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INSTITUT POLYTECHNIQUE DE PARIS

NNT : 2021IPPAX028

Thèse de doctorat

ÉCOLE POLYTECHNIQUE

IP PARIS

Dynamique en temps long pour des équations non linéaires de type onde avec ou sans amortissement

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École polytechnique

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Thèse présentée et soutenue à Palaiseau, le 21 juin 2021, par

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献给我的父母和伊萌。
To my parents and Yimeng.

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Chapter 0

Introduction (version française)

0.1 Généralités sur les équations de type onde non-linéaires

Le but de cette thèse est d'étudier les phénomènes non linéaires dans le comportement dynamique des solutions de type multi-solitons de certaines équations d'ondes non-linéaires focalisantes. Une équation d'onde non-linéaire avec ou sans amortissement est de la forme:

$$\partial_t^2 u(t,x) + 2\alpha\partial_t u(t,x) - \Delta_x u(t,x) + mu(t,x) = f(u(t,x)), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{NW}$$

où $m = 0$ ou 1, $\alpha \geqslant 0$ et $f(\cdot)$ est une non-linéarité focalisante avec $f(0) = 0$.

La question du problème de Cauchy local ou global pour les différentes équations d'onde en fonction de la dimension d et de la non-linéarité f est bien connue et n'est pas incluse dans cette thèse. Nous considérons des solutions d'énergie finie et nous nous renvoyons à [12, 59, 71] pour les équations de Klein-Gordon sous-critiques avec ou sans amortissement et les équations d'onde critiques.

Lorsque $\alpha = 0$ dans (NW), l'équation est une *équation hamiltonienne* mais avec un espace de phase de *dimension infinie*, par exemple $C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d)$. Plus précisément, (NW) peut se réécrire sous la forme:

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = JH \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix},$$

où

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\Delta + m & 0 \\ 0 & 1 \end{pmatrix}.$$

De plus, si on note $F(u) = \int_0^u f(s)\mathrm{d}s$, l'énergie E et le moment I sont conservés, où

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^d} \nabla u(t,x)\partial_t u(t,x)\mathrm{d}x, \\ E(t) &= \int_{\mathbb{R}^d} \left\{ |\nabla u(t,x)|^2 + mu^2(t,x) + (\partial_t u(t,x))^2 - 2F(u(t,x)) \right\} \mathrm{d}x. \end{aligned}$$

Une question centrale du domaine est la *conjecture de résolution en solitons*, énoncée dans les années 70 et 80 suite à des simulations numériques (voir Fermi–Pasta–Ulam [54], Zabusky–Kruskal [115]) et à la théorie intégrable de l'équation de Korteweg–de Vries (KdV) (voir [50, 51, 106]). De façon informelle, la conjecture stipule que toute solution globale de l'équation non linéaire se comporte asymptotiquement en temps long comme une somme de solitons découplés (*i.e.* multi-solitons) et une partie dispersive (comportement linéaire). La conjecture est très difficile, même pour les *modèles complètement intégrables*. À la connaissance de l'auteur la conjecture a été entièrement démontrée pour les modèles complètement intégrables par Eckhaus-Schuur [51] pour KdV, par Schuur [106] et Chen-Liu [19] pour mKdV, par Borghese-Jenkins-McLaughlin [9] pour l'équation NLS cubique en 1D et aussi par Jenkins-Liu-Perry-Sulem [70] pour NLS dérivée, via la méthode du *scattering inverse*.

Notons de plus que très peu de résultats existent pour les modèles dispersifs non-linéaires et de type onde qui ne sont pas complètement intégrables.

Introduisons maintenant l'un des modèles de type onde les plus importants de cette thèse, l'équation des ondes non linéaire (NLW) avec non-linéarité critique pour l'énergie en dimension $d = 4, 5$ (il s'agit d'un modèle non intégrable) :

$$\partial_{tt} u - \Delta u - |u|^{\frac{d}{d-2}} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \tag{NLW}$$

Au cours des 15 dernières années, des progrès remarquables ont été accomplis vers la compréhension de la dynamique en temps long de l'équation NLW critique. Nous passons ici en revue plusieurs résultats liés à la conjecture de résolution en solitons qui a été démontrée pour l'équation NLW critique. Tout d'abord, un résultat complet de résolution en solitons pour l'équation NLW critique radiale 3D est démontré dans Duyckaerts–Kenig-Merle [46] par la méthode des *canaux d'énergie* pour l'équation des ondes linéaire. La conjecture a été prouvée plus tard dans le cas non radial pour une sous-suite de temps pour l'équation NLW critique en dimensions 3,4 et 5 dans [43]. La décomposition obtenue dans [46] a été étendue à tous les cas radiaux de dimension impaire d'espace dans [49]. Plus récemment, Duyckaerts–Kenig-Martel-Merle [44] ont prouvé la conjecture pour le cas radial 4D. Voir [23, 24, 25] pour les résultats liés à la conjecture dans le cas des *wave maps*. Notons qu'une résolution complète de la conjecture pour l'équation non linéaire de Klein-Gordon, une autre équation de type onde importante, semble hors de portée pour le moment.

Un autre exemple d'équation de type onde non-linéaire que nous étudions dans cette thèse est l'équation non linéaire de Klein-Gordon (NLKG) avec double non-linéarité focalisante/défocalisante en dimension $d = 1$:

$$\partial_t^2 u - \partial_x^2 u + u - |u|^{p-1} u + |u|^{q-1} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1DNLKG}$$

où $1 < p < q < \infty$. La double non-linéarité garantit que l'effet non linéaire est défocalisant pour les petites valeurs de u et focalisant pour les grandes valeurs de u.

Nous étudions également l'équation NLKG amortie avec une non-linéarité sous-critique en dimension $1 \leqslant d \leqslant 5$:

$$\partial_{tt} u + 2\alpha \partial_t u - \Delta u + u - |u|^{p-1} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \tag{dampedNLKG}$$

où $\alpha > 0$ et $2 < p < \infty$.

Les principaux résultats de cette thèse se concentrent sur la dynamique en temps long des multi-solitons et fournissent des avancées solides et des exemples pour la conjecture de résolution de solitons sous divers aspects. Plus précisément, nous divisons cette thèse en trois thématiques :

(1) Stabilité des multi-solitons pour l'équation 1D NLKG (§0.2).

(2) Construction de multi-solitons pour l'équation NLW critique (§0.3).

(3) Dynamique en temps long pour NLKG amortie (§0.4). Il s'agit d'un projet en collaboration avec Raphaël Côte (IRMA, France), également en partie avec Yvan Martel (CMLS, France) et Lifeng Zhao (USTC, China).

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- R. Côte and X. Yuan, Asymptotics of solutions with a compactness property for the nonlinear damped Klein-Gordon equation. Preprint, *arXiv*:2102.11178, soumis.
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- X. Yuan, On multi-solitons for the energy-critical wave equation in dimension 5. *Nonlinearity* **32** (2019), no. 12, 5017–5048.
- X. Yuan, Construction of excited multi-solitons for the 5D energy-critical wave equation. A paraître dans *J. Hyperbolic Differ. Equ.*
- X. Yuan, Construction of excited multi-solitons for the focusing 4D cubic wave equation. Preprint, *arXiv*:2103.04168, soumis.

0.2 L'équation 1D NLKG avec double non-linéarité

0.2.1 Contexte

Considérons l'équation 1D NLKG avec une double non-linéarité focalisante/défocalisante,

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + u - |u|^{p-1}u + |u|^{q-1}u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u_{|t=0} = u_0 \in H^1(\mathbb{R}), \quad \partial_t u_{|t=0} = u_1 \in L^2(\mathbb{R}), \end{cases}$$

où $1 < q < p < \infty$. Pour exhiber la forme hamiltonienne, nous réécrivons cette équation comme un système du premier ordre en temps pour la fonction $\vec{u} = (u_1, u_2)$,

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = JH \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(u_1) \end{pmatrix}, \tag{0.1}$$

où

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(u_1) = |u_1|^{p-1}u_1 - |u_1|^{q-1}u_1.$$

On définit $F(u) = \int_0^u f(\sigma)\mathrm{d}\sigma = \frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|u|^{q+1}$. Pour toute solution $\vec{u} = (u_1, u_2)$ de (0.1) dans $H^1 \times L^2$, l'énergie $E(\vec{u})$ et le moment $\mathcal{I}(\vec{u})$ sont conservés, où

$$E(\vec{u}) = \int_{\mathbb{R}} \left\{ (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right\} \mathrm{d}x, \quad \mathcal{I}(\vec{u}) = 2\int_{\mathbb{R}} (\partial_x u_1) u_2 \mathrm{d}x.$$

Une solution stationnaire, *i.e.* solution indépendante du temps $\vec{u}(t,x) = (q(x), 0)$ de (0.1) est caractérisée comme une solution faible dans H^1 de l'ODE du deuxième ordre suivante

$$q'' - q + f(q) = 0 \quad \text{sur} \quad \mathbb{R}. \tag{0.2}$$

Grâce à [7], nous savons qu'il existe une unique solution (à translation près) dans H^1 de (0.2) qui est positive et paire. Cette unique solution paire est appelée *état fondamental*, et nous la notons Q. En appliquant la transformation de Lorentz à $\vec{Q} = (Q, 0)$, on obtient des ondes solitaires ou *solitons* : si pour ℓ vérifiant $-1 < \ell < 1$ on définit

$$Q_\ell(x) = Q\left(\frac{x}{\sqrt{1-\ell^2}}\right), \quad \vec{Q}_\ell = \begin{pmatrix} Q_\ell \\ -\ell \partial_x Q_\ell \end{pmatrix},$$

alors $\vec{u}(t,x) = \vec{Q}_\ell(x - \ell t)$ est une solution de (0.1).

Mentionnons que les solutions ci-dessus sont *instables* si l'on perturbe les données initiales et admettent une direction d'instabilité exponentielle. Plus précisément, grâce à la théorie de Sturm-Liouville, l'opérateur

$$\mathcal{L} = -\partial_x^2 + 1 - f'(Q)$$

apparaissant après linéarisation de l'équation (0.1) autour de $\vec{Q} = (Q, 0)$ a une valeur propre négative unique $-\nu_0^2$ ($\nu_0 > 0$), à laquelle on peut associer une fonction propre lisse et paire Y. Posons

$$\vec{Y}^+ = \begin{pmatrix} Y \\ \nu_0 Y \end{pmatrix} \quad \text{et} \quad \vec{u}^+(t,x) = \exp(\nu_0 t)\vec{Y}^+(x).$$

Un calcul explicite montre que la fonction $\vec{u}^+(t,x)$ est solution du système linéarisé

$$\begin{cases} \partial_t u_1 = u_2 \\ \partial_t u_2 = -\mathcal{L} u_1. \end{cases}$$

Puisque $\nu_0 > 0$, la solution $\vec{u}^+$ illustre l'instabilité exponentielle de l'onde solitaire $\vec{Q}$ en temps positif. Plus généralement, pour $-1 < \ell < 1$, il existe une direction instable autour de $\vec{Q}_\ell$ donnée par

$$Y_\ell = Y\left(\frac{x}{\sqrt{1-\ell^2}}\right) \quad \text{et} \quad \vec{Z}_\ell^+ = \begin{pmatrix} (\ell \partial_x Y_\ell + \frac{\nu_0}{\sqrt{1-\ell^2}} Y_\ell) e^{\frac{\ell \nu_0}{\sqrt{1-\ell^2}} x} \\ Y_\ell e^{\frac{\ell \nu_0}{\sqrt{1-\ell^2}} x} \end{pmatrix}.$$

Concernant les *multi-solitons* (solutions contenant un nombre arbitraire de solitons), l'une des premières intrusions dans cette étude pour (0.1) est le travail de Côte-Muñoz [31] où ils ont construit un exemple de multi-solitons basé sur l'état fondamental pour (0.1). Plus précisément:

Théorème 0.1 (Existence de multi-solitons, [31]). *Soit $\ell_1, \cdots, \ell_N$ un ensemble de vitesses vérifiant $\ell_n \neq \ell_{n'}$ pour tout $n \neq n'$ et $x_1, \cdots, x_N$ des paramètres de décalage. Il existe un temps $T_0 > 0$, des constantes $C > 0$, et $\gamma_0 > 0$ dépendant uniquement de $(\ell_n)_n$ et $(x_n)_n$ et une solution $\vec{u}(t) = (u_1(t), u_2(t)) \in \mathcal{C}\left([T_0, \infty) : H^1 \times L^2\right)$ de (0.1), définie sur $[T_0, \infty)$ et satisfaisant pour tout $t \geqslant T_0$*

$$\left\|\vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_{\ell_n}(x - \ell_n t - x_n)\right\|_{H^1 \times L^2} \leqslant C e^{-\gamma_0 t}.$$

Remarque 0.2. Côte-Muñoz [31] ont étudié le cas plus général d'une dimension d'espace quelconque $d \in \mathbb{N}^+$ avec des non-linéarités plus générales, incluant le cas de la double non-linéarité. En parallèle, Bellazzini-Ghimenti-Le Coz [5] ont construit des multi-solitons stables à valeurs complexes pour l'équation NLKG. Plus tard, Côte-Martel [26] ont étendu le résultat d'existence de [31] à n'importe quel *état excité* pour l'équation NLKG avec non-linéarité de puissance pure.

0.2.2 Résultat

Au paragraphe §0.2.1, nous avons observé que le soliton unique est instable, ce qui implique que les multi-solitons construits dans [31] le sont également. Par conséquent, lorsque nous étudions la stabilité de multi-solitons solutions de (0.1) avec des vitesses différentes, on ne peut obtenir mieux que la *stabilité conditionnelle* : toute donnée initiale proche de celle d'un multi-soliton donné peut être modifiée le long de chaque direction exponentiellement instable de telle sorte que la solution associée à ces données initiales modifiées soit finalement stable en temps longs et converge vers une solution multi-soliton proche avec des vitesses distinctes.

Le résultat principal de cette section est la stabilité conditionnelle des multi-solitons pour (0.1), comme mentionné ci-dessus. Plus précisément:

Théorème 0.3 (Stabilité conditionnelle des multi-solitons, [113]). *Soit $N \geqslant 2$. Pour tous $n \in \{1, \dots, N\}$, soit $\sigma_n = \pm 1$, $\ell_n \in (-1, 1)$ avec $-1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1$. Il existe des constantes $L_0 > 0$, $C_0 > 0$, $\gamma_0 > 0$ et $\delta_0 > 0$ telles que ce qui suit est vrai. Soit $\vec{\varepsilon} \in H^1 \times L^2$ et $y_1^0 < \cdots < y_N^0$ tel qu'il existe $L > L_0$ et $0 < \delta < \delta_0$ avec*

$$\|\vec{\varepsilon}\|_{H^1 \times L^2} < \delta \quad \text{et} \quad y_{n+1}^0 - y_n^0 > L \quad \text{pour tous } n = 1, \dots, N-1.$$

Alors, il existe $h_1^+, \dots, h_N^+ \in \mathbb{R}$ satisfaisant

$$\sum_{n=1}^{N} |h_n^+| \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big),$$

tels que la solution $\vec{u} = (u_1, u_2)$ de (0.1) avec la donnée initiale

$$\vec{u}_0 = \sum_{n=1}^{N} \big(\sigma_n \vec{Q}_{\ell_n} + h_n^+ \vec{Z}_{\ell_n}^+\big)(\cdot - y_n^0) + \vec{\varepsilon}$$

est définie globalement dans $H^1 \times L^2$ pour $t \geqslant 0$ et, pour tout $t \geqslant 0$,

$$\Big\| \vec{u}(t) - \sum_{n=1}^{N} \sigma_n \vec{Q}_{\ell_n}(\cdot - y_n(t)) \Big\|_{H^1 \times L^2} \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big),$$

où $y_1(t), \dots, y_N(t)$ sont des fonctions de classe C^1 satisfaisant, pour tout $n = 1, \dots, N$, $t \geqslant 0$,

$$|y_n(0) - y_n^0| \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big), \quad |\dot{y}_n(t) - \ell_n| \leqslant C_0^2\big(\delta + e^{-\gamma_0 L}\big).$$

Remarque 0.4. La non-linéarité double puissance est un choix typique pour les équations dispersives ou d'onde, mais le Théorème 0.3 peut être étendu au cas plus général d'une fonction à valeurs réelles $f \in C^{1,\alpha}$ (avec $\alpha > 0$) satisfaisant :

(i) f est impaire et $f(0) = f'(0) = 0$,

(ii) l'ensemble des $s > 0$ tels que $F(s) - \frac{1}{2}s^2 = 0$ et $f(s) - s > 0$ admet un minimum $s_0 > 0$,

(iii) il existe $r_0 > 0$ tel que pour tout $s \in (-r_0, r_0)$, $sf(s) - 2F(s) \leqslant 0$,

où $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$ pour $s \in \mathbb{R}$. Les conditions (i)-(ii) sont liées à la condition nécessaire et suffisante pour l'existence d'une onde stationnaire unique dans [7, Theorem 5]. Suivant [7, Example 2, p. 318], si f représente la non-linéarité double puissance, alors l'existence d'un plus petit $s_0 > 0$ tel que $F(s_0) - \frac{1}{2}s_0^2 = 0$ est clair et de plus

$$\begin{aligned} f(s_0) - s_0 = s_0^p - s_0^q - s_0 &= \frac{1}{s_0}\left((p+1)F(s_0) + \frac{p-q}{q+1}s_0^{q+1} - s_0^2\right) \\ &= \frac{1}{s_0}\left(\frac{p-q}{q+1}s_0^{q+1} + \frac{p-1}{2}s_0^2\right) > 0. \end{aligned}$$

Cela justifie l'existence d'une onde stationnaire Q pour (0.2). La condition (iii) garantit que l'effet non linéaire est défocalisant pour les petites valeurs de u_1, ce qui est décisif dans notre argument du viriel. Rappelons que ce problème est déjà présent dans [89, 104] pour l'équation NLS, où une restriction plus faible, mais similaire est imposée sur la non-linéarité.

À la connaissance de l'auteur, ce résultat est le premier énoncé de stabilité (conditionnelle) des sommes de solitons découplés pour une équation de type onde. La preuve est basée sur une stratégie générale introduite dans Martel-Merle-Tsai [88, 89] pour l'équation de Korteweg-de Vries généralisée (gKdV) et l'équation de Schrödinger non linéaire (NLS) dans un contexte similaire. Cependant, l'adaptation de cette stratégie à une équation de type onde nécessite l'introduction d'une nouvelle fonctionnelle d'énergie adaptée à la transformée de Lorentz. Plus précisément, par rapport à la méthode d'énergie développée dans [89], nous utilisons une fonctionnelle basée sur des versions localisées de l'impulsion et de l'énergie autour de chaque soliton. De plus, une combinaison spécifique de ces grandeurs doit être utilisée dans le cas l'équation des ondes.

Comme nous l'avons remarqué dans §0.2.1, chaque soliton $\vec{u}(t,x) = \pm\vec{Q}_{\ell_n}(x-\ell_n t)$ solution de l'équation (0.1) possède exactement une direction d'instabilité exponentielle en temps positif. Rappelons qu'une telle instabilité a conduit à la construction d'une variété de codimension un centrée stable pour des données initiales proches de celles d'un soliton simple pour NLKG ou les équations d'onde dans [75, 100, 101]. Dans le cas d'une perturbation de la somme de N solitons découplés $\sum_{n=1}^{N}\sigma_n Q_{\ell_n}(\cdot - y_n)$, cela justifie la nécessité d'ajuster les paramètres libres $(h_n^+)_{n\in\{1,\dots,N\}}$ au moment du choix des données initiales pour obtenir l'existence globale et la stabilité dans le Théorème 0.3. De telles difficultés n'apparaissent pas lorsque les solitons sont stables, comme pour les équations (gKdV) et (NLS) dans les cas sous-critiques pour la masse.

L'étude de *la stabilité conditionnelle des multi-solitons pour 1DNLKG avec non-linéarité puissance pure* est toujours un problème ouvert. Dans [113], c'est l'utilisation de l'argument du viriel qui nous a conduit à choisir la non-linéarité de la double puissance. Cependant, la non-linéarité de puissance pure $f(u) = |u|^{p-1}u$ aurait probablement plus d'intérêt d'un point de vue physique. La preuve de la propriété de monotonie n'est pas connue dans ce cas. Notons que le contrôle des effets des non-linéarités focalisantes de faible puissance est une difficulté classique qui a déjà été rencontré dans des travaux antérieurs sur la stabilité asymptotique des solitons comme [13, 14]. Un autre problème ouvert lié au résultat de [113] est *l'extension du résultat de stabilité conditionnelle à l'équation NLKG en dimension supérieure.* Il convient de mentionner que la difficulté d'étudier le cas d'une dimension d'espace supérieure à 1 se situe dans configurations géométriques d'ondes solitaires. Cela conduit à une modification de la fonctionnelle d'énergie pour l'argument du viriel. De plus, mentionnons que l'étude de *la stabilité des paires kink-antikink pour les champs scalaires* $1+1$ est un problème ouvert. Plus précisément, pour une autre famille d'équations de type ondes, à savoir les *champs scalaires* $1+1$ (par exemple, le modèle ϕ_{2n} pour $n \in \mathbb{N}^+$), des solutions spéciales, nommées *kinks*, existent. La stabilité d'un seul *kink* dans l'espace énergétique est prouvée et il est donc naturel d'étudier la stabilité des paires *kink-antikink* découplées. Un point clé ici, similaire au cas de 1DNLKG avec une non-linéarité de puissance pure, sera de contrôler les non-linéarités de faible puissance (voir les problèmes et hypothèses associés dans [72, 73]).

0.3 L'équation NLW avec non-linéarité critique

0.3.1 Contexte

Considérons l'équation NLW avec une non-linéarité critique pour l'énergie en dimension $d = 4, 5$,

$$\begin{cases} \partial_{tt}u - \Delta u - |u|^{\frac{4}{d-2}}u = 0, \quad (t,x) \in [0,\infty)\times\mathbb{R}^d, \\ u_{|t=0} = u_0 \in \dot{H}^1(\mathbb{R}^d), \quad \partial_t u_{|t=0} = u_1 \in L^2(\mathbb{R}^d). \end{cases} \tag{0.3}$$

Soit $F(u) = \frac{d-2}{2d}|u|^{\frac{2d}{d-2}}$. Pour toute solution $\vec{u} = (u, \partial_t u)$ dans $\dot{H}^1 \times L^2$, l'énergie E et le moment P sont conservés le long du flot, où

$$E(u,\partial_t u) = \frac{1}{2}\int_{\mathbb{R}^d}\Big(|\nabla u|^2 + (\partial_t u)^2 - 2F(u)\Big)\mathrm{d}x, \quad P(u,\partial_t u) = \frac{1}{2}\int_{\mathbb{R}^d}(\partial_t u \nabla u)\mathrm{d}x.$$

La recherche de solutions stationnaires $u(t,x) = q(x)$ de (0.3) dans $\dot{H}^1(\mathbb{R}^d)$ se réduit à l'équation critique de Yamabe

$$\Delta q + f(q) = 0, \quad q \in \dot{H}^1(\mathbb{R}^d). \tag{0.4}$$

On définit l'ensemble des solutions de (0.4) d'énergie finie non-nulle par

$$\Sigma = \left\{ q \in \dot{H}^1(\mathbb{R}^d)/0 : \Delta q + f(q) = 0 \quad \text{sur } \mathbb{R}^d \right\}.$$

Pour $q \in \Sigma$ et $\boldsymbol{\ell} \in \mathbb{R}^5$ tels que $|\boldsymbol{\ell}| < 1$, soit

$$q_{\boldsymbol{\ell}}(x) = q\left(\left(\frac{1}{\sqrt{1-|\boldsymbol{\ell}|^2}} - 1 \right) \frac{\boldsymbol{\ell}(\boldsymbol{\ell}\cdot x)}{|\boldsymbol{\ell}|^2} + x \right), \tag{0.5}$$

alors $u(t,x) = q_{\boldsymbol{\ell}}(x - \boldsymbol{\ell} t)$ est une solution globale et bornée de (0.3).

On sait, grâce aux résultats de [3, 15, 102, 108] qu'au signe près et à dilatation et translation près, la seule solution stationnaire *positive* de (0.3) est *l'état fondamental* W, où W a la forme explicite suivante :

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}} \quad \text{sur } \mathbb{R}^d.$$

Les fonctions $q \in \Sigma$ avec $q \neq W$ (au signe, à dilatation et translation près) sont appelées *états excités*. L'existence d'états excités non-radiaux qui changent de signe et avec une énergie arbitrairement grande a été prouvée pour la première fois par Ding [42] en utilisant un argument variationnel. Plus tard, des constructions plus explicites de solutions changeant de signe ont été développées par del Pino-Musso-Pacard-Pistoia [36]. Plus spécifiquement, ils ont construit des solutions constituées d'un soliton centré couronné par des pointes négatives (i.e des solitons redimensionnés) aux sommets d'un polygone régulier de rayon 1. Puis dans [37], suivant une stratégie similaire à [36], ils ont construit des solutions non-radiales et changeant de signe pour l'équation de type Yamabe sur la sphère $\mathbb{S}^d$ ($d \geqslant 4$), l'énergie de ces solutions étant concentrée le long de sous-variétés spéciales de $\mathbb{S}^d$.

Nous étudions d'abord toutes les invariances de l'équation (0.4) en suivant la présentation de [48]. Elles sont au nombre de quatre :

(i) translation : Si $q \in \Sigma$ alors $q(x + x_0) \in \Sigma$ pour tout $x_0 \in \mathbb{R}^d$;

(ii) dilatation : Si $q \in \Sigma$ alors $\lambda^{\frac{d-2}{2}} q(\lambda x) \in \Sigma$ pour tout $\lambda \in (0, \infty)$;

(iii) transformation orthogonale: Si $q \in \Sigma$ alors $q(Ox) \in \Sigma$ pour tout $O \in \mathcal{O}_d$;

(iv) transformation de Kelvin: Si $q \in \Sigma$ alors $\mathrm{K}q \in \Sigma$ où $\mathrm{K}q(x) = \frac{1}{|x|^{d-2}} q\left(\frac{x}{|x|^2}\right)$.

On note $\mathcal{M}$ le groupe d'isométries de $\dot{H}^1$ généré par les quatre transformations précédentes. En suivant [48, Section 3], nous voyons que $\mathcal{M}$ définit une famille à N_d paramètres de transformations dans un voisinage de l'identité, où la dimension N_d est donné par

$$N_d = 2d + 1 + \frac{d(d-1)}{2}, \quad \text{for } d = 4, 5.$$

Pour $q \in \Sigma$, on définit l'opérateur linéarisé autour de q par

$$\mathcal{L}_q = -\Delta - f'(q) = -\Delta - \frac{d+2}{d-2}|q|^{\frac{4}{d-2}}.$$

On définit le noyau de $\mathcal{L}_q$

$$\mathcal{Z}_q = \left\{ f \in \dot{H}^1(\mathbb{R}^d) : \mathcal{L}_q f = 0 \right\}.$$

On note $\widetilde{\mathcal{Z}}_q$ le sous-espace de $\mathcal{Z}_q$ généré par les éléments de $\mathcal{M}$, c'est-à-dire

$$\widetilde{\mathcal{Z}}_q = \mathrm{Span}\Big\{\frac{d-2}{2}q + x\cdot\nabla q; (x_i\partial_{x_j}q - x_j\partial_{x_i}q), 1\leqslant i<j\leqslant d;$$
$$\partial_{x_i}q; (2-d)x_iq + |x|^2\partial_{x_i}q - 2x_i(x\cdot\nabla q), 1\leqslant i\leqslant d\Big\}.$$

Notons que la dimension de $\widetilde{\mathcal{Z}}_q$ est au plus N_d, mais elle peut être strictement inférieure à N_d si q a des symétries supplémentaires. Par exemple, dans le cas de la solution radiale positive $q = W$, on a

$$\widetilde{\mathcal{Z}}_W = \mathrm{Span}\left\{\frac{d-2}{2}W + x\cdot\nabla W, \partial_{x_i}W, 1\leqslant i\leqslant d\right\} \quad \text{et} \quad \dim\widetilde{\mathcal{Z}}_W = d+1.$$

Duyckaerts–Kenig–Merle [48] introduisent la définition suivante de la non-dégénérescence pour une solution de (0.4) :

Définition 0.5. Soit $q\in\Sigma$, q est un *état non-dégénéré* si $\mathcal{Z}_q = \widetilde{\mathcal{Z}}_q$.

Remarque 0.6. Le premier exemple non-trivial d'état non-dégénéré changeant de signe a été obtenu dans Musso-Wei [99]. Plus précisément, ils ont prouvé que les états changeant de signe construits dans [36] ne sont pas dégénérés. Plus tard, en adoptant une stratégie similaire, Medina-Musso-Wei [91] ont construit l'état non-dégénéré avec un rang maximal, i.e $\dim\widetilde{\mathcal{Z}}_q = N_d$, pour le cas $d=4$.

Remarque 0.7. L'hypothèse de non-dégénérescence joue un rôle important dans la preuve de la *conjecture de rigidité* pour les solutions de (0.3) avec la *propriété de compacité*. De manière informelle, la conjecture de rigidité dit que les ondes solitaires $q_{\boldsymbol{\ell}}$ avec $q\in\Sigma$ et $|\boldsymbol{\ell}|<1$ sont les seules solutions de (0.3) possédant la propriété de compacité. Elle n'a été prouvée que sous l'hypothèse de non-dégénérescence dans [48].

Nous introduisons également la définition suivante liée au taux de décroissance de $q\in\Sigma$.

Définition 0.8. Soit $q\in\Sigma$, q est un *état décroissant* si $|q(x)|\lesssim\langle x\rangle^{-(d-1)}$ sur $\mathbb{R}^d$.

Remarque 0.9. Comme remarqué dans [48, Remark 3.2], l'existence d'états décroissants se déduit par l'application de translations et de transformations de Kelvin à l'état changeant de signe construit dans [36].

Comme nous l'avons mentionné dans §0.1, la conjecture de résolution en solitons dans le cas radial 3D a été entièrement résolue dans Duyckaerts-Kenig-Merle [46], et a également été prouvée pour une sous-suite de temps en 3, 4 et 5D dans Duyckaerts-Jia–Kenig–Merle [43]. Récemment, Duyckaerts–Kenig–Martel-Merle [44] ont prouvé la conjecture pour le système wave-maps corotationnel critique et l'équation d'onde cubique radiale en dimension 4. Puisque cette partie de la thèse étudie les dimensions $d=4,5$ sans hypothèse radiale, nous ne rappelons ici que le résultat de la décomposition dans [43]. Plus précisément:

Théorème 0.10 (Résolution en Solitons pour une sous-suite de temps, [43])**.** *Soit $d=4,5$ dans (0.3). Soit $\vec{u}=(u,\partial_t u)\in C\big([0,\infty):\dot{H}^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\big)$, avec $u\in L_t^{\frac{d+2}{d-2}}L_x^{2\frac{d+2}{d-2}}([0,T)\times\mathbb{R}^d)$ pour tout $0<T<\infty$, une solution globale de (0.3) qui satisfait*

$$\sup_{t\in[0,\infty)}\|\vec{u}(t)\|_{\dot{H}^1\times L^2}<\infty.$$

Alors il existe une solution d'énergie finie $\vec{u}^L=(u^L,\partial_t u^L)$ à l'équation d'onde linéaire

$$\partial_{tt}u-\Delta u=0, \quad \textit{dans } \mathbb{R}\times\mathbb{R}^d,$$

et un entier $J_* \geqslant 0$; *une suite de temps* $t_n \to \infty$, *de mises à l'échelle* λ_n^j *avec* $\lambda_n^j > 0$ *et* $\lim_{n\to\infty} \frac{\lambda_n^j}{t_n} = 0$; *de positions* $\boldsymbol{c}_n^j \in \mathbb{R}^d$ *satisfaisant* $\boldsymbol{c}_n^j \in \mathcal{B}_{\beta t_n}(0)$ *pour un certain* $\beta \in (0,1)$ *avec* $\boldsymbol{\ell}_j = \lim_{n\to\infty} \dfrac{\boldsymbol{c}_n^j}{t_n}$ *bien défini*; *et* $Q^j \in \Sigma$ *pour* $1 \leqslant j \leqslant J_*$, *tel que*

$$\lim_{n\to\infty} \left\| \vec{u}(t_n) - \vec{u}^L(t_n) - \sum_{j=1}^{J_*} \vec{Q}_{\boldsymbol{\ell}_j}^{j,\lambda_n^j}\left(x - \boldsymbol{c}_n^j\right) \right\|_{\dot{H}^1 \times L^2} = 0,$$

où

$$\vec{Q}_{\boldsymbol{\ell}_j}^{j,\lambda_n^j} = \left(\frac{1}{(\lambda_n^j)^{\frac{d}{2}-1}} Q_{\boldsymbol{\ell}_j}^j \left(\frac{\cdot}{\lambda_n^j}\right), -\frac{1}{(\lambda_n^j)^{\frac{d}{2}}} \left(\boldsymbol{\ell}_n \cdot \nabla Q_{\boldsymbol{\ell}_j}^j\right) \left(\frac{\cdot}{\lambda_n^j}\right) \right).$$

Remarque 0.11. Si $J_* = 0$ dans l'énoncé ci-dessus, il n'y a pas de soliton dans la décomposition et la solution se comporte asymptotiquement comme une solution linéaire. La théorie de la diffusion de (0.3) a été développe avec succès en utilisant la méthode de concentration-compacité. Voir Kenig-Merle [71] et les références qui y figurent.

Une question naturelle, motivée par [43], est celle de la construction d'une solution globale de (0.3) avec comportement en temps long de type multi-solitons. Une telle construction pour l'équation d'onde à énergie critique est techniquement plus difficile que pour les autres modèles dispersifs et de type ondulatoire, en raison de la faible décroissance algébrique des solitons. Cette construction a été réalisée pour la première fois par Martel-Merle [85] pour le cas 5D. Plus précisément :

Théorème 0.12 (Existence de multi-solitons pour NLW critique 5D, [85])**.** *Soit* $d = 5$ *dans* (0.3) *et* $N \geqslant 2$. *Pour* $n \in \{1, \cdots, N\}$, *soit* $\lambda_n^\infty > 0$, $\boldsymbol{y}_n^\infty \in \mathbb{R}^5$, $\tau_n = \pm 1$ *et* $\boldsymbol{\ell}_n \in \mathbb{R}^5$ *avec* $|\boldsymbol{\ell}| < 1$, $\boldsymbol{\ell}_n \neq \boldsymbol{\ell}_{n'}$ *pour* $n \neq n'$. *Supposons que l'une des hypothèses suivantes soit vraie*

(A) Deux solitons : $N = 2$.

(B) Vitesses colinéaires : *Pour tout* $n \in \{1, \ldots, N\}$, $\boldsymbol{\ell}_n = \ell_n \mathbf{e}_1$ *où* $\ell_n \in (-1, 1)$.

Alors, il existe $T_0 > 0$ *et une solution* $\vec{u} = (u, \partial_t u)$ *de* (0.3) *sur* $[T_0, \infty)$ *dans l'espace énergétique tel que*

$$\lim_{t\to\infty} \left\| u(t) - \sum_{n=1}^{N} \frac{\tau_n}{(\lambda_n^\infty)^{\frac{3}{2}}} W_{\boldsymbol{\ell}_n} \left(\frac{\cdot - \boldsymbol{\ell}_n t - \boldsymbol{y}_n^\infty}{\lambda_n^\infty} \right) \right\|_{\dot{H}^1} = 0,$$

$$\lim_{t\to\infty} \left\| \partial_t u(t) + \sum_{n=1}^{N} \frac{\tau_n}{(\lambda_n^\infty)^{\frac{5}{2}}} \left(\boldsymbol{\ell}_n \cdot \nabla W_{\boldsymbol{\ell}_n}\right) \left(\frac{\cdot - \boldsymbol{\ell}_n t - \boldsymbol{y}_n^\infty}{\lambda_n^\infty} \right) \right\|_{L^2} = 0.$$

Remarque 0.13. Nous nous attendons à ce que le Théorème 0.12 soit toujours vrai pour l'équation d'onde critique pour des dimensions d'espace $d \geqslant 6$. Comme mentionné dans l'introduction de [85] tous les calculs importants de [85] peuvent être formellement reproduits dans le cas $d \geqslant 6$. Cependant, cette preuve ne fonctionne pas directement dans le cas $d \geqslant 6$ pour la principale raison que la dérivée de la non-linéarité n'est dans ce cas plus C^1. Rappelons que de telles difficultés ont déjà été rencontrées et surmontées pour le problème de Cauchy dans l'espace énergétique $\dot{H}^1 \times L^2$ dans [10].

Remarque 0.14. Dans l'introduction de [85], les auteurs conjecturent que dans le cas 3D et 4D il n'existe pas de solution N-soliton basée sur l'état fondamental comme dans Théorème 0.12, et ce pour tout $N \geqslant 2$. Leur principal argument est qu'en raison de la lente décroissance de l'état fondamental, l'interaction entre deux solitons de vitesses différentes est t^{2-d}, *i.e.* t^{-1} dans la dimension 3 et t^{-2} dans la dimension 4. En suivant la stratégie générale de [85], ces interactions seraient trop fortes et créeraient des termes divergents dans la construction envisagée. Cependant, pour prouver rigoureusement l'inexistence de solutions multi-solitons, il faudrait d'avantage d'informations préalables sur toute solution multi-soliton, ce qui constitue encore un problème ouvert pour $d \geqslant 3$.

0.3.2 Résultats

Dans ce domaine, mes deux premiers résultats concernent principalement la construction de deux types différents de multi-solitons pour l'équation NLW critique 5D, dans diverses conditions sur les vitesses de Lorentz, basées soit sur l'état fondamental, soit sur des états excités appropriés.

Le premier résultat est la construction de multi-solitons tels que chaque onde progressive est générée par la transformée de Lorentz de l'état fondamental, sous l'hypothèse que les vitesses de Lorentz satisfassent une condition de petitesse explicite, mais sans hypothèse géométrique comme (B) Théorème (0.12).

Théorème 0.15 (Existence de multi-solitons pour le cas 5D, [111]). *Soit $d = 5$ dans (0.3) et $N \geqslant 2$. Pour $n \in \{1, \dots, N\}$, soit $\lambda_n^\infty > 0$, $\mathbf{y}_n^\infty \in \mathbb{R}^5$, $\epsilon_n \in \{\pm 1\}$, $\boldsymbol{\ell}_n = \sum_{i=1}^5 \ell_{n,i}\mathbf{e}_i$ avec $|\boldsymbol{\ell}_n| < 1$ tels que $\boldsymbol{\ell}_n \neq \boldsymbol{\ell}_{n'}$ si $n \neq n'$, et*

$$\left(\sum_{i=1}^5 \left(\max_{n\in\{1,\dots,N\}} \ell_{n,i}^2\right)\right)^{\frac{1}{2}} < \frac{3}{5}. \tag{0.6}$$

Alors, il existe $T_0 > 0$ et une solution u de (0.3) sur $[T_0, +\infty)$ dans l'espace d'énergie telle que

$$\lim_{t\to+\infty} \left\| u(t) - \sum_{n=1}^N \frac{\epsilon_n}{(\lambda_n^\infty)^{3/2}} W_{\boldsymbol{\ell}_n}\left(\frac{\cdot - \boldsymbol{\ell}_n t - \mathbf{y}_n^\infty}{\lambda_n^\infty}\right)\right\|_{\dot{H}^1} = 0,$$

$$\lim_{t\to+\infty} \left\| \partial_t u(t) + \sum_{n=1}^N \frac{\epsilon_n}{(\lambda_n^\infty)^{5/2}} (\boldsymbol{\ell}_n \cdot \nabla W_{\boldsymbol{\ell}_n})\left(\frac{\cdot - \boldsymbol{\ell}_n t - \mathbf{y}_n^\infty}{\lambda_n^\infty}\right)\right\|_{L^2} = 0.$$

Remarque 0.16. La preuve du Théorème 0.15 est basée sur l'extension de la méthode de construction de multi-solitons asymptotiques utilisée dans [85, 86]. La condition de petitesse (0.6) est probablement une condition technique et nous conjecturons que le résultat d'existence est valable pour tout choix de vitesses. Cependant, pour notre méthode, (0.6) semble être la meilleure condition utilisant la solution approchée raffinée de [86]. De plus, construire des solutions approchées améliorant celle de [86] semble être une tâche difficile.

Ensuite, en nous inspirant de Côte-Martel [26], nous avons étudié la possibilité de construire des multi-solitons à partir d'états excités, ce qui conduit au deuxième résultat de cette partie. Plus précisément, nous avons construit des multi-solitons différents de ceux de [85, 111] : chaque onde progressive est générée par la transformée de Lorentz d'un état excité non dégénéré et décroissant, avec des vitesses de Lorentz distinctes mais colinéaires.

Théorème 0.17 (Existence d'états multi-excités pour le cas 5D, [112]). *Soit $d = 5$ dans (0.3), $N \geqslant 2$ et soit $\mathbf{e}$ un vecteur unitaire de $\mathbb{R}^5$. Pour $n \in \{1, \dots, N\}$, soit $\boldsymbol{\ell}_n \in \mathbb{R}^5$ tel que*

$$\boldsymbol{\ell}_n = \ell_n \mathbf{e} \text{ où } -1 < \ell_n < 1 \text{ et pour tout } n \neq n',\ \ell_n \neq \ell_{n'}.$$

Soit $q_1, \dots, q_N$ des états décroissants non dégénérés de (0.4). Alors il existe $T_0 > 0$ et une solution $\vec{u} = (u, \partial_t u)$ de (0.3) dans l'espace d'énergie $\dot{H}^1 \times L^2$, défini sur $[T_0, +\infty)$ tel que

$$\lim_{t\to+\infty} \left\| \vec{u}(t) - \sum_{n=1}^N \vec{Q}_n(t)\right\|_{\dot{H}^1\times L^2} = 0,$$

où pour $n = 1, \dots, N$,

$$\vec{Q}_n(t,x) = \begin{pmatrix} q_{n,\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \\ -\ell_n \partial_{x_1} q_{n,\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \end{pmatrix}.$$

Enfin, nous considérons le cas plus difficile de la dimension 4. Comme nous l'avons mentionné dans §0.3.1, nous nous attendons à ce qu'il n'existe pas de multi-soliton pour le cas 4D basé sur l'état fondamental. Ici, nous considérons un état non dégénéré spécifique plutôt que l'état fondamental. Plus précisément:

Théorème 0.18 (Existence de multi-solitons excités pour le cas 4D, [114]). *Soit $d = 4$ dans (0.3) et (0.4). Alors, il existe un état excité non dégénéré $Q \in \Sigma$ tel que ce qui suit est vrai. Soit $N \geqslant 2$ et $\mathbf{e}_1$ le premier vecteur de la base canonique de $\mathbb{R}^4$. Pour $n = 1, \cdots, N$, soit $\tau_n = \pm 1$ et $\boldsymbol{\ell}_n \in \mathbb{R}^4$ tels que*

$$\boldsymbol{\ell}_n = \ell_n \mathbf{e}_1 \text{ avec } -1 < \ell_n < 1 \text{ et pour tout } n \neq n',\ \ell_n \neq \ell_{n'}.$$

Alors, il existe $T_0 > 0$ et une solution $\vec{u} = (u, \partial_t u)$ de (0.3) sur $[T_0, +\infty)$ dans l'espace énergétique $\dot{H}^1 \times L^2$ tel que

$$\begin{aligned} \lim_{t \to +\infty} \left\| u(t) - \sum_{n=1}^{N} \tau_n Q_{\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \right\|_{\dot{H}^1} &= 0, \\ \lim_{t \to +\infty} \left\| \partial_t u(t) + \sum_{n=1}^{N} \tau_n \ell_n \partial_{x_1} Q_{\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \right\|_{L^2} &= 0. \end{aligned} \tag{0.7}$$

Remarque 0.19. L'état non dégénéré spécifique Q impliqué dans le Théorème 0.18 est obtenu à partir de l'état changeant de signe non dégénéré construit dans [36, 99]. En particulier, par rotation, translation et transformation de Kelvin sur un état changeant de signe non dégénéré, nous obtenons un état excité non dégénéré qui décroît assez rapidement au moins le long de $\mathbf{e}_1$. L'interaction de deux telles ondes progressives avec des vitesses différentes le long de $\mathbf{e}_1$, est formellement d'ordre t^{-4} pour t grand. Ce taux nous permet de fermer les estimations d'énergie. Cependant, la lente décroissance algébrique du noyau de l'opérateur linéarisé conduit à une autre difficulté technique. Pour la surmonter, nous avons besoin d'une compréhension pointue du comportement asymptotique du noyau de l'opérateur linéarisé.

Remarque 0.20. En effet, le Théorème d'existence 0.18 est également vrai pour le cas plus général où chaque onde progressive est basée sur q non dégénéré avec le même comportement asymptotique que Q. En particulier, il n'est pas nécessaire que chaque onde progressive soit basée sur le même état non dégénéré.

Les preuves des deux résultats d'existence mentionnés ci-dessus reposent sur la stratégie générale utilisant des estimations uniformes et la compacité introduite dans Merle [92] pour l'équation de Schrödinger non-linéaire L^2 critique (NLS) et Martel [83] pour l'équation de Korteweg-de Vries généralisé sous-critique et critique (gKdV). La stratégie de ces travaux a été étendue à d'autres modèles dispersifs et ondulatoires (voir par exemple [5, 26, 28, 31, 53, 84, 96]).

À la connaissance de l'auteur, le Théorème 0.18 est le premier résultat de l'existence de solutions multi-solitons pour l'équation des ondes critiques 4D. Comme nous l'avons mentionné dans §0.3.1, la construction de telles solutions pour l'équation des ondes critiques est en particulier motivée par les décompositions obtenues dans [44, 43, 46]. En effet, les résultats d'existence de [85, 111, 112, 114] complètent les énoncés de [43] en présentant des exemples de solutions contenant N solitons à $t \to +\infty$ pour tout $N \geqslant 2$. Dans la même direction, rappelons que pour l'équation d'onde critique 6D, Jendrej [67] a prouvé l'existence de solutions deux bulles radiales. Plus tard, pour l'équation d'onde critique 5D, Jendrej-Martel [69] ont prouvé l'existence de solutions N-bubble pour tout $N \geqslant 2$, en des points différents. Rappelons également que, pour le cas 4D, Hillairet-Raphaël [62] ont prouvé l'existence de solutions de type II contenant un seul soliton et explosant en temps fini.

On se réfère à [38, 39, 40] pour les résultats liés à l'existence de solutions de type II contenant une ou plusieurs bulles et explosant en temps fini pour l'équation de la chaleur critique. En particulier le profil d'explosion dans [40] est l'état changeant de signe non dégénéré construit dans [36, 99] plutôt que l'état fondamental W.

Il est à noter que la question de *la construction de solutions multi-bulles pour l'équation 4D NLW* est toujours un problème ouvert. Dans [69], Jendrej et Martel ont construit des solutions globales multi-bulles de l'équation NLW 5D critique, explosant en un temps infini en toute famille finie donnée de points $(z_k)_{1\leq k\leq K}$ de $\mathbb{R}^5$. Dans le cas 4D, en raison de la faible décroissance de l'état fondamental, on s'attend à l'existence d'une solution globale explosant en un temps infini le long de lignes de vitesses distinctes, plutôt qu'en un ensemble fini de points donnés. Le système d'ODE auto-similaires, généré par l'interaction de deux ondes progressives ayant des vitesses de Lorentz différentes admet formellement des solutions qui explosent en un temps infini. La difficulté de ce problème est de trouver une solution approchée raffinée et appropriée, puis de construire une solution qui lui est proche, en utilisant des estimations d'énergie et de la compacité. Un autre problème ouvert lié aux résultats de [69, 114] est *la construction de solutions multi-solitons ou multi-bulles pour l'équation d'onde critique 3D*. La principale difficulté est la décroissance algébrique lente des solitons, qui est encore plus faible que dans le cas 4D.

0.4 L'équation NLKG amortie avec non-linéarité sous-critique

0.4.1 Contexte

Considérons l'équation de Klein-Gordon amortie focalisante non linéaire

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0 \quad (t,x) \in [0,\infty)\times\mathbb{R}^d, \tag{0.8}$$

où $f(u) = |u|^{p-1}u$, $\alpha > 0$, $1 \leqslant d \leqslant 5$ et l'exposant p satisfait

$$2 < p < p^*(d) \quad \text{avec} \quad p^*(d) = \begin{cases} \infty & \text{si } d = 1,2, \\ \dfrac{d+2}{d-2} & \text{si } d = 3,4,5. \end{cases}$$

On définit $F(u) = \frac{1}{p+1}|u|^{p+1}$ et

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{|\nabla u|^2 + u^2 + (\partial_t u)^2 - 2F(u)\right\}\mathrm{d}x.$$

Pour toute solution $\vec{u} = (u, \partial_t u) \in H^1 \times L^2$ de (0.8), nous avons

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 \mathrm{d}t. \tag{0.9}$$

Les restrictions $1 \leqslant d \leqslant 5$ et $2 < p < \frac{d+2}{d-2}$ assurent une bonne théorie d'existence locale et une régularité suffisante sur la non-linéarité pour que le développement de Taylor autour de 0 soit bien défini jusqu'à l'ordre 2. Notre analyse inclut la situation la plus physiquement pertinente, i.e celle d'une non-linéarité cubique en dimension d'espace 3.

Rappelons d'abord, en suivant [17] (voir aussi les références qui y figurent), quelques caractéristiques de la solution stationnaire, à savoir une solution de l'équation elliptique

$$-\Delta q + q - f(q) = 0, \quad q \in H^1(\mathbb{R}^d). \tag{0.10}$$

Nous appelons les solutions de (0.10) *états stationnaires*, et notons $\mathcal{B}$ l'ensemble des états stationnaires:

$$\mathcal{B} = \{q : q \text{ est une solution non-triviale de } (0.10)\}.$$

La théorie elliptique standard (voir par exemple [58] ou [17, Théorème 8.1.1]) montre que si $q \in \mathcal{B}$, alors q est de classe $\mathcal{C}^2(\mathbb{R}^d)$ et a une décroissance exponentielle quand $|x| \to +\infty$, ainsi que ses dérivés premières et secondes.

Soit

$$\mathcal{W}(v) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{|\nabla v|^2 + v^2 - 2F(v)\right\} \mathrm{d}x, \quad \text{pour } v \in H^1.$$

Les solutions de (0.10) qui minimisent la fonctionnelle $\mathcal{W}$ sont appelés les *états fondamentaux* ; l'ensemble des états fondamentaux est noté $\mathcal{G}$

$$\mathcal{G} = \{q_0 \in \mathcal{B} : \forall q \in \mathcal{B},\ \mathcal{W}(q_0) \leqslant \mathcal{W}(q)\}.$$

Les états fondamentaux sont des objets très étudiés. Ils sont uniques à translation spatiale près (pour des non-linéarités assez générales). Autrement dit, il existe unique fonction radiale positive Q de classe $\mathcal{C}^2$, qui décroît exponentiellement (ainsi que ses dérivées premières et secondes) telle que

$$\mathcal{G} = \{Q(x - x_0) : x_0 \in \mathbb{R}^d\}.$$

On réfère à Berestycki-Lions [7], Gidas-Ni-Nirenberg [57], Kwong [77], Serrin-Tang [107]. Plus précisément, l'état fondamental Q peut se mettre sous la forme $Q(x) = q(|x|)$ où $q > 0$ satisfait

$$q'' + \frac{d-1}{r}q' - q + q^p = 0, \quad q'(0) = 0, \quad \lim_{r\to\infty} q(r) = 0. \tag{0.11}$$

Il est bien connu et facilement vérifiable que pour une certaine constante $\kappa > 0$ et pour tout $r > 1$, on a

$$\left|q(r) - \kappa r^{-\frac{d-1}{2}} e^{-r}\right| + \left|q'(r) + \kappa r^{-\frac{d-1}{2}} e^{-r}\right| \lesssim r^{-\frac{d+1}{2}} e^{-r}.$$

En particulier, en dimension $d = 1$, nous avons une formule explicite pour Q :

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}} \quad \text{sur } \mathbb{R}.$$

Il est également bien connu (voir par exemple Grillakis-Shatah-Strauss [60]) que l'état fondamental Q est instable dans l'espace d'énergie. Ce résultat est également connu en physique sous le nom de Théorème de Derrick [41]. Plus précisément, par un argument d'ODE, l'opérateur

$$\mathcal{L} = -\Delta + 1 - f'(Q)$$

apparaissant après linéarisation de l'équation (0.8) autour de $\vec{Q} = (Q, 0)$, a une valeur propre négative unique $-\nu_0^2$ ($\nu_0 > 0$). On note Y la fonction propre normalisée associée. En particulier, il résulte de calculs explicites que si l'on définit

$$\nu^{\pm} = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{et} \quad \vec{Y}^{\pm} = \begin{pmatrix} Y \\ \nu^{\pm} Y \end{pmatrix},$$

alors la fonction $\vec{\varepsilon}^{\pm}(t,x) = \exp(\nu^{\pm} t)\vec{Y}^{\pm}(x)$ est solution du problème linéarisé

$$\begin{cases} \partial_t \varepsilon = \eta \\ \partial_t \eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{0.12}$$

Puisque $\nu^+ > 0$, la solution $\vec{\varepsilon}^+$ illustre l'instabilité exponentielle de l'onde solitaire en temps positif. En particulier, on voit que la présence du terme d'amortissement $\alpha > 0$ dans l'équation ne supprime pas l'instabilité exponentielle de l'onde solitaire solution de Klein-Gordon.

En dimension $d = 1$, en raison d'arguments d'ODE standard, $\mathcal{B} = \mathcal{G}$. En revanche, d'après [17, Remark 8.1.16], pour tout $d \geqslant 2$, on a $\mathcal{G} \subsetneq \mathcal{B}$. Les fonctions $q \in \mathcal{B} \setminus \mathcal{G}$ sont appelés *états excités*. Rappelons que, Berestycki-Lions [8] ont montré l'existence d'une infinité de solutions radiales *nodales* (*i.e.* changeant de signe), voir aussi [61, 98] et leurs références. En réalité, on en sait assez peu sur les états excités, exception faite de certaines constructions spécifiques.

Comme nous l'avons fait au paragraphe §0.3.1, nous introduisons toutes les invariances possibles de (0.10):

(i) translation : si q est une solution de (0.10), alors $q(x+x_0)$ est aussi une solution de (0.10) pour tout $x_0 \in \mathbb{R}^d$;

(ii) transformation orthogonale : si q est une solution de (0.10), alors $q(Ox)$ est aussi une solution de (0.10) pour tout $O \in \mathcal{O}_d$.

En raison des invariances de l'équation, l'opérateur $\mathcal{L}_q$ a toujours un noyau : pour $d > 1$, notons Ω_{ij} les dérivées angulaires

$$\Omega_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i} \quad \text{pour } 1 \leqslant i < j \leqslant d, \tag{0.13}$$

et considérons l'espace vectoriel $\mathcal{Z}_q$ engendré par le générateur infinitésimal des invariances de l'équation appliquées à q :

$$\mathcal{Z}_q = \operatorname{Span}\left\{\partial_{x_n} q, n = 1, \cdots, d; \Omega_{ij} q, 1 \leqslant i < j \leqslant d\right\}. \tag{0.14}$$

On a toujours $\mathcal{Z}_q \subset \operatorname{Ker}\mathcal{L}_q$. Nous définissons maintenant les états non dégénérés et dégénérés.

Définition 0.21. Soit $q \in \mathcal{B}$.

(i) q est appelé un *état non-dégénéré* si $\mathcal{Z}_q = \operatorname{Ker}\mathcal{L}_q$.

(ii) q est appelé un *état dégénéré* si $\mathcal{Z}_q \subsetneq \operatorname{Ker}\mathcal{L}_q$.

L'exemple le plus pertinent est bien sûr l'état fondamental Q qui est non-dégénéré. De plus, les constructions de [2, 98] produisent également des états excités *non-dégénérés*. Il faut garder à l'esprit qu'on ne sait toujours pas si des états excités dégénérés existent. Pour l'instant, mentionnons simplement une façon de comprendre la dégénérescence (nous notons $'$ pour la différentielle de Gateaux). La condition que q est un état stationnaires s'écrit $E'(q) = 0$. Dans ce cas $\langle \mathcal{L}_q v, w\rangle = E''(q)\cdot(v,w)$, de telle sorte que la condition que q soit dégénéré équivaut au fait que pour un certain $\phi \notin \mathcal{Z}_q$, la forme linéaire $E''(q)\cdot(\phi,\cdot)$ soit identiquement nulle.

Nous rappelons maintenant deux résultats liés à la dynamique en temps longs des solutions globales. Premièrement, dans [12] en dimension $d \geq 2$, Burq, Raugel et Schlag ont prouvé la convergence de toute solution radiale globale vers un état stationnaires radial, pour toute la séquence de temps. Deuxièmement, sous certaines conditions sur d et p, les résultats de [53, 82] indiquent que pour toute suite de temps, toute solution globale bornée de (0.8) converge vers une somme d'états stationnaires découplés après extraction d'une sous-suite de temps. Plus précisément:

Théorème 0.22 ([53, 82]). *Soit $1 \leqslant d \leqslant 4$ avec $1 < p < p^*(d)$ ou $5 \leqslant d \leqslant 6$ avec $1 < p < p^{**}(d)$ dans (0.8) où*

$$p^*(d) = \begin{cases} \infty & \text{si } d = 1,2, \\ \dfrac{d+2}{d-2} & \text{si } d = 3,4. \end{cases} \quad \text{et} \quad p^{**}(d) = \frac{d}{d-2} \quad \text{si } 3 \leqslant d \leqslant 6.$$

Alors pour toute solution globale bornée $\vec{u} = (u, \partial_t u)$ de (0.8) et toute suite de temps $t_n \to \infty$, il existe $J \in \mathbb{N}^+$, une suite de points $x_{j,n_k} \in \mathbb{R}^N$ pour $j = 1, \ldots, N$, et des états stationnaires Q^j tels que pour une sous-suite (t_{n_k})

$$\lim_{k\to\infty} \left\{\|u(t_{n_k}) - Q^j(\cdot - x_{j,n_k})\|_{H^1} + \|\partial_t u(t_{n_k})\|_{L^2}\right\} = 0,$$

où $\{x_{j,n_k}\}$ satisfait la propriété de séparation

$$\lim_{k\to\infty} |x_{j,n_k} - x_{j',n_k}| = \infty, \quad \text{pour tout } j \neq j'.$$

Remarque 0.23. Le Théorème 0.22 a d'abord été prouvé dans Feireisl [53] pour le cas $3 \leqslant d \leqslant 6$ avec $1 < p < p^{**}(d)$. Plus tard, Li-Zhao [82] ont étendu le résultat au cas $1 \leqslant d \leqslant 4$ avec $1 < p < p^{*}(d)$ par la méthode compacité/concentration. Dans [53], Feireisl a également construit des solutions globales qui se comportent comme la somme d'un nombre pair d'états fondamentaux (i.e multi-solitons).

Remarque 0.24. Comme cela est discuté dans [12], les résultats de [12, 53, 82] sont intimement liés à la conjecture générale de résolution en solitons pour les solutions globales et bornées de problèmes dispersifs.

0.4.2 Résultats

Le Théorème 0.22, valable pour toute solution globale bornée et tout suite $t_n \to \infty$, est remarquable. Cependant, il ne décrit pas complètement le comportement asymptotique des solutions globales générales. Par exemple, il ne spécifie pas le signe des solitons, le comportement des paramètres de positions, si la convergence est valable pour tous les temps positifs, si le taux de convergence. Les principaux résultats de cette section visent à compléter le Théorème 0.22 en décrivant précisément le comportement asymptotique quand $t \to \infty$ de solutions solitons ou multi-solitons de (0.8). Les principaux résultats sont regroupés en trois parties.

Description et classification des 2-ondes solitaires

Dans cette partie, nous décrivons complètement les 2-ondes solitaires liées à l'état fondamental Q. Nous commençons par donner leur définition.

Définition 0.25. Une solution $\vec{u} \in \mathcal{C}([T,\infty), H^1 \times L^2)$ de (0.8), pour un certain $T \in \mathbb{R}$, est appelé une *2-ondes solitaire* s'il existe $\sigma_1, \sigma_2 = \pm 1$, une suite $t_n \to \infty$ et une suite $(\xi_{1,n}, \xi_{2,n}) \in \mathbb{R}^{2d}$ tels que

$$\lim_{n\to\infty} \left\{ \left\| u(t_n) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_{k,n}) \right\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \right\} = 0$$

et $\lim_{n\to\infty} |\xi_{1,n} - \xi_{2,n}| \to \infty$.

Remarque 0.26. On observe que si u est une solution globale de (0.8) satisfaisant

$$\lim_{t\to\infty} \left\| u(t) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k(t)) \right\|_{H^1} = 0 \tag{0.15}$$

où $\lim_{t\to\infty} |\xi_1(t) - \xi_2(t)| \to \infty$, alors u est une 2-ondes solitaire. En effet, il résulte de (0.9) et (0.15) que $t \mapsto E(\vec{u}(t))$ est bornée inférieurement. Ainsi, (0.9) implique $\int_0^\infty \|\partial_t u(t)\|_{L^2}^2 \mathrm{d}t < \infty$ et donc $\|\partial_t u(t_n)\|_{L^2} \to 0$ pour une suite $t_n \to \infty$.

Notre premier résultat concerne la non-existence d'2-ondes solitaire avec les mêmes signes.

Théorème 0.27 ([29]). *Il n'existe pas de 2-onde solitaire de (0.8) avec $\sigma_1 = \sigma_2$.*

Dans le résultat suivant, nous montrons que les 2-ondes solitaires de signe opposé satisfont un comportement asymptotique universel.

Théorème 0.28 (Description des 2-ondes solitaires, [29]). *Pour toute 2-ondes solitaires u de (0.8), il existe $\sigma_1 = \pm 1$, $\sigma_2 = -\sigma_1$, $z_\infty \in \mathbb{R}^d$, $\omega_\infty \in S_{\mathbb{R}^d}(1)$, $T > 0$ et $t \in [T,\infty) \mapsto (z_1(t), z_2(t))$ tels que pour tout $t \in [T,\infty)$,*

$$\left\| u(t) - \sum_{k=1,2} \sigma_k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{0.16}$$

et pour une certaine constante $c_0 = c_0(d,\alpha)$, pour tout $\theta < \min(p-1, 2)$ et pour $k = 1, 2$,

$$z_k = z_\infty + \frac{(-1)^k}{2} \left(\log t - \frac{1}{2}(d-1) \log\log t + c_0 \right) \omega_\infty + O(t^{1-\theta}). \tag{0.17}$$

Enfin, nous décrivons la famille complète des 2-ondes solitaires pour des données initiales proches de la somme de deux ondes solitaires éloignées.

Théorème 0.29 (Classification des 2-ondes solitaires, [29]). *Il existe $C, \delta > 0$ et une application Lipschitzienne*

$$H : (\mathbb{R}^d \setminus \bar{\mathcal{B}}_{\mathbb{R}^d}(10|\log\delta|)) \times \mathcal{B}_{H^1\times L^2}(\delta) \to \mathbb{R}^2, \quad (L, \vec{\phi}) \mapsto H(L, \vec{\phi})$$

tels que

$$|H(L, \vec{\phi})| < C\left(e^{-\frac{L}{2}} + \|\vec{\phi}\|_{H^1\times L^2}\right),$$

avec la propriété suivante. Etant donné $L, \vec{\phi}, h_1, h_2$ tels que

$$|L| > 10|\log\delta|, \quad \|\vec{\phi}\|_{H^1\times L^2} < \delta, \quad |h_1| + |h_2| < \delta,$$

la solution $\vec{u}$ de (0.8) avec les données initiales

$$\vec{u}(0) = \left(\vec{Q} + h_1\vec{Y}^+\right)\left(\cdot - \frac{L}{2}\right) - \left(\vec{Q} + h_2\vec{Y}^+\right)\left(\cdot + \frac{L}{2}\right) + \vec{\phi}$$

est une onde 2-solitaire si et seulement si $(h_1, h_2) = H(L, \vec{\phi})$.

Ce résultat signifie essentiellement que localement autour de la somme de deux solitons suffisamment séparés de signes opposés, les données initiales des 2-ondes solitaires forment une variété Lipschitzienne de codimension 2 (les directions instables étant dirigées par $\vec{Y}^+$ translatés autour de chaque soliton).

Il est à noter que l'étude de la *description et classification des 2-ondes solitaires pour l'équation NLKG amortie fractionnaire* est un problème intéressant lié au résultat de [29]. Pour être plus précis, nous considérons l'équation NLKG amortie fractionnaire,

$$\partial_{tt}u + 2\alpha\partial_t u + (-\Delta)^s u + u - |u|^\sigma u = 0, \quad (t, x) \in \mathbb{R}\times\mathbb{R}^d,$$

où $0 < s < 1$ et où l'exposant σ satisfait

$$0 < \sigma < \sigma_*(s, d), \quad \sigma_*(s, d) = \begin{cases} \dfrac{4s}{d-2s} & \text{pour } 0 < s < \dfrac{d}{2}, \\ +\infty & \text{pour } \dfrac{d}{2} \leqslant s < 1. \end{cases}$$

Dénotons *l'état fondamental* par Q_s, qui est l'unique solution radiale strictement positive dans $H^s(\mathbb{R}^d)$ de

$$(-\Delta)^s Q_s + Q_s - |Q_s|^\sigma Q_s = 0, \quad Q_s(x) \sim \frac{1}{|x|^{d+2s}} \quad \text{comme } |x| \to \infty.$$

L'existence et les propriétés de cette solution sont étudiées dans [55, 56]. Pour l'équation NLKG amortie fractionnaire, en raison de la lente décroissance de l'état fondamental, nous nous attendons à ce que les 2-ondes solitaires aient un comportement universel différent de celui de l'équation NLKG amortie. Plus précisément, l'ODE formellement obtenue pour l'interaction de deux ondes progressives à grande distance est $\dot{r} = \pm Cr^{-(d+2s)}$. Dans le cas $\dot{r} = -Cr^{-(d+2s)}$, il n'y a pas de solution globale telle que $r \gg 1$. En revanche, dans le cas $\dot{r} = Cr^{-(d+2s)}$, l'ODE admet une solution de la forme $r(t) = ct^{\frac{1}{d+2s+1}}$ comme $t \to \infty$.

Asymptotiques en temps longs pour l'équation NLKG amortie 1D

L'objectif de cette partie est de compléter le Théorème 0.22 en décrivant précisément le comportement asymptotique quand $t \to \infty$ de toute solution globale de (0.8) avec $d = 1$. Plus précisément:

Théorème 0.30 ([30]). *Pour toute solution globale $\vec{u} \in C\big([0,\infty), H^1 \times L^2\big)$ de (0.8) avec $d = 1$, l'un des trois scénarios suivants se produit:*

Dissipation: *$\vec{u}(t)$ converge exponentiellement vers 0 dans $H^1 \times L^2$ lorsque $t \to \infty$.*

Soliton unique: *Il existe $\sigma = \pm 1$, $\ell \in \mathbb{R}$ tel que $\vec{u}(t)$ converge exponentiellement vers $(\sigma Q(\cdot - \ell), 0)$ dans $H^1 \times L^2$ lorsque $t \to \infty$.*

Multi-soliton: *Il existe $K \geq 2$, $\sigma = \pm 1$, $\ell \in \mathbb{R}$ et des fonctions $z_k : [0,\infty) \to \mathbb{R}$ pour tout $k = 1, \dots, K$ tels que pour tout $t \in (0,\infty)$,*

$$\left\| u(t) - \sigma \sum_{k=1}^{K} (-1)^k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{0.18}$$

et pour tout $1 < \theta < \min(p-1, \frac{5}{4})$, $k = 1, \dots, K$,

$$z_k(t) = \left(k - \frac{K+1}{2}\right) \log t + \tau_k + \ell + O(t^{-\theta+1}), \tag{0.19}$$

quand $t \to \infty$, où τ_k sont les constantes définies uniquement par

$$\sum_{k=1}^{K} \tau_k = 0, \quad e^{-(\tau_{k+1} - \tau_k)} = \frac{2\alpha}{\kappa} \gamma_k \quad \text{où} \quad \gamma_k = \frac{k(K-k)}{2}, \tag{0.20}$$

et où $\kappa > 0$ ne dépend que de p.

Remarque 0.31. Le paramètre ℓ est lié à l'invariance par translation. Dans les cas de dissipation et de soliton simple, l'amortissement conduit à une convergence exponentielle quand $t \to \infty$. Dans le cas multi-soliton, du fait des interactions non linéaires entre les ondes solitaires, le comportement asymptotique (0.19) des centres de masse z_k est lié au système différentiel non linéaire suivant

$$\begin{cases} \dot{y}_1 = -\dfrac{\kappa}{2\alpha} e^{-(y_2 - y_1)}, \\ \dot{y}_k = \dfrac{\kappa}{2\alpha} \left(e^{-(y_k - y_{k-1})} - e^{-(y_{k+1} - y_k)} \right), \quad \text{pour } k = 2, \cdots, K-1, \\ \dot{y}_K = \dfrac{\kappa}{2\alpha} e^{-(y_K - y_{K-1})}. \end{cases} \tag{0.21}$$

Ce système est étudié dans [33, 93] dans le cadre de solutions explosant en temps fini de l'équation d'onde non linéaire. Les interactions non linéaires à courte distance entre les ondes solitaires implique également le taux de convergence plus lent t^{-1} dans (0.18). Voir [65, 90, 109, 110] pour d'autres exemples d'interactions fortes conduisant à des ondes solitaires distantes de $\log t$.

Le Théorème 0.30 est une version de la résolution en soliton sour les solutions globales de l'équation de Klein-Gordon non linéaire amortie unidimensionnelle (0.8), avec convergence pour toute la suite de temps et une description des paramètres des ondes solitaires. Le cas de l'équation de Klein-Gordon amortie sur un domaine borné pour des non-linéarités appropriées est traité dans [18, Theorem 9.5.3]; voir également les références qui y figurent.

Évidemment, $\vec{u} = (0,0)$ et $\vec{u} = (Q,0)$ sont des exemples des deux premiers scénarios du Théorème 0.30. Notre deuxième résultat donne des exemples du troisième scénario pour tout $K \geq 2$.

Théorème 0.32 ([30]). *Pour tout $K \geq 2$, $\sigma = \pm 1$ et $\ell \in \mathbb{R}$, il existe des solutions globales de* (0.8) *avec $d = 1$ satisfaisant* (0.18) *et* (0.19).

Remarque 0.33. Noter que les résultats de [12, 29, 53, 82] s'appliquent également au cas multidimensionnel, sous des restrictions appropriées sur l'exposant p de la non-linéarité, ou pour des solutions radialement symétriques. Cependant, pour des dimensions d'espace supérieures à 1, l'existence de solutions de $\Delta w - w + f(w) = 0$ distincts de l'état fondamental Q, ainsi que la possibilité qu'il existe des configurations géométriques d'ondes solitaires compliquent l'analyse. C'est pourquoi nous nous limitons à la dimension 1 dans cette partie.

Asymptotique des solutions avec une propriété de compacité

Dans cette partie, nous nous intéressons à la compréhension du comportement des solutions de (0.8) pour lesquelles *un seul* objet non linéaire apparaît en temps long, au moins pour une suite de temps. Plus précisément, nous définissons les solutions *attirée* comme suit.

Définition 0.34. Une solution maximale $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0, T_{\max}), H^1 \times L^2)$ de (0.8) avec $2 \leqslant d \leqslant 5$ est appelé une *solution attirée* s'il existe $(W_0, W_1) \in H^1 \times L^2$, et une suite de temps $t_n \to T_{\max}$ et de positions $y_n \in \mathbb{R}^d$ tels que

$$\lim_{n\to\infty} \{\|u(t_n) - W_0(\cdot - y_n)\|_{H^1} + \|\partial_t u(t_n) - W_1(\cdot - y_n)\|_{L^2}\} = 0. \tag{0.22}$$

On dit alors que $\vec{W} = (W_0, W_1)$ est un *point d'attraction* pour $\vec{u}$ à $(t_n, y_n)_n$.

Observons que tout point d'attraction (W_0, W_1) est en fait un état stationnaire $(q, 0)$. Plus précisément, la proposition suivante est vraie.

Proposition 0.35. *Soit $\vec{u} = (u, \partial_t u)$ une solution attirée de* (0.8). *Alors $\vec{u} \in \mathcal{C}([0, +\infty), H^1 \times L^2)$ est défini globalement pour les temps positifs, et si $(W_0, W_1) \in H^1 \times L^2$ est un point d'attraction pour $\vec{u}$ à $(t_n, y_n)_n$, alors $W_0 = q$ est un état lié de* (0.10) *et $W_1 = 0$. De plus, l'énergie est bornée inférieurement, $\partial_t u \in L^2([0, +\infty), L^2)$ et pour tout $t \geqslant 0$,*

$$E(\vec{u}(t)) - E(q, 0) = 2\alpha \int_t^{+\infty} \|\partial_t u(s)\|_{L^2}^2 \mathrm{d}s. \tag{0.23}$$

Noter qu'il n'est pas clair si toute une solution attirée est globalement bornée dans $H^1 \times L^2$. Rappelons que d'après les arguments de [16] – voir aussi [12] et [30] – si $p \leqslant \frac{d}{d-2}$, alors toute solution globale de (0.8) est globalement bornée dans $H^1 \times L^2$, mais ceci n'est pas connu pour des puissances supérieures de p.

Notre premier résultat est que si un point d'attraction d'une solution attirée est un état non dégénéré, alors la convergence est vérifiée pour tout temps positif et se produit à un taux exponentiel. Plus précisément,

Théorème 0.36 ([32]). *Soit $\vec{u} = (u, \partial_t u)$ une solution attirée de* (0.8), *avec le point d'attraction $q \in H^1$ à $(t_n, y_n)_n$. Si q est un état non dégénéré, alors la convergence est valable pour tous les temps et la décroissance est exponentielle, c'est-à-dire qu'il existe $\mu > 0$ et $z_\infty \in \mathbb{R}^d$ tels que*

$$\forall t \geqslant 0, \quad \|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim e^{-\mu t}.$$

Remarque 0.37. Observons tout d'abord que le Théorème 0.36 (et sa preuve) est aussi valide en dimension 1, mais bien sûr, c'est une conséquence directe de la description complète [30] des solutions globales en 1D (comme mentionné dans §0.4.1, les états excités n'existent que pour $d \geqslant 2$). C'est la raison pour laquelle nous nous limitons à la dimension $d \geqslant 2$ dans cette partie.

Ensuite, nous considérons des *états excités de degré 1* où $\mathrm{Ker}\mathcal{L}_q$ a une dimension supplémentaire sans lien avec les invariances de l'équation (0.10) et qui implique également une condition sur les différentiels de Gateau d'odre 3 de E, selon la définition suivante.

Définition 0.38. Soit q un état excité dégénéré, q est appelé un *état excité de degré 1* s'il existe $\phi \in H^1$ tel que

$$\mathrm{Ker}\mathcal{L}_q = \mathcal{Z}_q \oplus \mathrm{Span}\{\phi\} \quad \text{et} \quad E'''(q)\cdot(\phi,\phi,\phi) \neq 0. \tag{0.24}$$

Notre deuxième résultat concerne les points d'attraction qui sont des états excités de degré 1.

Théorème 0.39 ([32]). *Soit $\vec{u} = (u, \partial_t u)$ une solution de (0.8), avec le point de $q \in H^1$ à $(t_n, y_n)_n$. Si q est un état excité de degré 1, alors la convergence $\vec{u}(t) \to (q, 0)$ est valable pour tous les temps avec un taux algébrique,* i.e. *il existe $z_\infty \in \mathbb{R}^d$ tel que*

$$\forall t > 0, \quad \|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}.$$

Enfin, nous montrons que le taux de convergence dans le Théorème 0.39 peut être atteint : nous fournissons un exemple où la solution converge exactement au taux t^{-1} vers l'état excité de degré-1.

Théorème 0.40 ([32]). *Soit q un état excité de degré 1. Alors, il existe une solution globale $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0, +\infty), H^1 \times L^2)$ de (0.8) tel que*

$$\|u(t) - q\|_{H^1} + \|\partial_t u(t)\|_{L^2} \sim t^{-1} \quad \textit{comme} \quad t \to +\infty.$$

Remarque 0.41. Les états excités de degré 1 sont en quelque sorte les états bornés dégénérés les plus simples. A ce point déjà, on ne sait que très peu de choses, et à notre connaissance, les théorèmes 0.39 et 0.40 sont les premiers résultats décrivant précisément la dynamique dans un contexte dégénéré. De ce point de vue, la condition que $\dim \mathrm{Ker}\mathcal{L}_q = \dim(\mathcal{Z}_q) + 1$ est très naturelle. Concernant la condition supplémentaire $E'''(q)\cdot(\phi,\phi,\phi) \neq 0$, notons qu'elle est générique ; c'est équivalent au fait que $E'''(q)$ n'est pas identiquement nulle sur $(\mathrm{Ker}\mathcal{L}_q)^3$. Notons que *la construction de l'état excité de degré 1* est toujours un problème ouvert.

Remarque 0.42. Il est remarquable que l'on observe déjà un changement drastique dans la dynamique dû à la dégénérescence de degré 1, par rapport à la non-dégénérescence. La convergence ici est en effet simplement polynomiale en temps, ce qui est une surprise : un taux de convergence aussi lent est généralement observé en raison de l'interaction avec un autre objet non linéaire (comme dans [29, 30]), ce qui n'est pas le cas ici.

Il convient de mentionner que *la description complète de toutes les solutions attirées* est un problème ouvert intéressant, mais qui semble hors de portée pour le moment. En effet, nous ne comprenons pas la dynamique autour des états excités généraux, et le système des centres de masse des états stationnaires impliqués pourrait avoir une dynamique très complexe.

Chapter 1

Introduction (English version)

1.1 Generalities on nonlinear wave-type equations

The aim of this thesis is to study nonlinear phenomena in the dynamical behavior of multi-soliton solutions of some nonlinear focusing wave-type equations. The nonlinear wave-type equation with or without damping is of the form:

$$\partial_t^2 u(t,x) + 2\alpha\partial_t u(t,x) - \Delta_x u(t,x) + mu(t,x) = f(u(t,x)), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{NW}$$

where $m = 0$ or 1, $\alpha \geqslant 0$ and $f(\cdot)$ is a focusing nonlinearity with $f(0) = 0$.

The question of local or global well-posedness of the Cauchy problem for the various wave-type equations under suitable restriction of dimension d and nonlinearity f is well-known and is not included in this thesis. We mostly consider energy solutions and we refer to [12, 59, 71] for the sub-critical Klein-Gordon with or without damping and energy-critical wave equations.

When $\alpha = 0$ in (NW), the equation is a *Hamiltonian equation*. However with a *infinite dimensional* phase space, it can be taken for instance as $C_0^\infty(\mathbb{R}^d)\times C_0^\infty(\mathbb{R}^d)$. More specifically, (NW) can be rewritten in the form

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = JH \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\Delta + m & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, let $F(u) = \int_0^u f(s)\mathrm{d}s$. The energy E and momentum I are conserved, where

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^d} \nabla u(t,x)\partial_t u(t,x)\mathrm{d}x, \\ E(t) &= \int_{\mathbb{R}^d} \left\{ |\nabla u(t,x)|^2 + mu^2(t,x) + (\partial_t u(t,x))^2 - 2F(u(t,x)) \right\} \mathrm{d}x. \end{aligned}$$

A core question in the area is the so-called *soliton resolution conjecture* which arose in 70's and 80's from numerical simulations (see Fermi–Pasta–Ulam [54], Zabusky–Kruskal [115]) and the integrable theory of Korteweg–de Vries (KdV) equation (see [50, 51, 106]). Roughly speaking, the conjecture states that any global solution of the nonlinear equation behaves asymptotically in large time as a sum of decoupled solitons (*i.e.* multi-solitons) plus a dispersive part (linear behavior). The conjecture is very challenging, even for the *completely integrable models*. To the author's knowledge, for the completely integrable models, so far the conjecture was fully established by Eckhaus-Schuur [51] for KdV, by Schuur [106] and Chen-Liu [19] for mKdV, by Borghese-Jenkins-McLaughlin [9] for the 1D cubic NLS and also by Jenkins-Liu-Perry-Sulem [70] for the derivative NLS, via the method of *inverse*

scattering. It is worth mentioning that very few results exist for nonlinear dispersive and wave-type models that are not completely integrable.

Let us introduce one of the most important wave-type models to be studied in this thesis: the nonlinear wave equation (NLW) with the energy-critical power nonlinearity in dimension $d = 4, 5$ which is a non-integrable model:

$$\partial_{tt}u - \Delta u - |u|^{\frac{d}{d-2}}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{NLW}$$

In the last 15 years, there has been remarkable progress towards the understanding of long-time dynamics of the energy-critical NLW equation. Here, we review several results related to the soliton resolution conjecture that was proved for the energy-critical NLW equation. First, a complete result of soliton resolution for the 3D radial energy-critical NLW equation was proved in Duyckaerts–Kenig-Merle [46] by the method of *channels of energy* for the linear wave equation. The conjecture was later proved in the non-radial case for a subsequence of time for the 3,4 and 5D energy-critical NLW equation in [43]. The decomposition result in [46] was extended to all odd space dimension radial cases in [49]. More recently, Duyckaerts–Kenig–Martel-Merle [44] proved the conjecture for the 4D radial case. See [23, 24, 25] for results related to the conjecture for the case of wave maps. Note that a complete resolution of the conjecture for the nonlinear Klein-Gordon equation which is another important wave-type equation, seems out of reach for now.

Another example of nonlinear wave-type equation that we study in this thesis is the nonlinear Klein-Gordon equation (NLKG) with focusing-defocusing double power nonlinearity in dimension $d = 1$:

$$\partial_t^2 u - \partial_x^2 u + u - |u|^{p-1}u + |u|^{q-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1DNLKG}$$

where $1 < p < q < \infty$. The double power nonlinearity ensures that the nonlinear effect is defocusing for small values of u and focusing for large values of u.

We also study the damped NLKG equation with the sub-critical power nonlinearity in dimension $1 \leqslant d \leqslant 5$:

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - |u|^{p-1} = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{dampedNLKG}$$

where $\alpha > 0$ and $2 < p < \infty$.

The main results of this thesis focus on the long-time dynamics of multi-solitons and provide strong evidence and examples for the soliton resolution conjecture from various aspects. More precisely, we split this set of works into the following three more precise thematics:

(1) Stability of multi-solitons for the 1D NLKG equation (§1.2).

(2) Construction of multi-solitons for the energy-critical NLW equation (§1.3).

(3) Long time dynamics of the damped NLKG equation (§1.4). This is a joint project with Raphaël Côte (IRMA, France), also partly with Yvan Martel (CMLS, France) and Lifeng Zhao (USTC, China).

This thesis is composed of the following articles:

- R. Côte, Y. Martel and X. Yuan, Long-time Asymptotics of the One-dimensional Damped Nonlinear Klein–Gordon Equation. *Arch. Ration. Mech. Anal.* **239** (2021), no. 3, 1837-1874
- R. Côte, Y. Martel, X. Yuan and L. Zhao, Description and classification of 2-solitary waves for nonlinear damped Klein-Gordon equations. Preprint, *arXiv:*1908.09527, submitted.
- R. Côte and X. Yuan, Asymptotics of solutions with a compactness property for the nonlinear damped Klein-Gordon equation. Preprint, *arXiv*:2102.11178, submitted.
- X. Yuan, Conditional stability of multi-solitons for the 1D NLKG equation with double power nonlinearity. To appear in *Ann. Inst. H. Poincaré Anal. Non Linéaire.*

- X. Yuan, On multi-solitons for the energy-critical wave equation in dimension 5. *Nonlinearity* **32** (2019), no. 12, 5017–5048.
- X. Yuan, Construction of excited multi-solitons for the 5D energy-critical wave equation. To appear in *J. Hyperbolic Differ. Equ.*
- X. Yuan, Construction of excited multi-solitons for the focusing 4D cubic wave equation. Preprint, *arXiv*:2103.04168, submitted.

1.2 The 1D NLKG equation with double power nonlinearity

1.2.1 Background

Consider the 1D NLKG equation with a focusing-defocusing double power nonlinearity,

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + u - |u|^{p-1}u + |u|^{q-1}u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u_{|t=0} = u_0 \in H^1(\mathbb{R}), \quad \partial_t u_{|t=0} = u_1 \in L^2(\mathbb{R}), \end{cases}$$

where $1 < q < p < \infty$. To express this equation in Hamiltonian form, we rewrite it as a first order system in time for the function $\vec{u} = (u_1, u_2)$,

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = JH \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(u_1) \end{pmatrix}, \tag{1.1}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(u_1) = |u_1|^{p-1}u_1 - |u_1|^{q-1}u_1.$$

Let $F(u) = \int_0^u f(\sigma)\mathrm{d}\sigma = \frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|u|^{q+1}$. For any solution $\vec{u} = (u_1, u_2)$ of (1.1) in $H^1 \times L^2$, the energy $E(\vec{u})$ and momentum $\mathcal{I}(\vec{u})$ are conserved, where

$$E(\vec{u}) = \int_{\mathbb{R}} \left\{ (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right\} \mathrm{d}x, \quad \mathcal{I}(\vec{u}) = 2\int_{\mathbb{R}} (\partial_x u_1) u_2 \mathrm{d}x.$$

A stationary solution, *i.e.* time-independent solution $\vec{u}(t,x) = (q(x), 0)$ of (1.1) is characterized as an H^1 weak solution of the second-order ODE problem

$$q'' - q + f(q) = 0 \quad \text{on} \quad \mathbb{R}. \tag{1.2}$$

Following [7], we know that there exists a unique (up to translation) H^1 weak solution of (1.2) and the solution is positive and even. The unique even solution is called the *ground state*, which we denote by Q. Then, using the Lorentz transformation on $\vec{Q} = (Q, 0)$, one obtains traveling solitary waves or *solitons*: for $\ell \in \mathbb{R}$, with $-1 < \ell < 1$, let

$$Q_\ell(x) = Q\left(\frac{x}{\sqrt{1-\ell^2}}\right), \quad \vec{Q}_\ell = \begin{pmatrix} Q_\ell \\ -\ell \partial_x Q_\ell \end{pmatrix},$$

then $\vec{u}(t,x) = \vec{Q}_\ell(x - \ell t)$ is a solution of (1.1).

Let us mention that the above solutions are *unstable* with respect to perturbation of the initial data with one direction of exponential instability. More specifically, from Sturm–Liouville theory, the operator

$$\mathcal{L} = -\partial_x^2 + 1 - f'(Q)$$

appearing after linearization of equation (1.1) around $\vec{Q} = (Q, 0)$, has a unique negative eigenvalue $-\nu_0^2$ ($\nu_0 > 0$), with corresponding smooth even eigenfunction Y. Set

$$\vec{Y}^+ = \begin{pmatrix} Y \\ \nu_0 Y \end{pmatrix} \quad \text{and} \quad \vec{u}^+(t,x) = \exp(\nu_0 t)\vec{Y}^+(x).$$

From explicit computations, the function $\vec{u}^+(t,x)$ is solution of the linearized system

$$\begin{cases} \partial_t u_1 = u_2 \\ \partial_t u_2 = -\mathcal{L} u_1. \end{cases}$$

Since $\nu_0 > 0$, the solution $\vec{u}^+$ illustrates the exponential instability of the solitary wave $\vec{Q}$ in positive time. More generally, for $-1 < \ell < 1$, there exists the following unstable direction around $\vec{Q}_\ell$,

$$Y_\ell = Y\left(\frac{x}{\sqrt{1-\ell^2}}\right) \quad \text{and} \quad \vec{Z}_\ell^+ = \begin{pmatrix} (\ell \partial_x Y_\ell + \frac{\nu_0}{\sqrt{1-\ell^2}} Y_\ell) e^{\frac{\ell \nu_0}{\sqrt{1-\ell^2}} x} \\ Y_\ell e^{\frac{\ell \nu_0}{\sqrt{1-\ell^2}} x} \end{pmatrix}.$$

Concerning *multi-solitons* (solutions containing an arbitrary number of solitons), one of the first intrusions into this study for (1.1) is the work of Côte-Muñoz [31] where they constructed the first example of multi-solitons based on the ground state for (1.1). More precisely:

Theorem 1.1 (Existence of multi-solitons, [31]). *Let $\ell_1, \cdots, \ell_N$ be a set of different velocities with $\ell_n \neq \ell_{n'}$ for all $n \neq n'$ and $x_1, \cdots, x_N$ shift parameters. Then there exist a time $T_0 > 0$, constants $C > 0$, and $\gamma_0 > 0$ only depending on the set $(\ell_n)_n$, $(x_n)_n$ and a solution $\vec{u}(t) = (u_1(t), u_2(t)) \in \mathcal{C}\left([T_0, \infty) : H^1 \times L^2\right)$ of (1.1), globally defined for forward times and satisfying*

$$\left\|\vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_{\ell_n}(x - \ell_n t - x_n)\right\|_{H^1 \times L^2} \leqslant C e^{-\gamma_0 t}.$$

Remark 1.2. Côte-Muñoz [31] studied the more general case, *i.e.* for any space dimension $d \in \mathbb{N}^+$ and more general nonlinearities including double power nonlinearity. In parallel, Bellazzini-Ghimenti-Le Coz [5] constructed the complex-valued, stable multi-solitons for NLKG equation. Later, Côte-Martel [26] extended the existence result of [31] to any *bound state* for the NLKG equation with pure power nonlinearity.

1.2.2 Main result

From §1.2.1, we have observed that the single soliton is unstable, which implies that the multi-solitons constructed in [31] are also unstable. Therefore, when we study the stability property of multi-solitons with different speeds for (1.1), only *conditional stability* can be obtained: any initial data close to the initial data for a given multi-soliton solution can be appropriately adjusted along each exponentially unstable direction so that the solution launched by the modified initial data is eventually stable in time and converges to a nearby multi-soliton solution with different speeds.

The main result of this section is the conditional stability of multi-solitons for (1.1) as mentioned above. More precisely:

Theorem 1.3 (Conditional stability of multi-solitons, [113]). *Let $N \geqslant 2$. For all $n \in \{1, \ldots, N\}$, let $\sigma_n = \pm 1$, $\ell_n \in (-1, 1)$ with $-1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1$. There exist $L_0 > 0$, $C_0 > 0$, $\gamma_0 > 0$ and $\delta_0 > 0$ such that the following is true. Let $\vec{\varepsilon} \in H^1 \times L^2$ and $y_1^0 < \cdots < y_N^0$ be such that there exist $L > L_0$ and $0 < \delta < \delta_0$ with*

$$\|\vec{\varepsilon}\|_{H^1 \times L^2} < \delta \quad \text{and} \quad y_{n+1}^0 - y_n^0 > L \quad \text{for all } n = 1, \ldots, N-1.$$

Then, there exist $h_1^+, \dots, h_N^+ \in \mathbb{R}$ *satisfying*

$$\sum_{n=1}^{N} |h_n^+| \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big),$$

such that the solution $\vec{u} = (u_1, u_2)$ *of* (1.1) *with initial data*

$$\vec{u}_0 = \sum_{n=1}^{N} \big(\sigma_n \vec{Q}_{\ell_n} + h_n^+ \vec{Z}_{\ell_n}^+\big)(\cdot - y_n^0) + \vec{\varepsilon}$$

is globally defined in $H^1 \times L^2$ *for* $t \geqslant 0$ *and, for all* $t \geqslant 0$,

$$\Big\| \vec{u}(t) - \sum_{n=1}^{N} \sigma_n \vec{Q}_{\ell_n}(\cdot - y_n(t)) \Big\|_{H^1 \times L^2} \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big),$$

where $y_1(t), \dots, y_N(t)$ *are* C^1 *functions satisfying, for all* $n = 1, \dots, N$, $t \geqslant 0$,

$$|y_n(0) - y_n^0| \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big), \quad |\dot{y}_n(t) - \ell_n| \leqslant C_0^2\big(\delta + e^{-\gamma_0 L}\big).$$

Remark 1.4. The double power nonlinearity is a typical choice for dispersive or wave equations, but Theorem 1.3 can be extended to any general real-valued $C^{1,\alpha}$ function f (with $\alpha > 0$) satisfying the following:

(i) The function f is odd and $f(0) = f'(0) = 0$.

(ii) There exists a smallest $s_0 > 0$ such that $F(s_0) - \frac{1}{2}s_0^2 = 0$, and $f(s_0) - s_0 > 0$.

(iii) There exists $r_0 > 0$ such that for all $s \in (-r_0, r_0)$, $sf(s) - 2F(s) \leqslant 0$.

Here, $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$ for $s \in \mathbb{R}$. Conditions (i)-(ii) are related to the necessary and sufficient condition for existence of a unique standing wave in [7, Theorem 5]. Following [7, Example 2, p. 318], for the double power nonlinearity f, the existence of a smallest $s_0 > 0$ such that $F(s_0) - \frac{1}{2}s_0^2 = 0$ is clear and then

$$\begin{aligned} f(s_0) - s_0 = s_0^p - s_0^q - s_0 &= \frac{1}{s_0}\left((p+1)F(s_0) + \frac{p-q}{q+1}s_0^{q+1} - s_0^2\right) \\ &= \frac{1}{s_0}\left(\frac{p-q}{q+1}s_0^{q+1} + \frac{p-1}{2}s_0^2\right) > 0, \end{aligned}$$

which justifies the existence of the standing wave Q for (1.2). Condition (iii) ensures that the nonlinear effect is defocusing for small values of u_1, which is decisive in our virial argument. Recall that this issue is already present in [89, 104] for the NLS equation, where a weaker but similar restriction is imposed on the nonlinearity.

To the author's knowledge, this result is the first statement of (conditional) stability of the sums of decoupled solitons for a wave-type equation. The proof is based on the general strategy introduced in Martel-Merle-Tsai [88, 89] for the generalized Korteweg-de Vries (gKdV) and the nonlinear Schrödinger equation (NLS) equation in a similar context. However, the adaptation of this strategy to a wave-type equation requires the introduction of a new energy functional adapted to the Lorentz transform. More specifically, compared to the energy method developed in [89], we use a similar functional based on localized versions of the momentum and the energy around each soliton. Moreover, a specific combination of these quantities has to be used in the wave-type case.

As we remarked in §1.2.1, each soliton solution $\vec{u}(t,x) = \pm\vec{Q}_{\ell_n}(x - \ell_n t)$ of equation (1.1) has exactly one exponential instability direction in positive time. Recall that such an instability has led

to the construction of co-dimension one center-stable manifolds for initial data close to a single soliton for the NLKG or the wave equations in [75, 100, 101]. In the case of a perturbation of the sum of N decoupled solitons $\sum_{n=1}^{N} \sigma_n Q_{\ell_n}(\cdot - y_n)$, this justifies the need of adjusting the N free parameters $(h_n^+)_{n\in\{1,\dots,N\}}$ in the choice of the initial data to obtain global existence and stability in Theorem 1.3. Such difficulties do not appear when the solitons are stable, like for the (gKdV) and (NLS) equation in the mass sub-critical cases.

The study of *the conditional stability of multi-solitons for 1DNLKG with pure power nonlinearity* is still open. In [113], we chose the double power nonlinearity for the technical reason of the virial argument. However, pure power nonlinearities $f(u) = |u|^{p-1}u$ would probably make more sense in physics. The proof of the monotonicity property is open in that case. Note that to control the effect of low power focusing nonlinearities is a classical difficulty, and has already been encountered in earlier works on soliton asymptotic stability like [13, 14]. Another open problem related to the result in [113] is *how to extend the conditional stability result to the higher dimensional NLKG equation.* It is worth mentioning the difficulty of studying the case where the space dimension is greater than 1, lies in geometric configurations of solitary waves. It leads to adaptation of the energy functional for virial argument. Also, we mention that the study of *the stability of kink-antikink pairs for* $1+1$ *scalar fields* is an open problem. To be more specific, for another family of wave-type equations, namely the $1+1$ *scalar fields* (for example, the ϕ_{2n} model for $n \in \mathbb{N}^+$), special solutions, named *kinks*, exist. The stability of a single kink in the energy space is known and thus it is natural to study the stability of decoupling kink-antikink pairs. A key point here, which is similar to that in the case of 1DNLKG with pure power nonlinearity, will be to control low power nonlinear effects (see related problems and assumptions in [72, 73]).

1.3 The NLW equation with energy-critical nonlinearity

1.3.1 Background

Consider the NLW equation with energy-critical nonlinearity in dimension $d = 4, 5$,

$$\begin{cases} \partial_{tt}u - \Delta u - |u|^{\frac{4}{d-2}}u = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u_{|t=0} = u_0 \in \dot{H}^1(\mathbb{R}^d), & \partial_t u_{|t=0} = u_1 \in L^2(\mathbb{R}^d). \end{cases} \tag{1.3}$$

Let $F(u) = \frac{d-2}{2d}|u|^{\frac{2d}{d-2}}$. For any $\dot{H}^1 \times L^2$ solution $\vec{u} = (u, \partial_t u)$, the energy E and the momentum P are conserved along the flow, where

$$E(u,\partial_t u) = \frac{1}{2}\int_{\mathbb{R}^d} \Big(|\nabla u|^2 + (\partial_t u)^2 - 2F(u)\Big)\mathrm{d}x, \quad P(u,\partial_t u) = \frac{1}{2}\int_{\mathbb{R}^d} (\partial_t u \nabla u)\mathrm{d}x.$$

Looking for stationary solutions $u(t,x) = q(x)$ of (1.3) in $\dot{H}^1(\mathbb{R}^d)$, we reduce to the critical Yamabe equation

$$\Delta q + f(q) = 0, \quad q \in \dot{H}^1(\mathbb{R}^d). \tag{1.4}$$

Denote the set of non-zero finite energy solutions to (1.4) by

$$\Sigma = \left\{ q \in \dot{H}^1(\mathbb{R}^d)/0 : \Delta q + f(q) = 0 \quad \text{on } \mathbb{R}^d \right\}.$$

For $q \in \Sigma$ and $\boldsymbol{\ell} \in \mathbb{R}^5$ such that $|\boldsymbol{\ell}| < 1$, let

$$q_{\boldsymbol{\ell}}(x) = q\left(\left(\frac{1}{\sqrt{1-|\boldsymbol{\ell}|^2}} - 1\right)\frac{\boldsymbol{\ell}(\boldsymbol{\ell}\cdot x)}{|\boldsymbol{\ell}|^2} + x\right), \tag{1.5}$$

then $u(t,x) = q_{\boldsymbol{\ell}}(x - \boldsymbol{\ell}t)$ is a global, bounded solution of (1.3).

It is known by the results developed in [3, 15, 102, 108] that up to sign, dilation and translation, the only *positive* stationary solution of (1.3) is the *ground state* W, where W has the explicit form

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}} \quad \text{on } \mathbb{R}^d.$$

Functions $q \in \Sigma$ with $q \neq W$ (up to sign, dilation and translation) are called *excited states*. The existence of excited states that are nonradial sign-changing, and with arbitrary large energy was first proved by Ding [42] using a variational argument. Later, more explicit constructions of sign changing solutions have been obtained by del Pino-Musso-Pacard-Pistoia [36]. More specifically, they constructed solutions with a centered soliton crowned by negative spikes (rescaled solitons) at the vertices of a regular polygon of radius 1. Then, following a similar general strategy in [36], they constructed sign-changing, non-radial solutions to Yamabe-type equation on the sphere $\mathbb{S}^d$ ($d \geqslant 4$) whose energy is concentrated along special submanifolds of $\mathbb{S}^d$ in [37].

Following [48], we first discover all possible invariances of (1.4). More specifically, (1.4) is invariant under the following four transformations:

(i) translation: If $q \in \Sigma$ then $q(x + x_0) \in \Sigma$ for all $x_0 \in \mathbb{R}^d$;

(ii) dilation: If $q \in \Sigma$ then $\lambda^{\frac{d-2}{2}} q(\lambda x) \in \Sigma$ for all $\lambda \in (0, \infty)$;

(iii) orthogonal transformation: If $q \in \Sigma$ then $q(Ox) \in \Sigma$ for all $O \in \mathcal{O}_d$.

(iv) Kelvin transformation: If $q \in \Sigma$ then $\mathrm{K}q \in \Sigma$ where $\mathrm{K}q(x) = \frac{1}{|x|^{d-2}} q\left(\frac{x}{|x|^2}\right)$.

We denote by $\mathcal{M}$ the group of isometries of $\dot{H}^1$ generated by the previous four transformations. From [48, Section 3], we see that $\mathcal{M}$ defines a N_d-parameter family of transformations in a neighborhood of the identity, where the dimension N_d is given by

$$N_d = 2d + 1 + \frac{d(d-1)}{2}, \quad \text{for } d = 4, 5.$$

In other words, for $q \in \Sigma$, we denote the linearized operator around q by

$$\mathcal{L}_q = -\Delta - f'(q) = -\Delta - \frac{d+2}{d-2}|q|^{\frac{4}{d-2}}.$$

Set the null space of $\mathcal{L}_q$

$$\mathcal{Z}_q = \left\{ f \in \dot{H}^1(\mathbb{R}^d) : \mathcal{L}_q f = 0 \right\}.$$

We denote the subspace of $\mathcal{Z}_q$ generated by the family of transformations $\mathcal{M}$ by

$$\widetilde{\mathcal{Z}}_q = \mathrm{Span}\Big\{ \frac{d-2}{2} q + x \cdot \nabla q; (x_i \partial_{x_j} q - x_j \partial_{x_i} q), 1 \leqslant i < j \leqslant d; \\ \partial_{x_i} q; (2-d) x_i q + |x|^2 \partial_{x_i} q - 2x_i (x \cdot \nabla q), 1 \leqslant i \leqslant d \Big\}.$$

We note that the dimension of $\widetilde{\mathcal{Z}}_q$ is at most N_d, but in principle it could be strictly less than N_d if q has symmetries. For example, in the case of the positive radial solution $q = W$,

$$\widetilde{\mathcal{Z}}_W = \mathrm{Span}\left\{ \frac{d-2}{2} W + x \cdot \nabla W, \partial_{x_i} W, 1 \leqslant i \leqslant d \right\} \quad \text{and} \quad \dim \widetilde{\mathcal{Z}}_W = d + 1.$$

Duyckaerts–Kenig–Merle [48] introduced the following definition of nondegeneracy for a solution of (1.4).

Definition 1.5. Let $q \in \Sigma$. q is called a *non-degenerate state* if $\mathcal{Z}_q = \widetilde{\mathcal{Z}}_q$.

Remark 1.6. The first nontrivial example of non-degenerate sign-changing state was obtained in Musso-Wei [99]. More specifically, they proved that the sign-changing states constructed in [36] are non-degenerate. Later, following a similar strategy, Medina-Musso-Wei [91] constructed the non-degenerate state with maximal rank, *i.e.* $\dim\widetilde{\mathcal{Z}}_q = N_d$, for the case $d = 4$.

Remark 1.7. The nondegeneracy assumption plays an important role in the proof of the *rigidity conjecture* for solutions of (1.3) with the *compactness property*. Roughly speaking, the rigidity conjecture says that the solitary waves $Q_{\boldsymbol{\ell}}$ with $Q \in \Sigma$ and $|\boldsymbol{\ell}| < 1$ are the only solutions of (1.3) with the compactness property. The rigidity conjecture has only been proven under the nondegeneracy assumption in [48].

We also introduce the following definition related to the decay rate of $q \in \Sigma$ for further reference.

Definition 1.8. Let $q \in \Sigma$. q is called a *decaying state* if $|q(x)| \lesssim \langle x\rangle^{-(d-1)}$ on $\mathbb{R}^d$.

Remark 1.9. As noticed in [48, Remark 3.2], the existence of decaying states follows by translation and the Kelvin transformation for the sign-changing state constructed in [36].

As we mentioned in §1.1, the soliton resolution conjecture in the 3D radial case has been fully solved in Duyckaerts, Kenig, Merle [46], and has also been proven for a subsequence of time in 3, 4 and 5D in Duyckaerts-Jia–Kenig–Merle [43]. Recently, Duyckaerts–Kenig–Martel-Merle [44] proved the conjecture for critical co-rotational wave maps and 4D radial cubic wave equation. Since the framework in this part of the thesis is $d = 4, 5$ and is without radial assumption, here we only recall the decomposition result in [43]. More precisely:

Theorem 1.10 (Soliton resolution for a subsequence of time, [43])**.** *Let $d = 4, 5$ in (1.3). Let $\vec{u} = (u, \partial_t u) \in C\big([0,\infty) : \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\big)$ with $u \in L_t^{\frac{d+2}{d-2}} L_x^{2\frac{d+2}{d-2}}([0,T) \times \mathbb{R}^d)$ for any $0 < T < \infty$ be a global solution of (1.3) that satisfies*

$$\sup_{t\in[0,\infty)} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} < \infty.$$

Then there exist a finite energy solution $\vec{u}^L = (u^L, \partial_t u^L)$ to the linear wave equation

$$\partial_{tt} u - \Delta u = 0, \quad \text{in } (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

and an integer $J_* \geqslant 0$; a time sequence $t_n \to \infty$, scales λ_n^j with $\lambda_n^j > 0$ and $\lim_{n\to\infty} \frac{\lambda_n^j}{t_n} = 0$; positions $\boldsymbol{c}_n^j \in \mathbb{R}^d$ satisfying $\boldsymbol{c}_n^j \in \mathcal{B}_{\beta t_n}(0)$ for some $\beta \in (0,1)$ with $\boldsymbol{\ell}_j = \lim\limits_{n\to\infty} \dfrac{\boldsymbol{c}_n^j}{t_n}$ well defined; and $Q^j \in \Sigma$ for $1 \leqslant j \leqslant J_*$, such that

$$\lim_{n\to\infty} \left\| \vec{u}(t_n) - \vec{u}^L(t_n) - \sum_{j=1}^{J_*} \vec{Q}_{\boldsymbol{\ell}_j}^{j,\lambda_n^j}\left(x - \boldsymbol{c}_n^j\right) \right\|_{\dot{H}^1\times L^2} = 0,$$

where

$$\vec{Q}_{\boldsymbol{\ell}_j}^{j,\lambda_n^j} = \left(\frac{1}{(\lambda_n^j)^{\frac{d}{2}-1}} Q_{\boldsymbol{\ell}_j}^j\left(\frac{\cdot}{\lambda_n^j}\right), -\frac{1}{(\lambda_n^j)^{\frac{d}{2}}} \left(\boldsymbol{\ell}_n \cdot \nabla Q_{\boldsymbol{\ell}_j}^j\right)\left(\frac{\cdot}{\lambda_n^j}\right) \right).$$

Remark 1.11. If $J_* = 0$ in the above statement, there is no soliton in the decomposition result and the solution scatters to a linear solution. The scattering theory of (1.3) was successfully developed using the concentration-compactness method. See Kenig-Merle [71] and the references therein.

A natural question, motivated by [43], is whether we can construct a global solution of (1.3) with multi-solitons type long time behavior. Technically, such constructions for the energy-critical wave equation are more challenging than other dispersive and wave-type models, because of the weak algebraic decay of solitons. The construction is first successfully established by Martel-Merle [85] for the 5D case. More precisely:

Theorem 1.12 (Existence of multi-solitons for 5D critical NLW, [85]). *Let $d = 5$ in (1.3) and $N \geqslant 2$. For $n \in \{1, \cdots, N\}$, let $\lambda_n^\infty > 0$, $\boldsymbol{y}_n^\infty \in \mathbb{R}^5$, $\tau_n = \pm 1$ and $\boldsymbol{\ell}_n \in \mathbb{R}^5$ with $|\boldsymbol{\ell}| < 1$, $\boldsymbol{\ell}_n \neq \boldsymbol{\ell}_{n'}$ for $n \neq n'$. Assume that one of the following assumptions holds*

(A) Two-solitons ($N = 2$).

(B) Collinear speeds. *For all $n \in \{1, \ldots, N\}$, $\boldsymbol{\ell}_n = \ell_n \mathbf{e}_1$ where $\ell_n \in (-1, 1)$.*

Then, there exist $T_0 > 0$ and a solution $\vec{u} = (u, \partial_t u)$ of (1.3) on $[T_0, \infty)$ in the energy space such that

$$\lim_{t\to\infty} \left\| u(t) - \sum_{n=1}^{N} \frac{\tau_n}{(\lambda_n^\infty)^{\frac{3}{2}}} W_{\boldsymbol{\ell}_n}\left(\frac{\cdot - \boldsymbol{\ell}_n t - \boldsymbol{y}_n^\infty}{\lambda_n^\infty}\right)\right\|_{\dot{H}^1} = 0,$$
$$\lim_{t\to\infty} \left\| \partial_t u(t) + \sum_{n=1}^{N} \frac{\tau_n}{(\lambda_n^\infty)^{\frac{5}{2}}} (\boldsymbol{\ell}_n \cdot \nabla W_{\boldsymbol{\ell}_n})\left(\frac{\cdot - \boldsymbol{\ell}_n t - \boldsymbol{y}_n^\infty}{\lambda_n^\infty}\right)\right\|_{L^2} = 0.$$

Remark 1.13. We expect that Theorem 1.12 still holds true for the energy-critical wave equation for space dimensions $d \geqslant 6$. As mentioned in the introduction of [85], at the formal level, all the important computations of [85] can be reproduced for $d \geqslant 6$. However, this proof does not carry over directly to higher dimensions $d \geqslant 6$. The main reason for this is that for $d \geqslant 6$, the derivative of the nonlinearity is no longer C^1 continuous. Recall that such difficulties were overcome for the Cauchy problem in the energy space $\dot{H}^1 \times L^2$ in [10].

Remark 1.14. In the Introduction of [85], the authors also conjecture that in the 3D and 4D case, there exists no N-soliton solutions based on the ground state as in Theorem 1.12, for any $N \geqslant 2$. The main reason for this is that, due to the slow decay of ground state, the interaction between two solitons of different speeds is t^{2-d}, *i.e.* t^{-1} in dimension 3 and t^{-2} in dimension 4. Following the general strategy in [85], these interactions are too strong and create diverging terms in the construction. However, to prove the nonexistence of multi-soliton solutions rigorously, one would need enough prior information on any multi-soliton, which is an open problem for any dimension $d \geqslant 3$.

1.3.2 Main results

In this area, my first two results are mainly about the construction of two different types of multi-solitons for the 5D critical NLW equation, under various conditions on the Lorentz speeds, either based on the ground state or on suitable excited states, respectively.

The first result is the construction of multi-solitons such that each traveling wave is generated by the Lorentz transform from the ground state, given the assumption that the Lorentz speeds satisfy an explicit smallness condition.

Theorem 1.15 (Existence of multi-solitons for 5D case, [111]). *Let $d = 5$ in (1.3) and $N \geqslant 2$. For $n \in \{1, \ldots, N\}$, let $\lambda_n^\infty > 0$, $\mathbf{y}_n^\infty \in \mathbb{R}^5$, $\epsilon_n \in \{\pm 1\}$, $\boldsymbol{\ell}_n = \sum_{i=1}^5 \ell_{n,i}\mathbf{e}_i$ with $|\boldsymbol{\ell}_n| < 1$. Assume that $\boldsymbol{\ell}_n \neq \boldsymbol{\ell}_{n'}$ for $n \neq n'$, and*

$$\left(\sum_{i=1}^{5}\left(\max_{n\in\{1,\ldots,N\}} \ell_{n,i}^2\right)\right)^{\frac{1}{2}} < \frac{3}{5}. \tag{1.6}$$

Then, there exist $T_0 > 0$ and a solution u of (1.3) on $[T_0, +\infty)$ in the energy space such that

$$\lim_{t\to+\infty} \left\| u(t) - \sum_{n=1}^{N} \frac{\epsilon_n}{(\lambda_n^\infty)^{3/2}} W_{\boldsymbol{\ell}_n}\left(\frac{\cdot - \boldsymbol{\ell}_n t - \mathbf{y}_n^\infty}{\lambda_n^\infty}\right)\right\|_{\dot H^1} = 0,$$
$$\lim_{t\to+\infty} \left\| \partial_t u(t) + \sum_{n=1}^{N} \frac{\epsilon_n}{(\lambda_n^\infty)^{5/2}} (\boldsymbol{\ell}_n \cdot \nabla W_{\boldsymbol{\ell}_n})\left(\frac{\cdot - \boldsymbol{\ell}_n t - \mathbf{y}_n^\infty}{\lambda_n^\infty}\right)\right\|_{L^2} = 0.$$

Remark 1.16. The proof of Theorem 1.15 is based on extending the method of construction of asymptotic multi-solitons from [85, 86]. The smallness condition (1.6) is probably technical and we conjecture that the existence result holds for any choice of speeds. However, by our method, (1.6) seems to be the best condition using the refined approximate solution from [86]. Moreover, constructing approximate solutions improving the one in [86] seems a difficult task.

Then, inspired by Côte-Martel [26], we investigated the possibility of constructing multi-solitons based on excited states which leads to the second result of this part. More precisely, we constructed multi-solitons that are different from the ones in [85, 111]: each traveling wave is generated by Lorentz transform from a non-degenerate and decaying excited state, with different but collinear Lorentz speeds.

Theorem 1.17 (Existence of multi-excited states for 5D case, [112])**.** *Let $d = 5$ in (1.3), $N \geqslant 2$ and let $\mathbf{e}$ be a unit vector of $\mathbb{R}^5$. For $n \in \{1, \dots, N\}$, let $\boldsymbol{\ell}_n \in \mathbb{R}^5$ be such that*

$$\boldsymbol{\ell}_n = \ell_n \mathbf{e} \text{ where } -1 < \ell_n < 1 \text{ and for all } n \neq n',\ \ell_n \neq \ell_{n'}.$$

Let $q_1, \dots, q_N$ be non-degenerate decaying states of (1.4). Then there exist $T_0 > 0$ and a solution $\vec{u} = (u, \partial_t u)$ of (1.3) in the energy space $\dot H^1 \times L^2$, defined on $[T_0, +\infty)$ such that

$$\lim_{t\to+\infty} \left\| \vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\right\|_{\dot H^1 \times L^2} = 0,$$

where for $n = 1, \dots, N$,

$$\vec{Q}_n(t,x) = \begin{pmatrix} q_{n,\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \\ -\ell_n \partial_{x_1} q_{n,\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \end{pmatrix}.$$

Last, we consider the more challenging 4D case. As we mentioned in §1.3.1, we expect there exists no multi-soliton for the 4D case based on the ground state. Here, we consider a specific non-degenerate state rather than the ground state. More precisely:

Theorem 1.18 (Existence of excited multi-solitons for 4D case, [114])**.** *Let $d = 4$ in (1.3) and (1.4). Then, there exists a non-degenerate excited state $Q \in \Sigma$ such that the following is true. Let $N \geqslant 2$ and $\mathbf{e}_1$ be the first vector of the canonical basis of $\mathbb{R}^4$. For $n = 1, \cdots, N$, let $\tau_n = \pm 1$ and $\boldsymbol{\ell}_n \in \mathbb{R}^4$ be such that*

$$\boldsymbol{\ell}_n = \ell_n \mathbf{e}_1 \text{ with } -1 < \ell_n < 1 \text{ and for all } n \neq n',\ \ell_n \neq \ell_{n'}.$$

Then, there exist $T_0 > 0$ and a solution $\vec{u} = (u, \partial_t u)$ of (1.3) on $[T_0, +\infty)$ in the energy space $\dot H^1 \times L^2$ such that

$$\begin{aligned} &\lim_{t\to+\infty} \left\| u(t) - \sum_{n=1}^{N} \tau_n Q_{\boldsymbol{\ell}_n}(\cdot - \boldsymbol{\ell}_n t)\right\|_{\dot H^1} = 0, \\ &\lim_{t\to+\infty} \left\| \partial_t u(t) + \sum_{n=1}^{N} \tau_n \ell_n \partial_{x_1} Q_{\boldsymbol{\ell}_n}(\cdot - \boldsymbol{\ell}_n t)\right\|_{L^2} = 0. \end{aligned} \tag{1.7}$$

Remark 1.19. The specific non-degenerate state Q involved in Theorem 1.18 is obtained from the non-degenerate sign-changing state constructed in [36, 99]. In particular, by rotation, translation and Kelvin transformation on a non-degenerate sign-changing state, we obtain a non-degenerate excited state that decays fast enough at least along $\mathbf{e}_1$. The interaction of two such traveling waves with different speeds along $\mathbf{e}_1$, is formally of order t^{-4} for large time. This rate allows us to close the energy estimates. However, the slow algebraic decay of the linearized operator's kernel leads to another technical difficulty. To overcome it, we need a sharp understanding of the asymptotic behavior of the the kernel of linearized operator.

Remark 1.20. Indeed, the existence Theorem 1.18 is also true for the more general case where each traveling wave is based on non-degenerate q with the same asymptotic behavior as Q. In particular, we don't need each traveling wave to be based on the same non-degenerate state.

The proofs of the existence results mentioned above both rely on the general strategy using uniform estimates and compactness introduced in Merle [92] for the L^2 critical nonlinear Schrödinger equation (NLS) and Martel [83] for the subcritical and critical generalized Korteweg-de Vries equations (gKdV). The strategy of these works has been extended to other dispersive and wave-type models (see for example [5, 26, 28, 31, 53, 84, 96]).

To the author's knowledge, Theorem 1.18 is the first result of existence of multi-soliton solutions for the 4D energy-critical wave equation. As we mentioned in §1.3.1, the construction of such solutions for the energy-critical wave equation is especially motivated by the decomposition results in [44, 43, 46]. Indeed, the existence results in [85, 111, 112, 114] complement the statements in [43] by exhibiting examples of solutions containing N solitons at $t \to +\infty$ for any $N \geqslant 2$. In the same direction, recall that for the 6D energy-critical wave equation, Jendrej [67] proved the existence of radial two-bubble solutions. Later, for the 5D energy-critical wave equation, Jendrej-Martel [69] proved the existence of N-bubble solutions for any $N \geqslant 2$. Recall also that, for the 4D case, Hillairet-Raphaël [62] proved the existence of type-II finite time blow-up solitons containing single soliton.

We refer to [38, 39, 40] for results related to the existence of type-II blow up solutions containing single or multi-bubble of the energy-critical heat equation. Particularly related to [114], the blow up profile in [40] is the non-degenerate sign-changing state constructed in [36, 99] rather than the ground state W.

It is worth mentioning that the question of *the construction of multi-bubble solutions for the 4D NLW equation* is still an open problem. In [69], Jendrej and Martel have constructed global multi-bubble solutions of the 5D energy-critical NLW equation, blowing up in infinite time at any K given points z_k of $\mathbb{R}^5$. In the 4D case, due to the weak decay of the ground state, we expect the existence of global solution blowing up in infinite time along lines of different speeds, instead of at given points. The self-similar ODE system, generated by the interaction of two traveling wave with different Lorentz speeds formally admits solutions that blow up in infinite time. The difficulties of this problem are to find a suitable refined approximate solution and then to construct a solution close to this approximate solution, using energy estimates and compactness. Another open problem related to the results in [69, 114] is *the construction of multi-soliton or multi-bubble solutions for 3D energy-critical wave equation.* The main difficulty there is the slow algebraic decay of solitons, which is even lower than that of the 4D case.

1.4 The damped NLKG equation with energy sub-critical nonlinearity

1.4.1 Background

Consider the nonlinear focusing damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0 \quad (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{1.8}$$

where $f(u) = |u|^{p-1}u$, $\alpha > 0$, $1 \leqslant d \leqslant 5$ and the exponent p satisfies

$$2 < p < p^*(d) \quad \text{with} \quad p^*(d) = \begin{cases} \infty & \text{if } d = 1, 2, \\ \dfrac{d+2}{d-2} & \text{if } d = 3, 4, 5. \end{cases}$$

Setting $F(u) = \frac{1}{p+1}|u|^{p+1}$ and

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{|\nabla u|^2 + u^2 + (\partial_t u)^2 - 2F(u)\right\} \mathrm{d}x,$$

for any $H^1 \times L^2$ solution $\vec{u} = (u, \partial_t u)$ of (1.8), we have that

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 \mathrm{d}t. \tag{1.9}$$

The restrictions $1 \leqslant d \leqslant 5$ and $2 < p < \frac{d+2}{d-2}$ ensure a suitable local well-posedness theory and sufficient smoothness on the non-linearity so that the Taylor expansion around 0 makes sense up to the second order. Our analysis encompasses the most physically relevant nonlinearity, the 3D cubic one.

Let us first recall from [17] (see also references therein) some features of stationary solution, namely a solution to the elliptic equation

$$-\Delta q + q - f(q) = 0, \quad q \in H^1(\mathbb{R}^d). \tag{1.10}$$

We call the solutions of (1.10) *bound states*, and denote $\mathcal{B}$ the set of bound states:

$$\mathcal{B} = \{q : q \text{ is a nontrivial solution of (1.10)}\}.$$

Standard elliptic arguments (see *e.g.* [58] or [17, Theorem 8.1.1]) show that if $q \in \mathcal{B}$, then q is of class $\mathcal{C}^2(\mathbb{R}^d)$ and has exponential decay as $|x| \to +\infty$, as well as its first and second-order derivatives.

Let

$$\mathcal{W}(v) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{|\nabla v|^2 + v^2 - 2F(v)\right\} \mathrm{d}x, \quad \text{for } v \in H^1.$$

We call the solutions of (1.10) that minimize the functional $\mathcal{W}$ *ground states*; the set of ground states is denoted by $\mathcal{G}$

$$\mathcal{G} = \{q_0 \in \mathcal{B} : \forall q \in \mathcal{B},\ \mathcal{W}(q_0) \leqslant \mathcal{W}(q)\}.$$

Ground states are well-studied objects. They are unique up to space translation (for rather general nonlinearities). That is, there exists a radial positive function Q of class $\mathcal{C}^2$, exponentially decreasing, along with its first and second-order derivatives, such that

$$\mathcal{G} = \{Q(x - x_0) : x_0 \in \mathbb{R}^d\}.$$

We refer to Berestycki-Lions [7], Gidas-Ni-Nirenberg [57], Kwong [77], Serrin-Tang [107]. More specifically, the ground state Q can be rewritten as $Q(x) = q(|x|)$ where $q > 0$ satisfies

$$q'' + \frac{d-1}{r}q' - q + q^p = 0, \quad q'(0) = 0, \quad \lim_{r\to\infty} q(r) = 0. \tag{1.11}$$

It is well-known and easily checked that for a constant $\kappa > 0$, for all $r > 1$,

$$\left|q(r) - \kappa r^{-\frac{d-1}{2}} e^{-r}\right| + \left|q'(r) + \kappa r^{-\frac{d-1}{2}} e^{-r}\right| \lesssim r^{-\frac{d+1}{2}} e^{-r}.$$

In particular, in dimension $d = 1$, we have the explicit formula for Q:

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}} \quad \text{on } \mathbb{R}.$$

It is also well-known (see *e.g.* Grillakis-Shatah-Strauss [60]) that the ground state Q is unstable in the energy space. This result was also known in the physics literature as the Derrick's Theorem [41]. More specifically, from ODE argument, the operator

$$\mathcal{L} = -\Delta + 1 - f'(Q)$$

appearing after linearization of equation (1.8) around $\vec{Q} = (Q, 0)$ has a unique negative eigenvalue $-\nu_0^2$ ($\nu_0 > 0$). We denote by Y the corresponding normalized eigenfunction. In particular, it follows from explicit computations that setting

$$\nu^{\pm} = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{and} \quad \vec{Y}^{\pm} = \begin{pmatrix} Y \\ \nu^{\pm} Y \end{pmatrix},$$

the function $\vec{\varepsilon}^{\pm}(t,x) = \exp(\nu^{\pm} t)\vec{Y}^{\pm}(x)$ is solution of the linearized problem

$$\begin{cases} \partial_t \varepsilon = \eta \\ \partial_t \eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{1.12}$$

Since $\nu^+ > 0$, the solution $\vec{\varepsilon}^+$ illustrates the exponential instability of the solitary wave in positive time. In particular, we see that the presence of the damping $\alpha > 0$ in the equation does not remove the exponential instability of the Klein-Gordon solitary wave.

In dimension $d = 1$, due to standard ODE arguments, $\mathcal{B} = \mathcal{G}$. In contrast, following [17, Remark 8.1.16], for any $d \geqslant 2$, $\mathcal{G} \subsetneq \mathcal{B}$. Functions $q \in \mathcal{B} \setminus \mathcal{G}$ are referred to as *excited states*. Recall that, Berestycki-Lions [8] showed the existence of infinitely many radial *nodal* (*i.e.* sign changing) solutions (see also [61, 98] and the references therein). As a matter of fact, much less is known about excited states except for specific constructions.

Similar as §1.3.1, we introduce all possible invariances of (1.10):

(i) translation: If q is a solution of (1.10), then $q(x + x_0)$ is also a solution of (1.10) for all $x_0 \in \mathbb{R}^d$;

(ii) orthogonal transformation: If q is a solution of (1.10), then $q(Ox)$ is also a solution of (1.10) for all $O \in \mathcal{O}_d$.

Due to the invariances of equations, $\mathcal{L}_q$ always has a kernel: for $d > 1$, denote by Ω_{ij} the angular derivatives

$$\Omega_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i} \quad \text{for } 1 \leqslant i < j \leqslant d, \tag{1.13}$$

and consider the vector space $\mathcal{Z}_q$ spanned by the infinitesimal generator of the invariance of the equation on q:

$$\mathcal{Z}_q = \operatorname{Span}\left\{\partial_{x_n} q, n = 1, \cdots, d; \Omega_{ij} q, 1 \leqslant i < j \leqslant d\right\}. \tag{1.14}$$

One always has $\mathcal{Z}_q \subset \operatorname{Ker}\mathcal{L}_q$. Then we define the non-degenerate and degenerate states.

Definition 1.21. Let $q \in \mathcal{B}$.

(i) q is called a *non-degenerate state* if $\mathcal{Z}_q = \operatorname{Ker}\mathcal{L}_q$.

(ii) q is called a *degenerate state* if $\mathcal{Z}_q \subsetneq \operatorname{Ker}\mathcal{L}_q$.

The most relevant example is of course the ground state Q which is non-degenerate. But one should keep in mind that degenerate excited states are not known to exist. As a matter of fact, the constructions in [2, 98] yield *non-degenerate* excited states as well. For now, let us simply mention one way to understand degeneracy (we denote $'$ for Gateaux differential). The condition that q is a bound state writes $E'(q) = 0$. Then $\langle \mathcal{L}_q v, w\rangle = E''(q)\cdot(v,w)$, so that the condition that q is degenerate, is equivalent to the fact that for some $\phi \notin \mathcal{Z}_q$, the linear form $E''(q)\cdot(\phi,\cdot) = 0$.

We now recall two results related to the long time dynamics of global solutions. First, in [12], for dimension $d \geq 2$, Burq, Raugel and Schlag proved the convergence of any global radial solution to one radial bound state, for the whole sequence of time. Second, under some conditions on d and p, results in [53, 82] state that for any sequence of time, any global bounded solution of (1.8) converges to a sum of decoupled bound states after extraction of a subsequence of times. More precisely:

Theorem 1.22 ([53, 82]). *Let $1 \leqslant d \leqslant 4$ with $1 < p < p^*(d)$ or $5 \leqslant d \leqslant 6$ with $1 < p < p^{**}(d)$ in (1.8) where*

$$p^*(d) = \begin{cases} \infty & \text{if } d = 1,2, \\ \dfrac{d+2}{d-2} & \text{if } d = 3,4. \end{cases} \quad \text{and} \quad p^{**}(d) = \frac{d}{d-2} \quad \text{if } 3 \leqslant d \leqslant 6.$$

Then for any global bounded solution $\vec{u} = (u, \partial_t u)$ of (1.8) and time sequence $t_n \to \infty$, up to subsequence, there exist $J \in \mathbb{N}^+$, a space sequence $x_{j,n} \in \mathbb{R}^N$ for $j = 1, \dots, N$, and bound states Q^j such that

$$\lim_{n\to\infty} \left\{ \|u(t_n) - Q^j(\cdot - x_{j,n})\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \right\} = 0,$$

where $\{x_{j,n}\}$ satisfies the separation property:

$$\lim_{n\to\infty} |x_{j,n} - x_{j',n}| = \infty, \quad \text{for all } j \neq j'.$$

Remark 1.23. Theorem 1.22 was first proved in Feireisl [53] for the case $3 \leqslant d \leqslant 6$ with $1 < p < p^{**}(d)$. Later, Li-Zhao [82] extended the result to the case $1 \leqslant d \leqslant 4$ with $1 < p < p^*(d)$ by concentration compact method. In [53], Feireisl also constructed global solutions that behave as sum of an even number of ground states (*i.e.* multi-solitons).

Remark 1.24. As discussed in [12], such results in [12, 53, 82] are closely related to the general soliton resolution conjecture for global bounded solutions of dispersive problems.

1.4.2 Main results

Theorem 1.22, valid on any global bounded solution, is remarkable. However, it does not fully describe the asymptotic behavior of general global solutions. For example, it does not specify the signs of solitons, the behavior of the position parameters, whether the convergence holds for all positive time, and the convergence rate. The main results of this section are to complement Theorem 1.22 by describing precisely the asymptotic behavior as $t \to \infty$ of solitons or multi-soliton solutions of (1.8). The main results are split into the following three parts.

Description and classification of 2-solitary waves

In this part, we describe completely 2-solitary waves related to the ground state Q. We start with the definition of 2-solitary waves.

Definition 1.25. A solution $\vec{u} \in \mathcal{C}([T,\infty), H^1 \times L^2)$ of (1.8), for some $T \in \mathbb{R}$, is called a 2-*solitary wave* if there exist $\sigma_1, \sigma_2 = \pm 1$, a sequence $t_n \to \infty$ and a sequence $(\xi_{1,n}, \xi_{2,n}) \in \mathbb{R}^{2d}$ such that

$$\lim_{n\to\infty} \left\{ \Big\| u(t_n) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_{k,n}) \Big\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \right\} = 0$$

and $\lim_{n\to\infty} |\xi_{1,n} - \xi_{2,n}| \to \infty$.

Remark 1.26. We observe that if u is a global solution of (1.8) satisfying

$$\lim_{t\to\infty}\left\|u(t)-\sum_{k=1,2}\sigma_k Q(\cdot-\xi_k(t))\right\|_{H^1}=0 \tag{1.15}$$

where $\lim_{t\to\infty}|\xi_1(t)-\xi_2(t)|\to\infty$, then u is a 2-solitary wave. Indeed, it follows from (1.9) and (1.15) that $t\mapsto E(\vec{u}(t))$ is lower bounded. Thus, from (1.9), it holds $\int_0^\infty \|\partial_t u(t)\|_{L^2}^2 dt<\infty$ and so $\|\partial_t u(t_n)\|_{L^2}\to 0$ for some sequence $t_n\to\infty$.

Our first result concerns the non-existence of 2-solitary waves with the same signs.

Theorem 1.27 ([29]). *There exists no 2-solitary wave of (1.8) with $\sigma_1=\sigma_2$.*

In the next result, we show that 2-solitary waves with opposite sign satisfy a universal asymptotic behavior.

Theorem 1.28 (Description of 2-solitary waves, [29]). *For any 2-solitary wave u of (1.8), there exist $\sigma_1=\pm1$, $\sigma_2=-\sigma_1$, $z_\infty\in\mathbb{R}^d$, $\omega_\infty\in S_{\mathbb{R}^d}(1)$, $T>0$ and $t\in[T,\infty)\mapsto(z_1(t),z_2(t))$ such that for all $t\in[T,\infty)$,*

$$\left\|u(t)-\sum_{k=1,2}\sigma_k Q(\cdot-z_k(t))\right\|_{H^1}+\|\partial_t u(t)\|_{L^2}\lesssim t^{-1}, \tag{1.16}$$

and for some constant $c_0=c_0(d,\alpha)$, for any $\theta<\min(p-1,2)$ and for $k=1,2$,

$$z_k=z_\infty+\frac{(-1)^k}{2}\left(\log t-\frac{1}{2}(d-1)\log\log t+c_0\right)\omega_\infty+O(t^{1-\theta}). \tag{1.17}$$

Finally, we describe the full family of 2-solitary waves for initial data close to the sum of two remote solitary waves.

Theorem 1.29 (Classification of 2-solitary waves, [29]). *There exist $C,\delta>0$ and a Lipschitz map*

$$H:(\mathbb{R}^d\setminus\bar{\mathcal{B}}_{\mathbb{R}^d}(10|\log\delta|))\times\mathcal{B}_{H^1\times L^2}(\delta)\to\mathbb{R}^2,\quad (L,\vec{\phi})\mapsto H(L,\vec{\phi})$$

such that

$$|H(L,\vec{\phi})|<C\left(e^{-\frac{L}{2}}+\|\vec{\phi}\|_{H^1\times L^2}\right),$$

with the following property. Given any $L,\vec{\phi},h_1,h_2$ such that

$$|L|>10|\log\delta|,\quad \|\vec{\phi}\|_{H^1\times L^2}<\delta,\quad |h_1|+|h_2|<\delta,$$

the solution $\vec{u}$ of (1.8) with initial data

$$\vec{u}(0)=\left(\vec{Q}+h_1\vec{Y}^+\right)\left(\cdot-\frac{L}{2}\right)-\left(\vec{Q}+h_2\vec{Y}^+\right)\left(\cdot+\frac{L}{2}\right)+\vec{\phi}$$

is a 2-solitary wave if and only if $(h_1,h_2)=H(L,\vec{\phi})$.

This result essentially means that locally around the sum of two sufficiently separated solitons with opposite signs, the initial data of 2-solitary waves form a codimension-2 Lipschitz manifold (the unstable directions being directed by $\vec{Y}^+$ translated around each soliton).

It is worth mentioning that the study of *description and classification of 2-solitary waves for fractional damped NLKG equation* is an interesting problem related to the result in [29]. To be more specific, we consider the fractional damped NLKG equation,

$$\partial_{tt}u+2\alpha\partial_t u+(-\Delta)^s u+u-|u|^\sigma u=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$

where $0 < s < 1$ and where the exponent σ satisfies

$$0 < \sigma < \sigma_*(s,d), \quad \sigma_*(s,d) = \begin{cases} \dfrac{4s}{d-2s} & \text{for } 0 < s < \dfrac{d}{2}, \\ +\infty & \text{for } \dfrac{d}{2} \leqslant s < 1. \end{cases}$$

Denote the *ground state* by Q_s, which is the unique radial nonnegative solution in $H^s(\mathbb{R}^d)$ of

$$(-\Delta)^s Q_s + Q_s - |Q_s|^\sigma Q_s = 0, \quad Q_s(x) \sim \frac{1}{|x|^{d+2s}} \quad \text{as } |x| \to \infty.$$

The existence and properties of this solution are studied in [55, 56]. For the fractional damped NLKG equation, due to the slow decay of ground state, we expect 2-solitary waves to have a universal behavior that is different from that of the damped NLKG equation. Specifically, the ODE formally observed for the interaction of two traveling waves with large distance is $\dot{r} = \pm C r^{-(d+2s)}$. In the case $\dot{r} = -Cr^{-(d+2s)}$, there is no global solution such that $r \gg 1$. In contrast, in the case $\dot{r} = Cr^{-(d+2s)}$, the ODE admits a solution of the form $r(t) = ct^{\frac{1}{d+2s+1}}$ as $t \to \infty$.

Long-time asymptotics of the 1D damped NLKG equation

The objective of the present part is to complement Theorem 1.22 by describing precisely the asymptotic behavior as $t \to \infty$ of any global solution of (1.8) with $d = 1$. More precisely:

Theorem 1.30 ([30]). *For any global solution $\vec{u} \in C\big([0,\infty), H^1 \times L^2\big)$ of (1.8) with $d = 1$, one of the following three scenarios occurs:*

Vanishing: *$\vec{u}(t)$ converges exponentially to 0 in $H^1 \times L^2$ as $t \to \infty$.*

Single soliton: *There exist $\sigma = \pm 1$, $\ell \in \mathbb{R}$ such that $\vec{u}(t)$ converges exponentially to $(\sigma Q(\cdot - \ell), 0)$ in $H^1 \times L^2$ as $t \to \infty$.*

Multi-soliton: *There exist $K \geq 2$, $\sigma = \pm 1$, $\ell \in \mathbb{R}$ and functions $z_k : [0,\infty) \to \mathbb{R}$, for all $k = 1, \dots, K$ such that for all $t \in (0,\infty)$,*

$$\left\| u(t) - \sigma \sum_{k=1}^{K} (-1)^k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{1.18}$$

and for any $1 < \theta < \min(p-1, \frac{5}{4})$, $k = 1, \dots, K$,

$$z_k(t) = \left(k - \frac{K+1}{2}\right) \log t + \tau_k + \ell + O(t^{-\theta+1}), \tag{1.19}$$

as $t \to \infty$, where τ_k are the constants uniquely defined by

$$\sum_{k=1}^{K} \tau_k = 0, \quad e^{-(\tau_{k+1} - \tau_k)} = \frac{2\alpha}{\kappa} \gamma_k \quad \text{where} \quad \gamma_k = \frac{k(K-k)}{2}, \tag{1.20}$$

and $\kappa > 0$ only depends on p.

Remark 1.31. The parameter ℓ is related to the translation invariance. In the vanishing and the single soliton cases, the damping leads to exponential convergence as $t \to \infty$. In the multi-soliton case,

due to the nonlinear interactions between the solitary waves, the asymptotic behavior (1.19) of the centers of mass z_k is related to the following nonlinear differential system

$$\begin{cases} \dot{y}_1 = -\dfrac{\kappa}{2\alpha} e^{-(y_2-y_1)}, \\ \dot{y}_k = \dfrac{\kappa}{2\alpha}\left(e^{-(y_k-y_{k-1})} - e^{-(y_{k+1}-y_k)}\right), \quad \text{for } k = 2, \cdots, K-1, \\ \dot{y}_K = \dfrac{\kappa}{2\alpha} e^{-(y_K-y_{K-1})}. \end{cases} \tag{1.21}$$

This system is studied in [33, 93] in the context of blowup solutions of the nonlinear wave equation. The nonlinear interactions at short distances between the solitary waves also yield the slower rate of convervence t^{-1} in (1.18). See [65, 90, 109, 110] for other examples of strong interactions leading to $\log t$ distant solitary waves.

Theorem 1.30 is a version of the soliton resolution for global solutions of the one-dimensional damped nonlinear Klein-Gordon equation (1.8), with convergence for the whole sequence of time and a description of the parameters of the solitary waves. The case of the damped Klein-Gordon equation on a bounded domain for suitable nonlinearities is addressed in [18, Theorem 9.5.3]; see also references therein.

Obviously, $\vec{u} = (0, 0)$ and $\vec{u} = (Q, 0)$ are examples of the first two scenarios of Theorem 1.30. Our second result gives examples of the third scenario for any $K \geq 2$.

Theorem 1.32 ([30]). *For any $K \geq 2$, $\sigma = \pm 1$ and $\ell \in \mathbb{R}$, there exist global solutions of (1.8) with $d = 1$ satisfying (1.18) and (1.19).*

Remark 1.33. Note that [12, 29, 53, 82] also apply to the multi-dimensional case, under suitable restrictions on the exponent p of the nonlinearity, or for radially symmetric solutions. However, for space dimensions greater than 1, the existence of bound states solutions of $\Delta w - w + f(w) = 0$ other than the ground state Q, together with the possibility of involved geometric configurations of solitary waves, complicate the analysis. This is why we restrict to dimension 1 in the present part.

Asymptotics of solutions with a compactness property

In this part, we are instead interested in understanding the behavior of solutions to (1.8) for which *only one* nonlinear object appears for large times, at least for a sequence of time. More precisely, we define *packed* solutions as follows.

Definition 1.34. A maximal solution $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0, T_{\max}), H^1 \times L^2)$ of (1.8) with $2 \leqslant d \leqslant 5$ is called a *packed solution* if there exist $(W_0, W_1) \in H^1 \times L^2$, and a time sequence $t_n \to T_{\max}$ and a position sequence $y_n \in \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \{\|u(t_n) - W_0(\cdot - y_n)\|_{H^1} + \|\partial_t u(t_n) - W_1(\cdot - y_n)\|_{L^2}\} = 0. \tag{1.22}$$

We say that $\vec{W} = (W_0, W_1)$ is a *cluster point* for $\vec{u}$ at $(t_n, y_n)_n$.

Observe that any cluster point (W_0, W_1) is actually a bound state $(q, 0)$. More precisely, the following Proposition holds true.

Proposition 1.35. *Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (1.8). Then $\vec{u} \in \mathcal{C}([0, +\infty), H^1 \times L^2)$ is globally defined for positive times, and if $(W_0, W_1) \in H^1 \times L^2$ is a cluster point for $\vec{u}$ at $(t_n, y_n)_n$, then $W_0 = q$ is a bound state of (1.10) and $W_1 = 0$. Furthermore, the energy is bounded below, $\partial_t u \in L^2([0, +\infty), L^2)$ and for all $t \geqslant 0$,*

$$E(\vec{u}(t)) - E(q, 0) = 2\alpha \int_t^{+\infty} \|\partial_t u(s)\|_{L^2}^2 \mathrm{d}s. \tag{1.23}$$

Notice that it is unclear whether a packed solution is globally bounded in $H^1 \times L^2$. Recall that from the arguments of [16] – see also [12] and [30] – if $p \leqslant \frac{d}{d-2}$, then any global solution to (1.8) is globally bounded in $H^1 \times L^2$, but this is not known for higher powers of p.

Our first result is that if one cluster point of a packed solution is a non-degenerate state, then the convergence holds of all positive time, and occurs with an exponential rate. More precisely, we have the following.

Theorem 1.36 ([32]). *Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (1.8), with cluster point $q \in H^1$ at $(t_n, y_n)_n$. If q is a non-degenerate state, we have convergence holding for all time and exponential decay, i.e. there exist $\mu > 0$ and $z_\infty \in \mathbb{R}^d$ such that*

$$\forall t \geqslant 0, \quad \|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim e^{-\mu t}.$$

Remark 1.37. Let us first observe that Theorem 1.36 (and its proof) holds also in dimension 1, but of course, this is a direct consequence of the complete description [30] of global solutions in 1D (as mentioned in §1.4.1, excited states only exist for $d \geqslant 2$). This is the reason why we restrict to dimension $d \geqslant 2$ in this part.

Next, we consider *degree-1 excited states* where $\mathrm{Ker}\mathcal{L}_q$ has one extra dimension not related to the geometric invariances of the equation (1.10) and which also involves a condition on the third-order Gateau differentials of E, according to the next definition.

Definition 1.38. Let q be a degenerate excited state. q is called a *degree-1 excited state* if there exists $\phi \in H^1$ such that

$$\mathrm{Ker}\mathcal{L}_q = \mathcal{Z}_q \oplus \mathrm{Span}\{\phi\} \quad \text{and} \quad E'''(q) \cdot (\phi, \phi, \phi) \neq 0. \tag{1.24}$$

Our second result is concerned with cluster points which are degree-1 excited states.

Theorem 1.39 ([32]). *Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (1.8), with cluster point $q \in H^1$ at $(t_n, y_n)_n$. If q is a degree-1 excited state, then the convergence $\vec{u}(t) \to (q, 0)$ holds for all time as $t \to +\infty$, and the rate of convergence has algebraic decay, i.e. there exists $z_\infty \in \mathbb{R}^d$ such that*

$$\forall t > 0, \quad \|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}.$$

Last, we show that the convergence rate in Theorem 1.39 can be sharp: we provide an example where the solution converges exactly at the rate t^{-1} to the degree-1 excited state.

Theorem 1.40 ([32]). *Let q be a degree-1 excited state. Then, there exists a global solution $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0, +\infty), H^1 \times L^2)$ of (1.8) such that*

$$\|u(t) - q\|_{H^1} + \|\partial_t u(t)\|_{L^2} \sim t^{-1} \quad \text{as} \quad t \to +\infty.$$

Remark 1.41. Degree-1 excited states are somehow the simplest degenerate bound states. Already here, very little is known, and to our knowledge, Theorem 1.39 and Theorem 1.40 are the first results describing precisely the dynamics in a degenerate setting. From this point of view, the condition that $\dim \mathrm{Ker}\mathcal{L}_q = \dim(\mathcal{Z}_q) + 1$ is very natural. Regarding the extra condition $E'''(q) \cdot (\phi, \phi, \phi) \neq 0$, let us note that it is generic; it is equivalent with $E'''(q)$ being non-identically 0 on $(\mathrm{Ker}\mathcal{L}_q)^3$. Note that *the construction of degree-1 excited state* is still an open problem.

Remark 1.42. It is remarkable that one already observes a drastic change in the dynamic in degree-1 degeneracy, when compared to non-degeneracy. The convergence here is indeed merely polynomial in time, which is a surprise: such a slow rate of convergence is usually observed due to the interaction with another nonlinear object (as in [29, 30]), but this is not the case here.

It is worth mentioning that *the complete description of all packed solutions* is an interesting open problem, but it seems out of reach for now. This is because we lack the understanding of the dynamics around general excited states, and the system of centers of mass of the involved bound states may have very intricate dynamics.

Part I

Stability of multi-solitons for the 1D NLKG equation

Chapter 2

Conditional stability of multi-solitons for the 1D NLKG equation with double power nonlinearity

Abstract: We consider the one-dimensional nonlinear Klein-Gordon equation with a double power focusing-defocusing nonlinearity

$$\partial_t^2 u - \partial_x^2 u + u - |u|^{p-1}u + |u|^{q-1}u = 0, \quad \text{on } [0,\infty) \times \mathbb{R},$$

where $1 < q < p < \infty$. The main result states the stability in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ of the sums of decoupled solitary waves with different speeds, up to the natural instabilities. The proof is inspired by the techniques developed for the generalized Korteweg-de Vries equation and the nonlinear Schrödinger equation in a similar context by Martel, Merle and Tsai [88, 89]. However, the adaptation of this strategy to a wave-type equation requires the introduction of a new energy functional adapted to the Lorentz transform.

2.1 Introduction

2.1.1 Main result

We consider the one-dimensional nonlinear Klein-Gordon equation (NLKG) with a double power nonlinearity

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + u - |u|^{p-1}u + |u|^{q-1}u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u_{|t=0} = u_0 \in H^1, \quad \partial_t u_{|t=0} = u_1 \in L^2, \end{cases} \tag{2.1}$$

where $1 < q < p < \infty$. This equation also rewrites as a first order system in time for the function $\vec{u} = (u_1, u_2)$,

$$\begin{cases} \partial_t u_1 = u_2 \\ \partial_t u_2 = \partial_x^2 u_1 - u_1 + f(u_1), \end{cases} \tag{2.2}$$

where

$$f(u_1) = |u_1|^{p-1}u_1 - |u_1|^{q-1}u_1. \tag{2.3}$$

We recall that the Cauchy problem for equation (2.2) is locally well-posed in the energy space $H^1 \times L^2$. See *e.g.* [59]. Denote $F(u_1) = \frac{1}{p+1}|u_1|^{p+1} - \frac{1}{q+1}|u_1|^{q+1}$. For any solution $\vec{u} = (u_1, u_2)$ of (2.2) in $H^1 \times L^2$, the energy $E(\vec{u})$ and momentum $\mathcal{I}(\vec{u})$ are conserved, where

$$E(\vec{u}) = \int_{\mathbb{R}} \left\{ (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right\} \mathrm{d}x, \quad \mathcal{I}(\vec{u}) = 2\int_{\mathbb{R}} (\partial_x u_1) u_2 \mathrm{d}x.$$

Denote by Q the *ground state*, which is the unique positive even solution in H^1 of the equation

$$Q'' - Q + f(Q) = 0 \quad \text{on} \quad \mathbb{R}. \tag{2.4}$$

The existence and properties of this solution are studied in [7, Section 6] (see also Remark 2.4 and §2.2.1). The ground state generates the standing wave solution $\vec{Q} = (Q, 0)$ of (2.2). Moreover, using the Lorentz transformation on $\vec{Q}$, one obtains traveling solitary waves or *solitons*: for $\ell \in \mathbb{R}$, with $-1 < \ell < 1$, let

$$Q_\ell(x) = Q\left(\frac{x}{\sqrt{1-\ell^2}}\right), \quad \vec{Q}_\ell = \begin{pmatrix} Q_\ell \\ -\ell \partial_x Q_\ell \end{pmatrix},$$

then $\vec{u}(t,x) = \vec{Q}_\ell(x - \ell t)$ is a solution of (2.2).

It is well-known (see precise statements and references in §2.2.1) that the operator

$$\mathcal{L} = -\partial_x^2 + 1 - f'(Q)$$

appearing after linearization of equation (2.2) around $\vec{Q} = (Q, 0)$, has a unique negative eigenvalue $-\nu_0^2$ ($\nu_0 > 0$), with corresponding smooth even eigenfunction Y. Set

$$\vec{Y}^+ = \begin{pmatrix} Y \\ \nu_0 Y \end{pmatrix} \quad \text{and} \quad \vec{Z}^+ = \begin{pmatrix} \nu_0 Y \\ Y \end{pmatrix}.$$

From explicit computations, the function $\vec{u}^+(t,x) = \exp(\nu_0 t)\vec{Y}^+(x)$ is solution of the linearized system

$$\begin{cases} \partial_t u_1 = u_2 \\ \partial_t u_2 = -\mathcal{L} u_1. \end{cases} \tag{2.5}$$

Since $\nu_0 > 0$, the solution $\vec{u}^+$ illustrates the (one-dimensional) exponential instability of the solitary wave $\vec{Q}$ in positive time. An equivalent formulation of instability is obtained by observing that for any solution $\vec{u}$ of (2.5), it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} a^+ = \nu_0 a^+ \quad \text{where} \quad a^+(t) = \left(\vec{u}(t), \vec{Z}^+\right)_{L^2}.$$

More generally, for $-1 < \ell < 1$, set

$$Y_\ell = Y\left(\frac{x}{\sqrt{1-\ell^2}}\right) \quad \text{and} \quad \vec{Z}_\ell^+ = \begin{pmatrix} (\ell\partial_x Y_\ell + \frac{\nu_0}{\sqrt{1-\ell^2}} Y_\ell) e^{\frac{\ell\nu_0}{\sqrt{1-\ell^2}}x} \\ Y_\ell e^{\frac{\ell\nu_0}{\sqrt{1-\ell^2}}x} \end{pmatrix}.$$

The main purpose of this article is to study the conditional stability of multi-solitons with different speeds for (2.1). More precisely, the main result is the following.

Theorem 2.1. *Let $N \geqslant 2$. For all $n \in \{1, \dots, N\}$, let $\sigma_n = \pm 1$, $\ell_n \in (-1, 1)$ with $-1 < \ell_1 < \ell_2 < \dots < \ell_N < 1$. There exist $L_0 > 0$, $C_0 > 0$, $\gamma_0 > 0$ and $\delta_0 > 0$ such that the following is true. Let $\vec{\varepsilon} \in H^1 \times L^2$ and $y_1^0 < \dots < y_N^0$ be such that there exist $L > L_0$ and $0 < \delta < \delta_0$ with*

$$\|\vec{\varepsilon}\|_{H^1\times L^2} < \delta \quad \text{and} \quad y_{n+1}^0 - y_n^0 > L \quad \text{for all } n = 1, \dots, N-1.$$

Then, there exist $h_1^+, \dots, h_N^+ \in \mathbb{R}$ satisfying

$$\sum_{n=1}^N |h_n^+| \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big),$$

such that the solution $\vec{u} = (u_1, u_2)$ of (2.2) with initial data

$$\vec{u}_0 = \sum_{n=1}^N \big(\sigma_n \vec{Q}_{\ell_n} + h_n^+ \vec{Z}_{\ell_n}^+\big)(\cdot - y_n^0) + \vec{\varepsilon}$$

is globally defined in $H^1 \times L^2$ for $t \geqslant 0$ and, for all $t \geqslant 0$,

$$\Big\|\vec{u}(t) - \sum_{n=1}^N \sigma_n \vec{Q}_{\ell_n}(\cdot - y_n(t))\Big\|_{H^1\times L^2} \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big),$$

where $y_1(t), \dots, y_N(t)$ are C^1 functions satisfying, for all $n = 1, \dots, N$, $t \geqslant 0$,

$$|y_n(0) - y_n^0| \leqslant C_0\big(\delta + e^{-\gamma_0 L}\big), \quad |\dot{y}_n(t) - \ell_n| \leqslant C_0^2\big(\delta + e^{-\gamma_0 L}\big).$$

Remark 2.2. As remarked before, each soliton solution $\vec{u}(t,x) = \pm\vec{Q}_{\ell_n}(x - \ell_n t)$ of equation (2.2) has exactly one exponential instability direction in positive time. Recall that such instability has led to the construction of co-dimension one center-stable manifolds for initial data close to a single soliton for the NLKG or the wave equations in [75, 100, 101]. In the case of a perturbation of the sum of N decoupled solitons $\sum_{n=1}^N \sigma_n Q_{\ell_n}(\cdot - y_n)$, this justifies the need of adjusting the N free parameters $(h_n^+)_{n\in\{1,\dots,N\}}$ in the choice of the initial data to obtain global existence and stability in Theorem 2.1. Such difficulty does not appear when the solitons are stable, like for the mass subcritical generalized Korteweg-de Vries (gKdV) and nonlinear Schrödinger equation (NLS) discussed in the next remark.

Remark 2.3. Historically, multi-soliton solutions were first studied extensively for integrable equations, like the Korteweg-de Vries equation and the one-dimensional cubic Schrödinger equation. In the non-integrable cases, for dispersive or wave equations, the first result concerning stability and asymptotic stability of multi-soliton solutions was given by Perelman [104] for some NLS equations, extending the single soliton case treated by Buslaev and Perelman [11]. The present work is inspired by [88, 89] proving the stability of the sums of decoupled solitons for the mass subcritical gKdV equations and some NLS equations. See also [105] and [6, 22, 79, 95] for other related models.

Such stability results are closely related to the existence of *asymptotic pure multi-solitons*, established in the case of the NLKG equation by Côte and Muñoz [31]. For gKdV, NLS, and the energy-critical wave equation for both stable and unstable solitons, we refer to [20, 28, 83, 84, 85, 111, 112, 114].

Remark 2.4. The double power nonlinearity (2.3) is a typical choice for dispersive or wave equations, but Theorem 2.1 can be extended to any general real-valued $C^{1,\alpha}$ function f (with $\alpha > 0$) satisfying the following:

(i) The function f is odd and $f(0) = f'(0) = 0$.

(ii) There exists a smallest $s_0 > 0$ such that $F(s_0) - \frac{1}{2}s_0^2 = 0$, and $f(s_0) - s_0 > 0$.

(iii) There exists $r_0 > 0$ such that for all $s \in (-r_0, r_0)$, $sf(s) - 2F(s) \leqslant 0$.

Here, $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$ for $s \in \mathbb{R}$. Conditions (i)-(ii) are related to the necessary and sufficient condition for existence of a unique standing wave in [7, Theorem 5]. Note that for f defined in (2.3), the existence of a smallest $s_0 > 0$ such that $F(s_0) - \frac{1}{2}s_0^2 = 0$ is clear and then

$$\begin{aligned} f(s_0) - s_0 = s_0^p - s_0^q - s_0 &= \frac{1}{s_0}\left((p+1)F(s_0) + \frac{p-q}{q+1}s_0^{q+1} - s_0^2\right) \\ &= \frac{1}{s_0}\left(\frac{p-q}{q+1}s_0^{q+1} + \frac{p-1}{2}s_0^2\right) > 0, \end{aligned}$$

which justifies the existence of the standing wave Q for (2.1) (see also [7, Example 2, p. 318]).

Condition (iii) ensures that the nonlinear effect is defocusing for small values of u_1, which is decisive in our proof (more precisely, in the proof of the monotonicity result Lemma 2.20). Recall that this issue is already present in [89, 104] for the NLS equation, where a weaker but similar restriction is imposed on the nonlinearity.

To our knowledge, this article proves the first statement of (conditional) stability of the sums of decoupled solitons for a wave-type equation. Observe that the Lorentz transform involved in the propagation of solitons has quite different properties than the Galilean transform for the NLS equation or the natural propagation phenomenon related to the solitons of the gKdV equation. Compared to the energy method developed in [89], we use a similar functional based on localized versions of the momentum and the energy around each soliton (respectively defined in (2.54) and (2.55)), but a specific combination of these quantities has to be used in the wave case (see (2.57) and the definition of the coefficients c_n in (2.50)); moreover, the introduction of an additional lower order term F_n (see (2.56)) was necessary to remove diverging terms. See also Remark 2.16.

This paper is organized as follows. Section 2.2 introduces technical tools involved in a dynamical approach to the N-soliton problem for (2.1): estimates of the nonlinear interactions between solitons, decomposition by modulation and parameter estimates. Energy estimates and monotonicity properties are proved in Section 2.3. Finally, Theorem 2.1 is proved in Section 2.4.

2.1.2 Notation

We denote $(\cdot,\cdot)_{L^2}$ the L^2 scalar product for real-valued functions $u, v \in L^2$,

$$(u,v)_{L^2} := \int_{\mathbb{R}} u(x)v(x)\mathrm{d}x.$$

For

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

denote

$$(\vec{u},\vec{v})_{L^2} := \sum_{k=1,2} (u_k, v_k)_{L^2}, \quad \|\vec{u}\|_{\mathcal{H}}^2 := \|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2.$$

For $f \in L^2$ and $\ell \in (-1,1)$, set

$$f_\ell(x) = f(x_\ell), \quad \text{where} \quad x_\ell = \frac{x}{\sqrt{1-\ell^2}}.$$

2.2 Preliminaries

2.2.1 Spectral theory

It is well-known that the unique positive even solution Q of (2.4) is smooth and satisfies the following estimate on $\mathbb{R}$,

$$|Q^{(\alpha)}(x)| \lesssim e^{-|x|} \quad \text{for any } \alpha \in \mathbb{N}. \tag{2.6}$$

In this section, we recall the spectral properties of the linearized operator around Q_ℓ. First, for $-1 < \ell < 1$, let

$$\mathcal{L}_\ell = -(1-\ell^2)\partial_x^2 + 1 - f'(Q_\ell).$$

We recall the following standard spectral properties for $\mathcal{L}$ and $\mathcal{L}_\ell$ (see *e.g.* [31, Lemma 1 and Corollary 1]).

Lemma 2.5. (i) Spectral properties. *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1,+\infty)$, its kernel is spanned by Q' and it has a unique negative eigenvalue $-\nu_0^2$ ($\nu_0 > 0$), with corresponding smooth even eigenfunction Y. Moreover, on $\mathbb{R}$,*

$$\left|Y^{(\alpha)}(x)\right| \lesssim e^{-\sqrt{1+\nu_0^2}|x|} \quad \textit{for any } \alpha \in \mathbb{N}. \tag{2.7}$$

(ii) Coercivity property of $\mathcal{L}$. *There exists $\nu > 0$ such that, for all $v \in H^1$,*

$$\big(\mathcal{L}v, v\big)_{L^2} \geqslant \nu\|v\|_{H^1}^2 - \nu^{-1}\big((v,Q')_{L^2}^2 + (v,Y)_{L^2}^2\big).$$

(iii) Coercivity property of $\mathcal{L}_\ell$. *Let $\ell \in (-1,1)$. There exists $\nu > 0$ such that, for all $v \in H^1$,*

$$\big(\mathcal{L}_\ell v, v\big)_{L^2} \geqslant \nu\|v\|_{H^1}^2 - \nu^{-1}\big((v,\partial_x Q_\ell)_{L^2}^2 + (v,Y_\ell)_{L^2}^2\big).$$

Second, we define

$$\mathcal{H}_\ell = \begin{pmatrix} -\partial_x^2 + 1 - f'(Q_\ell) & -\ell\partial_x \\ \ell\partial_x & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$\vec{Z}_\ell^0 = \begin{pmatrix} \partial_x Q_\ell \\ -\ell\partial_x^2 Q_\ell \end{pmatrix}, \quad \vec{Z}_\ell^\pm = \begin{pmatrix} \Upsilon_{\ell,1}^\pm \\ \Upsilon_{\ell,2}^\pm \end{pmatrix}$$

where

$$\Upsilon_{\ell,1}^\pm = \left(\ell\partial_x Y_\ell \pm \frac{\nu_0}{\sqrt{1-\ell^2}} Y_\ell\right) e^{\pm\frac{\ell\nu_0}{\sqrt{1-\ell^2}}x} \quad \text{and} \quad \Upsilon_{\ell,2}^\pm = Y_\ell e^{\pm\frac{\ell\nu_0}{\sqrt{1-\ell^2}}x}.$$

The functions $\vec{Z}_\ell^\pm$ are eigenfunctions of the linearized matrix operator $\mathcal{H}_\ell J$ and are explicitly determined by solving the eigenvalue problem $\mathcal{H}_\ell J\vec{Z} = \lambda\vec{Z}$.

Lemma 2.6 ([31]). *Let $\ell \in (-1,1)$.*

(i) Decay properties of $\Upsilon_{\ell,1}^\pm$ and $\Upsilon_{\ell,2}^\pm$. *It holds, on $\mathbb{R}$, for $k = 1,2$,*

$$\left|\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\Upsilon_{\ell,k}^\pm(x)\right| \lesssim e^{-|x|} \quad \textit{for any } \alpha \in \mathbb{N}. \tag{2.8}$$

(ii) Properties of $\mathcal{H}_\ell$ and $\mathcal{H}_\ell J$. *It holds*

$$\mathcal{H}_\ell\vec{Z}_\ell^0 = 0, \quad \big(\vec{Z}_\ell^0, \vec{Z}_\ell^\pm\big)_{L^2} = 0 \quad \textit{and} \quad \mathcal{H}_\ell J\big(\vec{Z}_\ell^\pm\big) = \mp\nu_0(1-\ell^2)^{\frac{1}{2}}\vec{Z}_\ell^\pm. \tag{2.9}$$

(iii) Coercivity property of $\mathcal{H}_\ell$. *There exists $\nu > 0$ such that, for all $\vec{v} \in H^1 \times L^2$,*

$$\big(\mathcal{H}_\ell\vec{v}, \vec{v}\big)_{L^2} \geqslant \nu\|\vec{v}\|_{\mathcal{H}}^2 - \nu^{-1}\big((\vec{v},\vec{Z}_\ell^0)_{L^2}^2 + (\vec{v},\vec{Z}_\ell^+)_{L^2}^2 + (\vec{v},\vec{Z}_\ell^-)_{L^2}^2\big). \tag{2.10}$$

Proof. We prove (i) for $\alpha = 0$; the estimates for $\alpha \geq 1$ are proved similarly. By (2.7), we have, for $k = 1, 2$, on $\mathbb{R}$,

$$|\Upsilon^{\pm}_{\ell,k}(x)| \lesssim e^{-\frac{\sqrt{1+\nu_0^2}-|\ell|\nu_0}{\sqrt{1-\ell^2}}|x|}.$$

But an elementary computation yields

$$\left(\sqrt{1+\nu_0^2} - |\ell|\nu_0\right)^2 = \left(|\ell|\sqrt{1+\nu_0^2} - \nu_0\right)^2 + 1 - \ell^2,$$

and so $\sqrt{1+\nu_0^2} - |\ell|\nu_0 \geq \sqrt{1-\ell^2}$, which implies $|\Upsilon^{\pm}_{\ell,k}(x)| \lesssim e^{-|x|}$.

For (ii) and (iii), we refer to Proposition 1 and the proofs of Lemma 2, Lemma 3 and Proposition 2 in [31]. □

2.2.2 Decomposition of the solution around multi-solitary waves

We recall general results on solutions of (2.2) that are close to the sum of $N \geqslant 2$ decoupled solitary waves. For any $n \in \{1, \dots, N\}$, let $\sigma_n = \pm 1$ and $\ell_n \in (-1, 1)$ satisfy

$$-1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1.$$

Let $t \mapsto y_n(t) \in \mathbb{R}$ be C^1 functions such that

$$y_{n+1} - y_n \gg 1 \quad \text{for any } n = 1, \dots N-1. \tag{2.11}$$

For $n \in \{1, \dots, N\}$, define

$$Q_n = \sigma_n Q_{\ell_n}(\cdot - y_n), \quad \vec{Q}_n = \begin{pmatrix} Q_n \\ -\ell_n \partial_x Q_n \end{pmatrix}$$

and

$$\vec{Z}^0_n = \sigma_n \vec{Z}^0_{\ell_n}(\cdot - y_n), \quad \vec{Z}^{\pm}_n = \vec{Z}^{\pm}_{\ell_n}(\cdot - y_n) = \begin{pmatrix} \Upsilon^{\pm}_{n,1} \\ \Upsilon^{\pm}_{n,2} \end{pmatrix}.$$

We recall a decomposition result for solutions of (2.2).

Lemma 2.7. *There exist $L_0 > 0$ and $0 < \delta_0 \ll 1$ such that if $\vec{u} = (u_1, u_2)$ is a solution of (2.2) on $[0, T_0]$, where $T_0 > 0$, such that for all $t \in [0, T_0]$*

$$\inf_{z_{n+1} - z_n > L_0} \|\vec{u}(t) - \sum_{n=1}^{N} \sigma_n \vec{Q}_{\ell_n}(\cdot - z_n)\|_{\mathcal{H}} < \delta_0, \tag{2.12}$$

then there exist C^1 functions $y_1(t), \dots, y_N(t)$ on $[0, T_0]$ such that, $\vec{\varphi}$ being defined by

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \vec{u} = \sum_{n=1}^{N} \vec{Q}_n + \vec{\varphi}, \tag{2.13}$$

it satisfies

$$\left(\vec{\varphi}, \vec{Z}^0_n\right)_{L^2} = 0, \quad \text{for } n = 1, \dots, N, \tag{2.14}$$

and

$$\|\vec{\varphi}\|_{\mathcal{H}} \lesssim \delta_0, \quad y_{n+1} - y_n \geqslant \frac{3}{4} L_0 \quad \text{for } n = 1, \dots, N-1. \tag{2.15}$$

Proof. The proof of the decomposition lemma relies on a standard argument based on the Implicit function Theorem (see *e.g.* Lemma 5 and Appendix B in [31]) and we omit it. □

Set

$$\vec{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \sum_{n=1}^{N} \vec{Q}_n \quad \text{and} \quad G = f(U_1) - \sum_{n=1}^{N} f(Q_n). \tag{2.16}$$

Lemma 2.8 (Equation of $\vec{\varphi}$). *The function $\vec{\varphi}$ satisfies*

$$\begin{cases} \partial_t \varphi_1 = \varphi_2 + \mathrm{Mod}_1, \\ \partial_t \varphi_2 = \partial_x^2 \varphi_1 - \varphi_1 + f\left(U_1 + \varphi_1\right) - f\left(U_1\right) + G + \mathrm{Mod}_2, \end{cases} \tag{2.17}$$

where

$$\mathrm{Mod}_1 = \sum_{n=1}^{N} (\dot{y}_n - \ell_n) \partial_x Q_n, \quad \mathrm{Mod}_2 = -\sum_{n=1}^{N} (\dot{y}_n - \ell_n) \ell_n \partial_x^2 Q_n. \tag{2.18}$$

Proof. First, from the definition of $\vec{\varphi} = (\varphi_1, \varphi_2)$ in (2.13),

$$\partial_t \varphi_1 = \partial_t u_1 - \partial_t U_1 = \varphi_2 + U_2 - \partial_t U_1 = \varphi_2 + \sum_{n=1}^{N} (\dot{y}_n - \ell_n) \partial_x Q_n.$$

Second, using (2.2),

$$\partial_t \varphi_2 = \partial_t u_2 - \partial_t U_2 = \partial_x^2 u_1 - u_1 + f(u_1) - \sum_{n=1}^{N} \dot{y}_n \big(\ell_n \partial_x^2 Q_n\big).$$

We observe from (2.13) and $-(1 - \ell_n^2)\partial_x^2 Q_n + Q_n - f(Q_n) = 0$,

$$\partial_x^2 u_1 - u_1 + f(u_1) = \partial_x^2 \varphi_1 - \varphi_1 + f(U_1 + \varphi_1) - \sum_{n=1}^{N} f(Q_n) + \sum_{n=1}^{N} \ell_n^2 \partial_x^2 Q_n.$$

Therefore, from the definition of G and Mod_2, we obtain (2.17). □

For future reference, we present in Lemmas 2.9 and 2.10 some technical results. First, we give several pointwise estimates concerning the nonlinearity f.

Lemma 2.9. *For any $(s_n)_{n=1,\dots,N} \in \mathbb{R}^N$, the following estimates hold.*

(i) *For $1 < q < p \leqslant 3$,*

$$\left| f'\left(\sum_{n=1}^{N} s_n\right) - \sum_{n=1}^{N} f'(s_n) \right| \lesssim \sum_{n \neq n'} |s_n|^{\frac{q-1}{2}} |s_{n'}|^{\frac{q-1}{2}} + \sum_{n \neq n'} |s_n|^{\frac{p-1}{2}} |s_{n'}|^{\frac{p-1}{2}}. \tag{2.19}$$

(ii) *For $1 < q \leqslant 3 < p < \infty$,*

$$\left| f'\left(\sum_{n=1}^{N} s_n\right) - \sum_{n=1}^{N} f'(s_n) \right| \lesssim \sum_{n \neq n'} |s_n|^{\frac{q-1}{2}} |s_{n'}|^{\frac{q-1}{2}} + \sum_{n \neq n'} |s_n|^{p-2} |s_{n'}|. \tag{2.20}$$

(iii) *For $3 < q < p < \infty$,*

$$\left| f'\left(\sum_{n=1}^{N} s_n\right) - \sum_{n=1}^{N} f'(s_n) \right| \lesssim \sum_{n \neq n'} |s_n|^{q-2} |s_{n'}| + \sum_{n \neq n'} |s_n|^{p-2} |s_{n'}|. \tag{2.21}$$

(iv) *For* $1 < q < p < \infty$,

$$\left| f\left(\sum_{n=1}^{N} s_n\right) - \sum_{n=1}^{N} f(s_n) \right| \lesssim \sum_{n \neq n'} |s_n|^{q-1}|s_{n'}| + \sum_{n \neq n'} |s_n|^{p-1}|s_{n'}|, \tag{2.22}$$

$$\left| F\left(\sum_{n=1}^{N} s_n\right) - \sum_{n=1}^{N} F(s_n) \right| \lesssim \sum_{n \neq n'} |s_n|^{q}|s_{n'}| + \sum_{n \neq n'} |s_n|^{p}|s_{n'}|. \tag{2.23}$$

(v) *For* $1 < q < p < \infty$,

$$\left| f(s_1 + s_2) - f(s_1) - f'(s_1)s_2 \right| \lesssim |s_2|^2 + |s_2|^q + |s_2|^p, \tag{2.24}$$

$$\left| F(s_1 + s_2) - F(s_1) - f(s_1)s_2 - \frac{1}{2}f'(s_1)s_2^2 \right| \lesssim |s_2|^3 + |s_2|^{q+1} + |s_2|^{p+1}. \tag{2.25}$$

Proof. Proof of (i)-(ii)-(iii). Recall that $f'(s) = p|s|^{p-1} - q|s|^{q-1}$ for p, q such that $1 < q < p < \infty$. Let $(s_n)_{n=1,\dots,N} \in \mathbb{R}^N$ and let $m \in \{1, \dots, N\}$ be such that

$$|s_m| = \max_{n=1,\dots,N} |s_n|.$$

By the triangle inequality,

$$\left| \left| \sum_{n=1}^{N} s_n \right|^{p-1} - \sum_{n=1}^{N} |s_n|^{p-1} \right| \lesssim \sum_{n \neq m} |s_n|^{p-1} + \left| \left| \sum_{n=1}^{N} s_n \right|^{p-1} - |s_m|^{p-1} \right|.$$

First, by the definition of m, we observe that

$$\sum_{n \neq m} |s_n|^{p-1} \lesssim \begin{cases} |s_m|^{\frac{p-1}{2}} \sum_{n \neq m} |s_n|^{\frac{p-1}{2}} & \text{if } 1 < p \leqslant 3 \\ |s_m|^{p-2} \sum_{n \neq m} |s_n| & \text{if } 3 < p. \end{cases}$$

Second, if $|s_m| \leqslant 2 \sum\limits_{n \neq m} |s_n|$, then

$$\left| \left| \sum_{n=1}^{N} s_n \right|^{p-1} - |s_m|^{p-1} \right| \lesssim |s_m|^{p-1} \lesssim \begin{cases} |s_m|^{\frac{p-1}{2}} \sum_{n \neq m} |s_n|^{\frac{p-1}{2}} & \text{if } 1 < p \leqslant 3 \\ |s_m|^{p-2} \sum_{n \neq m} |s_n| & \text{if } 3 < p. \end{cases}$$

Third, if $|s_m| > 2\sum_{n \neq m} |s_n|$ then by the inequality $|(1+\mu)^{p-1} - 1| \lesssim |\mu|$ for any $|\mu| \leq \frac{1}{2}$, we have

$$\left| \left| \sum_{n=1}^{N} s_n \right|^{p-1} - |s_m|^{p-1} \right| = |s_m|^{p-1} \left| \left(1 + \frac{1}{s_m} \sum_{n \neq m} s_n\right)^{p-1} - 1 \right| \lesssim |s_m|^{p-2} \sum_{n \neq m} |s_n|.$$

Moreover, note that if $1 \leq p \leq 3$ then $p - 2 \leq \frac{p-1}{2}$ and thus by the definition of m,

$$|s_m|^{p-2} \sum_{n \neq m} |s_n| \lesssim |s_m|^{\frac{p-1}{2}} \sum_{n \neq m} |s_n|^{\frac{p-1}{2}}.$$

Proceeding similarly for the term $|s|^{q-1}$ in $f'(s)$, we obtain (i), (ii) and (iii).

The proofs of (iv) and (v) follow from the same technique. □

Second, using the pointwise estimates of Lemma 2.9, we derive estimates related to the equation of $\vec{\varphi}$ and the nonlinear interaction term G defined in (2.16). Denote

$$q^* = \begin{cases} \frac{q-1}{2} & \text{for } 1 < q \leqslant 3 \\ q-2 & \text{for } 3 < q < \infty \end{cases} \quad \text{and} \quad p^* = \begin{cases} \frac{p-1}{2} & \text{for } 1 < p \leqslant 3 \\ p-2 & \text{for } 3 < p < \infty \end{cases}$$

and observe that $0 < q^* < p^*$. Fix

$$\gamma_0 = \frac{1}{8}\min(q^*, 1) > 0 \quad \text{and let} \quad \theta = \sum_{n=1}^{N-1} e^{-4\gamma_0(y_{n+1}-y_n)}. \tag{2.26}$$

The introduction of γ_0 and θ will allow us to keep track of the exponent of exponential interactions between solitons (for instance, see the next lemma). We will not keep track of this dependence in multiplicative constants.

Lemma 2.10. *Assume* (2.11), *we have*

$$\sum_{n\neq n'} \int_{\mathbb{R}} \left(|Q_n Q_{n'}| + |\partial_x Q_n \partial_x Q_{n'}| + \left|\partial_x^2 Q_n \partial_x^2 Q_{n'}\right|\right) \mathrm{d}x \lesssim \theta, \tag{2.27}$$

$$\sum_{n\neq n'} \int_{\mathbb{R}} \left(\left|\partial_x Q_n \Upsilon_{n',1}\right| + \left|\partial_x^2 Q_n \Upsilon_{n',2}\right|\right) \mathrm{d}x \lesssim \theta, \tag{2.28}$$

$$\int_{\mathbb{R}} \left| F(U_1) - \sum_{n=1}^{N} F(Q_n) \right| \mathrm{d}x \lesssim \theta, \tag{2.29}$$

$$\|G\|_{L^2} + \sum_{n\neq n'} \|f'(Q_n)\Upsilon_{n',2}\|_{L^2} + \|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\|_{L^2} \lesssim \theta. \tag{2.30}$$

Remark 2.11. For convenience, all the terms above are bounded by the same error term $\sum_{n=1}^{N-1} e^{-4\gamma_0(y_{n+1}-y_n)}$. Of course, in view of the definition of γ_0, these estimates are far from being optimal.

Proof of Lemma 2.10. Proof of (2.27). First, by change of variable, for any $n \neq n'$,

$$\int_{\mathbb{R}} |Q_n Q_{n'}| \mathrm{d}x = H_1 + H_2,$$

where

$$H_1 = (1-\ell_n^2)^{\frac{1}{2}} \int_{I_1} Q(x) Q\left(\frac{(y_n - y_{n'}) + (1-\ell_n^2)^{\frac{1}{2}} x}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right) \mathrm{d}x,$$
$$H_2 = (1-\ell_n^2)^{\frac{1}{2}} \int_{I_2} Q(x) Q\left(\frac{(y_n - y_{n'}) + (1-\ell_n^2)^{\frac{1}{2}} x}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right) \mathrm{d}x,$$

and

$$I_1 = \left\{x \in \mathbb{R} : (1-\ell_n^2)^{\frac{1}{2}}|x| < \frac{1}{2}|y_n - y_{n'}|\right\},$$
$$I_2 = \left\{x \in \mathbb{R} : (1-\ell_n^2)^{\frac{1}{2}}|x| \geqslant \frac{1}{2}|y_n - y_{n'}|\right\}.$$

From the decay properties of Q in (2.6) and then $\gamma_0 \leqslant \frac{1}{8} \leqslant \frac{1}{8}(1-\ell_n^2)^{-\frac{1}{2}}$, for any $x \in I_2$, we have

$$|Q(x)| \lesssim e^{-\frac{1}{2}\frac{|y_n - y_{n'}|}{\sqrt{1-\ell_n^2}}} \lesssim e^{-4\gamma_0|y_n - y_{n'}|},$$

and so

$$H_2 \lesssim e^{-4\gamma_0|y_n - y_{n'}|} \int_{I_2} Q\left(\frac{(y_n - y_{n'}) + (1-\ell_n^2)^{\frac{1}{2}}x}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right) \mathrm{d}x \lesssim e^{-4\gamma_0|y_n - y_{n'}|} \lesssim \theta.$$

Similarly, for any $x \in I_1$,

$$\left|Q\left(\frac{(y_n - y_{n'}) + (1-\ell_n^2)^{\frac{1}{2}}x}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right)\right| \lesssim e^{-4\gamma_0|y_n - y_{n'}|},$$

and so

$$H_1 \lesssim e^{-4\gamma_0|y_n - y_{n'}|} \int_{I_1} Q(x)\mathrm{d}x \lesssim e^{-4\gamma_0|y_n - y_{n'}|} \lesssim \theta.$$

This proves estimate (2.27) for $Q_n Q_{n'}$. Using (2.6), the rest of the proof of (2.27) is the same.

Proof of (2.28). From (2.6), (2.8) and $(1-\ell_n^2)^{-\frac{1}{2}} \geqslant 1$, we have

$$\sum_{n\neq n'} \left|\partial_x Q_n \Upsilon_{n',1}\right| + \left|\partial_x^2 Q_n \Upsilon_{n',2}\right| \lesssim \sum_{n\neq n'} e^{-|x-y_n|} e^{-|x-y_{n'}|}.$$

Arguing as in the proof of (2.27), using $\gamma_0 = \frac{1}{8}\min(q^*, 1) \leqslant \frac{1}{8}$, we obtain (2.28).

Proof of (2.29). From (2.6), (2.23) and $1 < q < p < \infty$, we infer

$$\left|F(U_1) - \sum_{n=1}^{N} F(Q_n)\right| \lesssim \sum_{n\neq n'} |Q_n|^q |Q_{n'}| + \sum_{n\neq n'} |Q_n|^p |Q_{n'}| \lesssim \sum_{n\neq n'} |Q_n||Q_{n'}|.$$

Thus, estimate (2.29) follows from (2.27).

Proof of (2.30). From (2.22), we have

$$|G| \lesssim \sum_{n\neq n'} |Q_n|^{q-1}|Q_{n'}| + \sum_{n\neq n'} |Q_n|^{p-1}|Q_{n'}|.$$

By $1 < q < p$, (2.6) and (2.8),

$$\sum_{n\neq n'} \left|f'(Q_n)\Upsilon_{n',2}\right| \lesssim \sum_{n\neq n'} |Q_n|^{q-1} e^{-|x-y_{n'}|}.$$

Moreover, from (2.19), (2.20), (2.21), we infer, for $1 < q < p \leq 3$,

$$\left|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\right| \lesssim \sum_{n\neq n'} |Q_n|^{\frac{q-1}{2}} |Q_{n'}|^{\frac{q-1}{2}} + \sum_{n\neq n'} |Q_n|^{\frac{p-1}{2}} |Q_{n'}|^{\frac{p-1}{2}},$$

for $1 < q \leq 3 < p$,

$$\left|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\right| \lesssim \sum_{n\neq n'} |Q_n|^{\frac{q-1}{2}} |Q_{n'}|^{\frac{q-1}{2}} + \sum_{n\neq n'} |Q_n|^{p-2}|Q_{n'}|,$$

and last, for $3 < q < p$,

$$\left|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\right| \lesssim \sum_{n\neq n'} |Q_n|^{q-2}|Q_{n'}| + \sum_{n\neq n'} |Q_n|^{p-2}|Q_{n'}|.$$

Arguing as in the proof of (2.27), using $\gamma_0 = \frac{1}{8}\min(q^*, 1)$, we obtain (2.30). □

Third, we derive the control of $\dot{y}_n$ from the orthogonality conditions (2.14).

Lemma 2.12 (Control of $\dot{y}_n$)**.** *In the context of Lemma 2.7, we have*

$$\sum_{n=1}^{N} |\dot{y}_n - \ell_n| \lesssim \|\vec{\varphi}\|_{\mathcal{H}} + \theta. \tag{2.31}$$

Proof. First, we rewrite the equation of $\vec{\varphi} = (\varphi_1, \varphi_2)$ as

$$\partial_t \vec{\varphi} = \vec{\mathcal{L}}\vec{\varphi} + \vec{\mathrm{Mod}} + \vec{G} + \vec{R}_1 + \vec{R}_2, \tag{2.32}$$

where

$$\vec{\mathcal{L}} = \begin{pmatrix} 0 & 1 \\ \partial_x^2 - 1 + \sum_{n=1}^{N} f'(Q_n) & 0 \end{pmatrix}, \quad \vec{\mathrm{Mod}} = \begin{pmatrix} \mathrm{Mod}_1 \\ \mathrm{Mod}_2 \end{pmatrix},$$

$$\vec{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}, \quad \vec{R}_1 = \begin{pmatrix} 0 \\ R_1 \end{pmatrix} = \begin{pmatrix} 0 \\ f(U_1 + \varphi_1) - f(U_1) - f'(U_1)\varphi_1 \end{pmatrix},$$

and

$$\vec{R}_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \left(f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\right)\varphi_1 \end{pmatrix}.$$

Second, from the orthogonality conditions (2.14),

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\varphi}, \vec{Z}_n^0\right)_{L^2} = \left(\partial_t\vec{\varphi}, \vec{Z}_n^0\right)_{L^2} + \left(\vec{\varphi}, \partial_t\vec{Z}_n^0\right)_{L^2}.$$

Thus, using (2.32),

$$\begin{aligned} 0 = &\left(\vec{\mathcal{L}}\vec{\varphi}, \vec{Z}_n^0\right)_{L^2} + \left(\vec{R}_1, \vec{Z}_n^0\right)_{L^2} + \left(\vec{R}_2, \vec{Z}_n^0\right)_{L^2} + \left(\vec{G}, \vec{Z}_n^0\right)_{L^2} + \left(\vec{\mathrm{Mod}}, \vec{Z}_n^0\right)_{L^2} \\ &- (\dot{y}_n - \ell_n)\left(\vec{\varphi}, \partial_x\vec{Z}_n^0\right)_{L^2} - \ell_n\left(\vec{\varphi}, \partial_x\vec{Z}_n^0\right)_{L^2}. \end{aligned}$$

By integration by parts and the decay properties of Q in (2.6), the first term is

$$\left(\vec{\mathcal{L}}\vec{\varphi}, \vec{Z}_n^0\right)_{L^2} = \left(\vec{\varphi}, \vec{\mathcal{L}}^t\vec{Z}_n^0\right)_{L^2} = O(\|\vec{\varphi}\|_{\mathcal{H}}).$$

Next, by (2.24), we infer

$$|R_1| \lesssim |\varphi_1|^2 + |\varphi_1|^q + |\varphi_1|^p,$$

and by the Sobolev embedding theorem,

$$\left|\left(\vec{R}_1, \vec{Z}_n^0\right)_{L^2}\right| \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 + \|\vec{\varphi}\|_{\mathcal{H}}^q + \|\vec{\varphi}\|_{\mathcal{H}}^p.$$

Using the Cauchy-Schwarz inequality and (2.30), we obtain

$$\left|\left(\vec{R}_2, \vec{Z}_n^0\right)_{L^2}\right| \lesssim \|\vec{\varphi}\|_{\mathcal{H}}\|f'(U_1) - \sum_{n'=1}^{N} f'(Q_{n'})\|_{L^2} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 + \theta^2.$$

Then, using (2.30), we have

$$\left|\left(\vec{G}, \vec{Z}_n^0\right)_{L^2}\right| \lesssim \|G\|_{L^2} \lesssim \theta.$$

Next, using the expression of $\vec{\mathrm{Mod}}$,

$$\left(\vec{\mathrm{Mod}}, \vec{Z}_n^0\right)_{L^2} = (\dot{y}_n - \ell_n)\left(\vec{Z}_n^0, \vec{Z}_n^0\right)_{L^2} + \sum_{n' \neq n}(\dot{y}_{n'} - \ell_{n'})\left(\vec{Z}_{n'}^0, \vec{Z}_n^0\right)_{L^2}.$$

Moreover, from (2.27),

$$\sum_{n'\neq n}(\dot{y}_{n'}-\ell_{n'})\left(\vec{Z}^0_{n'},\vec{Z}^0_n\right)_{L^2}=O\Big(\theta\sum_{n'\neq n}|\dot{y}_{n'}-\ell_{n'}|\Big).$$

Last, by the Cauchy-Schwarz inequality,

$$\left|(\dot{y}_n-\ell_n)\left(\vec{\varphi},\partial_x\vec{Z}^0_n\right)_{L^2}\right|+\left|\ell_n\left(\vec{\varphi},\partial_x\vec{Z}^0_n\right)_{L^2}\right|\lesssim\|\vec{\varphi}\|_{\mathcal{H}}\left(1+|\dot{y}_n-\ell_n|\right).$$

Gathering above estimates, we obtain

$$\left|\dot{y}_n-\ell_n\right|\lesssim\|\vec{\varphi}\|_{\mathcal{H}}+\theta+O\Big((\theta+\|\vec{\varphi}\|_{\mathcal{H}})\sum_{n'=1}^{N}|\dot{y}_{n'}-\ell_{n'}|\Big).$$

Summing over n, we find

$$\sum_{n=1}^{N}\left|\dot{y}_n-\ell_n\right|\lesssim\|\vec{\varphi}\|_{\mathcal{H}}+\theta+O\Big((\theta+\|\vec{\varphi}\|_{\mathcal{H}})\sum_{n'=1}^{N}|\dot{y}_{n'}-\ell_{n'}|\Big),$$

which implies (2.31) for δ_0 small enough and L_0 large enough. □

Last, we consider the equation of the exponential directions.

Lemma 2.13 (Exponential directions)**.** *In the context of Lemma 2.7, let* $a_n^\pm=\left(\vec{\varphi},\vec{Z}_n^\pm\right)_{L^2}$ *for all* $n=1,\dots,N$*. It holds*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}a_n^\pm\mp\alpha_n a_n^\pm\right|\lesssim\|\vec{\varphi}\|_{\mathcal{H}}^2+\|\vec{\varphi}\|_{\mathcal{H}}^q+\theta,\tag{2.33}$$

where $\alpha_n=\nu_0(1-\ell_n^2)^{\frac{1}{2}}$.

Proof. First, using (2.32), we compute,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_1^\pm&=\left(\partial_t\vec{\varphi},\vec{Z}_1^\pm\right)_{L^2}+\left(\vec{\varphi},\partial_t\vec{Z}_1^\pm\right)_{L^2}\\&=\left(\vec{\mathcal{L}}\vec{\varphi},\vec{Z}_1^\pm\right)_{L^2}-\ell_1\left(\vec{\varphi},\partial_x\vec{Z}_1^\pm\right)_{L^2}+\left(\vec{G},\vec{Z}_1^\pm\right)_{L^2}\\&\quad+\left(\vec{R}_1,\vec{Z}_1^\pm\right)_{L^2}+\left(\vec{R}_2,\vec{Z}_1^\pm\right)_{L^2}+\left(\vec{\mathrm{Mod}},\vec{Z}_1^\pm\right)_{L^2}-(\dot{y}_1-\ell_1)\left(\vec{\varphi},\partial_x\vec{Z}_1^\pm\right)_{L^2}.\end{aligned}$$

Note that, from (2.9) and integration by parts,

$$\begin{aligned}\left(\vec{\mathcal{L}}\vec{\varphi},\vec{Z}_1^\pm\right)_{L^2}-\ell_1\left(\vec{\varphi},\partial_x\vec{Z}_1^\pm\right)_{L^2}&=-\left(\vec{\varphi},\left(\mathcal{H}_{\ell_1}J\vec{Z}_{\ell_1}^\pm\right)(\cdot-y_1)\right)_{L^2}\\&\quad+\sum_{n=2}^{N}\left(\varphi_1,f'(Q_n)\Upsilon_{1,2}^\pm\right)_{L^2}\\&=\pm\alpha_1a_1^\pm+\sum_{n=2}^{N}\left(\varphi_1,f'(Q_n)\Upsilon_{1,2}^\pm\right)_{L^2}.\end{aligned}$$

Using (2.30) and the Cauchy-Schwarz inequality,

$$\left|\sum_{n=2}^{N}\left(\varphi_1,f'(Q_n)\Upsilon_{1,2}^\pm\right)_{L^2}\right|\lesssim\sum_{n=2}^{N}\|f'(Q_n)\Upsilon_{1,2}^\pm\|_{L^2}\|\varphi_1\|_{L^2}\lesssim\|\varphi_1\|_{L^2}^2+\theta^2.$$

Next, using (2.30), (2.24) and the Sobolev embedding theorem

$$\left|\left(\vec{R}_1, \vec{Z}_1^{\pm}\right)_{L^2}\right| \lesssim \int_{\mathbb{R}} \left(|\varphi_1|^2 + |\varphi_1|^q + |\varphi_1|^p\right) \mathrm{d}x \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 + \|\vec{\varphi}\|_{\mathcal{H}}^q + \|\vec{\varphi}\|_{\mathcal{H}}^p,$$

$$\left|\left(\vec{R}_2, \vec{Z}_1^{\pm}\right)_{L^2}\right| \lesssim \|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\|_{L^2} \|\varphi_1\|_{L^2} \leqslant \|\vec{\varphi}\|_{\mathcal{H}}^2 + \theta^2,$$

and

$$\left|\left(\vec{G}, \vec{Z}_1^{\pm}\right)_{L^2}\right| \lesssim \|G\|_{L^2} \lesssim \theta.$$

Moreover, from (2.9), (2.28) and (2.31),

$$\left(\vec{\mathrm{Mod}}, \vec{Z}_1^{\pm}\right)_{L^2} = \sum_{n=2}^{N} (\dot{y}_n - \ell_n) \left(\vec{Z}_n^0, \vec{Z}_1^{\pm}\right)_{L^2} = O\left(\|\vec{\varphi}\|_{\mathcal{H}}^2 + \theta^2\right).$$

Last, from (2.31),

$$\left|(\dot{y}_1 - \ell_1)(\vec{\varphi}, \partial_x \vec{Z}_1^{\pm})_{L^2}\right| \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 + \theta^2.$$

Gathering the above estimates and proceeding similarly for $a_n^{\pm}$ for $n = 2, \dots, N$, we obtain (2.33). □

2.3 Monotonicity property for the 1D Klein-Gordon equation

2.3.1 Bootstrap setting

Fix

$$\gamma_1 = \frac{1}{8} \min\left(\ell_2 - \ell_1, \cdots, \ell_N - \ell_{N-1}\right) > 0. \tag{2.34}$$

We introduce the following bootstrap estimates: for $C_0 \gg 1$ to be chosen later,

$$\begin{cases} \|\vec{\varphi}(t)\|_{\mathcal{H}} \leqslant C_0 \left(\delta + e^{-\gamma_0 L}\right), \\ \min\{(y_{n+1} - y_n)(t);\ n = 1, \dots, N\} \geqslant \dfrac{9}{10} L + 2\gamma_1 t, \\ \displaystyle\sum_{n=1}^{N} |a_n^+(t)|^2 \leqslant C_0^{\frac{3}{2}} \left(\delta^2 + e^{-2\gamma_0 L}\right), \\ \displaystyle\sum_{n=1}^{N} |a_n^-(t)|^2 \leqslant C_0^{\frac{7}{4}} \left(\delta^2 + e^{-2\gamma_0 L}\right). \end{cases} \tag{2.35}$$

Recall that γ_0 is defined in (2.26). Let $\vec{u}_0$ satisfy (2.12) and (2.35) at $t = 0$, let $\vec{u}$ be the corresponding solution of (2.2) and define

$$T_*(\vec{u}_0) = \sup\{t \in [0, \infty) : \vec{u} \text{ satisfies (2.12) and (2.35) holds on } [0, t]\}, \tag{2.36}$$

Note that, from (2.35), for all $t \in [0, T_*(\vec{u}_0)]$,

$$\theta(t) = \sum_{n=1}^{N-1} e^{-4\gamma_0 (y_{n+1}(t) - y_n(t))} \lesssim e^{-\left(\frac{18}{5} L + 8\gamma_1 t\right)} \lesssim e^{-3\gamma_0 L}. \tag{2.37}$$

Note also that, for C_0 large enough, from (2.31) and (2.35), for all $n = 1, \dots, N$ and $t \in [0, T_*(\vec{u}_0)]$,

$$|\dot{y}_n - \ell_n| \leqslant C_0 \Big(\|\vec{\varphi}\|_{\mathcal{H}} + \theta\Big) \leqslant 2C_0^2 \left(\delta + e^{-\gamma_0 L}\right) \leqslant \gamma_1,$$

for L large enough and δ small enough (depending on C_0). Integrating on $[0, t]$ for any $t \in [0, T_*(\vec{u}_0)]$, we obtain, for all $n = 1, \dots, N$,

$$|y_n(t) - \ell_n t - y_n(0)| \leqslant \gamma_1 t. \tag{2.38}$$

2.3.2 Monotonicity property

First, we fix suitable cut-off functions. Let χ be a C^3 function such that

$$\chi' \geqslant 0, \quad \sqrt{\chi} \in C^1, \quad \sqrt{1-\chi} \in C^1, \quad \chi(x) = \begin{cases} 0 & \text{for } x \leqslant -1, \\ 1 & \text{for } x > 1. \end{cases}$$

Set

$$\beta_n = \frac{\ell_{n-1} + \ell_n}{2}, \quad \bar{y}_n^0 = \frac{y_{n-1}(0) + y_n(0)}{2}, \quad \chi_n(t,x) = \chi\left(\frac{x - \beta_n t - \bar{y}_n^0}{(t+a)^\alpha}\right),$$

for $n = 2, \dots, N$, where α and a are chosen so that

$$\frac{1}{2} < \alpha < \frac{4}{7} \quad \text{and} \quad a = \left(\frac{L}{10}\right)^{\frac{1}{\alpha}}.$$

We also set

$$\beta_1 = 0, \quad \chi_1 = 1, \quad \chi_{N+1} = 0.$$

Let

$$\psi_n(t,x) = \chi_n(t,x) - \chi_{n+1}(t,x) \quad \text{for } n = 1, \dots, N.$$

Note that $\psi_n \equiv 1$ around the solitary wave Q_n, and $\psi_n \equiv 0$ around the solitary waves $Q_{n'}$ for $n' \neq n$. Moreover,

$$\sum_{n=1}^{N} \psi_n = 1 \quad \text{and} \quad \chi_n = \sum_{n'=n}^{N} \psi_{n'} \quad \text{for} \quad n = 1, \dots, N. \tag{2.39}$$

Set

$$\Omega_n(t) = \left\{x \in \mathbb{R} : |x - \beta_n t - \bar{y}_n^0| \leqslant (t+a)^\alpha\right\} \quad \text{for } t \in [0, T_*(\vec{u}_0)].$$

From the definition of χ_n and Ω_n, we have the following estimates ($\mathbf{1}_I$ denotes the characteristic function of the interval I)

$$|\partial_x \chi_n| \lesssim \frac{1}{(t+a)^\alpha} \mathbf{1}_{\Omega_n}, \tag{2.40}$$

$$|\partial_x^2 \chi_n| + |\partial_{tx} \chi_n| \lesssim \frac{1}{(t+a)^{2\alpha}} \mathbf{1}_{\Omega_n}, \quad |\partial_x^3 \chi_n| \lesssim \frac{1}{(t+a)^{3\alpha}} \mathbf{1}_{\Omega_n}. \tag{2.41}$$

We give technical estimates related to the cut-off functions.

Lemma 2.14. *For all $n = 1, \dots, N$ and all $t \in [0, T_*(\vec{u}_0)]$, the following holds.*

(i) Estimates related to χ_n. *We have*

$$\sum_{n'=1}^{n-1} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2\right) \chi_n \mathrm{d}x \lesssim e^{-4\gamma_0 L}, \tag{2.42}$$

$$\sum_{n'=1}^{n-1} \left(\|\chi_n Q_{n'}\|_{L^2} + \|\chi_n \partial_x Q_{n'}\|_{L^2}\right) \lesssim e^{-2\gamma_0 L}. \tag{2.43}$$

(ii) Estimates related to $(1 - \chi_n)$. *We have*

$$\sum_{n'=n}^{N} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2\right) (1 - \chi_n) \mathrm{d}x \lesssim e^{-4\gamma_0 L}, \tag{2.44}$$

$$\sum_{n'=n}^{N} \left(\|(\chi_n - 1) Q_{n'}\|_{L^2} + \|(\chi_n - 1) \partial_x Q_{n'}\|_{L^2}\right) \lesssim e^{-2\gamma_0 L}. \tag{2.45}$$

(iii) Estimates related to Ω_n. *We have*

$$\sum_{n'=1}^{N} (\|\mathbf{1}_{\Omega_n} Q_{n'}\|_{L^2} + \|\mathbf{1}_{\Omega_n} \partial_x Q_{n'}\|_{L^2}) \lesssim e^{-2(\gamma_0 L + \gamma_1 t)}. \tag{2.46}$$

(iv) Estimates related to $\psi_{n'}$. *We have*

$$\sum_{n' \neq n} \|f'(Q_n)\psi_{n'}\|_{L^2} \lesssim e^{-2\gamma_0 L}, \tag{2.47}$$

$$\sum_{n' \neq n} \|\Upsilon_{n,1}\psi_{n'}\|_{L^2} + \sum_{n' \neq n} \|\Upsilon_{n,2}\psi_{n'}\|_{L^2} \lesssim e^{-2\gamma_0 L}, \tag{2.48}$$

$$\sum_{n' \neq n} \|\partial_x Q_n \psi_{n'}\|_{L^2} + \sum_{n' \neq n} \|\partial_x^2 Q_n \psi_{n'}\|_{L^2} \lesssim e^{-2\gamma_0 L}. \tag{2.49}$$

Proof. Proof of (i). We prove (2.42); (2.43) is proved similarly. By the definition of χ_n, for any $x \in \operatorname{supp} \chi_n$,

$$x - \beta_n t - \bar{y}_n^0 \geq -(t+a)^\alpha,$$

and so using (2.38), for any $1 \leqslant n' \leqslant n-1$,

$$x - y_{n'}(t) \geqslant (\beta_n - \ell_{n'})t - \gamma_1 t + (\bar{y}_n^0 - y_{n'}(0)) - (t+a)^\alpha.$$

By the definition of γ_1 and (2.35) at $t=0$, we have $\beta_n - \ell_{n'} \geq \frac{1}{2}(\ell_n - \ell_{n-1}) \geqslant 4\gamma_1$,

$$\bar{y}_n^0 - y_{n'}(0) = \frac{1}{2}\left(y_n(0) + y_{n-1}(0)\right) - y_{n'}(0) \geq \frac{1}{2}(y_n(0) - y_{n-1}(0)) \geq \frac{9L}{20},$$

and by the definition of $a^\alpha = \frac{L}{10}$, $0 < \alpha < 1$,

$$(t+a)^\alpha \leq t^\alpha + a^\alpha \leq t^\alpha + \frac{L}{10} \leq \gamma_1 t + \frac{L}{5},$$

for L large enough. Therefore, from $\gamma_0 \leqslant \frac{1}{8}$, for any $x \in \operatorname{supp} \chi_n$,

$$x - y_{n'}(t) \geqslant 4\gamma_1 t - \gamma_1 t + \frac{9L}{20} - \gamma_1 t - \frac{L}{5} \geqslant 2\gamma_1 t + \frac{L}{4} \geqslant 2(\gamma_0 L + \gamma_1 t),$$

for L large enough. Based on the decay properties of Q in (2.6) and above estimate,

$$\begin{aligned}\sum_{n'=1}^{n-1} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2\right) \chi_n \mathrm{d}x &\lesssim \sum_{n'=1}^{n-1} \int_{x - y_{n'}(t) \geqslant 2(\gamma_0 L + \gamma_1 t)} e^{-\frac{2|x - y_{n'}(t)|}{\sqrt{1-\ell_{n'}^2}}} \mathrm{d}x \\ &\lesssim \int_{x \geqslant 2(\gamma_0 L + \gamma_1 t)} e^{-2x} \mathrm{d}x \lesssim e^{-4(\gamma_0 L + \gamma_1 t)},\end{aligned}$$

which implies (2.42).

The proof of (ii) is the same.

Proof of (iii). Let any $1 \leqslant n \leqslant N$ and $1 \leqslant n' \leqslant N$. For any $x \in \Omega_n$, by the triangle inequality, $\gamma_1 = \frac{1}{8} \min_{n' \neq n} |\ell_n - \ell_{n'}|$, (2.35) at $t=0$ and (2.38),

$$\begin{aligned}|x - y_{n'}(t)| &\geqslant |(\ell_{n'} - \beta_n)t + y_{n'}(0) - \bar{y}_n^0| - |y_{n'}(t) - \ell_{n'} t - y_{n'}(0)| - |x - \beta_n t - \bar{y}_n^0| \\ &\geqslant |\ell_{n'} - \beta_n| t + \left|y_{n'}(0) - \bar{y}_n^0\right| - \gamma_1 t - (t+a)^\alpha \geqslant 3\gamma_1 t + \frac{9L}{20} - (t+a)^\alpha.\end{aligned}$$

Thus, using similar arguments as in the proof of (2.42), we obtain (2.46).

Proof of (iv). We prove (2.47); from (2.6) and (2.8), we see that (2.48) and (2.49) are proved similarly. Let $1 \leqslant n \leqslant N$ and $1 \leqslant n' \leqslant n-1$. From the definition of $\psi_{n'}$, we have, for any $x \in \operatorname{supp} \psi_{n'}$,

$$x - \beta_{n'+1} t - \bar{y}^0_{n'+1} \leqslant (t+a)^\alpha.$$

Thus, by $\gamma_1 = \frac{1}{8} \min_{n' \neq n} |\ell_n - \ell_{n'}|$, (2.35) at $t=0$, (2.38) and $a^\alpha = \frac{L}{10}$, for any $x \in \operatorname{supp} \psi_{n'}$

$$\begin{aligned} x - y_n(t) &\leqslant (\beta_{n'+1} - \ell_n)\, t + (\bar{y}_{n'+1}(0) - y_n(0)) + \gamma_1 t + (t+a)^\alpha \\ &\leqslant -\frac{1}{2}(\ell_n - \ell_{n-1})t - \frac{1}{2}(y_n(0) - y_{n-1}(0)) + \gamma_1 t + t^\alpha + a^\alpha \\ &\leqslant -3\gamma_1 t - \frac{9L}{20} + t^\alpha + \frac{L}{10} \leqslant -3\gamma_1 t - \frac{9L}{20} + \gamma_1 t + \frac{L}{5} \leqslant -\frac{L}{4}, \end{aligned}$$

for L large enough. Let $1 \leqslant n \leqslant N$ and $n' \geqslant n+1$. From the definition of $\psi_{n'}$, we have, for any $x \in \operatorname{supp} \psi_{n'}$,

$$x - \beta_{n'} t - \bar{y}^0_{n'} \geqslant -(t+a)^\alpha.$$

Using again $\gamma_1 \leqslant \frac{1}{8} \min_{n' \neq n} |\ell_n - \ell_{n'}|$, (2.35) at $t=0$, (2.38) and $a^\alpha = \frac{L}{10}$, for any $x \in \operatorname{supp} \psi_{n'}$,

$$\begin{aligned} x - y_n(t) &\geqslant (\beta_{n'} - \ell_n)t + (\bar{y}^0_{n'} - y_n(0)) - \gamma_1 t - (t+a)^\alpha \\ &\geqslant \frac{1}{2}(\ell_{n+1} - \ell_n)t + \frac{1}{2}(y_{n+1}(0) - y_n(0)) - \gamma_1 t - t^\alpha - a^\alpha \\ &\geqslant 3\gamma_1 t + \frac{9L}{20} - t^\alpha - \frac{L}{10} \geqslant 3\gamma_1 t + \frac{9L}{20} - \gamma_1 t - \frac{L}{5} \geqslant \frac{L}{4}. \end{aligned}$$

Thus, using similar arguments as in the proof of (2.42) and $\frac{p-1}{4} \leqslant 2\gamma_0$, we obtain (2.47). □

Second, let

$$c_1 = \ell_1, \quad c_2 = \frac{\ell_2 - \ell_1}{1 - \beta_2 \ell_2}, \quad c_n = \left(\frac{\ell_n - \ell_{n-1}}{1 - \beta_n \ell_n} \right) \prod_{n'=2}^{n-1} \left(\frac{1 - \beta_{n'} \ell_{n'-1}}{1 - \beta_{n'} \ell_{n'}} \right), \tag{2.50}$$

for $n = 3, \dots, N$. Denote

$$\tilde{c}_1 = 1 \quad \text{and} \quad \tilde{c}_n = 1 + \sum_{n'=2}^{n} c_{n'} \beta_{n'} \quad \text{for } n = 2, \dots, N. \tag{2.51}$$

Such specific choices are explained in Remark 2.16. By direct computation, one proves the following lemma.

Lemma 2.15. *For $n = 2, \dots, N$,*

$$\sum_{n'=1}^{n} c_{n'} = \tilde{c}_n \ell_n \quad \textit{and} \quad \tilde{c}_n = \prod_{n'=2}^{n} \left(\frac{1 - \beta_{n'} \ell_{n'-1}}{1 - \beta_{n'} \ell_{n'}} \right). \tag{2.52}$$

Proof. We prove (2.52) by induction.

Step 1. For $n = 2$. By direct computation,

$$\tilde{c}_2 = 1 + c_2 \beta_2 = 1 + \frac{\beta_2 \ell_2 - \beta_2 \ell_1}{1 - \beta_2 \ell_2} = \frac{1 - \beta_2 \ell_1}{1 - \beta_2 \ell_2},$$

$$c_1 + c_2 = \ell_1 + \frac{\ell_2 - \ell_1}{1 - \beta_2 \ell_2} = \frac{\ell_2 - \beta_2 \ell_1 \ell_2}{1 - \beta_2 \ell_2} = \tilde{c}_2 \ell_2,$$

which implies (2.52) for $n = 2$.

Step 2. Assuming that (2.52) is true for $n = k$ with $2 \leq k \leq N-1$, we prove that it is also true for $n = k+1$. From the definition of c_n for $n \geqslant 3$, we obtain

$$\begin{aligned}\tilde{c}_{k+1} &= \prod_{n'=2}^{k}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right) + \left(\frac{\beta_{k+1}(\ell_{k+1}-\ell_k)}{1-\beta_{k+1}\ell_{k+1}}\right)\prod_{n'=2}^{k}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right)\\ &= \left(1+\frac{\beta_{k+1}(\ell_{k+1}-\ell_k)}{1-\beta_{k+1}\ell_{k+1}}\right)\prod_{n'=2}^{k}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right) = \prod_{n'=2}^{k+1}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{n'=1}^{k+1} c_{n'} &= \tilde{c}_k\ell_k + c_{k+1}\\ &= \ell_k\prod_{n'=2}^{k}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right) + \left(\frac{\ell_{k+1}-\ell_k}{1-\beta_{k+1}\ell_{k+1}}\right)\prod_{n'=2}^{k}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right)\\ &= \ell_{k+1}\prod_{n'=2}^{k+1}\left(\frac{1-\beta_{n'}\ell_{n'-1}}{1-\beta_{n'}\ell_{n'}}\right) = \tilde{c}_{k+1}\ell_{k+1},\end{aligned}$$

which means that (2.52) is true for $n = k+1$. By the induction argument, (2.52) is proved for any $n = 2, \dots, N$. □

Third, we introduce the following quantities, for $n = 2, \dots, N$,

$$\mathcal{J}_n(\vec{u}) = \mathcal{I}_n(\vec{u}) + \beta_n E_n(\vec{u}) + \beta_n F_n(\vec{u}) \tag{2.53}$$

where

$$\mathcal{I}_n(\vec{u}) = 2\int_{\mathbb{R}}\left(\chi_n\partial_x u_1 + \frac{1-\beta_n^2}{2}(\partial_x\chi_n)u_1\right)u_2\mathrm{d}x, \tag{2.54}$$

$$E_n(\vec{u}) = \int_{\mathbb{R}}\left((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)\right)\chi_n\mathrm{d}x, \tag{2.55}$$

$$F_n(\vec{u}) = -\frac{\alpha}{t+a}\int_{\mathbb{R}}\left(x-\beta_n t-\bar{y}_n^0\right)u_1u_2(\partial_x\chi_n)\mathrm{d}x. \tag{2.56}$$

Last, we set

$$\mathcal{E}(\vec{u}) = E(\vec{u}) + c_1\mathcal{I}(\vec{u}) + \sum_{n=2}^{N} c_n\mathcal{J}_n(\vec{u}) \tag{2.57}$$

where, as defined in the Introduction,

$$E(\vec{u}) = \int_{\mathbb{R}}\left((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)\right)\mathrm{d}x, \quad \mathcal{I}(\vec{u}) = 2\int_{\mathbb{R}}(\partial_x u_1)u_2\mathrm{d}x.$$

Remark 2.16. (i) The quantity $\mathcal{I}_n(\vec{u})$ is a standard localized version of the momentum for a wave-type equation, while $E_n(\vec{u})$ is a standard localized version of the energy.

(ii) The expression $E(\vec{u}) + c_1\mathcal{I}(\vec{u}) + \sum_{n=2}^{N} c_n\left(\mathcal{I}_n(\vec{u}) + \beta_n E_n(\vec{u})\right)$ should be the correct functional for stability of the sum of N solitons. Indeed, due to the specific choices of c_n in (2.50), locally around each solitary wave Q_n this functional is analogous to the quantity $\tilde{c}_n\left(E(\vec{u}) + 2\ell_n\mathcal{I}(\vec{u})\right)$ (see Lemma 2.17). The functional $E(\vec{u}) + 2\ell\mathcal{I}(\vec{u})$ is a usual quantity to prove stability of a soliton Q_ℓ with speed $\ell \in (-1, 1)$.

(iii) For technical reasons, we need to add the refined term $F_n(\vec{u})$ to the definition of $\mathcal{J}_n$.

By expanding $\vec{u}(t) = \sum_{n=1}^{N} \vec{Q}_n(t) + \vec{\varphi}(t)$, we obtain the following formula.

Lemma 2.17. *For all $t \in [0, T_*(\vec{u}_0)]$, we have*

$$\begin{aligned}\mathcal{E}(\vec{u}) = &\sum_{n=1}^{N} \tilde{c}_n (1-\ell_n^2)^{\frac{1}{2}} E(\vec{Q}) + \sum_{n=1}^{N} \tilde{c}_n H_n(\vec{\varphi}, \vec{\varphi}) \\ &+ O\left(\frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L} + \|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{q+1} + e^{-3\gamma_0 L}\right),\end{aligned} \tag{2.58}$$

where

$$H_n(\vec{\varphi}, \vec{\varphi}) = \int_{\mathbb{R}} \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 + 2\ell_n (\partial_x \varphi_1)\varphi_2 - f'(Q_n)\varphi_1^2\right)\psi_n \mathrm{d}x,$$

for $n = 1, \dots, N$.

Remark 2.18. Error terms are estimated by $e^{-3\gamma_0 L}$ in this section. Observe that some loss is needed with respect to Section 2: Lemmas 2.10, 2.12 and 2.13.

Proof. **Step 1.** Expansion of $E(\vec{u})$. We prove the following estimate

$$\begin{aligned}E(\vec{u}) = &\sum_{n=1}^{N} (1-\ell_n^2)^{\frac{1}{2}} E(\vec{Q}) \\ &+ 2\sum_{n=1}^{N} \int_{\mathbb{R}} (\ell_n \partial_x Q_n)\left(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2\right) \mathrm{d}x \\ &+ \int_{\mathbb{R}} \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n=1}^{N} f'(Q_n)\varphi_1^2\right) \mathrm{d}x \\ &+ O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{q+1} + e^{-3\gamma_0 L}\right).\end{aligned} \tag{2.59}$$

Recall that $\vec{U} = \sum_{n=1}^{N} \vec{Q}_n$. First, using the decomposition (2.13), the definition of G, the equation $-(1-\ell_n^2)\partial_x^2 Q_n + Q_n - f(Q_n) = 0$ and integration by parts, we find

$$\begin{aligned}E(\vec{u}) = E(\vec{U}) &+ 2\sum_{n=1}^{N} \int_{\mathbb{R}} (\ell_n \partial_x Q_n)\big(\ell_n \partial_x \varphi_1 - \varphi_2\big) \mathrm{d}x \\ &+ \int_{\mathbb{R}} \big((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n=1}^{N} f'(Q_n)\varphi_1^2\big) \mathrm{d}x \\ &+ \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3,\end{aligned}$$

where

$$\begin{aligned}\tilde{E}_1 &= -2\int_{\mathbb{R}} \big(F(U_1 + \varphi_1) - F(U_1) - f(U_1)\varphi_1 - \frac{1}{2} f'(U_1)\varphi_1^2\big) \mathrm{d}x, \\ \tilde{E}_2 &= -2\int_{\mathbb{R}} G\varphi_1 \mathrm{d}x, \\ \tilde{E}_3 &= -\int_{\mathbb{R}} \big(f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\big)\varphi_1^2 \mathrm{d}x.\end{aligned}$$

By (2.25), $1 < q < p < \infty$ and the Sobolev embedding Theorem,

$$\left|\tilde{E}_1\right| \lesssim \int_{\mathbb{R}} \left(|\varphi_1|^{p+1} + |\varphi_1|^{q+1} + |\varphi_1|^3\right) \mathrm{d}x \lesssim \|\varphi_1\|_{H^1}^3 + \|\varphi_1\|_{H^1}^{q+1}.$$

By (2.30), (2.37), the Cauchy-Schwarz inequality and the Young's inequality, we have

$$\left|\tilde{E}_2\right| \lesssim \|G\|_{L^2}\|\varphi_1\|_{L^2} \lesssim \|\varphi_1\|_{L^2}^3 + \theta^{\frac{3}{2}} \lesssim \|\varphi_1\|_{L^2}^3 + e^{-\frac{9}{2}\gamma_0 L},$$

and

$$\left|\tilde{E}_3\right| \lesssim \left\|f'(U_1) - \sum_{n=1}^N f'(Q_n)\right\|_{L^2}\|\varphi_1\|_{H^1}^2 \lesssim \|\varphi_1\|_{H^1}^3 + \theta^3 \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-9\gamma_0 L}.$$

Second, by direct computation,

$$\begin{aligned} E(\vec{U}) = &\sum_{n=1}^N (1-\ell_n^2)^{\frac{1}{2}} E(\vec{Q}) + 2\sum_{n=1}^N \int_{\mathbb{R}} \left(\ell_n \partial_x Q_n\right)^2 \mathrm{d}x \\ &+ \sum_{n\neq n'} \int_{\mathbb{R}} \left[(1+\ell_n\ell_{n'})(\partial_x Q_n)(\partial_x Q_{n'}) + Q_n Q_{n'}\right]\mathrm{d}x \\ &- 2\int_{\mathbb{R}} \left(F(U_1) - \sum_{n=1}^N F(Q_n)\right)\mathrm{d}x. \end{aligned}$$

Moreover, using (2.27), (2.29) and (2.37),

$$\sum_{n\neq n'} \int_{\mathbb{R}} \left(|\partial_x Q_n \partial_x Q_{n'}| + |Q_n Q_{n'}|\right)\mathrm{d}x + \int_{\mathbb{R}} \left|F(U_1) - \sum_{n=1}^N F(Q_n)\right|\mathrm{d}x \lesssim \theta \lesssim e^{-3\gamma_0 L}.$$

We see that (2.59) follows from above estimates.

Step 2. Expansion of $\mathcal{I}(\vec{u})$. We claim

$$\begin{aligned} \mathcal{I}(\vec{u}) = &-2\sum_{n=1}^N \int_{\mathbb{R}} (\partial_x Q_n)\left(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2\right)\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} (\partial_x \varphi_1)\varphi_2 \mathrm{d}x + O\left(e^{-3\gamma_0 L}\right). \end{aligned} \tag{2.60}$$

By direct computation and (2.13),

$$\begin{aligned} \mathcal{I}(\vec{u}) = &-2\sum_{n=1}^N \int_{\mathbb{R}} (\partial_x Q_n)\left(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2\right)\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} (\partial_x \varphi_1)\varphi_2 \mathrm{d}x - 2\sum_{n\neq n'} \ell_{n'} \int_{\mathbb{R}} (\partial_x Q_n)(\partial_x Q_{n'})\,\mathrm{d}x. \end{aligned}$$

Thus, from (2.27) and (2.37), we obtain (2.60).

Step 3. Expansion of $\mathcal{I}_n(\vec{u})$. We claim

$$\begin{aligned} \mathcal{I}_n(\vec{u}) = &-2\sum_{n'=n}^N \int_{\mathbb{R}} (\partial_x Q_{n'})\left(\ell_{n'} \partial_x Q_{n'} + \ell_{n'} \partial_x \varphi_1 - \varphi_2\right)\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} \chi_n (\partial_x \varphi_1)\varphi_2 \mathrm{d}x + O\left(\frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L} + \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}\right). \end{aligned} \tag{2.61}$$

We decompose

$$\mathcal{I}_n(\vec{u}) = \mathcal{I}_n^1(\vec{u}) + (1-\beta_n^2)\mathcal{I}_n^2(\vec{u}),$$

where

$$\mathcal{I}_n^1(\vec{u}) = 2\int_{\mathbb{R}} (\chi_n \partial_x u_1)\, u_2 \mathrm{d}x, \quad \mathcal{I}_n^2(\vec{u}) = \int_{\mathbb{R}} (\partial_x \chi_n)\, u_1 u_2 \mathrm{d}x.$$

Estimate on $\mathcal{I}_n^1$. We claim

$$\begin{aligned}\mathcal{I}_n^1(\vec{u}) = &-2\sum_{n'=n}^{N}\int_{\mathbb{R}} (\partial_x Q_{n'})\,(\ell_{n'}\partial_x Q_{n'} + \ell_{n'}\partial_x\varphi_1 - \varphi_2)\,\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} \chi_n(\partial_x\varphi_1)\varphi_2 \mathrm{d}x + O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}\right).\end{aligned} \tag{2.62}$$

By direct computation and (2.13),

$$\begin{aligned}\mathcal{I}_n^1(\vec{u}) = &-2\sum_{n'=n}^{N}\int_{\mathbb{R}} (\partial_x Q_{n'})\,(\ell_{n'}\partial_x\varphi_1 + \ell_{n'}\partial_x Q_{n'} - \varphi_2)\,\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} \chi_n(\partial_x\varphi_1)\varphi_2 \mathrm{d}x + \mathcal{I}_n^{1,1} + \mathcal{I}_n^{1,2} + \mathcal{I}_n^{1,3},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_n^{1,1} &= -2\sum_{n'\neq n''}\ell_{n'}\int_{\mathbb{R}} \chi_n\,(\partial_x Q_{n'})\,(\partial_x Q_{n''})\,\mathrm{d}x, \\ \mathcal{I}_n^{1,2} &= -2\sum_{n'=1}^{n-1}\int_{\mathbb{R}} (\chi_n\partial_x Q_{n'})\,(\ell_{n'}\partial_x\varphi_1 + \ell_{n'}\partial_x Q_{n'} - \varphi_2)\,\mathrm{d}x, \\ \mathcal{I}_n^{1,3} &= -2\sum_{n'=n}^{N}\int_{\mathbb{R}} (\chi_n - 1)\,(\partial_x Q_{n'})\,(\ell_{n'}\partial_x\varphi_1 + \ell_{n'}\partial_x Q_{n'} - \varphi_2)\,\mathrm{d}x.\end{aligned}$$

From (2.27) and (2.37),

$$\left|\mathcal{I}_n^{1,1}\right| \lesssim \sum_{n'\neq n''}\int_{\mathbb{R}} |\partial_x Q_{n'}||\partial_x Q_{n''}|\mathrm{d}x \lesssim \theta \lesssim e^{-3\gamma_0 L}.$$

By (2.42), (2.43), (2.44), (2.45), the Cauchy-Schwarz inequality and the Young's inequality, we have

$$\begin{aligned}\left|\mathcal{I}_n^{1,2}\right| &\lesssim \sum_{n'=1}^{n-1}\left(\|\vec{\varphi}\|_{\mathcal{H}}\|\chi_n\partial_x Q_{n'}\|_{L^2}\right) + \sum_{n'=1}^{n-1}\int_{\mathbb{R}}(\partial_x Q_{n'})^2\chi_n \mathrm{d}x \\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n'=1}^{n-1}\|\chi_n\partial_x Q_{n'}\|_{L^2}^{\frac{3}{2}} + \sum_{n'=1}^{n-1}\int_{\mathbb{R}}(\partial_x Q_{n'})^2\chi_n \mathrm{d}x \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L},\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{I}_n^{1,3}\right| &\lesssim \sum_{n'=n}^{N}\left(\|\vec{\varphi}\|_{\mathcal{H}}\|(\chi_n - 1)\partial_x Q_{n'}\|_{L^2}\right) + \sum_{n'=n}^{N}\int_{\mathbb{R}}(\partial_x Q_{n'})^2(1-\chi_n)\mathrm{d}x \\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n'=n}^{N}\left(\|(\chi_n - 1)\partial_x Q_{n'}\|_{L^2}^{\frac{3}{2}} + \int_{\mathbb{R}}(\partial_x Q_{n'})^2(1-\chi_n)\mathrm{d}x\right) \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}.\end{aligned}$$

We see that (2.62) follows from above estimates.

Estimate on $\mathcal{I}_n^2$. We decompose

$$\mathcal{I}_n^2(\vec{u}) = \mathcal{I}_n^{2,1} + \mathcal{I}_n^{2,2} + \mathcal{I}_n^{2,3},$$

where

$$\mathcal{I}_n^{2,1} = -\sum_{n',n''=1}^{N} \ell_{n'} \int_{\mathbb{R}} (\partial_x \chi_n)(\partial_x Q_{n'}) Q_{n''} \mathrm{d}x,$$
$$\mathcal{I}_n^{2,2} = \sum_{n'=1}^{N} \int_{\mathbb{R}} (\partial_x \chi_n) \left(-\ell_{n'}(\partial_x Q_{n'})\varphi_1 + Q_{n'}\varphi_2\right) \mathrm{d}x,$$
$$\mathcal{I}_n^{2,3} = \int_{\mathbb{R}} (\partial_x \chi_n)\varphi_1\varphi_2 \mathrm{d}x.$$

By (2.46), the Cauchy-Schwarz inequality and the Young's inequality,

$$\left|\mathcal{I}_n^{2,1}\right| \lesssim \sum_{n'=1}^{N} \int_{\mathbb{R}} \left(|\partial_x Q_{n'}|^2 + |Q_{n'}|^2\right) \mathbf{1}_{\Omega_n} \mathrm{d}x \lesssim e^{-4\gamma_0 L},$$

and

$$\begin{aligned}\left|\mathcal{I}_n^{2,2}\right| &\lesssim \sum_{n'=1}^{N} \|\vec{\varphi}\|_{\mathcal{H}} \left(\|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2} + \|Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}\right) \\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n'=1}^{N} \left(\|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^{\frac{3}{2}} + \|Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^{\frac{3}{2}}\right) \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}.\end{aligned}$$

Moreover, from (2.40) and $a^\alpha = \frac{L}{10}$,

$$\left|\mathcal{I}_n^{2,3}\right| \lesssim \frac{1}{(t+a)^\alpha} \int_{\mathbb{R}} |\varphi_1||\varphi_2| \mathbf{1}_{\Omega_n} \mathrm{d}x \lesssim \frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{(t+a)^\alpha} \lesssim \frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L}.$$

From above estimates, we conclude that

$$\left|\mathcal{I}_n^2\right| \lesssim \frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L} + \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}. \tag{2.63}$$

We see that (2.61) follows from (2.62) and (2.63).

Step 4. Expansion of $E_n(\vec{u})$. We claim

$$\begin{aligned}E_n(\vec{u}) = &\sum_{n'=n}^{N} (1-\ell_{n'}^2)^{\frac{1}{2}} E(\vec{Q}) \\ &+ 2\sum_{n'=n}^{N} \int_{\mathbb{R}} (\ell_{n'}\partial_x Q_{n'}) \left(\ell_{n'}\partial_x Q_{n'} + \ell_{n'}\partial_x\varphi_1 - \varphi_2\right) \mathrm{d}x \\ &+ \int_{\mathbb{R}} \left((\partial_x\varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n'=n}^{N} f'(Q_{n'})\varphi_1^2\right) \chi_n \mathrm{d}x \\ &+ O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{q+1} + e^{-3\gamma_0 L}\right).\end{aligned} \tag{2.64}$$

First, from (2.13), $-(1-\ell_{n'}^2)\partial_x^2 Q_{n'} + Q_{n'} - f(Q_{n'}) = 0$, integration by parts and an elementary computation,

$$\begin{aligned}E_n(\vec{u}) =& E_n(\vec{U}) + 2\sum_{n'=n}^{N} \int_{\mathbb{R}} (\ell_{n'}\partial_x Q_{n'}) \left(\ell_{n'}\partial_x\varphi_1 - \varphi_2\right) \mathrm{d}x \\ &+ \int_{\mathbb{R}} \left((\partial_x\varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n'=n}^{N} f'(Q_{n'})\varphi_1^2\right) \chi_n \mathrm{d}x \\ &+ \tilde{E}_n^1 + \tilde{E}_n^2 + \tilde{E}_n^3 + \tilde{E}_n^4 + \tilde{E}_n^5 + \tilde{E}_n^6,\end{aligned}$$

where

$$\tilde{E}_n^1 = -2\int_{\mathbb{R}} \Big(F(U_1+\varphi_1) - F(U_1) - f(U_1)\varphi_1 - \frac{1}{2}f'(U_1)\varphi_1^2\Big)\chi_n \mathrm{d}x,$$
$$\tilde{E}_n^2 = -2\int_{\mathbb{R}} G\varphi_1\chi_n \mathrm{d}x,$$
$$\tilde{E}_n^3 = -\int_{\mathbb{R}} \Big(f'(U_1) - \sum_{n'=n}^{N} f'(Q_{n'})\Big)\varphi_1^2\chi_n \mathrm{d}x,$$

and

$$\tilde{E}_n^4 = -2\sum_{n'=1}^{N}(1-\ell_{n'}^2)\int_{\mathbb{R}} \big(\partial_x Q_{n'}\big)\big(\partial_x\chi_n\big)\varphi_1 \mathrm{d}x,$$
$$\tilde{E}_n^5 = 2\sum_{n'=1}^{n-1}\int_{\mathbb{R}} \big(\ell_{n'}\partial_x Q_{n'}\big)\big(\ell_{n'}\partial_x\varphi_1 - \varphi_2\big)\chi_n \mathrm{d}x,$$
$$\tilde{E}_n^6 = 2\sum_{n'=n}^{N}\int_{\mathbb{R}} \big(\ell_{n'}\partial_x Q_{n'}\big)\big(\ell_{n'}\partial_x\varphi_1 - \varphi_2\big)(\chi_n - 1) \mathrm{d}x.$$

Using again (2.25), $1<q<p<\infty$ and the Sobolev embedding Theorem, we obtain

$$\big|\tilde{E}_n^1\big| \lesssim \int_{\mathbb{R}} \big(|\varphi_1|^3 + |\varphi_1|^{q+1} + |\varphi_1|^{p+1}\big)\,\mathrm{d}x \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{q+1}.$$

By (2.30), (2.37), the Cauchy-Schwarz inequality and the Young's inequality,

$$\Big|\tilde{E}_2\Big| \lesssim \|G\|_{L^2}\|\varphi_1\|_{L^2} \lesssim \|\varphi_1\|_{L^2}^3 + \theta^{\frac{3}{2}} \lesssim \|\varphi_1\|_{L^2}^3 + e^{-\frac{9}{2}\gamma_0 L},$$

Recall that, for any $x \in \mathrm{supp}\chi_n$ and $1 \leqslant n' \leqslant n-1$,

$$|x - y_{n'}(t)| \geqslant \frac{1}{4}(\ell_n - \ell_{n-1})t + \frac{L}{4} \geqslant \frac{L}{4},$$

for L large enough. Thus, from $1<q<p<\infty$ and $\gamma_0 \leqslant \frac{p-1}{8}$, we have

$$\sum_{n'=1}^{n-1}\|f'(Q_{n'})\chi_n\|_{L^\infty} \lesssim e^{-\frac{1}{4}(q-1)L} \lesssim e^{-2\gamma_0 L}.$$

Therefore, using again (2.30), (2.37), the Sobolev embedding Theorem and the Young's inequality,

$$\begin{aligned}
|\tilde{E}_n^3| &\lesssim \int_{\mathbb{R}} \Big(\big|f'(U_1) - \sum_{n'=1}^{N} f'(Q_{n'})\big| + \sum_{n'=1}^{n-1}|f'(Q_{n'})|\Big)\varphi_1^2|\chi_n| \mathrm{d}x \\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 \Big(\|f'(U_1) - \sum_{n'=1}^{N} f'(Q_{n'})\|_{L^2} + \sum_{n'=1}^{n-1}\|f'(Q_{n'})\chi_n\|_{L^\infty}\Big) \\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \theta^3 + \sum_{n'=1}^{n-1}\|f'(Q_{n'})\chi_n\|_{L^\infty}^3 \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-6\gamma_0 L}.
\end{aligned}$$

Next, by (2.43), (2.45), (2.46), the Cauchy-Schwarz inequality and the Young's inequality, we have

$$|\tilde{E}_n^4| \lesssim \sum_{n'=1}^{N} \|\vec{\varphi}\|_{\mathcal{H}} \|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n'=1}^{N} \|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^{\frac{3}{2}} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L},$$

$$|\tilde{E}_n^5| \lesssim \sum_{n'=1}^{n-1} \|\vec{\varphi}\|_{\mathcal{H}} \|\chi_n \partial_x Q_{n'}\|_{L^2} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n'=1}^{n-1} \|\chi_n \partial_x Q_{n'}\|_{L^2}^{\frac{3}{2}} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L},$$

$$|\tilde{E}_n^6| \lesssim \sum_{n'=n}^{N} \|\vec{\varphi}\|_{\mathcal{H}} \|(\chi_n - 1)\partial_x Q_{n'}\|_{L^2} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n'=n}^{N} \|(\chi_n - 1)\partial_x Q_{n'}\|_{L^2}^{\frac{3}{2}} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}.$$

Second, by direct computation,

$$\begin{aligned} E_n(\vec{U}) = &\sum_{n'=n}^{N} (1 - \ell_{n'}^2)^{\frac{1}{2}} E(\vec{Q}) + 2 \sum_{n'=n}^{N} \int_{\mathbb{R}} (\ell_{n'} \partial_x Q_{n'})^2 \, \mathrm{d}x \\ &+ \tilde{E}_n^7 + \tilde{E}_n^8 + \tilde{E}_n^9 + \tilde{E}_n^{10}, \end{aligned}$$

where

$$\tilde{E}_n^7 = -2 \int_{\mathbb{R}} \Big(F(U_1) - \sum_{n'=1}^{N} F(Q_{n'}) \Big) \chi_n \mathrm{d}x,$$

$$\tilde{E}_n^8 = \sum_{n'=1}^{n-1} \int_{\mathbb{R}} \left((1 + \ell_{n'}^2)(\partial_x Q_{n'})^2 + Q_{n'}^2 - 2F(Q_{n'}) \right) \chi_n \mathrm{d}x,$$

and

$$\tilde{E}_n^9 = \sum_{n'=n}^{N} \int_{\mathbb{R}} \left((1 + \ell_{n'}^2)(\partial_x Q_{n'})^2 + Q_{n'}^2 - 2F(Q_{n'}) \right) (\chi_n - 1) \, \mathrm{d}x,$$

$$\tilde{E}_n^{10} = \sum_{n' \neq n''} \int_{\mathbb{R}} \left((1 + \ell_{n'} \ell_{n''})(\partial_x Q_{n'})(\partial_x Q_{n''}) + Q_{n'} Q_{n''} \right) \chi_n \mathrm{d}x.$$

By (2.29) and (2.37),

$$|\tilde{E}_n^7| \lesssim \int_{\mathbb{R}} \Big| F(U_1) - \sum_{n'=1}^{N} F(Q_{n'}) \Big| \mathrm{d}x \lesssim \theta \lesssim e^{-3\gamma_0 L}.$$

From $1 < q < p < \infty$, (2.42) and (2.44),

$$\begin{aligned} |\tilde{E}_n^8| &\lesssim \sum_{n'=1}^{n-1} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2 + |Q_{n'}|^{p+1} + |Q_{n'}|^{q+1} \right) \chi_n \mathrm{d}x \\ &\lesssim \sum_{n'=1}^{n-1} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2 \right) \chi_n \mathrm{d}x \lesssim e^{-4\gamma_0 L}, \end{aligned}$$

and

$$\begin{aligned} |\tilde{E}_n^9| &\lesssim \sum_{n'=n}^{N} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2 + |Q_{n'}|^{p+1} + |Q_{n'}|^{q+1} \right) (1 - \chi_n) \mathrm{d}x \\ &\lesssim \sum_{n'=n}^{N} \int_{\mathbb{R}} \left((\partial_x Q_{n'})^2 + Q_{n'}^2 \right) (1 - \chi_n) \mathrm{d}x \lesssim e^{-4\gamma_0 L}. \end{aligned}$$

Last, using again (2.27) and (2.37), we have

$$|\tilde{E}_n^{10}| \lesssim \sum_{n' \neq n''} \int_{\mathbb{R}} (|(\partial_x Q_{n'})(\partial_x Q_{n''})| + |Q_{n'} Q_{n''}|)\, \mathrm{d}x \lesssim \theta \lesssim e^{-3\gamma_0 L}.$$

We see that (2.64) follows from above estimates.

Step 5. Estimate of $F_n(\vec{u})$. We claim

$$|F_n(\vec{u})| \lesssim \frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L} + \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}. \tag{2.65}$$

From (2.13), we decompose

$$F_n(\vec{u}) = F_n^1 + F_n^2 + F_n^3,$$

where

$$\begin{aligned}
F_n^1 &= -\frac{\alpha}{t+a} \int_{\mathbb{R}} \left(x - \beta_n t - \bar{y}_n^0\right) (\varphi_1 \varphi_2)(\partial_x \chi_n) \mathrm{d}x, \\
F_n^2 &= \frac{\alpha}{t+a} \sum_{n',n''=1}^{N} \ell_{n''} \int_{\mathbb{R}} \left(x - \beta_n t - \bar{y}_n^0\right) (Q_{n'} \partial_x Q_{n''}) (\partial_x \chi_n) \mathrm{d}x, \\
F_n^3 &= -\frac{\alpha}{t+a} \sum_{n'=1}^{N} \int_{\mathbb{R}} \left(x - \beta_n t - \bar{y}_n^0\right) (\varphi_2 Q_{n'} - \ell_{n'} \varphi_1 \partial_x Q_{n'}) (\partial_x \chi_n) \mathrm{d}x.
\end{aligned}$$

First, from (2.40) and $a = \left(\frac{L}{10}\right)^{\frac{1}{\alpha}} > \frac{L}{10}$,

$$\left|F_n^1\right| \lesssim \frac{1}{(t+a)} \int_{\mathbb{R}} |\varphi_1||\varphi_2| \mathrm{d}x \lesssim \frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L}.$$

Next, using again (2.40), (2.46), the AM-GM inequality and the Young's inequality,

$$\begin{aligned}
\left|F_n^2\right| &\lesssim \frac{1}{(t+a)} \sum_{n',n''=1}^{N} \int_{\mathbb{R}} |Q_{n'}|\, |\partial_x Q_{n''}|\, \mathbf{1}_{\Omega_n} \mathrm{d}x \\
&\lesssim \sum_{n'=1}^{N} \left(\|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^2 + \|Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^2\right) \lesssim e^{-4\gamma_0 L},
\end{aligned}$$

and

$$\begin{aligned}
\left|F_n^3\right| &\lesssim \frac{1}{(t+a)} \sum_{n'=1}^{N} \int_{\mathbb{R}} (|\varphi_2 Q_{n'}| + |\varphi_1 \partial_x Q_{n'}|)\, \mathbf{1}_{\Omega_n} \mathrm{d}x \\
&\lesssim \sum_{n'=1}^{N} \|\vec{\varphi}\|_{\mathcal{H}} \left(\|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2} + \|Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}\right) \\
&\lesssim \sum_{n'=1}^{N} \left(\|\partial_x Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^{\frac{3}{2}} + \|Q_{n'} \mathbf{1}_{\Omega_n}\|_{L^2}^{\frac{3}{2}} + \|\vec{\varphi}\|_{\mathcal{H}}^3\right) \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-3\gamma_0 L}.
\end{aligned}$$

Gathering above estimates, we obtain (2.65).

Step 6. Conclusion. First, we claim

$$E(\vec{u}) + \sum_{n=2}^{N} c_n \beta_n E_n(\vec{u}) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{q+1} + e^{-3\gamma_0 L}\right). \tag{2.66}$$

where

$$\mathcal{E}_1 = \sum_{n=1}^{N} \tilde{c}_n (1-\ell_n^2)^{\frac{1}{2}} E(\vec{Q}),$$
$$\mathcal{E}_2 = 2\sum_{n=1}^{N} \tilde{c}_n \ell_n \int_{\mathbb{R}} (\partial_x Q_n)\left(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2\right) \mathrm{d}x,$$
$$\mathcal{E}_3 = \sum_{n=1}^{N} \tilde{c}_n \int_{\mathbb{R}} \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - f'(Q_n)\varphi_1^2\right)\psi_n \mathrm{d}x.$$

Indeed, from (2.51), (2.59) and (2.64) and direct computation,

$$\begin{aligned}
&E(\vec{u}) + \sum_{n=2}^{N} c_n \beta_n E_n(\vec{u}) \\
&= \mathcal{E}_1 + \mathcal{E}_2 + \int_{\mathbb{R}} \left(1 + \sum_{n=1}^{N} c_n \beta_n \chi_n\right)\left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2\right) \mathrm{d}x \\
&\quad - \sum_{n=1}^{N} \left[\int_{\mathbb{R}} \left(1 + \sum_{n'=1}^{n} c_{n'} \beta_{n'} \chi_{n'}\right) f'(Q_n)\varphi_1^2 \mathrm{d}x\right] + O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{q+1} + e^{-3\gamma_0 L}\right).
\end{aligned}$$

Observe that, from (2.39) and the definition of $\tilde{c}_n$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(1 + \sum_{n=1}^{N} c_n \beta_n \chi_n\right)\left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2\right) \mathrm{d}x \\
&\qquad = \sum_{n=1}^{N} \tilde{c}_n \int_{\mathbb{R}} \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2\right)\psi_n \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{N} \left[\int_{\mathbb{R}} \left(1 + \sum_{n'=2}^{n} c_{n'} \beta_{n'} \chi_{n'}\right) f'(Q_n)\varphi_1^2 \mathrm{d}x\right] \\
&\quad = \sum_{n=1}^{N} \tilde{c}_n \int_{\mathbb{R}} f'(Q_n)\varphi_1^2 \psi_n \mathrm{d}x + \sum_{n=1}^{N} \sum_{n' \neq n} \left[\left(1 + \sum_{k=1}^{(n')^+} (c_k \beta_k)\right) \int_{\mathbb{R}} f'(Q_n)\varphi_1^2 \psi_{n'} \mathrm{d}x\right],
\end{aligned}$$

where $(n')^+ = \min(n', n)$. Note that, by (2.47), the Cauchy-Schwarz inequality, the Young's inequality and the Sobolev embedding Theorem,

$$\begin{aligned}
\sum_{n' \neq n} \left| \int_{\mathbb{R}} f'(Q_n)\varphi_1^2 \psi_{n'} \mathrm{d}x \right| &\lesssim \sum_{n' \neq n} \|\vec{\varphi}\|_{\mathcal{H}}^2 \|f'(Q_n)\psi_{n'}\|_{L^2} \\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \sum_{n' \neq n} \|f'(Q_n)\psi_{n'}\|_{L^2}^3 \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-6\gamma_0 L}.
\end{aligned}$$

We see that (2.66) follows from combining these identities and estimates.

Second, we claim

$$\begin{aligned} c_1\mathcal{I}+\sum_{n=2}^{N}c_n\mathcal{I}_n &= -2\sum_{n=1}^{N}\left[\left(\sum_{n'=1}^{n}c_{n'}\right)\int_{\mathbb{R}}(\partial_x Q_n)\left(\ell_n\partial_x Q_n+\ell_n\partial_x\varphi_1-\varphi_2\right)\mathrm{d}x\right] \\ &\quad +2\sum_{n=1}^{N}\left[\left(\sum_{n'=1}^{n}c_{n'}\right)\int_{\mathbb{R}}(\partial_x\varphi_1)\varphi_2\psi_n\mathrm{d}x\right] \\ &\quad +O\left(\frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L}+\|\vec{\varphi}\|_{\mathcal{H}}^3+e^{-3\gamma_0 L}\right). \end{aligned} \tag{2.67}$$

Indeed, from (2.60) and (2.61),

$$\begin{aligned} &c_1\mathcal{I}+\sum_{n=2}^{N}c_n\mathcal{I}_n \\ &= -2\sum_{n=1}^{N}\int_{\mathbb{R}}\left(\sum_{n'=1}^{n}c_{n'}\right)(\partial_x Q_n)(\ell_n\partial_x Q_n+\ell_n\partial_x\varphi_1-\varphi_2)\mathrm{d}x \\ &\quad +2\int_{\mathbb{R}}\left(c_1+\sum_{n=2}^{N}c_n\chi_n\right)(\partial_x\varphi_1)\varphi_2\mathrm{d}x+O\left(\frac{\|\vec{\varphi}\|_{\mathcal{H}}^2}{L}+\|\vec{\varphi}\|_{\mathcal{H}}^3+e^{-3\gamma_0 L}\right). \end{aligned}$$

Using again (2.39), we have

$$2\int_{\mathbb{R}}\left(c_1+\sum_{n=2}^{N}c_n\chi_n\right)(\partial_x\varphi_1)\varphi_2\mathrm{d}x=2\sum_{n=1}^{N}\left[\left(\sum_{n'=1}^{n}c_{n'}\right)\int_{\mathbb{R}}(\partial_x\varphi_1)\varphi_2\psi_n\mathrm{d}x\right],$$

which implies (2.67).

Combining (2.52), (2.53), (2.57), (2.65), (2.66) and (2.67), we obtain (2.58). □

Next, using the standard localized argument, we obtain the following coercivity result.

Lemma 2.19 (Coercivity). *There exist $0<\mu\ll 1$ such that for all $t\in[0,T_*(\vec{u}_0)]$,*

$$\mu\|\vec{\varphi}\|_{\mathcal{H}}^2\leqslant\sum_{n=1}^{N}\tilde{c}_n H_n(\vec{\varphi},\vec{\varphi})+\mu^{-1}\left[\sum_{n=1}^{N}(a_n^-)^2+\sum_{n=1}^{N}(a_n^+)^2\right]+e^{-3\gamma_0 L}. \tag{2.68}$$

Proof. First, we prove the localized coercivity of $H_n(\vec{\varphi},\vec{\varphi})$ from the coercivity property (2.10) around one solitary wave with the orthogonality properties (2.14). By direct computation, the defintion of ψ_n and $\sqrt{\chi},\sqrt{1-\chi}\in C^1$,

$$\begin{aligned} &\int_{\mathbb{R}}\left((\partial_x\varphi_1)^2+2\ell_n(\partial_x\varphi_1)\varphi_2\right)\psi_n\mathrm{d}x \\ &=\int_{\mathbb{R}}\left(\left(\partial_x\left(\varphi_1\sqrt{\psi_n}\right)\right)^2+2\ell_n\left(\partial_x\left(\varphi_1\sqrt{\psi_n}\right)\right)\varphi_2\sqrt{\psi_n}\right)\mathrm{d}x+O\left(\frac{1}{L}\|\vec{\varphi}\|_{\mathcal{H}}^2\right). \end{aligned} \tag{2.69}$$

We define

$$\vec{\varphi}_n=(\varphi_{1,n},\varphi_{2,n})=\left(\varphi_1(\cdot+y_n)\sqrt{\psi_n(\cdot+y_n)},\varphi_2(\cdot+y_n)\sqrt{\psi_n(\cdot+y_n)}\right),$$

for all $n=1,\dots,N$. From (2.10), (2.69) and the translation $x\to x+y_n$, there exists $\nu_n>0$ such that

$$\begin{aligned} \mathcal{H}_n(\vec{\varphi},\vec{\varphi}) &= (\mathcal{H}_{\ell_n}\vec{\varphi}_n,\vec{\varphi}_n)_{L^2}+O\left(\frac{1}{L}\|\vec{\varphi}\|_{\mathcal{H}}^2\right) \\ &\geqslant \nu_n\|\vec{\varphi}_n\|_{\mathcal{H}}^2-\nu_n^{-1}\left(\left(\vec{\varphi}_n,\vec{Z}_{\ell_n}^0\right)_{L^2}^2+\sum_{\pm}\left(\vec{\varphi}_n,\vec{Z}_{\ell_n}^{\pm}\right)_{L^2}^2\right)+O\left(\frac{1}{L}\|\vec{\varphi}\|_{\mathcal{H}}^2\right). \end{aligned}$$

By the definition of ψ_n and the translation $x \to x - y_n$,

$$\|\vec{\varphi}_n\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2\right) \psi_n \mathrm{d}x + O\left(\frac{1}{L}\|\vec{\varphi}\|_{\mathcal{H}}^2\right).$$

Note that, from $0 \leqslant \psi_n \leqslant 1$, we have $1 - \sqrt{\psi_n} \leqslant 1 - \psi_n$. Thus, using (2.48), (2.49), and $\sum_{n=1}^N \psi_n = 1$, we have

$$\begin{aligned}
&\|(1-\sqrt{\psi_n})\partial_x Q_n\|_{L^2} + \|(1-\sqrt{\psi_n})\partial_x^2 Q_n\|_{L^2} \\
\lesssim &\|(1-\psi_n)\partial_x Q_n\|_{L^2} + \|(1-\psi_n)\partial_x^2 Q_n\|_{L^2} \\
\lesssim &\sum_{n' \neq n} \|\partial_x Q_n \psi_{n'}\|_{L^2} + \sum_{n' \neq n} \|\partial_x^2 Q_n \psi_{n'}\|_{L^2} \lesssim e^{-2\gamma_0 L},
\end{aligned}$$

and

$$\begin{aligned}
&\|(1-\sqrt{\psi_n})\Upsilon_{n,1}\|_{L^2} + \|(1-\sqrt{\psi_n})\Upsilon_{n,2}\|_{L^2} \\
\lesssim &\sum_{n' \neq n} \|\Upsilon_{n,1} \psi_{n'}\|_{L^2} + \sum_{n' \neq n} \|\Upsilon_{n,2} \psi_{n'}\|_{L^2} \lesssim e^{-2\gamma_0 L}.
\end{aligned}$$

Based on above estimates, (2.14), the definition of a_n, the translation $x \to x - y_n$ and the Young's inequality, we have

$$\begin{aligned}
\left(\vec{\varphi}_n, \vec{Z}_{\ell_n}^0\right)_{L^2}^2 &= \left(\vec{\varphi}, (1-\sqrt{\psi_n})\vec{Z}_n^0\right)_{L^2}^2 \\
&= O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + \|(1-\sqrt{\psi_n})\partial_x Q_n\|_{L^2}^6 + \|(1-\sqrt{\psi_n})\partial_x^2 Q_n\|_{L^2}^6\right) \\
&= O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-12\gamma_0 L}\right),
\end{aligned}$$

and

$$\begin{aligned}
\left(\vec{\varphi}_n, \vec{Z}_{\ell_n}^{\pm}\right)_{L^2}^2 &= \left(a_n^{\pm} + \left(\vec{\varphi}, (1-\sqrt{\psi_n})\vec{Z}_n^{\pm}\right)_{L^2}\right)^2 \\
&= (a_n^{\pm})^2 + O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + \|(1-\sqrt{\psi_n})\Upsilon_{n,1}\|_{L^2}^3 + \|(1-\sqrt{\psi_n})\Upsilon_{n,2}\|_{L^2}^3\right) \\
&= (a_n^{\pm})^2 + O\left(\|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-6\gamma_0 L}\right).
\end{aligned}$$

Gathering above estimates, we obtain the following localized coercivity of $H_n(\vec{\varphi}, \vec{\varphi})$,

$$\begin{aligned}
H_n(\vec{\varphi}, \vec{\varphi}) \geqslant \nu_n \int_{\mathbb{R}} \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2\right) \psi_n \mathrm{d}x - \nu_n^{-1}\left((a_n^-)^2 + (a_n^+)^2\right) \\
+ O\left(\frac{1}{L}\|\vec{\varphi}\|_{\mathcal{H}}^2 + \|\vec{\varphi}\|_{\mathcal{H}}^3 + e^{-6\gamma_0 L}\right),
\end{aligned} \tag{2.70}$$

for all $n = 1, \ldots, N$. Last, combining $\tilde{c}_n > 0$, (2.70), $\sum_{n=1}^N \psi_n = 1$ and taking L large enough, we obtain the coercivity property (2.68). □

Last, we prove the following almost monotonicity property for $\mathcal{J}_n$.

Lemma 2.20. *There exist C_1 such that for any $n = 1, \ldots, N$ and $t \in [0, T_*(\vec{u}_0)]$,*

$$\mathcal{J}_n(t) - \mathcal{J}_n(0) \leqslant \frac{C_1}{L^{2\alpha-1}} \sup_{s \in [0,t]} \|\vec{\varphi}(s)\|_{\mathcal{H}}^2 + C_1 e^{-3\gamma_0 L}. \tag{2.71}$$

Proof. **Step 1**. Time variation of $\mathcal{I}_n$. We claim

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_n = &-2\int_{\mathbb{R}}(\partial_x u_1+\beta_n u_2)^2\partial_x\chi_n\mathrm{d}x\\ &+(1-\beta_n^2)\int_{\mathbb{R}}\big(u_1 f(u_1)-2F(u_1)\big)\partial_x\chi_n\mathrm{d}x\\ &+\beta_n^2\int_{\mathbb{R}}\big((\partial_x u_1)^2+u_1^2+u_2^2\big)\partial_x\chi_n\mathrm{d}x\\ &+2\beta_n\int_{\mathbb{R}}(\partial_x u_1)u_2\partial_x\chi_n\mathrm{d}x-2\beta_n^2\int_{\mathbb{R}}F(u_1)\partial_x\chi_n\mathrm{d}x\\ &-\frac{\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^\alpha}\right)\big(2(\partial_x u_1)u_2\big)\partial_x\chi_n\mathrm{d}x\\ &+O\left(\frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}\right).\end{aligned}\tag{2.72}$$

First, using (2.2) and integrating by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}2\big(\chi_n\partial_x u_1\big)u_2\mathrm{d}x = &-\int_{\mathbb{R}}\big((\partial_x u_1)^2-u_1^2+u_2^2+2F(u_1)\big)\partial_x\chi_n\mathrm{d}x\\ &+2\int_{\mathbb{R}}(\partial_t\chi_n)(\partial_x u_1)u_2\mathrm{d}x.\end{aligned}$$

Observe that

$$\partial_t\chi_n = -\beta_n\partial_x\chi_n-\frac{\alpha}{(t+a)^{1-\alpha}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^\alpha}\right)(\partial_x\chi_n).\tag{2.73}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}2\big(\chi_n\partial_x u_1\big)u_2\mathrm{d}x = &-\int_{\mathbb{R}}\big((\partial_x u_1)^2-u_1^2+u_2^2+2\beta_n(\partial_x u_1)u_2\big)\partial_x\chi_n\mathrm{d}x\\ &-\frac{\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^\alpha}\right)\big(2(\partial_x u_1)u_2\big)\partial_x\chi_n\mathrm{d}x\\ &-2\int_{\mathbb{R}}F(u_1)\partial_x\chi_n\mathrm{d}x.\end{aligned}\tag{2.74}$$

Second, using again (2.2) and integrating by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}u_1u_2\partial_x\chi_n\mathrm{d}x = &\int_{\mathbb{R}}\big(-(\partial_x u_1)^2-u_1^2+u_2^2+u_1 f(u_1)\big)\partial_x\chi_n\mathrm{d}x\\ &+\int_{\mathbb{R}}(u_1u_2)\partial_{tx}\chi_n\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}u_1^2\partial_x^3\chi_n\mathrm{d}x.\end{aligned}$$

From (2.13), (2.41), (2.46) and the AM-GM inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}(u_1u_2)\partial_{tx}\chi_n\mathrm{d}x\right| &\lesssim \frac{1}{(t+a)^{2\alpha}}\int_{\mathbb{R}}\left(\sum_{n'=1}^N(Q_{n'})^2+\sum_{n'=1}^N(\partial_x Q_{n'})^2+\varphi_1^2+\varphi_2^2\right)\mathbf{1}_{\Omega_n}\mathrm{d}x\\ &\lesssim \frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+\sum_{n'=1}^N\big(\|\partial_x Q_{n'}\mathbf{1}_{\Omega_n}\|_{L^2}^2+\|Q_{n'}\mathbf{1}_{\Omega_n}\|_{L^2}^2\big)\\ &\lesssim \frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)},\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}} u_1^2\partial_x^3\chi_n \mathrm{d}x\right| &\lesssim \frac{1}{(t+a)^{3\alpha}}\int_{\mathbb{R}}\left(\sum_{n'=1}^{N}(Q_{n'})^2+\sum_{n'=1}^{N}(\partial_x Q_{n'})^2+\varphi_1^2\right)\mathbf{1}_{\Omega_n}\mathrm{d}x\\ &\lesssim \frac{1}{(t+a)^{3\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+\sum_{n'=1}^{N}\left(\|\partial_x Q_{n'}\mathbf{1}_{\Omega_n}\|_{L^2}^2+\|Q_{n'}\mathbf{1}_{\Omega_n}\|_{L^2}^2\right)\\ &\lesssim \frac{1}{(t+a)^{3\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u_1u_2\partial_x\chi_n\mathrm{d}x &= \int_{\mathbb{R}}\left(-(\partial_x u_1)^2-u_1^2+u_2^2+u_1 f(u_1)\right)\partial_x\chi_n\mathrm{d}x\\ &\quad+O\left(\frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}\right).\end{aligned}\tag{2.75}$$

Gathering (2.74) and (2.75), we obtain (2.72).

Step 2. Time variation of E_n. We claim

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_n &= -2\int_{\mathbb{R}}\left((\partial_x u_1)u_2\right)\partial_x\chi_n\mathrm{d}x-\beta_n\int_{\mathbb{R}}\left((\partial_x u_1)^2+u_1^2+u_2^2\right)\partial_x\chi_n\mathrm{d}x\\ &\quad+2\beta_n\int_{\mathbb{R}}F(u_1)\partial_x\chi_n\mathrm{d}x\\ &\quad+\frac{2\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)F(u_1)\partial_x\chi_n\mathrm{d}x\\ &\quad-\frac{\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left((\partial_x u_1)^2+u_1^2+u_2^2\right)\partial_x\chi_n\mathrm{d}x.\end{aligned}\tag{2.76}$$

First, from (2.2),

$$\partial_t\left[(\partial_x u_1)^2+u_1^2+u_2^2-2F(u_1)\right]=2(\partial_x u_1)\partial_x u_2+2(\partial_x^2 u_1)u_2=2\partial_x(u_2\partial_x u_1).$$

Therefore, by integration by parts,

$$\int_{\mathbb{R}}\left(\partial_t\left[(\partial_x u_1)^2+u_1^2+u_2^2-2F(u_1)\right]\right)\chi_n\mathrm{d}x=-2\int_{\mathbb{R}}\left((\partial_x u_1)u_2\right)\partial_x\chi_n\mathrm{d}x.$$

Second, from (2.73),

$$\begin{aligned}&\int_{\mathbb{R}}\left((\partial_x u_1)^2+u_1^2+u_2^2-2F(u_1)\right)\partial_t\chi_n\mathrm{d}x\\ &\quad=-\beta_n\int_{\mathbb{R}}\left((\partial_x u_1)^2+u_1^2+u_2^2-2F(u_1)\right)\partial_x\chi_n\mathrm{d}x\\ &\qquad-\frac{\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left((\partial_x u_1)^2+u_1^2+u_2^2-2F(u_1)\right)\partial_x\chi_n\mathrm{d}x.\end{aligned}$$

Gathering these identities, we obtain (2.76).

Step 3. Time variation of F_n. We claim

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_n &= \frac{-\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left[-(\partial_x u_1)^2-u_1^2+u_2^2+u_1 f(u_1)\right]\partial_x\chi_n\mathrm{d}x\\ &\quad+O\left(\frac{1}{(t+a)^{1+\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}\right).\end{aligned}\tag{2.77}$$

First, from (2.2), $\partial_t(u_1u_2) = u_2^2 - u_1^2 + u_1 f(u_1) + u_1(\partial_x^2 u_1)$. Therefore, by integration by parts,

$$\begin{aligned}
&\frac{1}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left[\partial_t(u_1u_2)\right]\partial_x\chi_n\mathrm{d}x\\
&\quad=\frac{1}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left(-(\partial_x u_1)^2-u_1^2+u_2^2+u_1f(u_1)\right)\partial_x\chi_n\mathrm{d}x\\
&\qquad+\frac{1}{t+a}\int_{\mathbb{R}}u_1^2\partial_x^2\chi_n\mathrm{d}x+\frac{1}{2(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)u_1^2\partial_x^3\chi_n\mathrm{d}x.
\end{aligned}$$

Observe that, from (2.13), (2.41), (2.46) and the AM-GM inequality, we have

$$\begin{aligned}
\left|\frac{1}{t+a}\int_{\mathbb{R}}u_1^2\partial_x^2\chi_n\mathrm{d}x\right| &\lesssim \frac{1}{(t+a)^{1+2\alpha}}\int_{\mathbb{R}}\left(\sum_{n'=1}^{N}Q_{n'}^2+\varphi_1^2\right)\mathbf{1}_{\Omega_n}\mathrm{d}x\\
&\lesssim \frac{1}{(t+a)^{1+2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+\sum_{n'=1}^{N}\|Q_{n'}\mathbf{1}_{\Omega_n}\|_{L^2}^2\\
&\lesssim \frac{1}{(t+a)^{1+2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0L+\gamma_1t)},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{2(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)u_1^2\partial_x^3\chi_n\mathrm{d}x\right|\\
&\quad\lesssim \frac{1}{(t+a)^{1+2\alpha}}\int_{\mathbb{R}}\left(\sum_{n'=1}^{N}Q_{n'}^2+\varphi_1^2\right)\mathbf{1}_{\Omega_n}\mathrm{d}x\\
&\quad\lesssim \frac{1}{(t+a)^{1+2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0L+\gamma_1t)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left[\partial_t(u_1u_2)\right]\partial_x\chi_n\mathrm{d}x\\
&\quad=\frac{1}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\left[-(\partial_x u_1)^2-u_1^2+u_2^2+u_1f(u_1)\right]\partial_x\chi_n\mathrm{d}x\\
&\qquad+O\left(\frac{1}{(t+a)^{1+2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0L+\gamma_1t)}\right).
\end{aligned}\tag{2.78}$$

Second, by direct computation and taking partial derivative of (2.73) with respect to x,

$$\begin{aligned}
\partial_t\left[\left(\frac{x-\beta_n t-\bar{y}_n^0}{t+a}\right)\partial_x\chi_n\right] &= -\frac{\beta_n}{(t+a)}\partial_x\chi_n-\frac{1}{(t+a)^{2-\alpha}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\partial_x\chi_n\\
&\quad-\frac{\beta_n}{(t+a)^{1-\alpha}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\partial_x^2\chi_n\\
&\quad-\frac{\alpha}{(t+a)^{2-2\alpha}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)^2\partial_x^2\chi_n.
\end{aligned}$$

Therefore, from (2.40), (2.41) and $\frac{1}{2}<\alpha<\frac{4}{7}$,

$$\left|\partial_t\left[\left(\frac{x-\beta_n t-y_n^0}{t+a}\right)\partial_x\chi_n\right]\right|\lesssim\frac{1}{(t+a)^{1+\alpha}}\mathbf{1}_{\Omega_n}.$$

Based on (2.13), (2.46), the AM-GM inequality and above inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(u_1u_2)\partial_t\left[\left(\frac{x-\beta_n t-\bar{y}_n^0}{t+a}\right)\partial_x\chi_n\right]\mathrm{d}x\right| \\
&\lesssim \frac{1}{(t+a)^{1+\alpha}}\int_{\mathbb{R}}\left(\sum_{n'=1}^{N}Q_{n'}^2+\sum_{n'=1}^{N}(\partial_x Q_{n'})^2+\varphi_1^2+\varphi_2^2\right)\mathbf{1}_{\Omega_n}\mathrm{d}x \\
&\lesssim \frac{1}{(t+a)^{1+\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}.
\end{aligned} \tag{2.79}$$

We see that (2.77) follows from (2.78) and (2.79).

Step 4. Conclusion. Note that from (2.72), (2.76), (2.77) and $\frac{1}{2}<\alpha<\frac{4}{7}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_n &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_n+\beta_n\frac{\mathrm{d}}{\mathrm{d}t}E_n+\beta_n\frac{\mathrm{d}}{\mathrm{d}t}F_n \\
&= \mathcal{F}_1+\mathcal{F}_2+O\left(\frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1 = &-2\int_{\mathbb{R}}(\partial_x u_1+\beta_n u_2)^2\partial_x\chi_n\mathrm{d}x \\
&-\frac{2\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)(u_2(\partial_x u_1+\beta_n u_2))\,\partial_x\chi_n\mathrm{d}x,
\end{aligned}$$

and

$$\mathcal{F}_2=\int_{\mathbb{R}}\left[1-\beta_n^2-\frac{\alpha\beta_n}{(t+a)^{1-\alpha}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)\right](u_1 f(u_1)-2F(u_1))\,\partial_x\chi_n\mathrm{d}x.$$

Estimates of $\mathcal{F}_1$. We claim

$$\mathcal{F}_1\leqslant\frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-3(\gamma_0 L+\gamma_1 t)}. \tag{2.80}$$

Indeed, by the AM-GM inequality and $1-\alpha=(1-\frac{5}{4}\alpha)+\frac{1}{4}\alpha$,

$$\begin{aligned}
&\left|\frac{2\alpha}{(t+a)^{1-\alpha}}\int_{\mathbb{R}}\left(\frac{x-\beta_n t-\bar{y}_n^0}{(t+a)^{\alpha}}\right)(u_2(\partial_x u_1+\beta_n u_2))\,\partial_x\chi_n\mathrm{d}x\right| \\
&\lesssim \frac{1}{(t+a)^{2-\frac{5}{2}\alpha}}\int_{\mathbb{R}}u_2^2\partial_x\chi_n\mathrm{d}x+\frac{1}{(t+a)^{\frac{\alpha}{2}}}\int_{\mathbb{R}}(\partial_x u_1+\beta_n u_2)^2\partial_x\chi_n\mathrm{d}x.
\end{aligned}$$

Moreover, using (2.13), (2.40), (2.46) and the AM-GM inequality,

$$\begin{aligned}
\left|\frac{1}{(t+a)^{2-\frac{5}{2}\alpha}}\int_{\mathbb{R}}u_2^2\partial_x\chi_n\mathrm{d}x\right| &\lesssim \frac{1}{(t+a)^{2-\frac{3}{2}\alpha}}\int_{\mathbb{R}}\left(\sum_{n'=1}^{N}(\partial_x Q_{n'})^2+\varphi_2^2\right)\mathbf{1}_{\Omega_n}\mathrm{d}x \\
&\lesssim \frac{1}{(t+a)^{2-\frac{3}{2}\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-4(\gamma_0 L+\gamma_1 t)}.
\end{aligned}$$

Note that, from $\frac{1}{2}<\alpha<\frac{4}{7}$, we have $2-\frac{3}{2}\alpha>2\alpha$. Thus, from $a=\left(\frac{L}{10}\right)^{\frac{1}{\alpha}}$ and taking L large enough,

$$\begin{aligned}
\mathcal{F}_1\leqslant &-2\int_{\mathbb{R}}(\partial_x u_1+\beta_n u_2)^2\partial_x\chi_n\mathrm{d}x+\frac{1}{L^{\frac{1}{4}}}\int_{\mathbb{R}}(\partial_x u_1+\beta_n u_2)^2\partial_x\chi_n\mathrm{d}x \\
&+\frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}\|_{\mathcal{H}}^2+e^{-3(\gamma_0 L+\gamma_1 t)},
\end{aligned}$$

which implies (2.80).

Estimates of $\mathcal{F}_2$. We claim

$$\mathcal{F}_2 \leqslant 0. \tag{2.81}$$

First, observe that, for L large enough, for any $x \in \Omega_n$,

$$1 - \beta_n^2 - \frac{\alpha\beta_n}{(t+a)^{1-\alpha}}\left(\frac{x - \beta_n t - y_n^0}{(t+a)^\alpha}\right) \geqslant 1 - \beta_n^2 - |\alpha\beta_n|\left|\frac{10}{L}\right|^{\frac{1}{\alpha}-1} > 0.$$

Second, recall that, for any $x \in \Omega_n$ and $n' = 1, \cdots, N$,

$$\begin{aligned}|x - y_{n'}(t)| &\geqslant |\ell_{n'} - \beta_n|t + |y_{n'}(0) - \bar{y}_n^0| - \gamma_1 t - t^\alpha - \frac{L}{10}\\ &\geqslant 8\gamma_1 t + \frac{9L}{20} - \left(2\gamma_1 t + \frac{L}{5}\right) \geqslant \frac{L}{4},\end{aligned}$$

for L large enough. Thus, from the decay properties of Q in (2.6), for any $t \in [0, T_*(\vec{u}_0)]$,

$$\sum_{n'=1}^{N} |Q_{n'}(t,x)|\mathbf{1}_{\Omega_n} \lesssim \sum_{n'=1}^{N} e^{\frac{-|x-y_{n'}(t)|}{\sqrt{1-\ell_{n'}^2}}}\mathbf{1}_{\Omega_n} \lesssim e^{-\frac{L}{4}}.$$

Based on above estimate, (2.35) and $\gamma_0 \leqslant \frac{1}{8}$, we have, for any $x \in \Omega_n$,

$$\begin{aligned}|u_1(t,x)| &\lesssim \sum_{n'=1}^{N} |Q_{n'}(t,x)|\mathbf{1}_{\Omega_n} + |\varphi_1(t,x)|\\ &\lesssim e^{-\frac{L}{4}} + \|\vec{\varphi}\|_{\mathcal{H}} \lesssim C_0\left(e^{-\gamma_0 L} + \delta\right).\end{aligned}$$

Therefore, for L large enough and δ small enough, we obtain for any $x \in \Omega_n$,

$$u_1 f(u_1) - 2F(u_1) = -\frac{q-1}{q+1}|u_1|^{q+1} + \frac{p-1}{p+1}|u_1|^{p+1} \leqslant 0,$$

since $1 < q < p < \infty$. We conclude (2.81) from above estimates.

Gathering estimates (2.80) and (2.81), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_n(t) \lesssim \frac{1}{(t+a)^{2\alpha}}\|\vec{\varphi}(t)\|_{\mathcal{H}}^2 + e^{-3(\gamma_0 L + \gamma_1 t)}.$$

Integrating on $[0,t]$ for any $t \in [0, T_*(\vec{u}_0)]$, we obtain

$$\mathcal{J}_n(t) - \mathcal{J}_n(0) \lesssim \frac{1}{L^{2\alpha-1}}\max_{s\in[0,t]}\|\vec{\varphi}(s)\|_{\mathcal{H}}^2 + e^{-3\gamma_0 L},$$

which implies (2.71). □

2.4 Proof of Theorem 2.1

In this section, we prove Theorem 2.1 using a bootstrap argument. We start with a technical result that will allow us to adjust the initial value with N free parameters.

Lemma 2.21 (The initial unstable mode)**.** *Let $N \geqslant 2$. For $n \in \{1, \ldots, N\}$, let $\sigma_n = \pm 1$ and $-1 < \ell_n < 1$ with $-1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1$. There exist $L_0 \gg 1$ and $0 < \delta_0 \ll 1$ such that the following is true. Let $y_1^0 < \cdots < y_N^0$ be such that*

$$L = \min(y_{n+1}^0 - y_n^0, n = 1, \ldots, N-1) > L_0,$$

and $\vec{\varepsilon} \in H^1 \times L^2$, $\boldsymbol{a}^+ = (a_n^+)_{n\in\{1,\dots,N\}} \in \mathbb{R}^N$ *be such that*

$$\|\vec{\varepsilon}\|_{\mathcal{H}} < \delta < \delta_0 \quad \text{and} \quad \boldsymbol{a}^+ \in \bar{B}_{\mathbb{R}^N}(r) \quad \text{where} \quad r = C_0^{\frac{3}{4}} \left(\delta^2 + e^{-2\gamma_0 L}\right)^{\frac{1}{2}}, \tag{2.82}$$

C_0 *is defined in the bootstrap* (2.35) *and to be taken large enough. Then, there exist* $\tilde{y}_1^0 < \cdots < \tilde{y}_N^0$ *and* $h_1^+, \dots, h_N^+ \in \mathbb{R}$ *satisfying*

$$\sum_{n=1}^{N} \left(|h_n^+| + |\tilde{y}_n^0 - y_n^0|\right) \leqslant C_0^{\frac{13}{16}} (\delta + e^{-\gamma_0 L}), \tag{2.83}$$

such that the initial value defined by

$$\vec{u}_0 = \sum_{n=1}^{N} \left(\sigma_n \vec{Q}_{\ell_n} + h_n^+ \vec{Z}_{\ell_n}^+\right)(\cdot - y_n^0) + \vec{\varepsilon}$$

rewrites as:

$$\vec{u}_0 = \sum_{n=1}^{N} \sigma_n \vec{Q}_{\ell_n}(\cdot - \tilde{y}_n^0) + \vec{\varphi}(0) \tag{2.84}$$

where $\vec{\varphi}(0)$ *satisfies for all* $n = 1, \dots, N$,

$$\left(\vec{\varphi}(0), \vec{Z}_{\ell_n}^0(\cdot - \tilde{y}_n^0)\right)_{L^2} = 0, \quad a_n^+(0) = \left(\vec{\varphi}(0), \vec{Z}_{\ell_n}^+(\cdot - \tilde{y}_n^0)\right)_{L^2} = a_n^+. \tag{2.85}$$

Moreover, the initial data (2.84) *is modulated in the sense of Lemma 2.7 with* $y_n(0) = \tilde{y}_n^0$, *for all* $n = 1, \dots, N$.

Proof. Let

$$\Gamma^0 = \left(\mathbf{0}, y_1^0, \dots, y_N^0\right) \in \mathbb{R}^{2N}, \quad \Gamma = \left(h_1^+, \dots, h_N^+, \tilde{y}_1^0, \dots, \tilde{y}_N^0\right) \in \mathbb{R}^{2N}.$$

Consider the map

$$\begin{aligned} \Psi : \quad & X \to \mathbb{R}^{2N} \\ & \left(\vec{\varepsilon}, \boldsymbol{a}^+, \Gamma\right) \mapsto \left(\Psi_1^a, \dots, \Psi_N^a, \Psi_1^0, \dots, \Psi_N^0\right), \end{aligned}$$

where $X = \left(H^1 \times L^2\right) \times \mathbb{R}^N \times \mathbb{R}^{2N}$, and for $n = 1, \dots, N$,

$$\begin{aligned} \Psi_n^a = & \sum_{n'=1}^{N} \sigma_{n'} \left(\left(\vec{Q}_{\ell_{n'}}(\cdot - y_{n'}^0) - \vec{Q}_{\ell_{n'}}(\cdot - \tilde{y}_{n'}^0)\right), \vec{Z}_{\ell_n}^+(\cdot - \tilde{y}_n^0)\right)_{L^2} \\ & + \sum_{n'=1}^{N} h_{n'}^+ \left(\vec{Z}_{\ell_{n'}}^+(\cdot - y_{n'}^0), \vec{Z}_{\ell_n}^+(\cdot - \tilde{y}_n^0)\right)_{L^2} + \left(\vec{\varepsilon}, \vec{Z}_{\ell_n}^+(\cdot - \tilde{y}_n^0)\right)_{L^2} - a_n^+, \end{aligned}$$

$$\begin{aligned} \Psi_n^0 = & \sum_{n'=1}^{N} \sigma_{n'} \left(\left(\vec{Q}_{\ell_{n'}}(\cdot - y_{n'}^0) - \vec{Q}_{\ell_{n'}}(\cdot - \tilde{y}_{n'}^0)\right), \vec{Z}_{\ell_n}^0(\cdot - \tilde{y}_n^0)\right)_{L^2} \\ & + \sum_{n'=1}^{N} h_{n'}^+ \left(\vec{Z}_{\ell_{n'}}^+(\cdot - y_{n'}^0), \vec{Z}_{\ell_n}^0(\cdot - \tilde{y}_n^0)\right)_{L^2} + \left(\vec{\varepsilon}, \vec{Z}_{\ell_n}^0(\cdot - \tilde{y}_n^0)\right)_{L^2}. \end{aligned}$$

From (2.84),

$$\vec{\varphi}(0) = \sum_{n=1}^{N} \sigma_n \left(\vec{Q}_{\ell_n}(\cdot - y_n^0) - \vec{Q}_{\ell_n}(\cdot - \tilde{y}_n^0)\right) + \sum_{n=1}^{N} h_n^+ \vec{Z}_{\ell_n}^+(\cdot - y_n^0) + \vec{\varepsilon}, \tag{2.86}$$

and thus the set of conditions in (2.85) is equivalent to $\Psi\left(\vec{\varepsilon}, \boldsymbol{a}^+, \Gamma\right) = \mathbf{0} \in \mathbb{R}^{2N}$. We solve this nonlinear system by the Implicit Function Theorem. First, it is easy to check that

$$\Psi(\vec{0}, \mathbf{0}, \Gamma^0) = 0.$$

Second, by direct computation and integration by parts, for any $(\vec{\varepsilon}, \boldsymbol{a}^+, \Gamma)$ around $\left(\vec{0}, \mathbf{0}, \Gamma^0\right)$,

$$D_\Gamma \Psi(\vec{\varepsilon}, \boldsymbol{a}^+, \Gamma) = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where

$$\begin{aligned}
A &= \left(\left(\vec{Z}^+_{\ell_n}(\cdot - y^0_n), \vec{Z}^+_{\ell_{n'}}(\cdot - y^0_{n'})\right)_{L^2}\right)_{n,n' \in \{1,\dots,N\}} + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0|\right), \\
B &= \left(\left(\sigma_n \vec{Z}^0_{\ell_n}(\cdot - y^0_n), \vec{Z}^+_{\ell_{n'}}(\cdot - y^0_{n'})\right)_{L^2}\right)_{n,n' \in \{1,\dots,N\}} + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0|\right), \\
C &= \left(\left(\sigma_n \vec{Z}^0_{\ell_n}(\cdot - y^0_n), \vec{Z}^+_{\ell_{n'}}(\cdot - y^0_{n'})\right)_{L^2}\right)_{n,n' \in \{1,\dots,N\}} + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0|\right), \\
D &= \left(\left(\sigma_n \vec{Z}^0_{\ell_n}(\cdot - y^0_n), \sigma_{n'} \vec{Z}^0_{\ell_{n'}}(\cdot - y^0_{n'})\right)_{L^2}\right)_{n,n' \in \{1,\dots,N\}} + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0|\right).
\end{aligned}$$

Moreover, from $\left(\vec{Z}^0_\ell, \vec{Z}^+_\ell\right)_{L^2} = 0$ and similar argument as in the proof of (2.27), we obtain

$$\begin{aligned}
A &= \operatorname{diag}\left(\left(\vec{Z}^+_{\ell_1}, \vec{Z}^+_{\ell_1}\right)_{L^2}, \dots, \left(\vec{Z}^+_{\ell_N}, \vec{Z}^+_{\ell_N}\right)_{L^2}\right) + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0| + e^{-2\gamma_0 L}\right), \\
B &= O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0| + e^{-2\gamma_0 L}\right), \\
C &= O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0| + e^{-2\gamma_0 L}\right), \\
D &= \operatorname{diag}\left(\left(\vec{Z}^0_{\ell_1}, \vec{Z}^0_{\ell_1}\right)_{L^2}, \dots, \left(\vec{Z}^0_{\ell_N}, \vec{Z}^0_{\ell_N}\right)_{L^2}\right) + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + |\Gamma - \Gamma^0| + e^{-2\gamma_0 L}\right).
\end{aligned}$$

Thus, $D_\Gamma \Psi(\vec{\varepsilon}, \boldsymbol{a}^+, \Gamma)$ is an invertible matrix for $L > L_0$ large enough, with a lower bound uniform around $\left(\vec{0}, \mathbf{0}, \Gamma^0\right)$. Therefore, by the uniform variant of the implicit function theorem, there exist $0 < \delta_1 \ll 1$ (independent with the choose of $y^0_1 < \dots < y^0_N$) and C^1 map

$$\Pi : \ B_{H^1 \times L^2}(\vec{0}, \delta_1) \times B_{\mathbb{R}^N}(\mathbf{0}, \delta_1) \to B_{\mathbb{R}^{2N}}(\Gamma^0, \delta_1),$$

such that for all $\vec{\varepsilon} \in B_{H^1 \times L^2}(\vec{0}, \delta_1)$, $\boldsymbol{a}^+ \in B_{\mathbb{R}^N}(\mathbf{0}, \delta_1)$ and $\Gamma \in B_{\mathbb{R}^{2N}}(\Gamma^0, \delta_1)$,

$$\Psi\left(\vec{\varepsilon}, \boldsymbol{a}^+, \Gamma\right) = 0 \quad \text{if and only if} \quad \Gamma = \Pi\left(\vec{\varepsilon}, \boldsymbol{a}^+\right).$$

Moreover, taking $0 < \delta < \delta_0 \ll 1$ small enough, $L > L_0$ large enough, for any $(\vec{\varepsilon}, \boldsymbol{a}^+)$ satisfying (2.82), we have

$$\left|\Gamma - \Gamma^0\right| = \left|\Pi\left(\vec{\varepsilon}, \boldsymbol{a}^+\right) - \Pi\left(\vec{0}, \mathbf{0}\right)\right| \lesssim \|\vec{\varepsilon}\|_{H^1 \times L^2} + \|\boldsymbol{a}^+\| \lesssim \delta + C_0^{\frac{3}{4}}\left(\delta^2 + e^{-2\gamma_0 L}\right)^{\frac{1}{2}},$$

which implies (2.83) for taking C_0 large enough (independent with δ_0 and L_0). The proof of Lemma 2.21 is complete. □

We are in a position to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\vec{\varepsilon} \in H^1 \times L^2$ and $y^0_1 < \dots < y^0_N$ as in the statement of Theorem 2.1. For all $\boldsymbol{a}^+(0) = \boldsymbol{a}^+ = (a^+_1, \dots, a^+_N) \in \bar{B}_{\mathbb{R}}(r)$, we consider the solution $\vec{u} = (u_1, u_2)$ with the initial data as defined in Lemma 2.21

$$\vec{u}_0 = \sum_{n=1}^N \left(\sigma_n \vec{Q}_{\ell_n} + h^+_n \vec{Z}^+_{\ell_n}\right)(\cdot - y^0_n) + \vec{\varepsilon} = \sum_{n=1}^N \sigma_n \vec{Q}_{\ell_n}(\cdot - \tilde{y}^0_n) + \vec{\varphi}(0). \tag{2.87}$$

Note that, for the proof of Theorem 2.1, we just need to prove the existence of $\vec{u}_0$ such that $T_*(\vec{u}_0) = \infty$. We start by closing all bootstrap estimates except the one for the instable modes. Last, we prove the existence of suitable parameters $\boldsymbol{a}^+ = (a_1^+, \dots, a_N^+)$ using a topological argument. Denote $q_0 = \min(3, q+1)$.

Step 1. Closing the estimates in $\vec{\varphi}$. First, from (2.83) and (2.87),

$$\begin{aligned}\|\vec{\varphi}(0)\|_{\mathcal{H}}^2 &\lesssim \sum_{n=1}^N \|\vec{Q}_{\ell_n}(\cdot - y_n^0) - \vec{Q}_{\ell_n}(\cdot - \tilde{y}_n^0)\|_{\mathcal{H}}^2 + \sum_{n=1}^N \|h_n^+ \vec{Z}_{\ell_n}^+(\cdot - y_n^0)\|_{\mathcal{H}}^2 + \|\vec{\varepsilon}\|_{\mathcal{H}}^2 \\ &\lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^2 + \sum_{n=1}^N \left(|h_n^+|^2 + |\tilde{y}_n^0 - y_n^0|^2\right) \lesssim C_0^{\frac{13}{8}}(\delta^2 + e^{-2\gamma_0 L}).\end{aligned} \tag{2.88}$$

Second, from (2.71), conservation of energy $E(\vec{u}(t))$ and momentum $\mathcal{I}(\vec{u}(t))$, for any $t \in [0, T_*(\vec{u}_0(0))]$,

$$\begin{aligned}\mathcal{E}(\vec{u}(t)) - \mathcal{E}(\vec{u}(0)) &= \sum_{n=2}^N c_n \left(\mathcal{J}_n(t) - \mathcal{J}_n(0)\right) \\ &\leqslant C_1 \left(\sum_{n=2}^N c_n\right) \left(\frac{1}{L^{2\alpha-1}} \sup_{s\in[0,t]} \|\vec{\varphi}(s)\|_{\mathcal{H}}^2 + e^{-3\gamma_0 L}\right).\end{aligned}$$

Note that, from (2.58),

$$\begin{aligned}&\sum_{n=1}^N \tilde{c}_n H_n(\vec{\varphi}(t), \vec{\varphi}(t)) \\ &\quad = \mathcal{E}(\vec{u}(t)) - \mathcal{E}(\vec{u}(0)) + O\left(\frac{\|\vec{\varphi}(t)\|_{\mathcal{H}}^2}{L} + \|\vec{\varphi}(t)\|_{\mathcal{H}}^{q_0} + e^{-3\gamma_0 L} + \|\vec{\varphi}(0)\|_{\mathcal{H}}^2\right).\end{aligned}$$

Therefore, using (2.35), (2.68), (2.88) and $\frac{1}{2} < \alpha < \frac{4}{7}$,

$$\begin{aligned}\mu\|\vec{\varphi}(t)\|_{\mathcal{H}}^2 &\lesssim \frac{1}{L^{2\alpha-1}} \sup_{s\in[0,t]} \|\vec{\varphi}(s)\|_{\mathcal{H}}^2 + \frac{1}{L}\|\vec{\varphi}(t)\|_{\mathcal{H}}^2 + \|\vec{\varphi}(t)\|_{\mathcal{H}}^{q_0} \\ &\quad + \left(C_0^{\frac{3}{2}} + C_0^{\frac{7}{4}} + C_0^{\frac{13}{8}}\right)(\delta^2 + e^{-2\gamma_0 L}) + e^{-3\gamma_0 L} \\ &\lesssim \left(\frac{C_0^2}{L^{2\alpha-1}} + C_0^{\frac{7}{4}}\right)\left(\delta^2 + e^{-2\gamma_0 L}\right) + C_0^{q_0}\left(\delta^{q_0} + e^{-q_0\gamma_0 L}\right) + e^{-3\gamma_0 L}.\end{aligned}$$

This strictly improves the estimate on $\vec{\varphi}$ in (2.35) for taking L, C_0 large enough and δ small enough.

Step 2. Closing the estimates in $\min_n(y_{n+1} - y_n)$. From (2.38), (2.83) and $\gamma_1 = \frac{1}{8}\min_n(\ell_{n+1} - \ell_n)$, for any $n = 1, \dots, N-1$ and $t \in [0, T_*(\vec{u}_0)]$,

$$\begin{aligned}y_{n+1}(t) - y_n(t) &\geqslant (\ell_{n+1} - \ell_n)t + (y_{n+1}^0 - y_n^0) - 2\gamma_1 t - |\tilde{y}_{n+1}^0 - y_{n+1}^0| - |\tilde{y}_n^0 - y_n^0| \\ &\geqslant 8\gamma_1 t + L - 2\gamma_1 t + O\left(C_0^{\frac{13}{16}}(\delta + e^{-\gamma_0 L})\right),\end{aligned}$$

which strictly improves the estimate on $\min_n(y_{n+1} - y_n)$ in (2.35) for taking L large enough and δ small enough.

Step 3. Closing the estimates in $\boldsymbol{a}^- = (a_n^-)_{n\in\{1,\dots,N\}}$. Note that, from (2.88),

$$\sum_{n=1}^N (a_n^-(0))^2 \lesssim \|\vec{\varphi}(0)\|_{\mathcal{H}}^2 \lesssim C_0^{\frac{13}{8}}(\delta^2 + e^{-2\gamma_0 L}).$$

By direct computation, (2.33), (2.35) and (2.37), for $n = 1, \dots, N$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\alpha_n t}(a_n^-(t))^2\right) &= 2e^{2\alpha_n t}a_n^-(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}a_n^-(t) + \alpha_n a_n^-(t)\right)\\ &= e^{2\alpha_n t}O\left(|a_n^-|^{q_0} + \|\vec{\varphi}\|_{\mathcal{H}}^{q_0} + \theta\right)\\ &= e^{2\alpha_n t}O\left(C_0^{q_0}\left(\delta^{q_0} + e^{-q_0\gamma_0 L}\right) + e^{-3\gamma_0 L}\right).\end{aligned}$$

Integrating on $[0, t]$, for any $t \in [0, T_*(\vec{u}_0)]$ and any $n = 1, \dots, N$, we obtain

$$\begin{aligned}(a_n^-(t))^2 &\lesssim e^{-2\alpha_n t}(a_n^-(0))^2 + e^{-2\alpha_n t}\int_0^t e^{2\alpha_n s}\left(C_0^{q_0}\left(\delta^{q_0} + e^{-q_0\gamma_0 L}\right) + e^{-3\gamma_0 L}\right)\mathrm{d}s\\ &\lesssim C_0^{\frac{13}{8}}\left(\delta^2 + e^{-2\gamma_0 L}\right) + C_0^{q_0}\left(\delta^{q_0} + e^{-q_0\gamma_0 L}\right) + e^{-3\gamma_0 L},\end{aligned}$$

which strictly improves the estimate on $\boldsymbol{a}^- = (a_n^-)_{n\in\{1,\dots,N\}}$ in (2.35) for taking L large enough and δ small enough.

Step 4. Final argument on the unstable parameters. Let

$$b(t) = \sum_{n=1}^N (a_n^+(t))^2 \quad \text{and} \quad \bar{\alpha} = \min_n \alpha_n.$$

We claim, for any time $t \in [0, T_*(\vec{u}_0)]$ where it holds $b(t) = C_0^{\frac{3}{2}}\left(\delta^2 + e^{-2\gamma_0 L}\right)$, the following transversality property holds,

$$\frac{\mathrm{d}}{\mathrm{d}t}b(t) \geqslant \bar{\alpha}C_0^{\frac{3}{2}}\left(\delta^2 + e^{-2\gamma_0 L}\right) > 0. \tag{2.89}$$

From (2.33), (2.35) and (2.37), taking C_0 large enough, for any time $t \in [0, T_*]$ where it holds $b(t) = C_0^{\frac{3}{2}}\left(\delta^2 + e^{-2\gamma_0 L}\right)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}b(t) &= 2\sum_{n=1}^N \alpha_n (a_n^+(t))^2 + O\left[\left(\sum_{n=1}^N |a_n^+(t)|\right)\left(\|\vec{\varphi}\|_{\mathcal{H}}^{q_0-1} + \theta\right)\right]\\ &= 2\sum_{n=1}^N \alpha_n (a_n^+(t))^2 + O\left(\sum_{n=1}^N |a_n^+|^{q_0} + \|\vec{\varphi}\|_{\mathcal{H}}^{q_0} + e^{-3\gamma_0 L}\right)\\ &\geqslant 2\bar{\alpha}C_0^{\frac{3}{2}}\left(\delta^2 + e^{-2\gamma_0 L}\right) - C_0^{q_0+1}\left(\delta^{q_0} + e^{-q_0\gamma_0 L}\right) - C_0 e^{-3\gamma_0 L},\end{aligned}$$

which implies (2.89) for taking δ small enough and L large enough (depending on C_0). The transversality relation (2.89) is enough to justify the existence of at least a couple $\boldsymbol{a}^+(0) = (a_1^+(0), \dots, a_N^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^N}(r)$ such that $T_*(\vec{u}_0) = \infty$ where $r = C_0^{\frac{3}{4}}\left(\delta^2 + e^{-2\gamma_0 L}\right)^{\frac{1}{2}}$.

The proof is by contradiction, we assume that for all $\boldsymbol{a}^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^N}(r)$, it holds $T_*(\vec{u}_0) < \infty$. Then, a contradiction follows from the following discussion (see for instance more details in [28] and [31, Section 3.1]).

Continuity of $T_*(\vec{u}_0)$. The above transversality condition (2.89) implies that the map

$$\boldsymbol{a}^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^N}(r) \mapsto T_*(\vec{u}_0) \in [0, \infty)$$

is continuous and

$$T_*(\vec{u}_0) = 0 \quad \text{for} \quad \boldsymbol{a}^+(0) \in \mathcal{S}_{\mathbb{R}^N}(r).$$

Construction of a retraction. We define

$$\begin{aligned}\mathcal{M}:\ \bar{\mathcal{B}}_{\mathbb{R}^N}(r) &\mapsto \mathcal{S}_{\mathbb{R}^N}(r)\\ \boldsymbol{a}^+(0) &\mapsto \boldsymbol{a}^+(T_*(\vec{u}_0)).\end{aligned}$$

From what precedes, $\mathcal{M}$ is continuous. Moreover, $\mathcal{M}$ restricted to $\mathcal{S}_{\mathbb{R}^N}(r)$ is the identity. The existence of such a map is contradictory with the no retraction theorem for continuous maps from the ball to the sphere.

We have proved the existence of $\boldsymbol{a}^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^N}(r)$, associated to a global solution $\vec{u} = (u_1, u_2)$ of (2.2) with initial data defined in Lemma 2.21, which also satisfies (2.12) and (2.35) for all $t \in [0, \infty)$. The proof of Theorem 2.1 is thus complete. $\square$

Part II

Construction of multi-solitons for the NLW equation

Chapter 3

On multi-solitons for the energy-critical wave equation in dimension 5

Abstract: We construct K-solitons of the focusing energy-critical nonlinear wave equation in five-dimensional space, i.e. solutions u of the equation such that

$$\|\nabla_{t,x} u(t) - \nabla_{t,x}\big(\sum_{k=1}^{K} W_k(t)\big)\|_{L^2} \to 0 \quad \text{as } t \to \infty,$$

where $K \geq 2$ and for any $k \in \{1, \dots, K\}$, W_k is the Lorentz transform of the explicit standing soliton $W(x) = (1 + |x|^2/15)^{-3/2}$, with any speed $\boldsymbol{\ell}_k \in \mathbb{R}^5$, $|\boldsymbol{\ell}_k| < 1$ satisfying $\boldsymbol{\ell}_{k'} \neq \boldsymbol{\ell}_k$ for $k' \neq k$, and an explicit smallness condition. The proof extends the refined method of construction of asymptotic multi-solitons from [85, 86].

3.1 Introduction

3.1.1 Statement of the main result.

In this paper, we are interested in the construction of multi-soliton solutions for the focusing energy-critical wave equation in five-dimensional space:

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{3}} u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^5, \\ u_{|t=0} = u_0 \in \dot{H}^1, \quad \partial_t u_{|t=0} = u_1 \in L^2. \end{cases} \tag{3.1}$$

Recall that the Cauchy problem for equation (3.1) is locally well-posed in the energy space $\dot{H}^1 \times L^2$, using suitable Strichartz estimates. See *e.g.* [71] and references therein. In particular, for any initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, there exists a maximal interval of existence (T_-, T_+), $-\infty \leqslant T_- < 0 < T_+ \leqslant +\infty$, and a unique solution $(u, \partial_t u) \in C\left((T_-, T_+); \dot{H}^1 \times L^2\left(\mathbb{R}^5\right)\right) \cap L^{\frac{7}{3}}_{\text{loc}}\left((T_-, T_+); L^{\frac{14}{3}}(\mathbb{R}^5)\right)$. For $\dot{H}^1 \times L^2$ solution, the energy $E(u(t), \partial_t u(t))$ and momentum $M(u(t), \partial_t u(t))$ are conserved, where

$$E(u,v) = \frac{1}{2}\int v^2 + \frac{1}{2}\int |\nabla u|^2 - \frac{3}{10}\int |u|^{\frac{10}{3}}, \quad M(u,v) = \int v \nabla u.$$

For a function $u : \mathbb{R}^5 \to \mathbb{R}$ and $\lambda > 0$, we denote

$$u_\lambda(x) = \frac{1}{\lambda^{\frac{3}{2}}} u(\frac{x}{\lambda}).$$

A change of variables shows that

$$E((u_0)_\lambda, \lambda^{-1}(u_1)_\lambda) = E(u_0, u_1).$$

Equation (3.1) is called *energy-critical* since it is invariant under the same scaling: if $(u, \partial_t u)$ is a solution of (3.1) and $\lambda > 0$, then

$$(t,x) \to \left(\frac{1}{\lambda^{\frac{3}{2}}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda^{\frac{5}{2}}} \partial_t u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)\right)$$

is also a solution with initial data $\left((u_0)_\lambda, \lambda^{-1}(u_1)_\lambda\right)$ at time $t = 0$.

Recall that the function W defined by

$$W(x) = \left(1 + \frac{|x|^2}{15}\right)^{-\frac{3}{2}}, \quad \Delta W + W^{\frac{7}{3}} = 0, \quad x \in \mathbb{R}^5, \tag{3.2}$$

is a stationary solution of (3.1), called here *ground state*, or *soliton*. By scaling, translation invariances and change of sign, we obtain a family of stationary solutions of (3.1) defined by $W_{\lambda, x_0, \pm}(x) = \pm\lambda^{-\frac{3}{2}} W\left(\lambda^{-1}(x - x_0)\right)$, where $\lambda > 0$ and $x_0 \in \mathbb{R}^5$.

Using the Lorentz transformation, we obtain *traveling waves*. For $\boldsymbol{\ell} \in \mathbb{R}^5$, with $|\boldsymbol{\ell}| < 1$, let

$$W_{\boldsymbol{\ell}}(x) = W\left(\left(\frac{1}{\sqrt{1 - |\boldsymbol{\ell}|^2}} - 1\right) \frac{\boldsymbol{\ell}(\boldsymbol{\ell} \cdot x)}{|\boldsymbol{\ell}|^2} + x\right); \tag{3.3}$$

then $u(t,x) = \pm W_{\boldsymbol{\ell}}(x - \boldsymbol{\ell} t)$ is solution of (3.1).

In this paper, we prove the existence of solutions of (3.1) which display non trivial asymptotic behavior in the nonradial case. Indeed, multi-solitons are canonical objects behaving exactly as the sum of decoupled traveling solitons as $t \to +\infty$.

Theorem 3.1 (Existence of multi-solitons)**.** *Let $K \geqslant 2$. For $k \in \{1, \dots, K\}$, let $\lambda_k^\infty > 0$, $\mathbf{y}_k^\infty \in \mathbb{R}^5$, $\epsilon_k \in \{\pm 1\}$, $\boldsymbol{\ell}_k = \sum_{i=1}^5 \ell_{k,i} \mathbf{e}_i$ with $|\boldsymbol{\ell}_k| < 1$. Assume that $\boldsymbol{\ell}_k \neq \boldsymbol{\ell}_{k'}$ for $k \neq k'$, and*

$$\left(\sum_{i=1}^5 \left(\max_{k \in \{1,\dots,K\}} \ell_{k,i}^2 \right) \right)^{\frac{1}{2}} < \frac{3}{5}. \tag{3.4}$$

Then, there exist $T_0 > 0$ and a solution u of (3.1) on $[T_0, +\infty)$ in the energy space such that

$$\lim_{t \to +\infty} \left\| u(t) - \sum_{k=1}^K \frac{\epsilon_k}{(\lambda_k^\infty)^{3/2}} W_{\boldsymbol{\ell}_k} \left(\frac{\cdot - \boldsymbol{\ell}_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right) \right\|_{\dot{H}^1} = 0, \tag{3.5}$$

$$\lim_{t \to +\infty} \left\| \partial_t u(t) + \sum_{k=1}^K \frac{\epsilon_k}{(\lambda_k^\infty)^{5/2}} (\boldsymbol{\ell}_k \cdot \nabla W_{\boldsymbol{\ell}_k}) \left(\frac{\cdot - \boldsymbol{\ell}_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right) \right\|_{L^2} = 0. \tag{3.6}$$

The construction of multi-solitons for non integrable dispersive and wave equations has been the subject of several previous works. First, the existence of multi-solitons for the nonlinear Schrödinger (NLS) and the generalized Korteweg-de Vries (gKdV) equations was studied in the mass critical and subcritical cases by Merle [92], Martel [83], and Martel and Merle [84]. The proofs are partly based on standard stability properties of solitary waves (see references in these articles). A key point in [83] and [84] is the introduction of localized versions of the energy and mass functionals to deal with solutions containing several solitons. Such functionals are reminiscent of Martel, Merle and Tsai [88], where the stability of the sum of several solitons was studied in the energy space. For (gKdV), the technique also allows to prove the uniqueness of multi-solitons [83], property which is not available for (NLS). Next, the strategy of these works was extended to the case of exponentially unstable solitons: see Côte, Martel and Merle [28] for the construction of multi-solitons for supercritical (gKdV) and (NLS), and Combet [20] for a classification result for supercritical (gKdV). In these papers, the exponential instability of each soliton is controlled through a simple topological argument. For the nonlinear Klein-Gordon equation, the strategy was adapted by Côte and Muñoz [31] (for real-valued, unstable solitons) and Bellazzini, Ghimenti and Le Coz [5] (for complex-valued, stable solitons). For the water-wave system, see the work of Ming, Rousset and Tzvetkov [96].

Closely related to the present work, for the energy-critical wave equation in dimension 5, Martel and Merle [85] proved the existence of K-solitons for any $K \geqslant 2$ but only in the case where the speeds $\{\boldsymbol{\ell}_k\}_{k \in \{1,\cdots,k\}}$ are collinear. The Lorentz transform allows to remove this assumption in the special case $K = 2$. Technically, such constructions for the energy critical wave equation are more challenging than for (gKdV) and (NLS) (for which no limitation on the speeds appears) because of the weak decay of the solitons (3.2). Specific localized energy-momentum functionals are introduced to control solitons interactions in this context. In [85], the nature of the localization does not allow non-collinear multi-solitons, even under a smallness condition. The main improvements in the present work are the following: (1) the use of a refined approximate solution from [86]; (2) the introduction of a new localized energy functional to deal with non-collinear solitons. The smallness condition (3.4) is still probably technical and we conjecture that the result holds for any choice of speeds. However, by our method, (3.4) seems to be the best condition using the refined approximate solution from [86]. Moreover, constructing approximate solutions improving the one in [86] seems a difficult task.

The construction of K-solitons for the energy critical wave equation is especially motivated by the soliton resolution conjecture, fully solved in the 3D radial case by Duyckaerts, Kenig and Merle [46], and for a subsequence of time in 3, 4 and 5D by Duyckaerts, Jia, Kenig and Merle [43]. Indeed, the existence of such multi-solitons complements the statements in [43, 46] by exhibiting examples of solutions containing K solitons as $t \to +\infty$ for any $K \geqslant 2$. In the same direction, recall that for the energy-critical wave equation in dimension 6, Jendrej [67] proved the existence of radial two-bubble solutions.

3.1.2 Outline of the proof

The strategy of the proof of Theorem 3.1 combines the use of a refined approximate solution for the K-soliton problem from [86], with uniform estimates and compactness, as in [85]. See also references in [85, 86] for earlier works.

First, we introduce the refined approximate solution of the form $\vec{\mathbf{W}} = \sum_{k=1}^{K}(\vec{W}_k + c_k\vec{v}_k)$ for large $t > 0$, where for any $k \in \{1, \dots, K\}$, $\vec{W}_k$ is a soliton with two time-dependent parameters regarding scaling and translation respectively, and $\vec{v}_k$ is a correction term [86] improving the simpler approximate solution used in [85]. These correction terms of size t^{-2} in the energy space are solutions of non-homogeneous wave equations whose source terms are the main order of the nonlinear interactions of size t^{-3} between the K-solitons. In this way, $\vec{\mathbf{W}}$ is an approximate solution of the K-solitons problem at order t^{-4}. This is decisive for the construction under the smallness condition (3.4).

Let $S_n \to +\infty$ as $n \to +\infty$ and, for each n, let u_n be the backwards solution of (3.1) with initial data at time S_n

$$\vec{u}_n(S_n, x) \sim \sum_{k=1}^{K}\left(\vec{W}_k^{\infty}(S_n, x) + c_k\vec{V}_k^{\infty}(S_n, x)\right) \tag{3.7}$$

(See (3.49) for a precise definition of $\vec{u}_n(S_n)$). The goal is to prove that there exists a time T_0 independent of n, such that the following uniform estimates (3.8) hold on $[T_0, S_n]$,

$$\left\|\vec{u}_n(t) - \sum_{k=1}^{K}\vec{W}_k^{\infty}(t) - \sum_{k=1}^{K}c_k\vec{V}_k^{\infty}(t)\right\|_{\dot{H}^1\times L^2} \lesssim \frac{1}{t}. \tag{3.8}$$

The existence of a multi-soliton then follows easily from standard compactness arguments (note that $\vec{u}_n(T_0)$ converges strongly in the energy space $\dot{H}^1 \times L^2$, see §5). Thus, we now focus on the proof of (3.8).

We introduce

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}, \quad \vec{u}_n = \begin{pmatrix} u_n \\ \partial_t u_n \end{pmatrix} = \vec{\mathbf{W}} + \vec{\varepsilon},$$

where

$$\vec{\mathbf{W}} = \begin{pmatrix} \mathbf{W} \\ \mathbf{X} \end{pmatrix} = \sum_{k=1}^{K}\left(\vec{W}_k + c_k\vec{v}_k\right).$$

By a standard procedure, we choose modulation parameters $\lambda_k(t)$ and $\mathbf{y}_k(t)$ close to λ_k^{∞} and $\mathbf{y}_k^{\infty}$, and obtain suitable orthogonality conditions on $\vec{\varepsilon}$. The equation of $\vec{\varepsilon}$ is thus coupled with equations of $\lambda_k(t)$ and $\mathbf{y}_k(t)$. See Lemma 3.7.

To prove (3.8), we introduce the following energy functional

$$\mathcal{H} = \int \left\{|\nabla\varepsilon|^2 + |\eta|^2 - 2(F(\mathbf{W}+\varepsilon) - F(\mathbf{W}) - f(\mathbf{W})\varepsilon)\right\} + 2\int(\vec{\chi}\cdot\nabla\varepsilon)\eta$$

where the bounded function $\vec{\chi}$ is equal to $\boldsymbol{\ell}_k$ in a neighborhood of the soliton W_k and close to $\frac{x}{t}$ in "transition regions" between K solitons (see §5.3 for a precise definition). The specific choice of function $\vec{\chi}$ is the main novelty of this paper compared to [85, 86].

The functional $\mathcal{H}(t)$ has the following two important properties (see Lemma 3.12 for more precise statements):

(1) $\mathcal{H}$ is coercive, in the sense that (up to unstable directions, to be controlled separately) it controls the size of $\vec{\varepsilon}$ in the energy space

$$\mathcal{H} \sim \|\varepsilon\|_{\dot{H}^1}^2 + \|\eta\|_{L^2}^2.$$

(2) The variation of $\mathcal{H}(t)$ is controlled on $[T_0, S_n]$ in the following sense, for any $\delta > 0$ small enough,

$$-\frac{d}{dt}\left(t^{6-3\delta}\mathcal{H}\right) \lesssim \frac{1}{t^{1+\delta}}. \tag{3.9}$$

Therefore, integrating (3.9) on $[t, S_n]$, from (3.7), we find the uniform bound, for any $t \in [T_0, S_n]$,

$$\|\varepsilon\|_{\dot{H}^1}^2 + \|\eta\|_{L^2}^2 \lesssim t^{-3+\delta}. \tag{3.10}$$

Using time integration of the equation of the parameters, we can obtain (3.11) from the above estimate

$$|\lambda_k(t) - \lambda_k^\infty| \lesssim \frac{1}{t}, \quad |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \lesssim \frac{1}{t}. \tag{3.11}$$

Then (3.8) follows from (3.10), (3.11).

3.2 Notation and preliminaries

3.2.1 Notation

We denote

$$(g, \tilde{g})_{L^2} = \int g\tilde{g}, \quad \|g\|_{L^2}^2 = \int |g|^2, \quad (g, \tilde{g})_{\dot{H}^1} = \int \nabla g \cdot \nabla \tilde{g}, \quad \|g\|_{\dot{H}^1}^2 = \int |\nabla g|^2.$$

For

$$\vec{g} = \begin{pmatrix} g \\ h \end{pmatrix}, \ \vec{\tilde{g}} = \begin{pmatrix} \tilde{g} \\ \tilde{h} \end{pmatrix},$$

set

$$(\vec{g}, \vec{\tilde{g}})_{L^2} = (g, \tilde{g})_{L^2} + (h, \tilde{h})_{L^2}, \quad (\vec{g}, \vec{\tilde{g}})_{\dot{H}^1 \times L^2} = (g, \tilde{g})_{\dot{H}^1} + (h, \tilde{h})_{L^2}.$$

Set $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Let

$$\Lambda = \frac{3}{2} + x \cdot \nabla, \quad \widetilde{\Lambda} = \frac{5}{2} + x \cdot \nabla, \quad \widetilde{\Lambda}\nabla = \nabla\Lambda, \quad \vec{\Lambda} = \begin{pmatrix} \widetilde{\Lambda} \\ \Lambda \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $\boldsymbol{\ell} \in \mathbb{R}^5$ with $|\boldsymbol{\ell}| < 1$, denote

$$(g, \tilde{g})_{\dot{H}^1_{\boldsymbol{\ell}}} = \int (\nabla g \cdot \nabla \tilde{g} - (\boldsymbol{\ell} \cdot \nabla g)(\boldsymbol{\ell} \cdot \nabla \tilde{g}), \quad \|g\|_{\dot{H}^1_{\boldsymbol{\ell}}} = \|g\|_{\dot{H}^1}^2 - \|\boldsymbol{\ell} \cdot \nabla g\|_{L^2}$$

Note that if we define

$$g_{\boldsymbol{\ell}}(x) = g\left(\left(\frac{1}{\sqrt{1 - |\boldsymbol{\ell}|^2}} - 1\right) \frac{\boldsymbol{\ell}(\boldsymbol{\ell} \cdot x)}{|\boldsymbol{\ell}|^2} + x\right)$$

and similarly for $\tilde{g}$, $\tilde{g}_{\boldsymbol{\ell}}$, then

$$(g_{\boldsymbol{\ell}}, \tilde{g}_{\boldsymbol{\ell}})_{\dot{H}^1_{\boldsymbol{\ell}}} = (1 - |\boldsymbol{\ell}|^2)^{\frac{1}{2}} (g, \tilde{g})_{\dot{H}^1}.$$

Set

$$x_{\boldsymbol{\ell}} = \left(\frac{1}{\sqrt{1 - |\boldsymbol{\ell}|^2}} - 1\right) \frac{\boldsymbol{\ell}(\boldsymbol{\ell} \cdot x)}{|\boldsymbol{\ell}|^2} + x - \frac{\boldsymbol{\ell} t}{\sqrt{1 - |\boldsymbol{\ell}|^2}}, \quad A_{\boldsymbol{\ell}} = \partial_t + \boldsymbol{\ell} \cdot \nabla,$$

$$B_{\boldsymbol{\ell}} = \partial_t^2 - (\boldsymbol{\ell} \cdot \nabla) \cdot (\boldsymbol{\ell} \cdot \nabla) = A_{\boldsymbol{\ell}}^2 - 2(\boldsymbol{\ell} \cdot \nabla) A_{\boldsymbol{\ell}},$$

$$\Lambda_{\boldsymbol{\ell}} = \frac{3}{2} + (x - \boldsymbol{\ell} t) \cdot \nabla, \quad \Delta_{\boldsymbol{\ell}} = \Delta - (\boldsymbol{\ell} \cdot \nabla)(\boldsymbol{\ell} \cdot \nabla).$$

For $\alpha > 0$ small to be fixed later, set

$$\varphi_\alpha(x) = (1+|x|^2)^{-\alpha}. \tag{3.12}$$

We recall standard Sobolev and Hölder inequalities

$$\|u\|_{L^{10/3}} \lesssim \|u\|_{\dot H^1}, \quad \|u\|_{L^{10}} \lesssim \|u\|_{\dot H^2}, \tag{3.13}$$

$$\int |u||v||w|^{\frac{4}{3}} \lesssim \|u\|_{L^{10/3}}\|v\|_{L^{10/3}}\|w\|_{L^{10/3}}^{4/3} \lesssim \|u\|_{\dot H^1}\|v\|_{\dot H^1}\|w\|_{\dot H^1}^{4/3}, \tag{3.14}$$

$$\|uv\|_{L^{10/7}} \lesssim \|u\|_{L^{10/3}}\|u\|_{L^{5/2}}, \quad \|uvw\|_{L^{10/7}} \lesssim \|u\|_{L^{10/3}}\|v\|_{L^{10/3}}\|w\|_{L^{10}}. \tag{3.15}$$

Denote

$$f(u) = |u|^{\frac{4}{3}}u, \quad F(u) = \frac{3}{10}|u|^{\frac{10}{3}}.$$

3.2.2 Energy linearization around W

Let

$$L = -\Delta - \frac{7}{3}W^{\frac{4}{3}}, \quad (Lg,g) = \int \left(|\nabla g|^2 - \frac{7}{3}W^{\frac{4}{3}}g^2\right),$$

$$H = \begin{pmatrix} L & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \quad (H\vec{g},\vec{g}) = (Lg,g) + \|h\|_{L^2}^2 \quad \text{for} \quad \vec{g} = \begin{pmatrix} g \\ h \end{pmatrix}.$$

For $\vec{g}$ small in the energy space, we recall the expansion of the energy

$$E(W+g,h) = E(W,0) + \frac{1}{2}(H\vec{g},\vec{g})_{L^2} + O(\|g\|_{\dot H^1}^3). \tag{3.16}$$

We recall some technical properties of the operator L.

Lemma 3.2 ([85]). (i) Spectrum. *The operator L on L^2 with domain H^2 is a self-adjoint operator with essential spectrum $[0,+\infty)$, no positive eigenvalue and only one negative eigenvalue $-\lambda_0$, with a smooth radial positive eigenfunction $Y \in \mathcal{S}(\mathbb{R}^5)$. Moreover,*

$$L(\Lambda W) = L(\partial_{x_j} W) = 0, \quad \textit{for any } j = 1,\dots,5.$$

(ii) Localized coercivity. *For $\alpha > 0$ small enough, there exists $\mu > 0$ such that, for all $g \in \dot H^1$, the following holds.*

$$\int |\nabla g|^2\varphi^2 - f'(W)g^2 \geq \mu \int |\nabla g|^2\varphi^2 - \frac{1}{\mu}\left((g,\Lambda W)_{\dot H^1}^2 + \sum_{j=1}^5 (g,\partial_{x_j}W)_{\dot H^1}^2 + (g,Y)_{L^2}^2\right).$$

3.2.3 Energy linearization around $W_{\boldsymbol{\ell}}$

For $\boldsymbol{\ell} \in \mathbb{R}^5$ with $|\boldsymbol{\ell}| < 1$, $W_{\boldsymbol{\ell}}$ defined in (3.3) solves

$$\Delta W_{\boldsymbol{\ell}} - \boldsymbol{\ell}\cdot\nabla(\boldsymbol{\ell}\cdot\nabla W_{\boldsymbol{\ell}}) + W_{\boldsymbol{\ell}}^{\frac{7}{3}} = 0. \tag{3.17}$$

so that $u(t,x) = W_{\boldsymbol{\ell}}(x - \boldsymbol{\ell}t)$ is a solution of (3.1). Note that

$$E(W_{\boldsymbol{\ell}}, -\boldsymbol{\ell}\cdot\nabla W_{\boldsymbol{\ell}}) - \int |\boldsymbol{\ell}\cdot\nabla W_{\boldsymbol{\ell}}|^2 = (1-|\boldsymbol{\ell}|^2)^{\frac{1}{2}}E(W,0). \tag{3.18}$$

The following operators are related to the linearization of the energy around $W_{\boldsymbol{\ell}}$. Let

$$\begin{aligned}
&L_{\boldsymbol{\ell}} = -\Delta + \boldsymbol{\ell}\cdot\nabla(\boldsymbol{\ell}\cdot\nabla) - f'(W_{\boldsymbol{\ell}}),\\
&(L_{\boldsymbol{\ell}}g, g)_{L^2} = \int \nabla g\cdot\nabla g - (\boldsymbol{\ell}\cdot\nabla g)\cdot(\boldsymbol{\ell}\cdot\nabla g) - f'(W_{\boldsymbol{\ell}})g^2,\\
&H_{\boldsymbol{\ell}} = \begin{pmatrix} -\Delta - f'(W_{\boldsymbol{\ell}}) & -\boldsymbol{\ell}\cdot\nabla \\ \boldsymbol{\ell}\cdot\nabla & \mathrm{Id} \end{pmatrix}, \quad (H_{\boldsymbol{\ell}}\vec{g},\vec{g})_{L^2} = (L_{\boldsymbol{\ell}}g, g)_{L^2} + \|\boldsymbol{\ell}\cdot\nabla g + h\|_{L^2}^2
\end{aligned}$$

Indeed, proceeding as in (3.16), using (3.17) and (3.18), we obtain

$$\begin{aligned}
&E(W_{\boldsymbol{\ell}} + g, -\boldsymbol{\ell}\cdot\nabla W_{\boldsymbol{\ell}} + h) + \int \boldsymbol{\ell}\cdot\nabla(W_{\boldsymbol{\ell}} + g)(-\boldsymbol{\ell}\cdot\nabla W_{\boldsymbol{\ell}} + h)\\
&= (1-|\boldsymbol{\ell}|^2)^{\frac{1}{2}}E(W,0) + \frac{1}{2}(H_{\boldsymbol{\ell}}\vec{g},\vec{g})_{L^2}^2 + O(\|g\|_{\dot{H}^1}^3).
\end{aligned}$$

Set

$$\vec{Z}_{\boldsymbol{\ell}}^{\Lambda} = \begin{pmatrix} \Lambda W_{\boldsymbol{\ell}} \\ -\boldsymbol{\ell}\cdot\nabla(\Lambda W_{\boldsymbol{\ell}}) \end{pmatrix}, \quad \vec{Z}_{\boldsymbol{\ell}}^{\nabla_j} = \begin{pmatrix} \partial_{x_j} W_{\boldsymbol{\ell}} \\ -\boldsymbol{\ell}\cdot\nabla(\partial_{x_j} W_{\boldsymbol{\ell}}) \end{pmatrix}, \quad \vec{Z}_{\boldsymbol{\ell}}^{W} = \begin{pmatrix} W_{\boldsymbol{\ell}} \\ -\boldsymbol{\ell}\cdot\nabla W_{\boldsymbol{\ell}} \end{pmatrix},$$

$$Y_{\boldsymbol{\ell}} = Y\left(\left(\frac{1}{\sqrt{1-|\boldsymbol{\ell}|^2}} - 1\right)\frac{\boldsymbol{\ell}(\boldsymbol{\ell}\cdot x)}{|\boldsymbol{\ell}|^2} + x\right), \quad \vec{Z}_{\boldsymbol{\ell}}^{\pm} = \begin{pmatrix} \left(\boldsymbol{\ell}\cdot\nabla Y_{\boldsymbol{\ell}} \pm \frac{\sqrt{\lambda_0}}{\sqrt{1-|\boldsymbol{\ell}|^2}} Y_{\boldsymbol{\ell}}\right) e^{\pm\frac{\sqrt{\lambda_0}}{\sqrt{1-|\boldsymbol{\ell}|^2}}\boldsymbol{\ell}\cdot x} \\ Y_{\boldsymbol{\ell}} e^{\pm\frac{\sqrt{\lambda_0}}{\sqrt{1-|\boldsymbol{\ell}|^2}}\boldsymbol{\ell}\cdot x} \end{pmatrix}.$$

We recall following several technical facts.

Lemma 3.3 ([86]). *For $\boldsymbol{\ell}\in\mathbb{R}^5$ with $|\boldsymbol{\ell}|<1$,*

(i) Properties of $L_{\boldsymbol{\ell}}$.

$$L_{\boldsymbol{\ell}}(\Lambda W_{\boldsymbol{\ell}}) = L_{\boldsymbol{\ell}}(\partial_{x_j} W_{\boldsymbol{\ell}}) = 0, \quad L_{\boldsymbol{\ell}} Y_{\boldsymbol{\ell}} = -\lambda_0 Y_{\boldsymbol{\ell}}, \quad L_{\boldsymbol{\ell}} W_{\boldsymbol{\ell}} = -\frac{4}{3} W_{\boldsymbol{\ell}}^{\frac{7}{3}}.$$

(ii) Properties of $H_{\boldsymbol{\ell}}$ and $H_{\boldsymbol{\ell}}J$.

$$\begin{aligned}
&H_{\boldsymbol{\ell}}\vec{Z}_{\boldsymbol{\ell}}^{\Lambda} = H_{\boldsymbol{\ell}}\vec{Z}_{\boldsymbol{\ell}}^{\nabla_j} = 0, \quad H_{\boldsymbol{\ell}}\vec{Z}_{\boldsymbol{\ell}}^{W} = -\frac{4}{3}\begin{pmatrix} W_{\boldsymbol{\ell}}^{\frac{7}{3}} \\ 0 \end{pmatrix},\\
&\left(H_{\boldsymbol{\ell}}\vec{Z}_{\boldsymbol{\ell}}^{W}, \vec{Z}_{\boldsymbol{\ell}}^{W}\right) = -\frac{4}{3}\int W_{\boldsymbol{\ell}}^{\frac{10}{3}}, \quad -H_{\boldsymbol{\ell}}J(\vec{Z}_{\boldsymbol{\ell}}^{\pm}) = \pm\sqrt{\lambda_0}(1-|\boldsymbol{\ell}|^2)^{\frac{1}{2}}\vec{Z}_{\boldsymbol{\ell}}^{\pm},\\
&\left(\vec{Z}_{\boldsymbol{\ell}}^{\Lambda}, \vec{Z}_{\boldsymbol{\ell}}^{W}\right)_{\dot{H}_{\boldsymbol{\ell}}^1\times L^2} = \left(\vec{Z}_{\boldsymbol{\ell}}^{\nabla_j}, \vec{Z}_{\boldsymbol{\ell}}^{W}\right)_{\dot{H}_{\boldsymbol{\ell}}^1\times L^2} = 0, \quad \left(\vec{Z}_{\boldsymbol{\ell}}^{\Lambda}, \vec{Z}_{\boldsymbol{\ell}}^{\pm}\right) = \left(\vec{Z}_{\boldsymbol{\ell}}^{\nabla_j}, \vec{Z}_{\boldsymbol{\ell}}^{\pm}\right) = 0.
\end{aligned}$$

(iii) Coercivity. *For $\alpha>0$ small enough, there exists $\mu>0$ such that, for all $\vec{g}\in\dot{H}^1\times L^2$,*

$$\begin{aligned}
&\int\left(|\nabla g|^2\varphi_\alpha^2 - \frac{7}{3}W_{\boldsymbol{\ell}}^{\frac{4}{3}}g^2 + h^2\varphi_\alpha^2 + 2(\boldsymbol{\ell}\cdot\nabla g)h\varphi_\alpha^2\right)\\
&\qquad \geq \mu\int\left(|\nabla g|^2 + h^2\right)\varphi_\alpha^2 - \frac{1}{\mu}\left\{(g,\Lambda W_{\boldsymbol{\ell}})_{\dot{H}_{\boldsymbol{\ell}}^1}^2 + |(g,\nabla W_{\boldsymbol{\ell}})_{\dot{H}_{\boldsymbol{\ell}}^1}|^2 + (\vec{g},\vec{Z}_{\boldsymbol{\ell}}^{+})_{L^2}^2 + (\vec{g},\vec{Z}_{\boldsymbol{\ell}}^{-})_{L^2}^2\right\}.
\end{aligned}$$

3.2.4 Approximate solution to a non-homogeneous linearized equation

Let $|\boldsymbol{\ell}| < 1$ and F, G be defined by

$$F = W^{\frac{4}{3}} + \kappa_{\boldsymbol{\ell}} \Lambda W, \quad G = \kappa_{\boldsymbol{\ell}} (1 - |\boldsymbol{\ell}|^2)^{-\frac{1}{2}} \boldsymbol{\ell} \cdot \nabla \Lambda W, \quad \kappa_{\boldsymbol{\ell}} = -(1 - |\boldsymbol{\ell}|^2) \frac{(W^{\frac{4}{3}}, \Lambda W)}{\|\Lambda W\|_{L^2}^2} > 0.$$

Set

$$w_{\boldsymbol{\ell}}(t,x) = W(x_{\boldsymbol{\ell}}), \quad F_{\boldsymbol{\ell}}(t,x) = F(x_{\boldsymbol{\ell}}), \quad G_{\boldsymbol{\ell}}(t,x) = G(x_{\boldsymbol{\ell}}).$$

We recall following technical lemma in [86].

Lemma 3.4. *There exists a smooth function $v_{\boldsymbol{\ell}}$ such that, for all $0 < \delta < 1$ and all $t \geq 1$,*

$$\|(v_{\boldsymbol{\ell}}, \partial_t v_{\boldsymbol{\ell}})(t)\|_{\dot{H}^1 \times L^2} \lesssim t^{-2}, \quad \|v_{\boldsymbol{\ell}}(t)\|_{L^2} \lesssim t^{-\frac{3}{2}+\delta}, \quad \|\mathcal{E}_{\boldsymbol{\ell}}(t)\|_{H^1} \lesssim t^{-4+\delta},$$

where

$$\mathcal{E}_{\boldsymbol{\ell}} = \partial_t^2 v_{\boldsymbol{\ell}} - \Delta v_{\boldsymbol{\ell}} - \frac{7}{3} w_{\boldsymbol{\ell}}^{\frac{4}{3}} v_{\boldsymbol{\ell}} - f_{\boldsymbol{\ell}} - g_{\boldsymbol{\ell}}, \quad f_{\boldsymbol{\ell}} = t^{-3} F_{\boldsymbol{\ell}}, \quad g_{\boldsymbol{\ell}} = t^{-2} G_{\boldsymbol{\ell}}.$$

Moreover, for all $m \geq 0$, $|\alpha| = 1$, $|\alpha'| \geq 2$, $t \geq 1$, $x \in \mathbb{R}^5$,

$$\begin{aligned} |A_{\boldsymbol{\ell}}^m v_\ell(t,x)| &\lesssim (t + \langle x_{\boldsymbol{\ell}} \rangle)^{-1} t^{-(1+m)} \langle x_{\boldsymbol{\ell}} \rangle^{-2+\delta}, \\ |A_{\boldsymbol{\ell}}^m \partial^\alpha v_{\boldsymbol{\ell}}(t,x)| &\lesssim (t + \langle x_{\boldsymbol{\ell}} \rangle)^{-1} t^{-(1+m)} \langle x_{\boldsymbol{\ell}} \rangle^{-3+\delta}, \\ |A_{\boldsymbol{\ell}}^m \partial^{\alpha'} v_{\boldsymbol{\ell}}(t,x)| &\lesssim (t + \langle x_{\boldsymbol{\ell}} \rangle)^{-1} t^{-(1+m)} \langle x_{\boldsymbol{\ell}} \rangle^{-4+\delta}, \end{aligned} \tag{3.19}$$

$$|A_{\boldsymbol{\ell}}^m \partial_t v_{\boldsymbol{\ell}}| \lesssim |A_{\boldsymbol{\ell}}^{m+1} v_{\boldsymbol{\ell}}| + |A_{\boldsymbol{\ell}}^m \nabla v_{\boldsymbol{\ell}}| \lesssim t^{-(2+m)} \langle x_{\boldsymbol{\ell}} \rangle^{-3+\delta}, \tag{3.20}$$

and

$$\begin{aligned} &|A_{\boldsymbol{\ell}}^m \mathcal{E}_{\boldsymbol{\ell}}(t,x)| \lesssim t^{-(4+m)+\delta} \langle x_{\boldsymbol{\ell}} \rangle^{-3}, \quad |A_{\boldsymbol{\ell}}^m \partial^\alpha \mathcal{E}_{\boldsymbol{\ell}}(t,x)| \lesssim t^{-(4+m)+\delta} \langle x_{\boldsymbol{\ell}} \rangle^{-4}, \\ &|A_{\boldsymbol{\ell}}^m \partial^{\alpha'} \mathcal{E}_{\boldsymbol{\ell}}(t,x)| \lesssim t^{-(4+m)+\delta} \langle x_{\boldsymbol{\ell}} \rangle^{-5}. \end{aligned} \tag{3.21}$$

Proof. See proof of Lemma 3.1 in [86]. Although this lemma is proved only for $\boldsymbol{\ell} = \ell \mathbf{e}_1$, by rotation, we see that it holds for any $\boldsymbol{\ell} \in \mathbb{R}^5$ with $|\boldsymbol{\ell}| < 1$. □

3.3 Refined approximate solution of K-solitons problem

In this section, we construct a refined approximate solution to the K-soliton problem. Let $K \geqslant 2$ and for any $k \in \{1, \dots, K\}$, let

$$\lambda_k^\infty > 0, \quad \mathbf{y}_k^\infty \in \mathbb{R}^5, \quad \epsilon_k = \pm 1, \quad \boldsymbol{\ell}_k \in \mathbb{R}^5, \quad \text{where } |\boldsymbol{\ell}_k| < 1.$$

Let $C_0 \gg 1$ and $T_0 \gg 1$ to be fixed and $I \subset [T_0, +\infty)$ be an interval of $\mathbb{R}$. We assume that these functions satisfy, for all $t \in I$,

$$|\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \leq C_0 t^{-1}. \tag{3.22}$$

For $k \in \{1, \dots, K\}$, we consider $\mathcal{C}^1$ functions $\lambda_k > 0$, $\mathbf{y}_k \in \mathbb{R}^5$ defined on I. For $\vec{G} = (G, H)$, define

$$(\theta_k G)(t,x) = \frac{\epsilon_k}{\lambda_k^{\frac{3}{2}}(t)} G\left(\frac{x - \boldsymbol{\ell}_k t - \mathbf{y}_k(t)}{\lambda_k(t)}\right), \quad \vec{\theta}_k \vec{G} = \begin{pmatrix} \theta_k G \\ \frac{\theta_k}{\lambda_k} H \end{pmatrix}, \quad \vec{\tilde{\theta}}_k \vec{G} = \begin{pmatrix} \frac{\theta_k}{\lambda_k} G \\ \theta_k H \end{pmatrix}.$$

In particular, set

$$W_k = \theta_k W_{\boldsymbol{\ell}_k}, \quad X_k = -\boldsymbol{\ell}_k \cdot \nabla W_k, \quad \vec{W}_k = \begin{pmatrix} W_k \\ X_k \end{pmatrix}. \tag{3.23}$$

3.3.1 Main interaction terms

Expanding the nonlinearity $|u|^{\frac{4}{3}}u$ at $u = \sum_{k=1}^{K} W_k$, we prove that the main order of the nonlinear interactions is equivalent to the form $t^{-3}\sum_{k=1}^{K} c_k|W_k|^{\frac{4}{3}}$. The remaining error term is of size t^{-4}.

Lemma 3.5. *Assume* (3.22). *For* $k, k' \in \{1, 2, \dots, K\}$, $k \neq k'$, *let*

$$\boldsymbol{\sigma}_{k,k'} = \left(\frac{1}{\sqrt{1-|\boldsymbol{\ell}_{k'}|^2}} - 1\right)\frac{\boldsymbol{\ell}_{k'}(\boldsymbol{\ell}_{k'}\cdot(\boldsymbol{\ell}_k - \boldsymbol{\ell}_{k'}))}{|\boldsymbol{\ell}_{k'}|^2} + \boldsymbol{\ell}_k - \boldsymbol{\ell}_{k'},$$

and $c_k = \frac{7}{3}(15)^{\frac{3}{2}}\sum_{k'\neq k}\epsilon_{k'}(\lambda_{k'}^{\infty})^{\frac{3}{2}}|\boldsymbol{\sigma}_{k,k'}|^{-3}$. *Then,*

$$\left|\sum_{k=1}^{K} W_k\right|^{\frac{4}{3}}\left(\sum_{k=1}^{K} W_k\right) - \sum_{k=1}^{K}|W_k|^{\frac{4}{3}}W_k = t^{-3}\sum_{k=1}^{K} c_k|W_k|^{\frac{4}{3}} + \mathbf{R}_{\Sigma},$$

where, for all $t \in I$,

$$\|\mathbf{R}_{\Sigma}\|_{H^1} \lesssim t^{-4}. \tag{3.24}$$

Proof. The proof is similar to that of Lemma 3.1 in [86]. Let $\sigma = \frac{1}{10}\min_{k_1\neq k_2}|\boldsymbol{\ell}_{k_1} - \boldsymbol{\ell}_{k_2}|$ and $B_k(t) = \{x,\ |x - \boldsymbol{\ell}_k t| \leq \sigma t\}$, $B(t) = \cup_k B_k(t)$.

First, we claim, $\forall k \in \{1, \dots, K\}$ and $k \neq k'$

$$\|W_k^{\frac{7}{3}}\|_{L^2(B_k^c)} \lesssim t^{-4}, \quad \|W_k^{\frac{4}{3}}\|_{L^2(B_k^c)} \lesssim t^{-\frac{9}{2}} \tag{3.25}$$

and

$$\||W_k|^{\frac{1}{3}}|W_{k'}|^2\|_{L^2(B_k)} \lesssim t^{-\frac{9}{2}}, \quad \|W_{k'}^{\frac{7}{3}}\|_{L^2(B_k)} \lesssim t^{-\frac{9}{2}} \tag{3.26}$$

Note that, $\forall x \in B_k^c$, we have

$$|W_k|^{\frac{7}{3}} \lesssim \langle x - \boldsymbol{\ell}_k t\rangle^{-7} \lesssim t^{-4}\langle x - \boldsymbol{\ell}_k t\rangle^{-3}, \quad |W_k|^{\frac{4}{3}} \lesssim \langle x - \boldsymbol{\ell}_k t\rangle^{-4} \lesssim t^{-1}\langle x - \boldsymbol{\ell}_k t\rangle^{-3},$$

and from $x \mapsto \langle x\rangle^{-3} \in L^2(\mathbb{R}^5)$, we obtain (3.25). Note that, $\forall x \in B_k$ and $k' \neq k$

$$|W_k|^{\frac{1}{3}}|W_{k'}|^2 \lesssim t^{-6}\langle x - \boldsymbol{\ell}_k t\rangle, \quad |W_{k'}|^{\frac{7}{3}} \lesssim t^{-\frac{9}{2}} \tag{3.27}$$

Integrating (3.27) on $B_k(t)$, we obtain (3.26).

Second, we claim, $\forall\ k \in \{1, \dots, K\}$,

$$\||\sum_{k'=1}^{K} W_{k'}|^{\frac{4}{3}}(\sum_{k'=1}^{K} W_{k'}) - |W_k|^{\frac{4}{3}}W_k - \frac{7}{3}|W_k|^{\frac{4}{3}}\sum_{k'\neq k} W_{k'}\|_{L^2(B_k)} \lesssim t^{-\frac{9}{2}} \tag{3.28}$$

From Taylor expansion, we have,

$$\begin{aligned}&\left|\sum_{k'=1}^{K} W_{k'}\right|^{\frac{4}{3}}(\sum_{k'=1}^{K} W_{k'})\\ &= |W_k|^{\frac{4}{3}}W_k + \frac{7}{3}|W_k|^{\frac{4}{3}}\sum_{k'\neq k} W_{k'} + O\left(|W_k|^{\frac{1}{3}}\sum_{k'\neq k}|W_{k'}|^2\right) + O\left(\sum_{k'\neq k}|W_{k'}|^{\frac{7}{3}}\right).\end{aligned} \tag{3.29}$$

Using (3.26) and (3.29), we obtain (3.28).

Third, we claim, $k' \neq k$,

$$\left\||W_k|^{\frac{4}{3}}\left(W_{k'}(t,x) - W_{k'}(t,\boldsymbol{\ell}_k t)\right)\right\|_{L^2(B_k)} \lesssim t^{-4}. \tag{3.30}$$

Indeed, for $x \in B_k$,

$$|W_{k'}(t,x) - W_{k'}(t,\boldsymbol{\ell}_k t)| \lesssim \sup_{B_k} |\nabla W_{k'}(t)| \cdot |x - \boldsymbol{\ell}_k t| \lesssim t^{-4}|x - \boldsymbol{\ell}_k t|,$$

and so,

$$|W_k|^{\frac{4}{3}} |W_{k'}(t,x) - W_{k'}(t,\boldsymbol{\ell}_k t)| \lesssim t^{-4}\langle x - \boldsymbol{\ell}_k t\rangle^{-3},$$

which implies (3.30).

Then, note that, from the explict expression (3.2) of W, we have,

$$|W(x) - 15^{\frac{3}{2}}|x|^{-3}| \lesssim |x|^{-5}, \quad \text{for } |x| \gg 1. \tag{3.31}$$

Thus, using the assumption on the parameters (3.22) and the definition (3.23) of $W_{k'}$ and (3.31), we obtain

$$\begin{aligned} W_{k'}(t,\boldsymbol{\ell}_k t) &= \frac{\epsilon_{k'}}{\lambda_{k'}^{\frac{3}{2}}(t)} W_{\ell_{k'}}\left(\frac{(\boldsymbol{\ell}_k - \boldsymbol{\ell}_{k'})t - \mathbf{y}_{k'}(t)}{\lambda_{k'}(t)}\right) = \frac{\epsilon_{k'}}{(\lambda_{k'}^{\infty})^{\frac{3}{2}}} W_{\ell_{k'}}\left(\frac{(\boldsymbol{\ell}_k - \boldsymbol{\ell}_{k'})t}{\lambda_{k'}^{\infty}}\right) + O(t^{-4}) \\ &= \frac{\epsilon_{k'}}{(\lambda_{k'}^{\infty})^{\frac{3}{2}}} W\left(\frac{\boldsymbol{\sigma}_{k,k'}t}{\lambda_{k'}^{\infty}}\right) + O(t^{-4}) = 15^{\frac{3}{2}}\epsilon_{k'}(\lambda_{k'}^{\infty})^{\frac{3}{2}}|\boldsymbol{\sigma}_{k,k'}|^{-3}t^{-3} + O(t^{-4}). \end{aligned}$$

Gathering above estimates, we obtain the L^2 estimate of $\mathbf{R}_\Sigma$ in (3.24).

Now, we observe for $j \in \{1, \cdots, 5\}$,

$$\begin{aligned} \partial_{x_j}\mathbf{R}_\Sigma = \frac{7}{3}\left|\sum_{k=1}^K W_k\right|^{\frac{4}{3}}\left(\sum_{k=1}^K \partial_{x_j} W_k\right) - \frac{7}{3}\sum_{k=1}^K \left(|W_k|^{\frac{4}{3}}\,\partial_{x_j} W_k\right) \\ - \frac{4}{3}t^{-3}\sum_{k=1}^K c_k\left(|W_k|^{-\frac{2}{3}}\,W_k\right)\partial_{x_j} W_k. \end{aligned}$$

From Taylor expansion and the decay of W and $\partial_{x_j} W$, we have, for $k \in \{1, \cdots, K\}$,

$$\begin{aligned} &\left|\sum_{k'=1}^K W_{k'}\right|^{\frac{4}{3}}\left(\sum_{k'=1}^K \partial_{x_j} W_{k'}\right) \\ &= |W_k|^{\frac{4}{3}}\partial_{x_j} W_k + \frac{4}{3}\left((|W_k|^{-\frac{2}{3}}W_k)\partial_{x_j} W_k\right)\left(\sum_{k'\neq k} W_{k'}\right) \\ &+ O\big(|W_k|\sum_{k'\neq k}|W_{k'}|^{\frac{4}{3}}\big) + O\big(|W_k|^{\frac{4}{3}}\sum_{k'\neq k}|\partial_{x_j} W_{k'}|\big) + O\big(\sum_{k'\neq k}|W_{k'}|^{\frac{7}{3}}\big). \end{aligned}$$

And then, using similar arguments, we prove $\dot{H}^1$ estimate of $\mathbf{R}_\Sigma$ in (3.24). □

3.3.2 The approximate solution $\vec{\mathbf{W}}$

We recall a result on approximate solution $\vec{\mathbf{W}}$ in 5D from [86]. To remove the main interaction terms $\sum_{k=1}^K c_k t^{-3}|W_k|^{\frac{4}{3}}$ computed in Lemma 3.5, we define suitably rescaled versions of the function $v_{\boldsymbol{\ell}_k}$ given by Lemma 3.4. Let

$$v_k(t,x) = \frac{1}{\lambda_k^3} v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x - \mathbf{y}_k}{\lambda_k}\right), \tag{3.32}$$

$$z_k(t,x) = \frac{1}{\lambda_k^4}(\partial_t v_{\boldsymbol{\ell}_k})\left(\frac{t}{\lambda_k}, \frac{x - \mathbf{y}_k}{\lambda_k}\right) + \frac{\kappa_{\boldsymbol{\ell}_k}\epsilon_k}{2\lambda_k^{\frac{1}{2}}t^2}\Lambda_k W_k(t,x) \tag{3.33}$$

where $\Lambda_k = \frac{3}{2} + (x - \boldsymbol{\ell}_k t - \mathbf{y}_k)\cdot\nabla$, and

$$\vec{v}_k = \begin{pmatrix} v_k \\ z_k \end{pmatrix}, \quad \kappa_{\boldsymbol{\ell}_k} = -(1-|\boldsymbol{\ell}_k|^2)\frac{(W^{\frac{4}{3}}, \Lambda W)}{\|\Lambda W\|_{L^2}^2}, \quad a_k = -\frac{c_k \kappa_{\boldsymbol{\ell}_k}\epsilon_k}{2}.$$

Set

$$\vec{\mathbf{W}} = \begin{pmatrix} \mathbf{W} \\ \mathbf{X} \end{pmatrix} = \sum_{k=1}^{K}\left(\vec{W}_k + c_k \vec{v}_k\right). \tag{3.34}$$

Lemma 3.6. *The function $\vec{\mathbf{W}}$ satisfies on $I\times\mathbb{R}^5$*

$$\begin{cases} \partial_t \mathbf{W} = \mathbf{X} - \mathrm{Mod}_{\mathbf{W}} - \mathrm{Mod}_{\mathbf{V}} \\ \partial_t \mathbf{X} = \Delta \mathbf{W} + |\mathbf{W}|^{\frac{4}{3}}\mathbf{W} - \mathrm{Mod}_{\mathbf{X}} - \mathrm{Mod}_{\mathbf{Z}} - \mathbf{R}_{\mathbf{X}} \end{cases} \tag{3.35}$$

where

$$\mathrm{Mod}_{\mathbf{W}} = \sum_{k=1}^{K}\left(\frac{\dot\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2}\right)\Lambda_k W_k + \sum_{k=1}^{K}\dot{\mathbf{y}}_k\cdot\nabla W_k$$

$$\mathrm{Mod}_{\mathbf{X}} = -\sum_{k=1}^{K}\left(\frac{\dot\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2}\right)(\boldsymbol{\ell}_k\cdot\nabla)\Lambda_k W_k - \sum_{k=1}^{K}(\dot{\mathbf{y}}_k\cdot\nabla)(\boldsymbol{\ell}_k\cdot\nabla)W_k,$$

and

$$\begin{aligned}
\mathrm{Mod}_{\mathbf{V}} = &\sum_{k=1}^{K}\left(3\frac{\dot\lambda_k}{\lambda_k^4} v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right) + \frac{\dot\lambda_k}{\lambda_k^4}\frac{t}{\lambda_k} A_{\boldsymbol{\ell}_k} v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right)\right) \\
&+ \sum_{k=1}^{K}\left(\frac{\dot\lambda_k}{\lambda_k^4}\left(\frac{x-\boldsymbol{\ell}_k t-\mathbf{y}_k}{\lambda_k}\right)\cdot\nabla v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right) + \frac{\dot{\mathbf{y}}_k}{\lambda_k^4}\cdot\nabla v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right)\right), \\
\mathrm{Mod}_{\mathbf{Z}} = &\sum_{k=1}^{K}\left(\frac{\dot{\mathbf{y}}_k}{\lambda_k^5}\cdot\nabla\partial_t v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right) + 4\frac{\dot\lambda_k}{\lambda_k^5}\partial_t v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right)\right) \\
&+ \sum_{k=1}^{K}\left(\frac{\dot\lambda_k}{\lambda_k^5}\frac{t}{\lambda_k} A_{\boldsymbol{\ell}_k}\partial_t v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right) + \frac{\dot\lambda_k}{\lambda_k^5}\left(\frac{x-\boldsymbol{\ell}_k t-\mathbf{y}_k}{\lambda_k}\right)\cdot\nabla\partial_t v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k}, \frac{x-\mathbf{y}_k}{\lambda_k}\right)\right) \\
&+ \sum_{k=1}^{K}\left(\frac{\kappa_{\boldsymbol{\ell}_k}\epsilon_k}{2t^2\lambda_k^{\frac{1}{2}}}\frac{\dot\lambda_k}{\lambda_k}\left(\frac{1}{2}\Lambda_k W_k + \Lambda_k^2 W_k\right) + \frac{\kappa_{\boldsymbol{\ell}_k}\epsilon_k}{2t^2\lambda_k^{\frac{1}{2}}}\dot{\mathbf{y}}_k\cdot\nabla\Lambda_k W_k\right),
\end{aligned}$$

$$\vec{\mathbf{R}} = \begin{pmatrix} 0 \\ \mathbf{R}_{\mathbf{X}} \end{pmatrix}, \quad \|\vec{\mathbf{R}}\|_{\dot H^1\times L^2} + \|\nabla\vec{\mathbf{R}}\|_{\dot H^1\times L^2} \lesssim t^{-4+\delta}. \tag{3.36}$$

Moreover, for all $0<\delta<1$,

$$\begin{aligned}
&|\mathbf{W}| + \langle x-\boldsymbol{\ell}_k t\rangle|\nabla\mathbf{W}| \lesssim \sum_{k=1}^{K}\left(\langle x-\boldsymbol{\ell}_k t\rangle^{-3} + t^{-1}\langle x-\boldsymbol{\ell}_k t\rangle^{-3+\delta}\right), \\
&|\mathbf{X}| \lesssim \sum_{k=1}^{K}\left(\langle x-\boldsymbol{\ell}_k t\rangle^{-4} + t^{-2}\langle x-\boldsymbol{\ell}_k t\rangle^{-3+\delta}\right).
\end{aligned} \tag{3.37}$$

Proof. See proof of Lemma 4.3 in [86]. Although the original argument in this lemma only holds for $\boldsymbol{\ell}_k = \ell_k\mathbf{e}_1$ and $K=2$, after using Lemma 2.3 and Lemma 3.1, we easily check that the argument still holds for any $K\geqslant 2$ and any $\boldsymbol{\ell}_k\in\mathbb{R}^5$ with $|\boldsymbol{\ell}_k|<1$. □

3.4 Decomposition around refined approximate solution

We prove in this section a general decomposition around refined approximate solution. Let $K \geqslant 2$ and for any $k \in \{1, \dots, K\}$, let $\lambda_k^\infty > 0$, $\mathbf{y}_k^\infty \in \mathbb{R}^5$, $\boldsymbol{\ell}_k \in \mathbb{R}^5$, $|\boldsymbol{\ell}_k| < 1$ with $\boldsymbol{\ell}_k \neq \boldsymbol{\ell}_{k'}$ for $k' \neq k$. For $\vec{G} = (G, H)$, set

$$(\theta_k^\infty G)(t,x) = \frac{\epsilon_k}{(\lambda_k^\infty)^{\frac{3}{2}}} G\left(\frac{x - \boldsymbol{\ell}_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty}\right), \quad \vec{\theta}_k^\infty \vec{G} = \begin{pmatrix} \theta_k^\infty G \\ \frac{\theta_k^\infty}{\lambda_k^\infty} H \end{pmatrix},$$

$$W_k^\infty = \theta_k^\infty W_{\boldsymbol{\ell}_k}, \quad X_k^\infty = -\boldsymbol{\ell}_k \cdot \nabla W_k^\infty, \quad \vec{W}_k^\infty = \begin{pmatrix} W_k^\infty \\ X_k^\infty \end{pmatrix}, \quad \Lambda_k^\infty = \frac{3}{2} + (x - \boldsymbol{\ell}_k t - \mathbf{y}_k^\infty).$$

We also set

$$v_k^\infty = \frac{1}{(\lambda_k^\infty)^3} v_{\boldsymbol{\ell}_k}\left(\frac{t}{\lambda_k^\infty}, \frac{x - \mathbf{y}_k^\infty}{\lambda_k^\infty}\right),$$
$$z_k^\infty = \frac{1}{(\lambda_k^\infty)^4} (\partial_t v_{\boldsymbol{\ell}_k})\left(\frac{t}{\lambda_k^\infty}, \frac{x - \mathbf{y}_k^\infty}{\lambda_k^\infty}\right) + \frac{\kappa_{\boldsymbol{\ell}_k} \epsilon_k}{2(\lambda_k^\infty)^{\frac{1}{2}} t^2} \Lambda_k^\infty W_k^\infty(t,x), \quad \vec{V}_k^\infty = \begin{pmatrix} v_k^\infty \\ z_k^\infty \end{pmatrix}.$$

Lemma 3.7 (Properties of the decomposition)**.** *There exist $T_0 \gg 1$ and $0 < \delta_0 \ll 1$ such that if $u(t)$ is a solution of (3.1) which satisfies on I,*

$$\left\| \vec{u} - \sum_{k=1}^K \left(\vec{W}_k^\infty + c_k \vec{V}_k^\infty\right) \right\|_{\dot{H}^1 \times L^2} < \delta_0, \tag{3.38}$$

then there exist $\mathcal{C}^1$ functions $\lambda_k > 0$, $\mathbf{y}_k$ on I such that, $\vec{\varepsilon}(t)$ being defined by

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \vec{\mathbf{W}} + \vec{\varepsilon}, \tag{3.39}$$

the following hold on I, for $k \in \{1, \dots, K\}$.

(i) First properties of the decomposition. *For $j = 1, \dots, 5$,*

$$(\varepsilon, \Lambda_k W_k)_{\dot{H}^1_{\boldsymbol{\ell}_k}} = \left(\varepsilon, \partial_{x_j} W_k\right)_{\dot{H}^1_{\boldsymbol{\ell}_k}} = 0, \tag{3.40}$$

$$|\lambda_k - \lambda_k^\infty| + |\mathbf{y}_k - \mathbf{y}_k^\infty| + \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} \lesssim \left\| \vec{u} - \sum_{k=1}^K \left(\vec{W}_k^\infty + c_k \vec{V}_k^\infty\right) \right\|_{\dot{H}^1 \times L^2} \tag{3.41}$$

(ii) Equation of $\vec{\varepsilon}$.

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_{\mathbf{W}} + \mathrm{Mod}_{\mathbf{V}} \\ \partial_t \eta = \Delta \varepsilon + |\mathbf{W} + \varepsilon|^{\frac{4}{3}} (\mathbf{W} + \varepsilon) - |\mathbf{W}|^{\frac{4}{3}} \mathbf{W} + \mathrm{Mod}_{\mathbf{X}} + \mathrm{Mod}_{\mathbf{Z}} + \mathbf{R}_{\mathbf{X}}. \end{cases} \tag{3.42}$$

(iii) Parameter estimates. *For any $0 < \delta < 1$,*

$$\left| \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2} \right| + |\dot{\mathbf{y}}_k| \lesssim \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} + \frac{1}{t^4}. \tag{3.43}$$

(iv) Unstable directions. *Let* $z_k^{\pm} = (\vec{\varepsilon}, \vec{\tilde{\theta}}_k \vec{Z}_{\boldsymbol{\ell}_k}^{\pm})$. *Then, for any* $0 < \delta < 1$,

$$\left| \frac{d}{dt} z_k^{\pm} \mp \frac{\sqrt{\lambda_0}}{\lambda_k} (1 - |\boldsymbol{\ell}_k|^2)^{\frac{1}{2}} z_k^{\pm} \right| \lesssim \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2}^2 + t^{-1} \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} + t^{-4+\delta} \tag{3.44}$$

Proof. **Step 1.** Decomposition. The existence of parameters λ_k and $\mathbf{y}_k$ such that (3.40) and (3.41) hold is proved similarly as (i) of Lemma 3.1 in [85].

Step 2. Equation of $\vec{\varepsilon}$. We formally derive the equation of $\vec{\varepsilon}(t)$, $\lambda_k(t)$ and $\mathbf{y}_k(t)$ from (3.1) and (3.35).

$$\varepsilon_t = \partial_t u - \partial_t \mathbf{W} = \eta + \mathbf{X} - \partial_t \mathbf{W} = \eta + \mathrm{Mod}_{\mathbf{W}} + \mathrm{Mod}_{\mathbf{V}}.$$

Second, since $\eta = \partial_t u - \mathbf{X}$, we have

$$\begin{aligned} \partial_t \eta &= \partial_t^2 u - \partial_t \mathbf{X} = \Delta u + |u|^{\frac{4}{3}} u - \Delta \mathbf{W} - |\mathbf{W}|^{\frac{4}{3}} \mathbf{W} + \mathrm{Mod}_{\mathbf{X}} + \mathrm{Mod}_{\mathbf{Z}} + \mathbf{R}_{\mathbf{X}} \\ &= \Delta \varepsilon + |\mathbf{W} + \varepsilon|^{\frac{4}{3}} (\mathbf{W} + \varepsilon) - |\mathbf{W}|^{\frac{4}{3}} \mathbf{W} + \mathrm{Mod}_{\mathbf{X}} + \mathrm{Mod}_{\mathbf{Z}} + \mathbf{R}_{\mathbf{X}}. \end{aligned}$$

We also denote

$$\mathbf{R}_{\mathrm{NL}} = |\mathbf{W} + \varepsilon|^{\frac{4}{3}} (\mathbf{W} + \varepsilon) - |\mathbf{W}|^{\frac{4}{3}} \mathbf{W} - \frac{7}{3} \sum_k |W_k|^{\frac{4}{3}} \varepsilon = \mathbf{R}_1 + \mathbf{R}_2,$$

$$\mathbf{R}_1 = \frac{7}{3} \left(|\mathbf{W}|^{\frac{4}{3}} - \sum_k |W_k|^{\frac{4}{3}} \right) \varepsilon, \quad \mathbf{R}_2 = |\mathbf{W} + \varepsilon|^{\frac{4}{3}} (\mathbf{W} + \varepsilon) - |\mathbf{W}|^{\frac{4}{3}} \mathbf{W} - \frac{7}{3} |\mathbf{W}|^{\frac{4}{3}} \varepsilon,$$

and

$$\vec{\mathcal{L}} = \begin{pmatrix} 0 & 1 \\ \Delta + \frac{7}{3} \sum_k |W_k|^{\frac{4}{3}} & 0 \end{pmatrix}, \quad \vec{\mathrm{Mod}}_1 = \begin{pmatrix} \mathrm{Mod}_{\mathbf{W}} \\ \mathrm{Mod}_{\mathbf{X}} \end{pmatrix}, \quad \vec{\mathrm{Mod}}_2 = \begin{pmatrix} \mathrm{Mod}_{\mathbf{V}} \\ \mathrm{Mod}_{\mathbf{Z}} \end{pmatrix},$$

$$\vec{\mathbf{R}} = \begin{pmatrix} 0 \\ \mathbf{R}_{\mathbf{X}} \end{pmatrix}, \quad \vec{\mathbf{R}}_{\mathrm{NL}} = \begin{pmatrix} 0 \\ \mathbf{R}_{\mathrm{NL}} \end{pmatrix}, \quad \vec{\mathbf{R}}_1 = \begin{pmatrix} 0 \\ \mathbf{R}_1 \end{pmatrix}, \quad \vec{\mathbf{R}}_2 = \begin{pmatrix} 0 \\ \mathbf{R}_2 \end{pmatrix}.$$

With this notation, the system (3.42) rewrites

$$\partial_t \vec{\varepsilon} = \vec{\mathcal{L}} \vec{\varepsilon} + \vec{\mathrm{Mod}}_1 + \vec{\mathrm{Mod}}_2 + \vec{\mathbf{R}} + \vec{\mathbf{R}}_{\mathrm{NL}} = \vec{\mathcal{L}} \vec{\varepsilon} + \vec{\mathrm{Mod}}_1 + \vec{\mathrm{Mod}}_2 + \vec{\mathbf{R}} + \vec{\mathbf{R}}_1 + \vec{\mathbf{R}}_2. \tag{3.45}$$

We claim the following estimates on $\mathbf{R}_1$ and $\mathbf{R}_2$

$$\|\mathbf{R}_1\|_{L^{\frac{10}{7}}} \lesssim t^{-1} \|\varepsilon\|_{L^{\frac{10}{3}}}, \quad |\mathbf{R}_2| \lesssim |\mathbf{W}|^{\frac{1}{3}} \varepsilon^2 + |\varepsilon|^{\frac{7}{3}}, \quad \|\mathbf{R}_2\|_{L^{\frac{10}{7}}} \lesssim \|\varepsilon\|_{L^{\frac{10}{3}}}^2 \lesssim \|\varepsilon\|_{\dot{H}^1}^2. \tag{3.46}$$

The estimate on $\mathbf{R}_2$ follows from (3.15). To prove the estimate on $\mathbf{R}_1$, we first recall the inequality, for $p > 1$, for any reals (r_k),

$$\left| \left| \sum r_k \right|^p - \sum |r_k|^p \right| \lesssim \sum_{k' \neq k} |r_{k'}| |r_k|^{p-1}.$$

Therefore,

$$\begin{aligned} \left| |\mathbf{W}|^{\frac{4}{3}} - \sum_{k=1}^K |W_k|^{\frac{4}{3}} \right| &\lesssim \left| |\mathbf{W}|^{\frac{4}{3}} - \left| \sum_{k=1}^K W_k \right|^{\frac{4}{3}} \right| + \left| \left| \sum_{k=1}^K W_k \right|^{\frac{4}{3}} - \sum_{k=1}^K |W_k|^{\frac{4}{3}} \right| \\ &\lesssim \left(\sum_{k=1}^K |v_k| \right) \left(\sum_{k=1}^K (|W_k| + |v_k|) \right)^{\frac{1}{3}} + \sum_{k' \neq k} |W_k| |W_{k'}|^{\frac{1}{3}}, \end{aligned}$$

and thus

$$|\mathbf{R}_1| \lesssim |\varepsilon| \left(\sum_{k=1}^{K} |v_k|\right) \left(\sum_{k=1}^{K} (|W_k| + |v_k|)\right)^{\frac{1}{3}} + |\varepsilon| \sum_{k' \neq k} |W_k||W_{k'}|^{\frac{1}{3}}.$$

By (3.15), we obtain

$$\|\mathbf{R}_1\|_{L^{\frac{10}{7}}} \lesssim \|\varepsilon\|_{L^{\frac{10}{3}}} \left(\sum_{k=1}^{K} \|v_k\|_{L^{\frac{10}{3}}}\right) \left(\sum_{k=1}^{K} \left(\|W_k\|_{L^{\frac{10}{3}}}^{\frac{1}{3}} + \|v_k\|_{L^{\frac{10}{3}}}^{\frac{1}{3}}\right)\right) + \|\varepsilon\|_{L^{\frac{10}{3}}} \sum_{k' \neq k} \left\| W_k |W_{k'}|^{\frac{1}{3}} \right\|_{L^{\frac{5}{2}}}.$$

By (3.19), we have $\|v_k\|_{L^{\frac{10}{3}}} \lesssim t^{-2}$. Moreover $\|W_k|W_{k'}|^{\frac{1}{3}}\|_{L^{\frac{5}{2}}} \lesssim t^{-1}$ is a consequence of the following technical result.

Claim 3.8 (Claim 2 in [85]). *Let $0 < r_2 \leq r_1$ be such that $r_1 + r_2 > \frac{5}{3}$. For t large, if $r_1 > \frac{5}{3}$ then $\int |W_1|^{r_1}|W_2|^{r_2} \lesssim t^{-3r_2}$, whereas if $r_1 \leq \frac{5}{3}$ then $\int |W_1|^{r_1}|W_2|^{r_2} \lesssim t^{5-3(r_1+r_2)}$.*

Step 3. Parameter estimates. Now, we derive the equations of λ_k and $\mathbf{y}_k$ from the orthogonality conditions (3.40). First,

$$\frac{d}{dt}(\varepsilon, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} = (\partial_t \varepsilon, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} + (\varepsilon, \partial_t (\Lambda_1 W_1))_{\dot{H}^1_{\boldsymbol{\ell}_1}} = 0.$$

Thus, using the first line of (3.42), and the expression of $\mathrm{Mod}_{\mathbf{W}}$ in Lemma 3.6,

$$\begin{aligned} 0 = &(\eta, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} - (\varepsilon, \boldsymbol{\ell}_1 \cdot \nabla(\Lambda_1 W_1))_{\dot{H}^1_{\boldsymbol{\ell}_1}} - \frac{\dot{\lambda}_1}{\lambda_1} (\varepsilon, \Lambda_1^2 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} - (\varepsilon, \dot{\mathbf{y}}_1 \cdot \nabla \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} \\ &+ \left(\frac{\dot{\lambda}_1}{\lambda_1} - \frac{a_1}{\lambda_1^{\frac{1}{2}} t^2}\right) (\Lambda_1 W_1, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} + (\dot{\mathbf{y}}_1 \cdot \nabla W_1, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} \\ &+ \sum_{k=2}^{K} \left(\frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2}\right) (\Lambda_k W_k, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} + \sum_{k=2}^{K} (\dot{\mathbf{y}}_k \cdot \nabla W_k, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} + (\mathrm{Mod}_{\mathbf{V}}, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}}. \end{aligned}$$

By the decay properties of W, we obtain

$$\left|(\eta, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}}\right| + \left|(\varepsilon, \nabla(\Lambda_1 W_1))_{\dot{H}^1_{\boldsymbol{\ell}_1}}\right| + \left|(\varepsilon, \Lambda_1^2 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}}\right| + \left|(\varepsilon, \nabla \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}}\right| \lesssim \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2}.$$

Next, $(\Lambda_1 W_1, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} = (1 - |\boldsymbol{\ell}_1|^2)^{\frac{1}{2}} \|\Lambda W\|_{\dot{H}^1}^2$ and by parity, $(\nabla W_1, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} = 0$. Using Claim 3.8, we have, for $k \in \{2, \ldots, K\}$,

$$\left|(\Lambda_k W_k, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}}\right| + \left|(\nabla W_k, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}}\right| \lesssim t^{-3}.$$

And then, using the expression of $\mathrm{Mod}_{\mathbf{V}}$ and (3.19), we obtain

$$\left|(\mathrm{Mod}_{\mathbf{V}}, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}}}\right| \lesssim \frac{1}{t^2} \sum_{k=1}^{K} \left(\left|\frac{\dot{\lambda}_k}{\lambda_k}\right| + |\dot{\mathbf{y}}_k|\right) \lesssim \frac{1}{t^2} \sum_{k=1}^{K} \left(\left|\frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2}\right| + |\dot{\mathbf{y}}_k|\right) + t^{-4}.$$

In conclusion of the previous estimates, the orthogonality condition $(\varepsilon, \Lambda_1 W_1)_{\dot{H}^1_{\boldsymbol{\ell}_1}} = 0$, gives the following

$$\left|\frac{\dot{\lambda}_1}{\lambda_1} - \frac{a_1}{\lambda_2^{\frac{1}{2}} t^2}\right| \lesssim \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} + \left(\|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} + t^{-2}\right) \sum_{k=1}^{K} \left(\left|\frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2}\right| + |\dot{\mathbf{y}}_k|\right) + t^{-4}.$$

Using the other orthogonality conditions, we obtain similarly,

$$\begin{aligned}&\sum_{k=1}^{K}\left(\left|\frac{\dot\lambda_k}{\lambda_k}-\frac{a_k}{\lambda_k^{\frac12}t^2}\right|+|\dot{\mathbf{y}}_k|\right)\\&\lesssim\|\vec\varepsilon\|_{\dot H^1\times L^2}+\left(\|\vec\varepsilon\|_{\dot H^1\times L^2}+t^{-2}\right)\sum_{k=1}^{K}\left(\left|\frac{\dot\lambda_k}{\lambda_k}-\frac{a_k}{\lambda_k^{\frac12}t^2}\right|+|\dot{\mathbf{y}}_k|\right)+t^{-4}.\end{aligned}$$

Therefore, for δ_0 small enough and T_0 large enough, we find (3.43).

Step 4. Equations of the unstable directions. We prove the case of z_1^+. The other case is similar. Recall that $\vec Z_{\boldsymbol\ell_k}^{\pm}\in\mathcal{S}$ by their definition in §3.2.3. By (3.45), we have

$$\begin{aligned}\frac{d}{dt}z_1^+&=\frac{d}{dt}\left(\vec\varepsilon,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)=\left(\partial_t\vec\varepsilon,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)+\left(\vec\varepsilon,\partial_t\left(\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)\right)\\&=\left(\vec{\mathcal{L}}\vec\varepsilon,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)-\frac{\boldsymbol\ell_1}{\lambda_1}\cdot\left(\vec\varepsilon,\vec{\tilde\theta}_1\nabla\vec Z_{\ell_1}^+\right)-\frac{\dot\lambda_1}{\lambda_1}\left(\vec\varepsilon,\vec{\tilde\theta}_1\vec\Lambda\vec Z_{\boldsymbol\ell_1}^+\right)-\frac{\dot{\mathbf{y}}_1}{\lambda_1}\cdot\left(\vec\varepsilon,\vec{\tilde\theta}_1\nabla\vec Z_{\boldsymbol\ell_1}^+\right)\\&\quad+\left(\vec{\mathrm{Mod}}_1,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)+\left(\vec{\mathrm{Mod}}_2,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)+\left(\vec{\mathbf{R}},\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)+\left(\vec{\mathbf{R}}_{\mathrm{NL}},\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right).\end{aligned}$$

First, by direct computations, using (ii) of Lemma 3.3,

$$\begin{aligned}\left(\vec{\mathcal{L}}\vec\varepsilon,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)-\frac{\boldsymbol\ell_1}{\lambda_1}\cdot\left(\vec\varepsilon,\vec{\tilde\theta}_1\nabla\vec Z_{\boldsymbol\ell_1}^+\right)&=\frac{1}{\lambda_1}\left(\vec\varepsilon,\vec{\tilde\theta}_1\left(-H_{\boldsymbol\ell_1}J\vec Z_{\boldsymbol\ell_1}^+\right)\right)+\sum_{k\geqslant2}\left(\varepsilon,f'(W_k)(\theta_1Z_{\boldsymbol\ell_1}^+)\right)\\&=\frac{\sqrt{\lambda_0}}{\lambda_1}(1-|\boldsymbol\ell_1|^2)^{\frac12}z_1^++\sum_{k\geqslant2}\left(\varepsilon,f'(W_k)(\theta_1Z_{\boldsymbol\ell_1}^+)\right).\end{aligned}$$

By the decay properties of $\vec Z_{\boldsymbol\ell_1}^+$ and Claim 3.8,

$$\sum_{k\geqslant2}\left|\left(\varepsilon,f'(W_k)(\theta_1Z_{\boldsymbol\ell_1}^{\pm})\right)\right|\lesssim t^{-4}\|\varepsilon\|_{\dot H^1}.$$

Next, by (3.43),

$$\left|\frac{\dot\lambda_1}{\lambda_1}\right|\left|\left(\vec\varepsilon,\vec{\tilde\theta}_1\vec\Lambda\vec Z_{\boldsymbol\ell_1}^+\right)\right|+\left|\frac{\dot{\mathbf{y}}_1}{\lambda_1}\cdot\left(\vec\varepsilon,\vec{\tilde\theta}_1\nabla\vec Z_{\boldsymbol\ell_1}^+\right)\right|\lesssim\|\vec\varepsilon\|_{\dot H^1\times L^2}^2+t^{-2}\|\vec\varepsilon\|_{\dot H^1\times L^2}.$$

Concerning the term with $\vec{\mathrm{Mod}}_1$. From (ii) of Lemma 3.3, we obtain

$$(\vec\theta_1\vec Z_{\boldsymbol\ell_1}^\Lambda,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+)=(\vec Z_{\boldsymbol\ell_1}^\Lambda,\vec Z_{\boldsymbol\ell_1}^+)=0,\quad(\vec\theta_1\vec Z_{\boldsymbol\ell_1}^\nabla,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+)=(\vec Z_{\boldsymbol\ell_1}^\nabla,\vec Z_{\boldsymbol\ell_1}^+)=0.$$

Moreover, by Claim 3.8, we have

$$\sum_{k=2}^{K}\left(\left|\left(\vec\theta_k\vec Z_{\boldsymbol\ell_k}^\Lambda,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)\right|+\left|\left(\vec\theta_k\vec Z_{\boldsymbol\ell_k}^\nabla,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)\right|\right)\lesssim t^{-3},$$

and thus, by (3.43),

$$\sum_{k=2}^{K}\left(\left|\frac{\dot\lambda_k}{\lambda_k}-\frac{a_k}{\lambda_k^{\frac12}t^2}\right|\left|\left(\vec\theta_k\vec Z_{\boldsymbol\ell_k}^\Lambda,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)\right|+\left|\frac{\dot{\mathbf{y}}_k}{\lambda_k}\cdot\left(\vec\theta_k\vec Z_{\boldsymbol\ell_k}^\nabla,\vec{\tilde\theta}_1\vec Z_{\boldsymbol\ell_1}^+\right)\right|\right)\lesssim t^{-3}\|\vec\varepsilon\|_{\dot H^1\times L^2}+t^{-7}.$$

Concerning the term with $\vec{\mathrm{Mod}}_2$. From (3.19), (3.43) and the definition of $\vec{\mathrm{Mod}}_2$, we obtain

$$\left|\left(\vec{\mathrm{Mod}}_2, \vec{\tilde{\theta}}_1 \vec{Z}^+_{\boldsymbol{\ell}_1}\right)\right| \lesssim \frac{1}{t^2} \sum_{k=1}^{K} \left(\left|\frac{\dot{\lambda}_k}{\lambda_k}\right| + |\dot{\mathbf{y}}_k| \right) \lesssim t^{-2} \|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} + t^{-4}$$

Finally, we claim

$$\left|\left(\vec{\mathbf{R}}, \vec{\tilde{\theta}}_1 \vec{Z}^+_{\boldsymbol{\ell}_1}\right)\right| + \left|\left(\vec{\mathbf{R}}_{\mathrm{NL}}, \vec{\tilde{\theta}}_1 \vec{Z}^+_{\boldsymbol{\ell}_1}\right)\right| \lesssim t^{-4+\delta} + t^{-1}\|\varepsilon\|_{\dot{H}^1} + \|\varepsilon\|^2_{\dot{H}^1}.$$

Indeed, from (3.36), we have $|(\vec{\mathbf{R}}, \vec{\tilde{\theta}}_1 \vec{Z}^+_{\boldsymbol{\ell}_1})| \lesssim t^{-4+\delta}$. Second, by (3.46) and the decay of Y, we have

$$\left|\left(\vec{\mathbf{R}}_1, \vec{\tilde{\theta}}_1 \vec{Z}^+_{\boldsymbol{\ell}_1}\right)\right| \lesssim \|\mathbf{R}_1\|_{\frac{10}{7}} \lesssim t^{-1}\|\varepsilon\|_{\dot{H}^1} \quad \text{and} \quad \left|\left(\vec{\mathbf{R}}_2, \vec{\tilde{\theta}}_1 \vec{Z}^+_{\boldsymbol{\ell}_1}\right)\right| \lesssim \|\mathbf{R}_2\|_{\frac{10}{7}} \lesssim \|\varepsilon\|^2_{\dot{H}^1}$$

Gathering these estimates, we obtain (3.44). The proof of Lemma 3.7 is complete. □

3.5 Proof of Theorem 3.1

To construct the K-soliton solution at $+\infty$, we follow the strategy of [85] using the refined approximate solution $\vec{\mathbf{W}}$ defined in the previous section. We argue by compactness and obtain the solution $u(t)$ as the limit of a sequence of approximate multi-solitons $u_n(t)$.

Proposition 3.9. *There exist $T_0 > 0$ and a solution $u(t)$ of (3.1) on $[T_0, +\infty)$ satisfying, for all $0 < \delta < 1$, for all $t \in [T_0, +\infty)$,*

$$\|\nabla u(t) - \nabla \mathbf{W}(t)\|_{L^2} + \|\partial_t u(t) - \mathbf{X}(t)\|_{L^2} \lesssim t^{-3+\delta} \tag{3.47}$$

where $\lambda_k(t)$, $\mathbf{y}_k(t)$ are such that, for all $t \in [T_0, +\infty)$,

$$|\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \lesssim t^{-1}. \tag{3.48}$$

This section is devoted to the proof of Proposition 3.9. Note that Proposition 3.9 implies Theorem 3.1.

Let $S_n \to +\infty$. Let $\zeta^\pm_{k,n} \in \mathbb{R}$ small to be determined later. These free parameters correspond to two exponentially stable/unstable directions for each soliton - see statements of Proposition 3.10, Claim 3.11 and Lemma 3.14. For any large n, we consider the solution u_n of

$$\begin{cases} \partial_t^2 u_n - \Delta u_n - |u_n|^{\frac{4}{3}} u_n = 0 \\ (u_n(S_n), \partial_t u_n(S_n))^\mathsf{T} = \displaystyle\sum_{k=1}^{K} \left(\vec{W}_k^\infty(S_n) + c_k \vec{V}_k^\infty(S_n) + \sum_{\pm} \zeta^\pm_{k,n} (\vec{\theta}_k^\infty \vec{Z}^\pm_{\boldsymbol{\ell}_k})(S_n) \right). \end{cases} \tag{3.49}$$

Note that since $(u_n(S_n), \partial_t u_n(S_n)) \in \dot{H}^1 \times L^2$, the solution $\vec{u}_n$ is well-defined in $\dot{H}^1 \times L^2$ at least on a small interval of time around S_n.

Now, we state uniform estimates on u_n backwards in time up to some uniform $T_0 \gg 1$.

Proposition 3.10. *There exist $n_0 > 0$ and $T_0 > 0$ such that, for any $n \geq n_0$, there exist $(\zeta^\pm_{k,n})_{k \in \{1,\dots,K\}} \in \mathbb{R}^K \times \mathbb{R}^K$, with*

$$\sum_{k=1}^{K} |\zeta^+_{k,n}|^2 + \sum_{k=1}^{K} |\zeta^-_{k,n}|^2 \lesssim S_n^{-7},$$

and such that the solution $\vec{u}_n = (u_n, \partial_t u_n)^\mathsf{T}$ *of* (3.49) *is well-defined in* $\dot{H}^1 \times L^2$ *on the time interval* $[T_0, S_n]$ *and satisfies, for all* $t \in [T_0, S_n]$,

$$\left\| \vec{u}_n(t) - \vec{\mathbf{W}}_n(t) \right\|_{\dot{H}^1 \times L^2} \lesssim t^{-3+\delta} \tag{3.50}$$

and for all $\nu > 0$, *there exists* $M > 0$ *such that, for all* $n \geq n_0$,

$$\|\vec{u}_n(T_0)\|_{(\dot{H}^1 \times L^2)(|x|>M)} < \nu, \tag{3.51}$$

where $\vec{\mathbf{W}}_n(t,x) = \vec{\mathbf{W}}\left(t, x; \{\lambda_{k,n}(t)\}, \{\mathbf{y}_{k,n}(t)\}\right)$ *is defined in §3.4, and*

$$|\lambda_{k,n}(t) - \lambda_k^\infty| + |\mathbf{y}_{k,n}(t) - \mathbf{y}_k^\infty| \lesssim t^{-1}, \quad \left|\dot{\lambda}_{k,n}(t)\right| + |\dot{\mathbf{y}}_{k,n}(t)| \lesssim t^{-2}. \tag{3.52}$$

Moreover, $\vec{u}_n \in \mathcal{C}([T_0, S_n], \dot{H}^2 \times \dot{H}^1)$ *and satisfies, for all* $t \in [T_0, S_n]$, $\|\vec{u}_n(t)\|_{\dot{H}^2 \times \dot{H}^1} \lesssim 1$.

3.5.1 Proof of Proposition 3.9, assuming Proposition 3.10

In view of the estimates obtained in Proposition 3.10 on $(\vec{u}_n(T_0))$ and (3.51), up to the extraction of a subsequence, $(\vec{u}_n(T_0))$ converges strongly in $\dot{H}^1 \times L^2$ to some $(u_0, u_1)^\mathsf{T}$ as $n \to +\infty$. Consider the solution $u(t)$ of (3.1) associated to the initial data $(u_0, u_1)^\mathsf{T}$ at $t = T_0$. Then, by the uniform bounds (3.50) and the continuous dependence of the solution of (3.1) with respect to its initial data in the energy space $\dot{H}^1 \times L^2$ (see *e.g.* [71] and references therein), the solution u is well-defined in the energy space on $[T_0, \infty)$.

Recall that we denote by $\lambda_{k,n}$ and $\mathbf{y}_{k,n}$ the parameters of the decomposition of u_n on $[T_0, S_n]$. By the uniform estimates in (3.52), using Ascoli's theorem and a diagonal argument, it follows that there exist continuous functions λ_k and $\mathbf{y}_k$ such that up to the extraction of a subsequence, $\lambda_{k,n} \to \lambda_k$, $\mathbf{y}_{k,n} \to \mathbf{y}_k$ uniformly on compact sets of $[T_0, +\infty)$. Moreover on $[T_0, +\infty)$,

$$|\lambda_k(t) - \lambda_k^\infty| \lesssim t^{-1}, \quad |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \lesssim t^{-1}.$$

Passing to the limit in (3.50) for any $t \in [T_0, +\infty)$, we finish the proof of Proposition 3.9.

The rest of this section is devoted to the proof of Proposition 3.10.

3.5.2 Bootstrap setting

We denote by $\mathcal{B}_{\mathbb{R}^K}(r)$ (respectively, $\mathcal{S}_{\mathbb{R}^K}(r)$) the open ball (respectively, the sphere) of $\mathbb{R}^K$ of center 0 and of radius $r > 0$, for the norm $|(\xi_k)_k| = (\sum_{k=1}^K \xi_k^2)^{1/2}$.

For $t = S_n$ and for $t < S_n$ as long as $u_n(t)$ is well-defined in $\dot{H}^1 \times L^2$ and satisfies (3.38), we decompose $u_n(t)$ as in Lemma 3.7. In particular, we denote by (ε, η), $(\lambda_k)_k$, $(\mathbf{y}_k)_k$, $(z_k^\pm)_k$ the parameters of the decomposition of u_n.

We start with a technical result similar to Lemma 3 in [28]. This claim will allow us to adjust the initial values of $(z_k^\pm(S_n))_k$ from the choice of $\zeta_{k,n}^\pm$ in (3.49).

Claim 3.11 (Choosing the initial unstable modes)**.** *There exist* $n_0 > 0$ *and* $C > 0$ *such that, for all* $n \geq n_0$, *for any* $(\xi_k)_{k \in \{1,\dots,K\}} \in \overline{\mathcal{B}}_{\mathbb{R}^K}(S_n^{-7/2})$, *there exists a unique* $(\zeta_{k,n}^\pm)_{k \in \{1,\dots,K\}} \in \mathcal{B}_{\mathbb{R}^{2K}}(CS_n^{-7/2})$ *such that the decomposition of* $u_n(S_n)$ *satisfies*

$$z_k^-(S_n) = \xi_k, \quad z_k^+(S_n) = 0, \tag{3.53}$$

$$|\lambda_k(S_n) - \lambda_k^\infty| + |\mathbf{y}_k(S_n) - \mathbf{y}_k^\infty| + \|\vec{\varepsilon}(S_n)\|_{\dot{H}^1 \times L^2} + \|\vec{\varepsilon}(S_n)\|_{\dot{H}^2 \times \dot{H}^1} \lesssim S_n^{-7/2}. \tag{3.54}$$

Sketch of the proof of Claim 3.11. The proof of existence of $(\zeta_{k,n}^\pm)_k$ in Claim 3.11 is similar to Lemma 3 in [28] and we omit it. Estimates in (3.54) are consequences of (3.41). □

The proof of Proposition 3.10 is based on the following bootstrap estimates, for some $0 < \delta \ll 1$ to be fixed later, $k \in \{1, \dots, K\}$ and C_0 to be chosen,

$$\begin{cases} |\lambda_k(t) - \lambda_k^\infty| \leq C_0 t^{-1}, \quad |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \leq C_0 t^{-1}, \\ |z_k^\pm(t)|^2 \leq t^{-7}, \quad \|\vec{\varepsilon}(t)\|_{\dot{H}^1 \times L^2} \leq C_0 t^{-3+\delta}. \end{cases} \tag{3.55}$$

Set

$$T^* = T_n^*((\xi_k)_k) = \inf\{t \in [T_0, S_n] \ ; \ u_n \text{ satisfies (3.38) and (3.55) holds on } [t, S_n]\}. \tag{3.56}$$

In what follows, we will prove that there exists T_0 large enough and at least one choice of $(\xi_k)_k \in \mathcal{B}_{\mathbb{R}^K}(S_n^{-7/2})$ so that $T^* = T_0$, which is enough to finish the proof of Proposition 3.10. For this, we derive general estimates for any $(\xi_k)_k \in \overline{\mathcal{B}}_{\mathbb{R}^K}(S_n^{-7/2})$ (see Lemma 3.13) and use a topological argument (see Lemma 3.14) to control the unstable directions, in order to strictly improve (3.55) on $[T^*, S_n]$.

As a consequence of the bootstrap estimates (3.43) and (3.55), we have, for $k \in \{1, \dots, K\}$,

$$\left| \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}} t^2} \right| + |\dot{\mathbf{y}}_k| \lesssim \|\vec{\varepsilon}(t)\|_{\dot{H}^1 \times L^2} + t^{-4+\delta} \lesssim C_0 t^{-3+\delta}.$$

In particular, from the expression of $\mathrm{Mod}_{\mathbf{V}}$ and $\mathrm{Mod}_{\mathbf{Z}}$ in Lemma 3.6, (3.19), (3.20), (3.43) and (3.55), we have

$$\|\mathrm{Mod}_{\mathbf{V}}\|_{\dot{H}^2 \cap \dot{H}^1} \leqslant t^{-4} \quad \text{and} \quad \|\mathrm{Mod}_{\mathbf{Z}}\|_{H^1} \leqslant t^{-4+\delta}. \tag{3.57}$$

From the expression of $\mathrm{Mod}_{\mathbf{W}}$ and $\mathrm{Mod}_{\mathbf{X}}$ in Lemma 3.6, for all $\alpha \in \mathbb{N}^5$,

$$|\partial_x^\alpha \mathrm{Mod}_{\mathbf{W}}(t)| \lesssim t^{-3+\delta} \sum_{k=1}^K |W_k|^{1+\frac{|\alpha|}{3}}, \quad |\partial_x^\alpha \mathrm{Mod}_{\mathbf{X}}(t)| \lesssim t^{-3+\delta} \sum_{k=1}^K |W_k|^{1+\frac{1+|\alpha|}{3}}. \tag{3.58}$$

3.5.3 Energy functional

Recall that $f(u) = |u|^{\frac{4}{3}} u$ and $F(u) = \frac{3}{10}|u|^{\frac{10}{3}}$. Let

$$\forall k \in \{1, \dots, K\}, \quad \boldsymbol{\ell}_k = \sum_{i=1}^5 \ell_{k,i} \mathbf{e}_i \quad \text{where} \quad |\boldsymbol{\ell}_k| < 1, \tag{3.59}$$

and denote (see (3.4))

$$\overline{\ell} = \Big(\sum_{i=1}^5 \big(\max_{k \in \{1, \cdots, K\}} \ell_{k,i}^2 \big) \Big)^{\frac{1}{2}} < \frac{3}{5}. \tag{3.60}$$

We fix $\delta < \frac{1}{100}$ small enough, such that

$$\begin{aligned} &\overline{\ell} \leqslant \left(\frac{(2-4\delta)(8-4\delta)}{(6-4\delta)^2} \right)^{\frac{1}{2}}, \quad \overline{\ell} \leqslant \left(\frac{(1-4\delta)(9-4\delta)}{(5-4\delta)^2} \right)^{\frac{1}{2}}, \\ &\overline{\ell} \leqslant \frac{(5-4\delta)}{(6-4\delta)}, \quad \overline{\ell} \leqslant \left(\frac{4-4\delta}{6-4\delta} \right)^{\frac{1}{2}}, \quad \overline{\ell} \leqslant \left(\frac{(3-4\delta)(7-4\delta)}{(6-4\delta)^2} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.61}$$

Moreover, we set,

$$\forall i \in \{1, \dots, 5\} \quad -1 < \overline{\ell}_{1,i} < \cdots < \overline{\ell}_{K_i,i} < 1 \quad \text{such that } \{\ell_{1,i}, \cdots, \ell_{K,i}\} = \{\overline{\ell}_{1,i} \cdots, \overline{\ell}_{K_i,i}\}.$$

We denote $I = \{i | i \in \{1, \cdots, 5\} \text{ and } K_i \geqslant 2\}$. For

$$0 < \sigma < \frac{1}{10} \min_{i \in I} (\overline{\ell}_{k+1,i} - \overline{\ell}_{k,i})$$

small enough to be fixed, we set

$$\begin{aligned} \text{for } k = 1, \ldots, K_i - 1, \quad & \overline{\ell}^+_{k,i} = \overline{\ell}_{k,i} + \sigma(\overline{\ell}_{k+1,i} - \overline{\ell}_{k,i}), \\ \text{for } k = 2, \ldots, K_i, \quad & \overline{\ell}^-_{k,i} = \overline{\ell}_{k,i} - \sigma(\overline{\ell}_{k,i} - \overline{\ell}_{k-1,i}), \end{aligned}$$

and for $t > 0$, we denote,

$$\Omega_{1,i}(t) = ((\overline{\ell}^+_{1,i} t, \overline{\ell}^-_{2,i} t) \cup \ldots \cup (\overline{\ell}^+_{K_i-1,i} t, \overline{\ell}^-_{K_i,i} t)), \quad \Omega_{0,i}(t) = \mathbb{R} \setminus \Omega_{1,i}(t) \quad \text{for } i \in I.$$

and

$$\Omega_{1,i}(t) = \emptyset, \quad \Omega_{0,i}(t) = \mathbb{R} \quad \text{for } i \notin I.$$

When $i \in I$, we consider the continuous function $\chi_i(t,x) = \chi_i(t, x_i)$ defined as follows, for all $t > 0$,

$$\begin{cases} \chi_i(t,x) = \overline{\ell}_{1,i} \text{ for } x_i \in (-\infty, \overline{\ell}^+_{1,i} t], \\ \chi_i(t,x) = \overline{\ell}_{k,i} \text{ for } x_i \in [\overline{\ell}^-_{k,i} t, \overline{\ell}^+_{k,i} t], \text{ for } k \in \{2, \ldots, K_i - 1\}, \\ \chi_i(t,x) = \overline{\ell}_{K_i,i} \text{ for } x_i \in [\overline{\ell}^-_{K_i,i} t, +\infty), \\ \chi_i(t,x) = \dfrac{x_i}{(1-2\sigma)t} - \dfrac{\sigma}{1-2\sigma}(\overline{\ell}_{k+1,i} + \overline{\ell}_{k,i}) \text{ for } x_i \in [\overline{\ell}^+_{k,i} t, \overline{\ell}^-_{k+1,i} t],\ k \in \{1, \ldots, K_i - 1\}. \end{cases} \tag{3.62}$$

In particular,

$$\begin{cases} \partial_t \chi_i(t,x) = 0, \quad \nabla \chi_i(t,x) = 0, \quad \text{on } \mathbb{R}^{i-1} \times \Omega_{0,i}(t) \times \mathbb{R}^{5-i}, \\ \partial_{x_i} \chi_i(t,x) = \dfrac{1}{(1-2\sigma)t} \quad \text{for } x \in \mathbb{R}^{i-1} \times \Omega_{1,i}(t) \times \mathbb{R}^{5-i}, \\ \partial_t \chi_i(t,x) = -\dfrac{1}{t} \dfrac{x_i}{(1-2\sigma)t} \quad \text{for } x \in \mathbb{R}^{i-1} \times \Omega_{1,i}(t) \times \mathbb{R}^{5-i}. \end{cases} \tag{3.63}$$

When $i \notin I$, we consider the continuous function $\chi_i(t,x) = \chi_i(t,x_i) = \ell_{1,i}$, for all $t > 0$.
We denote

$$\Omega = \bigcup_{\sum_i s_i \neq 0} \Omega_{s_1,1} \times \cdots \times \Omega_{s_5,5} \quad \text{and} \quad \vec{\chi} = (\chi_1, \cdots, \chi_5).$$

The choice of $\vec{\chi}$ in this paper is different from that in [85, 86] to take into account non-colinear speeds.

We define (see [88, 85, 86, 31] for similar energy functional)

$$\mathcal{H} = \int \left\{ |\nabla \varepsilon|^2 + |\eta|^2 - 2(F(\mathbf{W} + \varepsilon) - F(\mathbf{W}) - f(\mathbf{W})\varepsilon) \right\} + 2 \int (\vec{\chi} \cdot \nabla \varepsilon) \eta.$$

Lemma 3.12. *There exists $\mu > 0$ such that, for $t \in [T^*, S_n]$, the following hold.*

(i) Bound.

$$|\mathcal{H}(t)| \leq \frac{\|\vec{\varepsilon}(t)\|^2_{\dot{H}^1 \times L^2}}{\mu}. \tag{3.64}$$

(ii) Coercivity.

$$\mathcal{H}(t) \geq \mu \|\vec{\varepsilon}(t)\|^2_{\dot{H}^1 \times L^2} - \frac{t^{-7}}{\mu}. \tag{3.65}$$

(iii) Time variation. *For all $0 < \delta < \frac{1}{100}$ small enough to satisfy (3.61),*

$$-\frac{d}{dt}\left(t^{6-3\delta} \mathcal{H}\right) \leqslant \frac{1}{\mu} C_0 t^{-1-\delta}. \tag{3.66}$$

Proof of Lemma 3.12. **Proof of (3.64).** Since

$$|F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon| \lesssim |\varepsilon|^{\frac{10}{3}}+|\varepsilon|^2|\mathbf{W}|^{\frac{4}{3}},$$

estimate (3.64) on $\mathcal{H}$ follows from (3.13), (3.14) and $\|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}+\|\mathbf{W}\|_{\dot{H}^1}\lesssim 1$.

Proof of (3.65). Set

$$\mathcal{N}_\Omega(t)=\int_\Omega\left(|\nabla\varepsilon(t)|^2+\eta^2(t)+2(\vec{\chi}(t)\cdot\nabla\varepsilon(t))\eta(t)\right),\quad \mathcal{N}_{\Omega^C}(t)=\int_{\Omega^C}\left(|\nabla\varepsilon(t)|^2+\eta^2(t)\right).$$

Note that, since $|\vec{\chi}|<\bar{\ell}$,

$$\mathcal{N}_\Omega\geqslant\int_\Omega\left(|\nabla\varepsilon|^2+\eta^2\right)-2\bar{\ell}\int_\Omega|\nabla\varepsilon||\eta|\geqslant(1-\bar{\ell})\int_\Omega\left(|\nabla\varepsilon|^2+\eta^2\right). \tag{3.67}$$

To obtain (3.65), we claim the following estimates, for some small $\alpha>0$

$$\mathcal{H}(t)\geq\mathcal{N}_\Omega(t)+\mu\mathcal{N}_{\Omega^C}(t)-\frac{t^{-7}}{\mu}-\frac{t^{-4\alpha}}{\mu}\|\vec{\varepsilon}\|^2_{\dot{H}^1\times L^2}-\frac{t^{-1}}{\mu}\|\vec{\varepsilon}\|^2_{\dot{H}^1\times L^2}-\frac{1}{\mu}\|\vec{\varepsilon}\|^3_{\dot{H}^1\times L^2}. \tag{3.68}$$

To prove (3.68), we decompose $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3$, where

$$\begin{aligned}
\mathcal{H}_1&=\int|\nabla\varepsilon|^2-\sum_{k=1}^K\int f'(W_k)\varepsilon^2+\int\eta^2+2\int(\vec{\chi}\cdot\nabla\varepsilon)\eta,\\
\mathcal{H}_2&=-2\int\left(F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon-\frac{1}{2}f'(\mathbf{W})\varepsilon^2\right),\\
\mathcal{H}_3&=\int\left(\sum_{k=1}^K f'(W_k)-f'(\mathbf{W})\right)\varepsilon^2.
\end{aligned}$$

We claim the following estimates

$$\mathcal{H}_1\geq\mathcal{N}_\Omega+\mu\mathcal{N}_{\Omega^C}-\frac{t^{-7}}{\mu}-\frac{t^{-4\alpha}}{\mu}\|\vec{\varepsilon}\|^2_{\dot{H}^1\times L^2}, \tag{3.69}$$

$$|\mathcal{H}_2|+|\mathcal{H}_3|\lesssim\|\vec{\varepsilon}\|^3_{\dot{H}^1\times L^2}+t^{-1}\|\vec{\varepsilon}\|^2_{\dot{H}^1\times L^2}. \tag{3.70}$$

which imply (3.68) for T_0 large enough.

Proof of (3.69). For φ_α defined in (3.12), set

$$\varphi_k(t,x)=\varphi_\alpha\left(\frac{x-\boldsymbol{\ell}_k t-\mathbf{y}_k(t)}{\lambda_k(t)}\right).$$

We decompose $\mathcal{H}_1$ as follows

$$\begin{aligned}
\mathcal{H}_1=\mathcal{N}_\Omega&+\sum_{k=1}^K\left(\int|\nabla\varepsilon|^2\varphi_k^2-\int f'(W_k)\varepsilon^2+\int\eta^2\varphi_k^2+2\int(\vec{\chi}\cdot\nabla\varepsilon)\eta\varphi_k^2\right)\\
&+\int_{\Omega^C}\left(|\nabla\varepsilon|^2+\eta^2+2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right)\left(1-\sum_{k=1}^K\varphi_k^2\right)\\
&-\int_\Omega\left(|\nabla\varepsilon|^2+\eta^2+2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right)\left(\sum_{k=1}^K\varphi_k^2\right)\\
&+2\sum_{k=1}^K\int\left((\vec{\chi}-\boldsymbol{\ell}_k)\cdot\nabla\varepsilon\right)\eta\varphi_k^2=\mathcal{N}_\Omega+\mathcal{H}_{1,1}+\mathcal{H}_{1,2}+\mathcal{H}_{1,3}+\mathcal{H}_{1,4}.
\end{aligned}$$

By Lemma 3.3 (iii), the orthogonality conditions on $\vec{\varepsilon}$ and a change of variable, we have

$$\mathcal{H}_{1,1} \geq \mu_0 \int \left(|\nabla\varepsilon|^2 + \eta^2\right) \left(\sum_{k=1}^{K} \varphi_k^2\right) - \frac{1}{\mu_0} \sum_{k=1}^{K} \left((z_k^-)^2 + (z_k^+)^2\right).$$

Thus, using (3.55),

$$\mathcal{H}_{1,1} \geq \mu_0 \int \left(|\nabla\varepsilon|^2 + \eta^2\right) \left(\sum_{k=1}^{K} \varphi_k^2\right) - \frac{1}{\mu_0} t^{-7} \geq \mu_0 \int_{\Omega^C} \left(|\nabla\varepsilon|^2 + \eta^2\right) \left(\sum_{k=1}^{K} \varphi_k^2\right) - \frac{1}{\mu_0} t^{-7}.$$

Next, note that if x is such that $\varphi_k(t,x) > \frac{1}{K}$, then $\varphi_{k'}^2(x) \lesssim t^{-4\alpha}$ for $k' \neq k$. Thus, there exists μ_1 the estimate $1 - \sum_{k=1}^{K} \varphi_k^2 \geqslant -\frac{t^{-4\alpha}}{\mu_1}$ holds on $\mathbb{R}$. By direct computations (with the notation $v_+ = \max(0, v)$,

$$\begin{aligned}
\mathcal{H}_{1,2} &= \bar{\ell} \int_{\Omega^C} \left|\frac{\vec{\chi}\cdot\nabla\varepsilon}{\bar{\ell}} + \eta\right|^2 \left(1 - \sum_{k=1}^{K} \varphi_k^2\right) + (1-\bar{\ell}) \int_{\Omega^C} \eta^2 \left(1 - \sum_{k=1}^{K} \varphi_k^2\right) \\
&\quad + \int_{\Omega^C} \left(|\nabla\varepsilon|^2 - \frac{|\vec{\chi}\cdot\varepsilon|^2}{\bar{\ell}}\right) \left(1 - \sum_{k=1}^{K} \varphi_k^2\right) \\
&\geq (1-\bar{\ell}) \int_{\Omega^C} \left(|\nabla\varepsilon|^2 + \eta^2\right) \left(1 - \sum_{k=1}^{K} \varphi_k^2\right)_+ - \frac{\|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}^2}{\mu_1} t^{-4\alpha}.
\end{aligned}$$

Last, by the definition of $\vec{\chi}$, the decay property of φ_α and (3.55), we have

$$\|\varphi_k^2\|_{L^\infty(\Omega)} \lesssim t^{-4\alpha} \quad \text{and} \quad \|(\vec{\chi} - \boldsymbol{\ell}_k)\varphi_k^2\|_{L^\infty} \lesssim t^{-4\alpha}.$$

Thus, $|\mathcal{H}_{1,3}| \lesssim t^{-4\alpha} \|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}^2$ and $|\mathcal{H}_{1,4}| \lesssim t^{-4\alpha} \|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}^2$. Therefore, for some $\mu > 0$, and T_0 large enough, we have

$$\mathcal{H}_{1,1} + \mathcal{H}_{1,2} + \mathcal{H}_{1,3} + \mathcal{H}_{1,4} \geq \mu \mathcal{N}_{\Omega^C} - \frac{1}{\mu} t^{-7} - \frac{1}{\mu} t^{-4\alpha} \|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}^2.$$

Proof of (3.70). Using (3.13), (3.14), (3.37) and (3.55), we have

$$|\mathcal{H}_2| \lesssim \int |\varepsilon|^{\frac{10}{3}} + |\varepsilon|^3 |\mathbf{W}|^{\frac{1}{3}} \lesssim \|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}^3.$$

Last, we observe that by (3.46), $|\mathcal{H}_3| \lesssim \|\mathbf{R}_1\|_{L^{\frac{10}{7}}} \|\varepsilon\|_{L^{\frac{10}{3}}}^2 \lesssim t^{-1} \|\vec{\varepsilon}\|_{\dot{H}^1\times L^2}^2$.

Proof of (3.66). We decompose

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \int \partial_t \left\{|\nabla\varepsilon|^2 + |\eta|^2 - 2(F(\mathbf{W}+\varepsilon) - F(\mathbf{W}) - f(\mathbf{W})\varepsilon)\right\} \\
&\quad + 2\int \vec{\chi}\cdot\partial_t\left((\nabla\varepsilon)\eta\right) + 2\int (\partial_t\vec{\chi})\cdot(\nabla\varepsilon)\eta = \mathbf{g_1} + \mathbf{g_2}.
\end{aligned}$$

We claim the following estimates

$$\begin{aligned}
\mathbf{g_1} &= 2\int \varepsilon\left(-\Delta\mathrm{Mod}_{\mathbf{W}} - f'(\mathbf{W})\mathrm{Mod}_{\mathbf{W}}\right) + 2\int \eta\mathrm{Mod}_{\mathbf{X}} \\
&\quad + 2\int \left(\sum_{k=1}^{K} \boldsymbol{\ell}_k\cdot\nabla W_k\right)\left(f(\mathbf{W}+\varepsilon) - f(\mathbf{W}) - f'(\mathbf{W})\varepsilon\right) + O\left(C_0 t^{-7+2\delta}\right),
\end{aligned}$$

$$\begin{aligned}\mathbf{g_2} = &\int(-\eta^2+|\nabla\varepsilon|^2)\mathrm{div}\vec{\chi} - 2\sum_{j,k=1}^{5}(\partial_{x_k}\varepsilon)(\partial_{x_j}\varepsilon)\partial_{x_j}\chi_k + 2\sum_{k=1}^{5}(\partial_{x_k}\varepsilon)\eta\partial_t\chi_k\\ &-2\int\vec{\chi}\cdot\left(\sum_{k=1}^{K}\nabla W_k\right)\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right)\\ &+2\int(\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{W}})\eta - 2\int\varepsilon(\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{X}}) + O\left(C_0t^{-7+2\delta}\right).\end{aligned}$$

Estimate on $\mathbf{g_1}$. From direct differentiation and integration by parts, we have

$$\begin{aligned}\mathbf{g_1} = &\,2\int(\partial_t\varepsilon)\left(-\Delta\varepsilon-(f(\mathbf{W}+\varepsilon)-f(\mathbf{W}))\right)+2\int(\partial_t\eta)\eta\\ &-2\int(\partial_t\mathbf{W})\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right).\end{aligned}$$

Using (3.35) and (3.42)

$$\begin{aligned}\mathbf{g_1} = &\,2\int\left(-\Delta\varepsilon-f'(\mathbf{W})\varepsilon\right)\mathrm{Mod}_{\mathbf{W}}+2\int\eta\mathrm{Mod}_{\mathbf{X}}\\ &+2\int\left(-\Delta\varepsilon-f'(\mathbf{W})\varepsilon\right)\mathrm{Mod}_{\mathbf{V}}+2\int\eta\mathbf{R}_{\mathbf{X}}+2\int\eta\mathrm{Mod}_{\mathbf{Z}}\\ &-2\int\mathbf{X}\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right)=\mathbf{g_{1,1}}+\mathbf{g_{1,2}}+\mathbf{g_{1,3}}.\end{aligned}$$

We integrate by parts terms in $\mathbf{g_{1,1}}$. Next, by (3.13), (3.14), (3.36), (3.37), (3.55) and (3.57), we obtain

$$\begin{aligned}|\mathbf{g_{1,2}}| \lesssim &\,\|\varepsilon\|_{\dot{H}^1}\|\mathrm{Mod}_{\mathbf{V}}\|_{\dot{H}^1}+\|\eta\|_{L^2}\|\mathrm{Mod}_{\mathbf{Z}}\|_{L^2}\\ &+\|\eta\|_{L^2}\|\mathbf{R}_{\mathbf{X}}\|_{L^2}+\|\varepsilon\|_{L^{\frac{10}{3}}}\|\mathbf{W}\|^{\frac{4}{3}}_{L^{\frac{10}{3}}}\|\mathrm{Mod}_{\mathbf{V}}\|_{L^{\frac{10}{3}}}\lesssim C_0t^{-7+2\delta}.\end{aligned}$$

Recall from (3.34) that $\mathbf{X}=\sum_{k=1}^{K}(-\boldsymbol{\ell}_k\cdot\nabla W_k+c_kz_k)$. Moreover, from (3.19) and the definition of z_k in (3.33), it follows that $\|z_k\|_{L^{\frac{10}{3}}}\lesssim\|\nabla z_k\|_{L^2}\lesssim t^{-2}$. Thus,

$$\begin{aligned}&\left|\mathbf{g_{1,3}}-2\int\left(\sum_{k=1}^{K}\boldsymbol{\ell}_k\cdot\nabla W_k\right)\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right)\right|\\ &\qquad\lesssim\int\left(\sum_{k=1}^{K}|z_k|\right)\left(|\varepsilon|^{\frac{7}{3}}+\varepsilon^2|\mathbf{W}|^{\frac{1}{3}}\right)\lesssim\sum_{k=1}^{K}\|z_k\|_{L^{\frac{10}{3}}}(\|\varepsilon\|^{\frac{7}{3}}_{L^{\frac{10}{3}}}+\|\varepsilon\|^2_{L^{\frac{10}{3}}})\lesssim C_0t^{-7+2\delta}.\end{aligned}$$

Estimate on $\mathbf{g_2}$.

$$\begin{aligned}\mathbf{g_2} &= 2\int(\vec{\chi}\cdot\nabla\partial_t\varepsilon)\eta+2\int(\vec{\chi}\cdot\nabla\varepsilon)\partial_t\eta++2\int(\partial_t\vec{\chi})\cdot(\nabla\varepsilon)\eta\\ &= 2\int(\vec{\chi}\cdot\nabla\eta)\eta+2\int(\vec{\chi}\cdot\nabla\varepsilon)\left[\Delta\varepsilon+(f(\mathbf{W}+\varepsilon)-f(\mathbf{W}))\right]+2\int(\partial_t\vec{\chi})\cdot(\nabla\varepsilon)\eta\\ &\quad+2\int(\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{W}})\eta+2\int(\vec{\chi}\cdot\nabla\varepsilon)\mathrm{Mod}_{\mathbf{X}}+2\int(\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{V}})\eta\\ &\quad+2\int(\vec{\chi}\cdot\nabla\varepsilon)\mathrm{Mod}_{\mathbf{Z}}+2\int(\vec{\chi}\cdot\nabla\varepsilon)\mathbf{R}_{\mathbf{X}}.\end{aligned}$$

Note that by integration by parts

$$\begin{aligned}&2\int(\vec{\chi}\cdot\nabla\eta)\eta+2\int(\vec{\chi}\cdot\nabla\varepsilon)\Delta\varepsilon+2\int(\partial_t\vec{\chi})\cdot(\nabla\varepsilon)\eta\\ &=(-\eta^2+|\nabla\varepsilon|^2)\mathrm{div}\vec{\chi}-2\sum_{j,k=1}^{5}(\partial_{x_k}\varepsilon)(\partial_{x_j}\varepsilon)\partial_{x_j}\chi_k+2\sum_{k=1}^{5}(\partial_{x_k}\varepsilon)\eta\partial_t\chi_k.\end{aligned}$$

Next, we observe

$$\int (\vec{\chi}\cdot\nabla\varepsilon)\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})\varepsilon\right)$$
$$=\int \vec{\chi}\cdot\nabla(F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon)-\int \vec{\chi}\cdot(\nabla\mathbf{W})\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right).$$

Integrating by parts and using (3.63),

$$|-\int \vec{\chi}\cdot\nabla(F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon)|=|\int_\Omega (\mathrm{div}\vec{\chi})\left(F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon\right)|$$
$$\lesssim \frac{1}{(1-2\sigma)t}\int_\Omega |\left(F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon\right)|.$$

Thus, by (3.55) and

$$\|\mathbf{W}\|_{L^{\frac{10}{3}}(\Omega)}\lesssim \sum_{k=1}^K\left(\|W_k\|_{L^{\frac{10}{3}}(\Omega)}+\|v_k\|_{\dot{H}^1}\right)\lesssim t^{-\frac{3}{2}},$$

we obtain

$$\left|\int \vec{\chi}\cdot\nabla(F(\mathbf{W}+\varepsilon)-F(\mathbf{W})-f(\mathbf{W})\varepsilon)\right|\lesssim t^{-1}\int_\Omega\left(|\varepsilon|^{\frac{10}{3}}+\mathbf{W}^{\frac{4}{3}}|\varepsilon|^2\right)$$
$$\lesssim t^{-1}(\|\varepsilon\|^{\frac{10}{3}}_{L^{\frac{10}{3}}}+\|\varepsilon\|^2_{L^{\frac{10}{3}}}\|\mathbf{W}\|^{\frac{4}{3}}_{L^{\frac{10}{3}}(\Omega)})$$
$$\lesssim t^{-1}\left((C_0)^{\frac{10}{3}}t^{-10+\frac{10}{3}\delta}+(C_0^2t^{-8+2\delta}\right)\lesssim C_0t^{-7+2\delta}.$$

Moreover, again by (3.19) and (3.55)

$$\left|\int \vec{\chi}\cdot\left(\nabla\mathbf{W}-\nabla\sum_{k=1}^K W_k\right)\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right)\right|$$
$$=\left|\int \vec{\chi}\cdot\nabla\left(\sum_{k=1}^K c_kv_k\right)\left(f(\mathbf{W}+\varepsilon)-f(\mathbf{W})-f'(\mathbf{W})\varepsilon\right)\right|$$
$$\lesssim\sum_{k=1}^K\int|\nabla v_k|\left(|\varepsilon|^2\mathbf{W}^{\frac{1}{3}}+|\varepsilon|^{\frac{7}{3}}\right)\lesssim\left(\sum_{k=1}^K\|\nabla v_k\|_{L^{\frac{10}{3}}}\right)\|\varepsilon\|^2_{L^{\frac{10}{3}}}\lesssim(C_0)^2t^{-8+2\delta}\lesssim C_0t^{-7+2\delta}.$$

Next, integrating by parts,

$$2\int(\vec{\chi}\cdot\nabla\varepsilon)\mathrm{Mod}_{\mathbf{X}}=-2\int(\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{X}})\varepsilon+O\left(C_0t^{-7+2\delta}\right),$$

since by (3.55), (3.58) and (3.63)

$$\left|\int(\mathrm{div}\vec{\chi})\varepsilon\mathrm{Mod}_{\mathbf{X}}\right|\lesssim t^{-4+\delta}\int_\Omega|\varepsilon|\left(\sum_{k=1}^K|W_k|^{\frac{4}{3}}\right)\lesssim t^{-4+\delta}\|\varepsilon\|_{L^{\frac{10}{3}}}\sum_{k=1}^K\|W_k\|^{\frac{4}{3}}_{L^{\frac{40}{21}}(\Omega)}$$
$$\lesssim C_0t^{-7+2\delta}.$$

We finish the estimate of $\mathbf{g_2}$ by observing that (3.36), (3.55) and (3.57) yield

$$\left|\int(\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{V}})\eta\right|+\left|\int(\vec{\chi}\cdot\nabla\varepsilon)\mathrm{Mod}_{\mathbf{Z}}\right|+\left|\int(\vec{\chi}\cdot\nabla\varepsilon)\mathbf{R}_{\mathbf{X}}\right|$$
$$\lesssim\|\eta\|_{L^2}\|\nabla\mathrm{Mod}_{\mathbf{V}}\|_{L^2}+\|\nabla\varepsilon\|_{L^2}\|\mathrm{Mod}_{\mathbf{Z}}\|_{L^2}+\|\nabla\varepsilon\|_{L^2}\|\mathbf{R}_{\mathbf{X}}\|_{L^2}\lesssim C_0t^{-7+2\delta}.$$

Gathering the estimates on $\mathbf{g_1}$ and $\mathbf{g_2}$, we rewrite

$$\frac{d}{dt}\mathcal{H} = \mathbf{h_1} + \mathbf{h_2} + \mathbf{h_3} + \mathbf{h_4} + O\left(C_0 t^{-7+2\delta}\right),$$

where

$$\begin{aligned}
\mathbf{h_1} &= \int (-\eta^2 + |\nabla\varepsilon|^2)\mathrm{div}\vec{\chi} - 2\sum_{j,k=1}^{5}(\partial_{x_k}\varepsilon)(\partial_{x_j}\varepsilon)\partial_{x_j}\chi_k + 2\sum_{k=1}^{5}(\partial_{x_k}\varepsilon)\eta\partial_t\chi_k,\\
\mathbf{h_2} &= 2\int\left(\sum_k(\boldsymbol{\ell}_k - \vec{\chi})\cdot\nabla W_k\right)\left(f(\mathbf{W}+\varepsilon) - f(\mathbf{W}) - f'(\mathbf{W})\varepsilon\right),\\
\mathbf{h_3} &= 2\int \eta\left(\mathrm{Mod}_{\mathbf{X}} + \vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{W}}\right),\\
\mathbf{h_4} &= 2\int \varepsilon\left(-\Delta\mathrm{Mod}_{\mathbf{W}} - \vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{X}} - f'(\mathbf{W})\mathrm{Mod}_{\mathbf{W}}\right).
\end{aligned}$$

Estimate on $\mathbf{h_1}$. We claim the following estimate

$$-(1-2\sigma)t\mathbf{h_1} \leqslant (6-4\delta+C\sigma)\mathcal{N}_\Omega(t). \tag{3.71}$$

Let

$$\mathrm{I} = -(1-2\sigma)t\left((-\eta^2 + |\nabla\varepsilon|^2)\mathrm{div}\vec{\chi} - 2\sum_{j,k=1}^{5}(\partial_{x_k}\varepsilon)(\partial_{x_j}\varepsilon)\partial_{x_j}\chi_k + 2\sum_{k=1}^{5}(\partial_{x_k}\varepsilon)\eta\partial_t\chi_k\right).$$

To obtain (3.71), we will actually prove the following stronger property

$$\mathrm{I} \leqslant (6-4\delta)\left(|\nabla\varepsilon|^2 + \eta^2 + 2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right) + C\sigma(|\nabla\varepsilon|^2 + \eta^2) \quad \forall x\in\Omega(t). \tag{3.72}$$

Without loss of generality, we consider the following five cases,
Case 1: Let $x \in \Omega_{1,1}\times\Omega_{0,2}\times\Omega_{0,3}\times\Omega_{0,4}\times\Omega_{0,5}$. From (3.63), we obtain

$$\begin{cases}
\partial_{x_1}\chi_1(t,x) = \dfrac{1}{(1-2\sigma)t}, \quad \partial_t\chi_1(t,x) = -\dfrac{x_1}{t}\dfrac{1}{(1-2\sigma)t},\\
\partial_t\chi_i(t,x) = 0, \quad \nabla\chi_i(t,x) = 0 \quad \forall i\neq 1.
\end{cases} \tag{3.73}$$

From direct computations and (3.73), we obtain

$$\begin{aligned}
\mathrm{I} = \eta^2 - |\nabla\varepsilon|^2 + 2|\partial_{x_1}\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta - 2(5-4\delta)(\partial_{x_1}\varepsilon\cdot\chi_1)\eta\\
- 2(6-4\delta)\sum_{i=2}^{5}(\partial_{x_i}\varepsilon\cdot\chi_i)\eta + 2\eta\partial_{x_1}\varepsilon\left(\frac{x_1}{t} - \chi_1\right).
\end{aligned} \tag{3.74}$$

Finally, using (3.60), (3.61), (3.62), (3.74) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathrm{I} &\leqslant \eta^2 + |\nabla\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta + C\sigma(|\nabla\varepsilon|^2 + \eta^2)\\
&\quad + (6-4\delta)|\overline{\ell}|\left(\frac{5-4\delta}{|\overline{\ell}|(6-4\delta)}|\nabla\varepsilon|^2 + \frac{|\overline{\ell}|(6-4\delta)}{(5-4\delta)}\eta^2\right)\\
&\leqslant (6-4\delta)\left(|\nabla\varepsilon|^2 + \eta^2 + 2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right) + C\sigma(|\nabla\varepsilon|^2 + \eta^2).
\end{aligned}$$

Case 2: Let $x \in \Omega_{1,1}\times\Omega_{1,2}\times\Omega_{0,3}\times\Omega_{0,4}\times\Omega_{0,5}$. From (3.63), we obtain

$$\begin{cases}
\partial_{x_i}\chi_i(t,x) = \dfrac{1}{(1-2\sigma)t}, \quad \partial_t\chi_i(t,x) = -\dfrac{x_i}{t}\dfrac{1}{(1-2\sigma)t} \quad \text{for } i\in\{1,2\},\\
\partial_t\chi_i(t,x) = 0, \quad \nabla\chi_i(t,x) = 0 \quad \text{for } i\in\{3,4,5\}.
\end{cases} \tag{3.75}$$

From direct computations and (3.75), we obtain

$$\begin{aligned}\mathrm{I} = 2\eta^2 - 2|\nabla\varepsilon|^2 + 2\sum_{i=1}^{2}|\partial_{x_i}\varepsilon|^2 + 2(6-4\delta)\left(\vec{\chi}\cdot\nabla\varepsilon\right)\eta - 2(5-4\delta)\sum_{i=1}^{2}\left(\partial_{x_i}\varepsilon\cdot\chi_i\right)\eta \\ - 2(6-4\delta)\sum_{i=3}^{5}\left(\partial_{x_i}\varepsilon\cdot\chi_i\right)\eta + 2\sum_{i=1}^{2}\eta\partial_{x_i}\varepsilon\left(\frac{x_i}{t}-\chi_i\right).\end{aligned} \tag{3.76}$$

Finally, using (3.60), (3.61), (3.62), (3.76) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathrm{I} &\leqslant 2\eta^2 + 2(6-4\delta)\left(\vec{\chi}\cdot\nabla\varepsilon\right)\eta + (6-4\delta)|\bar{\ell}|\left(\frac{1}{|\bar{\ell}|}|\nabla\varepsilon|^2 + |\bar{\ell}|\eta^2\right) + C\sigma\left(|\nabla\varepsilon|^2+\eta^2\right) \\ &\leqslant (6-4\delta)\left(|\nabla\varepsilon|^2+\eta^2+2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right) + C\sigma\left(|\nabla\varepsilon|^2+\eta^2\right).\end{aligned}$$

Case 3: Let $x \in \Omega_{1,1}\times\Omega_{1,2}\times\Omega_{1,3}\times\Omega_{0,4}\times\Omega_{0,5}$. From (3.63), we obtain

$$\begin{cases}\partial_{x_i}\chi_i(t,x) = \dfrac{1}{(1-2\sigma)t}, \quad \partial_t\chi_i(t,x) = -\dfrac{x_i}{t}\dfrac{1}{(1-2\sigma)t} \quad \text{for } i\in\{1,2,3\}, \\ \partial_t\chi_i(t,x) = 0, \quad \nabla\chi_i(t,x) = 0 \quad \text{for } i\in\{4,5\}.\end{cases} \tag{3.77}$$

From direct computations and (3.77), we obtain

$$\begin{aligned}\mathrm{I} = 3\eta^2 - 3|\nabla\varepsilon|^2 + 2\sum_{i=1}^{3}|\partial_{x_i}\varepsilon|^2 + 2(6-4\delta)\left(\vec{\chi}\cdot\nabla\varepsilon\right)\eta \\ - 2(5-4\delta)\sum_{i=1}^{3}\left(\partial_{x_i}\varepsilon\cdot\chi_i\right)\eta - 2(6-4\delta)\sum_{i=4}^{5}\left(\partial_{x_i}\varepsilon\cdot\chi_i\right)\eta + 2\sum_{i=1}^{3}\eta\partial_{x_i}\varepsilon\left(\frac{x_i}{t}-\chi_i\right).\end{aligned} \tag{3.78}$$

Finally, using (3.60), (3.61), (3.62), (3.78) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathrm{I} &\leqslant 3\eta^2 - |\nabla\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta + C\sigma(|\nabla\varepsilon|^2+\eta^2) \\ &\quad + (6-4\delta)|\bar{\ell}|\left(\frac{(7-4\delta)}{(6-4\delta)|\bar{\ell}|}|\nabla\varepsilon|^2 + \frac{(6-4\delta)|\bar{\ell}|}{(7-4\delta)}\eta^2\right) \\ &\leqslant (6-4\delta)\left(|\nabla\varepsilon|^2+\eta^2+2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right) + C\sigma\left(|\nabla\varepsilon|^2+\eta^2\right).\end{aligned}$$

Case 4: Let $x \in \Omega_{1,1}\times\Omega_{1,2}\times\Omega_{1,3}\times\Omega_{1,4}\times\Omega_{0,5}$. From (3.63), we obtain

$$\begin{cases}\partial_{x_i}\chi_i(t,x) = \dfrac{1}{(1-2\sigma)t}, \quad \partial_t\chi_i(t,x) = -\dfrac{x_i}{t}\dfrac{1}{(1-2\sigma)t} \quad \text{for } i\in\{1,2,3,4\}, \\ \partial_t\chi_i(t,x) = 0, \quad \nabla\chi_i(t,x) = 0 \quad \text{for } i=5.\end{cases} \tag{3.79}$$

From direct computations and (3.79), we obtain

$$\begin{aligned}\mathrm{I} = 4\eta^2 - 4|\nabla\varepsilon|^2 + 2\sum_{i=1}^{4}|\partial_{x_i}\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta - 2(5-4\delta)\sum_{i=1}^{4}(\partial_{x_i}\varepsilon\cdot\chi_i)\eta \\ - 2(6-4\delta)(\partial_{x_5}\varepsilon\cdot\chi_5)\eta + 2\sum_{i=1}^{4}\eta\partial_{x_i}\varepsilon\left(\frac{x_i}{t}-\chi_i\right).\end{aligned} \tag{3.80}$$

Finally, using (3.60), (3.61), (3.62), (3.80) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathrm{I} &\leqslant 4\eta^2 - 2|\nabla\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta + C\sigma(|\nabla\varepsilon|^2+\eta^2) \\ &\quad + (6-4\delta)|\bar{\ell}|\left(\frac{(8-4\delta)}{(6-4\delta)|\bar{\ell}|}|\nabla\varepsilon|^2 + \frac{|\bar{\ell}|(6-4\delta)}{(8-4\delta)}\eta^2\right) \\ &\leqslant (6-4\delta)\left(|\nabla\varepsilon|^2+\eta^2+2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right) + C\sigma\left(|\nabla\varepsilon|^2+\eta^2\right).\end{aligned}$$

Case 5: Let $x \in \Omega_{1,1} \times \Omega_{1,2} \times \Omega_{1,3} \times \Omega_{1,4} \times \Omega_{1,5}$. From (3.63), we obtain

$$\partial_{x_i}\chi_i(t,x) = \frac{1}{(1-2\sigma)t}, \quad \partial_t\chi_i(t,x) = -\frac{x_i}{t}\frac{1}{(1-2\sigma)t} \quad \text{for } i \in \{1,2,3,4,5\}. \tag{3.81}$$

From direct computations and (3.81), we obtain

$$\begin{aligned} \mathrm{I} = 5\eta^2 - 5|\nabla\varepsilon|^2 + 2\sum_{i=1}^{5}|\partial_{x_i}\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta \\ - 2(5-4\delta)\left(\nabla\varepsilon\cdot\vec{\chi}\right)\eta + 2\sum_{i=1}^{5}\eta\partial_{x_i}\varepsilon\left(\frac{x_i}{t} - \chi_i\right). \end{aligned} \tag{3.82}$$

Finally, using (3.60), (3.61), (3.62), (3.78) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mathrm{I} &\leqslant 5\eta^2 - 3|\nabla\varepsilon|^2 + 2(6-4\delta)(\vec{\chi}\cdot\nabla\varepsilon)\eta + C\sigma(|\nabla\varepsilon|^2 + \eta^2) \\ &+ (5-4\delta)|\bar{\ell}|\left(\frac{(9-4\delta)}{|\bar{\ell}|(5-4\delta)}|\nabla\varepsilon|^2 + \frac{|\bar{\ell}|(5-4\delta)}{(9-4\delta)}\eta^2\right) \\ &\leqslant (6-4\delta)\left(|\nabla\varepsilon|^2 + \eta^2 + 2(\vec{\chi}\cdot\nabla\varepsilon)\eta\right) + C\sigma\left(|\nabla\varepsilon|^2 + \eta^2\right). \end{aligned}$$

Estimate on $\mathbf{h_2}$. We observe that by the definition of $\vec{\chi}$ in (3.63) and the decay of ∇W and W,

$$\|(\boldsymbol{\ell}_k - \vec{\chi})\cdot\nabla W_k\|_{L^{\frac{10}{3}}} \lesssim t^{-\frac{5}{2}}.$$

Thus, by (3.14), $|\mathbf{h_2}| \lesssim t^{-\frac{5}{2}}\|\varepsilon\|^2_{L^{\frac{10}{3}}} \lesssim C_0 t^{-\frac{17}{2}+2\delta} \lesssim C_0 t^{-7+2\delta}$.

Estimate on $\mathbf{h_3}$. Denote

$$M_k = \left(\frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{\frac{1}{2}}t^2}\right)\Lambda W_k + \dot{\mathbf{y}}_k\cdot\nabla W_k$$

so that $\mathrm{Mod}_{\mathbf{W}} = \sum_k M_k$ and $\mathrm{Mod}_{\mathbf{X}} = -\sum_k \boldsymbol{\ell}_k\cdot\nabla M_k$. Using (3.37) and the definition of $\vec{\chi}$ (see (3.63)), we have $\|(\boldsymbol{\ell}_k - \vec{\chi})\cdot\nabla M_k\|_{L^2} \lesssim t^{-\frac{9}{2}+\delta}$. It follows from (3.43) that

$$\|\mathrm{Mod}_{\mathbf{X}} + \vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{W}}\|_{L^2} \lesssim t^{-\frac{9}{2}+\delta},$$

and thus

$$|\mathbf{h_3}| = \left|\int \eta\left(\mathrm{Mod}_{\mathbf{X}} + \vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{W}}\right)\right| \lesssim t^{-\frac{9}{2}+\delta}\|\eta\|_{L^2} \lesssim C_0 t^{-7+2\delta}.$$

Estimate on $\mathbf{h_4}$. By (i) of Lemma 3.3, $-\Delta M_k + (\boldsymbol{\ell_k}\cdot\nabla)(\boldsymbol{\ell_k}\cdot\nabla)M_k - f'(W_k)M_k = 0$. Thus,

$$\begin{aligned} &\left|-\Delta M_k + (\vec{\chi}\cdot\nabla)\cdot(\boldsymbol{\ell_k}\cdot\nabla)M_k - f'(\mathbf{W})M_k\right| \\ &\lesssim |((\vec{\chi} - \boldsymbol{\ell_k})\cdot\nabla)\cdot(\boldsymbol{\ell_k}\cdot\nabla)M_k| + \left|f'(\mathbf{W}) - f'(W_k)\right||M_k|. \end{aligned}$$

As before, by (3.37), $\|((\vec{\chi} - \boldsymbol{\ell_k})\cdot\nabla)\cdot(\boldsymbol{\ell_k}\cdot\nabla)M_k\|_{L^{\frac{10}{7}}} \lesssim t^{-\frac{9}{2}+\delta}$. Moreover, by (3.19)

$$\begin{aligned} &\left||\mathbf{W}|^{\frac{4}{3}} - |W_k|^{\frac{4}{3}}\right||M_k| \\ &\lesssim C_0 t^{-3+\delta}\left(\sum|W_k| + \sum|v_k|\right)^{\frac{1}{3}}\left(\sum_{k'\neq k}|W_{k'}| + \sum|v_k|\right)|W_k| \\ &\lesssim C_0 t^{-3+\delta}\Bigg[\sum_{k'\neq k''}|W_{k'}|^{\frac{4}{3}}|W_{k''}| + t^{-2+\delta}\sum_{k'}|W_{k'}|^2 \\ &\qquad + t^{-\frac{2}{3}+\frac{\delta}{3}}\sum_{k'\neq k''}|W_{k'}|^{1+\frac{2}{9}}|W_{k''}| + t^{-\frac{8}{3}+\frac{4}{3}\delta}\sum_{k'}|W_{k'}|^{\frac{17}{9}}\Bigg], \end{aligned}$$

and from Claim 3.8,

$$\left\|\left(f'(\mathbf{W})-f'(W_k)\right)M_k\right\|_{L^{\frac{10}{7}}}\lesssim t^{-5+2\delta}.$$

Therefore, using (3.43),

$$\left\|-\Delta\mathrm{Mod}_{\mathbf{W}}-\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{X}}-f'(\mathbf{W})\mathrm{Mod}_{\mathbf{W}}\right\|_{L^{\frac{10}{7}}}\lesssim t^{-\frac{9}{2}+\delta}.$$

It follows that by (3.55),

$$|\mathbf{h_4}|\lesssim\|\varepsilon\|_{L^{\frac{10}{3}}}\left\|-\Delta\mathrm{Mod}_{\mathbf{W}}-\vec{\chi}\cdot\nabla\mathrm{Mod}_{\mathbf{X}}-f'(\mathbf{W})\mathrm{Mod}_{\mathbf{W}}\right\|_{L^{\frac{10}{7}}}\lesssim C_0t^{-7+2\delta}.$$

In conclusion, using (3.65), for σ small, and T_0 large,

$$-\frac{d}{dt}\mathcal{H}\leq\frac{(6-4\delta+C\sigma)}{t}\mathcal{N}_\Omega+O(C_0t^{-7+2\delta})\leq\frac{6-3\delta}{t}\mathcal{H}+O(C_0t^{-7+2\delta})$$

and the proof of Lemma 3.12 is complete. □

3.5.4 Parameters and energy estimates

The following result, mainly based on Lemma 3.12, improves all the estimates in (3.55), except the ones on $(z_k^-)_k$.

Lemma 3.13 (Closing estimates except $(z_k^-)_k$). *For $C_0>0$ large enough, for all $t\in[T^*,S_n]$,*

$$\begin{cases}|\lambda_k(t)-\lambda_k^\infty|\leq\dfrac{C_0}{2}t^{-1},\quad|\mathbf{y}_k(t)-\mathbf{y}_k^\infty|\leq\dfrac{C_0}{2}t^{-1},\\ |z_k^+(t)|^2\leq\dfrac{1}{2}t^{-7},\quad\|\vec{\varepsilon}(t)\|_{\dot{H}^1\times L^2}\leq\dfrac{C_0}{2}t^{-3+\delta}.\end{cases}\tag{3.83}$$

Proof. **Step 1.** Estimate on $(\lambda_k(t),\mathbf{y}_k(t))_{k\in\{1,\dots,K\}}$. From (3.43) and (3.55), we have

$$\left|\frac{\dot{\lambda}_k}{\lambda_k}\right|+|\dot{\mathbf{y}}_k|\leq Ct^{-2}\tag{3.84}$$

where the constant C depends on the parameters of the K solitons, but not on C_0. Using (3.54) and (3.84) and taking C_0 large enough, we obtain

$$\begin{aligned}|\lambda_k(t)-\lambda_k^\infty|&\leqslant|\lambda_k(t)-\lambda(S_n)|+|\lambda(S_n)-\lambda_k^\infty|\\&\leqslant|\int_t^{S_n}\dot{\lambda}_k(s)ds|+\frac{1}{S_n^{\frac{7}{2}}}\leqslant\int_t^{S_n}\frac{C}{s^2}ds+\frac{1}{S_n^{\frac{7}{2}}}\leqslant\frac{C_0}{2t}.\end{aligned}$$

Using again (3.54) and (3.84) and taking C_0 large enough, we obtain

$$\begin{aligned}|\mathbf{y}_k(t)-\mathbf{y}_k^\infty|&\leqslant|\mathbf{y}_k(t)-\mathbf{y}(S_n)|+|\mathbf{y}(S_n)-\mathbf{y}_k^\infty|\\&\leqslant|\int_t^{S_n}\dot{\mathbf{y}}_k(s)ds|+\frac{1}{S_n^{\frac{7}{2}}}\leqslant\int_t^{S_n}\frac{C}{s^2}ds+\frac{1}{S_n^{\frac{7}{2}}}\leqslant\frac{C_0}{2t}.\end{aligned}$$

Step 2. Bound on the energy norm. From (3.54) and (3.64), we have

$$\mathcal{H}(S_n)\lesssim S_n^{-7}.\tag{3.85}$$

Integrating (3.66) on $[t,S_n]$, and using (3.85), we obtain,

$$t^{6-3\delta}\mathcal{H}(t)=-\int_t^{S_n}\frac{d}{ds}(s^{6-3\delta}\mathcal{H}(s))ds+S_n^{6-3\delta}\mathcal{H}(S_n)\lesssim\int_t^{S_n}\frac{C_0}{s^{1+\delta}}ds\lesssim\frac{C_0}{t^\delta}+\frac{1}{S_n^{1+3\delta}}.$$

It follows that, $\mathcal{H}(t) \lesssim C_0 t^{-6+2\delta}$. Using (3.65), we conclude that $\|\vec{\varepsilon}\|_{\dot{H}^1 \times L^2} \lesssim \sqrt{C_0} t^{-3+\delta}$. The estimate follows for C_0 large enough.

Step 3. Estimate on $\left(z_k^+(t)\right)_{k\in\{1,\dots,K\}}$. Let $\beta_k = \frac{\sqrt{\lambda_0}}{\lambda_k^\infty}(1-|\boldsymbol{\ell}_k|^2)^{1/2} > 0$. Then, from (3.44) and (3.55),

$$\frac{d}{dt}\left(e^{-\beta_k t} z_k^+\right) \lesssim C_0 e^{-\beta_k t} t^{-4+\delta}. \tag{3.86}$$

Integrating (3.86) on $[t, S_n]$, and using (3.53), we obtain $-z_k^+(t) \lesssim t^{-4+\delta}$. Using the same estimate for $-e^{-\beta_k t} z_k^+$, we obtain the conclusion for T_0 large enough. □

As in [28, 31, 85, 86], the parameters z_k^- require a specific argument.

Lemma 3.14 (Control of unstable directions in [86])**.** *There exist $(\xi_{k,n})_k \in \mathcal{B}_{\mathbb{R}^K}(S_n^{-7/2})$ such that, for $C^* > 0$ large enough, $T^*((\xi_{k,n})_k) = T_0$. In particular, let $(\zeta_n^\pm)$ be given by Claim 3.11 from such $(\xi_{k,n})_k$, then the solution u_n of (3.49) satisfies (3.50).*

Proof. See proof of Lemma 5.8 in [86]. □

Estimates (3.50) follow directly from the estimates (3.55) on $\varepsilon(t)$, $\lambda_k(t)$, $\mathbf{y}_k(t)$.

3.5.5 End of the proof of Proposition 3.10

We refer to [86] for the proofs of the $\dot{H}^2 \times \dot{H}^1$ bound and (3.51) (§5.4 and §5.5 of [85]).

Chapter 4

Construction of excited multi-solitons for the 5D energy-critical wave equation

Abstract: For the 5D energy-critical wave equation, we construct excited N-solitons with collinear speeds, *i.e.* solutions u of the equation such that

$$\lim_{t\to+\infty}\left\|\nabla_{t,x}u(t)-\nabla_{t,x}\left(\sum_{n=1}^{N}Q_n(t)\right)\right\|_{L^2}=0,$$

where for $n=1,\dots,N$, $Q_n(t,x)$ is the Lorentz transform of a non-degenerate and sufficiently decaying excited state, each with different but collinear speeds. The existence proof follows the ideas of Martel-Merle [85] and Côte-Martel [26] developed for the energy-critical wave and nonlinear Klein-Gordon equations. In particular, we rely on an energy method and on a general coercivity property for the linearized operator.

4.1 Introduction

4.1.1 Main result

We consider the energy-critical focusing wave equation in dimension 5,

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{3}} u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^5, \\ u_{|t=0} = u_0 \in \dot{H}^1, \quad \partial_t u_{|t=0} = u_1 \in L^2. \end{cases} \tag{4.1}$$

Recall that the Cauchy problem for equation (4.1) is locally well-posed in the energy space $\dot{H}^1 \times L^2$. See *e.g.* [71] and references therein. Let $f(u) = |u|^{\frac{4}{3}} u$ and $F(u) = \frac{3}{10}|u|^{\frac{10}{3}}$. For any $\dot{H}^1 \times L^2$ solution $\vec{u} = (u, \partial_t u)$, the energy E and the momentum P are conserved along the flow, where

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^5} \Big(|\nabla u|^2 + |\partial_t u|^2 - 2F(u) \Big) \mathrm{d}x, \quad P(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^5} (\partial_t u \nabla u) \mathrm{d}x.$$

Looking for stationary solutions $u(t,x) = q(x)$ of (4.1) in $\dot{H}^1$, we reduce to the Yamabe equation

$$\Delta q + f(q) = 0 \quad \text{in } \mathbb{R}^5. \tag{4.2}$$

Denote

$$\Sigma = \Big\{ q \in \dot{H}^1(\mathbb{R}^5)/0 : q \text{ satisfies (4.2)} \Big\}.$$

For $q \in \Sigma$ and $\boldsymbol{\ell} \in \mathbb{R}^5$ such that $|\boldsymbol{\ell}| < 1$, let

$$q_{\boldsymbol{\ell}}(x) = q\left(\left(\frac{1}{\sqrt{1 - |\boldsymbol{\ell}|^2}} - 1 \right) \frac{\boldsymbol{\ell}(\boldsymbol{\ell} \cdot x)}{|\boldsymbol{\ell}|^2} + x \right), \tag{4.3}$$

then $u(t,x) = q_{\boldsymbol{\ell}}(x - \boldsymbol{\ell} t)$ is a global, bounded solution of (4.1).

It is known (see *e.g.* [3, 108]) that the unique (up to scaling invariance and sign change) *radially symmetric* element of Σ is the *ground state* W given explicitly by

$$W(x) = \left(1 + \frac{|x|^2}{15} \right)^{-\frac{3}{2}}.$$

The existence of non-radially symmetric, sign changing elements of Σ with arbitrary large energy was first proved by Ding [42], using variational arguments. Functions $q \in \Sigma$ with $q \neq W$ (up to invariances) are usually called *excited states*.

For any $q \in \Sigma$, we denote the linearized operator around q by

$$\mathcal{L} = -\Delta - f'(q).$$

Set

$$\mathcal{Z}_q = \Big\{ f \in \dot{H}^1 \text{ such that } \mathcal{L} f = 0 \Big\},$$

and

$$\widetilde{\mathcal{Z}}_q = \mathrm{Span} \Big\{ \frac{3}{2} q + x \cdot \nabla q; (x_i \partial_{x_j} q - x_j \partial_{x_i} q), 1 \leqslant i < j \leqslant 5; \\ \partial_{x_i} q; -3 x_i q + |x|^2 \partial_{x_i} q - 2 x_i (x \cdot \nabla q), 1 \leqslant i \leqslant 5 \Big\}.$$

The function space $\widetilde{\mathcal{Z}}_q \subset \mathcal{Z}_q$ is the null space of $\mathcal{L}$ that is generated by a family of explicit transformations (see [48, Lemma 3.8] and §4.2.1).

We will need the following definitions.

Definition 4.1. Let $q \in \Sigma$.

(i) q is called a *non-degenerate state* if $\mathcal{Z}_q = \widetilde{\mathcal{Z}}_q$.

(ii) q is called a *decaying state* if $|q(x)| \lesssim \langle x \rangle^{-4}$ for all $x \in \mathbb{R}^5$.

Remark 4.2. The existence of non-degenerate sign-changing states was proved in [36, 37, 99]. As noticed in [48, Remark 3.2], the existence of *non-degenerate, decaying states* then follows by translation and the Kelvin transformation.

The main goal of this article is to construct N-excited states.

Theorem 4.3 (Existence of multi-excited states)**.** *Let $N \geqslant 2$ and let $\mathbf{e}$ be a unit vector of $\mathbb{R}^5$. For $n \in \{1, \dots, N\}$, let $\boldsymbol{\ell}_n \in \mathbb{R}^5$ be such that*

$$\boldsymbol{\ell}_n = \ell_n \mathbf{e} \text{ where } -1 < \ell_n < 1 \text{ and for all } n \neq n',\ \ell_n \neq \ell_{n'}.$$

Let $q_1, \dots, q_N$ be non-degenerate decaying states of (4.2). Then there exist $T_0 > 0$ and a solution $\vec{u} = (u, \partial_t u)$ of (4.1) in the energy space $\dot{H}^1 \times L^2$, defined on $[T_0, +\infty)$ such that

$$\lim_{t \to +\infty} \left\| \vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t) \right\|_{\dot{H}^1 \times L^2} = 0, \tag{4.4}$$

where for $n = 1, \dots, N$,

$$\vec{Q}_n(t,x) = \begin{pmatrix} q_{n,\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \\ -\ell_n \partial_{x_1} q_{n,\boldsymbol{\ell}_n}(x - \boldsymbol{\ell}_n t) \end{pmatrix}.$$

Remark 4.4. By invariance by rotation in $\mathbb{R}^5$, we assume without loss of generality that $\mathbf{e}$ is $\mathbf{e}_1$, the first vector of the canonical basis of $\mathbb{R}^5$.

Remark 4.5. The *non-degeneracy* condition ensures that the null space of $\mathcal{L}$ is generated by 21-parameter transformations, which is needed in our proof to obtain cancellation properties for the linearized operator around the excited states. See more details in §4.2.1 and §4.2.2.

Remark 4.6. The *decay* condition ensures that the nonlinear interactions between two excited states of different speeds is of order at most t^{-4}. This rate allows us to close the energy estimates (see more details in §4.4.2 and §4.4.3). At this point, we do not known how to prove Theorem 4.3 without this condition. In particular, we do not construct multi-solitons partly based on *non-degenerate decaying state* and ground state, since the interaction caused by the ground state would be t^{-3}.

Remark 4.7. Using the Lorentz transformation, the existence result extends to the case of 2-solitons for any different, possibly noncollinear speeds $\boldsymbol{\ell}_1$, $\boldsymbol{\ell}_2$. See [85, Section 5].

The constructions of *asymptotic multi-solitons* for dispersive and wave equations have been the subject of several previous works, for both stable and unstable solitons. First results in non-integrable contexts were given by Merle [92] for the L^2 critical nonlinear Schrödinger equation (NLS) and Martel [83] for the subcritical and critical generalized Korteweg-de Vries equations (gKdV). Next, the strategy of these works was extended to the case of exponentially unstable solitons: see Côte, Martel and Merle [28] for the construction of multi-solitons for supercritical (gKdV) and (NLS), and Combet [20] for a classification result for supercritical (gKdV). We refer to [26, 85] for results on the existence of multi-solitons for the nonlinear Klein-Gordon (NLKG) and 5D energy critical wave equation that inspired the present work. See also [31, 84, 96, 111] for other existence results.

The article is organized as follows. Section 4.2 describes the spectral theory for any *non-degenerate decaying state* q. Section 4.3 introduces technical tools involved in a dynamical approach to the N-soliton problem for (4.1): estimates of the non-linear interactions between solitons, decomposition by modulation and parameter estimates. Finally, Theorem 4.3 is proved in Section 4.4 by energy estimates and a suitable compactness argument.

4.1.2 Notation

We denote

$$(u,v)_{L^2} = \int_{\mathbb{R}^5} uv\mathrm{d}x, \quad (u,v)_{\dot{H}^1} = \int_{\mathbb{R}^5} (\nabla u \cdot \nabla v)\mathrm{d}x.$$

For

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

denote

$$(\vec{u},\vec{v})_{L^2} = \sum_{k=1,2} (u_k, v_k)_{L^2}, \quad \|\vec{u}\|_{L^2} = (\vec{u},\vec{u})_{L^2},$$

$$(\vec{u},\vec{v})_{\mathcal{H}} = (u_1,v_1)_{\dot{H}^1} + (u_2,v_2)_{L^2}, \quad \|\vec{u}\|_{\mathcal{H}} = (\vec{u},\vec{u})_{\mathcal{H}}.$$

When x_1 is seen as a specific coordinate, denote

$$x' = (x_2,x_3,x_4,x_5), \quad \overline{\nabla} = (\partial_{x_2},\partial_{x_3},\partial_{x_4},\partial_{x_5}), \quad \text{and} \quad \bar{\Delta} = \sum_{i=2}^{5} \partial_{x_i}^2.$$

For $-1 < \ell < 1$, set

$$x_\ell = \left(\frac{x_1}{\sqrt{1-\ell^2}}, x'\right) \quad \text{for } x = (x_1,x') \in \mathbb{R}^5.$$

Let $X_1(\mathbb{R}^5) = \dot{H}^2(\mathbb{R}^5) \cap \dot{H}^1(\mathbb{R}^5)$ and $X = X_1(\mathbb{R}^5) \times H^1(\mathbb{R}^5)$.

Denote by $\mathcal{O}_5$ be the orthogonal group in dimension 5. Let $\mathcal{SO}_5$ be the special orthogonal group, *i.e.* the subgroup of the elements of $\mathcal{O}_5$ with determinant 1.

4.2 Spectral theory for non-degenerate state

4.2.1 Transforms of stationary solutions

Following [48], we recall that (4.2) is invariant under the following four transformations:

(1) translations: If $q \in \Sigma$ then $q(x+a) \in \Sigma$, for all $a \in \mathbb{R}^5$;

(2) dilation: If $q \in \Sigma$ then $\lambda^{\frac{3}{2}} q(\lambda x) \in \Sigma$ for all $\lambda > 0$;

(3) orthogonal transformation: If $q \in \Sigma$ then $q(Px) \in \Sigma$ where $P \in \mathcal{O}_5$;

(4) Kelvin transformation: If $q \in \Sigma$ then $|x|^{-3} q\big(\frac{x}{|x|^2}\big) \in \Sigma$.

Let M be the group of one-to-one maps of $\mathbb{R}^5$ generated by the above four transformations. Then, from [48, Section 3], M generates a 21-parameter family of transformations in a neighborhood of the identity. More precisely, we give explictly the formula for this 21-parameter family of transformations. Set the one-to-one map

$$\tau : \big\{(i,j) \in \mathbb{N}^2 : 1 \leqslant i < j \leqslant 5\big\} \to \{1,2,\dots,10\},$$

with

$$\tau(i,j) = 3i + j - \frac{(i-1)(i-2)}{2} - 4.$$

For $c = (c_1, c_2, \dots, c_{10})$, set

$$A_c = \big[a_{i,j}^c\big]_{1 \leqslant i \leqslant j \leqslant 5}, \quad e^{A_c} = \sum_{n=0}^{\infty} \frac{A_c^n}{n!} \in \mathcal{SO}_5,$$

where $a^c_{i,i} = 0$, $a_{i,j} = c_{\tau(i,j)}$ if $i < j$ and $a_{i,j} = -c_{\tau(j,i)}$ if $j < i$. Note that this definition provides a parametrization of $\mathcal{SO}_5$ by $\mathbb{R}^{10}$ in a neighborhood of the identity matrix.

For $\mathcal{A} = (\lambda, \xi, a, c) \in (0, \infty) \times \mathbb{R}^5 \times \mathbb{R}^5 \times \mathbb{R}^{10}$, we introduce the following transform $\theta_{\mathcal{A}} \in M$,

$$\theta_{\mathcal{A}}(f)(x) = \lambda^{\frac{3}{2}} \left| \frac{x}{|x|} - a|x| \right|^{-3} f\left(\xi + \frac{\lambda e^{A_c}\left(x - a|x|^2\right)}{1 - 2\langle a, x\rangle + |a|^2|x|^2} \right) \quad \text{for all } f \in \dot{H}^1.$$

Observe that for all $q \in \Sigma$, $\widetilde{\mathcal{Z}}_q$ is generated by taking partial derivatives of $\theta_{\mathcal{A}}(q)$ with respect to $\mathcal{A} = (\lambda, \xi, a, c)$ at $\mathcal{A} = (1, 0, 0, 0)$ (see details in [48, Lemma 3.8]).

4.2.2 Spectral analysis of $\mathcal{L}$

Following [48], we gather some spectral properties of the linearized operator. For q be a non-degenerate decaying state and collinear speeds $\boldsymbol{\ell} = \ell \mathbf{e}_1$, let

$$q_\ell = q(x_\ell) = q\left(\frac{x_1}{\sqrt{1-\ell^2}}, x' \right), \quad -(1-\ell^2)\partial^2_{x_1} q_\ell - \bar{\Delta} q_\ell + |q_\ell|^{\frac{4}{3}} q_\ell = 0. \tag{4.5}$$

Define the following operator,

$$\mathcal{L} = -\Delta - f'(q), \quad \mathcal{L}_\ell = -(1-\ell^2)\partial^2_{x_1} - \bar{\Delta} - f'(q_\ell), \tag{4.6}$$

and

$$\mathcal{H}_\ell = \begin{pmatrix} -\Delta - f'(q_\ell) & -\ell\partial_{x_1} \\ \ell\partial_{x_1} & 1 \end{pmatrix}, \quad \mathrm{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.7}$$

Lemma 4.8 (Spectral properties of $\mathcal{L}$). (i) Spectrum. *The self-adjoint operator $\mathcal{L}$ has essential spectrum $[0, +\infty)$, a finite number $J \geqslant 1$ of negative eigenvalues (counted with multiplicity), and its kernel is $\widetilde{\mathcal{Z}}_q$. Let $(Y_j)_{j\in\{1,\dots,J\}}$ be an L^2 orthogonal family of eigenvectors of $\mathcal{L}$ corresponding to the eigenvalues $(-\lambda_j^2)_{j\in\{1,\dots,J\}}$, i.e. for $j, j' \in \{1, \dots, J\}$*

$$\mathcal{L}Y_j = -\lambda_j^2 Y_j, \quad \lambda_j > 0 \quad \textit{and} \quad \left(Y_j, Y_{j'}\right)_{L^2} = \delta_{jj'}.$$

Let Ψ_k be an $\dot{H}^1$-orthogonal basis of $\mathrm{Ker}\mathcal{L}$*, i.e. for any $k, k' \in \{1, \dots, K\}$*

$$(\Psi_k, \Psi_{k'})_{\dot{H}^1} = \delta_{kk'} \quad \textit{and} \quad \mathrm{Span}(\Psi_1, \dots, \Psi_K) = \widetilde{\mathcal{Z}}_q.$$

It holds, for all $k = 1, \dots, K$, $j = 1, \dots, J$ and $\alpha \in \mathbb{N}^5$ with $|\alpha| \leqslant 2$, on $\mathbb{R}^5$,

$$|\partial_x^\alpha Y_j(x)| \lesssim e^{-\lambda_j|x|} \quad \textit{and} \quad |\partial_x^\alpha \Psi_k(x)| \lesssim \langle x\rangle^{-(3+\alpha)}.$$

(ii) Nonnegativity under $(Y_j)_{j=1,\dots,J}$ orthogonality. *It holds*

$$(\mathcal{L}v, v)_{L^2} \geqslant 0 \quad \textit{for all } v \in N^\perp,$$

where

$$N^\perp = \left\{ v \in \dot{H}^1 : (v, Y_j)_{L^2} = 0, \ \textit{for any } j = 1, \dots, J \right\}.$$

(iii) Cancellation. *It holds*

$$\int_{\mathbb{R}} f''(q)\Psi_k\Psi_{k'}\Psi_{k''}\mathrm{d}x = 0 \quad \textit{for all } k, k', k'' = 1, \dots, K.$$

Proof. Proof of (i). For the spectral properties of $\mathcal{L}$, see the proof of [48, Claim 3.5]. The algebraic decay of kernel functions Ψ_k directly follows from the non-degenerate decaying condition. The exponential decay of eigenfunctions Y_j follows from standard elliptic arguments. See *e.g.* [94] and [48, Proposition 3.9].

Proof of (ii). The proof relies on a standard argument based on the min-max principle (see *e.g.* [48, Proposition 3.6]) and we omit it.

Proof of (iii). Without loss of generality, we consider

$$\Psi_{k''} = -3x_i q + |x|^2 \partial_{x_i} q - 2x_i x \cdot \nabla q.$$

For all $k' = 1, \dots, K$, we have

$$-\Delta \Psi_{k'} - f'(q)\Psi_{k'} = 0.$$

We consider the transformation $T_a = \theta_{\mathcal{A}}$ with $\theta_{\mathcal{A}} = (1, 0, a, 0)$ for the above identity,

$$-\Delta T_a \Psi_{k'} - f'(T_a q) T_a \Psi_{k'} = 0.$$

Taking the derivative of above identity with respect to a_i, and then letting $a = 0$, we obtain

$$f''(q)\Psi_{k'}\Psi_{k''} = -\Delta \tilde{\Psi}_{k'} - f'(q)\tilde{\Psi}_{k'} = \mathcal{L}\tilde{\Psi}_{k'},$$

where

$$\tilde{\Psi}_{k'} = -3x_i \Psi_{k'} + |x|^2 \partial_{x_i} \Psi_{k'} - 2x_i (x \cdot \nabla \Psi_{k'}).$$

Therefore, by integration by parts, for any $k = 1, \dots, K$,

$$\int_{\mathbb{R}^5} \Psi_k \left(f''(q)\Psi_{k'}\Psi_{k''} \right) \mathrm{d}x = \int_{\mathbb{R}^5} \Psi_k \left(\mathcal{L}\tilde{\Psi}_{k'} \right) \mathrm{d}x = \int_{\mathbb{R}^5} \left(\mathcal{L}\Psi_k \right) \tilde{\Psi}_{k'} \mathrm{d}x = 0.$$

Using the non-degenerate condition and proceeding similarly for all the parameters in the transformation $\theta_{\mathcal{A}}$, we complete the proof of (iii). □

4.2.3 Coercivity of $\mathcal{H}_\ell$

For $-1 < \ell < 1$, let

$$\vec{\Psi}_{k,\ell} = \begin{pmatrix} \Psi_k(x_\ell) \\ -\ell \partial_{x_1} \Psi_k(x_\ell), \end{pmatrix} \quad \text{for } k = 1, \dots, K,$$

and for $j = 1, \dots, J$

$$Y_{j,\ell} = Y_j(x_\ell), \quad \vec{Y}^0_{j,\ell} = \begin{pmatrix} Y_{j,\ell} \\ 0 \end{pmatrix}, \quad \vec{Y}^{\pm}_{j,\ell} = \begin{pmatrix} Y_{j,\ell} e^{\mp \frac{\ell \lambda_j}{\sqrt{1-\ell^2}} x_1} \\ -(\ell \partial_{x_1} Y_{j,\ell} \mp \frac{\lambda_j}{\sqrt{1-\ell^2}} Y_{j,\ell}) e^{\mp \frac{\ell \lambda_j}{\sqrt{1-\ell^2}} x_1} \end{pmatrix}.$$

Following [27], we define the function $\vec{W}_{j,\ell}$ by

$$\vec{W}_{j,\ell} = \vec{Y}^+_{j,\ell} + \vec{Y}^-_{j,\ell}, \quad \text{for } j = 1, \dots, J.$$

First, we prove the following technical identities of $\vec{Y}^{\pm}_{j,\ell}$ and $\vec{W}_{j,\ell}$.

Lemma 4.9. *The functions $\vec{Y}^{\pm}_{j,\ell}$ and $\vec{W}_{j,\ell}$ satisfy the following properties.*

(i) *For $j = 1, \dots, J$,*

$$\mathcal{H}_\ell \vec{Y}^{\pm}_{j,\ell} = \mp \lambda_j (1 - \ell^2)^{\frac{1}{2}} \mathrm{J} \vec{Y}^{\pm}_{j,\ell}. \tag{4.8}$$

(ii) *For* $j = 1, \dots, J$,

$$\left(\mathcal{H}_\ell \vec{W}_{j,\ell}, \vec{W}_{j,\ell}\right)_{L^2} = -4\lambda_j^2 (1-\ell^2)^{\frac{1}{2}}. \tag{4.9}$$

(iii) *For* $j, j' = 1, \dots, J$ *with* $j \neq j'$,

$$\left(\mathcal{H}_\ell \vec{W}_{j,\ell}, \vec{W}_{j',\ell}\right)_{L^2} = 0. \tag{4.10}$$

Proof. Proof of (i). On the one hand, from (i) of Lemma 4.8 and direct computation,

$$\begin{aligned}
&\left(-\Delta - f'(q_\ell)\right) Y_{j,\ell} e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} + \ell \partial_{x_1} \left(\left(\ell \partial_{x_1} Y_{j,\ell} \mp \frac{\lambda_j}{\sqrt{1-\ell^2}} Y_{j,\ell} \right) e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} \right) \\
&= \left(\mathcal{L}_\ell Y_{j,\ell}\right) e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} \pm \ell (1-\ell^2)^{\frac{1}{2}} \lambda_j (\partial_{x_1} Y_{j,\ell}) e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} \\
&= \pm \lambda_j (1-\ell^2)^{\frac{1}{2}} \left(\ell \partial_{x_1} Y_{j,\ell} \mp \frac{\lambda_j}{\sqrt{1-\ell^2}} Y_{j,\ell} \right) e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1}.
\end{aligned}$$

On the other hand, by direct computation,

$$\begin{aligned}
&\ell \partial_{x_1} \left(Y_{j,\ell} e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} \right) - \left(\ell \partial_{x_1} Y_{j,\ell} \mp \frac{\lambda_j}{\sqrt{1-\ell^2}} Y_{j,\ell} \right) e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} \\
&= \left(\pm \frac{\lambda_j}{\sqrt{1-\ell^2}} \mp \frac{\ell^2 \lambda_j}{\sqrt{1-\ell^2}} \right) Y_{j,\ell} e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1} = \pm \lambda_j (1-\ell^2)^{\frac{1}{2}} Y_{j,\ell} e^{\mp \frac{\ell\lambda_j}{\sqrt{1-\ell^2}} x_1}.
\end{aligned}$$

Gathering above identities, we obtain (4.8).

Proof of (ii). To prove (4.9), we first show that for $j, j' = 1, \dots, J$,

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^+_{j',\ell}\right)_{L^2} = \left(\mathcal{H}_\ell \vec{Y}^-_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = 0. \tag{4.11}$$

On the one hand, by (4.8), for $j, j' = 1, \dots, J$,

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^+_{j',\ell}\right)_{L^2} = -\lambda_j (1-\ell^2)^{\frac{1}{2}} \left(\mathrm{J}\vec{Y}^+_{j,\ell}, \vec{Y}^+_{j',\ell}\right)_{L^2}.$$

On the other hand, using again (4.8), for $j, j' = 1, \dots, J$,

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^+_{j',\ell}\right)_{L^2} = \left(\vec{Y}^+_{j,\ell}, \mathcal{H}_\ell \vec{Y}^+_{j',\ell}\right)_{L^2} = \lambda_{j'} (1-\ell^2)^{\frac{1}{2}} \left(\mathrm{J}\vec{Y}^+_{j,\ell}, \vec{Y}^+_{j',\ell}\right)_{L^2}.$$

Since $\lambda_j, \lambda_{j'} > 0$, this implies $\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^+_{j',\ell}\right)_{L^2} = 0$. The proof of (4.11) for $\left(\mathcal{H}_\ell \vec{Y}^-_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = 0$ follows from similar arguments and it is omitted.

Now, we start to prove (4.9). From (4.11), we have

$$\left(\mathcal{H}_\ell \vec{W}_{j,\ell}, \vec{W}_{j,\ell}\right)_{L^2} = 2\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j,\ell}\right)_{L^2}.$$

Using (4.8) and the explicit expression of $\vec{Y}^\pm_{j,\ell}$, we compute

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j,\ell}\right)_{L^2} = -\lambda_j (1-\ell^2)^{\frac{1}{2}} \left(\mathrm{J}\vec{Y}^+_{j,\ell}, \vec{Y}^-_{j,\ell}\right)_{L^2} = -2\lambda_j^2 (1-\ell^2)^{\frac{1}{2}} \left(Y_j, Y_j\right)_{L^2}.$$

Therefore, the identity (4.9) follows from the normalization $(Y_j, Y_j)_{L^2} = 1$.

Proof of (iii). To prove (4.10), we first show that for $j, j' = 1, \dots, J$ with $j \neq j'$,

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = 0 \quad \text{when} \quad \lambda_j \neq \lambda_{j'}, \tag{4.12}$$

and

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} + \left(\mathcal{H}_\ell \vec{Y}^+_{j',\ell}, \vec{Y}^-_{j,\ell}\right)_{L^2} = 0 \quad \text{when} \quad \lambda_j = \lambda_{j'}. \tag{4.13}$$

Let $j \neq j'$. In the case where $\lambda_j \neq \lambda_{j'}$. On the one hand, by (4.8), for $j \neq j'$,

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = -\lambda_j (1-\ell^2)^{\frac{1}{2}} \left(\mathrm{J}\vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2}.$$

On the other hand, using again (4.8),

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = \left(\vec{Y}^+_{j,\ell}, \mathcal{H}_\ell \vec{Y}^-_{j',\ell}\right)_{L^2} = -\lambda_{j'} (1-\ell^2)^{\frac{1}{2}} \left(\mathrm{J}\vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2}.$$

Since $\lambda_j \neq \lambda_{j'}$, this implies (4.12).

In the case where $\lambda_j = \lambda_{j'}$. From (4.8) and the explicit expression of $\vec{Y}^\pm_{j,\ell}$, $\left(Y_j, Y_{j'}\right)_{L^2} = 0$ and integration by parts, we compute

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = -\lambda_j (1-\ell^2)^{\frac{1}{2}} \left(\mathrm{J}\vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} = -2\ell\lambda_j (1-\ell^2)^{\frac{1}{2}} \left(\partial_{x_1} Y_j, Y_{j'}\right)_{L^2}.$$

Therefore, by (4.11) and integration by parts,

$$\left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} + \left(\mathcal{H}_\ell \vec{Y}^+_{j',\ell}, \vec{Y}^-_{j,\ell}\right)_{L^2} = 0,$$

which proves (4.13).

Now, we start to prove (4.10). Let $j \neq j'$. From (4.11),

$$\left(\mathcal{H}_\ell \vec{W}_{j,\ell}, \vec{W}_{j',\ell}\right)_{L^2} = \left(\mathcal{H}_\ell \vec{Y}^+_{j,\ell}, \vec{Y}^-_{j',\ell}\right)_{L^2} + \left(\mathcal{H}_\ell \vec{Y}^+_{j',\ell}, \vec{Y}^-_{j,\ell}\right)_{L^2}.$$

Thus, the identity (4.10) directly follows from (4.12) and (4.13). □

Second, we prove the following coercivity property.

Proposition 4.10. *There exist $\mu > 0$ such that, for all $\vec{v} \in \dot{H}^1 \times L^2$*

$$(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geqslant \mu \|\vec{v}\|^2_{\mathcal{H}} - \mu^{-1} \left(\sum_{k=1}^{K} \left(\vec{v}, \vec{\Psi}_{k,\ell}\right)^2_{\mathcal{H}} + \sum_{\pm, j=1}^{J} \left(\mathcal{H}_\ell \vec{v}, \vec{Y}^\pm_{j,\ell}\right)^2_{L^2} \right). \tag{4.14}$$

Proof. By a standard argument, it suffices to prove that there exists $\mu > 0$ such that for any $\vec{v} \in \dot{H}^1 \times L^2$ with orthogonality conditions $\left(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell}\right)_{L^2} = \left(\vec{v}, \vec{\Psi}_{k,\ell}\right)_{\mathcal{H}} = 0$ for all $j = 1, \dots, J$ and $k = 1, \dots, K$ there holds

$$(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geqslant \mu \|\vec{v}\|^2_{\mathcal{H}}. \tag{4.15}$$

Step 1. Negative direction. Note that, for $\vec{v} = (v, z) \in \dot{H}^1 \times L^2$,

$$\mathcal{H}_\ell \vec{v} = \begin{pmatrix} -\Delta v - f'(q_\ell) v - \ell \partial_{x_1} z \\ \ell \partial_{x_1} v + z \end{pmatrix}, \quad (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} = (\mathcal{L}_\ell v, v)_{L^2} + \|\ell \partial_{x_1} v + z\|^2_{L^2}. \tag{4.16}$$

Therefore, from (ii) of Lemma 4.8, for any $\vec{v} \in \dot{H}^1 \times L^2$,

$$\left(\vec{v}, \vec{Y}^0_{j,\ell}\right)_{L^2} = 0 \text{ for } j = 1, \dots, J \implies (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geqslant 0. \tag{4.17}$$

We claim, for any $\vec{v} \in \dot{H}^1 \times L^2$,

$$\left(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell}\right)_{L^2} = 0 \text{ for } j = 1, \dots, J \implies (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geqslant 0. \tag{4.18}$$

For the sake of contradiction, assume that there exists a vector $\vec{v} \in \dot{H}^1 \times L^2$ satisfying the orthogonality conditions in (4.18) and $(\mathcal{H}_\ell \vec{v}, \vec{v}) < 0$. For any $(c_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}/\mathbf{0}$, set

$$\vec{h} = c_0 \vec{v} + \sum_{j=1}^{J} c_j \vec{W}_{j,\ell} \in \operatorname{Span}\left(\vec{v}, \vec{W}_{1,\ell}, \dots, \vec{W}_{J,\ell}\right).$$

By direct computation, (4.9), (4.10) and the orthogonality conditions in (4.18),

$$\begin{aligned}\left(\mathcal{H}_\ell \vec{h}, \vec{h}\right)_{L^2} &= c_0^2 \left(\mathcal{H}_\ell \vec{v}, \vec{v}\right)_{L^2} + \sum_{j=1}^{J} c_j^2 \left(\mathcal{H}_\ell \vec{W}_{j,\ell}, \vec{W}_{j,\ell}\right)_{L^2} \\ &= c_0^2 \left(\mathcal{H}_\ell \vec{v}, \vec{v}\right)_{L^2} - 4(1-\ell^2)^{\frac{1}{2}} \sum_{j=1}^{J} c_j^2 \lambda_j^2 < 0.\end{aligned}$$

It follows that $(\mathcal{H}_\ell \cdot, \cdot)_{L^2} < 0$ on $\operatorname{Span}\left(\vec{v}, \vec{W}_{1,\ell}, \dots, \vec{W}_{J,\ell}\right)$ and

$$\dim \operatorname{Span}\left(\vec{v}, \vec{W}_{1,\ell}, \dots, \vec{W}_{J,\ell}\right) = J+1.$$

This is contradictory with (4.17) which says that $(\mathcal{H}_\ell \cdot, \cdot)_{L^2}$ is nonnegative under only J orthogonality conditions. The proof of (4.18) is complete.

Step 2. Null direction. Note that, from (4.16), for any $\vec{v} = (v, z) \in \operatorname{Ker}\mathcal{H}_\ell$,

$$-\Delta v - f'(q_\ell) v - \ell \partial_{x_1} z = 0, \quad \ell \partial_{x_1} v + z = 0,$$

which is equivalent to

$$-\Delta v + \ell^2 \partial_{x_1}^2 v - f'(q_\ell) v = 0, \quad z = -\ell \partial_{x_1} v.$$

Thus,

$$\operatorname{Ker}\mathcal{H}_\ell = \operatorname{Span}\left(\vec{\Psi}_{k,\ell}, k = 1, \dots, K\right).$$

We claim, for all $\vec{v} = (v, z) \in \dot{H}^1 \times L^2$ with orthogonality conditions $(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell})_{L^2} = (\vec{v}, \vec{\Psi}_{k,\ell})_{\mathcal{H}} = 0$ for $j = 1, \dots, J$ and $k = 1, \dots, K$,

$$\vec{v} = 0 \quad \text{or} \quad (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} > 0. \tag{4.19}$$

Denote

$$\mathcal{N}_\ell^\perp = \left\{\vec{v} = (v, z) \in \dot{H}^1 \times L^2 : \left(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell}\right)_{L^2} = 0 \ \text{ for } j = 1, \dots, J\right\}.$$

Indeed, it suffices to prove that for any $\vec{v} \in \mathcal{N}_\ell^\perp$

$$(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} = 0 \implies \vec{v} \in \operatorname{Ker}\mathcal{H}_\ell.$$

Fix $\vec{v} \in \mathcal{N}_\ell^\perp$ with $(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} = 0$. From $(\mathcal{H}_\ell \cdot, \cdot)_{L^2} \geqslant 0$ on $\mathcal{N}_\ell^\perp$ and the Cauchy-Schwarz inequality for $(\mathcal{H}_\ell \cdot, \cdot)_{L^2}$,

$$(\mathcal{H}_\ell \vec{v}, \vec{f})_{L^2} = 0 \quad \text{for all } \vec{f} = (f, g) \in \mathcal{N}_\ell^\perp. \tag{4.20}$$

For any $\vec{\tilde{v}} = (\tilde{v}, \tilde{z}) \in C_0^\infty(\mathbb{R}^5) \times C_0^\infty(\mathbb{R}^5)$, we decompose

$$\vec{\tilde{v}} = \vec{f} + \sum_{j=1}^{J} \alpha_j \vec{W}_{j,\ell} \quad \text{where } \vec{f} \in \mathcal{N}_\ell^\perp \ \text{ and } \ \alpha_j = -\frac{1}{4} \lambda_j^{-2} (1-\ell^2)^{-\frac{1}{2}} \left(\mathcal{H}_\ell \vec{\tilde{v}}, \vec{W}_{j,\ell}\right)_{L^2}.$$

Thus, from (4.20),

$$\left(\mathcal{H}_\ell \vec{v}, \vec{\tilde{v}}\right)_{L^2} = \left(\mathcal{H}_\ell \vec{v}, \vec{f}\right)_{L^2} + \sum_{j=1}^{J} \alpha_j \left(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell}\right)_{L^2} = 0.$$

It follows that $\mathcal{H}_\ell \vec{v} = 0$ in the sense of distribution, *i.e.* $\vec{v} \in \mathrm{Ker}\mathcal{H}_\ell$. The proof of (4.19) is complete.

Step 3. Conclusion. Now, we prove (4.15) by contradiction and using standard compactness argument. For the sake of contradiction, assume that there exists a sequence

$$\{\vec{v}_n = (v_n, z_n)\}_{n\in\mathbb{N}} \in \dot{H}^1 \times L^2$$

such that

$$\vec{v}_n \in \mathcal{N}_\ell^\perp, \quad (\vec{v}_n, \vec{\Psi}_{k,\ell})_{\mathcal{H}} = 0 \quad \text{for } k = 1, \dots, K, \tag{4.21}$$

and

$$0 < (\mathcal{H}_\ell \vec{v}_n, \vec{v}_n)_{L^2} < \frac{1}{n}\|\vec{v}_n\|_{\mathcal{H}}^2, \quad \int_{\mathbb{R}^5} f'(q_\ell) v_n^2 = 1 \quad \text{for any } n \in \mathbb{N}. \tag{4.22}$$

From the above inequalities and (4.16), the sequence $\{\vec{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\dot{H}^1 \times L^2$. Upon extracting a subsequence, we can assume

$$\vec{v}_n = (v_n, z_n) \rightharpoonup \vec{v} = (v, z) \in \dot{H}^1 \times L^2 \quad \text{as } n \to \infty. \tag{4.23}$$

On the one hand, by the Rellich theorem, we have

$$\int_{\mathbb{R}^5} f'(q_\ell) v^2 \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^5} f'(q_\ell) v_n^2 \mathrm{d}x = 1,$$

which implies $\vec{v} \neq 0$. On the other hand, from $\vec{v}_n = (v_n, z_n) \rightharpoonup \vec{v} = (v, z) \in \dot{H}^1 \times L^2$,

$$\int_{\mathbb{R}^5} (\ell \partial_{x_1} v + z)^2 \mathrm{d}x \leqslant \varliminf_{n\to\infty} \int_{\mathbb{R}^5} (\ell \partial_{x_1} v_n + z_n)^2 \mathrm{d}x,$$

$$\int_{\mathbb{R}^5} \left((1-\ell^2)(\partial_{x_1} v)^2 + |\overline{\nabla} v|^2\right) \mathrm{d}x \leqslant \varliminf_{n\to\infty} \int_{\mathbb{R}^5} \left((1-\ell^2)(\partial_{x_1} v_n)^2 + |\overline{\nabla} v_n|^2\right) \mathrm{d}x.$$

Combining the above inequalities with (4.16) and (4.22),

$$(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \leqslant \varliminf_{n\to\infty} (\mathcal{H}_\ell \vec{v}_n, \vec{v}_n)_{L^2} \leqslant \varliminf_{n\to\infty} \frac{1}{n}\|\vec{v}_n\|_{\mathcal{H}}^2 \leqslant 0.$$

Using again (4.23) and taking limit in (4.21), we have

$$\vec{v} \in \mathcal{N}^\perp, \quad (\vec{v}, \vec{\Psi}_{k,\ell})_{\mathcal{H}} = 0 \quad \text{for } k = 1, \dots, K,$$

Therefore, from (4.19), we obtain $\vec{v} = 0$ which is a contradiction. The proof of (4.15) is complete. □

4.3 Decomposition around the sum of N solitons

We prove in this section a general decomposition result around the sum of N solitons. Let $N \geqslant 1$ and for any $n \in \{1, \dots, N\}$, let $\boldsymbol{\ell}_n = \ell_n \mathbf{e}_1$ where $-1 < \ell_n < 1$ and $\ell_n \neq \ell_{n'}$ for $n \neq n'$. Let $q_1, \dots, q_N$ be any non-degenerate decaying excited states. Denote by I and I^0 the following two sets of indices

$$I^0 = \{(n,k) : n = 1, \dots, N,\ k = 1, \dots, K_n\}, \quad |I^0| = \mathrm{Card}\, I^0 = \sum_{n=1}^N K_n,$$

$$I = \{(n,j) : n = 1, \dots, N,\ j = 1, \dots, J_n\}, \quad |I| = \mathrm{Card}\, I = \sum_{n=1}^N J_n.$$

We denote by $(\lambda_{n,j})_{(n,j)\in I}$, $\left(Y_{(n,j)}\right)_{(n,j)\in I}$, $\left(\Psi_{(n,k)}\right)_{(n,k)\in I^0}$ the negative eigenvalues, corresponding eigenfunctions and kernel functions for q_n as defined in Lemma 4.8.

For $n = 1, \dots, N$, set

$$Q_n(t,x) = q_{n,\ell_n}(x - \boldsymbol{\ell}_n t), \quad \vec{Q}_n(t,x) = \begin{pmatrix} Q_n(t,x) \\ -\ell_n \partial_{x_1} Q_n(t,x) \end{pmatrix}.$$

Similarly, for $(n,k) \in I^0$,

$$\Psi_{n,k}(t,x) = \Psi_{(n,k),\ell_n}(x - \boldsymbol{\ell}_n t), \quad \vec{\Psi}_{n,k}(t,x) = \vec{\Psi}_{(n,k),\ell_n}(x - \boldsymbol{\ell}_n t),$$

and for $(n,j) \in I$,

$$\vec{Y}^{\pm}_{n,j}(t,x) = \vec{Y}^{\pm}_{(n,j),\ell_n}(x - \boldsymbol{\ell}_n t), \quad \vec{Z}^{\pm}_{n,j} = \mathcal{H}_n \vec{Y}^{\pm}_{n,j},$$

where

$$\mathcal{H}_n = \begin{pmatrix} -\Delta - f'(Q_n) & -\ell_n \partial_{x_1} \\ \ell_n \partial_{x_1} & 1 \end{pmatrix}.$$

Consider a time dependent C^1 function $\boldsymbol{b}$ of the form

$$\boldsymbol{b} = (b_{n,k})_{(n,k)\in I^0} \in \mathbb{R}^{|I_0|} \quad \text{with} \quad |\boldsymbol{b}| \ll 1.$$

We introduce

$$U = \sum_{n=1}^{N} Q_n, \quad V = \sum_{(n,k)\in I^0} b_{n,k} \Psi_{n,k},$$

$$G = f(U+V) - \sum_{n=1}^{N} f(Q_n) - \sum_{(n,k)\in I^0} b_{n,k} f'(Q_n) \Psi_{n,k}.$$

First, we start with a technical lemma.

Lemma 4.11. *Let W_1 and W_2 be continuous functions such that,*

$$|W_1(x)| + |W_2(x)| \lesssim \langle x \rangle^{-4} \quad \text{on } \mathbb{R}^5.$$

Define

$$W_{n_1}(t,x) = W_1(x - \boldsymbol{\ell}_{n_1} t), \quad W_{n_2}(t,x) = W_2(x - \boldsymbol{\ell}_{n_2} t).$$

Let $0 < \alpha_1 \leqslant \alpha_2$ be such that $\alpha_1 + \alpha_2 > \frac{5}{4}$. There exists $T_0 \gg 1$ such that, for all $n_1, n_2 \in \{1, \dots, N\}$ with $n_1 \neq n_2$ and $t \geqslant T_0$, the following hold.

(i) *If $\alpha_2 > \frac{5}{4}$,*

$$\int_{\mathbb{R}^5} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-4\alpha_1}. \tag{4.24}$$

(ii) *If $\alpha_2 \leqslant \frac{5}{4}$,*

$$\int_{\mathbb{R}^5} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{5-4(\alpha_1+\alpha_2)}. \tag{4.25}$$

Proof. For $k = 1, 2$, we denote

$$\rho_k = x - \boldsymbol{\ell}_{n_k} t, \quad \Omega_k(t) = \left\{ x \in \mathbb{R}^5 : |\rho_k| \leqslant 10^{-1} |\boldsymbol{\ell}_{n_1} - \boldsymbol{\ell}_{n_2}| t \right\}.$$

Let $T_0 \gg 1$ large enough. For $t \geqslant T_0$, from the decay properties of W_1 and W_2,

$$|W_{n_2}(t,x)| \lesssim \langle \rho_2 \rangle^{-4} \lesssim (\langle \rho_1 \rangle + t)^{-4}, \quad \text{for } x \in \Omega_1, \tag{4.26}$$

$$|W_{n_1}(t,x)| \lesssim \langle \rho_1 \rangle^{-4} \lesssim (\langle \rho_2 \rangle + t)^{-4}, \quad \text{for } x \in \Omega_2, \tag{4.27}$$

$$|W_{n_1}(t,x)| \lesssim (\langle \rho_1 \rangle + t)^{-4} \lesssim t^{-4}, \qquad \text{for } x \in \Omega_1^C, \tag{4.28}$$

$$|W_{n_2}(t,x)| \lesssim (\langle \rho_2 \rangle + t)^{-4} \lesssim t^{-4}, \qquad \text{for } x \in \Omega_2^C. \tag{4.29}$$

Proof of (i). *Case* $\alpha_1 > \frac{5}{4}, \alpha_2 > \frac{5}{4}$. From (4.26) and (4.28), we obtain

$$\int_{\Omega_1} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-4\alpha_2} \int_{\Omega_1} |W_{n_1}|^{\alpha_1} \mathrm{d}x \lesssim t^{-4\alpha_2},$$

$$\int_{\Omega_1^C} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-4\alpha_1} \int_{\Omega_1^C} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-4\alpha_1},$$

which implies (4.24).

Case $0 < \alpha_1 \leqslant \frac{5}{4}, \alpha_2 > \frac{5}{4}$. By (4.26) and change of variable,

$$\begin{aligned}\int_{\Omega_1} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x &\lesssim \int_{\mathbb{R}^5} \langle \rho_1 \rangle^{-4\alpha_1} (\langle \rho_1 \rangle + t)^{-4\alpha_2} \mathrm{d}x \\ &\lesssim \int_{\mathbb{R}^5} \langle x \rangle^{-4\alpha_1} \left(\langle x \rangle + t\right)^{-4\alpha_2} \mathrm{d}x \\ &\lesssim t^{5-4(\alpha_1+\alpha_2)} \lesssim t^{-4\alpha_1}.\end{aligned}$$

Using again (4.28),

$$\int_{\Omega_1^C} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-4\alpha_1} \int_{\Omega_1^C} |W_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-4\alpha_1}.$$

These estimates imply (4.24).

Proof of (ii). First, from (4.26), (4.27) and change of variable, as before,

$$\begin{aligned}&\int_{\Omega_1} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x + \int_{\Omega_2} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x \\ &\lesssim \int_{\mathbb{R}^5} \left(\langle x \rangle^{-4\alpha_1} \left(\langle x \rangle + t\right)^{-4\alpha_2} + \langle x \rangle^{-4\alpha_2} \left(\langle x \rangle + t\right)^{-4\alpha_1} \right) \mathrm{d}x \lesssim t^{5-4(\alpha_1+\alpha_2)}.\end{aligned}$$

Second, by (4.28), (4.29), the Hölder's inequality and change of variable,

$$\begin{aligned}\int_{(\Omega_1 \cup \Omega_2)^C} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \mathrm{d}x &\lesssim \left(\int_{\Omega_1^C} |W_{n_1}|^{\alpha_1+\alpha_2} \right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left(\int_{\Omega_2^C} |W_{n_2}|^{\alpha_1+\alpha_2} \right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \\ &\lesssim \int_{\mathbb{R}^5} \left(\langle x \rangle + t\right)^{-4(\alpha_1+\alpha_2)} \mathrm{d}x \lesssim t^{5-4(\alpha_1+\alpha_2)}.\end{aligned}$$

Gathering above estimates, we obtain (4.25). □

Second, we introduce some pointwise estimates following from Taylor expansion, and omit its proof.

Lemma 4.12. *The following estimates hold.*

(i) *For all* $n = 1, \ldots, N$,

$$\left| f'(U) - f'(Q_n) \right| \lesssim \sum_{n' \neq n} \left(|Q_{n'}| |Q_n|^{\frac{1}{3}} + |Q_{n'}|^{\frac{4}{3}} \right). \tag{4.30}$$

(ii) *We have*

$$\left| f'(U+V) - f'(U) \right| \lesssim \sum_{n=1}^{N} |V| \, |Q_n|^{\frac{1}{3}} + |V|^{\frac{4}{3}}. \tag{4.31}$$

(iii) *For all* $s \in \mathbb{R}$,

$$\big|f(U+V+s) - f(U+V) - f'(U+V)s\big| \lesssim \Big(|U|^{\frac{1}{3}} + |V|^{\frac{1}{3}}\Big)|s|^2 + |s|^{\frac{7}{3}}, \tag{4.32}$$

$$|F(U+V+s) - F(U+V) - f(U+V)s| \lesssim \Big(|U|^{\frac{4}{3}} + |V|^{\frac{4}{3}}\Big)|s|^2 + |s|^{\frac{10}{3}}. \tag{4.33}$$

Third, we state some preliminary estimates related to the nonlinear interaction. We decompose G as

$$G = G_1 + G_2 = G_{1,1} + G_{1,2} + G_{1,3} + G_2,$$

where

$$\begin{aligned} G_{1,1} &= f(U+V) - f(U) - f'(U)V - \frac{1}{2}f''(U)V^2, \\ G_{1,2} &= f'(U)V - \sum_{(n,k)\in I^0} b_{n,k} f'(Q_n)\Psi_{n,k}, \\ G_{1,3} &= f(U) - \sum_{n=1}^{N} f(Q_n) \quad \text{and} \quad G_2 = \frac{1}{2}f''(U)V^2. \end{aligned}$$

For $n = 1, \dots, N$, set

$$G_{3,n} = \frac{1}{2}\sum_{k,k'=1}^{K_n} b_{n,k}b_{n,k'} f''(Q_n)\Psi_{n,k}\Psi_{n,k'} \quad \text{and} \quad G_3 = \sum_{n=1}^{N} G_{3,n}.$$

Lemma 4.13. *There exists* $T_0 \gg 1$ *such that the following estimates hold.*

(i) Estimates on G_1. *For* $t \geqslant T_0$,

$$\|G_1\|_{L^2} \lesssim \|G_{1,1}\|_{L^2} + \|G_{1,2}\|_{L^2} + \|G_{1,3}\|_{L^2} \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^{\frac{7}{3}} + t^{-4}. \tag{4.34}$$

(ii) Expansion of G_2. *For* $t \geqslant T_0$,

$$\|G_2 - G_3\|_{L^2} \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^{\frac{7}{3}} + t^{-4}. \tag{4.35}$$

(iii) Estimate on G. *For* $t \geqslant T_0$,

$$\|G\|_{L^2} \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}. \tag{4.36}$$

Proof. Proof of (i). From Taylor formula, we have

$$\big|G_{1,1}\big| \lesssim \big|V\big|^{\frac{7}{3}} \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^{\frac{7}{3}}\big|\Psi_{n,k}\big|^{\frac{7}{3}}, \quad \big|G_{1,3}\big| \lesssim \sum_{n'\neq n} |Q_n|^{\frac{4}{3}}|Q_{n'}|,$$

$$\big|G_{1,2}\big| \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|\Big[\sum_{n'\neq n} |Q_n|^{\frac{1}{3}}|Q_{n'}||\Psi_{n,k}| + \sum_{n'\neq n} |Q_{n'}|^{\frac{4}{3}}|\Psi_{n,k}|\Big].$$

Based on above pointwise estimates, Lemma 4.11 (i), the Cauchy-Schwarz inequality and the Young's inequality, we obtain (4.34).

Proof of (ii). Observe that

$$V^2 = \sum_{n=1}^{N}\sum_{k,k'=1}^{K_n} b_{n,k}b_{n,k'}\Psi_{n,k}\Psi_{n,k'} + \sum_{n\neq n'}\sum_{k=1}^{K_n}\sum_{k'=1}^{K_{n'}} b_{n,k}b_{n,k'}\Psi_{n,k}\Psi_{n',k'}.$$

Thus, we can decompose

$$G_2 = G_3 + G_{2,1} + G_{2,2},$$

where

$$G_{2,1} = \frac{1}{2} \sum_{n \neq n'} \sum_{k=1}^{K_n} \sum_{k'=1}^{K_{n'}} b_{n,k} b_{n',k'} f''(U) \Psi_{n,k} \Psi_{n',k'},$$

$$G_{2,2} = \frac{1}{2} \sum_{n=1}^{N} \sum_{k,k'=1}^{K_n} b_{n,k} b_{n,k'} \big(f''(U) - f''(Q_n) \big) \Psi_{n,k} \Psi_{n,k'}.$$

By (i) of Lemma 4.11, we have

$$\begin{aligned} \|G_{2,1}\|_{L^2} &\lesssim \sum_{n \neq n'} \sum_{k=1}^{K_n} \sum_{k'=1}^{K_{n'}} |b_{n,k} b_{n',k'}| \||Q_n|^{\frac{1}{3}} \Psi_{n,k} \Psi_{n',k'}\|_{L^2} \\ &\quad + \sum_{n \neq n'} \sum_{k'=1}^{K_{n'}} b_{n',k'}^2 \||Q_n|^{\frac{1}{3}} \Psi_{n',k'}^2\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^{\frac{7}{3}} + t^{-4}. \end{aligned}$$

From Taylor formula,

$$\big| f''(U) - f''(Q_n) \big| \lesssim \sum_{n' \neq n} |Q_{n'}|^{\frac{1}{3}} \quad \text{for all } n = 1, \dots, N.$$

Thus, using again (i) of Lemma 4.11,

$$\|G_{2,2}\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 \Big(\sum_{n' \neq n} \||Q_{n'}|^{\frac{1}{3}} \Psi_{n,k}^2\|_{L^2} \Big) \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^{\frac{7}{3}} + t^{-4}.$$

Gathering above estimates, we obtain (4.35).

Proof of (iii). Estimate (4.36) is a consequence of (4.34) and (4.35). □

Last we prove a standard decomposition result around the sum of N solitons.

Proposition 4.14 (Properties of the decomposition). *There exist $T_0 \gg 1$ and $0 < \delta_0 \ll 1$ such that if $\vec{u}(t) = (u(t), \partial_t u(t))$ is a solution of (4.1) on $[T_1, T_2]$, where $T_0 \leqslant T_1 < T_2 < +\infty$, such that for any $t \in [T_1, T_2]$*

$$\|\vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\|_{\mathcal{H}} \leqslant \delta_0, \tag{4.37}$$

then there exists C^1 functions $\boldsymbol{b} = (b_{n,k})_{(n,k) \in I^0}$ on $[T_1, T_2]$ such that, $\vec{\varepsilon}(t)$ being defined by

$$\vec{\varepsilon}(t) = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t) - \sum_{(n,k) \in I^0} b_{n,k} \vec{\Psi}_{n,k} \tag{4.38}$$

the following hold on $[T_1, T_2]$,

(i) First properties of the decomposition. *For all $(n,k) \in I^0$ and $t \in [T_1, T_2]$,*

$$(\vec{\varepsilon}(t), \vec{\Psi}_{n,k}(t))_{\mathcal{H}} = 0, \quad |b_{n,k}(t)| \lesssim \|\vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\|_{\mathcal{H}}. \tag{4.39}$$

(ii) Equation of $\vec{\varepsilon}$. *It holds*

$$\begin{cases} \partial_t \varepsilon = \eta - \mathrm{Mod}_1, \\ \partial_t \eta = \Delta \varepsilon + f(U + V + \varepsilon) - f(U + V) - \mathrm{Mod}_2 + G \end{cases} \tag{4.40}$$

where

$$\mathrm{Mod}_1 = \sum_{(n,k)\in I^0} \dot{b}_{n,k}\Psi_{n,k} \quad \textit{and} \quad \mathrm{Mod}_2 = -\sum_{(n,k)\in I^0} \dot{b}_{n,k}\ell_n\partial_{x_1}\Psi_{n,k}.$$

(iii) Estiamtes for $\boldsymbol{b}$. *For* $t \in [T_1, T_2]$, *we have*

$$\sum_{(n,k)\in I^0} |\dot{b}_{n,k}(t)| \lesssim \|\vec{\varepsilon}(t)\|_{\mathcal{H}} + \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}. \tag{4.41}$$

(iv) Unstable directions. *Let* $a^{\pm}_{n,j} = \left(\vec{\varepsilon}, \vec{Z}^{\pm}_{n,j}\right)_{L^2}$ *for all* $(n,j) \in I$. *Then*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}a^{\pm}_{n,j}(t) \pm \alpha_{n,j}a^{\pm}_{n,j}(t)\right| \lesssim \|\vec{\varepsilon}(t)\|^2_{\mathcal{H}} + \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}. \tag{4.42}$$

where $\alpha_{n,j} = \lambda_{n,j}(1-\ell_n^2)^{\frac{1}{2}} > 0$.

Proof. Proof of (i). Let $T_0 \gg 1$, fix $t \geqslant T_0$. Note that, the orthogonality conditions in (4.39) is equivalent to the following matrix identity,

$$\mathcal{M}\boldsymbol{b} = \left(\left(\vec{u} - \sum_{n=1}^{N}\vec{Q}_n, \vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right)_{(n,k)\in I^0},$$

where $\boldsymbol{b} = (b_{n,k})_{(n,k)\in I^0}$ written in one row and

$$\mathcal{M} = \left(\left(\vec{\Psi}_{n,k}, \vec{\Psi}_{n',k'}\right)_{\mathcal{H}}\right)_{(n,k),(n',k')\in I^0}.$$

Moreover, from (ii) of Lemma 4.11 and the decay property of $\vec{\Psi}_{n,k}$,

$$\mathcal{M} = \mathrm{diag}\,(\mathcal{M}_1, \ldots, \mathcal{M}_N) + O(t^{-2}) \quad \text{where } \mathcal{M}_n = \left((\vec{\Psi}_{n,k}, \vec{\Psi}_{n,k'})_{\mathcal{H}}\right)_{k,k'=1,\ldots,K_n}.$$

Note that for fixed n, the family $\left(\vec{\Psi}_{n,k}\right)_{k=1,\ldots,K_n}$ being linearly independent, the Gram matrix $\mathcal{M}_n$ is invertible. Hence, $\mathcal{M}$ is invertible, for T_0 large enough, and $\mathcal{M}^{-1}$ has uniform norm in $t \geqslant T_0$. Therefore, we obtain

$$\boldsymbol{b} = \mathcal{M}^{-1}\left(\left(\vec{u} - \sum_{n=1}^{N}\vec{Q}_n, \vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right)_{(n,k)\in I^0}$$

and

$$|\boldsymbol{b}| \leqslant \|\mathcal{M}^{-1}\| \left\|\left(\left(\vec{u} - \sum_{n=1}^{N}\vec{Q}_n, \vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right)_{(n,k)\in I^0}\right\| \lesssim \left\|\vec{u}(t) - \sum_{n=1}^{N}\vec{Q}_n(t)\right\|_{\mathcal{H}}.$$

Proof of (ii). First, by the definition of $\vec{\varepsilon}$ in (4.38)

$$\begin{aligned}\partial_t\varepsilon &= \partial_t u - \sum_{n=1}^{N}\partial_t Q_n - \sum_{(n,k)\in I^0} b_{n,k}\partial_t\Psi_{n,k} - \sum_{(n,k)\in I^0}\dot{b}_{n,k}\Psi_{n,k}\\ &= \eta - \sum_{(n,k)\in I^0}\dot{b}_{n,k}\Psi_{n,k}.\end{aligned}$$

Second, using again (4.38) and (4.1),

$$\begin{aligned}\partial_t\eta &= \partial_t^2 u + \sum_{n=1}^{N}\partial_t\big(\ell_n\partial_{x_1}Q_n\big) + \sum_{(n,k)\in I^0} b_{n,k}\partial_t(\ell_n\partial_{x_1}\Psi_{n,k}) + \sum_{(n,k)\in I^0}\dot b_{n,k}\big(\ell_n\partial_{x_1}\Psi_{n,k}\big)\\ &= \Delta u + f(u) - \sum_{n=1}^{N}\ell_n^2\partial_{x_1}^2 Q_n - \sum_{(n,k)\in I^0} b_{n,k}\ell_n^2\partial_{x_1}^2\Psi_{n,k} + \sum_{(n,k)\in I^0}\dot b_{n,k}(\ell_n\partial_{x_1}\Psi_{n,k}).\end{aligned}$$

From (4.38), $-(1-\ell_n^2)\partial_{x_1}^2 Q_n - \bar\Delta Q_n - f(Q_n) = 0$, $-(1-\ell_n^2)\partial_{x_1}^2\Psi_{n,k} - \bar\Delta\Psi_{n,k} - f'(Q_n)\Psi_{n,k} = 0$ and the definition of G,

$$\begin{aligned}&\Delta u + f(u) - \sum_{n=1}^{N}\ell_n^2\partial_{x_1}^2 Q_n - \sum_{(n,k)\in I^0} b_{n,k}\ell_n^2\partial_{x_1}^2\Psi_{n,k}\\ &= \Delta\varepsilon + f(U+V+\varepsilon) - f(U+V) + G.\end{aligned}$$

Therefore,

$$\partial_t\eta = \Delta\varepsilon + f(U+V+\varepsilon) - f(U+V) + \sum_{(n,k)\in I^0}\dot b_{n,k}\ell_n\partial_{x_1}\Psi_{n,k} + G.$$

Proof of (iii). First, we decompose

$$f(U+V+\varepsilon) - f(U+V) = \sum_{n=1}^{N} f'(Q_n)\varepsilon + R_1 + R_2 + R_3,$$

where

$$R_1 = f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon,$$

$$R_2 = \big(f'(U+V) - f'(U)\big)\,\varepsilon, \quad R_3 = \left(f'(U) - \sum_{n=1}^{N} f'(Q_n)\right)\varepsilon.$$

Therefore, from (4.40), we obtain

$$\partial_t\vec\varepsilon = \vec{\mathcal{L}}\vec\varepsilon - \vec{\mathrm{Mod}} + \vec G + \vec R_1 + \vec R_2 + \vec R_3, \tag{4.43}$$

where

$$\vec{\mathcal{L}} = \begin{pmatrix} 0 & 1\\ \Delta + \sum_{n=1}^{N} f'(Q_n) & 0\end{pmatrix}, \quad \vec{\mathrm{Mod}} = \begin{pmatrix}\mathrm{Mod}_1\\ \mathrm{Mod}_2\end{pmatrix},$$

and

$$\vec G = \begin{pmatrix}0\\ G\end{pmatrix}, \quad \vec R_1 = \begin{pmatrix}0\\ R_1\end{pmatrix}, \quad \vec R_2 = \begin{pmatrix}0\\ R_2\end{pmatrix}, \quad \vec R_3 = \begin{pmatrix}0\\ R_3\end{pmatrix}.$$

We differentiate the orthogonality condition $(\vec\varepsilon, \vec\Psi_{n,k})_{\mathcal{H}} = 0$ in (4.39) and using (4.43),

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}t}(\vec\varepsilon, \vec\Psi_{n,k})_{\mathcal{H}} =& (\partial_t\vec\varepsilon, \vec\Psi_{n,k})_{\mathcal{H}} - (\vec\varepsilon, \partial_t\vec\Psi_{n,k})_{\mathcal{H}}\\ =& (\vec{\mathcal{L}}\vec\varepsilon, \vec\Psi_{n,k})_{\mathcal{H}} - \sum_{(n',k')\in I^0}\dot b_{n',k'}(\vec\Psi_{n,k}, \vec\Psi_{n',k'})_{\mathcal{H}} + \left(\vec G, \vec\Psi_{n,k}\right)_{\mathcal{H}}\\ &+ \left(\vec R_1, \vec\Psi_{n,k}\right)_{\mathcal{H}} + \left(\vec R_2 + \vec R_3, \vec\Psi_{n,k}\right)_{\mathcal{H}} - \left(\vec\varepsilon, \ell_n\partial_{x_1}\vec\Psi_{n,k}\right)_{\mathcal{H}}.\end{aligned}$$

By integration by parts and the decay properties of $\vec\Psi_{n,k}$, the first term is

$$(\vec{\mathcal{L}}\vec\varepsilon, \vec\Psi_{n,k})_{\mathcal{H}} = \left(\mathrm{J}\vec\varepsilon, \vec{\mathcal{L}}\mathrm{J}\vec\Psi_{n,k}\right)_{\mathcal{H}} = O\left(\|\vec\varepsilon\|_{\mathcal{H}}\right).$$

From (4.36) and the Cauchy-Schwarz inequality,

$$\left|\left(\vec{G}, \vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right| \lesssim \|G\|_{L^2} \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}.$$

Next, from (4.30), (4.31), (4.32), the Sobolev embedding theorem and the decay properties of $\vec{\Psi}_{n,k}$,

$$\left|\left(\vec{R}_1, \vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right| \lesssim \left|\left((|U|^{\frac{1}{3}} + |V|^{\frac{1}{3}})\varepsilon^2 + |\varepsilon|^{\frac{7}{3}}, \partial_{x_1}\Psi_{n,k}\right)_{L^2}\right| \lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^2,$$

$$\left|\left(\vec{R}_2 + \vec{R}_3, \vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right| + \left|\left(\vec{\varepsilon}, \ell_n\partial_{x_1}\vec{\Psi}_{n,k}\right)_{\mathcal{H}}\right| \lesssim \|R_2\|_{L^2} + \|R_3\|_{L^2} + \|\vec{\varepsilon}\|_{\mathcal{H}} \lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}.$$

Gathering above estimates and proceeding similarly for all $(n,k) \in I^0$,

$$\mathcal{M}\dot{\boldsymbol{b}} = O\left(\|\vec{\varepsilon}\|_{\mathcal{H}} + \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}\right),$$

where $\dot{\boldsymbol{b}} = \left(\dot{b}_{n,k}\right)_{(n,k)\in I^0}$ written in one row. Therefore, from the matrix $\mathcal{M}^{-1}$ being uniformly bounded, we obtain (4.41).

Proof of (iv). Using (4.43), we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_{n,j}^{\pm} &= \left(\partial_t\vec{\varepsilon}, \vec{Z}_{n,j}^{\pm}\right)_{L^2} + \left(\vec{\varepsilon}, \partial_t\vec{Z}_{n,j}^{\pm}\right)_{L^2}\\ &= \left(\vec{\mathcal{L}}\vec{\varepsilon}, \vec{Z}_{n,j}^{\pm}\right)_{L^2} - \ell_n\left(\vec{\varepsilon}, \partial_{x_1}\vec{Z}_{n,j}^{\pm}\right)_{L^2} - \left(\vec{\mathrm{Mod}}, \vec{Z}_{n,j}^{\pm}\right)_{L^2}\\ &+ \left(\vec{G}, \vec{Z}_{n,j}^{\pm}\right)_{L^2} + \left(\vec{R}_1, \vec{Z}_{n,j}^{\pm}\right)_{L^2} + \left(\vec{R}_2, \vec{Z}_{n,j}^{\pm}\right)_{L^2} + \left(\vec{R}_3, \vec{Z}_{n,j}^{\pm}\right)_{L^2}.\end{aligned}$$

From (4.8), integration by parts and $-\mathrm{J}^2$ is identity matrix, we have

$$\begin{aligned}\left(\vec{\mathcal{L}}\vec{\varepsilon}, \vec{Z}_{n,j}^{\pm}\right)_{L^2} - \ell_n\left(\vec{\varepsilon}, \partial_{x_1}\vec{Z}_{n,j}^{\pm}\right)_{L^2} &= -\left(\vec{\varepsilon}, \mathcal{H}_n\mathrm{J}\vec{Z}_{n,j}^{\pm}\right)_{L^2} + \sum_{n'\neq n}\left(\varepsilon, f'(Q_{n'})Z_{n,j}^{\pm}\right)_{L^2}\\ &= \mp\alpha_{n,j}a_{n,j}^{\pm} + \sum_{n'\neq n}\left(\varepsilon, f'(Q_{n'})Z_{n,j}^{\pm}\right)_{L^2},\end{aligned}$$

where

$$Z_{n,j}^{\pm} = \pm\alpha_{n,j}\left(Y_{(n,j),\ell_n}e^{\mp\frac{\ell_n\lambda_{n,j}}{\sqrt{1-\ell_n^2}}x_1}\right)(\cdot - \boldsymbol{\ell}_n t).$$

By Sobolev embedding theorem and (i) of Lemma 4.11,

$$\sum_{n'\neq n}\left|\left(\varepsilon, f'(Q_{n'})Z_{n,j}^{\pm}\right)_{L^2}\right| \lesssim \sum_{n'\neq n}\|\varepsilon\|_{L^{\frac{10}{3}}}\left\|f'(Q_{n'})Z_{n,j}^{\pm}\right\|_{L^{\frac{10}{7}}} \lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^2 + t^{-4}.$$

Note that, for all $(n,k) \in I^0$ and $(n,j) \in I$, we have

$$\left(\vec{\Psi}_{n,k}, \mathcal{H}_n\vec{Y}_{n,j}^{\pm}\right)_{L^2} = \left(\mathcal{H}_n\vec{\Psi}_{n,k}, \vec{Y}_{n,j}^{\pm}\right)_{L^2} = 0.$$

Therefore, from (i) of Lemma 4.11, (4.41) and concerning the term with $\vec{\mathrm{Mod}}$,

$$\left(\vec{\mathrm{Mod}}, \vec{Z}_{n,j}^{\pm}\right)_{L^2} = -\sum_{n'\neq n}\sum_{k=1}^{K_{n'}}\dot{b}_{n',k}\left(\vec{\Psi}_{n',k}, \vec{Z}_{n,j}^{\pm}\right)_{L^2} = O\left(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + \sum_{(n,k)\in I^0}|b_{n,k}|^2 + t^{-4}\right).$$

Next, from (4.36) and the Cauchy Schwarz inequality,

$$\left|\left(\vec{G}, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| \lesssim \|G\|_{L^2} \lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}.$$

Last, from (4.30), (4.31), (4.32), (i) of Lemma 4.11 and decay properties of $Z^{\pm}_{n,j}$,

$$\left|\left(\vec{R}_1, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| \lesssim \left|\left((|U|^{\frac{1}{3}} + |V|^{\frac{1}{3}})|\varepsilon|^2 + |\varepsilon|^{\frac{7}{3}}, Z^{\pm}_{n,j}\right)_{L^2}\right| \lesssim \|\vec{\varepsilon}\|^2_{\mathcal{H}},$$

$$\left|\left(\vec{R}_2, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| \lesssim \|\varepsilon\|_{L^{\frac{10}{3}}} \|f'(U+V) - f'(U)\|_{L^2} \lesssim \|\vec{\varepsilon}\|^2_{\mathcal{H}} + \sum_{(n,k)\in I^0} |b_{n,k}|^2,$$

$$\left|\left(\vec{R}_3, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| \lesssim \|\varepsilon\|_{L^{\frac{10}{3}}} \left\| \sum_{n'\neq n} \left(|Q_{n'}||Q_n|^{\frac{1}{3}} + |Q_{n'}|^{\frac{4}{3}}\right) Z^{\pm}_{n,j} \right\|_{L^{\frac{10}{7}}} \lesssim \|\vec{\varepsilon}\|^2_{\mathcal{H}} + t^{-4}.$$

Gathering above estimates and proceeding similarly for all $(n,j) \in I$, we obtain (4.42). □

4.4 Proof of Theorem 4.1

In this section, we prove the existence of a solution $\vec{u}(t)$ of (4.1) satisfying (4.4) in Theorem 4.3. We argue by compactness and obtain $\vec{u}(t)$ as the limit of suitable approximate multi-solitons $\vec{u}_m(t)$.

We start with a technical lemma which constructs well-prepared initial data at $t = T \gg 1$ with sufficient freedom related to unstable directions.

Lemma 4.15. *Let $T \gg 1$ and $C \gg 1$. For any $\boldsymbol{a} = (a_{n,j})_{(n,j)\in I} \in \mathbb{R}^{|I|}$, there exists*

$$A = (\tilde{a}_{n,j})_{(n,j)\in I} \in \mathbb{R}^{|I|}, \quad B = (\tilde{b}_{n,k})_{(n,k)\in I^0} \in \mathbb{R}^{|I_0|},$$

satisfying

$$\sum_{(n,j)\in I} |\tilde{a}_{n,j}| + \sum_{(n,k)\in I} |\tilde{b}_{n,k}| \leqslant C \sum_{(n,j)\in I} |a_{n,j}|, \tag{4.44}$$

such that the function $\vec{\varepsilon}(T)$ defined by

$$\vec{\varepsilon}(T) = \sum_{(n,j)\in I} \tilde{a}_{n,j}(\boldsymbol{a}) \vec{Z}^{+}_{n,j}(T) + \sum_{(n,k)\in I^0} \tilde{b}_{n,k}(\boldsymbol{a}) \vec{\Psi}_{n,k}(T), \tag{4.45}$$

satisfies for all $(n,k) \in I^0$, $(n,j) \in I$,

$$\left(\vec{\varepsilon}(T), \vec{\Psi}_{n,k}(T)\right)_{\mathcal{H}} = 0, \quad \left(\vec{\varepsilon}(T), \vec{Z}^{+}_{n,j}(T)\right)_{L^2} = a_{n,j}. \tag{4.46}$$

Moreover, the initial data defined by $\vec{u}(T) = \sum_{n=1}^{N} \vec{Q}_n(T) + \vec{\varepsilon}(T)$ is modulated in the sense of Proposition 4.14 with $b_{n,k}(T) = 0$ for all $(n,k) \in I^0$ and $a^{+}_{n,j}(T) = a_{n,j}$ for all $(n,j) \in I$.

Proof. Our goal is to solve for $A = (\tilde{a}_{n,j})_{(n,j)\in I}$ and $B = (\tilde{b}_{n,k})_{(n,k)\in I^0}$ in terms of $\boldsymbol{a} = (a_{n,j})_{(n,j)\in I}$. From the relations, for all $n = 1, \dots, N$,

$$\left(\vec{\Psi}_{n,k}, \vec{Z}^{+}_{n,j}\right)_{L^2} = \left(\mathcal{H}_n \vec{\Psi}_{n,k}, \vec{Y}^{+}_{n,j}\right)_{L^2} = 0$$

and for all $n \neq n'$,

$$\left(\vec{\Psi}_{n,k}(T), \vec{\Psi}_{n',k'}(T)\right)_{\mathcal{H}} = O(T^{-1}), \quad \left(\vec{\Psi}_{n,k}(T), \vec{Z}^{+}_{n',j}(T)\right)_{\mathcal{H}} = O(T^{-1}),$$

$$\left(\vec{Z}^+_{n,j}(T), \vec{\Psi}^+_{n',k}(T)\right)_{L^2} = O(T^{-1}), \quad \left(\vec{Z}^+_{n,j}(T), \vec{Z}^+_{n',j}(T)\right)_{L^2} = O(T^{-1}),$$

the conditions in (4.46) are equivalent to a linear relation between A and B of the following form, for all $(n,k) \in I^0$ and $(n,j) \in I$,

$$\begin{aligned}&\sum_{j=1}^{J_n} \left(\vec{\Psi}_{n,k}(T), \vec{Z}^+_{n,j}(T)\right)_{\mathcal{H}} \tilde{a}_{n,j} + \sum_{k'=1}^{K_n} \left(\vec{\Psi}_{n,k}(T), \vec{\Psi}_{n,k'}(T)\right)_{\mathcal{H}} \tilde{b}_{n,k'} \\ =&O\Big(\sum_{n'\neq n}\Big(\sum_{j'=1}^{J_{n'}} |\tilde{a}_{n',j'}| + \sum_{k'=1}^{K_{n'}} |\tilde{b}_{n',k'}|\Big)T^{-1}\Big),\end{aligned}$$

$$\sum_{j'=1}^{J_n} \left(\vec{Z}^+_{n,j'}, \vec{Z}^+_{n,j}\right)_{L^2} \tilde{a}_{n,j'} = a_{n,j} + O\Big(\sum_{n'\neq n}\Big(\sum_{j'=1}^{J_{n'}} |\tilde{a}_{n',j'}| + \sum_{k'=1}^{K_{n'}} |\tilde{b}_{n',k'}|\Big)T^{-1}\Big).$$

Since the families $\left(\vec{\Psi}_{n,k}\right)_{(n,k)\in I^0}$ and $\left(\vec{Z}^+_{n,j}\right)_{(n,j)\in I}$ are linear independent, the above linear system is invertible for T large enough. We obtain the existence and desired properties of $A = (\tilde{a}_{n,j})_{(n,j)\in I}$ and $B = \left(\tilde{b}_{n,k}\right)_{(n,k)\in I^0}$ for T large enough. □

To obtain weak convergence on any finite time interval, not just at initial time, we recall a proposition from [67, 69] which associates with weak continuity of the flow.

Proposition 4.16. *There exists a constant $\epsilon > 0$ such that the following holds. Let $\mathrm{K} \subset \dot{H}^1 \times L^2$ be a compact set and let $\vec{u}_m : [T_1, T_2] \to \dot{H}^1 \times L^2$ be a sequence of solutions of (4.1) such that*

$$\operatorname{dist}(\vec{u}_m(t), \mathrm{K}) \leqslant \epsilon, \quad \textit{for all } m \in \mathbb{N}^+ \textit{ and } t \in [T_1, T_2].$$

Suppose that $\vec{u}_m(T_1) \rightharpoonup \vec{u}_0$ weakly in $\dot{H}^1 \times L^2$. Then the solution $\vec{u}(t)$ of (4.1) with the initial data $\vec{u}(T_1) = \vec{u}_0$ is defined for $t \in [T_1, T_2]$ and

$$\vec{u}_m(T_1) \rightharpoonup \vec{u}(t), \quad \textit{weakly in } \dot{H}^1 \times L^2 \textit{ for all } t \in [T_1, T_2].$$

Proof. The proof relies on a standard argument based on the result of profile decomposition stated in [45, Proposition 2.8]. See more details in [67, Appendix A.2] and [69, Appendix]. □

Let $T_m = m$ for all $m \in \mathbb{N}^+$. For $\boldsymbol{a}_m = (a^m_{n,j})_{(n,j)\in I} \in \mathbb{R}^{|I|}$ small to be determined later, we consider the solution $\vec{u}_m \in C\big(T^m_{\max}; \dot{H}^1 \times L^2\big)$ with the initial data $\vec{u}_m(T_m)$ given by Lemma 4.15. Since $\vec{u}_m(T_m) \in X$, by persistence of regularity (see for instance [66, Appendix B]), we have $\vec{u}_m \in C\big(T^m_{\max}; X\big)$. Such regularity will allow energy computations without density argument, see §4.4.3.

The following Proposition is the main part of the proof of Theorem 4.3.

Proposition 4.17 (Uniform estimates)**.** *Under the assumptions of Theorem 4.3, there exist $m_0 \in \mathbb{N}^+$ and $T_0 \gg 1$ such that, for any $m \geqslant m_0$, there exist $\boldsymbol{a}_m = (a^m_{n,j})_{(n,j)\in I} \in \mathbb{R}^{|I|}$ such that the solution $\vec{u}_m$ of (4.1) with initial data $\vec{u}_m(T_m)$ given by Lemma 4.15 is well-defined in $\dot{H}^1 \times L^2$ on the time interval $[T_0, T_m]$ and satisfies*

$$\forall t \in [T_0, T_m], \quad \Big\|\vec{u}_m(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\Big\|_{\mathcal{H}} \leqslant t^{-\frac{8}{5}}. \tag{4.47}$$

4.4.1 Proof of Theorem 4.3 from Proposition 4.17

We follow the strategy by uniform estimates and compactness introduced in [69, 83, 84]. Consider the solution $\vec{u}_m(t)$ given by Proposition 4.17. On the interval $[T_0, T_m]$, from (4.47), we know that, there exists $T_0 \gg 1$ such that

$$\forall t \in [T_0, T_m], \quad \left\| \vec{u}_m(t) - \sum_{n=1}^{N} \vec{Q}_n(t) \right\|_{\mathcal{H}} \leqslant \epsilon,$$

where $\epsilon > 0$ is the constant of Proposition 4.16.

From the uniform estimates obtained in (4.47) at $T = T_0$, up to the extraction of a subsequence, there exists $\vec{u}_0 = (u_0, u_1) \in \dot{H}^1 \times L^2$ such that

$$\vec{u}_m(T_0) \rightharpoonup \vec{u}_0 \quad \text{in } \dot{H}^1 \times L^2 - weak \quad \text{as } m \to \infty.$$

Fix $T > T_0$. From Proposition 4.16 applied to the compact set

$$\mathrm{K} = \left\{ \sum_{n=1}^{N} \vec{Q}_n(t) \in \dot{H}^1 \times L^2, t \in [T_0, T] \right\},$$

the solution $\vec{u}(t)$ of (4.1) corresponding to $\vec{u}(T_0) = \vec{u}_0$ is well-defined and it holds $\vec{u}_m(t) \rightharpoonup \vec{u}(t)$ weakly in $\dot{H}^1 \times L^2$ on $[T_0, T]$. By (4.47) and the properties of weak convergence, the solution $\vec{u}$ satisfies, for all $t \in [T_0, T]$,

$$\left\| \vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t) \right\|_{\mathcal{H}} \leqslant \varliminf_{m \to \infty} \left\| \vec{u}_m(t) - \sum_{n=1}^{N} \vec{Q}_n(t) \right\|_{\mathcal{H}} \leqslant t^{-\frac{8}{5}}.$$

Since $T \geqslant T_0$ is arbitrary, the solution $\vec{u}$ is defined and satisfies the conclusion of Theorem 4.3 on $[T_0, \infty)$. The proof of Theorem 4.3 from Proposition 4.17 is complete.

The rest of this section is devoted to the proof of Proposition 4.17.

4.4.2 Bootstrap setting

For $0 < t \leqslant T_m$, as long as $\vec{u}_m(t)$ is well defined in $\dot{H}^1 \times L^2$ and satisfies (4.37), we decompose $\vec{u}_m(t)$ as in Proposition 4.14. In particular, we denote by $\vec{\varepsilon} = (\varepsilon, \eta)$, $\boldsymbol{b} = (b_{n,k})_{(n,k)\in I^0}$, $(a^{\pm}_{n,j})_{(n,j)\in I}$ the parameters of the decomposition of $\vec{u}_m$.

To prove Proposition 4.17, we introduce the following bootstrap estimates:

$$\begin{aligned} \|\vec{\varepsilon}(t)\|_{\mathcal{H}} \leqslant t^{-\frac{29}{10}}, \qquad \sum_{(n,k)\in I^0} |b_{n,k}(t)| \leqslant t^{-\frac{9}{5}}, \\ \sum_{(n,j)\in I} |a^{+}_{n,j}(t)|^2 \leqslant t^{-6}, \qquad \sum_{(n,j)\in I} |a^{-}_{n,j}(t)|^2 \leqslant t^{-\frac{59}{10}}. \end{aligned} \tag{4.48}$$

For $\boldsymbol{a}_m \in B_{\mathbb{R}^{|I|}}(T_m^{-3})$, set

$$T_*(\boldsymbol{a}_m) = \inf\{t \in [T_0, T_m]; \vec{u}_m \text{ satisfies (4.37) and (4.48) holds on } [t, T_m]\}. \tag{4.49}$$

Note that, for the proof of Proposition 4.17, it suffices to prove that there exists $T_0 \gg 1$ (independent with m) large enough and at least one choice of $\boldsymbol{a}_m \in B_{\mathbb{R}^{|I|}}(T_m^{-3})$ such that $T_*(\boldsymbol{a}_m) = T_0$.

4.4.3 Energy functional

Without loss of generality

$$-1 < \ell_1 < \cdots < \ell_N < 1.$$

Denote

$$\ell^* = \max_n(|\ell_n|) < 1.$$

For

$$0 < \delta < \frac{1}{100}\min_n(\ell_{n+1} - \ell_n)$$

small enough to be chosen later, we denote

$$\begin{aligned} \bar{\ell}_n &= \ell_n + \delta(\ell_{n+1} - \ell_n), \quad \text{for } n = 1, \dots, N-1, \\ \underline{\ell}_n &= \ell_n - \delta(\ell_n - \ell_{n-1}), \quad \text{for } n = 2, \dots, N. \end{aligned}$$

For $t > 0$, denote

$$\Omega(t) = \big((\bar{\ell}_1 t, \underline{\ell}_2 t) \cup \ldots \cup (\bar{\ell}_{N-1} t, \underline{\ell}_N t)\big) \times \mathbb{R}^4, \quad \Omega^C(t) = \mathbb{R}^5 \setminus \Omega(t).$$

We define the continuous function $\chi_N(t,x) = \chi_N(t,x_1)$ as follow (see [85, 111] for a similar choice of cut-off function), for $t > 0$,

$$\begin{cases} \chi_N(t,x) = \ell_1 \text{ for } x_1 \in (-\infty, \bar{\ell}_1 t], \\ \chi_N(t,x) = \ell_n \text{ for } x_1 \in [\underline{\ell}_n t, \bar{\ell}_n t], \text{ for } n \in \{2, \dots, N-1\}, \\ \chi_N(t,x) = \ell_N \text{ for } x_1 \in [\underline{\ell}_N t, +\infty), \\ \chi_N(t,x) = \dfrac{x_1}{(1-2\delta)t} - \dfrac{\delta}{1-2\delta}(\ell_{n+1} + \ell_n) \text{ for } x_1 \in [\bar{\ell}_n t, \underline{\ell}_{n+1} t],\ n \in \{1, \dots, N-1\}. \end{cases} \tag{4.50}$$

Note that

$$\begin{cases} \partial_{x_1}\chi_N(t,x) = \dfrac{1}{(1-2\delta)t} \quad \text{for } x \in \Omega(t), \\ \partial_t \chi_N(t,x) = -\dfrac{1}{t}\dfrac{x_1}{(1-2\delta)t} \quad \text{for } x \in \Omega(t), \\ \partial_t \chi_N(t,x) = 0, \quad \nabla\chi_N(t,x) = 0, \quad \text{for } x \in \Omega^C(t). \end{cases} \tag{4.51}$$

We define (see [85, 88, 111] for similar energy functional)

$$\mathcal{K}(t) = \mathcal{F}(t) - \mathcal{G}(t) = \mathcal{E}(t) + \mathcal{P}(t) - \mathcal{G}(t)$$

where

$$\mathcal{E}(t) = \int_{\mathbb{R}^5} \big(|\nabla\varepsilon|^2 + \eta^2 - 2(F(U+V+\varepsilon) - F(U+V) - f(U+V)\varepsilon)\big)\,\mathrm{d}x,$$

$$\mathcal{P}(t) = 2\int_{\mathbb{R}^5} \big(\chi_N(t,x)\partial_{x_1}\varepsilon(t,x)\big)\eta(t,x)\mathrm{d}x \quad \text{and} \quad \mathcal{G}(t) = 2\int_{\mathbb{R}^5} \varepsilon(t,x)G_3(t,x)\mathrm{d}x.$$

We start with the following technical lemma.

Lemma 4.18. *Let W be a continuous function such that*

$$|W(x)| \lesssim \langle x\rangle^{-(4+\alpha)} \quad \textit{for all } x \in \mathbb{R}^5, \tag{4.52}$$

where $\alpha \geqslant 0$. For all $n = 1, \dots, N$, the following estimates hold.

(i) Estimate of $L^{\frac{10}{7}}$ norm.

$$\|(\ell_n - \chi_N)W(x - \boldsymbol{\ell}_n t)\|_{L^{\frac{10}{7}}} \lesssim t^{-(\frac{1}{2}+\alpha)}. \tag{4.53}$$

(ii) Estimate of L^2 norm.

$$\|(\ell_n - \chi_N)W(x - \boldsymbol{\ell}_n t)\|_{L^2} \lesssim t^{-(\frac{3}{2}+\alpha)}. \tag{4.54}$$

(iii) Estimate of $L^{\frac{10}{3}}$ norm.

$$\|(\ell_n - \chi_N)W(x - \boldsymbol{\ell}_n t)\|_{L^{\frac{10}{3}}} \lesssim t^{-(\frac{5}{2}+\alpha)}. \tag{4.55}$$

Proof. From (4.52), the definition of χ_N and change of variable,

$$\begin{aligned}\int_{\mathbb{R}^5} |\ell_n - \chi_N|^{\frac{10}{7}} |W(x - \boldsymbol{\ell}_n t)|^{\frac{10}{7}} \mathrm{d}x &\lesssim \int_{|x|\geqslant \delta^2 t} \langle x\rangle^{-\frac{10}{7}(4+\alpha)} \mathrm{d}x \\ &\lesssim \int_{\delta^2 t}^{\infty} \frac{r^4}{(1+r^2)^{\frac{5}{7}(4+\alpha)}} \mathrm{d}r \lesssim t^{-\frac{10}{7}(\frac{1}{2}+\alpha)},\end{aligned}$$

which implies (4.53). The proof of (4.54) and (4.55) follows from similar arguments and it is omitted. □

Under the bootstrap setting (4.48), we prove the following estimates.

Proposition 4.19. *There exists $0 < \nu \ll 1$ such that the following hold.*

(i) Coercivity.

$$\nu \|\vec{\varepsilon}(t)\|_{\mathcal{H}}^2 \leqslant \mathcal{K}(t) + \nu^{-1} t^{-\frac{59}{10}}. \tag{4.56}$$

(ii) Time variation of $\mathcal{K}$.

$$-\frac{\mathrm{d}}{\mathrm{d}t}(t^2 \mathcal{K})(t) \leqslant \nu^{-1} t^{-\frac{49}{10}}. \tag{4.57}$$

Proof. Proof of (i). We set

$$\mathcal{F}_\Omega(t) = \int_\Omega \left(|\nabla \varepsilon|^2 + \eta^2 + 2(\chi_N \partial_{x_1}\varepsilon)\eta\right) \mathrm{d}x, \quad \mathcal{F}_{\Omega^C}(t) = \int_{\Omega^C} \left(|\nabla\varepsilon|^2 + \eta^2\right) \mathrm{d}x.$$

First, from $|\chi_N| \leqslant \bar{\ell} < 1$,

$$\begin{aligned}\mathcal{F}_\Omega &= \bar{\ell} \int_\Omega \left(\frac{\chi_N}{\bar{\ell}} \partial_{x_1}\varepsilon + \eta\right)^2 \mathrm{d}x + \int_\Omega \left(\left(1 - \frac{\chi_N^2}{\bar{\ell}}\right)(\partial_{x_1}\varepsilon)^2 + (1-\bar{\ell})\eta^2 + |\overline{\nabla}\varepsilon|^2\right) \mathrm{d}x \\ &\geqslant \bar{\ell} \int_\Omega \left(\frac{\chi_N}{\bar{\ell}} \partial_{x_1}\varepsilon + \eta\right)^2 \mathrm{d}x + (1-\bar{\ell}) \int_\Omega \left(|\nabla\varepsilon|^2 + \eta^2\right) \mathrm{d}x.\end{aligned} \tag{4.58}$$

Second, by (4.48) and Young's inequality for product,

$$|\mathcal{G}(t)| \lesssim \left(\sum_{(n,k)\in I^0} |b_{n,k}|^2\right) \|\vec{\varepsilon}\|_{\mathcal{H}} \lesssim t^{-\frac{13}{2}}. \tag{4.59}$$

Therefore the coercivity property for $\mathcal{K}$ is a consequence of the following stronger coercivity property, for some $\alpha > 0$,

$$\mathcal{F}(t) \geqslant \mathcal{F}_\Omega(t) + \mu \mathcal{F}_{\Omega^C}(t) - \mu^{-1} t^{-\frac{59}{10}} - \mu^{-1} t^{-\alpha} \|\vec{\varepsilon}\|_{\mathcal{H}}^2 - \mu^{-1} \|\vec{\varepsilon}\|_{\mathcal{H}}^3. \tag{4.60}$$

The coercivity property (4.60) is a standard consequence of the localized coercivity property around one excited solitary wave q_ℓ in Proposition 4.10 with the orthogonality relations (4.39), and an elementary localization argument. We refer to the proof of [85, Proposition 4.2 (ii)] and [111, Lemma 5.4 (ii)] for a similar proof.

Proof of (ii). **Step 1.** Time variation of $\mathcal{E}$. We claim

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} =& 2\int_{\mathbb{R}^5} \mathrm{Mod}_1\left(\Delta\varepsilon + f'(U)\varepsilon\right)\mathrm{d}x - 2\int_{\mathbb{R}^5}\eta\mathrm{Mod}_2\mathrm{d}x + 2\int_{\mathbb{R}^5}\eta G_3\mathrm{d}x\\ &+2\sum_{n=1}^{N}\int_{\mathbb{R}^5}\ell_n\partial_{x_1}Q_n\left(f(U+V+\varepsilon)-f(U+V)-f'(U+V)\varepsilon\right)\mathrm{d}x + O\big(t^{-\frac{69}{10}}\big).\end{aligned} \tag{4.61}$$

We decompose

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4,$$

where

$$\begin{aligned}\mathcal{I}_1 &= -2\int_{\mathbb{R}^5}\partial_t U\big(f(U+V+\varepsilon)-f(U+V)-f'(U+V)\varepsilon\big)\mathrm{d}x,\\ \mathcal{I}_2 &= -2\int_{\mathbb{R}^5}\partial_t V\big(f(U+V+\varepsilon)-f(U+V)-f'(U+V)\varepsilon\big)\mathrm{d}x,\\ \mathcal{I}_3 &= 2\int_{\mathbb{R}^5}\partial_t\varepsilon\big(-\Delta\varepsilon - f(U+V+\varepsilon)+f(U+V)\big)\mathrm{d}x, \quad \mathcal{I}_4 = 2\int_{\mathbb{R}^5}\eta\partial_t\eta\mathrm{d}x.\end{aligned}$$

Estimate on $\mathcal{I}_1$. We claim

$$\mathcal{I}_1 = 2\sum_{n=1}^{N}\int_{\mathbb{R}^5}\ell_n\partial_{x_1}Q_n\big(f(U+V+\varepsilon)-f(U+V)-f'(U+V)\varepsilon\big)\mathrm{d}x. \tag{4.62}$$

By direct computation, we obtain

$$\partial_t U = \sum_{n=1}^{N}\partial_t Q_n = -\sum_{n=1}^{N}\ell_n\partial_{x_1}Q_n,$$

which implies (4.62).

Estimate on $\mathcal{I}_2$. We claim

$$|\mathcal{I}_2| \lesssim t^{-7}. \tag{4.63}$$

By direct computation,

$$\partial_t V = \sum_{(n,k)\in I^0}\dot{b}_{n,k}\Psi_{n,k} - \sum_{(n,k)\in I^0} b_{n,k}\ell_n\partial_{x_1}\Psi_{n,k}.$$

Thus, from (4.32), (4.41), (4.48), and the decay properties of $\Psi_{n,k}$,

$$\begin{aligned}|\mathcal{I}_2| &\lesssim \sum_{(n,k)\in I^0}\int_{\mathbb{R}^5}\Big(|\dot{b}_{n,k}||\Psi_{n,k}| + |b_{n,k}||\ell_n\partial_{x_1}\Psi_{n,k}|\Big)\Big(\big(|U|^{\frac{1}{3}}+|V|^{\frac{1}{3}}\big)|\varepsilon|^2+|\varepsilon|^{\frac{7}{3}}\Big)\mathrm{d}x\\ &\lesssim \sum_{(n,k)\in I^0}\left(\|\vec{\varepsilon}\|_{\mathcal{H}}+|b_{n,k}|+t^{-4}\right)\|\vec{\varepsilon}\|_{\mathcal{H}}^2 \lesssim t^{-\frac{87}{10}}+t^{-\frac{38}{5}}+t^{-7}\lesssim t^{-7},\end{aligned}$$

which means (4.63).

Estimate on $\mathcal{I}_3$. We prove the following estimate

$$\begin{aligned}\mathcal{I}_3 =& -2\int_{\mathbb{R}^5}\eta\big(\Delta\varepsilon + f(U+V+\varepsilon)-f(U+V)\big)\mathrm{d}x\\ &+2\int_{\mathbb{R}^5}\mathrm{Mod}_1\big(\Delta\varepsilon+f'(U)\varepsilon\big)\mathrm{d}x + O(t^{-7}).\end{aligned} \tag{4.64}$$

By direct computation and (4.40),

$$\begin{aligned}\mathcal{I}_3 =& -2\int_{\mathbb{R}^5}\eta\big(\Delta\varepsilon + f(U+V+\varepsilon) - f(U+V)\big)\mathrm{d}x\\ &+2\int_{\mathbb{R}^5}\mathrm{Mod}_1\big(\Delta\varepsilon + f'(U)\varepsilon\big)\mathrm{d}x + \mathcal{I}_{3,1} + \mathcal{I}_{3,2},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{3,1} &= 2\int_{\mathbb{R}^5}\mathrm{Mod}_1\left(f'(U+V) - f'(U)\right)\varepsilon\mathrm{d}x,\\ \mathcal{I}_{3,2} &= 2\int_{\mathbb{R}^5}\mathrm{Mod}_1\left(f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\right)\mathrm{d}x.\end{aligned}$$

By the Cauchy-Schwarz inequality, (4.31), (4.32), (4.41) and (4.48),

$$\begin{aligned}|\mathcal{I}_{3,1}| &\lesssim \|\mathrm{Mod}_1\|_{L^{\frac{10}{3}}}\|f'(U+V) - f'(U)\|_{L^{\frac{5}{2}}}\|\varepsilon\|_{L^{\frac{10}{3}}}\\ &\lesssim \sum_{(n,k)\in I^0}|b_{n,k}|\Big(\|\vec{\varepsilon}\|_{\mathcal{H}} + \sum_{(n',k')\in I^0}|b_{n',k'}|^2 + t^{-4}\Big)\|\vec{\varepsilon}\|_{\mathcal{H}}\\ &\lesssim t^{-\frac{9}{5}}\left(t^{-\frac{29}{10}} + t^{-\frac{18}{5}} + t^{-4}\right)t^{-\frac{29}{10}} \lesssim t^{-7},\end{aligned}$$

$$\begin{aligned}|\mathcal{I}_{3,2}| &\lesssim \|\mathrm{Mod}_1\|_{L^{\frac{10}{3}}}\|f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\|_{L^{\frac{10}{7}}}\\ &\lesssim \Big(\|\vec{\varepsilon}\|_{\mathcal{H}} + \sum_{(n,k)\in I^0}|b_{n,k}|^2 + t^{-4}\Big)\|\vec{\varepsilon}\|_{\mathcal{H}}^2 \lesssim \left(t^{-\frac{29}{10}} + t^{-\frac{18}{5}} + t^{-4}\right)t^{-\frac{29}{5}} \lesssim t^{-7}.\end{aligned}$$

We see that (4.64) follows from above estimates.

Estimate on $\mathcal{I}_4$. We claim

$$\begin{aligned}\mathcal{I}_4 =& 2\int_{\mathbb{R}^5}\eta\big(\Delta\varepsilon + f(U+V+\varepsilon) - f(U+V)\big)\mathrm{d}x\\ &-2\int_{\mathbb{R}^5}\eta\mathrm{Mod}_2\mathrm{d}x + 2\int_{\mathbb{R}^5}\eta G_3\mathrm{d}x + O\big(t^{-\frac{69}{10}}\big).\end{aligned}\tag{4.65}$$

We compute, using again (4.40),

$$\begin{aligned}\mathcal{I}_4 =& 2\int_{\mathbb{R}^5}\eta\big(\Delta\varepsilon + f(U+V+\varepsilon) - f(U+V)\big)\mathrm{d}x\\ &-2\int_{\mathbb{R}^5}\eta\mathrm{Mod}_2\mathrm{d}x + 2\int_{\mathbb{R}^5}\eta G_3\mathrm{d}x + 2\int_{\mathbb{R}^5}\eta G_1\mathrm{d}x + 2\int_{\mathbb{R}^5}\eta(G_2 - G_3)\mathrm{d}x.\end{aligned}$$

Indeed, from (4.34), (4.35) and (4.48), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^5}\eta G_1\mathrm{d}x\right| + \left|\int_{\mathbb{R}^5}\eta(G_2 - G_3)\mathrm{d}x\right| &\lesssim \|\eta\|_{L^2}\left(\|G_1\|_{L^2} + \|G_2 - G_3\|_{L^2}\right)\\ &\lesssim \|\eta\|_{L^2}\Big(\sum_{(n,k)\in I^0}|b_{n,k}|^{\frac{7}{3}} + t^{-4}\Big) \lesssim t^{-\frac{69}{10}},\end{aligned}$$

which implies (4.65).

In conclusion of estimates (4.62), (4.63), (4.64) and (4.65), we obtain (4.61).

Step 2. Time variation of $\mathcal{P}$. We claim

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} =& -\frac{1}{(1-2\delta)t}\int_\Omega\big(\eta^2 + (\partial_{x_1}\varepsilon)^2 + 2\frac{x_1}{t}(\partial_{x_1}\varepsilon)\eta - |\overline{\nabla}\varepsilon|^2\big)\mathrm{d}x\\ &-2\sum_{n=1}^N\int_{\mathbb{R}^5}\chi_N\partial_{x_1}Q_n\big(f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\big)\mathrm{d}x\\ &-2\int_{\mathbb{R}^5}\chi_N\left(\eta\partial_{x_1}\mathrm{Mod}_1 + \mathrm{Mod}_2\partial_{x_1}\varepsilon\right)\mathrm{d}x + 2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)G_3\mathrm{d}x + O(t^{-\frac{69}{10}}).\end{aligned}\tag{4.66}$$

We decompose

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{P} = 2\int_{\mathbb{R}^5}(\partial_t\chi_N)(\partial_{x_1}\varepsilon)\eta\mathrm{d}x + 2\int_{\mathbb{R}^5}\chi_N\partial_t\big((\partial_{x_1}\varepsilon)\eta\big)\mathrm{d}x = \mathcal{I}_5 + \mathcal{I}_6. \tag{4.67}$$

Estimate on $\mathcal{I}_5$. From (4.51), we obtain

$$\mathcal{I}_5 = -\frac{2}{(1-2\delta)t}\int_\Omega \frac{x_1}{t}(\partial_{x_1}\varepsilon)\eta\mathrm{d}x. \tag{4.68}$$

Estimate on $\mathcal{I}_6$. We claim

$$\begin{aligned}\mathcal{I}_6 = &-\frac{1}{(1-2\delta)t}\int_\Omega\big(\eta^2 + (\partial_{x_1}\varepsilon)^2 - |\overline{\nabla}\varepsilon|^2\big)\mathrm{d}x\\ &-2\sum_{n=1}^N\int_{\mathbb{R}^5}\chi_N\partial_{x_1}Q_n\big(f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\big)\mathrm{d}x\\ &-2\int_{\mathbb{R}^5}\chi_N\left(\eta\partial_{x_1}\mathrm{Mod}_1 + \mathrm{Mod}_2\partial_{x_1}\varepsilon\right)\mathrm{d}x + 2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)G_3\mathrm{d}x + O(t^{-\frac{69}{10}}).\end{aligned} \tag{4.69}$$

By direct computation, we decompose

$$\mathcal{I}_6 = 2\int_{\mathbb{R}^5}\chi_N\left(\partial_{x_1}(\partial_t\varepsilon)\right)\eta\mathrm{d}x + 2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)\partial_t\eta\mathrm{d}x = \mathcal{I}_{6,1} + \mathcal{I}_{6,2}.$$

From (4.40), (4.51), and integration by parts,

$$\mathcal{I}_{6,1} = -\frac{1}{(1-2\delta)t}\int_\Omega\eta^2\mathrm{d}x - 2\int_{\mathbb{R}^5}\chi_N\left(\eta\partial_{x_1}\mathrm{Mod}_1\right)\mathrm{d}x. \tag{4.70}$$

Using again (4.40), (4.51), and integration by parts,

$$\begin{aligned}\mathcal{I}_{6,2} = &-\frac{1}{(1-2\delta)t}\int_\Omega\left((\partial_{x_1}\varepsilon)^2 - |\overline{\nabla}\varepsilon|^2\right)\mathrm{d}x - 2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)\mathrm{Mod}_2\mathrm{d}x\\ &+2\int_{\mathbb{R}^5}\chi_N\left(\partial_{x_1}\varepsilon\right)G_3\mathrm{d}x + 2\int_{\mathbb{R}^5}\chi_N\partial_{x_1}\varepsilon\left(f(U+V+\varepsilon) - f(U+V)\right)\mathrm{d}x\\ &+2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)G_1\mathrm{d}x + 2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)(G_2 - G_3)\mathrm{d}x.\end{aligned} \tag{4.71}$$

Note that, by integration by parts,

$$\begin{aligned}&2\int_{\mathbb{R}^5}\chi_N\partial_{x_1}\varepsilon\left(f(U+V+\varepsilon) - f(U+V)\right)\mathrm{d}x\\ =&-2\sum_{n=1}^N\int_{\mathbb{R}^5}\chi_N\partial_{x_1}Q_n\big(f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\big)\mathrm{d}x + \mathcal{I}_6^1 + \mathcal{I}_6^2,\end{aligned}$$

where

$$\mathcal{I}_6^1 = -\frac{2}{(1-2\delta)t}\int_\Omega\left(F(U+V+\varepsilon) - F(U+V) - f(U+V)\varepsilon\right)\mathrm{d}x,$$

$$\mathcal{I}_6^2 = -2\sum_{(n,k)\in I^0}b_{n,k}\int_{\mathbb{R}^5}\chi_N\partial_{x_1}\Psi_{n,k}\left(f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\right)\mathrm{d}x.$$

From (4.33), (4.48), the decay properties of Q_n and $\Psi_{n,k}$ and Sobolev embedding theorem,

$$\begin{aligned}|\mathcal{I}_6^1| &\lesssim t^{-1}\int_\Omega\left((|U|^{\frac{4}{3}} + |V|^{\frac{4}{3}})\varepsilon^2 + \varepsilon^{\frac{10}{3}}\right)\mathrm{d}x\\ &\lesssim t^{-1}\left(t^{-1}\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\frac{10}{3}}\right) \lesssim t^{-1}(t^{-\frac{34}{5}} + t^{-\frac{29}{3}}) \lesssim t^{-7}.\end{aligned}$$

From (4.32), (4.48) and Sobolev embedding theorem,

$$\begin{aligned}|\mathcal{I}_6^2| &\lesssim \sum_{(n,k)\in I^0} |b_{n,k}| \int_{\mathbb{R}^5} |\partial_{x_1}\Psi_{n,k}| \left((|U|^{\frac{1}{3}}+|V|^{\frac{1}{3}})|\varepsilon|^2+|\varepsilon|^{\frac{7}{3}}\right) \mathrm{d}x \\ &\lesssim \sum_{(n,k)\in I^0} |b_{n,k}| \left(\|\vec{\varepsilon}\|_{\mathcal{H}}^2+\|\vec{\varepsilon}\|_{\mathcal{H}}^{\frac{7}{3}}\right) \lesssim t^{-\frac{9}{5}}(t^{-\frac{29}{5}}+t^{-\frac{20}{3}}) \lesssim t^{-\frac{38}{5}}.\end{aligned}$$

Next, by the Cauchy-Schwarz inequality, (4.34), (4.35) and (4.48),

$$\begin{aligned}&\left|\int_{\mathbb{R}^5} \chi_N(\partial_{x_1}\varepsilon)G_1\mathrm{d}x\right| + \left|\int_{\mathbb{R}^5} \chi_N(\partial_{x_1}\varepsilon)(G_2-G_3)\mathrm{d}x\right| \\ &\lesssim \|\partial_{x_1}\varepsilon\|_{L^2}\left(\|G_1\|_{L^2}+\|G_2-G_3\|_{L^2}\right) \\ &\lesssim \|\partial_{x_1}\varepsilon\|_{L^2}\left(\sum_{(n,k)\in I^0}|b_{n,k}|^{\frac{7}{3}}+t^{-4}\right) \lesssim t^{-\frac{29}{10}}\left(t^{-\frac{21}{5}}+t^{-4}\right) \lesssim t^{-\frac{69}{10}}.\end{aligned}$$

Gathering above estimates, we obtain (4.69).

In conclusion of estimates (4.68) and (4.69), we obtain (4.66).

Step 3. Time variation of $\mathcal{G}$. We claim

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} = 2\int_{\mathbb{R}^5} \eta G_3 \mathrm{d}x + 2\sum_{n=1}^{N}\int_{\mathbb{R}^5} \ell_n(\partial_{x_1}\varepsilon)G_{3,n}\mathrm{d}x + O(t^{-7}). \tag{4.72}$$

By direct computation and integration by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} = 2\int_{\mathbb{R}^5} \eta G_3 \mathrm{d}x + 2\sum_{n=1}^{N}\int_{\mathbb{R}^5} \ell_n(\partial_{x_1}\varepsilon)G_{3,n}\mathrm{d}x + \mathcal{I}_7+\mathcal{I}_8+\mathcal{I}_9,$$

where

$$\mathcal{I}_7 = \sum_{n=1}^{N}\sum_{k,k'=1}^{K_n} \dot{b}_{n,k}b_{n,k'}\int_{\mathbb{R}^5} \varepsilon\big(f''(Q_n)\Psi_{n,k}\Psi_{n,k'}\big)\mathrm{d}x,$$

$$\mathcal{I}_8 = -\sum_{n=1}^{N}\sum_{k=1}^{K_n}\dot{b}_{n,k}\int_{\mathbb{R}^5}\Psi_{n,k}G_{3,n}\mathrm{d}x \quad \text{and} \quad \mathcal{I}_9 = -\sum_{n\neq n'}\sum_{k=1}^{K_n}\dot{b}_{n,k}\int_{\mathbb{R}^5}\Psi_{n,k}G_{3,n'}\mathrm{d}x.$$

First, from (4.41), (4.48) and Sobolev embedding,

$$\begin{aligned}|\mathcal{I}_7| &\lesssim \sum_{n=1}^{N}\sum_{k,k'=1}^{K_n}\left|\dot{b}_{n,k}b_{n,k'}\right|\int_{\mathbb{R}^5}|\varepsilon|\left|f''(Q_n)\Psi_{n,k}\Psi_{n,k'}\right|\mathrm{d}x \\ &\lesssim \sum_{(n,k)\in I^0}|b_{n,k}|\|\vec{\varepsilon}\|_{\mathcal{H}}\left(\|\vec{\varepsilon}\|_{\mathcal{H}}+\sum_{(n,k)\in I^0}b_{n,k}^2+t^{-4}\right) \\ &\lesssim t^{-\frac{9}{5}}\left(t^{-\frac{29}{10}}+t^{-\frac{18}{5}}+t^{-4}\right)t^{-\frac{29}{10}} \lesssim t^{-7}.\end{aligned}$$

Next, from (iii) of Lemma 4.8, we have for all $(n,k)\in I^0$,

$$\begin{aligned}\int_{\mathbb{R}^5}\Psi_{n,k}G_{3,n}\mathrm{d}x &= \frac{1}{2}\sum_{k',k''=1}^{K_n} b_{n,k'}b_{n,k''}\int_{\mathbb{R}^5}\Psi_{n,k}\left(f''(Q_n)\Psi_{n,k'}\Psi_{n,k''}\right)\mathrm{d}x \\ &= \frac{1}{2}\sum_{k',k''=1}^{K} b_{n,k'}b_{n,k''}(1-\ell_n^2)^{\frac{1}{2}}\int_{\mathbb{R}^5} f''(Q)\left(\Psi_k\Psi_{k'}\Psi_{k''}\right)\mathrm{d}x = 0.\end{aligned}$$

It follows that $\mathcal{I}_8 = 0$. Last, from Lemma 4.11, (4.41) and (4.48),

$$\begin{aligned}
|\mathcal{I}_9| &\lesssim \sum_{n'\neq n}\sum_{k=1}^{K_n}\sum_{k',k''=1}^{K_{n'}} \left|\dot{b}_{n,k} b_{n',k'} b_{n',k''}\right| \int_{\mathbb{R}^5} \left|\Psi_{n,k}\left(f''(Q_{n'})\Psi_{n',k'}\Psi_{n',k''}\right)\right| \mathrm{d}x \\
&\lesssim t^{-1} \sum_{(n,k)\in I^0} b_{n,k}^2 \left(\|\vec{\varepsilon}\|_{\mathcal{H}} + \sum_{(n,k)\in I^0} b_{n,k}^2 + t^{-4}\right) \\
&\lesssim t^{-1}\left(t^{-\frac{29}{10}} + t^{-\frac{18}{5}} + t^{-4}\right) t^{-\frac{18}{5}} \lesssim t^{-7}.
\end{aligned}$$

Gathering above estimates, we obtain (4.72).

Step 4. Conclusion. Combining estimates (4.61), (4.66) and (4.72), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5 + O\big(t^{-\frac{69}{10}}\big).$$

where

$$\begin{aligned}
\mathcal{J}_1 &= -\frac{1}{(1-2\delta)t}\int_\Omega \left(\eta^2 + (\partial_{x_1}\varepsilon)^2 + 2\frac{x_1}{t}(\partial_{x_1}\varepsilon)\eta - |\overline{\nabla}\varepsilon|^2\right)\mathrm{d}x, \\
\mathcal{J}_2 &= 2\int_{\mathbb{R}^5}\left(\Delta\varepsilon + f'(U)\varepsilon\right)\mathrm{Mod}_1\mathrm{d}x - 2\int_{\mathbb{R}^5}\chi_N(\partial_{x_1}\varepsilon)\mathrm{Mod}_2\mathrm{d}x, \\
\mathcal{J}_3 &= 2\sum_{n=1}^N \int_{\mathbb{R}^5}\left(\ell_n - \chi_N\right)\partial_{x_1}Q_n\left(f(U+V+\varepsilon) - f(U+V) - f'(U+V)\varepsilon\right)\mathrm{d}x, \\
\mathcal{J}_4 &= -2\int_{\mathbb{R}^5}\eta\left(\mathrm{Mod}_2 + \chi_N\partial_{x_1}\mathrm{Mod}_1\right)\mathrm{d}x, \quad \mathcal{J}_5 = 2\sum_{n=1}^N\int_{\mathbb{R}^5}\left(\chi_N - \ell_n\right)(\partial_{x_1}\varepsilon)G_{3,n}\mathrm{d}x.
\end{aligned}$$

Estimate on $\mathcal{J}_1$. For $0<\delta\ll 1$ small enough and T_0 large enough, we claim

$$-\mathcal{J}_1 \leqslant 2t^{-1}\mathcal{K} + O\big(t^{-\frac{69}{10}}\big). \tag{4.73}$$

From (4.56), (4.58), (4.59) and (4.60), we have

$$\begin{aligned}
-(1-2\delta)t\mathcal{J}_1 =& \bar{\ell}\int_\Omega\left(\frac{\chi_N}{\bar{\ell}}\partial_{x_1}\varepsilon + \eta\right)^2\mathrm{d}x + \int_\Omega\left(\left(1-\frac{\chi_N^2}{\bar{\ell}}\right)(\partial_{x_1}\varepsilon)^2 - |\overline{\nabla}\varepsilon|^2\right)\mathrm{d}x \\
&+ (1-\bar{\ell})\int_\Omega \eta^2\mathrm{d}x + 2\int_\Omega\left(\frac{x_1}{t} - \chi_N\right)(\partial_{x_1}\varepsilon)\eta\mathrm{d}x \\
\leqslant& \mathcal{F}_\Omega + O\left(\delta\|\vec{\varepsilon}\|^2_{\mathcal{H}(\Omega)}\right) \leqslant (1 + O(\delta + T_0^{-\alpha}))\mathcal{K} + O\big(t^{-\frac{59}{10}}\big).
\end{aligned}$$

Letting δ small enough and T_0 large enough in above estimate, we obtain (4.73).

Estimate on $\mathcal{J}_2$. By integration by parts, (4.51) and $-(1-\ell_n^2)\partial_{x_1}^2\Psi_{n,k} - \bar{\Delta}\Psi_{n,k} - f'(Q_n)\Psi_{n,k} = 0$ for all $(n,k)\in I^0$, we have

$$\mathcal{J}_2 = \mathcal{J}_{2,1} + \mathcal{J}_{2,2} + \mathcal{J}_{2,3},$$

where

$$\begin{aligned}
\mathcal{J}_{2,1} &= \frac{2}{(1-2\delta)t}\int_\Omega \varepsilon\mathrm{Mod}_2\mathrm{d}x, \\
\mathcal{J}_{2,2} &= 2\sum_{(n,k)\in I^0}\dot{b}_{n,k}\int_{\mathbb{R}^5}\varepsilon\left(\ell_n - \chi_N\right)\ell_n\partial_{x_1}^2\Psi_{n,k}\mathrm{d}x, \\
\mathcal{J}_{2,3} &= 2\sum_{(n,k)\in I^0}\dot{b}_{n,k}\int_{\mathbb{R}^5}\varepsilon\left(f'(U) - f'(Q_n)\right)\Psi_{n,k}\mathrm{d}x.
\end{aligned}$$

From the Cauchy-Schwarz inequality, (4.41), (4.48), (4.53) and the decay properties of $\Psi_{n,k}$,

$$\begin{aligned}|\mathcal{J}_{2,1}| &\lesssim t^{-1}\|\varepsilon\|_{L^{\frac{10}{3}}}\sum_{(n,k)\in I^0}|\dot{b}_{n,k}|\|\partial_{x_1}\Psi_{n,k}\|_{L^{\frac{10}{7}}(\Omega)}\\ &\lesssim t^{-\frac{3}{2}}\|\vec{\varepsilon}\|_{\mathcal{H}}\Big(\|\vec{\varepsilon}\|_{\mathcal{H}}+\sum_{(n,k)\in I^0}|b_{n,k}|^2+t^{-4}\Big)\lesssim t^{-7},\\ |\mathcal{J}_{2,2}| &\lesssim \|\varepsilon\|_{L^{\frac{10}{3}}}\sum_{(n,k)\in I^0}|\dot{b}_{n,k}|\left\|(\ell_n-\chi_N)\partial_{x_1}^2\Psi_{n,k}\right\|_{L^{\frac{10}{7}}}\\ &\lesssim t^{-\frac{3}{2}}\|\vec{\varepsilon}\|_{\mathcal{H}}\Big(\|\vec{\varepsilon}\|_{\mathcal{H}}+\sum_{(n,k)\in I^0}|b_{n,k}|^2+t^{-4}\Big)\lesssim t^{-7}.\end{aligned}$$

From (i) of Lemma 4.11 and (4.32),

$$\begin{aligned}&\left\|\left(f'(U)-f'(Q_n)\right)\Psi_{n,k}\right\|_{L^{\frac{10}{7}}}\\ &\lesssim\sum_{n'\neq n}\left\||Q_n|^{\frac{1}{3}}|Q_{n'}|\Psi_{n,k}\right\|_{L^{\frac{10}{7}}}+\sum_{n'\neq n}\left\||Q_{n'}|^{\frac{4}{3}}\Psi_{n,k}\right\|_{L^{\frac{10}{7}}}\lesssim t^{-2}.\end{aligned}$$

Thus, by (4.41), (4.48) and Cauchy-Schwarz inequality,

$$\begin{aligned}|\mathcal{J}_{2,3}| &\lesssim \|\varepsilon\|_{L^{\frac{10}{3}}}\sum_{(n,k)\in I^0}|\dot{b}_{n,k}|\left\|\left(f'(U)-f'(Q_n)\right)\Psi_{n,k}\right\|_{L^{\frac{10}{7}}}\\ &\lesssim t^{-2}\|\vec{\varepsilon}\|_{\mathcal{H}}\Big(\|\vec{\varepsilon}\|_{\mathcal{H}}+\sum_{(n,k)\in I^0}|b_{n,k}|^2+t^{-4}\Big)\lesssim t^{-7}.\end{aligned}$$

Gathering above estimates, we obtain

$$|\mathcal{J}_2|\lesssim|\mathcal{J}_{2,1}|+|\mathcal{J}_{2,2}|+|\mathcal{J}_{2,3}|\lesssim t^{-7}. \tag{4.74}$$

Estimate on $\mathcal{J}_3$. From (4.32), (4.48), (4.55), Cauchy-Schwarz inequality and the decay properties of Q_n,

$$\begin{aligned}|\mathcal{J}_3| &\lesssim\sum_{n=1}^{N}\|(\ell_n-\chi_N)\partial_{x_1}Q_n\|_{L^{\frac{10}{3}}}\left\|\left(|U|^{\frac{1}{3}}+|V|^{\frac{1}{3}}\right)|\varepsilon|^2+|\varepsilon|^{\frac{7}{3}}\right\|_{L^{\frac{10}{7}}}\\ &\lesssim\|\vec{\varepsilon}\|_{\mathcal{H}}^2\sum_{n=1}^{N}\|(\ell_n-\chi_N)\partial_{x_1}Q_n\|_{L^{\frac{10}{3}}}\lesssim t^{-7}.\end{aligned} \tag{4.75}$$

Estimate on $\mathcal{J}_4$. Note that,

$$\mathrm{Mod}_2+\chi_N\partial_{x_1}\mathrm{Mod}_1=-\sum_{(n,k)\in I^0}\dot{b}_{n,k}\left(\ell_n-\chi_N\right)\partial_{x_1}\Psi_{n,k}.$$

Therefore, by Cauchy-Schwarz inequality, (4.41), (4.48) and (4.54),

$$\begin{aligned}|\mathcal{J}_4| &\lesssim\|\eta\|_{L^2}\sum_{(n,k)\in I^0}|\dot{b}_{n,k}|\|\left(\ell_n-\chi_N\right)\partial_{x_1}\Psi_{n,k}\|_{L^2}\\ &\lesssim t^{-\frac{3}{2}}\|\vec{\varepsilon}\|_{\mathcal{H}}\Big(\|\vec{\varepsilon}\|_{\mathcal{H}}+\sum_{(n,k)\in I^0}|b_{n,k}|^2+t^{-4}\Big)\lesssim t^{-7}.\end{aligned} \tag{4.76}$$

Estimate on $\mathcal{J}_5$. From (4.48), (4.54) and Cauchy-Schwarz inequality,

$$|\mathcal{J}_5|\lesssim\|\partial_{x_1}\varepsilon\|_{L^2}\sum_{(n,k)\in I^0}|b_{n,k}|^2\|(\ell_n-\chi_N)f''(Q_n)\Psi_{n,k}^2\|_{L^2}\lesssim t^{-7}. \tag{4.77}$$

In conclusion of estimates (4.73), (4.74), (4.75), (4.76) and (4.77), for δ small enough and T_0 large enough, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} \leqslant 2t^{-1}\mathcal{K} + O\big(t^{-\frac{69}{10}}\big), \tag{4.78}$$

which implies (4.57). □

4.4.4 End of the proof of Proposition 4.17

We start by improving all the estimates in (4.48) except the ones for the unstable directions $(a^+_{n,j})_{(n,j)\in I}$.

Lemma 4.20 (Closing estimates except $(a^+_{n,j})_{(n,j)\in I}$). *For all $t \in [T_*, T_m]$,*

$$\|\vec{\varepsilon}(t)\|_{\mathcal{H}} \leqslant \frac{1}{2}t^{-\frac{29}{10}}, \qquad \sum_{(n,k)\in I^0} |b_{n,k}(t)| \leqslant \frac{1}{2}t^{-\frac{9}{5}}, \qquad \sum_{(n,j)\in I} |a^-_{n,j}(t)|^2 \leqslant \frac{1}{2}t^{-\frac{59}{10}}. \tag{4.79}$$

Proof. **Step 1.** Bound on the energy norm. First, from (4.44), (4.45) and $\boldsymbol{a}_m \in B_{\mathbb{R}^{|I|}}\big(T_m^{-3}\big)$, we have

$$|\mathcal{K}(T_m)| \lesssim |\mathcal{E}(T_m)| + |\mathcal{P}(T_m)| + |\mathcal{G}(T_m)| \lesssim \|\vec{\varepsilon}(T_m)\|^2_{\mathcal{H}} \lesssim T_m^{-6}.$$

Thus, integrating (4.57) on $[t, T_m]$ for all $t \in [T_*, T_m]$,

$$\mathcal{K}(t) \leqslant (t/T_m)^{-2}\mathcal{K}(T_m) + \nu^{-1}t^{-\frac{59}{10}} \lesssim t^{-\frac{59}{10}}.$$

Using (4.56), we have

$$\|\vec{\varepsilon}(t)\|^2_{\mathcal{H}} \lesssim \mathcal{K}(t) + t^{-\frac{59}{10}} \lesssim t^{-\frac{59}{10}}.$$

Letting T_0 large enough, we conclude (4.79) on energy norm.

Step 2. Parameter estimates. Using again (4.44), (4.45) and $\boldsymbol{a}_m \in B_{\mathbb{R}^{|I|}}\big(T_m^{-3}\big)$,

$$\sum_{(n,k)\in I^0} |b_{n,k}(T_m)| + \sum_{(n,j)\in I} |a^-_{n,j}(T_m)| \lesssim T_m^{-3}. \tag{4.80}$$

Integrating (4.41) on $[t, T_m]$ for all $t \in [T_*, T_m]$ and using (4.48), (4.80), we obtain

$$\sum_{(n,k)\in I^0} |b_{n,k}(t)| \lesssim \sum_{(n,k)\in I^0} |b_{n,k}(T_m)| + \int_t^{T_m} s^{-\frac{29}{10}}\mathrm{d}s \lesssim t^{-\frac{19}{10}}.$$

Letting T_0 large enough, we conclude (4.79) for $(b_{n,k})_{(n,k)\in I^0}$.

Now we prove the bound on $(a^-_{n,j}(t))_{(n,j)\in I}$. By direct computation, (4.42) and (4.48),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\alpha_{n,j}t}(a^-_{n,j}(t))^2\right) &= e^{-2\alpha_{n,j}t}a^-_{n,j}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}a^-_{n,j} - \alpha_{n,j}a^-_{n,j}\right) \\
&= e^{-2\alpha_{n,j}t}O\bigg(|a^-_{n,j}|\big(\|\vec{\varepsilon}\|^2_{\mathcal{H}} + \sum_{(n,k)\in I^0}|b_{n,k}|^2 + t^{-4}\big)\bigg) \\
&= e^{-2\alpha_{n,j}t}O\left(t^{-\frac{29}{10}}\left(t^{-\frac{29}{5}} + t^{-\frac{18}{5}} + t^{-4}\right)\right) = e^{-2\alpha_{n,j}t}O\left(t^{-6}\right).
\end{aligned}$$

Integrating on $[t, T_m]$ for all $t \in [T_*, T_m]$ and using (4.80), we obtain

$$(a^-_{n,j}(t))^2 \lesssim e^{-2\alpha_{n,j}(T_m - t)}(a^-_{n,j}(T_m))^2 + \int_t^{T_m} e^{2\alpha_{n,j}(t-s)}s^{-6}\mathrm{d}s \lesssim t^{-6},$$

which implies the estimates on $(a^-_{n,j})_{(n,j)\in I}$ in (4.79) for taking T_0 large enough. □

Last, we prove the existence of suitable parameters $\boldsymbol{a}_m = (a^m_{n,j})_{(n,j)\in I}$ by contradiction, using a topological argument.

Lemma 4.21 (Control of unstable directions)**.** *There exist $\boldsymbol{a}_m = (a^m_{n,j}) \in B_{\mathbb{R}^{|I|}}(T_m^{-3})$ such that, for T_0 large enough, $T_*(\boldsymbol{a}_m) = T_0$. In particular, the solution $\vec{u}_m$ with the initial data $\vec{u}_m(T_m)$ given by Lemma 4.15 satisfies* (4.47)*.*

Note that Lemma 4.21 completes the proof of Proposition 4.17.

Proof. Let

$$a(t) = \sum_{(n,j)\in I} (a^+_{n,j}(t))^2 \quad \text{and} \quad \bar{\alpha} = \min_{(n,j)\in I} \alpha_{n,j} > 0.$$

We claim the following transversality property, for any $t \in [T_*, T_m]$ where it holds $a(t) = t^{-6}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^6 a(t)) \leqslant -\bar{\alpha}. \tag{4.81}$$

Indeed, from (4.42) and (4.48), for any $t \in [T_*, T_m]$ where it holds $a(t) = t^{-6}$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}a(t) &= -2\sum_{(n,j)\in I} \alpha_{n,j}(a^+_{n,j}(t))^2 + O\left(\|\vec{\varepsilon}\|_{\mathcal{H}}\left(\|\vec{\varepsilon}\|^2_{\mathcal{H}} + \sum_{(n,k)\in I^0} b^2_{n,k} + t^{-4}\right)\right) \\
&= -2\sum_{(n,j)\in I} \alpha_{n,j}(a^+_{n,j}(t))^2 + O\left(t^{-\frac{29}{10}}\left(t^{-\frac{29}{5}} + t^{-\frac{18}{5}} + t^{-4}\right)\right) \\
&\leqslant -2\bar{\alpha}t^{-6} + O(t^{-\frac{13}{2}}) \leqslant -\frac{3}{2}\bar{\alpha}t^{-6}.
\end{aligned}$$

which implies (4.81) for T_0 large enough. This transversality property is enough to justify the existence of at least a couple $\boldsymbol{a}_m = (a^m_{n,j}) \in B_{\mathbb{R}^{|I|}}(T_m^{-3})$ such that $T_*(\boldsymbol{a}_m) = T_0$.

The proof is by contradiction, we assume that for any $\boldsymbol{a}_m = (a^m_{n,j}) \in B_{\mathbb{R}^{|I|}}(T_m^{-3})$, it holds $T_0 < T_*(\boldsymbol{a}_m) \leqslant T_m$. Then, a construction follows from the following discussion (see for instance more details in [28, 31] and [85, Lemma 4.2]).

Continuity of T_*. The above transversality property (4.81) implies that the map

$$\boldsymbol{a}_m \in B_{\mathbb{R}^{|I|}}(T_m^{-3}) \mapsto T_*(\boldsymbol{a}_m) \in (T_0, T_m]$$

is continuous and

$$T_*(\boldsymbol{a}_m) = T_m \quad \text{for} \quad \boldsymbol{a}_m \in S_{\mathbb{R}^{|I|}}(T_m^{-3}).$$

Construction of a retraction. We define

$$\begin{aligned}
\Gamma : \bar{B}_{\mathbb{R}^{|I|}}(T_m^{-3}) &\mapsto S_{\mathbb{R}^{|I|}}(T_m^{-3}) \\
\boldsymbol{a}_m &\mapsto \left(\frac{T_*(\boldsymbol{a}_m)}{T_m}\right)^3 (a^+_{n,j}(T_*(\boldsymbol{a}_m)))_{(n,j)\in I}.
\end{aligned}$$

From what precedes, Γ is continuous. Moreover, Γ restricted to $S_{\mathbb{R}^{|I|}}(T_m^{-3})$ is the identity. The existence of such a map is contradictory with the no retraction theorem for continuous maps from the ball to the sphere. Therefore, the existence of $\boldsymbol{a}_m \in B_{\mathbb{R}^{|I|}}(T_m^{-3})$ have proved. Then, the uniform estimates (4.47) is a consequence of Lemma 4.20. The proof of Lemma 4.21 is complete. □

Chapter 5

Construction of excited multi-solitons for the focusing 4D cubic wave equation

Abstract: Consider the focusing 4D cubic wave equation

$$\partial_{tt}u - \Delta u - u^3 = 0, \quad \text{on } (t,x) \in [0,\infty) \times \mathbb{R}^4.$$

The main result states the existence in energy space $\dot{H}^1 \times L^2$ of multi-solitary waves where each traveling wave is generated by Lorentz transform from a specific excited state, with different but collinear Lorentz speeds. The specific excited state is deduced from the non-degenerate sign-changing state constructed in Musso-Wei [99]. The proof is inspired by the techniques developed for the 5D energy-critical wave equation and the nonlinear Klein-Gordon equation in a similar context by Martel-Merle [85] and Côte-Martel [26].

The main difficulty originates from the strong interactions between solutions in the 4D case compared to other dispersive and wave-type models. To overcome the difficulty, a sharp understanding of the asymptotic behavior of the excited states involved and of the kernel of their linearized operator is needed.

5.1 Introduction

5.1.1 Main result

Consider the focusing 4D cubic wave equation,

$$\begin{cases} \partial_{tt}u - \Delta u - u^3 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^4, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2. \end{cases} \tag{5.1}$$

Recall that the Cauchy problem for equation (5.1) is locally well-posed in the energy space $\dot{H}^1 \times L^2$. See *e.g.* [71] and references therein. For any $\dot{H}^1 \times L^2$ solution, the energy E and momentum P are conserved, where

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^4} \left(|\nabla u|^2 + (\partial_t u)^2 - \frac{1}{4} u^4 \right) \mathrm{d}x, \quad P(u, \partial_t u) = \int_{\mathbb{R}^4} \partial_t u \nabla u \mathrm{d}x.$$

Denote by Σ the set of non-zero stationary solutions of (5.1),

$$\Sigma = \left\{ q \in \dot{H}^1(\mathbb{R}^4) \setminus 0 : -\Delta q - q^3 = 0 \ \text{ on } \mathbb{R}^4 \right\}. \tag{5.2}$$

For $q \in \Sigma$ and $\boldsymbol{\ell} \in \mathbb{R}^4$ with $|\boldsymbol{\ell}| < 1$, let

$$q_{\boldsymbol{\ell}}(x) = q\left(\left(\frac{1}{\sqrt{1 - |\boldsymbol{\ell}|^2}} - 1 \right) \frac{\boldsymbol{\ell}(\boldsymbol{\ell} \cdot x)}{|\boldsymbol{\ell}|^2} + x \right),$$

then $u(t,x) = q_{\boldsymbol{\ell}}(x - \boldsymbol{\ell} t)$ is a global, bounded solution of (5.1).

Standard elliptic arguments (see *e.g.* [58]) show that if $q \in \Sigma$, then q is of class C^∞. It is known by the results developed in [3, 15, 102, 108] that up to dilation and translation, the only *positive* stationary solution of (5.1) is the *ground state* W, where W has the explicit form

$$W(x) = \left(1 + \frac{|x|^2}{8} \right)^{-1} \quad \text{on } \mathbb{R}^4.$$

Functions $q \in \Sigma$ with $q \neq W$ (up to sign, dilation and translation) are called *excited states*. The existence of excited states that are nonradial, sign-changing and with arbitrary large energy was first proved by Ding [42] using variational arguments. Later, more explicit constructions of sign-changing solutions have been obtained by del Pino-Musso-Pacard-Pistoia [36]. More specifically, they constructed solutions with a centered soliton crowned by negative spikes (rescaled solitons) at the vertices of a regular polygon of radius 1. Then, following a similar general strategy in [36], they constructed in [37] sign changing, non radial solutions to the Yamabe-type equation on the sphere $\mathbb{S}^d$ ($d \geqslant 4$) whose energy is concentrated along special submanifolds of $\mathbb{S}^d$.

For any $q \in \Sigma$, we denote the linearized operator around q by

$$\mathcal{L}_q = -\Delta - 3q^2.$$

Set

$$\mathcal{Z}_q = \left\{ f \in \dot{H}^1 \text{ such that } \mathcal{L}_q f = 0 \right\},$$

and

$$\widetilde{\mathcal{Z}}_q = \mathrm{Span}\Big\{ q + x \cdot \nabla q;\ (x_i \partial_{x_j} q - x_j \partial_{x_i} q), 1 \leqslant i < j \leqslant 4; \\ \partial_{x_i} q;\ -2x_i q + |x|^2 \partial_{x_i} q - 2x_i (x \cdot \nabla q), 1 \leqslant i \leqslant 4 \Big\}.$$

The function space $\widetilde{\mathcal{Z}}_q \subset \mathcal{Z}_q$ is the null space of $\mathcal{L}_q$ that is generated by a family of explicit transformations (see [48, Lemma 3.8] and §5.2.1). Following Duyckaerts-Kenig-Merle [48], we call $q \in \Sigma$ a *non-degenerate state* if $\mathcal{Z}_q = \widetilde{\mathcal{Z}}_q$. The first nontrivial example of non-degenerate sign-changing state was obtained in Musso-Wei [99] who proved that the sign-changing states constructed in [36] are non-degenerate. Later, following a similar strategy, Medina-Musso-Wei [91] constructed a non-degenerate state with maximal rank, *i.e.* $\dim\widetilde{\mathcal{Z}}_q = 15$.

The main goal of this article is to construct multi-solitons of (5.1) based on a specific non-degenerate excited state.

Theorem 5.1 (Existence of excited multi-solitons)**.** *There exists a non-degenerate excited state $Q \in \Sigma$ such that the following is true. Let $N \geqslant 2$ and $\mathbf{e}_1$ be the first vector of the canonical basis of $\mathbb{R}^4$. For $n = 1, \dots, N$, let $\tau_n = \pm 1$ and $\boldsymbol{\ell}_n \in \mathbb{R}^4$ be such that*

$$\boldsymbol{\ell}_n = \ell_n \mathbf{e}_1 \text{ with } -1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1.$$

Then, there exist $T_0 > 0$ and a solution $\vec{u} = (u, \partial_t u)$ of (5.1) on $[T_0, +\infty)$ in the energy space $\dot{H}^1 \times L^2$ such that

$$\begin{aligned} &\lim_{t\to+\infty} \left\| u(t) - \sum_{n=1}^{N} \tau_n Q_{\boldsymbol{\ell}_n}(\cdot - \boldsymbol{\ell}_n t) \right\|_{\dot{H}^1} = 0, \\ &\lim_{t\to+\infty} \left\| \partial_t u(t) + \sum_{n=1}^{N} \tau_n \ell_n \partial_{x_1} Q_{\boldsymbol{\ell}_n}(\cdot - \boldsymbol{\ell}_n t) \right\|_{L^2} = 0. \end{aligned} \tag{5.3}$$

Remark 5.2. The specific non-degenerate state Q involved in Theorem 5.1 is obtained from the non-degenerate sign-changing state constructed in [36, 99]. Indeed, by rotation, translation and Kelvin transformation on a non-degenerate sign-changing state, we obtain a non-degenerate excited state that decays fast enough at least along $\mathbf{e}_1$ (see more details in §5.2.1). The interaction of two such traveling waves with different speeds along $\mathbf{e}_1$, is formally of order t^{-4} for large time. This rate allows us to close the energy estimates (see more details in §5.4.1 and §5.4.2). However, the slow algebraic decay of the linearized operator's kernel leads to another technical difficulty. To overcome it, we need to refine the energy estimates (see more details in §5.3.2 and §5.4.2).

Remark 5.3. The Theorem 5.1 is also true for the more general case where each traveling wave is based on non-degenerate q with the same asymptotic behavior as Q (see Lemma 5.6 and Lemma 5.7). In particular, we do not need each traveling wave to be based on the same non-degenerate state.

Remark 5.4. Following [85, §1.2], we conjecture that in the 4D case, there exists no multi-soliton based on ground state W in the sence of Theorem 5.1, for any $N \geqslant 2$. Heuristically, from $W(x) \sim |x|^{-2}$ as $|x| \to \infty$, the interaction between two solitons of different speeds is t^{-2}. Following the modulation and energy method, we find these interactions are too strong and formally that can create diverging terms of the geometric parameters. However, such nonexistence result is open problem for any space dimension $d \geqslant 3$.

Historically, the existence of multi-soliton solutions for non-integrable cases was first studied by Merle [92] for the L^2 critical nonlinear Schrödinger equation (NLS) and Martel [83] for the subcritical and critical generalized Korteweg-de Vries equations (gKdV). Later, the strategy of these works was extended to other dispersive or wave-type models (see [5, 28, 31, 84, 96, 111, 112]). In particular, the present work is inspired by the proof of the existence of multi-soliton solutions for the nonlinear Klein-Gordon and 5D energy-critical wave equations in [26, 85]. Technically, such constructions of multi-solitons of the energy-critical wave equation are more challenging than for other dispersive and wave-type models, because of the weak algebraic decay of solitons.

To our knowledge, this article proves the first statement of the existence of multi-soliton solutions for the 4D energy-critical wave equation. The construction of such solutions for the energy-critical

wave equation is especially motivated by the soliton resolution conjecture, fully solved in the 3D radial case by Duyckaerts-Kenig-Merle [46], and for a subsequence of time in 3, 4 and 5D by Duyckaerts-Jia-Kenig-Merle [43]. Recently, the conjecture in the 4D radial case is proved by Duyckaerts-Kenig-Martel-Merle [44]. Indeed, the existence of such multi-solitons complements the statements in [43] by exhibiting examples of solutions containing N solitons at $t \to +\infty$ for any $N \geqslant 2$. In the same direction, recall that for the 6D energy-critical wave equation, Jendrej [67] proved the existence of radial two-bubble solutions. Later, for the 5D energy-critical wave equation, Jendrej-Martel [69] proved the existence of N-bubble solutions for any $N \geqslant 2$. For the 3 and 5D cases, Krieger-Nakanishi-Schlag [75] constructed a center-stable manifold of the ground state. Recall also that, for the 4D case, Hillairet-Raphaël [62] proved the existence of type-II finite time blow-up solutions containing single soliton.

We refer to [38, 39, 40] for results of the existence of type-II blow up solutions containing single or multi-bubble of the energy-critical heat equation. Particularly related to the present article, the blow up profile in [40] is the non-degenerate sign-changing state constructed in [36, 99] rather than the ground state W.

The article is organized as follows. Section 5.2 introduces the technical tools involved in the choice of specific excited state Q: the spectral theory of the linearized operator, the asymptotic behavior of Q and of the kernel of its linearized operator. Section 5.3 introduces the technical tools involved in a dynamical approach to the N-soliton problem for (5.1): estimates of the nonlinear interactions between solitons, decomposition by modulation and parameter estimates. Finally, Theorem 5.1 is proved in Section 5.4 by energy estimates and a suitable compactness argument.

5.1.2 Notation

We fix a smooth, even function $\xi : \mathbb{R} \to \mathbb{R}$ satisfying:

$$\xi \equiv 1 \text{ on } [0,1], \quad \xi \equiv 0 \text{ on } [2,\infty), \quad \text{and} \quad \xi' \leqslant 0 \leqslant \xi \leqslant 1 \text{ on } [0,\infty). \tag{5.4}$$

For future reference, we state the first-order Taylor expansion involving C^∞ function. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a C^∞ function,

$$f(x) = f(0) + \sum_{i=1}^{4} \partial_{x_i} f(0) x_i + O(|x|^2) \quad \text{for all } |x| < 1. \tag{5.5}$$

Recall that for two C^∞ functions $f_1, f_2 : \mathbb{R}^4 \to \mathbb{R}$, Leibniz's formula writes:

$$\partial_x^\alpha (f_1 f_2) = \sum_{\beta+\gamma=\alpha} C_\alpha^\beta (\partial_x^\beta f_1)(\partial_x^\gamma f_2), \quad \text{for all } \alpha \in \mathbb{N}^4. \tag{5.6}$$

We denote

$$(f,g)_{L^2} := \int_{\mathbb{R}^4} f(x)g(x)\mathrm{d}x, \quad (f,g)_{\dot{H}^1} := \int_{\mathbb{R}^4} \nabla f(x) \cdot \nabla g(x)\mathrm{d}x.$$

For

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

denote

$$\left(\vec{f}, \vec{g}\right)_{L^2} := \sum_{k=1,2} (f_k, g_k)_{L^2}, \quad \|\vec{f}\|_{\mathcal{H}}^2 := \|f_1\|_{\dot{H}^1}^2 + \|f_2\|_{L^2}^2.$$

When x_1 is seen as the specific coordinate, denote

$$\bar{x} = (x_2, x_3, x_4), \quad \overline{\nabla} = (\partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \quad \text{and} \quad \bar{\Delta} = \sum_{i=2}^{4} \partial_{x_i}^2.$$

For any $-1 < \ell < 1$ and $f : \mathbb{R}^4 \mapsto \mathbb{R}$, set

$$f_\ell(x) = f(x_\ell) \quad \text{where} \quad x_\ell = \left(\frac{x_1}{\sqrt{1-\ell^2}}, \bar{x} \right).$$

We set the Ω_{ij} are the angular derivatives that are

$$\Omega_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i} \quad \text{for } 1 \leqslant i < j \leqslant 4.$$

We also set

$$\Lambda = \mathrm{Id} + x \cdot \nabla \quad \text{and} \quad \widetilde{\Omega}_i = -2x_i + |x|^2 \partial_{x_i} - 2x_i x \cdot \nabla \quad \text{for } i = 1, \dots, 4.$$

Note that, for $q \in \Sigma$,

$$\widetilde{\mathcal{Z}}_q = \mathrm{Span}\left\{ \Lambda q; \partial_{x_i} q, \widetilde{\Omega}_i q, 1 \leqslant i \leqslant 4; \Omega_{ij} q, 1 \leqslant i < j \leqslant 4 \right\}.$$

Denote $\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$ for $x \in \mathbb{R}^4$. For some $0 < \gamma \ll 1$ to be fixed, set

$$\zeta(x) = \langle x \rangle^{-\gamma} \quad \text{on } \mathbb{R}^4. \tag{5.7}$$

For $J \in \mathbb{N}^+$ and $r > 0$, we denote by $\mathcal{B}_{\mathbb{R}^J}(r)$ (respectively, $\mathcal{S}_{\mathbb{R}^J}(r)$) be the ball (respectively, the sphere) of $\mathbb{R}^J$ of center 0 and of radius r.

Recall the Sobolev and Hardy inequalities in dimension 4, for any $f \in \dot{H}^1$,

$$\|f\|_{L^4(\mathbb{R}^4)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^4)}, \tag{5.8}$$

$$\left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^4)}. \tag{5.9}$$

5.2 Preliminaries

5.2.1 Transforms and Asymptotic behavior

Following [48], we first recall that Σ is invariant under the following four transformations:

(i) translation: If $q \in \Sigma$ then $q(x + x_0) \in \Sigma$ for all $x_0 \in \mathbb{R}^4$;

(ii) dilation: If $q \in \Sigma$ then $\lambda q(\lambda x) \in \Sigma$ for all $\lambda \in (0, \infty)$;

(iii) orthogonal transformation: If $q \in \Sigma$ then $q(Ox) \in \Sigma$ for all $O \in \mathcal{O}_4$.

(iv) Kelvin transformation: If $q \in \Sigma$ then $\mathrm{K}q \in \Sigma$ where $\mathrm{K}q(x) = \frac{1}{|x|^2} q\left(\frac{x}{|x|^2}\right)$.

We denote by $\mathcal{M}$ the group of isometries of $\dot{H}^1$ generated by the preceding transformations. From [48, Section 3], it is known $\mathcal{M}$ defines a 15-parameter family of transformations in a neighborhood of the identity.

For $\theta = (\theta_{ij})_{1 \leqslant i < j \leqslant 4} \in \mathbb{R}^6$, set

$$A_\theta = \left[a^\theta_{i,j} \right]_{1 \leqslant i \leqslant j \leqslant 4}, \quad \mathcal{R}_\theta = e^{A_\theta} = \sum_{n=0}^{\infty} \frac{A^n_\theta}{n!},$$

where $a^\theta_{i,i} = 0$, $a^\theta_{i,j} = \theta_{ij}$ if $i < j$, $a^\theta_{j,i} = -\theta_{ij}$ if $i < j$.

For $\mathcal{A} = (\lambda, z, \xi, \theta) \in (0, \infty) \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^6$, we introduce the following explicit formula of transform $\mathcal{T}_{\mathcal{A}} \in \mathcal{M}$,

$$\mathcal{T}_{\mathcal{A}}(f)(x) = \lambda \left| \frac{x}{|x|} - z|x| \right|^{-2} f\left(\xi + \frac{\lambda \mathcal{R}_\theta(x - z|x|^2)}{1 - 2\langle z, x\rangle + |z|^2|x|^2} \right) \quad \text{for } f \in \dot{H}^1.$$

Recall that for all $q \in \Sigma$, $\widetilde{\mathcal{Z}}_q$ is generated by taking partial derivatives of $\mathcal{T}_{\mathcal{A}}(q)$ with respect to $\mathcal{A} = (\lambda, z, \xi, \theta)$ at $\mathcal{A} = (1, 0, 0, 0)$. Also, following [48, Page 734], we see that if q is non-degenerate then $\mathcal{T}q$ is also non-degenerate for any $\mathcal{T} \in \mathcal{M}$, *i.e.* if $\mathcal{Z}_q = \widetilde{\mathcal{Z}}_q$ then $\mathcal{Z}_{\mathcal{T}q} = \widetilde{\mathcal{Z}}_{\mathcal{T}q}$ for any $\mathcal{T} \in \mathcal{M}$.

We first recall the following existence result proved in [99].

Theorem 5.5 ([99]). *There exists $q \in \Sigma$ which is a non-degenerate sign-changing state.*

The non-degenerate sign-changing state q plays a crucial role in our proof. Indeed, the specific non-degenerate excited state Q mentioned in Theorem 5.1 is constructed by q via rotation, translation, and Kelvin transformation. More precisely:

Lemma 5.6. *There exists non-degenerate excited state $Q \in \Sigma$ such that the following estimates hold.*

(i) Asymptotic behavior of Q. *For all $|x| > 1$,*

$$\left| Q(x) - \frac{x_4}{|x|^4} \right| \lesssim \frac{1}{|x|^4}. \tag{5.10}$$

(ii) Bound of $\partial_x^\alpha Q$. *For all $\alpha \in \mathbb{N}^4$,*

$$|\partial_x^\alpha Q(x)| \lesssim \langle x \rangle^{-(3+|\alpha|)} \quad \text{on } \mathbb{R}^4. \tag{5.11}$$

Proof. Let $q \in \Sigma$ be the non-degenerate sign-changing state in Theorem 5.5. Without loss of generality, we assume $q(0) > 0$. Set

$$r = \sup \{ R > 0; q(x) > 0 \text{ holds on } \mathcal{B}_{\mathbb{R}^4}(R) \}.$$

Since q is sign-changing state, we know that $r > 0$ is well-defined finite, and there exists $x_0 \in \mathcal{S}_{\mathbb{R}^4}(r)$ such that $q(x_0) = 0$. Moreover, by the Hopf's Lemma (see for instance [58, Lemma 3.4]), we have $\nabla q(x_0) \neq 0$. Considering translation $x \to x + x_0$, dilation and rotating for q, we construct a non-degenerate excited state $\tilde{q} \in \Sigma$ satisfying

$$\tilde{q}(0) = \partial_{x_1}\tilde{q}(0) = \partial_{x_2}\tilde{q}(0) = \partial_{x_3}\tilde{q}(0) = 0 \quad \text{and} \quad \partial_{x_4}\tilde{q}(0) = 1. \tag{5.12}$$

Next, we consider the Kelvin transform K to $\tilde{q}$ and denote it by Q, *i.e.*

$$Q(x) = (\mathrm{K}\tilde{q})(x) = \frac{1}{|x|^2}\tilde{q}\left(\frac{x}{|x|^2}\right) \quad \text{for } x \neq 0.$$

Following [48, Section 3], we know that Q is also a non-degenerate state, *i.e.*

$$-\Delta Q - Q^3 = 0 \ \text{ with } Q \in C^\infty, \quad \text{and} \quad \mathcal{Z}_Q = \widetilde{\mathcal{Z}}_Q.$$

Moreover, from [48, Lemma 3.8] and $\mathrm{K}^2 = \mathrm{Id}$, we have, for $i = 1, \ldots, 4$,

$$\widetilde{\Omega}_i Q(x) = (\mathrm{K}(\partial_{x_i}\tilde{q}))(x) = \frac{1}{|x|^2}\left((\partial_{x_i}\tilde{q})\left(\frac{x}{|x|^2}\right)\right) \quad \text{for } x \neq 0. \tag{5.13}$$

Now we start to prove Q satisfying (i) and (ii).

Proof of (i). Applied the Taylor expansion (5.5) of $\tilde{q}$ at $x=0$ and then using (5.12), we have

$$\tilde{q}(x)=\tilde{q}(0)+\sum_{i=1}^{4}\partial_{x_i}\tilde{q}(0)x_i+O(|x|^2)=x_4+O(|x|^2)\quad\forall\ |x|<1.$$

Replace x with $\frac{x}{|x|^2}$ in the above identity,

$$\tilde{q}\left(\frac{x}{|x|^2}\right)=\frac{x_4}{|x|^2}+O\left(\frac{1}{|x|^2}\right)\quad\forall\ |x|>1. \tag{5.14}$$

Therefore, from the definition of Kelvin transform in §5.2.1, we have

$$Q(x)=\frac{1}{|x|^2}\tilde{q}\left(\frac{x}{|x|^2}\right)=\frac{x_4}{|x|^4}+O\left(\frac{1}{|x|^4}\right)\quad\forall\ |x|>1,$$

which means (5.10).

Proof of (ii). On the one hand, from (5.14), we know that

$$\left|\tilde{q}\left(\frac{x}{|x|^2}\right)\right|\lesssim\frac{|x_4|}{|x|^2}+O\left(\frac{1}{|x|^2}\right)\lesssim\langle x\rangle^{-1},\quad\forall\ |x|>1. \tag{5.15}$$

Then, from $\tilde{q}\in C^\infty$ and chain rule, for any $i=1,\dots,4$ and $|x|>1$, we have

$$\left|\partial_{x_i}\tilde{q}\left(\frac{x}{|x|^2}\right)\right|=\left|\sum_{j=1}^{4}\left((\partial_{x_j}\tilde{q})\left(\frac{x}{|x|^2}\right)\right)\partial_{x_i}\left(\frac{x_j}{|x|^2}\right)\right|\lesssim\langle x\rangle^{-2}.$$

Using a similar argument as above for all $\gamma\in\mathbb{N}^4$ and combining with (5.15),

$$\left|\partial_x^\gamma\tilde{q}\left(\frac{x}{|x|^2}\right)\right|\lesssim\langle x\rangle^{-(1+|\gamma|)}\quad\forall\ |x|>1. \tag{5.16}$$

On the other hand, by direct computation and chain rule,

$$\left|\partial_x^\beta\left(\frac{1}{|x|^2}\right)\right|\lesssim\langle x\rangle^{-(2+|\beta|)}\quad\forall\ \beta\in\mathbb{N}^4\text{ and }|x|>1. \tag{5.17}$$

Based on (5.16), (5.17) and Leibniz's formula (5.6), for any $\alpha\in\mathbb{N}^4$ and $|x|>1$,

$$\begin{aligned}\partial_x^\alpha Q(x)&=\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\left(\partial_x^\gamma\tilde{q}\left(\frac{x}{|x|^2}\right)\right)\left(\partial_x^\beta\left(\frac{1}{|x|^2}\right)\right)\\&=\sum_{\beta+\gamma=\alpha}O\left(\langle x\rangle^{-(1+|\gamma|)}\langle x\rangle^{-(2+|\beta|)}\right)=O\left(\langle x\rangle^{-(3+|\alpha|)}\right).\end{aligned}$$

Combining above estimates with $Q\in C^\infty$, we complete the proof of (ii). □

Next, we introduce the following sharp understanding of the kernel of linearized operator around Q.

Lemma 5.7 (Asymptotic behavior of kernel functions). *There exist $K\in\mathbb{N}^+$ and linear independent C^∞ functions $\psi,\phi_1,\dots,\phi_K\in\dot{H}^1$ such that*

$$\widetilde{\mathcal{Z}}_Q=\operatorname{Span}\{\psi,\phi_1,\dots,\phi_K\}.$$

Moreover, we have

$$\left|\psi(x)-|x|^{-2}\right|\lesssim\langle x\rangle^{-3}\quad\forall\ |x|\geqslant 1, \tag{5.18}$$

$$\langle x\rangle^{-1}|\partial_x^\alpha\psi(x)|+\sum_{k=1}^{K}|\partial_x^\alpha\phi_k(x)|\lesssim\langle x\rangle^{-(3+\alpha)}\quad\forall\ \alpha\in\mathbb{N}^4,\ x\in\mathbb{R}^4. \tag{5.19}$$

Remark 5.8. Indeed, whatever is the decay on Q, its kernel contains the function ψ which has the slow algebraic decay $|x|^{-2}$. This is the main difficulty in our proof. Similar difficulty in [62] is due to the function ΛW where $W \in \Sigma$ is the ground state. In [78], for the wave maps, such similar difficulty is due to ΛQ where Q is the harmonic map.

Proof. We choose $\psi = \widetilde{\Omega}_4 Q$ and $\{\phi_1, \dots, \phi_K\}$ to be a basis of the function space

$$\mathrm{Span}\left\{\Lambda Q; \partial_{x_i} Q, 1 \leqslant i \leqslant 4; \Omega_{ij} Q, 1 \leqslant i < j \leqslant 4; \widetilde{\Omega}_i Q, 1 \leqslant i \leqslant 3\right\}.$$

Note that, from the definition of ψ and $\{\phi_1, \dots, \phi_K\}$, we have

$$\widetilde{\mathcal{Z}}_Q = \mathrm{Span}\left\{\psi, \phi_1, \dots, \phi_K\right\}.$$

It remains to show that the C^∞ functions $\{\psi, \phi_1, \dots, \phi_K\}$ satisfy (5.18) and (5.19).

First, from (ii) of Lemma 5.6, for any kernel function

$$\phi \in \mathrm{Span}\left\{\Lambda Q; \partial_{x_i} Q, 1 \leqslant i \leqslant 4; \Omega_{ij} Q, 1 \leqslant i < j \leqslant 4\right\},$$

$\alpha \in \mathbb{N}^4$ and $x \in \mathbb{R}^4$, we have

$$|\partial_x^\alpha \phi(x)| \lesssim |\partial_x^\alpha Q(x)| + \langle x \rangle \sum_{i=1}^4 |\partial_x^\alpha \partial_{x_i} Q(x)| \lesssim \langle x \rangle^{-(3+|\alpha|)}. \tag{5.20}$$

Second, we claim

$$\left|\widetilde{\Omega}_4 Q(x) - |x|^{-2}\right| \lesssim |x|^{-3} \quad \forall\, |x| > 1, \tag{5.21}$$

and for any $\alpha \in \mathbb{N}^4$,

$$\sum_{i=1}^3 \left|\partial_x^\alpha \widetilde{\Omega}_i Q(x)\right| + \langle x \rangle^{-1} \left|\partial_x^\alpha \widetilde{\Omega}_4 Q(x)\right| \lesssim \langle x \rangle^{-(3+|\alpha|)} \quad \text{on } \mathbb{R}^4. \tag{5.22}$$

Applied the Taylor expansion (5.5) for $\partial_{x_i} \tilde{q}$ at $x = 0$ and using (5.12), we have, for $i = 1, \dots, 4$,

$$(\partial_{x_i} \tilde{q})\left(\frac{x}{|x|^2}\right) = \delta_{i4} + O\left(\frac{1}{|x|}\right) \quad \forall\, |x| > 1. \tag{5.23}$$

Moreover, for $i = 1, \dots, 4$ and any $\gamma \in \mathbb{N}^4$, we have

$$\left|\partial_x^\gamma \left((\partial_{x_i} \tilde{q})\left(\frac{x}{|x|^2}\right)\right)\right| \lesssim \langle x \rangle^{-(1+|\gamma|)+\delta_{i4}} \quad \forall\, |x| > 1. \tag{5.24}$$

Recall that, for $i = 1, \dots, 4$,

$$\widetilde{\Omega}_i Q(x) = \frac{1}{|x|^2}\left((\partial_{x_i} \tilde{q})\left(\frac{x}{|x|^2}\right)\right) \quad \forall\, |x| > 1.$$

Based on above identity and (5.23),

$$\widetilde{\Omega}_4 Q(x) = |x|^{-2} + O\left(|x|^{-3}\right) \quad \forall\, |x| > 1,$$

which means (5.21). Then, using (5.24) and Leibniz's formula (5.6), for any $\alpha \in \mathbb{N}^4$, $|x| > 1$ and $i = 1, \dots, 4$,

$$\begin{aligned}\left|\partial_x^\alpha \widetilde{\Omega}_i Q(x)\right| &= \sum_{\beta+\gamma=\alpha} C_\alpha^\beta \partial_x^\beta \left(\frac{1}{|x|^2}\right) \partial_x^\gamma \left((\partial_{x_i} \tilde{q})\left(\frac{x}{|x|^2}\right)\right) \\ &= \sum_{\beta+\gamma=\alpha} O\left(\langle x \rangle^{-(2+|\beta|)} \langle x \rangle^{-(1+\gamma)+\delta_{i4}}\right) = O\left(\langle x \rangle^{-(3+\alpha)+\delta_{i4}}\right).\end{aligned}$$

Based on above estimates and $Q \in C^\infty$, we obtain (5.22). Note that, the C^∞ functions $\{\psi, \phi_1, \dots, \phi_K\}$ satisfying (5.18) and (5.19) is consequence of (5.20), (5.21) and (5.22). □

5.2.2 Spectral theory

In this section, we introduce the spectral properties of the linearized operator around Q and Q_ℓ. For $-1<\ell<1$, we define

$$\mathcal{L}=-\Delta-3Q^2,\quad \mathcal{L}_\ell=-(1-\ell^2)\partial_{x_1}^2-\bar{\Delta}-3Q_\ell^2.$$

Lemma 5.9 (Spectral properties of $\mathcal{L}$). (i) Spectrum. *The self-adjoint operator $\mathcal{L}$ has essential spectrum $[0,\infty)$, a finite number $J\geqslant 1$ of negative eigenvalues (counted with multiplicity), and its kernel is $\widetilde{\mathcal{Z}}_Q$. Let $(Y_j)_{j=1,\dots,J}$ be an L^2 orthogonal family of eigenvectors of $\mathcal{L}$ corresponding to the eigenvalues $(-\lambda_j^2)_{j=1,\dots,J}$, i.e. for $j,j'=1,\dots,J$,*

$$\big(Y_j,Y_{j'}\big)_{L^2}=\delta_{jj'}\quad \textit{and}\quad \mathcal{L}Y_j=-\lambda_j^2Y_j,\quad \lambda_j>0.$$

It holds, for all $j=1,\dots,J$ and $\alpha\in\mathbb{N}^4$,

$$|\partial_x^\alpha Y_j(x)|\lesssim e^{-\lambda_j|x|}\quad \textit{on } \mathbb{R}^4. \tag{5.25}$$

(ii) Cancellation. *We have*

$$(f_1f_2f_3,Q)_{L^2}=0\quad \textit{for all } f_1,f_2,f_3\in\widetilde{\mathcal{Z}}_Q. \tag{5.26}$$

Proof. Proof of (i). The spectral properties of $\mathcal{L}$ are consequence of [48, Claim 3.5] and [34, Theorem 5.8.1]. The exponential decay of eigenfunctions Y_j follows from standard elliptic arguments, see [1, Theorem 3.2].

Proof of (ii). The cancellation properties (5.26) follows from taking twice partial derivatives of the identity $-\mathcal{T}_{\mathcal{A}}Q-(\mathcal{T}_{\mathcal{A}}Q)^3=0$ with respect to $\mathcal{A}=(\lambda,z,\xi,\theta)$ at $\mathcal{A}=(1,0,0,0)$. See details in the proof of [112, Lemma 2.1 (iii)]. □

For $-1<\ell<1$, let

$$\mathcal{H}_\ell=\begin{pmatrix}-\Delta-3Q_\ell^2 & -\ell\partial_{x_1}\\ \ell\partial_{x_1} & 1\end{pmatrix},\quad \mathrm{J}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Similarly, for $k=,1\dots,K$,

$$\vec{\psi}_\ell=\begin{pmatrix}\psi(x_\ell)\\ -\ell\partial_{x_1}\psi(x_\ell)\end{pmatrix}\quad\text{and}\quad \vec{\phi}_{k,\ell}=\begin{pmatrix}\phi_k(x_\ell)\\ -\ell\partial_{x_1}\phi_k(x_\ell)\end{pmatrix},$$

and for $j=1,\dots,J$,

$$Y_{j,\ell}=Y_j(x_\ell),\quad \vec{\Upsilon}_{j,\ell}^{\pm}=\begin{pmatrix}\Upsilon_{j,\ell}^{\pm,1}\\ \Upsilon_{j,\ell}^{\pm,2}\end{pmatrix},\quad \vec{Z}_{j,\ell}^{\pm}=\mathcal{H}_\ell\vec{\Upsilon}_{j,\ell}^{\pm},$$

where

$$\Upsilon_{j,\ell}^{\pm,1}=Y_{j,\ell}e^{\mp\frac{\ell\lambda_j}{\sqrt{1-\ell^2}}x_1},\quad \Upsilon_{j,\ell}^{\pm,2}=-\left(\ell\partial_{x_1}Y_{j,\ell}\mp\frac{\lambda_j}{\sqrt{1-\ell^2}}Y_{j,\ell}\right)e^{\mp\frac{\ell\lambda_j}{\sqrt{1-\ell^2}}x_1}.$$

We recall the following spectral properties of $\mathcal{H}_\ell$.

Lemma 5.10. *Let $-1<\ell<1$.*

(i) Decay properties of $\Upsilon_{j,\ell}^{\pm,1}$ and $\Upsilon_{j,\ell}^{\pm,2}$. *We have, on $\mathbb{R}^4$, for $j=1,\dots,J$,*

$$\left|\Upsilon_{j,\ell}^{\pm,1}\right|+\left|\Upsilon_{j,\ell}^{\pm,2}\right|\lesssim e^{-\frac{1}{2}(1-|\ell|)^{\frac{1}{2}}\lambda_j|x|}. \tag{5.27}$$

(ii) Kernel of $\mathcal{H}_\ell$. *We have*

$$\mathrm{Ker}\mathcal{H}_\ell=\mathrm{Span}\left(\vec{\psi}_\ell,\vec{\phi}_{1,\ell},\dots,\vec{\phi}_{K,\ell}\right). \tag{5.28}$$

(iii) Identities of $\vec{Z}^{\pm}_{j,\ell}$ and $\vec{Y}^{\pm}_{j,\ell}$. *For $j = 1, \dots, J$, we have*

$$\vec{Z}^{\pm}_{j,\ell} = \mp\lambda_j (1-\ell^2)^{\frac{1}{2}} \mathrm{J}\vec{\Upsilon}^{\pm}_{j,\ell}. \tag{5.29}$$

(iv) Coercivity of $\mathcal{H}_\ell$. *There exists $c > 0$ such that, for all $\vec{v} \in \dot{H}^1 \times L^2$,*

$$(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geqslant c\|\vec{v}\|^2_{\mathcal{H}} - c^{-1}\left(\sum_{\phi\in\widetilde{\mathcal{Z}}_Q}\left(\vec{v},\vec{\phi}_\ell\right)^2_{\mathcal{H}} + \sum_{\pm,j=1}^{J}\left(\vec{v},\vec{Z}^{\pm}_{j,\ell}\right)^2_{L^2}\right). \tag{5.30}$$

Proof. Proof of (i). We prove (i) for $\Upsilon^{\pm,1}_{j,\ell}$; the estimate for $\Upsilon^{\pm,2}_{j,\ell}$ are proved similarly. From the exponential decay property of Y_j in (5.25) and the definition of $\Upsilon^{\pm,1}_{j,\ell}$,

$$\left|\Upsilon^{\pm,1}_{j,\ell}\right| \lesssim e^{-\lambda_j\left(\left(\frac{x_1^2}{1-\ell^2}+|\bar{x}|^2\right)^{\frac{1}{2}} - \frac{|\ell|}{\sqrt{1-\ell^2}}|x_1|\right)}, \quad \text{on } \mathbb{R}^4.$$

By an elementary computation, we have

$$\begin{aligned}
&\left(\left(\frac{x_1^2}{1-\ell^2}+|\bar{x}|^2\right)^{\frac{1}{2}} - \frac{|\ell|}{\sqrt{1-\ell^2}}|x_1|\right)^2 \\
&= \frac{1+\ell^2}{1-\ell^2}x_1^2 + |\bar{x}|^2 - 2|\ell|\frac{|x_1|}{\sqrt{1-\ell^2}}\left(\frac{x_1^2}{1-\ell^2}+|\bar{x}|^2\right)^{\frac{1}{2}} \\
&\geqslant \frac{(1-|\ell|)^2}{1-\ell^2}x_1^2 + (1-|\ell|)|\bar{x}|^2 > \frac{1}{4}(1-|\ell|)|x|^2,
\end{aligned}$$

which implies (5.27) for $\Upsilon^{\pm,1}_{j,\ell}$.

Proof of (ii) and (iii). The proof of the identities directly follows from an elementary computation (see *e.g.* [112, Lemma 2.4 (i)] and [85, Claim 1 (ii)]) and we omit it.

Proof of (iv). The proof of the coercivity property relies on an argument based on the spectral theory of $\mathcal{L}$ and compactness argument (see *e.g.* [27] and [112, Proposition 2.3]), and we omit it. □

Third, we introduce the following localized coercivity property of $\mathcal{H}_\ell$. The proof is similar to [85, Lemma 2.2 (ii)], but it is given for the sake of completeness and the readers' convenience.

Lemma 5.11 (Localized coercivity of $\mathcal{H}_\ell$)**.** *For $\gamma > 0$ small enough in (5.7), there exists $\mu > 0$ such that for any $\vec{v} = (v, z) \in \dot{H}^1 \times L^2$,*

$$\begin{aligned}
&\int_{\mathbb{R}^4}\left(|\nabla v|^2\zeta^2 - 3Q_\ell^2 v^2 + z^2\zeta^2 + 2\ell(\partial_{x_1}v)z\zeta^2\right)\mathrm{d}x \\
&\geqslant \mu\int_{\mathbb{R}^4}\left(|\nabla v|^2 + z^2\right)\zeta^2\mathrm{d}x - \mu^{-1}\left(\sum_{\phi\in\widetilde{\mathcal{Z}}_Q}\left(\vec{v},\vec{\phi}_\ell\right)^2_{\mathcal{H}} + \sum_{\pm,j=1}^{J}\left(\vec{v},\vec{Z}^{\pm}_{j,\ell}\right)^2_{L^2}\right).
\end{aligned} \tag{5.31}$$

Proof. First, we apply (5.30) on $\vec{v}\zeta = (v\zeta, z\zeta) \in \dot{H}^1 \times L^2$,

$$(\mathcal{H}_\ell(\vec{v}\zeta), \vec{v}\zeta)_{L^2} \geqslant c\|\vec{v}\zeta\|^2_{\mathcal{H}} - c^{-1}\left(\sum_{\phi\in\widetilde{\mathcal{Z}}_Q}\left(\vec{v}\zeta,\vec{\phi}_\ell\right)^2_{\mathcal{H}} + \sum_{\pm,j=1}^{J}\left(\vec{v}\zeta,\vec{Z}^{\pm}_{j,\ell}\right)^2_{L^2}\right). \tag{5.32}$$

Note that, from the definition of $\mathcal{H}_\ell$ and integration by parts,

$$\begin{aligned}(\mathcal{H}_\ell(\vec{v}\zeta),\vec{v}\zeta)_{L^2} = &\int_{\mathbb{R}^4}\left(|\nabla v|^2\zeta^2 - 3Q_\ell^2 v^2 + z^2\zeta^2 + 2\ell(\partial_{x_1}v)z\zeta^2\right)\mathrm{d}x \\ &- \int_{\mathbb{R}^4} v^2\zeta\Delta\zeta\mathrm{d}x + 2\ell\int_{\mathbb{R}^4} vz\zeta\partial_{x_1}\zeta\mathrm{d}x + 3\int_{\mathbb{R}^4} Q_\ell^2 v^2(1-\zeta^2)\mathrm{d}x.\end{aligned} \tag{5.33}$$

Note also that, from the definition of ζ in (5.7),

$$\partial_{x_1}\zeta = -\gamma\frac{x_1}{1+|x|^2}\zeta \quad \text{and} \quad \Delta\zeta = -\gamma((2-\gamma)|x|^2+4)\frac{\zeta}{(1+|x|^2)^2}.$$

Therefore, by the Hardy inequality (5.9), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^4} v^2\zeta\Delta\zeta\mathrm{d}x\right| &\lesssim \gamma\int_{\mathbb{R}^4}\frac{(v\zeta)^2}{\langle x\rangle^2}\mathrm{d}x \lesssim \gamma\|v\zeta\|_{\dot{H}^1}^2, \\ \left|\int_{\mathbb{R}^4} vz\zeta\partial_{x_1}\zeta\mathrm{d}x\right| &\lesssim \gamma\int_{\mathbb{R}^4}\left|\frac{v\zeta}{\langle x\rangle}\right||z\zeta|\mathrm{d}x \lesssim \gamma\|\vec{v}\zeta\|_{\mathcal{H}}^2.\end{aligned}$$

Then, by the decay property of Q in Lemma 5.6 and the Hardy inequality (5.9),

$$\begin{aligned}\left|\int_{\mathbb{R}^4} Q_\ell^2 v^2(1-\zeta^2)\mathrm{d}x\right| &\lesssim \|Q_\ell^2\langle x\rangle^{2+\gamma}(1-(1+|x|^2)^{-\gamma})\|_{L^\infty}\int_{\mathbb{R}^4}\frac{(v\zeta)^2}{\langle x\rangle^2}\mathrm{d}x \\ &\lesssim C(\gamma)\|v\zeta\|_{\dot{H}^1}^2 \quad \text{where } C(\gamma)\to 0 \text{ as } \gamma\to 0.\end{aligned}$$

Combining above three estimates, for $\gamma > 0$ small enough, we have

$$\left|-\int_{\mathbb{R}^4} v^2\zeta\Delta\zeta\mathrm{d}x + 2\ell\int_{\mathbb{R}^4} vz\zeta\partial_{x_1}\zeta\mathrm{d}x + 3\int_{\mathbb{R}^4} Q_\ell^2 v^2(1-\zeta^2)\mathrm{d}x\right| \leqslant \frac{c}{2}\|\vec{v}\zeta\|_{\mathcal{H}}^2. \tag{5.34}$$

Using similar arguments to $\|\vec{v}\zeta\|_{\mathcal{H}}^2$, $(\vec{v}\zeta,\vec{\phi}_\ell)_{\mathcal{H}}^2$ and $(\vec{v}\zeta,\vec{Z}_{j,\ell}^{\pm})_{L^2}^2$, we have

$$\begin{aligned}&\int_{\mathbb{R}^4}(|\nabla v|^2+z^2)\zeta^2\mathrm{d}x \leqslant 2\|\vec{v}\zeta\|_{\mathcal{H}}^2, \\ &(\vec{v}\zeta,\vec{\phi}_\ell)_{\mathcal{H}}^2 + (\vec{v}\zeta,\vec{Z}_{j,\ell}^{\pm})_{L^2}^2 \leqslant 2(\vec{v},\vec{\phi}_\ell)_{\mathcal{H}}^2 + 2(\vec{v},\vec{Z}_{j,\ell}^{\pm})_{L^2}^2 + \frac{c^2}{4}\|\vec{v}\zeta\|_{\mathcal{H}}^2,\end{aligned} \tag{5.35}$$

for γ small enough. Combining (5.32), (5.33), (5.34) and (5.35), we obtain (5.31). □

5.3 Dynamics close to the sum of N solitons

Let Q be the non-degenerate state chosen in Lemma 5.6 and $N \geqslant 2$. For any $n = 1,\dots,N$, let $\tau_n = \pm 1$ and $\boldsymbol{\ell}_n = \ell_n\mathbf{e}_1$ where $-1 < \ell_n < 1$ and $\ell_n \neq \ell_{n'}$ for any $n \neq n'$. Denote by I and I^0 the following two sets of indices,

$$\begin{aligned}I &= \{(n,j) : n = 1,\dots,N, j = 1,\dots,J\}, \qquad |I| = \mathrm{Card}I = NJ, \\ I^0 &= \{(n,k) : n = 1,\dots,N, k = 1,\dots,K\}, \quad |I^0| = \mathrm{Card}I^0 = NK.\end{aligned}$$

Define

$$Q_n(t,x) = \tau_n Q_{\ell_n}(x-\boldsymbol{\ell}_n t), \quad \vec{Q}_n = \begin{pmatrix} Q_n \\ -\ell_n\partial_{x_1}Q_n \end{pmatrix}.$$

Similarly, we set,

$$\Psi_n(t,x) = \psi_{\ell_n}(x-\boldsymbol{\ell}_n t), \quad \vec{\Psi}_n(t,x) = \vec{\psi}_{\ell_n}(x-\boldsymbol{\ell}_n t),$$

for $(n,k) \in I^0$,

$$\Phi_{n,k}(t,x) = \phi_{k,\ell_n}(x - \boldsymbol{\ell}_n t), \quad \vec{\Phi}_{n,k}(t,x) = \vec{\phi}_{k,\ell_n}(x - \boldsymbol{\ell}_n t),$$

and for $(n,j) \in I$,

$$\vec{Z}^{\pm}_{n,j}(t,x) = \vec{Z}^{\pm}_{j,\ell_n}(x - \boldsymbol{\ell}_n t) = \mp\lambda_j (1-\ell_n^2)^{\frac{1}{2}} \mathrm{J}\vec{\Upsilon}^{\pm}_{j,\ell_n}(x - \boldsymbol{\ell}_n t).$$

Consider time dependent C^1 functions $\boldsymbol{a}$ and $\boldsymbol{b}$ of the forms,

$$\boldsymbol{a} = (a_n)_{n=1,\dots,N} \in \mathbb{R}^N, \quad \boldsymbol{b} = (b_{n,k})_{(n,k)\in I^0} \in \mathbb{R}^{NK} \quad \text{with } |\boldsymbol{a}| + |\boldsymbol{b}| \ll 1.$$

We introduce

$$R = \sum_{n=1}^{N} Q_n, \quad U = \sum_{n=1}^{N} a_n \Psi_n, \quad V = \sum_{(n,k)\in I^0} b_{n,k}\Phi_{n,k},$$

and nonlinear interactions term

$$G = (R+U+V)^3 - \sum_{n=1}^{N} Q_n^3 - 3\sum_{n=1}^{N} a_n Q_n^2 \Psi_n - 3\sum_{(n,k)\in I^0} b_{n,k} Q_n^2 \Phi_{n,k}. \tag{5.36}$$

5.3.1 Nonlinear interactions

In this section, we introduce the estimates on the nonlinear interaction terms. In particular, due to sufficient decay along with the direction $\mathbf{e}_1$, the order of the main term of interactions is t^{-4} (see Lemma 5.15).

First, we expand G of the components of the R, U and V by an elementary computation.

Lemma 5.12. *We have*

$$G = G_1 + G_2 + G_3 = G_1 + \sum_{n=1}^{N} G_{2,n} + \sum_{i=1}^{4} G_{3,i}, \tag{5.37}$$

where

$$G_1 = 3\sum_{n\neq n'} Q_n^2 Q_{n'} + 6 \sum_{n_1<n_2<n_3} Q_{n_1}Q_{n_2}Q_{n_3},$$

$$G_{2,n} = 3Q_n\Big(a_n\Psi_n + \sum_{k=1}^{K} b_{n,k}\Phi_{n,k}\Big)^2 \quad \textit{for } n = 1,\dots,N,$$

and

$$G_{3,1} = 3\sum_{n\neq n'} Q_n\Big(a_{n'}\Psi_{n'} + \sum_{k=1}^{K} b_{n',k}\Phi_{n',k}\Big)^2,$$

$$G_{3,2} = (U+V)^3 = U^3 + V^3 + 3U^2V + 3UV^2,$$

$$G_{3,3} = 3\sum_{n\neq n'} Q_n^2\Big(a_{n'}\Psi_{n'} + \sum_{k=1}^{K} b_{n',k}\Phi_{n',k}\Big) + 3\sum_{n\neq n'} Q_nQ_{n'}(U+V),$$

$$G_{3,4} = 3R\sum_{n\neq n'}\Big(a_n\Psi_n + \sum_{k=1}^{K} b_{n,k}\Phi_{n,k}\Big)\Big(a_{n'}\Psi_{n'} + \sum_{k'=1}^{K} b_{n',k'}\Phi_{n',k'}\Big).$$

Remark 5.13. Note that G_1 is a pure interaction term and G_3 is a combination of the interaction terms and cubic terms, while G_2 is a pure non-interaction quadratic term and each component $G_{2,n}$ satisfies the traveling equation $\partial_t G_{2,n} \approx -\ell_n \partial_{x_1} G_{2,n}$ at the main order.

Proof. First, we decompose

$$\begin{aligned} G =&(U+V)^3 + \left(R^3 - \sum_{n=1}^N Q_n^3\right) + 3R(U+V)^2 \\ &+ 3\left(R^2(U+V) - \sum_{n=1}^N a_n Q_n^2 \Psi_n - \sum_{(n,k)\in I^0} b_{n,k} Q_n^2 \Phi_{n,k}\right). \end{aligned}$$

For the first two terms, by an elementary computation,

$$\begin{aligned} (U+V)^3 &= U^3 + V^3 + 3U^2V + 3UV^2, \\ R^3 - \sum_{n=1}^N Q_n^3 &= 3\sum_{n\neq n'} Q_n^2 Q_{n'} + 6 \sum_{n_1<n_2<n_3} Q_{n_1}Q_{n_2}Q_{n_3}. \end{aligned}$$

For the third term, by the definition of U and V, we decompose

$$\begin{aligned} (U+V)^2 =& \sum_{n=1}^N \left(a_n\Psi_n + \sum_{k=1}^K b_{n,k}\Phi_{n,k}\right)^2 \\ &+ \sum_{n\neq n'} \left(a_n\Psi_n + \sum_{k=1}^K b_{n,k}\Phi_{n,k}\right)\left(a_{n'}\Psi_{n'} + \sum_{k'=1}^K b_{n',k'}\Phi_{n',k'}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} 3R(U+V)^2 =& 3\sum_{n=1}^N Q_n \left(a_n\Psi_n + \sum_{k=1}^K b_{n,k}\Phi_{n,k}\right)^2 \\ &+ 3\sum_{n\neq n'} Q_n \left(a_{n'}\Psi_{n'} + \sum_{k=1}^K b_{n',k}\Phi_{n',k}\right)^2 \\ &+ 3R\sum_{n\neq n'} \left(a_n\Psi_n + \sum_{k=1}^K b_{n,k}\Phi_{n,k}\right)\left(a_{n'}\Psi_{n'} + \sum_{k'=1}^K b_{n',k'}\Phi_{n',k'}\right). \end{aligned}$$

For the last term, we have

$$R^2(U+V) = \sum_{n=1}^N Q_n^2(U+V) + \sum_{n\neq n'} Q_nQ_{n'}(U+V).$$

Thus, from the definition of U and V,

$$\begin{aligned} &3\left(R^2(U+V) - \sum_{n=1}^N a_n Q_n^2\Psi_n - \sum_{(n,k)\in I^0} b_{n,k}Q_n^2\Phi_{n,k}\right) \\ &= 3\sum_{n\neq n'} Q_n^2\left(a_{n'}\Psi_{n'} + \sum_{k=1}^K b_{n',k}\Phi_{n',k}\right) + 3\sum_{n\neq n'} Q_nQ_{n'}(U+V). \end{aligned}$$

Combining the above identities, we obtain the decomposition (5.37). □

Second, we introduce a technical lemma for future reference.

Lemma 5.14. *Let f_1 and f_2 be continuous functions such that*

$$|f_1(x)| + |f_2(x)| \lesssim \langle x\rangle^{-2} \quad \text{on } \mathbb{R}^4. \tag{5.38}$$

Define

$$f_{n_1}(t,x) = f_1\left(x - \boldsymbol{\ell}_{n_1} t\right), \quad f_{n_2}(t,x) = f_2\left(x - \boldsymbol{\ell}_{n_1} t\right).$$

Let $0 < \alpha_1 \leqslant \alpha_2$ be such that $\alpha_1 + \alpha_2 > 2$. There exists $T_0 \gg 1$ such that, for all $n_1, n_2 = 1, \dots, N$ with $n_1 \neq n_2$ and $t \geqslant T_0$, the following hold.

(i) *If $\alpha_2 > 2$,*

$$\int_{\mathbb{R}^4} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \, \mathrm{d}x \lesssim t^{-2\alpha_1}. \tag{5.39}$$

(ii) *If $\alpha_2 < 2$,*

$$\int_{\mathbb{R}^4} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \, \mathrm{d}x \lesssim t^{4-2(\alpha_1+\alpha_2)}. \tag{5.40}$$

(iii) *If $\alpha_2 = 2$,*

$$\int_{\mathbb{R}^4} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \, \mathrm{d}x \lesssim t^{-2\alpha_1} \mathrm{log} t. \tag{5.41}$$

Proof. For $k = 1, 2$, set

$$r_k = x - \boldsymbol{\ell}_{n_k} t \quad \text{and} \quad \Omega_{n_k}(t,x) = \left\{x \in \mathbb{R}^4 : |r_k| \leqslant 10^{-1} \left|\boldsymbol{\ell}_{n_1} - \boldsymbol{\ell}_{n_2}\right| t\right\}.$$

Let $T_0 \gg 1$ large enough. For $t \geqslant T_0$, from the decay property of f_k in (5.38),

$$|f_{n_1}(t,x)| \lesssim \langle r_1\rangle^{-2} \lesssim t^{-2}, \qquad \text{for } x \in \Omega_{n_1}^C(t,x), \tag{5.42}$$

$$|f_{n_2}(t,x)| \lesssim \langle r_2\rangle^{-2} \lesssim t^{-2}, \qquad \text{for } x \in \Omega_{n_2}^C(t,x), \tag{5.43}$$

$$|f_{n_1}(t,x)| \lesssim \langle r_1\rangle^{-2} \lesssim \left(\langle r_2\rangle + t\right)^{-2}, \quad \text{for } x \in \Omega_{n_2}(t,x), \tag{5.44}$$

$$|f_{n_2}(t,x)| \lesssim \langle r_2\rangle^{-2} \lesssim \left(\langle r_1\rangle + t\right)^{-2}, \quad \text{for } x \in \Omega_{n_1}(t,x). \tag{5.45}$$

Proof of (i). *Case* $0 < \alpha_1 \leqslant 2 < \alpha_2$. From (5.42),

$$\int_{\Omega_{n_1}^C} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-2\alpha_1} \int_{\Omega_{n_1}^C} |f_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-2\alpha_1}.$$

Next, by (5.38), (5.45) and change of variable,

$$\begin{aligned}
\int_{\Omega_{n_1}} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \mathrm{d}x &\lesssim \int_{\Omega_{n_1}} \langle r_1\rangle^{-2\alpha_1} \left(\langle r_1\rangle + t\right)^{-2\alpha_2} \mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^4} \langle x\rangle^{-2\alpha_1} \left(\langle x\rangle + t\right)^{-2\alpha_2} \mathrm{d}x \\
&\lesssim t^{4-2(\alpha_1+\alpha_2)} \mathrm{log} t \lesssim t^{-2\alpha_1}.
\end{aligned}$$

Combining the above estimates, we obtain (5.39).

Case $2 < \alpha_1 \leqslant \alpha_2$. By (5.42) and (5.45), we have

$$\begin{aligned}
&\int_{\Omega_{n_1}^C} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-2\alpha_1} \int_{\Omega_{n_1}^C} |f_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-2\alpha_1}, \\
&\int_{\Omega_{n_1}} |f_{n_1}|^{\alpha_1} |f_{n_2}|^{\alpha_2} \mathrm{d}x \lesssim t^{-2\alpha_2} \int_{\Omega_{n_1}} |f_{n_1}|^{\alpha_1} \mathrm{d}x \lesssim t^{-2\alpha_2} \lesssim t^{-2\alpha_1},
\end{aligned}$$

which implies (5.39).

Proof of (ii). First, by (5.38), (5.44) and (5.45), as before,

$$\begin{aligned}&\int_{\Omega_{n_1}}|f_{n_1}|^{\alpha_1}|f_{n_2}|^{\alpha_2}\mathrm{d}x+\int_{\Omega_{n_2}}|f_{n_1}|^{\alpha_1}|f_{n_2}|^{\alpha_2}\mathrm{d}x\\&\lesssim\int_{\mathbb{R}^4}\left(\langle x\rangle^{-2\alpha_1}(\langle x\rangle+t)^{-2\alpha_2}+\langle x\rangle^{-2\alpha_2}(\langle x\rangle+t)^{-2\alpha_1}\right)\mathrm{d}x\lesssim t^{4-2(\alpha_1+\alpha_2)}.\end{aligned}$$

Next, by the Hölder inequality, we have

$$\begin{aligned}\int_{(\Omega_{n_1}\cup\Omega_{n_2})^C}|f_{n_1}|^{\alpha_1}|f_{n_2}|^{\alpha_2}\mathrm{d}x&\lesssim\left(\int_{\Omega_{n_1}^C}|f_{n_1}|^{\alpha_1+\alpha_2}\mathrm{d}x\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\int_{\Omega_{n_2}^C}|f_{n_2}|^{\alpha_1+\alpha_2}\mathrm{d}x\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\\&\lesssim\int_{|x|>\frac{1}{10}|\boldsymbol{\ell}_{n_1}-\boldsymbol{\ell}_{n_2}|t}\langle x\rangle^{-2(\alpha_1+\alpha_2)}\mathrm{d}x\lesssim t^{4-2(\alpha_1+\alpha_2)}.\end{aligned}$$

Combining the above estimates, we obtain (5.40).

Proof of (iii). Using (5.38), (5.44) and (5.45) again,

$$\begin{aligned}&\int_{\Omega_{n_1}}|f_{n_1}|^{\alpha_1}|f_{n_2}|^{2}\mathrm{d}x+\int_{\Omega_{n_2}}|f_{n_1}|^{\alpha_1}|f_{n_2}|^{2}\mathrm{d}x\\&\lesssim\int_{\mathbb{R}^4}\left(\langle x\rangle^{-2\alpha_1}(\langle x\rangle+t)^{-4}+\langle x\rangle^{-4}(\langle x\rangle+t)^{-2\alpha_1}\right)\mathrm{d}x\lesssim t^{-2\alpha_1}\log t.\end{aligned}$$

Using a similar argument as in the proof of (ii), we have

$$\int_{(\Omega_{n_1}\cup\Omega_{n_2})^C}|f_{n_1}|^{\alpha_1}|f_{n_2}|^{2}\mathrm{d}x\lesssim\int_{|x|>\frac{1}{10}|\boldsymbol{\ell}_{n_1}-\boldsymbol{\ell}_{n_2}|t}\langle x\rangle^{-2(\alpha_1+2)}\mathrm{d}x\lesssim t^{-2\alpha_1}.$$

Combining the above estimates, we obtain (5.41). □

Now, we prove the following estimates on the nonlinear interaction terms.

Lemma 5.15. *There exists $T_0\gg 1$, such that the following estimates hold for $t\geqslant T_0$.*

(i) Estimate on G_1. *We have*

$$\|G_1\|_{L^2}\lesssim t^{-4}. \tag{5.46}$$

(ii) Estimate on G_2. *We have*

$$\|G_2\|_{L^2}\lesssim|\boldsymbol{a}|^2+|\boldsymbol{b}|^2. \tag{5.47}$$

(iii) Estimate on G_3. *We have*

$$\|G_3\|_{L^2}\lesssim|\boldsymbol{a}|^2+|\boldsymbol{b}|^3+t^{-4}. \tag{5.48}$$

(iv) Estimate on G. *We have*

$$\|G\|_{L^2}\lesssim|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+t^{-4}. \tag{5.49}$$

Remark 5.16. As mentioned in Remark 5.13, G_2 has a slow decay different from G_1 and G_3. This leads us to introduce an additional technical quantity $\mathcal{G}$ in the refined energy functional later (see also Remark 5.25). Since the algebraic decay of ψ and ϕ_k are different in Lemma 5.7, the decay of interaction terms related to $\boldsymbol{a}$ and $\boldsymbol{b}$ are different, which implies the power of $\boldsymbol{a}$ and $\boldsymbol{b}$ are different in (5.48).

Proof. Proof of (i). Let $t \geqslant T_0 \gg 1$. We prove the following estimate,

$$\sum_{n \neq n'} \|Q_n^2 Q_{n'}\|_{L^2} \lesssim t^{-4}. \tag{5.50}$$

Note that (5.46) is a consequence of (5.50) and the AM-GM inequality.

First, for any $n \neq n'$, we consider change of variable

$$\left(\frac{x_1 - \ell_n t}{\sqrt{1-\ell_n^2}}, \bar{x}\right) \mapsto (y_1, \bar{y}) \in \mathbb{R}^4.$$

It follows that

$$\int_{\mathbb{R}^4} Q_n^4 Q_{n'}^2 \mathrm{d}x = H_1 + H_2,$$

where

$$H_1 = (1-\ell_n^2)^{\frac{1}{2}} \int_{\mathbb{R}^3} \int_{I_1} Q^4(x_1, \bar{x}) Q^2 \left(\frac{(1-\ell_n^2)^{\frac{1}{2}} x_1 - (\ell_{n'} - \ell_n)t}{\sqrt{1-\ell_{n'}^2}}, \bar{x}\right) \mathrm{d}x_1 \mathrm{d}\bar{x},$$

$$H_2 = (1-\ell_n^2)^{\frac{1}{2}} \int_{\mathbb{R}^3} \int_{I_2} Q^4(x_1, \bar{x}) Q^2 \left(\frac{(1-\ell_n^2)^{\frac{1}{2}} x_1 - (\ell_{n'} - \ell_n)t}{\sqrt{1-\ell_{n'}^2}}, \bar{x}\right) \mathrm{d}x_1 \mathrm{d}\bar{x},$$

and

$$I_1 = \left\{x_1 \in \mathbb{R} : (1-\ell_n^2)^{\frac{1}{2}} |x_1| \geqslant \frac{1}{2}|\ell_n - \ell_{n'}|t\right\},$$

$$I_2 = \left\{x_1 \in \mathbb{R} : (1-\ell_n^2)^{\frac{1}{2}} |x_1| < \frac{1}{2}|\ell_n - \ell_{n'}|t\right\}.$$

Estimate on H_1. First, from (5.11), for any $x = (x_1, \bar{x}) \in I_1 \times \mathbb{R}^3$,

$$|Q(x_1, \bar{x})| \lesssim (t + |\bar{x}|)^{-3} \lesssim t^{-3}.$$

Based on above estimate and $Q \in L^2$, we have

$$H_1 \lesssim \|Q\|_{L^\infty(I_1 \times \mathbb{R}^3)}^4 \int_{\mathbb{R}^3} \int_{I_1} Q^2 \left(\frac{(1-\ell_n^2)^{\frac{1}{2}} x_1 - (\ell_{n'} - \ell_n)t}{\sqrt{1-\ell_{n'}^2}}, \bar{x}\right) \mathrm{d}x_1 \mathrm{d}\bar{x} \lesssim t^{-12}.$$

Estimate on H_2. Note that, for any $x_1 \in I_2$,

$$\left|\frac{(1-\ell_n^2)^{\frac{1}{2}} x_1 - (\ell_{n'} - \ell_n)t}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right| \geqslant \left|\frac{(\ell_{n'} - \ell_n)t}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right| - \left|\frac{(1-\ell_n^2)^{\frac{1}{2}} x_1}{(1-\ell_{n'}^2)^{\frac{1}{2}}}\right| \geqslant \frac{|\ell_{n'} - \ell_n|t}{2(1-\ell_{n'}^2)^{\frac{1}{2}}}.$$

By (5.10), for any $x = (x_1, \bar{x}) \in I_2 \times \mathbb{R}^3$, we have

$$\left|Q\left(\frac{(1-\ell_n^2)^{\frac{1}{2}} x_1 - (\ell_{n'} - \ell_n)t}{\sqrt{1-\ell_{n'}^2}}, \bar{x}\right)\right| \lesssim \frac{|\bar{x}|}{|\bar{x}|^4 + t^4} + t^{-4} \lesssim t^{-4}(|x| + 1).$$

Based on above estimate and (5.11),

$$H_2 \lesssim t^{-8} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \langle x \rangle^{-12} (|x|^2 + 1) \mathrm{d}x_1 \mathrm{d}\bar{x} \lesssim t^{-8}.$$

Combining the estimates on H_1 and H_2, we obtain (5.50).

Proof of (ii). First, from (5.11) and (5.19), we know that

$$|Q|\psi^2 + \sum_{k=1}^{K} |Q|\phi_k^2 \in L^2.$$

Next, from the AM-GM inequality,

$$|G_2| \lesssim \sum_{n=1}^{N} |a_n|^2 |Q_n||\Psi_n|^2 + \sum_{(n,k)\in I} |Q_n||b_{n,k}|^2|\Phi_{n,k}|^2.$$

It follows that,

$$\|G_2\|_{L^2} \lesssim \left(\sum_{n=1}^{N} a_n^2 + \sum_{(n,k)\in I^0} b_{n,k}^2\right)\left(\|Q\psi^2\|_{L^2} + \sum_{k=1}^{K} \|Q\phi_k^2\|_{L^2}\right) \lesssim |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2.$$

Proof of (iii). **Step 1.** Estimate on $G_{3,1}$. We claim

$$\|G_{3,1}\|_{L^2} \lesssim |\boldsymbol{a}|^8 + |\boldsymbol{b}|^8 + t^{-4}. \tag{5.51}$$

From the AM-GM inequality,

$$|G_{3,1}| \lesssim \sum_{n\neq n'} |Q_n|\left(a_{n'}^2\Psi_{n'}^2 + \sum_{k=1}^{K} b_{n',k}^2\Phi_{n',k}^2\right).$$

Note that, by (5.11), (5.19), (5.39) and the Young's inequality,

$$\sum_{n\neq n'} \|a_{n'}^2\Psi_{n'}^2 Q_n\|_{L^2} \lesssim |\boldsymbol{a}|^2 t^{-3} \lesssim |\boldsymbol{a}|^8 + t^{-4},$$

$$\sum_{n\neq n'}\sum_{k=1}^{K} \|b_{n',k}^2\Phi_{n',k}^2 Q_n\|_{L^2} \lesssim |\boldsymbol{b}|^2 t^{-3} \lesssim |\boldsymbol{b}|^8 + t^{-4}.$$

Combining above estimates, we find (5.51).

Step 2. Estimate on $G_{3,2}$. We claim

$$\|G_{3,2}\|_{L^2} \lesssim |\boldsymbol{a}|^3 + |\boldsymbol{b}|^3. \tag{5.52}$$

First, from (5.20), we know that

$$|\psi| + \sum_{k=1}^{K} |\phi_k| \in L^6.$$

Note that, from the AM-GM inequality,

$$|G_{3,2}| \lesssim |U|^3 + |V|^3 \lesssim \sum_{n=1}^{N} |a_n|^3|\Psi_n|^3 + \sum_{(n,k)\in I^0} |b_{n,k}|^3|\Phi_{n,k}|^3.$$

It follows that,

$$\|G_{3,2}\|_{L^2} \lesssim (|\boldsymbol{a}|^3 + |\boldsymbol{b}|^3)\left(\|\psi\|_{L^6}^3 + \sum_{k=1}^{K} \|\phi_k\|_{L^6}^3\right) \lesssim |\boldsymbol{a}|^3 + |\boldsymbol{b}|^3,$$

which means (5.52).

Step 3. Estimate on $G_{3,3}$. We claim

$$\|G_{3,3}\|_{L^2} \lesssim |\boldsymbol{a}|^2 + |\boldsymbol{b}|^4 + t^{-4}. \tag{5.53}$$

First, using the AM-GM inequality again,

$$\begin{aligned}|G_{3,3}| \lesssim \sum_{n\neq n'} |a_{n'} Q_n^2 \Psi_{n'}| + \sum_{n\neq n'} |a_n Q_n Q_{n'} \Psi_n| \\ + \sum_{k=1}^{K} \sum_{n\neq n'} |b_{n',k} Q_n^2 \Phi_{n',k}| + \sum_{k=1}^{K} \sum_{n\neq n'} |b_{n,k} Q_n Q_{n'} \Phi_{n,k}|.\end{aligned}$$

Second, using (5.11), (5.19), (5.39) and the Young's inequality again,

$$\sum_{n\neq n'} \|a_{n'} Q_n^2 \Psi_{n'}\|_{L^2} + \sum_{n\neq n'} \|a_n Q_n Q_{n'} \Psi_n\|_{L^2} \lesssim |\boldsymbol{a}|(t^{-2} + t^{-3}) \lesssim |\boldsymbol{a}|^2 + t^{-4},$$

$$\sum_{k=1}^{K} \sum_{n\neq n'} \|b_{n',k} Q_n^2 \Phi_{n',k}\|_{L^2} + \sum_{k=1}^{K} \sum_{n\neq n'} \|b_{n,k} Q_n Q_{n'} \Phi_{n,k}\|_{L^2} \lesssim |\boldsymbol{b}| t^{-3} \lesssim |\boldsymbol{b}|^4 + t^{-4}.$$

Combining above estimates, we obtain (5.53).

Step 4. Estimate on $G_{3,4}$. We claim

$$\|G_{3,4}\|_{L^2} \lesssim |\boldsymbol{a}|^4 + |\boldsymbol{b}|^4 + t^{-4}. \tag{5.54}$$

By the AM-GM inequality and the definition of R,

$$\begin{aligned}|G_{3,4}| \lesssim \sum_{n\neq n'} |Q_n| \Big(a_{n'}^2 \Psi_{n'}^2 + \sum_{k=1}^{K} b_{n',k}^2 \Phi_{n',k}^2 \Big) \\ + \sum_{n\neq n'} |a_n||Q_n||\Psi_n| \Big(|a_{n'}||\Psi_{n'}| + \sum_{k=1}^{K} |b_{n',k}||\Phi_{n',k}| \Big) \\ + \sum_{n\neq n'} \sum_{k=1}^{K} |b_{n,k}||Q_n||\Phi_{n,k}| \Big(|a_{n'}||\Psi_{n'}| + \sum_{k=1}^{K} |b_{n',k'}||\Phi_{n',k'}| \Big).\end{aligned}$$

Arguing as in the Step 2 and Step 3, using (5.11), (5.19), (5.39) and the Young's inequality, we obtain (5.54).

We see that (5.48) follows from (5.51), (5.52), (5.53) and (5.54).

Proof of (iv). Estimate (5.49) is a consequence of (5.46), (5.47) and (5.48). □

5.3.2 Decomposition of the solution around multi-solitary wave

In this section, we recall general results on solutions of (5.1) that are close to sum of $N \geqslant 2$ decoupled solitary waves.

First, we recall a decomposition result for solutions of (5.1).

Proposition 5.17. *There exist $T_0 \gg 1$ and $0 < \gamma_0 \ll 1$ such that if $\vec{u}(t) = (u(t), \partial_t u(t))$ is a solution of (5.1) on $[T_1, T_2]$, where $1 \ll T_0 \leqslant T_1 < T_2 < \infty$, such that*

$$\sup_{t\in[T_1,T_2]} \|\vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\| < \gamma_0, \tag{5.55}$$

then there exist C^1 functions $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{b} = (b_{n,k})_{(n,k)\in I^0}$ such that, $\vec{\varphi}$ being defined by

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \vec{u} - \sum_{n=1}^{N} \vec{Q}_n - \sum_{n=1}^{N} a_n \vec{\Psi}_n - \sum_{(n,k)\in I^0} b_{n,k} \vec{\Phi}_{n,k}, \tag{5.56}$$

it satisfies

$$\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}| + |\boldsymbol{b}| \lesssim \gamma_0,$$

and

$$\begin{aligned} \left(\vec{\varphi}, \vec{\Phi}_{n',k}\right)_{\mathcal{H}} &= 0, \quad \forall (n',k) \in I^0, \\ \left(\vec{\varphi}, \vec{\Psi}_n\right)_{\mathcal{H}} &= 0, \quad \forall\ n = 1, \dots, N. \end{aligned} \tag{5.57}$$

Proof. The proof of the decomposition proposition relies on a standard argument based on Lemma 5.7, (ii) and (iii) of Lemma 5.14 and the invertibility of Gram matrix (see *e.g.* [26, Lemma 3] and [112, Proposition 3.4]), and we omit it. □

Set

$$\begin{aligned} \mathrm{Mod}_1 &= \mathrm{Mod}_{1,1} + \mathrm{Mod}_{1,2}, \\ \mathrm{Mod}_2 &= \mathrm{Mod}_{2,1} + \mathrm{Mod}_{2,2}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{Mod}_{1,1} &= \sum_{n=1}^{N} \dot{a}_n \Psi_n, & \mathrm{Mod}_{2,1} &= -\sum_{n=1}^{N} \dot{a}_n \ell_n \partial_{x_1} \Psi_n, \\ \mathrm{Mod}_{1,2} &= \sum_{(n,k)\in I^0} \dot{b}_{n,k} \Phi_{n,k}, & \mathrm{Mod}_{2,2} &= -\sum_{(n,k)\in I^0} \dot{b}_{n,k} \ell_n \partial_{x_1} \Phi_{n,k}. \end{aligned}$$

We deduce the equation of $\vec{\varphi}$ from (5.1) and (5.56).

Lemma 5.18 (Equation of $\vec{\varphi}$)**.** *The function $\vec{\varphi}$ satisfies*

$$\begin{cases} \partial_t \varphi_1 = \varphi_2 - \mathrm{Mod}_1, \\ \partial_t \varphi_2 = \Delta \varphi_1 + (R + U + V + \varphi_1)^3 - (R + U + V)^3 - \mathrm{Mod}_2 + G. \end{cases} \tag{5.58}$$

Proof. First, from the definition of Q_n, Ψ_n and $\Phi_{n,k}$,

$$\begin{aligned} \partial_t Q_n &= -\ell_n \partial_{x_1} Q_n, \\ \partial_t (a_n \Psi_n) &= -\ell_n \partial_{x_1} \Psi_n + \dot{a}_n \Psi_n, \\ \partial_t (b_{n,k} \Phi_{n,k}) &= -\ell_n \partial_{x_1} \Phi_{n,k} + \dot{b}_{n,k} \Phi_{n,k}, \end{aligned}$$

for $n = 1, \dots, N$ and $(n,k) \in I^0$. Therefore, by the definition of $\vec{\varphi} = (\varphi_1, \varphi_2)$ in (5.56), we have

$$\begin{aligned} \partial_t \varphi_1 &= \partial_t u - \sum_{n=1}^{N} \partial_t Q_n - \sum_{n=1}^{N} \partial_t (a_n \Psi_n) - \sum_{(n,k)\in I^0} \partial_t (b_{n,k} \Phi_{n,k}) \\ &= \varphi_2 - \sum_{n=1}^{N} \dot{a}_n \Psi_n - \sum_{(n,k)\in I^0} \dot{b}_{n,k} \Phi_{n,k}. \end{aligned}$$

Second, using the definition of Q_n, Ψ_n and $\Phi_{n,k}$ again,

$$\begin{aligned} \partial_t (\ell_n \partial_{x_1} Q_n) &= -\ell_n^2 \partial_{x_1}^2 Q_n, \\ \partial_t (a_n \ell_n \partial_{x_1} \Psi_n) &= -a_n \ell_n^2 \partial_{x_1}^2 \Psi_n + \dot{a}_n \ell_n \partial_{x_1} \Psi_n, \\ \partial_t (b_{n,k} \ell_n \partial_{x_1} \Phi_{n,k}) &= -b_{n,k} \ell_n^2 \partial_{x_1}^2 \Phi_{n,k} + \dot{b}_{n,k} \ell_n \partial_{x_1} \Phi_{n,k}, \end{aligned}$$

for $n = 1, \dots, N$ and $(n,k) \in I^0$. Thus, by (5.1) and the definition of Mod_2,

$$\begin{aligned}\partial_t \varphi_2 &= \partial_{tt} u + \sum_{n=1}^N \partial_t(\ell_n \partial_{x_1} Q_n) + \sum_{n=1}^N \partial_t(a_n \ell_n \partial_{x_1} \Psi_n) + \sum_{(n,k)\in I^0} \partial_t(b_{n,k} \ell_n \partial_{x_1} \Phi_{n,k}) \\ &= \Delta u + u^3 - \sum_{n=1}^N \ell_n^2 \partial_{x_1}^2 Q_n - \sum_{n=1}^N a_n \ell_n^2 \partial_{x_1}^2 \Psi_n - \sum_{(n,k)\in I^0} b_{n,k} \ell_n^2 \partial_{x_1}^2 \Phi_{n,k} - \mathrm{Mod}_2.\end{aligned}$$

Note that, from $-(1-\ell_n^2)\partial_{x_1}^2 Q_n - \bar{\Delta} Q_n - Q_n^3 = 0$, $-(1-\ell_n^2)\partial_{x_1}^2 \Psi_n - \bar{\Delta}\Psi_n - 3Q_n^2 \Psi_n = 0$, $-(1-\ell_n^2)\partial_{x_1}^2 \Phi_{n,k} - \bar{\Delta}\Phi_{n,k} - 3Q_n^2 \Phi_{n,k} = 0$ and the definition of G in (5.36),

$$\begin{aligned}&\Delta u + u^3 - \sum_{n=1}^N \ell_n^2 \partial_{x_1}^2 Q_n - \sum_{n=1}^N a_n \ell_n^2 \partial_{x_1}^2 \Psi_n - \sum_{(n,k)\in I^0} b_{n,k} \ell_n^2 \partial_{x_1}^2 \Phi_{n,k} \\ &= \Delta \varphi_1 + (R+U+V+\varphi_1)^3 - (R+U+V)^3 + G.\end{aligned}$$

Based on the above identities, we obtain

$$\partial_t \varphi_2 = \Delta \varphi_1 + (R+U+V+\varphi_1)^3 - (R+U+V)^3 - \mathrm{Mod}_2 + G.$$

□

We decompose

$$(R+U+V+\varphi_1)^3 - (R+U+V)^3 = 3\sum_{n=1}^N Q_n^2 \varphi_1 + R_1 + R_2,$$

where

$$\begin{aligned}R_1 &= 3(R+U+V)\varphi_1^2 + \varphi_1^3, \\ R_2 &= 3\left((U+V)^2 + 2R(U+V)\right)\varphi_1 + 3\sum_{n\neq n'} Q_n Q_{n'} \varphi_1.\end{aligned}$$

Based on the above decomposition, the equation of $\vec{\varphi}$ in (5.58) can be rewritten as

$$\partial_t \vec{\varphi} = \vec{\mathcal{L}}\vec{\varphi} - \vec{\mathrm{Mod}} + \vec{G} + \vec{R}_1 + \vec{R}_2, \tag{5.59}$$

where

$$\vec{\mathcal{L}} = \begin{pmatrix} 0 & 1 \\ \Delta + 3\sum_{n=1}^N Q_n^2 & 0 \end{pmatrix}, \quad \vec{\mathrm{Mod}} = \begin{pmatrix} \mathrm{Mod}_1 \\ \mathrm{Mod}_2 \end{pmatrix},$$

and

$$\vec{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}, \quad \vec{R}_1 = \begin{pmatrix} 0 \\ R_1 \end{pmatrix}, \quad \vec{R}_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix}.$$

Now, we derive the control of $\dot{\boldsymbol{a}}$ and $\dot{\boldsymbol{b}}$ from the orthogonality conditions (5.57).

Lemma 5.19. *In the context of Proposition 5.17, we have*

$$|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}| \lesssim \|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}. \tag{5.60}$$

Proof. From (5.59) and the orthogonality condition (5.57), we have

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\varphi}, \vec{\Psi}_n\right)_{\mathcal{H}} &= \left(\partial_t \vec{\varphi}, \vec{\Psi}_n\right)_{\mathcal{H}} + \left(\vec{\varphi}, \partial_t \vec{\Psi}_n\right)_{\mathcal{H}} \\ &= \left(\vec{\mathcal{L}}\vec{\varphi}, \vec{\Psi}_n\right)_{\mathcal{H}} - \left(\vec{\mathrm{Mod}}, \vec{\Psi}_n\right)_{\mathcal{H}} + \left(\vec{G}, \vec{\Psi}_n\right)_{\mathcal{H}} \\ &\quad + \left(\vec{R}_1, \vec{\Psi}_n\right)_{\mathcal{H}} + \left(\vec{R}_2, \vec{\Psi}_n\right)_{\mathcal{H}} - \ell_n \left(\vec{\varphi}, \partial_{x_1} \vec{\Psi}_n\right)_{\mathcal{H}}.\end{aligned}$$

By integration by parts and the decay property of ψ in (5.19), the first term is

$$\left(\vec{\mathcal{L}}\vec{\varphi}, \vec{\Psi}_n\right)_{\mathcal{H}} = \left(\varphi_1, \left(\Delta + 3\sum_{n'=1}^{N} Q_{n'}^2\right)(-\ell_n \partial_{x_1}\Psi_n)\right)_{L^2} - (\varphi_2, \Delta\Psi_n)_{L^2} = O\left(\|\vec{\varphi}\|_{\mathcal{H}}\right).$$

Next, from the decay properties of ψ and ϕ in (5.19), and Lemma 5.14 (ii) and (iii),

$$\left(\vec{\mathrm{Mod}}, \vec{\Psi}_n\right)_{\mathcal{H}} = \dot{a}_n \left(\vec{\Psi}_n, \vec{\Psi}_n\right)_{\mathcal{H}} + \sum_{k=1}^{K} \dot{b}_{n,k} \left(\vec{\Phi}_{n,k}, \vec{\Psi}_n\right)_{\mathcal{H}} + O\left(t^{-2}(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|)\right).$$

Using (5.49), the decay property of ψ and the Cauchy-Schwarz inequality,

$$\left|\left(\vec{G}, \vec{\Psi}_n\right)_{\mathcal{H}}\right| \lesssim \|G\|_{L^2}\|\partial_{x_1}\Psi_n\|_{L^2} \lesssim |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}.$$

Then, from the expansion of R_1 and R_2, and the Sobolev inequality (5.8),

$$\begin{aligned}
\left|\left(\vec{R}_1, \vec{\Psi}_n\right)_{\mathcal{H}}\right| &\lesssim \|\varphi_1\|_{L^4}^2 \|\partial_{x_1}\Psi_n\|_{L^4} \left(\|\varphi_1\|_{L^4} + \|R + U + V\|_{L^4}\right) \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2, \\
\left|\left(\vec{R}_2, \vec{\Psi}_n\right)_{\mathcal{H}}\right| &\lesssim \|\varphi_1\|_{L^4} \|\partial_{x_1}\Psi_n\|_{L^4} \left(\|U\|_{L^4}^2 + \|V\|_{L^4}^2 + \|Q\|_{L^4}^2\right) \lesssim \|\vec{\varphi}\|_{\mathcal{H}}.
\end{aligned}$$

Last, using the Cauchy-Schwarz inequality again,

$$\left|\left(\vec{\varphi}, \partial_{x_1}\vec{\Psi}_n\right)_{\mathcal{H}}\right| \lesssim \left(\|\nabla\varphi_1\|_{L^2} + \|\varphi_2\|_{L^2}\right)\|\partial_{x_1}\Psi_n\|_{H^1} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}.$$

Gathering the above estimates, we have

$$\begin{aligned}
&\dot{a}_n \left(\vec{\Psi}_n, \vec{\Psi}_n\right)_{\mathcal{H}} + \sum_{k=1}^{K} \dot{b}_{n,k} \left(\vec{\Phi}_{n,k}, \vec{\Psi}_n\right)_{\mathcal{H}} + O\left(t^{-2}(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|)\right) \\
&= O\left(\|\vec{\varphi}\|_{\mathcal{H}} + t^{-4} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right).
\end{aligned}$$

Proceeding similarly for all $n = 1, \dots, N$ and $(n,k) \in I^0$, and then using the invertibility of Gram matrix, we obtain (5.60). □

Fix

$$\sigma = \frac{1}{10}\min_{n \neq n'} |\ell_n - \ell_{n'}| > 0. \tag{5.61}$$

For $n = 1, \dots, N$, let

$$\sigma_n = 2\pi^2(1 - \ell_n^2)^{\frac{1}{2}}, \quad c_n(t) = \frac{(\varphi_1, \Psi_n\xi_n)_{L^2}}{\sigma_n \log t}, \tag{5.62}$$

where

$$\xi_n(t,x) = \xi\left((\sigma t)^{-1}\left|\left(\frac{x_1 - \ell_n t}{\sqrt{1-\ell_n^2}}, \bar{x}\right)\right|\right), \quad \text{for } n = 1, \dots, N. \tag{5.63}$$

For $n = 1, \dots, N$, we also set

$$\Omega_n(t,x) = \left\{x = (x_1, \bar{x}) \in \mathbb{R}^4 : \sigma t \leqslant \left|\left(\frac{x_1 - \ell_n t}{\sqrt{1-\ell_n^2}}, \bar{x}\right)\right| \leqslant 2\sigma t\right\}.$$

Note that, from the definition of ξ_n and Ω_n, we have the following estimates ($\mathbf{1}_K$ denotes the characteristic function of the compact set K),

$$|\partial_t \xi_n(t,x)| + |\partial_{x_1}\xi_n(t,x)| \lesssim t^{-1}\mathbf{1}_{\Omega_n}. \tag{5.64}$$

Note also that, from the decay property of ψ in (5.19) and the definition of Ψ_n and ξ_n, we have the following estimates,

$$\|\Psi_n\xi_n\|_{L^2} \lesssim \left(\int_{|x|<2\sigma t} \langle x\rangle^{-4}\mathrm{d}x\right)^{\frac{1}{2}} \lesssim \left(\int_0^{2\sigma t} \frac{r^3}{1+r^4}\mathrm{d}r\right)^{\frac{1}{2}} \lesssim \log^{\frac{1}{2}} t, \tag{5.65}$$

$$\|\Psi_n\xi_n\|_{L^{\frac{4}{3}}} \lesssim \left(\int_{|x|<2\sigma t} \langle x\rangle^{-\frac{8}{3}}\mathrm{d}x\right)^{\frac{3}{4}} \lesssim \left(\int_0^{2\sigma t} \frac{r^3}{1+r^{\frac{8}{3}}}\mathrm{d}r\right)^{\frac{3}{4}} \lesssim t, \tag{5.66}$$

$$\|\Psi_n\mathbf{1}_{\Omega_n}\|_{L^{\frac{4}{3}}} \lesssim \left(\int_{\sigma t<|x|<2\sigma t} \langle x\rangle^{-\frac{8}{3}}\mathrm{d}x\right)^{\frac{3}{4}} \lesssim \left(\int_{\sigma t}^{2\sigma t} \frac{r^3}{1+r^{\frac{8}{3}}}\mathrm{d}r\right)^{\frac{3}{4}} \lesssim t. \tag{5.67}$$

We introduce the following technical lemma.

Lemma 5.20. *For $n = 1, \dots, N$ and $n' \neq n$, we have*

$$(\Psi_{n'}, \Psi_n\xi_n)_{L^2} = O(1), \tag{5.68}$$

$$(\Psi_n, \Psi_n\xi_n)_{L^2} = \sigma_n \log t + O(1). \tag{5.69}$$

Proof. Proof of (5.68). For any $n' \neq n$, we consider the change of variable,

$$\left(\frac{x_1 - \ell_n t}{\sqrt{1-\ell_n^2}}, \bar{x}\right) \mapsto (y_1, \bar{y}).$$

It follows that

$$(\Psi_{n'}, \Psi_n\xi_n)_{L^2} = (1-\ell_n^2)^{\frac{1}{2}} \int_{\mathbb{R}^4} \psi(x)\xi\left(\frac{|x|}{\sigma t}\right)\psi\left(\frac{(1-\ell_n^2)^{\frac{1}{2}}x_1 - (\ell_{n'} - \ell_n)t}{\sqrt{1-\ell_{n'}^2}}, \bar{x}\right)\mathrm{d}x.$$

From the definitions of ξ, σ and ξ_n respectively in (5.4), (5.61) and (5.63), for any $x = (x_1, \bar{x}) \in \mathrm{Supp}\xi((\sigma t)^{-1}|\cdot|)$, we have

$$\left|(1-\ell_n^2)^{\frac{1}{2}}x_1 - (\ell_{n'} - \ell_n)t\right| \geqslant \left(|\ell_{n'} - \ell_n| - 2\sigma(1-\ell_n^2)^{\frac{1}{2}}\right)t \geqslant \frac{1}{2}|\ell_{n'} - \ell_n|t.$$

Based on the above inequality and the decay property of ψ in (5.19), for any $x = (x_1, \bar{x}) \in \mathrm{Supp}\xi((\sigma t)^{-1}|\cdot|)$,

$$\left|\psi\left(\frac{(1-\ell_n^2)^{\frac{1}{2}}x_1 - (\ell_{n'} - \ell_n)t}{\sqrt{1-\ell_{n'}^2}}, \bar{x}\right)\right| \lesssim |(1-\ell_n^2)^{\frac{1}{2}}x_1 - (\ell_{n'} - \ell_n)t|^{-2} \lesssim t^{-2}.$$

Therefore, using the decay property of ψ again, we have

$$|(\Psi_{n'}, \Psi_n\xi_n)_{L^2}| \lesssim t^{-2}\int_{|x|\leqslant 2\sigma t} |\psi(x)|\mathrm{d}x \lesssim t^{-2}\int_{|x|\leqslant 2\sigma t} \langle x\rangle^{-2}\mathrm{d}x \lesssim 1,$$

which means (5.68).

Proof of (5.69). For $n = 1, \dots, N$, we consider the change of variable again,

$$\left(\frac{x_1 - \ell_n t}{\sqrt{1-\ell_n^2}}, \bar{x}\right) \mapsto (y_1, \bar{y}).$$

It follows that

$$(\Psi_n, \Psi_n\xi_n)_{L^2} = (1-\ell_n^2)^{\frac{1}{2}} \int_{\mathbb{R}^4} \psi^2(x)\xi\left(\frac{|x|}{\sigma t}\right)\mathrm{d}x.$$

Using the definition of ξ in (5.4) again, we decompose

$$\int_{\mathbb{R}^4} \psi^2(x)\xi\left(\frac{|x|}{\sigma t}\right)\mathrm{d}x = H_3 + H_4 + H_5 + H_6,$$

where

$$H_3 = \int_{|x|<1} \psi^2(x)\mathrm{d}x, \qquad H_4 = \int_{\sigma t<|x|<2\sigma t} \psi^2(x)\xi\left(\frac{|x|}{\sigma t}\right)\mathrm{d}x,$$
$$H_5 = \int_{1<|x|<\sigma t} |x|^{-4}\mathrm{d}x, \quad H_6 = \int_{1<|x|<\sigma t} \left(\psi^2(x) - |x|^{-4}\right)\mathrm{d}x.$$

By the asymptotic and decay properties of ψ in (5.18) and (5.19), we have

$$\begin{aligned}|H_3| + |H_4| + |H_6| &\lesssim \int_{|x|<1} \langle x\rangle^{-4}\mathrm{d}x + \int_{\sigma t<|x|<2\sigma t} \langle x\rangle^{-4}\mathrm{d}x + \int_{1<|x|<\sigma t} \langle x\rangle^{-5}\mathrm{d}x \\ &\lesssim \int_0^1 \frac{r^3}{1+r^4}\mathrm{d}r + \int_{\sigma t}^{2\sigma t} \frac{r^3}{1+r^4}\mathrm{d}r + \int_1^{\sigma t} \frac{r^3}{1+r^5}\mathrm{d}r \lesssim 1.\end{aligned}$$

Next, using the spherical coordinate transformation in dimension 4,

$$H_5 = 2\pi^2 \int_1^{\sigma t} \frac{1}{r}\mathrm{d}r = 2\pi^2 \log(\sigma t) = 2\pi^2 \log t + O(1).$$

Combining above estimates and using the definition of σ_n in (5.62), we obtain (5.69). □

Now, we derive a refined equation of $\dot{\boldsymbol{a}}$ from Lemma 5.19 and Lemma 5.20.

Lemma 5.21 (Refined equation of $\dot{\boldsymbol{a}}$). *For $n = 1, \dots, N$, we have*

$$|\dot{a}_n + \dot{c}_n| \lesssim \|\vec{\varphi}\|_{\mathcal{H}} \log^{-\frac{1}{2}} t + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}. \tag{5.70}$$

Proof. First, by (5.58) and direct computation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_1, \Psi_n\xi_n)_{L^2} &= (\partial_t\varphi_1, \Psi_n\xi_n)_{L^2} + (\varphi_1, \Psi_n\partial_t\xi_n)_{L^2} - \ell_n(\varphi_1, \xi_n\partial_{x_1}\Psi_n)_{L^2} \\ &= (\varphi_2, \Psi_n\xi_n)_{L^2} - (\mathrm{Mod}_{1,1}, \Psi_n\xi_n)_{L^2} - (\mathrm{Mod}_{1,2}, \Psi_n\xi_n)_{L^2} \\ &\quad + (\varphi_1, \Psi_n\partial_t\xi_n)_{L^2} - \ell_n(\varphi_1, \xi_n\partial_{x_1}\Psi_n)_{L^2}.\end{aligned}$$

From the Cauchy-Schwarz inequality and (5.65),

$$|(\varphi_2, \Psi_n\xi_n)_{L^2}| \lesssim \|\varphi_2\|_{L^2}\|\Psi_n\xi_n\|_{L^2} \lesssim \|\vec{\varphi}\|_{\mathcal{H}} \log^{\frac{1}{2}} t.$$

Next, by (5.60), (5.68) and (5.69),

$$\begin{aligned}(\mathrm{Mod}_{1,1}, \Psi_n\xi_n)_{L^2} &= \dot{a}_n(\Psi_n, \Psi_n\xi_n)_{L^2} + \sum_{n'\neq n} \dot{a}_{n'}(\Psi_{n'}, \Psi_n\xi_n)_{L^2} \\ &= (\sigma_n \log t)\,\dot{a}_n + O\left(\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right).\end{aligned}$$

Using (5.60), the Young's inequality and the decay properties of ψ and ϕ_k in (5.19),

$$\begin{aligned}\left|(\mathrm{Mod}_{1,2}, \Psi_n\xi_n)_{L^2}\right| &\lesssim |\dot{\boldsymbol{b}}| \sum_{(n',k)\in I^0} \int_{\mathbb{R}^4} \left(|\Phi_{n',k}|^{\frac{5}{3}} + |\Psi_n|^{\frac{5}{2}}\right)\mathrm{d}x \\ &\lesssim \left(\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right) \int_{\mathbb{R}^4} \langle x\rangle^{-5}\mathrm{d}x \\ &\lesssim \left(\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right).\end{aligned}$$

Note that, from (5.64), (5.67) and the Sobolev inequality (5.8),

$$|(\varphi_1, \Psi_n \partial_t \xi_n)_{L^2}| \lesssim t^{-1} \|\varphi_1\|_{L^4} \|\Psi_n \mathbf{1}_{\Omega_n}\|_{L^{\frac{4}{3}}} \lesssim \|\vec{\varphi}\|_{\mathcal{H}}.$$

Then, by integration by parts,

$$(\varphi_1, \xi_n \partial_{x_1} \Psi_n)_{L^2} = -(\partial_{x_1} \varphi_1, \Psi_n \xi_n)_{L^2} - (\varphi_1, \Psi_n \partial_{x_1} \xi_n)_{L^2}.$$

Thus, using (5.64), (5.65), (5.67), the Sobolev inequality (5.8) and the Cauchy-Schwarz inequality again,

$$\begin{aligned} |(\varphi_1, \xi_n \partial_{x_1} \Psi_n)_{L^2}| &\lesssim |(\partial_{x_1} \varphi_1, \Psi_n \xi_n)_{L^2}| + |(\varphi_1, \Psi_n \partial_{x_1} \xi_n)_{L^2}| \\ &\lesssim \|\nabla \varphi_1\|_{L^2} \|\Psi_n \xi_n\|_{L^2} + t^{-1} \|\varphi_1\|_{L^4} \|\Psi_n \mathbf{1}_{\Omega_n}\|_{L^{\frac{4}{3}}} \lesssim \|\vec{\varphi}\|_{\mathcal{H}} \log^{\frac{1}{2}} t. \end{aligned}$$

Gathering above estimates, we have

$$\left| (\sigma_n \log t) \dot{a}_n + \frac{\mathrm{d}}{\mathrm{d}t} (\varphi_1, \Psi_n \xi_n)_{L^2} \right| \lesssim \|\vec{\varphi}\|_{\mathcal{H}} \log^{\frac{1}{2}} t + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}. \tag{5.71}$$

Last, using (5.66),

$$\begin{aligned} \dot{c}_n &= (\sigma_n \log t)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} (\varphi_1, \Psi_n \xi_n)_{L^2} - \left(\sigma_n t \log^2 t\right)^{-1} (\varphi_1, \Psi_n \xi_n)_{L^2} \\ &= (\sigma_n \log t)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} (\varphi_1, \Psi_n \xi_n)_{L^2} + O\left(\|\vec{\varphi}\|_{\mathcal{H}} \log^{-2} t\right). \end{aligned} \tag{5.72}$$

Dividing both sides of the inequality (5.71) by $\sigma_n \log t$ and then using (5.72), we obtain (5.70). □

Last, we consider the equation of the exponential directions.

Lemma 5.22 (Exponential directions). *In the context of Proposition 5.17, for $(n, j) \in I$, let $z^{\pm}_{n,j} = \left(\vec{\varphi}, \vec{Z}^{\pm}_{n,j}\right)_{L^2}$ and $\alpha_{n,j} = \lambda_j (1 - \ell_n^2)^{\frac{1}{2}} > 0$. Then, we have*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} z^{\pm}_{n,j} \pm \alpha_{n,j} z^{\pm}_{n,j} \right| \lesssim \|\vec{\varphi}\|^2_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}. \tag{5.73}$$

Proof. First, by (5.59), we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} z^{\pm}_{n,j} &= \left(\vec{\mathcal{L}} \vec{\varphi}, \vec{Z}^{\pm}_{n,j}\right)_{L^2} - \ell_n \left(\vec{\varphi}, \partial_{x_1} \vec{Z}^{\pm}_{n,j}\right)_{L^2} \\ &\quad + \left(\vec{G}, \vec{Z}^{\pm}_{n,j}\right)_{L^2} + \left(\vec{R}_1, \vec{Z}^{\pm}_{n,j}\right)_{L^2} + \left(\vec{R}_2, \vec{Z}^{\pm}_{n,j}\right)_{L^2} - \left(\vec{\mathrm{Mod}}, \vec{Z}^{\pm}_{n,j}\right)_{L^2}. \end{aligned}$$

From (5.29) and integration by parts,

$$\begin{aligned} &\left(\vec{\mathcal{L}} \vec{\varphi}, \vec{Z}^{\pm}_{n,j}\right)_{L^2} - \ell_n \left(\vec{\varphi}, \partial_{x_1} \vec{Z}^{\pm}_{n,j}\right)_{L^2} \\ &= -\left(\vec{\varphi}, \left(\mathcal{H}_{\ell_n} \mathrm{J} \vec{Z}^{\pm}_{j,\ell_n}\right)(\cdot - \boldsymbol{\ell}_n t)\right)_{L^2} + 3 \sum_{n' \neq n} \left(\varphi_1, Q^2_{n'} \Upsilon^{\pm}_{n,j}\right)_{L^2} \\ &= \mp \alpha_{n,j} z^{\pm}_{n,j} + 3 \sum_{n' \neq n} \left(\varphi_1, Q^2_{n'} \Upsilon^{\pm}_{n,j}\right)_{L^2}, \end{aligned}$$

where

$$\Upsilon^{\pm}_{n,j} = \pm \lambda_j (1 - \ell_n^2)^{\frac{1}{2}} \Upsilon^{\pm,1}_{j,\ell_n} (\cdot - \boldsymbol{\ell}_n t).$$

Note that, by Lemma 5.14 (i) and the decay property of Q and $\Upsilon^{\pm,1}_{j,\ell}$ in (5.11) and (5.27),

$$\sum_{n'\neq n}\left|\left(\varphi_1, Q^2_{n'}\Upsilon^{\pm}_{n,j}\right)_{L^2}\right| \lesssim \sum_{n'\neq n}\|\varphi_1\|_{L^4}\|Q^2_{n'}\Upsilon^{\pm}_{n,j}\|_{L^{\frac{4}{3}}} \lesssim \|\vec{\varphi}\|^2_{\mathcal{H}} + t^{-12}.$$

Next, from (5.27), (5.49) and the expression of R_1,

$$\begin{aligned}
\left|\left(\vec{G}, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| &\lesssim \|G\|_{L^2}\|\Upsilon^{\pm}_{n,j}\|_{L^2} \lesssim |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4},\\
\left|\left(\vec{R}_1, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| &\lesssim \|\varphi_1\|^2_{L^4}\left(\|\varphi_1\|_{L^4} + \|R+U+V\|_{L^4}\right)\|\Upsilon^{\pm}_{n,j}\|_{L^4} \lesssim \|\vec{\varphi}\|^2_{\mathcal{H}}.
\end{aligned}$$

Using the decay property of Q in (5.11) and Lemma 5.14 again,

$$\begin{aligned}
\left|\left(\vec{R}_2, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\right| &\lesssim \|\varphi_1\|_{L^4}\|\Upsilon^{\pm}_{n,j}\|_{L^2}\left(\|U+V\|^2_{L^8} + \|R(U+V)\|_{L^4}\right)\\
&\quad + \sum_{n'\neq n''}\|\varphi_1\|_{L^4}\|\Upsilon^{\pm}_{n,j}\|_{L^2}\|Q_{n'}Q_{n''}\|_{L^4}\\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}\left(|\boldsymbol{a}| + |\boldsymbol{b}| + t^{-3}\right) \lesssim \|\vec{\varphi}\|^2_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-6}.
\end{aligned}$$

Last, using (5.28), (5.29), change of variable and integration by parts,

$$\begin{aligned}
\left(\vec{\Psi}_n, \vec{Z}^{\pm}_{n,j}\right)_{L^2} &= \left(\vec{\psi}_{\ell_n}, \mathcal{H}_{\ell_n}\vec{\Upsilon}^{\pm}_{j,\ell_n}\right)_{L^2} = \left(\mathcal{H}_{\ell_n}\vec{\psi}_{\ell_n}, \vec{\Upsilon}^{\pm}_{j,\ell_n}\right)_{L^2} = 0,\\
\left(\vec{\Phi}_{n,k}, \vec{Z}^{\pm}_{n,j}\right)_{L^2} &= \left(\vec{\phi}_{k,\ell_n}, \mathcal{H}_{\ell_n}\vec{\Upsilon}^{\pm}_{j,\ell_n}\right)_{L^2} = \left(\mathcal{H}_{\ell_n}\vec{\phi}_{k,\ell_n}, \vec{\Upsilon}^{\pm}_{j,\ell_n}\right)_{L^2} = 0.
\end{aligned}$$

Therefore, by (5.19), (5.27), (5.60) and the Lemma 5.14 (i),

$$\begin{aligned}
\left(\vec{\mathrm{Mod}}, \vec{Z}^{\pm}_{n,j}\right)_{L^2} &= \sum_{n'\neq n}\dot{a}_{n'}\left(\vec{\Psi}_{n'}, \vec{Z}^{\pm}_{n,j}\right)_{L^2} + \sum_{n'\neq n}\sum_{k=1}^{K}\dot{b}_{n',k}\left(\vec{\Phi}_{n',k}, \vec{Z}^{\pm}_{n,j}\right)_{L^2}\\
&= O\left((\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4})t^{-2}\right)\\
&= O\left(\|\vec{\varphi}\|^2_{\mathcal{H}} + |\boldsymbol{a}|^4 + |\boldsymbol{b}|^4 + t^{-4}\right).
\end{aligned}$$

Gathering above estimates, we obtain (5.73). □

5.4 Proof of Theorem 5.1

In this section, we prove the existence of a solution $\vec{u} = (u, \partial_t u)$ of (5.1) satisfying (5.3) in Theorem 5.1. We argue by compactness and obtain the solution $\vec{u}$ as the weak limit of a sequence of approximate excited multi-solitons $\vec{u}_m$.

First, we recall a technical lemma which constructs well-prepared initial data at $t = T \gg 1$ with sufficient freedom related to unstable directions.

Lemma 5.23. *There exists $T_0 \gg 1$ such that for any $T \geqslant T_0$ the following is true. For any $\boldsymbol{z} = (z_{n,j})_{(n,j)\in I} \in \mathcal{B}_{\mathbb{R}^{|I|}}(T^{-\frac{7}{2}})$, there exist*

$$Z = (\tilde{z}_{n,j})_{(n,j)\in I}, \quad A = (\tilde{a}_n)_{n=1,\dots,N} \quad \text{and} \quad B = (\tilde{b}_{n,k})_{(n,k)\in I^0},$$

satisfying

$$\sum_{(n,j)\in I}|\tilde{z}_{n,j}| + \sum_{n=1}^{N}|\tilde{a}_n| + \sum_{(n,k)\in I^0}\left|\tilde{b}_{n,k}\right| \lesssim T^{-\frac{7}{2}}, \tag{5.74}$$

such that the function $\vec{\varphi}(T) = (\varphi_1(T), \varphi_2(T)) \in \dot{H}^1 \times L^2$ *defined by*

$$\vec{\varphi}(T) = \sum_{(n,j)\in I} \tilde{z}_{n,j} \vec{Z}^+_{n,j}(T) + \sum_{n=1}^{N} \tilde{a}_n \vec{\Psi}_n(T) + \sum_{(n,k)\in I^0} \tilde{b}_{n,k} \vec{\Phi}_{n,k}(T) \tag{5.75}$$

satisfies for all $m = 1, \dots, K$, $(n,k) \in I^0$ *and* $(n,j) \in I$,

$$(\vec{\varphi}(T), \vec{\Psi}_n(T))_{\mathcal{H}} = (\vec{\varphi}(T), \vec{\Phi}_{n,k}(T))_{\mathcal{H}} = 0 \quad \text{and} \quad \left(\vec{\varphi}(T), \vec{Z}^+_{n,j}(T)\right)_{L^2} = z_{n,j}.$$

Moreover, the initial data defined by $\vec{u}(T) = \sum_{n=1}^{N} \vec{Q}_n(T) + \vec{\varphi}(T)$ *is modulated in the sense of Proposition 5.17 with* $a_n(T) = b_{n,k}(T) = 0$ *and* $z^+_{n,j}(T) = z_{n,j}$ *for any* $n = 1, \dots, N$, $(n,k) \in I^0$ *and* $(n,j) \in I$.

Proof. The proof of the technical lemma relies on a standard argument based on the Lemma 5.23 and the invertibility of Gram matrix (see *e.g.* [112, Lemma 4.1]), and we omit it. □

Let $S_m = m^2$ for any $m \in \mathbb{N}$. For $\boldsymbol{z}_m = (z^m_{n,j})_{(n,j)} \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$, we consider the backward solution $\vec{u}_m \in C\left([T_{\min}(m), S_m]; \dot{H}^1 \times L^2\right)$ of (5.1) with the initial data $\vec{u}_m(S_m)$ given by the Lemma 5.23. Since $\vec{u}_m(S_m) \in \dot{H}^2 \times \dot{H}^1$, by persistence of regularity (see for example [71, Remark 2.9]), we also have $\vec{u}_m \in C\left([T_{\min}(m), S_m]; \dot{H}^2 \times \dot{H}^1\right)$. Such regularity will allow energy computations in §5.4.2 without a density argument.

The following Proposition is the main part of the proof of Theorem 5.1.

Proposition 5.24 (Uniform estimates)**.** *Under the assumptions of Theorem 5.1, there exist* $m_0 \in \mathbb{N}$, $T_0 > 0$ *and* $C > 0$ *such that the following is true. For any* $m \geqslant m_0$, *there exists* $\boldsymbol{z}_m \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$ *such that the solution* $\vec{u}_m$ *of* (5.1) *with the initial data* $\vec{u}_m(S_m)$ *given by the Lemma 5.23 is well-defined* $\dot{H}^1 \times L^2$ *on the time interval* $[T_0, S_m]$. *Moreover, the solution* $\vec{u}_m$ *satisfies*

$$\|\vec{u}_m(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\|_{\mathcal{H}} \leqslant C t^{-2}, \quad \text{for all } t \in [T_0, S_m]. \tag{5.76}$$

The rest of the section is organized as follows. First, in §5.4.1-§5.4.3, we devote to the proof of Proposition 5.24 based on the energy method. Last, in §5.4.4, we prove the Theorem 5.1 from Proposition 5.24 by compactness argument.

5.4.1 Bootstrap setting

For $0 < t \leqslant S_m$, as long as $\vec{u}_m(t)$ is well-defined in $\dot{H}^1 \times L^2$ and satisfies (5.55), we decompose $\vec{u}_m(t)$ as in Proposition 5.17. In particular, we denote by $\vec{\varphi}(t)$ and $(a_n(t), b_{n,k}(t))_{(n,k)\in I^0}$ the remainder term and the parameters of the decomposition of $\vec{u}_m(t)$.

The proof of Proposition 5.24 is based on the following bootstrap estimates: for $C_0 \gg 1$ to be chosen,

$$\begin{cases} |\boldsymbol{a}| \leqslant C_0^2 t^{-2} \log^{-\frac{1}{2}} t, \\ |\boldsymbol{b}| \leqslant C_0^2 t^{-2}, \quad \|\vec{\varphi}\|_{\mathcal{H}} \leqslant C_0 t^{-3}, \\ \displaystyle\sum_{(n,j)\in I} |z^-_{n,j}|^2 \leqslant t^{-6}, \quad \sum_{(n,j)\in I} |z^+_{n,j}|^2 \leqslant t^{-7}. \end{cases} \tag{5.77}$$

For any $\boldsymbol{z}_m \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$, set

$$T^* = T^*_m(\boldsymbol{z}_m) = \inf\left\{t \in [T_0, S_m]; \vec{u}_m \text{ satisfies (5.55) and (5.77) on } [t, S_m]\right\}.$$

Note that, for the proof of Proposition 5.24, we just need to prove that there exist $T_0 \gg 1$ (independent with m) large enough and at least one choice of $\boldsymbol{z}_m = (z^m_{n,j})_{(n,j)\in I} \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$ such that $T^* = T_0$.

In the rest of the section, the implied constants in $\lesssim$ and O do not depend on the constant C_0 appearing in the bootstrap assumption (5.77).

5.4.2 Energy functional

Recall that,

$$-\bar{\ell} < \ell_1 < \ell_2 < \cdots < \ell_N < \bar{\ell} \quad \text{where} \quad \bar{\ell} = \max\left(|\ell_1|, |\ell_N|\right) < 1.$$

Fix

$$0 < \delta = \frac{1}{40}(1-\bar{\ell}) \times \min(\ell_2 - \ell_1, \ldots, \ell_N - \ell_{N-1}) < \frac{1}{20}. \tag{5.78}$$

We denote

$$\begin{aligned} \bar{\ell}_n &= \ell_n + \delta(\ell_{n+1} - \ell_n), \quad \text{for } n = 1, \ldots, N-1, \\ \underline{\ell}_n &= \ell_n - \delta(\ell_{n+1} - \ell_n), \quad \text{for } n = 2, \ldots, N. \end{aligned}$$

For $t > 0$, denote

$$\Omega(t) = \left((\bar{\ell}_1 t, \underline{\ell}_2 t) \cup \ldots \cup (\bar{\ell}_{N-1} t, \underline{\ell}_N t)\right) \times \mathbb{R}^3, \quad \Omega^C(t) = \mathbb{R}^4 \setminus \Omega(t). \tag{5.79}$$

We define the continuous function $\chi_N(t,x) = \chi_N(t,x_1)$ as follow (see [85, section 4.3] for a similar choice of cut-off function), for $t > 0$,

$$\begin{cases} \chi_N(t,x) = \ell_1 \text{ for } x_1 \in (-\infty, \bar{\ell}_1 t], \\ \chi_N(t,x) = \ell_n \text{ for } x_1 \in [\underline{\ell}_n t, \bar{\ell}_n t], \text{ for } n \in \{2, \ldots, N-1\}, \\ \chi_N(t,x) = \ell_N \text{ for } x_1 \in [\underline{\ell}_N t, +\infty), \\ \chi_N(t,x) = \dfrac{x_1}{(1-2\delta)t} - \dfrac{\delta}{1-2\delta}(\ell_{n+1} + \ell_n) \text{ for } x_1 \in [\bar{\ell}_n t, \underline{\ell}_{n+1} t],\ n \in \{1, \ldots, N-1\}. \end{cases} \tag{5.80}$$

Note that

$$\begin{cases} \overline{\nabla}\chi_N(t,x) = 0, \quad \text{for } x \in \Omega(t), \\ \partial_{x_1}\chi_N(t,x) = \dfrac{1}{(1-2\delta)t}, \quad \text{for } x \in \Omega(t), \\ \partial_t \chi_N(t,x) = -\dfrac{1}{t}\dfrac{x_1}{(1-2\delta)t}, \quad \text{for } x \in \Omega(t), \\ \partial_t \chi_N(t,x) = 0, \quad \nabla\chi_N(t,x) = 0, \quad \text{for } x \in \Omega^C(t). \end{cases} \tag{5.81}$$

We define

$$\mathcal{K}(t) = \mathcal{E}(t) + \mathcal{P}(t) + \mathcal{G}(t) + \mathcal{J}(t),$$

where

$$\begin{aligned} \mathcal{G}(t) &= -2\int_{\mathbb{R}^4} \varphi_1(t,x) G_2(t,x) \mathrm{d}x, \\ \mathcal{P}(t) &= 2\int_{\mathbb{R}^4} \left(\chi_N(t,x)\partial_{x_1}\varphi_1(t,x)\right)\varphi_2(t,x)\mathrm{d}x, \\ \mathcal{E}(t) &= \int_{\mathbb{R}^4} \Big\{ |\nabla\varphi_1|^2 + \varphi_2^2 - \frac{1}{2}\left(R+U+V+\varphi_1\right)^4 \\ &\qquad + \frac{1}{2}\left(R+U+V\right)^4 + 2(R+U+V)^3\varphi_1 \Big\} \mathrm{d}x, \end{aligned}$$

and

$$\mathcal{J} = \sum_{n=1}^{N} \mathcal{J}_n, \quad \text{with } \mathcal{J}_n = 2a_n \int_{\mathbb{R}^4} \left(\ell_n \partial_{x_1}\varphi_1 - \varphi_2\right)(\ell_n - \chi_N)\partial_{x_1}\Psi_n \mathrm{d}x.$$

Remark 5.25. The quantity $\mathcal{E}$ and $\mathcal{P}$ are standard energy and momentum functionals for the remainder term $\vec{\varphi}$. We introduce $\mathcal{G}$ here to remove the effect of G_2 on the time variation of the quantity $\mathcal{E}+\mathcal{P}$ (see Lemma 5.28 (iii)). Moreover, we introduce $\mathcal{J}$ to remove the effect of $(\Psi_n)_{n=1,\dots,N}$ which is the main difficulty in the energy estimate for the 4D case (see Lemma 5.28 (iv)).

Denote

$$\mathcal{N}_\Omega = \int_\Omega \left(|\nabla\varphi_1|^2 + \varphi_2^2 + 2\chi_N(\partial_{x_1}\varphi_1)\varphi_2\right)\mathrm{d}x, \quad \mathcal{N}_{\Omega^C} = \int_{\Omega^C}\left(|\nabla\varphi_1|^2+\varphi_2^2\right)\mathrm{d}x.$$

For $n=1,\dots,N$, we set

$$\zeta_n(t,x) = \zeta(x_1-\ell_n t,\bar{x}) = \left(1+\left((x_1-\ell_n t)^2+|\bar{x}|^2\right)^{\frac{1}{2}}\right)^{-\gamma},$$

where γ is introduced in Lemma 5.11.

Note that, from $|\chi_N| \leqslant \bar{\ell} < 1$,

$$\mathcal{N}_\Omega \geqslant (1-\bar{\ell})\int_\Omega \left(|\nabla\varphi_1|^2+\varphi_2^2\right)\mathrm{d}x. \tag{5.82}$$

Note also that, from the definition of χ_N and ζ_n,

$$\sum_{n=1}^N \|(\chi_N-\ell_n)\zeta_n^2\|_{L^\infty} + \sum_{n=1}^N \|\zeta_n^2\|_{L^\infty(\Omega)} + \left(\sum_{n=1}^N \zeta_n^2 - 1\right) \lesssim t^{-2\gamma}. \tag{5.83}$$

First, we introduce a technical lemma for future reference.

Lemma 5.26. *Let f be a continuous function such that*

$$|f(x)| \lesssim \langle x\rangle^{-(3+\alpha)} \quad \text{on } \mathbb{R}^4, \tag{5.84}$$

where $\alpha \geqslant 0$. For all $n=1,\dots,N$, we have

$$\|f_n\|_{L^2(\Omega)}^2 + \|(\ell_n-\chi_N)f_n\|_{L^2}^2 + \int_{\mathbb{R}^4}|\ell_n-\chi_N|f_n^2\mathrm{d}x \lesssim t^{-(2+2\alpha)}, \tag{5.85}$$

where $f_n(t,x) = f(x_1-\ell_n t,\bar{x})$.

Proof. We prove the estimate (5.85) for $\|f_n\|_{L^2(\Omega)}^2$; other estimates are proved same. First, from the definition of δ and Ω respectively in (5.78) and (5.79), we have

$$|x_1-\ell_n t| \geqslant \delta^2 t, \quad \text{for any } x=(x_1,\bar{x})\in\Omega.$$

Therefore, from the decay assumption of f in (5.84) and change of variables,

$$\begin{aligned}\|f_n\|_{L^2(\Omega)}^2 &\lesssim \int_{\mathbb{R}^3}\int_{|x_1-\ell_n t|\geqslant\delta^2 t} f^2(x_1-\ell_n t,\bar{x})\mathrm{d}x_1\mathrm{d}\bar{x}\\ &\lesssim \int_{|x|\geqslant\delta^2 t}\langle x\rangle^{-(6+2\alpha)}\mathrm{d}x \lesssim \int_{\delta^2 t}^\infty r^{-(3+2\alpha)}\mathrm{d}r \lesssim t^{-(2+2\alpha)}.\end{aligned}$$

The proof of (5.85) is complete. □

Second, we introduce the coercivity property of $\mathcal{K}$.

Lemma 5.27. *There exists $\nu>0$ such that the following estimates hold.*

(i) Bound on $\mathcal{G}$ and $\mathcal{J}$. *We have*

$$|\mathcal{G}| + |\mathcal{J}| \lesssim C_0^5 t^{-7} + C_0^3 t^{-6} \log^{-\frac{1}{2}} t. \tag{5.86}$$

(ii) Coercivity of $\mathcal{E} + \mathcal{P}$. *We have*

$$\mathcal{E} + \mathcal{P} \geqslant \mathcal{N}_\Omega + \nu \mathcal{N}_{\Omega^C} - \nu^{-1} \left(t^{-6} + C_0^4 t^{-(6+2\gamma)} \right). \tag{5.87}$$

(iii) Coercivity of $\mathcal{K}$. *We have*

$$\mathcal{K} \geqslant \mathcal{N}_\Omega + \nu \mathcal{N}_{\Omega^C} - \nu^{-1} \left(t^{-6} + C_0^5 t^{-6} \log^{-\frac{1}{2}} t \right). \tag{5.88}$$

Proof. Proof of (i). Note that, from (5.77), (5.85), the definition of G_2 in Lemma 5.12 and the Sobolev inequality (5.8), we have

$$|\mathcal{G}| \lesssim \|\varphi_1\|_{L^4} \|G_2\|_{L^{\frac{4}{3}}} \lesssim \left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 \right) \|\vec{\varphi}\|_{\mathcal{H}} \lesssim C_0^5 t^{-7},$$

$$|\mathcal{J}| \lesssim \sum_{n=1}^{N} |\boldsymbol{a}| \left(\|\nabla \varphi_1\|_{L^2} + \|\varphi_2\|_{L^2} \right) \|(\ell_n - \chi_N) \partial_{x_1} \Psi_n\|_{L^2} \lesssim C_0^3 t^{-6} \log^{-\frac{1}{2}} t,$$

which implies (5.86).

Proof of (ii). We decompose

$$\mathcal{E} + \mathcal{P} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3,$$

where

$$\mathcal{F}_1 = -\frac{1}{2} \int_{\mathbb{R}^4} \varphi_1^3 \left(\varphi_1 + 4(R + U + V) \right) \mathrm{d}x,$$

$$\mathcal{F}_2 = -3 \int_{\mathbb{R}^4} \varphi_1^2 \Big(\sum_{n \neq n'} Q_n Q_{n'} + 2R(U+V) + (U+V)^2 \Big) \mathrm{d}x,$$

$$\mathcal{F}_3 = \int_{\mathbb{R}^4} \left(|\nabla \varphi_1|^2 - 3 \sum_{n=1}^{N} Q_n^2 \varphi_1^2 + \varphi_2^2 + 2\chi_N (\partial_{x_1} \varphi_1) \varphi_2 \right) \mathrm{d}x.$$

Estimate on $\mathcal{F}_1$. From (5.77) and the Sobolev inequality (5.8),

$$|\mathcal{F}_1| \lesssim \|\varphi_1\|_{L^4}^3 \left(\|\varphi_1\|_{L^4} + \|R + U + V\|_{L^4} \right) \lesssim C_0^3 t^{-9}.$$

Estimate on $\mathcal{F}_2$. Note that, using (5.77), the Sobolev inequality (5.8), the Lemma 5.14 and the decay property of Q in (5.11) again,

$$\begin{aligned} |\mathcal{F}_2| &\lesssim \|\varphi_1\|_{L^4}^2 \Big(\sum_{n \neq n'} \|Q_n Q_{n'}\|_{L^2} + \|R\|_{L^4} \|U+V\|_{L^4} + \|U+V\|_{L^4}^2 \Big) \\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 \left(t^{-3} + |\boldsymbol{a}| + |\boldsymbol{b}| \right) \lesssim C_0^2 t^{-6} \left(t^{-3} + C_0^2 t^{-2} \log^{-\frac{1}{2}} t + C_0^2 t^{-2} \right) \lesssim C_0^4 t^{-8}. \end{aligned}$$

Estimate on $\mathcal{F}_3$. We decompose

$$\mathcal{F}_3 = \mathcal{N}_\Omega + \mathcal{F}_{3,1} + \mathcal{F}_{3,2} + \mathcal{F}_{3,3} + \mathcal{F}_{3,4},$$

where

$$\mathcal{F}_{3,1} = 2 \sum_{n=1}^{N} \int_{\mathbb{R}^4} (\chi_N - \ell_n)(\partial_{x_1} \varphi_1) \varphi_2 \zeta_n^2 \mathrm{d}x,$$

$$\mathcal{F}_{3,2} = -\int_{\Omega} \left(|\nabla \varphi_1|^2 + \varphi_2^2 + 2\chi_N (\partial_{x_1} \varphi_1) \varphi_2 \right) \Big(\sum_{n=1}^{N} \zeta_n^2 \Big) \mathrm{d}x,$$

and

$$\mathcal{F}_{3,3} = \int_{\Omega^C} \left(|\nabla\varphi_1|^2 + \varphi_2^2 + 2\chi_N(\partial_{x_1}\varphi_1)\varphi_2\right)\left(1 - \sum_{n=1}^N \zeta_n^2\right)\mathrm{d}x,$$

$$\mathcal{F}_{3,4} = \sum_{n=1}^N \int_{\mathbb{R}^4} \left(|\nabla\varphi_1|^2\zeta_n^2 - 3Q_n^2\varphi_1^2 + \varphi_2^2\zeta_n^2 + 2\ell_n(\partial_{x_1}\varphi_1)\varphi_2\zeta_n^2\right)\mathrm{d}x.$$

Note that, from (5.77) and (5.83),

$$|\mathcal{F}_{3,1}| + |\mathcal{F}_{3,2}| \lesssim \sum_{n=1}^N \left(\|(\chi_N - \ell_n)\zeta_n^2\|_{L^\infty} + \|\zeta_n^2\|_{L^\infty(\Omega)}\right)\|\vec{\varphi}\|_{\mathcal{H}}^2 \lesssim C_0^2 t^{-(6+2\gamma)}.$$

Then, using (5.77) and (5.83) again,

$$\mathcal{F}_{3,3} \geqslant \int_{\Omega^C} \left(|\nabla\varphi_1|^2 + \varphi_2^2 + 2\chi_N(\partial_{x_1}\varphi_1)\varphi_2\right)\left|1 - \sum_{n=1}^N \zeta_n^2\right|\mathrm{d}x + O\left(C_0^2 t^{-(6+2\gamma)}\right).$$

Last, from (5.31), (5.57), (5.77) for $\mathcal{F}_{3,4}$ and $|\chi_N| < 1$, we have

$$\begin{aligned}\mathcal{F}_{3,4} &\geqslant \mu \int_{\mathbb{R}^4} \left(|\nabla\varphi_1|^2 + \varphi_2^2\right)\left(\sum_{n=1}^N \zeta_n^2\right)\mathrm{d}x - \mu^{-1}t^{-6} \\ &\geqslant \frac{\mu}{2}\int_{\Omega^C} \left(|\nabla\varphi_1|^2 + \varphi_2^2 + 2\chi_N(\partial_{x_1}\varphi_1)\varphi_2\right)\left(\sum_{n=1}^N \zeta_n^2\right)\mathrm{d}x - \mu^{-1}t^{-6}.\end{aligned}$$

Combining above estimates and taking $\nu > 0$ small enough, we obtain (5.87).

Proof of (iii). Estimate (5.88) is a consequence of (5.86) and (5.87). □

Third, we derive the time variation of $\mathcal{K}$.

Lemma 5.28. *There exists T_0 large enough such that the following estimates hold.*

(i) Time variation of $\mathcal{E}$. *We have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = &-2\sum_{n=1}^N \dot{a}_n \int_{\mathbb{R}^4} (\ell_n\partial_{x_1}\varphi_1 - \varphi_2)\ell_n\partial_{x_1}\Psi_n\mathrm{d}x \\ &+ 2\int_{\mathbb{R}^4} \varphi_1\left(\Delta\mathrm{Mod}_{1,2} + 3R^2\mathrm{Mod}_{1,2}\right)\mathrm{d}x - 2\int_{\mathbb{R}^4}\varphi_2\mathrm{Mod}_{2,2}\mathrm{d}x \\ &+ 2\int_{\mathbb{R}^4}\varphi_2 G_2\mathrm{d}x + 6\sum_{n=1}^N \int_{\mathbb{R}^4}\left(\ell_n Q_n\partial_{x_1}Q_n\right)\varphi_1^2\mathrm{d}x + O(C_0t^{-7}).\end{aligned} \tag{5.89}$$

(ii) Time variation of $\mathcal{P}$. *We have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} = &-\frac{1}{(1-2\delta)t}\int_\Omega \left((\partial_{x_1}\varphi_1)^2 + \varphi_2^2 + 2\frac{x_1}{t}(\partial_{x_1}\varphi_1)\varphi_2 - |\overline{\nabla}\varphi_1|^2\right)\mathrm{d}x \\ &- 6\sum_{n=1}^N \int_{\mathbb{R}^4}\ell_n(Q_n\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x + 2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)G_2\mathrm{d}x \\ &- 2\int_{\mathbb{R}^4}\chi_N\mathrm{Mod}_{2,2}\partial_{x_1}\varphi_1\mathrm{d}x - 2\int_{\mathbb{R}^4}\left(\chi_N\partial_{x_1}\mathrm{Mod}_{1,2}\right)\varphi_2\mathrm{d}x \\ &+ 2\sum_{n=1}^N \dot{a}_n\int_{\mathbb{R}^4}\left(\ell_n\partial_{x_1}\varphi_1 - \varphi_2\right)\chi_N\partial_{x_1}\Psi_n\mathrm{d}x + O(C_0t^{-7}).\end{aligned} \tag{5.90}$$

(iii) Time variation of $\mathcal{G}$. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} = -2\int_{\mathbb{R}^4} \varphi_2 G_2 \mathrm{d}x - 2\int_{\mathbb{R}^4} \chi_N(\partial_{x_1}\varphi_1) G_2 \mathrm{d}x + O(t^{-7}). \tag{5.91}$$

(iv) Time variation of $\mathcal{J}_n$. *For $n = 1, \dots, N$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_n = 2\dot{a}_n \int_{\mathbb{R}^4} \left(\ell_n \partial_{x_1}\varphi_1 - \varphi_2\right)\left(\ell_n - \chi_N\right)\partial_{x_1}\Psi_n \mathrm{d}x + O(t^{-7}). \tag{5.92}$$

Proof. Proof of (i). We decompose

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$\begin{aligned}
\mathcal{I}_1 &= 2\int_{\mathbb{R}^4} \left(\nabla\varphi_1 \cdot \nabla\partial_t\varphi_1 + \varphi_2\partial_t\varphi_2\right)\mathrm{d}x,\\
\mathcal{I}_2 &= -2\int_{\mathbb{R}^4} \partial_t\varphi_1\big(\,(R+U+V+\varphi_1)^3 - (R+U+V)^3\,\big)\mathrm{d}x,\\
\mathcal{I}_3 &= -2\int_{\mathbb{R}^4} \left(\partial_t R + \partial_t U + \partial_t V\right)\left(3(R+U+V)\varphi_1^2 + \varphi_1^3\right)\mathrm{d}x.
\end{aligned}$$

Estimate on $\mathcal{I}_1$. We claim

$$\begin{aligned}
\mathcal{I}_1 = &-2\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4}\left(\ell_n\partial_{x_1}\varphi_1 - \varphi_2\right)\ell_n\partial_{x_1}\Psi_n\mathrm{d}x + 2\int_{\mathbb{R}^4}\varphi_2 G_2\mathrm{d}x\\
&-6\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4} Q_n^2\Psi_n\varphi_1\mathrm{d}x + 2\int_{\mathbb{R}^4}\varphi_1\Delta\mathrm{Mod}_{1,2}\mathrm{d}x - 2\int_{\mathbb{R}^4}\varphi_2\mathrm{Mod}_{2,2}\mathrm{d}x\\
&+2\int_{\mathbb{R}^4}\varphi_2\left((R+U+V+\varphi_1)^3 - (R+U+V)^3\right)\mathrm{d}x + O(C_0t^{-7}).
\end{aligned} \tag{5.93}$$

Indeed, from (5.58) and integration by parts,

$$\begin{aligned}
\mathcal{I}_1 =& 2\int_{\mathbb{R}^4}\varphi_1\Delta\mathrm{Mod}_{1,1}\mathrm{d}x - 2\int_{\mathbb{R}^4}\varphi_2\mathrm{Mod}_{2,1}\mathrm{d}x\\
&+2\int_{\mathbb{R}^4}\varphi_1\Delta\mathrm{Mod}_{1,2}\mathrm{d}x - 2\int_{\mathbb{R}^4}\varphi_2\mathrm{Mod}_{2,2}\mathrm{d}x + 2\int_{\mathbb{R}^4}\varphi_2 G_2\mathrm{d}x\\
&+2\int_{\mathbb{R}^4}\varphi_2\left((R+U+V+\varphi_1)^3 - (R+U+V)^3\right)\mathrm{d}x + 2\sum_{i=1,3}\int_{\mathbb{R}^4}\varphi_2 G_i\mathrm{d}x.
\end{aligned}$$

By integration by parts, $-(1-\ell_n^2)\partial_{x_1}^2\Psi_n - \bar{\Delta}\Psi_n - 3Q_n^2\Psi_n = 0$ and the definition of $\mathrm{Mod}_{1,1}$ and $\mathrm{Mod}_{2,1}$,

$$\begin{aligned}
&2\int_{\mathbb{R}^4}\varphi_1\Delta\mathrm{Mod}_{1,1}\mathrm{d}x - 2\int_{\mathbb{R}^4}\varphi_2\mathrm{Mod}_{2,1}\mathrm{d}x\\
&= -2\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4}\left(\ell_n\partial_{x_1}\varphi_1 - \varphi_2\right)\ell_n\partial_{x_1}\Psi_n\mathrm{d}x - 6\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4}Q_n^2\Psi_n\varphi_1\mathrm{d}x.
\end{aligned}$$

Then, from (5.46), (5.48), (5.77) and the Cauchy-Schwarz inequality, for T_0 large enough (depending on C_0), we have

$$\begin{aligned}
\sum_{i=1,3}\left|\int_{\mathbb{R}^4}\varphi_2 G_i\mathrm{d}x\right| &\lesssim \|\varphi_2\|_{L^2}\left(\|G_1\|_{L^2} + \|G_3\|_{L^2}\right)\\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^3 + t^{-4}\right)\\
&\lesssim C_0t^{-3}\left(C_0^4t^{-4}\log^{-1}t + C_0^6t^{-6} + t^{-4}\right) \lesssim C_0t^{-7}.
\end{aligned}$$

We see that (5.93) follows from above identities and estimate.

Estimate on $\mathcal{I}_2$. We claim

$$\begin{aligned}\mathcal{I}_2 =& -2\int_{\mathbb{R}^4}\varphi_2\left((R+U+V+\varphi_1)^3-(R+U+V)^3\right)\mathrm{d}x\\ &+6\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4}Q_n^2\Psi_n\varphi_1\mathrm{d}x+6\int_{\mathbb{R}^4}\left(R^2\mathrm{Mod}_{1,2}\right)\varphi_1\mathrm{d}x+O(t^{-7}).\end{aligned}\tag{5.94}$$

First, using (5.58) again, we decompose,

$$\begin{aligned}\mathcal{I}_2 =& -2\int_{\mathbb{R}^4}\varphi_2\left((R+U+V+\varphi_1)^3-(R+U+V)^3\right)\mathrm{d}x\\ &+6\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4}Q_n^2\Psi_n\varphi_1\mathrm{d}x+6\int_{\mathbb{R}^4}\left(R^2\mathrm{Mod}_{1,2}\right)\varphi_1\mathrm{d}x+\mathcal{I}_{2,1}+\mathcal{I}_{2,2}+\mathcal{I}_{2,3},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{2,1} &= 6\sum_{n=1}^{N}\dot{a}_n\int_{\mathbb{R}^4}\Psi_n(R^2-Q_n^2)\varphi_1\mathrm{d}x,\\ \mathcal{I}_{2,2} &= 2\int_{\mathbb{R}^4}\mathrm{Mod}_1\varphi_1^2\left(\varphi_1+3(R+U+V)\right)\mathrm{d}x,\\ \mathcal{I}_{2,3} &= 6\int_{\mathbb{R}^4}\mathrm{Mod}_1\varphi_1\left(2(U+V)R+(U+V)^2\right)\mathrm{d}x.\end{aligned}$$

By an elementary computation,

$$\mathcal{I}_{2,1}=6\sum_{n=1}^{N}\sum_{n'\neq n}\dot{a}_n\int_{\mathbb{R}^4}\left(2\Psi_nQ_nQ_{n'}+\Psi_nQ_{n'}^2\right)\varphi_1\mathrm{d}x.$$

Therefore, from (5.60), (5.77), the Lemma 5.14 (i), the Sobolev inequality (5.8) and the decay properties of ψ and Q respectively in (5.11) and (5.19), for T_0 large enough,

$$\begin{aligned}|\mathcal{I}_{2,1}| &\lesssim \sum_{n\neq n'}|\dot{\boldsymbol{a}}|\|\varphi_1\|_{L^4}\left(\|\Psi_nQ_nQ_{n'}\|_{L^{\frac{4}{3}}}+\|\Psi_nQ_{n'}^2\|_{L^{\frac{4}{3}}}\right)\\ &\lesssim t^{-2}\|\vec{\varphi}\|_{\mathcal{H}}\left(\|\vec{\varphi}\|_{\mathcal{H}}+|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+t^{-4}\right)\\ &\lesssim C_0t^{-5}\left(C_0t^{-3}+C_0^4t^{-4}\log^{-1}t+C_0^4t^{-4}+t^{-4}\right)\lesssim t^{-7}.\end{aligned}$$

Then, using (5.60), (5.77) and the Sobolev inequality in (5.8) again,

$$\begin{aligned}|\mathcal{I}_{2,2}| &\lesssim \|\varphi_1\|_{L^4}^2\|\mathrm{Mod}_1\|_{L^4}\left(\|\varphi_1\|_{L^4}+\|R+U+V\|_{L^4}\right)\\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2\left(\|\vec{\varphi}\|_{\mathcal{H}}+|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+t^{-4}\right)\\ &\lesssim C_0^2t^{-6}\left(C_0t^{-3}+C_0^4t^{-4}\log^{-1}t+C_0^4t^{-4}+t^{-4}\right)\lesssim t^{-7},\end{aligned}$$

$$\begin{aligned}|\mathcal{I}_{2,3}| &\lesssim \|\varphi_1\|_{L^4}\|\mathrm{Mod}_1\|_{L^4}\|U+V\|_{L^4}\left(\|R\|_{L^4}+\|U+V\|_{L^4}\right)\\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}(|\boldsymbol{a}|+|\boldsymbol{b}|)\left(\|\vec{\varphi}\|_{\mathcal{H}}+|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+t^{-4}\right)\\ &\lesssim C_0^3t^{-5}\left(C_0t^{-3}+C_0^4t^{-4}\log^{-1}t+C_0^4t^{-4}+t^{-4}\right)\lesssim t^{-7}.\end{aligned}$$

Combining above estimates, we obtain (5.94).

Estimate on $\mathcal{I}_3$. We claim

$$\mathcal{I}_3=6\sum_{n=1}^{N}\int_{\mathbb{R}^4}\left(\ell_nQ_n\partial_{x_1}Q_n\right)\varphi_1^2\mathrm{d}x+O(t^{-7}).\tag{5.95}$$

From $\partial_t R = -\sum_{n=1}^N \ell_n \partial_{x_1} Q_n$ and an elementary computation, we decompose

$$\mathcal{I}_3 = 6\sum_{n=1}^N \int_{\mathbb{R}^4} (\ell_n Q_n \partial_{x_1} Q_n)\, \varphi_1^2 \mathrm{d}x + \mathcal{I}_{3,1} + \mathcal{I}_{3,2} + \mathcal{I}_{3,3},$$

where

$$\begin{aligned}
\mathcal{I}_{3,1} &= 6\sum_{n\neq n'} \int_{\mathbb{R}^4} (\ell_n Q_{n'} \partial_{x_1} Q_n)\, \varphi_1^2 \mathrm{d}x,\\
\mathcal{I}_{3,2} &= -6\int_{\mathbb{R}^4} R\varphi_1^2\, (\partial_t U + \partial_t V)\, \mathrm{d}x,\\
\mathcal{I}_{3,3} &= -2\int_{\mathbb{R}^4} \varphi_1^2\, (\partial_t R + \partial_t U + \partial_t V)\, (3(U+V) + \varphi_1)\mathrm{d}x.
\end{aligned}$$

By (5.77), the Lemma 5.14 (i) and the Sobolev inequality (5.8),

$$|\mathcal{I}_{3,1}| \lesssim \sum_{n\neq n'} \|Q_{n'}\partial_{x_1} Q_n\|_{L^2} \|\varphi_1\|_{L^4}^2 \lesssim C_0^2 t^{-9}.$$

Recall that,

$$\begin{aligned}
\partial_t V &= -\sum_{(n,k)\in I^0} b_{n,k}\ell_n \partial_{x_1}\Phi_{n,k} + \mathrm{Mod}_{1,2},\\
\partial_t U &= -\sum_{n=1}^N a_n \ell_n \partial_{x_1}\Psi_n + \mathrm{Mod}_{1,1}, \quad \partial_t R = -\sum_{n=1}^N \ell_n \partial_{x_1} Q_n.
\end{aligned}$$

Therefore, using (5.60), (5.77) and the Sobolev inequality (5.8) again,

$$\begin{aligned}
|\mathcal{I}_{3,2}| &\lesssim \|R\|_{L^4}\|\varphi_1\|_{L^4}^2 \|\partial_t U + \partial_t V\|_{L^4}\\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 (\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}| + |\boldsymbol{b}| + t^{-4})\\
&\lesssim C_0^2 t^{-6}\left(C_0 t^{-3} + C_0^2 t^{-2} + t^{-4}\right) \lesssim C_0^4 t^{-8},\\
|\mathcal{I}_{3,3}| &\lesssim \|\varphi_1\|_{L^4}^2 \left(\|U+V\|_{L^4} + \|\varphi_1\|_{L^4}\right)\left(\|\partial_t R + \partial_t U + \partial_t V\|_{L^4}\right)\\
&\lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 (|\boldsymbol{a}| + |\boldsymbol{b}| + \|\vec{\varphi}\|_{\mathcal{H}}) \lesssim C_0^4 t^{-8}.
\end{aligned}$$

Combining the above estimates, we obtain (5.95) for T_0 large enough.

We see that (5.89) follows from (5.93), (5.94) and (5.95).

Proof of (ii). We decompose

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} =& 2\int_{\mathbb{R}^4} (\partial_t \chi_N)\, (\partial_{x_1}\varphi_1)\varphi_2 \mathrm{d}x + 2\int_{\mathbb{R}^4} \chi_N\, (\partial_{x_1}\partial_t\varphi_1)\, \varphi_2 \mathrm{d}x\\
&+ 2\int_{\mathbb{R}^4} \chi_N\, (\partial_{x_1}\varphi_1)\, \partial_t\varphi_2 \mathrm{d}x = \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6.
\end{aligned}$$

Estimate on $\mathcal{I}_4$. By (5.81), we have

$$\mathcal{I}_4 = -\frac{1}{(1-2\delta)t}\int_\Omega 2\frac{x_1}{t}(\partial_{x_1}\varphi_1)\varphi_2 \mathrm{d}x. \tag{5.96}$$

Estimate on $\mathcal{I}_5$. By (5.58), (5.81) and integration by parts,

$$\begin{aligned}
\mathcal{I}_5 &= 2\int_{\mathbb{R}^4} \chi_N(\partial_{x_1}\varphi_2)\varphi_2 \mathrm{d}x - 2\int_{\mathbb{R}^4} \chi_N\varphi_2\, (\partial_{x_1}\mathrm{Mod}_{1,1} + \partial_{x_1}\mathrm{Mod}_{1,2})\, \mathrm{d}x\\
&= -\frac{1}{(1-2\delta)t}\int_\Omega \varphi_2^2 \mathrm{d}x + 2\int_{\mathbb{R}^4} \varphi_2\, (-\chi_N\partial_{x_1}\mathrm{Mod}_{1,2})\, \mathrm{d}x\\
&\quad - 2\sum_{n=1}^N \dot{a}_n \int_{\mathbb{R}^4} \varphi_2\, (\chi_N\partial_{x_1}\Psi_n)\, \mathrm{d}x.
\end{aligned} \tag{5.97}$$

Estimate on $\mathcal{I}_6$. We claim

$$\begin{aligned}\mathcal{I}_6 =& -\frac{1}{(1-2\delta)t}\int_\Omega \left((\partial_{x_1}\varphi_1)^2 - |\overline{\nabla}\varphi_1|^2\right)\mathrm{d}x + 2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)G_2\mathrm{d}x \\ &+ 2\sum_{n=1}^N \dot{a}_n\int_{\mathbb{R}^4}\left(\ell_n\partial_{x_1}\varphi_1\right)\left(\chi_N\partial_{x_1}\Psi_n\right)\mathrm{d}x - 2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)\mathrm{Mod}_{2,2}\mathrm{d}x \\ &- 6\sum_{n=1}^N\int_{\mathbb{R}^4}\left(\ell_n Q_n\partial_{x_1}Q_n\right)\varphi_1^2\mathrm{d}x + O(C_0t^{-7}).\end{aligned} \tag{5.98}$$

First, using (5.58) again, we decompose,

$$\begin{aligned}\mathcal{I}_6 =& 2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)\Delta\varphi_1\mathrm{d}x + 6\int_{\mathbb{R}^4}\chi_N R^2(\partial_{x_1}\varphi_1)\varphi_1\mathrm{d}x \\ &+ 2\int_{\mathbb{R}^4}\chi_N\left(\partial_{x_1}\varphi_1\right)G_2\mathrm{d}x - 2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)\mathrm{Mod}_{2,2}\mathrm{d}x \\ &+ 2\sum_{n=1}^N\dot{a}_n\int_{\mathbb{R}^4}\left(\ell_n\partial_{x_1}\varphi_1\right)\left(\chi_N\partial_{x_1}\Psi_n\right)\mathrm{d}x + \mathcal{I}_{6,1} + \mathcal{I}_{6,2} + \mathcal{I}_{6,3},\end{aligned}$$

where

$$\mathcal{I}_{6,1} = 2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)\varphi_1^3\mathrm{d}x, \quad \mathcal{I}_{6,2} = 2\sum_{i=1,3}\int_{\mathbb{R}^4}\chi_N\left(\partial_{x_1}\varphi_1\right)G_i\mathrm{d}x,$$

$$\mathcal{I}_{6,3} = 6\int_{\mathbb{R}^4}\chi_N\varphi_1(\partial_{x_1}\varphi_1)\left((U+V)^2 + 2R(U+V) + (R+U+V)\varphi_1\right)\mathrm{d}x.$$

Note that, from (5.81) and integration by parts, we have

$$2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)\Delta\varphi_1\mathrm{d}x = -\frac{1}{(1-2\delta)t}\int_\Omega\left((\partial_{x_1}\varphi_1)^2 - |\overline{\nabla}\varphi_1|^2\right)\mathrm{d}x. \tag{5.99}$$

Using integration by parts again,

$$\begin{aligned}&6\int_{\mathbb{R}^4}\chi_N R^2(\partial_{x_1}\varphi_1)\varphi_1\mathrm{d}x \\ =& -6\int_{\mathbb{R}^4}\chi_N(R\partial_{x_1}R)\varphi_1^2\mathrm{d}x - 3\int_{\mathbb{R}^N}(\partial_{x_1}\chi_N)R^2\varphi_1^2\mathrm{d}x \\ =& -6\sum_{n=1}^N\int_{\mathbb{R}^4}(\ell_n Q_n\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x - \frac{3}{(1-2\delta)t}\int_\Omega R^2\varphi_1^2\mathrm{d}x \\ &+ 6\sum_{n=1}^N\int_{\mathbb{R}^4}(\ell_n-\chi_N)(Q_n\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x - 6\sum_{n\neq n'}\int_{\mathbb{R}^4}\chi_N(Q_{n'}\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x.\end{aligned}$$

From (5.77), (5.85), the Lemma 5.14 (i), the decay property of Q in (5.11) and the Sobolev inequality (5.8),

$$\left|\frac{3}{(1-2\delta)t}\int_\Omega R^2\varphi_1^2\mathrm{d}x\right| \lesssim \sum_{n=1}^N t^{-1}\|\varphi_1\|_{L^4}^2\|Q_n^2\|_{L^2(\Omega)} \lesssim C_0^2t^{-11},$$

$$\left|\sum_{n\neq n'}\int_{\mathbb{R}^4}\chi_N(Q_{n'}\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x\right| \lesssim \sum_{n\neq n'}\|\varphi_1\|_{L^4}^2\|Q_{n'}\partial_{x_1}Q_n\|_{L^2} \lesssim C_0^2t^{-9},$$

$$\left|\sum_{n=1}^N\int_{\mathbb{R}^4}(\ell_n-\chi_N)(Q_n\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x\right| \lesssim \sum_{n=1}^N\|\varphi_1\|_{L^4}^2\|(\ell_n-\chi_N)(Q_n\partial_{x_1}Q_n)\|_{L^2} \lesssim C_0^2t^{-11}.$$

Based on the above estimates, for T_0 large enough, we have

$$6\int_{\mathbb{R}^4}\chi_N R^2(\partial_{x_1}\varphi_1)\varphi_1\mathrm{d}x=-6\sum_{n=1}^{N}\int_{\mathbb{R}^4}(\ell_n Q_n\partial_{x_1}Q_n)\varphi_1^2\mathrm{d}x+O(t^{-7}). \tag{5.100}$$

Then, using (5.77), (5.81), integration by parts and the Sobolev inequality (5.8) again,

$$\mathcal{I}_{6,1}=-\frac{1}{(1-2\delta)t}\int_{\Omega}\varphi_1^4\mathrm{d}x=O\left(t^{-1}\|\vec{\varphi}\|_{\mathcal{H}}^4\right)=O\left(C_0^4t^{-13}\right). \tag{5.101}$$

From (5.46), (5.48) and (5.77), for T_0 large enough

$$\begin{aligned}|\mathcal{I}_{6,2}|&\lesssim\sum_{i=1,3}\|\partial_{x_1}\varphi_1\|_{L^2}\|G_i\|_{L^2}\\&\lesssim\|\vec{\varphi}\|_{\mathcal{H}}\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^3+t^{-4}\right)\\&\lesssim C_0t^{-3}\left(C_0^4t^{-4}\log^{-1}t+C_0^6t^{-6}+t^{-4}\right)\lesssim C_0t^{-7}.\end{aligned} \tag{5.102}$$

Last, using (5.77) and the Sobolev inequality (5.8) again,

$$\begin{aligned}|\mathcal{I}_{6,3}|&\lesssim\|\varphi_1\|_{L^4}\|\partial_{x_1}\varphi_1\|_{L^2}\|U+V\|_{L^8}\left(\|U+V\|_{L^8}+\|R\|_{L^8}\right)\\&\quad+\|\varphi_1\|_{L^4}^2\|\partial_{x_1}\varphi_1\|_{L^2}\|R+U+V\|_{L^\infty}\\&\lesssim\|\vec{\varphi}\|_{\mathcal{H}}^2\left(|\boldsymbol{a}|+|\boldsymbol{b}|+\|\vec{\varphi}\|_{\mathcal{H}}\right)\lesssim C_0^4t^{-8}.\end{aligned} \tag{5.103}$$

Gathering estimates (5.99), (5.100), (5.101), (5.102) and (5.103), we obtain (5.98) for T_0 large enough.

We see that (5.90) follows from (5.96), (5.97) and (5.98).

Proof of (iii). We decompose

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}=-2\int_{\mathbb{R}^4}(\partial_t\varphi_1)G_2\mathrm{d}x-2\int_{\mathbb{R}^4}\varphi_1\partial_tG_2\mathrm{d}x=\mathcal{I}_7+\mathcal{I}_8. \tag{5.104}$$

Estimate on $\mathcal{I}_7$. We claim

$$\mathcal{I}_7=-2\int_{\mathbb{R}^4}\varphi_2G_2\mathrm{d}x+O(t^{-7}). \tag{5.105}$$

Indeed, from (5.58),

$$\mathcal{I}_7=-2\int_{\mathbb{R}^4}\varphi_2G_2\mathrm{d}x+2\int_{\mathbb{R}^4}\mathrm{Mod}_1G_2\mathrm{d}x.$$

Note that, by (5.26) and change of variables, we have

$$2\int_{\mathbb{R}^4}\mathrm{Mod}_1G_2\mathrm{d}x=2\sum_{n\neq n'}\dot{a}_{n'}\int_{\mathbb{R}^4}\Psi_{n'}G_{2,n}\mathrm{d}x+2\sum_{n\neq n'}\sum_{k=1}^{K}\dot{b}_{n',k}\int_{\mathbb{R}^4}\Phi_{n',k}G_{2,n}\mathrm{d}x.$$

Therefore, using (5.60), (5.77), the Lemma 5.14 (i) and the decay property of ψ and ϕ_k in (5.19), for any $n\neq n'$ and $k=1,\dots,K$, we have

$$\begin{aligned}\left|\dot{a}_{n'}\int_{\mathbb{R}^4}\Psi_{n'}G_{2,n}\mathrm{d}x\right|&\lesssim t^{-2}(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2)(\|\vec{\varphi}\|_{\mathcal{H}}+|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+t^{-4})\\&\lesssim C_0^4t^{-6}\left(C_0t^{-3}+C_0^4t^{-4}\log^{-1}t+C_0^4t^{-4}+t^{-4}\right)\lesssim C_0^8t^{-9},\end{aligned}$$

$$\begin{aligned}\left|\dot{b}_{n',k}\int_{\mathbb{R}^4}\Phi_{n',k}G_{2,n}\mathrm{d}x\right|&\lesssim t^{-3}(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2)(\|\vec{\varphi}\|_{\mathcal{H}}+|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+t^{-4})\\&\lesssim C_0^4t^{-7}\left(C_0t^{-3}+C_0^4t^{-4}\log^{-1}t+C_0^4t^{-4}+t^{-4}\right)\lesssim C_0^8t^{-10}.\end{aligned}$$

Combining above estimates and taking T_0 large enough, we obtain (5.105).

Estimate on $\mathcal{I}_8$. We claim

$$\mathcal{I}_8 = -2\int_{\mathbb{R}^4} \chi_N(\partial_{x_1}\varphi_1)G_2\mathrm{d}x + O(t^{-7}). \tag{5.106}$$

Note that, from the definition of $G_{2,n}$ in Lemma 5.12,

$$\begin{aligned}\partial_t G_{2,n} =& -\ell_n\partial_{x_1}G_{2,n} + 6\sum_{k=1}^{K}(\dot{a}_n b_{n,k} + a_n\dot{b}_{n,k})Q_n\Psi_n\Phi_{n,k}\\ &+ 6\dot{a}_n a_n Q_n\Psi_n^2 + 6\sum_{k,k'=1}^{K}\dot{b}_{n,k}b_{n,k'}Q_n\Phi_{n,k}\Phi_{n,k'}.\end{aligned}$$

Therefore, by integration by parts, we have

$$\mathcal{I}_8 = -2\int_{\mathbb{R}^4}\chi_N(\partial_{x_1}\varphi_1)G_2\mathrm{d}x + \mathcal{I}_{8,1} + \mathcal{I}_{8,2} + \mathcal{I}_{8,3} + \mathcal{I}_{8,4},$$

where

$$\begin{aligned}\mathcal{I}_{8,1} &= 2\sum_{n=1}^{N}\int_{\mathbb{R}^4}(\chi_N - \ell_n)(\partial_{x_1}\varphi_1)G_{2,n}\mathrm{d}x,\\ \mathcal{I}_{8,2} &= -12\sum_{n=1}^{N}\dot{a}_n a_n\int_{\mathbb{R}^4}\chi_N\varphi_1(Q_n\Psi_n\Psi_n)\mathrm{d}x,\\ \mathcal{I}_{8,3} &= -12\sum_{n=1}^{N}\sum_{k,k'=1}^{K}\dot{b}_{n,k}b_{n,k'}\int_{\mathbb{R}^4}\chi_N\varphi_1(Q_n\Phi_{n,k}\Phi_{n,k'})\mathrm{d}x,\\ \mathcal{I}_{8,4} &= -12\sum_{(n,k)\in I^0}(\dot{a}_n b_{n,k} + a_n\dot{b}_{n,k})\int_{\mathbb{R}^4}\chi_N\varphi_1(Q_n\Psi_n\Phi_{n,k})\mathrm{d}x.\end{aligned}$$

Using (5.77), (5.85), the definition of $G_{2,n}$ in Lemma 5.12 and the decay properties of Q, ψ and ϕ_k in (5.11) and (5.19), we have

$$\begin{aligned}|\mathcal{I}_{8,1}| \lesssim& \sum_{n=1}^{N}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)\|\nabla\varphi_1\|_{L^2}\|(\chi_N - \ell_n)Q_n\Psi_n^2\|_{L^2}\\ &+ \sum_{(n,k)\in I^0}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)\|\nabla\varphi_1\|_{L^2}\|(\chi_N - \ell_n)Q_n\Phi_{n,k}^2\|_{L^2}\\ \lesssim& C_0 t^{-8}\left(C_0^4 t^{-4}\log^{-1}t + C_0^4 t^{-4}\right) \lesssim C_0^5 t^{-12}.\end{aligned}$$

From (5.60), (5.77) and the Sobolev inequality (5.8),

$$\begin{aligned}|\mathcal{I}_{8,2}| + |\mathcal{I}_{8,3}| + |\mathcal{I}_{8,4}| \lesssim& \left(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|\right)(|\boldsymbol{a}| + |\boldsymbol{b}|)\|\varphi_1\|_{L^4}\left(\|Q\psi^2\|_{L^{\frac{4}{3}}} + \sum_{k=1}^{K}\|Q\phi_k^2\|_{L^{\frac{4}{3}}}\right)\\ \lesssim& \|\vec{\varphi}\|_{\mathcal{H}}(|\boldsymbol{a}| + |\boldsymbol{b}|)(\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}) \lesssim C_0^7 t^{-8}.\end{aligned}$$

Combining above estimates and taking T_0 large enough, we obtain (5.106).

We see that (5.91) follows from (5.105) and (5.106).

Proof of (iv). We decompose

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_n = 2\dot{a}_n\int_{\mathbb{R}^4}(\ell_n\partial_{x_1}\varphi_1 - \varphi_2)(\ell_n - \chi_N)\partial_{x_1}\Psi_n\mathrm{d}x + \mathcal{I}_9 + \mathcal{I}_{10} + \mathcal{I}_{11} + \mathcal{I}_{12},$$

where

$$\mathcal{I}_9 = 2a_n \int_{\mathbb{R}^4} (\partial_t \varphi_2)(\chi_N - \ell_n)\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{10} = 2a_n \int_{\mathbb{R}^4} (\ell_n \partial_{x_1}\partial_t \varphi_1)(\ell_n - \chi_N)\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{11} = 2a_n \int_{\mathbb{R}^4} (\ell_n \varphi_1 - \varphi_2)(\ell_n - \chi_N)\partial_t\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{12} = 2a_n \int_{\mathbb{R}^4} (\ell_n \varphi_1 - \varphi_2)(\partial_t(\ell_n - \chi_N))\partial_{x_1}\Psi_n \mathrm{d}x.$$

Estimate on $\mathcal{I}_9$. We claim

$$\mathcal{I}_9 = O(t^{-7}). \tag{5.107}$$

Indeed, from (5.58), we have

$$\mathcal{I}_9 = \mathcal{I}_{9,1} + \mathcal{I}_{9,2} + \mathcal{I}_{9,3} + \mathcal{I}_{9,4} + \mathcal{I}_{9,5},$$

where

$$\mathcal{I}_{9,1} = 2a_n \int_{\mathbb{R}^4} G(\chi_N - \ell_n)\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{9,2} = 2a_n \int_{\mathbb{R}^4} \mathrm{Mod}_2(\ell_n - \chi_N)\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{9,3} = 2a_n \int_{\mathbb{R}^4} (\Delta\varphi_1)(\chi_N - \ell_n)\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{9,4} = 6a_n \int_{\mathbb{R}^4} \left((R+U+V)^2\varphi_1\right)(\chi_N - \ell_n)\partial_{x_1}\Psi_n \mathrm{d}x,$$
$$\mathcal{I}_{9,5} = 2a_n \int_{\mathbb{R}^4} \left(3(R+U+V)\varphi_1^2 + \varphi_1^3\right)(\chi_N - \ell_n)\partial_{x_1}\Psi_n \mathrm{d}x.$$

First, by (5.49), (5.77), (5.85), the decay property of ψ in (5.19) and the Cauchy-Schwarz inequality,

$$\begin{aligned}|\mathcal{I}_{9,1}| &\lesssim |\boldsymbol{a}|\|G\|_{L^2}\|(\chi_N - \ell_n)\partial_{x_1}\Psi_n\|_{L^2} \\ &\lesssim C_0^2 t^{-3}\log^{-\frac{1}{2}} t\left(C_0^4 t^{-4}\log^{-1} t + C_0^4 t^{-4} + t^{-4}\right) \lesssim C_0^6 t^{-7}\log^{-\frac{1}{2}} t.\end{aligned}$$

Second, from the expression of Mod_2, we decompose

$$\mathcal{I}_{9,2} = \mathcal{I}_{9,2,1} + \mathcal{I}_{9,2,2} + \mathcal{I}_{9,2,3} + \mathcal{I}_{9,2,4},$$

where

$$\begin{aligned}\mathcal{I}_{9,2,1} =& 2a_n\dot{a}_n \int_{\mathbb{R}^4} \ell_n(\chi_N - \ell_n)\left(\partial_{x_1}\Psi_n\right)^2 \mathrm{d}x, \\ \mathcal{I}_{9,2,2} =& 2\sum_{n\neq n'} a_n\dot{a}_{n'} \int_{\mathbb{R}^4} \ell_{n'}(\chi_N - \ell_n)(\partial_{x_1}\Psi_n)(\partial_{x_1}\Psi_{n'})\mathrm{d}x, \\ \mathcal{I}_{9,2,3} =& 2\sum_{k=1}^{K} a_n\dot{b}_{n,k} \int_{\mathbb{R}^4} \ell_n(\chi_N - \ell_n)\left(\partial_{x_1}\Psi_n\right)\left(\partial_{x_1}\Phi_{n,k}\right)\mathrm{d}x, \\ \mathcal{I}_{9,2,4} =& 2\sum_{n\neq n'}\sum_{k=1}^{K} a_n\dot{b}_{n',k} \int_{\mathbb{R}^4} \ell_{n'}(\chi_N - \ell_n)(\partial_{x_1}\Psi_n)(\partial_{x_1}\Phi_{n',k})\mathrm{d}x.\end{aligned}$$

Using (5.60), (5.77), (5.85), the Lemma 5.14 (i), (iii) and the decay properties of ϕ_k and ψ in (5.19), we have

$$\begin{aligned}|\mathcal{I}_{9,2,1}| &\lesssim |\boldsymbol{a}|\left(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|\right)\int_{\mathbb{R}^4} |\chi_N - \ell_n|\left(\partial_{x_1}\Psi_n\right)^2 \mathrm{d}x \\ &\lesssim C_0^2 t^{-4}\log^{-\frac{1}{2}} t\left(C_0 t^{-3} + C_0^4 t^{-4}\log^{-1} t + C_0^4 t^{-4} + t^{-4}\right) \lesssim C_0^6 t^{-7}\log^{-\frac{1}{2}} t,\end{aligned}$$

$$
\begin{aligned}
|\mathcal{I}_{9,2,2}| &\lesssim |\boldsymbol{a}| \left(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|\right) \sum_{n' \neq n} \int_{\mathbb{R}^4} |\partial_{x_1} \Psi_n| \, |\partial_{x_1} \Psi_{n'}| \, \mathrm{d}x \\
&\lesssim C_0^2 t^{-4} \log^{-\frac{1}{2}} t \left(C_0 t^{-3} + C_0^4 t^{-4} \log^{-1} t + C_0^4 t^{-4} + t^{-4}\right) \lesssim C_0^6 t^{-7} \log^{-\frac{1}{2}} t, \\
|\mathcal{I}_{9,2,3}| &\lesssim |\boldsymbol{a}| \left(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|\right) \sum_{k=1}^{K} \int_{\mathbb{R}^4} |\chi_N - \ell_n| \, |\partial_{x_1} \Psi_n| \, |\partial_{x_1} \Phi_{n,k}| \, \mathrm{d}x \\
&\lesssim C_0^2 t^{-5} \log^{-\frac{1}{2}} t \left(C_0 t^{-3} + C_0^4 t^{-4} \log^{-1} t + C_0^4 t^{-4} + t^{-4}\right) \lesssim C_0^6 t^{-8} \log^{-\frac{1}{2}} t,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{I}_{9,2,4}| &\lesssim |\boldsymbol{a}| \left(|\dot{\boldsymbol{a}}| + |\dot{\boldsymbol{b}}|\right) \sum_{n' \neq n} \sum_{k=1}^{K} \int_{\mathbb{R}^4} |\partial_{x_1} \Psi_n| \, \left|\partial_{x_1} \Phi_{n',k}\right| \mathrm{d}x \\
&\lesssim C_0^2 t^{-5} \log^{\frac{1}{2}} t \left(C_0 t^{-3} + C_0^4 t^{-4} \log^{-1} t + C_0^4 t^{-4} + t^{-4}\right) \lesssim C_0^6 t^{-8} \log^{\frac{1}{2}} t.
\end{aligned}
$$

Third, by integration by parts and (5.81), we decompose

$$
\mathcal{I}_{9,3} = \mathcal{I}_{9,3,1} + \mathcal{I}_{9,3,2},
$$

where

$$
\begin{aligned}
\mathcal{I}_{9,3,1} &= -\, 2a_n \int_{\mathbb{R}^4} (\partial_{x_1} \varphi_1) \, (\partial_{x_1} \chi_N) \partial_{x_1} \Psi_n \mathrm{d}x, \\
\mathcal{I}_{9,3,2} &= -\, 2a_n \sum_{i=1}^{4} \int_{\mathbb{R}^4} (\partial_{x_i} \varphi_1) \, (\chi_N - \ell_n) \partial_{x_i} \partial_{x_1} \Psi_n \mathrm{d}x.
\end{aligned}
$$

From (5.77), (5.81), (5.85) and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|\mathcal{I}_{9,3,1}| &\lesssim t^{-1} |\boldsymbol{a}| \|\nabla \varphi_1\|_{L^2} \|\partial_{x_1} \Psi_n\|_{L^2(\Omega)} \lesssim C_0^3 t^{-7} \log^{-\frac{1}{2}} t, \\
|\mathcal{I}_{9,3,2}| &\lesssim \sum_{i=1}^{4} |\boldsymbol{a}| \|\nabla \varphi_1\|_{L^2} \|(\chi_N - \ell_n) \partial_{x_i} \partial_{x_1} \Psi_n\|_{L^2} \lesssim C_0^3 t^{-7} \log^{-\frac{1}{2}} t.
\end{aligned}
$$

Fourth, by an elementary computation, we decompose

$$
\mathcal{I}_{9,4} = \mathcal{I}_{9,4,1} + \mathcal{I}_{9,4,2},
$$

where

$$
\begin{aligned}
\mathcal{I}_{9,4,1} &= 6a_n \int_{\mathbb{R}^4} \varphi_1 (\chi_N - \ell_n) R^2 \partial_{x_1} \Psi_n \mathrm{d}x, \\
\mathcal{I}_{9,4,2} &= 6a_n \int_{\mathbb{R}^4} \left(2R(U+V) + (U+V)^2\right) \varphi_1 (\chi_N - \ell_n) \partial_{x_1} \Psi_n \mathrm{d}x.
\end{aligned}
$$

Note that, from (5.77), (5.85), the Lemma 5.14 (i) and the Sobolev inequality (5.8),

$$
\begin{aligned}
|\mathcal{I}_{9,4,1}| &\lesssim |\boldsymbol{a}| \|\varphi_1\|_{L^4} \|Q\|_{L^4} \left(\|(\chi_N - \ell_n) Q_n \partial_{x_1} \Psi_n\|_{L^2} + \sum_{n' \neq n} \|Q_{n'} \partial_{x_1} \Psi_n\|_{L^2} \right) \\
&\lesssim C_0^3 t^{-5} \log^{-\frac{1}{2}} t \left(t^{-4} + t^{-3}\right) \lesssim C_0^3 t^{-8} \log^{-\frac{1}{2}} t.
\end{aligned}
$$

Note also that, using (5.77) and (5.85) again,

$$
\begin{aligned}
|\mathcal{I}_{9,4,2}| &\lesssim |\boldsymbol{a}| \|\varphi_1\|_{L^4} \|U+V\|_{L^8} \|(\chi_N - \ell_n) \partial_{x_1} \Psi_n\|_{L^2} \left(\|R\|_{L^8} + \|U+V\|_{L^8}\right) \\
&\lesssim |\boldsymbol{a}| (|\boldsymbol{a}| + |\boldsymbol{b}|) \|\vec{\varphi}\|_{\mathcal{H}} \|(\chi_N - \ell_n) \partial_{x_1} \Psi_n\|_{L^2} \lesssim C_0^5 t^{-8} \log t^{-\frac{1}{2}} t.
\end{aligned}
$$

Last, using (5.77) and the Sobolev inequality (5.8) again,

$$|\mathcal{I}_{9,5}| \lesssim |\boldsymbol{a}|\|\varphi_1\|_{L^4}^2\|\partial_{x_1}\Psi_n\|_{L^4}\left(\|\varphi_1\|_{L^4}+\|R+U+V\|_{L^4}\right) \lesssim C_0^4 t^{-8}\log^{-\frac{1}{2}}t.$$

Combining above estimates and taking T_0 large enough, we obtain (5.107).

Estimates on $\mathcal{I}_{10}$. We claim

$$\mathcal{I}_{10} = O(t^{-7}). \tag{5.108}$$

First, from (5.58) and integration by parts, we decompose

$$\mathcal{I}_{10} = \mathcal{I}_{10,1} + \mathcal{I}_{10,2} + \mathcal{I}_{10,3},$$

where

$$\begin{aligned}
\mathcal{I}_{10,1} &= 2a_n\ell_n \int_{\mathbb{R}^4} \varphi_2(\partial_{x_1}\chi_N)(\partial_{x_1}\Psi_n)\mathrm{d}x,\\
\mathcal{I}_{10,2} &= 2a_n\ell_n \int_{\mathbb{R}^4} \varphi_2(\chi_N-\ell_n)\partial_{x_1}^2\Psi_n\mathrm{d}x,\\
\mathcal{I}_{10,3} =& 2a_n\ell_n \sum_{n'=1}^{N} \dot{a}_{n'} \int_{\mathbb{R}^4} (\chi_N-\ell_n)(\partial_{x_1}\Psi_n)(\partial_{x_1}\Psi_{n'})\mathrm{d}x\\
&+ 2a_n\ell_n \sum_{(n',k)\in I^0} \dot{b}_{n',k} \int_{\mathbb{R}^4} (\chi_N-\ell_n)(\partial_{x_1}\Psi_n)(\partial_{x_1}\Phi_{n',k})\mathrm{d}x.
\end{aligned}$$

From (5.77), (5.81) and (5.85),

$$\begin{aligned}
|\mathcal{I}_{10,1}| &\lesssim t^{-1}|\boldsymbol{a}|\|\varphi_2\|_{L^2}\|\partial_{x_1}\Psi_n\|_{L^2(\Omega)} \lesssim C_0^3 t^{-7}\log^{-\frac{1}{2}}t,\\
|\mathcal{I}_{10,2}| &\lesssim |\boldsymbol{a}|\|\varphi_2\|_{L^2}\|(\chi_N-\ell_n)\partial_{x_1}^2\Psi_n\|_{L^2} \lesssim C_0^3 t^{-7}\log^{-\frac{1}{2}}t.
\end{aligned}$$

Then, using the same arguments as the estimates of $\mathcal{I}_{9,2}$, we also have

$$|\mathcal{I}_{10,3}| \lesssim C_0^6 t^{-7}\log^{-\frac{1}{2}}t + C_0^6 t^{-8}\log^{\frac{1}{2}}t.$$

Combining above estimates and taking T_0 large enough, we obtain (5.108).

Estimate on $\mathcal{I}_{11}$. Recall that

$$\partial_t\partial_{x_1}\Psi_n = -\ell_n\partial_{x_1}^2\Psi_n \quad \text{for all } n=1,\dots,N.$$

Thus, using (5.77) and (5.85) again,

$$|\mathcal{I}_{11}| \lesssim |\boldsymbol{a}|\left(\|\nabla\varphi_1\|_{L^2}+\|\varphi_2\|_{L^2}\right)\|(\chi_N-\ell_n)\partial_{x_1}^2\Psi_n\|_{L^2} \lesssim C_0^3 t^{-7}\log^{-\frac{1}{2}}t. \tag{5.109}$$

Estimate on $\mathcal{I}_{12}$. From (5.77), (5.81) and (5.85), we have

$$|\mathcal{I}_{12}| \lesssim t^{-1}|\boldsymbol{a}|\left(\|\nabla\varphi_1\|_{L^2}+\|\varphi_2\|_{L^2}\right)\|\partial_{x_1}\Psi_n\|_{L^2(\Omega)} \lesssim C_0^3 t^{-7}\log^{-\frac{1}{2}}t. \tag{5.110}$$

We see that (5.92) follows from (5.107), (5.108), (5.109) and (5.110). □

Last, we conclude the time variation of $\mathcal{K}$ from Lemma 5.28.

Lemma 5.29 (Time variation of $\mathcal{K}$). *There exists T_0 large enough such that*

$$-\frac{\mathrm{d}}{\mathrm{d}t}(t^2\mathcal{K}) \lesssim C_0 t^{-5}. \tag{5.111}$$

Proof. Combining the estimates (5.89), (5.90), (5.91) and (5.92), we decompose

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} + \sum_{n=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_n = \mathcal{I}_{13} + \mathcal{I}_{14} + \mathcal{I}_{15} + O(C_0 t^{-7}).$$

where

$$\begin{aligned}
\mathcal{I}_{13} &= -2\int_{\mathbb{R}^4} \varphi_2 \left(\mathrm{Mod}_{2,2} + \chi_N \partial_{x_1}\mathrm{Mod}_{1,2}\right)\mathrm{d}x,\\
\mathcal{I}_{14} &= 2\int_{\mathbb{R}^4} \varphi_1 \left(\Delta\mathrm{Mod}_{1,2} + 3R^2\mathrm{Mod}_{1,2}\right)\mathrm{d}x - 2\int_{\mathbb{R}^4} \chi_N \mathrm{Mod}_{2,2}\partial_{x_1}\varphi_1 \mathrm{d}x,\\
\mathcal{I}_{15} &= -\frac{1}{(1-2\delta)t}\int_{\Omega}\left((\partial_{x_1}\varphi_1)^2 + \varphi_2^2 + 2\frac{x_1}{t}(\partial_{x_1}\varphi_1)\varphi_2 - |\overline{\nabla}\varphi_1|^2\right)\mathrm{d}x.
\end{aligned}$$

Note that, from the definition of $\mathrm{Mod}_{1,2}$, $\mathrm{Mod}_{2,2}$ and χ_N,

$$\mathcal{I}_{13} = 2\sum_{(n,k)\in I^0}\dot{b}_{n,k}\int_{\mathbb{R}^4}\varphi_2\left(\ell_n - \chi_N\right)\partial_{x_1}\Phi_{n,k}\mathrm{d}x.$$

Therefore, from (5.77), (5.85) and the decay property of ϕ_k in (5.19),

$$\begin{aligned}
|\mathcal{I}_{13}| &\lesssim \sum_{(n,k)\in I^0}|\dot{\boldsymbol{b}}|\|\varphi_2\|_{L^2}\|\left(\ell_n - \chi_N\right)\partial_{x_1}\Phi_{n,k}\|_{L^2}\\
&\lesssim t^{-2}\|\vec{\varphi}\|_{\mathcal{H}}\left(\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right) \lesssim C_0^5 t^{-8}.
\end{aligned}$$

Note also that, by $-(1-\ell_n^2)\partial_{x_1}^2\Phi_{n,k} - \bar{\Delta}\Phi_{n,k} - 3Q_n^2\Phi_{n,k} = 0$ and integration by parts,

$$\begin{aligned}
\mathcal{I}_{14} =& 2\sum_{(n,k)\in I^0}\dot{b}_{n,k}\int_{\mathbb{R}^4}\ell_n\partial_{x_1}\varphi_1\left(\chi_N - \ell_n\right)\partial_{x_1}\Phi_{n,k}\mathrm{d}x\\
&+ 6\sum_{(n,k)\in I^0}\sum_{n'\neq n}\dot{b}_{n,k}\int_{\mathbb{R}^4}\varphi_1\Phi_{n,k}\left(2Q_nQ_{n'} + Q_{n'}^2\right)\mathrm{d}x
\end{aligned}$$

Therefore, using (5.60), (5.77), (5.85), the Lemma 5.14 and the decay property of ϕ_k in (5.19) again, we have

$$\begin{aligned}
|\mathcal{I}_{14}| \lesssim& \sum_{(n,k)\in I^0}|\dot{\boldsymbol{b}}|\|\nabla\varphi_1\|_{L^2}\|\left(\ell_n - \chi_N\right)\partial_{x_1}\Phi_{n,k}\|_{L^2}\\
&+ \sum_{(n,k)\in I^0}\sum_{n'\neq n}|\dot{\boldsymbol{b}}|\|\varphi_1\|_{L^4}\left(\|Q_nQ_{n'}\Phi_{n,k}\|_{L^{\frac{4}{3}}} + \|Q_{n'}^2\Phi_{n,k}\|_{L^{\frac{4}{3}}}\right)\\
\lesssim& t^{-2}\|\vec{\varphi}\|_{\mathcal{H}}\left(\|\vec{\varphi}\|_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right) \lesssim C_0^5 t^{-8}.
\end{aligned}$$

From $0 < \delta < \frac{1}{20}$ and the definition of χ_N, we have, for all $x = (x_1, \bar{x}) \in \Omega$,

$$\left|\frac{x_1}{t} - \chi_N\right| \leqslant \frac{2\delta}{1-2\delta}\left|\frac{x_1}{t}\right| + \frac{2\bar{\ell}\delta}{1-2\delta} \leqslant 8\delta.$$

Thus, from (5.82), (5.88) and the definition of δ and $\mathcal{N}_\Omega$,

$$\begin{aligned}
-t\mathcal{I}_{15} &\leqslant \frac{1}{1-2\delta}\left(\mathcal{N}_\Omega + 2\int_\Omega\left(\frac{x_1}{t} - \chi_N\right)(\partial_{x_1}\varphi_1)\varphi_2\right)\mathrm{d}x\\
&\leqslant \frac{10}{9}\left(1 + 16\delta(1-\bar{\ell})^{-1}\right)\mathcal{N}_\Omega \leqslant 2\mathcal{K} + 2\nu^{-1}\left(t^{-6} + C_0^5 t^{-6}\log^{-\frac{1}{2}}t\right).
\end{aligned}$$

Gathering above estimates we have,

$$-\frac{\mathrm{d}}{\mathrm{d}t}(t^2\mathcal{K}) \lesssim C_0 t^{-5} + C_0^5 t^{-6} + C_0^5 t^{-5}\log^{-\frac{1}{2}}t,$$

which implies (5.111) for T_0 large enough. □

5.4.3 End of the proof of Propostion 5.24

Recall that, for the proof of Proposition 5.24, we just need to prove the existence of $\boldsymbol{z}_m \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$ such that $T^*(\boldsymbol{z}_m) = T_0$. We start by closing all bootstrap estimates except the one for the unstable modes. Last, we prove the existence of suitable parameters $\boldsymbol{z}_m = (z_{n,j}^m)_{(n,j)\in I} \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$ using a topological argument.

Proof of Proposition 5.24. For all $\boldsymbol{z}_m = (z_{n,j}^m)_{(n,j)\in I} \in \mathcal{B}_{\mathbb{R}^{|I|}}(S_m^{-\frac{7}{2}})$, we consider the solution $\vec{u}$ with the initial data at $T = S_m$ as defined in Lemma 5.23. From the definition of the initial data, we have

$$\|\vec{\varphi}(S_m)\|_{\mathcal{H}} \lesssim S_m^{-\frac{7}{2}} \quad \text{and} \quad a_n(S_m) = b_{n,j}(S_m) = 0, \tag{5.112}$$

for all $n = 1, \dots, N$ and $(n,k) \in I^0$.

Step 1. Closing the estimate in $\vec{\varphi}$. Integrating (5.111) on $[t, S_m]$ for any $t \in [T^*, S_m]$ and using (5.112), we have

$$\mathcal{K}(t) \lesssim C_0 t^{-6} + \left(\frac{S_m}{t}\right)^2 |\mathcal{K}(S_m)| \lesssim C_0 t^{-6}.$$

Thus, from (5.82) and (5.88), for T_0 large enough (depending on C_0),

$$\|\vec{\varphi}(t)\|_{\mathcal{H}}^2 \lesssim \mathcal{K}(t) + t^{-6} + C_0^5 t^{-6} \log^{-\frac{1}{2}} t \lesssim C_0 t^{-6} + C_0^5 t^{-6} \log^{-\frac{1}{2}} t \lesssim C_0 t^{-6}.$$

This strictly improves the estimate on $\vec{\varphi}$ in (5.77) for C_0 large enough.

Step 2. Closing the estimate in $\boldsymbol{a}$. First, from (5.66) and (5.77), for all $t \in [T^*, S_m]$,

$$\sum_{n=1}^N |c_n(t)| \lesssim \log^{-1} t \left(\|\varphi_1\|_{L^4} \|\Psi_n \xi_n\|_{L^{\frac{4}{3}}} \right) \lesssim C_0 t^{-2} \log^{-1} t. \tag{5.113}$$

Then, from (5.70) and (5.77), we have

$$\sum_{n=1}^N |\dot{a}_n + \dot{c}_n| \lesssim C_0 t^{-3} \log^{-\frac{1}{2}} t + C_0^2 t^{-4}.$$

Integrating above estimate on $[t, S_m]$ for any $t \in [T^*, S_m]$ and then using (5.113),

$$\begin{aligned} \sum_{n=1}^N |a_n(t)| &\lesssim \sum_{n=1}^N (|c_n(t)| + |c_n(S_m)|) + C_0 t^{-2} \log^{-\frac{1}{2}} t + C_0^2 t^{-3} \\ &\lesssim C_0 t^{-2} \log^{-1} t + C_0 t^{-2} \log^{-\frac{1}{2}} t + C_0^2 t^{-3} \lesssim C_0 t^{-2} \log^{-\frac{1}{2}} t, \end{aligned}$$

for T_0 large enough. This strictly improves the estimate on $\boldsymbol{a}$ in (5.77) for taking C_0 large enough.

Step 3. Closing the estimate in $\boldsymbol{b}$. From (5.60) and (5.77), we have

$$\sum_{(n,k)\in I^0} |\dot{b}_{n,k}| \lesssim C_0 t^{-3} + C_0^4 t^{-4}.$$

Integrating above estimate on $[t, S_m]$ for any $t \in [T^*, S_m]$ and then using (5.112),

$$\sum_{(n,k)\in I^0} |b_{n,k}(t)| \lesssim C_0 t^{-2} + C_0^4 t^{-3} \lesssim C_0 t^{-2},$$

for T_0 large enough. This strictly improves the estimate on $\boldsymbol{b}$ in (5.77) for taking C_0 large enough.

Step 4. Closing the estimate in $(z^-_{n,j})_{(n,j)\in I}$. By (5.73) and (5.77), for all $(n,j)\in I$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(z^-_{n,j}\right)^2 &= 2\alpha_{n,j}\left(z^-_{n,j}\right)^2 + O\left(\|\vec{\varphi}\|_{\mathcal{H}}\left(\|\vec{\varphi}\|^2_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right)\right)\\ &= 2\alpha_{n,j}\left(z^-_{n,j}\right)^2 + O\left(C_0^3 t^{-9} + C_0^5 t^{-7}\right) = 2\alpha_{n,j}\left(z^-_{n,j}\right)^2 + O\left(C_0^5 t^{-7}\right).\end{aligned}$$

Integrating above estimate on $[t, S_m]$ for any $t\in[T^*, S_m]$ and then using (5.112)

$$\left(z^-_{n,j}\right)^2(t) \lesssim e^{-2\alpha_{n,j}(S_m-t)}\left(z^-_{n,j}\right)^2(S_m) + C_0^5\int_t^{S_m} e^{2\alpha_{n,j}(t-s)}s^{-7}\mathrm{d}s \lesssim C_0^5 t^{-7}.$$

This strictly improves the estimate on $(z^-_{n,j})_{(n,j)\in I}$ in (5.77) for taking T_0 and C_0 large enough.

Step 5. Final argument on the unstable parameters. Let

$$\mathcal{A}(t) = \sum_{(n,j)\in I}\left(z^+_{n,j}\right)^2 \quad \text{and} \quad \bar{\alpha} = \min_{(n,j)\in I}\alpha_{n,j} > 0.$$

We claim, for any time $t\in[T^*, S_m]$ where it holds $\mathcal{A}(t) = t^{-7}$, the following transversality property holds,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^7\mathcal{A}(t)\right) \leqslant -\bar{\alpha}. \tag{5.114}$$

Indeed, from (5.73) and (5.77), for any $t\in[T^*, S_m]$ where it holds $\mathcal{A}(t) = t^{-7}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t) &= -2\sum_{(n,j)\in I}\alpha_{n,j}\left(z^+_{n,j}\right)^2 + O\Big(\sum_{(n,j)\in I}|z^+_{n,j}|\left(\|\vec{\varphi}\|^2_{\mathcal{H}} + |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + t^{-4}\right)\Big)\\ &\leqslant -2\bar{\alpha}t^{-7} + O\left(C_0^2 t^{-\frac{19}{2}} + C_0^4 t^{-\frac{15}{2}}\right) \leqslant -2\bar{\alpha}t^{-7} + C_0^5 t^{-\frac{15}{2}},\end{aligned}$$

which implies (5.114) for taking T_0 large enough. The transversality relation (5.114) is enough to justify the existence of at least a couple $\boldsymbol{z}_m = (z^m_{n,j})_{(n,j)\in I}\in\mathcal{B}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)$ such that $T^*(\boldsymbol{z}_m) = T_0$.

The proof is by contradiction, we assume that for all $\boldsymbol{z}_m = (z^m_{n,j})_{(n,j)\in I}\in\mathcal{B}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)$, it holds $T_0 < T_*(\boldsymbol{z}_m)$. Then, a contradiction follows from the following discussion (see for instance more details in [31, Section 3.1] and [85, Lemma 4.2]).

Continuity of $T^*(\boldsymbol{z}_m)$. The above transversality condition (5.114) implies that the map

$$\boldsymbol{z}_m\in\mathcal{B}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)\mapsto T^*(\boldsymbol{z}_m)\in[T_0, S_m]$$

is continuous and

$$T^*(\boldsymbol{z}_m) = S_m \quad \text{for} \quad \boldsymbol{z}_m\in\mathcal{S}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right).$$

Construction of a retraction. We define

$$\begin{aligned}\mathcal{M}:\ &\bar{\mathcal{B}}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)\mapsto\mathcal{S}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)\\ &\boldsymbol{z}_m = (z^m_{n,j})_{(n,j)\in I}\mapsto\left(\frac{T^*(\boldsymbol{z}_m)}{S_m}\right)^{\frac{7}{2}}\left(z^+_{n,j}\left(T^*(\boldsymbol{z}_m)\right)\right)_{(n,j)\in I}.\end{aligned}$$

From what precedes, $\mathcal{M}$ is continuous. Moreover, $\mathcal{M}$ restricted to $\mathcal{S}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)$ is the identity. The existence of such a map is contradictory with the no retraction theorem for continuous maps from

the ball to the sphere. Therefore, the existence of $\boldsymbol{z}_m \in \mathcal{B}_{\mathbb{R}^{|I|}}\left(S_m^{-\frac{7}{2}}\right)$ have proved. Then, the uniform estimate (5.76) is a consequence of bootstrap estimates (5.77). The proof of Proposition 5.24 is complete.

□

5.4.4 Proof of Theorem 5.1 from Proposition 5.24

The proof is based on a standard compactness argument (see for example [26, 69, 112]). The main point is the following proposition of weak continuity of the flow near a compact set.

Proposition 5.30. *There exists a constant $\varepsilon > 0$ such that the following holds. Let $\mathcal{S} \subset \dot{H}^1 \times L^2$ be a compact set and let $\vec{u}_m : [T_1, T_2] \to \dot{H}^1 \times L^2$ be a sequence of solutions of (5.1) such that*

$$\sup_{m\in\mathbb{N}} \left(\max_{t\in[T_1,T_2]} \operatorname{dist}\left(\vec{u}_m(t), \mathcal{S}\right) \right) \leqslant \varepsilon.$$

Suppose that $\vec{u}_m(T_0) \rightharpoonup \vec{u}_0$ weakly in $\dot{H}^1 \times L^2$. Then the solution $\vec{u}(t)$ of (5.1) with the initial data $\vec{u}(T_1) = \vec{u}_0$ is well-defined on $[T_1, T_2]$ and

$$\vec{u}_m(t) \rightharpoonup \vec{u}(t), \quad \text{weakly in } \dot{H}^1 \times L^2 \quad \text{for all } t \in [T_1, T_2].$$

Proof. The proof relies on an argument based on the result of profile decomposition stated in [45, Proposition 2.8]. See more details in [67, Appendix A.2] and [69, Appendix]. □

We are in a position to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. We consider the sequence of solutions $(\vec{u}_m)_{m\in\mathbb{N}}$ given by Proposition 5.24. On the time interval $[T_0, S_m]$, the solution $\vec{u}_m$ is well-defined. Moreover, for T_0 large enough, we have

$$\max_{t\in[T_0,S_m]} \|\vec{u}_m(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\|_{\mathcal{H}} \leqslant Ct^{-2} \leqslant \varepsilon,$$

where $\varepsilon > 0$ is the constant of Proposition 5.30.

Since the sequence $(\vec{u}_m(T_0))_{m\in\mathbb{N}}$ is bounded in $\dot{H}^1 \times L^2$, up to the extraction of a subsequence, there exists $\vec{u}_0 \in \dot{H}^1 \times L^2$ such that $\vec{u}_m(T_0) \rightharpoonup \vec{u}_0$ weakly in $\dot{H}^1 \times L^2$. Fix $S_M = M^2 > T_0$ for $M \in \mathbb{N}$. Applied the Proposition 5.30 to the following compact set

$$\mathcal{S}_M = \left\{ \sum_{n=1}^{N} \vec{Q}_n(t) \in \dot{H}^1 \times L^2, t \in [T_0, S_M] \right\},$$

we have, the solution $\vec{u}(t)$ of (5.1) with the initial data $\vec{u}(T_0) = \vec{u}_0$ is well-defined on $[T_0, S_M]$ and $\vec{u}_m(t) \rightharpoonup \vec{u}(t)$ weakly in $\dot{H}^1 \times L^2$ for all $t \in [T_0, S_M]$. From (5.76) and the properties of weak convergence, we have

$$\max_{t\in[T_0,S_M]} \left(\|\vec{u}(t) - \sum_{n=1}^{N} \vec{Q}_n(t)\|_{\mathcal{H}} \right) \leqslant Ct^{-2}.$$

Since $S_M = M^2 > T_0$ is arbitrary, the solution $\vec{u}(t)$ is well-defined and satisfies the conclusion of Theorem 5.1 on $[T_0, \infty)$. The proof of Theorem 5.1 is complete. □

Part III

Long time dynamics for the damped NLKG equation

Chapter 6

Description and classification of 2-solitary waves for nonlinear damped Klein-Gordon equations

Abstract: We describe completely 2-solitary waves related to the ground state of the nonlinear damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - |u|^{p-1}u = 0$$

on $\mathbb{R}^N$, for $1 \leq N \leq 5$ and energy subcritical exponents $p > 2$. The description is twofold.

First, we prove that 2-solitary waves with same sign do not exist. Second, we construct and classify the full family of 2-solitary waves in the case of opposite signs. Close to the sum of two remote solitary waves, it turns out that only the components of the initial data in the unstable direction of each ground state are relevant in the large time asymptotic behavior of the solution. In particular, we show that 2-solitary waves have a universal behavior: the distance between the solitary waves is asymptotic to $\log t$ as $t \to \infty$. This behavior is due to damping of the initial data combined with strong interactions between the solitary waves.

6.1 Introduction

6.1.1 Setting of the problem

We consider the nonlinear focusing damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0 \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{6.1}$$

where $f(u) = |u|^{p-1}u$, $\alpha > 0$, $1 \leqslant N \leq 5$, and the exponent p corresponds to the energy sub-critical case, *i.e.*

$$2 < p < \infty \text{ for } N = 1,2 \text{ and } 2 < p < \frac{N+2}{N-2} \text{ for } N = 3,4,5. \tag{6.2}$$

The restriction $p > 2$ is discussed in Remark 6.6.

This equation also rewrites as a first order system for $\vec{u} = (u, \partial_t u) = (u, v)$

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - u + f(u) - 2\alpha v. \end{cases}$$

It follows from [12, Theorem 2.3] that the Cauchy problem for (6.1) is locally well-posed in the energy space: for any initial data $(u_0, v_0) \in H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$, there exists a unique (in some class) maximal solution $u \in \mathcal{C}([0,T_{\max}), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T_{\max}), L^2(\mathbb{R}^N))$ of (6.1). Moreover, if the maximal time of existence $T_{\max}$ is finite, then $\lim_{t\uparrow T_{\max}} \|\vec{u}(t)\|_{H^1\times L^2} = \infty$.

Setting $F(u) = \frac{1}{p+1}|u|^{p+1}$ and

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^N} \left\{|\nabla u|^2 + u^2 + (\partial_t u)^2 - 2F(u)\right\} \mathrm{d}x,$$

for any $H^1\times L^2$ solution $\vec{u}$ of (6.1), it holds

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t. \tag{6.3}$$

In this paper, we are interested in the dynamics of 2-solitary waves related to the ground state Q, which is the unique positive, radial H^1 solution of

$$-\Delta Q + Q - f(Q) = 0, \quad x \in \mathbb{R}^N. \tag{6.4}$$

(See [7, 77].) The ground state generates the stationary solution $\vec{Q} = \binom{Q}{0}$ of (6.1). The function $-\vec{Q}$ as well as any translate $\vec{Q}(\cdot - z_0)$ are also solutions of (6.1).

The question of the existence of multi-solitary waves for (6.1) was first addressed by Feireisl in [53, Theorem 1.1], under suitable conditions on N and p, for an even number of solitary waves with specific geometric and sign configurations. His construction is based on variational and symmetry arguments to treat the instability direction of the solitary waves. The goal of the present paper is to fully understand 2-solitary waves by proving non-existence, existence and classification results using dynamical arguments.

6.1.2 Main results

First let us introduce a few basic notation. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ denote the canonical basis of $\mathbb{R}^N$. We denote by $\mathcal{B}_{\mathbb{R}^N}(\rho)$ (respectively, $\mathcal{S}_{\mathbb{R}^N}(\rho)$) the open ball (respectively, the sphere) of $\mathbb{R}^N$ of center the origin and of radius $\rho > 0$, for the usual norm $|\xi| = (\sum_{j=1}^N \xi_j^2)^{1/2}$. We denote by $\mathcal{B}_{H^1\times L^2}(\rho)$ the ball of $H^1\times L^2$ of center the origin and of radius $\rho > 0$ for the norm $\|\binom{\varepsilon}{\eta}\|_{H^1\times L^2} = \left(\|\varepsilon\|_{H^1}^2 + \|\eta\|_{L^2}^2\right)^{1/2}$. We

denote $\langle \cdot, \cdot \rangle$ the L^2 scalar product for real valued functions u_i or vector-valued functions $\vec{u}_i = (u_i, v_i)$ $(i = 1, 2)$:

$$\langle u_1, u_2 \rangle := \int u_1(x) u_2(x) \, \mathrm{d}x, \quad \langle \vec{u}_1, \vec{u}_2 \rangle := \int u_1(x) u_2(x) \, \mathrm{d}x + \int v_1(x) v_2(x) \, \mathrm{d}x.$$

It is well-known that the operator

$$\mathcal{L} = -\Delta + 1 - pQ^{p-1}$$

appearing after linearization of equation (6.1) around $\vec{Q}$, has a unique negative eigenvalue $-\nu_0^2$ $(\nu_0 > 0)$. We denote by Y the corresponding normalized eigenfunction (see Lemma 6.7 for details and references). In particular, it follows from explicit computations that setting

$$\nu^{\pm} = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{and} \quad \vec{Y}^{\pm} = \begin{pmatrix} Y \\ \nu^{\pm} Y \end{pmatrix},$$

the function $\vec{\varepsilon}^{\pm}(t, x) = \exp(\nu^{\pm} t) \vec{Y}^{\pm}(x)$ is solution of the linearized problem

$$\begin{cases} \partial_t \varepsilon = \eta \\ \partial_t \eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{6.5}$$

Since $\nu^+ > 0$, the solution $\vec{\varepsilon}^+$ illustrates the exponential instability of the solitary wave in positive time. In particular, we see that the presence of the damping $\alpha > 0$ in the equation does not remove the exponential instability of the Klein-Gordon solitary wave. An equivalent formulation of instability is obtained by setting

$$\zeta^{\pm} = \alpha \pm \sqrt{\alpha^2 + \nu_0^2} \quad \text{and} \quad \vec{Z}^{\pm} = \begin{pmatrix} \zeta^{\pm} Y \\ Y \end{pmatrix}$$

and observing that for any solution $\vec{\varepsilon}$ of (6.5),

$$a^{\pm} = \langle \vec{\varepsilon}, \vec{Z}^{\pm} \rangle \quad \text{satisfies} \quad \frac{\mathrm{d}a^{\pm}}{\mathrm{d}t} = \nu^{\pm} a^{\pm}.$$

We start with the definition of 2-solitary waves.

Definition 6.1. A solution $\vec{u} \in \mathcal{C}([T, \infty), H^1 \times L^2)$ of (6.1), for some $T \in \mathbb{R}$, is called a 2-*solitary wave* if there exist $\sigma_1, \sigma_2 = \pm 1$, a sequence $t_n \to \infty$ and a sequence $(\xi_{1,n}, \xi_{2,n}) \in \mathbb{R}^{2N}$ such that

$$\lim_{n \to \infty} \left\{ \left\| u(t_n) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_{k,n}) \right\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \right\} = 0$$

and $\lim_{n \to \infty} |\xi_{1,n} - \xi_{2,n}| \to \infty$.

Remark 6.2. We observe that if u is a global solution of (6.1) satisfying

$$\lim_{t \to \infty} \left\| u(t) - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k(t)) \right\|_{H^1} = 0 \tag{6.6}$$

where $\lim_{t \to \infty} |\xi_1(t) - \xi_2(t)| \to \infty$, then u is a 2-solitary wave. Indeed, it follows from (6.3) and (6.6) that $t \mapsto E(\vec{u}(t))$ is lower bounded. Thus, from (6.3), it holds $\int_0^\infty \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t < \infty$ and so $\lim_{n \to \infty} \|\partial_t u(t_n)\|_{L^2} = 0$ for some sequence $t_n \to \infty$.

Our first result concerns the non-existence of 2-solitary waves with same signs.

Theorem 6.3. *There exists no 2-solitary wave of (6.1) with $\sigma_1 = \sigma_2$.*

In the next result, we show that 2-solitary waves with opposite sign satisfy a universal asymptotic behavior.

Theorem 6.4 (Description of 2-solitary waves). *For any 2-solitary wave u of (6.1), there exist $\sigma_1 = \pm 1$, $\sigma_2 = -\sigma_1$, $T > 0$ and $t \in [T, \infty) \mapsto (z_1(t), z_2(t))$ such that for all $t \in [T, \infty)$,*

$$\Big\| u(t) - \sum_{k=1,2} \sigma_k Q(\cdot - z_k(t)) \Big\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{6.7}$$

and for some constant $c_0 = c_0(N)$,

$$\lim_{t\to\infty} \left\{ |z_1(t) - z_2(t)| - \left(\log t - \frac{1}{2}(N-1)\log\log t + c_0 \right) \right\} = 0. \tag{6.8}$$

Moreover, there exists $\omega^\infty \in \mathcal{S}_{\mathbb{R}^N}(1)$ such that

$$\lim_{t\to\infty} \frac{z_1(t) - z_2(t)}{\log t} = 2 \lim_{t\to\infty} \frac{z_1(t)}{\log t} = -2 \lim_{t\to\infty} \frac{z_2(t)}{\log t} = \omega^\infty. \tag{6.9}$$

Finally, we describe the full family of 2-solitary waves for initial data close to the sum of two remote solitary waves.

Theorem 6.5 (Classification of 2-solitary waves). *There exist $C, \delta > 0$ and a Lipschitz map*

$$H : (\mathbb{R}^N \setminus \bar{\mathcal{B}}_{\mathbb{R}^N}(10|\log\delta|)) \times \mathcal{B}_{H^1\times L^2}(\delta) \to \mathbb{R}^2, \quad (L, \vec{\phi}) \mapsto H(L, \vec{\phi})$$

such that

$$|H(L, \vec{\phi})| < C \left(e^{-\frac{L}{2}} + \|\vec{\phi}\|_{H^1\times L^2} \right),$$

with the following property. Given any $L, \vec{\phi}, h_1, h_2$ such that

$$|L| > 10|\log\delta|, \quad \|\vec{\phi}\|_{H^1\times L^2} < \delta, \quad |h_1| + |h_2| < \delta,$$

the solution $\vec{u}$ of (6.1) with initial data

$$\vec{u}(0) = \left(\vec{Q} + h_1 \vec{Y}^+\right)\left(\cdot - \frac{L}{2}\right) - \left(\vec{Q} + h_2 \vec{Y}^+\right)\left(\cdot + \frac{L}{2}\right) + \vec{\phi}$$

is a 2-solitary wave if and only if $(h_1, h_2) = H(L, \vec{\phi})$.

This result essentially means that locally around the sum of two sufficiently separated solitons with opposite signs, the initial data of 2-solitary waves form a codimension-2 Lipschitz manifold (the unstable directions being directed by $\vec{Y}^+$ translated around each soliton).

We refer to §6.2.4 for a formal discussion on the dynamics of 2-solitary waves of (6.1) justifying the main results of Theorems 6.3, 6.4 and 6.5.

Remark 6.6. We discuss the condition on p in (6.2). The energy sub-criticality condition is necessary for the existence of solitary waves and allows to work in the framework of finite energy solutions. The condition $p > 2$ could be waived for some of the above results, but it would complicate the analysis and weaken the results. Keeping in mind that the most relevant case is $p = 3$, we will not pursue further here the question of lowering p.

6.1.3 Previous results

The question of the long time asymptotic behavior of solutions of the damped Klein-Gordon equation in relation with the bound states was addressed in several articles; see *e.g.* [12, 52, 53, 63, 82]. Notably, under some conditions on N and p, results in [53, 82] state that for any sequence of time, any global bounded solution of (6.1) converges to a sum of decoupled bound states after extraction of a subsequence of times. (Note that such result would allow us to weaken the definition of 2-solitary wave given in Definition 6.1; however, we have preferred a stronger definition valid in any case.) In [12], for radial solutions in dimension $N \geq 2$, the convergence of any global solution to one equilibrium is proved to hold for the whole sequence of time. As discussed in [12], such results are closely related to the general soliton resolution conjecture for global bounded solutions of dispersive problems; see [46, 43] for details and results related to this conjecture for the undamped energy critical wave equation.

The existence and properties of multi-solitary waves is a classical question for integrable models (see for instance [97] for the Korteweg-de Vries equation and [116] for the 1D cubic Schrödinger equation). As mentioned above, [53] gave the first construction of such solutions for (6.1). Since then the same question has been addressed for various non-integrable and undamped nonlinear dispersive and wave equations. We refer to [20, 28, 31, 85] for the generalized Korteweg-de Vries equation, the nonlinear Schrödinger equation, the Klein-Gordon equation and the wave equation, in situations where unstable ground states are involved. See also references therein for previous works related to stable ground states. In those works, the distance between two traveling waves is asymptotic to Ct for $C > 0$, as $t \to \infty$. The more delicate case of multi-solitary waves with logarithmic distance is treated in [65, 90, 109, 110] for Korteweg-de Vries and Schrödinger type equations and systems, both in stable and unstable cases. Note that the logarithmic distance in the latter works is non-generic while it is the universal behavior for the damped equation (6.1). See also [64, 67, 68] for works on the non-existence, existence and classification of radial two-bubble solutions for the energy critical wave equation in large dimensions.

The construction of (center-) stable manifolds in the neighborhood of unstable ground state was addressed in several situations, see *e.g.* [4, 74, 76, 75, 87, 100].

While the initial motivation and several technical tools originate from some of the above mentioned papers, we point out that the present article is self-contained except for the local Cauchy theory for (6.1) (see [12]) and elliptic theory for (6.4) and its linearization (we refer to [7, 31, 77]).

This paper is organized as follows. Section 6.2 introduces all the technical tools involved in a dynamical approach to the 2-solitary wave problem for (6.1): computation of the nonlinear interaction, modulation, parameter estimates and energy estimates. Theorems 6.3 and 6.4 are proved in Section 6.3. Finally, Theorem 6.5 is proved in Section 6.4.

6.1.4 Recollection on the ground state

The ground state Q rewrites $Q(x) = q(|x|)$ where $q > 0$ satisfies

$$q'' + \frac{N-1}{r} q' - q + q^p = 0, \quad q'(0) = 0, \quad \lim_{r\to\infty} q(r) = 0. \tag{6.10}$$

It is well-known and easily checked that for a constant $\kappa > 0$, for all $r > 1$,

$$\left| q(r) - \kappa r^{-\frac{N-1}{2}} e^{-r} \right| + \left| q'(r) + \kappa r^{-\frac{N-1}{2}} e^{-r} \right| \lesssim r^{-\frac{N+1}{2}} e^{-r}. \tag{6.11}$$

Due to the radial symmetry, there hold the following cancellation (which we will use repetitively):

$$\forall i \neq j, \quad \int \partial_{x_i} Q(x) \partial_{x_j} Q(x) \, \mathrm{d}x = 0. \tag{6.12}$$

Let

$$\mathcal{L} = -\Delta + 1 - pQ^{p-1}, \quad \langle \mathcal{L}\varepsilon, \varepsilon \rangle = \int \left\{ |\nabla \varepsilon|^2 + \varepsilon^2 - pQ^{p-1}\varepsilon^2 \right\} \mathrm{d}x.$$

We recall standard properties of the operator $\mathcal{L}$ (see *e.g.* [31, Lemma 1]).

Lemma 6.7. (i) Spectral properties. *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1,\infty)$, its kernel is* $\operatorname{span}\{\partial_{x_j}Q : j = 1,\dots,N\}$ *and it has a unique negative eigenvalue $-\nu_0^2$, with corresponding smooth normalized radial eigenfunction Y ($\|Y\|_{L^2} = 1$) Moreover, on $\mathbb{R}^N$,*

$$\left|\partial_x^\beta Y(x)\right| \lesssim e^{-\sqrt{1+\nu_0^2}|x|} \quad \textit{for any } \beta = (\beta_1,\dots,\beta_N) \in \mathbb{N}^N.$$

(ii) Coercivity property. *There exists $c > 0$ such that, for all $\varepsilon \in H^1$,*

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geqslant c\|\varepsilon\|_{H^1}^2 - c^{-1}\left(\langle \varepsilon, Y\rangle^2 + \sum_{j=1}^N \langle \varepsilon, \partial_{x_j}Q\rangle^2\right).$$

6.2 Dynamics of two solitary waves

We prove in this section a general decomposition result close to the sum of two decoupled solitary waves. Let any $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$ and denote $\sigma = \sigma_1\sigma_2$. Consider time dependent $\mathcal{C}^1$ parameters $(z_1, z_2, \ell_1, \ell_2) \in \mathbb{R}^{4N}$ with $|\ell_1| \ll 1$, $|\ell_2| \ll 1$ and $|z| \gg 1$ where

$$z = z_1 - z_2 \quad \text{and} \quad \ell = \ell_1 - \ell_2.$$

Define the modulated ground state solitary waves, for $k = 1, 2$,

$$Q_k = \sigma_k Q(\cdot - z_k) \quad \text{and} \quad \vec{Q}_k = \begin{pmatrix} Q_k \\ -(\ell_k \cdot \nabla)Q_k \end{pmatrix}. \tag{6.13}$$

Set

$$R = Q_1 + Q_2, \qquad \vec{R} = \vec{Q}_1 + \vec{Q}_2,$$

and the nonlinear interaction term

$$G = f(Q_1 + Q_2) - f(Q_1) - f(Q_2). \tag{6.14}$$

The following functions are related to the exponential instabilities around each solitary wave:

$$Y_k = \sigma_k Y(\cdot - z_k), \quad \vec{Y}_k^\pm = \sigma_k \vec{Y}^\pm(\cdot - z_k), \quad \vec{Z}_k^\pm = \sigma_k \vec{Z}^\pm(\cdot - z_k).$$

6.2.1 Nonlinear interactions

A key of the understanding of the dynamics of 2-solitary waves is the computation of the first order of the projections of the nonlinear interaction term G on the directions ∇Q_1 and ∇Q_2 (see *e.g.* [110, Lemma 7]).

Lemma 6.8. *The following estimates hold for $|z| \gg 1$.*

(i) Bounds. *For any $0 < m' < m$,*

$$\int |Q_1 Q_2|^m \lesssim e^{-m'|z|}, \tag{6.15}$$

$$\int |F(R) - F(Q_1) - F(Q_2) - f(Q_1)Q_2 - f(Q_2)Q_1| \lesssim e^{-\frac{5}{4}|z|}. \tag{6.16}$$

(ii) Sharp bounds. *For any* $m > 0$,

$$\int |Q_1||Q_2|^{1+m} \lesssim q(|z|), \tag{6.17}$$

$$\|G\|_{L^2} \lesssim \|Q_1^{p-1}Q_2\|_{L^2} + \|Q_1Q_2^{p-1}\|_{L^2} \lesssim q(|z|). \tag{6.18}$$

(iii) Asymptotics. *It holds*

$$|\langle f(Q_2), Q_1\rangle - \sigma c_1 g_0 q(|z|)| \leq |z|^{-1}q(|z|) \tag{6.19}$$

where

$$g_0 = \frac{1}{c_1}\int Q^p(x)e^{-x_1}\,\mathrm{d}x > 0, \quad c_1 = \|\partial_{x_1}Q\|_{L^2}^2. \tag{6.20}$$

(iv) Sharp asymptotics. *There exists a smooth function* $g : [0,\infty) \to \mathbb{R}$ *such that, for any* $0 < \theta < \min(p-1, 2)$ *and* $r > 1$

$$|g(r) - g_0 q(r)| \lesssim r^{-1}q(r) \tag{6.21}$$

and

$$\left|\langle G, \nabla Q_1\rangle - \sigma c_1 \frac{z}{|z|}g(|z|)\right| \lesssim e^{-\theta|z|}, \tag{6.22}$$

$$\left|\langle G, \nabla Q_2\rangle + \sigma c_1 \frac{z}{|z|}g(|z|)\right| \lesssim e^{-\theta|z|}. \tag{6.23}$$

Proof. (i) By (6.11), $|Q(y)| \lesssim e^{-|y|}$, and thus

$$\int |Q_1Q_2|^m\,\mathrm{d}y \lesssim \int e^{-m|y|}e^{-m'|y+z|}\,\mathrm{d}y \lesssim e^{-m'|z|}\int e^{-(m-m')|y|}\,\mathrm{d}y \lesssim e^{-m'|z|}.$$

Then, we observe that using $p \geq 2$, by Taylor expansion:

$$|F(Q_1+Q_2) - F(Q_1) - F(Q_2) - f(Q_1)Q_2 - f(Q_2)Q_1| \lesssim |Q_1Q_2|^{\frac{3}{2}},$$

which reduces the proof of (6.16) to applying (6.15) with $m = \frac{3}{2}$ and $m' = \frac{5}{4}$.

(ii) We estimate

$$\begin{aligned}\int |Q_1||Q_2|^{1+m}\,\mathrm{d}x &= \int Q(y-z)Q^{1+m}(y)\,\mathrm{d}y \\ &\lesssim q(|z|)\int_{|y|<\frac{3}{4}|z|} e^{|y|}Q^{1+m}(y)\,\mathrm{d}y + e^{-|z|}\int_{|y|>\frac{3}{4}|z|} e^{|y|}Q^{1+m}(y)\,\mathrm{d}y \lesssim q(|z|).\end{aligned}$$

Next, using

$$|G| \lesssim |Q_1|^{p-1}|Q_2| + |Q_1||Q_2|^{p-1},$$

and (using $p > 2$)

$$\begin{aligned}\int Q_1^2|Q_2|^{2(p-1)}\,\mathrm{d}x &= \int Q^2(y-z)Q^{2(p-1)}(y)\,\mathrm{d}y \\ &\lesssim [q(|z|)]^2\int_{|y|<\frac{3}{4}|z|} e^{2|y|}Q^{2(p-1)}(y)\,\mathrm{d}y + e^{-2|z|}\int_{|y|>\frac{3}{4}|z|} e^{2|y|}Q^{2(p-1)}(y)\,\mathrm{d}y \\ &\lesssim [q(|z|)]^2.\end{aligned}$$

(iii) We claim the following estimate

$$\left|\int Q^p(y)Q(y+z)\,\mathrm{d}y - c_1 g_0 \kappa |z|^{-\frac{N-1}{2}}e^{-|z|}\right| \lesssim |z|^{-1}q(|z|). \tag{6.24}$$

Observe that (6.19) follows directly from (6.24) and (6.11).

Proof of (6.24). First, for $|y| < \frac{3}{4}|z|$ (and so $|y+z| \geq |z| - |y| \geq \frac{1}{4}|z| \gg 1$), we have using (6.11),

$$\left|Q(y+z) - \kappa|y+z|^{-\frac{N-1}{2}} e^{-|y+z|}\right| \lesssim |y+z|^{-\frac{N+1}{2}} e^{-|y+z|} \lesssim |z|^{-\frac{N+1}{2}} e^{-|z|} e^{|y|}.$$

In particular,

$$\left|\int_{|y|<\frac{3}{4}|z|} Q^p(y)\Big[Q(y+z) - \kappa|y+z|^{-\frac{N-1}{2}} e^{-|y+z|}\Big]\,\mathrm{d}y\right| \lesssim |z|^{-\frac{N+1}{2}} e^{-|z|}. \tag{6.25}$$

Moreover, for $|y| < \frac{3}{4}|z|$, we have the expansions

$$\left||y+z|^{-\frac{N-1}{2}} - |z|^{-\frac{N-1}{2}}\right| \lesssim |z|^{-\frac{N+1}{2}}|y|,$$
$$\left||y+z| - |z| - \frac{y\cdot z}{|z|}\right| \lesssim |z|^{-1}|y|^2,$$

and so

$$\left||y+z|^{-\frac{N-1}{2}} e^{-|y+z|} - |z|^{-\frac{N-1}{2}} e^{-|z|-\frac{y\cdot z}{|z|}}\right| \lesssim |z|^{-\frac{N+1}{2}} e^{-|z|}(1+|y|^2)e^{|y|}.$$

Inserted into (6.25), this yields

$$\left|\int_{|y|<\frac{3}{4}|z|} Q^p(y)\Big[Q(y+z) - \kappa|z|^{-\frac{N-1}{2}} e^{-|z|-\frac{y\cdot z}{|z|}}\Big]\,\mathrm{d}y\right| \lesssim |z|^{-\frac{N+1}{2}} e^{-|z|}.$$

Next, using (6.11), we observe

$$\int_{|y|>\frac{3}{4}|z|} Q^p(y)Q(y+z)\,\mathrm{d}y \lesssim e^{-\frac{3}{4}p|z|},$$

and

$$\int_{|y|>\frac{3}{4}|z|} Q^p(y)e^{-|z|-\frac{y\cdot z}{|z|}}\,\mathrm{d}y \lesssim e^{-|z|}\int_{|y|>\frac{3}{4}|z|} Q^p(y)e^{|y|}\,\mathrm{d}y \lesssim e^{-\frac{3}{4}p|z|}.$$

Gathering these estimates, we have proved

$$\left|\int Q^p(y)Q(y+z)\,\mathrm{d}y - \kappa|z|^{-\frac{N-1}{2}} e^{-|z|}\int Q^p(y)e^{-\frac{y\cdot z}{|z|}}\,\mathrm{d}y\right| \lesssim |z|^{-\frac{N+1}{2}} e^{-|z|}.$$

Last, the identity $\int Q^p(y)e^{-\frac{y\cdot z}{|z|}}\,\mathrm{d}y = \int Q^p(y)e^{-y_1}\,\mathrm{d}y$ (recall that Q is radially symmetric) and the definition of g_0 imply (6.24).

(iv) First, using the Taylor formula, it holds

$$|G - p|Q_1|^{p-1}Q_2| \lesssim |Q_1|^{p-2}Q_2^2 + |Q_2|^{p-1}|Q_1|.$$

Thus, using (6.15), we obtain for any $1 < \theta < \min(p-1, 2)$,

$$|\langle G, \nabla Q_1\rangle - \sigma_1\sigma_2 H(z)| \lesssim \int Q^2(y)Q^{p-1}(y+z)\,\mathrm{d}y \lesssim e^{-\theta|z|} \tag{6.26}$$

where we set $H(z) = \int \nabla(Q^p)(y)Q(y+z)\,\mathrm{d}y$. Second, we claim that there exists a function $g : [0,\infty) \to \mathbb{R}$ such that $H(z) = c_1\frac{z}{|z|}g(|z|)$. Indeed, remark by using the change of variable $y = 2x_1\mathbf{e}_1 - x$ that

$$\begin{aligned} H(r\mathbf{e}_1) &= p\int \frac{y}{|y|}q'(|y|)q^{p-1}(|y|)q(|y+r\mathbf{e}_1|)\,\mathrm{d}y \\ &= p\int \frac{y_1\mathbf{e}_1}{|y|}q'(|y|)q^{p-1}(|y|)q(|y+r\mathbf{e}_1|)\,\mathrm{d}y. \end{aligned}$$

Thus, we set

$$g(r) = \frac{\mathbf{e}_1 \cdot H(r\mathbf{e}_1)}{c_1} \quad \text{so that} \quad H(r\mathbf{e}_1) = c_1 \mathbf{e}_1 g(r).$$

Let $\omega \in \mathcal{S}_{\mathbb{R}^N}(1)$ be such that $z = |z|\omega$ and let U be an orthogonal matrix of size N such that $U\mathbf{e}_1 = \omega$. Then, using the change of variable $y = Ux$,

$$\begin{aligned} H(z) &= p \int \frac{y}{|y|} q'(|y|) q^{p-1}(|y|) q(|y+z|) \, \mathrm{d}y \\ &= p \int \frac{Ux}{|x|} q'(|x|) q^{p-1}(|x|) q(|x + |z|\mathbf{e}_1|) \, \mathrm{d}x = UH(|z|\mathbf{e}_1) = c_1 \frac{z}{|z|} g(z). \end{aligned}$$

Together with (6.26), this proves (6.22). The proof of (6.23) is the same but it is important to notice the change of sign due to $H(-z) = -H(z)$.

Last, we observe that proceeding as in the proof of (6.24), it holds

$$\left| \mathbf{e}_1 \cdot H(r\mathbf{e}_1) - \kappa r^{-\frac{N-1}{2}} e^{-r} \int \partial_{x_1}(Q^p)(y) e^{-y_1} \, \mathrm{d}y \right| \lesssim r^{-\frac{N+1}{2}} e^{-r}.$$

Moreover, by integration by parts, $\int \partial_{x_1}(Q^p)(y) e^{-y_1} \, \mathrm{d}y = \int Q^p(y) e^{-y_1} \, \mathrm{d}y$, which proves (6.21). □

6.2.2 Decomposition around the sum of two solitary waves

The following quantity measures the proximity of a function $\vec{u} = (u, v)$ to the sum of two distant solitary waves, for $\gamma > 0$,

$$d(\vec{u}; \gamma) = \inf_{|\xi_1 - \xi_2| > |\log \gamma|} \Big\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k) \Big\|_{H^1} + \|v\|_{L^2}.$$

We state a decomposition result for solutions of (6.1).

Lemma 6.9. *There exists $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$, $T_1 \leq T_2$, and any solution $\vec{u} = (u, \partial_t u)$ of (6.1) on $[T_1, T_2]$ satisfying*

$$\sup_{t \in [T_1, T_2]} d(\vec{u}(t); \gamma) < \gamma, \tag{6.27}$$

there exist unique $\mathcal{C}^1$ functions

$$t \in [T_1, T_2] \mapsto (z_1, z_2, \ell_1, \ell_2)(t) \in \mathbb{R}^{4N},$$

such that the solution $\vec{u}$ decomposes on $[T_1, T_2]$ as

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \vec{Q}_1 + \vec{Q}_2 + \vec{\varepsilon}, \quad \vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \tag{6.28}$$

with the following properties on $[T_1, T_2]$.

(i) Orthogonality and smallness. *For any $k = 1, 2$, $j = 1, \dots, N$,*

$$\langle \varepsilon, \partial_{x_j} Q_k \rangle = \langle \eta, \partial_{x_j} Q_k \rangle = 0 \tag{6.29}$$

and

$$\|\vec{\varepsilon}\|_{H^1 \times L^2} + \sum_{k=1,2} |\ell_k| + e^{-2|z|} \lesssim \gamma. \tag{6.30}$$

(ii) Equation of $\vec{\varepsilon}$.

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_\varepsilon \\ \partial_t \eta = \Delta\varepsilon - \varepsilon + f(R+\varepsilon) - f(R) - 2\alpha\eta + \mathrm{Mod}_\eta + G \end{cases} \tag{6.31}$$

where

$$\mathrm{Mod}_\varepsilon = \sum_{k=1,2} (\dot{z}_k - \ell_k) \cdot \nabla Q_k,$$
$$\mathrm{Mod}_\eta = \sum_{k=1,2} \left(\dot{\ell}_k + 2\alpha\ell_k\right) \cdot \nabla Q_k - \sum_{k=1,2} (\ell_k \cdot \nabla)(\dot{z}_k \cdot \nabla) Q_k.$$

(iii) Equations of the geometric parameters. *For* $k = 1, 2$,

$$|\dot{z}_k - \ell_k| \lesssim \|\vec{\varepsilon}\|^2_{H^1 \times L^2} + \sum_{k=1,2} |\ell_k|^2, \tag{6.32}$$

$$|\dot{\ell}_k + 2\alpha\ell_k| \lesssim \|\vec{\varepsilon}\|^2_{H^1 \times L^2} + \sum_{k=1,2} |\ell_k|^2 + q(|z|). \tag{6.33}$$

(iv) Refined equation for ℓ_k. *For any* $1 < \theta < \min(p-1, 2)$, $k = 1, 2$,

$$\left|\dot{\ell}_k + 2\alpha\ell_k - (-1)^k \sigma \frac{z}{|z|} g(|z|)\right| \lesssim \|\vec{\varepsilon}\|^2_{H^1 \times L^2} + \sum_{k=1,2} |\ell_k|^2 + e^{-\theta|z|}. \tag{6.34}$$

(v) Equations of the exponential directions. *Let*

$$a_k^\pm = \langle \vec{\varepsilon}, \vec{Z}_k^\pm \rangle. \tag{6.35}$$

Then,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} a_k^\pm - \nu^\pm a_k^\pm\right| \lesssim \|\vec{\varepsilon}\|^2_{H^1 \times L^2} + \sum_{k=1,2} |\ell_k|^2 + q(|z|). \tag{6.36}$$

Remark 6.10. We see on estimate (6.33) the damping of the Lorentz parameters ℓ_k. The more precise estimate (6.34) involves the nonlinear interactions which becomes preponderant for large time.

Proof. Proof of (i). The existence and uniqueness of the geometric parameters is proved for fixed time. Let $0 < \gamma \ll 1$. First, for any $u \in H^1$ such that

$$\inf_{|\xi_1 - \xi_2| > |\log\gamma|} \left\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k) \right\|_{H^1} \leq \gamma, \tag{6.37}$$

we consider $z_1(u)$ and $z_2(u)$ achieving the infimum

$$\left\| u - \sum_{k=1,2} \sigma_k Q(\cdot - z_k(u)) \right\|_{L^2} = \inf_{|\xi_1 - \xi_2| > \frac{3}{4}|\log\gamma|} \left\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k) \right\|_{L^2}.$$

Let

$$\varepsilon(x) = u(x) - \sigma_1 Q(x - z_1(u)) - \sigma_2 Q(x - z_2(u)), \quad \|\varepsilon\|_{L^2} \leq \gamma.$$

By standard arguments since $|z_1(u) - z_2(u)| > \frac{3}{4}|\log\gamma|$, it holds

$$\langle \varepsilon, \partial_{x_j} Q(\cdot - z_1(u)) \rangle = \langle \varepsilon, \partial_{x_j} Q(\cdot - z_2(u)) \rangle = 0. \tag{6.38}$$

For u and $\tilde{u}$ as in (6.37), we compare the corresponding z_k, $\tilde{z}_k$ and ε, $\tilde{\varepsilon}$. First, for ζ, $\tilde{\zeta} \in \mathbb{R}^N$, setting $\check{\zeta} = \zeta - \tilde{\zeta}$, we observe the following estimates

$$Q(\cdot - \zeta) - Q(\cdot - \tilde{\zeta}) = -(\check{\zeta} \cdot \nabla)Q(\cdot - \zeta) + O_{H^1}(|\check{\zeta}|^2), \tag{6.39}$$

$$\nabla Q(\cdot - \zeta) - \nabla Q(\cdot - \tilde{\zeta}) = -(\check{\zeta} \cdot \nabla^2)Q(\cdot - \zeta) + O_{H^1}(|\check{\zeta}|^2). \tag{6.40}$$

Thus, denoting $\check{u} = u - \tilde{u}$, $\check{z}_k = z_k - \tilde{z}_k$, $\check{\varepsilon} = \varepsilon - \tilde{\varepsilon}$, we obtain

$$\check{u} = \sum_{k=1,2} \sigma_k(\check{z}_k \cdot \nabla)Q(\cdot - z_k) + \check{\varepsilon} + O_{H^1}(|\check{z}_1|^2 + |\check{z}_2|^2).$$

(In the O_{H^1}, there is no dependence in $\check{u}$ or $\check{\varepsilon}$). Projecting on each $\nabla Q(\cdot - z_k)$, using (6.38) and the above estimates, we obtain

$$|\check{z}_1| + |\check{z}_2| \lesssim \|\check{u}\|_{L^2} + (|\check{z}_1| + |\check{z}_2|)(e^{-\frac{1}{2}|z|} + \|\check{\varepsilon}\|_{L^2} + |\check{z}_1| + |\check{z}_2|)$$

and thus, for γ, $\check{z}_1$ and $\check{z}_2$ small,

$$\|\check{\varepsilon}\|_{H^1} \lesssim \|\check{u}\|_{H^1}, \quad |\check{z}_1| + |\check{z}_2| \lesssim \|\check{u}\|_{L^2}. \tag{6.41}$$

Therefore, for γ small enough, this proves uniqueness and Lipschitz continuity of z_1 and z_2 with respect to u in L^2.

Now, let $v \in L^2$ and z_1, z_2 be such that $\|v\|_{L^2} < \gamma$ and $|z_1 - z_2| > \frac{3}{4}|\log\gamma|$. Set

$$\eta(x) = v(x) + \sigma_1(\ell_1 \cdot \nabla)Q(x - z_1) + \sigma_2(\ell_2 \cdot \nabla)Q(x - z_2).$$

Then, it is easy to check that the $2N$ conditions

$$\langle \eta, \partial_{x_j} Q(\cdot - z_k) \rangle = 0 \tag{6.42}$$

for $j = 1, \dots, N$, $k = 1, 2$, are equivalent to a linear system in the components of ℓ_1 and ℓ_2 whose matrix a perturbation of the identity (for γ small) up to a multiplicative constant. In particular, it is invertible and the existence and uniqueness of parameters $\ell_1(v, z_1, z_2), \ell_2(v, z_1, z_2) \in \mathbb{R}^N$ satisfying (6.42) and $|\ell_1| + |\ell_2| \lesssim \|v\|_{L^2}$ is clear. Moroever, with similar notation as before, it holds

$$\|\check{\eta}\|_{L^2} + |\check{\ell}_1| + |\check{\ell}_2| \lesssim \|\check{v}\|_{L^2} + |\check{z}_1| + |\check{z}_2|. \tag{6.43}$$

Estimate (6.30) is now proved. In the rest of this proof, we formally derive the equations of $\vec{\varepsilon}$ and the geometric parameters from the equation of u. This derivation can be justified rigorously and used to prove by the Cauchy-Lipschitz theorem that the parameters are $\mathcal{C}^1$ functions of time (see for instance [21, Proof of Lemma 2.7]).

Proof of (ii). First, by the definition of ε and η,

$$\partial_t \varepsilon = \partial_t u - \sum_{k=1,2} \partial_t Q_k = \eta + \sum_{k=1,2} (\dot{z}_k - \ell_k) \cdot \nabla Q_k.$$

Second,

$$\begin{aligned} \partial_t \eta &= \partial_{tt} u + \sum_{k=1,2} \partial_t \left(\ell_k \cdot \nabla Q_k\right) \\ &= \Delta u - u + f(u) - 2\alpha \partial_t u + \sum_{k=1,2} \dot{\ell}_k \cdot \nabla Q_k - \sum_{k=1,2} \left(\ell_k \cdot \nabla\right)\left(\dot{z}_k \cdot \nabla\right) Q_k. \end{aligned}$$

By (6.28), $\Delta Q_k - Q_k + f(Q_k) = 0$ (from (6.4)) and the definition of G,

$$\begin{aligned}\Delta u - u + f(u) - 2\alpha\partial_t u = \Delta\varepsilon - \varepsilon + f(R+\varepsilon) - f(R) - 2\alpha\eta \\ + 2\alpha \sum_{k=1,2} \ell_k \cdot \nabla Q_k + G.\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_t \eta = \Delta\varepsilon - \varepsilon + f(R+\varepsilon) - f(R) - 2\alpha\eta \\ + \sum_{k=1,2} (\dot{\ell}_k + 2\alpha\ell_k) \cdot \nabla Q_k - \sum_{k=1,2} (\ell_k \cdot \nabla)(\dot{z}_k \cdot \nabla) Q_k + G.\end{aligned}$$

Proof of (iii)-(iv). We derive (6.32) from (6.29). For any $j = 1, \dots, N$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \varepsilon, \partial_{x_j} Q_1\rangle = \langle \partial_t \varepsilon, \partial_{x_j} Q_1\rangle + \langle \varepsilon, \partial_t(\partial_{x_j} Q_1)\rangle.$$

Thus (6.31) gives

$$\langle \eta, \partial_{x_j} Q_1\rangle + \langle \mathrm{Mod}_\varepsilon, \partial_{x_j} Q_1\rangle - \langle \varepsilon, \dot{z}_1 \cdot \nabla \partial_{x_j} Q_1\rangle = 0.$$

The first term is zero due to the orthogonality (6.29). Hence,

$$|\dot{z}_{1,j} - \ell_{1,j}| \|\partial_{x_j} Q\|_{L^2}^2 \lesssim |\dot{z}_2 - \ell_2| \int |\nabla Q_2(x)||\nabla Q_1(x)| \,\mathrm{d}x + |\dot{z}_1| \, \|\varepsilon\|_{L^2}.$$

Thus, also using (6.15) with $m = 1$ and $m' = \frac{1}{2}$, we obtain

$$|\dot{z}_1 - \ell_1| \lesssim |\dot{z}_2 - \ell_2| e^{-\frac{1}{2}|z|} + |\dot{z}_1 - \ell_1| \, \|\vec{\varepsilon}\|_{H^1 \times L^2} + |\ell_1| \, \|\vec{\varepsilon}\|_{H^1 \times L^2}.$$

Since $\|\vec{\varepsilon}\|_{H^1 \times L^2} \lesssim \gamma$, this yields

$$|\dot{z}_1 - \ell_1| \lesssim |\dot{z}_2 - \ell_2| e^{-\frac{1}{2}|z|} + |\ell_1| \, \|\vec{\varepsilon}\|_{H^1 \times L^2}.$$

Similarly, it holds

$$|\dot{z}_2 - \ell_2| \lesssim |\dot{z}_1 - \ell_1| e^{-\frac{1}{2}|z|} + |\ell_2| \, \|\vec{\varepsilon}\|_{H^1 \times L^2},$$

and thus, for large $|z|$,

$$\sum_{k=1,2} |\dot{z}_k - \ell_k| \lesssim (|\ell_1| + |\ell_2|) \, \|\vec{\varepsilon}\|_{H^1 \times L^2},$$

which implies (6.32).

Next, we derive (6.33)-(6.34). From (6.29), it holds

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, \partial_{x_j} Q_1\rangle = \langle \partial_t \eta, \partial_{x_j} Q_1\rangle + \langle \eta, \partial_t(\partial_{x_j} Q_1)\rangle.$$

Thus, by (6.29) and (6.31), we have

$$\begin{aligned}0 = \langle \Delta\varepsilon - \varepsilon + f'(Q_1)\varepsilon, \partial_{x_j} Q_1\rangle + \langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_{x_j} Q_1\rangle \\ + \langle (f'(R) - f'(Q_1))\varepsilon, \partial_{x_j} Q_1\rangle \\ + \langle \mathrm{Mod}_\eta, \partial_{x_j} Q_1\rangle + \langle G, \partial_{x_j} Q_1\rangle - \langle \eta, (\dot{z}_1 \cdot \nabla)\partial_{x_j} Q_1\rangle.\end{aligned}$$

Since $\partial_{x_j} Q_1$ satisfies $\Delta \partial_{x_j} Q_1 - \partial_{x_j} Q_1 + f'(Q_1)\partial_{x_j} Q_1 = 0$, the first term is zero. Next, by Taylor expansion (as f is $\mathcal{C}^2$), we have

$$f(R+\varepsilon) - f(R) - f'(R)\varepsilon = \varepsilon^2 \int_0^1 (1-\theta) f''(R + \theta\varepsilon) \,\mathrm{d}\theta,$$

and by the H^1 sub-criticality of the exponent $p > 2$, we infer

$$\left|\langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_{x_j} Q_1\rangle\right| \lesssim \|\varepsilon\|_{H^1}^2. \tag{6.44}$$

Then, again by Taylor expansion and $p > 2$,

$$|f'(R) - f'(Q_1)||\partial_{x_j} Q_1| \lesssim |Q_2||Q_1|^{p-1} + |Q_1||Q_2|^{p-1}.$$

Thus, using also the Cauchy-Schwarz inequality and (6.18),

$$|\langle (f'(R) - f'(Q_1))\varepsilon, \partial_{x_j} Q_1\rangle| \lesssim \|\varepsilon\|_{L^2}^2 + [q(|z|)]^2. \tag{6.45}$$

Direct computations show that

$$\begin{aligned}\langle \mathrm{Mod}_\eta, \partial_{x_j} Q_1\rangle &= \big(\dot\ell_{1,j} + 2\alpha\ell_{1,j}\big)\|\partial_{x_j} Q_1\|_{L^2}^2 + \langle (\dot\ell_2 + 2\alpha\ell_2)\cdot\nabla Q_2, \partial_{x_j} Q_1\rangle \\ &\quad - \sum_{k=1,2} \langle (\ell_k\cdot\nabla)(\dot z_k\cdot\nabla) Q_k, \partial_{x_j} Q_1\rangle.\end{aligned}$$

Thus, using Lemma 6.8 and (6.12), for any $m \in (0,1)$,

$$\begin{aligned}\langle \mathrm{Mod}_\eta, \partial_{x_j} Q_1\rangle &= \big(\dot\ell_{1,j} + 2\alpha\ell_{1,j}\big)\|\partial_{x_j} Q_1\|_{L^2}^2 \\ &\quad + O\left(\big|\dot\ell_2 + 2\alpha\ell_2\big| e^{-m|z|}\right) + O\left(|\ell_2||\dot z_2| e^{-m|z|}\right).\end{aligned}$$

Using also (6.32), it follows that

$$\begin{aligned}\langle \mathrm{Mod}_\eta, \partial_{x_j} Q_1\rangle &= \left(\dot\ell_{1,j} + 2\alpha\ell_{1,j}\right)\|\partial_{x_j} Q_1\|_{L^2}^2 \\ &\quad + O\left(\big|\dot\ell_2 + 2\alpha\ell_2\big| e^{-m|z|}\right) + O\Big(\sum_{k=1,2} |\ell_k|^2\Big) + O(\|\vec\varepsilon\|_{H^1\times L^2}^2).\end{aligned}$$

By (6.22) and the definition of c_1 in (6.20), we have, for any $1 < \theta < \min(p-1, 2)$,

$$\left|\frac{\langle G, \partial_{x_j} Q_1\rangle}{\|\partial_{x_1} Q\|_{L^2}^2} - \sigma\frac{z_j}{|z|} g(|z|)\right| \lesssim e^{-\theta|z|}.$$

Finally, by (6.32) again, we have

$$|\langle \eta, (\dot z_1\cdot\nabla)\partial_{x_j} Q_1\rangle| \lesssim (|\dot z_1 - \ell_1| + |\ell_1|)\|\vec\varepsilon\|_{H^1\times L^2} \lesssim \|\vec\varepsilon\|_{H^1\times L^2}^2 + \sum_{k=1,2} |\ell_k|^2.$$

Combining the above estimates, we have obtained

$$\left|\big(\dot\ell_1 + 2\alpha\ell_1\big) + \sigma\frac{z}{|z|} g(|z|)\right| \lesssim \big|\dot\ell_2 + 2\alpha\ell_2\big| e^{-m|z|} + \sum_{k=1,2} |\ell_k|^2 + \|\vec\varepsilon\|_{H^1\times L^2}^2 + e^{-\theta|z|}.$$

Similarly, from $(\eta, \partial_{x_j} Q_2) = 0$, we check

$$\left|\big(\dot\ell_2 + 2\alpha\ell_2\big) - \sigma\frac{z}{|z|} g(|z|)\right| \lesssim \big|\dot\ell_1 + 2\alpha\ell_1\big| e^{-m|z|} + \sum_{k=1,2} |\ell_k|^2 + \|\vec\varepsilon\|_{H^1\times L^2}^2 + e^{-\theta|z|}.$$

These estimates imply (6.34); (6.33) follows readily using (6.21).

Proof of (v). By (6.31), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} a_1^\pm &= \langle \partial_t\vec\varepsilon, \vec Z_1^\pm\rangle + \langle \vec\varepsilon, \partial_t \vec Z_1^\pm\rangle \\ &= (\zeta^\pm - 2\alpha)\langle \eta, Y_1\rangle + \langle \Delta\varepsilon - \varepsilon + f'(Q_1)\varepsilon, Y_1\rangle \\ &\quad + \langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, Y_1\rangle + \langle (f'(R) - f'(Q_1))\varepsilon, Y_1\rangle \\ &\quad + \langle G, Y_1\rangle + \zeta^\pm\langle \mathrm{Mod}_\varepsilon, Y_1\rangle + \langle \mathrm{Mod}_\eta, Y_1\rangle - \langle \vec\varepsilon, \dot z_1\cdot\nabla\vec Z_1^\pm\rangle.\end{aligned}$$

Using $\zeta^{\pm} - 2\alpha = \nu^{\pm}$ and (6.35), $\mathcal{L}Y = -\nu_0^2 Y$ and $\nu_0^2 = \nu^{\pm}\zeta^{\pm}$, we observe that

$$(\zeta^{\pm} - 2\alpha)\langle \eta, Y_1\rangle + \langle \Delta\varepsilon - \varepsilon + f'(Q_1)\varepsilon, Y_1\rangle = \nu^{\pm} a_1^{\pm}.$$

Using the decay properties of Y in Lemma 6.7 and proceeding as before for (6.44) and (6.45),

$$|\langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, Y_1\rangle| + |\langle (f'(R) - f'(Q_1))\varepsilon, Y_1\rangle| \lesssim \|\varepsilon\|_{L^2}^2 + e^{-\frac{3}{2}|z|}.$$

Next, by (6.18), $|\langle G, Y_1\rangle| \lesssim q(|z|)$. Last, by (6.32) and (6.33),

$$|\langle \mathrm{Mod}_{\varepsilon}, Y_1\rangle| + |\langle \mathrm{Mod}_{\eta}, Y_1\rangle| + |\langle \vec{\varepsilon}, \dot{z}_1 \cdot \nabla \vec{Z}_1^{\pm}\rangle| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 + q(|z|).$$

Gathering these estimates, and proceeding similarly for $a_2^{\pm}$, (6.36) is proved. □

6.2.3 Energy estimates

For $\mu > 0$ small to be chosen, we denote $\rho = 2\alpha - \mu$. Consider the nonlinear energy functional

$$\mathcal{E} = \int \left\{ |\nabla\varepsilon|^2 + (1-\rho\mu)\varepsilon^2 + (\eta + \mu\varepsilon)^2 - 2[F(R+\varepsilon) - F(R) - f(R)\varepsilon] \right\}. \tag{6.46}$$

Lemma 6.11. *There exists $\mu > 0$ such that in the context of Lemma 6.9, the following hold.*

(i) Coercivity and bound.

$$\mu\|\vec{\varepsilon}\|_{H^1\times L^2}^2 - \frac{1}{2\mu}\sum_{k=1,2}\left((a_k^+)^2 + (a_k^-)^2\right) \le \mathcal{E} \le \frac{1}{\mu}\|\vec{\varepsilon}\|_{H^1\times L^2}^2. \tag{6.47}$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leqslant -2\mu\mathcal{E} + \frac{1}{\mu}\|\vec{\varepsilon}\|_{H^1\times L^2}\left[\|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2 + q(|z|)\right]. \tag{6.48}$$

Remark 6.12. The above lemma is valid for any small enough $\mu > 0$. For future needs, we assume further

$$\mu \le \min(1, \alpha, |\nu_-|). \tag{6.49}$$

One checks that a usual linearized energy, corresponding to $\mu = 0$ in the definition of $\mathcal{E}$, would only gives damping for the component η. This is the reason why we introduce the modified energy $\mathcal{E}$. For the simplicity of notation, the same small constant $\mu > 0$ is used in (6.47) and (6.48) though in the former estimate the small constant is related to the coercivity constant c of Lemma 6.7, while in the latter it is related to the damping α.

Proof. Proof of (i). The upper bound on $\mathcal{E}$ in (6.47) easily follows from the energy subcriticality of p. The coercivity is proved for fixed time and so we omit the time dependency. By translation invariance, we assume without loss of generality that $z_1 = -z_2 = \frac{z}{2}$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the following properties

$$\chi = 1 \text{ on } [0,1], \quad \chi = 0 \text{ on } [2,+\infty), \quad \chi' \le 0 \text{ on } \mathbb{R}.$$

For $\lambda = \frac{|z|}{4} \gg 1$, let

$$\chi_1(x) = \chi\left(\frac{|x - z_1|}{\lambda}\right), \quad \chi_2(x) = (1 - \chi_1^2(x))^{\frac{1}{2}}.$$

We define $\varepsilon_k = \varepsilon(\cdot + z_k)\chi_k(\cdot + z_k)$ for $k = 1, 2$, so that

$$\int |\nabla \varepsilon_k|^2 = \int |\nabla \varepsilon|^2 \chi_k^2 - \int \varepsilon^2 (\Delta \chi_k)\chi_k,$$

and thus

$$\int |\nabla \varepsilon|^2 = \int |\nabla \varepsilon_1|^2 + |\nabla \varepsilon_2|^2 + O(\lambda^{-2}\|\varepsilon\|_{L^2}^2). \tag{6.50}$$

Next, using (6.29),

$$\langle \varepsilon_k, \partial_{x_j} Q\rangle = \sigma_k \langle \varepsilon\chi_k, \partial_{x_j} Q_k\rangle = \sigma_k \langle \varepsilon, (\chi_k - 1)\partial_{x_j} Q_k\rangle = O(e^{-\frac{|z|}{4}}\|\varepsilon\|_{L^2}),$$

and

$$\langle \varepsilon_k, Y\rangle = \sigma_k \langle \varepsilon\chi_k, Y_k\rangle = \sigma_k \langle \varepsilon, Y_k\rangle + O(e^{-\frac{|z|}{4}}\|\varepsilon\|_{L^2}).$$

Thus, applying (ii) of Lemma 6.7 to ε_k, one obtains

$$\langle \mathcal{L}\varepsilon_k, \varepsilon_k\rangle \geq c\|\varepsilon_k\|_{H^1}^2 - C\langle \varepsilon, Y_k\rangle^2 - C(e^{-\frac{|z|}{4}} + \lambda^{-2})\|\varepsilon\|_{L^2}^2.$$

Using (6.50), we obtain for z large

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geq \frac{c}{2}\|\varepsilon\|_{H^1}^2 - C\sum_{k=1,2}\langle \varepsilon, Y_k\rangle^2.$$

The estimate

$$\int \left\{|\nabla \varepsilon|^2 + \varepsilon^2 + \eta^2 - f'(R)\varepsilon^2\right\} \geq \mu\|\vec{\varepsilon}\|_{H^1\times L^2}^2 - C(a_k^+)^2 - C(a_k^-)^2$$

then follows from

$$\langle \varepsilon, Y_k\rangle = \frac{\langle \vec{\varepsilon}, \vec{Z}_k^+ - \vec{Z}_k^-\rangle}{\zeta^+ - \zeta^-} = \frac{a_k^+ - a_k^-}{\zeta^+ - \zeta^-}.$$

Using

$$|F(R+\varepsilon) - F(R) - f(R)\varepsilon - f'(R)\varepsilon| \lesssim |\varepsilon|^3 + |\varepsilon|^{p+1},$$

and $\int |\varepsilon|^3 + |\varepsilon|^{p+1} \lesssim \|\varepsilon\|_{H^1}^3 + \|\varepsilon\|_{H^1}^{p+1}$ by energy subcriticality of p, for γ and μ small, we have proved (6.47).

Proof of (ii). From direct computations and integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = &\int \partial_t\varepsilon\left[-\Delta\varepsilon + (1-\rho\mu)\varepsilon - f(R+\varepsilon) + f(R)\right] + (\partial_t\eta + \mu\partial_t\varepsilon)(\eta + \mu\varepsilon)\\ &+ \int \sum_{k=1,2}(\dot{z}_k\cdot\nabla Q_k)\left[f(R+\varepsilon) - f(R) - f'(R)\varepsilon\right] = \mathbf{g_1} + \mathbf{g_2}.\end{aligned}$$

Using (6.31), integration by parts and $2\alpha = \rho + \mu$, we compute

$$\begin{aligned}\mathbf{g_1} = &-\mu\int\left\{|\nabla\varepsilon|^2 + (1-\rho\mu)\varepsilon^2 - \varepsilon[f(R+\varepsilon) - f(R)]\right\} - \rho\int(\eta+\mu\varepsilon)^2\\ &+ \int \mathrm{Mod}_\varepsilon\left\{-\Delta\varepsilon + (1-\rho\mu)\varepsilon - \left[f(R+\varepsilon) - f(R)\right]\right\}\\ &+ \int(\eta+\mu\varepsilon)[\mathrm{Mod}_\eta + \mu\mathrm{Mod}_\varepsilon] + \int(\eta+\mu\varepsilon)G\\ = &\ \mathbf{g_{1,1}} + \mathbf{g_{1,2}} + \mathbf{g_{1,3}} + \mathbf{g_{1,4}}.\end{aligned}$$

Note that by $0 < \mu < \alpha$, one has $\rho - \mu = 2(\alpha - \mu) > 0$ and so

$$\begin{aligned}\mathbf{g_{1,1}} &= -\mu\mathcal{E} - 2\mu \int \left[F(R+\varepsilon) - F(R) - f(R)\varepsilon - \tfrac{1}{2}f'(R)\varepsilon^2\right] \\ &\quad + \mu \int \varepsilon[f(R+\varepsilon) - f(R) - f'(R)\varepsilon] - (\rho - \mu)\int (\eta + \mu\varepsilon)^2 \leqslant -\mu\mathcal{E} + C\|\varepsilon\|_{H^1}^3,\end{aligned}$$

where we have estimated, using $p > 2$, Hölder inequality, the sub-criticality of p and Sobolev embedding,

$$\begin{aligned}&\int \left|F(R+\varepsilon) - F(R) - f(R)\varepsilon - \tfrac{1}{2}f'(R)\varepsilon^2\right| \\ &\quad + \int |\varepsilon[f(R+\varepsilon) - f(R) - f'(R)\varepsilon]| \lesssim \int |\varepsilon|^3|R|^{p-2} + |\varepsilon|^{p+1} \lesssim \|\varepsilon\|_{H^1}^3.\end{aligned}$$

Using the Cauchy-Schwarz inequality and (6.32)-(6.33), we also derive the following estimates

$$|\mathbf{g_{1,2}}| \lesssim (|\dot z_1 - \ell_1| + |\dot z_2 - \ell_2|)\,\|\varepsilon\|_{H^1} \lesssim \|\vec\varepsilon\|_{H^1\times L^2}\left[\|\vec\varepsilon\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2\right],$$

and (also using the orthogonality conditions (6.29))

$$\begin{aligned}|\mathbf{g_{1,3}}| &= \left|\int (\eta + \mu\varepsilon)\Big(\sum_{k=1,2}(\ell_k\cdot\nabla)(\dot z_k\cdot\nabla)Q_k\Big)\right| \\ &\lesssim (|\ell_1| + |\ell_2|)(|\dot z_1 - \ell_1| + |\dot z_2 - \ell_2| + |\ell_1| + |\ell_2|)(\|\varepsilon\|_{L^2} + \|\eta\|_{L^2}) \\ &\lesssim \|\vec\varepsilon\|_{H^1\times L^2}\left[\|\vec\varepsilon\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|^2\right].\end{aligned}$$

Next, from (6.18) and the Cauchy-Schwarz inequality,

$$|\mathbf{g_{1,4}}| \lesssim \|G\|_{L^2}(\|\varepsilon\|_{L^2} + \|\eta\|_{L^2}) \lesssim q(|z|)\|\vec\varepsilon\|_{H^1\times L^2}.$$

Last, by (6.32), proceeding as before, we see that

$$\begin{aligned}|\mathbf{g_2}| &\lesssim (|\dot z_1| + |\dot z_2|)\|\varepsilon\|_{H^1}^2 \lesssim (|\dot z_1 - \ell_1| + |\dot z_2 - \ell_2| + |\ell_1| + |\ell_2|)\|\vec\varepsilon\|_{H^1\times L^2}^2 \\ &\lesssim \|\vec\varepsilon\|_{H^1\times L^2}^2\left[\|\vec\varepsilon\|_{H^1\times L^2}^2 + \sum_{k=1,2}|\ell_k|\right].\end{aligned}$$

Gathering the above estimates, (6.48) is proved, taking μ small enough. □

6.2.4 Trichotomy of evolution

Estimates (6.32), (6.33), (6.34), (6.36) and (6.48) give basic information on the evolution of the various components of a solution in the framework of the decomposition introduced in Lemma 6.9.

We introduce notation related to modified parameters that allow us to justify the following trichotomy in the evolution of the solution.

ODE behavior for $|z|$ The distance $z = z_1 - z_2$ formally satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(|z|)}\right] = -\frac{\sigma g_0}{\alpha}.$$

Note that for $\sigma = -1$, $\log t$ is an approximate solution of this ODE. This justifies the rigidity results in Theorem 6.4. In contrast, when $\sigma = 1$, there are no solution such that $|z| \to \infty$, which explains the non-existence result in Theorem 6.3.

Exponential growth The parameters a_k^+ are related to the forward exponential instability of the solitary wave. They will require a specific approach, involving backward in time arguments. The existence of exactly one direction of instability for each solitary wave justifies Theorem 6.5.

Damped evolution The parameters ℓ_k, a_k^- and the remainder $\vec{\varepsilon}$ (without its unstable components a_k^+) enjoy exponential damping; they will be easily estimated provided that the other parameters are locked, see Proposition 6.16.

First, we set

$$y = z + \frac{\ell}{2\alpha}, \quad r = |y|. \tag{6.51}$$

Second, we define

$$b = \sum_{k=1,2} (a_k^+)^2. \tag{6.52}$$

Third, we introduce notation for the damped components:

$$\mathcal{F} = \mathcal{E} + \mathcal{G}, \quad \mathcal{G} = \sum_{k=1,2} |\ell_k|^2 + \frac{1}{2\mu} \sum_{k=1,2} (a_k^-)^2, \tag{6.53}$$

and for all the components of the solution

$$\mathcal{N} = \left[\|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 \right]^{\frac{1}{2}}. \tag{6.54}$$

Last, we define

$$\mathcal{M} = \frac{1}{\mu^2} \left(\mathcal{F} - \frac{b}{2\nu^+} \right). \tag{6.55}$$

In the following lemma, we rewrite the estimates of Lemmas 6.9 and 6.11 in terms of these new parameters.

Lemma 6.13. *In the context of Lemma 6.9, the following hold.*

(i) Comparison with original variables.

$$\big|r - |z|\big| \leq |y - z| \lesssim \mathcal{N}, \quad |g(|z|) - g_0 q(r)| \lesssim q(r)(\mathcal{N} + r^{-1}), \tag{6.56}$$

$$\mu \mathcal{N}^2 \leq \mu \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\ell_k|^2 \leq \mathcal{F} + \frac{b}{2\mu} \lesssim \mathcal{N}^2. \tag{6.57}$$

(ii) ODE behavior for the distance. *For some $K > 0$,*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{q(r)} \right] + \frac{\sigma g_0}{\alpha} \right| \leq \frac{K}{q(r)} \left(\mathcal{N}^2 + r^{-1} q(r) \right). \tag{6.58}$$

(iii) Exponential instability.

$$|\dot{b} - 2\nu^+ b| \lesssim \mathcal{N}^3 + q(r)\mathcal{N}. \tag{6.59}$$

(iv) Damped components.

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F} + 2\mu \mathcal{F} \lesssim \mathcal{N}^3 + q(r)\mathcal{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{G} + 2\mu \mathcal{G} \lesssim \mathcal{N}^3 + q(r)\mathcal{N}. \tag{6.60}$$

(v) Liapunov type functional.

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M} \leq -\mathcal{N}^2 + C[q(r)]^2. \tag{6.61}$$

(vi) Refined estimates for the distance. *Setting*

$$R^+ = \frac{1}{q(r)} \exp(K\mathcal{M}) \quad \textit{and} \quad R^- = \frac{1}{q(r)} \exp(-K\mathcal{M}),$$

(K is given in (6.58)), it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} R^+ \leq \left(-\frac{\sigma g_0}{\alpha} + 2Kr^{-1}\right) \exp(K\mathcal{M}), \tag{6.62}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} R^- \geq \left(-\frac{\sigma g_0}{\alpha} - 2Kr^{-1}\right) \exp(-K\mathcal{M}). \tag{6.63}$$

Proof. Proof of (6.56). It follows from the triangle inequality that

$$\big|r - |z|\big| \leq |y - z| \leq \frac{|\ell|}{2\alpha} \lesssim \mathcal{N}.$$

The second part of (6.56) then follows from (6.21).

Proof of (6.57). It follows readily from (6.47).

Proof of (6.58). First, from (6.32), (6.34) and (6.56), we note (θ is as in (6.34))

$$\begin{aligned} \dot{y} = \dot{z} + \frac{\dot{\ell}}{2\alpha} &= -\frac{\sigma}{\alpha}\frac{z}{|z|} g(|z|) + O(\mathcal{N}^2 + e^{-\theta|z|}) \\ &= -\frac{\sigma g_0}{\alpha}\frac{y}{r} q(r) + O(\mathcal{N}^2 + r^{-1}q(r)). \end{aligned} \tag{6.64}$$

Hence,

$$\dot{r} = \frac{\dot{y}\cdot y}{r} = -\frac{\sigma g_0}{\alpha} q(r) + O(\mathcal{N}^2 + r^{-1}q(r)).$$

Using also $|q'(r) + q(r)| \lesssim r^{-1}q(r)$ (from (6.11)), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(r)}\right] = -\frac{\dot{r}q'(r)}{[q(r)]^2} = -\frac{\sigma g_0}{\alpha} + \frac{1}{q(r)} O(\mathcal{N}^2 + r^{-1}q(r)).$$

Proof of (6.59). It follows from (6.36) and $|a_k^+| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2} \leq \mathcal{N}$.

Proof of (6.60). From the expression of $\mathcal{F}$ and then (6.48), (6.33) and (6.36)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + 2\sum_{k=1,2} \dot{\ell}_k \cdot \ell_k + \frac{1}{\mu}\sum_{k=1,2} \dot{a}_k^- a_k^- \\ &\leq -2\mu\mathcal{E} - 4\alpha \sum_{k=1,2} |\ell_k|^2 + \frac{\nu^-}{\mu}\sum_{k=1,2} (a_k^-)^2 + O(\mathcal{N}^3 + q(r)\mathcal{N}). \end{aligned}$$

Since $0 < \mu < \alpha$ and $0 < \mu < |\nu^-|$ (see (6.49)) we obtain (6.60) for $\mathcal{F}$. The proof for $\mathcal{G}$ is the same.

Proof of (6.61). First, it follows from combining (6.59) and (6.60) that

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leq -2\mu\left(\mathcal{F} + \frac{b}{2\mu}\right) + O(\mathcal{N}^3 + q(r)\mathcal{N}),$$

Thus, from (6.57), we observe that

$$\mathcal{N}^2 \leq \frac{1}{\mu}\left(\mathcal{F} + \frac{b}{2\mu}\right) \leq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} + O(\mathcal{N}^3 + q(r)\mathcal{N}).$$

and (6.61) follows for $\mathcal{N}$ small enough.

Proof of (6.62)-(6.63). It follows from direct computation and (6.58) and (6.61) that (for r large enough)

$$\exp(-K\mathcal{M})\frac{\mathrm{d}}{\mathrm{d}t}R^+ = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{q(r)}\right] + \frac{K}{q(r)}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leq -\frac{\sigma g_0}{\alpha} + 2Kr^{-1}.$$

The computation for $\frac{d}{dt}R^-$ is similar. □

6.2.5 Energy of a 2-solitary wave

We observe from the definition and the energy property (6.3) that a 2-solitary wave u of (6.1) satisfies

$$\lim_{t\to\infty} E(\vec{u}(t)) = \int |\nabla Q|^2 + Q^2 - 2F(Q) = 2E(Q,0). \tag{6.65}$$

More precisely, we expand the energy for a solution close to a 2-solitary wave.

Lemma 6.14. *In the context of Lemma 6.9, the following holds*

$$E(\vec{u}) = 2E(Q,0) - \sigma c_1 g_0 q(r) + O(r^{-1}q(r)) + O(q(r)\mathcal{N}) + O(\mathcal{N}^2). \tag{6.66}$$

Proof. Expanding $E(u,\partial_t u)$ using the decomposition (6.28), integration by parts, the equation $-\Delta Q_k + Q_k - f(Q_k) = 0$ and the definition of G in (6.14), we find

$$\begin{aligned} 2E(u,\partial_t u) = &\int |\partial_t u|^2 + 2E\left(R,0\right) - 2\int G\varepsilon \\ &+ \int \left(|\nabla\varepsilon|^2 + \varepsilon^2 - 2F(R+\varepsilon) + 2F(R) + 2f(R)\varepsilon\right). \end{aligned}$$

Thus, using (6.18), the subcriticality of p and Sobolev embedding, there hold

$$2E(u,\partial_t u) = \int |\partial_t u|^2 + 2E\left(R,0\right) + O\big(q(r)\|\vec{\varepsilon}\|_{H^1\times L^2} + \|\vec{\varepsilon}\|^2_{H^1\times L^2}\big).$$

Note that $\partial_t u = \eta - \sum_{k=1,2}(\ell_k\cdot\nabla)Q_k$ implies $\|\partial_t u\|_{L^2} \lesssim \mathcal{N}$. Next, by direct computation, $-\Delta Q_1 + Q_1 - f(Q_1) = 0$ and then (6.19), (6.21) and (6.16)

$$\begin{aligned} E(Q_1+Q_2,0) &= 2E(Q,0) + \int (\nabla Q_1\cdot\nabla Q_2 + Q_1Q_2 - f(Q_1)Q_2 - f(Q_2)Q_1) \\ &\quad - \int (F(R) - F(Q_1) - F(Q_2) - f(Q_1)Q_2 - f(Q_2)Q_1) \\ &= 2E(Q,0) - \int f(Q_2)Q_1 + O(e^{-\frac{5}{4}|z|}) \\ &= 2E(Q,0) - \sigma c_1 g(|z|) + O(|z|^{-1}g(|z|)). \end{aligned}$$

Using (6.56), $g(|z|) = g_0 q(r) + O(r^{-1}q(r)) + O(q(r)\mathcal{N})$ and the proof is complete. □

As a consequence of (6.3), (6.65) and (6.66), we obtain the following estimate.

Corollary 6.15. *Let $\vec{u}$ be a 2-solitary wave solution of (6.1) satisfying the decomposition of Lemma 6.9 on $[t,\infty)$, for some $t\in\mathbb{R}$. Then,*

$$\int_t^{+\infty} \|\partial_t u(s)\|^2_{L^2}\,\mathrm{d}s \lesssim q(r(t)) + \mathcal{N}^2(t). \tag{6.67}$$

6.3 Proofs of Theorems 6.3 and 6.4

6.3.1 General estimates

Proposition 6.16. *There exists $C>0$ and $\delta_1>0$ such that the following hold. For any $0<\delta<\delta_1$ and any 2-solitary wave $\vec{u}$ of (6.1), there exists $T_\delta\in\mathbb{R}$ such that $\vec{u}(t)$ admits a decomposition as in Lemma 6.9 in a neighborhood of T_δ with*

$$\mathcal{N}(T_\delta) \le \delta \quad \text{and} \quad q(r(T_\delta)) \le \delta^2. \tag{6.68}$$

Moreover, for any such T_δ, it holds:

Same sign case *if $\sigma = 1$ then there exists $T_* > T_\delta$ such that $\vec{u}$ admits a decomposition as in Lemma 6.9 on $[T_\delta, T_*]$ with*

$$\forall t \in [T_\delta, T_*], \quad \mathcal{N}(t) \leq C\delta \quad \text{and} \quad q(r(T_*)) = \delta^{\frac{3}{2}}. \tag{6.69}$$

Opposite sign case *if $\sigma = -1$ then $\vec{u}$ admits a decomposition as in Lemma 6.9 for all $t \geq T_\delta$ and satisfies*

$$\forall t \in [T_\delta, \infty), \quad \mathcal{N}(t) \leq C\delta \quad \text{and} \quad \left| \frac{1}{q(r(t))} - \frac{g_0}{\alpha} t \right| \lesssim \frac{t}{|\log \delta|} + \tilde{C} \tag{6.70}$$

for some $\tilde{C} > 0$.

Proof. Let $\vec{u}$ be a 2-solitary wave of (6.1). For δ to be taken small enough, the existence of T_δ satisfying (6.68) is a consequence of Definition 6.1 and Lemma 6.9.

For a constant $C > 1$ to be taken large enough, we introduce the following bootstrap estimate

$$\mathcal{N} \leq C\delta, \quad q(r) \leq \delta^{\frac{3}{2}}, \quad b \leq C\delta^2. \tag{6.71}$$

Set

$$T_* = \sup \left\{ t \in [T_\delta, \infty) \text{ such that (6.71) holds on } [T_\delta, t] \right\} > T_\delta.$$

In the remainder of the proof, the implied constants in $\lesssim$ or O *do not depend* on the constant C appearing in the bootstrap assumption (6.71). We start by improving the bootstrap assumption on $\mathcal{N}$ and b.

Estimate on $\mathcal{N}$. We now improve the bootstrap assumption (6.71) on $\mathcal{N}$.

From (6.60) and (6.71), it holds on $[T_\delta, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[e^{2\mu t} \mathcal{F} \right] \lesssim C^3 \delta^3 e^{2\mu t} + C\delta^{\frac{5}{2}} e^{2\mu t} \lesssim \delta^2 e^{2\mu t},$$

for $\delta > 0$ small enough. From (6.57) and (6.68), we have $\mathcal{F}(T_\delta) \lesssim \delta^2$. Thus, integrating the above estimate on $[T_\delta, t]$, for any $t \in [T_\delta, T_*)$, it holds $\mathcal{F} \lesssim \delta^2$. In particular, by (6.57), we obtain

$$\|\vec{\varepsilon}\|^2_{H^1 \times L^2} \lesssim \mathcal{F} + b \lesssim C\delta^2. \tag{6.72}$$

Arguing similarly for the quantity $\mathcal{G}$, we have

$$\sum_{k=1,2} |\ell_k|^2 + \sum_{k=1,2} (a_k^-)^2 \lesssim \mathcal{G} \lesssim \delta^2. \tag{6.73}$$

Hence we obtain

$$\forall t \in [T_\delta, T_*), \quad \mathcal{N}(t) \lesssim \sqrt{C}\delta.$$

For C large enough, this strictly improves the estimate (6.71) of $\mathcal{N}$ on $[T_\delta, T_*)$.

Estimate on b. We now prove that for C large enough, it holds

$$\forall t \in [T_\delta, T_*), \quad b(t) \leqslant \frac{C}{2}\delta^2.$$

From (6.67) in Corollary 6.15 , we have

$$\int_{T_\delta}^{+\infty} \|\partial_t u(s)\|^2_{L^2} \, \mathrm{d}s \lesssim q(r(T_\delta)) + \mathcal{N}^2(T_\delta) \lesssim \delta^2. \tag{6.74}$$

By (6.68), we have $b(T_\delta) \lesssim \delta^2$.

For the sake of contradiction, take C large and assume that there exists $t_2 \in [T_\delta, T_*)$ such that

$$b(t_2) = \frac{C}{2}\delta^2, \quad b(t) < \frac{C}{2}\delta^2 \quad \text{on } [T_\delta, t_2).$$

On the one hand, by continuity of b, there exists $t_1 \in [T_\delta, t_2)$ such that

$$b(t_1) = \frac{C}{4}\delta^2 \quad \text{and} \quad b(t) > \frac{C}{4}\delta^2 \quad \text{on } (t_1, t_2].$$

Using (6.59) and (6.71), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b = 2\nu^+ b + O(C^3\delta^3 + C\delta^{\frac{5}{2}})$$

which implies (for δ small enough with respect to $1/C$)

$$t_2 - t_1 = \frac{\log 2}{2\nu^+} + O(\delta^{\frac{1}{2}}), \tag{6.75}$$

and thus

$$\int_{t_1}^{t_2} b \gtrsim C\delta^2. \tag{6.76}$$

On the other hand, by (6.13), (6.28), $\partial_t u = \eta - \sum_{k=1,2}(\ell_k \cdot \nabla)Q_k$ and (6.73) we have $\|\eta\|_{L^2}^2 \lesssim \|\partial_t u\|_{L^2}^2 + \delta^2$ on $[T_\delta, T_*]$. Thus, using (6.74) and (6.75),

$$\int_{t_1}^{t_2} \|\eta(t)\|_{L^2}^2 \,\mathrm{d}t \lesssim \delta^2. \tag{6.77}$$

By the definition of $a_k^\pm$, one has

$$a_k^+ = \zeta^+\langle \varepsilon, Y_k\rangle + \langle \eta, Y_k\rangle, \quad a_k^- = \zeta^-\langle \varepsilon, Y_k\rangle + \langle \eta, Y_k\rangle$$

and thus

$$a_k^+ = \frac{\zeta^+}{\zeta^-} a_k^- + \frac{\zeta^- - \zeta^+}{\zeta^-}\langle \eta, Y_k\rangle.$$

Combining (6.73), (6.75) and (6.77), we find $\displaystyle\int_{t_1}^{t_2} b \lesssim \delta^2$, a contradiction with (6.76) for C large enough.

We can now prove the two statements of Proposition 6.16 themselves. First observe that (6.72) and (6.73) gives on $[T_\delta, T_*)$

$$\exp(\pm K\mathcal{M}) = 1 + O(C\delta^2).$$

Same sign case. We use the quantity R^+ defined in Lemma 6.13. From (6.62), for r large,

$$\frac{\mathrm{d}}{\mathrm{d}t} R^+ \le \left(-\frac{g_0}{\alpha} + 2Kr^{-1}\right)\exp\left(K\mathcal{M}\right) \le -\frac{g_0}{2\alpha}.$$

Therefore, for all $t \in [T_\delta, T_*]$, it holds

$$R^+(t) \le R^+(T_\delta) - \frac{g_0}{2\alpha}(t - T_\delta).$$

Assuming $T_* = \infty$, $R^+(t)$ becomes negative for some time, which is absurd. Since all the estimates in (6.71) have been strictly improved on $[T_\delta, T_*]$ except the one for $q(r)$, it follows by a continuity argument that $q(r(T_*)) = \delta^{\frac{3}{2}}$.

Opposite sign case. Here we use both R^+ and R^-. First notice that on $[T_\delta, T_*]$,

$$R^-(1+O(C\delta^2)) \leqslant \frac{1}{q(r)} \leqslant R^+(1+O(C\delta^2)). \tag{6.78}$$

From (6.62),

$$\frac{\mathrm{d}}{\mathrm{d}t}R^+ \leq \frac{g_0}{\alpha}\left(1+\frac{O(K)}{|\log\delta|}\right)(1+O(C\delta^2)) \leq \frac{g_0}{\alpha}\left(1+\frac{O(K)}{|\log\delta|}\right),$$

for $\delta > 0$ small enough with respect to C. Thus, for any $t \in [T_\delta, T_*)$, it holds by integration on $[T_\delta, t]$

$$R^+(t) \leq R^+(T_\delta) + \frac{g_0}{\alpha}\left(1+\frac{O(K)}{|\log\delta|}\right)(t-T_\delta). \tag{6.79}$$

Similarly, we check that

$$R^-(t) \geq R^-(T_\delta) + \frac{g_0}{\alpha}\left(1-\frac{O(K)}{|\log\delta|}\right)(t-T_\delta). \tag{6.80}$$

Note that (6.68), (6.78) and (6.80) imply that

$$\forall t \in [T_\delta, T_*), \quad \frac{1}{q(r(t))} \geq \frac{1}{2}\left(\delta^{-2} + \frac{g_0}{\alpha}(t-T_\delta)\right).$$

In particular, the estimate on $q(r)$ in (6.71) is strictly improved for δ small enough, As a consequence $T_* = \infty$.

Last, we observe that (6.78), (6.79) and (6.80) imply (6.70) where $\tilde{C}$ can be written in terms of $R^\pm(T_\delta)$, T_δ and δ. □

6.3.2 Proof of Theorem 6.3

Let $\vec{u}$ be a 2-solitary wave of (6.1) with $\sigma_1 = \sigma_2$.

Let $\delta > 0$ to be fixed later, and let T_δ and T_* be as Proposition 6.16. Using (6.66) at $t = T_*$ and (6.69), we have

$$E(u(T_*), \partial_t u(T_*)) = 2E(Q,0) - c_1 g_0 \delta^{\frac{3}{2}} + O(|\log\delta|^{-1}\delta^{\frac{3}{2}}).$$

This allows to fix a $\delta > 0$ small enough so that

$$E(u(T_*), \partial_t u(T_*)) < 2E(Q,0).$$

This contradicts the fact that the energy is decreasing and converges to $2E(Q,0)$ as $t \to \infty$.

6.3.3 Proof of Theorem 6.4

Let $\vec{u}$ be a 2-solitary wave of (6.1) with $\sigma_1 = -\sigma_2$.

Proposition 6.17. *There exists $T > 0$ such that the decomposition of $\vec{u}$ satisfies*

$$\forall t \geqslant T, \quad q(r(t)) \lesssim t^{-1}, \quad \mathcal{N}(t) \lesssim t^{-1}. \tag{6.81}$$

Proof. Let $0 < \delta < \delta_1$ in the framework of Proposition 6.16. From (6.70), there exists $T > 0$ large enough such that

$$\forall t \geqslant T/2, \quad \mathcal{N}(t) \leq C\delta, \quad q(r(t)) \leq \frac{2g_0}{\alpha t}.$$

In particular, from (6.59) and (6.60)

$$\left|\frac{\mathrm{d}b}{\mathrm{d}t} - 2\nu^+ b\right| \lesssim \mathcal{N}^3 + t^{-1}\mathcal{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3 + t^{-1}\mathcal{N}. \tag{6.82}$$

Our goal is to obtain the decay rate of $\mathcal{N}$. The above bounds are not quite enough for this, due to the cubic term in $\mathcal{N}$ for which we have no decay yet, only smallness. In order to get around this, we will work on a modification $\tilde{b}$ of b. Recall (6.57):

$$0 \leqslant \mathcal{N}^2 \lesssim \mathcal{F} + \frac{b}{2\mu}.$$

For $0 < \epsilon \ll 1$ to be fixed, we observe that

$$\mathcal{N}^3 + t^{-1}\mathcal{N} \lesssim \epsilon^2\mathcal{N}^2 + \epsilon^{-2}t^{-2} \lesssim \epsilon^2\left(\mathcal{F} + \frac{b}{2\mu}\right) + \epsilon^{-2}t^{-2}. \tag{6.83}$$

(Here and below the implied constants do not depend on ε). Set $\tilde{b} = b - \epsilon\mathcal{F}$ and observe that

$$\tilde{b} = b - \epsilon\mathcal{F} = \left(1 + \frac{\epsilon}{2\mu}\right)b - \epsilon\left(\mathcal{F} + \frac{1}{2\mu}b\right) \leq \left(1 + \frac{\epsilon}{2\mu}\right)b.$$

Therefore, using (6.82),

$$\begin{aligned}\frac{\mathrm{d}\tilde{b}}{\mathrm{d}t} &\geq 2\nu^+ b + 2\epsilon\mu\mathcal{F} - C\epsilon^2\left(\mathcal{F} + \frac{b}{2\mu}\right) - C\epsilon^{-2}t^{-2}\\ &\geq (2\nu^+ - \epsilon)b + \left(2\epsilon\mu - C\epsilon^2\right)\left(\mathcal{F} + \frac{b}{2\mu}\right) - C\epsilon^{-2}t^{-2}\\ &\geq \nu^+\tilde{b} - C\epsilon^{-2}t^{-2},\end{aligned}$$

where $\epsilon > 0$ is taken small enough so that

$$(2\nu^+ - \epsilon)\left(1 + \frac{\epsilon}{2\mu}\right)^{-1} \geq \nu^+, \quad 2\epsilon\mu - C\epsilon^2 > 0.$$

This bound is suitable for our purpose. Integrating on $[t, \tau] \subset [T/2, +\infty)$, we obtain

$$\tilde{b}(t) - e^{-\nu^+(\tau - t)}\tilde{b}(\tau) \lesssim \epsilon^{-2}\int_t^\tau e^{-\nu^+(s-t)}s^{-2}\,\mathrm{d}s \lesssim \epsilon^{-2}t^{-2}.$$

Passing to the limit as $\tau \to \infty$, one obtains for all $t \geq T/2$,

$$\tilde{b}(t) \lesssim \epsilon^{-2}t^{-2}, \quad b(t) \leq C\epsilon^{-2}t^{-2} + \epsilon\mathcal{F}(t).$$

Inserting this information into the equations (6.82) of $\mathcal{F}$ and b and using (6.83), it holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F} + \frac{b}{2\mu}\right) &\leq -2\mu\mathcal{F} + \frac{\nu^+}{\mu}b + C\epsilon^2\left(\mathcal{F} + \frac{b}{2\mu}\right) + C\epsilon^{-2}t^{-2}\\ &\leq -(2\mu - C\epsilon^2)\left(\mathcal{F} + \frac{b}{2\mu}\right) + \left(1 + \frac{\nu^+}{\mu}\right)b + C\epsilon^{-2}t^{-2}\\ &\leq -\mu\left(\mathcal{F} + \frac{b}{2\mu}\right) + C\epsilon^{-2}t^{-2},\end{aligned}$$

by possibly choosing a smaller $\epsilon > 0$. Integrating on $[T/2, t]$, we obtain

$$\left(\mathcal{F} + \frac{b}{2\mu}\right)(t) - e^{-\mu(t - \frac{T}{2})}\left(\mathcal{F} + \frac{b}{2\mu}\right)(T/2) \lesssim \int_{\frac{T}{2}}^t e^{-\mu(t-s)}s^{-2}\,\mathrm{d}s$$

and so

$$\mathcal{N}^2(t) \lesssim \left(\mathcal{F} + \frac{b}{2\mu}\right)(t) \lesssim t^{-2} + \delta^2 e^{\mu T/2}e^{-\mu t}$$

For $t \geqslant T > 0$, the last term is bounded by $e^{-\mu t/2} \lesssim t^{-2}$, and this proves (6.81). □

We complete the proof of Theorem 6.4. By (6.28) and (6.81), estimate (6.7) is proved. Since (6.70) holds for any $\delta > 0$, it means that

$$\lim_{t\to\infty} \frac{1}{tq(r(t))} = \frac{g_0}{\alpha},$$

and, using the expansion (6.11) of q, this is equivalent to

$$\lim_{t\to\infty} \frac{1}{t} r(t)^{\frac{N-1}{2}} e^{r(t)} = \frac{\kappa g_0}{\alpha}.$$

This implies

$$\lim_{t\to\infty} \left\{ r(t) - \left(\log t - \frac{1}{2}(N-1)\log\log t + c_0 \right) \right\} = 0,$$

for $c_0 = \log\frac{\kappa g_0}{\alpha}$. As $\mathcal{N}(t) \to 0$ (due to (6.81)) and in view of (6.56), we obtain (6.8).

Finally, we prove (6.9). Let $1 < \theta < \min(p-1, 2)$. We observe from (6.64), $|y - z| \lesssim \mathcal{N}$ and (6.81) that

$$\dot{y} = \frac{1}{\alpha}\frac{z}{|z|} g(|z|) + O(\mathcal{N}^2 + e^{-\theta|z|}) = \frac{1}{\alpha}\frac{y}{|z|} g(|z|) + O(t^{-\theta}).$$

Set $y = r\omega$, where $\omega \in \mathcal{S}_{\mathbb{R}^N}(1)$. We have $\dot{y} = \dot{r}\omega + r\dot{\omega}$, and using $\dot{\omega}\cdot\omega = 0$, we find the estimate $r|\dot{\omega}| = O(t^{-\theta})$. Thus, there exists $\omega^\infty \in \mathcal{S}_{\mathbb{R}^N}(1)$ such that $\lim_{t\to\infty} \omega = \omega^\infty$. Since $\dfrac{z}{\log t} = \dfrac{y}{\log t} + O(\mathcal{N})$, we have proved $\displaystyle\lim_{t\to\infty} \frac{z(t)}{\log t} = \omega^\infty$.

From (6.32), (6.34) and (6.81), we also observe that

$$|2\dot{z}_1 - \dot{z}| + |2\dot{z}_2 + \dot{z}| \lesssim t^{-\theta},$$

which is sufficient to complete the proof of (6.9).

6.4 Proof of Theorem 6.5

6.4.1 Preliminary result

We use the notation from the beginning of Section 6.2. We also use the constant

$$\beta := \frac{1}{2\sqrt{\alpha^2 + \nu_0^2}} = \langle \vec{Y}^+, \vec{Z}^+ \rangle^{-1} > 0. \tag{6.84}$$

Lemma 6.18. *For any $(z_1, z_2, \ell_1, \ell_2) \in \mathbb{R}^{4N}$ with $|z|$ large enough, there exist linear maps*

$$B : \mathbb{R}^2 \to \mathbb{R}^2, \quad V_j : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{for } j = 1, \dots, N,$$

smooth in $(z_1, z_2, \ell_1, \ell_2)$, satisfying

$$\|B - \beta\,\mathrm{Id}\| \lesssim e^{-\frac{1}{2}|z|}, \quad \|V_j\| \lesssim e^{-\frac{1}{2}|z|},$$

and such that the function $W(a_1, a_2) : \mathbb{R}^N \to \mathbb{R}$ defined by

$$W(a_1, a_2) := \sum_{k=1,2} \left\{ B_k(a_1, a_2) Y_k + \sum_{j=1}^N V_{k,j}(a_1, a_2) \partial_{x_j} Q_k \right\},$$

satisfies, for all $k = 1, 2$, $j = 1, \dots, N$,

$$\langle W(a_1, a_2), \partial_{x_j} Q_k \rangle = 0, \quad \langle W(a_1, a_2), Y_k \rangle = \beta a_k.$$

In particular, setting

$$\vec{W}(a_1, a_2) = \begin{pmatrix} W(a_1, a_2) \\ \nu^+ W(a_1, a_2) \end{pmatrix} \quad \text{it holds} \quad \langle \vec{W}(a_1, a_2), \vec{Z}_k^+ \rangle = a_k.$$

Proof. Define

$$W(a_1,a_2)=\sum_{k=1,2}\Big\{b_kY_k+\sum_{j=1}^N v_{k,j}\partial_{x_j}Q_k\Big\},$$

our goal is to solve for b_k, $v_{k,j}$ in function of a_1,a_2. Using the relations $\langle Y_1,\partial_{x_j}Q_1\rangle=0$, $\langle\partial_{x_{j'}}Q_1,\partial_{x_j}Q_1\rangle=0$ for $j\neq j'$, and the estimate $\langle\partial_{x_{j'}}Q_2,\partial_{x_j}Q_1\rangle=O(e^{-\frac12|z|})$ for any j,j' (see (6.15)), we observe that the condition $\langle W(a_1,a_2),\partial_{x_j}Q_1\rangle=0$ is equivalent to a linear relation between the coefficients in the definition of $W(a_1,a_2)$ of the form

$$\|\partial_{x_j}Q\|_{L^2}^2v_{1,j}=O(e^{-\frac12|z|})b_2+\sum_{j'=1}^N O(e^{-\frac12|z|})v_{2,j'}.$$

Similarly, the condition $\langle W(a_1,a_2),\partial_{x_j}Q_2\rangle=0$ is equivalent to

$$\|\partial_{x_j}Q\|_{L^2}^2v_{2,j}=O(e^{-\frac12|z|})b_1+\sum_{j'=1}^N O(e^{-\frac12|z|})v_{1,j'}.$$

Moreover, since $\langle Y,Y\rangle=1$ (see Lemma 6.7) and $\langle Y_1,Y_2\rangle=O(e^{-\frac12|z|})$, the conditions $\langle W(a_1,a_2),Y_k\rangle=\beta a_k$ write

$$b_1=\beta a_1+O\Big\{\Big(|b_2|+\sum_{j'=1}^N|v_{2,j'}|\Big)e^{-\frac12|z|}\Big\},$$
$$b_2=\beta a_2+O\Big\{\Big(|b_1|+\sum_{j'=1}^N|v_{1,j'}|\Big)e^{-\frac12|z|}\Big\}.$$

The existence and desired properties of b_k and $v_{k,j}$ for $k=1,2$ and $j=1,\dots,N$ thus follow from inverting a linear system for $|z|$ large enough. □

6.4.2 Construction of a family of 2-solitary waves

Proposition 6.19. *Assume $\sigma=-1$. For $\delta>0$ small enough, for any*

$$\begin{cases}(z_1(0),z_2(0))\in\mathbb{R}^{2N}\ \text{with}\ |z_1(0)-z_2(0)|>5|\log\delta|,\\ (\ell_1(0),\ell_2(0))\in\mathcal{B}_{\mathbb{R}^{2N}}(\delta),\\ \vec{\varepsilon}_\perp(0)\in\mathcal{B}_{H^1\times L^2}(\delta)\ \text{with (6.29) and}\ \langle\vec{\varepsilon}_\perp(0),\vec{Z}_k^+(0)\rangle=0\ \ \text{for}\ k=1,2,\end{cases}\tag{6.85}$$

there exists $(a_1^+(0),a_2^+(0))\in\bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac54})$ such that the solution $\vec{u}$ of (6.1) with the initial data

$$\vec{u}(0)=\vec{Q}_1(0)+\vec{Q}_2(0)+\vec{W}(a_1^+(0),a_2^+(0))+\vec{\varepsilon}_\perp(0)$$

is a 2-solitary wave.

Proof. Decomposition. For any $t\geq0$ such that the solution $\vec{u}(t)$ satisfies (6.27), we decompose it according to Lemma 6.9. Note that by the properties of the function W in Lemma 6.18 and the orthogonality properties (6.29) of $\vec{\varepsilon}_\perp(0)$ assumed in (6.85), the initial data $\vec{u}(0)$ is modulated, in the sense that $(z_1(0),z_2(0),\ell_1(0),\ell_2(0))$ and

$$\vec{\varepsilon}(0)=\vec{W}(a_1^+(0),a_2^+(0))+\vec{\varepsilon}_\perp(0),$$

are the parameters of the decomposition of $\vec{u}(0)$. In particular, it holds from (6.85)

$$\mathcal{N}(0) \lesssim \delta, \quad q(r(0)) \lesssim \delta^2. \tag{6.86}$$

Moreover, by Lemma 6.18, for $k = 1, 2$,

$$\langle \vec{\varepsilon}(0), \vec{Z}_k^+(0) \rangle = \langle \vec{W}(a_1^+(0), a_2^+(0)), Z_k^+ \rangle = a_k^+(0),$$

which is consistent with the definition of a_k^+ in (v) of Lemma 6.9.

Bootstrap estimates. We introduce the following bootstrap estimates

$$\mathcal{N} \leq \delta^{\frac{3}{4}}, \quad q(r) \leq \delta^{\frac{3}{2}}, \quad b \leq \delta^{\frac{5}{2}}. \tag{6.87}$$

We set

$$T_* = \sup \{ t \in [0, \infty) \text{ such that (6.87) holds on } [0, t] \} \geq 0.$$

Estimates on the damped components. The estimate on $\mathcal{N}$ is strictly improved on $[0, T_*]$ as in the proof of Proposition 6.16. In particular, $\mathcal{N} \lesssim \delta$ on $[0, T_*]$.

Estimate on the distance. Note that $r(t) \gtrsim \log\delta$ and thus, from (6.62),

$$\frac{\mathrm{d}R^-}{\mathrm{d}t} \geq \frac{g_0}{\alpha} \left(1 - \frac{C}{|\log\delta|} \right) (1 - C\delta^2) \geq \frac{g_0}{2\alpha}.$$

Thus, by integration on $[0, t]$, for any $t \in [0, T_*]$, it holds $R^-(t) \geq R^-(0) + \frac{g_0 t}{2\alpha}$. Using (6.78) and next (6.86), we obtain

$$\frac{1}{q(r(t))} \geq \left(\frac{1 - C\delta^2}{q(r(0))} + \frac{g_0 t}{2\alpha} \right) (1 - C\delta^2) \gtrsim \delta^{-2}.$$

This strictly improves the estimate of $q(r)$ in (6.71).

Transversality condition. From (6.59) and $\mathcal{N} \lesssim \delta$, we observe that for any time $t \in [0, T_*]$ where it holds $b(t) = \delta^{\frac{5}{2}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t) \geq 2\nu^+ b(t) - C\delta^3 \geq 2\nu^+ \delta^{\frac{5}{2}} - C\delta^3 \geq \nu^+ \delta^{\frac{5}{2}} > 0,$$

for $\delta > 0$ small enough. This transversality condition is enough to justify the existence of at least a couple $(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$ such that $T_* = \infty$.

Indeed, for the sake of contradiction assume that for all $(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$, it holds $T_* < \infty$. Then, a contradiction follows from the following observations (see for instance more details in [28] or in [31, Section 3.1]).

Continuity of T_* the above transversality condition proves that the map

$$(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}}) \mapsto T_* \in [0, +\infty)$$

is continuous and that $T_* = 0$ when $(a_1^+(0), a_2^+(0)) \in \mathcal{S}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$.

Construction of a retraction As a consequence, the map

$$(a_1^+(0), a_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta^{\frac{5}{4}}) \mapsto (a_1^+(T_*), a_2^+(T_*)) \in \mathcal{S}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$$

is continuous and its restriction to the sphere $\mathcal{S}_{\mathbb{R}^2}(\delta^{\frac{5}{4}})$ is the identity.

This is a contradiction with the no retraction theorem for continuous maps from the ball to the sphere. □

Remark 6.20. The use of initial data similar to the ones in Proposition 6.19 allows to correct a flaw in the articles [28, 31, 85] dealing with the construction of multi-solitons in several contexts. For example, in [31], the initial data U_0 chosen page 18 is not exactly modulated, and this is why Lemma 6 provides the modulation keeping track of the free parameters necessary for the topological argument. However, this modulation involves a translation parameter denoted by $\tilde{\mathbf{y}}$ in [31] (similar to (z_1, z_2) in the present paper) on which the topological argument finally depends. To close the choice of the free parameters for initial data as chosen in [31], an extra argument would be needed (like a fixed point result). It is simpler though to define an initial data already modulated as in Proposition 6.19, or in [67, Lemma 3.1].

6.4.3 Lipschitz estimate and uniqueness

The heart of Theorem 6.5 is the following proposition.

Proposition 6.21. *For $\delta > 0$ small enough, let $(z_1(0), z_2(0), \ell_1(0), \ell_2(0), \vec{\varepsilon}_\perp(0))$ and $(\tilde{z}_1(0), \tilde{z}_2(0), \tilde{\ell}_1(0))$, $(\tilde{\ell}_2(0), \tilde{\vec{\varepsilon}}_\perp(0))$ be as in (6.85), and let $(a_1^+(0), a_2^+(0)) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$, $(\tilde{a}_1^+(0), \tilde{a}_2^+(0)) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$ be such that the solutions $\vec{u}(t)$ and $\tilde{\vec{u}}(t)$ of (6.1) with initial data*

$$\vec{u}(0) = \vec{Q}_1(0) + \vec{Q}_2(0) + \vec{W}(a_1^+(0), a_2^+(0)) + \vec{\varepsilon}_\perp(0),$$
$$\tilde{\vec{u}}(0) = \tilde{\vec{Q}}_1(0) + \tilde{\vec{Q}}_2(0) + \vec{W}(\tilde{a}_1^+(0), \tilde{a}_2^+(0)) + \tilde{\vec{\varepsilon}}_\perp(0),$$

are 2-solitary waves. Then it holds

$$\sum_{k=1,2} |a_k^+(0) - \tilde{a}_k^+(0)| \lesssim \delta^{\frac{1}{4}} \Big(\|\vec{\varepsilon}_\perp(0) - \tilde{\vec{\varepsilon}}_\perp(0)\|_{H^1 \times L^2} + \sum_{k=1,2} \Big\{ |z_k(0) - \tilde{z}_k(0)| + |\ell_k(0) - \tilde{\ell}_k(0)| \Big\} \Big).$$

Proof. We will split the proof in several steps. For $\delta > 0$ small enough, from Proposition 6.16 and the assumption on their initial data $\vec{u}(0)$, $\tilde{\vec{u}}(0)$, the 2-solitary waves $\vec{u}$ and $\tilde{\vec{u}}$ decompose for any $t \geq 0$ as in Lemma 6.9 $\vec{u} = \vec{Q}_1 + \vec{Q}_2 + \vec{\varepsilon}$, $\tilde{\vec{u}} = \tilde{\vec{Q}}_1 + \tilde{\vec{Q}}_2 + \tilde{\vec{\varepsilon}}$ and the parameters of their decompositions $(z_k, \ell_k, \mathcal{N})_{k=1,2}$ and $(\tilde{z}_k, \tilde{\ell}_k, \tilde{\mathcal{N}})_{k=1,2}$ respectively satisfy (6.70) for all $t \geq 0$. Denote

$$\check{z}_k = z_k - \tilde{z}_k, \quad \check{\ell}_k = \ell_k - \tilde{\ell}_k, \quad \check{Q}_k = Q_k - \tilde{Q}_k, \quad \check{\varepsilon} = \varepsilon - \tilde{\varepsilon}, \quad \check{\eta} = \eta - \tilde{\eta}, \tag{6.88}$$

$$\check{\mathcal{N}} = \Big[\|\check{\vec{\varepsilon}}\|_{H^1 \times L^2}^2 + \sum_{k=1,2} |\check{z}_k|^2 + \sum_{k=1,2} |\check{\ell}_k|^2 \Big]^{\frac{1}{2}}. \tag{6.89}$$

(Notice the extra term in $\check{\mathcal{N}}$).

If $\sum_{k=1,2} |\check{z}_k(0)| > \delta^{\frac{3}{4}}$, since $(a_1^+(0), a_2^+(0))$, $(\tilde{a}_1^+(0), \tilde{a}_2^+(0)) \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$, the desired estimate is satisfied. Thus, we assume now $\sum_{k=1,2} |\check{z}_k(0)| \leq \delta^{\frac{3}{4}}$, and we work only for time t such that $\sum_{k=1,2} |\check{z}_k(t)|$ is small (see (6.103)).

Equation of $\check{\vec{\varepsilon}}$. By direct computation from (6.31) for $\vec{\varepsilon}$ and $\tilde{\vec{\varepsilon}}$, we have

$$\begin{cases} \partial_t \check{\varepsilon} = \check{\eta} + \mathrm{Mod}_{\check{\varepsilon},1} + \mathrm{Mod}_{\check{\varepsilon},2} \\ \partial_t \check{\eta} = \Delta \check{\varepsilon} - \check{\varepsilon} + f'(R)\check{\varepsilon} - 2\alpha\check{\eta} + \mathrm{Mod}_{\check{\eta},1} + \mathrm{Mod}_{\check{\eta},2} + \check{G} \end{cases} \tag{6.90}$$

where

$$\begin{aligned}
\mathrm{Mod}_{\check{\varepsilon},1} &= \sum_{k=1,2} \left(\dot{\check{z}}_k - \check{\ell}_k\right) \cdot \nabla Q_k, \quad \mathrm{Mod}_{\check{\varepsilon},2} = \sum_{k=1,2} \left(\dot{\check{z}}_k - \tilde{\ell}_k\right) \cdot \nabla \check{Q}_k, \\
\mathrm{Mod}_{\check{\eta},1} &= \sum_{k=1,2} \left(\dot{\check{\ell}}_k + 2\alpha \check{\ell}_k\right) \cdot \nabla Q_k \\
&\quad - \sum_{k=1,2} (\check{\ell}_k \cdot \nabla)(\dot{z}_k \cdot \nabla) Q_k - \sum_{k=1,2} (\tilde{\ell}_k \cdot \nabla)(\dot{\check{z}}_k \cdot \nabla) Q_k, \\
\mathrm{Mod}_{\check{\eta},2} &= \sum_{k=1,2} \left(\dot{\check{\ell}}_k + 2\alpha \tilde{\ell}_k\right) \cdot \nabla \check{Q}_k - \sum_{k=1,2} (\tilde{\ell}_k \cdot \nabla)(\dot{\check{z}}_k \cdot \nabla) \check{Q}_k
\end{aligned}$$

and

$$\check{G} = \left[f(R+\varepsilon) - f(R) - f'(R)\varepsilon\right] - \left[f(\tilde{R}+\tilde{\varepsilon}) - f(\tilde{R}) - f'(R)\tilde{\varepsilon}\right] + G - \tilde{G}.$$

Step 1. We claim the following estimate on the nonlinear term $\check{G}$.

Lemma 6.22.

$$\|\check{G}\|_{L^2} \lesssim \check{\mathcal{N}}(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}). \tag{6.91}$$

Proof. We will in fact derive pointwise bounds. We decompose $\check{G} = \sum_{j=1}^{5} \check{G}_j$ where

$$\begin{aligned}
\check{G}_1 &= [f'(R) - f'(\tilde{R})]\tilde{\varepsilon}, \\
\check{G}_2 &= [f(R+\varepsilon) - f(R) - f'(R)\varepsilon] - [f(R+\tilde{\varepsilon}) - f(R) - f'(R)\tilde{\varepsilon}], \\
\check{G}_3 &= [f(R+\tilde{\varepsilon}) - f(R) - f'(R)\tilde{\varepsilon}] - [f(\tilde{R}+\tilde{\varepsilon}) - f(\tilde{R}) - f'(\tilde{R})\tilde{\varepsilon}], \\
\check{G}_4 &= [f(Q_1+Q_2) - f(Q_2)] - [f(Q_1+\tilde{Q}_2) - f(\tilde{Q}_2)], \\
\check{G}_5 &= [f(Q_1+\tilde{Q}_2) - f(Q_1)] - [f(\tilde{Q}_1+\tilde{Q}_2) - f(\tilde{Q}_1)].
\end{aligned}$$

For $\check{G}_1$, using $|f'(R) - f'(\tilde{R})| \lesssim \left(|Q_1|^{p-1} + |Q_2|^{p-1}\right) \sum_{k=1,2} |\check{z}_k|$, we have

$$|\check{G}_1| \lesssim |\tilde{\varepsilon}| \left(|Q_1|^{p-1} + |Q_2|^{p-1}\right) \sum_{k=1,2} |\check{z}_k|.$$

For $\check{G}_2$, by Taylor formular, we have

$$\begin{aligned}
\check{G}_2 &= f(R+\tilde{\varepsilon}+\check{\varepsilon}) - f(R+\tilde{\varepsilon}) - f'(R)\check{\varepsilon} \\
&= \check{\varepsilon} \int_0^1 f'(R+\tilde{\varepsilon}+\theta\check{\varepsilon}) - f'(R)\,\mathrm{d}\theta = \check{\varepsilon} \int_0^1 (\tilde{\varepsilon}+\theta\check{\varepsilon}) \int_0^1 f''(R+\rho(\tilde{\varepsilon}+\theta\check{\varepsilon}))\,\mathrm{d}\rho\,\mathrm{d}\theta \\
&= \check{\varepsilon} \int_0^1 \int_0^1 (\theta\varepsilon + (1-\theta)\tilde{\varepsilon}) f''(R+\rho(\tilde{\varepsilon}+\theta\check{\varepsilon}))\,\mathrm{d}\rho\,\mathrm{d}\theta.
\end{aligned}$$

As $|f''(x+y)| = p(p-1)|x+y|^{p-2} \lesssim |x|^{p-2} + |y|^{p-2}$, we get

$$|\check{G}_2| \lesssim |\check{\varepsilon}|(|\varepsilon| + |\tilde{\varepsilon}|) \left(|R|^{p-2} + |\tilde{\varepsilon}|^{p-2} + |\varepsilon|^{p-2}\right).$$

For $\check{G}_3$: by Taylor formula,

$$\begin{aligned}
\left|[f(R+\tilde{\varepsilon}) - f(R)] - [f(\tilde{R}+\tilde{\varepsilon}) - f(\tilde{R})]\right| &= \left|\tilde{\varepsilon} \int_0^1 [f'(R+\theta\tilde{\varepsilon}) - f'(\tilde{R}+\theta\tilde{\varepsilon})]\,\mathrm{d}\theta\right| \\
&\lesssim (|\check{z}_1| + |\check{z}_2|)|\tilde{\varepsilon}|\left(|R|^{p-2} + |\tilde{R}|^{p-2} + |\tilde{\varepsilon}|^{p-2}\right),
\end{aligned}$$

and

$$\left|[f'(R) - f'(\tilde{R})]\tilde{\varepsilon}\right| \lesssim (|\check{z}_1| + |\check{z}_2|)|\tilde{\varepsilon}|\left(|R|^{p-2} + |\tilde{R}|^{p-2}\right).$$

It follows that

$$|\check{G}_3| \lesssim (|\check{z}_1| + |\check{z}_2|)|\tilde{\varepsilon}|\big(|R|^{p-2} + |\tilde{R}|^{p-2} + |\tilde{\varepsilon}|^{p-2}\big).$$

For $\check{G}_4$: Taylor formula gives similarly

$$\begin{aligned}|\check{G}_4| &= \left|Q_1 \int_0^1 \left[f'(Q_2 + \theta Q_1) - f'(\tilde{Q}_2 + \theta Q_1)\right] \mathrm{d}\theta\right| \\ &\lesssim |Q_1||Q_2 - \tilde{Q}_2| \left(|Q_1|^{p-2} + |Q_2|^{p-2}\right) \lesssim |\check{z}_2| \left(|Q_1|^{p-1}|Q_2| + |Q_1||Q_2|^{p-1}\right).\end{aligned}$$

For $\check{G}_5$, a similar argument yields

$$|\check{G}_5| \lesssim |\check{z}_1|(|Q_1|^{p-1}|Q_2| + |Q_1||Q_2|^{p-1}).$$

Estimate (6.91) then follows from (6.18) and Sobolev embeddings. □

Step 2. Equations for the parameters.

Lemma 6.23.

(i) Equations of the geometric parameters. *For* $k = 1, 2$,

$$|\dot{\check{z}}_k - \check{\ell}_k| \lesssim \check{\mathcal{N}}(\mathcal{N} + \tilde{\mathcal{N}}), \tag{6.92}$$

$$|\dot{\check{\ell}}_k + 2\alpha\check{\ell}_k| \lesssim \check{\mathcal{N}}(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}). \tag{6.93}$$

(ii) Equations of the exponential directions. *Let* $\check{a}_k^{\pm} = \langle \vec{\tilde{\varepsilon}}, \vec{Z}_k^{\pm}\rangle$. *Then,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\check{a}_k^{\pm} - \nu^{\pm}\check{a}_k^{\pm}\right| \lesssim \check{\mathcal{N}}(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}). \tag{6.94}$$

Proof. (i) From the orthogonality conditions in (6.29), we derive the following relations: for all $k = 1, 2$ and $j = 1, \dots, N$,

$$\langle \check{\varepsilon}, \partial_{x_j} Q_k\rangle + \langle \tilde{\varepsilon}, \partial_{x_j}\check{Q}_k\rangle = \langle \check{\eta}, \partial_{x_j} Q_k\rangle + \langle \tilde{\eta}, \partial_{x_j}\check{Q}_k\rangle = 0. \tag{6.95}$$

We derive the equations of $\check{z}_k$ and $\check{\ell}_k$ from these relations.

First, from (6.95) for $\check{\varepsilon}$ with $k = 1$ and $j = 1, \dots, N$,

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \check{\varepsilon}, \partial_{x_j} Q_1\rangle + \langle \tilde{\varepsilon}, \partial_{x_j}\check{Q}_1\rangle\right) \\ &= \langle \partial_t\check{\varepsilon}, \partial_{x_j} Q_1\rangle + \langle \check{\varepsilon}, \partial_t\partial_{x_j} Q_1\rangle + \langle \partial_t\tilde{\varepsilon}, \partial_{x_j}\check{Q}_1\rangle + \langle \tilde{\varepsilon}, \partial_t\partial_{x_j}\check{Q}_1\rangle.\end{aligned}$$

Thus, using (6.90) for $\partial_t\check{\varepsilon}$ and (6.31) for $\partial_t\tilde{\varepsilon}$,

$$\begin{aligned}0 &= \langle \check{\eta}, \partial_{x_j} Q_1\rangle + (\dot{\check{z}}_{1,j} - \check{\ell}_{1,j})\|\partial_{x_j} Q_1\|_{L^2}^2 + \langle (\dot{\check{z}}_2 - \check{\ell}_2)\cdot\nabla Q_2, \partial_{x_j} Q_1\rangle \\ &\quad + \sum_{k=1,2}\langle (\dot{\tilde{z}}_k - \tilde{\ell}_k)\cdot\nabla\check{Q}_k, \partial_{x_j} Q_1\rangle - \langle \check{\varepsilon}, (\dot{z}_1\cdot\nabla)\,\partial_{x_j} Q_1\rangle \\ &\quad + \langle \tilde{\eta}, \partial_{x_j}\check{Q}_1\rangle + \sum_{k=1,2}\langle (\dot{\tilde{z}}_k - \tilde{\ell}_k)\cdot\nabla\tilde{Q}_k, \partial_{x_j}\check{Q}_1\rangle \\ &\quad - \langle \tilde{\varepsilon}, (\dot{\tilde{z}}_1\cdot\nabla)\partial_{x_j} Q_1\rangle - \langle \tilde{\varepsilon}, (\dot{\tilde{z}}_1\cdot\nabla)\partial_{x_j}\check{Q}_1\rangle.\end{aligned}$$

Using (6.15), (6.32), (6.95) and the estimates

$$|\langle \partial_{x_{j'}}\check{Q}_1, \partial_{x_j} Q_1\rangle| + |\langle \partial_{x_{j'}}\check{Q}_2, \partial_{x_j} Q_1\rangle| + |\langle \partial_{x_{j'}} Q_2, \partial_{x_j}\check{Q}_1\rangle| \lesssim |\check{z}_1| + |\check{z}_2|,$$

it follows that

$$\begin{aligned}|\dot{\check{z}}_{1,j} - \check{\ell}_{1,j}| \lesssim e^{-\frac{1}{2}|z|}|\dot{\check{z}}_2 - \check{\ell}_2| + (|\check{z}_1| + |\check{z}_2|)\tilde{\mathcal{N}}^2 + (|\ell_1| + \mathcal{N}^2)\check{\mathcal{N}} \\ + \tilde{\mathcal{N}}(|\check{\ell}_1| + |\dot{\check{z}}_1 - \check{\ell}_1|) + \tilde{\mathcal{N}}|\check{z}_1|(\mathcal{N}^2 + |\ell_1|).\end{aligned}$$

By symmetry, an analogous estimate holds for $|\dot{\check{z}}_{2,j} - \check{\ell}_{2,j}|$. Summing up both gives (6.92).

Second, using (6.95) for $\check{\eta}$ with $k = 1$ and $j = 1, \dots, N$, we have

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \check{\eta}, \partial_{x_j} Q_1\rangle + \langle \tilde{\eta}, \partial_{x_j}\check{Q}_1\rangle\right) \\ &= \langle \partial_t\check{\eta}, \partial_{x_j} Q_1\rangle + \langle \check{\eta}, \partial_t\partial_{x_j} Q_1\rangle + \langle \partial_t\tilde{\eta}, \partial_{x_j}\check{Q}_1\rangle + \langle \tilde{\eta}, \partial_t\partial_{x_j}\check{Q}_1\rangle.\end{aligned}$$

Using (6.90), one has

$$\begin{aligned}&\langle \partial_t\check{\eta}, \partial_{x_j} Q_1\rangle \\ &\quad = \langle \Delta\check{\varepsilon} - \check{\varepsilon} + f'(Q_1)\check{\varepsilon}, \partial_{x_j} Q_1\rangle - 2\alpha\langle \check{\eta}, \partial_{x_j} Q_1\rangle \\ &\qquad + \langle [f'(R) - f'(Q_1)]\check{\varepsilon}, \partial_{x_j} Q_1\rangle + \langle \check{G}, \partial_{x_j} Q_1\rangle \\ &\qquad + (\dot{\check{\ell}}_{1,j} + 2\alpha\check{\ell}_{1,j})\|\partial_{x_j} Q_1\|_{L^2}^2 + \langle (\dot{\check{\ell}}_2 + 2\alpha\check{\ell}_2)\cdot\nabla Q_2, \partial_{x_j} Q_1\rangle \\ &\qquad - \sum_{k=1,2}\langle (\check{\ell}_k\cdot\nabla)(\dot{z}_k\cdot\nabla)Q_k, \partial_{x_j} Q_1\rangle - \sum_{k=1,2}\langle (\tilde{\ell}_k\cdot\nabla)(\dot{\check{z}}_k\cdot\nabla)Q_k, \partial_{x_j} Q_1\rangle \\ &\qquad + \sum_{k=1,2}\langle (\dot{\check{\ell}}_k + 2\alpha\tilde{\ell}_k)\cdot\nabla\check{Q}_k, \partial_{x_j} Q_1\rangle - \sum_{k=1,2}\langle (\tilde{\ell}_k\cdot\nabla)(\dot{\check{z}}_k\cdot\nabla)\check{Q}_k, \partial_{x_j} Q_1\rangle.\end{aligned}$$

Next,

$$\begin{aligned}\langle \check{\eta}, \partial_t\partial_{x_j} Q_1\rangle &= -\langle \check{\eta}, (\dot{z}_1\cdot\nabla)\partial_{x_j} Q_1\rangle \\ &= -\langle \check{\eta}, ((\dot{z}_1 - \ell_1)\cdot\nabla)\partial_{x_j} Q_1\rangle - \langle \check{\eta}, (\ell_1\cdot\nabla)\partial_{x_j} Q_1\rangle = O(\check{\mathcal{N}}\mathcal{N}).\end{aligned}$$

Then, using (6.31),

$$\begin{aligned}\langle \partial_t\tilde{\eta}, \partial_{x_j}\check{Q}_1\rangle &= \langle \Delta\tilde{\varepsilon} - \tilde{\varepsilon} + f'(\tilde{Q}_1)\tilde{\varepsilon}, \partial_{x_j}\check{Q}_1\rangle - 2\alpha\langle \tilde{\eta}, \partial_{x_j}\check{Q}_1\rangle \\ &\quad + \langle f(\tilde{R} + \tilde{\varepsilon}) - f(\tilde{R}) - f'(\tilde{Q}_1)\tilde{\varepsilon}, \partial_{x_j}\check{Q}_1\rangle \\ &\quad + \sum_{k=1,2}\langle (\dot{\check{\ell}}_k + 2\alpha\tilde{\ell}_k)\cdot\nabla\tilde{Q}_k, \partial_{x_j}\check{Q}_1\rangle \\ &\quad - \sum_{k=1,2}\langle (\tilde{\ell}_k\cdot\nabla)(\dot{\check{z}}_k\cdot\nabla)\tilde{Q}_k, \partial_{x_j}\check{Q}_1\rangle + \langle \tilde{G}, \partial_{x_j}\check{Q}_1\rangle.\end{aligned}$$

Last,

$$\begin{aligned}\langle \tilde{\eta}, \partial_t\partial_{x_j}\check{Q}_1\rangle &= -\langle \tilde{\eta}, (\dot{\check{z}}_1\cdot\nabla)\partial_{x_j} Q_1\rangle - \langle \tilde{\eta}, (\dot{\check{z}}_1\cdot\nabla)\partial_{x_j}\check{Q}_1\rangle \\ &= -\langle \tilde{\eta}, (\dot{\check{z}}_1 - \check{\ell}_1)\cdot\nabla)\partial_{x_j} Q_1\rangle - \langle \tilde{\eta}, \check{\ell}_1\cdot\nabla)\partial_{x_j} Q_1\rangle \\ &\quad - \langle \tilde{\eta}, ((\dot{\check{z}}_1 - \tilde{\ell}_1)\cdot\nabla)\partial_{x_j}\check{Q}_1\rangle - \langle \check{\eta}, \tilde{\ell}_1\cdot\nabla)\partial_{x_j} Q_1\rangle \\ &= O(\tilde{\mathcal{N}}|\dot{\check{z}}_1 - \check{\ell}_1|) + O(\check{\mathcal{N}}\tilde{\mathcal{N}}).\end{aligned}$$

By (6.91), we have $|\langle \check{G}, \nabla Q_k\rangle| \lesssim \check{\mathcal{N}}\big(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}\big)$.

Combining these estimates with $\Delta\partial_{x_j} Q_1 - \partial_{x_j} Q_1 + f'(Q_1)\partial_{x_j} Q_1 = 0$, (6.95) for $\check{\eta}$, we obtain

$$|\dot{\check{\ell}}_{1,j} + 2\alpha\check{\ell}_{1,j}| \lesssim \left(\check{\mathcal{N}} + \sum_{k=1,2}\left\{|\dot{\check{z}}_k - \check{\ell}_k| + |\dot{\check{\ell}}_k + 2\alpha\check{\ell}_k|\right\}\right)\big(\tilde{\mathcal{N}} + \mathcal{N} + e^{-\frac{1}{2}|z|}\big).$$

We argue similarly to obtain the same bound on $|\dot{\check{\ell}}_{2,j} + 2\alpha\check{\ell}_{2,j}|$. Summing up and using (6.92), we obtain (6.93).

(ii) The proof of (6.94) is similar. See also the proof of (v) in Lemma 6.9. □

Step 3. Energy estimates. For $\mu > 0$ small to be chosen, denote $\rho = 2\alpha - \mu$, and consider the following energy functional

$$\check{\mathcal{E}} = \int \left\{ |\nabla \check{\varepsilon}|^2 + (1-\rho\mu)\check{\varepsilon}^2 + (\check{\eta} + \mu\check{\varepsilon})^2 - f'(R)\check{\varepsilon}^2 \right\}. \tag{6.96}$$

Lemma 6.24. *There exists $\mu > 0$ such that the following hold.*

(i) Coercivity and bound.

$$2\mu \|\vec{\check{\varepsilon}}\|_{H^1\times L^2}^2 - \frac{1}{2\mu}\left[\tilde{\mathcal{N}}^2 \check{\mathcal{N}}^2 + \sum_{k=1,2} \left((\check{a}_k^+)^2 + (\check{a}_k^-)^2 \right) \right] \leq \check{\mathcal{E}} \leq \frac{1}{\mu} \|\vec{\check{\varepsilon}}\|_{H^1\times L^2}^2. \tag{6.97}$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{\mathcal{E}} \leq -2\mu\check{\mathcal{E}} + \frac{1}{\mu}\check{\mathcal{N}}^2(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|}).$$

Proof. (i) The upper bound in (6.97) is clear. Next, adapting the proof of (i) of Lemma 6.11, one checks

$$\check{\mathcal{E}} \geqslant 2\mu \|\vec{\check{\varepsilon}}\|_{H^1\times L^2}^2 - \frac{1}{2\mu}\left[\sum_{k,j} |\langle \check{\varepsilon}, \partial_{x_j} Q_k\rangle|^2 + \sum_{k=1,2} \left((\check{a}_k^+)^2 + (\check{a}_k^-)^2 \right) \right].$$

The estimate $|\langle \check{\varepsilon}, \partial_{x_j} Q_k\rangle| = |\langle \check{\varepsilon}, \partial_{x_j} \check{Q}_k\rangle| \lesssim \tilde{\mathcal{N}}\check{\mathcal{N}}$ from (6.95) then implies the lower bound in (6.97).

(ii) We follow the computation of the proof of (ii) of Lemma 6.11. First,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\check{\mathcal{E}} = &\int \partial_t \check{\varepsilon} \left[-\Delta\check{\varepsilon} + (1-\rho\mu)\check{\varepsilon} - f'(R)\check{\varepsilon} \right] + (\partial_t \check{\eta} + \mu\partial_t\check{\varepsilon})\ (\check{\eta} + \mu\check{\varepsilon}) \\ &+ \int \sum_{k=1,2} (\dot{z}_k \cdot \nabla Q_k) f''(R)\check{\varepsilon}^2 = \mathbf{g_1} + \mathbf{g_2}. \end{aligned}$$

Second, using (6.90) and integration by parts,

$$\begin{aligned} \mathbf{g_1} = &-\mu \int \left[|\nabla\check{\varepsilon}|^2 + (1-\rho\mu)\check{\varepsilon}^2 - f'(R)\check{\varepsilon}^2 \right] - \rho \int (\check{\eta} + \mu\check{\varepsilon})^2 \\ &+ \sum_{k=1,2} \int \check{\varepsilon} \left[-\Delta \mathrm{Mod}_{\check{\varepsilon},k} + (1-\rho\mu)\mathrm{Mod}_{\check{\varepsilon},k} - f'(R)\mathrm{Mod}_{\check{\varepsilon},k} \right] \\ &+ \sum_{k=1,2} \int (\check{\eta} + \mu\check{\varepsilon}) \left[\mathrm{Mod}_{\check{\eta},k} + \mu \mathrm{Mod}_{\check{\varepsilon},k} \right] + \int (\check{\eta} + \mu\check{\varepsilon})\check{G}. \end{aligned}$$

The first line in the expression of $\mathbf{g_1}$ above is less than $-\mu\check{\mathcal{E}}$ (taking $\mu \leq \rho$). Next, using (6.32), (6.33), (6.91), (6.92) and (6.93), we check that the remaining terms in $\mathbf{g_1}$ and $\mathbf{g_2}$ are estimated by $\check{\mathcal{N}}^2(\mathcal{N} + \tilde{\mathcal{N}} + e^{-\frac{1}{2}|z|})$. □

Define

$$\check{y}_k = \check{z}_k + \frac{\check{\ell}_k}{2\alpha}, \quad \check{\mathcal{F}} = \check{\mathcal{E}} + \sum_{k=1,2} |\check{\ell}_k|^2 + \frac{1}{2\mu}\sum_{k=1,2} (\check{a}_k^-)^2, \quad \check{b} = \sum_{k=1,2} (\check{a}_k^+)^2. \tag{6.98}$$

Lemma 6.25. *For $\mu > 0$ defined in Lemma 6.24, it holds.*

(i) Comparison.

$$\mu\check{\mathcal{N}}^2 \leq \mu\|\vec{\check{\varepsilon}}\|_{H^1\times L^2} + \sum_{k=1,2} |\check{\ell}_k|^2 + \sum_{k=1,2} |\check{z}_k|^2 \leq \check{\mathcal{F}} + \frac{\check{b}}{2\mu} + \sum_{k=1,2} |\check{z}_k|^2 \lesssim \check{\mathcal{N}}^2. \tag{6.99}$$

(ii) Positions.

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\check{y}_k\right| \lesssim \check{\mathcal{N}}(\mathcal{N}+\check{\mathcal{N}}+e^{-\frac{1}{2}|z|}) \lesssim \delta\check{\mathcal{N}}. \tag{6.100}$$

(iii) Damped components.

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{\mathcal{F}}+2\mu\check{\mathcal{F}} \lesssim \check{\mathcal{N}}^2(\mathcal{N}+\check{\mathcal{N}}+e^{-\frac{1}{2}|z|}) \lesssim \delta\check{\mathcal{N}}^2. \tag{6.101}$$

Proof. (i) follows from (6.97) (see also the proof of (6.57) in Lemma 6.13, (i)).
(ii) Estimate (6.100) follows from (6.92) and (6.93).
(iii) is a consequence of Lemmas 6.23 and 6.24 (see also Lemma 6.13), (iv)). □

Step 4. For the sake of contradiction, assume

$$\delta^{\frac{1}{2}}\check{\mathcal{N}}^2(0) < \check{b}(0). \tag{6.102}$$

We introduce the following bootstrap estimate

$$\delta^{\frac{3}{4}}\check{\mathcal{N}}^2 < \check{b} \tag{6.103}$$

and we define

$$T_* = \sup\{t>0 \text{ such that (6.103) holds on } [0,t]\} > 0.$$

By (6.94) and then (6.103), there holds on $[0,T_*]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{b} \geq 2\nu^+\check{b} - C\delta\check{\mathcal{N}}\sqrt{\check{b}} \geq \nu^+\check{b}. \tag{6.104}$$

In particular, $\check{b}$ is positive, increasing on $[0,T_*)$ and

$$\forall t\in[0,T_*), \quad b(t) \geqslant e^{\nu^+ t}\check{b}(0). \tag{6.105}$$

By (6.101) and next (6.103),

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^{2\mu t}\check{\mathcal{F}}] \lesssim e^{2\mu t}\delta\check{\mathcal{N}}^2 \lesssim e^{2\mu t}\delta^{\frac{1}{4}}\check{b}.$$

Integrating on $[0,t]$, for any $t\in[0,T_*)$ and using that $\check{b}$ is increasing, we obtain

$$\check{\mathcal{F}}(t) \lesssim e^{-2\mu t}\check{\mathcal{F}}(0) + \delta^{\frac{1}{4}}\check{b}(t),$$

and thus using (6.102),

$$\delta^{\frac{3}{4}}\check{\mathcal{F}}(t) \lesssim \delta^{\frac{3}{4}}\check{\mathcal{F}}(0) + \delta\check{b}(t) \lesssim \delta^{\frac{1}{4}}(\check{b}(0)+\check{b}(t)) \lesssim \delta^{\frac{1}{4}}\check{b}(t).$$

Using similar argument, we check that

$$\delta^{\frac{3}{4}}\left|\ell_k(t)\right|^2 \lesssim \delta^{\frac{1}{4}}\check{b}(t) \quad \text{for } k=1,2. \tag{6.106}$$

From (6.100), next (6.103) and then (6.104),

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|\check{y}_k|^2\right| \lesssim \delta\check{\mathcal{N}}^2 \lesssim \delta^{\frac{1}{4}}\check{b} \lesssim \delta^{\frac{1}{4}}\frac{\mathrm{d}}{\mathrm{d}t}\check{b}.$$

Integrating on $[0,t]$, for any $t\in[0,T_*)$,

$$|\check{y}_k(t)|^2 \lesssim |\check{y}_k(0)|^2 + \delta^{\frac{1}{4}}\check{b}(t).$$

Thus, using (6.102) and (6.106),

$$\delta^{\frac{3}{4}}|\check{z}_k(t)|^2 \lesssim \delta^{\frac{3}{4}}|\check{y}_k(t)|^2 + \delta^{\frac{3}{4}}|\check{\ell}_k(t)|^2 \lesssim \delta^{\frac{1}{4}}\check{b}(t).$$

Using (6.99), we obtain

$$\delta^{3/4}\check{\mathcal{N}}^2 \lesssim \delta^{1/4}\check{b}$$

on $[0, T_*)$, and so we have strictly improved estimate (6.103) for δ small enough.

By a continuity argument it follows that $T_* = +\infty$. This is contradictory since the exponential growth (6.105) would lead to unbounded $\check{b}$.

In conclusion, we have just proved that $\check{b}(0) \leq \delta^{\frac{1}{2}}\check{\mathcal{N}}^2(0)$. Observing

$$|a_k^+(0) - \tilde{a}_k^+(0)| \lesssim |\check{a}_k^+(0)| + |\langle \vec{\check{\varepsilon}}(0), (\vec{Z}_k^+(0) - \vec{\tilde{Z}}_k^+(0)\rangle| \lesssim \delta^{\frac{1}{4}}\check{\mathcal{N}}(0),$$

the proof of Proposition 6.21 is complete. □

6.4.4 Modulation at initial time

We introduce some notation related to initial data as written in the statement of Theorem 6.5

$$\Omega = (L, \vec{\phi}), \quad \|\Omega\| = |L| + \|\vec{\phi}\|_{H^1\times L^2}, \quad h = (h_1, h_2),$$

and as written in Propositions 6.19 and 6.21

$$\Gamma = (z_1, z_2, \ell_1, \ell_2, \vec{\varepsilon}_\perp), \quad \|\Gamma\| = |(z_1, z_2)| + |(\ell_1, \ell_2)| + \|\vec{\varepsilon}_\perp\|_{H^1\times L^2}, \quad a^+ = (a_1^+, a_2^+).$$

For $\delta > 0$, we denote by $\mathcal{V}_\delta^\perp$ the set of $\Omega = (L, \vec{\phi})$ satisfying

$$\begin{cases} L \in \mathbb{R}^N \text{ with } |L| > 10|\log\delta|, \\ \vec{\phi} \in \mathcal{B}_{H^1\times L^2}(\delta) \text{ with } \langle \vec{\phi}, \vec{Z}^+(\cdot \pm \frac{L}{2})\rangle = 0, \end{cases}$$

and by $\mathcal{W}_\delta^\perp$ the set of $\Gamma = (z_1, z_2, \ell_1, \ell_2, \vec{\varepsilon}_\perp)$ satisfying

$$\begin{cases} (z_1, z_2) \in \mathbb{R}^{2N} \text{ with } |z_1 - z_2| > 5|\log\delta|, \\ (\ell_1, \ell_2) \in \mathcal{B}_{\mathbb{R}^{2N}}(\delta), \\ \vec{\varepsilon}_\perp \in \mathcal{B}_{H^1\times L^2}(\delta) \text{ with (6.29) and } \langle \vec{\varepsilon}_\perp, \vec{Z}_k^+\rangle = 0 \text{ for } k = 1, 2. \end{cases}$$

In the statement of Theorem 6.5, we do not require the orthogonality conditions $\langle \vec{\phi}, \vec{Z}^+(\cdot \pm \frac{L}{2})\rangle = 0$ as in the definition of the set $\mathcal{V}_\delta^\perp$. Those conditions are obtained by adjusting (h_1, h_2) in a second step, see the proof of Theorem 6.5. We start with constructing the "manifold" for data in $\mathcal{V}_\delta^\perp$, which somehow corresponds to the tangent space and where the discrepancy is superlinear in δ.

Lemma 6.26. *Assume $\sigma_1 = 1$ and $\sigma_2 = -1$. There exists $C > 0$ such that for all $\delta > 0$ small enough, and for any $(\Omega, h) \in \mathcal{V}_\delta^\perp \times \mathcal{B}_{\mathbb{R}^2}(\delta)$, there exist unique $\Gamma[\Omega, h] \in \mathcal{W}_{C\delta}^\perp$ and $a^+[\Omega, h] \in \mathcal{B}_{\mathbb{R}^2}(C\delta)$ such that*

$$(\vec{Q} + h_1\vec{Y}^+)(\cdot - \tfrac{L}{2}) - (\vec{Q} + h_2\vec{Y}^+)(\cdot + \tfrac{L}{2}) + \vec{\phi} = \vec{Q}_1 + \vec{Q}_2 + \vec{W}(a^+) + \vec{\varepsilon}_\perp, \tag{6.107}$$

$$|\beta a^+ - h| \lesssim \delta^2. \tag{6.108}$$

Moreover, for any (Ω, h), $(\tilde{\Omega}, \tilde{h}) \in \mathcal{V}_\delta^\perp \times \mathcal{B}_{\mathbb{R}^2}(\delta)$, with $|L - \tilde{L}| < \delta$, it holds

$$\|\Gamma[\Omega, h] - \Gamma[\tilde{\Omega}, \tilde{h}]\| \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|, \tag{6.109}$$

$$|(\beta a^+[\Omega, h] - h) - (\beta a^+[\tilde{\Omega}, \tilde{h}] - \tilde{h})| \lesssim \delta(\|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|). \tag{6.110}$$

Proof. First, (i) of Lemma 6.9 implies the existence of $z_1, z_2, \ell_1, \ell_2, \vec{\varepsilon}$ such that

$$(\vec{Q} + h_1\vec{Y}^+)(\cdot - \tfrac{L}{2}) - (\vec{Q} + h_2\vec{Y}^+)(\cdot + \tfrac{L}{2}) + \vec{\phi} = \vec{Q}_1 + \vec{Q}_2 + \vec{\varepsilon} \tag{6.111}$$

where

$$\|\vec{\varepsilon}\|_{H^1\times L^2} + \sum_{k=1,2} |\ell_k| + e^{-2|z_1-z_2|} \lesssim \delta$$

and $\vec{\varepsilon}$ satisfies the orthogonality relations (6.29). Using (6.39) and projecting (6.111) on $\partial_{x_j} Q_k$ for $k = 1, 2$, we find

$$|z_1 - \tfrac{L}{2}| + |z_2 + \tfrac{L}{2}| \lesssim \delta.$$

Hence $|z| \geqslant L - 2\delta^{\frac{1}{2}} \geqslant 5|\log\delta|$ for $\delta > 0$ small enough.

Moreover, (6.41) and (6.43) provide the Lipschitz estimates

$$\sum_{k=1,2} \{|z_k - \tilde{z}_k| + |\ell_k - \tilde{\ell}_k|\} + \|\vec{\varepsilon} - \vec{\tilde{\varepsilon}}\|_{H^1\times L^2} \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|. \tag{6.112}$$

Second, define

$$a_k^+ = \langle \vec{\varepsilon}, \vec{Z}_k^+ \rangle \quad \text{and} \quad \vec{\varepsilon}_\perp = \vec{\varepsilon} - \vec{W}(a_1^+, a_2^+) = \vec{\varepsilon} - \vec{W}(a^+),$$

so that $\vec{\varepsilon}_\perp$ satisfies $\langle \vec{\varepsilon}_\perp, \vec{Z}_k^+ \rangle = 0$ for $k = 1, 2$. Using the definition of $\vec{\varepsilon}_\perp$ and (6.112), we obtain

$$\begin{aligned}
&\sum_{k=1,2} \{|z_k - \tilde{z}_k| + |\ell_k - \tilde{\ell}_k|\} + \|\vec{\varepsilon}_\perp - \vec{\tilde{\varepsilon}}_\perp\|_{H^1\times L^2} \\
&\lesssim \sum_{k=1,2} \{|z_k - \tilde{z}_k| + |\ell_k - \tilde{\ell}_k|\} + \|\vec{\varepsilon} - \vec{\tilde{\varepsilon}}\|_{H^1\times L^2} \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|,
\end{aligned}$$

which is (6.109). Let us now project (6.111) on $\vec{Z}_1^+$. Using the expansion

$$Q(\cdot - \tfrac{L}{2}) - Q_1 = (z_1 - \tfrac{L}{2}) \cdot \nabla Q(\cdot - \tfrac{L}{2}) + O_{L^2}(\delta^2),$$

the fact that $\langle \partial_{x_j} Q, Y \rangle = 0$ for all $j = 1, \dots, N$, we get

$$|\langle \vec{Q}(\cdot - \tfrac{L}{2}) - \vec{Q}_1, \vec{Z}_1^+ \rangle| \lesssim \delta^2.$$

Similarly,

$$|\langle \vec{Q}(\cdot + \tfrac{L}{2}) - \vec{Q}_2, \vec{Z}_1^+ \rangle| = \langle (z_2 + \tfrac{L}{2}) \cdot \nabla Q(\cdot + \tfrac{L}{2}), \vec{Z}_1^+ \rangle + O(\delta^2) \lesssim \delta e^{-\frac{1}{2}|z|} + O(\delta^2) \lesssim \delta^2.$$

Using the assumption $\langle \vec{\phi}, \vec{Z}^+(\cdot \pm \frac{L}{2}) \rangle = 0$, we infer

$$\left| a_1^+ - h_1 \langle \vec{Y}^+, \vec{Z}^+ \rangle \right| \lesssim \delta^2.$$

One can argue in the same way upon projecting (6.111) on $\vec{Z}_2^+$, and so derive (6.108).

It remains to prove (6.110). From (6.109), we have

$$\sum_{k=1,2} |a_k^+ - \tilde{a}_k^+| + \|\vec{\varepsilon}_\perp - \vec{\tilde{\varepsilon}}_\perp\|_{H^1\times L^2} \lesssim \|\Omega - \tilde{\Omega}\| + |h - \tilde{h}|.$$

Using (6.39)-(6.40), observe as above that the following relation holds:

$$\begin{aligned}
&\big[-\tfrac{\check{L}}{2} \cdot (\nabla\vec{Q} + h_1 \nabla\vec{Y}^+) + \check{h}_1 \vec{Y}^+ \big](\cdot - \tfrac{\tilde{L}}{2}) + \big[\tfrac{\check{L}}{2} \cdot (\nabla\vec{Q} + h_2 \nabla\vec{Y}^+) - \check{h}_2 \vec{Y}^+ \big](\cdot + \tfrac{\tilde{L}}{2}) \\
&= \begin{pmatrix} \check{z}_1 \cdot \nabla \\ -(\check{z}_1 \cdot \nabla)(\check{\ell}_1 \cdot \nabla) + (\check{\ell}_1 \cdot \nabla) \end{pmatrix} Q(x - z_1) - \begin{pmatrix} \check{z}_2 \cdot \nabla \\ -(\check{z}_2 \cdot \nabla)(\check{\ell}_2 \cdot \nabla) + (\check{\ell}_2 \cdot \nabla) \end{pmatrix} Q(x - z_2) \\
&\quad + \vec{W}(\check{a}^+) - \vec{\tilde{W}}(\tilde{a}^+) + \vec{\tilde{\varepsilon}}_\perp - \vec{\tilde{\phi}} + O_{L^2}(|\check{L}|^2 + |\check{h}|^2 + |(\check{z}_1, \check{z}_2)|^2).
\end{aligned}$$

Note that we use the notation from (6.88) and

$$\check{W}(a) := \sum_{k=1,2} \left\{ B_k(a)\check{Y}_k + \sum_{j=1}^{N} V_{k,j}(a)\partial_{x_j}\check{Q}_k \right\}, \quad \vec{\check{W}}(a) := \begin{pmatrix} \check{W}(a) \\ \nu\check{W}(a) \end{pmatrix}.$$

Projecting this relation on $\vec{Z}_k^+$, using the orthogonality relations on $\vec{\varepsilon}_\perp$, $\vec{\tilde{\varepsilon}}_\perp$, $\vec{\phi}$, $\vec{\tilde{\phi}}$ and the same argument as in the proof of (6.108), we find

$$|\beta(a_k^+ - \tilde{a}_k^+) - (h_k - \tilde{h}_k)| \lesssim \delta \left(\|\Omega - \tilde{\Omega}\| + |h - \tilde{h}| \right) \quad \text{for } k = 1, 2,$$

which completes the proof of the lemma. □

6.4.5 End of the proof of Theorem 6.5

We first classify 2-solitary wave whose initial data is chosen so that $\Omega = (L, \vec{\phi}) \in \mathcal{V}_\delta^\perp$.

Proposition 6.27. *For all $\delta > 0$ small enough, there exists a map $H : \mathcal{V}_\delta^\perp \to \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$ such that, given $\Omega = (L, \vec{\phi}) \in \mathcal{V}_\delta^\perp$ and $h = (h_1, h_2) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$, the solution $\vec{u}$ with initial data*

$$\vec{u}(0) = (\vec{Q} + h_1\vec{Y}^+)(\cdot - \tfrac{L}{2}) - (\vec{Q} + h_2\vec{Y}^+)(\cdot + \tfrac{L}{2}) + \vec{\phi}$$

is a 2-solitary wave if and only if $h = H(\Omega)$. Moreover, for any Ω, $\tilde{\Omega} \in \mathcal{V}_\delta^\perp$ such that $\|\Omega - \tilde{\Omega}\| < \delta$,

$$|H(\Omega) - H(\tilde{\Omega})| \lesssim \delta^{\frac{1}{4}} \|\Omega - \tilde{\Omega}\|. \tag{6.113}$$

Proof. First observe that from Propositions 6.19 and 6.21, for any $\Gamma(0) \in \mathcal{W}_\delta^\perp$, there exists a unique $A^+(\Gamma(0)) \in \mathcal{B}_{\mathbb{R}^2}(C\delta^{\frac{5}{4}})$ such that the solution $\vec{v}$ of (6.1) with initial data

$$\vec{v}(0) = \vec{Q}_1(0) + \vec{Q}_2(0) + \vec{W}(a_1^+(0), a_2^+(0)) + \vec{\varepsilon}_\perp(0)$$

with $(a_1^+(0), a_2^+(0)) \in \mathcal{B}_{\mathbb{R}^2}(\delta)$ is a 2-solitary wave if and only if $(a_1^+(0), a_2^+(0)) = A^+(\Gamma(0))$. Moreover, the following estimate holds

$$|A^+(\Gamma(0)) - A^+(\tilde{\Gamma}(0))| \lesssim \delta^{\frac{1}{4}} \|\Gamma(0) - \tilde{\Gamma}(0)\|.$$

Using the notation of Lemma 6.26, this means that

$$\vec{u} \text{ is a 2-solitary wave if and only if } a^+[\Omega, h] = A^+(\Gamma[\Omega, h]).$$

Let us show that this condition can be written $h = H(\Omega)$ for some function H.

For $\Omega \in \mathcal{V}_\delta^\perp$ fixed, and $h \in \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$, define

$$F_\Omega(h) = h - \beta \left\{ a^+[\Omega, h] - A^+(\Gamma[\Omega, h]) \right\}.$$

First, we observe that $F_\Omega : \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta) \to \bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$ is a contraction for $\delta > 0$ small enough. Indeed, by Lemma 6.26 and Proposition 6.19,

$$|F_\Omega(h)| \leq |h - \beta a^+[\Omega, h]| + |A^+(\Gamma[\Omega, h])| \lesssim \delta^{\frac{5}{4}},$$

and

$$\begin{aligned} |F_\Omega(h) - F_\Omega(\tilde{h})| &\leq |(h - \tilde{h}) - \beta(a^+[\Omega, h] - a^+[\Omega, \tilde{h}])| + \beta|A^+(\Gamma[\Omega, h]) - A^+(\Gamma[\Omega, \tilde{h}])| \\ &\lesssim \delta|h - \tilde{h}| + \delta^{\frac{1}{4}} \|\Gamma[\Omega, h] - \Gamma[\Omega, \tilde{h}]\| \lesssim \delta^{\frac{1}{4}} |h - \tilde{h}|. \end{aligned}$$

Therefore, by the fixed point theorem, there exists a unique $H = H(\Omega)$ in $\bar{\mathcal{B}}_{\mathbb{R}^2}(\delta)$ such that $F_\Omega(H) = H$, which is equivalent to $a^+[\Omega, H] = A^+(\Gamma[\Omega, H])$.

Second, for any Ω, $\tilde{\Omega} \in \mathcal{V}_\delta^\perp$,

$$\begin{aligned}|H(\Omega) - H(\tilde{\Omega})| &= |F_\Omega(H(\Omega)) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))| \\ &\leq |F_\Omega(H(\Omega)) - F_\Omega(H(\tilde{\Omega}))| + |F_\Omega(H(\tilde{\Omega})) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))| \\ &\leq \frac{1}{2}|H(\Omega) - H(\tilde{\Omega})| + |F_\Omega(H(\tilde{\Omega})) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))|\end{aligned}$$

and so

$$|H(\Omega) - H(\tilde{\Omega})| \leq 2|F_\Omega(H(\tilde{\Omega})) - F_{\tilde{\Omega}}(H(\tilde{\Omega}))|.$$

By definition of $F_\Omega(h)$, Lemma 6.26 and Proposition 6.19, one has

$$\begin{aligned}|F_\Omega(h) - F_{\tilde{\Omega}}(h)| &\leq \beta\left(|a^+[\Omega, h] - a^+[\tilde{\Omega}, h]| + |A^+(\Gamma[\Omega, h]) - A^+(\Gamma[\tilde{\Omega}, h])|\right) \\ &\lesssim \delta\|\Omega - \tilde{\Omega}\| + \delta^{\frac{1}{4}}\|\Gamma[\Omega, h] - \Gamma[\tilde{\Omega}, h]\| \lesssim \delta^{\frac{1}{4}}\|\Omega - \tilde{\Omega}\|,\end{aligned}$$

which yields $|H(\Omega) - H(\tilde{\Omega})| \lesssim \delta^{\frac{1}{4}}\|\Omega - \tilde{\Omega}\|$. □

Proof of Theorem 6.5. The map H was constructed in Proposition 6.27 (locally) on a subspace. Our goal is now to extend it to the full open set given in the statement of the Theorem. Let $\delta > 0$ to be fixed later. Given $(L, \vec{\phi})$ such that $|L| \geqslant 10|\log\delta|$ and $\|\vec{\phi}\|_{H^1\times L^2} < \delta$, we decompose

$$\vec{\phi} = -h_{1,\|}\vec{Y}^+(\cdot - \tfrac{L}{2}) + h_{2,\|}\vec{Y}^+(\cdot + \tfrac{L}{2}) + \vec{\phi}_\perp, \quad \langle\vec{\phi}_\perp, \vec{Z}^+(\cdot \pm \tfrac{L}{2})\rangle = 0.$$

These conditions are a linear system on $h_{k,\|} = h_{k,\|}(L, \vec{\phi})$ of the form

$$\begin{cases} h_{1,\|} = -\beta\langle\vec{\phi}, \vec{Z}^+(\cdot - \frac{L}{2})\rangle + O(e^{-L/2}h_{2,\|}) \\ h_{2,\|} = \beta\langle\vec{\phi}, \vec{Z}^+(\cdot + \frac{L}{2})\rangle + O(e^{-L/2}h_{1,\|}) \end{cases}$$

which can be inverted for $\delta > 0$ small enough, and furthermore, for any such δ, one has

$$|h_{1,\|}| + |h_{2,\|}| + \|\vec{\phi}_\perp\|_{H^1\times L^2} \leqslant C\|\vec{\phi}\|_{H^1\times L^2}, \tag{6.114}$$

and the Lipschitz estimates

$$\sum_{k=1,2} |h_{k,\|} - \tilde{h}_{k,\|}| + \|\vec{\phi}_\perp - \vec{\tilde{\phi}}_\perp\|_{H^1\times L^2} \leqslant C\left(\left|L - \tilde{L}\right| + \|\vec{\phi} - \vec{\tilde{\phi}}\|_{H^1\times L^2}\right), \tag{6.115}$$

for some constant C independent of small $\delta > 0$. In particular, $\vec{\phi}_\perp \in \mathcal{V}_{C\delta}^\perp$ and up to lowering δ, we can assume that Proposition 6.27 applies on that set. Observe that our initial data can be written

$$\vec{u}(0) = \left(\vec{Q} + (h_1 - h_{1,\|})\vec{Y}^+\right)(\cdot - \tfrac{L}{2}) - \left(\vec{Q} + (h_2 - h_{2,\|})\vec{Y}^+\right)(\cdot + \tfrac{L}{2}) + \vec{\phi}_\perp.$$

Proposition 6.27 asserts that $\vec{u}$ is a 2-solitary wave if and only if $H(L, \vec{\phi}_\perp) = (h_1 - h_{1,\|}, h_2 - h_{2,\|})$ or equivalently that $(h_{1,\|}, h_{2,\|}) + H(L, \vec{\phi}_\perp) = (h_1, h_2)$.

We are therefore led to define the extension of H as

$$H(L, \vec{\phi}) := (h_{1,\|}, h_{2,\|}) + H(L, \vec{\phi}_\perp).$$

(where $H(L, \vec{\phi}_\perp)$ is given in Proposition 6.27). Due to (6.113), (6.114) and (6.115), $H(L, \vec{\phi})$ is a Lipschitz map. In conclusion, H meets the requirements of Theorem 6.5. □

Chapter 7

Long-time asymptotics of the one-dimensional damped nonlinear Klein-Gordon equation

Abstract: For the one-dimensional nonlinear damped Klein-Gordon equation

$$\partial_t^2 u + 2\alpha\partial_t u - \partial_x^2 u + u - |u|^{p-1}u = 0 \quad \text{on } \mathbb{R}\times\mathbb{R},$$

with $\alpha > 0$ and $p > 2$, we prove that any global finite energy solution either converges to 0 or behaves asymptotically as $t \to \infty$ as the sum of $K \geq 1$ decoupled solitary waves. In the multi-soliton case $K \geq 2$, the solitary waves have alternate signs and their distances are of order $\log t$.

7.1 Introduction

7.1.1 Main results

We consider the one-dimensional nonlinear focusing damped Klein-Gordon equation

$$\partial_t^2 u + 2\alpha\partial_t u - \partial_x^2 u + u - f(u) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \quad f(u) = |u|^{p-1}u, \tag{7.1}$$

with $\alpha > 0$ and $p > 2$. It follows from standard arguments that the Cauchy problem for (7.1) is locally well-posed in the energy space $H^1 \times L^2$ (see references in §7.2.1). Moreover, the existence of solutions blowing up in finite time is well-known, [12]. Denote $F(u) = \frac{1}{p+1}|u|^{p+1}$. Defining the energy of a solution $\vec{u} = (u, \partial_t u)$ by

$$E(\vec{u}) = \frac{1}{2}\int \left\{(\partial_t u)^2 + (\partial_x u)^2 + u^2 - 2F(u)\right\} \mathrm{d}x, \tag{7.2}$$

it holds formally

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t. \tag{7.3}$$

It is also well-known that up to sign and translation, the only stationary solution of (7.1) is the solitary wave $(Q, 0)$, where Q is the explicit ground state

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}}, \tag{7.4}$$

which solves the equation

$$Q'' - Q + Q^p = 0 \quad \text{on } \mathbb{R}. \tag{7.5}$$

Remarkable results and techniques developed in [12, 16, 53, 80, 81] provide information on the long-time asymptotic behavior of global solutions of (7.1) and of higher dimensional variants of this model for suitable nonlinearities. From techniques in [12, 16], all global solutions are proved to be bounded in the energy space (see Theorem 7.6). From [53] and the concentration-compactness principle as stated in [80, 81], any global solution either converges to zero in the energy space, or decomposes along a subsequence of time into a sum of decoupled ground states (see Theorem 7.7 and Remark 7.9). For space dimension $2 \le N \le 6$, it is proved in [12] that any global radially symmetric solution converges either to 0 or to a single solitary wave as $t \to \infty$. In [29], in space dimension $1 \leqslant N \leqslant 5$ and for energy subcritical nonlinearities, global solutions of the damped Klein-Gordon equation containing two solitary waves are described and classified.

The objective of the present article is to complement those works by describing precisely the asymptotic behavior as $t \to \infty$ of any global solution of (7.1). The choice of considering the one-dimensional model is discussed in Remark 7.10.

Theorem 7.1. *For any global solution $\vec{u} \in \mathcal{C}\big([0,\infty), H^1 \times L^2\big)$ of (7.1), one of the following three scenarios occurs:*

Vanishing *$\vec{u}(t)$ converges exponentially to 0 in $H^1 \times L^2$ as $t \to \infty$.*

Single soliton *There exist $\sigma = \pm 1$, $\ell \in \mathbb{R}$ such that $\vec{u}(t)$ converges exponentially to $(\sigma Q(\cdot - \ell), 0)$ in $H^1 \times L^2$ as $t \to \infty$.*

Multi-soliton *There exist $K \ge 2$, $\sigma = \pm 1$, $\ell \in \mathbb{R}$ and functions $z_k : [0,\infty) \to \mathbb{R}$, for all $k = 1, \dots, K$ such that for all $t \in (0,\infty)$,*

$$\left\| u(t) - \sigma \sum_{k=1}^{K} (-1)^k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}, \tag{7.6}$$

and for any $1 < \theta < \min(p-1, \frac{5}{4})$, $k = 1, \dots, K$,

$$z_k(t) = \left(k - \frac{K+1}{2}\right) \log t + \tau_k + \ell + O(t^{-\theta+1}), \tag{7.7}$$

as $t \to \infty$, *where* τ_k *are the constants uniquely defined by*

$$\sum_{k=1}^{K} \tau_k = 0, \quad e^{-(\tau_{k+1}-\tau_k)} = \frac{2\alpha}{\kappa}\gamma_k \quad \textit{where} \quad \gamma_k = \frac{k(K-k)}{2}, \tag{7.8}$$

and $\kappa > 0$ *defined in* (7.38) *only depends on* p.

Remark 7.2. The parameter ℓ is related to the translation invariance. In the vanishing and the single soliton cases, the damping leads to exponential convergence as $t \to \infty$. In the multi-soliton case, due to the nonlinear interactions between the solitary waves, the asymptotic behavior (7.7) of the centers of mass z_k is related to the following nonlinear differential system

$$\begin{cases} \dot{y}_1 = -\dfrac{\kappa}{2\alpha} e^{-(y_2-y_1)}, \\ \dot{y}_k = \dfrac{\kappa}{2\alpha}\left(e^{-(y_k-y_{k-1})} - e^{-(y_{k+1}-y_k)}\right), \quad \text{for } k = 2, \cdots, K-1, \\ \dot{y}_K = \dfrac{\kappa}{2\alpha} e^{-(y_K-y_{K-1})}. \end{cases} \tag{7.9}$$

This system is studied in [33, 93] in the context of blowup solutions of the nonlinear wave equation. The nonlinear interactions at short distances between the solitary waves also yield the slower rate of convervence t^{-1} in (7.6). See [65, 109, 110] for other examples of strong interactions leading to $\log t$ distant solitary waves.

Theorem 7.1 is a version of the soliton resolution for global solutions of the one-dimensional damped nonlinear Klein-Gordon equation (7.1), with convergence for the whole sequence of time and a description of the parameters of the solitary waves. The case of the damped Klein-Gordon equation on a bounded domain for suitable nonlinearities is addressed in [18, Theorem 9.5.3]; see also references therein.

We refer to [23, 46, 43] for results related to the soliton resolution conjecture for the challenging case of the undamped energy critical wave type equation.

Obviously, $\vec{u} = (0, 0)$ and $\vec{u} = (Q, 0)$ are examples of the first two scenarios of Theorem 7.1. Our second result gives examples of the third scenario for any $K \geq 2$.

Theorem 7.3. *For any* $K \geq 2$, $\sigma = \pm 1$ *and* $\ell \in \mathbb{R}$, *there exist global solutions of* (7.1) *satisfying* (7.6) *and* (7.7).

7.1.2 Notation and basic results on the solitary wave

We denote $\langle \cdot, \cdot \rangle$ the L^2 scalar product for real-valued functions u_i or vector-valued functions $\vec{u}_i = (u_i, v_i)$ $(i = 1, 2)$

$$\langle u_1, u_2 \rangle := \int u_1(x) u_2(x)\, \mathrm{d}x, \quad \langle \vec{u}_1, \vec{u}_2 \rangle := \int u_1(x) u_2(x)\, \mathrm{d}x + \int v_1(x) v_2(x)\, \mathrm{d}x.$$

We see from the explicit expression of Q in (7.4) that, as $x \to \infty$,

$$Q(x) = c_Q e^{-x} + O(e^{-2x}), \quad Q'(x) = -c_Q e^{-x} + O(e^{-2x}) \tag{7.10}$$

where $c_Q = (2p+2)^{\frac{1}{p-1}}$. Note that by (7.5), it holds $\int (\partial_x Q)^2 + Q^2 - Qf(Q) = 0$ and so

$$E(Q, 0) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int Q^{p+1} > 0. \tag{7.11}$$

Linearizing equation (7.1) around the static solution Q and discarding nonlinear terms yields the following linear equation

$$\partial_t^2\varepsilon + 2\alpha\partial_t\varepsilon - \partial_x^2\varepsilon + \varepsilon - pQ^{p-1}\varepsilon = 0. \tag{7.12}$$

This motivates the introduction of the *undamped* linearized operator $\mathcal{L}$ and the associated quadratic form:

$$\mathcal{L} = -\partial_x^2 + 1 - pQ^{p-1}, \quad \langle \mathcal{L}\varepsilon, \varepsilon\rangle = \int \left\{|\partial_x\varepsilon|^2 + \varepsilon^2 - pQ^{p-1}\varepsilon^2\right\} \mathrm{d}x.$$

With such notation, we rewrite (7.12) for the variable $\vec{\varepsilon} = (\varepsilon, \eta)$ as

$$\begin{cases} \partial_t\varepsilon = \eta \\ \partial_t\eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{7.13}$$

We recall some standard properties of the operator $\mathcal{L}$ (see *e.g.* [31, Lemma 1]).

Lemma 7.4. *The following properties hold.*

(i) Spectral properties. *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1, \infty)$, its kernel is $\mathrm{span}\{Q'\}$ and it has a unique negative eigenvalue $-\nu_0^2$, with corresponding smooth normalized eigenfunction Y ($\|Y\|_{L^2} = 1$). Moreover, on $\mathbb{R}$,*

$$|Y^{(n)}(x)| \lesssim e^{-\sqrt{1+\nu_0^2}|x|} \quad \textit{for any } n \in \mathbb{N}.$$

(ii) Coercivity property. *There exists $c > 0$ such that, for all $\varepsilon \in H^1$,*

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geqslant c\|\varepsilon\|_{H^1}^2 - c^{-1}\left(\langle \varepsilon, Y\rangle^2 + \langle \varepsilon, Q'\rangle^2\right).$$

Recall that the unique negative eigenvalue of $\mathcal{L}$ is related to an instability of the solitary wave for the equation (7.1), described by the following functions:

$$\nu^\pm = -\alpha \pm \sqrt{\alpha^2 + \nu_0^2}, \quad \vec{Y}^\pm = \begin{pmatrix} Y \\ \nu^\pm Y \end{pmatrix}. \tag{7.14}$$

Indeed, the functions $\vec{\varepsilon}^\pm(t,x) = \exp(\nu^\pm t)\vec{Y}^\pm(x)$ are solutions of the linearized problem (7.13). Since $\nu^+ > 0$, the solution $\vec{\varepsilon}^+$ illustrates the forward exponential instability of the solitary wave despite the presence of the damping $\alpha > 0$ in the equation. An equivalent formulation of instability is obtained by setting

$$\zeta^\pm = \alpha \pm \sqrt{\alpha^2 + \nu_0^2}, \quad \vec{Z}^\pm = \begin{pmatrix} \zeta^\pm Y \\ Y \end{pmatrix}. \tag{7.15}$$

Then, for any solution $\vec{\varepsilon}$ of (7.13), the time-dependent quantity $a^\pm = \langle \vec{\varepsilon}, \vec{Z}^\pm\rangle$ satisfies

$$\frac{\mathrm{d}a^\pm}{\mathrm{d}t} = \nu^\pm a^\pm.$$

This formulation will be used for the nonlinear problem, see (iv) of Lemma 7.13.

7.2 General properties of finite energy solutions

In this section, we gather some known material on finite energy solutions of (7.1). We repeat some proofs for the sake of completeness.

7.2.1 Cauchy problem in the energy space

It is well-known (see for instance [18, Chapter 9.5]) that the linear problem

$$\partial_t^2 u + 2\alpha\partial_t u - \partial_x^2 u + u = 0 \quad (t,x) \in \mathbb{R}\times\mathbb{R}$$

generates a strongly continuous semigroup of contractions $(S_\alpha(t))_{t\geq 0}$ in $H^1\times L^2$ or $L^2\times H^{-1}$ satisfying, for some $C\geq 1$, $\gamma>0$,

$$\|S_\alpha(t)\|_{\mathcal{L}(H^1\times L^2)} \leq Ce^{-\gamma t}, \quad \|S_\alpha(t)\|_{\mathcal{L}(L^2\times H^{-1})} \leq Ce^{-\gamma t}, \tag{7.16}$$

for all $t\geq 0$. Recall also that the map $u\mapsto f(u)$ is Lipschitz continuous from bounded sets of H^1 to L^2. In particular, the standard theory of semilinear evolution equations (see for instance [18, Chapter 4.3] or [103]) yields the following result.

Proposition 7.5. *For any initial data $(u_0,v_0)\in H^1\times L^2$, there exists a unique maximal solution*

$$\vec{u} = (u,\partial_t u) \in \mathcal{C}([0,T_{\max}),H^1\times L^2)\cap\mathcal{C}^1([0,T_{\max}),L^2\times H^{-1})$$

of (7.1) satisfying $\vec{u}(0)=(u_0,v_0)$. If the maximal time of existence $T_{\max}$ is finite, then $\lim_{t\uparrow T_{\max}}\|\vec{u}(t)\|_{H^1\times L^2}=\infty$.

Moreover, the map $T_{\max}:(u_0,v_0)\in H^1\times L^2\mapsto(0,\infty]$ is lower semicontinuous, and if $(u_{0,n},v_{0,n})\to_{n\to\infty}(u_0,v_0)$ in $H^1\times L^2$ then, for any $0<T<T_{\max}$,

$$(u_n,\partial_t u_n)\to(u,\partial_t u) \quad \text{in } \mathcal{C}([0,T],H^1\times L^2),$$

where $(u_n,\partial_t u_n)$ is the solution of (7.1) corresponding to $(u_{0,n},v_{0,n})$.

In this paper, we systematically work in the framework of such maximal finite energy solutions, for which it is standard to check that the relation (7.3) holds. We call *global solution* a solution for which $T_{\max}=\infty$. We do not consider solutions of (7.1) backwards in time (*i.e.* for negative values of t).

7.2.2 Bound on global solutions

Gathering the arguments of [16] and [12, Proof of Lemma 2.7], we recall the following bound on global solutions of (7.1). Note that [16] is devoted to the undamped Klein-Gordon equation, but as suggested in the Introduction of [12], the proof extends to the damped case.

Theorem 7.6 ([12, 16])**.** *Any global solution of (7.1) is bounded in $H^1\times L^2$.*

Proof. Let $\vec{u}$ be a global solution of (7.1). Together with the energy functional $E(t):=E(\vec{u}(t))$ defined in (7.2) and satisfying (7.3), we will use the following quantities

$$M(t) := \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s,$$
$$W(t) := \frac{1}{2}\left(\|\partial_t u(t)\|_{L^2}^2 + \|\partial_x u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\right).$$

By direct computations using (7.1) and (7.2), we check the following relations

$$M'(t) = \int u(t)\partial_t u(t)\,\mathrm{d}x + \alpha\|u(t)\|_{L^2}^2 \tag{7.17}$$
$$= \int u(t)\partial_t u(t)\,\mathrm{d}x + 2\alpha\int_0^t\int u(s)\partial_t u(s)\,\mathrm{d}x\,\mathrm{d}s + \alpha\|u(0)\|_{L^2}^2, \tag{7.18}$$
$$M''(t) = \frac{p+3}{2}\|\partial_t u(t)\|_{L^2}^2 + \frac{p-1}{2}\left(\|\partial_x u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\right) - (p+1)E(t), \tag{7.19}$$
$$W'(t) = -2\alpha\|\partial_t u(t)\|_{L^2}^2 + \int f(u(t))\partial_t u(t)\,\mathrm{d}x. \tag{7.20}$$

In particular, from (7.17) and the Cauchy-Schwarz inequality,

$$|M'(t)| \leq (1+2\alpha)W(t). \tag{7.21}$$

Moreover, by (7.3) and (7.19),

$$M''(t) \geq (p-1)W(t) - (p+1)E(0). \tag{7.22}$$

The proof of the global bound now proceeds in three steps.

Step 1. We prove that

$$\liminf_{t\to\infty} M'(t) < \infty. \tag{7.23}$$

Proof of (7.23). We argue by contradiction, proving that $\lim_\infty M' = \infty$ implies the following inequality, for all t large enough,

$$(1+\epsilon)[M'(t)]^2 < M''(t)M(t) \quad \text{where} \quad \epsilon > 0. \tag{7.24}$$

Then, we reach a contradiction by a standard argument. Indeed, remark that (7.24) implies $\frac{\mathrm{d}^2}{\mathrm{d}t^2}[M^{-\epsilon}(t)] < 0$, and $\lim_\infty M' = \infty$ also implies $\lim_\infty M^{-\epsilon} = 0$. Thus, there exists $t_1 > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\epsilon}(t_1)] < 0$, and for all $t \geq t_1$,

$$0 \leq M^{-\epsilon}(t) \leq M^{-\epsilon}(t_1) + (t-t_1)\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\epsilon}(t_1)],$$

which is absurd for $t \geq t_1$ large enough.

Thus, we only need to prove (7.24) assuming $\lim_\infty M' = \infty$. On the one hand, by (7.18) and the Cauchy-Schwarz inequality, it holds

$$|M'| \leq \|u\|_{L^2}\|\partial_t u\|_{L^2} + 2\alpha\left(\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t \|\partial_t u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \alpha\|u(0)\|_{L^2}^2.$$

Let $\epsilon > 0$ to be chosen later, we estimate

$$\begin{aligned}|M'|^2 \leq (1+\epsilon)&\left[\|u\|_{L^2}\|\partial_t u\|_{L^2} + 2\alpha\left(\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t \|\partial_t u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right]^2\\ &+\left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|_{L^2}^4.\end{aligned}$$

Thus,

$$\begin{aligned}|M'|^2 &\leq (1+\epsilon)\left[\frac{1}{2}\|u\|_{L^2}^2 + \alpha\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s\right]\left[2\|\partial_t u\|_{L^2}^2 + 4\alpha\int_0^t \|\partial_t u(s)\|_{L^2}^2\,\mathrm{d}s\right]\\ &\quad+\left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|_{L^2}^4\\ &\leq (1+\epsilon)M\left[2\|\partial_t u\|_{L^2}^2 + 4\alpha\int_0^t \|\partial_t u(s)\|_{L^2}^2\,\mathrm{d}s\right] + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|_{L^2}^4.\end{aligned}$$

On the other hand, by (7.3) and (7.19),

$$\begin{aligned}M'' &= 2\|\partial_t u\|_{L^2}^2 + (p-1)W + 2\alpha(p+1)\int_0^t \|\partial_t u(s)\|_{L^2}^2\,\mathrm{d}s - (p+1)E(0)\\ &\geq (1+\epsilon)^3\left[2\|\partial_t u\|_{L^2}^2 + 4\alpha\int_0^t \|\partial_t u(s)\|_{L^2}^2\,\mathrm{d}s\right] + \frac{p-1}{2}W(t) - (p+1)E(0),\end{aligned}$$

by fixing any ϵ such that

$$0 < \epsilon < \left(\frac{p+7}{8}\right)^{\frac{1}{3}} - 1.$$

In particular, since $\lim_\infty W = \infty$ by (7.21) and the assumption $\lim_\infty M' = \infty$, we have for t large enough,

$$M'' \geq (1+\epsilon)^3 \left[2\|\partial_t u\|_{L^2}^2 + 4\alpha \int_0^t \|\partial_t u(s)\|^2 \, \mathrm{d}s \right].$$

Thus,

$$(1+\epsilon)^2 |M'|^2 \leq MM'' + \left(1 + \frac{1}{\epsilon}\right) \alpha^2 \|u(0)\|_{L^2}^4,$$

and using again $\lim_\infty M' = \infty$ we obtain (7.24) for any t large enough.

Step 2. We prove that

$$\sup_{t\in[0,\infty)} |M'(t)| < \infty. \tag{7.25}$$

Proof of (7.25). Combining (7.21) and (7.22), we obtain

$$M''(t) \geq \frac{p-1}{1+2\alpha} |M'(t)| - (p+1)E(0).$$

Let

$$H(t) = \frac{p-1}{1+2\alpha} M'(t) - (p+1)E(0).$$

Then, $H'(t) = \frac{p-1}{1+2\alpha} M''(t) \geq \frac{p-1}{1+2\alpha} H(t)$. If there exists $t \geq 0$ such that $H(t) > 0$, then $\lim_\infty H = \infty$, contradicting (7.23). It follows that for all $t \geq 0$,

$$M'(t) \leq \frac{1+2\alpha}{p-1} (p+1)E(0).$$

Similarly, let

$$K(t) = -\frac{p-1}{1+2\alpha} M'(t) + (p+1)E(0).$$

Then, $K'(t) = -\frac{p-1}{1+2\alpha} M''(t) \leq -\frac{p-1}{1+2\alpha} K(t)$. It follows that $K(t) \leq e^{-\frac{p-1}{1+2\alpha} t} K(0)$, for all $t \geq 0$. Thus,

$$M'(t) \geq -\frac{1+2\alpha}{p-1} \left((p+1)E(0) + |K(0)| \right).$$

and (7.25) is proved.

Step 3. Last, we prove the global bound

$$\sup_{t\in[0,\infty)} |W(t)| < \infty. \tag{7.26}$$

Proof of (7.26). We rewrite (7.22) as

$$W(t) \leq \frac{1}{p-1} M''(t) + \frac{p+1}{p-1} E(0).$$

Integrating on $(t, t+1)$ and using (7.25), we observe that

$$\sup_{t\geq 0} \int_t^{t+1} W(s) \, \mathrm{d}s < \infty. \tag{7.27}$$

Moreover, by (7.20),

$$W' \leq -2\alpha \|\partial_t u\|_{L^2}^2 + \int |u|^p |\partial_t u| \leq \frac{1}{2} \|\partial_t u\|_{L^2}^2 + \frac{1}{2} \int |u|^{2p} \leq W + \frac{1}{2} \int |u|^{2p}.$$

For $t \geq 1$ and $\tau \in (0,1)$, integrating on $(t-\tau, t)$, we find

$$\begin{aligned} W(t) &\leq W(t-\tau) + \int_{t-\tau}^{t} W(s)\,\mathrm{d}s + \frac{1}{2}\int_{t-\tau}^{t}\int |u(s)|^{2p}\,\mathrm{d}x\,\mathrm{d}s \\ &\leq W(t-\tau) + \int_{t-1}^{t} W(s)\,\mathrm{d}s + \frac{1}{2}\int_{t-1}^{t}\int |u(s)|^{2p}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Using the Sobolev inequality (in space-time) for the last term, we obtain, for some constants $C > 0$,

$$\begin{aligned} W(t) &\leq W(t-\tau) + \int_{t-1}^{t} W(s)\,\mathrm{d}s + C\|u\|^{2p}_{H^1((t-1,t)\times\mathbb{R})} \\ &\leq W(t-\tau) + \int_{t-1}^{t} W(s)\,\mathrm{d}s + C\left(\int_{t-1}^{t} W(s)\,\mathrm{d}s\right)^{p}. \end{aligned}$$

Integrating in $\tau \in (0,1)$ and using (7.27), we find (7.26). □

7.2.3 Decomposition of any global solution along a subsequence

Theorem 7.7 ([53, 80, 81]). *Any global solution $\vec{u}$ of (7.1)*

- *either converges to* 0, i.e. $\lim_{t\to\infty} \|\vec{u}(t)\|_{H^1\times L^2} = 0$;
- *or is asymptotically a (multi-)solitary wave along a subsequence of time: there exist $K \geq 1$, a sequence $t_n \to \infty$, a sequence $(\xi_{k,n})_{k\in\{1,\dots,K\}} \in \mathbb{R}^K$ and signs $\sigma_k = \pm 1$, for any $k \in \{1,\dots,K\}$, such that*

$$\lim_{n\to\infty}\left\{\left\|u(t_n) - \sum_{k=1}^{K}\sigma_k Q(\cdot - \xi_{k,n})\right\|_{H^1} + \|\partial_t u(t_n)\|_{L^2}\right\} = 0 \tag{7.28}$$

and in the case $K \geq 2$,

$$\lim_{n\to\infty} \xi_{k+1,n} - \xi_{k,n} = \infty \quad \textit{for any } 1 \leq k \leq K-1.$$

Remark 7.8. It is clear that if a global solution $\vec{u}$ satisfies (7.28) for two different sequences $(t_n)_n$ and $(t'_n)_n$, then the number $K \geq 1$ of solitary waves is the same for both sequences. Indeed, by monotonicity of the energy (7.3) and (7.11), it holds

$$\lim_{t\to\infty} E(\vec{u}(t)) = K\,E(Q,0) > 0. \tag{7.29}$$

Remark 7.9. The following stronger result holds in the framework of Theorem 7.7: for any sequence $(t_n)_n$ with $t_n \to \infty$, the multi-solitary wave behavior (7.28) is satisfied for a subsequence of $(t_n)_n$. This result, valid on any global solution of (7.1), is quite remarkable. However, it does not fully describe the asymptotic behavior of global solutions as $t \to \infty$, which is the objective of Theorem 7.1.

Remark 7.10. Note that [12, 16, 29, 53, 80, 81, 82] also apply to the multi-dimensional case, under suitable restrictions on the exponent p of the nonlinearity, or for radially symmetric solutions. However, for space dimensions greater than 1, the existence of bound states solutions of $\Delta w - w + f(w) = 0$ other than the ground state Q, together with the possibility of involved geometric configurations of solitary waves, complicate the analysis. This is why we restrict to dimension 1 in the present paper.

Proof. Let $\vec{u}$ be a global solution of (7.1); in particular, by Theorem 7.6, it is bounded in $H^1 \times L^2$. The proof proceeds in two steps.

Step 1. We prove that

$$\lim_{t\to\infty}\left\{\|\partial_t u(t)\|_{L^2} + \|\partial_t^2 u(t)\|_{H^{-1}}\right\} = 0. \tag{7.30}$$

With the notation of §7.2.1, the function $\vec{v}(t) = (v(t), \partial_t v(t)) = (\partial_t u(t), \partial_t^2 u(t))$ satisfies

$$\vec{v}(t) = S_\alpha(t)\vec{v}(0) + \int_0^t S_\alpha(t-s)(0, p|u(s)|^{p-1}v(s))\,\mathrm{d}s.$$

By the bound in $H^1 \times L^2$ and (7.3), it follows that $v \in L^2((0,\infty)\times\mathbb{R})$. Moreover, using estimate (7.16), $\|\cdot\|_{H^{-1}} \lesssim \|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{H^1}$, we have, for all $t \geq 0$,

$$\|\vec{v}(t)\|_{L^2\times H^{-1}} \lesssim e^{-\gamma t}\|\vec{v}(0)\|_{L^2\times H^{-1}} + \|u\|^{p-1}_{L^\infty([0,\infty),H^1)} \int_0^t e^{-\gamma(t-s)}\|v(s)\|_{L^2}\,\mathrm{d}s.$$

Splitting the integral $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$ in the last term and using the Cauchy-Schwarz inequality

$$\int_0^t e^{-\gamma(t-s)}\|v(s)\|_{L^2}\,\mathrm{d}s \lesssim e^{-\gamma t/2}\|v\|_{L^2((0,\infty)\times\mathbb{R})} + \|v\|_{L^2((t/2,\infty)\times\mathbb{R})},$$

which implies $\lim_{t\to\infty}\|\vec{v}(t)\|_{L^2\times H^{-1}} = 0$ and thus (7.30).

Step 2. Let $(t_n)_n$ be any sequence such that $t_n \to \infty$ and let $u_n(x) = u(t_n, x)$. Then, by (7.30) and equation (7.1), it follows that

$$\lim_{n\to\infty}\|\partial_x^2 u_n - u_n + |u_n|^{p-1}u_n\|_{H^{-1}} = 0.$$

Moreover, the sequence $(u_n)_n$ is bounded in H^1. Then, the alternative stated in the Theorem follows directly from results from concentration-compactness arguments in [80, Appendix A] and [81, Theorem III.4]. In the present framework, we use [81, Theorem III.4] in space dimension 1 and with the constant coefficient elliptic operator $-\partial_x^2 + 1$, which simplifies the statement. Observe that in dimension 1, we enjoy the fact that the only non trivial solutions of $\partial_x^2 w - w + f(w) = 0$ are $w = \pm Q$, up to space translation.

In the case where $\lim_{n\to\infty}\|\vec{u}(s_n)\|_{H^1\times L^2} = 0$, for some sequence of time $(s_n)_n$, $s_n \to \infty$, then it follows from (7.3) that $\lim_{t\to\infty} E(\vec{u}(t)) = 0$. Thus, by (7.11) and the previous arguments applied to any sequence $(t_n)_n$, with $t_n \to \infty$, there exists a subsequence $(t_{n'})_{n'}$ such that $\lim_{n'\to\infty}\|\vec{u}(t_{n'})\|_{H^1\times L^2} = 0$. This implies that $\lim_{t\to\infty}\|\vec{u}(t)\|_{H^1\times L^2} = 0$ as stated in the first part of the alternative. □

7.3 Dynamics close to decoupled solitary waves

In this section, we prove general results on solutions of (7.1) close to the sum of $K \geq 1$ decoupled solitary waves. For any $k \in \{1, \cdots, K\}$, let $\sigma_k = \pm 1$ and let $t \mapsto (z_k(t), \ell_k(t)) \in \mathbb{R}^2$ be $\mathcal{C}^1$ functions such that

$$\sum_{k=1}^K |\ell_k| \ll 1 \quad \text{and, if } K \geq 2, \text{ for any } k = 1, \ldots, K-1, \quad z_{k+1} - z_k \gg 1. \tag{7.31}$$

For $k \in \{1, \cdots, K\}$, define

$$Q_k = \sigma_k Q(\cdot - z_k), \quad \vec{Q}_k = \begin{pmatrix} Q_k \\ -\ell_k \partial_x Q_k \end{pmatrix}, \tag{7.32}$$

and similarly (see (7.14)-(7.15))

$$Y_k = \sigma_k Y(\cdot - z_k), \quad \vec{Y}_k^\pm = \sigma_k \vec{Y}^\pm(\cdot - z_k), \quad \vec{Z}_k^\pm = \sigma_k \vec{Z}^\pm(\cdot - z_k).$$

Set

$$R = \sum_{k=1}^K Q_k, \quad \vec{R} = \sum_{k=1}^K \vec{Q}_k, \quad G = f\left(\sum_{k=1}^K Q_k\right) - \sum_{k=1}^K f(Q_k). \tag{7.33}$$

Remark 7.11. To study precisely the dynamics of a solution of (7.1) containing K solitons, it is of course necessary to introduce a time-dependent position parameter, denoted by z_k, for each soliton Q_k ($k = 1, \dots, K$). As it appears clearly in (7.48), the additional parameter ℓ_k is related to the time derivative of z_k, *i.e.* the speed of the soliton Q_k. In the case where $K \geq 2$, the dynamics of ℓ_k is mainly driven by the interactions between the various solitons, see (7.52), (7.53). This is a main point of this paper. The parameters $(\ell_k)_k$ are thus introduced to track this nonlinear effect, and then its consequences on the positions of the solitons. This strategy of introducing time-dependent parameters related to orthogonality conditions (see (7.46)) to study the dynamics of solitons is often called *modulation*.

Note that for wave-type equations, the displacement of solitons with speed ℓ is related to the Lorentz transformation, which involves a space dilation by a factor $\frac{1}{\sqrt{1-\ell^2}}$. In the present paper, $\lim_{t\to\infty} \ell_k(t) = 0$ for all k and thus the dilation in the formula of Q_k is neglected. This will induce error terms (like $\sum_k \ell_k^2$ in (7.48), (7.49)) that can be handled.

7.3.1 Leading order of the nonlinear interactions

Lemma 7.12. *Assuming* (7.31), *for any* $k, k' \in \{1, \dots, K\}$, $k' \neq k$, *it holds.*

(i) Bounds. *For any* $0 < m' < m$,

$$\int |Q_{k'} Q_k|^m \lesssim e^{-m'|z_{k'}-z_k|}, \quad \int |Q_{k'}||Q_k|^{1+m} \lesssim e^{-|z_{k'}-z_k|}, \tag{7.34}$$

$$\int \left| F(R) - \sum_{k=1}^{K} F(Q_k) - \sum_{k\neq k'} f(Q_k) Q_{k'} \right| \lesssim \sum_{k=1}^{K-1} e^{-\frac{5}{4}(z_{k+1}-z_k)}, \tag{7.35}$$

$$\|G\|_{L^2} \lesssim \sum_{k'\neq k} \|Q_{k'}^{p-1} Q_k\|_{L^2} \lesssim \sum_{k=1}^{K-1} e^{-(z_{k+1}-z_k)}. \tag{7.36}$$

(ii) Asymptotics.

$$\left| \langle f(Q_k), Q_{k'} \rangle - \sigma_k \sigma_{k'} c_1 \kappa e^{-|z_k - z_{k'}|} \right| \lesssim e^{-\frac{3}{2}|z_k - z_{k'}|} \tag{7.37}$$

where

$$\kappa := \frac{c_Q}{c_1} \int Q^p(x) e^{-x} \, \mathrm{d}x > 0 \quad \textit{and} \quad c_1 := \|Q'\|_{L^2}^2. \tag{7.38}$$

(iii) Leading order interactions. *Let any* $1 < \theta < \min(p-1, \frac{3}{2})$.

- *If* $K = 1$ *then* $G = 0$;
- *If* $K \geq 2$ *then*

$$\left| \langle G, \partial_x Q_1 \rangle + c_1 \kappa \sigma_1 \sigma_2 e^{-(z_2 - z_1)} \right| \lesssim \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}; \tag{7.39}$$

$$\left| \langle G, \partial_x Q_K \rangle - c_1 \kappa \sigma_{K-1} \sigma_K e^{-(z_K - z_{K-1})} \right| \lesssim \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}; \tag{7.40}$$

- *If* $K \geq 3$ *then, for any* $k \in \{2, \dots, K-1\}$,

$$\begin{aligned} &\left| \langle G, \partial_x Q_k \rangle - c_1 \kappa \sigma_k \left[\sigma_{k-1} e^{-(z_k - z_{k-1})} - \sigma_{k+1} e^{-(z_{k+1}-z_k)} \right] \right| \\ &\lesssim \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}. \end{aligned} \tag{7.41}$$

Proof. Proof of (i). These estimates are direct consequences of the decay properties of Q in (7.10) and $p > 2$. See details in [29, proof of Lemma 2.1].

Proof of (ii). We claim the following estimate for $z \gg 1$.

$$\left| \int Q^p(y)Q(y+z)\,\mathrm{d}y - c_1\kappa e^{-z} \right| \lesssim e^{-\frac{3}{2}z}. \tag{7.42}$$

Observe that (7.37) follows directly from (7.42).

Now, we prove (7.42). First, for $|y| < \frac{3}{4}z$, using (7.10), we have

$$\left|Q(y+z) - c_Q e^{-(y+z)}\right| \lesssim e^{-2(y+z)} \lesssim e^{-2z}e^{2|y|},$$

and so

$$\left| \int_{|y|<\frac{3}{4}z} Q^p(y)\left[Q(y+z) - c_Q e^{-(y+z)}\right]\mathrm{d}y \right| \lesssim e^{-2z}.$$

Second, using (7.10) and $p > 2$, it holds

$$\int_{|y|>\frac{3}{4}z} Q^p(y)Q(y+z)\,\mathrm{d}y + \int_{|y|>\frac{3}{4}z} Q^p(y)e^{-(y+z)}\,\mathrm{d}y \lesssim e^{-\frac{3}{2}z}.$$

Gathering these estimates, we have proved (7.42).

Proof of (iii). We treat the case $k \in \{2, \dots, K-1\}$ for $K \geq 3$. Other cases are similar. On the one hand, using Taylor formula, it holds

$$\left| G - p|Q_k|^{p-1} \sum_{k'\neq k} Q_{k'} \right| \lesssim |Q_k|^{p-2} \sum_{k'\neq k} |Q_{k'}|^2 + \sum_{k'\neq k, l\neq k'} |Q_{k'}|^{p-1}|Q_l|.$$

Thus, using (7.34), we have for any $1 < \theta < \min(p-1, 2)$,

$$\begin{aligned} \left| \langle G, \partial_x Q_k\rangle - \sum_{k'\neq k} \langle p|Q_k|^{p-1}Q_{k'}, \partial_x Q_k\rangle \right| &\lesssim \sum_{l\neq k'} \int |Q_{k'}|^{p-1}|Q_l|^2\,\mathrm{d}x \\ &\lesssim \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}. \end{aligned}$$

On the other hand, by direct computation, integrating by parts and using the proof of (7.42), we obtain, for any $1 < \theta < \min(p-1, \frac{3}{2})$,

$$\begin{aligned} &\sum_{k'\neq k} \langle p|Q_k|^{p-1}Q_{k'}, \partial_x Q_k\rangle \\ &= -\sum_{k'\neq k} \sigma_k\sigma_{k'} \int Q^p(y)\partial_x Q(y + z_k - z_{k'})\,\mathrm{d}y \\ &= \sum_{k'<k} \sigma_{k'}\sigma_k c_1\kappa e^{-(z_k - z_{k'})} - \sum_{k'>k} \sigma_k\sigma_{k'} c_1\kappa e^{-(z_{k'}-z_k)} + O\left(\sum_{l=1}^{K-1} e^{-\frac{3}{2}(z_{l+1}-z_l)}\right) \\ &= \sigma_{k-1}\sigma_k c_1\kappa e^{-(z_k - z_{k-1})} - \sigma_k\sigma_{k+1}c_1\kappa e^{-(z_{k+1}-z_k)} + O\left(\sum_{l=1}^{K-1} e^{-\frac{3}{2}(z_{l+1}-z_l)}\right). \end{aligned}$$

Indeed, for example, if $k \geq 3$, one has

$$z_k - z_{k-2} = (z_k - z_{k-1}) + (z_{k-1} - z_{k-2}) \geq 2\min(z_k - z_{k-1}; z_{k-1} - z_{k-2}), \tag{7.43}$$

which allows us to consider interactions between to non neighbor solitary waves as error terms. Gathering the above estimates, we find (7.41). □

7.3.2 Decomposition close to the sum of solitary waves

Lemma 7.13. *Let $\vec{u} = (u, \partial_t u)$ be a solution of (7.1) on the interval $[T_1, T_2]$ and let $K \geqslant 1$. Assume that*

$$\sup_{t\in[T_1,T_2]} \left\{ \inf_{\xi_{k+1}-\xi_k>|\log\gamma|} \left\| u(t) - \sum_{k=1}^{K} \sigma_k Q(\cdot - \xi_k) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} \right\} < \gamma, \tag{7.44}$$

for some small $\gamma > 0$. Then, there exist unique $\mathcal{C}^1$ functions

$$t \in [T_1, T_2] \mapsto (z_k(t), \ell_k(t))_{k\in\{1,\dots,K\}} \in \mathbb{R}^{2K},$$

such that the solution $\vec{u}$ decomposes as

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \sum_{k=1}^{K} \vec{Q}_k + \vec{\varepsilon}, \quad \vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \tag{7.45}$$

with the following properties on $[T_1, T_2]$.

(i) Orthogonality and smallness. *For any $k = 1, \dots, K$,*

$$\langle \varepsilon, \partial_x Q_k \rangle = \langle \eta, \partial_x Q_k \rangle = 0 \tag{7.46}$$

and

$$\|\vec{\varepsilon}\|_{H^1\times L^2} + \sum_{l=1}^{K} |\ell_l| + \sum_{l=1}^{K-1} e^{-2(z_{l+1}-z_l)} \lesssim \gamma.$$

(ii) Equation of $\vec{\varepsilon}$.

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_\varepsilon \\ \partial_t \eta = \partial_x^2 \varepsilon - \varepsilon + f(R+\varepsilon) - f(R) - 2\alpha\eta + \mathrm{Mod}_\eta + G \end{cases} \tag{7.47}$$

where

$$\mathrm{Mod}_\varepsilon = \sum_{k=1}^{K} (\dot{z}_k - \ell_k)\, \partial_x Q_k,$$
$$\mathrm{Mod}_\eta = \sum_{k=1}^{K} \left(\dot{\ell}_k + 2\alpha\ell_k\right) \partial_x Q_k - \sum_{k=1}^{K} \ell_k \dot{z}_k \partial_x^2 Q_k.$$

(iii) Control of the geometric parameters. *For $k = 1, \dots, K$,*

$$|\dot{z}_k - \ell_k| \lesssim \|\vec{\varepsilon}\|^2_{H^1\times L^2} + \sum_{l=1}^{K} |\ell_l|^2, \tag{7.48}$$

$$|\dot{\ell}_k + 2\alpha\ell_k| \lesssim \|\vec{\varepsilon}\|^2_{H^1\times L^2} + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-(z_{l+1}-z_l)}. \tag{7.49}$$

(iv) Control of the exponential directions. *For $k = 1, \cdots, K$, let*

$$a_k^{\pm} = \langle \vec{\varepsilon}, \vec{Z}_k^{\pm} \rangle. \tag{7.50}$$

Then,

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} a_k^{\pm} - \nu^{\pm} a_k^{\pm} \right| \lesssim \|\vec{\varepsilon}\|^2_{H^1\times L^2} + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-(z_{l+1}-z_l)}. \tag{7.51}$$

Remark 7.14. In the above estimates, if $K = 1$ then the terms $\sum_{l=1}^{K-1}$ are 0.

Proof. See [29, proof of Lemma 2.2]. □

Lemma 7.15 (Refined equation for ℓ_k)**.** *In the context of Lemma 7.13, assume that $K \geq 2$. Let $1 < \theta < \min(p-1, \frac{3}{2})$. Then,*

$$\left|\dot{\ell}_1 + 2\alpha\ell_1 - \kappa\sigma_1\sigma_2 e^{-(z_2-z_1)}\right| + \left|\dot{\ell}_K + 2\alpha\ell_K + \kappa\sigma_{K-1}\sigma_K e^{-(z_K-z_{K-1})}\right| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K}|\ell_l|^2 + \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}. \quad (7.52)$$

Moreover, if $K \geq 3$ then for any $k \in \{2, \dots, K-1\}$,

$$\left|\dot{\ell}_k + 2\alpha\ell_k + \kappa\sigma_k\left[\sigma_{k-1}e^{-(z_k-z_{k-1})} - \sigma_{k+1}e^{-(z_{k+1}-z_k)}\right]\right| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K}|\ell_l|^2 + \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}. \quad (7.53)$$

Proof. We treat the case $k \in \{2, \dots, K-1\}$ for $K \geq 3$. Other cases are similar. First,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, \partial_x Q_k\rangle = \langle \partial_t\eta, \partial_x Q_k\rangle + \langle \eta, \partial_t\partial_x Q_k\rangle = 0.$$

Thus, using (7.46) and (7.47),

$$\begin{aligned} 0 =& \langle \partial_x^2\varepsilon - \varepsilon + f'(Q_k)\varepsilon, \partial_x Q_k\rangle + \langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_x Q_k\rangle \\ &+ \langle \big(f'(R) - f'(Q_k)\big)\varepsilon, \partial_x Q_k\rangle + \langle G, \partial_x Q_k\rangle + \langle \mathrm{Mod}_\eta, \partial_x Q_k\rangle - \langle \eta, \dot{z}_k\partial_x^2 Q_k\rangle. \end{aligned}$$

Since $\partial_x Q_k$ satisfies $\partial_x^2\partial_x Q_k - \partial_x Q_k + f'(Q_k)\partial_x Q_k = 0$, by integration by parts, the first term is zero. Next, by Taylor expansion (using $p > 2$), we have the pointwise estimate

$$\left|f(R+\varepsilon) - f(R) - f'(R)\varepsilon\right| \lesssim |\varepsilon|^p + |\varepsilon|^2\sum_{l=1}^{K}|Q_l|^{p-2}$$

and

$$\left|f'(R) - f'(Q_k)\right| \lesssim |Q_k|^{p-2}\sum_{k'\neq k}|Q_{k'}| + \sum_{k'\neq k}\left|Q_{k'}\right|^{p-1}.$$

Thus, using $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{H^1}$,

$$\left|\langle f(R+\varepsilon) - f(R) - f'(R)\varepsilon, \partial_x Q_k\rangle\right| \lesssim \|\varepsilon\|_{H^1}^2$$

and by the Cauchy-Schwarz inequality and (7.36),

$$\left|\langle (f'(R) - f'(Q_k))\varepsilon, \partial_x Q_k\rangle\right| \lesssim \|\varepsilon\|_{H^1}^2 + \sum_{k=1}^{K-1} e^{-2(z_{k+1}-z_k)}.$$

By direct computation, we obtain

$$\begin{aligned} \langle \mathrm{Mod}_\eta, \partial_x Q_k\rangle =& \big(\dot{\ell}_k + 2\alpha\ell_k\big)\|Q'\|_{L^2}^2 + \sum_{k'\neq k}\big(\dot{\ell}_{k'} + 2\alpha\ell_{k'}\big)\langle \partial_x Q_{k'}, \partial_x Q_k\rangle \\ &- \sum_{k'\neq k}\big(\ell_{k'}\dot{z}_{k'}\big)\langle \partial_x^2 Q_{k'}, \partial_x Q_k\rangle. \end{aligned}$$

Thus, using the equation of Q, (7.34), (7.38) and (7.48), we obtain

$$\langle \mathrm{Mod}_\eta, \partial_x Q_k\rangle = c_1\big(\dot\ell_k + 2\alpha\ell_k\big) + O\Big(\sum_{k'\neq k} |\dot\ell_{k'} + 2\alpha\ell_{k'}| e^{-\frac{3}{4}|z_{k'}-z_k|}\Big) + O\Big(\|\vec\varepsilon\|^2_{H^1\times L^2} + \sum_{l=1}^K |\ell_l|^2\Big).$$

Note that, by (7.48) and the Cauchy-Schwarz inequality, we obtain

$$\big|\langle \eta, \dot z_k \partial_x^2 Q_k\rangle\big| \lesssim \big(|\dot z_k - \ell_k| + |\ell_k|\big)\|\vec\varepsilon\|_{H^1\times L^2} \lesssim \|\vec\varepsilon\|^2_{H^1\times L^2} + \sum_{k=1}^K |\ell_k|^2.$$

Gathering above estimates and using (7.39)-(7.40)-(7.41), we obtain

$$\begin{aligned}&\Big|\dot\ell_k + 2\alpha\ell_k + \kappa\sigma_k\Big[\sigma_{k-1}e^{-(z_k-z_{k-1})} - \sigma_{k+1}e^{-(z_{k+1}-z_k)}\Big]\Big| \\ &\lesssim \|\vec\varepsilon\|^2_{H^1\times L^2} + \sum_{l=1}^K |\ell_l|^2 + \sum_{k'\neq k} |\dot\ell_{k'} + 2\alpha\ell_{k'}| e^{-\frac{3}{4}|z_{k'}-z_k|} + \sum_{l=1}^{K-1} e^{-\theta(z_{l+1}-z_l)}.\end{aligned} \tag{7.54}$$

We obtain (7.53) by combining (7.54) for all $k \in \{1,\dots,K\}$. □

7.3.3 Energy estimates

For $\mu > 0$ small to be chosen, we denote $\rho = 2\alpha - \mu$. Consider the nonlinear energy functional

$$\mathcal{E} = \int \big\{(\partial_x\varepsilon)^2 + (1-\rho\mu)\varepsilon^2 + (\eta+\mu\varepsilon)^2 - 2[F(R+\varepsilon) - F(R) - f(R)\varepsilon]\big\}.$$

We recall the following energy estimates.

Lemma 7.16. *There exists $\mu > 0$ such that in the context of Lemma 7.13, the following hold.*

(i) Coercivity and bound.

$$\mu\|\vec\varepsilon\|^2_{H^1\times L^2} - \frac{1}{2\mu}\sum_{k=1}^K \big((a_k^+)^2 + (a_k^-)^2\big) \le \mathcal{E} \le \frac{1}{\mu}\|\vec\varepsilon\|^2_{H^1\times L^2}. \tag{7.55}$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leqslant -2\mu\mathcal{E} + \frac{1}{\mu}\|\vec\varepsilon\|_{H^1\times L^2}\Big[\|\vec\varepsilon\|^2_{H^1\times L^2} + \sum_{k=1}^K |\ell_k|^2 + \sum_{k=1}^{K-1} e^{-(z_{k+1}-z_k)}\Big]. \tag{7.56}$$

Remark 7.17. The above lemma is valid for any $\mu > 0$ small enough. For future needs, we fix such a $\mu > 0$ satisyfing the following additional smallness condition

$$\mu \le \min\left(1, \alpha, |\nu_-|\right). \tag{7.57}$$

Remark 7.18. In the case where $K = 1$, the term $\sum_{k=1}^{K-1}$ due to the nonlinear interactions does not appear in (7.56).

Proof. See [29, proof of Lemma 2.4]. □

7.3.4 Time evolution analysis

We introduce new parameters and functionals to analyse the time evolution of solutions in the framework of Lemma 7.13. First, we set, for $k = 1, \dots, K$,

$$y_k = z_k + \frac{\ell_k}{2\alpha},$$

and for $k = 1, \dots, K-1$ (when $K \geq 2$),

$$r_k = y_{k+1} - y_k \gg 1.$$

Second, we define

$$\mathcal{K}_+ = \{k = 1, \dots, K-1 : \sigma_k = \sigma_{k+1}\}, \quad F_+ = \sum_{k \in \mathcal{K}_+} e^{-r_k},$$
$$\mathcal{K}_- = \{k = 1, \dots, K-1 : \sigma_k = -\sigma_{k+1}\}, \quad F_- = \sum_{k \in \mathcal{K}_-} e^{-r_k}.$$

Third, we introduce notation for the damped components of the solution

$$\mathcal{F} = \mathcal{E} + \mathcal{G}, \quad \mathcal{G} = \sum_{k=1}^{K} |\ell_k|^2 + \frac{1}{2\mu} \sum_{k=1}^{K} (a_k^-)^2,$$

and for all the components except distances

$$\mathcal{N} = \left[\|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{k=1}^{K} |\ell_k|^2 \right]^{\frac{1}{2}}.$$

Last, we define

$$b = \sum_{k=1}^{K} (a_k^+)^2, \quad \mathcal{M} = \frac{1}{\mu^2} \left(\mathcal{F} - \frac{b}{2\nu^+} \right).$$

We rewrite the estimates of Lemmas 7.13 and 7.16 using such notation.

Lemma 7.19. *Assume $K \geq 2$. Let any $1 < \theta < \min(p-1, \frac{5}{4})$. In the context of Lemma 7.13, the following hold.*

(i) Comparison with original variables. *For $k = 1, \dots, K-1$,*

$$\left| r_k - (z_{k+1} - z_k) \right| \lesssim \mathcal{N}, \tag{7.58}$$

$$\mu \mathcal{N}^2 \leq \mu \|\vec{\varepsilon}\|_{H^1 \times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 \leq \mathcal{F} + \frac{b}{2\mu} \lesssim \mathcal{N}^2. \tag{7.59}$$

(ii) ODE for the distances between solitary waves. *The equation for the evolution of y_k is*

$$\begin{cases} \dot{y}_1 = \dfrac{\kappa}{2\alpha} \sigma_1 \sigma_2 e^{-r_1} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big), \\ \dot{y}_k = -\dfrac{\kappa}{2\alpha} \sigma_{k-1} \sigma_k e^{-r_{k-1}} + \dfrac{\kappa}{2\alpha} \sigma_k \sigma_{k+1} e^{-r_k} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big), \\ \dot{y}_K = -\dfrac{\kappa}{2\alpha} \sigma_{K-1} \sigma_K e^{-r_{K-1}} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big), \end{cases} \tag{7.60}$$

for $2 \leqslant k \leqslant K-1$. Moreover, there exists $\lambda > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{F_+} \right] \leqslant -\lambda + \frac{1}{\lambda F_+} \left(\mathcal{N}^2 + F_+^\theta + F_-^\theta \right), \tag{7.61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{F_-} \right] \geqslant \lambda - \frac{1}{\lambda F_-} \left(\mathcal{N}^2 + F_+^\theta + F_-^\theta \right). \tag{7.62}$$

(iii) Exponential instability.

$$|\dot{b} - 2\nu^+ b| \lesssim \mathcal{N}^3 + \mathcal{N}(F_+ + F_-). \tag{7.63}$$

(iv) Damped components.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} &\lesssim \mathcal{N}^3 + \mathcal{N}(F_+ + F_-), \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} + 2\mu\mathcal{G} &\lesssim \mathcal{N}^3 + \mathcal{N}(F_+ + F_-). \end{aligned} \tag{7.64}$$

(v) Liapunov type functional.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leq -\mathcal{N}^2 + C\big(F_+^2 + F_-^2\big). \tag{7.65}$$

(vi) Refined estimates for the distance. *Setting*

$$R_+ = \frac{1}{F_+}\exp\left(\lambda^{-1}\mathcal{M}\right) \quad \textit{and} \quad R_- = \frac{1}{F_-}\exp\left(-3\lambda^{-1}\mathcal{M}\right),$$

where λ *is given in* (ii)*, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}R_+ \leq \left(-\lambda + \frac{2}{\lambda F_+}\left(F_+^\theta + F_-^\theta\right)\right)\exp\left(\lambda^{-1}\mathcal{M}\right), \tag{7.66}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}R_- \geq \left(\lambda + \frac{2}{\lambda F_-}\mathcal{N}^2 - \frac{2}{\lambda F_-}\left(F_+^\theta + F_-^\theta\right)\right)\exp\left(-3\lambda^{-1}\mathcal{M}\right). \tag{7.67}$$

Remark 7.20. When using the quantities $F_\pm$ and $R_\pm$, we tacitly assume that $\mathcal{K}_\pm$ is not empty. Otherwise, the respective quantities are ignored. For example, if the set $\mathcal{K}_+$ is empty, then (7.67) rewrites:

$$\frac{\mathrm{d}}{\mathrm{d}t}R_- \geq \left(\lambda + \frac{2}{\lambda}\frac{\mathcal{N}^2}{F_-} - \frac{2}{\lambda}F_-^{\theta-1}\right)\exp\left(-3\lambda^{-1}\mathcal{M}\right).$$

When $K = 1$, the quantities $F_\pm$ and $R_\pm$ are systematically 0 and the variables r_k are not defined. The estimates (7.59), (7.63), (7.64) and (7.65) hold in this context.

Remark 7.21. Lemmas 7.13, 7.15 and 7.16 provide the main order of the time evolution of the parameters z_k, ℓ_k, $a_k^\pm$ as well as a control on the energy-type quantity on $\vec{\varepsilon}$. The new parameters introduced above are variants of these quantities, whose equations are better suited to control error terms. For example, the introduction of y_k allows to simplify the system in (z_k, ℓ_k) by discarding the time derivative of ℓ_k, which is of lower order in this context. The parameter b gathers the K parameters related to the instability of the solitons, while the quantities $\mathcal{F}$ and $\mathcal{G}$ gather the damped components of the solution. The evolution of the distances r_k between the solitons is analysed through the parameters F_+ and F_-. We see clearly from (7.61) and (7.62) that consecutive solitons with same signs will essentially attract each other, while consecutive solitons with opposite signs will repel each other. The quantities $R_\pm$ are similar to $1/F_\pm$, but their equations will be easier to handle. Indeed, comparing (7.64) with (7.66), (7.67), we see that the introduction of $\mathcal{M}$ (similar to a Liapunov functional for $\vec{\varepsilon}$) in a multiplicative exponential factor, removes the contribution of the quantity $\mathcal{N}$ in the error terms. Such estimates will provide a good understanding of the evolution of the distances between the solitons based on a weak information on the size of $\vec{\varepsilon}$, for example in the proof of the alternate sign property Proposition 7.25.

Proof. Proof of (i): (7.58) follows from triangle inequality and the definition of $\mathcal{N}$ and (7.59) follows directly from (7.55).

Proof of (ii). In this computation, we use the convention that terms involving y_k or σ_k for $k \leq 0$ or $k \geq K+1$ are zero, for example by setting $\sigma_0 = \sigma_{K+1} = 0$. Similarly, when e^{-r_0} and e^{-r_K} appears,

the corresponding term has to be ignored. By direct computation and using (7.48), (7.52)-(7.53), we obtain for $k = 1, \dots, K$,

$$\dot{y}_k = \dot{z}_k + \frac{\dot{\ell}_k}{2\alpha} = -\frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} + \frac{\kappa}{2\alpha}\sigma_k\sigma_{k+1}e^{-r_k} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big).$$

It follows that, for any $k = 1, \dots, K-1$,

$$\begin{aligned}\dot{r}_k = \dot{y}_{k+1} - \dot{y}_k = &-\frac{\kappa}{\alpha}\sigma_k\sigma_{k+1}e^{-r_k} + \frac{\kappa}{2\alpha}\sigma_{k+1}\sigma_{k+2}e^{-r_{k+1}} + \frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} \\ &+ O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big).\end{aligned}$$

On the right-hand side of the above expression, the first term is always present for $k = 1, \dots, K-1$, while the second and third terms might be zero depending on the value of k. For $k \in \mathcal{K}_+$, it holds $\sigma_k = \sigma_{k+1}$ and one sees that

$$\dot{r}_k = -\frac{\kappa}{\alpha}e^{-r_k} + \frac{\kappa}{2\alpha}\sigma_{k+1}\sigma_{k+2}e^{-r_{k+1}} + \frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} + O\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big),$$

with the same observation concerning the second and third terms on the right-hand side. Thus,

$$\sum_{k\in\mathcal{K}_+} \dot{r}_k e^{-r_k} = -\frac{\kappa}{2\alpha}S_+ + O\big(F_+\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big)\big). \tag{7.68}$$

where S_+ denotes

$$S_+ = \sum_{k\in\mathcal{K}_+}\left[2e^{-2r_k} - \sigma_{k+1}\sigma_{k+2}e^{-(r_k+r_{k+1})} - \sigma_{k-1}\sigma_k e^{-(r_{k-1}+r_k)}\right].$$

We claim that there exists $\tilde{\lambda} > 0$ such that S_+ satisfies

$$S_+ \geq \tilde{\lambda}\sum_{k\in\mathcal{K}_+} e^{-2r_k}. \tag{7.69}$$

Indeed, first, recall that the symmetric matrix of size n

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

is definite positive by the Sylvester criterion since for any $j \in \{1, \dots n\}$, the jth leading principal minor of this matrix, *i.e.* the determinant of its upper-left $j \times j$ sub-matrix, is positive (its value is $j+1$).

Second, observe that in the sum defining S_+, for given $k \in \mathcal{K}_+$, if $k-1 \notin \mathcal{K}_+$, then $\sigma_{k-1}\sigma_k = 1$ (if $k \geq 2$) or 0 (if $k = 1$) and thus the corresponding term is positive or zero and can be ignored in establishing a lower bound for S_+. The same property is true for the term corresponding to $\sigma_{k+1}\sigma_{k+2}$ if $k+1 \notin \mathcal{K}_+$. Letting $n = \mathrm{card}(\mathcal{K}_+)$, this observation justifies that S_+ is lower bounded by the quadratic form associated to the matrix A_n taken at the vector of $\mathbb{R}^n$ of components $\{e^{-r_k} : k \in \mathcal{K}_+\}$. This shows that (7.69) holds for some $\tilde{\lambda} > 0$.

It follows from (7.68) and (7.69) that there exists $\lambda > 0$ such that

$$\sum_{k\in\mathcal{K}_+} \dot{r}_k e^{-r_k} \leqslant -\lambda F_+^2 + \frac{1}{\lambda}F_+\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big).$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{F_+}\right] = \frac{1}{F_+^2}\sum_{k\in\mathcal{K}_+}\dot{r}_k e^{-r_k} \leqslant -\lambda + \frac{1}{\lambda F_+}\big(\mathcal{N}^2 + F_+^\theta + F_-^\theta\big).$$

The estimate (7.62) concerning $\frac{\mathrm{d}}{\mathrm{d}t}[\frac{1}{F_-}]$ is proved similarly.

Proof of (iii). It follows from (7.51) and the bound $|a_k^+| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2} \leq \mathcal{N}$.

Proof of (iv). By the definition of $\mathcal{F}$ and (7.49), (7.51), (7.56), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + 2\sum_{k=1}^K \dot{\ell}_k\ell_k + \frac{1}{\mu}\sum_{k=1}^K \dot{a}_k^- a_k^- \\ &\leqslant -2\mu\mathcal{E} - 4\alpha\sum_{k=1}^K \ell_k^2 + \frac{\nu^-}{\mu}\sum_{k=1}^K (a_k^-)^2 + O\big(\mathcal{N}^3 + \mathcal{N}\big(F_+ + F_-\big)\big).\end{aligned}$$

Since $0 < \mu \leq \alpha$ and $0 < \mu \leq |\nu^-|$ (see (7.57)), we obtain (7.64) for $\mathcal{F}$. The proof for $\mathcal{G}$ is the same.

Proof of (v). From (7.63) and (7.64), we estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} = \frac{1}{\mu^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} - \frac{1}{2\nu^+}\frac{\mathrm{d}}{\mathrm{d}t}b\right) \leqslant -\frac{2}{\mu}\left(\mathcal{F} + \frac{b}{2\mu}\right) + O\big(\mathcal{N}^3 + \mathcal{N}\big(F_+ + F_-\big)\big).$$

Thus, from (7.59) and then for $\mathcal{N}$ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leqslant -2\mathcal{N}^2 + O\big(\mathcal{N}^3 + \mathcal{N}\big(F_+ + F_-\big)\big) \leq -\mathcal{N}^2 + O\big(F_+^2 + F_-^2\big).$$

Proof of (vi). From (7.61) and (7.65), we estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_+ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{F_+}\right] + \frac{1}{\lambda F_+}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}\right)\exp(\lambda^{-1}\mathcal{M}) \\ &\leqslant \left(-\lambda + \frac{1}{\lambda F_+}\big(F_+^\theta + F_-^\theta + CF_+^2 + CF_-^2\big)\right)\exp(\lambda^{-1}\mathcal{M}),\end{aligned}$$

which implies (7.66). The estimate for $\frac{\mathrm{d}}{\mathrm{d}t}R_-$ is proved similarly. Note that the coefficient 3 of the factor $\exp(-3\lambda^{-1}\mathcal{M})$ in the definition R_- allows us to obtain a positive factor $\mathcal{N}^2$ in the right-hand side of (7.67). □

7.3.5 Long-time energy asymptotics

Lemma 7.22. *Let any $1 < \theta < \min(p-1, \frac{5}{4})$. In the context of Lemmas 7.13 and 7.19, it holds*

$$E(\vec{u}) = KE(Q,0) - c_1\kappa F_+ + c_1\kappa F_- + O(\mathcal{N}^2 + F_+^\theta + F_-^\theta). \tag{7.70}$$

Proof. Expanding $E(u,\partial_t u)$ using the decomposition (7.45), integration by parts, the equation $-\partial_x^2 Q + Q - f(Q) = 0$ and the definition of G in (7.33), we find

$$\begin{aligned}2E(u,\partial_t u) &= \int |\partial_t u|^2 + 2E\left(R,0\right) - 2\int G\varepsilon \\ &\quad + \int \left(|\partial_x\varepsilon|^2 + \varepsilon^2 - 2F(R+\varepsilon) + 2F(R) + 2f(R)\varepsilon\right).\end{aligned}$$

Thus, using (7.36), the Cauchy-Schwarz and Sobolev inequalities, it holds

$$2E(u,\partial_t u) = \int |\partial_t u|^2 + 2E\left(R,0\right) + O\big(\mathcal{N}^2 + F_+^2 + F_-^2\big).$$

Note also that (7.45) implies

$$\int |\partial_t u|^2 \lesssim \int \left(|\eta|^2 + \sum_{k=1}^{K} \left|\ell_k \partial_x Q_k\right|^2 \right) \lesssim \mathcal{N}^2.$$

Then, by direct computation, next $-\partial_x^2 Q + Q - f(Q) = 0$ and (7.35),

$$\begin{aligned} E(R,0) &= KE(Q,0) + \sum_{k<k'} \int [(\partial_x Q_k)(\partial_x Q_{k'}) + Q_k Q_{k'} - f(Q_k)Q_{k'} - f(Q_{k'})Q_k] \\ &\quad - \int \left(F(R) - \sum_{k=1}^{K} F(Q_k) - \sum_{k\neq k'} f(Q_k)Q_{k'} \right) \\ &= KE(Q,0) - \sum_{k<k'} \langle f(Q_k), Q_{k'} \rangle + O(F_+^\theta + F_-^\theta). \end{aligned}$$

Last, from (7.37), (7.58) and the definition of F_+ and F_-, we observe that

$$\sum_{k<k'} \langle f(Q_k), Q_{k'} \rangle = c_1 \kappa F_+ - c_1 \kappa F_- + O(\mathcal{N}^2 + F_+^\theta + F_-^\theta)$$

Indeed, in the above double sum $\sum_{k<k'}$ in k and k', the terms corresponding to $k' = k+1$ contribute to $\pm c_1 \kappa F_\pm$ (depending on $k \in \mathcal{K}_\pm$) and the other terms (*i.e.* $k' \geq k+2$) only contribute to the error term (see also (7.43)).

Gathering all the above estimates, we have proved (7.70). □

Combining (7.70) with (7.3) and (7.29), we obtain the following result.

Corollary 7.23. *Let $\vec{u}$ be a global solution of (7.1) satisfying the decomposition given by Lemma 7.13 on $[t,\infty)$, for some $t \geq 0$. Let $1 < \theta < \min(p-1, \frac{5}{4})$. Then*

$$2\alpha \int_t^\infty \|\partial_t u(s)\|_{L^2}^2 \, \mathrm{d}s = -c_1 \kappa F_+(t) + c_1 \kappa F_-(t) + O(\mathcal{N}^2(t) + F_+^\theta(t) + F_-^\theta(t)). \tag{7.71}$$

7.3.6 General estimates for global solutions

The following result is similar to Proposition 3.1 in [29]. We repeat the proof for the sake of completeness.

Proposition 7.24. *There exists a universal constant $\delta_1 > 0$ such that the following holds. Let $0 < \delta < \delta_1$ and $\vec{u}$ be a global solution of (7.1) satisfying (7.28) with $K \geq 1$. Let $T_\delta \gg 1$ be such that $\vec{u}$ admits a decomposition as in Lemma 7.13 in a neighborhood of T_δ, with*

$$[\mathcal{N}(T_\delta)]^2 + b(T_\delta) + F_+(T_\delta) + F_-(T_\delta) \leqslant \delta^2. \tag{7.72}$$

Then, for all $t \geq T_\delta$, it holds

$$[\mathcal{N}(t)]^2 + b(t) + F_+(t) + F_-(t) \lesssim \delta^2, \tag{7.73}$$

and

$$F_-(t) \leq 3\left(\delta^{-2} + \lambda(t - T_\delta)\right)^{-1}. \tag{7.74}$$

Proof. For a constant $C > 1$ to be taken large enough, we introduce the following bootstrap estimates

$$\mathcal{N} \leqslant C\delta, \quad b \leqslant C\delta^2, \quad F_+ \leqslant \delta^{\frac{3}{2}}, \quad F_- \leqslant \delta^{\frac{3}{2}}, \tag{7.75}$$

and we set

$$T_* = \sup\left\{t \in [T_\delta, \infty) \text{ such that (7.75) holds on } [T_\delta, t]\right\} > T_\delta.$$

We prove $T_* = \infty$ by strictly improving the bootstrap estimate (7.75) on $[T_\delta, T_*)$ for $C > 1$ large enough. In the rest of the proof, the implicit constants in $\lesssim$ or $O(\cdot)$ do not depend on the constant C of the bootstrap estimate (7.75).

Estimate on $\mathcal{N}$. From (7.64) and (7.75), it holds on $[T_\delta, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2\mu t}\mathcal{F}\right] \lesssim C^3\delta^3 e^{2\mu t} + C\delta^{\frac{5}{2}} e^{2\mu t} \leq \delta^2 e^{2\mu t},$$

for $\delta > 0$ small enough (depending on C). From (7.59) and (7.72), $\mathcal{F}(T_\delta) \lesssim \delta^2$. Thus, integrating the above estimate on $[T_\delta, t]$, for any $t \in [T_\delta, T_*)$, $\mathcal{F}(t) \lesssim \delta^2$. In particular, by (7.59), we obtain

$$\|\vec{\varepsilon}\|^2_{H^1\times L^2} \lesssim \mathcal{F} + b \lesssim C\delta^2.$$

Arguing similarly for the quantity $\mathcal{G}$, we have

$$\sum_{k=1}^{K} |\ell_k|^2 + \sum_{k=1}^{K} (a_k^-)^2 \lesssim \mathcal{G} \lesssim \delta^2. \tag{7.76}$$

Hence we obtain, for all $t \in [T_\delta, T_*)$,

$$\mathcal{N}(t) \lesssim \sqrt{C}\delta. \tag{7.77}$$

For C large enough, this strictly improves the bootstrap estimate (7.75) on $\mathcal{N}$ on the interval $[T_\delta, T_*)$.

Estimate on b. Now, we prove that for C large enough, it holds for all $t \in [T_\delta, T_*)$,

$$b(t) \leqslant \frac{C}{2}\delta^2. \tag{7.78}$$

From (7.71) in Corollary 7.23 and (7.72), we have

$$\int_{T_\delta}^{\infty} \|\partial_t u(s)\|^2_{L^2}\,\mathrm{d}s \lesssim \mathcal{N}^2(T_\delta) + F_+(T_\delta) + F_-(T_\delta) \lesssim \delta^2. \tag{7.79}$$

By (7.72), we have $b(T_\delta) \leq \delta^2$. For the sake of contradiction, take $C > 4$ large and assume that there exists $t_2 \in [T_\delta, T_*)$ such that

$$b(t_2) = \frac{C}{2}\delta^2, \quad b(t) < \frac{C}{2}\delta^2 \quad \text{on } [T_\delta, t_2).$$

On the one hand, by continuity of b, there exists $t_1 \in [T_\delta, t_2)$ such that

$$b(t_1) = \frac{C}{4}\delta^2 \quad \text{and} \quad b(t) > \frac{C}{4}\delta^2 \quad \text{on } (t_1, t_2].$$

Using (7.63) and the bootstrap estimates (7.75), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b = 2\nu^+ b + O(C^3\delta^3 + C\delta^{\frac{5}{2}})$$

which implies (for δ small enough depending on C)

$$t_2 - t_1 = \frac{\log 2}{2\nu^+} + O(\delta^{\frac{1}{3}}), \tag{7.80}$$

and thus

$$\int_{t_1}^{t_2} b(s)\,\mathrm{d}s \gtrsim C\delta^2. \tag{7.81}$$

On the other hand, by (7.32), (7.45), (7.76), (7.79) and (7.80),

$$\int_{t_1}^{t_2} \|\eta(t)\|_{L^2}^2\,\mathrm{d}t \lesssim \int_{t_1}^{t_2} \left(\|\partial_t u(t)\|_{L^2}^2 + \sum_{k=1}^{K} \|\ell_k \partial_x Q_k(t)\|_{L^2}^2\right)\mathrm{d}t \lesssim \delta^2. \tag{7.82}$$

By the definition of $a_k^{\pm}$, one has

$$a_k^+ = \zeta^+\langle \varepsilon, Y_k\rangle + \langle \eta, Y_k\rangle, \quad a_k^- = \zeta^-\langle \varepsilon, Y_k\rangle + \langle \eta, Y_k\rangle$$

and thus

$$a_k^+ = \frac{\zeta^+}{\zeta^-}a_k^- + \frac{\zeta^- - \zeta^+}{\zeta^-}\langle \eta, Y_k\rangle.$$

Combining (7.76), (7.80) and (7.82), we find the estimate $\int_{t_1}^{t_2} b(s)\,\mathrm{d}s \lesssim \delta^2$, which contradicts (7.81) for C large enough. This proves (7.78).

Estimate on F_-. Let $1 < \theta < \min(p-1, \frac{5}{4})$. First, from (7.71) in Corollary 7.23 and (7.75), we have

$$F_+ \leqslant 2F_- + O(\mathcal{N}^2) \quad \text{and} \quad F_+^\theta \leqslant 2F_-^\theta + \mathcal{N}^2. \tag{7.83}$$

Second, from (7.67), (7.75) and (7.83), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_- &\geq \left(\lambda + \frac{2}{\lambda F_-}\left(\mathcal{N}^2 - F_+^\theta - F_-^\theta\right)\right)\exp\left(-3\lambda^{-1}\mathcal{M}\right)\\ &\geq \left(\lambda - \frac{6}{\lambda}F_-^{\theta-1}\right)\exp\left(-3\lambda^{-1}\mathcal{M}\right) \geq \frac{\lambda}{2}.\end{aligned}$$

By integration on $[T_\delta, t]$ for any $t \in [T_\delta, T_*)$, it holds

$$R_-(t) \geq R_-(T_\delta) + \frac{\lambda}{2}(t - T_\delta). \tag{7.84}$$

By the definition of R_- and (7.72), we have $R_-(T_\delta) \geq \frac{1}{2}\delta^{-2}$. Thus, (7.84) implies that, for any $t \in [T_\delta, T_*)$

$$F_-(t) \leq \frac{3}{2R_-} \leq \frac{3}{2}\left(\frac{1}{2}\delta^{-2} + \frac{\lambda}{2}(t - T_\delta)\right)^{-1} \leqslant 3\delta^2.$$

This is strictly improves the bootstrap estimate (7.75) of F_- on the interval $[T_\delta, T_*)$ and proves (7.74).

Estimate on F_+. From (7.77), (7.83) and (7.74), we observe that

$$F_+ \leq 2F_- + O(\mathcal{N}^2) \leq O(C\delta^2).$$

For δ small enough (depending on C), this strictly improves the estimate (7.75) of F_+ on $[T_\delta, T_*)$.

The previous estimates prove that $T_* = \infty$ and that (7.73) holds on $[T_\delta, \infty)$. □

7.4 Alternate signs property for neighbor solitary waves

In this Section, we prove the following property.

Proposition 7.25. *Let $\vec{u}$ be a global solution of (7.1) such that $K \geq 2$ in (7.28) of Theorem 7.7. Then,*

$$\sigma_k = -\sigma_{k+1} \quad \textit{for all } k \in \{1, \dots, K-1\}. \tag{7.85}$$

Proof. We perform computations in the context of Proposition 7.24. Assuming that $\mathcal{K}_+$ is not empty, we reach a contradiction. In this context, if $\mathcal{K}_-$ is empty, we have $F_- = 0$. Otherwise, we may use the estimate (7.74) on F_- proved in Proposition 7.24. Using (7.66), (7.73) and (7.74), we obtain that for any $t \in [T_\delta, \infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} R_+ \leq -\frac{3}{4}\lambda + \frac{3}{\lambda} F_+^{\theta-1} + \frac{2}{\lambda} R_+ F_-^\theta \leq -\frac{\lambda}{2} + \frac{18}{\lambda}(\delta^{-2} + \lambda(t - T_\delta))^{-\theta} R_+. \tag{7.86}$$

Define the auxiliary function

$$\widetilde{R}^+(t) = R_+(t) \exp\left[\frac{18}{\lambda^2(\theta-1)}(\delta^{-2} + \lambda(t - T_\delta))^{1-\theta}\right], \quad \text{for } t \in [T_\delta, \infty).$$

By direct computation and (7.86), we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{R}^+ \leq -\frac{\lambda}{2} \exp\left[\frac{18}{\lambda^2(\theta-1)}(\delta^{-2} + \lambda(t - T_\delta))^{1-\theta}\right] \leq -\frac{\lambda}{2}.$$

By integrating the above estimate on $[T_\delta, t]$, we obtain

$$\widetilde{R}^+(t) \leq \widetilde{R}^+(T_\delta) - \frac{\lambda}{2}(t - T_\delta),$$

which is contradictory with $\widetilde{R}^+(t) \geq 0$ for large t. This means that $\mathcal{K}_+ = \emptyset$ and so $\mathcal{K}_- = \{1, \ldots, K-1\}$: the signs of the solitary waves alternate. □

7.5 Description of long-time asymptotics

We consider any global solution $\vec{u}$ of (7.1). We prove Theorem 7.1 by considering separately the no soliton case, and then the cases $K = 1$ and $K \geq 2$ in Theorem 7.7.

7.5.1 No soliton case

If $\vec{u}(t)$ converges to 0 as $t \to \infty$, as a consequence of (7) of Theorem 2.3 in [12], $\vec{u}$ converges exponentially to 0 in $H^1 \times L^2$. Alternatively, one can use the energy functional

$$\int \left\{(\partial_x u)^2 + (1 - \rho\mu)u^2 + (\partial_t u + \mu u)^2\right\}$$

where $\mu > 0$ is small and $\rho = 2\alpha - \mu$ as in §7.3.3 to prove the exponential convergence.

7.5.2 Single soliton case

Assume that $\vec{u}$ follows the solitary wave scenario in Theorem 7.7 with $K = 1$. Let $\delta > 0$ to be chosen small enough. Following Lemma 7.13 and Proposition 7.24, there exists $T_\delta > 0$ such that estimates (7.73)-(7.74) hold on $[T_\delta, \infty)$. When $K = 1$, by convention

$$F_+(t) = F_-(t) = 0, \quad \text{for } t \geqslant 0,$$

and the estimates proved in the previous Sections simplify; see Remarks 7.14, 7.18 and 7.20. In particular, from (7.63) and (7.64), it holds

$$|\dot{b} - 2\nu^+ b| \lesssim \mathcal{N}^3, \tag{7.87}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} + 2\mu\mathcal{G} \lesssim \mathcal{N}^3, \tag{7.88}$$

and from (7.59) and (7.75),

$$\mathcal{N}^3 \leqslant C\delta\mathcal{N}^2 \lesssim C\delta\left(\mathcal{F}+\frac{b}{2\mu}\right). \tag{7.89}$$

Set $\tilde{b}=b-\delta^{\frac{1}{2}}\mathcal{F}$, observe that

$$\tilde{b}=\left(1+\frac{\delta^{\frac{1}{2}}}{2\mu}\right)b-\delta^{\frac{1}{2}}\left(\mathcal{F}+\frac{b}{2\mu}\right)\leqslant\left(1+\frac{\delta^{\frac{1}{2}}}{2\mu}\right)b.$$

Therefore, using (7.87), (7.88) and (7.89),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{b}&\geqslant 2\nu^+b+2\mu\delta^{\frac{1}{2}}\mathcal{F}-C\delta^{\frac{3}{4}}\left(\mathcal{F}+\frac{b}{2\mu}\right)\\&\geqslant(2\nu^+-\delta^{\frac{1}{2}})b+(2\mu\delta^{\frac{1}{2}}-C\delta^{\frac{3}{4}})\left(\mathcal{F}+\frac{b}{2\mu}\right)\geqslant\nu^+\tilde{b},\end{aligned}$$

where $\delta>0$ small enough such that

$$(2\nu^+-\delta^{\frac{1}{2}})\left(1+\frac{\delta^{\frac{1}{2}}}{2\mu}\right)\geqslant\nu^+,\quad 2\mu\delta^{\frac{1}{2}}-C\delta^{\frac{3}{4}}>0.$$

Integrating on $[t,s]\in[T_\delta,\infty)$,

$$\tilde{b}(t)\leqslant e^{-\nu^+(s-t)}\tilde{b}(s). \tag{7.90}$$

Let $s\to\infty$ in (7.90) and using (7.75), we obtain

$$\tilde{b}(t)\leqslant 0,\quad b(t)\leqslant\delta^{\frac{1}{2}}\mathcal{F}(t)\quad\text{for } t\geq T_\delta.$$

Thus, using (7.87), (7.88) and (7.89) again,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}+\frac{b}{2\mu}\right)&\leqslant-2\mu\mathcal{F}+\frac{\nu^+}{\mu}b+C\delta^{\frac{3}{4}}\left(\mathcal{F}+\frac{b}{2\mu}\right)\\&\leqslant-(2\mu-C\delta^{\frac{3}{4}})\left(\mathcal{F}+\frac{b}{2\mu}\right)+\left(1+\frac{\nu^+}{\mu}\right)b\leqslant-\mu\left(\mathcal{F}+\frac{b}{2\mu}\right),\end{aligned}$$

by possibly choosing $\delta>0$ small enough. Integrating on $[T_\delta,t]$, we obtain

$$\left(\mathcal{F}+\frac{b}{2\mu}\right)(t)\leqslant e^{-\mu t}e^{\mu T_\delta}\left(\mathcal{F}+\frac{b}{2\mu}\right)(T_\delta)\lesssim e^{-\mu t}e^{\mu T_\delta}\delta^2.$$

Therefore, using again (7.59),

$$\mathcal{N}^2(t)\lesssim\left(\mathcal{F}+\frac{b}{2\mu}\right)(t)\lesssim e^{-\mu t}. \tag{7.91}$$

From (7.48) and (7.91), we have $|\dot{z}_1|\leq e^{-\frac{\mu}{2}t}$, which proves that $z_1(t)$ converges exponentially to its limit as $t\to\infty$. In view of the decomposition of $\vec{u}$ in (7.45), the proof in the case $K=1$ is complete.

7.5.3 Multi-soliton case

Assume that $\vec{u}$ follows the multi-solitary wave scenario in Theorem 7.7 with $K\geq 2$. Let $\delta>0$ to be chosen small enough. Following Lemma 7.13 and Proposition 7.24, there exists $T_\delta>0$ such that estimates (7.73)-(7.74) hold on $[T_\delta,\infty)$. Recall that from Proposition 7.25, the set $\mathcal{K}_+$ is empty. In particular, following Remark 7.20, we use the estimates of Lemma 7.19 ignoring the quantity F_+. We start by showing that the quantity $\mathcal{N}(t)$ decays as t^{-1}.

Proposition 7.26. *There exists $T > 0$ such that the decomposition of $\vec{u}$ satisfies, for all $t \geq T$*

$$F_-(t) \lesssim t^{-1}, \quad \mathcal{N}(t) \lesssim t^{-1}. \tag{7.92}$$

Proof. The proof is inspired by that of [29, Proposition 3.2]. Let $0 < \delta < \delta_1$ in the context of Proposition 7.24. From (7.73) and (7.74), there exists $T > 0$ large enough (fix any $T \geqslant 4T_\delta$) such that, for all $t \geq T/2$,

$$\mathcal{N}(t) \leq C\delta, \quad F_-(t) \leq \frac{6}{\lambda} t^{-1}.$$

In particular, from (7.63) and (7.64)

$$\left| \frac{\mathrm{d}b}{\mathrm{d}t} - 2\nu^+ b \right| \lesssim \mathcal{N}^3 + t^{-1}\mathcal{N}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \lesssim \mathcal{N}^3 + t^{-1}\mathcal{N}. \tag{7.93}$$

Our goal is to obtain the decay rate of $\mathcal{N}$. The above bounds are not quite enough because of the term $\mathcal{N}^3$ for which only smallness is known at this point. This is the reason why we will work on a modification $\tilde{b}$ of b. Recall (7.59):

$$0 \leqslant \mathcal{N}^2 \lesssim \mathcal{F} + \frac{b}{2\mu}.$$

For $0 < \omega \ll 1$ to be chosen later, observe that, for δ small,

$$\mathcal{N}^3 + t^{-1}\mathcal{N} \lesssim \omega^2\mathcal{N}^2 + \omega^{-2}t^{-2} \lesssim \omega^2 \left(\mathcal{F} + \frac{b}{2\mu} \right) + \omega^{-2}t^{-2}. \tag{7.94}$$

(Here and below the implied constants do not depend on ω). Set $\tilde{b} = b - \omega\mathcal{F}$ and observe that

$$\tilde{b} = b - \omega\mathcal{F} = \left(1 + \frac{\omega}{2\mu}\right) b - \omega \left(\mathcal{F} + \frac{1}{2\mu} b \right) \leq \left(1 + \frac{\omega}{2\mu}\right) b.$$

Therefore, using (7.93) and (7.94),

$$\begin{aligned} \frac{\mathrm{d}\tilde{b}}{\mathrm{d}t} &\geq 2\nu^+ b + 2\omega\mu\mathcal{F} - C\omega^2 \left(\mathcal{F} + \frac{b}{2\mu} \right) - C\omega^{-2}t^{-2} \\ &\geq (2\nu^+ - \omega) b + \left(2\omega\mu - C\omega^2\right) \left(\mathcal{F} + \frac{b}{2\mu} \right) - C\omega^{-2}t^{-2} \\ &\geq \nu^+ \tilde{b} - C\omega^{-2}t^{-2}, \end{aligned}$$

where $\omega > 0$ is taken small enough such that

$$(2\nu^+ - \omega) \left(1 + \frac{\omega}{2\mu}\right)^{-1} \geq \nu^+, \quad 2\omega\mu - C\omega^2 > 0.$$

Integrating on $[t, s] \subset [T/2, \infty)$, we obtain

$$\tilde{b}(t) - e^{-\nu^+(s-t)}\tilde{b}(s) \leqslant C\omega^{-2} \int_t^s e^{-\nu^+(\tau - t)} \tau^{-2} \, \mathrm{d}\tau \lesssim \omega^{-2}t^{-2}. \tag{7.95}$$

Let $s \to \infty$ in (7.95) and using (7.75), we obtain for all $t \geq T/2$,

$$\tilde{b}(t) \lesssim \omega^{-2}t^{-2}, \quad b(t) \leq C\omega^{-2}t^{-2} + \omega\mathcal{F}(t).$$

Thus using (7.93) and (7.94) again, it holds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathcal{F} + \frac{b}{2\mu} \right) &\leq -2\mu\mathcal{F} + \frac{\nu^+}{\mu} b + C\omega^2 \left(\mathcal{F} + \frac{b}{2\mu} \right) + C\omega^{-2}t^{-2} \\ &\leq -(2\mu - C\omega^2) \left(\mathcal{F} + \frac{b}{2\mu} \right) + \left(1 + \frac{\nu^+}{\mu}\right) b + C\omega^{-2}t^{-2} \\ &\leq -\mu \left(\mathcal{F} + \frac{b}{2\mu} \right) + C\omega^{-2}t^{-2}, \end{aligned}$$

by possibly choosing $\omega > 0$ small enough. Integrating on $[T/2, t]$, we obtain

$$\left(\mathcal{F} + \frac{b}{2\mu}\right)(t) - e^{-\mu(t-\frac{T}{2})}\left(\mathcal{F} + \frac{b}{2\mu}\right)(T/2) \lesssim \int_{\frac{T}{2}}^{t} e^{-\mu(t-s)} s^{-2}\,\mathrm{d}s$$

Therefore, using again (7.59),

$$\mathcal{N}^2(t) \lesssim \left(\mathcal{F} + \frac{b}{2\mu}\right)(t) \lesssim t^{-2} + \delta^2 e^{\mu T/2} e^{-\mu t} \leqslant t^{-2},$$

which proves (7.92). □

We continue the proof of Theorem 7.1. In view of the alternate signs property (7.85) and the decay estimate (7.92), the system (7.60) rewrites as, for $k = 2, \cdots, K-1$, and any $t \geqslant T$,

$$\begin{cases} \dot{y}_1 = -\dfrac{\kappa}{2\alpha} e^{-(y_2 - y_1)} + O\left(t^{-\theta}\right), \\ \dot{y}_k = \dfrac{\kappa}{2\alpha}\left(e^{-(y_k - y_{k-1})} - e^{-(y_{k+1} - y_k)}\right) + O\left(t^{-\theta}\right), \\ \dot{y}_K = \dfrac{\kappa}{2\alpha} e^{-(y_K - y_{K-1})} + O(t^{-\theta}). \end{cases} \tag{7.96}$$

This system of ODEs is studied in [93] and [33], where it appears naturally in a different context (the description of characteristic blowup points of the semilinear wave equation), with slightly different perturbation terms. For the convenience of the reader, we provide a study of the dynamics.

We introduce an explicit solution to the unperturbed ODE system (7.9)

$$\bar{y}_k(t) := \left(k - \frac{K+1}{2}\right) \log t + \tau_k, \quad \text{for } k = 1, \dots, K, \tag{7.97}$$

where $(\tau_k)_{k=1,\dots,K}$ are constants uniquely defined by (7.8) (see also [33, p. 1549]).

From (7.96), we observe that $\sum_{k=1}^{K} \dot{y}_k = O\left(t^{-\theta}\right)$. Since $\theta > 1$, there exist $\bar{y}_\infty \in \mathbb{R}$ such that

$$\frac{1}{K}\sum_{k=1}^{K} y_k = \bar{y}_\infty + O\left(t^{-\theta+1}\right). \tag{7.98}$$

We introduce, for $k = 1, \dots, K$,

$$\xi_k(t) := y_k - \bar{y}_k - \bar{y}_\infty.$$

Observe that, for $k = 1, \cdots, K-1$,

$$y_{k+1} - y_k = \xi_{k+1} - \xi_k + \log t + (\tau_{k+1} - \tau_k),$$

so that using (7.8),

$$\frac{\kappa}{2\alpha} e^{-(y_{k+1} - y_k)} = \frac{\kappa}{2\alpha} e^{-(\xi_{k+1} - \xi_k) - \log t - (\tau_{k+1} - \tau_k)} = \gamma_k t^{-1} e^{-(\xi_{k+1} - \xi_k)}.$$

Therefore, from (7.96) and $\gamma_k - \gamma_{k-1} = \frac{K+1}{2} - k$, it holds $(2 \leqslant k \leqslant K-1)$

$$\begin{cases} \dot{\xi}_1 = -t^{-1}\gamma_1\left(e^{-(\xi_2 - \xi_1)} - 1\right) + O\left(t^{-\theta}\right), \\ \dot{\xi}_k = t^{-1}\left(\gamma_{k-1}(e^{-(\xi_k - \xi_{k-1})} - 1) - \gamma_k(e^{-(\xi_{k+1} - \xi_k)} - 1)\right) + O\left(t^{-\theta}\right), \\ \dot{\xi}_K = t^{-1}\gamma_{K-1}(e^{-(\xi_K - \xi_{K-1})} - 1) + O(t^{-\theta}). \end{cases} \tag{7.99}$$

To complete the proof of Theorem 7.1, it suffices to show that for any $k = 1, \dots, K$,

$$\xi_k(t) = O(t^{-\theta+1}). \tag{7.100}$$

First, we prove a bound on ξ_k.

Lemma 7.27. *There exists $M > 0$ such that for all $k = 1, \dots, K$ and for all $t \geq T_\delta$, it holds $|\xi_k(t)| \leqslant M$.*

Proof. Set

$$\zeta_0 = \zeta_K = 0 \quad \text{and} \quad \zeta_k = \xi_{k+1} - \xi_k \quad \text{for } k = 1, \cdots, K-1.$$

From (7.99), it holds for $k = 1, \dots, K-1$ (with $\gamma_0 = \gamma_K = 0$)

$$\dot{\zeta}_k = t^{-1}\left(-\gamma_{k+1}(e^{-\zeta_{k+1}} - 1) + 2\gamma_k(e^{-\zeta_k} - 1) - \gamma_{k-1}(e^{-\zeta_{k-1}} - 1)\right) + O\left(t^{-\theta}\right). \tag{7.101}$$

Claim 7.28. *There exists $M_1 > 0$ such that for all $k = 1, \dots, K-1$, and for all $t \geqslant T_\delta$, it holds $|\zeta_k(t)| \leq M_1$.*

Proof Claim 7.28. First, we prove a lower bound $\zeta_k(t) \geq -M_1$, for some $M_1 > 0$. Fix $\vartheta = \frac{1+\theta}{2}$ and for $D_1 > 0$ to be fixed later, denote for $k = 1, \cdots, K-1$,

$$\rho_k = \gamma_k(e^{-\zeta_k} - 1) + D_1 t^{-\vartheta+1}, \quad \rho_+ = \max_k \rho_k.$$

Let us prove that ρ_+ is non increasing for large enough times using a bootstrap argument. For $C_1 > 0$ and $T_1 \geqslant T_\delta$ to be chosen later, let

$$T_* = \sup\{t \in [T_1, \infty) \text{ such that } \rho_+ \leqslant C_1 \text{ on } [T_1, t]\}.$$

Let $t \in [T_1, T_*)$, and consider an index k such that $\rho_k(t) = \rho_+(t)$. Observe that

$$\gamma_k e^{-\zeta_k(t)} \leqslant \rho_k(t) + \gamma_k \leqslant C_1 + K^2, \tag{7.102}$$

$$2\gamma_k\big(e^{-\zeta_k(t)} - 1\big) \geqslant \gamma_{k+1}\big(e^{-\zeta_{k+1}(t)} - 1\big) + \gamma_{k-1}\big(e^{-\zeta_{k-1}(t)} - 1\big). \tag{7.103}$$

Gathering (7.101), (7.102) and (7.103), there exist $C_0 > 0$ such that at t,

$$\begin{aligned}
\dot{\rho}_k &= -\gamma_k e^{-\zeta_k}\dot{\zeta}_k - D_1(\vartheta - 1)t^{-\vartheta} \\
&= -\gamma_k e^{-\zeta_k} t^{-1}\left(-\gamma_{k+1}(e^{-\zeta_{k+1}} - 1) + 2\gamma_k(e^{-\zeta_k} - 1) - \gamma_{k-1}(e^{-\zeta_{k-1}} - 1)\right) \\
&\quad + O(\gamma_k e^{-\tau_k(t)} t^{-\theta}) - D_1(\vartheta - 1)t^{-\vartheta} \\
&\leqslant C_0 \gamma_k e^{-\zeta_k(t)} t^{-\theta} - D_1(\vartheta - 1)t^{-\vartheta} \\
&\leqslant \left((C_1 + K^2)C_0 - D_1(\vartheta - 1)T_1^{\theta-\vartheta}\right) t^{-\theta}.
\end{aligned}$$

Fix $T_1 \geqslant T_\delta$ and then D_1, C_1 such that

$$(\vartheta - 1)T_1^{\theta-\vartheta} \geqslant C_0 + 1,$$
$$D_1 \geqslant C_0(\rho_+(T_1) + 1 + K^2) + 1, \quad C_1 = \rho_+(T_1) + D_1 + 1.$$

Then, there holds

$$\begin{aligned}
(C_1 + K^2)C_0 - D_1(\vartheta - 1)T_1^{\theta-\vartheta} &\leqslant (\rho_+(T_1) + D_1 + 1 + K^2)C_0 - D_1(C_0 + 1) \\
&\leqslant (\rho_+(T_1) + 1 + K^2)C_0 - D_1 \leqslant -1.
\end{aligned}$$

Since $\rho_+(T_1) < C_1$, by continuity, $T_* > T_1$. We also have $\dot{\rho}_k(t) < 0$, so that ρ_k is decreasing at t; note that this property holds for any index k such that $\rho_k(t) = \rho_+(t)$. If j is an index such that $\rho_j(t) < \rho_+(t)$, then by continuity this inequality holds on a neighborhood of t. Thus ρ_+ is decreasing at t, for any $t \in [T_1, T^*)$. In particular, $\rho_+(t) \leqslant \rho_+(T_1) \leqslant C_1$. By continuity, we obtain $T_* = \infty$ and so for all $t \geqslant T_1$, $\rho_+(t) \leqslant C_1$.

By continuity, there exists C_2 such that for all $t \geqslant T_\delta$, $\rho_+(t) \leqslant C_2$, and so for all $k = 1, \dots, K-1$, for all $t \geqslant T_\delta$, $e^{-\zeta_k(t)} \leqslant 2C_2 + 1$, since $D_1 > 0$ and $\gamma_k \geqslant 1/2$. Therefore, for $M_1 = \log(2C_2 + 1)$, we have proved the lower bound on $\zeta_k(t)$.

Arguing similarly using the minimum of $\tilde{\rho}_k = \gamma_k(e^{-\zeta_k} - 1) - D_1 t^{-\vartheta+1}$, one also proves an upper bound on ζ_k. □

By (7.98) and $\sum_{k=1}^K \bar{y}_k = \sum_{k=1}^K \tau_k = 0$, there exists $M_2 > 0$ such that for all $t \geqslant T_\delta$, $\left|\sum_{k=1}^K \xi_k(t)\right| \leqslant M_2$. By contradiction, assume that for some k_0 and $t \geqslant T_\delta$

$$\xi_{k_0}(t) \geqslant M \quad \text{where} \quad M := \frac{2M_2}{K} + \frac{K-1}{2} M_1.$$

Then,

$$\xi_k(t) = \xi_{k_0}(t) + \sum_{j=k_0}^{k-1} \zeta_j \geqslant M - |k-k_0| M_1 \quad \text{for } 1 \leqslant k_0 < k \leqslant K,$$
$$\xi_k(t) = \xi_{k_0}(t) - \sum_{j=k}^{k_0-1} \zeta_j \geqslant M - |k-k_0| M_1 \quad \text{for } 1 \leqslant k < k_0 \leqslant K,$$

so that (in view of the definition of M)

$$\sum_{k=1}^K \xi_k(t) \geqslant KM - M_1 \sum_{k=1}^K |k-k_0| \geqslant K\left(M - \frac{K-1}{2} M_1\right) \geqslant 2M_2,$$

which is a contradiction. Therefore, for all $k = 1, \dots, K$ and $t \geqslant T_\delta$, $\xi_k(t) \leqslant M$. One argues similarly to show that $\xi_k(t) \geqslant -M$. □

Now, we consider the unperturbed ODE system for the $(\xi_k)_{k=1,\dots,K}$, that is

$$\dot{\boldsymbol{\varpi}} = t^{-1}\Phi(\boldsymbol{\varpi}), \tag{7.104}$$

where $\boldsymbol{\varpi} = (\varpi_k)_{k=1,\dots,K}$ and $\Phi : \mathbb{R}^K \to \mathbb{R}^K$ is defined by

$$\begin{cases} \Phi_1(\boldsymbol{\varpi}) = -\gamma_1(e^{-(\varpi_2-\varpi_1)} - 1), \\ \Phi_k(\boldsymbol{\varpi}) = \gamma_{k-1}(e^{-(\varpi_k-\varpi_{k-1})} - 1) - \gamma_k(e^{-(\varpi_{k+1}-\varpi_k)} - 1), \text{ for } 2 \leqslant k \leqslant K-1, \\ \Phi_K(\boldsymbol{\varpi}) = \gamma_{K-1}(e^{-(\varpi_K-\varpi_{K-1})} - 1). \end{cases}$$

This system is studied in [33]. Observe that setting $\boldsymbol{e}_1 = \frac{1}{\sqrt{K}}(1,\dots,1)^T$, it holds

$$\text{for all } t, t_0 \geqslant T_\delta, \quad \frac{1}{K}\sum_{j=1}^K \varpi_j(t) = \frac{1}{\sqrt{K}}(\boldsymbol{\varpi}, \boldsymbol{e}_1) = \frac{1}{K}\sum_{j=1}^K \varpi_j(t_0).$$

Moreover, $D\Phi(0)$ is the $K \times K$ matrix with entries

$$m_{1,1} = -\gamma_1, \quad m_{K,K} = -\gamma_{K-1}, \quad m_{k,k} = -(\gamma_{k-1} + \gamma_k), \quad \text{for } k = 2, \cdots, K-1,$$
$$m_{k,k-1} = \gamma_{k-1}, \quad m_{k,k+1} = \gamma_k, \quad m_{k,k'} = 0, \quad \text{if } |k-k'| \geqslant 2.$$

We recall the following properties.

Proposition 7.29 ([33]). *It holds $D\Phi(0)\boldsymbol{e}_1 = 0$ and*

$$\textit{for all } \boldsymbol{x} \in \boldsymbol{e}_1^\perp, \quad (D\Phi(0)\boldsymbol{x}, \boldsymbol{x}) \leqslant -\|\boldsymbol{x}\|^2. \tag{7.105}$$

Furthermore, for any $M > 0$, there exists $C(M) > 0$ such that for any $t_0 > 0$, if $|\boldsymbol{\varpi}(t_0)| \leqslant M$, then

$$\textit{for all } t \geqslant t_0, \quad \|\boldsymbol{\varpi}(t) - (\boldsymbol{\varpi}(t_0), \boldsymbol{e}_1)\boldsymbol{e}_1\| \leqslant C(M) t_0 t^{-1}. \tag{7.106}$$

Proof. See [33, Lemma 2.8] for the coercivity (7.105) of $D\Phi(0)$ and [33, Proposition 2.5] for the convergence (7.106) (written there in the variable $\tau = \log t$). □

Now, we complete the proof of Theorem 7.1 by showing the estimate (7.100).

Proof of (7.100). We summarise what was obtained so far. Let $\boldsymbol{\xi} := (\xi_k)_{k=1,\dots,K}$. There exists C_θ such that for $t \geqslant T_\delta$,

(i) $\dot{\boldsymbol{\xi}} = t^{-1}\Phi(\boldsymbol{\xi}) + \boldsymbol{r}(t)$ where $\|\boldsymbol{r}(t)\| \leqslant C_\theta t^{-\theta}$,

(ii) $|(\boldsymbol{\xi}, \boldsymbol{e}_1)| \leqslant C_\theta t^{-\theta+1}$,

(iii) $\|\boldsymbol{\xi}(t)\| \leqslant \sqrt{K}M$, for $t \geqslant T_\delta$.

Moreover, by (7.105), there exist $\epsilon, C > 0$ such that if $\|\boldsymbol{x}\| \leqslant \epsilon$, then

$$(\Phi(\boldsymbol{x}), \boldsymbol{x}) \leqslant -\frac{1}{2}\|\boldsymbol{x}\|^2 + C(\boldsymbol{x}, \boldsymbol{e}_1)^2. \tag{7.107}$$

Step 1. We claim that there exists $T_\epsilon \geqslant T_\delta$ such that

$$\text{for all } t \geqslant T_\epsilon, \quad \|\boldsymbol{\xi}(t)\| \leqslant \epsilon. \tag{7.108}$$

Let $L = 4\epsilon^{-1}C(M) + 1$, where $C(M)$ given by Proposition 7.29. Set

$$A = 3 + \sup\{\|D\Phi(\boldsymbol{x})\| : \|\boldsymbol{x}\| \leqslant \max(C(M), \sqrt{K}M)\}.$$

By *(i)* and *(ii)*, we fix t_0 large enough such that

$$\sup_{t \geqslant t_0}\big(t \sup_{s \geqslant t}\|\boldsymbol{r}(s)\|\big) \leqslant \frac{\epsilon}{4}L^{-A} \quad \text{and} \quad t_0^{-\theta+1} \leqslant \frac{\epsilon}{2}C_\theta^{-1}.$$

For any $t_1 \geqslant t_0$, we denote $\boldsymbol{\varpi}_{t_1}$ the solution of (7.104) with data $\boldsymbol{\varpi}_{t_1}(t_1) = \boldsymbol{\xi}(t_1)$ at time t_1. On the one hand, by standard Gronwall estimates, it holds for any $t \geqslant t_1$,

$$\|\boldsymbol{\xi}(t) - \boldsymbol{\varpi}_{t_1}(t)\| \leqslant t^A \int_{t_1}^t s^{-A}\|\boldsymbol{r}(s)\|\,\mathrm{d}s \leqslant t^A t_1^{-A+1} \sup_{s \geqslant t_1}\|\boldsymbol{r}(s)\|.$$

Let $t = Lt_1$, using the definition of t_0, we obtain

$$\|\boldsymbol{\xi}(Lt_1) - \boldsymbol{\varpi}_{t_1}(Lt_1)\| \leqslant L^A\big(t_1 \sup_{s \geqslant t_1}\|\boldsymbol{r}(s)\|\big) \leqslant \frac{\epsilon}{4}. \tag{7.109}$$

On the other hand, using (7.106), *(ii)* and the definition of t_0,

$$\|\boldsymbol{\varpi}_{t_1}(Lt_1)\| \leqslant \|\big(\boldsymbol{\xi}(t_1), \boldsymbol{e}_1\big)\boldsymbol{e}_1\| + C(M)t_1(Lt_1)^{-1} \leqslant \frac{3}{4}\epsilon. \tag{7.110}$$

Let $T_\epsilon = Lt_0$, from (7.109) and (7.110), we obtain (7.108).

Step 2. Using *(i)*, *(ii)*, (7.107) and (7.108), we infer that for $t \geqslant T_\epsilon$,

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{\xi}\|^2 &= 2t^{-1}(\Phi(\boldsymbol{\xi}), \boldsymbol{\xi}) + 2(r, \boldsymbol{\xi}) \\ &\leqslant -t^{-1}\|\boldsymbol{\xi}\|^2 + O(t^{1-2\theta}) + O(t^{-\theta}\|\boldsymbol{\xi}\|) \\ &\leqslant -\left(\theta - \frac{1}{2}\right)t^{-1}\|\boldsymbol{\xi}\|^2 + O(t^{1-2\theta}).\end{aligned}$$

A direct integration and $1 < \theta < 3/2$ yield, for some $C > 0$,

$$\|\boldsymbol{\xi}(t)\|^2 \leqslant \left(\frac{T_\epsilon}{t}\right)^{\theta-\frac{1}{2}}\|\boldsymbol{\xi}(T_\epsilon)\|^2 + Ct^{-2\theta+2} \lesssim t^{-2\theta+2},$$

which is (7.100). □

7.6 Construction of multi-solitary waves

In this Section, we prove Theorem 7.3, adapting arguments from [29, Section 4]. Here, $\mathcal{B}_{H^1\times L^2}(\delta)$ denotes the open ball of $H^1\times L^2$ of center 0 and radius δ and $\mathcal{B}_{\mathbb{R}^K}(\delta)$ (respectively, $\bar{\mathcal{B}}_{\mathbb{R}^K}(\delta)$) denotes the open ball (respectively, closed ball) of $\mathbb{R}^K$ of center 0 and radius δ. Last, $\mathcal{S}_{\mathbb{R}^K}(\delta)$ denotes the sphere of $H^1\times L^2$ of center 0 and radius δ. We also use the notation from §7.3 and set

$$\beta := \frac{1}{2\sqrt{\alpha^2+\nu_0^2}} = \langle \vec{Y}^+, \vec{Z}^+\rangle^{-1} > 0.$$

We recall the following preliminary result (for the proof, see [29, Lemma 4.1]).

Lemma 7.30. *Let $(z_k,\ell_k)_{k=1,\dots,K}\in\mathbb{R}^{2K}$ be such that*

$$r = \min(z_{k+1}-z_k, k=1,\dots,K-1)$$

is large enough. There exist linear maps

$$B:\mathbb{R}^K\to\mathbb{R}^K,\quad V:\mathbb{R}^K\to\mathbb{R}^K$$

smooth in $(z_k,\ell_k)_{k=1,\dots,K}$, satisfying

$$\|B-\beta\,\mathrm{Id}\,\|\lesssim e^{-\frac12 r},\quad \|V_j\|\lesssim e^{-\frac12 r},$$

and such that the function $W(\boldsymbol{a}):\mathbb{R}\to\mathbb{R}$ defined by

$$W(\boldsymbol{a})(x) := \sum_{k=1}^K\Big\{B_k(\boldsymbol{a})Y_k(x)+V_k(\boldsymbol{a})\partial_x Q_k(x)\Big\},$$

for any $\boldsymbol{a}=(a_1,\dots,a_k)$, satisfies, for all $k=1,\dots,K$,

$$\langle W(\boldsymbol{a}),\partial_x Q_k\rangle = 0,\quad \langle W(\boldsymbol{a}),Y_k\rangle = \beta a_k.$$

In particular, setting

$$\vec{W}(\boldsymbol{a}) = \begin{pmatrix} W(\boldsymbol{a})\\ \nu^+W(\boldsymbol{a})\end{pmatrix}\quad \text{it holds}\quad \langle\vec{W}(\boldsymbol{a}),\vec{Z}_k^+\rangle = a_k.$$

The next proposition and the invariance by translation of (7.1) imply Theorem 7.3.

Proposition 7.31. *Let $K\geqslant 2$ and $\sigma=\pm1$. For $\delta>0$ small enough, let any*

$$\begin{cases}(\ell_k(0))_{k=1,\dots,K}\in\mathcal{B}_{\mathbb{R}^K}(\delta),\\ (z_k(0))_{k=1,\dots,K}\in\mathbb{R}^K\\ \qquad \text{satisfying } \min\{z_{k+1}(0)-z_k(0), k=1,\dots,K-1\}>5|\log\delta|,\\ \vec{\varepsilon}_\perp(0)\in\mathcal{B}_{H^1\times L^2}(\delta)\\ \qquad \text{satisfying (7.46) and } \langle\vec{\varepsilon}_\perp(0),\vec{Z}_k^+(0)\rangle = 0 \text{ for } k=1,\dots,K.\end{cases}\tag{7.111}$$

There exists $\boldsymbol{a}_\sharp^+(0)=(a_{\sharp,k}^+(0))_{k=1,\dots,K}\in\bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac54})$ such that the solution $\vec{u}_\sharp$ of (7.1) with the initial data

$$\vec{u}_\sharp(0) = \sigma\sum_{k=1}^K(-1)^k(Q(\cdot-z_k(0)),0)+\vec{W}(\boldsymbol{a}_\sharp^+(0))+\vec{\varepsilon}_\perp(0)$$

is global and satisfies (7.6) where, for $k=1,\dots,K$, $z_k=\bar{y}_k+y_\sharp+O(t^{-\theta+1})$, for some $y_\sharp\in\mathbb{R}$ and $\bar{y}_k$ being defined in (7.97).

Remark 7.32. From the proof of Proposition 7.31, there exist even solutions of (7.1) with any odd number $K \geq 3$ of solitary waves.

Proof. Given $\boldsymbol{a}^+(0) = (a_k^+(0))_{k=1,\dots,K} \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$, we consider the solution $\vec{u}(t)$ of (7.1) with initial data

$$\vec{u}(0) = \sigma \sum_{k=1}^{K} (-1)^k (Q(\cdot - z_k(0)), 0) + \vec{W}(\boldsymbol{a}^+(0)) + \vec{\varepsilon}_\perp(0).$$

Decomposition. For any $t \geq 0$ such that $\vec{u}(t)$ is defined and satisfies (7.44), we consider its decomposition according to Lemma 7.13. Following §7.3.4, we introduce the notation y_k $(k = 1, \dots, K)$, $\mathcal{N}$, $\mathcal{M}$ and

$$F = \sum_{k=1}^{K-1} e^{-(y_{k+1} - y_k)}, \quad R = \frac{1}{F} \exp(-3\lambda^{-1}\mathcal{M})$$

(due to the choice of signs in the decomposition of $\vec{u}(0)$, $F = F_-$ and $F_+ = 0$). Note that by the properties of the function W in Lemma 7.30 and the orthogonality properties (7.46) of $\vec{\varepsilon}_\perp(0)$ assumed in (7.111), the initial data $\vec{u}(0)$ is modulated. Indeed, $(z_k(0), \ell_k(0))_{k=1,\dots,K}$ and

$$\vec{\varepsilon}(0) = \vec{W}(\boldsymbol{a}^+(0)) + \vec{\varepsilon}_\perp(0),$$

are the parameters of the decomposition of $\vec{u}(0)$. In particular, it holds from (7.111)

$$\mathcal{N}(0) \lesssim \delta, \quad F(0) \lesssim \delta^2.$$

Moreover, by Lemma 7.30, for $k = 1, \cdots, K$, it holds

$$\langle \vec{\varepsilon}(0), \vec{Z}_k^+(0) \rangle = \langle \vec{W}(\boldsymbol{a}^+(0)), Z_k^+ \rangle = a_k^+(0),$$

which is consistent with the definition of a_k^+ in (v) of Lemma 7.13.

Bootstrap estimates. We introduce the following bootstrap estimates

$$\mathcal{N} \leq \delta^{\frac{3}{4}}, \quad F \leq \delta^{\frac{3}{2}}, \quad b \leq \delta^{\frac{5}{2}} \tag{7.112}$$

and we set $T_* = \sup\{t \in [0, \infty) \text{ such that (7.112) holds on } [0, t]\} \geq 0$.

Estimates on the damped components. The estimate on $\mathcal{N}$ is strictly improved on $[0, T_*]$ as in the proof of Proposition 7.24. In particular, $\mathcal{N} \lesssim \delta$ on $[0, T_*]$.

Estimate on the distance. Inequality (7.67) rewrites

$$\frac{\mathrm{d}R}{\mathrm{d}t} \geqslant \left(\lambda + \frac{2}{\lambda F}(\mathcal{N}^2 - F^\theta)\right) \exp(-3\lambda^{-1}\mathcal{M}).$$

Now for $t \in [0, T_*]$, in view of (7.112), $\exp(-3\lambda^{-1}\mathcal{M}) = 1 + O(\mathcal{N}^2) = 1 + O(\delta^{3/2})$ and $F^{\theta-1} = O(\delta^{3(\theta-1)/2}) \leqslant \frac{1}{6\lambda^2}$ so that

$$\frac{\mathrm{d}R}{\mathrm{d}t} \geqslant \frac{2}{3}\lambda(1 - C\delta^{3/2}) \geqslant \frac{\lambda}{2}.$$

Integrating on $[0, t] \subset [0, T_*]$, it holds $R(t) \geqslant R(0) + \frac{\lambda}{2}t$. Hence, as we also have $\exp(3\lambda^{-1}\mathcal{M}) = 1 + O(\delta^{3/2})$ and $R(0)^{-1} \leqslant 2F(0) \lesssim \delta^2$,

$$F = \frac{1}{R}\exp(3\lambda^{-1}\mathcal{M}) \leqslant \frac{1}{R(0) + \frac{\lambda}{2}t}(1 + O(\delta^{3/2})) \lesssim \delta^2.$$

This strictly improves the estimate of F in (7.112), and as in the proof of Proposition 7.26, we also obtain the decay $F \lesssim t^{-1}$.

Transversality condition. From (7.63) and $\mathcal{N} \lesssim \delta$, we observe that for any time $t \in [0, T_*]$ where it holds $b(t) = \delta^{\frac{5}{2}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t) \geq 2\nu^+ b(t) - C\delta^3 \geq 2\nu^+ \delta^{\frac{5}{2}} - C\delta^3 \geq \nu^+ \delta^{\frac{5}{2}} > 0,$$

for $\delta > 0$ small enough. This transversality condition is enough to justify the existence of at least a point $\boldsymbol{a}_\sharp^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$ such that $T_* = \infty$.

Indeed, for the sake of contradiction assume that for all $\boldsymbol{a}^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$, it holds $T_* < \infty$. Then, a contradiction follows from the following observations (see for instance more details in [28] or in [31, Section 3.1]).

Continuity of T_*. The above transversality condition implies that the map

$$\boldsymbol{a}^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{5}{4}}) \mapsto T_* \in [0, \infty)$$

is continuous and that $T_* = 0$ for $\boldsymbol{a}^+(0) \in \mathcal{S}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$.

Construction of a retraction. As a consequence, the map

$$\boldsymbol{a}^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{5}{4}}) \mapsto \boldsymbol{a}^+(T_*) \in \mathcal{S}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$$

is continuous and its restriction to the sphere $\mathcal{S}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$ is the identity.

This is a contradiction with the no retraction theorem for continuous maps from the ball to the sphere.

At this point, we have proved the existence of $\boldsymbol{a}_\sharp^+(0) \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{5}{4}})$, associated with a global solution $\vec{u}_\sharp \in \mathcal{C}([0, \infty), H^1)$ of (7.1), which also satisfies (7.112) for all $t \geqslant 0$ (and $F = F_-$, $F_+ = 0$). Applying the results of §7.5 to $\vec{u}_\sharp$, we infer that $\mathcal{N} \lesssim t^{-1}$ (so that (7.6) holds) and that there exists $y_\sharp \in \mathbb{R}$ with, for $k = 1, \ldots, K$,

$$y_k = \bar{y}_k + y_\sharp + O(t^{-\theta+1})$$

As $z_k = y_k - \frac{\ell_k}{2\alpha}$ and $|\ell_k| \leqslant \mathcal{N} \lesssim t^{-1}$, $u_\sharp$ has the requested properties and the proof is complete. $\square$

Chapter 8

Asymptotics of solutions with a compactness property for the nonlinear damped Klein-Gordon equation

Abstract: We consider the nonlinear damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - |u|^{p-1}u = 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}^N$$

with $\alpha > 0$, $2 \leqslant N \leqslant 5$ and energy subcritical exponents $p > 2$. We study the behavior of solutions for which it is supposed that only one nonlinear object appears asymptotically for large times, at least for a sequence of times.

We first prove that the nonlinear object is necessarily a bound state. Next, we show that when the nonlinear object is a non-degenerate state or a degenerate excited state satisfying a simplicity condition, the convergence holds for all positive times, with an exponential or algebraic rate respectively. Last, we provide an example where the solution converges exactly at the rate t^{-1} to the excited state.

8.1 Introduction

8.1.1 Setting of the problem

We consider the nonlinear focusing damped Klein-Gordon equation

$$\partial_{tt}u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0 \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{8.1}$$

where $f(u) = |u|^{p-1}u$, $\alpha > 0$, $2 \leqslant N \leqslant 5$ and the exponent p satisfies

$$2 < p < p^*(N) \quad \text{with} \quad p^*(N) = \begin{cases} \infty & \text{if } N = 2, \\ \dfrac{N+2}{N-2} & \text{if } N = 3,4,5. \end{cases}$$

It follows from [12, Theorem 2.3] that the Cauchy problem for (8.1) is locally well-posed in the energy space: for any initial data $(u_0, v_0) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, there exists a unique (in some class) maximal solution $u \in \mathcal{C}([0,T_{\max}), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T_{\max}), L^2(\mathbb{R}^N))$ of (8.1). Moreover, if the maximal time of existence $T_{\max}$ is finite, then $\lim_{t\uparrow T_{\max}} \|\vec{u}(t)\|_{H^1\times L^2} = \infty$.

Setting $F(u) = \frac{1}{p+1}|u|^{p+1}$ and

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^N} \left\{|\nabla u|^2 + u^2 + (\partial_t u)^2 - 2F(u)\right\}\mathrm{d}x,$$

for any $H^1 \times L^2$ solution $\vec{u} = (u, \partial_t u)$ of (8.1), it holds

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 \mathrm{d}t. \tag{8.2}$$

One can easily construct finite time blow-up solutions by adequately truncating a constant in space solution, whose initial data lead to finite time blow-up for the inferred ODE $y'' + 2\alpha y' + y - f(y) = 0$ (and using finite speed of propagation). On the other hand, solutions to (8.1) which are globally defined for positive time, that is for which $T_{\max} = +\infty$, are believed to possess much more structure, in the spirit of a soliton resolution: it roughly asserts that any global solution (maybe under a genericity condition) splits for large times into a sum of decoupled rigid nonlinear objects, which should be here stationary solutions, especially in view of decay of energy (8.2).

Let us first recall from [17] (see also references therein) some features on stationary solution, namely a solution to the elliptic equation

$$-\Delta q + q - f(q) = 0, \quad q \in H^1(\mathbb{R}^N). \tag{8.3}$$

We call the solutions of (8.3) *bound states*, and denote $\mathcal{B}$ the set of bound states:

$$\mathcal{B} = \{q : q \text{ is a nontrivial solution of (8.3)}\}.$$

Standard elliptic arguments (see *e.g.* [58] or [17, Theorem 8.1.1]) show that if $q \in \mathcal{B}$, then q is of class $\mathcal{C}^2(\mathbb{R}^N)$ and has exponential decay as $|x| \to +\infty$, as well as its first and second-order derivatives.

Let

$$\mathcal{W}(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left\{|\nabla v|^2 + v^2 - 2F(v)\right\}\mathrm{d}x, \quad \text{for } v \in H^1.$$

We call the solutions of (8.3) which minimize the functional $\mathcal{W}$ by *ground states*; the set of ground states is denoted by $\mathcal{G}$

$$\mathcal{G} = \{q_0 \in \mathcal{B} : \forall q \in \mathcal{B},\ \mathcal{W}(q_0) \leqslant \mathcal{W}(q)\}.$$

Ground states are well studied objects. They are unique up to space translation (for rather general nonlinearities): there exists a radial positive function q_0 of class $\mathcal{C}^2$, exponentially decreasing, along with its first and second-order derivatives, such that

$$\mathcal{G} = \{q_0(x - x_0) : x_0 \in \mathbb{R}^N\}.$$

We refer to Berestycki-Lions [7], Gidas-Ni-Nirenberg [57], Kwong [77], Serrin-Tang [107] (however, a positive bound state may not be a ground state, see [35]). It is well-known (see *e.g.* Grillakis-Shatah-Strauss [60]) that the ground state q_0 is unstable in the energy space. This result was also known in the physics literature as the Derrick's Theorem [41].

In dimension 1, $\mathcal{B} = \mathcal{G}$ (due to ODE arguments). In contrast, for any $N \geqslant 2$, $\mathcal{G} \subsetneq \mathcal{B}$: see [17, Remark 8.1.16]. Functions $q \in \mathcal{B} \setminus \mathcal{G}$ are referred to as *excited states*. As a matter of fact, much less is known about excited states.

Here are some references on the construction of excited states. Berestycki-Lions [8] showed the existence of infinitely many radial *nodal* (*i.e.* sign changing) solutions (see also [61, 98] and the references therein). For the massless version of equation (8.3), the existence of excited states that are nonradial sign-changing and with arbitrary large energy was first proved in Ding [42] by variational argument. Later, del Pino-Musso-Pacard-Pistoia [36] constructed more explicit solutions to the massless equation (8.3) with a centered soliton crowned with negative spikes (rescaled solitons) at the vertices of a regular polygon of radius 1. Then, following similar general strategy in [36], they constructed sign changing, non radial solutions to (8.3) on the sphere $\mathbb{S}^N$ ($N \geqslant 4$) whose energy is concentrated along special submanifolds of $\mathbb{S}^N$ in [37].

We can now go back to (8.1), and recall some previous results related to the long time dynamics of global solutions.

Under some conditions on N and p, results in [53, 82] state that for any sequence of time, any global bounded solution of (8.1) converges to a sum of decoupled bound states after extraction of a subsequence of times. Also in [53], Feireisl constructed global solutions that behave as sum of an *even* number of ground states (*i.e.* multi-solitons).

In [12], for dimension $N \geq 2$, Burq, Raugel and Schlag proved the convergence of any global *radial* solution to one (radial) bound state, for the whole sequence of time.

In [29], it is given a complete description of 2-soliton solutions (that is, solutions which, on at least a sequence of time, behave as the sum of two decoupled ground states), in dimension $N \leqslant 5$. Building on the tools developed there, [30] gave a complete description of global solutions in dimension $N = 1$, that is, the soliton resolution in that case.

We aim at considering the behavior of solution without conditions on symmetry (like radiality). A complete description seems out of reach, because of the lack of understanding of the dynamics around general excited states, and because the system of centers of mass of the involved bound states may have itself a very intricate dynamics.

8.1.2 Main results

In this paper, we are instead interested in understanding the behavior of solutions to (8.1) for which *only one* nonlinear object appears for large times, at least for a sequence of time. More precisely, we define *packed* solutions as follows.

Definition 8.1. A maximal solution $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0, T_{\max}), H^1 \times L^2)$ of (8.1) is called a *packed solution* if there exist $(W_0, W_1) \in H^1 \times L^2$, and a time sequence $t_n \to T_{\max}$ and a position sequence $y_n \in \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \{\|u(t_n) - W_0(\cdot - y_n)\|_{H^1} + \|\partial_t u(t_n) - W_1(\cdot - y_n)\|_{L^2}\} = 0. \tag{8.4}$$

We say that $\vec{W} = (W_0, W_1)$ is a cluster point for $\vec{u}$ at $(t_n, y_n)_n$.

Observe that any cluster point (W_0, W_1) is actually a bound state $(q, 0)$. More precisely, the following Proposition holds true.

Proposition 8.2. *Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (8.1). Then $\vec{u} \in \mathcal{C}([0, +\infty), H^1 \times L^2)$ is globally defined for positive times, and if $(W_0, W_1) \in H^1 \times L^2$ is a cluster point for $\vec{u}$ at $(t_n, y_n)_n$, then $W_0 = q$ is a bound state of (8.3) and $W_1 = 0$. Furthermore, the energy is bounded below, $\partial_t u \in L^2([0, +\infty), L^2)$ and for all $t \geqslant 0$,*

$$E(\vec{u}(t)) - E(q, 0) = 2\alpha \int_t^{+\infty} \|\partial_t u(s)\|_{L^2}^2 \mathrm{d}s. \tag{8.5}$$

Notice that it is unclear whether a packed solution is globally bounded in $H^1 \times L^2$ (recall that from arguments of [16] – see also [12] and [30] – if $p \leqslant \frac{N}{N-2}$, then any global solution to (8.1) is globally bounded in $H^1 \times L^2$, but this is not known for higher powers of p).

It turns out that the description of the convergence depends deeply on the bound state. More specifically, consider the linearized operator $\mathcal{L}_q$ of the energy around a bound state q:

$$\mathcal{L}_q = -\Delta + 1 - f'(q), \quad \langle \mathcal{L}_q v, v \rangle = \int_{\mathbb{R}^N} \left\{ |\nabla v|^2 + v^2 - f'(q) v^2 \right\} \mathrm{d}x. \tag{8.6}$$

Due to the invariances of equations, $\mathcal{L}_q$ always has a important kernel: denote the Ω_{ij} are the angular derivatives that are

$$\Omega_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i} \quad \text{for } 1 \leqslant i < j \leqslant N, \tag{8.7}$$

and consider the vector space $\mathcal{Z}_q$ spanned by the infinitesimal generator of the invariance of the equation on q:

$$\mathcal{Z}_q = \mathrm{Span}\left\{ \partial_{x_n} q, n = 1, \cdots, N; \Omega_{ij} q, 1 \leqslant i < j \leqslant N \right\}. \tag{8.8}$$

One always has $\mathcal{Z}_q \subset \mathrm{Ker}\mathcal{L}_q$. Then we define non-degenerate and degenerate state.

Definition 8.3. Let $q \in \mathcal{B}$.

(i) q is called a *non-degenerate state* if $\mathcal{Z}_q = \mathrm{Ker}\mathcal{L}_q$.

(ii) q is called a *degenerate state* if $\mathcal{Z}_q \subsetneq \mathrm{Ker}\mathcal{L}_q$.

The most relevant example is of course the ground state q_0 which is non-degenerate. We will comment further on degenerate excited state in the comment paragraph below. For now, let us simply mention one way to understant degeneracy (we denote $'$ for Gateau differentials). The condition that q is a bound state writes $E'(q) = 0$. Then $\langle \mathcal{L}_q v, w \rangle = E''(q) \cdot (v, w)$, so that the condition that q is degenerate, is equivalent to the fact that for some $\phi \notin \mathcal{Z}_q$, the linear form $E''(q) \cdot (\phi, \cdot) = 0$.

Our first result is that if one cluster point of a packed solution is a non-degenerate state, then the convergence holds of all positive time, and occurs with an exponential rate. More precisely, we have the following.

Theorem 8.4. *Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (8.1), with cluster point $q \in H^1$ at $(t_n, y_n)_n$. If q is a non-degenerate state, we have convergence holding for all time and exponential decay, i.e. there exist $\mu > 0$ and $z_\infty \in \mathbb{R}^N$ such that*

$$\forall t \geqslant 0, \quad \|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim e^{-\mu t}.$$

Next, we consider *degree-1 excited states* where $\mathrm{Ker}\mathcal{L}_q$ has one extra dimension not related to the geometric invariances of the equation (8.1) and which also involves a condition on the third-order Gateau differentials of E, according to the next definition.

Definition 8.5. Let q be a degenerate excited state. q is called a *degree-1 excited state* if there exists $\phi \in H^1$ such that

$$\mathrm{Ker}\mathcal{L}_q = \mathcal{Z}_q \oplus \mathrm{Span}\{\phi\} \quad \text{and} \quad E'''(q) \cdot (\phi, \phi, \phi) \neq 0. \tag{8.9}$$

Again, we will comment on this definition in the paragraph below, the main point being that degree-1 excited states are somehow the simplest degenerate bounds states.

Our second result is concerned with cluster points which are degree-1 excited states.

Theorem 8.6. *Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (8.1), with cluster point $q \in H^1$ at $(t_n, y_n)_n$. If q is a degree-1 excited state, then the convergence $\vec{u}(t) \to (q, 0)$ holds for all time as $t \to +\infty$, and the rate of convergence has algebraic decay, i.e. there exists $z_\infty \in \mathbb{R}^N$ such that*

$$\forall t > 0, \quad \|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim t^{-1}.$$

Last, we show that the convergence rate in Theorem 8.6 can be sharp: we provide an example where the solution converges exactly at the rate t^{-1} to the degree-1 excited state.

Theorem 8.7. *Let q be a degree-1 excited state. Then, there exists a global solution $\vec{u} = (u, \partial_t u) \in \mathcal{C}([0, +\infty), H^1 \times L^2)$ of (8.1) such that*

$$\|u(t) - q\|_{H^1} + \|\partial_t u(t)\|_{L^2} \sim t^{-1} \quad \text{as} \quad t \to +\infty.$$

8.1.3 Comments

Let us first observe that Theorem 8.4 (and its proof) holds also in dimension 1, but of course, they are in that case a direct consequence of the complete description [30] of global solutions in 1D (as mentioned above, excited states only exist for $N \geqslant 2$). This is the only reason why we restrict to dimension $N \geqslant 2$. The restrictions to $N \leqslant 5$ and $2 < p < \frac{N+2}{N-2}$ are to ensure a nice local well posedness theory, and sufficient smoothness on the non-linearity so that Taylor expansion make sense up to order 2. In this perspective let us remind that our analysis encompasses the most physically relevant nonlinearity, the 3D cubic one.

Regarding Theorem 8.4: the ground state is of course non-degenerate, but one should keep in mind that is not so easy to construct degenerate excited states. As a matter of fact, the constructions in [2, 98] (see also [36, 99] for the massless case) yield *non-degenerate* excited states as well. This means that the scope of Theorem 8.4 is rather large and does certainly not restrict to the ground state.

We now discuss degree-1 excited states: as we mentioned, they should be understood as the simplest degenerate case. Already here, very little is known, and to our knowledge, our results are the first describing precisely the dynamics in a degenerate setting. From this point of view, the condition that $\dim \mathrm{Ker}\mathcal{L}_q = \dim(\mathcal{Z}_q) + 1$ is very natural. Regarding the extra condition $E'''(q) \cdot (\phi, \phi, \phi) \neq 0$, let us note that it is generic; as we will see in Lemma 8.8, it is equivalent with $E'''(q)$ being non identically 0 on $(\mathrm{Ker}\mathcal{L}_q)^3$.

It is remarkable that one already observes a drastic change in the dynamic in degree-1 degeneracy, when compared to non-degeneracy. The convergence here is indeed merely polynomial in time, which is a surprise: such slow rate of convergence is usually observed due to the interaction with another nonlinear object (as in [29, 30]), and this is not the case here. As it is seen in the proofs, the derivation of the main bootstrap regime is noticeably more involved in degree-1 degeneracy, and relies on the very specific algebra of the main ODE system at leading order (see §8.3).

One setting where excited states are better understood is the case of radial functions. Among these, radial bound states q are either non-degenerate or satisfy the first condition in the degree-1 degeneracy definition (8.9): indeed, among radial functions, the geometric kernel $\mathcal{Z}_q$ is trivial and $\dim \mathrm{Ker}_{\mathrm{rad}}\mathcal{L}_q \leqslant 1$, see for example [12, Section 2.3].

All the arguments in the proofs below can taken word for word to the radial setting, and so our results hold for any packed *radial* solution converging to a radial bound state q which is either non-degenerate (Theorem 8.4) or such that $E'''(q)|_{(\mathrm{Ker}\mathcal{L}_q)^3} \neq 0$ (Theorems 8.6 and 8.7).

8.2 Preliminaries

8.2.1 Proof of Proposition 8.2

Proof of Proposition 8.2. Denote $\vec{W}(t) = (W(t), \partial_t W(t))$ the solution to (8.1) with initial data $\vec{W}(0) = (W_0, W_1)$ and $\vec{u}_n(t,x) = \vec{u}(t_n + t, x + y_n)$ the solution to (8.1) with initial data $\vec{u}_n(0,x) = \vec{u}(t_n, x + y_n)$. We can assume that $\vec{W} \in \mathcal{C}([0,T_0], H^1 \times L^2)$ for some $T_0 > 0$.

As the (8.1) flow is continuous in $H^1 \times L^2$ and $\vec{u}_n(0) \to \vec{W}(0)$, we infer that $\vec{u}_n$ is defined on $[0,T_0]$ for n large enough and that

$$\vec{u}_n \to \vec{W} \quad \text{in} \quad \mathcal{C}([0,T_0], H^1 \times L^2). \tag{8.10}$$

This immediately prove that $\vec{u}$ is globally defined for positive times. Indeed, if $T_{\max} < +\infty$, then for large enough n, $T_{\max} \geqslant t_n + T_0 \to T_{\max} + T_0$, a contradiction: hence $T_{\max} = +\infty$.

As $\vec{E}(\vec{u}(t_n)) \to \vec{E}(W_0, W_1)$ we infer from the energy dissipation identity that $\partial_t u \in L^2([0,+\infty), L^2)$. Assume that $\vec{W}$ is not a stationnary solution. Then we can furthermore assume that $\partial_t W \neq 0$ on $[0,T_0] \times \mathbb{R}^N$, so that $\|\partial_t W\|_{L^2([0,T_0]),L^2)} > 0$. In particular, from the convergence (8.10), we conclude that

$$\|\partial_t u\|_{L^2([t_n,t_n+T_0],L^2)} = \|\partial_t u_n\|_{L^2([0,T_0]),L^2)} \to \|\partial_t W\|_{L^2([0,T_0]),L^2)} \quad \text{as} \quad n \to +\infty.$$

Let t'_n be a subsequence of t_n such that for all $n \in \mathbb{N}$, $t'_{n+1} \geqslant t'_n + T_0$ and

$$\|\partial_t u\|_{L^2([t'_n,t'_n+T_0],L^2)} \geqslant \frac{1}{2}\|\partial_t W\|_{L^2([0,T_0]),L^2)}.$$

There holds

$$\|\partial_t u\|^2_{L^2([0,+\infty),L^2)} \geqslant \sum_n \|\partial_t u\|^2_{L^2([t'_n,t'_n+T_0],L^2)} \geqslant \sum_n \frac{1}{4}\|\partial_t W\|^2_{L^2([0,T_0]),L^2)} = +\infty,$$

which is a contradiction. As a consequence, $\vec{W}$ is a stationary solution, which means that for all $t \geqslant 0$, $\partial_t W(t) = 0$ and $W(t) = q$ for some bound state q. In particular, $W_0 = q$ and $W_1 = 0$.

Furthermore, the energy dissipation identity writes for all $0 \leqslant t \leqslant t_n$:

$$E(\vec{u}(t)) - E(\vec{u}(t_n)) = 2\alpha \int_t^{t_n} \|\partial_t u(s)\|^2_{L^2} \mathrm{d}s.$$

Letting $n \to +\infty$, we see that the left-hand side has a limit $E(\vec{u}(t)) - E(q,0)$, and so $\partial_t u \in L^2([t,+\infty), L^2)$. This completes the proof of Proposition 8.2. □

8.2.2 Notation

Let q be a bound state. Let I_q be a subset of $\{(i,j) : 1 \leqslant i < j \leqslant N\}$ such that

$$\{\partial_{x_n} q, 1 \leqslant n \leqslant N; \Omega_{ij} q, (i,j) \in I_q\} \text{ is a basis of } \mathcal{Z}_q.$$

For any $(i,j) \in I_q$ and $\vartheta \in \mathbb{R}$, we recall the Givens rotation:

$$G_{ij}(\vartheta) = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cos\vartheta & \cdots & -\sin\vartheta & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \sin\vartheta & \cdots & \cos\vartheta & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}, \tag{8.11}$$

where $\cos\vartheta$ and $\sin\vartheta$ appear at the intersections ith and jth rows and columns. That is, the non-zero elements of the Givens matrix $G_{i,j}(\vartheta) = (g_{nm})_{nm}$ are given by:

$$g_{nn} = 1 \quad \text{for } n \neq i,j, \quad g_{ii} = g_{jj} = \cos\vartheta, \quad \text{and} \quad g_{ij} = -g_{ji} = -\sin\vartheta.$$

For $K \in \mathbb{N}^*$ and $r > 0$, we denote by $\mathcal{B}_{\mathbb{R}^K}(r)$ (respectively, $\mathcal{S}_{\mathbb{R}^K}(r)$) be the ball (respectively, the sphere) of $\mathbb{R}^K$ of center 0 and of radius r.

We denote $\langle \cdot, \cdot \rangle$ the L^2 scalar product for real-valued functions $u, v \in L^2$,

$$\langle u, v \rangle := \int_{\mathbb{R}^N} u(x)v(x)\mathrm{d}x.$$

For vector-valued functions

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

the notation $\langle \cdot, \cdot \rangle$ is also the L^2 scalar product,

$$\langle \vec{u}, \vec{v} \rangle := \sum_{k=1,2} \langle u_k, v_k \rangle, \quad \|\vec{u}\|_{\mathcal{H}}^2 := \|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2.$$

We also define $\bar{p} = \min\{3, p\} > 2$ (recall that the nonlinearity power $p > 2$).

8.2.3 Spectral theory of linearized operator

In this section, we introduce some spectral properties of the linearized operator for any bound state $q \in \mathcal{B}$.

For $\theta = (\theta_{ij})_{(i,j)\in I_q} \in \mathbb{R}^{\#I_q}$, denote the rotation

$$R_\theta = G_{i_1 j_1}(\theta_{i_1 j_1}) \cdots G_{i_{\#I_q} j_{\#I_q}}(\theta_{i_{\#I_q} j_{\#I_q}}).$$

For $(z, \theta) \in \mathbb{R}^{N+\#I_q}$, we introduce the following transformation $\mathcal{T}_{(z,\theta)}$ linked to the symmetries of (8.1): for $f \in L^2$,

$$\mathcal{T}_{(z,\theta)} f := f(R_\theta(\cdot - z)).$$

Observe that for all $q \in \mathcal{B}$, $\mathcal{Z}_q$ is generated by taking partial derivatives of $\mathcal{T}_{(z,\theta)} q$ with respect to (z, θ) at $(z, \theta) = (\mathbf{0}, \mathbf{0})$:

$$\partial_{x_n} q = -\frac{\partial}{\partial z_n} \mathcal{T}_{(z,\theta)} q|_{(z,\theta)=(\mathbf{0},\mathbf{0})}, \quad \Omega_{ij} q = \frac{\partial}{\partial \theta_{ij}} \mathcal{T}_{(z,\theta)} q|_{(z,\theta)=(\mathbf{0},\mathbf{0})}. \tag{8.12}$$

First, we recall standard properties of the linearized operator $\mathcal{L}_q$.

Lemma 8.8. (i) Spectral properties. *The self-adjoint operator $\mathcal{L}_q$ has essential spectrum $[1,+\infty)$, a finite number $K \geqslant 1$ of negative eigenvalues and its kernel is of finite dimension M with $M \geqslant N$. Let $(Y_k)_{k=1,\cdots,K}$ be an L^2 orthogonal family of eigenfunctions of $\mathcal{L}_q$ with negative eigenvalues $(-\lambda_k^2)_{k=1,\cdots,K}$, i.e.*

$$\langle Y_k, Y_{k'}\rangle = \delta_{kk'} \quad \text{and} \quad \mathcal{L}_q Y_k = -\lambda_k^2 Y_k, \quad \lambda_k > 0. \tag{8.13}$$

(ii) Coercivity. *Denote Π_q the L^2-orthogonal projection on $\mathrm{Ker}\mathcal{L}_q$. There exists $c > 0$ such that for all $\eta \in H^1$,*

$$\langle \mathcal{L}_q\eta, \eta\rangle \geqslant c\|\eta\|_{H^1}^2 - c^{-1}\Big(\|\Pi_q\eta\|_{L^2}^2 + \sum_{k=1}^{K}\langle \eta, Y_k\rangle^2\Big). \tag{8.14}$$

(iii) Cancellation. *We have, for all $\psi_1, \psi_2 \in \mathrm{Ker}\mathcal{L}_q$ and $\psi_3 \in \mathcal{Z}_q$*

$$\langle f''(q)\psi_1\psi_2, \psi_3\rangle = 0. \tag{8.15}$$

Proof. Proof of (i) and (ii). See the proof of [26, Lemma 1].

Proof of (iii). Without loss of generality, we first consider

$$\psi_3 = \Omega_{ij} q \quad \text{for } (i,j) \in I_q.$$

For any $\psi_1 \in \mathrm{Ker}\mathcal{L}_q$, we have

$$-\Delta\psi_1 + \psi_1 - f'(q)\psi_1 = 0.$$

Consider the transformation $\mathcal{T}_{(z,\theta)}$ with $(z,\theta) = (\mathbf{0},\theta)$ for the above identity,

$$-\Delta(\mathcal{T}_{(z,\theta)}\psi_1) + \mathcal{T}_{(z,\theta)}\psi_1 - f'(\mathcal{T}_{(z,\theta)}q)(\mathcal{T}_{(z,\theta)}\psi_1) = 0.$$

Note that, from $p > 2$, we can take the derivative of above identity with respect to θ_{ij}, and then let $\theta = \mathbf{0}$. It follows that

$$f''(q)\psi_1\psi_3 = -\Delta\tilde{\psi}_1 + \tilde{\psi}_1 - f'(q)\tilde{\psi}_1 = \mathcal{L}_q\tilde{\psi}_1 \quad \text{where } \tilde{\psi}_1 = \Omega_{ij}\psi_1.$$

Thus, by integration by parts and $\psi_2 \in \mathrm{Ker}\mathcal{L}_q$,

$$\langle f''(q)\psi_1\psi_2, \psi_3\rangle = \langle f''(q)\psi_1\psi_3, \psi_2\rangle = \langle \mathcal{L}_q\tilde{\psi}_1, \psi_2\rangle = \langle \tilde{\psi}_1, \mathcal{L}_q\psi_2\rangle = 0.$$

Proceeding similarly for all the parameters in the transformation $\mathcal{T}_{(z,\theta)}$, we complete the proof of (iii). □

As a consequence of the above Lemma, in the case when q is furthermore assumed to be a degree-1 excited state, we now choose ϕ (introduced in Definition 8.5) with more rigid properties: namely we claim that there exists (a unique) $\phi \in H^1$ such that for all $n = 1, \cdots, N$, $(i,j) \in I_q$,

$$\langle \phi, \partial_{x_n} q\rangle = \langle \phi, \Omega_{ij} q\rangle = 0 \quad \text{and} \quad \mathrm{Ker}\mathcal{L}_q = \mathcal{Z}_q \oplus \mathrm{Span}\{\phi\}, \tag{8.16}$$

and such that

$$-(4\alpha\|\phi\|_{L^2}^2)^{-1}E'''(\phi,\phi,\phi) = 1. \tag{8.17}$$

(8.16) essentially means that ϕ is the L^2-orthogonal supplement of $\mathcal{Z}_q$. Moreover, due to (8.15) and the fact that $E'''(q)$ is not identically 0 on $\mathcal{L}_q^3$ (in view of (8.9)), for such a ϕ one has $E'''(q)(\phi,\phi,\phi) \neq 0$: and so, by considering the transformations $\phi \to -\phi$ and $\phi \to \lambda\phi$, the condition (8.17) can be met.

8.2.4 Modulation around a bound state

Let $q \in \mathcal{B}$ be a degree-1 degenerate bound state. Given time dependent $\mathscr{C}^1$ functions z, θ, a, with values in $\mathbb{R}^N$, $\mathbb{R}^{\#I_q}$ and $\mathbb{R}$, we denote

$$Q = \mathcal{T}_{(z,\theta)} q = q\left(R_\theta(\cdot - z)\right), \quad \Phi(t,x) = \mathcal{T}_{(z,\theta)}\phi = \phi\left(R_\theta(\cdot - z)\right), \tag{8.18}$$

$$V(t,x) = a(t)\Phi(t,x), \quad G = f\left(Q+V\right) - f(Q) - f'(Q)V. \tag{8.19}$$

It will convenient to encompass both non-degenerate and degree-1 degenerate cases at once by setting $a \equiv 0$ if q is non-degenerate.

For all $(i,j) \in I_q$ and $(i',j') \in I_q$, we denote as follows the derivatives:

$$\Psi_{ij} = \frac{\partial Q}{\partial \theta_{ij}}, \quad \Phi_{ij} = \frac{\partial \Phi}{\partial \theta_{ij}}, \quad \Psi_{ij}^{i'j'} = \frac{\partial \Psi_{ij}}{\partial \theta_{i'j'}}. \tag{8.20}$$

Finally, we introduce the exponential directions. For $k = 1, \cdots, K$, we denote

$$\Upsilon_k = \mathcal{T}_{(z,\theta)} Y_k, \quad \Upsilon_k^{ij} = \frac{\partial \Upsilon_k}{\partial \theta_{ij}},$$

and

$$\nu_k^\pm = -\alpha \pm \sqrt{\alpha^2 + \lambda_k^2}, \quad \zeta_k^\pm = \alpha \pm \sqrt{\alpha^2 + \lambda_k^2} \quad \text{and} \quad \vec{Z}_k^\pm = \begin{pmatrix} \zeta_k^\pm \Upsilon_k \\ \Upsilon_k \end{pmatrix}. \tag{8.21}$$

The importance of $\vec{Z}_k^\pm$ come from the following observation: if $\vec{v} = (v_1, v_2)$ is a solution to the linearized (8.1) equation

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -2\alpha v_2 - \mathcal{L}_q v_1 \end{pmatrix},$$

then with $a_k^\pm := \langle \vec{v}, \vec{Z}_k^\pm \rangle$, there hold $\frac{\mathrm{d}}{\mathrm{d}t} a_k^\pm = \nu_k^\pm a_k^\pm$.

Observe that all the function introduced are at least of class $\mathscr{C}^1$ and have pointwise exponential decay.

By direct computation and the definition of Givens rotations in (8.11), we have, for all $n = 1, \cdots, N$ and $(i,j) \in I_q$,

$$\begin{aligned} \Psi_{ij} &\in \operatorname{Span}\left\{\left(\Omega_{ij} q\right)\left(R_\theta(\cdot - z)\right) : (i,j) \in I_q\right\}, \\ \partial_{x_n} Q &\in \operatorname{Span}\left\{\left(\partial_{x_1} q\right)\left(R_\theta(\cdot - z)\right), \cdots, \left(\partial_{x_N} q\right)\left(R_\theta(\cdot - z)\right)\right\}. \end{aligned} \tag{8.22}$$

Moreover, by the chain rule, we have, for all $(i,j) \in I_q$ and $k = 1, \cdots, K$,

$$\begin{aligned} \partial_t Q &= -\dot{z} \cdot \nabla Q + \sum_{(i,j) \in I_q} \dot{\theta}_{ij} \Psi_{ij}, \\ \partial_t \Phi &= -\dot{z} \cdot \nabla \Phi + \sum_{(i,j) \in I_q} \dot{\theta}_{ij} \Phi_{ij}, \\ \partial_t \Upsilon_k &= -\dot{z} \cdot \nabla \Upsilon_k + \sum_{(i,j) \in I_q} \dot{\theta}_{ij} \Upsilon_k^{ij}, \\ \partial_t \Psi_{ij} &= -\dot{z} \cdot \nabla \Psi_{ij} + \sum_{(i',j') \in I_q} \dot{\theta}_{i'j'} \Psi_{ij}^{i'j'}. \end{aligned} \tag{8.23}$$

As a consequence of (8.12), we have the following expansions for small $\theta \in \mathbb{R}^{\#I_q}$.

Lemma 8.9. *For $|\theta| \ll 1$ small, we have, for all $(i,j) \in I_q$, $(i',j') \in I_q$, $k = 1, \dots, K$ and $n = 1, \dots, N$,*

$$\begin{aligned}
\Psi_{ij} &= (\Omega_{ij} q)\,(R_\theta(\cdot - z)) + O_{H^1}(|\theta|),\\
\Phi_{ij} &= (\Omega_{ij} \phi)\,(R_\theta(\cdot - z)) + O_{H^1}(|\theta|),\\
\Upsilon_k^{ij} &= (\Omega_{ij} Y_k)\,(R_\theta(\cdot - z)) + O_{H^1}(|\theta|),\\
\partial_{x_n} Q &= (\partial_{x_n} q)\,(R_\theta(\cdot - z)) + O_{H^1}(|\theta|),\\
\Psi_{ij}^{i'j'} &= \left(\Omega_{i'j'}\Omega_{ij} q\right)(R_\theta(\cdot - z)) + O_{H^1}(|\theta|).
\end{aligned} \tag{8.24}$$

If q is a non-degenerate bound state, we use same notations as above for degree-1 degenerate bound states, but with $a = 0$ and $\phi = 0$.

For future reference, we state the following Taylor formulas involving the functions F and f, and omit its proof.

Lemma 8.10. *For all $s \in \mathbb{R}$, and $x \in \mathbb{R}^N$, we have*

$$\left|f'(Q+s) - f'(Q)\right| \lesssim |s| + |s|^{p-1}, \tag{8.25}$$

$$\left|f(Q+s) - f(Q) - f'(Q)s\right| \lesssim s^2 + |s|^p, \tag{8.26}$$

$$\left|f(Q+s) - f(Q) - f'(Q)s - \frac{1}{2}f''(Q)s^2\right| \lesssim |s|^3 + |s|^p, \tag{8.27}$$

$$\left|f(Q+V+s) - f(Q+V) - f'(Q+V)s\right| \lesssim s^2 + |s|^p, \tag{8.28}$$

$$|F(Q+V+s) - F(Q+V) - f(Q+V)s| \lesssim |Q+V|^{p-1}s^2 + |s|^{p+1}, \tag{8.29}$$

$$\left|F(Q+V+s) - F(Q+V) - f(Q+V)s - \frac{1}{2}f'(Q+V)s^2\right| \lesssim |s|^3 + |s|^{p+1}, \tag{8.30}$$

where all the implied constants in the $\lesssim$ are uniform in the space variable of Q or V.

First, we introduce the standard modulation result around the non-degenerate state or degree-1 excited state q.

Proposition 8.11 (Properties of the modulation)**.** *There exists $0 < \gamma_0 \ll 1$ such that for any $0 < \gamma < \gamma_0$, $T_1 \leqslant T_2$, and any solution $\vec{u} = (u, \partial_t u)$ of (8.1) on $[T_1, T_2]$ satisfying*

$$\sup_{t \in [T_1,T_2]} \left\{ \inf_{\xi \in \mathbb{R}^N} \|u(t) - q(\cdot - \xi)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \right\} < \gamma, \tag{8.31}$$

there exist unique $\mathscr{C}^1$ functions

$$\begin{array}{rcl}
[T_1, T_2] & \to & \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{\#I_q} \times \mathbb{R}^{\#I_q} \times \mathbb{R} \times \mathbb{R}\\
t & \mapsto & (z(t), \ell(t), \theta(t), \beta(t), a(t), b(t))
\end{array}$$

such that, if we define $\vec{\varphi} = (\varphi_1, \varphi_2)$ by

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} Q \\ -\ell \cdot \nabla Q \end{pmatrix} + \sum_{(i,j) \in I_q} \beta_{ij} \begin{pmatrix} 0 \\ \Psi_{ij} \end{pmatrix} + \begin{pmatrix} a\Phi \\ b\Phi \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{8.32}$$

where $\beta = (\beta_{ij})_{(i,j) \in I_q}$, it satisfies, for all $t \in [T_1, T_2]$,

$$\begin{aligned}
\|\vec{\varphi}(t)\|_{\mathcal{H}} + |\theta(t)| &\lesssim \|u(t) - q\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim \gamma,\\
|\ell(t)| + |\beta(t)| + |a(t)| + |b(t)| &\lesssim \|u(t) - q\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim \gamma,
\end{aligned} \tag{8.33}$$

and for all $n = 1, \cdots, N$ and $(i,j) \in I_q$,

$$\langle \varphi_1, \partial_{x_n} Q \rangle = \langle \varphi_1, \Psi_{ij} \rangle = \langle \varphi_1, \Phi \rangle = 0, \tag{8.34}$$

$$\langle \varphi_2, \partial_{x_n} Q \rangle = \langle \varphi_2, \Psi_{ij} \rangle = \langle \varphi_2, \Phi \rangle = 0. \tag{8.35}$$

Proof. The proof of the decomposition result relies on a standard argument based on the Implicit function Theorem (See *e.g.* [31, Appendix B]) and we omit it. □

Second, we derive the equation of $\vec{\varphi}$ from (8.1) and (8.32).

Lemma 8.12 (Equation of $\vec{\varphi}$)**.** *In the contex of Proposition 8.11, we have*

$$\begin{cases} \partial_t \varphi_1 = \varphi_2 + \mathrm{Mod}_1 + G_1, \\ \partial_t \varphi_2 = \Delta \varphi_1 - \varphi_1 - 2\alpha\varphi_2 + f(Q+V+\varphi_1) - f(Q+V) + \mathrm{Mod}_2 + G_2 + G, \end{cases} \tag{8.36}$$

where

$$\begin{aligned} \mathrm{Mod}_1 &:= \left(\dot{z} - \ell\right) \cdot \nabla Q - \sum_{(i,j)\in I_q} \left(\dot{\theta}_{ij} - \beta_{ij}\right) \Psi_{ij} - (\dot{a} - b)\Phi, \\ \mathrm{Mod}_2 &:= \left(\dot{\ell} + 2\alpha\ell\right) \cdot \nabla Q - \sum_{(i,j)\in I_q} \left(\dot{\beta}_{ij} + 2\alpha\beta_{ij}\right) \Psi_{ij} - \left(\dot{b} + 2\alpha b\right) \Phi, \end{aligned}$$

and $G = f(Q+V) - f(Q) - f'(Q)V$ *is defined in* (8.19),

$$\begin{aligned} G_1 :=& a\dot{z} \cdot \nabla\Phi - a \sum_{(i,j)\in I_q} \dot{\theta}_{ij} \Phi_{ij}, \\ G_2 :=& \sum_{(i,j)\in I_q} \dot{\theta}_{ij} \left(\ell \cdot \nabla \Psi_{ij}\right) + \sum_{(i,j)\in I_q} \beta_{ij} \left(\dot{z} \cdot \nabla \Psi_{ij}\right) - \left(\ell \cdot \nabla\right)\left(\dot{z} \cdot \nabla\right) Q \\ & - \sum_{(i',j')\in I_q} \sum_{(i,j)\in I_q} \dot{\theta}_{i'j'} \beta_{ij} \Psi_{ij}^{i'j'} + b\dot{z} \cdot \nabla\Phi - \sum_{(i,j)\in I_q} b\dot{\theta}_{ij} \Phi_{ij}. \end{aligned}$$

Proof. First, from (8.23) and (8.32),

$$\begin{aligned} \partial_t \varphi_1 =& \partial_t u - \partial_t Q - a\partial_t \Phi - \dot{a}\Phi \\ =& \varphi_2 + \left(\dot{z} - \ell\right) \cdot \nabla Q - \sum_{(i,j)\in I_q} \left(\dot{\theta}_{ij} - \beta_{ij}\right) \Psi_{ij} - (\dot{a} - b)\Phi \\ & + a\dot{z} \cdot \nabla\Phi - a \sum_{(i,j)\in I_q} \dot{\theta}_{ij} \Phi_{ij} = \varphi_2 + \mathrm{Mod}_1 + G_1. \end{aligned}$$

Using (8.23) and (8.32) again,

$$\begin{aligned} \partial_t \varphi_2 =& \partial_{tt} u + \dot{\ell} \cdot \nabla Q - \sum_{(i,j)\in I_q} \dot{\beta}_{ij} \Psi_{ij} - \dot{b}\Phi \\ & + \ell \cdot \nabla \partial_t Q - \sum_{(i,j)\in I_q} \beta_{ij} \partial_t \Psi_{ij} - b\partial_t \Phi \\ =& \partial_{tt} u + \dot{\ell} \cdot \nabla Q - \sum_{(i,j)\in I_q} \dot{\beta}_{ij} \Psi_{ij} - \dot{b}\Phi + G_2. \end{aligned}$$

From (8.1), (8.32), $-\Delta Q + Q - f(Q) = 0$ and $-\Delta\Phi + \Phi - f'(Q)\Phi = 0$,

$$\begin{aligned} \partial_{tt} u =& \Delta u - u - 2\alpha\partial_t u + f(u) \\ =& \Delta\varphi_1 - \varphi_1 - 2\alpha\varphi_2 + f(Q+V+\varphi_1) - f(Q+V) \\ & + 2\alpha\ell \cdot \nabla Q - 2\alpha \sum_{(i,j)\in I_q} \beta_{ij} \Psi_{ij} - 2\alpha b\Phi + G. \end{aligned}$$

Therefore,

$$\begin{aligned}\partial_t\varphi_2 = \Delta\varphi_1 - \varphi_1 - 2\alpha\varphi_2 + f(Q+V+\varphi_1) - f(Q+V) + G_2 + G \\ + \left(\dot\ell + 2\alpha\ell\right)\cdot\nabla Q - \sum_{(i,j)\in I_q}\left(\dot\beta_{ij} + 2\alpha\beta_{ij}\right)\Psi_{ij} - \left(\dot b + 2\alpha b\right)\Phi.\end{aligned}$$ □

Third, we derive the control of geometric parameters from orthogonality conditions (8.34) and (8.35). Our goal here is to get an ODE system on the modulations parameters, at leading order, with bounds on the remainder terms as squares or higher powers of $|a|$ and

$$\mathcal{N} := \|\vec\varphi\|_{\mathcal{H}} + |\ell| + |\beta| + |b|. \tag{8.37}$$

Recall that we defined $\bar p = \min\{3, p\} > 2$.

Lemma 8.13. *In the context of Proposition 8.11, the following holds.*

(i) Control of non-degenerate directions. *We have*

$$|\dot z - \ell| + \left|\dot\theta - \beta\right| \lesssim \mathcal{N}^2 + |a|\mathcal{N}, \tag{8.38}$$

$$|\dot\ell + 2\alpha\ell| + |\dot\beta + 2\alpha\beta| \lesssim \mathcal{N}^2 + |a|\mathcal{N} + |a|^{\bar p}. \tag{8.39}$$

(ii) Control of extra direction. *For q be a degree-1 excited state, we have*

$$|\dot a - b| \lesssim \mathcal{N}^2 + |a|\mathcal{N}, \tag{8.40}$$

$$|\dot b + 2\alpha b + 2\alpha a^2| \lesssim \mathcal{N}^2 + |a|\mathcal{N} + |a|^{\bar p}. \tag{8.41}$$

Proof. Proof of (i). First, we differentiate the orthogonality $\langle\varphi_1, \partial_{x_n}Q\rangle = 0$ in (8.34),

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi_1, \partial_{x_n}Q\rangle = \langle\partial_t\varphi_1, \partial_{x_n}Q\rangle + \langle\varphi_1, \partial_t\partial_{x_n}Q\rangle.$$

Using (8.34) and (8.36),

$$\langle\partial_t\varphi_1, \partial_{x_n}Q\rangle = \langle \mathrm{Mod}_1, \partial_{x_n}Q\rangle + \langle G_1, \partial_{x_n}Q\rangle.$$

From (8.16), (8.22), (8.24), (8.33), the expression of Mod_1 and change of variables,

$$\begin{aligned}&\langle \mathrm{Mod}_1, \partial_{x_n}Q\rangle \\ &= \langle(\dot z - \ell)\cdot\nabla Q, \partial_{x_n}Q\rangle - \sum_{(i,j)\in I_q}\left(\dot\theta_{ij} - \beta_{ij}\right)\langle\Psi_{ij}, \partial_{x_n}Q\rangle - (\dot a - b)\langle\Phi, \partial_{x_n}Q\rangle \\ &= \langle(\dot z - \ell)\cdot\nabla q, \partial_{x_n}q\rangle - \sum_{(i,j)\in I_q}\left(\dot\theta_{ij} - \beta_{ij}\right)\langle\Omega_{ij}q, \partial_{x_n}q\rangle + O(\gamma(|\dot z - \ell| + |\dot\theta - \beta|)).\end{aligned}$$

From the expression of G_1 and (8.33),

$$\begin{aligned}|\langle G_1, \partial_{x_n}Q\rangle| &\lesssim |a|\left(|\dot z - \ell| + |\ell| + |\dot\theta - \beta| + |\beta|\right) \\ &\lesssim \gamma\left(|\dot z - \ell| + |\dot\theta - \beta|\right) + |a|\mathcal{N}.\end{aligned}$$

Next, using again (8.23),

$$\partial_t\partial_{x_n}Q = \partial_{x_n}\partial_t Q = -\dot z\cdot\nabla\partial_{x_n}Q + \sum_{(i,j)\in I_q}\dot\theta_{ij}\partial_{x_n}\Psi_{ij}. \tag{8.42}$$

Thus, from (8.33) and the Sobolev embedding Theorem,

$$\begin{aligned}|\langle\varphi_1,\partial_t\partial_{x_n}Q\rangle| &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}\left(|\dot{z}-\ell|+|\ell|+|\dot{\theta}-\beta|+|\beta|\right)\\ &\lesssim \gamma\left(|\dot{z}-\ell|+|\dot{\theta}-\beta|\right)+\mathcal{N}^2.\end{aligned}$$

Combining above estimates, we have

$$\begin{aligned}&\langle(\dot{z}-\ell)\cdot\nabla q,\partial_{x_n}q\rangle-\sum_{(i,j)\in I_q}\left(\dot{\theta}_{ij}-\beta_{ij}\right)\langle\Omega_{ij}q,\partial_{x_n}q\rangle\\ &=O\left(\gamma\left(|\dot{z}-\ell|+|\dot{\theta}-\beta|\right)+\mathcal{N}^2+|a|\mathcal{N}\right).\end{aligned}$$

This gives N inequalities. Proceeding similarly for the $\#I_q$ orthogonality conditions $\langle\varphi_1,\Omega_{ij}Q\rangle=0$ in (8.34), and using the fact that the family

$$\{\partial_{x_n}q, 1\leqslant n\leqslant N;\Omega_{ij}q,(i,j)\in I_q\}$$

is linearly independent, so that its Gram matrix is invertible, we obtain

$$|\dot{z}-\ell|+\left|\dot{\theta}-\beta\right|\lesssim\gamma\big(|\dot{z}-\ell|+\left|\dot{\theta}-\beta\right|\big)+\mathcal{N}^2+|a|\mathcal{N},$$

which implies (8.38), upon taking γ small enough.

Second, we differentiate the orthogonality $\langle\varphi_2,\partial_{x_n}Q\rangle=0$ in (8.35),

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi_2,\partial_{x_n}Q\rangle=\langle\partial_t\varphi_2,\partial_{x_n}Q\rangle+\langle\varphi_2,\partial_t\partial_{x_n}Q\rangle.$$

From (8.36), we have

$$\begin{aligned}\langle\partial_t\varphi_2,\partial_{x_n}Q\rangle&=\langle\Delta\varphi_1-\varphi_1+f'(Q)\varphi_1,\partial_{x_n}Q\rangle-2\alpha\langle\varphi_2,\partial_{x_n}Q\rangle\\ &\quad+\langle G_2,\partial_{x_n}Q\rangle+\langle R,\partial_{x_n}Q\rangle+\langle G,\partial_{x_n}Q\rangle+\langle\mathrm{Mod}_2,\partial_{x_n}Q\rangle,\end{aligned}$$

where

$$R=f(Q+V+\varphi_1)-f(Q+V)-f'(Q)\varphi_1.$$

Based on $-\Delta\partial_{x_n}q+\partial_{x_n}q-f'(q)\partial_{x_n}q=0$, integration by parts and (8.35),

$$\langle\Delta\varphi_1-\varphi_1+f'(Q)\varphi_1,\partial_{x_n}Q\rangle-2\alpha\langle\varphi_2,\partial_{x_n}Q\rangle=0.$$

Then, by the expression of G_2, (8.33) and (8.38),

$$\begin{aligned}|\langle G_2,\partial_{x_n}Q\rangle|&\lesssim\left(|\dot{z}-\ell|+|\dot{\theta}-\beta|+|\ell|+|\beta|\right)(|\ell|+|\beta|+|b|)\\ &\lesssim\mathcal{N}^2\left(\mathcal{N}+|a|+1\right)\lesssim\mathcal{N}^2.\end{aligned}$$

Next, using (8.25), (8.28) and (8.33),

$$\begin{aligned}|R|&\lesssim\left|f(Q+V+\varphi_1)-f(Q+V)-f'(Q+V)\varphi_1\right|+\left|(f'(Q+V)-f'(Q))\varphi_1\right|\\ &\lesssim|\varphi_1|^2+|\varphi_1|^p+(|V|+|V|^{p-1})|\varphi_1|\lesssim|\varphi_1|^2+|\varphi_1|^p+|a|\left(|\Phi|+|\Phi|^{p-1}\right)|\varphi_1|.\end{aligned}$$

It follows that

$$|\langle R,\partial_{x_n}Q\rangle|\lesssim\int_{\mathbb{R}^N}\left(|\varphi_1|^2+|\varphi_1|^p+|\varphi_1||V|\right)\mathrm{d}x\lesssim\mathcal{N}^2+|a|\mathcal{N}.$$

For q be a non-degenerate state, we have $a=0$ which implies $G=0$. For q be a degree-1 excited state, from (8.27), we have

$$\left|G-\frac{1}{2}f''(Q)V^2\right| \lesssim |V|^3+|V|^p \lesssim \left(|a|^3+|a|^p\right)\left(|\Phi|^3+|\Phi|^p\right).$$

It follows that

$$\begin{aligned}&\left|\langle G,\partial_{x_n}Q\rangle-\langle\frac{1}{2}f''(Q)V^2,\partial_{x_n}Q\rangle\right|\\ &\lesssim \left(|a|^3+|a|^p\right)\int_{\mathbb{R}^N}\left(|\Phi|^3+|\Phi|^p\right)|\partial_{x_n}Q|\mathrm{d}x \lesssim |a|^3+|a|^p.\end{aligned}$$

Note that, by (8.15), (8.22) and change of variables,

$$\langle\frac{1}{2}f''(Q)V^2,\partial_{x_n}Q\rangle=0.$$

Then, from (8.16), (8.22), (8.24), (8.33), the expression of Mod_2 and change of variables,

$$\begin{aligned}&\langle \mathrm{Mod}_2,\partial_{x_n}Q\rangle\\ &=\langle(\dot\ell+2\alpha\ell)\cdot\nabla Q,\partial_{x_n}Q\rangle-\sum_{(i,j)\in I_q}\left(\dot\beta_{ij}+2\alpha\beta_{ij}\right)\langle\Psi_{ij},\partial_{x_n}Q\rangle-(\dot b+2\alpha b)\langle\Phi,\partial_{x_n}Q\rangle\\ &=\langle(\dot\ell+2\alpha\ell)\cdot\nabla q,\partial_{x_n}q\rangle-\sum_{(i,j)\in I_q}\left(\dot\beta_{ij}+2\alpha\beta_{ij}\right)\langle\Omega_{ij}q,\partial_{x_n}q\rangle\\ &\quad+O\left(\gamma\left(|\dot\ell+2\alpha\ell|+|\dot\beta+2\alpha\beta|\right)\right).\end{aligned}$$

Last, from (8.33), (8.38) and (8.42),

$$|\langle\varphi_2,\partial_t\partial_{x_n}Q\rangle|\lesssim\|\vec\varphi\|_{\mathcal{H}}\left(|\dot z-\ell|+|\ell|+|\dot\theta-\beta|+|\beta|\right)\lesssim\mathcal{N}^2\left(\mathcal{N}+|a|+1\right)\lesssim\mathcal{N}^2.$$

In conclusion of the previous estimates, the orthogonality condition $\langle\varphi_2,\partial_{x_n}Q\rangle=0$ gives the following,

$$\begin{aligned}&\langle(\dot\ell+2\alpha\ell)\cdot\nabla q,\partial_{x_n}q\rangle-\sum_{(i,j)\in I_q}\left(\dot\beta_{ij}+2\alpha\beta_{ij}\right)\langle\Omega_{ij}q,\partial_{x_n}q\rangle\\ &+O\left(\gamma\left(|\dot\ell+2\alpha\ell|+|\dot\beta+2\alpha\beta|\right)\right)=O\left(\mathcal{N}^2+|a|\mathcal{N}+|a|^3+|a|^p\right).\end{aligned}$$

Proceeding similarly for the orthogonality conditions $\langle\phi_2,\Omega_{ij}Q\rangle=0$ in (8.35) and using again the fact that family $\{\partial_{x_n}q,1\leqslant n\leqslant N;\Omega_{ij}q,(i,j)\in I_q\}$ is linearly independent, we find (8.39) for γ small enough.

Proof of (ii). We prove (8.41); the proof of (8.40) is same as (8.38). We differentiate the orthogonality $\langle\varphi_2,\Phi\rangle=0$ in (8.35),

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi_2,\Phi\rangle=\langle\partial_t\varphi_2,\Phi\rangle+\langle\varphi_2,\partial_t\Phi\rangle.$$

Using again (8.36),

$$\begin{aligned}\langle\partial_t\varphi_2,\Phi\rangle&=\langle\Delta\varphi_1-\varphi_1+f'(Q)\varphi_1,\Phi\rangle-2\alpha\langle\varphi_2,\Phi\rangle\\ &\quad+\langle G_2,\Phi\rangle+\langle R,\Phi\rangle+\langle G+\mathrm{Mod}_2,\Phi\rangle,\end{aligned}$$

where

$$R=f(Q+V+\varphi_1)-f(Q+V)-f'(Q)\varphi_1.$$

From $-\Delta\Phi+\Phi-f'(Q)\Phi=0$, integration by parts and (8.35),

$$\langle\Delta\varphi_1-\varphi_1+f'(Q)\varphi_1,\Phi\rangle-2\alpha\langle\varphi_2,\Phi\rangle=0.$$

By the expression of G_2, (8.33) and (8.38),

$$|\langle G_2,\Phi\rangle|\lesssim\left(|\dot{z}-\ell|+|\dot{\theta}-\beta|+|\ell|+|\beta|\right)(|\ell|+|\beta|+|b|)\lesssim\mathcal{N}^2.$$

Recall that, from (8.25), (8.28) and (8.33),

$$|R|\lesssim|\varphi_1|^2+|\varphi_1|^p+|a|\left(|\Phi|+|\Phi|^{p-1}\right)|\varphi_1|.$$

Based on above inequality and the Sobolev embedding Theorem,

$$\begin{aligned}|\langle R,\Phi\rangle|&\lesssim\int_{\mathbb{R}^N}\left(|\varphi_1|^2+|\varphi_1|^p+|a|\left(|\Phi|+|\Phi|^{p-1}\right)|\varphi_1|\right)|\Phi|\mathrm{d}x\\&\lesssim\|\vec{\varphi}\|_{\mathcal{H}}^2+\|\vec{\varphi}\|_{\mathcal{H}}^p+|a|\|\vec{\varphi}\|_{\mathcal{H}}\lesssim\mathcal{N}^2+|a|\mathcal{N}.\end{aligned}$$

Then, from (8.27) and $p>2$,

$$\left|G-\frac{1}{2}f''(Q)V^2\right|\lesssim|V|^3+|V|^p\lesssim\left(|a|^3+|a|^p\right)\left(|\Phi|^3+|\Phi|^p\right).$$

It follows that

$$\left|\langle G-\frac{1}{2}f''(Q)V^2,\Phi\rangle\right|\lesssim\left(|a|^3+|a|^p\right)\int_{\mathbb{R}^N}\left(|\Phi|^4+|\Phi|^{p+1}\right)\mathrm{d}x\lesssim|a|^3+|a|^p.$$

Note that, by our normalization choice for ϕ (8.17), and a change of variables,

$$\langle\frac{1}{2}f''(Q)V^2,\Phi\rangle=\frac{a^2}{2}E'''(q)(\phi,\phi,\phi)=-2\alpha a^2\|\phi\|^2.$$

Thus, from (8.16), (8.22), the expression of Mod_2, the above estimates and change of variables,

$$\begin{aligned}&\langle\mathrm{Mod}_2+G,\Phi\rangle\\&=-(\dot{b}+2\alpha b)\langle\Phi,\Phi\rangle+\langle\frac{1}{2}f''(Q)V^2,\Phi\rangle+\langle G-\frac{1}{2}f''(Q)V^2,\Phi\rangle\\&=-(\dot{b}+2\alpha b+2\alpha a^2)\langle\phi,\phi\rangle+O\left(|a|^3+|a|^p\right).\end{aligned}$$

Last, from (8.23), (8.33), (8.38) and the Sobolev embedding Theorem,

$$|\langle\varphi_2,\partial_t\Phi\rangle|\lesssim\|\vec{\varphi}\|_{\mathcal{H}}\left(|\dot{z}-\ell|+|\dot{\theta}-\beta|+|\ell|+|\beta|\right)\lesssim\mathcal{N}^2\left(\mathcal{N}+|a|+1\right)\lesssim\mathcal{N}^2.$$

Combining the above estimates, we have

$$(\dot{b}+2\alpha b+2\alpha a^2)\langle\phi,\phi\rangle=O\left(\mathcal{N}^2+|a|\mathcal{N}+|a|^3+|a|^p\right),$$

which means (8.41). □

Last, we derive the control of exponential directions.

Lemma 8.14. *Let $a_k^{\pm}=\langle\vec{\varphi},\vec{Z}_k^{\pm}\rangle$ for $k=1,\cdots,K$. Then*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}a_k^{\pm}-\nu_k^{\pm}a_k^{\pm}\right|\lesssim\mathcal{N}^2+a^2.\tag{8.43}$$

Proof. By (8.36),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a_k^{\pm} =&\langle \partial_t\vec{\varphi}, \vec{Z}_k^{\pm}\rangle + \langle \vec{\varphi}, \partial_t \vec{Z}_k^{\pm}\rangle\\ =&\langle \Delta\varphi_1 - \varphi_1 + f'(Q)\varphi_1, \Upsilon_k\rangle + \left(\zeta_k^{\pm} - 2\alpha\right)\langle \varphi_2, \Upsilon_k\rangle + \langle R + G, \Upsilon_k\rangle\\ &+ \langle \zeta_k^{\pm}\,\mathrm{Mod}_1 + \mathrm{Mod}_2, \Upsilon_k\rangle + \langle \zeta_k^{\pm} G_1 + G_2, \Upsilon_k\rangle + \langle \vec{\varphi}, \partial_t \vec{Z}_k^{\pm}\rangle.\end{aligned}$$

From $\lambda_k^2 = \nu_k^{\pm}\zeta_k^{\pm}$, $\mathcal{L}_q Y_k = -\lambda_k^2 Y_k$ and integration by parts,

$$\langle \Delta\varphi_1 - \varphi_1 + f'(Q)\varphi_1, \Upsilon_k\rangle = \langle \varphi_1, \left(\Delta - 1 + f'(Q)\right)\Upsilon_k\rangle = \nu_k^{\pm}\zeta_k^{\pm}\langle \varphi_1, \Upsilon_k\rangle.$$

Combining above identity with $\nu_k^{\pm} = \zeta_k^{\pm} - 2\alpha$ and the definition of $a_k^{\pm}$,

$$\begin{aligned}&\langle \Delta\varphi_1 - \varphi_1 + f'(Q)\varphi_1, \Upsilon_k\rangle + \left(\zeta_k^{\pm} - 2\alpha\right)\langle \varphi_2, \Upsilon_k\rangle\\ &= \nu_k^{\pm}\left(\zeta_k^{\pm}\langle \varphi_1, \Upsilon_k\rangle + \langle \varphi_2, \Upsilon_k\rangle\right) = \nu_k^{\pm} a_k^{\pm}.\end{aligned}$$

Recall that, from (8.25), (8.26) and (8.28),

$$|R| + |G| \lesssim |\varphi_1|^2 + |\varphi_1|^p + |V|^2 + |V|^p.$$

Therefore, by the Sobolev embedding Theorem, and as Υ_k is exponentially localized, we have

$$|\langle R + G, \Upsilon_k\rangle| \lesssim \int_{\mathbb{R}^N} \left(\varphi_1^2 + |\varphi_1|^p + |V|^2 + |V|^p\right)|\Upsilon_k|\,\mathrm{d}x \lesssim \mathcal{N}^2 + a^2.$$

Note that, from $\langle \psi, Y_k\rangle = 0$ for any $\psi \in \mathrm{Ker}\mathcal{L}_q$ and $k = 1, \cdots, K$,

$$\langle \zeta_k^{\pm}\,\mathrm{Mod}_1 + \mathrm{Mod}_2, \Upsilon_k\rangle = 0.$$

Next, by the expression of G_1, G_2, (8.33), (8.38) and (8.39),

$$\begin{aligned}\left|\langle \zeta_k^{\pm} G_1 + G_2, \Upsilon_k\rangle\right| &\lesssim \left(|\dot{z} - \ell| + |\dot{\theta} - \beta| + |\ell| + |\beta|\right)\left(|\ell| + |\beta| + |a| + |b|\right)\\ &\lesssim \mathcal{N}\left(\mathcal{N} + |a|\right)\left(\mathcal{N} + |a| + 1\right) \lesssim \mathcal{N}^2 + a^2.\end{aligned}$$

Last, from (8.23) (8.33), (8.38) and the Sobolev embedding Theorem,

$$\begin{aligned}\left|\langle \vec{\varphi}, \partial_t \vec{Z}_k^{\pm}\rangle\right| &\lesssim \int_{\mathbb{R}^N} \left(|\varphi_1| + |\varphi_2|\right)|\partial_t \Upsilon_k|\mathrm{d}x\\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}\left(|\dot{z} - \ell| + |\dot{\theta} - \beta| + |\ell| + |\beta|\right) \lesssim \mathcal{N}^2\left(\mathcal{N} + |a| + 1\right) \lesssim \mathcal{N}^2.\end{aligned}$$

Gathering these estimates, and proceeding similarly for all $k = 1, \cdots, K$, we obtain (8.43). □

8.2.5 Energy estimates

Let q be a non-degenerate or degree-1 excited state. For $\mu > 0$ small to be chosen later, we denote $\rho = 2\alpha - \mu$, and consider the nonlinear energy functional

$$\begin{aligned}\mathcal{E} = &\int_{\mathbb{R}^N} \left\{|\nabla\varphi_1|^2 + (1 - \rho\mu)\varphi_1^2 + (\varphi_2 + \mu\varphi_1)^2\right\}\mathrm{d}x\\ &- 2\int_{\mathbb{R}^N} \left\{F(Q + V + \varphi_1) - F(Q + V) - f(Q + V)\varphi_1\right\}\mathrm{d}x.\end{aligned} \tag{8.44}$$

Analogous energy functionals were introduced in [29, 30], in order to take full advantage of the damping of (8.1), as is shown in the Lemma below. Recall that in the previous paragraph, in (8.37), we set

$$\mathcal{N} = \|\vec{\varphi}\|_{\mathcal{H}} + |\ell| + |\beta| + |b|.$$

Lemma 8.15. *In the context of Proposition 8.11, there exist* $0 < \gamma_0 \ll 1$ *and* $0 < \mu < \min_{k,\pm}(1, \alpha, \nu_k^{\pm})$ *such that the following hold, for all* $t \in [T_1, T_2]$.

(i) Coercivity and bound.

$$\mu\|\vec{\varphi}\|_{\mathcal{H}}^2 - \mu^{-1}\sum_{k=1}^{K}\left((a_k^+)^2 + (a_k^-)^2\right) \leqslant \mathcal{E} \leqslant \mu^{-1}\|\vec{\varphi}\|_{\mathcal{H}}^2. \tag{8.45}$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + 2\mu\mathcal{E} \leqslant \mu^{-1}\left(\mathcal{N}^3 + a^2\mathcal{N}\right). \tag{8.46}$$

Proof. Proof of (i). First, from (8.29) and the Sobolev embedding Theorem,

$$\begin{aligned}&\int_{\mathbb{R}^N} |F(Q+V+\varphi_1) - F(Q+V) - f(Q+V)\varphi_1|\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}^N}(|Q+V|^{p-1}|\varphi_1|^2 + |\varphi_1|^{p+1})\mathrm{d}x \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 + \|\vec{\varphi}\|_{\mathcal{H}}^{p+1}.\end{aligned}$$

It follows that the right-hand side of (8.45).

Second, from the definition of $\mathcal{E}$ in (8.44), we decompose

$$\mathcal{E} = \langle -\Delta\varphi_1 + \varphi_1 - f'(Q)\varphi_1, \varphi_1\rangle - \rho\mu\langle\varphi_1,\varphi_1\rangle + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,$$

where

$$\begin{aligned}\mathcal{E}_1 &= \int_{\mathbb{R}^N}(\varphi_2 + \mu\varphi_1)^2\mathrm{d}x, \quad \mathcal{E}_2 = -\int_{\mathbb{R}^N}\left(f'(Q+V) - f'(Q)\right)\varphi_1^2\mathrm{d}x,\\ \mathcal{E}_3 &= -2\int_{\mathbb{R}^N}\left\{F(Q+V+\varphi_1) - F(Q+V) - f(Q+V)\varphi_1 - \frac{1}{2}f'(Q+V)\varphi_1^2\right\}\mathrm{d}x.\end{aligned}$$

By (8.14) and (8.34),

$$\langle -\Delta\varphi_1 + \varphi_1 - f'(Q)\varphi_1, \varphi_1\rangle \geqslant c\|\varphi_1\|_{H^1}^2 - c^{-1}\sum_{k=1}^{K}\left((a_k^+)^2 + (a_k^-)^2\right).$$

From the AM-GM inequality, we have

$$\mathcal{E}_1 = \int_{\mathbb{R}^N}(\varphi_2 + \mu\varphi_1)^2\,\mathrm{d}x \geqslant \frac{1}{2}\int_{\mathbb{R}^N}\varphi_2^2\mathrm{d}x - \mu^2\int_{\mathbb{R}^N}\varphi_1^2\mathrm{d}x.$$

Then, using (8.25), (8.30), (8.33) and the Sobolev embedding Theorem, we have

$$\begin{aligned}|\mathcal{E}_2| &\lesssim \int_{\mathbb{R}^N}\left(|V| + |V|^{p-1}\right)|\varphi_1|^2\mathrm{d}x \lesssim \left(\gamma_0 + \gamma_0^{p-1}\right)\|\vec{\varphi}\|_{\mathcal{H}}^2,\\ |\mathcal{E}_3| &\lesssim \int_{\mathbb{R}^N}\left(|\varphi_1|^3 + |\varphi_1|^{p+1}\right)\mathrm{d}x \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^3 + \|\vec{\varphi}\|_{\mathcal{H}}^{p+1}.\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}\mathcal{E} \geqslant &\left(c - \rho\mu - \mu^2\right)\|\varphi_1\|_{H^1}^2 + \frac{1}{2}\|\varphi_2\|_{L^2}^2 - c^{-1}\sum_{k=1}^{K}\left((a_k^+)^2 + (a_k^-)^2\right)\\ &+ O\left(\left(\gamma_0 + \gamma_0^{p-1}\right)\|\vec{\varphi}\|_{\mathcal{H}}^2 + \|\vec{\varphi}\|_{\mathcal{H}}^3\right),\end{aligned}$$

which implies the left-hand side of (8.45) for $\gamma_0 > 0$ small enough and $\mu > 0$ small enough.

Proof of (ii). By direct computation and integration by parts, we decompose,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} =& 2\int_{\mathbb{R}^N} \partial_t\varphi_1\{-\Delta\varphi_1 + (1-\rho\mu)\varphi_1 - \left[f(Q+V+\varphi_1) - f(Q+V)\right]\}\mathrm{d}x \\ &- 2\int_{\mathbb{R}^N} \left\{\left(f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right)(\partial_t Q + \partial_t V)\right\}\mathrm{d}x \\ &+ 2\int_{\mathbb{R}^N} \{(\varphi_2 + \mu\varphi_1)(\partial_t\varphi_2 + \mu\partial_t\varphi_1)\}\,\mathrm{d}x = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.\end{aligned}$$

Estimate on $\mathcal{I}_1$. We claim

$$\begin{aligned}\mathcal{I}_1 =& -2\int_{\mathbb{R}^N} \{\Delta\varphi_1 - \varphi_1 + [f(Q+V+\varphi_1) - f(Q+V)]\}\,\varphi_2\mathrm{d}x \\ &- 2\rho\mu\int_{\mathbb{R}^N} \varphi_1\varphi_2\mathrm{d}x + O\left(\mathcal{N}^3 + |a|\mathcal{N}^2\right).\end{aligned} \tag{8.47}$$

By (8.34), (8.36), $-\Delta\,\mathrm{Mod}_1 + \mathrm{Mod}_1 - f'(Q)\,\mathrm{Mod}_1 = 0$ and integration by parts,

$$\begin{aligned}\mathcal{I}_1 =& 2\int_{\mathbb{R}^N} \{-\Delta\varphi_1 + (1-\rho\mu)\varphi_1 - [f(Q+V+\varphi_1) - f(Q+V)]\}\,\varphi_2\mathrm{d}x \\ &+ \mathcal{I}_{1,1} + \mathcal{I}_{1,2} + \mathcal{I}_{1,3} + \mathcal{I}_{1,4},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{1,1} &= -2\int_{\mathbb{R}^N} [\Delta\varphi_1 - (1-\rho\mu)\varphi_1]\,G_1\mathrm{d}x, \\ \mathcal{I}_{1,2} &= -2\int_{\mathbb{R}^N} [f(Q+V+\varphi_1) - f(Q+V)]\,G_1\mathrm{d}x, \\ \mathcal{I}_{1,3} &= -2\int_{\mathbb{R}^N} \left[\left(f'(Q+V) - f'(Q)\right)\varphi_1\right]\mathrm{Mod}_1\,\mathrm{d}x, \\ \mathcal{I}_{1,4} &= -2\int_{\mathbb{R}^N} \left[f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right]\mathrm{Mod}_1\,\mathrm{d}x.\end{aligned}$$

Using (8.28), (8.33), (8.38) and integration by parts, we have

$$\begin{aligned}|\mathcal{I}_{1,1}| &\lesssim \int_{\mathbb{R}^N} (|\Delta G_1| + |G_1|)\,|\varphi_1|\mathrm{d}x \\ &\lesssim |a|\mathcal{N}\left(|\dot{z} - \ell| + |\dot{\theta} - \beta| + \mathcal{N}\right) \lesssim |a|\mathcal{N}^2\left(\mathcal{N} + |a| + 1\right) \lesssim |a|\mathcal{N}^2, \\ |\mathcal{I}_{1,2}| &\lesssim \int_{\mathbb{R}^N} |G_1|(|\varphi_1| + |\varphi_1|^p)\mathrm{d}x \\ &\lesssim |a|\mathcal{N}\left(|\dot{z} - \ell| + |\dot{\theta} - \beta| + \mathcal{N}\right) \lesssim |a|\mathcal{N}^2\left(\mathcal{N} + |a| + 1\right) \lesssim |a|\mathcal{N}^2.\end{aligned}$$

Then, using (8.25), (8.28), (8.33) and (8.38), we have

$$\begin{aligned}|\mathcal{I}_{1,3}| &\lesssim \int_{\mathbb{R}^N} \left(|V| + |V|^{p-1}\right)|\varphi_1||\,\mathrm{Mod}_1\,|\mathrm{d}x \\ &\lesssim \|\vec{\varphi}\|_{\mathcal{H}}\left(|a| + |a|^{p-1}\right)\left(|\dot{z} - \ell| + |\dot{\theta} - \beta| + |\dot{a} - b|\right) \\ &\lesssim \mathcal{N}^2\left(|a| + |a|^{p-1}\right)\left(\mathcal{N} + |a|\right) \lesssim \mathcal{N}^3 + |a|\mathcal{N}^2,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{I}_{1,4}| &\lesssim \int_{\mathbb{R}^N} \left(\varphi_1^2 + |\varphi_1|^p\right)|\,\mathrm{Mod}_1\,|\mathrm{d}x \\ &\lesssim \left(\mathcal{N}^2 + \mathcal{N}^p\right)\left(|\dot{z} - \ell| + |\dot{\theta} - \beta| + |\dot{a} - b|\right) \lesssim \mathcal{N}\left(\mathcal{N}^2 + \mathcal{N}^p\right)\left(\mathcal{N} + |a|\right) \lesssim \mathcal{N}^3.\end{aligned}$$

Gathering the above estimates, we obtain (8.47).

Estimate on $\mathcal{I}_2$. We claim

$$|\mathcal{I}_2| \lesssim \mathcal{N}^3. \tag{8.48}$$

By (8.23),

$$\partial_t Q + \partial_t V = -\ell \cdot \nabla Q + \sum_{(i,j)\in I_q} \beta_{ij}\Psi_{ij} + b\Phi - \mathrm{Mod}_1 - G_1.$$

Based on above identity, we decompose

$$\mathcal{I}_2 = \mathcal{I}_{2,1} + \mathcal{I}_{2,2} + \mathcal{I}_{2,3} + \mathcal{I}_{2,4} + \mathcal{I}_{2,5},$$

where

$$\mathcal{I}_{2,1} = 2\int_{\mathbb{R}^N} \left(f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right)(-b\Phi)\,\mathrm{d}x,$$
$$\mathcal{I}_{2,2} = 2\int_{\mathbb{R}^N} \left(f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right)G_1\mathrm{d}x,$$

and

$$\mathcal{I}_{2,3} = 2\int_{\mathbb{R}^N} \left(f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right)\mathrm{Mod}_1\,\mathrm{d}x,$$
$$\mathcal{I}_{2,4} = 2\int_{\mathbb{R}^N} \left(f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right)\left(\ell\cdot\nabla Q\right)\mathrm{d}x,$$
$$\mathcal{I}_{2,5} = -2\sum_{(i,j)\in I_q}\beta_{ij}\int_{\mathbb{R}^N} \left(f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right)\Psi_{ij}\mathrm{d}x.$$

From (8.28) and the Sobolev embedding Theorem,

$$|\mathcal{I}_{2,1}| \lesssim |b|\int_{\mathbb{R}^N}\varphi_1^2|\Phi|\mathrm{d}x \lesssim \mathcal{N}^3.$$

Using (8.28), (8.33), (8.38), (8.40) and the Sobolev embedding Theorem, we have

$$\begin{aligned}
|\mathcal{I}_{2,2}| &\lesssim \int_{\mathbb{R}^N}\left(\varphi_1^2 + |\varphi_1|^p\right)|G_1|\mathrm{d}x\\
&\lesssim \left(\mathcal{N}^2+\mathcal{N}^p\right)\Big(|a|\mathcal{N} + |a||\dot z - \ell| + |a||\dot\theta - \beta|\Big)\\
&\lesssim \left(\mathcal{N}^2+\mathcal{N}^p\right)\left(|a|\mathcal{N} + |a|\mathcal{N}(|a|+\mathcal{N})\right) \lesssim \mathcal{N}^3,
\end{aligned}$$

$$\begin{aligned}
|\mathcal{I}_{2,3}| &\lesssim \int_{\mathbb{R}^N}\left(\varphi_1^2 + |\varphi_1|^p\right)|\,\mathrm{Mod}_1\,|\mathrm{d}x\\
&\lesssim \left(\mathcal{N}^2+\mathcal{N}^p\right)\Big(\dot z - \ell| + |\dot\theta - \beta| + |\dot a - b|\Big)\\
&\lesssim \left(\mathcal{N}^2+\mathcal{N}^p\right)(|a|+\mathcal{N})\,\mathcal{N} \lesssim \mathcal{N}^3,
\end{aligned}$$

$$|\mathcal{I}_{2,4}| \lesssim \int_{\mathbb{R}^N}\left(\varphi_1^2 + |\varphi_1|^p\right)|\ell\cdot\nabla Q|\mathrm{d}x \lesssim \mathcal{N}^3,$$

and

$$|\mathcal{I}_{2,5}| \lesssim \sum_{(i,j)\in I_q}\int_{\mathbb{R}^N}\left(\varphi_1^2 + |\varphi_1|^p\right)|\beta_{ij}\Psi_{ij}|\,\mathrm{d}x \lesssim \mathcal{N}^3.$$

Combining above estimates with (8.33), we obtain (8.48).

Estimate on $\mathcal{I}_3$. We claim

$$\begin{aligned}
\mathcal{I}_3 =& -2\mu\mathcal{E} + 2\int_{\mathbb{R}^N}\left\{\Delta\varphi_1 - \varphi_1 + [f(Q+V+\varphi_1) - f(Q+V)]\right\}\varphi_2\mathrm{d}x\\
&+ 2\rho\mu\int_{\mathbb{R}^N}\varphi_1\varphi_2\mathrm{d}x - 2(\rho-\mu)\int_{\mathbb{R}^N}(\varphi_2+\mu\varphi_1)^2\mathrm{d}x + O\left(\mathcal{N}^3 + a^2\mathcal{N}\right).
\end{aligned} \tag{8.49}$$

By direct computation and (8.36), we decompose,

$$\begin{aligned}\mathcal{I}_3 = &-2\mu\mathcal{E} + 2\int_{\mathbb{R}^N}\{\Delta\varphi_1 - \varphi_1 + [f(Q+V+\varphi_1) - f(Q+V)]\}\,\varphi_2 \mathrm{d}x \\ &+ 2\rho\mu\int_{\mathbb{R}^N}\varphi_1\varphi_2\mathrm{d}x - 2(\rho-\mu)\int_{\mathbb{R}^N}(\varphi_2+\mu\varphi_1)^2\mathrm{d}x \\ &+ \mathcal{I}_{3,1} + \mathcal{I}_{3,2} + \mathcal{I}_{3,3} + \mathcal{I}_{3,4} + \mathcal{I}_{3,5},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{3,1} &= 2\int_{\mathbb{R}^N}(\varphi_2+\mu\varphi_1)\,G\mathrm{d}x, \\ \mathcal{I}_{3,2} &= 2\int_{\mathbb{R}^N}(\varphi_2+\mu\varphi_1)\,(G_2+\mu G_1)\,\mathrm{d}x, \\ \mathcal{I}_{3,3} &= 2\int_{\mathbb{R}^N}(\varphi_2+\mu\varphi_1)\,(\mathrm{Mod}_2+\mu\,\mathrm{Mod}_1)\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{3,4} &= 2\mu\int_{\mathbb{R}^N}\left[f(Q+V+\varphi_1) - f(Q+V) - f'(Q+V)\varphi_1\right]\varphi_1\mathrm{d}x, \\ \mathcal{I}_{3,5} &= -4\mu\int_{\mathbb{R}^N}\left[F(Q+V+\varphi_1) - F(Q+V) - f(Q+V)\varphi_1 - \frac{1}{2}f'(Q+V)\varphi_1^2\right]\mathrm{d}x.\end{aligned}$$

Using (8.26),

$$|\mathcal{I}_{3,1}| \lesssim \int_{\mathbb{R}^N}(|\varphi_2|+|\varphi_1|)\,V^2\mathrm{d}x \lesssim a^2\mathcal{N}.$$

Next, from (8.33), (8.38) and the AM-GM inequality,

$$\begin{aligned}|\mathcal{I}_{3,2}| &\lesssim \int_{\mathbb{R}^N}(|\varphi_1|+|\varphi_2|)\,(|G_1|+|G_2|)\,\mathrm{d}x \\ &\lesssim \mathcal{N}\,(|a|+|b|+|\ell|+|\beta|)\left(|\dot{z}-\ell|+|\dot{\theta}-\beta|+|\ell|+|\beta|\right) \\ &\lesssim \mathcal{N}^2\,(|a|+\mathcal{N})\,(\mathcal{N}+|a|+1) \lesssim a^2\mathcal{N}+\mathcal{N}^3.\end{aligned}$$

Note that, from (8.34) and (8.35), we have

$$\mathcal{I}_{3,3} = 2\langle\varphi_2, \mathrm{Mod}_2+\mu\,\mathrm{Mod}_1\rangle + 2\mu\langle\varphi_1, \mathrm{Mod}_2+\mu\,\mathrm{Mod}_1\rangle = 0.$$

Last, from (8.28), (8.30) and the Sobolev embedding Theorem,

$$\begin{aligned}|\mathcal{I}_{3,4}| &\lesssim \int_{\mathbb{R}^N}\left(|\varphi_1|^2+|\varphi_1|^p\right)|\varphi_1|\mathrm{d}x \lesssim \mathcal{N}^3+\mathcal{N}^{p+1}, \\ |\mathcal{I}_{3,5}| &\lesssim \int_{\mathbb{R}^N}\left(|\varphi_1|^3+|\varphi_1|^{p+1}\right)\mathrm{d}x \lesssim \mathcal{N}^3+\mathcal{N}^{p+1}.\end{aligned}$$

Gathering the above estimates, we obtain (8.49).

Last, combining (8.47), (8.48) and (8.49), we conclude (8.46). □

We observe from the definition and the energy property (8.2) that a packed solution $\vec{u}$ with degree-1 excited state cluster point q satisfies

$$\lim_{t\to\infty} E(\vec{u}(t)) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla q|^2 + q^2 - 2F(q)\right)\mathrm{d}x = E(q,0). \tag{8.50}$$

More precisely, we expand the energy for a solution close to degree-1 excited state.

Lemma 8.16. *In the context of Proposition 8.11, for the case q be a degree-1 excited state, we have*

$$E(\vec{u}) = E(q,0) + \frac{2}{3}\alpha\|\phi\|_{L^2}^2 a^3 + O\left(\mathcal{N}^2 + |a|^{\bar{p}+1}\right). \tag{8.51}$$

Proof. By (8.32) and an elementary computation, we find

$$\begin{aligned}
|u|^2 &= |Q|^2 + |V|^2 + |\varphi_1|^2 + 2Q(V+\varphi_1) + 2V\varphi_1,\\
|\nabla u|^2 &= |\nabla Q|^2 + |\nabla V|^2 + |\nabla\varphi_1|^2 + 2\nabla Q\cdot\nabla(V+\varphi_1) + 2\nabla V\cdot\nabla\varphi_1,
\end{aligned}$$

and

$$\begin{aligned}
F(u) =& F(Q) + f(Q)(V+\varphi_1) + \frac{1}{2}f'(Q)(V^2+\varphi_1^2+2V\varphi_1) + \frac{1}{6}f''(Q)V^3\\
&+ F(Q+V+\varphi_1) - F(Q) - f(Q)(V+\varphi_1) - \frac{1}{2}f'(Q)(V+\varphi_1)^2\\
&- \frac{1}{6}f''(Q)(V+\varphi_1)^3 + \frac{1}{6}f''(Q)\left((V+\varphi_1)^3 - V^3\right).
\end{aligned}$$

Based on the above identities and change of variables, we have

$$E(\vec{u}) = E(q,0) - \frac{1}{6}\int_{\mathbb{R}^N} f''(Q)V^3\mathrm{d}x + E_1 + E_2 + E_3 + E_4 + E_5 + E_6,$$

where

$$\begin{aligned}
E_1 &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla V|^2 + |V|^2 - f'(Q)V^2\right)\mathrm{d}x,\\
E_2 &= \int_{\mathbb{R}^N}\left(\nabla V\cdot\nabla\varphi_1 + V\varphi_1 - f'(Q)V\varphi_1\right)\mathrm{d}x,\\
E_3 &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla\varphi_1|^2 + |\varphi_1|^2 - f'(Q)\varphi_1^2 + (\partial_t u)^2\right)\mathrm{d}x,\\
E_4 &= \int_{\mathbb{R}^N}\left(\nabla Q\cdot\nabla(V+\varphi_1) + Q(V+\varphi_1) - f(Q)(V+\varphi_1)\right)\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
E_5 &= -\frac{1}{3}\int_{\mathbb{R}^N} f''(Q)\left((V+\varphi_1)^3 - V^3\right)\mathrm{d}x,\\
E_6 &= -2\int_{\mathbb{R}^N}\Big(F(Q+V+\varphi_1) - F(Q) - f(Q)(V+\varphi_1)\\
&\qquad - \frac{1}{2}f'(Q)(V+\varphi_1)^2 - \frac{1}{6}f''(Q)(V+\varphi_1)^3\Big)\mathrm{d}x.
\end{aligned}$$

First, from (8.17) we have

$$E'''(\phi,\phi,\phi) = -4\alpha\|\phi\|_{L^2}^2.$$

Therefore, by change of variables,

$$-\frac{1}{6}\int_{\mathbb{R}^N} f''(Q)V^3\mathrm{d}x = -\frac{a^3}{6}E'''(\phi,\phi,\phi) = \frac{2}{3}\alpha\|\phi\|_{L^2}^2 a^3.$$

Then, from $-\Delta\Phi + \Phi - f'(Q)\Phi = 0$, $V = a\Phi$ and integration by parts,

$$\begin{aligned}
E_1 &= \frac{a^2}{2}\int_{\mathbb{R}^N}\left(-\Delta\Phi + \Phi - f'(Q)\Phi\right)\Phi\mathrm{d}x = 0,\\
E_2 &= a\int_{\mathbb{R}^N}\left(-\Delta\Phi + \Phi - f'(Q)\Phi\right)\varphi_1\mathrm{d}x = 0.
\end{aligned}$$

By the Sobolev embedding theorem and the AM-GM inequality,

$$|E_3| \lesssim \|\vec{\varphi}\|_{\mathcal{H}}^2 + |\ell|^2 + |\beta|^2 + b^2 \lesssim \mathcal{N}^2.$$

Next, from integration by parts and $-\Delta Q + Q - f(Q) = 0$, we have

$$E_4 = \int_{\mathbb{R}^N} (-\Delta Q + Q - f(Q))\,(V + \varphi_1)\mathrm{d}x = 0.$$

Last, from the Sobolev embedding Theorem and the AM-GM inequality, we have

$$|E_5| \lesssim \int_{\mathbb{R}^N} |f''(Q)| \left(|\varphi_1|V^2 + |V|\varphi_1^2 + |\varphi_1|^3\right) \mathrm{d}x \lesssim a^4 + \mathcal{N}^2,$$
$$|E_6| \lesssim \int_{\mathbb{R}^N} \left(|V + \varphi_1|^4 + |V + \varphi_1|^{p+1}\right) \mathrm{d}x \lesssim a^4 + |a|^{p+1} + \mathcal{N}^4 + \mathcal{N}^{p+1}.$$

Gathering above estimates, we find (8.51). □

8.2.6 Time evolution analysis

We introduce new parameters and functionals to analyze the time evolution of solutions in the framework of Proposition 8.11. It is convenient to consider together all the damped components, as follows:

$$\mathcal{S} = |\ell|^2 + |\beta|^2 + \sum_{k=1}^{K} (a_k^-)^2 + b^2 \quad \text{and} \quad \mathcal{F} = \mathcal{E} + \mu^{-1}\mathcal{S}. \tag{8.52}$$

We also define the unstable component:

$$\mathcal{A} = \sum_{k=1}^{K} (a_k^+)^2. \tag{8.53}$$

Finally, for the analysis of the main ODE system in the degree-1 degenerate case, we will crucially relie on the following quantities

$$\mathcal{R}_1 = a + \frac{b}{2\alpha} \quad \text{and} \quad \mathcal{R}_2 = \frac{2\alpha}{3}a^3 + \frac{1}{2\alpha}ab^2 + a^2 b. \tag{8.54}$$

Indeed, $\mathcal{R}_1$ and $\mathcal{R}_2$ enjoy monotonicity related properties as it is made explicit in the following lemma, in which we also rewrite the estimates of Lemma 8.13, 8.14 and 8.15, using our new notations.

Lemma 8.17. *In the context of Proposition 8.11, there exist $C_0 > 1$ such that the following hold.*

(i) Coercivity and bound. *We have*

$$C_0^{-1}\mathcal{N}^2 \leqslant \mathcal{F} + \mu^{-1}\mathcal{A} \leqslant C_0\mathcal{N}^2. \tag{8.55}$$

(ii) Liapunov evolution of $\mathcal{R}_1$. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_1 + \frac{1}{2}a^2 \leqslant C_0\mathcal{N}^2, \tag{8.56}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_2 + \alpha a^4 \leqslant C_0\mathcal{N}^3, \tag{8.57}$$
$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_1 + a^2\right| \leqslant C_0\left(\mathcal{N}^2 + |a|\mathcal{N} + |a|^{\bar{p}}\right). \tag{8.58}$$

(iii) Damped evolution. *There exists* $\nu_1 > 2\mu$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} + \nu_1 \mathcal{S} \leqslant C_0 \left(\mathcal{N}^3 + a^2 \mathcal{N}\right), \tag{8.59}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu \mathcal{F} \leqslant C_0 \left(\mathcal{N}^3 + a^2 \mathcal{N}\right). \tag{8.60}$$

(iv) Exponential growth. *There exists* $\nu_2 \geqslant \nu_3 > 2\mu$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} - \nu_2 \mathcal{A} \leqslant C_0 \left(\mathcal{N}^3 + a^2 \mathcal{N}\right), \tag{8.61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} - \nu_3 \mathcal{A} \geqslant -C_0 \left(\mathcal{N}^3 + a^2 \mathcal{N}\right). \tag{8.62}$$

Proof. Proof of (i). Due to the bound from below in (8.45),

$$\mathcal{F} + \mu^{-1}\mathcal{A} \geqslant \mu \|\vec{\varphi}\|_{\mathcal{H}}^2 + \mu^{-1}(|\ell|^2 + |\beta|^2 + b^2) \gtrsim \mathcal{N}^2.$$

Now $|a_k^{\pm}| = |\langle \vec{\varphi}, \vec{Z}_k^{\pm} \rangle| \leqslant \|\vec{\varphi}\|_{\mathcal{H}} \leqslant \mathcal{N}$, so that using the bound from above in (8.45),

$$\mathcal{F} + \mu^{-1}\mathcal{A} \lesssim \mathcal{E} + |\ell|^2 + |\beta|^2 + b^2 + \sum_{k=1}^{K} (a_k^-)^2 + (a_k^+)^2 \lesssim \mathcal{N}^2.$$

Proof of (ii). First, from (8.40) and (8.41), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_1 = \dot{a} + \frac{\dot{b}}{2\alpha} = -a^2 + O(\mathcal{N}^2 + |a|\mathcal{N} + |a|^3 + |a|^p).$$

Based on (8.33), the above estimate and the AM-GM inequality, we obtain (8.56) and (8.58) for C_0 large enough. Second, using again (8.40), (8.41), we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}a^3 = 3a^2 b + O\left(a^2\mathcal{N}^2 + |a|^3\mathcal{N}\right), \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(ab^2\right) = -4\alpha a b^2 + O\left(\mathcal{N}^3 + |a|^3\mathcal{N}\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(a^2 b) = -2\alpha a^4 + 2ab^2 - 2\alpha a^2 b + O\left(\mathcal{N}^4 + |a|^3\mathcal{N} + |a|^5 + |a|^{p+2}\right).$$

Combining the above estimates, we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_2 &= \frac{2\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}t}a^3 + \frac{1}{2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(ab^2) + \frac{\mathrm{d}}{\mathrm{d}t}(a^2 b) \\ &= -2\alpha a^4 + O\left(\mathcal{N}^3 + |a|^3\mathcal{N} + |a|^5 + |a|^{p+2}\right),\end{aligned}$$

Using the AM-GM inequality (to see $|a|^3\mathcal{N} \leqslant \mathcal{N}^{5/2} + |a|^5$, $a^2\mathcal{N}^2 \leqslant \mathcal{N}^3 + |a|^5$), and as $p + 2 > 4$, this implies (8.57) for C_0 large enough.

Proof of (iii) and (iv). The estimates (8.59), (8.60), (8.61) and (8.62) are consequences of (8.39), (8.41), (8.43), (8.46) and taking $0 < \mu < \min_{k,\pm}(1, \alpha, \nu_k^{\pm})$ in Lemma 8.15. □

8.3 Proof of Theorem 8.4 and Theorem 8.6

In this section, we prove Theorem 8.4 and Theorem 8.6. First, we prove the following trapping Proposition for packed solution.

Proposition 8.18. *There exists $\delta_0 > 0$ such that the following holds. Let $\vec{u} = (u, \partial_t u)$ be a packed solution of (8.1), with cluster point $(q, 0)$ where q is a non-degenerate state or degree-1 excited state. Due to Proposition 8.2, for any $0 < \delta < \delta_0$, there exists $T_\delta > 0$ and $\xi_\delta \in \mathbb{R}^N$ such that*

$$\|u(T_\delta) - q(\cdot - \xi_\delta)\|_{H^1} + \|\partial_t u(t)\|_{L^2} + \left[\int_{T_\delta}^{\infty} \|\partial_t u(t)\|_{L^2}^2 \mathrm{d}t\right]^{\frac{1}{2}} < \delta. \tag{8.63}$$

Then $\vec{u}$ admits a decomposition as in Proposition 8.11 for all $t \in [T_\delta, \infty)$ and satisfies

$$\forall t \geqslant T_\delta, \quad \|u(t) - q(\cdot - \xi_\delta)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim \delta^{\frac{1}{2}}. \tag{8.64}$$

Proof. For $\delta < \gamma$ small enough, the existence of T_δ is a direct consequence of Proposition 8.2: $\vec{u}(T_\delta)$ admits a decomposition as in Proposition 8.11 and satisfies

$$|z(T_\delta) - \xi_\delta| + |\theta(T_\delta)| + |a(T_\delta)| + \mathcal{N}(T_\delta) \lesssim \delta. \tag{8.65}$$

For a constant $C > 1$ to be chosen later, we introduce the following bootstrap estimate

$$|z - \xi_\delta| + |\theta| \leqslant \delta^{\frac{1}{2}}, \quad |a| \leqslant C\delta^{\frac{2}{3}}, \quad \mathcal{N} \leqslant C\delta, \quad \mathcal{A} \leqslant 2C\delta^2. \tag{8.66}$$

Set

$$T^* = \sup\left\{t \in [T_\delta, \infty) \text{ such that (8.31) and (8.66) holds on } [T_\delta, t]\right\}.$$

We prove thqat $T^* = \infty$ by strictly improving the bootstrap assumption (8.66) on $[T_\delta, T^*)$ (upon chosing C large enough). In the remainder of the proof, the implied constants in $\lesssim$ or O do not depend on δ nor on the constant C appearing in the bootstrap assumption (8.66). Recall that, in the case when q is a non-degenerate state, we denote $a = b = 0$.

Step 1. Preliminary bounds. Let $t \in [T_\delta, T_*)$. Integrating (8.62) on $[T_\delta, t]$, and using (8.65), we have

$$\begin{aligned}\int_{T_\delta}^{t} \mathcal{A}(s)\mathrm{d}s &\lesssim \mathcal{A}(t) + \int_{T_\delta}^{t} \mathcal{N}^3(s)\mathrm{d}s + \int_{T_\delta}^{t} |a|^3(s)\mathrm{d}s \\ &\lesssim C\delta^2 + C\delta \int_{T_\delta}^{t} \mathcal{N}^2(s)\mathrm{d}s + C\delta^{\frac{2}{3}} \int_{T_\delta}^{t} a^2(s)\mathrm{d}s,\end{aligned}$$

Similarly, integrating(8.60), we get

$$\begin{aligned}\int_{T_\delta}^{t} \mathcal{F}(s)\mathrm{d}s &\lesssim |\mathcal{F}(t)| + |\mathcal{F}(T_\delta)| + \int_{T_\delta}^{t} \mathcal{N}^3(s)\mathrm{d}s + \int_{T_\delta}^{t} |a|^3(s)\mathrm{d}s \\ &\lesssim C^2\delta^2 + C\delta \int_{T_\delta}^{t} \mathcal{N}^2(s)\mathrm{d}s + C\delta^{\frac{2}{3}} \int_{T_\delta}^{t} a^2(s)\mathrm{d}s,\end{aligned}$$

and integrating (8.56), there hold

$$\frac{1}{2}\int_{T_\delta}^{t} a^2(s)\mathrm{d}s - C_0 \int_{T_\delta}^{t} \mathcal{N}^2(s)\mathrm{d}s \leqslant R_1(T_\delta) - R_1(t) \lesssim C\delta^{\frac{2}{3}} + C\delta.$$

Now recall that due to the coercivity (8.55), $\mathcal{F} + \mu^{-1}\mathcal{A} \geqslant C_0^{-1}\mathcal{N}^2$, hence

$$\begin{aligned}&\int_{T_\delta}^{t} \left(\mathcal{N}^2(s) + a^2(s)\right) \mathrm{d}s \\ &\quad\leqslant \int_{T_\delta}^{t} a^2(s)\mathrm{d}s - C_0 \int_{T_\delta}^{t} \mathcal{N}^2(s)\mathrm{d}s + C_0(C_0+1) \int_{T_\delta}^{t} (\mathcal{F}(s) + \mu^{-1}\mathcal{A}(s))\mathrm{d}s \\ &\quad\lesssim C\delta^{\frac{2}{3}} + C\delta + C^2\delta^2 + C\delta \int_{T_\delta}^{t} \mathcal{N}^2(s)\mathrm{d}s + C\delta^{\frac{2}{3}} \int_{T_\delta}^{t} a^2(s)\mathrm{d}s.\end{aligned}$$

Taking $0 < \delta \ll 1$ small enough, we infer

$$\int_{T_\delta}^{t} \left(\mathcal{N}^2(s) + a^2(s)\right) \mathrm{d}s \lesssim C\delta^{\frac{2}{3}} + C\delta + C^2\delta^2 \lesssim C\delta^{\frac{2}{3}} + C^2\delta^2.$$

Step 2. Estimate on z and θ. By (8.38) and (8.39), we have

$$\begin{aligned}
|z(t) - \xi_\delta| + |\theta(t)| &\lesssim \left|z(t) - \xi_\delta + \frac{\ell(t)}{2\alpha}\right| + \left|\theta(t) + \frac{\ell(t)}{2\alpha}\right| + |\ell(t)| + |\beta(t)| \\
&\lesssim \mathcal{N}(t) + \mathcal{N}(T_\delta) + |z(T_\delta) - \xi_\delta| + |\theta(T_\delta)| + \int_{T_\delta}^{t} \left(\mathcal{N}^2(s) + a^2(s)\right) \mathrm{d}s \\
&\lesssim C\delta + C\delta^{\frac{2}{3}} + C^2\delta^2,
\end{aligned}$$

which strictly improves the estimate (8.66) of z and θ on $[T_\delta, T^*)$ for δ small enough.

Step 3. Estimate on a. Due to the energy dissipation (8.5) and the initial assumption (8.63), note that we have

$$E(q, 0) \leqslant E(\vec{u}(t)) \leqslant E(q, 0) + O(\delta^2).$$

Using the expansion of the energy (8.51) (and the bootstrap bounds (8.66)), we deduce that

$$\begin{aligned}
|a|^3 &\lesssim \delta^2 + \mathcal{N}^2 + a^4 + |a|^{p+1} \\
&\lesssim C^2\delta^2 + C^{\bar{p}+1}\delta^{\frac{2}{3}(\bar{p}+1)}.
\end{aligned}$$

It follows that

$$|a| \lesssim \delta^{\frac{2}{3}} + C^{\frac{2}{3}}\delta^{\frac{2}{3}} + C^{\frac{\bar{p}+1}{3}}\delta^{\frac{2}{9}(\bar{p}+1)}.$$

which strictly improves the estimate (8.66) of a on $[T_\delta, T^*)$ for C large enough and δ small enough.

Step 4. Estimate on $\mathcal{N}$. Under the bootstrap assumption (8.66), estimate (8.60) writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \lesssim C^3\delta^{\frac{7}{3}} + C^3\delta^3 \lesssim C^3\delta^{\frac{7}{3}}.$$

Let $t \in [T_\delta, T^*)$. Integrating the above estimate on $[T_\delta, t]$, using also the initial time assumption (8.65), yields

$$\mathcal{F}(t) \lesssim e^{-2\mu(t-T_\delta)}\mathcal{F}(T_\delta) + C^3 \int_{T_\delta}^{t} e^{-2\mu(t-s)}\delta^{\frac{7}{3}}\mathrm{d}s \lesssim \delta^2 + C^3\delta^{\frac{7}{3}}. \tag{8.67}$$

We now use once more the coercivity (8.55), the (8.66) on $\mathcal{A}$ so that the estimate (8.67) above gives, for C large enough

$$\mathcal{N}^2(t) \lesssim \mathcal{F} + \mathcal{A} \lesssim \delta^2 + C^3\delta^{\frac{7}{3}} + C\delta^2 \leqslant 2C\delta^2.$$

This strictly improves the estimate (8.66) of $\mathcal{N}$ on $[T_\delta, T^*)$.

Step 5. Estimate on $\mathcal{A}$. Let us first rewrite the estimates (8.59), (8.61), (8.62) in the context of the bootstrap assumption (8.66): for all $t \in [T_\delta, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} + \nu_1\mathcal{S} \leqslant C_0\left(C^3\delta^{\frac{7}{3}} + C^3\delta^3\right) \leqslant C_0C^3\delta^{\frac{7}{3}}, \tag{8.68}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} - \nu_2\mathcal{A} \leqslant C_0\left(C^3\delta^{\frac{7}{3}} + C^3\delta^3\right) \leqslant C_0C^3\delta^{\frac{7}{3}}, \tag{8.69}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} - \nu_3\mathcal{A} \geqslant -C_0\left(C^3\delta^{\frac{7}{3}} + C^3\delta^3\right) \geqslant -C_0C^3\delta^{\frac{7}{3}}. \tag{8.70}$$

Then for $t \in [T_\delta, T^*)$, we have (integrating (8.68) on $[T_\delta, t]$)

$$\mathcal{S}(t) \lesssim e^{-\nu_1(t-T_\delta)}S(T_\delta) + C^3 \int_{T_\delta}^{t} e^{-2\mu(t-s)}\delta^{\frac{7}{3}}\mathrm{d}s \lesssim \delta^2 + C^3\delta^{\frac{7}{3}}. \tag{8.71}$$

We now prove by contradiction that for C large enough, it holds

$$\forall t \in [T_\delta, T_*), \quad \mathcal{A}(t) < 2C\delta^2. \tag{8.72}$$

For the sake of contradiction, assume that there exists $t_2 \in [T_\delta, T^*)$ such that

$$\mathcal{A}(t_2) = 2C\delta^2 \quad \text{and for all } t \in [T_\delta, t_2), \quad \mathcal{A}(t) < 2C\delta^2.$$

On the one hand, by continuity of $\mathcal{A}$, there exists $t_1 \in [T_\delta, t_2]$ such that

$$\mathcal{A}(t_1) = C\delta^2 \quad \text{and for all } t \in (t_1, t_2), \quad C\delta^2 < \mathcal{A}(t) < 2C\delta^2. \tag{8.73}$$

For $t \in [t_1, t_2]$, divide (8.69) and (8.70) by $\mathcal{A}$ (using (8.73)), and then integrate on $[t_1, t_2]$: it follows that

$$\frac{\log 2}{\nu_2} + O(C^2\delta^{\frac{1}{3}}) \leqslant t_2 - t_1 \leqslant \frac{\log 2}{\nu_3} + O(C^2\delta^{\frac{1}{3}}). \tag{8.74}$$

Therefore, using (8.73) again

$$\int_{t_1}^{t_2} \mathcal{A}(t)\mathrm{d}t \geqslant C\delta^2(t_2 - t_1) \geqslant C\delta^2 \left(\frac{\log 2}{\nu_2} + O(C^2\delta^{\frac{1}{3}}) \right). \tag{8.75}$$

On the other hand, by the definition of φ_2 in (8.32) and the bound on $\mathcal{S}$ (8.71), for any $t \in [t_1, t_2]$

$$\|\varphi_2(t)\|_{L^2}^2 \lesssim \|\partial_t u(t)\|_{L^2}^2 + \mathcal{S}(t) \lesssim \|\partial_t u(t)\|_{L^2}^2 + \delta^2 + C^3\delta^{\frac{7}{3}}.$$

Thus, from (8.63) and (8.74),

$$\int_{t_1}^{t_2} \|\varphi_2(t)\|_{L^2}^2 \mathrm{d}t \lesssim \int_{t_1}^{t_2} \left(\|\partial_t u(t)\|_{L^2}^2 + \delta^2 + C^3\delta^{\frac{7}{3}} \right) \mathrm{d}t \lesssim \delta^2 + C^3\delta^{\frac{7}{3}}. \tag{8.76}$$

By the definition of $a_k^\pm$, we have

$$a_k^+ = \zeta_k^+ \langle \varphi_1, \Upsilon_k \rangle + \langle \varphi_2, \Upsilon_k \rangle, \quad a_k^- = \zeta_k^- \langle \varphi_1, \Upsilon_k \rangle + \langle \varphi_2, \Upsilon_k \rangle$$

and thus for all $k = 1, \cdots, K$

$$a_k^+ = \frac{\zeta_k^+}{\zeta_k^-} a_k^- + \frac{\zeta_k^- - \zeta_k^+}{\zeta_k^-} \langle \varphi_2, \Upsilon_k \rangle,$$

from where we see that $\mathcal{A} \lesssim \mathcal{S} + \|\varphi_2\|_{L^2}^2$. Gathering estimates (8.71) and (8.76), we find

$$\int_{t_1}^{t_2} \mathcal{A}(t)\mathrm{d}t \lesssim \int_{t_1}^{t_2} \mathcal{S}(t)\mathrm{d}t + \int_{t_1}^{t_2} \|\varphi_2(t)\|_{L^2}^2 \mathrm{d}t \lesssim \delta^2 + C^3\delta^{\frac{7}{3}},$$

which is a contradiction with (8.75) for C large enough and δ small enough. This proves (8.72), and this strictly improves the estimate on $\mathcal{A}$

Step 6. Conclusion. As a consequence of improving the bootstrap assumption (8.66) on $z - \xi_\delta$, θ, $\mathcal{N}$ and $\mathcal{A}$, we conclude that $T^* = \infty$.

Finally, from (8.66), we know that, for all $t \in [T_\delta, +\infty)$,

$$\begin{aligned}
&\|u(t) - q(\cdot - \xi_\delta)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \\
&\lesssim \|u(t) - Q(t)\|_{H^1} + \|Q(t) - q(\cdot - \xi_\delta)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \\
&\lesssim \mathcal{N}(t) + |a(t)| + |z(t) - \xi_\delta| + |\theta(t)| \lesssim \delta^{\frac{1}{2}} + C\delta^{\frac{2}{3}} + C\delta,
\end{aligned}$$

which implies (8.64) for δ small enough. The proof of Proposition 8.18 is complete. □

Remark 8.19. Observe that the proof of Proposition 8.18 actually proves that

$$\theta(t), a(t), \mathcal{N}(t), \mathcal{A}(t) \to 0 \quad \text{as} \quad t \to +\infty,$$

a fact that we will use in the proofs of Theorem 8.4 and 8.6.

End of the proof of Theorem 8.4. We assume here that q is a non-degenerate bound state, and prove exponential convergence. Recall that, in this context, we have set $a = b = 0$.

First, from Proposition 8.18, we know that, for any $0 < \delta < \delta_0^2$, there exists $T_\delta^* \gg 1$ and $\xi_\delta \in \mathbb{R}^N$ such that, $\vec{u}$ admits a decomposition as in Proposition 8.11 for all $t \in [T_\delta^*, \infty)$ and satisfies

$$\|u(t) - q(\cdot - \xi_\delta)\|_{H^1} + \|\partial_t u(t)\|_{L^2} + |z(t) - \xi_\delta| + |\theta(t)| + \mathcal{N}(t) < \delta. \tag{8.77}$$

Step 1. We claim that

$$\forall t \geqslant T_\delta^*, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}(t) + \mu^{-1}\mathcal{A}(t)\right) \leqslant -\mu\left(\mathcal{F}(t) + \mu^{-1}\mathcal{A}(t)\right). \tag{8.78}$$

This will implies exponential decay of $\mathcal{F} + \mu^{-1}\mathcal{A}$. So as to prove the above differential inequality, it is convenient to introduce the auxiliary quantity $\tilde{\mathcal{A}} := \mathcal{A} - \delta^{\frac{1}{2}}\mathcal{F}$. First note that, from (8.55) and (8.77),

$$\mathcal{N}^3(t) \leqslant \delta\mathcal{N}^2(t) \leqslant \delta C_0\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right). \tag{8.79}$$

As a consequence,

$$\tilde{\mathcal{A}} = (1 + \delta^{\frac{1}{2}}\mu^{-1})\mathcal{A} - \delta^{\frac{1}{2}}\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right) \leqslant (1 + \delta^{\frac{1}{2}}\mu^{-1})\mathcal{A}.$$

Therefore, using the evolution equations (8.60) and (8.62) for $\mathcal{F}$ and $\mathcal{A}$, and in view of (8.79),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{A}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} - \delta^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} &\geqslant \left(\nu_3 - 2\delta^{\frac{1}{2}}\right)\mathcal{A} + 2\mu\delta^{\frac{1}{2}}\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right) - (1 + \delta^{\frac{1}{2}})C_0\mathcal{N}^3 \\ &\geqslant \left(\nu_3 - 2\delta^{\frac{1}{2}}\right)\mathcal{A} + \left(2\mu\delta^{\frac{1}{2}} - \delta C_0^2(1 + \delta^{\frac{1}{2}})\right)\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right) \geqslant \frac{1}{2}\nu_3\tilde{\mathcal{A}},\end{aligned}$$

where $\delta > 0$ is so small that

$$\left(\nu_3 - 2\delta^{\frac{1}{2}}\right) \geqslant \frac{1}{2}\nu_3(1 + \delta^{\frac{1}{2}}\mu^{-1}), \quad 2\mu\delta^{\frac{1}{2}} - \delta\mu^{-1}C_0(1 + \delta^{\frac{1}{2}}) \geqslant 0.$$

Integrating on $[t, s] \subseteq [T_\delta^*, \infty)$, we get

$$\tilde{\mathcal{A}}(t) \leqslant e^{\frac{1}{2}\nu_3(t-s)}\tilde{\mathcal{A}}(s).$$

We now take the limit $s \to \infty$ and using that $\tilde{\mathcal{A}}(s)$ is bounded due to (8.77), we obtain that $\tilde{\mathcal{A}}(t) \leqslant 0$, that is

$$\forall t \in [T_\delta^*, \infty), \quad \mathcal{A}(t) \leqslant \delta^{\frac{1}{2}}\mathcal{F}(t) \leqslant \delta^{\frac{1}{2}}\left(\mathcal{F}(t) + \mu^{-1}\mathcal{A}(t)\right). \tag{8.80}$$

Combining (8.60), (8.61), (8.79) and (8.80), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \mu^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} \\ &\leqslant -2\mu(\mathcal{F} + \mu^{-1}\mathcal{A}) + (\mu^{-1}\nu_2 + 2)\mathcal{A} + C_0(1 + \mu^{-1})\mathcal{N}^3 \\ &\leqslant (-2\mu + \delta^{\frac{1}{2}}(\mu^{-1}\nu_2 + 2) + \delta C_0^2(1 + \mu^{-1}))\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right),\end{aligned}$$

which implies (8.78) for δ small enough.

Step 2. Integrating (8.78) on $[T_\delta^*, t]$ and using (8.77), we have

$$\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right)(t) \leqslant e^{-\mu(t - T_\delta^*)}\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right)(T_\delta^*) \lesssim \delta^2 e^{-\mu(t - T_\delta^*)}.$$

Recall once again (8.55): it follows that

$$\mathcal{N}^2(t) \lesssim (\mathcal{F} + \mu^{-1}\mathcal{A})(t) \lesssim e^{-\mu t}. \tag{8.81}$$

Now we can integrate (8.38), using (8.81) and recalling that $a = 0$: this proves that there exists $z_\infty \in \mathbb{R}^N$ such that $z(t) \to z_\infty$ as $t \to +\infty$, and

$$\forall t \geqslant T_\delta, \quad |z(t) - z_\infty| + |\theta(t)| \lesssim \int_t^\infty \mathcal{N}(s)^2 \mathrm{d}s \lesssim e^{-\mu t}. \tag{8.82}$$

Gathering estimates (8.81) and (8.82), we obtain

$$\begin{aligned}
&\|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \\
&\quad \lesssim \|u(t) - Q(t)\|_{H^1} + \|Q(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \\
&\quad \lesssim \mathcal{N}(t) + |z(t) - z_\infty| + |\theta(t)| \lesssim e^{-\mu t}.
\end{aligned}$$

The proof of Theorem 8.4 is complete. □

End of the proof of Theorem 8.6. We here assume that q is a degree-1 excited state, and prove algebraic convergence of $\vec{u}$ to $(q, 0)$.

As earlier, we know from Proposition 8.18, that for any $0 < \delta < \delta_0^2$, there exists $T_\delta^* \gg 1$ (we can choose $T_\delta^* = T_{\delta^2/2}$) and $\xi_\delta \in \mathbb{R}^N$ such that, $\vec{u}$ admits a decomposition as in Proposition 8.11 for all $t \in [T_\delta^*, \infty)$ and satisfies

$$\|u(t) - q(\cdot - \xi_\delta)\|_{H^1} + \|\partial_t u(t)\|_{L^2} + |z(t) - \xi_\delta| + |a(t)| + |\theta(t)| + \mathcal{N}(t) < \delta. \tag{8.83}$$

Step 1. We claim that there exists $C_1 \geqslant 1$ (independent of δ) such that

$$\forall t \geqslant T_\delta^*, \quad \mathcal{A}(t) \leqslant C_1 \delta^{\frac{1}{4}} (a(t)^2 + \mathcal{N}(t)^2). \tag{8.84}$$

Again, we consider an auxiliary quantity, which is here $\tilde{\mathcal{A}} := \mathcal{A} - \delta^{\frac{1}{4}}\mathcal{F} - \delta^{-\frac{1}{2}}\mathcal{R}_2$. From the evolution equations (8.57), (8.60) and (8.62) of $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{R}_2$, we bound

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{A}} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} - \delta^{\frac{1}{4}}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} - \delta^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_2 \\
&\geqslant 2\mu\delta^{\frac{1}{4}}\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right) + (\nu_3 - 2\delta^{\frac{1}{4}})\mathcal{A} + \delta^{-\frac{1}{2}}\alpha a^4 \\
&\qquad - C_0(1 + \delta^{-\frac{1}{2}} + \delta^{\frac{1}{4}})\mathcal{N}^3 - C_0\left(1 + \delta^{\frac{1}{4}}\right)a^2\mathcal{N}. \\
&\geqslant 2\mu^2\delta^{\frac{1}{4}}\mathcal{N}^2 + (\nu_3 - 2\delta^{\frac{1}{4}})\mathcal{A} + \delta^{-\frac{1}{2}}\alpha a^4 - 3C_0\delta^{-\frac{1}{2}}\mathcal{N}^3 - 2C_0 a^2\mathcal{N}.
\end{aligned}$$

Notice that from the AM-GM inequality, we have

$$2a^2\mathcal{N} = 2(a^2\delta^{-\frac{3}{16}})(\mathcal{N}\delta^{\frac{3}{16}}) \leqslant a^4\delta^{-\frac{3}{8}} + \mathcal{N}^2\delta^{\frac{3}{8}}, \tag{8.85}$$

so that, rearranging the terms in the preceding inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{A}} \geqslant \left(2\mu^2\delta^{\frac{1}{4}} - 3C_0\delta^{\frac{1}{2}} - C_0\delta^{\frac{3}{8}}\right)\mathcal{N}^2 + (\nu_2 - 2\delta^{\frac{1}{4}})\mathcal{A} + (\delta^{-\frac{1}{2}}\alpha - \delta^{-\frac{3}{8}}C_0)a^4 \geqslant 0,$$

where δ is so small that

$$2\mu^2\delta^{\frac{1}{4}} - 3C_0\delta^{\frac{1}{2}} - C_0\delta^{\frac{3}{8}} > 0, \quad \nu_2 - 2\delta^{\frac{1}{4}} > 0 \quad \text{and} \quad \delta^{-\frac{1}{2}}\alpha - \delta^{-\frac{3}{8}}C_0 > 0.$$

Now, from Proposition 8.18, we know that $\tilde{\mathcal{A}}(t) \to 0$ as $t \to \infty$. As we just showed that $\tilde{\mathcal{A}}$ is non decreasing on $[T_\delta^*, \infty)$, we hence conclude that for all $t \geqslant T_\delta^*$, $\tilde{\mathcal{A}}(t) \leqslant 0$, or equivalently (due to the definition of $\tilde{\mathcal{A}}$), that for all $t \geqslant T_\delta^*$,

$$\begin{aligned}\mathcal{A}(t) &\leqslant \delta^{\frac{1}{4}}\mathcal{F}(t) + \delta^{-\frac{1}{2}}\mathcal{R}_2(t)\\ &\leqslant \delta^{\frac{1}{4}}\mathcal{F}(t) + \delta^{\frac{1}{2}}\left(\frac{2\alpha}{3}+1\right)a^2 + \frac{\delta^{\frac{1}{2}}}{2\alpha}\left(a^2+\mathcal{N}^2\right),\end{aligned}$$

As $\mathcal{F} \lesssim \mathcal{N}$, this implies (8.84) for δ small enough.

Step 2. We claim that there exists $C_2 \geqslant 1$ (independent of δ) such that,

$$\forall t \geqslant T_\delta^*, \quad |a(t)| \leqslant C_2\left((t - T_\delta^* + \delta^{-1})^{-1} + \mathcal{N}(t)\right). \tag{8.86}$$

We use yet another auxiliary quantity, namely $\widetilde{\mathcal{R}}_1 := \mathcal{R}_1 + 2C_0^2\mu^{-1}\left(\mathcal{F} + \mu^{-1}\mathcal{A}\right)$. Note that, from the AM-GM inequality and (8.83), there exists $C_2 \geqslant 1$ such that

$$\widetilde{\mathcal{R}}_1^2 \leqslant 2\mathcal{R}_1^2 + O\left(\mathcal{N}^4\right) \leqslant C_2(a^2+\mathcal{N}^2). \tag{8.87}$$

Furthermore, using the evolution equation (8.60) and (8.61) for $\mathcal{F}$ and $\mathcal{A}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}+\mu^{-1}\mathcal{A}\right) &\leqslant -2\mu\left(\mathcal{F}+\mu^{-1}\mathcal{A}\right) + (\nu_3+2)\mathcal{A} + C_0\left(1+\mu^{-1}\right)\left(\mathcal{N}^3 + a^2\mathcal{N}\right)\\ &\leqslant -\mu\left(\mathcal{F}+\mu^{-1}\mathcal{A}\right) + O(\delta^{\frac{1}{4}}(a^2+\mathcal{N}^2))\end{aligned}$$

where we also used (8.83) and (8.84). Therefore, using the evolution equation (8.56) for $\mathcal{R}_1$, we deduce that,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{R}}_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_1 + 2C_0^2\mu^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}+\mu^{-1}\mathcal{A}\right)\\ &\leqslant -\frac{1}{2}a^2 + C_0\mathcal{N}^2 + 2C_0^2\mu^{-1}\left(-\mu(\mathcal{F}+\mu^{-1}\mathcal{A})\right) + O(\delta^{\frac{1}{4}}(a^2+\mathcal{N}^2))\\ &\leqslant -\frac{1}{2}a^2 - C_0\mathcal{N}^2 + O(\delta^{\frac{1}{4}}(a^2+\mathcal{N}^2)).\end{aligned}$$

(We used the coercivity estimate (8.55): $\mathcal{F}+\mu^{-1}\mathcal{A} \geqslant C_0^{-1}\mathcal{N}$, and also that $|b| \lesssim \mathcal{N}$). For δ small enough, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{R}}_1 \leqslant -\frac{1}{4}a^2 - \frac{C_0}{2}\mathcal{N}^2 \leqslant -c_2\tilde{\mathcal{R}}_1^2.$$

for some universal constant $c_2 > 0$. Recall now that from Proposition 8.18, $\tilde{\mathcal{R}}_1(t) \to 0$ as $t \to +\infty$, and therefore, after dividing by $\tilde{\mathcal{R}}_1^2$, and integrating on $[T_\delta^*, t]$, we obtain

$$\forall t \geqslant T_\delta^*, \quad 0 \leqslant \tilde{\mathcal{R}}_1(t) \lesssim (t - T_\delta^* + \delta^{-1})^{-1}.$$

(Notice that $\tilde{\mathcal{R}}_1(T_\delta^*) \lesssim \delta$, in view of (8.83)). It can be rewritten as

$$-2C_0\mu^{-2}\left(\mathcal{F}(t)+\mu^{-1}\mathcal{A}(t)\right) \leqslant a(t) + \frac{b(t)}{2\alpha} \lesssim (t - T_\delta^* + \delta^{-1})^{-1}.$$

As $|b|, \mathcal{F}+\mu^{-1}\mathcal{A} \lesssim \mathcal{N}$, this implies (8.86) for C_2 large enough.

Step 3. Conclusion. Now we prove the algebraic decay rate by a bootstrap argument. For $C_3 \geqslant 1$ to be chosen later, we introduce the following bootstrap bounds

$$|a(t)| \leqslant \frac{C_3^2}{(t - T_\delta^* + \delta^{-1})}, \quad \mathcal{N}(t) \leqslant \frac{C_3}{(t - T_\delta^* + \delta^{-1})}. \tag{8.88}$$

Let

$$T^{**} = \sup\left\{t \in [T_\delta^*, \infty) \text{ such that } (8.88) \text{ holds on } [T_\delta^*, t]\right\}.$$

We will prove that $T^{**} = +\infty$ by strictly improving the bootstrap assumption (8.88) on $[T_\delta^*, T^{**})$, upon choosing δ small enough. In this bootstrap, the implied constants do not depend on δ and C_3, but can depend on C_0, C_1, C_2.

As $|a(T_\delta^*)| \leqslant \delta$ and $\mathcal{N}(T_\delta^*) \leqslant \delta$ due to (8.83), the bootstrap estimate (8.88) holds (strictly) at $t = T_\delta^*$, and so $T^{**} > T_\delta^*$.

Estimate on a. From (8.86) and the boostrap estimate (8.88) on $\mathcal{N}$, we have

$$|a(t)| \leqslant C_2\left(\frac{1}{t - T_\delta^* + \delta^{-1}} + \mathcal{N}(t)\right) \leqslant \frac{C_2(C_3+1)}{(t - T_\delta^* + \delta^{-1})},$$

which strictly improves the estimate (8.88) on a for taking C_3 large enough (depended on C_2).

Estimate on $\mathcal{N}$. First, we claim that

$$\forall t \in [T_\delta^*, T^{**}), \quad \mathcal{F}(t) \lesssim \frac{1}{(t - T_\delta^* + \delta^{-1})^2}. \tag{8.89}$$

Indeed, using (8.83) and the bootstrap bound (8.88), the evolution equation (8.60) on $\mathcal{F}$ writes

$$\forall t \in [T_\delta^*, T^{**}), \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \leqslant C_0\mathcal{N}(a^2 + \mathcal{N}^2) \lesssim \frac{\delta C_3^2(C_3^2+1)}{(t - T_\delta^* + \delta^{-1})^2}.$$

Let $t \in [T_\delta^*, T^{**})$, and integrate on $[T_\delta^*, t]$: for δ small enough (dependent on C_3), this gives

$$\begin{aligned}\mathcal{F}(t) &\lesssim e^{-2\mu(t-T_\delta^*)}\mathcal{F}(T_\delta^*) + \delta\int_{T_\delta^*}^t e^{-2\mu(t-s)}\frac{C_3^2(C_3^2+1)\mathrm{d}s}{(s - T_\delta^* + \delta^{-1})^2} \\ &\lesssim \delta^2 e^{-2\mu(t-T_\delta^*)} + \frac{\delta C_3^2(C_3^2+1)}{(t - T_\delta^* + \delta^{-1})^2} \lesssim \frac{1}{(t - T_\delta^* + \delta^{-1})^2},\end{aligned}$$

which means (8.89). Therefore, from the coercivity estimate (8.55), combining the bound (8.84) and (8.89) on $\mathcal{F}$ and $\mathcal{A}$ that we just obtained and the boostrap bounds (8.88), we get

$$\mathcal{N}^2 \lesssim \mathcal{F} + \mu^{-1}\mathcal{A} \lesssim \frac{1}{(t - T_\delta^* + \delta^{-1})^2} + \frac{\delta^{\frac{1}{4}}C_3^2(C_3^2+1)}{(t - T_\delta^* + \delta^{-1})^2} \lesssim \frac{1}{(t - T_\delta^* + \delta^{-1})^2},$$

which strictly improves the estimate (8.88) of $\mathcal{N}$, upon δ small enough.

As a consequence of improving the bootstrap assumption on a and $\mathcal{N}$, we conclude that $T^{**} = \infty$.

Finally, it suffices to bound the geometric parameters. First, recall the equation (8.39) on ℓ and β: proceeding as for $\mathcal{F}$ (using (8.88) for $t \geqslant T_\delta^*$), we get that

$$\forall t \geqslant T_\delta^*, \quad |\ell(t)| + |\beta(t)| \lesssim \frac{1}{(t - T_\delta^* + 1)^2}.$$

Then we consider z and θ: using the above estimate and (8.88), the equation (8.38) now writes

$$\begin{aligned}|\dot{z}| &\leqslant |\dot{z} - \ell| + |\ell| \lesssim |\ell| + \mathcal{N}^2 + |a|\mathcal{N} \lesssim \frac{1}{(t - T_\delta^* + 1)^2}, \\ |\dot{\theta}| &\leqslant |\dot{\theta} - \beta| + |\beta| \lesssim |\beta| + \mathcal{N}^2 + |a|\mathcal{N} \lesssim \frac{1}{(t - T_\delta^* + 1)^2}.\end{aligned}$$

This proves that $z(t) \to z_\infty$ as $t \to +\infty$, and that

$$\forall t \geqslant T_\delta^*, \quad |z(t) - z_\infty| \lesssim \frac{1}{t - T_\delta^* + 1}.$$

From Proposition 8.18, we already know that $\theta(t) \to 0$ as $t \to +\infty$, and we obtain as for z the convergence rate

$$\forall t \geqslant T_\delta^*, \quad |\theta(t)| \lesssim \frac{1}{t - T_\delta^* + 1}.$$

Finally, gathering the above estimates, we conclude that for all $t \geqslant T_0$,

$$\begin{aligned}
&\|u(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \\
&\quad \leqslant \|u(t) - Q(t)\|_{H^1} + \|Q(t) - q(\cdot - z_\infty)\|_{H^1} + \|\partial_t u(t)\|_{L^2} \\
&\quad \lesssim \mathcal{N}(t) + |a(t)| + |z(t) - z_\infty| + |\theta(t)| \lesssim \frac{1}{t - T_\delta^* + 1}.
\end{aligned}$$

The proof of Theorem 8.6 is complete. □

8.4 Proof of Theorem 8.7

In this section, we prove Theorem 8.7. Let q be a degree-1 excited state.

Proof of Theorem 8.7. Let $0 < \delta \ll 1$ to be chosen later. Given $\mathfrak{a}^+ = (\mathfrak{a}_k^+)_{k=1,\cdots,K} \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$, we consider the solution $\vec{u}$ of (8.1) with initial data

$$\vec{u}(0) = (q, 0) + (\delta\phi, 0) + \vec{W}(\mathfrak{a}_k^+), \tag{8.90}$$

where

$$\vec{W}(\mathfrak{a}^+) = \sum_{k=1}^{K} \frac{\mathfrak{a}_k^+}{(\zeta_k^+)^2 + 1} \begin{pmatrix} \zeta_k^+ Y_k \\ Y_k \end{pmatrix}. \tag{8.91}$$

Step 1. Decomposition. For any $t \geqslant 0$ such that $\vec{u}$ is defined and satisfies (8.31), we consider its decomposition according to Proposition 8.11: this gives the functions $z, \ell, \theta, \beta, a, b$ and $\vec{\varphi}$. Note that by the definition of $\vec{W}(\mathfrak{a}^+)$, the initial data $\vec{u}(0)$ is modulated in the sense that

$$\vec{\varphi}(0) = \vec{W}((\mathfrak{a}^+), \ z(0) = \ell(0) = 0, \ \theta(0) = \beta(0) = 0, \ b(0) = 0, \ a(0) = \delta. \tag{8.92}$$

Also, by. (8.13), and as $\vec{Z}_k^\pm(0) = \begin{pmatrix} \zeta_k^\pm Y_k \\ Y_k \end{pmatrix}$, we have

$$\forall k = 1, \ldots, K, \quad a_k^+(0) = \langle \vec{W}((\mathfrak{a}^+), \vec{Z}_k(0) \rangle = \mathfrak{a}_k^+. \tag{8.93}$$

We introduce the following bootstrap setting,

$$\begin{cases} \left|a(t) - (t + \delta^{-1})^{-1}\right| \leqslant (t + \delta^{-1})^{-\frac{8}{7}} + (t + \delta^{-1})^{-\frac{\bar{p}}{2}}, \\ |\theta(t)| \leqslant \delta^{\frac{1}{2}}, \quad \mathcal{N}(t) \leqslant (t + \delta^{-1})^{-\frac{5}{4}}, \quad \mathcal{A}(t) \leqslant (t + \delta^{-1})^{-3}. \end{cases} \tag{8.94}$$

Let $T_* = T_*(\mathfrak{a}^+)$ be the supremum of times $t \geqslant 0$ such that $\vec{u}$ is defined on $[0, t]$, satisfies the assumption (8.31) of Proposition 8.11 on $[0, t]$, and such that the bootstrap estimates (8.94) hold on $[0, t]$.

Our goal is to prove that there exists at least one choice of $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$ such that $T_*(\mathfrak{a}^+) = \infty$. For this, we start by closing all bootstrap estimates except the one for the instable modes, $\mathcal{A}$, given any $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$. Then we prove the existence of suitable parameters $\boldsymbol{a}^+ = (a_k^+)_{k=1,\cdots,K}$ for which $\mathcal{A}$ is controlled, using a topological argument. Before we proceed with the actual proof, let us emphasize a major difference between the argument here and the previous boostraps. In Theorem 8.7 the goal is *to construct* a solution, and our only choice is a good guess for the initial data as in (8.90); hence, we do not have any long time a priori knowledge, in particular, we have no way to ensure a priori that

$\partial_t u \in L^2([0,+\infty), L^2)$ (a bound which plays a key role in controlling the unstable modes): it will be consequence of our construction.

In the remainder of the proof, the implied constants in $\lesssim$ or O do not depend on the small parameter $\delta > 0$ which appears in the definition of the initial data (8.90) and in the bootstrap assumptions (8.94), nor on $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$.

Step 2. Closing the estimates in (8.94) *except for* $\mathcal{A}$.

Fix $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$. For simplicity of notations, we drop any references to $\mathfrak{a}^+$ in this step.

Estimate of a. From (8.94) and the definition of $\mathcal{R}_1$, we have, for any $t \in [0, T_*)$,

$$\left|\mathcal{R}_1(t)^2 - a(t)^2\right| \lesssim |a(t)|\mathcal{N}(t) + \mathcal{N}(t)^2 \lesssim (t+\delta^{-1})^{-\frac{9}{4}}. \tag{8.95}$$

Also,

$$\left|\mathcal{R}_1(t) - (t+\delta^{-1})^{-1}\right| \leqslant (t+\delta^{-1})^{-\frac{8}{7}} + (t+\delta^{-1})^{-\frac{\bar{p}}{2}} + (t+\delta^{-1})^{-\frac{5}{4}}, \tag{8.96}$$

and so $\mathcal{R}_1(t) > 0$ if $\delta > 0$ is small enough. By (8.58), (8.94) and (8.95),

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_1 + \mathcal{R}_1^2\right| &\lesssim \left|\mathcal{R}_1^2(t) - a^2(t)\right| + \mathcal{N}^2(t) + |a(t)|\mathcal{N}(t) + |a(t)|^{\bar{p}} \\ &\lesssim (t+\delta^{-1})^{-\frac{9}{4}} + (t+\delta^{-1})^{-\bar{p}}.\end{aligned}$$

Therefore, from (8.96), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}_1^{-1}\right)(t) - 1\right| \lesssim (t+\delta^{-1})^{-\frac{1}{4}} + (t+\delta^{-1})^{-\bar{p}+2}.$$

Integrating above estimate on $[0,t]$, and using (8.92), we obtain

$$\left|\mathcal{R}_1^{-1}(t) - (t+\delta^{-1})\right| \lesssim (t+\delta^{-1})^{\frac{3}{4}} + (t+\delta^{-1})^{-\bar{p}+3},$$

so that in particular $\mathcal{R}_1(t)^{-1} \geqslant \frac{1}{2}(t+\delta^{-1})$ (for δ small enough) and

$$\left|\mathcal{R}_1(t) - \frac{1}{t+\delta^{-1}}\right| = \frac{\left|\mathcal{R}_1^{-1}(t) - (t+\delta^{-1})\right|}{\mathcal{R}_1(t)^{-1}(t+\delta^{-1})} \lesssim (t+\delta^{-1})^{-\frac{5}{4}} + (t+\delta^{-1})^{-\bar{p}+1}.$$

From there, we infer

$$\begin{aligned}\left|a(t) - (t+\delta^{-1})^{-1}\right| &\lesssim \left|\mathcal{R}_1(t) - (t+\delta^{-1})^{-1}\right| + \mathcal{N}(t) \\ &\lesssim (t+\delta^{-1})^{-\frac{5}{4}} + (t+\delta^{-1})^{-\bar{p}+1},\end{aligned}$$

which strictly improves the estimate (8.94) of a for δ small enough (recall that $\bar{p} > 2$).

Estimate of θ. We use the evolution equations (8.38) and (8.39) for θ and β, under the bootstrap assumption (8.94): this gives, for all $t \in [0, T_*)$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\theta + \frac{\beta}{2\alpha}\right)(t)\right| \lesssim \mathcal{N}^2(t) + a^2(t) \lesssim (t+\delta^{-1})^{-2}.$$

Fix $t \in [0, T_*)$ and integrate the above estimate on $[0,t]$, using (8.92) and (8.94): it yields

$$|\theta(t)| \lesssim \mathcal{N}(t) + \left|\theta(t) + \frac{\beta(t)}{2\alpha}\right| \lesssim (t+\delta^{-1})^{-\frac{5}{4}} + \int_0^t (s+\delta^{-1})^{-2}\mathrm{d}s \lesssim \delta^{\frac{5}{4}} + \delta \lesssim \delta,$$

which strictly improves the estimate (8.94) of θ, for δ small enough.

Estimate of $\mathcal{N}$. We first derive a bound on $\mathcal{F}$ using its evolution equation (8.60): together with the boostrap assumption (8.94), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\mu\mathcal{F} \leqslant C_0\left((t+\delta^{-1})^{-\frac{15}{4}} + 2(t+\delta^{-1})^{-\frac{13}{4}}\right) \lesssim (t+\delta^{-1})^{-\frac{13}{4}}.$$

Fix $t \in [0, T_*)$ and integrate this on $[0,t]$ and using the initial bounds (8.91) and (8.92) to infer that

$$\begin{aligned}\mathcal{F}(t) &\lesssim e^{-2\mu t}\mathcal{F}(0) + e^{-2\mu t}\int_0^t e^{2\mu s}(t+\delta^{-1})^{-\frac{13}{4}}\,\mathrm{d}s \\ &\lesssim \delta^3 e^{-2\mu t} + (t+\delta^{-1})^{-\frac{13}{4}} \lesssim (t+\delta^{-1})^{-3}.\end{aligned}$$

To get a bound on $\mathcal{N}$, we now recall the coercivity bound (8.55) again and the bootstrap assumption (8.94) on $\mathcal{A}$, and conclude that

$$\mathcal{N}^2(t) \lesssim \mathcal{F}(t) + \mu^{-1}\mathcal{A}(t) \lesssim (t+\delta^{-1})^{-3},$$

which strictly improves the boostrap estimate (8.94) of $\mathcal{N}$ for δ small enough.

Step 3. Transversality and choice of $\mathfrak{a}^+$. Observe that for any time t where the bootstrap bounds (8.94) holds together with the equality $\mathcal{A}(t) = (t+\delta^{-1})^{-3}$, the evolution equation (8.62) on $\mathcal{A}$ yields the following transversality property:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left((t+\delta^{-1})^3\mathcal{A}(t)\right) &\geqslant \left(\nu_3(t+\delta^{-1})^3 + 3(t+\delta^{-1})^2\right)\mathcal{A}(t) \\ &\quad - C_0(t+\delta^{-1})^3\left(\mathcal{N}^3(t) + a^2(t)\mathcal{N}(t)\right) \\ &\geqslant \nu_3 + O\left((t+\delta^{-1})^{-\frac{1}{4}} + (t+\delta^{-1})^{-\frac{3}{4}} + (t+\delta^{-1})^{-1}\right) \geqslant \frac{1}{2}\nu_3,\end{aligned} \tag{8.97}$$

for $\delta > 0$ small enough. Estimate (8.97) is enough to justify the existence of at least K-uple $\mathfrak{a}^+ \in \bar{B}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$ such that $T_*(\mathfrak{a}^+) = \infty$.

The proof is by contradiction: assume for its sake that for all $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$, it holds $T_*(\mathfrak{a}^+) < \infty$. Then, a contradiction follows from the following discussion (see for instance more details in [28] and [31, Section 3.1]).

Continuity of the map $\mathfrak{a}^+ \mapsto T_*(\mathfrak{a}^+)$.

Let $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$: as $T_*(\mathfrak{a}^+) < +\infty$, then as we improved all other estimates in (8.94) in the previous step, necessarily, the equality $\mathcal{A}(t) = (t+\delta^{-1})^{-3}$ holds for $t = T_*(\mathfrak{a}^+)$, and so (8.97) holds at $t = T_*(\mathfrak{a}^+)$.

By continuity of the flow of (8.1) (and of the modulation technique), the above transversality property implies that the map

$$\bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}}) \to [0, +\infty), \quad \mathfrak{a}^+ \mapsto T_*(\mathfrak{a}^+)$$

is continuous and there is instantaneous exit for initial data on the boundary:

$$T_*(\mathfrak{a}^+) = 0 \quad \text{for all } \mathfrak{a}^+ \in \mathcal{S}_{\mathbb{R}^K}\left(\delta^{\frac{3}{2}}\right).$$

Construction of a retraction. As a consequence, the map giving the exit point (on the boundary)

$$\begin{aligned}\mathcal{M}:\ \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}}) &\to \mathcal{S}_{\mathbb{R}^K}(\delta^{\frac{3}{2}}) \\ \mathfrak{a}^+ &\mapsto \delta^{\frac{3}{2}}\left(T_*(\mathfrak{a}^+) + \delta^{-1}\right)^{\frac{3}{2}}\boldsymbol{a}^+\left(T_*(\mathfrak{a}^+)\right)\end{aligned}$$

is well defined, continuous, and moreover, the restriction of $\mathcal{M}$ to $\mathcal{S}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$ is the identity.

The existence of such a map $\mathcal{M}$ contradicts Brouwer's no retraction theorem for continuous maps from the ball to the sphere. We conclude that there exists at least one $\mathfrak{a}^+ \in \bar{\mathcal{B}}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$ such that $T_*(\mathfrak{a}^+) = \infty$.

Step 4. Conclusion. At this point, we have proved the existence of $\mathfrak{a}^+ \in \mathcal{B}_{\mathbb{R}^K}(\delta^{\frac{3}{2}})$, associated with a global solution $\vec{u} \in \mathcal{C}([0,+\infty), H^1 \times L^2)$ of (8.1) with initial data defined in (8.90), which also can be modulated (in the sense of (8.31)) and satisfies (8.94) for all $t \in [0,\infty)$. Let us now control the convergence of z and θ: using their evolution equations (8.82) (and (8.39)) under the (8.94) regime, we see that there exist $z(t) \to z_\infty$ and $\theta(t) \to \theta_\infty$ as $t \to +\infty$ and moreover

$$\begin{aligned}
|z(t) - z^\infty| + |\theta(t) - \theta^\infty| &\leqslant \left| z(t) + \frac{\ell}{2\alpha}(t) - z_\infty \right| + \left| \theta(t) + \frac{\beta}{2\alpha}(t) - \theta_\infty \right| + \frac{1}{2\alpha}\mathcal{N}(t) \\
&\leqslant \frac{1}{2\alpha}\mathcal{N}(t) + \int_t^\infty \left(\left| \dot{z} + \frac{\dot{\ell}}{2\alpha} \right| + \left| \dot{\theta} + \frac{\dot{\beta}}{2\alpha} \right| \right) ds \\
&\lesssim \mathcal{N}(t) + \int_t^\infty \left(\mathcal{N}^2(s) + |a(s)|\mathcal{N}(s) + |a|^{\bar{p}}(s) \right) \mathrm{d}s \\
&\lesssim (t+\delta^{-1})^{-\frac{5}{4}} + (t+\delta^{-1})^{-\frac{3}{2}} + (t+\delta^{-1})^{-\bar{p}+1}.
\end{aligned} \tag{8.98}$$

We now use the invariance of the equation to get a solution to (8.1) where $z_\infty = 0$ and $\theta_\infty = 0$ by setting

$$\vec{u}_*(t,x) = \vec{u}(t, R_{\theta_\infty}^{-1}(x + z_\infty)) \quad \text{for all} \quad (t,x) \in [0,\infty) \times \mathbb{R}^N.$$

Then, from (8.94) and (8.98), $\vec{u}_* \in \mathcal{C}([0,+\infty), H^1 \times L^2)$ is a solution of (8.1) which enjoys requested properties in the conclusion of Theorem 8.7, and more precisely,

$$\forall t \geqslant 0, \quad \|\vec{u}_*(t) - (q,0) - (t+\delta^{-1})^{-1}(\phi,0)\|_{H^1\times L^2} \lesssim (t+\delta^{-1})^{-\frac{5}{4}} + (t+\delta^{-1})^{-\bar{p}+1}.$$

The proof of Theorem 8.7 is complete. □

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Dynamique en temps long pour des équations non linéaires de type onde avec ou sans amortissement

Mots clés : soliton, équations aux dérivées partielles, dynamique en temps long, équations d'ondes non-linéaires, multi-soliton, amortissement

Résumé : Dans cette thèse, nous étudions le comportement qualitatif des solutions d'équations non linéaires de type onde, avec ou sans amortissement, en mettant un accent particulier sur la description de la dynamique en temps long des solutions dans l'espace d'énergie. Sur les exemples typiques de l'équation de Klein-Gordon non linéaire (NLKG), avec ou sans amortissement, et de l'équation d'onde non linéaire (NLW), nous étudions le comportement des solutions qui se décomposent en un seul soliton ou en une somme de solitons, quand le temps tend vers l'infini.

Premierement, nous montrons la stabilité conditionnelle des multi-solitons pour l'équation NLKG en dimension un d'espace et avec double non-linéarité puissance. Ensuite, pour l'équation NLW énergie critique, nous prouvons l'existence de différents types de multi-solitons basées soit sur l'état fondamental, soit sur des états excités bien choisis, sous diverses conditions sur la dimension spatiale et les vitesses de Lorentz. Finalement, nous étudions la dynamique en temps long des solitons et multi-solitons de l'équation NLKG sous-critique pour l'énergie, avec amortissement.

Title : Long time dynamics of nonlinear wave-type equations with or without damping

Keywords : soliton, partial differential equation, long time dynamics, nonlinear wave models, multi-soliton, damping

Abstract : In this thesis, we study the qualitative behavior of solutions of nonlinear wave-type equations, with or without damping, putting a special emphasis on the description of the long-time dynamics of solutions in the energy space. Through the typical examples of the nonlinear Klein-Gordon equation (NLKG) with or without damping and the nonlinear wave equation (NLW), we study the behavior of solutions that decompose into a single soliton or into sums of solitons, when time goes to infinity.

First, we show the conditional stability of multi-solitons for the 1D NLKG equation with double power nonlinearity. Second, for the energy-critical NLW equation, we prove the existence of different types of multi-solitons based on the ground state or on suitable excited states, under various conditions on the space dimension and Lorentz speeds. Finally, we study the long-time dynamics of solitons and multi-solitons of the damped energy sub-critical NLKG equation.

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