling one. For $n = 0$ case the intercept function in the strong coupling limit was analyzed in [48, 124–126] and then extended to the next orders by the QSC method in [22]. As we have already the numerical data for the intercept for the different values of conformal spin n, then using the numerical algorithm described in the Section 4.2 and assuming that the coefficients are some simple rational numbers and extrapolating the high precision numerical data by the inverse powers of $\lambda^{1/2}$ we extract

$$\begin{aligned} S(0,3) &= -3 + \frac{10}{\lambda^{1/2}} - \frac{25}{\lambda} + \frac{175}{4}\frac{1}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-2}), \quad (4.321) \\ S(0,2) &= -2 + \frac{4}{\lambda^{1/2}} - \frac{6}{\lambda} + \frac{9}{2}\frac{1}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-2}), \\ S(0,1) &= -1, \\ S(0,0) &= 0 - \frac{2}{\lambda^{1/2}} - \frac{1}{\lambda} + \frac{1}{4}\frac{1}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-2}), \end{aligned}$$

where λ is given by (4.231) and the result for $n = 0$ is taken from [22]. We see that the leading term is linear in n, sub-leading is quadratic and so on. This pattern is quite typical at strong coupling (see for example [61]). Assuming this polynomial pattern from the above data (4.321) we get

$$S(0,n) = -n + \frac{(n-1)(n+2)}{\lambda^{1/2}} - \\ - \frac{(n-1)(n+2)(2n-1)}{2\lambda} + \frac{(n-1)(n+2)(7n^2-9n-1)}{8\lambda^{3/2}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (4.322)$$

which we can cross-check with our slope-to-intercept function (4.233) by differentiating it at $n = 1$! Comparison with (4.233) shows us complete agreement. We also verified our result numerically by taking $n = 1.5$ and fitting the data with inverse powers of $\lambda^{1/2}$ we reproduced precisely the coefficients from (4.322).

Chapter 5

Conclusions and outlook

In this work we managed to reproduce the dimension of twist-2 operator of $\mathcal{N} = 4$ SYM theory in the 't Hooft limit in the leading order (LO) of the BFKL regime directly from the exact equations for the spectrum of local operators called the Quantum Spectral Curve – QSC. This result is a very non-trivial confirmation of the general validity of this QSC approach and of the whole program of integrability of the spectral problem in AdS/CFT – S-matrix and asymptotic Bethe ansatz, TBA, Y-system, FiNLIE equations, etc. In particular, this is one of a very few examples of all-loop calculations, with all wrapping corrections included, where the integrability result can be checked by direct Feynman graph summation of the original BFKL approach. An obvious step to do in this direction is to compute the NLO correction to the twist-2 dimension for finite Δ from QSC and compare to the direct BFKL computation of [2]. Many of the elements of the NLO construction, such as the NLO Baxter equation for the **Q**-functions, are present already in this paper, but the most difficult ingredient – the formula of the type (3.198) for the leading singularity, has yet to be derived. Of course, the ultimate goal of the BFKL approximation to QSC would be to find an algorithmic way of generation of any BFKL correction (NNLO, NNNLO, etc.) on Mathematica program, similarly to the one for the weak coupling expansion via QSC, proposed by [20]. It would be also very interesting to build numerically from the QSC the twist-2 dimension as a continuous function of spin $S \in \mathbb{R}$ qualitatively described in [127]. We also hope that our approach will allow to understand deeper the similarities and differences of $\mathcal{N} = 4$ SYM and the pure Yang-Mills theory (multicolor QCD) starting from the BFKL approximation, regarding the well known fact that, at least in the 't Hooft limit, $\mathcal{N} = 4$ SYM Feynman graphs capture an important part of all QCD graphs and in the LO BFKL the results simply coincide. We also hope that the methods of QSC presented here will be inspiring for construction of the systematic strong coupling expansion in $\mathcal{N} = 4$ SYM. A deeper insight into the structure of QSC will be needed to approach the whole circle of these complex problems.

BFKL regime is traditionally one of the "hard" problems of high-energy theoretical physics. QSC method allowed to make progress in this direction when the traditional perturbative methods become too complex to implement. In our work we extended the area of application of QSC even further, adding an additional parameter – conformal spin n. Importantly, we can deal with the situation when conformal spin takes an arbitrary real value. QSC can now deal with the situation when all three global charges corresponding to the 5-dimensional sphere S^5 – S_1, S_2 and Δ are non-integer. This required allowing

exponential asymptotics in two components of the gluing matrix. Gluing conditions are thus the main ingredient of analytical continuation of QSC presented in this paper.

As an illustration of our method we have computed two all-coupling quantities – slope of BFKL intercept at $\Delta = 0$ with respect to n around $n = 1$ and curvature of the twist-2 operator trajectory in the vicinity of the point $\Delta = 0$ and $n = 1$. We generated analytical perturbative and numerical data using the iterative procedures described in [31], which we had to modify to take into account exponential asymptotics for non-integer global charges. The iterative data helped us to fix NNLO and NNNLO BFKL intercept in terms of nested harmonic sums.

Several further directions of work come to mind immediately. First, the basis of nested harmonic sums seems to describe well the perturbative expansion of BFKL eigenvalue and in particular its intercept. Since the iterative procedure can be used up to arbitrary high order in g and for arbitrary odd n, it is just a question of time and computational power to fix BFKL eigenvalues at higher orders. The first task would be to find a closed expression for the NNLO BFKL eigenvalue for arbitrary n. It should then agree with the results of [58], where a procedure calculating NNLO BFKL eigenvalue for any particular given n was presented.

The second direction one can pursue is exploring the neighborhood of the BPS point $n = 1, \Delta = 0$. After computing the slope-to-intercept with respect to n, one can compute the next order term proportional to $(n-1)^2$. The computation should be similar in spirit to the computation of S^2 correction to the twist operator anomalous dimension [22].

Furthermore, one should study in a similar way the Odderon spectrum, which is expected to correspond to the length parameter to be taken $L = 3$. Most of the steps described in this paper should be applicable for arbitrary L and it would be very interesting to reproduce previously known perturbative results and extend them to finite coupling.

Finally, there are indications [128] that the structure constants can be also governed by the QSC Q-functions, which were evaluated in this paper in various regimes. So it would be interesting to compare and extend to finite coupling the results on the triple Pomeron vertex [129].

Chapter 6

Résumé en français

6.1 Introduction

La $\mathcal{N} = 4$ théorie de Super-Yang-Mills joue un rôle important dans notre compréhension des théories quantiques des champs, en particulier dans un contexte AdS/CFT. En raison du principe de transcendentalité maximale de Kotikov-Lipatov, [1, 2] certains des résultats obtenus dans cette théorie peuvent être directement exportés vers une QCD planaire plus réaliste. Dans ce travail, nous décrivons comment effectuer efficacement des calculs dans cette théorie pour l'une des principales observables de la QCD - le spectre BFKL, en utilisant l'intégrabilité à toute valeur du couplage 't Hooft λ, initialement découverte par Lipatov à la spectre LO BFKL [3], et s'est développé bien au-delà du régime perturbatif dans le $\mathcal{N} = 4$ SYM ces dernières années. Lev Nikolaevich a été l'un des principaux moteurs de ces progrès et nous sommes profondément attristés de savoir qu'il nous a quitté en Septembre 2017.

Au début, nous allons décrire brièvement la signification des quantités étudiées dans le présent travail dans le contexte de la diffusion de haute énergie. La section totale $\sigma(s)$ pour la diffusion à haute énergie de deux particules incolores A et B peut être écrite comme [4]

$$\sigma(s) = \int \frac{d^2q d^2q'}{(2\pi)^2 q^2 q'^2} \Phi_A(q)\Phi_B(q') \int\limits_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(q, q') \,, \tag{6.1}$$

où $\Phi_i(q_i)$ sont les facteurs d'impact, $G_\omega(q, q')$ est l'onde partielle t-channel pour la diffusion gluon-gluon, $s_0 = |q||q'|$ et dépendent des impulsions transversales et de $s = 2p_A p_B$, où p_A et p_B sont respectivement les 4-moments des particules A et B. Pour l'onde partielle t-channel, il existe l'équation de Bethe-Salpeter

$$\omega G_\omega(q, q_1) = \delta^{D-2}(q - q_1) + \int d^{D-2}q_2 K(q, q_2) G_\omega(q_2, q_1) \,. \tag{6.2}$$

Il semble possible de classer les valeurs propres ω du noyau BFKL K en utilisant deux nombres quantiques: entier n (spin conforme) et réel ν

$$\omega = \omega(n, \nu) \,. \tag{6.3}$$

La fonction $\omega(n,\nu)$ est appelée l'eigenvalue de Pomeron du noyau BFKL ou simplement l'eigenvalue de BFKL Pomeron et ses valeurs pour différents n et ν constituent le spectre BFKL. Pour les applications phénoménologiques des eigenvalues du noyau BFKL avec un spin conforme non nul, voir [5]. L'objet $\omega(n,\nu)$[1] dans le planaire $\mathcal{N}=4$ SYM sera étudié dans ce travail au moyen de l'intégrabilité.

L'étude des structures intégrables dans la théorie jauge 4d a une longue et intéressante histoire du développement. L'intégrabilité dans la théorie QCD et les théories supersymétriques de Yang-Mills sont apparues dans deux contextes. Premièrement, dans la théorie des jauges, à savoir la QCD, l'équation de Bartels-Kwiecinski-Praszalowicz (BKP) [7, 8] pour les états multi-reggeon a été reformulée par L.N. Lipatov [3] le modèle avec les hamiltoniens holomorphes et antiholomorphes, qui possède un ensemble d'opérateurs commutés mutuellement provenant de la matrice de monodromie satisfaisant à l'équation de Yang-Baxter. Après cela, L.D. Faddeev et G.P. Korchemsky dans [9] ont prouvé que ce modèle était complètement intégrable et équivalent au problème spectral pour $SL(2,\mathbb{C})$ XXX chaîne de spin de Heisenberg. Puis, dans le contexte de la diffusion à haute énergie, on a considéré une certaine classe d'opérateurs de cône de lumière dans les théories QCD et Yang-Mills supersymétriques et dans [10–13] Le problème de trouver les dimensions anormales des opérateurs de cônes de lumière a été formulé en termes de chaîne de spin de $SL(2,\mathbb{R})$ Heisenberg.

L'autre réalisation a été que la théorie supersymétrique $\mathcal{N}=4$ Yang-Mills en 4 dimensions, qui est double à $AdS_5\times S^5$ la théorie des supercordes de type IIB, était intégrable [14, 15]. L'étude de la structure d'intégrabilité de cette dernière théorie a permis d'explorer son spectre dans le régime non-perturbatif. La solution du problème spectral a été formulée en termes de la Courbe Spectrale Quantique (QSC) [16, 17]).

Néanmoins, jusqu'à récemment, on ne savait pas comment établir un pont entre l'intégrabilité dans la limite BFKL et l'intégrabilité trouvée dans le cadre AdS/CFT. Dans [30], la contribution asymptotique de Bethe Ansatz (ABA) à 4 boucles à la dimension anormale des opérateurs twist-2 $\mathfrak{sl}(2)$ a été poursuivie de manière analytique aux spins non entiers et comparée avec la prédiction correspondante des eigenvalues de BFKL Pomeron. Cette suite analytique aux spins non entiers a été incorporée au formalisme QSC dans [22] pour les opérateurs twist-2 du secteur $\mathfrak{sl}(2)$ et dans ce travail, nous expliquons comment dériver l'equation de Faddeev-Korchemsky Baxter [9] pour la chaîne de spin de Lipatov $SL(2,\mathbb{C})$ reproduisant correctement l'eigenvalue de premier ordre (LO) BFKL Pomeron. De plus, QSC a permis de calculer [31] de manière analytique l'eigenvalue BFKL de l'ordre suivant proche de l'avant-plan (NNLO) dans la théorie de Yang-Mills supersymétrique $\mathcal{N}=4$.

Considérons les opérateurs twist-2 $\mathfrak{sl}(2)$ de la forme

$$\mathcal{O}=\mathrm{tr}ZD_+^SZ+(\text{permutations}) \tag{6.4}$$

et rappelez-vous que des calculs perturbatifs dans la théorie de jauge pour le cas d'un entier pair S, nous connaissons la dimension de ces opérateurs Δ en fonction de S jusqu'à plusieurs ordres de boucles. Dans le cadre QSC, la solution de l'équation de Baxter pour le spectre de tels opérateurs dans le cas d'un spin conforme à n et d'un entier même spins S a été obtenue dans [16]. Puis, dans [33, 34], la solution de ces équations de Baxter était

[1] In [4] la fonction ω est utilisée avec les différents arguments $\gamma=1/2+i\nu$.

valide pour un spin arbitraire S, ce qui conduit à la dimension anormale des opérateurs twist-2 $\mathfrak{sl}(2)$ continué analytiquement pour le spin non entier S. Après avoir fait cette continuation analytique dans le régime BFKL, nous pouvons échanger les rôles de Δ et S pour obtenir $S+1=\omega(n=0,\nu)$, où $\nu=-i\Delta/2$ et Δ est la dimension de l'opérateur en question.

Nous démontrons ici l'application du QSC à un problème important – le calcul des dimensions conformes Δ des opérateurs d'un type (6.4) appartenant au secteur $\mathfrak{sl}(2)$ dans le Limite BFKL, correspondant à un double régime d'échelle de la petite constante 't Hooft $g\equiv\sqrt{\lambda}/(4\pi)\alpha 0$ et le spin de Lorentz S approchant à -1, alors que le ratio $\Lambda\equiv g^2/(S+1)$ est maintenu fixe. Nous reproduirons la fameuse formule de cette dimension, obtenue dans [2, 35, 36] à partir de la re-sommation directe des graphes de Feynman

$$\frac{1}{4\Lambda}=-\psi\left(\frac{1}{2}-\frac{\Delta}{2}\right)-\psi\left(\frac{1}{2}+\frac{\Delta}{2}\right)+2\psi(1)+\mathcal{O}(g^2)\,, \tag{6.5}$$

où $\psi(x)=\Gamma'(x)/\Gamma(x)$. Remarquablement, ce résultat est également connu pour être valide pour la théorie pure de Yang-Mills dans la limite planaire puisque seuls les gluons apparaissent à l'intérieur des diagrammes de Feynman de $\mathcal{N}=4$ SYM à LO! Notre méthode, conçue ici pour le cas de la singularité de Pomeron (un état lié de deux gluons reggized), devrait être applicable à l'étude d'un état lié de gluons régularisés par L.

infinitely many loops for a highly non-trivial non-BPS quantity.

Soulignons l'un des principaux résultats de ce travail – la reproduction correcte de la formule (1.7) du QSC – est un test très non-trivial du QSC ainsi que de l'approche intégrable de l'intégrabilité de le spectre AdS/CFT planaire. Il résume un nombre infini de corrections dites d'enveloppement infiniment de boucles pour une quantité non-BPS hautement non-triviale.

Dans le présent travail, nous considérons la généralisation permettant une valeur arbitraire du spin conforme. A savoir, nous considérons les opérateurs

$$\mathcal{O}=\mathrm{tr}ZD_{+}^{S_1}\partial_{\perp}^{S_2}Z+(\text{permutations})\,. \tag{6.6}$$

Pour les opérateurs (6.6) nous suivons la même stratégie que pour le cas du spin conforme à zéro. De manière analogue à ce cas, nous construisons la suite analytique dans les spins S_1 et S_2, qui sont identifiés respectivement avec le spin S et le spin conforme n. Illustrons cette suite analytique avec la figure 1.1. Les opérateurs physiques, pour lesquels la somme de l'entier non négatif $S=S_1$ et $n=S_2$ est égale, sont représentés avec les points. Ensuite, en retournant les rôles de la dimension Δ et $S=S_1$, on peut atteindre le régime BFKL décrit par la quantité $\omega(n=S_2,\nu)=S_1+1$, où $\nu=-i\Delta/2$.

La façon de procéder avec le problème en question est de généraliser l'approche QSC à des valeurs non-entières de S_1 (comme cela a déjà été fait dans [22]) et aussi à des valeurs non-entières de S_2. Nous décrivons les détails techniques de cette procédure dans la section 4.1. Cela permet de traiter $\omega(n,\nu)$ comme une fonction analytique de ses deux paramètres, ce qui simplifie les considérations analytiques et numériques. Cela donne un cadre universel pour étudier le spectre BFKL en généralité pour toutes les valeurs des paramètres sur un pied d'égalité dans le formalisme QSC étendu.

Après avoir formulé le problème comme une extension du QSC initial, un certain nombre de méthodes, initialement développées pour les opérateurs locaux, sont devenues

disponibles pour le problème BFKL. En particulier, nous pouvons utiliser un algorithme numérique très puissant [32] après quelques modifications. Comme nous prenons les spins S_1 comme variable continue, nous pouvons considérer à la place de la fonction $\Delta(S_1, S_2)$ la fonction $S_1(\Delta, S_2)$. Ensuite, en utilisant l'algorithme, nous construisons les trajectoires d'opérateur pour différentes valeurs de spin conforme S_2 et les dépendances du spin S_1 sur la constante de couplage g pour différentes valeurs de spin conforme et de dimension Δ (y compris un fonction d'intercept particulièrement intéressante correspondant à $\Delta = 0$). Ayant les résultats numériques pour les trajectoires d'opérateurs, nous avons pu ajuster les valeurs numériques des eigenvalues du noyau BFKL[2], qui ont été confirmées par une méthode différente dans [58].

Une autre méthode disponible dans le formalisme QSC est une expansion perturbative efficace développée dans [20, 31, 59–61]. Nous avons appliqué cette méthode pour trouver la valeur de l'intercept de Pomeron pour la valeur arbitraire du spin conforme jusqu'à des 3 boucles. Notre résultat est en plein accord avec [58] au niveau NNLO, mais nous donnons également une prédiction pour la prochaine commande NNNLO.

Ensuite, nous avons trouvé et étudié en détail un point particulièrement intéressant dans l'espace des paramètres du BFKL Pomeron. C'est le point "BPS" $\Delta = 0$ et $n = 1$. Comme nous l'avons confirmé numériquement et analytiquement, la trajectoire de l'opérateur passe par le point $S = -1$, $n = 1$ et $\Delta = 0$ pour toute valeur de la constante de couplage g. En étudiant la proximité de ce point, nous avons pu trouver deux grandeurs non perturbatives: la fonction "slope-to-intercept" et la "curvature function". La première fonction est la première dérivée de $S(\Delta, n)$ par rapport à n au point $\Delta = 0$, $n = 1$ et la deuxième fonction est la dérivée seconde de $S(\Delta, n)$ par rapport à Δ au même point. Nous avons utilisé les méthodes développées dans [22] pour calculer analytiquement ces quantités de manière non-perturbative dans tous les ordres de g.

Enfin, nous avons pu identifier la fonction d'intercept dans la forte expansion du couplage jusqu'au 4ème ordre. Pour l'obtenir, nous avons utilisé les dépendances de l'intercept sur la constante de couplage calculée par la méthode numérique QSC. En effectuant l'ajustement numérique de ces dépendances pour différentes valeurs de spin conforme n, nous prédisons la formule de l'expansion du couplage fort à l'origine jusqu'à la 4ème ordre pour un spin conforme arbitraire.

Présentons un bref résumé des quantités que nous avons calculées. Ils incluent la fonction d'intercept NNLO (6.41) et la partie non rationnelle de la fonction d'intercept NNNLO (6.43). Les autres grandeurs que nous avons calculées exactement pour tous les ordres dans la constante de couplage 't Hooft sont les fonctions slope-to-intercept (6.44) et la curvature (6.50) avec les extensions fortes de couplage de ces fonctions données par (6.48) et (6.53) respectivement. De plus, il a été écrit l'expansion de couplage fort (6.55) de la fonction d'intercept pour un spin conforme arbitraire n. Nous avons également implémenté la méthode numérique pour trouver les eigenvalues à des valeurs arbitraires des paramètres de `Mathematica`, les fichiers correspondants `code_for_arxiv.nb` et `BFKLdata.mx` peuvent être trouvés dans les pièces jointes à cette arXiv. soumission [62]. Voir le fichier `"description.txt"`pour la description.

[2]Voir la présentation de M.Alfimov au GATIS Training Event à DESY [57].

6.2 Description du cadre basé sur QSC

Dans cette Section, nous allons présenter le cadre que nous utilisons pour résoudre le QSC [16,17] et dont la dérivation est basée sur les propriétés analytiques et asymptotiques des Q-fonctions. Premièrement, nous reformulons le QSC en termes de matrice de collage. A savoir, nous partons des plusieurs axiomes concernant la structure analytique du Q-système et les symétries qui préservent les QQ-relations et en dérivent les conditions dites de collage. Ces conditions de collage apparaissaient déjà dans [17, 31] mais notre approche présentée ci-dessous n'utilise pas la notion de fonctions μ et ω pour obtenir la matrice de collage. Deuxièmement, en utilisant la connexion entre les asymptotiques de certains sous-ensembles de Q-fonctions et les charges globales avec leurs propriétés analytiques, le système de contraintes pour la matrice de collage est dérivé. Il semble possible de résoudre ces équations dans certains cas physiquement intéressants. À savoir, nous trouvons la matrice de collage pour le cas où les deux spins AdS S_1 et S_2 sont des entiers de la même parité et que sa forme semble très simple et en accord avec le résultat de [17]. Nous considérons ensuite un cas plus général des spires non entières AdS S_1 et S_2, ce qui est particulièrement intéressant pour l'exploration du régime BFKL. Dans ce cas, nous n'avons pas trouvé la solution générale pour la matrice de collage, mais nous avons trouvé la sous-classe de solutions qui semble s'appliquer à nos quantités d'intérêt. Nous suivons principalement l'article original [17], mais la discussion de la matrice de collage et de l'extension aux nombres quantiques non entiers est nouvelle.

6.2.1 Contraindre la matrice de collage

Dans la présente Subsection, nous allons résoudre l'ensemble des équations pour les éléments de la matrice de collage à partir des conditions de collage

$$\tilde{\mathbf{Q}}^i(u) = M^{ij}(u)\bar{\mathbf{Q}}_j \ , \quad \tilde{\mathbf{Q}}_i(u) = (M^{-t})_{ij}\bar{\mathbf{Q}}^j \ . \tag{6.7}$$

Rappelons brièvement le cadre QSC que nous utilisons, rappelons-nous les contraintes sur les matrices de collage connues à ce jour. Les matrices non dégénérées $M^{ij}(u)$ et $L^{ij}(u)$ satisfont à l'ensemble de contraintes suivant

$$\begin{aligned} &\bar{M}^{ji}(u) = M^{ij}(u) \ , \quad M^{ij}(u)\Omega^k_j(u) = -M^{kj}(u)\Omega^i_j(u) \ , \quad \left(\Omega^{-1}\right)^j_i(u) = \bar{\Omega}^j_i(u) \ , \\ &L^{il}(u) = M^{ij}(u)\Omega^k_j(u)\Theta^l_k(-u) \ , \quad L^{li}(-u) = L^{il}(u) \ , \quad \left(\Theta^{-1}\right)^k_j(u) = \Theta^k_j(-u) \ . \end{aligned} \tag{6.8}$$

Nous pouvons maintenant considérer la matrice de collage pour les cas où AdS spins S_1 et S_2 résolvent l'ensemble des contraintes (6.8).

Entier S_1 et S_2

Commençons notre étude à partir de la situation où tous les frais sauf la dimension Δ sont des nombres entiers. Plus précisément, dans la présente Subsection, nous traitons le cas où les spins S_1 et S_2 ont la même parité. Ceci est motivé par le fait que pour $S_2 = 0$, les états physiques ont même S_1[3]. Pour analyser les contraintes (6.8) de plus près, nous devons trouver les propriétés des matrices $\Omega^j_i(u)$ et $\Theta^j_i(u)$.

[3]Cela est dû à la contrainte de cyclicité sur les états de $\mathfrak{sl}(2)$ Heisenberg chaîne de spin, qui est uniquement compatible avec la distribution symétrique des racines menant à S_1 pair.

Tout d'abord, nous considérons la matrice $\Omega_i^j(u)$ et rappelons la conjugaison complexe des **Q**-fonctions. Si Re u tend vers $+\infty$, alors, comme $\bar{\mathbf{Q}}_i \simeq \bar{B}_i u^{\hat{M}_i-1}$, le élément diagonal $(\Omega_+^{(0)})_i^i$ est égal à $\bar{B}_i/B_i$. Mais si Re u tend vers $-\infty$, la situation est un peu plus subtile. Les fonctions $\mathbf{Q}_i(u)$ ont une échelle infinie de raccourcis descendant de l'axe réel, tandis que les fonctions $\bar{\mathbf{Q}}_i(u)$ ont la même échelle de coupure. Puis en prenant la limite de Re u pour $-\infty$, nous devons aller dans $-\infty$ le long du demi-cercle dans le UHP pour $\mathbf{Q}_i(u)$, soit $\mathbf{Q}_i(u) \simeq B_i e^{i\pi(\hat{M}_i-1)}(-u)^{\hat{M}_i-1}$ et le long du demi-cercle dans le LHP pour $\bar{\mathbf{Q}}_i(u)$, c'est-à-dire $\bar{\mathbf{Q}}_i(u) \simeq \bar{B}_i e^{-i\pi(\hat{M}_i-1)}(-u)^{\hat{M}_i-1}$, nous voyons donc que l'élément diagonal de $(\Omega_-^{(0)})_i^i$ est égal à $\bar{B}_i/B_i e^{-2i\pi\hat{M}_i}$. En résumé, nous obtenons

$$\Omega_i^j(u) = \begin{cases} \delta_i^j e^{2i\phi_{B_j}} + (\Omega_+^{(1)})_i^j e^{-2\pi u} + \mathcal{O}\left(e^{-4\pi u}\right), & \text{Re } u \gg 1 , \\ \delta_i^j e^{2i\phi_{B_j} - 2i\pi\hat{M}_j} + (\Omega_-^{(1)})_i^j e^{2\pi u} + \mathcal{O}\left(e^{4\pi u}\right), & \text{Re } u \ll -1 , \end{cases} \tag{6.9}$$

où $e^{2i\phi_{B_i}} = \bar{B}_i/B_i$.

Deuxièmement, l'analyse de la matrice $\Theta_i^j(u)$ de (6.8) est analogue. Ainsi, en appliquant les arguments du paragraphe précédent, on voit que chez Re u tendant vers $+\infty$ on doit faire le tour du demi-cercle dans UHP et $\mathbf{Q}_i(-u) \simeq B_i e^{i\pi(\hat{M}_i-1)} u^{\hat{M}_i-1}$ à Re u tendant à $-\infty$ nous avons $\mathbf{Q}_i(-u) \simeq B_i(-u)^{\hat{M}_i-1}$. Alors on obtient

$$\Theta_i^j(u) = \begin{cases} -\delta_i^j e^{i\pi\hat{M}_j} + (\Theta_+^{(1)})_i^j e^{-2\pi u} + \mathcal{O}\left(e^{-4\pi u}\right), & \text{Re } u \gg 1 , \\ -\delta_i^j e^{-i\pi\hat{M}_j} + (\Theta_-^{(1)})_i^j e^{2\pi u} + \mathcal{O}\left(e^{4\pi u}\right), & \text{Re } u \ll -1 . \end{cases} \tag{6.10}$$

Dans le cas de spins entiers S_1 et S_2 de même parité, on obtient la matrice de collage suivante

$$M^{ij} = \begin{pmatrix} 0 & M^{12} & 0 & 0 \\ \bar{M}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & M^{34} \\ 0 & 0 & \bar{M}^{34} & 0 \end{pmatrix} . \tag{6.11}$$

Nous sommes capables de fixer les phases des éléments matriciels non nuls de (4.80)

$$M^{12} = \left|M^{12}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_1}-\phi_{B_2}\right)} , \quad M^{34} = \left|M^{34}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_3}-\phi_{B_4}\right)} . \tag{6.12}$$

Commençons maintenant la prise en compte du cas où au moins un des spins n'est pas un nombre entier car cela est particulièrement intéressant pour la limite BFKL.

Non entier S_1 et S_2

Tout d'abord, à partir de l'asymptotique des **Q**-fonctions, nous voyons immédiatement que si au moins une des charges S_1 ou S_2 est non entière, alors ne pas violer la pureté des extensions asymptotiques du **Q**-fonctions les matrices $(\Omega_\pm^{(0)})_i^j$ et $(\Theta_\pm^{(0)})_i^j$ ne peut pas mélanger différentes **Q**-fonctions et être en diagonale. Cependant, comme S_1 ou S_2 ou les deux tours ne sont pas des nombres entiers, toutes les différences $\hat{M}_i - \hat{M}_j$ sont en général non entières, donc nous concluons que $M^{ij} = 0$. Ensuite, il faut modifier l'ansatz pour $M^{ij}(u)$.

La matrice $M^{ij}(u)$ est analytique et i-periodique, donc le choix minimal serait d'ajouter les termes proportionnels à $e^{2\pi u}$ et $e^{-2\pi u}$

$$M^{ij}(u) = M_1^{ij} + M_2^{ij} e^{2\pi u} + M_3^{ij} e^{-2\pi u} \tag{6.13}$$

et ceci est cohérent avec ce que nous savons de la considération de la limite de BFKL pour laquelle S_1 approche -1 et $S_2 = 0$ (voir [82]). D'après les conditions précédentes (6.8), les matrices $M_{1,2,3}^{ij}$ sont hermitiennes.

Résumant les résultats des contraintes que nous obtenons[4]

$$\begin{aligned} M = &\begin{pmatrix} M_1^{11} & M_1^{12} & M_1^{13} & M_1^{14} \\ \bar{M}_1^{12} & 0 & 0 & 0 \\ \bar{M}_1^{13} & 0 & M_1^{33} & M_1^{34} \\ \bar{M}_1^{14} & 0 & \bar{M}_1^{34} & M_1^{44} \end{pmatrix} + \\ &+ \begin{pmatrix} 0 & 0 & M_2^{13} & M_2^{14} \\ 0 & 0 & 0 & 0 \\ \bar{M}_2^{13} & 0 & 0 & 0 \\ \bar{M}_2^{14} & 0 & 0 & 0 \end{pmatrix} e^{2\pi u} + \begin{pmatrix} 0 & 0 & M_3^{13} & M_3^{14} \\ 0 & 0 & 0 & 0 \\ \bar{M}_3^{13} & 0 & 0 & 0 \\ \bar{M}_3^{14} & 0 & 0 & 0 \end{pmatrix} e^{-2\pi u} , \end{aligned} \tag{6.14}$$

où les éléments des matrices $M_{2,3}^{ij}$ sont soumis aux contraintes (6.8). Il suffit d'écrire uniquement les phases des éléments matriciels non nuls de la matrice M_2^{ij} extraits de (6.8)

$$M_2^{13} = \left|M_2^{13}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_1}-\phi_{B_3}\right)} , \quad M_2^{14} = \left|M_2^{14}\right| e^{i\left(\pm\frac{\pi}{2}+\phi_{B_1}-\phi_{B_4}\right)} . \tag{6.15}$$

Signalons que la construction présentée ci-dessus fournira une suite analytique à toutes les valeurs de S_2 à partir des valeurs entières $S_2 \geq 0$. Cependant, cette suite analytique décompose la symétrie $S_2 \to -S_2$, qui est naïvement présente dans le QSC, comme on peut le voir dans la relation asymptotique des **Q**-fonctions. La suite analytique, qui décrit un entier parfaitement positif S_2, produira des pôles à nombre entier négatif S_2. Cela peut paraître un peu déroutant, mais la résolution de ce paradoxe réside dans l'existence de la seconde solution pour la matrice de mixage, obtenue en réétiquetant les indices conformément à $S_2 \to -S_2$. En pratique, le résultat doit être égal à S_2 et il suffit de considérer $S_2 \geq 0$, il suffit donc d'utiliser la matrice de mixage présentée ci-dessus.

6.3 Solution numérique

Les équations de QSC sont particulièrement bien adaptées à l'analyse numérique: les propriétés analytiques simples des **P**-fonctions permettent de les paramétrer en termes de séries de Laurent tronquées puis de contraindre ces coefficients par la condition de collage. Des algorithmes numériques pour résoudre les équations QSC ont été développés et appliqués dans [32, 84, 87, 88]. Dans un cas non symétrique, tel que BFKL avec $S_2 = n \neq 0$, la procédure doit être modifiée de manière à être décrite ici. Nous avons joint un cahier `Mathematica` nommé `code_for_arxiv.nb` implémentant l'algorithme, que nous avons utilisé pour obtenir les résultats décrits dans cette Section.

[4]Dans le cas où le spin S_2 est un entier, la matrice de collage simplifie et nous avons $M^{14}(u) = M^{44}(u) = 0$ comme nous le verrons dans la section 6.3 décrivant l'algorithme numérique QSC appliqué.

En utilisant l'algorithme numérique proposé, nous avons réussi à calculer plusieurs quantités numériques pour les cas où n est non nul et même non entier. Sur la Figure 4.3, on peut trouver la trajectoire de l'opérateur longueur-2 pour $n = 1$.

Il est également possible de calculer numériquement la dépendance du spin S sur la constante de couplage g pour la dimension fixe Δ. Sur la Figure 4.4, vous pouvez voir la dépendance $S(g)$ pour $\Delta = 0.45$ et $n = 1$ en comparaison avec le même résultat calculé de manière perturbante comme la somme de LO et NLO Valeurs propres BFKL.

De plus, ce schéma numérique nous permet de comparer les valeurs numériques des eigenvalues du noyau BFKL avec les eigenvalues perturbatives connues des ordres LO et NLO. Dans la Table 4.1, les valeurs numériques de l'eigenvalue du noyau BFKL ajustées à partir des tracés de la Figure 4.4 sont écrites dans les quatre premiers ordres avec des résultats perturbatifs dans les deux premières commandes calculées pour $n = 1$ et $\Delta = 0,45$. Dans les ordres LO, NLO et NNLO, nous observons l'accord avec une précision de 22, 20 et 16 chiffres respectivement.

Désormais, concentrons-nous sur le calcul numérique de la fonction d'intercept. Sur la figure 4.5, on peut trouver les dépendances de l'intercept sur la constante de couplage g pour les différentes valeurs de spin conforme n. Les lignes pointillées sont tracées en fonction de la fonction d'intercept de la section 6.4 calculée dans le petit régime de couplage. Les lignes continues correspondent à l'expansion du couplage fort de la fonction d'intercept de la Section 6.6, ajustée à partir des données numériques obtenues dans la présente Section.

Dans la section suivante, nous allons analyser l'expansion du couplage faible de la fonction d'intercept. Pour ce faire, nous appliquons la méthode itérative appliquée dans [31].

6.4 Expansion du couplage faible

Dans cette section, nous explorons la fonction $S(\Delta, n)$ perturbativement au couplage faible pour un spin conforme à un nombre entier arbitraire n. En particulier, nous nous intéressons à l'intercept BFKL $j(n) = S(0, n) + 1$. Le calcul de cette quantité comprend deux étapes.

6.4.1 Limite BFKL du QSC pour les opérateurs de twist-2 avec des valeurs non nulles $S_2 = n$

Dans cette Subsection, nous considérons la limite de BFKL du QSC avec un spin conforme non nul. Rappelons d'abord brièvement quelle est la limite de BFKL. Nous allons étudier le régime en même temps que la constante de couplage $g \to 0$ et l'un des spins $S_1 = S \to -1$ tout en gardant le rapport $g^2/(S + 1)$ fini. LO BFKL dans cette limite correspond à la reprise de toutes les contributions de la forme $(g^2/(S + 1))^k$, NLO BFKL – aux contributions de la forme $(S + 1)(g^2/(S + 1))^k$ et ainsi de suite. Cependant, dans la présente Subsection, nous prenons le deuxième tour (spin conforme) $S_2 = n \neq 0$ et il apparaît des différences par rapport au régime BFKL avec un spin conforme nul.

LO solution

Dans la présente partie, nous trouverons la solution LO du $\mathbf{P}\mu$-système dans le régime BFKL. Tenant compte du fait que $\tilde{\mathbf{P}}$ est égal à $1/w^2$, dans le LO de w, on obtient pour les $\mathbf{P}$-fonctions

$$\begin{aligned}
&\mathbf{P}_1^{(0)} = \frac{1}{u^2}\ , \quad \mathbf{P}_2^{(0)} = \frac{1}{u}\ , \quad \mathbf{P}_3^{(0)} = A_3^{(0)}\ , \quad \mathbf{P}_4^{(0)} = A_4^{(0)} u + \frac{c_{4,1}^{(1)}}{\Lambda u}\ , \\
&\mathbf{P}^{(0)1} = A^{1(0)} u + \frac{c^{1,1(1)}}{\Lambda u}\ , \quad \mathbf{P}^{(0)2} = A^{2(0)}\ , \quad \mathbf{P}^{(0)3} = -\frac{1}{u}\ , \quad \mathbf{P}^{(0)4} = \frac{1}{u^2}\ ,
\end{aligned} \tag{6.16}$$

où nous avons impliqué $c_{4,1} = \frac{1}{(\Lambda w)^{3/2}}\left(c_{4,1}^{(1)} w + \mathcal{O}(w^2)\right)$, $c^{1,1} = \frac{1}{(\Lambda w)^{3/2}}\left(c^{1,1(1)} w + \mathcal{O}(w^2)\right)$ et

$$A_3^{(0)} = -\frac{i\left((\Delta-n)^2-1\right)\left((\Delta+n)^2-9\right)}{32}\ ,\ A_4^{(0)} = -\frac{i\left((\Delta+n)^2-1\right)\left((\Delta-n)^2-25\right)}{96}\ . \tag{6.17}$$

Examinons maintenant les μ-fonctions au voisinage du point $S=-1$. Dans le LO, quand $S=-1$, μ -fonctions avec les indices inférieurs à $u\to\infty$ ont les asymptotiques

$$(\mu_{12},\mu_{13},\mu_{14},\mu_{23},\mu_{24},\mu_{34}) \sim (u^0,u^1,u^2,u^2,u^3,u^4) e^{2\pi|u|}\ . \tag{6.18}$$

Pour trouver les μ-fonctions avec les indices les plus bas de la variable LO de manière analogue au cas symétrique gauche-droite, nous remarquons que $\mathbf{P}_{1,2}^{(0)}$ et $\mathbf{P}^{(0)3,4}$ sont singuliers, tandis que $\tilde{\mathbf{P}}_{1,2}$ et $\tilde{\mathbf{P}}^{3,4}$ sont réguliers à $u=0$. Cela peut être garanti si $\mu_{ab}^{(0)}$ sont réguliers à $u=0$ et ont un zéro de commande suffisante à ce stade. Par conséquent, comme dans le cas symétrique gauche-droite, nous pouvons supposer que dans la LO, la seule solution contribuant à $\mu_{ab}^{(0)}$ est celle avec l'asymptotique (6.18). Par conséquent, nous conjecturons $\mu_{ab}^{(0)}$ pour avoir la forme d'un polynôme multiplié par un facteur i. Comme pour les opérateurs de longueur-2, les fonctions μ_{ab}^{+} ont la certaine parité, on est conduit à ansatz

$$\begin{aligned}
&\mu_1^{(0)+} = \mathcal{P}(u) b_{1,1}\ , \quad \mu_2^{(0)+} = \mathcal{P}(u) b_{2,1} u\ , \\
&\mu_3^{(0)+} = \mathcal{P}(u)(b_{3,1}u^2 + b_{3,2})\ , \quad \mu_4^{(0)+} = \mathcal{P}(u)(b_{4,1}u^2 + b_{4,2})\ , \\
&\mu_5^{(0)+} = \mathcal{P}(u)(b_{5,1}u^3 + b_{5,2}u)\ , \quad \mu_6^{(0)+} = \mathcal{P}(u)(b_{6,1}u^4 + b_{6,2}u^2 + b_{6,3})\ .
\end{aligned} \tag{6.19}$$

où $\mathcal{P}(u)$ est une fonction i-periodic. Déterminons d'abord la partie polynomiale de la solution. En connectant les ansatz (6.19) aux équations $\mu_{ab}^{++} = \mu_{ab} + \mathbf{P}_a \mu_{bc} \tilde{\mathbf{P}}^c - \mathbf{P}_b \mu_{ac} \tilde{\mathbf{P}}^c$ nous fixons tous les coefficients en dehors de $b_{1,1}$ et $c^{1,1(1)} = -c_{4,1}^{(1)}$. Cette dernière égalité entraîne automatiquement la validité de l'exigence

$$\mathbf{P}_a^{(0)} \mathbf{P}^{(0)a} = 0 \tag{6.20}$$

dans le LO.

Revenons maintenant à la solution générale. Parce que dans le premier ordre, les équations pour μ sont homogènes, la solution est déterminée jusqu'à une fonction périodique i. Cette fonction périodique ne pousse pas plus vite que $e^{2\pi|u|}$, alors la réponse la plus générale pour elle est

$$B_{1,1} e^{2\pi u} + B_{1,2} e^{-2\pi u} + B_{1,3}\ . \tag{6.21}$$

Nous devons également nous souvenir des exigences de l'analytique, c'est-à-dire que les expressions suivantes ne comportent pas de coupes

$$\mathbf{P}+\tilde{\mathbf{P}}\ ,\quad \frac{\mathbf{P}-\tilde{\mathbf{P}}}{\sqrt{u^2-4\Lambda w}}\ ,\quad \mu+\tilde{\mu}\ ,\quad \frac{\mu-\tilde{\mu}}{\sqrt{u^2-4\Lambda w}}\ . \tag{6.22}$$

Les contraintes (6.22) nous amènent aux résultats

$$B_{1,2}=B_{1,1}\ ,\quad B_{1,3}=-2B_{1,1}\ ,\quad c^{(1)}_{4,1}=-\frac{i\Lambda}{96}\left(\left(\Delta^2-1\right)^2-2\left(\Delta^2+1\right)n^2+n^4\right)\ . \tag{6.23}$$

Avec ces conditions, (6.23) l'exigence de parité pour μ^+ est automatiquement satisfaite.

Résumant les résultats obtenus et définissant $B_{1,1}=-B_1/4$, toutes les exigences nous donnent la solution unique (jusqu'à une constante multiplicative) donnée par

$$\begin{aligned}
\mu^{(0)+}_{12}&=B_1\cosh^2(\pi u)\ , \qquad (6.24)\\
\mu^{(0)+}_{13}&=-B_1\frac{i}{16}\left(\left(\Delta^2-1\right)^2-2\left(\Delta^2+1\right)n^2+n^4\right)u\cosh^2(\pi u)\ ,\\
\mu^{(0)+}_{14}&=\mu^{(0)+}_{23}=-B_1\frac{i}{128}\left(\left(\Delta^2-1\right)^2-2\left(\Delta^2+1\right)n^2+n^4\right)\left(4u^2+1\right)\cosh^2(\pi u)\ ,\\
\mu^{(0)+}_{24}&=-B_1\frac{i}{192}\left(\left(\Delta^2-1\right)^2-2\left(\Delta^2+1\right)n^2+n^4\right)u\left(4u^2+1\right)\cosh^2(\pi u)\ ,\\
\mu^{(0)+}_{34}&=-B_1\frac{1}{49152}\left(\left(\Delta^2-1\right)^2-2\left(\Delta^2+1\right)n^2+n^4\right)^2\left(4u^2-3\right)\left(4u^2+1\right)\cosh^2(\pi u)\ .
\end{aligned}$$

Ayant obtenu la solution LO donnée par (6.16), (6.24) et (6.17) nous pouvons procéder à la recherche de la solution NLO du $\mathbf{P}\mu$-système avec le coefficient B_1, ce qui sera fait dans la prochaine Subsection.

Conditions de collage

Revenons maintenant aux conditions de collage du spin conforme à l'entier et notons deux d'entre elles contenant uniquement les $\mathbf{Q}$-fonctions dans le paramètre de mise à l'échelle w

$$\begin{aligned}
\tilde{\mathbf{Q}}^2&=\bar{M}^{(0)12}_1\bar{\mathbf{Q}}_1\ , \qquad (6.25)\\
\tilde{\mathbf{Q}}^4&=\bar{M}^{(0)34}_1\bar{\mathbf{Q}}_3\ .
\end{aligned}$$

Pour trouver $M^{(0)12}_1$ et $M^{(0)34}_1$ on peut utiliser la continuité des fonctions $\mathbf{Q}^2$ et $\mathbf{Q}^4$ sur la coupe $\tilde{\mathbf{Q}}^2(0)=\mathbf{Q}^2(0)$ et $\tilde{\mathbf{Q}}^4(0)=\mathbf{Q}^4(0)$. Le résultat est

$$M^{(0)12}_1=M^{(0)34}_1=\frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1}\ . \tag{6.26}$$

En ce qui concerne les $\mathbf{Q}$-fonctions avec l'asymptotique pur, nous obtenons les conditions de collage suivantes dans le LO

$$\tilde{\mathbf{Q}}^{(0)2,4}(u)=\frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1}\bar{\mathbf{Q}}^{(0)}_{1,3}(u)\ . \tag{6.27}$$

LO BFKL eigenvalue

Nous allons ici obtenir l'eigenvalue de LO BFKL de manière similaire à [82]. Pour commencer, notons l'équation de Baxter pour $\mathbf{Q}^{2,4}$ dans le NLO

$$\mathbf{Q}^{(1)2,4++} + \mathbf{Q}^{(1)2,4--} + \left(-2 + \frac{(\Delta+n)^2-1}{4u^2}\right)\mathbf{Q}^{(1)2,4} = \\ = -\frac{i}{2(u+i)}\mathbf{Q}^{(0)2,4++} + \frac{i}{2(u-i)}\mathbf{Q}^{(0)2,4--} + \frac{u^2-\Lambda(\Delta+n)^2-1)}{2u^4}\mathbf{Q}^{(0)2,4} \ . \quad (6.28)$$

D'un côté de l'équation de Baxter (6.28) il s'ensuit que

$$\frac{\mathbf{Q}^{(1)j}(u)}{\mathbf{Q}^{(0)j}(u)} = \frac{i}{2u} + \mathcal{O}(u^0) \ , \quad j = 2,4 \ . \quad (6.29)$$

De l'autre côté, on peut appliquer l'astuce suivante pour trouver la partie singulière de $\mathbf{Q}^j$ dans le NLO

$$\mathbf{Q}^j = \frac{\mathbf{Q}^j + \tilde{\mathbf{Q}}^j}{2} + \frac{\mathbf{Q}^j - \tilde{\mathbf{Q}}^j}{2\sqrt{u^2-4\Lambda w}}\sqrt{u^2-4\Lambda w} \ . \quad (6.30)$$

Pour $\mathbf{Q}^2$ on obtient

$$\frac{\mathbf{Q}^2 - \tilde{\mathbf{Q}}^2}{2\sqrt{u^2-4\Lambda w}} = \frac{1}{2u}\left(\mathbf{Q}^{(0)2}(u) - \frac{\cos\frac{\pi(\Delta+n)}{2}}{\cos\frac{\pi(\Delta-n)}{2}}\frac{(\Delta-n)^2-1}{(\Delta+n)^2-1}\bar{\mathbf{Q}}_1^{(0)}(u)\right) + \mathcal{O}(w) \ . \quad (6.31)$$

En combinant (6.31) et les résultats précédemment obtenus pour les solutions de Baxter (6.28), on obtient

$$\mathbf{Q}^{(1)2}(u) = \left(-\frac{i\mathbf{Q}^{(0)2}(0)(\Psi(\Delta+n)+\Psi(\Delta-n))\Lambda}{u} + \mathcal{O}(u^0)\right)w + \mathcal{O}(w^2) \ , \quad (6.32)$$

où

$$\Psi(\Delta) \equiv \psi\left(\frac{1}{2} - \frac{\Delta}{2}\right) + \psi\left(\frac{1}{2} + \frac{\Delta}{2}\right) - 2\psi(1) \ . \quad (6.33)$$

Ainsi, en comparant deux résultats indépendants (6.29) et (6.32), nous avons la relation

$$-2(\Psi(\Delta+n)+\Psi(\Delta-n))\Lambda = 1 \ . \quad (6.34)$$

Après quelques calculs, on obtient pour l'entier n

$$\frac{1}{4\Lambda} = \frac{1}{2}\left(\Psi(\Delta+n)+\Psi(\Delta-n)\right) + \mathcal{O}(w) = \\ = -\psi\left(\frac{1+n-\Delta}{2}\right) - \psi\left(\frac{1+n+\Delta}{2}\right) + 2\psi(1) + \mathcal{O}(w) \ . \quad (6.35)$$

En réécrivant le résultat en fonction des paramètres d'expansion habituels, nous obtenons la valeur propre du noyau LO BFKL bien connue pour un spin conforme à nombre entier non nul n [4]

$$S = -1 - 4g^2\left(\psi\left(\frac{1+n-\Delta}{2}\right) + \psi\left(\frac{1+n+\Delta}{2}\right) - 2\psi(1)\right) + \mathcal{O}(g^4) \ . \quad (6.36)$$

6.4.2 Multiloop expansion de la fonction d'intercept pour un spin conforme arbitraire

En utilisant la procédure itérative, nous avons calculé l'expansion de l'intercept de l'eigenvalue de BFKL pour n et $n = 91$ dans la limite de couplage faible jusqu'à l'ordre de g^8 (NNNLO). Ces données sont précieuses en elles-mêmes, car elles peuvent servir de test pour de futurs calculs d'ordre supérieur ou non-perturbatif. Ce qui est plus important, cependant, c'est que cela nous a permis de trouver NNLO et les eigenvalues de NNNLO BFKL partiellement comme une fonction du spin conforme n.

Nous commençons par remarquer que les intercepts de LO et NLO BFKL peuvent être représentées comme une combinaison linéaire de sommes harmoniques imbriquées de transcendance globale. En effet, les eigenvalues BFKL Pomeron LO et NLO peuvent être exprimées (voir, par exemple, [2]) à travers les sommes des harmoniques imbriquées décrites, par exemple, dans [48]

$$S_{a_1,a_2,\ldots,a_n}(x) = \sum_{y=1}^{x} \frac{\mathrm{sign}(a_1)^y}{y^{|a_1|}} S_{a_2,\ldots,a_n}(y) \;, \quad S_\emptyset(x) = 1 \;, \tag{6.37}$$

où x est un entier positif. On peut prendre $\Delta = 0$ dans ces eigenvalues, ce qui après une algèbre simple donne

$$j_{LO}(n) = 8S_1\left(\frac{n-1}{2}\right) \;, \tag{6.38}$$

$$j_{NLO}(n) = 4S_3\left(\frac{n-1}{2}\right) + 4S_{-3}\left(\frac{n-1}{2}\right) - 8S_{-2,1}\left(\frac{n-1}{2}\right) + \frac{2\pi^2}{3} S_1\left(\frac{n-1}{2}\right) \;.$$

On calcule ici et ci-dessous la transcendentalité comme suit: on suppose que la transcendance des produits est égale à la somme des transcendantalités des facteurs et que la transcendance des nombres est 0. Transcendance des $\log 2$ est 1 et transcendance des ζ_k est k. Puisque ζ_k pour meme k est proportionnel á π^2, il est facile de voir que la transcendantalité de π est 1.

Comme on le voit, l'argument de toutes les sommes harmoniques dans (6.38) est $(n-1)/2$. Cela conduit à une idée d'essayer de trouver des intercepts NNLO et NNNLO comme des combinaisons linéaires analogues de sommes harmoniques avec des coefficients transcendants de transcendance globale uniforme. Les coefficients de la combinaison linéaire peuvent être contraints en utilisant les données générées par la procédure itérative. Mais le nombre de sommes harmoniques de certaines transcendentalités augmente rapidement au fur et à mesure que la transcendentalité augmente. Heureusement, on peut réduire considérablement le nombre de sommes harmoniques dans les ansatz en conjecturant une certaine propriété du résultat que nous appelons la réciprocité.

Tout cela est directement applicable à notre cas et nous sommes en mesure de formuler un ansatz pour la fonction d'intercept NNLO. A partir des expressions LO et NLO (6.38), nous voyons que leurs extensions asymptotiques à grande échelle n sont même dans n. Puisque nous utilisons les sommes harmoniques de l'argument $M = (n-1)/2$, nous ne devons garder que l'invariant des sommes harmoniques sous la transformation $M \to -1 - M$ ou $n \to -n$ dans nos notations. Ce sont exactement les sommes binomiales

$$\mathbb{S}_{i_1,\ldots,i_k}(M) = (-1)^M \sum_{j=1}^{M} (-1)^j \left(\begin{array}{c} M \\ j \end{array}\right) \left(\begin{array}{c} M+j \\ j \end{array}\right) S_{i_1,\ldots,i_k}(j) \;. \tag{6.39}$$

Les expressions (6.38) pour les intercepts LO et NLO peuvent être facilement exprimées à travers elles

$$j_{LO} = 4\mathbb{S}_1 \,, \tag{6.40}$$
$$j_{NLO} = 8\left(\mathbb{S}_{2,1} + \mathbb{S}_3\right) + \frac{4\pi^2}{3}\mathbb{S}_1 \,,$$

où les arguments des sommes sont à nouveau $(n-1)/2$.

Pour trouver l'intercept NNLO, nous faisons un ansatz sous la forme d'une combinaison linéaire de sommes d'harmoniques binomiales avec des coefficients transcendants. Le principe de transcendentalité maximale, formulé par L.N. Lipatov et A.V. Kotikov [1, 2], vaut également pour l'intercept: chaque terme de la somme doit être de la transcendance totale 5.

En assimilant les coefficients devant chaque produit unique de constantes transcendantales dans ces deux expressions, on obtient un système linéaire pour les coefficients rationnels des ansatz. En le résolvant on obtient une expression étonnamment simple

$$j_{NNLO} = 32\left(\mathbb{S}_{1,4} - \mathbb{S}_{3,2} - \mathbb{S}_{1,2,2} - \mathbb{S}_{2,2,1} - 2\mathbb{S}_{2,3}\right) - \frac{16\pi^2}{3}\mathbb{S}_3 - \frac{32\pi^4}{45}\mathbb{S}_1 \,. \tag{6.41}$$

Le résultat (6.41) pour la fonction d'intercept pour n arbitraire peut être comparé aux autres quantités connues. Le premier d'entre eux est la valeur propre de NNLO BFKL Pomeron pour le spin conforme $n = 0$ calculé dans [31]. En prenant cette valeur propre $\Delta = 0$ et en la comparant avec (6.41) pour $n = 0$, nous voyons un accord parfait. Deuxièmement, pour les spins conformes non nuls, les formules pour les trajectoires de Pomeron ont été trouvées dans [58], à partir duquel nous pouvons extraire l'intercept pour n donné. Nous avons également vérifié que le résultat de ce travail coïncide avec notre résultat (6.41) pour plusieurs premiers spins conformes non négatifs n, ce qui représente une confirmation indépendante de l'exactitude de notre calcul.

Limitons-nous aux valeurs du spin conforme dans nos données égales à $n = 4k + 1$ avec $k = 0, 1, \ldots, 22$. Dans ces points, les valeurs des fonctions d'intercept sont données par

$$j_{NNNLO}(4k+1) = \pi^2 j^{\pi^2}_{NNNLO} + \pi^4 j^{\pi^4}_{NNNLO} + \pi^6 j^{\pi^6}_{NNNLO} + \pi^2\zeta_3 j^{\pi^2\zeta_3}_{NNNLO} + \\ + \zeta_3 j^{\zeta_3}_{NNNLO} + \zeta_5 j^{\zeta_5}_{NNNLO} + j^{\text{rat.}}_{NNNLO} \,, \tag{6.42}$$

où tous les coefficients devant les constantes transcendantales sur le RHS de (6.42) sont des fonctions rationnelles de k. Chaque coefficient du RHS de (6.42) est supposé être une combinaison linéaire avec des coefficients rationnels des sommes harmoniques binomiales avec la transcendance, complétant la transcendance des coefficients correspondants à 7. Nous avons pu adapter toutes les contributions sauf $j^{\zeta_3}_{NNNLO}$ et $j^{\text{rat.}}_{NNNLO}$, qui, comme les autres sommes harmoniques, prennent des valeurs rationnelles au points $n = 4k + 1$ pour entier $k \geq 0$. Cependant, nous avons constaté que le terme $j^{\zeta_3}_{NNNLO}$ ne peut pas être utilisé avec les ansatz qui consistent en des sommes harmoniques binomiales (6.39). Cela nous a motivés à essayer d'adapter cette contribution aux sommes harmoniques imbriquées (6.37). Cela semblait être le cas et nous avons réussi à adapter cette partie aux sommes harmoniques ordinaires, ce qui signifie que la réciprocité, à savoir la

symétrie $n \to -n$ dans l'expansion asymptotique, est violée. Pour la dernière contribution rationnelle $j^{\text{rat.}}_{NNNLO}$, Nous avons également constaté qu'elle n'était pas décrite par les sommes harmoniques binomiales. Malheureusement, le fait d'adapter cette contribution aux sommes harmoniques ordinaires ne nous a pas conduit à corriger complètement cette contribution en raison du manque de données. Par conséquent, en combinant les résultats obtenus, nous écrivons la partie non rationnelle de la réponse pour les points $n = 4k + 1$, qui est la somme des termes de le RHS de (6.42), à l'exception de $j^{\text{rat.}}_{NNNLO}$

$$\begin{aligned} j^{\text{non-rat.}}_{NNNLO}(4k+1) = &-\frac{32\pi^2}{3}\left(3\mathbb{S}_{1,4} - 3\mathbb{S}_{2,3} - \mathbb{S}_{3,2} + \mathbb{S}_{1,1,3} - 2\mathbb{S}_{1,2,2} + \mathbb{S}_{2,2,1} - \mathbb{S}_{3,1,1}\right) + \\ &+ \frac{16\pi^4}{15}\left(4\mathbb{S}_3 - \mathbb{S}_{2,1}\right) + \frac{56\pi^6}{135}\mathbb{S}_1 + \frac{32\pi^2\zeta_3}{3}\mathbb{S}_{1,1} + 224\zeta_5\mathbb{S}_{1,1} - 128\zeta_3\left(S_{-3,1} + 2S_{-2,2} - 5S_{1,-3} - \right. \\ &- 15S_{1,3} - 4S_{2,-2} - 12S_{2,2} - 15S_{3,1} - 4S_{-2,1,1} + 2S_{1,-2,1} + 8S_{1,1,-2} + 12S_{1,1,2} + \\ &\left. + 12S_{1,2,1} + 12S_{2,1,1} + S_{-4} + 9S_4\right) . \quad (6.43) \end{aligned}$$

On peut trouver les valeurs complètes de la fonction d'intercept NNNLO, y compris les termes rationnels pour les spins conformes $n = 4k + 1$ de 1 à 89 dans la soumission arXiv de l'article [62] dans le fichier `intercept_values_Nodd.mx`. Etant donné que la partie de la somme harmonique constituée des sommes harmoniques imbriquées de la transcendance 7, qui constitue la partie rationnelle aux points $n = 4k + 1$, nous ne pouvons pas écrire une expression pour le travail d'intercept NNNLO pour tous les spins conformes laissant cette tâche pour des études futures.

6.5 Près-BPS expansion de toutes les boucles

Dans cette Section, nous allons analyser les équations QSC proches du point BPS $\Delta = 0$, $S_1 = -1$ et $S_2 = 1$. Il apparaît qu'il est possible de calculer deux valeurs non-perturbatives sur ce point par les méthodes de QSC. Un autre point BPS $S_2 = 0$, $S_1 = 0$ a été analysé en détail dans [21, 22, 119]. Dans cette Section, nous suivons de près la méthode d'expansion près-BPS par [22].

6.5.1 Slope-to-intercept près du point BPS

La fonction d'intercept $j(n) = S(0, n) + 1$, où $S = S_1$ et $n = S_2$, joue un rôle particulièrement important dans les calculs de BFKL. Comme nous l'avons mentionné ci-dessus, le point $\Delta = 0$, $n = 1$ est BPS, ce qui signifie qu'il est fixé pour tout couplage 't Hooft. L'argument théorique du groupe expliquant ce phénomène devrait être basé sur la condition de raccourcissement. Du point de vue de QSC, les points BPS sont les points où $A^a A_a = 0$ simultanément pour tous les $a = 1, \ldots, 4$. Dans cette Subsection, nous étudions de petits écarts par rapport à ce point BPS et calculons la pente de $j(n)$ par rapport à n dans le point $n = 1$ dans tous les ordres de la constante de couplage g.

Résultat pour la slope-to-intercept fonction

De l'équation QSC, nous corrigeons θ, qui est maintenant égal à

$$\boxed{\theta(g) = 1 + \frac{I_1 I_2}{\sum\limits_{k=1}^{+\infty} (-1)^k I_k I_{k+1}}}\,, \tag{6.44}$$

et constitue notre résultat pour la slope-to-intercept fonction.

L'expansion du couplage faible du résultat obtenu (6.44) est donnée par

$$\theta(g) = -\frac{2\pi^2}{3} g^2 + \frac{4\pi^4}{9} g^4 - \frac{28\pi^6}{135} g^6 + \frac{8\pi^8}{405} g^8 + \mathcal{O}\left(g^{10}\right)\,. \tag{6.45}$$

Étant donné que la slope-to-intercept fonction par définition est la dérivée de la fonction d'intercept par rapport au spin conforme n à $n = 1$, nous pouvons comparer immédiatement les premiers coefficients de l'expansion du couplage faible (6.44) avec la dérivée de (6.38) et (6.41). Cela montrera également que ces expressions fournissent les formules compatibles avec notre continuation analytique dans n, loin des valeurs entières. Les dérivées de la fonction d'intercept (4.177) et (6.41) peuvent être calculées et nous trouvons qu'elles sont entièrement en accord avec (6.45)

$$\left.\frac{dj}{dn}\right|_{n=1} = -\frac{2\pi^2}{3} g^2 + \frac{4\pi^4}{9} g^4 - \frac{28\pi^6}{135} g^6 + \mathcal{O}\left(g^8\right)\,, \tag{6.46}$$

confirmant notre résultat (6.44).

Dans la Section suivante, nous calculons l'expansion du couplage fort de notre résultat pour la slope-to-intercept fonction. Comme nous le verrons, le calcul est moins simple qu'à couplage faible, même si le résultat est encore assez simple.

Expansion du couplage fort de la slope-to-intercept fonction

Nous obtenons la réponse suivante

$$\begin{aligned}\sum_{k=1}^{+\infty} \frac{(-1)^k I_k(4\pi g) I_{k+1}(4\pi g)}{I_1(4\pi g) I_2(4\pi g)} = \\ = -\frac{1}{2} - \frac{3}{4(4\pi g)} - \frac{3}{4(4\pi g)^2} - \frac{9}{32(4\pi g)^3} + \frac{9}{8(4\pi g)^4} + \frac{2331}{512(4\pi g)^5} + \mathcal{O}\left(\frac{1}{g^6}\right)\,.\end{aligned} \tag{6.47}$$

Ensuite, nous obtenons la forte expansion de couplage pour cela

$$\theta = -1 + \frac{3}{\lambda^{1/2}} - \frac{3}{2\lambda} - \frac{9}{8\lambda^{3/2}} - \frac{9}{4\lambda^2} - \frac{711}{128\lambda^{5/2}} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)\,. \tag{6.48}$$

Cette expansion (6.48) nous sera utile dans la Section 6.6 lorsque nous pourrons la comparer avec la dérivée de notre formule pour l'expansion du couplage fort de la fonction d'intercept pour un spin conforme arbitraire n pris au point $n = 1$, basé sur une analyse numérique intensive.

6.5.2 Fonction de courbure à proximité du point BPS

Comme il a été mentionné ci-dessus dans la Subsection 6.5.1, nous considérons l'expansion à proximité du point BPS $\Delta = 0$, $S_1 = -1$ et $S_2 = 1$. Dans la Subsection précédente, nous avons développé les puissances de $S_2 - 1$, cependant, pour trouver la courbure, nous conservons $S_2 = 1$ et développons les puissances de Δ. Le schéma de résolution des équations QSC dans ce cas est similaire mais avec une différence. Puisque la fonction $S(\Delta, 1)$ est une fonction paire de Δ, nous devons élargir l'ordre NLO dans Δ pour obtenir un résultat non-trivial.

Comme dans la section précédente, nous utiliserons une simplification de la fonction $\mathcal{Q}_{a|i}$ pour résoudre explicitement le Q-système.

Résultat de la fonction de courbure

Pour obtenir la fonction de courbure, il faut se rappeler que $c^{(1)}_{4,1}$ détermine le coefficient devant $1/x$ dans $\mathbf{P}^{(1)}_4$. Ensuite, nous trouvons la réponse

$$\gamma(g) = \frac{iH_0\left(\left(\Gamma\cdot v\cosh^v_-\right)(u) - u\left(\Gamma\cdot\cosh^v_-\right)(u)\right)}{2g^2 I_2} + \frac{iH_1\left(u\left(\Gamma\cdot v\cosh^v_-\right)(u) - \left(\Gamma\cdot v^2\cosh^v_-\right)(u)\right)}{4g^2 I_2} +$$

$$+ \frac{iH_2\left(\left(\Gamma\cdot v\cosh^v_-\right)(u) - u\left(\Gamma\cdot\cosh^v_-\right)(u)\right)}{4g^2 I_2} + \frac{iK_2\left(\cosh^u_-\left(\left(\Gamma\cdot v\cosh^v_-\right)(u) - u\left(\Gamma\cdot\cosh^v_-\right)(u)\right)\right)}{g^4 I_2}\,. \quad (6.49)$$

En élargissant les noyaux intégraux à grande échelle, nous pouvons réécrire la fonction de courbure sous une forme plus concise

$$\boxed{\begin{aligned}\gamma(g) = {}& \frac{1}{4\pi g^4 I_2^2}\oint_{-2g}^{2g} dv\left(v\cosh^v_-\left(\Gamma\cdot u\cosh^u_-\right)(v) - v^2\cosh^v_-\left(\Gamma\cdot\cosh^u_-\right)(v)\right)+\\ &+\frac{1}{16\pi g^5 I_2}\oint_{-2g}^{2g} dv\left(\frac{v^3\left(\Gamma\cdot\cosh^u_-\right)(v) - 2v^2\left(\Gamma\cdot u\cosh^u_-\right)(v) + v\left(\Gamma\cdot u^2\cosh^u_-\right)}{x_v - \frac{1}{x_v}}\right),\end{aligned}} \quad (6.50)$$

où

$$\left(\Gamma\cdot h(v)\right)(u) = \oint_{-2g}^{2g}\frac{dv}{2\pi i}\partial_u\log\frac{\Gamma\left[i(u-v)+1\right]}{\Gamma\left[-i(u-v)+1\right]}h(v)\,. \quad (6.51)$$

Après avoir obtenu (6.50), nous pouvons le comparer aux autres résultats connus. Dans les deux premiers ordres, nous connaissons les eigenvalues de BFKL Pomeron pour un spin conforme arbitraire incluant $n = 1$. Nous pouvons donc calculer la courbure à partir de ces eigenvalues dans ces deux premiers ordres. En comparant avec l'expansion de couplage faible de la formule (6.50) pour la fonction de courbure

$$\begin{aligned}\gamma(g) = {}& 2\zeta_3 g^2 + \left(-\frac{2\pi^2}{3}\zeta_3 - 35\zeta_5\right)g^4 + \left(\frac{16\pi^4}{45}\zeta_3 + \frac{22\pi^2}{3}\zeta_5 + 504\zeta_7\right)g^6 + \left(-\frac{28\pi^6}{135}\zeta_3 - \frac{8\pi^4}{3}\zeta_5 -\right.\\ &\left.-56\pi^2\zeta_7 - 6930\zeta_9\right)g^8 + \left(\frac{136\pi^8}{2835}\zeta_3 + \frac{668\pi^6}{189}\zeta_5 - \frac{112\pi^4}{3}\zeta_7 + 508\pi^2\zeta_9 + 93720\zeta_{11}\right)g^{10} +\\ &+\left(\frac{754\pi^{10}}{42525}\zeta_3 - \frac{1402\pi^8}{567}\zeta_5 - \frac{73\pi^6}{45}\zeta_7 + \frac{4618\pi^4}{3}\zeta_9 - 12969\pi^2\zeta_{11} - 1234233\zeta_{13}\right)g^{12} + O\left(g^{14}\right)\end{aligned} \quad (6.52)$$

Nous trouvons que les deux premiers termes de l'expansion (6.52) coïncident avec les valeurs obtenues à partir des eigenvalues de BFKL Pomeron, ce qui représente une vérification de notre résultat. Ayant calculé l'expansion du couplage faible (6.52), nous analysons dans la partie suivante l'autre limite intéressante, à savoir l'expansion du couplage fort, qui est une tâche de calcul distincte dans le cas de la fonction de courbure.

Expansion du couplage forte de la fonction de courbure

Pour le calcul de l'expansion du couplage fort de la fonction de courbure (6.50), nous utilisons la même méthode que celle utilisée dans [22].

L'application de la méthode numérique décrite ci-dessus nous donne la grande extension de g de la fonction de courbure

$$\gamma = \frac{1}{2\lambda^{1/2}} - \frac{1}{4\lambda} - \frac{33}{16\lambda^{3/2}} - \frac{81}{16\lambda^2} - \\ - \frac{2265}{256\lambda^{5/2}} + \left(\frac{1440\zeta_5}{64} - \frac{765}{64}\right)\frac{1}{\lambda^3} + \left(\frac{207360\zeta_5}{2048} - \frac{22545}{2048}\right)\frac{1}{\lambda^{7/2}} + \mathcal{O}\left(\frac{1}{\lambda^4}\right), \quad (6.53)$$

où λ est donné par $\lambda = (4\pi g)^2$. Avec la fonction de courbure (6.50) et sa expansion de couplage faible (6.52) la formule (6.53), contenant l'expansion de couplage fort de cette fonction, conclut la liste des résultats de la présente Section.

6.6 Fonction d'intercept à couplage fort

L'autre limite intéressante de la fonction d'intercept en plus de la limite de couplage petite considérée dans la Section 6.4 est celle du couplage fort. Pour $n = 0$ cas, la fonction d'intercept dans la limite de couplage fort a été analysée dans [48, 124–126], puis étendue aux prochains ordres par la méthode QSC dans [22]. Comme nous avons déjà les données numériques pour l'intercept des différentes valeurs du spin conforme n, alors en utilisant l'algorithme numérique décrit dans la Section 6.3 et en supposant que les coefficients sont des nombres rationnels simples et extrapolant la haute précision données numériques par les puissances inverses de $\lambda^{1/2}$ extraites

$$\begin{aligned} S(0,3) &= -3 + \frac{10}{\lambda^{1/2}} - \frac{25}{\lambda} + \frac{175}{4}\frac{1}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-2}) , \quad (6.54) \\ S(0,2) &= -2 + \frac{4}{\lambda^{1/2}} - \frac{6}{\lambda} + \frac{9}{2}\frac{1}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-2}) , \\ S(0,1) &= -1 , \\ S(0,0) &= 0 - \frac{2}{\lambda^{1/2}} - \frac{1}{\lambda} + \frac{1}{4}\frac{1}{\lambda^{3/2}} + \mathcal{O}(\lambda^{-2}) , \end{aligned}$$

où λ est donné par $\lambda = (4\pi g)^2$ et le résultat pour $n = 0$ est tiré de [22]. Nous voyons que le terme principal est linéaire en n, le subleading est quadratique et ainsi de suite. Ce modèle est assez typique au couplage fort (voir par exemple [61]). En supposant ce motif

polynomial à partir des données ci-dessus (6.54), nous obtenons

$$S(0,n) = -n + \frac{(n-1)(n+2)}{\lambda^{1/2}} - \\ - \frac{(n-1)(n+2)(2n-1)}{2\lambda} + \frac{(n-1)(n+2)(7n^2-9n-1)}{8\lambda^{3/2}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (6.55)$$

que nous pouvons contre-vérifier avec notre slope-to-intercept fonction vers (6.48) en la différenciant à $n = 1$! La comparaison avec (6.48) nous montre un accord complet. Nous avons également vérifié numériquement notre résultat en prenant $n = 1.5$ et en ajustant les données avec des puissances inverses de $\lambda^{1/2}$ nous avons reproduit précisément les coefficients de (6.55).

6.7 Conclusions et perspectives

Dans ce travail, nous avons réussi à reproduire la dimension de l'opérateur twist-2 de la théorie $\mathcal{N} = 4$ SYM dans la limite 't Hooft dans le premier ordre (LO) du régime BFKL directement à partir des équations exactes pour le spectre d'opérateurs locaux appelé la Courbe Spectrale Quantique - QSC. Ce résultat est une confirmation très non-triviale de la validité générale de cette approche QSC et de l'ensemble du programme d'intégrabilité du problème spectral dans les équations AdS/CFT S-matrice et asymptotique Bethe ansatz, TBA, Y-system, FiNLIE équations, etc. En particulier, il s'agit de l'un des très rares calculs de boucle, avec toutes les corrections d'emballage incluses, où le résultat de l'intégrabilité peut être vérifié par une sommation directe du graphe de Feynman de l'approche BFKL originale. Une étape évidente dans cette direction consiste à calculer la correction NLO à la dimension twist-2 pour Δ fini de QSC et à la comparer au calcul direct de BFKL de [2]. Bon nombre des éléments de la construction NLO, tels que l'équation NLO Baxter pour les **Q**-fonctions, sont déjà présents dans cet article, mais l'ingrédient le plus difficile – la formule du type (3.198) pour la singularité dominante, doit encore être dérivé. Bien entendu, le but ultime de l'approximation BFKL de QSC serait de trouver une méthode algorithmique de génération de toute correction BFKL (NNLO, NNNLO, etc.) sur le programme Mathematica, similaire à celle du couplage faible via QSC, proposée par [20]. Il serait également très intéressant de construire numériquement à partir du QSC la dimension twist-2 en tant que fonction continue du spin $S \in \mathbb{R}$ décrit qualitativement dans [127]. Nous espérons également que notre approche permettra de mieux comprendre les similitudes et les différences entre $\mathcal{N} = 4$ SYM et la théorie de Yang-Mills pure (multicolor QCD) à partir de l'approximation BFKL, en raison du fait bien connu que, du moins dans le 't Hooft limit, $\mathcal{N} = 4$ SYMgraphes de Feynman capturent une part importante de tous les graphes QCD et dans le LO BFKL les résultats coïncident simplement.

Nous espérons également que les méthodes de QSC présentées ici seront une source d'inspiration pour la construction de l'expansion systématique du couplage fort dans $\mathcal{N} = 4$ SYM. Un aperçu plus approfondi de la structure du QSC sera nécessaire pour aborder l'ensemble de ces problèmes complexes.

Le régime BFKL est traditionnellement l'un des problèmes "durs" de la physique théorique des hautes énergies. La méthode QSC a permis de progresser dans cette direction lorsque les méthodes perturbatives traditionnelles deviennent trop complexes à

mettre en œuvre. Dans notre travail, nous avons encore élargi la zone d'application de QSC en ajoutant un paramètre supplémentaire: le conformal spin n. Fait important, nous pouvons gérer la situation lorsque le spin conforme prend une valeur réelle arbitraire. QSC peut maintenant faire face à la situation où les trois charges globales correspondant à la sphère à 5 dimensions S^5 – S_1, S_2 et Δ sont non entières. Cela a nécessité une asymptotique exponentielle dans deux composants de la matrice de collage. Les conditions de collage constituent donc le principal ingrédient de la poursuite analytique de QSC présentée dans cet article.

Pour illustrer notre méthode, nous avons calculé deux quantités de tous les couplages – slope-to-intercept BFKL à $\Delta = 0$ par rapport à n autour de $n = 1$ et courbure de la trajectoire de l'opérateur twist-2 dans le voisinage du point $\Delta = 0$ et $n = 1$. Nous avons généré des données analytiques perturbatives et numériques en utilisant les procédures itératives décrites dans [31], que nous avons dû modifier pour prendre en compte les asymptotiques exponentielles pour les charges globales non entières. Les données itératives nous ont permis de corriger les intercepts NNLO et NNNLO BFKL en termes de sommes harmoniques imbriquées.

Plusieurs autres pistes de travail viennent immédiatement à l'esprit. Tout d'abord, la base des sommes harmoniques imbriquées semble bien décrire l'expansion perturbative de l'eigenvalue de BFKL et en particulier son intercept. Puisque la procédure itérative peut être utilisée pour un ordre élevé arbitraire dans g et pour un n impair arbitraire, il est juste question de temps et de puissance de calcul pour corriger les eigenvalues de BFKL à d'autres commandes. La première tâche serait de trouver une expression fermée pour l'eigenvalue NNLO BFKL pour n arbitraire. Il devrait alors être en accord avec les résultats de [58], qui a présenté une procédure de calcul de l'eigenvalue de NNLO BFKL pour un n donné.

La deuxième direction que l'on peut suivre est l'exploration du voisinage du point BPS $n = 1$, $\Delta = 0$. Après avoir calculé la slope-to-intercept par rapport à n, on peut calculer l'ordre suivant $(n-1)^2$. Le calcul devrait être semblable en esprit au calcul de la correction de S^2 à la dimension anormale de l'opérateur de torsion [22].

De plus, il convient d'étudier de manière similaire le spectre Odderon, qui devrait correspondre au paramètre de longueur à prendre $L = 3$. La plupart des étapes décrites dans cet article devraient être applicables à L arbitraire et il serait très intéressant de reproduire des résultats perturbatifs connus antérieurement et les étendre au couplage fini.

Enfin, il existe des indications [128] que les constantes de structure peuvent également être gouvernées par les Q-fonctions de QSC, qui ont été évaluées dans cet article sous différents régimes. Il serait donc intéressant de comparer et d'étendre au couplage fini les résultats sur le triple sommet (vertex) de Pomeron [129].

Appendix A

Appendices

A.1 Derivation of the 4th order Baxter equation for the Q-functions

Following the discussion in [17] we begin by picking a few QQ-relations out of many possible

$$\mathcal{Q}^{+}_{a|j} - \mathcal{Q}^{-}_{a|j} = \mathbf{P}_a \mathbf{Q}_j \ , \quad \mathbf{Q}_j \equiv -\mathbf{P}^a \mathcal{Q}^{\pm}_{a|j} \ . \tag{A.1}$$

The first one follows directly from (2.50c) with $A = I = \emptyset$. The second one is also an algebraic consequence of general QQ-relations (2.50a), (2.50b) and (2.50c). It is shown in [17] (see the equations (4.5)-(4.7) therein) that all Q-functions can be obtained in a simple way through 3 ones – $\mathbf{P}_a$, $\mathbf{Q}_j$ and $\mathcal{Q}_{a|j}$. Thus, expressing $\mathbf{P}^a \equiv \mathcal{Q}_{\bar{a}|1234}$, where $\bar{a} = \{1234\}\backslash a$ is a subset complementary to a, through these 3 types of Q-functions we can prove the second of equations (A.1).

We see that the function $\mathcal{Q}_{a|j}$ is designed to "rotate" $\mathbf{P}^a$ into $\mathbf{Q}_j$. The strategy is to exclude $\mathcal{Q}_{a|j}$ and relate $\mathbf{P}$- and $\mathbf{Q}$-functions directly. Shifting the argument u in the second of these two equations by $\pm i, \pm 2i$ and then using there the first equation to bring all shifted arguments in $\mathcal{Q}_{a|j}$ to the the same one we obtain a linear system of 4 equations

$$\begin{aligned}
&\mathbf{P}^{a[-3]} \mathcal{Q}_{a|j} = \mathbf{Q}_j^{[-3]} - \mathbf{Q}_j^{[-1]} (\mathbf{P}^{a[-3]} \mathbf{P}_a^{[-1]}) \ , \\
&\mathbf{P}^{a[-1]} \mathcal{Q}_{a|j} = \mathbf{Q}_j^{[-1]} \ , \\
&\mathbf{P}^{a[+1]} \mathcal{Q}_{a|j} = \mathbf{Q}_j^{[+1]} \ , \\
&\mathbf{P}^{a[+3]} \mathcal{Q}_{a|j} = \mathbf{Q}_j^{[+3]} - \mathbf{Q}_j^{[+1]} (\mathbf{P}^{a[+3]} \mathbf{P}_a^{[+1]}) \ ,
\end{aligned} \tag{A.2}$$

from which we can express $\mathcal{Q}_{a|j}$ in terms of $\mathbf{P}_a$, $\mathbf{P}^a$ and $\mathbf{Q}_j$. Now, taking the second of two equations (A.1) in the shifted form $\mathbf{Q}_j^{[5]} \equiv -\mathbf{P}^{a[5]} \mathcal{Q}_{a|j}^{[4]}$ and using there again the first equation to bring all shifted arguments in $\mathcal{Q}_{a|j}$ to the the same one, we get, together with (A.2), a system of 5 linear equations on only 4 functions $\mathcal{Q}_{a|j}$ (the second index is fixed in all equations). From their compatibility we obtain the closed 4-th order linear equation (2.51) with coefficients expressed only through $\mathbf{P}_a$ and $\mathbf{P}^a$.

A.2 Computation of the P-functions in the NLO in w

The asymptotics of μ's in this order contains a logarithmic correction. For example, from (3.162) it follows that, up to the exponential factor, $\mu_{12} \sim u^{-S-1} \sim 1 - w \log u$. This shows that μ_{ab} can no longer be a polynomial times exponentials. Nevertheless, we show here that the non-polynomial part can be identified easily and we will have to find a few coefficients in the polynomial part as we did for the leading order.

As we discuss in the main text, in the expression (3.187), relating μ_{ab} to ω_{ij} in the LO only the term with ω_{24} (or equivalently ω^{13}) survives since the other terms are suppressed by w^2. Therefore we can write (3.187) as

$$\mu_{ab}^{(0)} = \mathcal{Q}_{ab|13}^{(0)-} \omega^{(0)13} \,, \tag{A.3}$$

where, importantly, $\omega^{(0)13}$ is a periodic function, whereas $\mathcal{Q}_{ab|13}^{(0)-}$ is analytic in the upper half-plane. We denote the LO and NLO orders $\omega^{13} = \omega_{24} = w^{-2}\omega^{(0)13} + w^{-1}\omega^{(1)13} + \mathcal{O}(w^0)$ and $\mu_{ab} = w^{-2}\mu_{ab}^{(0)} + w^{-1}\mu_{ab}^{(1)} + \mathcal{O}(w^0)$ and represent (A.3) in the form

$$\frac{\mu_{ab}}{\omega^{13}} = \frac{\mu_{ab}^{(0)}}{\omega^{(0)13}} \left(1 + w \left(\frac{\mu_{ab}^{(1)}}{\mu_{ab}^{(0)}} - \frac{\omega^{(1)13}}{\omega^{(0)13}}\right)\right) + \mathcal{O}(w^2) = \mathcal{Q}_{ab|13}^{(0)-} + w\mathcal{Q}_{ab|13}^{(1)-} + \mathcal{O}(w^2) \,. \tag{A.4}$$

First, let us look at the leading term in the RHS: $\mu_{ab}^{(0)}/\omega_{(0)}^{13}$ should be analytic in the upper half-plane and it has a power-like asymptotics, as the RHS does. This means that $\omega^{(0)13} = B\sinh^2(\pi u)$ and thus $\mu_{ab}^{(0)}/\omega^{(0)13} \sim \mu_{ab}^{(0)}/\sinh^2(\pi u) = P_{ab}^-$ is a polynomial. So $\omega^{(1)13}/\omega^{(0)13}$ could contain in the upper half-plane a sum of poles of the second order and of the first order at all $u = in$, $n \in \mathbb{Z}$ with equal residues (because ω^{13} is periodic). To preserve the analyticity in the upper half-plane we should cancel these poles by the poles in $\mu_{ab}^{(1)}/\mu_{ab}^{(0)}$. Also we note that the ratio $\mu_{ab}^{(1)}(u+i/2)/\mu_{ab}^{(0)}(u+i/2)$ is an even function, which also fixes the pole structure in the lower half-plane. In addition $\mu_{ab}^{(1)}/\mu_{ab}^{(0)}$ could have a finite number of poles at zeros of the polynomial P_{ab}^-. In other words, the most general function with these properties can be written as

$$\frac{\mu_{ab}^{(1)}(u+i/2)}{\mu_{ab}^{(0)}(u+i/2)} = \frac{1}{\cosh^2(\pi u)} r_{ab} + p_{ab}\Psi(u) + \frac{\mathcal{R}_{ab}(u)}{P_{ab}(u)} \tag{A.5}$$

where r_{ab} and p_{ab} are some constants and $\mathcal{R}_{ab}(u)$ are regular functions. The first term represents an infinite series of the second order poles with equal residues and the second term gives an infinite series of the first order poles with equal residues since

$$\Psi(u) = \psi\left(\frac{1}{2} - iu\right) + \psi\left(\frac{1}{2} + iu\right) - 2\psi(1) \,. \tag{A.6}$$

The last term takes into account the possibility that there are extra poles cancelled by the ratio $\mu_{ab}^{(0)}/\omega^{(0)13}$ outside the brackets. We notice that $\mathcal{R}_{ab}$ can only be a polynomial of the same order as P_{ab}. Thus again we have a small number of constant coefficients in our ansatz to fix.

We can fix r_{ab}, p_{ab} and $\mathcal{R}_{ab}(u)$ in the same way as we did for the leading order, i.e. by applying (2.56) and the regularity conditions, telling that the combinations $\mu_{ab} + \tilde{\mu}_{ab}$

and $(\mu_{ab} - \tilde{\mu}_{ab})/\sqrt{u^2 - 4g^2}$ are regular at $u \sim 0$. This procedure leads to the following result

$$
\begin{aligned}
& r_{ab} = 2\pi^2\Lambda \,, \quad p_{ab} = -\frac{1}{2} \,, \\
& \mathcal{R}_{12} = \frac{1}{\pi^2\Lambda^2}\frac{8i\left(2\pi^2\left(\Delta^2-1\right)\Lambda+3\right)}{3\left(\Delta^2-1\right)^2} \,, \\
& \mathcal{R}_{13} = \frac{1}{\pi^2\Lambda^2}\frac{3-2\pi^2\left(\Delta^2-1\right)\Lambda}{6}u \,, \\
& \mathcal{R}_{14} = -\frac{1}{\pi^2\Lambda^2}\left(\frac{2\pi^2(\Delta^2-1)\Lambda+3}{48}(4u^2+1) - \frac{(\Delta^2-1)\Lambda}{3}\right) \,, \\
& \mathcal{R}_{24} = -\frac{1}{\pi^2\Lambda^2}\frac{\left(2\pi^2\left(\Delta^2-1\right)\Lambda+9\right)}{72}u(4u^2+1) \,, \\
& \mathcal{R}_{34} = \frac{1}{\pi^2\Lambda^2}\frac{i(\Delta^2-1)^2}{18432}(4u^2+1)((2\pi^2(\Delta^2-1)\Lambda+3)(4u^2-3)+72(\Delta^2-1)\Lambda) \,.
\end{aligned}
\tag{A.7}
$$

and also fixes the remaining coefficient in (3.178) to (3.179).

A.3 NLO solution of the second order Baxter equation

By making the Mellin transformation of $\mathbf{Q}(u)$ we converted the finite difference equation (3.184) into a second order PDE which we managed to solve and transform the solution back explicitly. The result we found reads

$$
\begin{aligned}
&\frac{\sqrt{w}(u^2-2\Lambda w)}{-iu-\frac{w}{4}+i\sqrt{2\Lambda w}}\frac{\Gamma\left(-iu+\frac{w}{4}+i\sqrt{2\Lambda w}\right)}{\Gamma\left(-iu-\frac{w}{4}-i\sqrt{2\Lambda w}\right)}\times \\
&\qquad\times\ {}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},-iu-\frac{w-i\sqrt{32\Lambda w}}{4};-\frac{w}{2},2i\sqrt{2\Lambda w}+1;1\right) \,.
\end{aligned}
\tag{A.8}
$$

Note that this solution contains $\sqrt{w}$ terms. As the initial equation is analytic in w changing the sign of $\sqrt{w}$ we get two linear independent solutions. Suitable combinations of these two solutions should give $\mathbf{Q}_1$ and $\mathbf{Q}_3$ with $\mathcal{O}(w)$ accuracy. As this result is not required for the leading order calculation of this paper these combinations will be published elsewhere.

The Baxter equation in the NLO

$$
\begin{aligned}
&\mathbf{Q}_j\left(\frac{\Delta^2-1-8u^2}{4u^2}+w\frac{\left(\Delta^2-1\right)\Lambda-u^2}{2u^4}\right)+ \\
&\qquad+\mathbf{Q}_j^{--}\left(1-\frac{iw}{2(u-i)}\right)+\mathbf{Q}_j^{++}\left(1+\frac{iw}{2(u+i)}\right)=0 \,, \quad j=1,3 \,.
\end{aligned}
\tag{A.9}
$$

Let's consider the NLO solution of this Baxter equation (for now we consider w to be

finite)

$$Q(u)=\frac{u^2-2\Lambda w}{iu-\frac{w}{4}-\frac{1}{4}\sqrt{w^2-2\Lambda w}}\frac{\Gamma\left(-iu-\frac{w}{4}+\frac{1}{4}\sqrt{w^2-32\Lambda w}\right)}{\Gamma\left(-iu+\frac{w}{4}+\frac{1}{4}\sqrt{w^2-32\Lambda w}\right)}\times$$
$$\times\ {}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},-iu-\frac{w}{4}+\frac{1}{4}\sqrt{w^2-32\Lambda w};-\frac{w}{2},\frac{1}{2}\sqrt{w^2-32\Lambda w}+1;1\right)\ . \quad (A.10)$$

Expanding this solution in w, we obtain

$$Q(u)=\frac{\Delta^2-1}{2w}iu\,{}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},-iu+1;1,2;1\right)+\mathcal{O}\left(\frac{1}{\sqrt{w}}\right)=$$
$$=\frac{\Delta^2-1}{4w}\mathbf{Q}_1^{(0)}(u)+\mathcal{O}\left(\frac{1}{\sqrt{w}}\right)\ . \quad (A.11)$$

Let us determine the poles. To do this, we expand the hypergeometric function in the series and find the large n limit of this series. This gives

$$-\frac{\cos\left(\frac{\pi\Delta}{2}\right)\Gamma\left(-\frac{w}{2}\right)\left(u^2-2\Lambda w\right)\Gamma\left(\frac{1}{2}\sqrt{w(w-32\Lambda)}+1\right)}{\pi\Gamma\left(iu+\frac{1}{4}\sqrt{w(w-32\Lambda)}-\frac{w}{4}+1\right)}\times$$
$$\times\,\zeta\left(iu+\frac{1}{4}\sqrt{w(w-32\Lambda)}-\frac{w}{4}+1\right)=$$
$$=\frac{i\cos\left(\frac{\pi\Delta}{2}\right)w\Gamma\left(-\frac{w}{2}\right)\left(\sqrt{w(w-32\Lambda)}-w\right)}{8\pi\left(u-\frac{1}{4}i\sqrt{w(w-32\Lambda)}+\frac{iw}{4}\right)}+\mathcal{O}\left(u-i\sqrt{w(w-32\Lambda)}+\frac{iw}{4}\right)\ . \quad (A.12)$$

Thus, the poles of the solution $Q(u)$ are situated in the points $u=-iw/4+i\sqrt{w(w-2\Lambda)}+in$, where $n=0,1,\ldots,+\infty$. Also we have the poles in the points $u=iw/4-i\sqrt{w(w-2\Lambda)}-in$ in the lower half-plane. And the solution has the series of zeroes in the points $u=-iw/4-i\sqrt{w(w-2\Lambda)}-i(n+1)$. For simplicity let us denote from this moment

$$a=-\frac{iw}{4}+i\sqrt{w(w-2\Lambda)}\ ,\quad b=-\frac{iw}{4}-i\sqrt{w(w-2\Lambda)}\ . \quad (A.13)$$

There is another solution of the NLO Baxter equation with the different sign of the square root $\sqrt{w(w-2\Lambda w)}$. But, in fact, if we expand this solution in the powers of w, we get the same functions, but with the different signs. This follows just from the following expansion

$$Q(u)=\frac{1}{w}Q_1^{(0)}(u)+\frac{1}{w^{\frac{1}{2}}}Q_3^{(0)}(u)+Q_1^{(1)}(u)+w^{\frac{1}{2}}Q_3^{(1)}(u)+\mathcal{O}(w). \quad (A.14)$$

Due to the fact that the Baxter equation is analytic in w, there exists the other solution with the opposite sign of the root

$$Q_c(u)=\frac{u^2-2\Lambda w}{iu-\frac{w}{4}+\frac{1}{4}\sqrt{w^2-2\Lambda w}}\frac{\Gamma\left(-iu-\frac{w}{4}-\frac{1}{4}\sqrt{w^2-2\Lambda w}\right)}{\Gamma\left(-iu+\frac{w}{4}-\frac{1}{4}\sqrt{w^2-2\Lambda w}\right)}\times$$
$$\times\ {}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},-iu-\frac{w}{4}-\frac{1}{4}\sqrt{w^2-2\Lambda w};-\frac{w}{2},-\frac{1}{2}\sqrt{w^2-2\Lambda w}+1;1\right). \quad (A.15)$$

This solution has two series of poles in the points $u = b + in$ and $u = -b - in$ and the series of zeroes in the points $u = a - i(n+1)$, where $n = 0, 1, \ldots, +\infty$.

The solution $Q(-u)$ has the same two series of poles as $Q(u)$, but with the different residues. But the key observation is that the ratio of the residues is constant for any pair of poles symmetric with respect to zero. Calculating these residues we obtain

$$\frac{\operatorname*{res}_{u=a+in} Q(u)}{\operatorname*{res}_{u=-a-in} Q(u)} = \frac{\cos\frac{\pi\Delta}{2}}{\sin\frac{\pi w}{2}} \,. \tag{A.16}$$

From the equation (A.16) we can obtain the solution which is analytic in the upper half-plane

$$\mathbf{Q}_+(u) = Q(u) + \frac{\cos\frac{\pi\Delta}{2}}{\sin\frac{\pi w}{2}} Q(-u) \,. \tag{A.17}$$

There is also an independent UHPA solution with the different sign of the square root

$$\mathbf{Q}_-(u) = Q_c(u) + \frac{\cos\frac{\pi\Delta}{2}}{\sin\frac{\pi w}{2}} Q_c(-u) \,. \tag{A.18}$$

Expanding the first of these functions, we derive

$$\mathbf{Q}_+(u) = -\frac{(\Delta^2-1)\cos\frac{\pi\Delta}{2}}{2\pi w^2} \mathbf{Q}_1^{(0)}(u) + \mathcal{O}\left(\frac{1}{w^{3/2}}\right) \,. \tag{A.19}$$

A.4 Finding ω_{ij}

Combining (3.188) and (3.194) we can extract

$$\begin{aligned}
\omega_{12}^{(0)} &= -\cos\left(\frac{\pi\Delta}{2}\right) \,, \\
\omega_{14^{(0)}} &= -i\frac{\cosh\left(\pi\left(u+\frac{i\Delta}{2}\right)\right)}{\sinh(\pi u)} \,, \\
\omega_{34}^{(0)} &= -\frac{\cos\left(\frac{\pi\Delta}{2}\right)}{\sinh^2(\pi u)} - 2i\tan\left(\frac{\pi\Delta}{2}\right)\frac{\cosh\left(\pi\left(u+\frac{i\Delta}{2}\right)\right)}{\sinh(\pi u)} \,.
\end{aligned} \tag{A.20}$$

Next, using the unit Pfaffian constraint we get

$$\omega_{12}^{(0)}\omega_{34}^{(0)} - \omega_{13}^{(0)}\omega_{24}^{(0)} + \left(\omega_{14}^{(0)}\right)^2 = 1 \,, \tag{A.21}$$

from where we obtain

$$\omega_{13}^{(0)}\omega_{24}^{(0)} = -2 \,. \tag{A.22}$$

As we discussed in Appendix A.2

$$\omega_{24} = \frac{B\sinh^2(\pi u)}{w^2} + \mathcal{O}(w^{-1}) \,, \tag{A.23}$$

i.e. from (A.22)

$$\omega_{13} = -\frac{2w^2}{B\sinh^2(\pi u)} + \mathcal{O}(w^3)\ . \tag{A.24}$$

To complete the calculation we have to find the constant B. We note that B can be extracted from the singularity of ω_{13} which can be computed independently from

$$\omega_{13} = \frac{\omega_{13}+\tilde\omega_{13}}{2} + \frac{\omega_{13}-\tilde\omega_{13}}{2\sqrt{u^2-4\Lambda w}}\sqrt{u^2-4\Lambda w}\ , \tag{A.25}$$

where the combinations $(\omega_{13}+\tilde\omega_{13})/2$ and $(\omega_{13}-\tilde\omega_{13})/(2\sqrt{u^2-4\Lambda w})$ are regular around $u=0$. We shall use that

$$\frac{\omega_{13}-\tilde\omega_{13}}{2\sqrt{u^2-4\Lambda w}} = \frac{\tilde{\mathbf{Q}}_1\mathbf{Q}_3 - \mathbf{Q}_1\tilde{\mathbf{Q}}_3}{2\sqrt{u^2-4\Lambda w}} = \left(-\frac{16\cos\frac{\pi\Delta}{2}}{\pi^2(\Delta^2-1)^2}u + \mathcal{O}(u^3)\right) + \mathcal{O}(w)\ . \tag{A.26}$$

Therefore for the leading singularity of ω_{13} we have

$$\omega_{13} = w^2\left(\frac{32\cos\frac{\pi\Delta}{2}}{\pi^2(\Delta^2-1)^2}\frac{\Lambda^2}{u^2} + \mathcal{O}\left(\frac{1}{u}\right)\right) + \mathcal{O}(w^3) \tag{A.27}$$

and comparing with (A.24) we get

$$B = -\frac{(\Delta^2-1)^2}{16\Lambda^2\cos\frac{\pi\Delta}{2}}\ . \tag{A.28}$$

A.5 Details of QSC construction

To show that the matrix $\Omega_i^j = \bar{\mathcal{Q}}^-_{a|i}C^a_b\mathcal{Q}^{b|j-}$ is i-periodic let us apply the conjugated equation (4.15) and the same equation for $\mathcal{Q}^{a|i}$ which is valid now in the whole complex plane as we know the functions $\bar{\mathbf{Q}}_i$ and $\mathbf{Q}^i$ on the sheet with the short cuts. We have

$$\begin{aligned}\Omega_i^{j++} - \Omega_i^j = -\bar{\mathbf{P}}_a\bar{\mathbf{Q}}_iC^a_b\mathcal{Q}^{b|j-} + \bar{\mathcal{Q}}^-_{a|i}C^a_b\mathbf{P}^b\mathbf{Q}^j = \\ = -\bar{\mathbf{Q}}_i\mathbf{P}_b\mathcal{Q}^{b|j-} - \bar{\mathcal{Q}}^-_{a|i}\bar{\mathbf{P}}^a\mathbf{Q}^j = -\bar{\mathbf{Q}}_i\mathbf{Q}^j + \bar{\mathbf{Q}}_i\mathbf{Q}^j = 0\ .\end{aligned} \tag{A.29}$$

To show that the matrix $\Theta_i^j(u) = (-1)^{a+1}\mathcal{Q}^-_{a|i}(-u)\mathcal{Q}^{b|j-}(u)$ is i-periodic let us apply the equation (4.15) with u replaced by $-u$ and the same equation for $\mathcal{Q}^{a|i}$ which is valid now in the whole complex plane as we know the functions $\mathbf{Q}_i(-u)$ and $\mathbf{Q}^i(u)$ on the sheet with the short cuts. We have

$$\begin{aligned}\Theta_i^{j++} - \Theta_i^j = -(-1)^{a+1}\mathbf{P}_a(-u)\mathbf{Q}_i(-u)\mathcal{Q}^{b|j-}(u) + (-1)^{a+1}\mathcal{Q}^-_{a|i}(-u)\mathbf{P}^b(u)\mathbf{Q}^j(u) = \\ = -\mathbf{Q}_i(-u)\mathbf{P}_b(u)\mathcal{Q}^{b|j-}(u) - \mathcal{Q}^-_{a|i}(-u)\mathbf{P}^a(-u)\mathbf{Q}^j(u) = \\ = -\mathbf{Q}_i(-u)\mathbf{Q}^j(u) + \mathbf{Q}_i(-u)\mathbf{Q}^j(u) = 0\ .\end{aligned} \tag{A.30}$$

A.6 NLO BFKL eigenvalue in terms of nested harmonic sums

In this Appendix we show how to rewrite the NNLO BFKL Pomeron eigenvalue given in [1, 2] in terms of nested harmonic sums. In the notations of the present work we have

$$S(\Delta, n) = -1 + g^2 \chi_{LO}(\Delta, n) + g^4 \chi_{NLO}(\Delta, n) + \mathcal{O}(g^6) \,, \tag{A.31}$$

where the LO BFKL Pomeron eigenvalue is known in terms of the harmonic sums

$$\chi_{LO}(\Delta, n) = -4 \left(S_1 \left(\frac{1-\Delta+n}{2} - 1 \right) + S_1 \left(\frac{1-\Delta+n}{2} - 1 \right) \right) \,. \tag{A.32}$$

Upon identification with the notations of [1,2] of the BFKL Pomeron eigenvalues $\chi_{LO,NLO}(\Delta, n) = 4\chi_{1,2}(n, (1+\Delta)/2)$ we write down the original expression, which reads as

$$\chi_2(n,\gamma) = -\frac{1}{4} \left[2\Phi(n,\gamma) + 2\Phi(n, 1-\gamma) + 2\zeta_2 \chi_1(n,\gamma) - \right.$$

$$\left. -6\zeta_3 - \psi'' \left(\gamma + \frac{n}{2} \right) - \psi'' \left(1 - \gamma + \frac{n}{2} \right) \right] \,, \tag{A.33}$$

where $\gamma = (1+\Delta)/2$ and

$$\Phi(n,\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+\gamma+n/2} \left[\psi'(k+n+1) - \psi'(k+1) + \right. \tag{A.34}$$

$$\left. + (-1)^{k+1} (\beta'(k+n+1) + \beta'(k+1)) + \frac{1}{k+\gamma+n/2} \left(\psi(k+n+1) - \psi(k+1) \right) \right]$$

and

$$\beta'(z) = \frac{1}{4} \left[\psi' \left(\frac{z+1}{2} \right) - \psi' \left(\frac{z}{2} \right) \right] = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{(z+k)^2} \,. \tag{A.35}$$

Let us consider the following function

$$\tilde{\Phi}(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(k+x)^2} (\psi(k+x+1) - \psi(1)) \,. \tag{A.36}$$

We can express it in terms of harmonic sums

$$\tilde{\Phi}(x) = S_{-2,1}(x-1) + \frac{5}{8}\zeta(3) \,. \tag{A.37}$$

Then we use the following identity for $\Phi(0,x)$[1] which can be found in [48]

$$\Phi(0,-x) + \Phi(0,1+x) = 2\tilde{\Phi}(-x) + 2\tilde{\Phi}(1+x) + \frac{\pi^3}{2\sin(\pi x)} \,. \tag{A.38}$$

[1] In our notations $\Phi(0,x)$ coicides with $\Phi(x)$ from [48].

Taking into account the relation

$$\frac{\pi}{\sin(\pi x)} = -S_{-1}(x) - S_{-1}(-x-1) - 2\log 2\,, \tag{A.39}$$

we obtain the following equation

$$\Phi(0,-x) + \Phi(0,1+x) = 2S_{-2,1}(x) + 2S_{-2,1}(-x-1) + \\ + \frac{5}{2}\zeta(3) - \frac{\pi^2}{2}(S_{-1}(x) + S_{-1}(-x-1) + 2\log 2)\,. \tag{A.40}$$

Thus, the NLO BFKL Pomeron eigenvalue

$$\chi_{NLO}(\Delta,n) = -2\Phi\left(n,\frac{1-\Delta}{2}\right) - 2\Phi\left(n,\frac{1+\Delta}{2}\right) - 2\zeta(2)\chi_{LO}(\Delta,n) + \\ + 6\zeta(3) + \psi''\left(\frac{1+n-\Delta}{2}\right) + \psi''\left(\frac{1+n+\Delta}{2}\right) \tag{A.41}$$

for $n = 0$

$$\chi_{NLO}(\Delta,0) = -2\Phi\left(0,\frac{1-\Delta}{2}\right) - 2\Phi\left(0,\frac{1+\Delta}{2}\right) - 2\zeta(2)\chi_{LO}(\Delta,0) + \\ + 6\zeta(3) + \psi''\left(\frac{1-\Delta}{2}\right) + \psi''\left(\frac{1+\Delta}{2}\right) \tag{A.42}$$

can be rewritten as follows

$$\chi_{NLO}(\Delta,0) = F_2\left(\frac{1-\Delta}{2}\right) + F_2\left(\frac{1+\Delta}{2}\right)\,, \tag{A.43}$$

where

$$F_2(x) = -\frac{3}{2}\zeta(3) + \pi^2\log 2 + \frac{\pi^2}{3}S_1(x-1) + \pi^2 S_{-1}(x-1) + 2S_3(x-1) - 4S_{-2,1}(x-1)\,. \tag{A.44}$$

Using these results, we are able to rewrite the NLO BFKL Pomeron eigenvalue for non-zero conformal spin in the following way

$$\chi_{NLO}(n,\Delta) = \frac{1}{2}(\chi_{NLO}(0,\Delta+n) + \chi_{NLO}(0,\Delta-n)) + R_n\left(\frac{1-\Delta}{2}\right) + R_n\left(\frac{1+\Delta}{2}\right)\,. \tag{A.45}$$

After some calculations, we derive

$$R_n(\gamma) + R_n(1-\gamma) = 2\Phi(n,\gamma) + 2\Phi(n,1-\gamma) - \Phi\left(0,\gamma+\frac{n}{2}\right) - \\ - \Phi\left(0,\gamma-\frac{n}{2}\right) - \Phi\left(0,1-\gamma+\frac{n}{2}\right) - \Phi\left(0,1-\gamma-\frac{n}{2}\right)\,. \tag{A.46}$$

To proceed we rewrite $\Phi(n,\gamma)$ in the following way

$$\Phi(n,\gamma) = -\beta'\left(\gamma+\frac{n}{2}\right)\sum_{p=1}^{n}\frac{1}{p-\gamma-\frac{n}{2}} + \sum_{p=1}^{n}\frac{\beta'(p)}{p-\gamma-\frac{n}{2}} + \sum_{k=n}^{+\infty}\frac{\beta'(k+1)}{k+\gamma-\frac{n}{2}} + \sum_{k=0}^{+\infty}\frac{\beta'(k+1)}{k+\gamma+\frac{n}{2}}\,. \tag{A.47}$$

Then, after simple calculations, we obtain the result

$$R_n(\gamma) = -2\beta' \left(\gamma + \frac{n}{2}\right) \sum_{p=1}^{n} \frac{1}{p - \gamma - \frac{n}{2}} = \\ = -2 \left(S_{-2}\left(\gamma + \frac{n}{2} - 1\right) + \frac{\pi^2}{12}\right) \left(S_1\left(\gamma + \frac{n}{2} - 1\right) - S_1\left(\gamma - \frac{n}{2} - 1\right)\right) . \quad \text{(A.48)}$$

To sum up, substituting into (A.45) the expressions from (A.42) with (A.44) and (A.48), we are able to write the NLO BFKL Pomeron eigenvalue with non-zero conformal spin in terms of nested harmonic sums.

A.7 Values of the NNNLO intercept

Using the iterative procedure we managed to calculate the values of NNNLO intercept for a set of values of the conformal spin n. They are written below

$$j_{N^3LO}(1) = 0 , \quad \text{(A.49)}$$

$$j_{N^3LO}(3) = -128\zeta_3^2 + \left(\frac{32\pi^4}{3} - 64\pi^2\right)\zeta_3 + \left(\frac{320\pi^2}{3} - 704\right)\zeta_5 + 2208\zeta_7 + \frac{544\pi^6}{189} - \frac{32\pi^4}{3} - \frac{128\pi^2}{3} ,$$

$$j_{N^3LO}(5) = \left(48\pi^2 - 864\right)\zeta_3 + 1008\zeta_5 + \frac{56\pi^6}{45} + \frac{4\pi^4}{5} - 12\pi^2 + 2832 ,$$

$$j_{N^3LO}(7) = -\frac{704\zeta_3^2}{3} + \left(\frac{16976}{9} - \frac{976\pi^2}{9} + \frac{32\pi^4}{3}\right)\zeta_3 + \left(\frac{320\pi^2}{3} - \frac{15536}{9}\right)\zeta_5 + \\ + 2208\zeta_7 + \frac{2992\pi^6}{567} + \frac{8476\pi^2}{729} - \frac{644\pi^4}{81} - \frac{5440444}{2187} ,$$

$$j_{N^3LO}(9) = \left(\frac{2500\pi^2}{27} - \frac{196750}{81}\right)\zeta_3 + \frac{17500\zeta_5}{9} + \frac{140\pi^6}{81} + \frac{95\pi^4}{54} - \frac{291275\pi^2}{5832} + \frac{76123175}{17496} ,$$

$$j_{N^3LO}(11) = -\frac{4384\zeta_3^2}{15} + \left(\frac{157729}{45} - \frac{33076\pi^2}{225} + \frac{32\pi^4}{3}\right)\zeta_3 + \left(\frac{320\pi^2}{3} - \frac{573836}{225}\right)\zeta_5 + \\ + 2208\zeta_7 + \frac{18632\pi^6}{2835} + \frac{959892043\pi^2}{18225000} - \frac{145853\pi^4}{20250} - \frac{20803468134991}{5467500000} .$$

A.8 Solution of the NLO finite difference equation for $\mathcal{Q}_{a|i}$

Rewriting the equation (4.278) in the form (4.280) we have

$$\mathcal{Q}_{a|i}^{(1)+} - \mathcal{Q}_{a|i}^{(1)-} = \begin{pmatrix} \frac{i\cosh_-}{2g^2 I_2} & \frac{u\cosh_-^2}{g^4 I_2^2} & -\frac{iu\cosh_-}{2g^2 I_2} & -\frac{\cosh_-^2}{g^4 I_2^2} \\ \frac{iu\cosh_-}{2g^2 I_2} & \frac{u^2\cosh_-^2}{g^4 I_2^2} & -\frac{i}{2} - \frac{i\left(u^2\cosh_- - g^2 I_2\right)}{2g^2 I_2} & -\frac{u\cosh_-^2}{g^4 I_2^2} \\ -\frac{1}{4} & \frac{iu\cosh_-}{2g^2 I_2} & \frac{u}{4} & -\frac{i\cosh_-}{2g^2 I_2} \\ -\frac{u}{4} & \frac{i}{2} + \frac{i\left(u^2\cosh_- - g^2 I_2\right)}{2g^2 I_2} & \frac{u^2}{4} & -\frac{iu\cosh_-}{2g^2 I_2} \end{pmatrix} . \quad \text{(A.50)}$$

The parts of the solution are given by

$$\left(\mathcal{Q}_{a|i}^{(1)+}\right)_{\text{pol}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{u}{2} & 0 \\ \frac{iu}{4} & 0 & -\frac{iu^2}{8} + \frac{u}{8} & 0 \\ \frac{iu^2}{8} - \frac{u}{8} & \frac{u}{2} & -\frac{iu^3}{12} + \frac{u^2}{8} + \frac{iu}{24} & 0 \end{pmatrix} \quad \text{(A.51)}$$

and

$$\left(\mathcal{Q}^{(1)+}_{a|i}\right)_U = \frac{i}{2g^2I_2}\Gamma_U \cdot \begin{pmatrix} \cosh^v_- & -\frac{2i}{g^2I_2}v\left(\cosh^v_-\right)^2 & -v\cosh^v_- & \frac{2i}{g^2I_2}\left(\cosh^v_-\right)^2 \\ v\cosh^v_- & -\frac{2i}{g^2I_2}v^2\left(\cosh^v_-\right)^2 & -v^2\cosh^v_- & \frac{2i}{g^2I_2}v\left(\cosh^v_-\right)^2 \\ 0 & v\cosh^v_- & 0 & -\cosh^v_- \\ 0 & v^2\cosh^v_- & 0 & -v\cosh^v_- \end{pmatrix}, \tag{A.52}$$

where we used the fact that $(\Gamma_U \cdot \text{const}) = 0$.

Using the large u expansion of the kernel Γ_U, it is possible to fix the coefficients $c_{a|i}$

$$c_{a|i} = \begin{pmatrix} 0 & 0 & \frac{7}{12} - \frac{i\pi}{4} & 0 \\ \frac{3}{4} + \frac{i\pi}{4} & 0 & -\frac{i}{4} & 0 \\ -\frac{1}{8} & \frac{3}{4} + \frac{i\pi}{4} & -\frac{ig^2I_4}{24I_2} + \frac{ig^2}{12} - \frac{g}{2}c^{(1)}_{4,1} + \frac{i}{48} & 0 \\ \frac{ig^2I_4}{24I_2} - \frac{ig^2}{12} + \frac{g}{2}c^{(1)}_{4,1} - \frac{i}{48} & \frac{i}{4} & 0 & \frac{7}{12} - \frac{i\pi}{4} \end{pmatrix}. \tag{A.53}$$

Rewriting equation (4.290) in the form (4.280) we have

$$\bar{\mathcal{Q}}^{(1)+}_{a|i} - \bar{\mathcal{Q}}^{(1)-}_{a|i} = \begin{pmatrix} \frac{i\cosh_-}{2g^2I_2} & -\frac{u\cosh^2_-}{g^4I_2^2} & -\frac{iu\cosh_-}{2g^2I_2} & \frac{\cosh^2_-}{g^4I_2^2} \\ \frac{iu\cosh_-}{2g^2I_2} & -\frac{u^2\cosh^2_-}{g^4I_2^2} & -\frac{i}{2} - \frac{i\left(u^2\cosh_- - g^2I_2\right)}{2g^2I_2} & \frac{u\cosh^2_-}{g^4I_2^2} \\ \frac{1}{4} & \frac{iu\cosh_-}{2g^2I_2} & -\frac{u}{4} & -\frac{i\cosh_-}{2g^2I_2} \\ \frac{u}{4} & \frac{i}{2} + \frac{i\left(u^2\cosh_- - g^2I_2\right)}{2g^2I_2} & -\frac{u^2}{4} & -\frac{iu\cosh_-}{2g^2I_2} \end{pmatrix}. \tag{A.54}$$

Parts of the solution are as follows

$$\left(\bar{\mathcal{Q}}^{(1)-}_{a|i}\right)_{\text{pol}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{u}{2} & 0 \\ -\frac{iu}{4} & 0 & \frac{iu^2}{8} + \frac{u}{8} & 0 \\ -\frac{iu^2}{8} - \frac{u}{8} & \frac{u}{2} & \frac{iu^3}{12} + \frac{u^2}{8} - \frac{iu}{24} & 0 \end{pmatrix} \tag{A.55}$$

and

$$\left(\bar{\mathcal{Q}}^{(1)-}_{a|i}\right)_D = \frac{i}{2g^2I_2}\Gamma_D \cdot \begin{pmatrix} \cosh^v_- & \frac{2i}{g^2I_2}v\left(\cosh^v_-\right)^2 & -v\cosh^v_- & -\frac{2i}{g^2I_2}\left(\cosh^v_-\right)^2 \\ v\cosh^v_- & \frac{2i}{g^2I_2}v^2\left(\cosh^v_-\right)^2 & -v^2\cosh^v_- & -\frac{2i}{g^2I_2}v\left(\cosh^v_-\right)^2 \\ 0 & v\cosh^v_- & 0 & -\cosh^v_- \\ 0 & v^2\cosh^v_- & 0 & -v\cosh^v_- \end{pmatrix}, \tag{A.56}$$

where we used the fact that $\Gamma_D\,(\text{const}) = 0$.

A.9 Fixing the constants $M_1^{(1)13}$, $M_1^{(1)11}$ and $M_1^{(1)33}$.

Let us rewrite the equation (4.304) once more introducing a new designation for the LHS of (4.298)

$$Z_{1,2} \equiv \tilde{R}_{1,2} + R_{1,2} + \frac{2i}{g^2I_2}\left(\cosh_+ + \cosh_-\right)R_{3,4} = 0\,, \tag{A.57}$$

where R_a, $a = 1, \dots, 4$ are given by (4.298). After the substitution of the **P**-functions, Q-functions with two indices and the gluing matrix we see that the integral kernels Γ_U and Γ_D in (A.57) combine into the difference

$$(\Gamma_U \cdot h)(u) - (\Gamma_D \cdot h)(u) = \oint\limits_{-2g}^{2g} \frac{dv}{2\pi i} \left(\frac{1}{v-u} + \pi \coth(\pi(u-v)) \right) h(v) \, . \tag{A.58}$$

As we are dealing in (A.57) with the functions $h(v)$ being the series in $x(v)$ with negative powers, the first integral in (A.58) with the kernel $1/(u-v)$ can be taken by using the residue theorem and we obtain

$$(\Gamma_U \cdot h)(u) - (\Gamma_D \cdot h)(u) = h(u) + \oint\limits_{-2g}^{2g} \frac{dv}{2\pi i} \pi \coth(\pi(u-v)) h(v) \, . \tag{A.59}$$

Rewriting $\coth(2\pi(u-v)) = (e^{2\pi u} + e^{2\pi v})/(e^{2\pi u} - e^{2\pi v})$ and changing the integration variable from v to Zhukovsky variable $x(v)$, we are able to derive relatively compact expressions for Z_1 and Z_2

$$Z_1 = -\frac{i\pi}{g^2 I_2} \sinh(2\pi u) - i M_1^{(1)33} u + i M_1^{(1)13} + \frac{i I_0}{g^2 I_2} \left(i \left(M_1^{(1)12} - M_1^{(1)34} \right) + \pi \right) + T_1 \, , \tag{A.60}$$

$$Z_2 = \frac{i\pi}{g^2 I_2} u \sinh(2\pi u) - \left(i M_1^{(1)13} + \frac{i I_0}{g^2 I_2} \left(i \left(M_1^{(1)12} - M_1^{(1)34} \right) + \pi \right) \right) u + i M_1^{(1)11} + T_2 \, ,$$

where

$$T_1 = -\frac{i\pi}{g^2 I_2^2} \oint\limits_{|x_v|=1} \frac{dx_v}{2\pi i} \frac{e^{2\pi u} + e^{2\pi v}}{e^{2\pi u} - e^{2\pi v}} \left(x_v - \frac{1}{x_v} \right) \left(x_u - \frac{1}{x_v} \right) \left(\frac{1}{x_v} - \frac{1}{x_u} \right) \times \tag{A.61}$$
$$\times \left(I_0 + \cosh_-^v - \sinh(2\pi u) \right) \cosh_-^v \, ,$$
$$T_2 = \frac{i\pi}{g I_2^2} \oint\limits_{|x_v|=1} \frac{dx_v}{2\pi i} \frac{e^{2\pi u} + e^{2\pi v}}{e^{2\pi u} - e^{2\pi v}} \left(x_v^2 - \frac{1}{x_v^2} \right) \left(x_u - \frac{1}{x_v} \right) \left(\frac{1}{x_v} - \frac{1}{x_u} \right) \times$$
$$\times \left(I_0 (1 - I_2 x_u x_v^2) + \cosh_-^v - (1 - I_2 x_u x_v^2) \sinh(2\pi u) \right) \cosh_-^v \, ,$$

where the integration contour goes clockwise. To calculate the integrals (A.61) we utilize the following trick. If in both integrals we make the inversion of the integration variable x_v this does not change their value. Thus, taking the half-sum of $T_{1,2}$ and $T_{1,2}$ with x_v replaced by $1/x_v$ leaves the integrals intact, but gives us the integrands which are much simpler to work with

$$T_1 = -\frac{i\pi}{2g^2 I_2^2} \oint\limits_{|x_v|=1} \frac{dx_v}{2\pi i} \left(x_v - \frac{1}{x_v} \right) \left(x_u - \frac{1}{x_v} \right) \left(\frac{1}{x_v} - \frac{1}{x_u} \right) \times \tag{A.62}$$
$$\times \left(\cosh_+^v - \cosh_-^v \right) \left(\sinh(2\pi u) + \sinh(2\pi v) \right) \, ,$$
$$T_2 = -\frac{i\pi}{2g I_2^2} \oint\limits_{|x_v|=1} \frac{dx_v}{2\pi i} \left(x_v^2 - \frac{1}{x_v^2} \right) \left(x_u - \frac{1}{x_v} \right) \left(\frac{1}{x_v} - \frac{1}{x_u} \right) \times$$
$$\times \left(\cosh_+^v - \cosh_-^v \right) \left(\sinh(2\pi u) + \sinh(2\pi v) \right) \, .$$

Calculation of the integrals (A.62) leads us to the result

$$T_1 = \frac{i\pi}{g^2 I_2}\sinh(2\pi u) + \frac{i\pi}{gI_2^2}u\sum_{k=1}^{+\infty} I_{2k}(I_{2k+1} - I_{2k-1})\ , \tag{A.63}$$

$$T_2 = -\frac{i\pi}{g^2 I_2}u\sinh(2\pi u) - \frac{i\pi}{gI_2^2}\sum_{k=1}^{+\infty} I_{2k}(I_{2k+3} + I_{2k+1} - I_{2k-1} - I_{2k-3})\ .$$

After the substitution of (A.63) into (A.60), we manage to fix the following coefficients of the gluing matrix in the NLO

$$M_1^{(1)13} = -\frac{I_0}{g^2 I_2}\left(i\left(M_1^{(1)12} - M_1^{(1)34}\right) + \pi\right)\ , \tag{A.64}$$

$$M_1^{(1)11} = \frac{\pi}{g^2 I_2}\sum_{k=1}^{+\infty} I_{2k}(I_{2k+3} + I_{2k+1} - I_{2k-1} - I_{2k-3})\ ,$$

$$M_1^{(1)33} = \frac{\pi}{g^2 I_2}\sum_{k=1}^{+\infty} I_{2k}(I_{2k+1} - I_{2k-1})\ .$$

Bibliography

[1] A. V. Kotikov and L. N. Lipatov, *DGLAP and BFKL evolution equations in the N=4 supersymmetric gauge theory*, in *35th Annual Winter School on Nuclear and Particle Physics Repino, Russia, February 19-25, 2001*, 2001, [hep-ph/0112346].

[2] A. Kotikov and L. Lipatov, *DGLAP and BFKL equations in the N=4 supersymmetric gauge theory*, *Nucl.Phys.* **B661** (2003) 19–61, [hep-ph/0208220].

[3] L. N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models*, *JETP Lett.* **59** (1994) 596–599, [hep-th/9311037]. [Pisma Zh. Eksp. Teor. Fiz.59,571(1994)].

[4] A. Kotikov and L. Lipatov, *NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories*, *Nucl.Phys.* **B582** (2000) 19–43, [hep-ph/0004008].

[5] O. Kepka, C. Marquet, and C. Royon, *Gaps between jets in hadronic collisions*, *Phys. Rev.* **D83** (2011) 034036, [arXiv:1012.3849].

[6] L. Brink, J. H. Schwarz, and J. Scherk, *Supersymmetric Yang-Mills Theories*, *Nucl. Phys.* **B121** (1977) 77–92.

[7] J. Bartels, *High-Energy Behavior in a Nonabelian Gauge Theory (II)*, *Nucl. Phys.* **B175** (1980) 365–401.

[8] J. Kwiecinski and M. Praszalowicz, *Three Gluon Integral Equation and Odd c Singlet Regge Singularities in QCD*, *Phys. Lett.* **B94** (1980) 413–416.

[9] L. D. Faddeev and G. P. Korchemsky, *High-energy QCD as a completely integrable model*, *Phys. Lett.* **B342** (1995) 311–322, [hep-th/9404173].

[10] V. M. Braun, S. E. Derkachov, and A. N. Manashov, *Integrability of three particle evolution equations in QCD*, *Phys. Rev. Lett.* **81** (1998) 2020–2023, [hep-ph/9805225].

[11] A. V. Belitsky, G. P. Korchemsky, and D. Mueller, *Integrability in Yang-Mills theory on the light cone beyond leading order*, *Phys. Rev. Lett.* **94** (2005) 151603, [hep-th/0412054].

[12] A. V. Belitsky, G. P. Korchemsky, and D. Mueller, *Integrability of two-loop dilatation operator in gauge theories*, *Nucl. Phys.* **B735** (2006) 17–83, [hep-th/0509121].

[13] A. V. Belitsky, G. P. Korchemsky, and D. Mueller, *Towards Baxter equation in supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B768** (2007) 116–134, [hep-th/0605291].

[14] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, *JHEP* **03** (2003) 013, [hep-th/0212208].

[15] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et. al.*, *Review of AdS/CFT Integrability: An Overview*, *Lett.Math.Phys.* **99** (2012) 3–32, [arXiv:1012.3982].

[16] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum spectral curve for AdS_5/CFT_4*, *Phys.Rev.Lett.* **112** (2014) 011602, [arXiv:1305.1939].

[17] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4*, *JHEP* **09** (2015) 187, [arXiv:1405.4857].

[18] N. Gromov, *Introduction to the Spectrum of $N = 4$ SYM and the Quantum Spectral Curve*, arXiv:1708.0364.

[19] V. Kazakov, *Quantum Spectral Curve of γ-twisted $\mathcal{N} = 4$ SYM theory and fishnet CFT*, arXiv:1802.0216.

[20] C. Marboe and D. Volin, *Quantum spectral curve as a tool for a perturbative quantum field theory*, *Nucl. Phys.* **B899** (2015) 810–847, [arXiv:1411.4758].

[21] B. Basso, *An exact slope for AdS/CFT*, arXiv:1109.3154.

[22] N. Gromov, F. Levkovich-Maslyuk, G. Sizov, and S. Valatka, *Quantum spectral curve at work: from small spin to strong coupling in $\mathcal{N} = 4$ SYM*, *JHEP* **07** (2014) 156, [arXiv:1402.0871].

[23] D. Correa, J. Henn, J. Maldacena, and A. Sever, *An exact formula for the radiation of a moving quark in N=4 super Yang Mills*, *JHEP* **1206** (2012) 048, [arXiv:1202.4455].

[24] D. Correa, J. Maldacena, and A. Sever, *The quark anti-quark potential and the cusp anomalous dimension from a TBA equation*, *JHEP* **1208** (2012) 134, [arXiv:1203.1913].

[25] N. Gromov and A. Sever, *Analytic Solution of Bremsstrahlung TBA*, *JHEP* **1211** (2012) 075, [arXiv:1207.5489].

[26] A. Cavaglià, D. Fioravanti, N. Gromov, and R. Tateo, *Quantum Spectral Curve of the $\mathcal{N} = 6$ Supersymmetric Chern-Simons Theory*, *Phys. Rev. Lett.* **113** (2014), no. 2 021601, [arXiv:1403.1859].

[27] D. Bombardelli, A. Cavaglià, D. Fioravanti, N. Gromov, and R. Tateo, *The full Quantum Spectral Curve for AdS_4/CFT_3*, *JHEP* **09** (2017) 140, [arXiv:1701.0047].

[28] N. Gromov and G. Sizov, *Exact Slope and Interpolating Functions in N=6 Supersymmetric Chern-Simons Theory*, *Phys. Rev. Lett.* **113** (2014), no. 12 121601, [arXiv:1403.1894].

[29] L. Bianchi, M. S. Bianchi, A. Bres, V. Forini, and E. Vescovi, *Two-loop cusp anomaly in ABJM at strong coupling*, *JHEP* **10** (2014) 013, [arXiv:1407.4788].

[30] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher, and V. N. Velizhanin, *Dressing and Wrapping*, *J. Stat. Mech.* **0710** (2007) P10003, [arXiv:0704.3586].

[31] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, *Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **115** (2015), no. 25 251601, [arXiv:1507.0401].

[32] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, *Quantum Spectral Curve and the Numerical Solution of the Spectral Problem in AdS5/CFT4*, *JHEP* **06** (2016) 036, [arXiv:1504.0664].

[33] N. Gromov and V. Kazakov, *Analytic continuation in spin of the baxter equation solutions for twist-2 operators*, *unpublished*.

[34] R. A. Janik, *Twist-two operators and the BFKL regime - nonstandard solutions of the Baxter equation*, *JHEP* **1311** (2013) 153, [arXiv:1309.2844].

[35] T. Jaroszewicz, *Gluonic Regge Singularities and Anomalous Dimensions in QCD*, *Phys.Lett.* **B116** (1982) 291.

[36] L. Lipatov, *The Bare Pomeron in Quantum Chromodynamics*, *Sov.Phys.JETP* **63** (1986) 904–912.

[37] V. Fadin, R. Fiore, A. Grabovsky, and A. Papa, *Connection between complete and Moebius forms of gauge invariant operators*, *Nucl.Phys.* **B856** (2012) 111–124, [arXiv:1109.6634].

[38] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Multi - Reggeon Processes in the Yang-Mills Theory*, *Sov.Phys.JETP* **44** (1976) 443–450.

[39] E. Kuraev, L. Lipatov, and V. S. Fadin, *The Pomeranchuk Singularity in Nonabelian Gauge Theories*, *Sov.Phys.JETP* **45** (1977) 199–204.

[40] I. Balitsky and L. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics*, *Sov.J.Nucl.Phys.* **28** (1978) 822–829.

[41] I. Balitsky, *Operator expansion for high-energy scattering*, *Nucl.Phys.* **B463** (1996) 99–160, [hep-ph/9509348].

[42] I. Balitsky and G. A. Chirilli, *High-energy amplitudes in N=4 SYM in the next-to-leading order*, *Phys.Lett.* **B687** (2010) 204–213, [arXiv:0911.5192].

[43] I. Balitsky, V. Kazakov, and E. Sobko, *Two-point correlator of twist-2 light-ray operators in N=4 SYM in BFKL approximation*, arXiv:1310.3752.

[44] I. Balitsky and G. A. Chirilli, *Conformal kernel for NLO BFKL equation in N = 4 SYM*, *Phys. Rev.* **D79** (2009) 031502, [arXiv:0812.3416].

[45] L. Cornalba, M. S. Costa, and J. Penedones, *Eikonal Methods in AdS/CFT: BFKL Pomeron at Weak Coupling*, *JHEP* **0806** (2008) 048, [arXiv:0801.3002].

[46] L. Cornalba, M. S. Costa, and J. Penedones, *Deep Inelastic Scattering in Conformal QCD*, *JHEP* **1003** (2010) 133, [arXiv:0911.0043].

[47] M. S. Costa, J. Drummond, V. Goncalves, and J. Penedones, *The role of leading twist operators in the Regge and Lorentzian OPE limits*, *JHEP* **1404** (2014) 094, [arXiv:1311.4886].

[48] M. S. Costa, V. Goncalves, and J. Penedones, *Conformal Regge theory*, *JHEP* **1212** (2012) 091, [arXiv:1209.4355].

[49] L. Lipatov, *Gauge invariant effective action for high-energy processes in QCD*, *Nucl.Phys.* **B452** (1995) 369–400, [hep-ph/9502308].

[50] B. Basso, S. Caron-Huot, and A. Sever, *Adjoint BFKL at finite coupling: a short-cut from the collinear limit*, *JHEP* **01** (2015) 027, [arXiv:1407.3766].

[51] B. Basso, A. Sever, and P. Vieira, *Spacetime and Flux Tube S-Matrices at Finite Coupling for N=4 Supersymmetric Yang-Mills Theory*, *Phys.Rev.Lett.* **111** (2013), no. 9 091602, [arXiv:1303.1396].

[52] H. De Vega and L. Lipatov, *Interaction of reggeized gluons in the Baxter-Sklyanin representation*, *Phys.Rev.* **D64** (2001) 114019, [hep-ph/0107225].

[53] S. E. Derkachov, G. Korchemsky, and A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD. 3. Quasiclassical approach*, *Nucl.Phys.* **B661** (2003) 533–576, [hep-th/0212169].

[54] S. E. Derkachov, G. Korchemsky, J. Kotanski, and A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD. 2. Quantization conditions and energy spectrum*, *Nucl.Phys.* **B645** (2002) 237–297, [hep-th/0204124].

[55] S. E. Derkachov, G. Korchemsky, and A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD: 1. Baxter Q operator and separation of variables*, *Nucl.Phys.* **B617** (2001) 375–440, [hep-th/0107193].

[56] V. S. Fadin and L. Lipatov, *BFKL pomeron in the next-to-leading approximation*, *Phys.Lett.* **B429** (1998) 127–134, [hep-ph/9802290].

[57] M. Alfimov, *QCD Pomeron with conformal spin from AdS/CFT Quantum Spectral Curve*, 29.02.2016. GATIS Training Event, DESY, Hamburg, https://indico.desy.de/getFile.py/access?contribId=3&resId=0&materialId=slides&confId=8491.

[58] S. Caron-Huot and M. Herranen, *High-energy evolution to three loops*, *JHEP* **02** (2018) 058, [arXiv:1604.0741].

[59] C. Marboe, V. Velizhanin, and D. Volin, *Six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory*, *JHEP* **07** (2015) 084, [arXiv:1412.4762].

[60] C. Marboe and D. Volin, *The full spectrum of AdS5/CFT4 I: Representation theory and one-loop Q-system*, arXiv:1701.0370.

[61] N. Gromov, V. Kazakov, G. Korchemsky, S. Negro, and G. Sizov, *Integrability of Conformal Fishnet Theory*, *JHEP* **01** (2018) 095, [arXiv:1706.0416].

[62] M. Alfimov, N. Gromov, and G. Sizov, *BFKL spectrum of $\mathcal{N} = 4$: non-zero conformal spin*, *JHEP* **07** (2018) 181, [arXiv:1802.0690].

[63] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[64] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[65] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[66] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, *Complete spectrum of long operators in N=4 SYM at one loop*, *JHEP* **07** (2005) 030, [hep-th/0503200].

[67] M. Staudacher, *The Factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054, [hep-th/0412188].

[68] N. Beisert and M. Staudacher, *Long-range psu(2,2|4) Bethe Ansatze for gauge theory and strings*, *Nucl. Phys.* **B727** (2005) 1–62, [hep-th/0504190].

[69] N. Beisert, B. Eden, and M. Staudacher, *Transcendentality and Crossing*, *J.Stat.Mech.* **0701** (2007) P01021, [hep-th/0610251].

[70] G. Arutyunov and S. Frolov, *On String S-matrix, Bound States and TBA*, *JHEP* **12** (2007) 024, [arXiv:0710.1568].

[71] J. Ambjorn, R. A. Janik, and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, *Nucl. Phys.* **B736** (2006) 288–301, [hep-th/0510171].

[72] R. A. Janik and T. Lukowski, *Wrapping interactions at strong coupling: The Giant magnon*, *Phys. Rev.* **D76** (2007) 126008, [arXiv:0708.2208].

[73] G. Arutyunov and S. Frolov, *String hypothesis for the AdS(5) x S**5 mirror*, *JHEP* **03** (2009) 152, [arXiv:0901.1417].

[74] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **103** (2009) 131601, [arXiv:0901.3753].

[75] A. Cavaglia, D. Fioravanti, and R. Tateo, *Extended Y-system for the AdS_5/CFT_4 correspondence*, *Nucl. Phys.* **B843** (2011) 302–343, [arXiv:1005.3016].

[76] J. Balog and A. Hegedus, *$AdS_5 \times S^5$ mirror TBA equations from Y-system and discontinuity relations*, *JHEP* **08** (2011) 095, [arXiv:1104.4054].

[77] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Solving the AdS/CFT Y-system*, *JHEP* **07** (2012) 023, [arXiv:1110.0562].

[78] N. Gromov, V. Kazakov, and Z. Tsuboi, *PSU(2,2|4) Character of Quasiclassical AdS/CFT*, *JHEP* **07** (2010) 097, [arXiv:1002.3981].

[79] B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* **0611** (2006) P11014, [hep-th/0603157].

[80] V. Kazakov, S. Leurent, and D. Volin, *T-system on T-hook: Grassmannian Solution and Twisted Quantum Spectral Curve*, *JHEP* **12** (2016) 044, [arXiv:1510.0210].

[81] N. Gromov, V. Kazakov, S. Leurent, and Z. Tsuboi, *Wronskian Solution for AdS/CFT Y-system*, *JHEP* **01** (2011) 155, [arXiv:1010.2720].

[82] M. Alfimov, N. Gromov, and V. Kazakov, *QCD Pomeron from AdS/CFT Quantum Spectral Curve*, *JHEP* **07** (2015) 164, [arXiv:1408.2530].

[83] A. V. Kotikov, A. Rej, and S. Zieme, *Analytic three-loop Solutions for N=4 SYM Twist Operators*, *Nucl. Phys.* **B813** (2009) 460–483, [arXiv:0810.0691].

[84] N. Gromov and F. Levkovich-Maslyuk, *Quark-anti-quark potential in $\mathcal{N} = 4$ SYM*, *JHEP* **12** (2016) 122, [arXiv:1601.0567].

[85] M. Beccaria and G. Macorini, *Quantum folded string in S^5 and the Konishi multiplet at strong coupling*, *Journal of High Energy Physics* **2011** (2011) 1–17, [arXiv:1108.3480].

[86] R. C. Brower, M. S. Costa, M. Djurić, T. Raben, and C.-I. Tan, *Strong Coupling Expansion for the Conformal Pomeron/Odderon Trajectories*, *JHEP* **02** (2015) 104, [arXiv:1409.2730].

[87] N. Gromov and F. Levkovich-Maslyuk, *Quantum Spectral Curve for a cusped Wilson line in $\mathcal{N} = 4$ SYM*, *JHEP* **04** (2016) 134, [arXiv:1510.0209].

[88] A. Hegedűs and J. Konczer, *Strong coupling results in the AdS_5 /CFT_4 correspondence from the numerical solution of the quantum spectral curve*, *JHEP* **08** (2016) 061, [arXiv:1604.0234].

[89] S. Leurent and D. Volin, *Multiple zeta functions and double wrapping in planar $N = 4$ SYM*, *Nucl.Phys.* **B875** (2013) 757–789, [arXiv:1302.1135].

[90] L. Anselmetti, D. Bombardelli, A. Cavaglià, and R. Tateo, *12 loops and triple wrapping in ABJM theory from integrability*, *JHEP* **10** (2015) 117, [arXiv:1506.0908].

[91] R. N. Lee and A. I. Onishchenko, *ABJM quantum spectral curve and Mellin transform*, arXiv:1712.0041.

[92] C. Duhr, *Mathematical aspects of scattering amplitudes*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: Journeys Through the Precision Frontier: Amplitudes for Colliders (TASI 2014): Boulder, Colorado, June 2-27, 2014*, pp. 419–476, 2015, [arXiv:1411.7538].

[93] A. V. Kotikov and V. N. Velizhanin, *Analytic continuation of the Mellin moments of deep inelastic structure functions*, hep-ph/0501274.

[94] D. I. Kazakov and A. V. Kotikov, *Total $\alpha^- s$ Correction to Deep Inelastic Scattering Cross-section Ratio, $R = \sigma^- l \,/\, \sigma^- t$ in QCD. Calculation of Longitudinal Structure Function*, *Nucl. Phys.* **B307** (1988) 721. [Erratum: Nucl. Phys.B345,299(1990)].

[95] C. Lopez and F. J. Yndurain, *Behavior at $x = 0, 1$, Sum Rules and Parametrizations for Structure Functions Beyond the Leading Order*, *Nucl. Phys.* **B183** (1981) 157–181.

[96] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, and V. N. Velizhanin, *Three-loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model*, *Phys. Lett.* **B595** (2004) 521–529, [hep-th/0404092].

[97] J. Blumlein, *Structural Relations of Harmonic Sums and Mellin Transforms up to Weight $w = 5$*, *Comput. Phys. Commun.* **180** (2009) 2218–2249, [arXiv:0901.3106].

[98] J. Ablinger, *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, arXiv:1011.1176.

[99] J. Ablinger, *Computer Algebra Algorithms for Special Functions in Particle Physics*. PhD thesis, Linz U., 2012-04. arXiv:1305.0687.

[100] J. Ablinger, J. Blümlein, and C. Schneider, *Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms*, *J. Math. Phys.* **54** (2013) 082301, [arXiv:1302.0378].

[101] J. Ablinger, J. Blumlein, and C. Schneider, *Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials*, *J. Math. Phys.* **52** (2011) 102301, [arXiv:1105.6063].

[102] E. Remiddi and J. A. M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys.* **A15** (2000) 725–754, [hep-ph/9905237].

[103] J. A. M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, *Int. J. Mod. Phys.* **A14** (1999) 2037–2076, [hep-ph/9806280].

[104] Yu. L. Dokshitzer, G. Marchesini, and G. P. Salam, *Revisiting parton evolution and the large-x limit*, *Phys. Lett.* **B634** (2006) 504–507, [hep-ph/0511302].

[105] Yu. L. Dokshitzer and G. Marchesini, *N=4 SUSY Yang-Mills: three loops made simple(r)*, *Phys. Lett.* **B646** (2007) 189–201, [hep-th/0612248].

[106] B. Basso and G. P. Korchemsky, *Anomalous dimensions of high-spin operators beyond the leading order*, *Nucl. Phys.* **B775** (2007) 1–30, [hep-th/0612247].

[107] L. F. Alday, A. Bissi, and T. Lukowski, *Large spin systematics in CFT*, *JHEP* **11** (2015) 101, [arXiv:1502.0770].

[108] V. N. Gribov and L. N. Lipatov, *Deep inelastic e p scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 438–450. [Yad. Fiz.15,781(1972)].

[109] V. N. Gribov and L. N. Lipatov, *e+ e- pair annihilation and deep inelastic e p scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 675–684. [Yad. Fiz.15,1218(1972)].

[110] M. Beccaria, Yu. L. Dokshitzer, and G. Marchesini, *Twist 3 of the sl(2) sector of N=4 SYM and reciprocity respecting evolution*, *Phys. Lett.* **B652** (2007) 194–202, [arXiv:0705.2639].

[111] M. Beccaria, V. Forini, T. Lukowski, and S. Zieme, *Twist-three at five loops, Bethe Ansatz and wrapping*, *JHEP* **03** (2009) 129, [arXiv:0901.4864].

[112] M. Beccaria and V. Forini, *Four loop reciprocity of twist two operators in N=4 SYM*, *JHEP* **03** (2009) 111, [arXiv:0901.1256].

[113] T. Lukowski, A. Rej, and V. N. Velizhanin, *Five-Loop Anomalous Dimension of Twist-Two Operators*, *Nucl. Phys.* **B831** (2010) 105–132, [arXiv:0912.1624].

[114] V. N. Velizhanin, *Six-Loop Anomalous Dimension of Twist-Three Operators in N=4 SYM*, *JHEP* **11** (2010) 129, [arXiv:1003.4717].

[115] V. N. Velizhanin, *Vanishing of the four-loop charge renormalization function in N=4 SYM theory*, *Phys. Lett.* **B696** (2011) 560–562, [arXiv:1008.2198].

[116] V. N. Velizhanin, *Double-logs, Gribov-Lipatov reciprocity and wrapping*, *JHEP* **08** (2011) 092, [arXiv:1104.4100].

[117] V. N. Velizhanin, *Twist-2 at five loops: Wrapping corrections without wrapping computations*, *JHEP* **06** (2014) 108, [arXiv:1311.6953].

[118] C. Marboe and V. Velizhanin, *Twist-2 at seven loops in planar $\mathcal{N} = 4$ SYM theory: full result and analytic properties*, *JHEP* **11** (2016) 013, [arXiv:1607.0604].

[119] N. Gromov, *On the Derivation of the Exact Slope Function*, *JHEP* **02** (2013) 055, [arXiv:1205.0018].

[120] N. Dorey, D. M. Hofman, and J. M. Maldacena, *On the singularities of the magnon S-matrix*, *Phys. Rev.* **D76** (2007) 025011, [hep-th/0703104].

[121] N. Beisert, R. Hernandez, and E. Lopez, *A crossing-symmetric phase for AdS(5) x S^5 strings*, *JHEP* **11** (2006) 070, [hep-th/0609044].

[122] P. Vieira and D. Volin, *Review of AdS/CFT Integrability, Chapter III.3: The Dressing factor*, *Lett.Math.Phys.* **99** (2012) 231–253, [arXiv:1012.3992].

[123] *http://wayback.cecm.sfu.ca/projects/EZFace/*, .

[124] A. V. Kotikov and L. N. Lipatov, *Pomeron in the N=4 supersymmetric gauge model at strong couplings*, *Nucl. Phys.* **B874** (2013) 889–904, [arXiv:1301.0882].

[125] R. C. Brower, M. Costa, M. Djuric, T. Raben, and C.-I. Tan, *Conformal Pomeron and Odderon in Strong Coupling*, in *International Workshop on Low X Physics (Israel 2013) Eilat, Israel, May 30-June 04, 2013*, 2013, [arXiv:1312.1419].

[126] R. A. Janik and P. Laskoś-Grabowski, *Approaching the BFKL pomeron via integrable classical solutions*, *JHEP* **01** (2014) 074, [arXiv:1311.2302].

[127] R. C. Brower, J. Polchinski, M. J. Strassler, and C.-I. Tan, *The Pomeron and gauge/string duality*, *JHEP* **0712** (2007) 005, [hep-th/0603115].

[128] A. Cavaglià, N. Gromov, and F. Levkovich-Maslyuk, *Quantum Spectral Curve and Structure Constants in N=4 SYM: Cusps in the Ladder Limit*, arXiv:1802.0423.

[129] J. Bartels, M. G. Ryskin, and G. P. Vacca, *On the triple pomeron vertex in perturbative QCD*, *Eur. Phys. J.* **C27** (2003) 101–113, [hep-ph/0207173].

Résumé

Dans la présente thèse, nous avons développé un cadre non-perturbatif général pour le spectre de Balitsky-Fadin-Kuraev-Lipatov (BFKL) de $\mathcal{N} = 4$ SYM planaire, basé sur la Courbe Spectrale Quantique (QSC). Il permet d'étudier le spectre dans toute la généralité pour les spins conformes arbitraires n. Notre approche ppermet de reproduire les résultats perturbatifs connus et obtenir les nouvelles prévisions.
En utilisant les méthodes du QSC provenant de l'intégrabilité de $\mathcal{N} = 4$ SYM, nous poursuivons analytiquement les dimensions d'échelle des opérateurs twist-2 et length-2 et reproduisons l'eigenvalue de BFKL Pomeron pour zéro et non zéro spins conformes. Aussi nous avons récupéré l'équation Baxter de Faddeev-Korchemsky pour la chaîne de spin de Lipatov dans les deux cas. Nos résultats sont un test non trivial de QSC décrivant le spectre exact de $\mathcal{N} = 4$ SYM planaire à l'infinité de boucles pour une quantité non-BPS et ouvrent la voie à une expansion systématique de le régime BFKL.
Nous obtenons de nouveaux résultats analytiques non-perturbatifs pour l'eigenvalue Pomeron prés de point $|n| = 1$ et la dimension $\Delta = 0$ et une formule explicite pour l'intercept BFKL pour un spin conforme arbitraire jusqu'à l'ordre des 3 et partiellement 4 boucles dans le limite du couplage petite. De plus, nous avons implémenté l'algorithme numérique QSC. Du résultat numérique nous avons dérivé une formule analytique pour l'expansion du couplage fort de l'intercept pour un spin conforme arbitraire.

Mots Clés

Théorie des champs intégrable
AdS/CFT correspondance
Chromodynamique quantique

Abstract

In the present thesis we developed a general non-perturbative framework for the Balitsky-Fadin-Kuraev-Lipatov (BFKL) spectrum of planar $\mathcal{N} = 4$ SYM, based on the Quantum Spectral Curve (QSC). It allows one to study the spectrum in the whole generality, extending previously known methods to arbitrary values of conformal spin n. We apply our approach to reproduce the known perturbative results and get new predictions.
Using the methods of the QSC originating from integrability of $\mathcal{N} = 4$ SYM we analytically continue the scaling dimensions of twist-2 and length-2 operators and reproduce the Pomeron eigenvalue of the BFKL equation for zero and non-zero conformal spins. Furthermore, we recovered the Faddeev-Korchemsky Baxter equation for Lipatov's spin chain in both cases. Our results provide a non-trivial test of QSC describing the exact spectrum in planar $\mathcal{N} = 4$ SYM at infinitely many loops for a highly non-trivial non-BPS quantity and also open a way for a systematic expansion in the BFKL regime.
We also get new non-perturbative analytic results for the Pomeron eigenvalue in the vicinity of $|n| = 1$ and dimension $\Delta = 0$ point and we obtained an explicit formula for the BFKL intercept function for arbitrary conformal spin up to the 3-loop order and partially 4-loop in the small coupling expansion. In addition, we implemented the QSC numerical algorithm. From the numerical result we managed to deduce an analytic formula for the strong coupling expansion of the intercept function for arbitrary conformal spin.

Keywords

Integrable field theory
AdS/CFT correspondence
Quantum Chromodynamics