Partial compactifications of the complement of an arrangement of lines and fundamental groups

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Compactifications partielles du complément d'un arrangement de droites et groupes fondamentaux

Partial compactifications of the complement of an arrangement of lines and fundamental groups

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Abstract. We study the fundamental group of certain partial compactifications of the complement of an arrangement of lines in the complex projective plane.

First, by modifying a method of Randell, we give a presentation for the fundamental group when the arrangement is defined by real linear forms. We use this to give a negative answer to a question posed by Eyssidieux to the effect that the first integral homology group of such surfaces is finite if and only if the fundamental group is finite.

After that, in order to study certain partial compactifications related to isotrivial fibrations to curves, a generalization of a structure theorem for the fundamental group of quotients of products of curves of Bauer-Catanese-Grunewald-Pignatelli is given.

Finally, we extend the presentation obtained in the case of a real arrangement to the case of a complex arrangement and to a more general type of partial compactifications. For one such surface, we compare this presentation with that of its fundamental group at infinity and we show that the first can be obtained from the latter by adding some relations. We obtain as a consequence a presentation for the fundamental group of certain homology planes constructed from arrangements of lines.

Résumé. Dans cette thèse, on étudie le groupe fondamental de certains compactifications partielles du complément d'un arrangement de droites dans le plan projectif complexe.

D'abord, on modifie une méthode utilisée par Randell pour obtenir une présentation du groupe fondamental de telles compactifications partielles quand l'arrangement est définie par des formes linéaires réels. On utilise cette présentation pour donner une réponse négative à une question posée par P. Eyssidieux demandant si le premier groupe d'homologie d'une telle surface est fini si et seulement si son groupe fondamental l'est.

Après, motivé par l'étude de certains compactifications partielles reliées à des fibrations isotriviales, on généralise un théorème de structure du groupe fondamental d'un quotient d'un produit de courbes dû à Bauer-Catanese-Grunewald-Pignatelli.

Finalement, on généralise la présentation obtenue dans le cas d'un arrangement réel au cas d'un arrangement complexe et à un type plus général de compactifications partielles. Pour une telle surface, on compare cette présentation avec celle du groupe fondamental à l'infini et on montre qu'on peut obtenir la première en ajoutant certains relations à la deuxième. On obtient comme conséquence une présentation pour le groupe fondamental de certains plans d'homologie provenant d'un arrangement de droites.

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Contents

Introduction

A Context

A.1 Some results about fundamental groups in algebraic geometry

Around 1930, Zariski [Zar29], influenced by the ideas of S. Lefschetz, related the fundamental group of the complement of a planar curve C to the existence of coverings of the complex projective plane $\mathbb{P}^2$ branched at C. Some consequences of this were obtained in subsequent work [Zar31],[Zar32] relating the structure of the fundamental group and the position of cusps in the plane in the former, and studying the role of the fundamental group in the classification of singularities of curves in the latter.

Thirty years later, Mumford studied the local fundamental group, the fundamental group of a pointed neighborhood B^* around an isolated singular point p in a normal complex surface S. He obtained a partial presentation for $\pi_1(B^*)$ and derived the following Theorem.

Theorem A.1 ([Mum61])**.** *If S is a manifold at p then $\pi_1(B^*) = 1$, if $\pi_1(B^*) = 1$ then p is a simple point of S.*

A more systematic study of the fundamental group of smooth complex algebraic varieties was started with the so-called Serre problem: determining which finitely presentable groups can arise as fundamental groups of smooth complex algebraic varieties or more generally (in the compact case) the study of fundamental group of compact Kähler manifolds. With the development of Hodge theory [Hod52],[Del71],[Del74], new perspectives were opened to carry-out this study: for some restrictions on the algebraic case see [Mor78],[JR87], for a survey of the Kähler case see [ABC$^+$96] or [Bur11]. For this work, it is important to remark that already in [ABC$^+$96, p.8] it is noted that a very few methods to study the Serre problem do not make any use of linear representations. A topic that is not included in *op. cit.*, that also depends heavily on linear representations, and that is indirectly related to the present work is the so-called Shafarevich conjecture to the effect that the universal cover of a smooth projective algebraic variety may be holomorphically convex. The following Theorem sets the conjecture in the affirmative case under certain hypothesis over the existence of linear resentations of the fundamental group.

Theorem A.2 ([EKPR12]). *Let X be a smooth complex projective variety. The Shafarevich conjecture for X holds whenever $\pi_1(X)$ has a faithful finite-dimensional complex linear representation.*

Note that there are some few examples of fundamental groups of projective varieties which are non-residually finite [Tol93],[BCC92], in particular non-linear by a result of Mal'cev, they both admit a non-faithful representation with infinite image and satisfy the Shafarevich conjecture.

A.2 Motivation and main object of study

As mentioned above, much of the study of the Serre problem is made through representations of the fundamental group. However, in some cases it can be tracked purely by presentations of this fundamental group, and in fewer cases some information about the group itself can be deduced from this presentation.

Here, we will be interested in questions related to the fundamental group of branched coverings of the complex projective plane $\mathbb{P}^2$ along an arrangement of lines $\mathscr{A}$.

Hirzebruch, motivated by the extremal case in the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$, studied in [Hir83] abelian branched coverings of $\mathbb{P}^2$ ramified over an arrangement of lines $\mathscr{A} \subset \mathbb{P}^2$ giving some examples where the equality hold. He needed very little information about the complement $\mathbb{P}^2 \setminus \mathscr{A}$ for his purposes (only the euler number). Later, E. Hironaka in [Hir93], uses in a crucial way a presentation of the fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ in order to compute the first Betti number of these coverings.

There is a good amount of information around these surfaces (see [BHH87], [Tre16]), however, as remarked in [Eys17], it seems to be no systematic study of their fundamental group. There, some sufficient conditions for the cover to be simply connected are given. Let us review the strategy followed there:

For $N \in \mathbb{N}^*$ the homomorphism $H_1(\mathbb{P}^2 \setminus \mathscr{A}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \mathscr{A}, \mathbb{Z}/N\mathbb{Z})$ induces an abelian covering $Y \to \mathbb{P}^2$ with Galois group $G = (\mathbb{Z}/N\mathbb{Z})^{|\mathscr{A}|-1}$. The surface Y is singular above the singular points of $\mathscr{A}$ of multiplicity 3 or higher. Denote by P the set of these points and by $\pi : \mathrm{Bl}_P \mathbb{P}^2 \to \mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ at P. The pull-back $\pi^* Y$ is the minimal desingularization of Y [Hir83, p. 122]. We can consider the orbifold $\mathcal{X}_N(\mathscr{A}) := [\pi^* Y / G]$, as a Deligne-Mumford stack and take its fundamental group (See [Noo05]). The main result of [Eys17] with respect to the fundamental group of the Hirzerbruch surfaces is the following.

Theorem A.3 ([Eys17]). *Let $\mathscr{A}$ be an arrangement with only double and triple points, then $\pi_1(\mathcal{X}_2(\mathscr{A})) = (\mathbb{Z}/2\mathbb{Z})^{|\mathscr{A}|-1}$.*

The surface $\pi^* Y$ can be identified with the covering of stacks given by the map $\eta : \pi_1(\mathcal{X}_N(\mathscr{A})) \to (\mathbb{Z}/N\mathbb{Z})^{|\mathscr{A}|-1}$, and therefore we can identify $\ker \eta$ with $\pi_1(\pi^* Y)$.

It is worthwhile to mention one of the main Theorems of op. cit.

Theorem A.4 ([Eys17]). *The Hirzerbruch covering surfaces corresponding to* $\ker \eta$ *satisfy the Shafarevich conjecture.*

By an appropriate version of the Seifert-van Kampen Theorem we can see that $\pi_1(\mathcal{X}_N(\mathscr{A}))$ can be computed by taking the quotient of $\pi_1(\mathrm{Bl}_P \mathbb{P}^2 \setminus \mathscr{A})$ by the normal subgroup generated by certain powers of the meridians around the irreducible components of $D = \pi^*\mathscr{A}$, with the powers depending on the isotropy at the generic point of these irreducible componentes. Therefore, once we have a presentation for $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ a presentation for $\pi_1(\mathcal{X}_N(\mathscr{A}))$ is obtained once we have the answer to the following question:

Question 1. *How does one obtain explicit expression of the meridians around the exceptional divisors coming from* P *in terms of certain meridians around the lines in* $\mathscr{A}$*?*

In fact, the answer to this question will give a presentation of the fundamental group of an orbifold $\mathcal{X}(\bar{X}, D, r = (r_1, \ldots, r_s))$ with non-trivial isotropy groups over the generic point of the irreducible components D_i of a divisor $D = \sum D_i \subset \bar{X}$ and arbitrary weights $r_i \in \mathbb{N}^*$ (See 1.1.2).

Already Hirzebruch, with all weights r_i being equal, remarked that his construction was related to some surfaces constructed by Deligne-Mostow [DM86] as quotients of the two ball by an uniform lattice in $PU(2,1)$ (to questions related to the arithmeticity of this constructions see [Hol86], [Der19]). By allowing unequal weights more surfaces of Deligne-Mostow were constructed starting from an arrangement of lines $\mathscr{A}$, see [BHH87],[Tre16].

This was one motivation to introduce the following surfaces whose fundamental group is the main subject of study in this work: Let $\mathscr{A} \subset \mathbb{P}^2$ be an arrangement of lines, let us denote by $\mathrm{Sing}\,\mathscr{A}$ its set of singular points. Consider the blow-up $\pi : \mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}} \mathbb{P}^2 \to \mathbb{P}^2$, the divisor $D = \pi^*\mathscr{A} = \sum_{i=1}^{s} D_i$ with $s = |\mathscr{A}| + |\mathrm{Sing}\,\mathscr{A}|$, and let $I \subset \{1, \ldots, s\}$.

Definition A.1. We call the surface $\mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}} \mathbb{P}^2 \setminus \sum_{i \notin I} D_i$ a Linear Arrangement (Partial) Compactification of $\mathbb{P}^2 \setminus \mathscr{A}$ or a *LAC surface*.

Another reason to study these surfaces is the research of possible instances of the following question. See [Eys18] for further motivation related to the Shafarevich conjecture.

Question 2. *Does there exists a smooth algebraic variety* X *with infinite fundamental group such that every linear representation* $\pi_1(X) \to GL_N(\mathbb{C})$ *has finite image?*

No example seems to be known (c.f. with the last paragraph of A.1). My advisor, P. Eyssidieux, didn't know what was the answer to the following question:

Question 3. *For a LAC surface* X*, if its first integral homology group* $H_1(X, \mathbb{Z})$ *is finite, does it follows that* $\pi_1(X)$ *is as well finite?*

B Results

B.1 Arrangement of lines defined over the reals

Context

The theory of hyperplanes arrangements is a very wide area of mathematics by itself. For a classical introduction we refer to [OT92], and for a more recent treatement see [Dim17].

Definition B.1. Let $\mathscr{A}^{\text{aff}} = \{H_1, \dots, H_n\} \subset \mathbb{C}^l$ be a finite collection of affine hyperplanes. We call $\mathscr{A}^{\text{aff}}$ an (affine) arrangement of hyperplanes.

By homogenizing the hyperplanes $H_1, \dots, H_n$, and considering the hyperplane at infinite H_{n+1}, we can construct a hyperplane arrangement $\mathscr{A} \subset \mathbb{P}^l$ such that $\mathbb{P}^l \setminus \mathscr{A} \cong \mathbb{C}^l \setminus \mathscr{A}^{\text{aff}}$. The following theorem of Zariski of Lefschetz type reduce the study of the fundamental group of $\mathbb{P}^l \setminus \mathscr{A}$ to the study of the complement of an arrangement of lines in the projective plane.

Theorem B.1 (Zariski)**.** *Let Y be a hypersurface of $\mathbb{P}^l$ and let H be a generic hyperplane with respect to Y. If $l > 2$ then $\pi_1(\mathbb{P}^l \setminus Y) \cong \pi_1(H \setminus H \cap Y)$.*

It was first stated in [Zar37], for a precise definition of the word generic, and more far-reaching generalizations see [HTL73], [Dim92].

When the lines of an arrangement $\mathscr{A}^{\text{aff}} \subset \mathbb{C}^2$ are defined by linear forms with real coefficients we say that $\mathscr{A}$ is a (complexified) *real arrangement*. Let us write $\operatorname{Sing} \mathscr{A}^{\text{aff}}$ for the set of singular points of $\mathscr{A}^{\text{aff}}$. In [Ran85], Randell constructed from $\mathscr{A}$ a planar oriented graph $\mathscr{A}^{\text{aff}}(\mathbb{R}) \subset \mathbb{R}^2$. Besides this, he also used implicitly the following construction: around every singular point $p \in \operatorname{Sing} \mathscr{A}^{\text{aff}}$ let B_p be a sufficiently small neighborhood and consider the link $\partial B_p \cap \mathscr{A}^{\text{aff}}$ inside ∂B_p, he used the presentation of $\pi_1(\partial B_p \setminus \partial B_p \cap \mathscr{A}^{\text{aff}})$ to relate the meridians around the segments of $\mathscr{A}^{\text{aff}}(\mathbb{R}) \setminus \operatorname{Sing} \mathscr{A}^{\text{aff}}$.

Theorem B.2 (Randell)**.** *The Wirtinger presentation of the spatial graph $\mathscr{A}^{aff}(\mathbb{R})$ and a presentation $\pi_1(\partial B_p \setminus \partial B_p \cap \mathscr{A}^{aff})$ for every $p \in \operatorname{Sing} \mathscr{A}^{aff}$ give a presentation for $\pi_1(\mathbb{C}^2 \setminus \mathscr{A}^{aff})$.*

Real arrangements

The first chapter 1 of this thesis is devoted to the following results and were published in the article [AA20a].

Let $\mathscr{A} = \{L_1, \dots, L_k\} \subset \mathbb{P}^2$ be a (complexified) real arrangement of lines. Let $\pi : \bar{X} = \operatorname{Bl}_{\operatorname{Sing} \mathscr{A}} \mathbb{P}^2 \to \mathbb{P}^2$ be the blow-up at the points $\operatorname{Sing} \mathscr{A}$ in $\mathbb{P}^2$, the divisor $D = \pi^* \mathscr{A} = \sum_{i=1}^{s} D_i$ and $r = (r_1, \dots, r_s) \in \mathbb{N}^s$. We can consider the orbifold $\pi_1(\mathcal{X}(\bar{X}, D, r))$ as in A.2.

Theorem B.3. *A presentation for $\pi_1(\mathcal{X}(\bar{X}, D, r))$ can be obtained by modifying the method of Randell and adding to his presentation powers of* explicit *words in its generators.*

Indeed, the method of Randell can be seen as sweeping a vertical line from right to left in $\mathbb{R}^2 \subset \mathbb{C}^2 \cong \mathbb{P}^2 \setminus L_k$, this is part of a pencil of lines based at a point lying in the line at infinity L_k, as we keep track of the relations and conjugations arising when crossing a singular point in $\operatorname{Sing} \mathscr{A}^{\text{aff}}$.

This method takes care of the singular ponts in $\mathbb{C}^2$ but not of these lying in the line at infinity L_k. Thus it is sufficient to apply the same method for a pencil based at a well-chosen point in $\mathbb{C}^2 \setminus \mathscr{A}^{\text{aff}}$ together with an appropriate choose of meridians.

In order to reduce the number of surfaces obtained as partial compactifications to be studied we prove in 1.3.2 that it suffices to partially compactify with respect to exceptional divisors up to having studied first all the arrangement of less lines.

Then, we present an answer to question 3.

Theorem B.4. *There exists a LAC surface Y with infinite fundamental group and finite first (integral) homology group.*

The fundamental group obtained is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and therefore has a faithful linear representation in $GL_2(\mathbb{C})$ and so it sheds no light to the more subtle question 2.

Theorem B.4 is obtained from a partial compactification of a very interesting-by-its-own arrangement called the complete quadrilateral $\mathcal{B}$ or Ceva(2) which can be seen as the union of the six lines passing by 4 points in general position in $\mathbb{P}^2$ and the application of Theorem B.3.

The arrangement $\mathcal{B}$ induces a fibration $\operatorname{Bl}_{4-\text{points}} \mathbb{P}^2 \to \mathbb{P}^1$ having 3 singular fibers, and it is therefore a multinet (c.f. B.2). This map can be extended to a map $Y \to \mathcal{X}(\mathbb{P}^1, D', r')$ from the LAC surface Y to the orbifold over $\mathbb{P}^1$ with divisor $D' = 0 + 1 + \infty$ and weights $r' = (2, 2, \infty)$, which induces an isomorphism in the fundamental groups.

Theorem B.3 also gives a presentation for the fundamental groups of the orbifolds $\mathcal{X}(\operatorname{Bl}_{4-\text{points}} \mathbb{P}^2, \pi^*\mathcal{B}, r)$ which for certain weights r can be seen as quotients of the ball B^2 by an uniform lattice as in A.2.

B.2 Structure theorems for the fundamental group of a quotient of a product

Context

There exist a class of arrangements of lines $\mathscr{A} \subset \mathbb{P}^2$ which are called nets, or more generally multinets (see [Yuz04],[FY07],[MB09]) that naturally gives a fibration from a blow-up of the projective plane $\pi : X = \operatorname{Bl}_P \mathbb{P}^2 \to \mathbb{P}^2$ with $P \subset \operatorname{Sing} \mathscr{A}$ to the projective line $p : X \to \mathbb{P}^1$ with the property that there exists a partition of $\mathscr{A} = \cup_{i=1}^k \mathscr{A}_i$ in subarrangements $\mathscr{A}_i$ such that the map p has the strict transform of the subarrangements $\mathscr{A}_i$ in X as singular fibers.

For every $n \in \mathbb{N}^*$ define the arrangement Ceva(n) to be the zero locus of the homogeneous polynomial $f(z_1, z_2, z_3) = (z_1^n - z_2^n)(z_1^n - z_3^n)(z_2^n - z_3^n)$ in

$\mathbb{P}^2$ with homogeneous coordinates $(z_1 : z_2 : z_3)$. Every Ceva(n) is a multinet and it seems natural to try to study its partial compactifications in order to generalize the results of B.1. We have two inconvenients: already Ceva(3) is not a real arrangement and the complexity of the computations increase with the number of lines. For these reasons another method other than the use of presentations had to be found.

It is easy to see that the fibration given by Ceva(n) is isotrivial, and therefore we can hope for an structure theorem for the fundamental group of its partial compactifications in the lines of [BCGP12] to exists.

We recall here some results of [BCGP12].

Definition B.2. Let C be a smooth projective curve of genus g. The group $\Pi_g = \pi_1(C)$ is called a surface group.

Let $C_1, \ldots, C_k$ be smooth projective curves and let G be a finite group acting on each curve C_i and freely in the product $C_1 \times \cdots \times C_k$ with the diagonal action. Then we have an exact sequence

$$1 \to \pi_1(C_1) \times \ldots \times \pi_1(C_k) \to \pi_1\left(\frac{C_1 \times \cdots \times C_k}{G}\right) \to G \to 1 \qquad (1)$$

If we assume that G acts faithfully on each factor C_i and remove the hypothesis of the action being free in the product, there is a priori, no reason for a sequence similar to (1) to hold. However, the following theorem was proved in [BCGP12].

Theorem B.5 ([BCGP12])**.** *Let $C_1, \ldots, C_k$ be smooth projective curves of genus greater or equal than two and let G be a finite group acting faithfully on each of the factors and diagonally in the product. Then the fundamental group $\pi_1((C_1 \times \ldots \times C_k)/G)$ admits a normal finite index subgroup $\mathcal{N} \cong \Pi_{h_1} \times \ldots \times \Pi_{h_k}$ for some $h_1, \ldots, h_k \in \mathbb{N}$ and $\Pi_{h_1}, \ldots, \Pi_{h_k}$ surface groups.*

The hypothesis of the action being faithful was later removed in [DP12].

Note that the quotient $(C_1 \times \ldots \times C_k)/G$ may be singular. In the case of only two curves C_1, C_2, if it happens that the divisors obtained by resolving the singularities of $(C_1 \times C_2)/G$ are strict transforms of lines of an arrangement $\mathscr{A} \subset \mathbb{P}^2$ some partial compactifications of the type $(C_1 \setminus \{k_1 - \text{points}\} \times C_2 \setminus \{k_2 - \text{points}\})/G$ could be studied by extending Theorem B.5 to the case of open surfaces.

We mention briefly the methods of proof of Theorem B.5 as in [BCGP12] with a slight change of terms by using the notations of stacks. The proof can be divided in two parts. To explain the first we can consider the orbifold fundamental group of the curve C_i/G, this is $\pi_1([C_i/G])$. There is an exact sequence

$$1 \to \pi_1(C_i) \to \pi_1([C_i/G]) \to G \to 1,$$

and therefore we can construct the fiber product $\mathbb{H} := \pi_1([C_1/G]) \times_G \ldots \times_G \pi_1([C_k/G])$. Let $\operatorname{Tors}\mathbb{H}$ denote the normal subgroup of $\mathbb{H}$ generated by the elements of torsion. The first main step was to show that

$$\pi_1((C_1 \times \ldots \times C_k)/G) \cong \mathbb{H}/\operatorname{Tors}\mathbb{H}. \tag{2}$$

To explain the second part we need the following definition.

Definition B.3. An (abstract) orbifold surface group is a group admitting a presentation of the form

$$T_g = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k \mid c_1^{n_1} = \ldots = c_k^{n_k} = 1, \prod_{i=1}^{g}[a_i, b_i]c_1 \ldots c_k = 1\rangle$$

with $n_i \in \mathbb{N}$ and $[a, b] = aba^{-1}b^{-1}$ the commutator of a and b.

Then the second step relies in the following Proposition whose proof is of group theoretic nature.

Proposition B.6 ([BCGP12, DP12])**.** *There exists an exact sequence of groups*

$$1 \to E \to \mathbb{H}/\operatorname{Tors}\mathbb{H} \to T \to 1$$

where E is a finite group and T is a finite-index subgroup of a product of orbifold surfaces group $\prod_{i=1}^{k} T_{h_i}$.

From this and (2) it is easy to show Theorem B.5 once the residual finiteness of $\mathbb{H}/\operatorname{Tors}\mathbb{H}$ is proven using methods of [GJZZ08].

Generalizing the structure theorem for quotient of product of curves

The second chapter of this thesis 2 is devoted to the following results appearing in the pre-publication [AA20b].

The Proposition B.6 can be generalized as follows: let $X_1, \ldots, X_k$ be topological spaces admitting an universal cover; they are connected, locally path-connected, and semi-locally simply connected. Denote the product of them by $X = X_1 \times \ldots \times X_k$, and let G be a finite group acting on each X_i for $i = 1, \ldots, k$, and consider the diagonal action on X. Let $I < G$ be the subgroup generated by the elements fixing a point in every X_i for $i = 1, \ldots, k$. Note that I is normal. Denote by X_i^g the subset of points of X_i that are fixed by $g \in G$.

Theorem B.7. *Let $X, X_1, \ldots, X_k, G$, and I as above. Suppose that the set $\pi_0 X_i^g$ of path connected components of X_i^g is finite for every $i = 1, \ldots, k$, and every $g \in G$. Then there exists a homomorphism*

$$\pi_1(X/G) \to \prod_{i=1}^{k} \pi_1([(X_i/I)/(G/I)])$$

whose image has finite index and whose kernel is finite.

The action of G/I over X_i/I is that induced by the one of G over X_i.

It is worthwhile to mention the case of taking the product of the same topological space, this is, $X_i = X_1$ for $i = 2, \ldots, k$. Note that in this case the action of G/I in X_i/I is free and therefore the fundamental group of the stack $[X_i/I/G/I]$ coincides with the topological fundamental group of $X_i/G/G/I$. Moreover, it can be seen that in this case, as topological spaces, we have $X_i/I/G/I \cong X_i/G$. We obtain therefore the following corollary.

Corollary B.8. *Let $X = X_1^k$, G, and I as above. Then the map $\pi_1(X/G) \to \pi_1(X_1/G)^k$ has finite kernel and its image has finite index in $\pi_1(X_1/G)^k$.*

For the proof of Theorem B.7 we extend the arguments of [BCGP12] to the stacky context. We consider the action of $\pi_1([X/G])$ over the universal cover of X, and compare it with the action of $\pi_1([X_1/G]) \times \ldots \times \pi_1([X_k/G])$. Under the hypothesis made over $\pi_0 X_i^g$, we can construct subgroups $C_i < \pi_1([X_i/G])$ that lift to subgroups of $\pi_1([X/G])$, and whose product $C_1 \times \ldots \times C_k$ has finite index in the subgroup of $N < \pi_1([X/G])$ generated by the elements having a fixed point in the universal cover of X. We conclude by a Theorem of Noohi [Noo08] to the effect that $\pi_1([X/G])/N \cong \pi_1(X/G)$ and by interpreting geometrically the projection of N over each $\pi_1([X_i/G])$.

An important instance of Theorem B.7 and Corollary B.8 lies in the study of fundamental group of smooth complex algebraic varieties, indeed, the fundamental group of a singular algebraic variety with only quotient singularities is the fundamental group of a smooth algebraic variety [Kol93].

There is also a generalization of Theorem B.5.

Theorem B.9. *Suppose that the hypothesis of Theorem B.7 are satisfied, and moreover that $\pi_1(X/G)$ is residually finite. Then there exists a subgroup $\mathcal{N} < \pi_1(X/G)$ isomorphic to a product $H_1 \times \ldots \times H_k$ of normal subgroups $H_i \lhd_{f.i.} \pi_1(X_i/I)$ of finite index.*

Following closely the arguments of [BCGP12], which use the group theoretic results of [GJZZ08], we can prove that $\pi_1(C_1 \times \ldots \times C_k/G)$ is residually finite for a finite group G, and smooth algebraic curves $C_1, \ldots, C_k$ not necessarily compact.

In order to apply the Theorem B.9 to the family Ceva(n) we prove the following proposition. Denote the group $H(n) = \mathbb{Z}/n\mathbb{Z}$, it acts over the Fermat curve of degree n defined by $F(n) = z_1^n + z_2^n + z_3^n$. Denote by $\bar{f}$ the fibration induced by the multinet Ceva(n).

Proposition B.10. *Consider the diagonal action of $H(n)$ in $F(n) \times F(n)$. Denote by S the minimal resolution of $F(n) \times F(n)/H(n)$.*

1. *The fibration $S \to (F(n) \times F(n))/H(n) \to F(n)/H(n) \cong \mathbb{P}^1$ is isomorphic to $\tilde{f}$.*

2. *Every singular point in $F(n) \times F(n)/H(n)$ corresponds to the contraction of the strict transform D_i of some line $L_i \in$ Ceva(n).*

3. *The contraction of the n lines corresponding to $\mathscr{A}_i$ lie in the line E_i which is the exceptional divisor corresponding to the unique singular point in $\mathscr{A}_i$.*

4. *E_i maps to a point via $F(n) \times F(n)/H(n) \to \mathbb{P}^1$.*

We give the proof in 2.3.3. This proof was not included in [AA20b].

By using proposition B.10 and Theorem B.9 we are able to study two families of partial compactifications of $\mathbb{P}^2 \setminus \text{Ceva}(n)$. In the first family we obtain that the fundamental group is at most finite, and in the second it has a subgroup isomorphic to $\mathbb{Z}$ as a finite-index normal subgroup. See examples 2.3.1 and 2.3.2.

B.3 Partial compactifications of the complement of a complex arrangement and boundary manifolds

In Theorem B.3, we have developed a method to obtain a presentation for the fundamental group of certain partial compactifications of the complement $\mathbb{P}^2 \setminus \mathscr{A}$ of an arrangement of lines $\mathscr{A} \subset \mathbb{P}^2$ under the hypothesis that the lines in $\mathscr{A}$ are defined by real linear forms.

Chapter 3 of this thesis is devoted to a generalization of this in two directions:

- to admit a general arrangement $\mathscr{A} \subset \mathbb{P}^2$ defined by complex linear forms and
- to admit a more general class $M(\mathscr{A}, I, P)$ of partial compactifications of $\mathbb{P}^2 \setminus \mathscr{A}$. See 3.1.3 for a precise definition.

We can proceed in two different ways: firstly, following [Arv92] and [CS97], whose work generalize [Ran85] to complex arrangements, we define a braided wiring diagram $\mathcal{W}$ that encodes some over or under-crossing of the lines in $\mathscr{A}$ arising by the complex nature of the forms defining them. The graph $\mathcal{W}$ encodes enough information to obtain a presentation of $\pi_1(M(\mathscr{A}, I, P))$.

Theorem B.11. *A presentation for $\pi_1(M(\mathscr{A}, I, P))$ can be obtained from $\mathcal{W}$. The set of generators are in correspondence with the set of lines in $\mathscr{A}$ and the set of relations has two types of them:*

- *those relations R_p coming from a singular point p of $\mathscr{A}$. These relations already appeared in a presentation of $\pi_1(M(\mathscr{A}))$ and*
- *for each element ι either in I or in P, a relation R_ι which is a product of conjugates of some generators depending on ι.*

As in the proof of Theorem B.3, a main step in the proof of Theorem B.11 consists in the explicit computation of an expression for the meridians around

certain exceptional divisors, obtained by blowing-up $\mathscr{A}$ in some singular points, in terms of the generators.

Secondly, let U denote a closed regular tubular neighborhood of $\mathscr{A}$ in $\mathbb{P}^2$. We call ∂U the boundary manifold of $\mathscr{A}$. In [FGBMB15], a presentation for $\pi_1(M(\mathscr{A}))$ is obtained from a presentation of $\pi_1(\partial U)$ by studying the map $\pi_1(\partial U) \to \pi_1(M(\mathscr{A}))$ induced by the inclusion $\partial U \hookrightarrow M(\mathscr{A})$. For the homological version see [GB13].

It turns out that their methods can also be applied to determine a presentation for the fundamental group of some partial compactifications $M(\mathscr{A}, I, P)$. However, in order to study the boundary manifolds ∂U_D of strict transforms D of $\mathscr{A}$ in some birational model of $\mathbb{P}^2$, we start from a different presentation for the boundary manifold ∂U of $\mathscr{A}$.

Indeed, when $D = \sum D_i$ is a connected, simple normal crossing divisor such that $\pi_1(D)$ is trivial, Mumford gave a presentation for $\pi_1(\partial U_D)$ in [Mum61]. This, together with the graph-manifold structure in the sense of Waldhausen [Wal67], permitted Westlund to give a presentation of $\pi_1(\partial U)$ in [Wes67] (see also [CS08]). Here, by a choice of a surface birational to $\mathbb{P}^2$ where the strict transform of $\mathscr{A}$ satisfies the hypothesis for the presentation of Mumford, we obtain the same presentation of Westlund. See Theorem 3.3.1. Following this construction, we are able to give a presentation for the fundamental group of a boundary manifold ∂U_D of a divisor D lying in a surface $\bar{X}$ obtained by successive blows-up of $\mathbb{P}^2$ such that $M(\mathscr{A}, I, P) = \bar{X} \setminus D$.

We obtain in Theorem 3.3.3 a presentation for $\pi_1(M(\mathscr{A}))$ by studying the map $i_* : \pi_1(\partial U) \to \pi_1(M(\mathscr{A}))$. Moreover, as the construction for $\pi_1(\partial U)$ depends of several choices, we can make them in such a way that the image under i of the meridians of the lines in $\mathscr{A}$ lying in ∂U, whose homotopy class are part of the generators of $\pi_1(\partial U)$, lie in the same homotopy class as the meridians constructed for Theorem B.11. From this, we do not only obtain that the presentation of Theorem 3.3.3 and B.11 are equivalent, but that the image of the set of relation in the presentation of $\pi_1(M(\mathscr{A}))$ coincides with the relations as in Theorem B.11. From this, we can obtain a presentation for partial compactifications $\pi_1(M(\mathscr{A}, I, P)$, see Theorem 3.3.14.

Finally, inspired by [tDP93], we present an example of a partial compactification $M(\mathscr{A}, I, P) = \bar{X} \setminus D$ that comes from an arrangement of 5 lines with two triples points and four double points such that $M(\mathscr{A}, I, P)$ is a $\mathbb{Q}$-homology plane.

Introduction (version française)

A Contexte

A.1 Quelques résultats sur le groupe fondamental en géométrie algébrique

Environ 1930, Zariski [Zar29], influencé par les idées de S. Lefschetz, a relié le groupe fondamental du complément d'une courbe plane C, à l'existence de revêtements du plan projectif $\mathbb{P}^2$ ramifié sur C. Quelques conséquences de ce travail ont étés obtenu par la suite [Zar31],[Zar32] reliant la structure du groupe fondamental et la position des cuspides dans le plan projectif dans le premier et étudiant le rôle du groupe fondamental dans la classification de singularités des courbes dans le deuxième.

Trente ans plus tard, Mumford a étudié le groupe fondamental local, le groupe fondamental d'un voisinage épointé B^* autour d'une singularité isolée p dans une surface normale S. Il a obtenu une présentation partielle pour $\pi_1(B^*)$ et comme conséquence il a obtenu le Théorème suivante.

Theorem A.1 ([Mum61])**.** *Si $\pi_1(B^*) = 1$ alors p est un point simple de S.*

Une étude plus systématique du groupe fondamental de variétés algébriques complexes lisses a commencé avec le problème de Serre: déterminer quels groupes peuvent apparaître comme groupe fondamental d'une variété algébrique complexe lisse ou plus généralement (dans le cas compacte) l'étude de variétés de Kähler compactes. Avec les développements de la théorie de Hodge [Hod52], [Del71], [Del74], de perspectives nouvelles ont été ouvertes pour réaliser cette étude: pour certaines restrictions dans le cas algébrique voir [Mor78],[JR87], pour un survey du cas Kählerienne voir [ABC$^+$96] or [Bur11].

Pour ce travail, il est important de mentionner que déjà dans [ABC$^+$96, p.8] il est remarqué que très peu de méthodes pour étudier le problème de Serre n'utilisent aucune représentation linéaire. Un sujet qui n'est pas traité dans *op. cit.*, qui dépend aussi fortement des représentations linéaires et qui est indirectement relié au travail actuel est la conjecture de Shafarevich, à savoir que le revêtement universel d'une variété algébrique complexe lisse pourrait être holomorphiquement convexe. Le Théorème suivant répond cette conjecture sur certains hypothèses sur l'existence d'une représentation linéaire du groupe fondamental.

Theorem A.2 ([EKPR12]). *Soit X une variété projective complexe lisse. La conjecture de Shafarevich pour X est vraie si $\pi_1(X)$ admet une représentation linéaire fidèle de dimension finie.*

On peut remarquer qu'il y a quelques exemples de groupes fondamentaux des variétés projectives lisses qui ne sont résiduellement finis [Tol93],[BCC92], en particulière non linéaires par un résultat de Mal'cev. Ces exemples admettent représentations linéaires avec image infinie et satisfont la conjecture de Shafarevich.

A.2 Motivation et surfaces à étudier

Comme mentionné plus haut, une grande partie de l'étude du problème de Serre se fait en utilisant les représentations linéaires du groupe fondamental. Cependant, dans certains cas ce problème peut être traité seulement avec une présentation du groupe fondamental et dans certains cas encore plus particuliers, on peut obtenir directement de cette présentation des propriétés sur le groupe.

Ici, on va s'intéresse aux certaines questions liées au groupe fondamental de revêtements du plan projectif $\mathbb{P}^2$ ramifiés au-dessus d'un arrangement de droites $\mathscr{A}$.

Hirzebruch, motivé par le cas extrémal de l'inégalité de Miyaoka-Yau $c_1^2 \leq 3c_2$, a étudié dans [Hir83] les revêtements abéliens de $\mathbb{P}^2$ ramifiés au-dessus d'un arrangement de droites $\mathscr{A} \subset \mathbb{P}^2$. Il a donné des exemples où l'égalité est obtenue. Il a eu besoin de très peu d'information sur le complément $\mathbb{P}^2 \setminus \mathscr{A}$ pour ces propos (seulement le nombre d'Euler). Après, E. Hironaka dans [Hir93], a utilisé dans d'une manière essentielle une présentation du groupe fondamental $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ afin de calculer le premier nombre de Betti de ces revêtements.

Il y a une bonne quantité d'information autour de ces surfaces (voir [BHH87], [Tre16]), néanmoins, comme indiqué dans [Eys17], il ne semble pas y avoir une étude systématique de leur groupe fondamental. On y trouve des conditions suffisantes pour que le revêtement soit simplement connexe. Expliquons la stratégie suivie:

Pour $N \in \mathbb{N}^*$ l'homomorphisme $H_1(\mathbb{P}^2 \setminus \mathscr{A}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \mathscr{A}, \mathbb{Z}/N\mathbb{Z})$ induit un revêtement abélien $Y \to \mathbb{P}^2$ avec groupe de Galois $G = (\mathbb{Z}/N\mathbb{Z})^{|\mathscr{A}|-1}$. La surface Y est singulière au-dessus des points singuliers de $\mathscr{A}$ de multiplicité 3 ou plus. Notons par P l'ensemble de ces points et par $\pi : \mathrm{Bl}_P \mathbb{P}^2 \to \mathbb{P}^2$ l'éclaté de $\mathbb{P}^2$ sur P. Le tire-en arrière π^*Y est la désingularisation minimale de Y [Hir83, p. 122]. On peut considérer l'orbifold $\mathcal{X}_N(\mathscr{A}) := [\pi^*Y/G]$, comme un champ de Deligne-Mumford et on peut prendre son groupe fondamental (voir [Noo05]). Le résultat principal de [Eys17] par rapport au groupe fondamental de surfaces de Hirzebruch est le suivant.

Theorem A.3 ([Eys17]). *Soit $\mathscr{A}$ un arrangement qui a seulement de points singuliers doubles et triples, alors $\pi_1(\mathcal{X}_2(\mathscr{A})) = (\mathbb{Z}/2\mathbb{Z})^{|\mathscr{A}|-1}$.*

La surface π^*Y peut être identifié avec le revêtement de champs donné par l'homomorphisme $\eta : \pi_1(\mathcal{X}_N(\mathscr{A})) \to (\mathbb{Z}/N\mathbb{Z})^{|\mathscr{A}|-1}$, et alors on peut identifier $\ker \eta$ avec $\pi_1(\pi^*Y)$.

Il est intéressant de mentionner l'un des principaux Théorèmes de l'*op. cit.*

Theorem A.4 ([Eys17])**.** *Les surfaces de Hirzerbruch correspondant à* $\ker \eta$ *satisfont la conjecture de Shafarevich.*

Par une version appropriée du Théorème de Seifert-van Kampen Theorem on peut voir que $\pi_1(\mathcal{X}_N(\mathscr{A}))$ peut être calculé en prenant le quotient de $\pi_1(\mathrm{Bl}_P \mathbb{P}^2 \setminus \mathscr{A})$ par le sous-groupe engendré par certaines puissances de méridiens autour des composants irréductibles de $D = \pi^*\mathscr{A}$, avec les puissances qui dépend de l'isotropie sur le point générique de ces composantes irréductibles. Ainsi, si on a une présentation pour $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$, on peut obtenir une présentation pour $\pi_1(\mathcal{X}_N(\mathscr{A}))$ si on connait la réponse à la question suivante:

Question 4. *Comme on obtient une expression explicite des méridiens autour des diviseurs exceptionnels provenant de* P *en termes de ces méridiens autour des droites dans* $\mathscr{A}$ *?*

En fait, la réponse à cette question donnera une présentation pour le groupe fondamental d'un orbifold $\mathcal{X}(\bar{X}, D, r = (r_1, \ldots, r_s))$ avec groupe d'isotropie non trivial au-dessus du point générique des composantes irréductibles D_i d'un diviseur $D = \sum D_i \subset \bar{X}$ et poids arbitraires $r_i \in \mathbb{N}^*$ (Voir 1.1.2).

Hirzebruch a déjà remarqué que si on met les poids r_i tous égaux, sa construction était reliée à d'autres surfaces construit par Deligne-Mostow [DM86] comme quotients de la boule par une réseau uniforme $PU(2,1)$ (pour des questions relies à la arithméticité de cette construction voir [Hol86], [Der19]). En admettant des poids inégaux, plus des surfaces de Deligne-Mostow ont était obtenus en commençant par un arrangement de droites $\mathscr{A}$, voir [BHH87],[Tre16].

Ceci, c'était l'une des motivations pour introduire les surfaces suivantes dont le groupe fondamental est l'objet principal d'étude dans ce travail: soit $\mathscr{A} \subset \mathbb{P}^2$ un arrangement des droites, notons par $\mathrm{Sing}\,\mathscr{A}$ l'ensemble de ces points singuliers. Considérons l'éclaté $\pi : \mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}} \mathbb{P}^2 \to \mathbb{P}^2$, le diviseur $D = \pi^*\mathscr{A} = \sum_{i=1}^{s} D_i$ avec $s = |\mathscr{A}| + |\mathrm{Sing}\,\mathscr{A}|$, et soit $I \subset \{1, \ldots, s\}$.

Definition A.1. On appelle la surface $\mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}} \mathbb{P}^2 \setminus \sum_{i \notin I} D_i$ une Linear Arrangement (Partial) Compactification de $\mathbb{P}^2 \setminus \mathscr{A}$ ou une *surface LAC*.

Une autre motivation pour l'étude de ces surfaces et la recherche de possibles exemples de la question suivante. Voir [Eys18] pour motivation supplémentaire lié à la conjecture de Shafarevich.

Question 5. *Existe-t-il une variété algébrique lisse* X *avec un groupe fondamental infini tel que toute représentation linéaire* $\pi_1(X) \to GL_N(\mathbb{C})$ *a une image finie?*

Aucun exemple ne semble être connu (c.f. le dernier paragraphe de A.1). Mon directeur de thèse, P. Eyssidieux, ne savait pas quelle était la réponse à la question suivante:

Question 6. *Pour une surface LAC X, si son premier groupe d'homologie entière $H_1(X, \mathbb{Z})$ est fini, s'ensuit-il que $\pi_1(X)$ est également fini?*

B Résultats

B.1 Arrangements de droites définit sur les réels

Contexte

La théorie d'arrangements d'hyperplans est un domaine très large en soi. Pour une introduction classique on renvoie à [OT92], pour un traitement plus récent voir [Dim17].

Definition B.1. Soit $\mathscr{A}^{\text{aff}} = \{H_1, \ldots, H_n\} \subset \mathbb{C}^l$ une collection finie des hyperplans affines. On appelle $\mathscr{A}^{\text{aff}}$ un arrangement (affine) des hyperplans.

Si on homogénéise les hyperplans $H_1, \ldots, H_n$ et on considère l'hyperplan à l'infinie H_{n+1}, on peut construire un arrangement d'hyperplans $\mathscr{A} \subset \mathbb{P}^l$ tel que $\mathbb{P}^l \setminus \mathscr{A} \cong \mathbb{C}^l \setminus \mathscr{A}^{\text{aff}}$. Le Théorème suivante de Zariski du type de Lefschetz réduit l'étude du groupe fondamental de $\mathbb{P}^l \setminus \mathscr{A}$ à l'étude du complément d'un arrangement de droites dans le plan projectif.

Theorem B.1 (Zariski)**.** *Soit Y une hypersurface de $\mathbb{P}^l$ et soit H un hyperplan générique par rapport à Y. Si $l > 2$ alors $\pi_1(\mathbb{P}^l \setminus Y) \cong \pi_1(H \setminus H \cap Y)$.*

Ce Théorème a été énoncé en première dans [Zar37], pour une définition précise du mot générique, un énoncé plus général et une preuve, voir [HTL73], [Dim92].

Quand les droites de l'arrangement $\mathscr{A}^{\text{aff}} \subset \mathbb{C}^2$ sont définit par formes linéaires avec coefficients réels on dit que $\mathscr{A}$ est une *arrangement réel* (complexifié). Notons $\operatorname{Sing} \mathscr{A}^{\text{aff}}$ pour l'ensemble des points singuliers de $\mathscr{A}^{\text{aff}}$. Dans [Ran85], Randell a construit à partir de $\mathscr{A}$ une graphe plaine orientée $\mathscr{A}^{\text{aff}}(\mathbb{R}) \subset \mathbb{R}^2$. De plus, il a utilisé implicitement la construction suivante: autour de chaque point singulier $p \in \operatorname{Sing} \mathscr{A}^{\text{aff}}$ soit B_p un voisinage suffisamment petit et considérons le entrelacs $\partial B_p \cap \mathscr{A}^{\text{aff}}$ dedans ∂B_p, il a utilisé la présentation de $\pi_1(\partial B_p \setminus \partial B_p \cap \mathscr{A}^{\text{aff}})$ pour relier les méridiens autour des segments de $\mathscr{A}^{\text{aff}}(\mathbb{R}) \setminus \operatorname{Sing} \mathscr{A}^{\text{aff}}$.

Theorem B.2 (Randell)**.** *La présentation de Wirtinger du graphe $\mathscr{A}^{aff}(\mathbb{R})$ et une présentation de $\pi_1(\partial B_p \setminus \partial B_p \cap \mathscr{A}^{aff})$ pour chaque $p \in \operatorname{Sing} \mathscr{A}^{aff}$ donne une présentation pour $\pi_1(\mathbb{C}^2 \setminus \mathscr{A}^{aff})$.*

Arrangements réels

Le premier chapitre 1 de cette thèse est consacré aux résultats suivants qui ont apparu dans [AA20a].

Soit $\mathscr{A} = \{L_1, \dots, L_k\} \subset \mathbb{P}^2$ un arrangement de droites réel (complexifié). Soit $\pi : \bar{X} = \mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}}\,\mathbb{P}^2 \to \mathbb{P}^2$ l'éclaté aux points $\mathrm{Sing}\,\mathscr{A}$ dans $\mathbb{P}^2$, le diviseur $D = \pi^*\mathscr{A} = \sum_{i=1}^{s} D_i$ et $r = (r_1, \dots, r_s) \in \mathbb{N}^s$. On peut considérer l'orbifold $\pi_1(\mathcal{X}(\bar{X}, D, r))$ as in A.2.

Theorem B.3. *Une présentation pour $\pi_1(\mathcal{X}(\bar{X}, D, r))$ peut être obtenu en modifiant la méthode de Randell et en rajoutant a cette présentation puissances des mots* explicites *en ses générateurs.*

En effet, la méthode de Randell peut se voir comme une ligne verticale qui se balaye de droite à gauche dans $\mathbb{R}^2 \subset \mathbb{C}^2 \cong \mathbb{P}^2 \setminus L_k$, elle est partie d'un pinceau de droites basé sur un point appartenant à la droite à l'infini L_k, tout en prenant en compte les relations et conjugaisons obtenus quand on croise un point singulière dans $\mathrm{Sing}\,\mathscr{A}^{\mathrm{aff}}$.

Cette méthode prend en compte les points singuliers dans $\mathbb{C}^2$ mais pas ceux qui se trouvent dans la ligne à l'infini L_k. Il suffit donc d'appliquer la même méthode pour un pinceau basé sur un point bien choisi dans $\mathbb{C}^2 \setminus \mathscr{A}^{\mathrm{aff}}$ ainsi comme un choix approprié de méridiens.

Afin de réduire le nombre de surfaces obtenues comme compactifications partielles à étudier, on prouve dans 1.3.2 qu'il suffit de compactifier par rapport à des diviseurs exceptionnels quitte à avoir étudié déjà les compactifications partielles pour un arrangement de moins droites.

Ensuite, nous présentons une réponse à la question 6.

Theorem B.4. *Il existe une surface LAC Y avec groupe fondamental infini et premier groupe d'homologie (entier) fini.*

Le groupe fondamental obtenu est isomorphe à $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ et donc, il admet une représentation linéaire fidèle dans $GL_2(\mathbb{C})$ et il n'éclaire pas donc la question plus subtile 5.

Le Théorème B.4 est obtenu à partir d'une compactification partielle provenant d'un arrangement très intéressant par lui-même appelé Ceva(2) qui peut être vu comme l'union des six droites passant par quatre points en position générale dans $\mathbb{P}^2$ et l'application du Théorème B.3.

L'arrangement Ceva(2) induit une fibration $\mathrm{Bl}_{4-\mathrm{points}}\,\mathbb{P}^2 \to \mathbb{P}^1$ ayant 3 fibres singulières, il est donc une multinet (c.f. B.2). Cette application peut être prolongé à une application $Y \to \mathcal{X}(\mathbb{P}^1, D', r')$ de la surface LAC Y à l'orbifold définit sur $\mathbb{P}^1$ avec diviseur $D' = 0 + 1 + \infty$ et poids $r' = (2, 2, \infty)$, qui induit un isomorphisme dans les groupes fondamentaux.

Le Théorème B.3 donne également une présentation pour les groupes fondamentaux des orbifolds $\mathcal{X}(\mathrm{Bl}_{4-\mathrm{points}}\,\mathbb{P}^2, \pi^*\mathcal{B}, r)$ qui pour certains poids r peuvent être considérés comme quotients de la boule B^2 par un réseau uniforme comme dans A.2.

B.2 Théorèmes de structure pour le groupe fondamental d'un quotient d'un produit

Contexte

Il existe une classe d'arrangements de droites $\mathscr{A} \subset \mathbb{P}^2$ qui s'appellent nets, ou plus généralement multinets (see [Yuz04],[FY07],[MB09]). Cette classe donne naturellement une fibration $p : X \to \mathbb{P}^1$ d'un éclate du plan projectif $\pi : X = \mathrm{Bl}_P \mathbb{P}^2 \to \mathbb{P}^2$ avec $P \subset \mathrm{Sing}\,\mathscr{A}$ vers la droite projective $\mathbb{P}^1$ avec la propriété qu'il existe une partition de $\mathscr{A} = \cup_{i=1}^k \mathscr{A}_i$ en sous-arrangements $\mathscr{A}_i$ tel que l'application p a la transformé stricte des sous-arrangements $\mathscr{A}_i$ dans X comme fibres singulières.

Pour chaque $n \in \mathbb{N}^*$ définissons l'arrangement Ceva(n) comme l'ensemble de zéros du polynôme homogène $f(z_1, z_2, z_3) = (z_1^n - z_2^n)(z_1^n - z_3^n)(z_2^n - z_3^n)$ dans $\mathbb{P}^2$ avec coordonnées homogènes $(z_1 : z_2 : z_3)$. Chaque Ceva(n) est une multinet et il semble naturel d'essayer d'étudier ses compactifications partielles afin de généraliser les résultats de B.1. On a deux inconvénients: déjà Ceva(3) n'est pas un arrangement réel et la complexité des calculs augmente avec le nombre de droites. Pour ces raisons une autre méthode différent des présentations a dû être utilisé.

Il est facile à voir que la fibration donnée par Ceva(n) est isotriviale, et donc elle peut être lié à un produit de courbes. Ainsi, on peut espérer qu'il existe un théorème de structure pour le groupe fondamental de certains compactifications partielles comme dans [BCGP12].

Rappelons quelques résultats de [BCGP12].

Definition B.2. Soit C une courbe projective lisse de genre g. Le groupe $\Pi_g = \pi_1(C)$ est appelé un groupe de surface.

Soient $C_1, \ldots, C_k$ courbes projectives lisses et soit G un groupe fini qui agit sur chaque courbe C_i et agit librement sur le produit $C_1 \times \cdots \times C_k$ avec l'action diagonal. Alors on a une suite exacte exact:

$$1 \to \pi_1(C_1) \times \ldots \times \pi_1(C_k) \to \pi_1\left(\frac{C_1 \times \cdots \times C_k}{G}\right) \to G \to 1 \qquad (3)$$

Si on suppose que G agit seulement de façon effective dans chaque facteur C_i et on enlève l'hypothèse que l'action est libre dans le produit, il n'y pas de raison, a priori, pour qu'une suite similaire à (3) existe. Néanmoins, le Théorème suivante à été prouvé dans [BCGP12].

Theorem B.5 ([BCGP12])**.** *Soient $C_1, \ldots, C_k$ courbes projectives lisses de genre supérieur ou égal à deux et soit G un groupe fini qui agit effectivement sur chaque facteur et diagonalement sur le produit. Alors le groupe fondamental $\pi_1((C_1 \times \ldots \times C_k)/G)$ admet un sous-groupe distingué d'indice fini $\mathcal{N} \cong \Pi_{h_1} \times \ldots \times \Pi_{h_k}$ pour certains $h_1, \ldots, h_k \in \mathbb{N}$ et $\Pi_{h_1}, \ldots, \Pi_{h_k}$ groupes de surfaces.*

L'hypothèse d'effectivité sur l'action a été enlevé après dans [DP12].

Notons que le quotient $(C_1 \times \ldots \times C_k)/G$ peut être singulière. Dans le cas où on a seulement deux courbes C_1, C_2, si les diviseurs obtenus en résolvant les singularités de $(C_1 \times C_2)/G$ sont transformées strictes de droites d'un arrangement $\mathscr{A} \subset \mathbb{P}^2$ certaines compactifications partielles de la forme $(C_1 \setminus \{k_1 - \text{points}\} \times C_2 \setminus \{k_2 - \text{points}\})/G$ pourront être étudié en généralisant le Théorème B.5 pour le cas de surfaces ouvertes.

On mentionne brièvement les méthodes de preuve du Théorème B.5 comme dans [BCGP12] avec un léger changement de langage en utilisant la notation des champs. La preuve peut être divisé en deux parties. Pour la première on peut considérer le groupe fondamental orbifolde de la courbe C_i/G, c'est-à-dire $\pi_1([C_i/G])$. On a une suite exacte

$$1 \to \pi_1(C_i) \to \pi_1([C_i/G]) \to G \to 1,$$

et donc, on peut construire le produit fibré $\mathbb{H} := \pi_1([C_1/G]) \times_G \ldots \times_G \pi_1([C_k/G])$. Notons par $\operatorname{Tors}\mathbb{H}$ le sous-groupe distingué de $\mathbb{H}$ engendré par ces éléments de torsion. Le premier pas consiste à montrer que

$$\pi_1((C_1 \times \ldots \times C_k)/G) \cong \mathbb{H}/\operatorname{Tors}\mathbb{H}. \tag{4}$$

On a besoin de la définition suivante pour la deuxième partie de la preuve.

Definition B.3. Un groupe de surface orbifolde (abstraite) est un groupe qui admet une présentation de la forme

$$T_g = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k \mid c_1^{n_1} = \ldots = c_k^{n_k} = 1, \prod_{i=1}^{g} [a_i, b_i] c_1 \ldots c_k = 1 \rangle$$

avec $n_i \in \mathbb{N}$ et $[a, b] = aba^{-1}b^{-1}$ le commutateur de a et b.

La deuxième partie de la preuve repose sur la proposition suivante dont la preuve est de nature de théorie de groupes.

Proposition B.6 ([BCGP12, DP12])**.** *Il existe une suite exacte de groupes*

$$1 \to E \to \mathbb{H}/\operatorname{Tors}\mathbb{H} \to T \to 1$$

avec E un groupe fini et T un sous-groupe d'indice fini dans un produit de groupes de surfaces orbifoldes $\prod_{i=1}^{k} T_{h_i}$.

A partir de cette proposition et de (4) on peut montrer facilement le Théorème B.5 dès qu'on sait que $\mathbb{H}/\operatorname{Tors}\mathbb{H}$ est résiduellement fini. On preuve la dernière assertion en utilisant les méthodes de[GJZZ08].

Généralisation du Théorème de structure pour les quotients d'un produit des courbes

Le seconde chapitre de cette travail 2 porte sur les résultats qui apparaissent dans la prépublication [AA20b].

La proposition B.6 peut être généralisé ainsi: soient $X_1, \ldots, X_k$ espaces topologiques admettent un revêtement universel; ils sont connexes, localement connexes par arcs, et délaçables. Notons par $X = X_1 \times \ldots \times X_k$ le produit d'eux, soit G un groupe fini qui agisse sur chaque X_i pour $i = 1, \ldots, k$, et considère l'action diagonal de G sur X. Soit $I < G$ le sous-groupe engendré par les éléments qui fixent un point dans chaque X_i pour $i = 1, \ldots, k$. On remarque que I est distingué. Notons par X_i^g le sous-ensemble de points de X_i qui sont fixés par $g \in G$.

Theorem B.7. *Soient $X, X_1, \ldots, X_k, G$, et I comme ci-dessus. Supposons que l'ensemble $\pi_0 X_i^g$ des composantes connexes par arcs de X_i^g est fini pour chaque $i = 1, \ldots, k$, et chaque $g \in G$. Alors il existe un morphisme*

$$\pi_1(X/G) \to \prod_{i=1}^{k} \pi_1([(X_i/I)/(G/I)])$$

dont l'image a indice fini et le noyau est fini.

L'action de G/I sur X_i/I est celle induite par G dans X_i.

Il est intéressant de mentionner le cas où on prenne le produit d'un même espace topologique $X_i = X_1$ pour $i = 2, \ldots, k$. Notons que dans ce cas l'action de G/I dans X_i/I est libre et donc le groupe fondamental du champ $[X_i/I/G/I]$ coïncide avec le groupe fondamental topologique de $X_i/G/G/I$. De plus, on peut voir que dans ce cas, comme espaces topologiques, on a $X_i/I/G/I \cong X_i/G$. On obtient ainsi le corollaire suivante.

Corollary B.8. *Soient $X = X_1^k$, G, et I comme ci-dessus. Alors le morphisme $\pi_1(X/G) \to \pi_1(X_1/G)^k$ a noyau fini et son image a indice fini dans $\pi_1(X_1/G)^k$.*

Pour prouver le Théorème B.7 on ramène les arguments de [BCGP12] au contexte des champs. On considère l'action de $\pi_1([X/G])$ dans le revêtement universel de X, et on le compare avec l'action de $\pi_1([X_1/G]) \times \ldots \times \pi_1([X_k/G])$. D'après les hypothèses faites sur $\pi_0 X_i^g$, on peut construire sous-groupes $C_i < \pi_1([X_i/G])$ qui se relèvent à sous-groupes de $\pi_1([X/G])$, et dont le produit $C_1 \times \ldots \times C_k$ a indice fini dans le sous-groupe $N < \pi_1([X/G])$ engendré par les éléments qui ont un point fixe dans le revêtement universel de X. On fini par un Théorème de Noohi [Noo08] qui montre que $\pi_1([X/G])/N \cong \pi_1(X/G)$ et en donnant une interprétation géométrique de la projection de N sur chaque $\pi_1([X_i/G])$.

Une application intéressant du Théorème B.7 et Corollaire B.8 se trouve dans l'étude du groupe fondamental de variétés algébriques lisses, en effet,

le groupe fondamental d'une variété algébrique singulier avec seulement singularités quotients est le groupe fondamental d'une variété algébrique lisse [Kol93].

Il y a aussi une généralisation du Théorème B.5.

Theorem B.9. *Supposons que les hypothèses du Théorème B.7 sont satisfaites, et de plus que $\pi_1(X/G)$ est résiduellement fini. Alors, il existe un sous-groupe $\mathcal{N} < \pi_1(X/G)$ isomorphe à un produit $H_1 \times \ldots \times H_k$ de sous-groupes distingués $H_i \lhd_{f.i.} \pi_1(X_i/I)$ d'indice fini.*

En suivant de près les arguments de [BCGP12], qui utilisent les résultats de la théorie de groupes de [GJZZ08], on peut montrer que $\pi_1(C_1 \times \ldots \times C_k/G)$ est résiduellement fini pour un groupe fini G, et courbes algébriques lisses $C_1, \ldots, C_k$ qui ne sont pas nécessairement compactes.

Pour appliquer le Théorème B.9 à la famille Ceva(n) on montre la proposition suivante. Notons $H(n) = \mathbb{Z}/n\mathbb{Z}$, il agit sur la courbe de Fermat de dégrée n définit par $F(n) = z_1^n + z_2^n + z_3^n$. Notons par $\bar{f}$ la fibration induite par le multinet Ceva(n).

Proposition B.10. *Considérons l'action diagonal de $H(n)$ sur $F(n) \times F(n)$. Notons par S la résolution minimale de $F(n) \times F(n)/H(n)$. Alors*

1. *La fibration $S \to (F(n) \times F(n))/H(n) \to F(n)/H(n) \cong \mathbb{P}^1$ es isomorphe à $\tilde{f}$.*

2. *Chaque fibre singulière de $F(n) \times F(n)/H(n)$ corresponds à la contraction de la transformé stricte D_i d'une certaine droite $L_i \in$ Ceva(n).*

3. *La contraction de n droites correspondant à $\mathscr{A}_i$ sont dans une droite E_i qui est un diviseur exceptionnel qui correspond à l'unique point singulier en $\mathscr{A}_i$.*

4. *E_i s'envoie vers un point via $F(n) \times F(n)/H(n) \to \mathbb{P}^1$.*

La preuve est donnée en 2.3.3. Cette preuve n'a pas apparu dans [AA20b].

En utilisant la proposition B.10 et le Théorème B.9 on a étudié deux familles de compactifications partielles de $\mathbb{P}^2 \setminus$ Ceva(n). Dans la première famille on obtient que le groupe fondamental est au plus fini, et dans le deuxième il a un sous-groupe isomorphe à $\mathbb{Z}$ comme un sous-groupe distingué d'indice fini. Voir les exemples 2.3.1 and 2.3.2.

B.3 Compactifications partielles du complément d'un arrangement de droites et variétés de frontière

Dans B.3, on a développé une méthode pour obtenir une présentation pour le groupe fondamental de certaines compactifications partielles du complément

$\mathbb{P}^2 \setminus \mathscr{A}$ d'un arrangement de droites $\mathscr{A} \subset \mathbb{P}^2$ sous l'hypothèse que les droites dans $\mathscr{A}$ sont définies par formes linéaires réelles.

Le chapitre 3 de cette thèse est dédié à la généralisation de ce Théorème dans deux directions:

- on considère un arrangement général $\mathscr{A} \subset \mathbb{P}^2$ défini par formes linéaires complexes et
- on considère une classe plus général $M(\mathscr{A}, I, P)$ des compactifications partielles de $\mathbb{P}^2 \setminus \mathscr{A}$. Voir 3.1.3 pour une définition précise.

Nous pouvons procéder de deux manières différentes: tout d'abord, en suivant [Arv92] et [CS97], dont les travaux généralisent [Ran85] pour les arrangements complexes, nous définissons un "braided wiring diagram" $\mathcal{W}$ qui encode certains sur ou sous-croisements de lignes en $\mathscr{A}$ provenant de la nature complexe des formes qui le définissent. Le diagramme $\mathcal{W}$ encode information suffisante pour obtenir une présentation de $\pi_1(M(\mathscr{A}, I, P))$.

Theorem B.11. *Une présentation de $\pi_1(M(\mathscr{A}, I, P))$ peut être obtenu de $\mathcal{W}$. L'ensemble de générateurs est en correspondance avec l'ensemble de droites dans $\mathscr{A}$ et l'ensemble de relations consiste de deux types:*

- *ces relations R_p venant d'un point singulier p de $\mathscr{A}$. Ces relations figuraient déjà dans une présentation de $\pi_1(M(\mathscr{A}))$ et*
- *pour chaque élément ι soit dans I ou P, une relation R_ι qui est un produit de conjuguées de certains générateurs qui dépend en ι.*

Comme dans la preuve du Théorème B.3, un pas principal pour la preuve du Théorème B.11 consiste en calculer explicitement une expression pour un méridien autour de certains diviseurs exceptionnels, qui sont obtenus en éclatant $\mathscr{A}$ en certains points singuliers, en termes de générateurs.

Pour la deuxième partie du chapitre trois, notons par U une voisinage tubulaire régulière fermé de $\mathscr{A}$ dans $\mathbb{P}^2$. On appelle ∂U la variété de frontière de $\mathscr{A}$. Dans [FGBMB15], une présentation pour $\pi_1(M(\mathscr{A}))$ a été obtenu en utilisant une présentation de $\pi_1(\partial U)$ en étudiant le morphisme $\pi_1(\partial U) \to \pi_1(M(\mathscr{A}))$ induit par l'inclusion $\partial U \hookrightarrow M(\mathscr{A})$. Pour la version homologique voir [GB13].

Il s'avère que leurs méthodes peuvent être appliqué aussi pour déterminer une présentation du groupe fondamental de certains compactifications partielles $M(\mathscr{A}, I, P)$. Néanmoins, pour étudier la variété de frontière ∂U_D de la transformé stricte D de $\mathscr{A}$ dans un modèle birationnel de $\mathbb{P}^2$, on travaille avec une présentation différente de la variété de frontière ∂U de $\mathscr{A}$.

En effet, quand $D = \sum D_i$ est connexe, un diviseur simple à croisements normaux tel que $\pi_1(D)$ est trivial, Mumford a donné une présentation pour $\pi_1(\partial U_D)$ dans [Mum61]. Cette présentation, avec la structure de variété graphé dans le sens de Waldhausen [Wal67], a été utilisé par Westlund pour donner

une présentation de $\pi_1(\partial U)$ dans [Wes67] (voir aussi [CS08]). Ici, en choisissant une surface birationnelle à $\mathbb{P}^2$ où la transformé stricte de $\mathscr{A}$ satisfait les hypothèses pour la présentation de Mumford, on donne une nouvelle preuve de la présentation de Westlund. Voir le Théorème 3.3.1. En suivant cette construction, on peut donner une présentation pour le groupe fondamental de la variété de frontière ∂U_D d'un diviseur D dans une surface $\bar{X}$ obtenu en éclatant successivement $\mathbb{P}^2$ de tel façon que $M(\mathscr{A}, I, P) = \bar{X} \setminus D$.

On obtient dans le Théorème 3.3.3 une présentation pour $\pi_1(M(\mathscr{A}))$ en étudiant aussi le morphisme $i_* : \pi_1(\partial U) \to \pi_1(M(\mathscr{A}))$. De plus, comme la présentation de $\pi_1(\partial U)$ dépend de plusieurs choix, on peut les faire d'une tel façon que l'image des méridiens des droites de $\mathscr{A}$ par i qui sont dans ∂U, dont leurs classes d'homotopie sont une partie des générateurs de $\pi_1(\partial U)$, se trouvent dans la même classe d'homotopie que les méridiens utilisés dans le Théorème B.11. Ainsi, on ne obtient pas seulement que les présentations du Théorème 3.3.3 et B.11 sont équivalents, mais que l'image de l'ensemble de relations dans la présentation de $\pi_1(M(\mathscr{A}))$ coïncide avec les relations du Théorème B.11. À partir de ces résultats, on peut obtenir une présentation pour les compactifications partielles $\pi_1(M(\mathscr{A}, I, P)$, voir Théorème 3.3.14.

Finalement, inspiré par [tDP93], on présent un exemple d'une compactification partielle $M(\mathscr{A}, I, P) = \bar{X} \setminus D$ qui provient d'un arrangement avec 5 droites, deux points triples et quatre points doubles tels que $M(\mathscr{A}, I, P)$ est une $\mathbb{Q}$-plan d'homologie.

Chapter 1

Fundamental groups of partial compactifications of the complement of a real arrangement

By modifying a method used by Randell [Ran85] to get a presentation of the complement of a (complexified) *real* arrangement of lines $\mathscr{A}$ in the complex projective plane $\mathbb{P}^2$, we get expressions for the meridians around exceptional divisors obtained by blowing up points in the singular set of the arrangement $\mathscr{A}$ in terms of meridians around the lines in $\mathscr{A}$. This gives a presentation for the fundamental group of certain partial compactifications of $\mathbb{P}^2 \setminus \mathscr{A}$. Some examples and applications of this presentation are given. These results appeared in [AA20a].

1.1 Preliminaries

We review the definitions and some properties of meridians and orbifolds. For the latter we follow [Eys17].

1.1.1 Meridians

Let M be a connected complex manifold, $H \subset M$ a hypersurface, D an irreducible component of H and $q \in M \setminus H$. Denote by $U = \{z \in \mathbb{C} \mid |z| < 2\}$ and let $f : U \to M$ be a holomorphic function such that:

1. $f^{-1}(H) = \{0\}$,
2. $f(0) = p$ is an smooth point of H and $p \in D$,
3. $f'(0) \notin T_pH$ where T_pH is the tangent space of H at p.

Then $f\mid_{S^1}\colon S^1 \to M \setminus H$ defines a free-homotopy class independent of f where $S^1 \subset U$ is the unit circle. A loop $\gamma \in \pi_1(M \setminus H, q)$ freely homotopic to $f|_{S^1}$ is called a *meridian* of D around p.

If D is smooth, any other meridian of D around a smooth point of H is a conjugate of γ. Denoting by $H' = H \setminus D$, we have that the inclusion $i : M \setminus H \hookrightarrow M \setminus H'$ induces a morphism $i_* : \pi_1(M \setminus H, q) \to \pi_1(M \setminus H', q)$ whose kernel is the normal subgroup of $\pi_1(M \setminus H, q)$ generated by γ. By Van Kampen's theorem the normal subgroup generated by the set of meridians around each irreducible component of H is the kernel of the map $\pi_1(M \setminus H, q) \to \pi_1(M, q)$ induced by the natural inclusion.

Suppose $H = D$ is smooth and let γ_D be a meridian. Denote by $\pi : \bar{M} \to M$ the blow up of M at some $p \in D$ and let E_p be the exceptional divisor. Then $\pi^{-1}(\gamma_D)$ is a meridian of E_p in $\bar{M}$.

1.1.2 Orbifolds

Let M be a complex manifold and D a smooth effective divisor. Let $r \in \mathbb{N}^*$ and consider $P \to M$ the principal $\mathbb{C}^*$-bundle attached to $\mathscr{O}_M(-D)$. The tautological section $\sigma_D \in H^0(M, \mathscr{O}_M(D))$ can be lifted to a holomorphic function $f_D : P \to \mathbb{C}$ satisfying $f_D(p \cdot \lambda) = \lambda f_D(p)$. Let $Y \subset P \times \mathbb{C}$ be the complex analytic space defined by the equation $z^r = f_D(p)$ where z is a coordinate for $\mathbb{C}$. Since D is smooth Y is smooth too. The action of $\mathbb{C}^*$ can be extended to Y in the following way: $(p, z) \cdot \lambda = (p \cdot \lambda^r, \lambda z)$. Then the complex analytic stack

$$M(\sqrt[r]{D}) := [Y_D/\mathbb{C}^*]$$

is an orbifold. The non-trivial isotropy groups lie over the points in D and are isomorphic to the group μ_r of r-roots of unity.

We allow also the weight $+\infty$ by considering the manifold $M \setminus D$ as a stack $[M \setminus D]$ and write

$$M(\sqrt[+\infty]{D}) := [M \setminus D].$$

Let $\bar{X}$ be a complex manifold and $D = \sum_{i=1}^{l} D_i$ be a simple normal crossing divisor, where D_i is an irreducible component of D. For any choice of weights $r := (r_1, \ldots, r_l) \in (\mathbb{N}^* \cup \{+\infty\})^l$ we can define the orbifold

$$\mathcal{X}(\bar{X}, D, r) := \bar{X}(\sqrt[r_1]{D_1}) \times_{\bar{X}} \cdots \times_{\bar{X}} \bar{X}(\sqrt[r_l]{D_l})$$

Denoting by $X = \bar{X} \setminus D$, we can view $\mathcal{X}(\bar{X}, D, r)$ as an orbifold (partial if some $r_i = +\infty$) compactification of X. Let $j_r : X \hookrightarrow \mathcal{X}(\bar{X}, D, r)$ denote the natural open immersion. By fixing $q \in X$, it turns out that we can define $\pi_1(\mathcal{X}(\bar{X}, D, r), q)$ and moreover it is the quotient of $\pi_1(X, q)$ by the normal subgroup generated by all $\gamma_i^{r_i}$, where γ_i is a meridian around D_i and $r_i \neq +\infty$. We obtain that $j_{r*} : \pi_1(X, q) \to \pi_1(\mathcal{X}(\bar{X}, D, r), q)$ is surjective. As a particular case we have that if $r = (1, \ldots, 1)$ then $\mathcal{X}(\bar{X}, D, r) = \bar{X}$.

Let $D_\infty := \sum D_j$ the sum of all irreducible component of D such that $r_j = +\infty$. We can regard $\mathcal{X}(\bar{X}, D, r)$ as $\mathcal{X}(\bar{X} \setminus D_\infty, D - D_\infty, r')$ where r' consists of the same finite values that r. In particular, if $r'_i = 1$ for all i we have that $\mathcal{X}(\bar{X}, D, r) = [\bar{X} \setminus D_\infty]$ and we write simply $\bar{X} \setminus D_\infty$.

Definition 1.1.1. Let X be a smooth algebraic variety, Y a projective curve, $D = \sum_{i=1}^{l} y_i$ a divisor on Y and $r \in (\mathbb{N}^*)^l$. Consider the orbifold $\mathcal{X}(Y, D, r)$. A dominant algebraic morphism $f : X \to Y$ is said to be an *orbifold morphism* if for all $y_i \in D$ the multiplicity of the fiber $f^*(y_i)$ is divisible by r_i.

1.2 Fundamental group

1.2.1 Modification of the method of Randell

Elementary geometric bases

Consider n real points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R} \subset \mathbb{C}$ such that $x_1 < x_2 < \ldots < x_n$. Fix $q \in \mathbb{R} \setminus \{x_1, \ldots, x_n\}$. Any oriented simple closed curve $C \subset \mathbb{C} \setminus \{x_1, \ldots, x_n\}$ is freely homotopic to a loop based at q. Moreover, if it contains at least one x_i in the bounded component that C determines, there exists a simple path θ connecting q and C satisfying:

$$\Im(\theta(t)) < 0 \text{ for } t \in (0, 1).$$

If $C \cap \{\Im(z) < 0\}$ is connected we call $C_q := \theta \cdot C \cdot \theta^{-1}$ an *elementary loop*. Here $\Im$ denotes the imaginary part of a complex number. (We suppose the curve C starts at a point with $\Im(z) \leq 0$).

Remark 1.2.1. We have made all the choices in order to have C_q unique in $\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}, q)$.

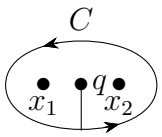


Figure 1.1: Elementary loop C_q.

The following definition is inspired from [MT88], [ABCRCA03].

Definition 1.2.1. An (ordered) *geometric base* $\Gamma = (\gamma_1, \ldots, \gamma_n)$ for the group $\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}, q)$ is an n-tuple such that γ_i is a meridian of x_i based at q and satisfying:

$$\gamma_n \cdot \gamma_{n-1} \cdots \gamma_1 = \partial B(0, M)_q$$

in $\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}, q)$, with $M > |x_i|$ for all $i = 1, \ldots, n$. The curve $\partial B(0, M)$ is a circle centered at 0 with radius M and oriented counterclockwise. We consider the product of loops from left to right.

Remark 1.2.2. The loop $\partial B(0, M)_q$ can be seen as the inverse of a meridian loop around the point at infinity.

By abuse of notation we will write $\Gamma \subset \mathbb{C}$.

Definition 1.2.2. An *elementary geometric base* $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is a geometric base such that every γ_i is an elementary loop.

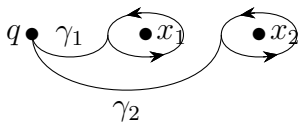


Figure 1.2: An elementary geometric base.

Lemma 1.2.3. *Given n real points and a base point as above, there is a unique* elementary geometric base Γ.

Proof. Is immediate by the ordering of Γ and the uniqueness of the elementary loops. □

Remark 1.2.4. The notion of geometric base for $\pi_1((L \otimes \mathbb{C}) \setminus P; q)$ depends only on the real oriented line L and $P = \{x_1, \ldots, x_n\} \in L(\mathbb{R}), q \in L(\mathbb{R})$.

Randell's pencil

Definition 1.2.3. A *complex arrangement of lines* is an algebraic set $\mathscr{A} \subset \mathbb{P}^2$ whose irreducible components are complex lines. The arrangement $\mathscr{A}$ is said to be *real* or *to be defined over the reals* if the coefficients of all linear forms defining each line can be taken to be real.

Denote by $M(\mathscr{A}) := \mathbb{P}^2 \setminus \mathscr{A}$. We are going to review and adapt a method to compute a presentation for $\pi_1(M(\mathscr{A}))$ when $\mathscr{A}$ is real as in [Ran85].

Associate to each (projective) arrangement $\mathscr{A}$ an affine one, defined as follows: fix a line $L_\infty \in \mathscr{A}$ and consider it as a line at infinity, then

$$\mathscr{A}^{\text{aff}} := \mathscr{A} \cap (\mathbb{P}^2 \setminus L_\infty) \cong \mathscr{A} \cap \mathbb{C}^2,$$

where we have chosen an homeomorphism $h : \mathbb{C}^2 \to \mathbb{P}^2 \setminus L_\infty$. If we denote $M(\mathscr{A}^{\text{aff}}) := \mathbb{C}^2 \setminus \mathscr{A}^{\text{aff}}$, we have the identification:

$$M(\mathscr{A}) = M(\mathscr{A}^{\text{aff}}).$$

Fixing $q \in M(\mathscr{A}^{\text{aff}})$ and denoting also by $q = h(q)$, we have:

$$\pi_1(M(\mathscr{A}), q) \cong \pi_1(M(\mathscr{A}^{\text{aff}}), q).$$

Moreover if the arrangement $\mathscr{A}$ is real, we can associate it a planar graph (allowing rays) in $\mathbb{R}^2$. Suppose $\mathscr{A}^{\text{aff}}$ is the associated affine arrangement, then all

multiple points lie in a real plane. Namely, if we consider $\mathbb{C}^2$ with coordinates $(z, w) = (x_1 + iy_1, x_2 + iy_2)$, the real plane is given by $\{(z, w) \in \mathbb{C} \mid y_1 = y_2 = 0\} \cong \mathbb{R}^2$. Set $\mathscr{A}(\mathbb{R}) := \mathscr{A}^{\text{aff}} \cap \mathbb{R}^2$ to be the set of real points of the arrangement $\mathscr{A}^{\text{aff}}$, denote by $M(\mathscr{A}(\mathbb{R})) := \mathbb{R}^2 \setminus \mathscr{A}(\mathbb{R})$. Suppose there is no vertical line in $\mathscr{A}(\mathbb{R})$. Denote by $\operatorname{Sing} \mathscr{A}^\bullet$ the multiple points of the corresponding arrangement $\mathscr{A}^\bullet = \mathscr{A}, \mathscr{A}^{\text{aff}}, \mathscr{A}(\mathbb{R})$.

Consider $\mathbb{R}^2$ with coordinates (x_1, x_2). We orient the non vertical lines in $\mathbb{R}^2$ taking the positive direction to be that of decreasing x_1.

Fix a base point $q = (q_1, q_2)$ in the lower right part of $M(\mathscr{A}(\mathbb{R}))$ further and lower than any point in $\operatorname{Sing} \mathscr{A}(\mathbb{R})$ and lower than any line. For a complex line $\Sigma \subset \mathbb{C}^2$ defined by an equation with real coefficients, denote by $\Sigma(\mathbb{R})$ its restriction to $\mathbb{R}^2$ and orient it as before if it is non-vertical. Set $\Sigma^{(0)} := \{(z, w) \mid z = q_1\}$, note that $\Sigma^{(0)}(\mathbb{R})$ is the vertical line passing through q, we orient it by taking as positive direction that of increasing x_2. For any triple $P \subset \Sigma(\mathbb{R}) \subset \Sigma$, where P is a finite set of points, $\Sigma(\mathbb{R})$ a real oriented line and Σ a complex line as before, we can consider an elementary geometric base $\Gamma \subset \Sigma$ of $\pi_1(\Sigma \setminus P, q)$ by fixing $q \in \Sigma(\mathbb{R})$.

As $\Sigma^{(0)}(\mathbb{R})$ intersects all the lines of $\mathscr{A}(\mathbb{R})$, we can number $P = \Sigma^{(0)}(\mathbb{R}) \cap \mathscr{A}(\mathbb{R})$ from bottom to top (given by the orientation chosen for $\Sigma^{(0)}(\mathbb{R})$) and denote $\Gamma^{(0)} = \{\gamma_1^{(0)}, \ldots, \gamma_n^{(0)}\} \subset \Sigma^{(0)}$ the associated elementary geometric base with base point q.

The idea to obtain a presentation for the fundamental group is to study how the elementary geometric base change when we rotate the line $\Sigma^{(0)}$ counterclockwise while fixing the base point q and keep track of the relations arising.

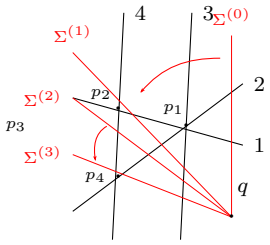


Figure 1.3: Base point

The set of lines passing through q, can be seen as $\mathbb{RP}^1$, which we parametrised by the angle with respect to the line $x_2 = 0$ (oriented in the positive sense), this is, a value in $[\pi/2, 3\pi/2[$. To every real line $\Sigma(\mathbb{R})$ passing through q we can associate its angle, which we denote by:

$$\theta(\Sigma(\mathbb{R})) \in [\pi/2, 3\pi/2[.$$

For $t \in [\pi/2, 3\pi/2[$, the line being parametrised by t will be denoted by Σ_t.

In particular, $\theta(\Sigma^{(0)}) = \pi/2$. The elementary geometric base $\Gamma^{(0)}$ varies in a continuous way as we vary t. There exists two types of directions where it changes:

S.1 Those $t \in [\pi/2, 3\pi/2[$ such that the associated Σ_t contains a point in $\operatorname{Sing}\mathscr{A}(\mathbb{R})$,

S.2 Those $t \in [\pi/2, 3\pi/2[$ such that Σ_t is parallel to a line in $\mathscr{A}(\mathbb{R})$, which correspond to the points in $\operatorname{Sing}\mathscr{A} \cap L_\infty$.

By a slight change of q, we can consider that no line passing through it contains two points of $\operatorname{Sing}\mathscr{A}$. Given $p \in \operatorname{Sing}\mathscr{A}$, denote by $\theta(p)$ the angle of the unique line passing through p and q. Given $p, p' \in \operatorname{Sing}\mathscr{A}$, we define a total order by

$$p < p' \quad \text{iff} \quad \theta(p) < \theta(p').$$

Let us write $\operatorname{Sing}\mathscr{A} = \{p_1, \ldots, p_s\}$ with this order.

Remark 1.2.5. Note that the original method in [Ran85] correspond to choose the base point q at the line at infinity an therefore, the lines passing through it are all parallel in $\mathbb{C}^2$. This method takes care of the singular point lying in $\operatorname{Sing}\mathscr{A}^{\text{aff}}$ but not of those in $\operatorname{Sing}\mathscr{A} \setminus \operatorname{Sing}\mathscr{A}^{\text{aff}}$, they do not intervene in the presentation of $M(\mathscr{A})$.

However, these singular points lying on the line at infinity are indispensable for the presentation of the surfaces to be introduced in Section 1.3 which are our main object of study in this note. This is why we have chosen a base point at finite distance.

Elementary geometric transition of regular fibers in Randell's pencil

Fix a point $p_i \in \operatorname{Sing}\mathscr{A}$. Denote by $t_i = \theta(p_i)$. Choose $\varepsilon > 0$ sufficiently small such that no $t \in [t_i - \varepsilon, t_i + \varepsilon] \setminus \{t_i\}$ is of type **S.1** or **S.2**. Let:

$$\Sigma^{(i-1)} := \Sigma_{t_i - \varepsilon}, \quad \Sigma^{(i)} := \Sigma_{t_i + \varepsilon}.$$

This is, $\Sigma^{(i-1)}$ lies to the right and $\Sigma^{(i)}$ to the left of p_i. Recall that $\Sigma^{(i-1)}(\mathbb{R})$ is an oriented real line and by intersecting with $\mathscr{A}(\mathbb{R})$ we can consider the elementary geometric base:

$$\Gamma^{(i-1)} = (\gamma_1^{(i-1)}, \gamma_2^{(i-1)}, \ldots, \gamma_n^{(i-1)}) \subset \Sigma^{(i-1)},$$

similarly

$$\Gamma^{(i)} = (\gamma_1^{(i)}, \gamma_2^{(i)}, \ldots, \gamma_n^{(i)}) \subset \Sigma^{(i)}.$$

A priori, we should take such geometric bases for every point p_i but as there is no direction between t_i and t_{i+1} in which the geometric base changes, by continuity we will still write $\Gamma^{((i+1)-1)} = \Gamma^{(i)}$.

Remark 1.2.6. In fact, as remarked above, only the points of type **S.1** play a role in the presentation of $\pi_1(M(\mathscr{A}))$. The points of type **S.2** do not modify the meridians who are about to cross a point in $\operatorname{Sing}\mathscr{A}(\mathbb{R})$, they only change their numeration in the geometric base. These points are studied in section 1.2.2 and they are needed for the explicit form of the exceptional meridians given in Section 1.2.3.

The description of the change in the geometric basis is given in the following Proposition when the meridians surround a point of type **S.1**. Let $p_i \in \operatorname{Sing} \mathscr{A}(\mathbb{R})$, $\Gamma^{(i-1)}$ and $\Gamma^{(i)}$ as above.

Proposition 1.2.7. *Let j be the first index for which the meridian $\gamma_j^{(i-1)}$ surrounds a line which passes through p_i and let k be the last such index. Then we have:*

$$\Gamma^{(i)} = (\gamma_1^{(i-1)}, \dots, \gamma_{j-1}^{(i-1)}, \gamma_j^{(i)}, \dots, \gamma_k^{(i)}, \gamma_{k+1}^{(i-1)}, \dots, \gamma_n^{(i-1)}),$$

where:

$$\begin{aligned}
\gamma_k^{(i)} &= \gamma_j^{(i-1)}, \\
\gamma_{k-1}^{(i)} &= {\gamma_j^{(i-1)}}^{-1} \gamma_{j+1}^{(i-1)} \gamma_j^{(i-1)}, \\
&\vdots \\
\gamma_j^{(i)} &= {\gamma_j^{(i-1)}}^{-1} {\gamma_{j+1}^{(i-1)}}^{-1} \cdots {\gamma_{k-1}^{(i-1)}}^{-1} \gamma_k^{(i-1)} \gamma_{k-1}^{(i-1)} \cdots \gamma_j^{(i-1)} \gamma_j^{(i-1)}.
\end{aligned}$$

And a set of relations in $\pi_1(M(\mathscr{A}), q)$:[1]

$$R_{p_i} = \left\{ \gamma_k^{(i-1)} \gamma_{k-1}^{(i-1)} \cdots \gamma_j^{(i-1)} = \gamma_{\sigma(k)}^{(i-1)} \gamma_{\sigma(k-1)}^{(i-1)} \cdots \gamma_{\sigma(j)}^{(i-1)} = \dots \right\} \tag{1.1}$$

where σ runs over the set of cyclic permutations of $k-j+1$ elements.

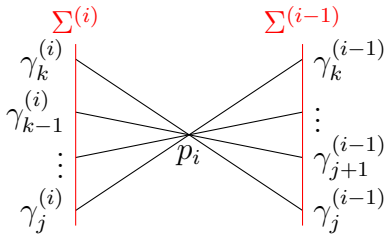


Figure 1.4: Conjugates

Proof. Let $B_{p_i,\varepsilon}$ be a 4-real ball of radius ε sufficiently small around p_i and consider the fundamental group G of $\partial B_{p_i,\varepsilon} \setminus (\partial B_{p_i,\varepsilon} \cap \mathscr{A})$ with base point q', where $\partial B_{p_i,\varepsilon}$ denotes the boundary of $B_{p_i,\varepsilon}$.

The group G is equivalent to the fundamental group of a $k-j+1$ Hopf link in a torus and by the Wirtinger presentation we obtain the relations stated above (See [OT92, Lemma 5.75]) for local meridians in $\partial B_{p_i,\varepsilon}$.

The local base point q' can be chosen in such a way that there exists a path $\gamma^{(i-1)}$ joining q and q' such that the base point change of the local meridians in $\partial B_{p_i,\varepsilon}$ to R via $\gamma^{(i-1)}$, coincide with the $\gamma_j^{(i-1)}$. □

[1] These relations are stated as in [Fal93] p.142, where in a footnote he points to an error of [Ran85].

Expressing every meridian in terms of the geometric base $\Gamma^{(0)}$ by means of Proposition 1.2.7 and replacing in (1.1) we obtain:

Theorem 1.2.8 ([Ran85])**.** *The fundamental group of $M(\mathscr{A})$ admits a presentation:*

$$\pi_1(M(\mathscr{A}), q) \cong \left\langle \gamma_1^{(0)}, \gamma_2^{(0)}, \ldots, \gamma_n^{(0)} \;\middle|\; \bigcup_{p_i \in \mathrm{Sing}(\mathscr{A}(\mathbb{R}))} R_{p_i} \right\rangle.$$

Proof. By choosing suitable open sets M_i of $M(\mathscr{A})$ in such a way that $\pi_1(M_i \cap M_{i+1})$ is the free group in $n = \left|\mathscr{A}^{\mathrm{aff}}\right|$ generators and M_i contains only the singular point p_i we can apply Van-Kampen and Proposition 1.2.7.

Note that by the choice of the base point q, all the points in $\mathrm{Sing}(\mathscr{A}(\mathbb{R}))$ lie to the left of $\Sigma^{(0)}$ and therefore when a meridian γ of a line L crosses a point at infinity of type **S.2**, it will not cross another singular point, hence it will give no further relations to the presentation of the group. □

Remark 1.2.9. An equivalent presentation can be obtained by studying the monodromy in the Randell pencil by a slight modification of the methods in [CS97].

Indeed by blowing up the base point R, we obtain a fibration with base $\mathbb{P}^1$ that restricted to $\mathbb{P}^2 \setminus \mathscr{A}$ has singular fibers whenever a point $p_i \in \mathrm{Sing}\,\mathscr{A}$ lies at such fiber. The conjugations arising when a geometric basis crosses a singular point p_i is given by a permutation braid μ_i called a half twist [CS97, p. 14] (c.f. with Proposition 1.2.7) and the local monodromy is given by μ_i^2 (c.f. [Hir93, p. 41] up to a different order). Finally, by keeping track of the global conjugations by means of a wiring diagram (see [CS97] for definitions) a presentation is obtained.

Adapting in an appropriate way the definitions of [CS97] or those of [Arv92], a presentation of $M(\mathscr{A})$, for $\mathscr{A}$ an arrangement not necessarily real, adapted to our purposes is possible.

We have preferred here to adapt the description of Randell because it seemed to us to be simpler and quicker for the heuristic methods of treating the question of Dimca-Eyssidieux case by case for arrangements of few lines.

1.2.2 Meridians crossing a point at infinity

Let us describe the change in the geometric base when it traverses a singular point at infinity. Let $p_i \in \mathrm{Sing}\,\mathscr{A} \cap L_\infty$ and $\Gamma^{(i-1)} \subset \Sigma^{(i-1)}$ and $\Gamma^{(i)} \subset \Sigma^{(i)}$ be given as in section 1.2.1. This is:

$$\Gamma^{(i-1)} = (\gamma_1^{(i-1)}, \ldots, \gamma_{n-k+1}^{(i-1)}, \ldots, \gamma_n^{(i-1)}),$$

and

$$\Gamma^{(i)} = (\gamma_1^{(i)}, \ldots, \gamma_k^{(i)}, \gamma_{k+1}^{(i)}, \ldots, \gamma_n^{(i)}).$$

Proposition 1.2.10. *Assume that there are exactly k parallel lines in $\mathscr{A}(\mathbb{R})$ whose corresponding lines in $\mathscr{A}$ intersect at p_i. Then these lines are associated to the last k meridians $\gamma_{n-k+1}^{(i-1)}, \ldots, \gamma_n^{(i-1)}$ of $\Gamma^{(i-1)}$.*

Proof. Let $t_i = \theta(p_i)$ and Σ_{t_i} be the line passing by q and p_i. Using the order of the real lines write $\Sigma^{(i-1)}(\mathbb{R}) \cap \mathscr{A}(\mathbb{R}) = \{y_1, \ldots, y_n\}$ and as no other point of $\operatorname{Sing}\mathscr{A}$ different from p_i lies in Σ_{t_i} we have that $\Sigma_{t_i} \cap \mathscr{A}(\mathbb{R}) = \{x_1, \ldots, x_{n-k}\}$. In fact, it must be the case that x_i and y_i are in the same line of $\mathscr{A}(\mathbb{R})$ otherwise a point of type **S.1** or **S.2** would lie between $\Sigma^{(i-1)}$ and Σ_{t_i} which can not happen. □

Corollary 1.2.11. *We have the following identifications in $\Sigma^{(i)}$:*

$$\gamma_{k+1}^{(i)} = \gamma_1^{(i-1)}, \quad \ldots \quad , \gamma_n^{(i)} = \gamma_{n-k}^{(i-1)}. \tag{1.2}$$

Proof. As we are turning counter-clockwise, by the orientation given to $\Sigma^{(i)}(\mathbb{R})$ it will intersect first the k parallel lines associated to p_i and then, by the same argument as in Proposition 1.2.10, the point in the position $k + j$ of $\Sigma^{(i)}(\mathbb{R}) \cap \mathscr{A}(\mathbb{R})$ lies in the same line as x_j . □

Proposition 1.2.12. *The last k meridians in $\Gamma^{(i-1)}$ invert their order to fit in the first k places of $\Gamma^{(i)}$. By doing so a conjugation for all the precedent meridians is needed (see figure 1.5). More precisely we have:*

$$\begin{aligned}
\gamma_k^{(i)} &= \gamma_1^{(i-1)^{-1}} \cdots \gamma_{n-k}^{(i-1)^{-1}} \; \gamma_{n-k+1}^{(i-1)} \; \gamma_{n-k}^{(i-1)} \cdots \gamma_1^{(i-1)}, \\
\gamma_{k-1}^{(i)} &= \gamma_1^{(i-1)^{-1}} \cdots \gamma_{n-k+1}^{(i-1)^{\;-1}} \; \gamma_{n-k+2}^{(i-1)} \; \gamma_{n-k+1}^{(i-1)} \cdots \gamma_1^{(i-1)}, \\
&\vdots \\
\gamma_1^{(i)} &= \gamma_1^{(i-1)^{-1}} \cdots \gamma_{n-1}^{(i-1)^{-1}} \; \gamma_n^{(i-1)} \; \gamma_{n-1}^{(i-1)} \cdots \gamma_1^{(i-1)}.
\end{aligned} \tag{1.3}$$

Proof. By writing the l-element $\gamma_l \in \Gamma = (\gamma_1, \ldots, \gamma_n)$ as $\gamma_l = \theta_l \cdot C_l \cdot \theta_l^{-1}$ as in 1.2.1, it follows that the element $\gamma_l' = \gamma_1^{-1} \cdots \gamma_{l-1}^{-1} \gamma_l \gamma_{l-1} \cdots \gamma_1$ described in the statement of the Proposition can be written as $\gamma_l' = \theta_l' \cdot C_l \theta_l'^{-1}$ with θ_l' homotopic to a loop such that $\Im(\theta_l'(t)) > 0$ for $t \in (0, 1)$, see Figure 1.5. These loops can then be moved to the firs k positions. The result follows by unicity of the elementary geometric base. □

1.2.3 Loops around singular points

Consider an arrangement $\mathscr{A}$ defined over the reals as in the precedent section. We have a canonical way of associating an elementary geometric base for every line Σ_t passing though q with $t \in [\pi/2, 3\pi/2[$ as in 1.2.1. We will write the elementary geometric base over the directions of the points **S.1** and **S.2** in terms of the elements of $\Gamma^{(i)}$.

This can be seen as finding elementary loops for the points in $\operatorname{Sing}\mathscr{A}$, which can be divided into finite distance points $\operatorname{Sing}\mathscr{A}^{\text{aff}} \cong \operatorname{Sing}\mathscr{A}(\mathbb{R})$ and infinite distance $\operatorname{Sing}\mathscr{A} \cap L_\infty$.

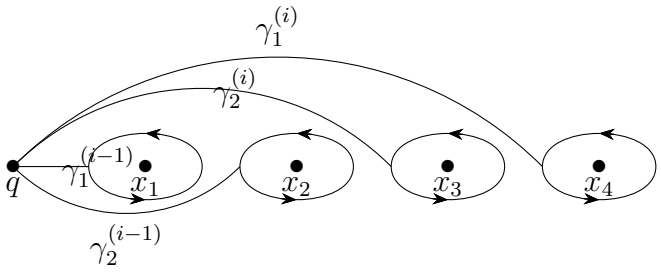


Figure 1.5: Loops crossing a point **S.2**.

Lemma 1.2.13. *The inverse of a meridian loop around the line at infinity L_∞ at the point $L_\infty \cap \Sigma^{(i)}$ is given by the product of the elements of the elementary base $\Gamma^{(i)}$, this is :*

$$(\gamma_\infty^{(i)})^{-1} = \gamma_n^{(i)} \cdot \gamma_{n-1}^{(i)} \cdots \gamma_1^{(i)}$$

Proof. This is a simple consequence on the definition of geometric base and the choice of the base point. □

This meridian can be seen as an *elementary* loop, as it is product of loops of this type.

Recall that $t_i = \theta(p_i)$ denotes the angle of the line Σ_{t_i} containing p_i and q.

Definition 1.2.4. A meridian γ'_{p_i} *around a singular point* $p_i \in \operatorname{Sing} \mathscr{A}(\mathbb{R})$, is a meridian of $p_i \in \Sigma_{t_i}$ (based at q).

We can consider the *elementary* meridian γ_{p_i} as the elementary loop of p_i in Σ_{t_i} (based at q). With the notation of Proposition 1.2.7 we have:

Lemma 1.2.14. *The elementary meridian γ_{p_i} can be obtained as a product of the elements of $\Gamma^{(i-1)}$ which surround the lines passing through p_i. Namely*

$$\gamma_{p_i} = \gamma_k^{(i-1)} \gamma_{k-1}^{(i-1)} \cdots \gamma_{j+1}^{(i-1)} \gamma_j^{(i-1)}.$$

Proof. An elementary geometric base Γ is constructed in such a way that the product of $k-j+1$ consecutive elements $(\gamma_j, \ldots, \gamma_k)$ of Γ equals an elementary loop C_q, where C is an oriented counter-clockwise simple closed curve that in the bounded part that it determines contains exactly $\{x_j, \ldots, x_k\}$. □

Next we determine the meridians around multiple points lying in the line at infinity.

Let $p_i \in \operatorname{Sing} \mathscr{A} \cap L_\infty$. Consider the line Σ_{t_i} passing through q and p_i. Suppose there are exactly k lines in $\mathscr{A}$ different from L_∞ passing through p_i then their real points are parallel lines to $\Sigma_{t_i}(\mathbb{R})$ in $\mathscr{A}(\mathbb{R})$. As is Section 1.2.2 we have:

$$\Sigma_{t_i}(\mathbb{R}) \cap \mathscr{A}(\mathbb{R}) = \{x_1, \ldots, x_{n-k}\},$$

with $k \geq 1$ depending on i. The order of the points x_i given by the orientation of $\Sigma_{t_i}(\mathbb{R})$. Hence we can take the elementary geometric base $\Gamma_{t_i} \subset \Sigma_{t_i}$

associated with $P = \{x_1, \ldots, x_{n-k}\}$ and the base point q. Suppose $\Gamma_{t_i} = (\gamma_1, \ldots, \gamma_{n-k})$.

Definition 1.2.5. For a point at infinity $p_i \in \operatorname{Sing} \mathscr{A} \cap L_\infty$, we say that a meridian γ_{p_i} at infinity of Σ_{t_i} is a meridian around the singular point p_i.

Lemma 1.2.15. *Let $\Gamma^{(i-1)} = (\gamma_1^{(i-1)}, \ldots, \gamma_n^{(i-1)})$ be as in Section 1.2.1. For every point $p_i \in \operatorname{Sing} \mathscr{A} \cap L_\infty$ the elementary meridian γ_{p_i} is given by any of the equivalent expressions*

$$\gamma_{p_i} = \gamma_\infty^{(i-1)} \cdot \gamma_n^{(i-1)} \cdots \gamma_{n-k+1}^{(i-1)}, \tag{1.4}$$

or

$$\gamma_{p_i}^{-1} = \gamma_{n-k}^{(i-1)} \cdots \gamma_1^{(i-1)}. \tag{1.5}$$

Remark 1.2.16. In (1.4) a similitude with the formula of Lemma 1.2.14 can be observed. Namely the product of the meridians of the lines crossing the point p_i give the meridian. In (1.5) we simply compute the meridian around the point at infinity in the line Σ_{t_i}, so it is closer to Lemma 1.2.13.

Proof. As no other point of $\operatorname{Sing} \mathscr{A}$ lies in Σ_{t_i} by continuity, Proposition 1.2.10 and the uniqueness of the elementary geometric base we have that

$$\Gamma_{p_i} = (\gamma_1^{(i-1)}, \ldots, \gamma_{n-k}^{(i-1)}),$$

by applying Lemma 1.2.13 we obtain (1.5).

In $\Sigma^{(i-1)}$ we have

$$\gamma_\infty^{(i-1)} = (\gamma_1^{(i-1)})^{-1} \cdots (\gamma_n^{(i-1)})^{-1},$$

therefore for the right hand side of (1.4)

$$\gamma_\infty^{(i-1)} \cdot \gamma_n^{(i-1)} \cdots \gamma_{n-k+1}^{(i-1)} = (\gamma_1^{(i-1)})^{-1} \cdots (\gamma_{n-k}^{(i-1)})^{-1}$$

which equals γ_{p_i} by (1.5). □

Remark 1.2.17. By the results of this Section, we have obtained meridians around every point $p \in \operatorname{Sing} \mathscr{A}$ (in the sense of definitions 1.2.4 and 1.2.5) with $\mathscr{A}$ defined over the reals. In [Gar03] Garber generalize a formula of Fujita [Fuj82] expressing locally the meridians around singular points as the product of the meridians of the irreducible components in the singular point. He then uses this result globally when the lines intersect transversally, this is, when there is no additional conjugation. Our method can be seen as a generalization of this by allowing multiple points of higher order.

1.3 LAC Surfaces

1.3.1 Construction

We will construct surfaces generalizing the complement of a hyperplane arrangement and obtain presentations of the fundamental group of these surfaces.

Fix $\mathscr{A} = \{L_1, \dots, L_k\}$ an arrangement of lines in $\mathbb{P}^2$. Let $\bar{X}$ be the blow up of $\mathbb{P}^2$ at $\operatorname{Sing}\mathscr{A} = \{p_1, \dots, p_s\}$ and $\pi : \bar{X} \to \mathbb{P}^2$ the projection map. Denote by $D_1, \dots, D_k$ the strict transform of the lines $L_1, \dots, L_k$ and by $D_{k+1}, \dots, D_{k+s}$ the exceptional divisors associated to the points $p_1, \dots, p_s$.

Given a subset $I \subset \{1, \dots, k+s\}$ we can define the orbifold $\mathcal{X}(\bar{X}, D, r_I)$ associated to the divisor $D = \sum D_i$ and the weights $r_I = (r_1, \dots, r_{k+s})$ where $r_i = 1$ if $i \in I$ and $r_i = +\infty$ if $i \notin I$. Then $\mathcal{X}(\bar{X}, D, r_I) = \bar{X} \setminus (D_I)_\infty$ where we have written D_I for D to emphasize the dependence on I.

Definition 1.3.1. We call $\bar{X} \setminus (D_I)_\infty$ a *(partial) Linear Arrangement Compactification* or LAC surface.

Remark 1.3.1. If $I = \varnothing$, $(D_I)_\infty = D$ and π restricted to $\bar{X} \setminus D$ is a biholomorphism with $M(\mathscr{A})$, from which it follows that

$$\pi_1(\bar{X} \setminus D) \cong \pi_1(M(\mathscr{A})), \tag{1.6}$$

showing that these surfaces are indeed generalizations of the complement of an arrangement.

1.3.2 Reduced LAC Surfaces

In [Eys17] a comment before Proposition 1.3 mentions that the log pair $(\bar{X}, D)$ has to be rigid if one wants the fundamental group to be very different from $\bar{X} \setminus D$. We prove here that we can reduce the study of LAC surfaces to partially compactify only with respect to exceptional divisors, this is, the subset of irreducible components of D with weight 1 are exceptional divisors.

We do so by showing that if a strict transform of a line L_i has weight 1, then we can find an arrangement of less lines whose associated LAC surface has the same fundamental group. In this process the double points lying in the line that we have removed create isolated points and we must allow to blow them up as well in order to cover the case when this exceptional divisor had weight 1 in $\mathscr{A}$. With this is mind we have the following definition.

Definition 1.3.2. A *LAC datum*, is a triple

$$(\mathscr{A}, S, I) := (\mathscr{A} = \{L_1, \dots, L_k\} \subset \mathbb{P}^2, S = \{p_1, \dots, p_s\} \subset \mathbb{P}^2, I \subset \{1, \dots, k+s\})$$

where $\mathscr{A}$ is an arrangement of lines in $\mathbb{P}^2$, S a finite set of points and I an index set.

Given a LAC datum $(\mathscr{A}, S, I)$ we can construct a surface $\bar{X} \setminus (D_I)_\infty$ as in 1.3.1. Consider $\bar{X}$ the blow up of $\mathbb{P}^2$ in the points S, call $D_1, \ldots, D_k$ the strict transform of the lines in $\mathscr{A}$, $D_{k+1}, \ldots, D_{k+s}$ the exceptional divisors and $(D_I)_\infty = \sum_{j \in J} D_j$ where $J := \{1, \ldots, k+s\} \setminus I$. As we can change the arrangement and the set of points to blow up, we prefer the notation $M(\mathscr{A}, S, I)$ for this surface.

Definition 1.3.3. Two LAC datum $(\mathscr{A}, S, I)$, $(\mathscr{A}', S', I')$ are said to be equivalent if and only if

$$\pi_1(M(\mathscr{A}, S, I)) \cong \pi_1(M(\mathscr{A}', S', I')).$$

In such a case we write $(\mathscr{A}, S, I) \sim (\mathscr{A}', S', I')$.

Definition 1.3.4. A LAC datum $(\mathscr{A}, S, I)$ such that $S \subset \operatorname{Sing} \mathscr{A}$ and $I = S$ is called *reduced*. In this case we write $(\mathscr{A}, I)$.

Theorem 1.3.2. *For every LAC datum $(\mathscr{A}, S, I)$ there is a canonical equivalent reduced LAC $(\mathscr{A}', I')$.*

We will need to prove first three reduction Lemmas.

Lemma 1.3.3. *Let $(\mathscr{A}, S, I)$ be a LAC datum. Suppose there exists $L_i \in \mathscr{A}$ such that $i \in I$, then*

$$(\mathscr{A}, S, I) \sim (\mathscr{A} \setminus L, S, I \setminus \{i\}).$$

Proof. Denote by $\bar{X} = \operatorname{Bl}_S \mathbb{P}^2$. As $M(\mathscr{A}, S, I) = \bar{X} \setminus (D_I)_\infty$ and

$$\{1, \ldots, s+k\} \setminus I = \{1, \ldots, \hat{i}, \ldots, s+k\} \setminus (I \setminus \{i\})$$

denoting by $(D'_{I \setminus \{i\}})_\infty$ the divisor to be removed given by $(\mathscr{A} \setminus L, S, I \setminus \{i\})$ we have that

$$(D_I)_\infty = (D'_{I \setminus \{i\}})_\infty$$

that implies

$$M(\mathscr{A}, S, I) = M(\mathscr{A} \setminus L, S, I \setminus \{i\}).$$

□

So we can suppose $I \subset \{1, \ldots, s\}$. The next step is to consider points lying outside $\mathscr{A}$.

Lemma 1.3.4. *Let $(\mathscr{A}, S, I)$ be a LAC datum such that there is $p_j \in S$ that lies in no line of $\mathscr{A}$.*

1. *If $j \in I$ then*
$$(\mathscr{A}, S, I) \sim (\mathscr{A}, S \setminus \{p_j\}, I \setminus \{j\}).$$
2. *If $j \notin I$ then*
$$(\mathscr{A}, S, I) \sim (\mathscr{A}, S \setminus \{p_j\}, I).$$

Proof. 1. The surface $M(\mathscr{A}, S, I)$ is the blowing up of $M(\mathscr{A}, S\setminus\{p_j\}, I\setminus\{j\})$ at the point p_j, as the fundamental group is invariant under blow ups we obtain the stated.

2. We have a biholomorphism given by restricting the blowing up map of $M(\mathscr{A}, S \setminus \{p_j\}, I)$ at p_j, to the complement of the exceptional divisor

$$M(\mathscr{A}, S, I) \xrightarrow{\sim} M(\mathscr{A}, S \setminus \{p_j\}, I) \setminus \{p_j\}$$

but as

$$\pi_1(M(\mathscr{A}, S \setminus \{p_j\}, I) \setminus \{p_j\}) \cong \pi_1(M(\mathscr{A}, S \setminus \{p_j\}, I))$$

the result follows.

□

The last reduction Lemma, can be divided into two parts. In the first case we show that it is only interesting when we blow up a point and do not remove the exceptional divisor. In the second part, a point of $p_j \in S$ that is a smooth point of $\mathscr{A}$ does not affect the fundamental group in either case $j \in I$ or $j \notin I$. By the last Lemma we can assume that every point in S lies in the arrangement $\mathscr{A}$.

Lemma 1.3.5. *Let $p_j \in S \subset \mathscr{A}$.*

1. *If $j \notin I$ then*
$$(\mathscr{A}, S, I) \sim (\mathscr{A}, S \setminus \{p_j\}, I).$$

2. *Suppose $p_j \in L$ for some line $L \in \mathscr{A}$. If $j \in I$ and $p_j \notin \operatorname{Sing}\mathscr{A}$, then*

$$(\mathscr{A}, S, I) \sim (\mathscr{A} \setminus \{L\}, S \setminus \{p_j\}, I \setminus \{j\}).$$

Proof. 1. Let $\bar{X} = \operatorname{Bl}_{S\setminus\{p_j\}} \mathbb{P}^2$ and $\bar{Y} = \operatorname{Bl}_{p_j} \bar{X}$. In $\bar{X}$ we have

$$(D_I)_\infty = \sum D_r \text{ with } r \in S \setminus (I \cup \{p_j\})$$

In $\bar{Y}$

$$(D'_I)_\infty = (D_I)_\infty + D'_j$$

Where we have denote also by $(D_I)_\infty$ the strict transform of the divisor with same notation in $\bar{X}$. Therefore we have a biholomorphism

$$\bar{Y}\setminus(D'_I)_\infty = M(\mathscr{A}, S, I) \xrightarrow{\sim} M(\mathscr{A}, S\setminus\{p_i\}, I)\setminus\{p_j\} = \bar{X}\setminus((D_I)_\infty \cup \{p_j\})$$

and the result follows.

2. If γ_{p_j} is a meridian at p_j of L, as p_j is a smooth point then it is also a meridian of the exceptional divisor D_{k+j} in $M(\mathscr{A}, S, I \setminus \{j\})$. As D_{k+j} is smooth, γ_{p_j} generates the kernel of

$$\pi_1(M(\mathscr{A}, S, I \setminus \{j\})) \to \pi_1(M(\mathscr{A}, S, I))$$

hence

$$\pi_1(M(\mathscr{A}, S, I \setminus \{j\}))/\left\langle\left\langle \gamma_{p_j} \right\rangle\right\rangle \cong \pi_1(M(\mathscr{A}, S, I)) \tag{1.7}$$

By the point 1 above we have that $(\mathscr{A}, S, I\setminus\{j\}) \sim (\mathscr{A}, S\setminus\{p_j\}, I\setminus\{j\})$. Replacing in (1.7) we obtain

$$\pi_1(M(\mathscr{A}, S \setminus \{p_j\}, I \setminus \{j\}))/\left\langle\left\langle \gamma_{p_j} \right\rangle\right\rangle \cong \pi_1(M(\mathscr{A}, S, I)) \tag{1.8}$$

But γ_{p_j} also generates the kernel of the map of fundamental group induced by the inclusion

$$M(\mathscr{A}, S \setminus \{p_j\}, I \setminus \{j\}) \hookrightarrow M(\mathscr{A} \setminus L, S \setminus \{p_j\}, I \setminus \{j\})$$

therefore

$$\pi_1(M(\mathscr{A}, S \setminus \{p_j\}, I \setminus \{j\}))/\left\langle\left\langle \gamma_{p_j} \right\rangle\right\rangle \cong \pi_1(M(\mathscr{A} \setminus \{L\}, S \setminus \{p_j\}, I \setminus \{j\}))$$

which together with (1.8) prove the statement.

□

Proof of Theorem 1.3.2 . Given an arbitrary LAC datum $(\mathscr{A}, S, I)$ by Lemma 1.3.3 we can suppose that $I \subset \{1, \ldots, s\}$. By Lemma 1.3.4 all those points in S not lying over $\mathscr{A}$ can be also discarded without changing the fundamental group.

By Proposition 1.3.5 1, we remove from S all points p_j such that $j \notin I$ so $S = I$, we will denote the LAC datum by (A, S) .

If there is a smooth point $p_j \in S$ such that $p_j \in L$ for some $L \in \mathscr{A}$ by Proposition 1.3.5 2, $(\mathscr{A}, S) \sim (\mathscr{A} \setminus \{L\}, S \setminus \{p_j\})$. This new LAC datum could have as well smooth points lying in $S \setminus \{p_j\}$, either coming from S or from double points in $\mathscr{A}$ lying in L. We repeatedly apply Proposition 1.3.5 2, until $I \subset \operatorname{Sing}\mathscr{A}$ or $\mathscr{A} = \emptyset$. As there are only a number finite of points and lines this process must end and we obtain an equivalent reduced LAC datum $(\mathscr{A}', I')$ as wanted.

□

1.3.3 A presentation for the orbifold fundamental group

Definition 1.3.5. Let $\mathscr{A} = \{L_1, \ldots, L_k\}$, $\bar{X}$ the blow up of $\mathbb{P}^2$ at $\operatorname{Sing}\mathscr{A} = \{p_1, \ldots, p_s\}$ and D_i as in section 1.3.1. The divisor $D = \sum D_i$ is simple normal crossing and for $r \in (\mathbb{N}^* \cup \{\infty\})^{k+s}$ the orbifold $\mathcal{X}(\bar{X}, D, r)$ is called a *weighted LAC Surface*.

Theorem 1.3.6. *Let $\mathscr{A} = \{L_1, \dots, L_k\}$ be a real arrangement, $\mathcal{X}(\bar{X}, D, r)$ a weighted LAC surface. Suppose we consider L_k as a line at infinity and $\mathscr{A}^{aff}$ has no vertical line. Choose a base point q and a canonical elementary geometric base $\Gamma^{(0)} = (\gamma_1, \dots, \gamma_{k-1})$ based at q and to the right of any vertex as in Section 1.2. Let $\operatorname{Sing}\mathscr{A} = \{p_1, \dots, p_s\}$ and γ_{p_j} be the elementary meridian around p_j. Then the γ_{p_j} can be expressed in terms of $\Gamma^{(0)}$ as in Lemmas 1.2.14 and 1.2.15 a presentation for $\pi_1(\mathcal{X}(\bar{X}, D, r), q)$ is given by*

$$\left\langle \gamma_1, \dots, \gamma_{k-1} \mid \bigcup_{p_l \in \operatorname{Sing}(\mathscr{A}(\mathbb{R}))} R_{p_l}, \gamma_i^{r_i}, \gamma_{p_j}^{r_{k+j}}, \begin{array}{l} i = 1, \dots, k, \\ j = 1, \dots, s \end{array} \right\rangle \tag{1.9}$$

where we omit the relation $\gamma^{r_m} = 1$ if $r_m = \infty$.

Proof. We find first a presentation for $\pi_1(\bar{X} \setminus D, q)$ and express the meridians around the D_r in terms of γ_i. As $\bar{X} \setminus D \cong M(\mathscr{A})$ by remark 1.3.1, we obtain that γ_i is a meridian of D_i in $\bar{X}$ and by Theorem 1.2.8 we have the following presentation for $\pi_1(\bar{X} \setminus D, q)$:

$$\pi_1(\bar{X} \setminus D, q) = \left\langle \gamma_1, \gamma_2, \dots, \gamma_{k-1} \mid \bigcup_{p \in \operatorname{Sing}(\mathscr{A}(\mathbb{R}))} R_p \right\rangle.$$

The elementary meridian γ_k around D_k is given by Lemma 1.2.13 as $\gamma_k = (\gamma_{k-1} \cdots \gamma_1)^{-1}$. The meridians around the exceptional divisor D_{k+j} are given by the Lemmas 1.2.14 and 1.2.15 in the following way: γ_{p_j} is a meridian around p_j lying completely in the line Σ_i, so after the blow up this meridian lies in the strict transform of Σ_i giving a meridian of D_{k+j}. Moreover, γ_{p_j} is expressed in terms of $\Gamma^{(0)}$. By [Eys17] p.3 dividing by the normal subgroup generated by $\gamma_i^{r_i}, \gamma_{p_j}^{r_{k+j}}$ we obtain the presentation. □

Corollary 1.3.7. *Let $(\mathscr{A}, I)$ be a reduced LAC surface with $\mathscr{A}$ real. A presentation for $\pi_1(\bar{X} \setminus (D_I)_\infty)$ is given by*

$$\pi_1(\bar{X} \setminus (D_I)_\infty) \cong \left\langle \gamma_1, \dots, \gamma_{k-1} \mid \bigcup_{p_r \in \operatorname{Sing}(\mathscr{A}(\mathbb{R}))} R_{p_r}, \gamma_{p_j}, \quad j \in I \right\rangle. \tag{1.10}$$

1.4 Applications

1.4.1 LAC Surface with infinite fundamental group and finite abelianization

Consider a set S of 4 points in general position in $\mathbb{P}^2$. The arrangement $\mathcal{B} = \{L_1, \dots, L_6\}$ of 6 lines connecting each pair of these is called *the complete quadrilateral* or Ceva(2). It has 4 triple points and 3 double points: $\operatorname{Sing}\mathcal{B} = \{p_1, \dots, p_7\}$ numbered as in Figure 1.6. It has the following equation $(z_1^2 - z_2^2)(z_1^2 - z_3^2)(z_2^2 - z_3^2) = 0$ for projective coordinates $(z_1 : z_2 : z_3)$.

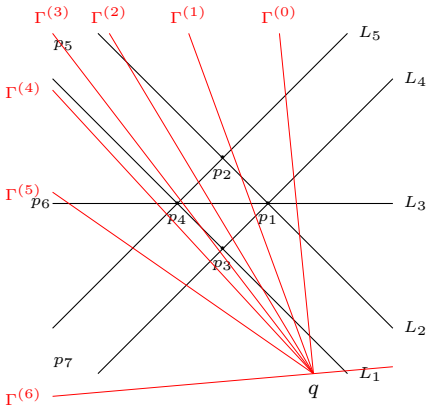


Figure 1.6: Complete quadrilateral.

If we consider L_6 as the line at infinity, after a small rotation in order to have no vertical lines, we obtain the real picture as in Fig. 1.6.
By the subsections 1.2.2 and 1.2.3 we have that the elementary geometric base (up to homotopy in $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}, q)$, replacing $\gamma_i^{(0)}$ by x_i and writing $x^y := y^{-1}xy$) are

$$\begin{aligned}
\Gamma^{(0)} &= (x_1, x_2, x_3, x_4, x_5) \\
\Gamma^{(1)} &= (x_1, x_4, x_3^{x_2}, x_2, x_5) \\
\Gamma^{(2)} &= (x_1, x_4, x_3^{x_2}, x_5, x_2) \\
\Gamma^{(3)} &= (x_4, x_1, x_3^{x_2}, x_5, x_2) \\
\Gamma^{(4)} &= (x_4, x_5, x_3^{x_2x_1}, x_1, x_2) \\
\Gamma^{(5)} &= (x_2^{x_3x_2x_1x_5x_4}, x_1^{x_3^{x_2}x_1x_5x_4}, x_4, x_5, x_3^{x_2x_1}) \\
\Gamma^{(6)} &= (x_3^{a^{-1}x_2x_1a}, x_2^{x_3x_2x_1x_5x_4}, x_1^{x_3^{x_2}x_1x_5x_4}, x_4, x_5)
\end{aligned} \tag{1.11}$$

where

$$a = (x_2x_1)^{x_3x_2x_1}x_5x_4$$

By Theorem 1.2.8 we obtain the following presentation:

$$G = \pi_1(\mathbb{P}^2 \setminus \mathcal{B}, q) \cong \langle x_1, \dots, x_5 \mid [x_4, x_1], [x_5, x_2], [x_4, x_3, x_2], [x_5, x_3^{x_2}, x_1] \rangle \tag{1.12}$$

which can be easily seen to be a semidirect product $\mathbb{F}_2 \ltimes \mathbb{F}_3$ where $\mathbb{F}_2 = \langle x_4, x_5 \rangle$ and $\mathbb{F}_3 := \langle x_1, x_2, x_3 \rangle$.

Let $\bar{X}$ denote the blow up of $\mathbb{P}^2$ at $\operatorname{Sing}\mathcal{B}$, to simplify denote $E_k = D_{6+k}$ the exceptional divisor coming from p_k. Consider the reduced LAC surface $M(\mathcal{B}, I)$ where I consists of three triple points and two double ones. The simplest case is $I = \{p_1, p_2, p_3, p_4, p_5\}$.

Theorem 1.4.1. *The reduced LAC surface $M(\mathcal{B}, I)$ has infinite fundamental group and finite first (integral) homology group.*

Proof. Consider the meridians γ_{p_j} around p_j for $j = 1, \ldots, 5$, which by Lemmas 1.2.14 and 1.2.15 are given by

$$\begin{array}{lll} \gamma_{p_1} = x_4x_3x_2, & \gamma_{p_3} = x_4x_1, & \gamma_{p_5} = x_3^{x_2x_1}x_5x_4, \\ \gamma_{p_2} = x_5x_2, & \gamma_{p_4} = x_5x_3^{x_2}x_1. & \end{array} \tag{1.13}$$

By the corollary 1.3.7 a presentation of $\pi_1(M(\mathcal{B}, I))$ can be obtained by

$$H := \pi_1(M(\mathcal{B}, I)) = \pi_1(\mathbb{P}^2 \setminus \mathcal{B}, q)/\langle\langle \gamma_{p_1}, \gamma_{p_2}, \gamma_{p_3}, \gamma_{p_4}, \gamma_{p_5} \rangle\rangle.$$

By making $\gamma_{p_2} = 1$ and $\gamma_{p_3} = 1$ we obtain $x_5 = x_2^{-1}$ and $x_4 = x_1^{-1}$, replacing them in (1.12) and (1.13), we obtain the following presentation for H

$$\langle x_1, x_2, x_3 | [x_1^{-1}, x_3, x_2], [x_2^{-1}, x_3^{x_2}, x_1], x_1 = x_3x_2, x_2 = x_3^{x_2}x_1, x_3^{x_2x_1} = x_1x_2 \rangle$$

By replacing x_1 by x_3x_2 the relation $[x_2^{-1}x_3^{-1}, x_3, x_2]$ becomes trivial. So we are left with:

$$H = \langle x_2, x_3 \mid [x_2^{-1}, x_3^{x_2}, x_3x_2],\ x_2 = x_3^{x_2}x_3x_2,\ x_3^{x_2x_3x_2} = x_3x_2x_2 \rangle$$

By writing down the relations:

$$x_2^{-2}(x_3x_2)^2 = x_2^{-1}x_3x_2x_3 = x_3x_2^{-1}x_3x_2 \tag{1.14}$$

$$x_2^2 = (x_3x_2)^2 \tag{1.15}$$

$$(x_3x_2)^2 = x_2(x_3x_2)^2x_2 \tag{1.16}$$

By replacing (1.15) in (1.16) we obtain that $x_2^2 = 1$, hence $(x_3x_2)^2 = 1$. Note that these two relations include all the precedent. Therefore we obtain the presentation

$$H = \langle x_2, x_3 \mid x_2^2 = 1, (x_3x_2)^2 = 1 \rangle$$

which can be seen either as $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ or as $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$, by this we see that H is infinite and its abelianization is finite. □

We can clarify this example geometrically by means of the following proposition.

Proposition 1.4.2. *There is an orbifold morphism from $M(\mathcal{B}, I)$ to $\mathcal{X}(\mathbb{P}^1, D, r)$ where $D = [0:1] + [1:-1] + [1:0]$ and $r = (2, +\infty, 2)$. The morphism comes from a pencil of conics and induces an isomorphism between orbifold fundamental groups.*

Proof. Consider a pencil $\mathscr{P}$ having 4 fixed points in general position, which we may assume to be $S = \{p_1, p_4, p_5, p_7\}$. If we let $Q_1 = (z_1^2 - z_2^2), Q_2 = (z_1^2 - z_3^2)$ and $Q_3 = (z_2^2 - z_3^2)$ we have that the complete quadrilateral $\mathscr{A}$ is given by $Q = Q_1Q_2Q_3 = 0$.

The pencil $\mathscr{P}$ can be written as $\mathscr{P} = aQ_1 - bQ_2$ with $a, b \in \mathbb{C}$ not both zero. Note that $Q_3 \in \mathscr{P}$ as $Q_3 = Q_2 - Q_1$. This pencil defines a rational map

$$f_{\mathscr{P}} : \mathbb{P}^2 \to \mathbb{P}^1,\ (z_1 : z_2 : z_3) \mapsto (Q_1(z_1 : z_2 : z_3), Q_2(z_1 : z_2 : z_3))$$

whose indeterminacy locus is S. By blowing it up, we obtain a regular map $\tilde{f} : \mathrm{Bl}_S \mathbb{P}^2 \to \mathbb{P}^1$ with fiber over $(a : b)$ the strict transform of $aQ_1 - bQ_2$.

As any point lying in two elements of the pencil is a fixed point of it, for any $x \in \mathbb{P}^2 \setminus S$ there is a unique curve $C \in \mathscr{P}$ passing through it. In particular for the double points $p_2 \in \{z_1 - z_2 = 0\} \cap \{z_1 + z_2 = 0\}$ and $p_3 \in \{z_1 - z_3 = 0\} \cap \{z_1 + z_3 = 0\}$ the curves are Q_1 and Q_2 respectively. This allows us to extend $\tilde{f}$ to the blow up of $\mathrm{Bl}_S \mathbb{P}^2$ at p_2, p_3 as $f : \mathrm{Bl}_{S \cup \{p_2, p_3\}} \to \mathbb{P}^1$. We have that $f(E_2) = (1 : 0)$ and $f(E_3) = (0 : 1)$. Let $X = \mathrm{Bl}_{S \cup \{p_2, p_3\}} \setminus \{Q \cup E_7\}$. Note that $f|_X : X \to \mathbb{P}^1 \setminus \{(1 : 1)\}$ as $f(Q_3) = (1, 1)$.

Moreover $f|_X$ has double fibers at $(0 : 1)$ and $(1 : 0)$. For any other $(a : b) \in \mathbb{P}^1 \setminus \{(1 : 1)\}$ the fiber is the strict transform of $aQ_1 - bQ_2$ minus one point (corresponding to the intersection with E_7). The former assertion can be seen by local computations: Consider $\mathbb{P}^2$ and $\mathbb{P}^1$ with coordinates $(z_1 : z_2 : z_3)$ and (u, v) respectively. Restricting to the standard open sets $W_3 = \{z_3 = 1\} \subset \mathbb{P}^2$ and $V_2 = \{v = 1\} \subset \mathbb{P}^1$ we have that

$$\tilde{f}|_{W_3} = \frac{z_1^2 - z_2^2}{z_1^2 - 1}$$

with $z_1^2 - 1 \neq 0$. Blowing up at $p_2 = (0, 0)$ and working in coordinates (z_1, Z_2) (where Z_2 is the coordinate in $U_1 \subset \mathbb{P}^1$) we have that

$$f|_{W_3 \cap U_1} = z_1^2 \frac{1 - Z_2^2}{z_1^2 - 1}.$$

Analogous computations for the other open sets and for p_3 show that the fibers are indeed double. The last part of the statement is then clear. □

There is a modification of Dimca's suggestion that may still hold.

Question 7. *Let X be a reduced LAC surface with finite first (integral) homology group $H_1(X)$ whose universal abelian cover has also finite first homology group. Is $\pi_1(X)$ finite?*

Remark 1.4.3. The complete quadrilateral belongs to a certain class of arrangements $\mathscr{A}$ called nets (more generally the arrangement is a multinet, see [FY07], [MB09]). Every net $\mathscr{A}$ can be seen as the union of the closure of singular fibers $\mathscr{A}_i := \overline{f^{-1}(x_i)}$ for a rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with $x_i \in \mathbb{P}^1$ and such that the irreducible components of $\mathscr{A}_i$ are lines.

The LAC surface obtained in Theorem 1.4.1 suggests that this class of arrangements could provide further examples of LAC surfaces with infinite fundamental group and finite first homology group. However, as the number

of lines increase, the computations become more involved and other methods are to be used. Indeed, in [AA20b] the fundamental group of some partial compactifications of an infinite family of nets called Ceva(n) defined by equations $(z_1^n - z_2^n)(z_1^n - z_3^n)(z_2^n - z_3^n) = 0$ with $n \in \mathbb{N}$ is treated. The complete quadrilateral then corresponds to $n = 2$, this is, to Ceva(2).

Remark 1.4.4. Let X be a quasi-projective surface. The space of characters $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \cong \mathrm{Hom}(H_1(X), \mathbb{C}^*)$ and a class of subvarieties V_k called characteristic varieties are object of much recent study, see for example [Dim10], [ABCAM13] and reference there-in for definitions and applications. The notion of orbifold morphism is used in [ABCAM13] to give a more precise description of the irreducible components of V_k, in particular of the isolated points.

By Proposition 1.4.2 the fundamental group of $M(\mathcal{B}, I)$ (with the notation used there) is isomorphic to that of $\pi_1(\mathcal{X}(, \mathbb{P}^1, D, r))$ and therefore their characteristic varieties are isomorphic. By [ABCAM13, Prop. 2.10] the first characteristic variety Σ_1 has an unique isolated torsion point v and therefore it correspond to that of point (1) in [ABCAM13, Thm. 1].

By construction, the LAC surfaces provide a potential class of quasi-projective surfaces where examples of isolated components of characteristic varieties could arise. Some further investigation on these lines could be pursued.

1.4.2 Presentation for a weighted complete quadrilateral

By considering weighted LAC surfaces $\mathcal{X}(\bar{X}, D, r)$ we can study the ramified covers of $\bar{X}$ over D. In the case where all the lines of D have the same weight Hirzebruch constructed a finite abelian cover in [Hir83]. If moreover we ask the cover to be a quotient of the ball, Deligne-Mostow have given weights (not necessarily equal) for this to hold [DM86].

Consider again the complete quadrilateral $\mathcal{B} = \{L_1, \ldots, L_6\}$ with the same notation as in 1.4.1, suppose L_6 is the line at infinity. Let $\tilde{X} = \mathrm{Bl}_S \mathbb{P}^2 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the four triple points $S = \{p_1, p_4, p_5, p_7\}$ and E_1, E_4, E_5, E_7 be the respective exceptional divisors.

Consider the elementary geometric base $\Gamma^{(0)} = (x_1, \ldots, x_5)$. A meridian x_6 for the line at infinity around the point $\Sigma^{(0)} \cap L_6$ (recall that $\Sigma^{(0)}$ is the line where $\Gamma^{(0)}$ lies) is given by Lemma 1.2.13

$$x_6 = (x_5 x_4 x_3 x_2 x_1)^{-1}. \tag{1.17}$$

Denote by γ_{p_i} the meridian around E_i. By Lemma 1.2.14, using respectively the elementary geometric bases $\Gamma^{(0)}$ and $\Gamma^{(3)}$ of (1.11), we obtain:

$$\begin{aligned} y_1 &:= \gamma_{p_1} = x_4 x_3 x_2 \\ y_2 &:= \gamma_{p_4} = x_5 x_3^{x_2} x_1 \end{aligned} \tag{1.18}$$

Finally, the meridians around the triple points lying in L_6 are given by Lemma 1.2.15 and bases $\Gamma^{(4)}$ and $\Gamma^{(6)}$ of (1.11).

$$\begin{aligned} y_3 &:= \gamma_{p_5} = x_3^{x_2x_1}x_5x_4 \\ y_4 &:= \gamma_{p_7} = x_4^{-1}x_5^{-1}ax_3^{a^{-1}x_2x_1a} \end{aligned} \tag{1.19}$$

where $a = (x_2x_1)^{x_3x_2x_1}x_5x_4$.

Proposition 1.4.5. *Let $\mathcal{B}$ be the complete quadrilateral, $\tilde{X}$, $\Gamma^{(0)} = (x_1, \dots, x_5)$ and y_i as above. For any $r = (m_1, \dots, m_4, n_1, \dots, n_6) \in (\mathbb{N}^* \cup +\infty)^{10}$ as in [Tre16] p.110, $D = E_1 + E_4 + E_5 + E_7 + \sum_{i=1}^{6} L_i$ we have a presentation for the fundamental group of the ball quotient $\mathcal{X}(\tilde{X}, D, r)$ given by*

$$\pi_1(\mathcal{X}(\tilde{X}, D, r)) = \langle x_1, \dots, x_5 | [x_4, x_1], [x_5, x_2], [x_4, x_3, x_2], [x_5, x_3^{x_2}, x_1], x_i^{n_i}, y_i^{m_i} \rangle$$

Chapter 2

The fundamental group of quotients of products of some topological spaces by a finite groups

In view of the previous chapter, we can try to study a certain class of arrangements closely related with fibrations over curves, and more specifically, with fibrations birationally equivalent to a quotient of product of curves.

In doing so, we noted that the results of [BCGP12] were valid in a more general setting. We present here the details of this generalizations as well as some applications. These results appeared in [AA20b].

2.1 Preliminaries

2.1.1 Properties of fundamental group of topological stacks

Let X be a connected, semi-locally simply connected and locally path-connected topological space and G a finite group acting continuously on it.

Fiber homotopy exact sequence

There exists a homotopy theory for stacks and the existence of the long exact sequence of homotopy, see [Noo14], is more general than what follows, however we only need the following case: consider the topological stack $\mathcal{X} = [X/G]$, a point $x \in X$ and denote by $\bar{x} \in \mathcal{X}$ the image of x. We have an associated fibration $G \to X \to \mathcal{X}$ and a long exact sequence of homotopy groups,

$$\ldots \to \pi_{n+1}(\mathcal{X}, \bar{x}) \to \pi_n(G, Id) \to \pi_n(X, x) \to \pi_n(\mathcal{X}, \bar{x}) \to \pi_{n-1}(G, \mathrm{Id}) \ldots$$

the map $\pi_n(G, Id) \to \pi_n(X, x)$ is induced by the orbit $G \cdot x \hookrightarrow X$.

Action on the universal cover

The hypothesis made on X ensures that there exists an universal cover $\tilde{X}$ and moreover, if we let $\mathcal{X} = [X/G]$ as in 2.1.1, we have an action of $\pi_1(\mathcal{X}, \bar{x})$ over $\tilde{X}$ (see 2.2.2). We will use several times the following lemma in what follows:

Lemma 2.1.1. *Consider the action of $\pi_1(\mathcal{X}, \bar{x})$ in $\tilde{X}$, let $y \in \tilde{X}$ and denote by I_y the isotropy group of the action. Then there exists a monomorphism $I_y \to G$.*

Proof. By 2.1.1 we obtain a short exact sequence

$$1 \to \pi_1(X, x) \to \pi_1(\mathcal{X}, \bar{x}) \to G \to 1,$$

as the action of $\pi_1(X, x)$ over $\tilde{X}$ is free, we obtain that the restriction of $\pi_1(\mathcal{X}, \bar{x}) \to G$ to I_y is injective. □

2.1.2 Product of topological spaces

Fundamental group of the quotient of a product

For $i = 1, \ldots, k$ let X_i as in 2.1.1 be a connected, semi-locally simply connected and locally path-connected topological space and G a finite group acting on each of them.

By 2.1.1 we have k exact sequences

$$1 \to \pi_1(X_i, x_i) \to \pi_1(\mathcal{X}_i, \bar{x}_i) \xrightarrow{\varphi_i} (G, \mathrm{Id}) \to 1 \tag{2.1}$$

where $\mathcal{X}_i = [X_i/G]$, $x_i \in X_i$ and its image in $\mathcal{X}_i$ is denoted by $\bar{x}_i$.

Denote by $\mathbb{H} := \pi_1(\mathcal{X}_1, x_1) \times_G \ldots \times_G \pi_k(\mathcal{X}_k, x_k)$. The exact sequences in (2.1) can be assembled as follows

$$1 \to \pi_1(X_1 \times \ldots \times X_k, x) \to \mathbb{H} \to G \to 1 \tag{2.2}$$

with $x = (x_1, \ldots, x_k)$. The geometric nature of $\mathbb{H}$ is shown in the following Lemma.

Lemma 2.1.2. *Let G act diagonally over $X = X_1 \times \ldots \times X_k$. Consider the stack $\mathcal{X} = [X/G]$ then $\pi_1(\mathcal{X}, \bar{x}) \cong \mathbb{H}$.*

Proof. We have natural projection maps $\mathcal{X} \to \mathcal{X}_i$ for $i = 1, \ldots, k$, which together with the morphisms $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \to G$ and the universal property of the fiber product give us a morphism $\pi_1(\mathcal{X}, \bar{x}) \to \mathbb{H}$. By the exact sequence of a fibration 2.1.1 applied to the action of G to $X_1 \times \ldots \times X_k$ and by (2.2) we obtain

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_1 \times \ldots \times X_k, x) & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \pi_1(X_1 \times \ldots \times X_k, x) & \longrightarrow & \mathbb{H} & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

which implies the result. □

Lemma 2.1.3. *Let X, X_i and G be as above. Then*

$$\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N \cong \pi_1(\mathcal{X}, \bar{x})/\mathbf{I}$$

where N is the normal subgroup generated by the image of the inertia groups I_x and $\mathbf{I}$ is the subgroup generated by the elements of $\pi_1(\mathcal{X})$ having fixed points in the universal cover of $X_1 \times \ldots \times X_k$.

Proof. By [Noo08, Thm 8.3 i)] we have that $\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N$.

The group $\pi_1(\mathcal{X}, \bar{x})$ acts over $\tilde{X} \cong \tilde{X_1} \times \ldots \times \tilde{X_k}$ the universal cover of $X_1 \times \ldots \times X_k$ in such a way that $[(\tilde{X_1} \times \ldots \times \tilde{X_k})/\pi_1(\mathcal{X}, \bar{x})] \cong \mathcal{X}$. As G is finite, by Lemma 2.1.1 any stabilizer I_x for the action of $\pi_1(\mathcal{X})$ over $\tilde{X}$ is finite, therefore it has the slice property and by [Noo08, Thm 9.1] we obtain that $\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/\mathbf{I}$.

□

2.2 The fundamental group of the product of topological spaces

2.2.1 Constructing the homomorphism

Finite index of the group in the product

Let I_y denote the isotropy at the point y in $\tilde{X}$ for the action of $\pi_1(\mathcal{X}, \bar{x})$. By Lemma 2.1.1 the map $\pi_1(\mathcal{X}, \bar{x}) \to G$ restricted to I_y is injective, therefore we can identify I_y with a subgroup of G. When we do such identification we will denote it by $I'_y < G$.

Now as $\pi_1(\mathcal{X}, \bar{x}) \cong \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \ldots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$, if $y = (y_1, \ldots, y_k)$ we define $I_i < \pi_1(\mathcal{X}_i, \bar{x}_i)$ as the image of I_y via the morphism $\pi_1(\mathcal{X}, \bar{x}) \to \pi_1(\mathcal{X}_i, \bar{x}_i)$.

Lemma 2.2.1. *We have that $I_y \cong I_i$ for all $i = 1, \ldots, k$ and $I_y = I_1 \times_{I'_y} \ldots \times_{I'_y} I_k$.*

Proof. For $\gamma = (\gamma_1, \ldots, \gamma_k) \in I_y$ note that $\gamma_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$ fixes $y_i \in \tilde{X}_i$, otherwise γ can not fix a point in $\tilde{X}$. As above, the restriction of $\pi_1(\mathcal{X}_i, \bar{x}_i) \to G$ to I_{y_i} is injective and as $I_i \subset I_{y_i}$ we have that $\gamma_i \neq \beta_i$ for $\gamma, \beta \in I_y \subset \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \ldots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$ with $\gamma \neq \beta$. Therefore we can construct an inverse to the projection. The result follows. □

Note that we obtain that $I_i < I_{y_i}$, but in general I_{y_i} can be bigger.

Define a homomorphism $I_y \to \prod I_{y_i}$ given by decomposing an element in its components. By Lemma 2.2.1 it is injective. Denote by N the subgroup in $\pi_1(\mathcal{X}, \bar{x})$ generated by all the I_y and by N'_i the subgroup in $\pi_1(\mathcal{X}_i, \bar{x}_i)$ generated by I_i.

Lemma 2.2.2. *The subgroup N'_i is normal in $\pi_1(\mathcal{X}_i, \bar{x}_i)$.*

Proof. Let $\gamma_i \in N_i'$ and $t_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$. We can write $\gamma_i = \gamma_{i_1} \cdots \gamma_{i_j}$ with each $\gamma_{i_l} \in I_{i_l}$ coming from $\gamma_l = (\gamma_{1_l}, \ldots, \gamma_{i_l}, \ldots, \gamma_{k_l}) \in I_{y_l} \subset \pi_1(\mathcal{X}, \bar{x})$ and the point $y_l = (y_{1_l}, \ldots, y_{i_l}, \ldots, y_{k_l}) \in X$ for $l = 1, \ldots, j$. As every $\pi_1(\mathcal{X}_j, \bar{x}_j) \to G$ is surjective, for $j = 1, \ldots, i-1, i+1, \ldots, k$, there exists $t_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$ such that $t = (t_1, \ldots, t_k) \in \pi_1(\mathcal{X}, \bar{x})$.

As $t \cdot \gamma_l \cdot t^{-1} \in I_{ty_l}$ it follows that $t_i \gamma_{i_l} t_i^{-1} \in N_i'$ and therefore

$$t_i \gamma_i t_i^{-1} = (t_i \gamma_{i_1} t_i^{-1}) \cdot t_i \cdots t_i^{-1} \cdot (t_i \gamma_{i_j} t_i^{-1}) \in N_i'.$$

□

Proposition 2.2.3. *There is an homomorphism $\pi_1(X/G, [x]) \to \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ such that the image has finite index.*

Proof. By Lemma 2.1.2 we have that $\pi_1(\mathcal{X}, \bar{x}) \cong \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \ldots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$. Therefore there is an injective homomorphism $\pi_1(\mathcal{X}, \bar{x}) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)$.

By Lemma 2.2.2 we obtain the exact sequence

$$1 \to \prod N_i' \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \to 1, \tag{2.3}$$

together with Lemma 2.1.3 we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & 1 & & 1 \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & N & \longrightarrow & \prod N_i' \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i) \\
 & & \downarrow & & \downarrow \\
 & & \pi_1(X/G, [x]) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \\
 & & \downarrow & & \downarrow \\
 & & 1 & & 1
\end{array} \tag{2.4}$$

This diagram provides a homomorphism $\pi_1(X/G, [x]) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ and shows that it is well defined.

We can not complete (2.4) to a commutative diagram of groups with short exact sequence in the rows because usually $\pi_1(\mathcal{X}, \bar{x})$ is not normal in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)$. It will be normal, for example, if G is abelian.

As G is finite we obtain that $\pi_1(\mathcal{X}, \bar{x})$ has finite index in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)$. In fact $[\prod \pi_1(\mathcal{X}_i, \bar{x}_i) : \pi_1(\mathcal{X}, \bar{x})] \leq |G|^{k-1}$: for each surjection $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \to G$ consider a lift $G_i \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$ of G with $|G_i| = |G|$. In $\prod G_i$ consider the equivalence relation $(g_1, \ldots, g_k) \sim (g_1', \ldots, g_k') \Leftrightarrow (\varphi_1(g_1), \ldots, \varphi_k(g_k)) = (g\varphi_1(g_1'), \ldots, g\varphi_k(g_k'))$ with $g \in G$. It is easily seen that $\prod G_i / \sim \cong (G \times \ldots \times G)/\Delta_G$ is a set of representatives of left cosets $(\prod \pi_1(\mathcal{X}_i), \bar{x}_i)/\pi_1(\mathcal{X}, \bar{x})$.

By considering as coset representatives in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ the image of $\prod G_i$ and using the diagram (2.4) we have that $\pi_1(X/G, [x])$ has finite index in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$.

□

2.2.2 The homomorphism has finite kernel

N_i' is finitely normally generated

Let X be a connected, semi-locally simply connected and locally path connected topological space. Let G be a discrete finite group acting on X, $x \in X$ and denote by $\bar{x} \in \mathcal{X} = [X/G]$ the image of the point x and by $p : X \to [X/G]$ the quotient map.

Let us briefly recall the description of $\pi_1(\mathcal{X}, \bar{x})$ as given in [Che01]. It can be defined as $\pi_0(\Omega(\mathcal{X}, \bar{x}))$ where $\Omega(\mathcal{X}, \bar{x})$ denote the space loop of $\mathcal{X}$ pointed at the constant loop of value $\bar{x}$. Every loop is given locally as a map from an open subset of S^1 to a given uniformization of an open subset of $\mathcal{X}_{top}$ and plus some gluing conditions. In our case of a global quotient, a more explicit description of $\Omega(\mathcal{X}, \bar{x})$ can be given as follows:

Let $P(X, x)$ consist of paths in X starting at x. As subspace of $\Lambda(X)$, the free loop space of X, it inherits a structure of a topological space. By considering the constant loop x of value $x \in X$, we obtain $(P(X, x), x)$ a pointed topological space. Define $P(X, G, x)$ as the subspace of $P(X, x) \times G$ consisting of the elements (γ, g) satisfying $\gamma(1) = g \cdot \gamma(0) = g \cdot x$. As a topological space it is pointed at (x, Id_G)

Lemma 2.2.4. *[Che01, Lemma 3.4.2] There exists a natural homeomorphism between the pointed topological spaces* $(\Omega(\mathcal{X}, \bar{x}), x)$ *and* $(P(X, G, x), (x, Id_G))$.

Remark 2.2.5. When $(\mathcal{X}, \bar{x})$ is a pointed topological stack there exists a pointed topological space $(B[R \rightrightarrows X], x')$, where $B[R \rightrightarrows X]$ is the classifying space of the topological grupoid $[R \rightrightarrows X]$, such that we can take $\pi_1(\mathcal{X}, \bar{x}) := \pi_1(B[R \rightrightarrows X], x')$. In the case of a global quotient $\mathcal{X} = [X/G]$ it happens that $B[R \rightrightarrows X]$ equals the Borel construction $X \times_G EG$, see [Noo12].

Now, the construction of Chen also gives a natural isomorphism of $\pi_1(\mathcal{X}, \bar{x})$ and $\pi_1(X \times_G EG, x')$ [Che01, Theorem 3.4.1] linking the two definitions.

There exists a canonical projection $(P(X, G, x), (x, Id_G)) \to (G, Id_G)$ given by sending (γ, g) to g. This map can be seen to be a fibration [Che01, Lemma 3.4.3] having as fiber at Id_G the space loop $\Omega(X, x)$ via the embedding $\Omega(X, x) \hookrightarrow P(X, G, x)$ where γ maps to (γ, Id_G).

With this description at hand, suppose there is $y \in X$ such that it is fixed by an element g, this is, $y \in X^g$. Denote by γ_y a path starting at x and finishing at y, then $\gamma_y(g\gamma_y^{-1}) \in P(X, G, x)$, where $g\gamma_y^{-1}$ denotes the action of g applied to each point of the path.

Lemma 2.2.6. *Let $I_y < G$ denote the inertia (stabilizer) of the action of G at $y \in X$. Every homotopy class of a path $[\gamma_y] \in \pi_1(X, x, y)$ induces an injective morphism $I_y \to \pi_1(\mathcal{X}, x)$.*

Proof. As G is discrete $g \mapsto \gamma_g(g\gamma_g^{-1})$ is continuous, with $g \in I_y$. Then by taking the functor π_0 we got a morphism of groups $\pi_0(I_y) \to \pi_0(P(X, G, x)) = \pi_1(\mathcal{X}, \bar{x})$. Finally, by composing with the projection $(\pi_0(P(X, G, x), x)) \to \pi_0((G, Id_G))$ we obtain that different points under $\pi_0(I_y) \to \pi_1(\mathcal{X}, \bar{x}) \to \pi_0(G)$ have different images, thus the morphism in injective. □

Lemma 2.2.7. *Let $Y \in \pi_0(X^g)$, $y_1, y_2 \in Y$ and let $\gamma_{y_1}, \gamma_{y_2}$ be paths starting at $x \in X$ and finishing at y_1 and y_2 respectively, then $\gamma_{y_1}(g\gamma_{y_1}^{-1})$ is a conjugate of $\gamma_{y_2}(g\gamma_{y_2}^{-1})$ in $\pi_1(\mathcal{X}, \bar{x})$ by elements of $\pi_1(X, x)$.*

Proof. There exists a path $\beta \subset Y$ connecting y_1 and y_2, therefore $\gamma_{y_1}\beta(g\beta^{-1}\gamma_{y_1}^{-1}) \in P(X, G, x)$ but as $g\beta = \beta$ passing to $\pi_0(P(X, G, x), x)$ it equals $[\gamma_{y_1}(g\gamma_{y_1}^{-1})]$.

Now consider the path γ_{y_2}. Note that $\theta := \gamma_{y_1}\beta\gamma_{y_2}^{-1} \in \Omega(X, x)$. There exists a continuous map $\# : P(X, G, x) \times P(X, G, x) \to P(X, G, x)$ which induces the multiplication in the fundamental group (see [Che01, Section 3.1]). The element $\theta\#(\gamma_{y_2}(g\gamma_{y_2}^{-1})\#\theta^{-1})$ can be seen to be $\theta(\gamma_{y_2}\cdot(g(\theta\cdot\gamma_{y_2})^{-1})) \in P(X, G, x)$. By passing to $\pi_1(\mathcal{X}, \bar{x}) = \pi_0(P(X, G, x), x)$ we have that $[\theta][\gamma_{y_2}(g\gamma_{y_2}^{-1})][\theta^{-1}] = [\gamma_{y_1}(g\gamma_{y_1}^{-1})]$. □

Recall that given (X, x) as above, we have a pointed universal cover map $p : (\tilde{X}, \tilde{x}) \to (X, x)$ where $\tilde{x}$ represents the constant loop of value x. Every element in $\gamma \in \pi_1(X)$ correspond to a point in $p^{-1}(x)$. So given a pointed map $p_\gamma : (\tilde{X}, \gamma) \to (X, x)$ it induces a deck transformation of $\tilde{X}$ in the following way: given $y \in \tilde{X}$ take a path $\alpha_y \subset \tilde{X}$ starting at γ and finishing at y. Consider the unique lift $\widetilde{p_\gamma(\alpha_y)} \subset \tilde{X}$ starting at x and assign to y the point $\widetilde{p_\gamma(\alpha_y)}(1)$. It can be seen to be a well-defined map (See [Hat00]).

Now, by the description given above of $\pi_1(\mathcal{X}, \bar{x})$, any $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ such that $\varphi(\gamma) = g$ (recall that $\varphi : \pi_1(\mathcal{X}, \bar{x}) \to G$) have as a representative an element in $P(X, G, x)$ which we still denote by γ. So γ starts at x and finishes at gx. Denote by $\tilde{\pi} : (\tilde{X}, \tilde{x}) \to (\mathcal{X}, \bar{x})$ the universal cover morphism, note that $\tilde{\pi}_\gamma : (\tilde{X}, \gamma) \to (\mathcal{X}, \bar{x})$ is also a cover morphism. By [Che01, Thm 4.1.6] we obtain a deck transformation in the following way: given $y \in \tilde{X}$ take a path $\alpha_y \subset \tilde{X}$ starting at γ and ending at y. Using the notation of the precedent paragraph, the path $p_\gamma(\alpha_y)$ starts at gx. Then the path $g^{-1}p_\gamma(\alpha_y)$ starts at x so we can lift it to $g^{-1}\widetilde{p_\gamma}(\alpha_y)$ in $(\tilde{X}, \tilde{x})$, the end point of this lift is then defined as the image of y. It is shown that it is a well defined map and does not depend on the path chosen.

Lemma 2.2.8. *Let $y \in X$ be fixed by $g \in G$, consider a path γ_y connecting x and y. Consider the action of $\pi_1(\mathcal{X}, \bar{x})$ on $\tilde{X}$ given by deck transformations $Deck(\tilde{X}, \mathcal{X})$, then the element $\gamma_y(g\gamma_y^{-1}) \in \pi_1(\mathcal{X}, \bar{x})$ fixes a point in $\tilde{X}$. Moreover, any element of $\pi_1(\mathcal{X}, \bar{x})$ fixing a point in $\tilde{X}$ is of this form.*

Proof. As the endpoint of $\gamma_y(g\gamma_y^{-1})$ is gx we have a pointed covering morphism $\tilde{\pi}_{\gamma_y(g\gamma_y^{-1})} : (\tilde{X}, \gamma_y(g\gamma_y^{-1})) \to (\mathcal{X}, \bar{x})$, we can consider $g\gamma_y$ as a path in $\tilde{X}$ connecting $\gamma_y(g\gamma_y^{-1})$ and γ_y as follows: define $f(t) = \gamma(g\gamma_y^{-1}) \cdot (g\gamma_y|_t)$ where $g\gamma_y|_t(t') := g\gamma_y(t'/t)$ denote the path starting at gx and finishing at $g\gamma_y(t)$ in time 1 for $t \neq 0$ and being the constant path with value gx if $t = 0$. We project then $f(t)$ to X and obtain $g\gamma_y$ which starts at gx and finishes at $\bar{y}$. By the discussion before the lemma, we obtain that it lifts to γ_y in $(\tilde{X}, \tilde{x})$, as g fixes y we obtain that the point $\gamma_y \in \tilde{X}$ is fixed by the induced deck transformation.

Consider the exact sequence

$$1 \to \pi_1(X, x) \to \pi_1(\mathcal{X}, \bar{x}) \xrightarrow{\varphi} G \to 1,$$

let $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ and $z \in \tilde{X}$ such that γ fixes z. Let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ be the projection, as it is φ-invariant we have that $\varphi(\gamma)p(z) = p(z)$. Then by considering the path in X corresponding to z, we can construct an element $z\varphi(\gamma)z^{-1}$, which fixes $z \in \tilde{X}$. As in the isotropy φ is injective by Lemma 2.1.1, we have that $z\varphi(\gamma)z^{-1} = \gamma$.

□

Proposition 2.2.9. *Suppose that there are only a finite number of elements in $\pi_0(X^g)$ for each $g \in G$, then there exists a finite set $L \subset \pi_1(\mathcal{X}, \bar{x})$ consisting of elements having fixed points in $\tilde{X}$ such that if $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ fixes a point in $\tilde{X}$ then it is conjugate to an element of L by elements in $\pi_1(X, x)$.*

Proof. By Lemma 2.2.7 for every element in $Y \in \pi_0(X^g)$ it suffices to fix an element $\gamma_y(g\gamma_y^{-1})$ with $y \in Y$. For every $g \in G$ and every element in $\pi_0(X^g)$ we pick such an element. We define L the set consisting of such elements. By Lemma 2.2.8 every such element fixes a point in $\tilde{X}$ and any other fixing a point will be conjugate of the element in L corresponding to its connected component. □

Proof that the homomorphism has finite kernel

Let us return to the case of k-topological spaces $X_1, \ldots, X_k$ and let G be a finite group acting on each one of them on the left as in 2.1.2. The Proposition 2.2.9 gives us k subsets $L(\mathcal{X}_i) \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$ whose elements correspond to the element of $\pi_0(X_i^g)$ with $g \in G$. Now consider the subsets $L_i \subset L(\mathcal{X}_i)$ consisting of elements corresponding to $\pi_0(X_i^g)$ where g fixes a point in X_i for $i = 1, \ldots, k$.

Recall that $N < \pi_1(\mathcal{X}, \bar{x})$ (with $\mathcal{X} = [(X_1 \times \ldots \times X_k)/G]$) is the subgroup generated by the inertia subgroups I_y given by the action of $\pi_1(\mathcal{X}, \bar{x})$ in $\tilde{X}$ and $N_i' < \pi_1(\mathcal{X}_i, \bar{x}_i)$ is the image of the i-projection of N. The following Lemma is immediate from Proposition 2.2.9

Lemma 2.2.10. *We have that $N_i' = \left\langle \gamma_i l_i \gamma_i^{-1} \mid l_i \in L_i, \gamma_i \in \pi_1(X_i, x_i) \right\rangle$ in $\pi_1(\mathcal{X}_i, \bar{x}_i)$ for $i = 1, \ldots, k$.*

Definition 2.2.1. Let us define

$$C_i = C_i(\pi_1(X_i), L_i) := \left\langle \left\langle \gamma_i l_i \gamma_i^{-1} l_i^{-1} \mid \gamma_i \in \pi_1(X_i, x_i), l_i \in L_i \right\rangle \right\rangle_{\pi_1(\mathcal{X}_i, \bar{x}_i)},$$

to be the normal subgroup generated by the commutators of elements in $\pi_1(X_i, x_i)$ and in L_i. Denote by $\mathbb{T}_i := \pi_1(\mathcal{X}_i, \bar{x}_i)/C_i$ and by $\hat{L}_i$ the image of L_i in $\mathbb{T}_i$.

Lemma 2.2.11. *It happens that $C_i < N_i'$ and moreover we can consider C_i as a subgroup of N via $\{e\} \times \ldots \times C_i \times \ldots \times \{e\}$ and $C_1 \times \ldots \times C_k < N$.*

Proof. Let $l_i \in L_i$ and $\gamma_i \in \pi_1(X_i, x_i)$, the elements of L_i were chosen such that there exists $l_j \in L_j$ and $y \in \tilde{X}$ such that $l = (l_1, \ldots, l_i, \ldots, l_k) \in I_y < N$. We have that $\gamma_i' = (e, \ldots, \gamma_i, \ldots, e) \in \pi_1(\mathcal{X}, \bar{x})$ and as N is normal in $\pi_1(\mathcal{X}, \bar{x})$ we have that $\gamma_i' l \gamma_i'^{-1} \in N$, so

$$\gamma_i' l \gamma_i'^{-1} l^{-1} = (e, \ldots, \gamma_i l_i \gamma_i^{-1} l_i^{-1}, \ldots, e) \in N$$

This element projects to $[\gamma_i, l_i] \in C_i$. Finally given $\beta_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$, as every φ_j is surjective, there exists $\beta_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$ such that $\varphi_i(\beta_i) = \varphi_j(\beta_j)$, so $\beta = (\beta_1, \ldots, \beta_k) \in \pi_1(\mathcal{X}, \bar{x})$ and every conjugate of $[\gamma_i, l_i]$ can be seen as an element of N.

Finally, by considering the product of the identification of the elements in C_j we have that $C_1 \times \ldots \times C_k < N$. □

Before stating the next lemma recall that $N < N_1' \times_G \ldots \times_G N_k'$.

Lemma 2.2.12. *The subgroup C_i has finite index in N_i', in particular $C_1 \times \ldots \times C_k$ has finite index in $N_1' \times \ldots \times N_k'$ hence also in N.*

Proof. First note that by Lemma 2.2.10 and by definition of $\mathbb{T}_i$ we have that $N_i'/C_i = \langle\langle L_i \rangle\rangle_{\pi_1(X,x)} / C_i \cong \langle\langle \hat{L}_i \rangle\rangle_{R_i} = \langle \hat{L}_i \rangle$, with R_i the image of $\pi_1(X_i, x_i)$ in $\mathbb{T}_i$.

Moreover as $\varphi(C_i) = \{e\}$ we have that $C_i < \ker \varphi \cong \pi_1(X_i, x_i)$. As $\pi_1(X_i, x_i)$ has finite index in $\pi_1(\mathcal{X}_i, \bar{x}_i)$, it follows that R_i has finite index in $\mathbb{T}_i$, which implies that $R_i \cap \langle \hat{L}_i \rangle$ has finite index in $\langle \hat{L}_i \rangle$. Note that $\langle \hat{L}_i \rangle$ is generated by a finite number of torsion elements and that by construction $R_i \cap \langle \hat{L}_i \rangle$ is a central group in $\langle \hat{L}_i \rangle$. As any group generated by a finite number of torsion elements and such that the center has finite index is finite (see [BCGP12, Lem. 4.6]) the result follows.

□

Theorem 2.2.13. *The homomorphism $\pi_1(X/G, [x]) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ has finite kernel.*

Proof. By composing the quotient map $\prod \pi_1(\mathcal{X}_i, \bar{x}_i) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ with the inclusion $\pi_1(\mathcal{X}, \bar{x}) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)$ we obtain $\pi_1(\mathcal{X}, \bar{x}) \to \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ with kernel $N_1' \times \ldots \times N_k' \cap \pi_1(\mathcal{X}, \bar{x}) = N_1' \times_G \ldots \times_G N_k'$ by the description of

$\pi_1(\mathcal{X}, \bar{x})$ as fiber product. We put this as a row in the following commutative diagram together with a vertical column given by Lemma 2.1.3 and complete to

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & N & \xrightarrow{\quad=\quad} & N & & \\
 & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N_1' \times_G \ldots \times_G N_k' & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \\
 & & \downarrow & & \downarrow & & \downarrow{=} \\
1 & \longrightarrow & N_1' \times_G \ldots \times_G N_k'/N & \longrightarrow & \pi_1(X/G, [x]) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
\end{array} \tag{2.5}$$

By Lemma 2.2.12 both $N_1' \times_G \ldots \times_G N_k'/C_1 \times \ldots \times C_k$ and $N/C_1 \times \ldots \times C_k$ are finite hence

$$\frac{N_1' \times_G \ldots \times_G N_k'/C_1 \times \ldots \times C_k}{N/C_1 \times \ldots \times C_k} \cong N_1' \times_G \ldots \times_G N_k'/N$$

is finite.

□

Geometric interpretation of the groups $\pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$

Let us denote by I, the subgroup of G generated by the elements having a fixed point in every X_i for $i = 1, \ldots, k$. Note that I is a normal subgroup.

Let x_i' denote the class of x_i in X/I and $\bar{x'}_i$ the image of x_i' in $[(X_i/I)/(G/I)]$.

Proposition 2.2.14. *There is an isomorphism*

$$\pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \xrightarrow{\sim} \pi_1([(X_i/I)/(G/I)], \bar{x'}_i).$$

Proof. Observe that the action of G on X_i descends to an action of G/I on X_i/I and therefore we can define $[(X_i/I)/(G/I)]$. Recall by the previous subsection 2.2.2 that $\pi_1(\mathcal{X}_i, \bar{x}_i)$ can be identified with the set of path-components of $P(X_i, G, x)$. Therefore an element $[\gamma] \in \pi_1(\mathcal{X}_i, \bar{x}_i)$ can be represented by a path γ in X_i starting at x_i and finishing at gx_i for some $g \in G$. Denote by $p_i : X_i \to X_i/I$ the quotient map. By considering $p_i(\gamma)$, we obtain a morphism between $\pi_1(\mathcal{X}_i, \bar{x}_i)$ and $\pi_1([(X_i/I/G/I)], \bar{x'}_i)$.

It is immediate to see that the paths coming from the inertia of I in X_i, this is, the elements of the form $\gamma_y(g\gamma_y^{-1})$ with $g \in I$ and $y \in X_i^g$, are sent to the trivial element in $\pi_1(X_i/I, x_i')$.

Now consider $\gamma \in \ker(\pi(\mathcal{X}_i, \bar{x}_i) \to \pi_1([(X_i/I)\,/\,(G/I)], \bar{x'}_i))$. Then γ is represented by a path in X_i, which we still denote by γ, starting at x_i and finishing at gx_i with $g \in G$. Note that moreover $g \in I$, otherwise by the projection $\pi_1([X_i/I\,/G/I], \bar{x'}_i) \to G$ the element would be sent to a non-zero element. Hence the image of γ lies in $\pi_1(X_i/I, x'_i)$ and it is trivial. By the exact sequence

$$1 \to N_{[X_i/I]} \to \pi_1([X_i/I], \bar{x}_i) \to \pi_1(X/I, x'_i) \to 1$$

and noticing that $N_{[X_i/I]} = N'_i$ we have that $\gamma \in N'_i$ which proves the result. □

2.3 Applications

2.3.1 Product of the same topological space

Now let us describe a case where N'_i equals the whole subgroup N_i generated by the elements having a fixed point in the universal cover.

Corollary 2.3.1. *Let $X_i = X_1$ for $i = 2, \dots, k$ and G finite acting on X_1. Then the morphism $\pi_1((X_1 \times \dots \times X_1)/G, [x]) \to \prod \pi_1(X_1/G, [x_i])$ has finite kernel.*

Proof. We only have to show that $N'_1 = N_1$ and then we obtain the result by applying Theorem 2.2.13. By construction we have that $N'_1 \subset N_1$. Let us show the inverse inclusion. Take $\gamma_1 \in N_1$, then we can write $\gamma_1 = \gamma_{1_1} \cdots \gamma_{1_l}$ such that there exists $y_{1_j} \in \tilde{X}_1$ satisfying $\gamma_{1_j} \in I_{y_{1_j}}$ for $j = 1, \dots, l$. As $\tilde{X} = \tilde{X}_1 \times \dots \times \tilde{X}_k$ by taking $y_j = (y_{1_j}, \dots, y_{1_j}) \in \tilde{X}$ we have that $\gamma^j = (\gamma_{1_j}, \dots, \gamma_{1_j}) \in I_{y_j}$ and therefore $\gamma = \gamma^1 \cdots \gamma^l \in N$ and the image of γ in N_1 equals γ_1. □

Another proof using Proposition 2.2.14 is as follows: The action of G/I is free in X_1/I and $X_1/G \cong X_1/I\,/G/I$ so $\pi_1([X_1/I\,/G/I] = \pi_1(X_1/G)$.

2.3.2 Second Main Theorem

Theorem 2.3.2. *Let $X_1, \dots, X_k$ admit a universal cover and let G be a finite group acting on each of them such that $|\pi_0(X_i^g)| < +\infty$ for every $g \in G$ and $i = 1, \dots, k$. Denote $X = X_1 \times \dots \times X_k$ and consider the diagonal action of G on it. Suppose $\pi_1(X/G, [x])$ is residually finite, then $\pi_1(X/G, [x])$ has a normal finite-index subgroup $\mathcal{N} \cong H_1 \times \dots \times H_k$ isomorphic to a product of normal finite index subgroups subgroups $H_i < \pi_1(X_i/I, [x_i])$.*

Proof. By Theorem B.7 we get a morphism $\Theta : \pi_1(X/G, [x]) \to \prod \pi_1([X_i/I/G/I])$ having finite kernel E. As $\pi_1(X/G, [x])$ is residually finite we can construct a finite-index normal subgroup $\Gamma \lhd \pi_1(X/G, [x])$ such that $\Gamma \cap E = \{e\}$.

The morphism $\Theta|_\Gamma : \Gamma \to \prod \pi_1([X_i/I\,/G/I], \bar{x'}_i)$ is therefore injective and moreover as $\Theta(\pi_1(X/G)) < \prod \pi_1([X_i/I\,/G/I], \bar{x'}_i)$ has finite index it follows that $\Theta(\Gamma) < \prod \pi_1([X_i/I\,/G/I], \bar{x'}_i)$ has finite index.

For every $i = 1 \ldots, k$, we have $\pi_1(X_i/I, [x_i]) < \pi_1([X_i/I\,/G/I], \bar{x'}_i)$ as a normal finite-index subgroup. Define the subgroup

$$\Theta(\Gamma)_i := \Theta(\Gamma) \cap \{\{e_1\} \times \ldots \times \pi_1(X_i/I, [x_i]) \times \ldots \times \{e_k\}\}$$

where $e_k \in \pi_1(X_j/I, [x_j])$ is the identity element. As $\Theta(\Gamma)_i$ has finite index in $\pi_1(X_i/I, [x_i])$, there exists a normal subgroup of finite index H_i of $\pi_1([X_i/I\,/G/I])$ contained in $\Theta(\Gamma)_i$. Set $H := H_1 \times \ldots \times H_k$, then $H \lhd \Theta(\Gamma)$ and it is a finite-index normal subgroup of $\prod \pi_1([X_i/I\,/G/I], \bar{x'}_i)$. The subgroup $\mathcal{N} := \Theta^{-1}(H) \cap \Gamma$ satisfies the stated properties. □

Case of smooth curves

Corollary 2.3.3. *Let $C_1, \ldots, C_k$ be smooth algebraic curves and let G be a finite group acting on each C_i. Denote $C = C_1 \times \ldots \times C_k$. Consider $\mathcal{C} = [C/G]$ with G acting diagonally on C. Then $\pi_1(C/G)$ has a normal subgroup $\mathcal{N}$ of finite index isomorphic to $\Pi_1 \times \ldots \times \Pi_k$ where Π_i is either a surface group or a finitely generated free group for $i = 1, \ldots, k$.*

By Theorem B.7 we have a morphism $\pi_1(C/G) \to \prod \pi_1([C_i/I\,/G/I])$ with finite kernel, however if the action of G/I is not faithful on C_i/I then $\pi_1([C_i/I\,/G/I]$ is not necessarily an orbifold surface group. This can be overcome as follows: let $K_i := \ker(G/I \to \operatorname{Aut} C_i/I)$ and $H_i := (G/I)/K_i$. Denote by $\mathcal{C}_i := [(C_i/I)/G/I]$ and by $\mathcal{C}'_i := [(C_i/I)/H_i]$, we have a canonical morphism $\mathcal{C}_i \to \mathcal{C}'_i$.

Lemma 2.3.4. *The induced homomorphism $q_i : \pi_1(\mathcal{C}_i) \to \pi_1(\mathcal{C}'_i)$ is surjective and has finite kernel.*

Proof. By choosing a point $x_i \in C_i$ and denoting by $\bar{x}_i$ its image in both $\mathcal{C}_i$ and $\mathcal{C}'_i$ we obtain a fibration $[pt/K, pt] \hookrightarrow (\mathcal{C}_i, \bar{x}_i) \to (\mathcal{C}'_i, \bar{x}_i)$. By taking the long homotopy exact sequence

$$\ldots \to \pi_2(\mathcal{C}'_i, \bar{x}_i) \to \pi_1(pt/K, pt) \to \pi_1(\mathcal{C}_i, \bar{x}_i) \to \pi_1(\mathcal{C}'_i, \bar{x}_i) \to 1,$$

as there is an isomorphism between $\pi_1(pt/K, pt)$ and $\pi_0(K, Id)$, the result follows. □

So by composing, we obtain a morphism $\Theta : \pi_1(C/G) \to \prod \pi_1(\mathcal{C}_i) \to \prod \pi_1(\mathcal{C}'_i)$, this allows us to prove the following Lemma, which together with Theorem 2.3.2 will imply Corollary 2.3.3.

Lemma 2.3.5. *The group $\pi_1(C/G)$ is residually finite.*

Proof. First note that as $\pi_1(\mathcal{C}'_i)$ is an orbifold surface group. In particular it is residually finite.

Now, it follows that $\Theta(\pi_1(C/G))$ is residually finite as it is a finite-index subgroup of a direct product of residually finite groups.

We need another property of these groups to continue. Let H be a group and let $\hat{H}$ be its profinite completion. A group H is called *good* if for each $k \geq 0$ and for each finite H-module M the natural homomorphism

$$H^k(\hat{H}, M) \to H^k(H, M)$$

is an isomorphism. In [GJZZ08, Lem. 3.2, 3.4, Prop. 3.6] it is shown that a finite-index subgroup of a good group is good, the product of good groups is good and that $\pi_1(\mathcal{C})$ for $\mathcal{C}$ an algebraic orbifold curve is good. We obtain therefore that $\Theta(\pi_1(C/G))$ is good.

Finally, [GJZZ08, Proposition 6.1] asserts that if T is a residually finite good group and $\varphi : H \to T$ is a surjective homomorphism with finite kernel then H is residually finite. Applying this to $\Theta' : \pi_1(C/G) \to \Theta(\pi_1(C/G))$ we obtain the result.

□

2.3.3 Ceva(n) as a product-quotient surface

Isotriviality of the family

Let $(z_1 : z_2 : z_3)$ be coordinates for $\mathbb{P}^2$ and $(\lambda : \beta)$ for $\mathbb{P}^1$. For Ceva(n), the fiber over (λ, β) of $\tilde{f}$ are isomorphic to

$$-\lambda(z_1^n - z_2^n) + \beta(z_2^n - z_3^n) = 0$$

having singular fibers only at the values $0, 1$ and ∞ in $\mathbb{P}^1$. Recall that a fibration is called *isotrivial* if its non-singular fibers are all isomorphic.

Lemma 2.3.6. *Let S be the base locus associated to the rational map f of the net* Ceva(n). *Then the fibration $\tilde{f} : \mathrm{Bl}_S(\mathbb{P}^2) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is isotrivial.*

Proof. This follows from the fact that every smooth curve in the pencil $-\lambda(z_1^n - z_2^n) + \beta(z_2^n - z_3^n) = 0$ is isomorphic to $z_1^n + z_2^n + z_3^n = 0$.

By working in the standard open subset $U_1 \subset \mathbb{P}^1$ where $\beta = 1$ we obtain the equation:

$$(-\lambda + 1)z_1^n + \lambda z_2^n - z_3^n = 0$$

for the fiber over the point $(\lambda : 1)$.

If $\lambda \neq 0, 1$ let ϵ, η such that $\epsilon^n = -\lambda + 1$ and $\eta^n = \lambda$. Then the linear biholomorphism of $\mathbb{P}^2$ given by

$$(z_1 : z_2 : z_3) \mapsto (\epsilon z_1 : \eta z_2 : e^{\frac{\pi i}{n}} z_3)$$

takes $z_1^n + z_2^n + z_3^n = 0$ to f_λ.

□

Birationality to a quotient of a product of curves

Any smooth proper family $h : X \to T$ of curves of genus $g \geq 1$ over an algebraic curve T can be rigidified in the following sense: there exists an étale cover $T' \to T$ such that $h' : X' := X \times_T T' \to T'$ can be seen as a pullback of the universal curve in the moduli space of n-level structures $U_g^{(n)} \to M_g^{(n)}$. When h is isotrivial this implies that X' is isomorphic to a product $T' \times C$ for some curve C.

Let us describe the curve T' and its galois group G.

The Fermat curve and its automorphisms

We call the curve $F(n) = \{z_1^n + z_2^n + z_3^n = 0\} \subset \mathbb{P}^2$ *the Fermat curve of degree* n. Denote by $G(n)$ the group of algebraic automorphisms of $F(n)$. Let S_3 be the group of permutations of 3 elements and let it act over $\mathbb{P}^2$ by interchanging coordinates. Then it also acts over $F(n)$.

Let $\mu(n)$ be the group of roots of unity of degree n. Then $H(n) := \mu_1(n) \oplus \mu_2(n) \oplus \mu_3(n)/(\mu_1\mu_2\mu_3 = 1)$ with $\mu_i \in \mu_i(n)$, acts on $F(n)$ by $(\mu_1, \mu_2, \mu_3)(z_1 : z_2 : z_3) = (\mu_1 z_1 : \mu_2 z_2 : \mu z_3)$. The following Theorem is due to Pavlos Tzermias in [Tze95]

Theorem 2.3.7 ([Tze95])**.** *Let $n \geq 4$, $F(n)$ and $G(n)$ be as above. Then $H(n)$ is normal in $G(n)$ and*

$$0 \to H(n) \to G(n) \to S_3 \to 0.$$

F(n) as a cyclic and abelian cover of the projective line

We recall two basic lemmas that will be used in the structure of the quotient $C \times F(n)/G$.

Lemma 2.3.8. *Let $F(n)$ be the Fermat curve of degree n. Then $F(n)$ can be seen as the n-cyclic covering of $\mathbb{P}^1$ ramified over the n-roots of unity.*

Proof. The n-cyclic covering of $\mathbb{P}^1$ ramified over the n-roots of unity has equation in $\mathbb{C}^2$

$$z_2^n = \prod_{i=1}^{n} (z_1 - \xi^i) = z_1^n - 1.$$

By homogenizing we obtain the equation $z_2^n = z_1^n - z_3^n$. □

Lemma 2.3.9. *Let $H(n)$ and $F(n)$ as above. Then $F(n)$ is an abelian cover of $\mathbb{P}^1$ ramified over three points with $H(n)$ as deck transformation group. It has ramification index equal to n over each point above the branching locus.*

Proof. We can realize the covering map via the morphism of $\mathbb{P}^2$: $(z_1 : z_2 : z_3) \mapsto (z_1^n : z_2^n : z_3^n)$ which sends the curve $F(n)$ to the line $\{z_1 + z_2 + z_3 = 0\}$ and ramifies over the three points $(0 : 1 : -1), (1 : 0 : -1), (1 : -1 : 0)$. □

By a version of the Riemann's existence Theorem (c.f. [BCGP12]), the covers above are completely determined by surjective group homomorphisms $\pi_1(\mathcal{X}(\mathbb{P}^1 : n, n, \ldots, n)) \to \mu(n)$ and $\varphi : \pi_1(\mathcal{X}(\mathbb{P}^1 : n, n, n)) \to H(n)$ and the ramification index is given by a Hurwitz's type formula.

Structure of the quotient

This subsection is devoted to prove the following proposition

Proposition 2.3.10. *Let $\pi' : X' \to (F(n) \times F(n))/H(n)$ be the minimal desingularization. Then the fibration $\tilde{f} : \mathrm{Bl}_{\mathrm{Sing}(\mathrm{Ceva}(n))} \to \mathbb{P}^1$ is isomorphic to $X' \to F(n)/H(n) \cong \mathbb{P}^1$.*

Let us first prove:

Lemma 2.3.11. *Consider the family of projective curves*

$$V = \{(\lambda, (z_1 : z_2 : z_3) \in \Delta \times \mathbb{P}^2 \mid \lambda z_1^n = z_2^n + z_3^n\} \to \Delta$$

with Δ denoting the unit disc with coordinate λ. Then the stable reduction $W'' \to \Delta$ of $V \to \Delta$ is isomorphic to $\Delta \times F(n) \to \Delta$ with projection in the first factor with coordinate t and $t^n = \lambda$.

Proof. The central fiber $V_0 = \sum_{i=0}^n L_i$ consists of n lines L_i. Blow up the point $p = (0, (1 : 0 : 0))$ and denote the total transform of V_0 in $\mathrm{Bl}_p V$ by $V_0' = \sum_{i=1}^n L_i + nE_1$ where we have used the same notation L_i for the strict transform of L_i.

Take a cover of degree n of the unit disc $\Delta \to \Delta$ given by $t \mapsto t^n$ and consider the fiber product $W := \Delta \times_\Delta \mathrm{Bl}_p V$. Let W' be the normalization of W. We have the following commutative diagram:

$$\begin{array}{ccccc} & & \overset{\tau}{\frown} & & \\ W' & \longrightarrow & W & \longrightarrow & \mathrm{Bl}_p V \\ \downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \lambda} \\ \Delta & \xrightarrow{=} & \Delta & \xrightarrow{t^n} & \Delta \end{array}$$

In the central fiber, every $w \in W_0$ is the center of local coordinates (x, y) with $t^n = x^a y^b$ being a local equation for W in $\mathbb{C}^3$. As $V_0' = \sum_{i=1}^n L_i + nE_1$ we have three different pairs of values for a and b:

- $a = 1, b = 0$: This happens if the image of w lies in a smooth point of V_0'. As every point of $t^n = x$ is smooth we have $W' = W$.
- $a = n, b = 0$: The image of w lies in E_1 and it is different from its intersections points with L_i. Let ε be a n-th root of unity, then $t^n - x^n = \prod_{i=1}^n (t - \varepsilon^i x) = 0$ and W' decomposes into n pieces with local equations $t = x, \ldots, t = \varepsilon^{n-1} x$ which make W' an unramified cyclic cover of degree n of $\mathrm{Bl}_p V$ locally around the image of w.

- $a = n, b = 1$: The image of w lies in $E_1 \cap L_i$ for some i. The study of the normalization of W amounts to study the existence domain of the function $\sqrt[n]{x^n y}$. Consider

$$U := \{(u, x_1, y_1) \in \mathbb{C}^3 \mid u = x_1 y_1\},$$

for points sufficiently close to the origin, we have a map $U \to W$ given by $(u, x_1, y_1) \mapsto (u, x_1, y_1^n)$. Note that the restriction of the map to $U \setminus \{x_1 y_1 = 0\} \to W \setminus \{xy = 0\}$ is injective and locally surjective. As U is smooth, by the universal property of normalization we obtain a morphism $U \to W'$, the restiction of $\sqrt[n]{x^n y}$ to $W \setminus \{xy = 0\}$ gives a global inverse and we obtain that $U \cong W'$. The morphism $W' \to \mathrm{Bl}_p V$ obtained by composing, realizes W' as a n cyclic cover ramified over L_i.

The action of $\mu(n)$ over Δ induces an action over W hence on W' which respects the fibrations. We have then that $W'/\mu(n) \to \Delta/\mu(n)$ is isomorphic to $\mathrm{Bl}_p V \to \Delta$.

This expresses $\tau : W' \to \mathrm{Bl}_p V$ as a cyclic covering of degree n branched at $D = \sum_{i=1}^n L_i$. As E_1 intersects the branch divisor D in the n-th roots of unity and τ has degree n, by Lemma 2.3.8 we have that $\tau^{-1}(E_1) \cong F(n)$. Therefore we can write as divisors

$$\tau^* V_0' = \sum_{i=1}^{n} n L_i + n F(n).$$

For the central fiber of W' we have $W_0' = \tau^* V_0'/n$, so we have now a reduced fiber. In order to make it stable we show that the L_i are in fact (-1)-curves. Any irreducible component of W_0' has zero intersection number with the whole W_0', this is: $L_i W_0' = 0$ for all i. As L_i intersects only $F(n)$ it follows that $L_i^2 = -1$. Denote by W'' the surface obtained from W' by blowing down the (-1)-curves. The family $W'' \to \Delta$ has all its fibers isomorphic to $F(n)$ and by the local-triviality Theorem of Grauert-Fischer we have that $W'' \cong \Delta \times F(n)$. $\square$

The action of $\mu(n)$ over W' descends to an action of W' respecting the fibers and by taking the quotient we obtain the following diagram

$$\begin{array}{ccc} W' & \longrightarrow & W'/\mu(n) \cong \mathrm{Bl}_p V \\ \updownarrow & & \updownarrow \\ W'' \cong \Delta \times F(n) & \longrightarrow & (\Delta \times F(n))/\mu(n) \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta/\mu(n) \end{array}$$

For $n \geq 3$ the action of $\mu(n)$ over $\Delta \times F(n)$ is diagonal. If $n > 3$ it is easy to show it: let $\xi \in \mu(n)$ and $(x, y) \in \Delta \times F(n)$, as the action is induced from that

of $\mu(n)$ in Δ we have that $\xi \cdot (x, y) = (\xi x, \phi_\xi(x)y)$, where $\phi_\xi(x) \in \mathrm{Aut}(F(n))$ which is finite by $n > 3$, as it must vary continuously on x it follows that it is constant. For $n = 3$ see [Beau, Lemma VI.10].

Note that $W' \to W''$ is a morphism, this induces a morphism $\mathrm{Bl}_p V \to (\Delta \times F(n))/\mu(n)$ which does not contract any (-1)-curves. As $\mathrm{Bl}_p V$ is smooth, by unicity of the minimal resolution of singularities it will follow that $\mathrm{Bl}_p V \to (\Delta \times F(n))/\mu(n)$ is such resolution. Indeed, $(\Delta \times F(n))/\mu(n)$ has n singular points in the central fiber. More generally we have:

Lemma 2.3.12. *Let $H(n)$ act diagonally in $F(n) \times F(n)$. Consider the morphism $g : (F(n) \times F(n))/H(n) \to F(n)/H(n) \cong \mathbb{P}^1$. Then:*

1. *The composition $g \circ \pi' : X' \to \mathbb{P}^1$ has three singular fibers isomorphic to each other.*

2. *Each singular fiber F of $g \circ \pi'$ has a central component Y isomorphic to $\mathbb{P}^1$ with multiplicity n.*

3. *The closure of $F \setminus Y$ has n irreducible disjoint components each one intersecting Y in one point and transversely. They are rational curves with self intersection $-n$ (the simplest Hirzerbruch-Jung strings) coming from the resolution of singularities of type $A_{n,1}$ over $(F(n) \times F(n))/H(n)$ lying over $\pi'(Y)$.*

4. *The reduced curve Y_{red} is a (-1)-curve.*

Proof. The first three points are the application of [Ser96, Theorem 2.1] and Lemmas 2.3.9 and 2.3.8.

Indeed, let $\bar{b} \in F(n)/H(n)$ and H_b the stabilizer of any element in $b \in F(n)$ in the preimage of $\bar{b}$. Then the fibers of $g : (F(n) \times F(n))/H(n) \to F(n)/H(n)$ are isomorphic to $F(n)/H_b$. By considering the projection to the other component $F(n)$ we can count the multiplicity of the fiber.

By Lemma 2.3.9 $F(n)/H(n)$ has only three branching points and the stabilizers are $\mu(n)$. By Lemma 2.3.8 $F(n)/\mu(n)$ has n branching points. The cyclic quotient is given by the unweighted action and therefore we obtain singularities $A_{n,1}$. The resolution of each of them is a rational curve of self intersection $-n$.

By Zariski's Lemma [BPVdVH04, Lemma III.8.2] we conclude that $Y_{red}^2 = -1$ in the following way:

$$0 = F^2 = (\sum_{i=1}^{m} E_i + Y)(\sum_{i=1}^{m} E_i + Y) = -m^2 + m^2 + m^2 + m^2 Y_{red}^2$$

□

Proof of Proposition 2.3.10. By Lemma 2.3.9 we can take $h : F(n) \to \mathbb{P}^1$ to be an abelian covering of $\mathbb{P}^1$ ramified over $0, 1$ and ∞, the three points of

the ramification locus of $\tilde{f}$. By Lemma 2.3.11 and as h looks locally around a branching point as $t \to t^n$, $\tilde{f}$ attains stable reduction over $F(n)$. Denote by W' the normalization of $F(n) \times_{\mathbb{P}^1} \mathrm{Bl}_{\mathrm{Sing}(\mathrm{Ceva}(n))} \mathbb{P}^2$ and by W'' the surface obtained by contracting all (-1)-curves, we have

$$\begin{array}{ccc} W' & \longrightarrow & \mathrm{Bl}_{\mathrm{Sing}(\mathrm{Ceva}(n))} \mathbb{P}^2 \\ \downarrow & & \\ W'' & & \downarrow \tilde{f} \\ \downarrow & & \\ F(n) & \xrightarrow{\quad h \quad} & \mathbb{P}^1 \end{array} \tag{2.6}$$

Let us show that $W'' \cong F(n) \times F(n)$ respecting the fibration.

Locally, around any point in the ramification locus, the fibration has monodromy $\mu(n)$ by Lemma 2.3.11. Over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ the fibration $\tilde{f}$ is a proper smooth family. We have that $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1\gamma_2\gamma_3 = 1 \rangle$ where every γ_i represents a loop going around $0, 1$ or ∞ in the positive direction for $i = 1, 2, 3$ respectively. The covering $F(n) \to \mathbb{P}^1$ is given by the surjective homomorphism $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to H(n) = \mu_1(n) \oplus \mu_2(n) \oplus \mu_3(n) / < \xi_1\xi_2\xi_3 = 1 >$ with $\xi_i \in \mu_i(n)$, where γ_i is sent to a generator of $\mu_i(n)$ for $i = 1, 2, 3$, So the fibration $W'' \to F(n)$ has trivial monodromy and it can be rigidified, by sending it to $M_g^{(n)}$ as in 2.3.3 we obtain that W'' is isomorphic to $F(n) \times F(n) \to F(n)$.

By Lemma 2.3.12 we can complete the diagram 2.6 as follows

$$\begin{array}{ccccc} W' & & \longrightarrow & & \mathrm{Bl}_{\mathrm{Sing}(\mathrm{Ceva}(n))} \mathbb{P}^2 \\ \downarrow & & & \swarrow & \\ W'' \cong F(n) \times F(n) & \longrightarrow & (F(n) \times F(n))/H(n) & & \downarrow \tilde{f} \\ \downarrow & & \downarrow & & \\ F(n) & \xrightarrow{\quad h \quad} & \mathbb{P}^1 & \xrightarrow{\quad = \quad} & \mathbb{P}^1 \end{array}$$

and by unicity of the minimal desingularization we obtain the result. □

Recall that for $S \to S'$ be a resolution of singularities of S', if S' has only quotient singularities, by [Kol93, Thm 7.8.1] we have that $\pi_1(S) \to \pi_1(S')$ is an isomorphism.

Example 2.3.1. Consider the surface $S_1 := (F(n) \times F(n) \setminus \{X_1, \ldots, X_n\})/H(n)$. The subgroup I generated by the elements of $H(n)$ having fixed points both in $F(n)$ and in $F(n) \setminus \{X_1, \ldots, X_n\})$ equals $H(n)$. As $F(n)/H(n) \cong \mathbb{P}^1$, $F(n) \setminus \{X_1, \ldots, X_n\})/H(n) \cong \mathbb{C}$ and by Theorem B.7 the morphism

$$\pi_1(S_1) \to \pi_1(\mathbb{P}^1) \times \pi_1(\mathbb{C})$$

has finite kernel, it follows that $\pi_1(S_1)$ is finite.

The minimal resolution of singularities $S_1' \to S_1$ can be identified with the following partial compactification of Ceva(n). Consider

$$J := \{1, \dots, n, 3n + n^3 + 1\} \subset \{1, \dots, 3n + n^2 + 3\}$$

then following the construction given in 1.3 we have that

$$\mathrm{Bl}_{\mathrm{Sing}\,\mathrm{Ceva}(n)}\,\mathbb{P}^2 \setminus \{\cup_{j\in J} D_j\} \cong S_1'.$$

This is from the surface $\mathrm{Bl}_{\mathrm{Sing}\,\mathrm{Ceva}(n)}\,\mathbb{P}^2$ we remove only the strict transform of $\mathscr{A}_1$ and the exceptional divisor coming from the singular point of $\mathscr{A}_1$. This can be identified with a singular fiber or $\tilde{f}$.

Example 2.3.2. Consider now $S_2 := (F(n) \times F(n) \setminus \{X_i, Y_i\})/H(n)$. In this case the subgroup I, defined as in the previous paragraph, is isomorphic to $\mu(n)$. As $F(n)/\mu(n) \cong \mathbb{P}^1$, $F(n) \setminus \{X_i, Y_i\})/\mu(n) \cong \mathbb{C}^*$ and by Theorem B.7 the morphism

$$\pi_1(S_2) \to \pi_1([\mathbb{P}^1/\mu(n)]) \times \pi_1([\mathbb{C}^*/\mu(n)])$$

has finite kernel and the image is a finite-index subgroup.

By Theorem 2.3.2 and Corollary 2.3.3, we have that $\mathbb{Z} \lhd_{f.i.} \pi_1(S_2)$. As in 2.3.1 the minimal resolution of singularities $S_2' \to S_2$ can be identified with $\mathrm{Bl}_{\mathrm{Sing}\,\mathrm{Ceva}(n)}\,\mathbb{P}^2$ minus two singular fibers of $\tilde{f}$.

Example 2.3.3. If we consider $S_3 := (F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\})/H(n)$ it can be identified with $\mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}}\,\mathbb{P}^2$ minus the three singular fibers of $\tilde{f}$. As $H(n)$ acts freely in $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\}$. By the long exact sequence of homotopy associated to the covering map $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\} \to S_3$ we have

$$1 \to \pi_1(F(n)) \times \pi_1(F(n) \setminus \{X_i, Y_i, Z_i\}) \to \pi_1(S_2) \to H(n) \to 1.$$

Remark 2.3.13. We can remove points also in the first component $F(n)$ of the product. However, we can not get more partial compactifications of Ceva(n) in this way. This can be shown by drawing the dual graph of the divisor $\pi^*\,\mathrm{Ceva}(n)$ and noticing that the lines obtained by removing points does not satisfy the intersection pattern of the graph.

Chapter 3

Partial compactifications of the complement of a complex arrangement and boundary manifolds

In this chapter we generalize the presentation obtained in Chapter 1 for a more general type of partial compactifications of the complement of a *complex* arrangement of lines $\mathscr{A} \subset \mathbb{P}^2$. We do so by using a diagram encoding the extra crossing of the lines in $\mathscr{A}$ that can arise when the arrangement is not longer real.

This diagram also permits to study the relations between the fundamental group of the boundary manifold of $\mathscr{A}$ and that of the $\mathbb{P}^2 \setminus \mathscr{A}$. We continue in this lines in order to study similar relations between the fundamental group of the boundary manifold of certain partial compactifications of $\mathbb{P}^2 \setminus \mathscr{A}$ and its fundamental group. Some detailed computations and applications to homology planes are given.

3.1 Preliminaries

3.1.1 Notations

We will denote by $\mathbb{P}^2$ the complex projective plane.

Let $\mathscr{A} = \{L_1, \ldots, L_{n+1}\}$ be an arrangement of $n+1$ lines in $\mathbb{P}^2$. The complement of the arrangement is denoted by $M(\mathscr{A}) := \mathbb{P}^2 \setminus \mathscr{A}$. Once a line $L \in \mathscr{A}$ is fixed, usually $L = L_{n+1}$, and after an identification of $\mathbb{P}^2 \setminus L$ with $\mathbb{C}^2$ we let $\mathscr{A}^{\text{aff}} := \mathscr{A} \cap \mathbb{C}^2$.

Let X be a complex manifold, for $p \in X$ we denote by $\pi : \mathrm{Bl}_p X \to X$ the blow up of X at p. If $D \subset X$ is a divisor, we denote by $|D|$ the reduced divisor with the same support as D and by $\operatorname{Sing} D$ the set of singular points of D.

We will denote by $a^b = b^{-1}ab$ if $a, b \in G$ with G a group. If $a \in G$ and $b \in \mathbb{Z}$, we denote as well by a^b the b-power of a.

3.1.2 Meridians

Let X be a complex manifold and $H \subset X$ a hypersurface. Let $p \in H$ be a smooth point and Δ a disc cutting transversaly H at p. A loop γ in $\pi_1(X \setminus H)$ freely homotopic to the boundary of Δ with the natural orientation is called a *meridian*.

The following Proposition is well-known, for a proof see [Shi07].

Proposition 3.1.1. *Let X be a complex manifold and $D = \sum D_i$ a divisor such that each irreducible reduced component $|D_i|$ of D is smooth. Let γ_i be a meridian of $|D_i|$, then every other meridian of $|D_i|$ is a conjugate of γ_i in $\pi_1(X \setminus D)$ and the kernel of the map $\pi_1(X \setminus D) \to \pi_1(X)$ is the normal subgroup generated by the meridians of its irreducible components.*

3.1.3 Dual graph of a divisor and partial compactifications of its complement

Let $\bar{X}$ be a projective smooth surface and let $D = \sum_{i=1}^{N} D_i \subset \bar{X}$ be a reduced simple normal crossing divisor with the D_i being the irreducible smooth components of D and denote by $w_i = D_i \cdot D_i$ the self-intersection number of D_i. Let Δ be the unoriented graph where the vertices $V(\Delta) := \{v_1, \ldots, v_N\}$ are in correspondence with the irreducible components D_i of D and the edges $\mathcal{E}(\Delta)$ correspond with the intersection of the irreducible components of D, this is, there is an unoriented edge joining v_i and v_j for each point in $D_i \cap D_j$. Denote by $X := \bar{X} \setminus D$.

In what follows, we will define some partial compactifications of X. The idea goes as follows: we choose a subset of irreducible components of D indexed by I which are not to be removed from $\bar{X}$, we then select a subset P of points in $\operatorname{Sing} \sum_{i \notin I} D_i$ to be blown-up and remove the *strict* transform of $\sum_{i \notin I} D_i$ in $\operatorname{Bl}_P \bar{X}$.

More precisely, let $I \subset \{1, \ldots, N\}$, $P = \{p_1, \ldots, p_{s_1}\} \subset \operatorname{Sing}(\sum_{i \notin I} D_i)$ and denote by $\pi : \operatorname{Bl}_P \bar{X} \to \bar{X}$ the composite of the blow-ups at the points in P. Denote by $\pi^* D = \sum_{i=1}^{N+s_1} D'_i$ the total transform of the divisor D in $\operatorname{Bl}_P \bar{X}$, suppose that for $i = 1, \ldots, N$, we have that D'_i is a strict transform of D_i and for $j = 1, \ldots, s_1$, the D'_{N+j} are exceptional divisors. Define the divisor

$$D'(I, P) = \pi^* D - \sum_{i \in I} D'_i - \sum_{N < j} D'_j.$$

Note that $\operatorname{Bl}_P \bar{X} \setminus \pi^* D \hookrightarrow X'(I, P) := \operatorname{Bl}_P \bar{X} \setminus D'(I, P)$. By restricting π, we obtain an isomorphism $\operatorname{Bl}_P \bar{X} \setminus \pi^* D \xrightarrow{\sim} X$. We call $X'(I, P)$ a *partial compactification* of $X = \bar{X} \setminus D$. By Proposition 3.1.1, the induced homomorphism $\pi_1(X) \to \pi_1(X'(P, I))$ is surjective.

We comment on the effects of this construction in the dual graph. Denote by $\Delta'(I,P)$ the dual graph of $D'(I,P)$. It is obtained from Δ by deleting the following vertices and edges: for the set I, we have a subset $V(I) \subset V(\Delta)$ of vertices corresponding to the lines D_i for $i \in I$, remove these vertices from Δ, together with all edges in $\mathcal{E}(\Delta)$ having an endpoint in $V(I)$. We also remove the edges corresponding to P: let $p_j \in P$, there exists $j_1, j_2 \in \{1,\dots,N\}$ such that $p_j = D_{j_1} \cap D_{j_2}$. In the dual graph of $\pi^* D$ the edge corresponding to p_j in Δ has been divided in two, with a vertex in between corresponding to the exceptional divisor coming from p_j.

Partial compactifications for an arrangement of lines

We can carry the above construction for a divisor $D \subset \bar{X}$ coming from an arrangement of lines $\mathscr{A} = \{L_1,\dots,L_{n+1}\} \subset \mathbb{P}^2$. In fact, this will be the only case we will be interested in.

Let $\mathscr{A} \subset \mathbb{P}^2$ be an arrangement of lines. Denote by $P_0 := \{p_1,\dots,p_{s_0}\} \subset \operatorname{Sing}\mathscr{A}$ the points with multiplicity strictly bigger that 2. Define $\pi : \bar{X} := \operatorname{Bl}_{P_0}\mathbb{P}^2 \to \mathbb{P}^2$ and denote by $D = |\pi^*\mathscr{A}| = \sum_{i=1}^{n+1+s_0} D_i$ the reduced total transform of $\mathscr{A}$ in $\bar{X}$. Note that D is simple normal crossing. For a divisor D where the irreducible components are smooth rational curves, the set of edges $\mathcal{E}(\Delta)$ of the dual graph Δ can be described as $\mathcal{E}(\Delta) = \{(i,j) \in \{1,\dots,n+1+s_0\}^2 \mid D_i \cap D_j \neq \varnothing, i<j\}$ once the irreducible components of D are numbered. We assume that D_i is the strict transform of L_i.

Let $I \subset \{1,\dots,N = n+1+s_0\}$ and $P = \{p'_1,\dots,p'_{s_1}\} \subset \operatorname{Sing}\sum_{i\notin I} D_i$. Consider $\pi' : \operatorname{Bl}_P \bar{X} \to \bar{X}$ and let $D' = \pi'^*(D) - \sum_{i\in I} D'_i - \sum_{N<j} D'_j$ as above. We write $M(\mathscr{A},I,P) := X'(I,P) = \operatorname{Bl}_P\bar{X} \setminus D'$ for a *partial compactification* of the complement of an arrangement $M(\mathscr{A}) = \mathbb{P}^2 \setminus \mathscr{A}$.

We can iterate this construction in the following way; consider a sequence of blow-ups:

$$\operatorname{Bl}_{P_k,\dots,P_1}\bar{X} \overset{\pi^{(k)}}{\to} \operatorname{Bl}_{P_{k-1},\dots,P_1}\bar{X} \overset{\pi^{(k-1)}}{\to} \dots \overset{\pi^{(2)}}{\to} \operatorname{Bl}_{P_1}\bar{X} \overset{\pi^{(1)}}{\to} \bar{X} \overset{\pi^{(0)}}{\to} \mathbb{P}^2$$

with $P_l \subset \operatorname{Sing}((\pi^{(0)}\circ\pi^{(1)}\circ\dots\circ\pi^{(l-1)})^*\mathscr{A})$ for $l = 1,\dots,k$ and $\pi^{(l)} : \operatorname{Bl}_{P_l,\dots,P_1}\bar{X} \to \operatorname{Bl}_{P_{l-1},\dots,P_1}\bar{X}$ denoting the blow-up of $\operatorname{Bl}_{P_{l-1},\dots,P_1}\bar{X}$ at P_l. We can suppose that the irreducible components of the reduced divisor

$$D' := \left|(\pi^{(0)}\circ\dots\circ\pi^{(k)})^*\mathscr{A}\right| = \sum_{1}^{|\mathscr{A}|} D'_i + \sum_{|\mathscr{A}|}^{|\mathscr{A}|+|P_0|} D'_j + \dots + \sum_{|\mathscr{A}|+|P_0|+\dots+|P_{k-1}|}^{|\mathscr{A}|+|P_0|+\dots+|P_k|} D'_l,$$

where $|P|$ denotes the cardinality of the set P, are ordered in such a way that $\pi^{(l)}\circ\dots\circ\pi^{(k)}$ contracts the curves D'_i with $i > |\mathscr{A}| + \dots + |P_{l-1}|$ for $l = 1,\dots,k$. Let $I \subset \{1,\dots,|\mathscr{A}|+\dots+|P_k|\}$ and define $M(\mathscr{A},I,P_1,\dots,P_k) := \operatorname{Bl}_{P_k,\dots,P_1}\bar{X} \setminus D' - \sum_{i\in I} D'_i$ as an *iterated partial compactification* of $M(\mathscr{A})$.

Lemma 3.1.2. *Let $(\bar{X}', D')$ be a smooth projective surface such that*

1. *the divisor D' is a simple normal crossing divisor,*

2. *there is a birational morphism $\bar{X}' \stackrel{\psi}{\to} \bar{X}$,*

3. *we have that $\psi^* D \supset D'$*

then there exists an iterated partial compactification $(\bar{X}'', D'')$ and a proper birational morphism $\bar{X}' \stackrel{\psi''}{\to} \bar{X}''$ such that $(\psi'')^{-1} D'' \supset D'$ and $\pi_1(\bar{X}'' \setminus D'') \stackrel{\sim}{\leftarrow} \pi_1(\bar{X}' \setminus D')$ is an isomorphism.

Here we will restrict the study to $M(\mathscr{A}, I, P)$ unless otherwise stated. The results are easily generalized to the above more general setting of iterated partial compactifications.

Remark 3.1.3. We have that $\pi_1(\bar{X}' \setminus D')$ is a quotient group of $\pi_1(M(\mathscr{A}))$ by proposition 3.1.1.

3.1.4 Boundary manifolds

Let $\bar{X}$ be a projective smooth surface and $D = \sum_{i=1}^{k} D_i \subset \bar{X}$ be a connected divisor. We can construct a regular tubular neighborhood U of D in $\bar{X}$ which comes with a surjective continuous retraction $\varphi : U \to D$ such that $\varphi|_D = id_D$. The boundary ∂U of U is an oriented, connected, closed 3-manifold (see [Mum61]). We call the 3-manifold ∂U the *boundary manifold* of D and denote by $\psi : \partial U \to D$ the restriction of φ to ∂U.

Suppose now that (X, D) is simple normal crossing and assume that:

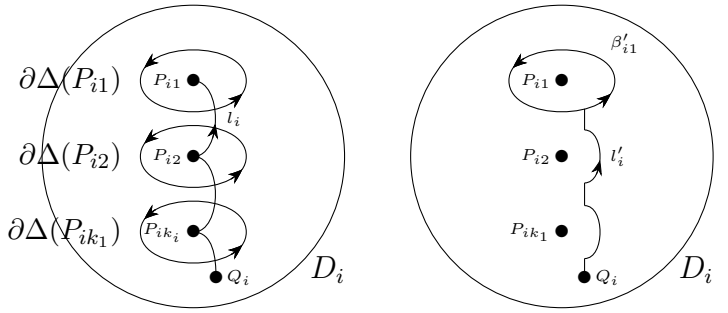
- the divisor D is connected,
- the irreducible components D_i of D are rational curves ,
- the dual graph of D has no cycles, in particular $\#D_i \cap D_j = 0$ or 1 if $i \neq j$. This dual graph is a tree that we denote by $\mathcal{T}$.

For such a pair, a presentation of $\pi_1(\partial U)$ is given in [Mum61, p. 235] (See also [Hir64]). As we shall need the notations, let us describe it.

Fix a base point $Q_i \in D_i \setminus \cup_{i \neq m} D_m$ in every rational curve $i = 1, \ldots, k$. Denote by P'_{im} the unique point in $D_i \cap D_m$. Select a simple contractible oriented curve $l_i \subset D_i$ containing Q_i and passing through every point $P'_{im} \in D_i$ as in figure 3.1a and denote by $l = \cup l_i \subset D$. We can construct a continuous map $h : l \to \partial U$ such that $\psi \circ h|_{l_i} = id_{l_i}$ and $h(l_i) \cap h(l_m) \neq \varnothing$ if $P'_{im} = D_i \cap D_m = l_i \cap l_m \neq \varnothing$.

It is easy to see that l is a homeomorphic image of a tree and deformation retracts to a point.

Label the points $P'_{im} \in D_i$ by the order they intersect l_i as $P_{i1}, \ldots, P_{ik_i}$, see Figure 3.1a. Denote by $\psi_i : \partial U_i \to D_i$ the boundary manifold of D_i. Let $D_i^* = D_i \setminus \cup_{m=1}^{k_i} \Delta(P_{im})$ with $\Delta(P_{im})$ a small open disk around P_{im} in D_i. Define $\partial U_i^* := \psi_i^{-1}(D_i^*)$. We may suppose that $\partial U \cap \partial U_i = \partial U_i^*$.



(a) Mumford generators (b) Paths in D_i^*

Figure 3.1: Generators

We may also assume that $Q_i \in D_i^*$. Define another contractible path $l'_i \subset D_i^*$ as follows: join every two connected components of $l_i \cap D_i^*$ touching the boundary of a disk $\partial\Delta(P_{im})$, by the segment of $\partial\Delta(P_{im})$ that connects these two points when traveled in the natural orientation, see figure 3.1b. We assume l_i and $\partial\Delta(P_{im})$ intersect transversally at two points for all $m = 1, \ldots, k_i - 1$.

Consider the circle $\partial\Delta(P_{im})$ traveled in the natural orientation and connect it to Q_i via a segment of l'_i. We obtain a path $\beta'_{im} \in \pi_1(D_i^*, Q_i)$, for $m = 1, \ldots, k_i$, see figure 3.1b. Note that $\beta'_{i1} \cdots \beta'_{ik_i} = 1$ in $\pi_1(D_i^*)$.

We can construct continuous maps $h_i : \cup_{m=1}^{k_i} \beta'_{im} \to \partial U_i^*$ such that $\psi_i \circ h_i|_{\beta'_{im}} = id_{\beta'_{im}}$ for every $i = 1, \ldots, k$. Let $h_i(Q_i)$ be a base point in ∂U_i^*, denote by $\gamma'_{im} = h_i(\beta'_{im})$ and let γ'_i be a fiber S^1 at Q_i of ∂U_i^* traveled in the natural orientation.

By using the long homotopy sequence of a fiber bundle, Mumford obtained the following presentation in [Mum61]. See also [Hir64].

Lemma 3.1.4 ([Mum61, p. 236-237])**.** *The fundamental group of ∂U_i^* is given by the following presentation*

$$\left\langle \gamma'_{i1}, \ldots, \gamma'_{ik_i}, \gamma'_i \mid [\gamma'_{im}, \gamma'_i]\ m = 1, \ldots, k_i, \gamma_i'^{-w_i} = \gamma'_{i1} \cdots \gamma'_{ik_i} \right\rangle \tag{3.1}$$

with $w_i = D_i \cdot D_i$ the self-intersection number of D_i.

Remark 3.1.5. Note that ∂U_i^* is non canonically homeomorphic to the trivial bundle $S^1 \times D_i^*$, but the image of the paths γ'_{im} are not longer identified with a path freely homotopic to one of the form $\{\text{point}\} \times \partial\Delta(P_{i_m})$. In fact, we need to twist this image by a multiple of γ'_i for it to be of such form. See [Mum61, p. 235].

Now, to globalize this construction to ∂U, we can use $h(l) \subset \partial U$ as a skeleton to define paths generating $\pi_1(\partial U)$. Let γ_i be the loop based at $h(Q_1)$ constructed as follows. Join $h(Q_1)$ to $h(Q_i)$ by a segment λ of $h(l)$, follow γ'_i and come back by λ^{-1}. Then it is homotopic to the canonical representative of γ'_i in $\pi_1(\partial U_i^* \cup h(l), h(Q_1))$ using the natural isomorphism $\pi_1(\partial U_i^* \cup h(l), h(Q_1)) \to \pi_1(\partial U_i^*, Q_i)$ thus obtained. Define similarly γ_{im} for $1 \leq m \leq k_i$. Then $\gamma_{im} = \gamma_{j_{\mathcal{T}}(i,m)}$ for some injective map $m \mapsto j_{\mathcal{T}}(i, m)$ from $\{1, \ldots, k_i\}$ to $\{1, \ldots, k\}$.

By gluing the ∂U_i^* together and by using van Kampen theorem, he obtained the following presentation for $\pi_1(\partial U)$.

Theorem 3.1.6 ([Mum61])**.** *With the notations and assumptions as above, a presentation for $\pi_1(\partial U)$ is given by:*

$$\pi_1(\partial U) = \left\langle \gamma_1, \ldots, \gamma_k \mid [\gamma_i, \gamma_{j_{\mathcal{T}}(i,m)}],\ m = 1, \ldots, k_i, \gamma_i^{-w_i} = \prod_{m=1}^{k_i} \gamma_{j_{\mathcal{T}}(i,m)}, 1 \leq i \leq k \right\rangle.$$

where $w_i = D_i \cdot D_i$, $[a, b] = aba^{-1}b^{-1}$, $a^0 = 1$, the identity of the group.

3.2 Wiring diagrams and a first presentation of the fundamental group of a partial compactification

We will describe the construction of a diagram permitting to express some meridians around the lines in $\mathscr{A}$ lying in a pencil of lines passing through a base point $R \in \mathbb{P}^2 \setminus \mathscr{A}$, in terms of a fixed set of meridians lying in a special fiber of this pencil.

As an application we obtain a first presentation for the fundamental group of a partial compactification $M(\mathscr{A}, I, P)$. To do that we will use a modification of the presentation of the fundamental group of $M(\mathscr{A})$ given in [Arv92] and [CS97].

This diagram will also carry the information to compute the image of the cycles in the boundary manifolds of $\mathscr{A}$ into $M(\mathscr{A})$. This will be done in section 3.3

3.2.1 Wiring diagram associated to a complex arrangement

Consider an arrangement of lines $\mathscr{A}$ in $\mathbb{P}^2$. Let us fix a base point $R \in \mathbb{P}^2 \setminus \mathscr{A}$ and denote by $\pi_R : \mathrm{Bl}_R \mathbb{P}^2 \to \mathbb{P}^2$ the blow-up at R. Let $\bar{f} : \mathrm{Bl}_R \mathbb{P}^2 \to \mathbb{P}^1$ be the morphism defined by the pencil of lines passing through R. In what follows, we assume that we have chosen R in such a way that $\bar{f}|_{\mathrm{Sing}\,\mathscr{A}} : \mathrm{Sing}\,\mathscr{A} \to \mathbb{P}^1$ is injective.

Let $* \in \mathbb{P}^1$, consider a simple piece-wise linear path $\beta : ([0, 1], 0) \to (\mathbb{P}^1, *)$ starting at $*$ and passing through every point $\bar{f}(p)$ for all $p \in \mathrm{Sing}\,\mathscr{A}$, being locally linear around these points.

By abuse of notation let us denote by $\mathscr{A}$ the union of the lines of arrangement in $\mathbb{P}^2$. As $R \in \mathbb{P}^2 \setminus \mathscr{A}$, the blow-up being an isomorphism outside $\pi_R^{-1}(R)$, we identify $\mathscr{A}$ and $\pi_R^* \mathscr{A}$.

Definition 3.2.1. The *wiring diagram* of $\mathscr{A}$ with respect to β is $\mathcal{W} = \bigcup_{t\in[0,1]}(\mathscr{A} \cap \bar{f}^{-1}(\beta(t))) \subset \mathrm{Bl}_R\,\mathbb{P}^2$. The i-wire W_i is $L_i \cap \mathcal{W}$. Here, we view $\mathscr{A}, L_i$ as subvarieties of $\mathrm{Bl}_R\,\mathbb{P}^2$ since $R \notin \mathscr{A}$.

By the choice of β, as it passes through the points $\bar{f}(p)$ for $p \in \mathrm{Sing}\,\mathscr{A}$, we have that $\mathrm{Sing}\,\mathscr{A} \subset \mathcal{W}$.

Lemma 3.2.1. *Every wire is a piece-wise linear simple curve.*

Proof. As no line in $\mathscr{A}$ passes through R, every $L_i \in \mathscr{A}$ induces a section of $\mathrm{Bl}_R\,\mathbb{P}^2 \to \mathbb{P}^1$ which is in fact an isomorphism. By the choice of β the result follows. □

Planar representation of the wiring diagram

By considering the pullback $\beta^*(\mathcal{W})$ and a trivialization $\beta^*\,\mathrm{Bl}_R\,\mathbb{P}^2 \cong [0,1]\times\mathbb{P}^1$, we can view $\beta^*(\mathcal{W})$ as a closed graph embedded in $[0,1]\times\mathbb{P}^1$. Sometimes we will continuing writing $\mathcal{W}$ for $\beta^*(\mathcal{W})$. Moreover we can remove the exceptional divisor $\pi_R^{-1}(R)$ from $[0,1]\times\mathbb{P}^1$ and we can view $\mathcal{W}$ as a closed graph embedded in $[0,1]\times\mathbb{C}$ via a piece-wise linear isomorphism.

There exists a complex coordinate z in $\mathbb{C}$ such that the projection $(p : [0,1]\times\mathbb{C} \to [0,1]\times\mathbb{R}, (t,z) \mapsto (t, \Re(z)))$ is generic, in the sense that the extra crossings in $p(\mathcal{W})$ arise as transversal intersection of only two wires $p(W_i)$ and $p(W_k)$ for certain $t \in [0,1]$ and wires W_i, W_k that do not intersect in $\bar{f}^{-1}(\beta(t))$. We call these crossings virtual vertices. We obtain a planar diagram which can be represented as in the figure 3.2.

We assume that the order of the lines $L_1, \ldots, L_n$ is such that, at the very right of the planar representation of $\mathcal{W}$, the wire W_1 is at the bottom of $\mathcal{W}$, above it is the wire W_2 and then W_3, continuing in this way until W_n.

Definition 3.2.2. Consider coordinates (t,x,y) in $\mathbb{R}^3$. We say that a wire W_i passes above W_k at a point $t' \in [0,1]$ if $(t',x,y_i) \in W_i, (t',x,y_k) \in W_k$ and $y_i < y_k$.

In order to distinguish the virtual vertices arising in the projection we mark the projection $p(W_i) \cap p(W_k)$ to indicate if the wires over or under crossed in $\beta^*\mathcal{W}$ as in Figure 3.5. We call the first a positive braiding (or positive virtual vertex) and the second a negative braiding (or negative virtual vertex).

Remark 3.2.2. As in the [CS97], we read the wiring diagram from right to left.

Example 3.2.1. Let $(z_1 : z_2 : z_3)$ be homogeneous coordinates of $\mathbb{P}^2$. Consider the arrangement consisting of two transverse pairs of parallel lines in $\mathbb{C}^2 \cong \mathbb{P}^2 \setminus \{z_3 = 0\}$, defined by the equation $(z_2-z_1)(z_2-z_1+z_3)(z_2+z_1)(z_2+z_1-z_3)z_3 = 0$. The wiring diagram associated to this arrangement is shown in Figure 3.2. There are no virtual vertices since the arrangement is real and β is a real segment.

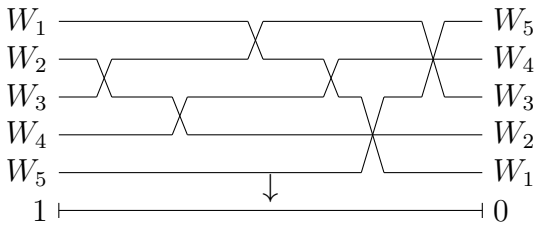


Figure 3.2: Wiring Diagram

Remark 3.2.3. When no under or overcrossing is marked in a wiring diagram $\mathcal{W}$, it coincides with the notion of wiring diagram in [BLVS$^+$99]. They are in correspondence with arrangement of "pseudo-lines", in particular there exists a wiring diagram of 9 wires that does not comes from an arrangement of lines (the so called non-Pappus arrangement, see [BLVS$^+$99, Proposition 8.3.1]), however for 8-wires or less they are in correspondence with the real arrangement of lines [BLVS$^+$99, Thm 6.3.1].

3.2.2 Using the diagram to obtain presentations

Algorithm for computing a presentation of the fundamental group of $M(\mathscr{A})$

We will use the following well-known Lemma.

Lemma 3.2.4. *Let $Z \subset X$ be an algebraic subvariety of an algebraic smooth surface X. Fix a point $R \in X \setminus Z$. Denote by $\pi_R : \mathrm{Bl}_R X \to X$, then $\pi_1(X \setminus Z) \cong \pi_1(\mathrm{Bl}_R X \setminus \pi_R^* Z)$.*

This allow us to compute $\pi_1(M(\mathscr{A}))$ in the total space of the fiber bundle $\bar{f} : \mathrm{Bl}_R \mathbb{P}^2 \to \mathbb{P}^1$. We will find suitable subspaces of the total space of this fiber bundle to apply the van-Kampen Theorem.

Let $\mathcal{W} \subset \mathrm{Bl}_R \mathbb{P}^2$ be a wiring diagram. Let $\beta^*(\mathcal{W}) \subset [0,1] \times \mathbb{P}^1$ be as in 3.2.1. Every vertical line $t \times \mathbb{P}^1$ in $[0,1] \times \mathbb{P}^1$ corresponds to the fiber $\bar{f}^{-1}(\beta(t))$. Recall that if $p \in \bar{f}^{-1}(\beta(t_p))$ for $p \in \mathrm{Sing}\,\mathscr{A}$ and $t_p \in [0,1]$, then no other point in $\mathrm{Sing}\,\mathscr{A}$ lies in the same fiber. Suppose that there are s points $t_{p_1}, \ldots, t_{p_s}$ corresponding to $p_1, \ldots, p_s$ in $\mathrm{Sing}\,\mathscr{A}$.

By fixing a planar representation $p(\beta^*(\mathcal{W}))$ of $\beta^*(\mathcal{W})$ as in 3.2.1, some under or over-crossing can arise. As the projection is generic, they correspond to a finite number $t'_1, \ldots, t'_\nu$ of elements of $[0,1]$ distinct from the t_{p_r}.

Order the set $\{t_{p_1}, \ldots, t_{p_s}, t'_1, \ldots, t'_\nu\}$ by increasing order and relabel them by t_κ for $\kappa = 1, \ldots, \nu + s$. Let $B_\kappa \subset \mathbb{P}^1$ be a neighborhood of $\beta(t_\kappa)$ homeomorphic to a disk in $\mathbb{C}$ such that $B_\kappa \cap B_j = \varnothing$ if $|\kappa - j| > 1$ and $B_\kappa \cap B_{\kappa+1}$ is homeomorphic to a disk. Consider $M_\kappa := \bar{f}^{-1}(B_\kappa) \subset \mathrm{Bl}_R(\mathbb{P}^2)$ for $\kappa = 1, \ldots, \nu + s$ and denote by $M_\kappa(\mathscr{A}) := M_\kappa \setminus M_\kappa \cap \pi_R^* \mathscr{A}$.

Lemma 3.2.5. *We have that*

$$\pi_1(M_\kappa(\mathscr{A}) \cap M_{\kappa+1}(\mathscr{A})) \cong F_n \text{ for } \kappa = 1, \dots, \nu+s-1$$

with F_n the free group in n generators.

Proof. First note that as $B_\kappa \cap B_{\kappa+1} \subset \mathbb{P}^1 \setminus \{\bar{f}(p) \mid p \in \operatorname{Sing} \mathscr{A}\}$ we have that $M_\kappa(\mathscr{A}) \cap M_{\kappa+1}(\mathscr{A}) = \bar{f}^{-1}(B_\kappa \cap B_{\kappa+1})$ is the restriction of a fiber bundle to a contractible base. The fundamental group of any fiber in $B_\kappa \cap B_{\kappa+1}$ is a free group in n generators. □

Proposition 3.2.6. *We have that*

$$\pi_1(M(\mathscr{A})) \cong \pi_1(M_1(\mathscr{A})) \underset{\pi_1(M_1(\mathscr{A}) \cap M_2(\mathscr{A}))}{*} \cdots \underset{\pi_1(M_{\nu+s-1}(\mathscr{A}) \cap M_{\nu+s}(\mathscr{A}))}{*} \pi_1(M_{\nu+s}(\mathscr{A})).$$

Proof. By Lemma 3.2.4, we have that the morphism $\operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A} \to M(\mathscr{A}) = \mathbb{P}^2 \setminus \mathscr{A}$ induces an isomorphism in the fundamental groups.

Denote the restriction of $\bar{f}$ to $\operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A}$ by $f : \operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A} \to \mathbb{P}^1$. Let $\infty \in \mathbb{P}^1 \setminus \cup_{\kappa=1}^{\nu+s} B_\kappa$ and note that $f^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ is the complement in $\operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A}$ of a smooth irreducible divisor D_∞ that is the restriction to $\operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A}$ of the strict transform of a line in $\mathbb{P}^2$ passing through R .

By Proposition 3.1.1, we have that

$$\pi_1(\operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A}) = \pi_1\left(f^{-1}(\mathbb{P}^1 \setminus \{\infty\})\right) / \langle\langle \gamma_{D_\infty} \rangle\rangle$$

where γ_{D_∞} is a meridian around D_∞.

Note that, as $R \in \mathbb{P}^2 \setminus \mathscr{A}$, we have that $\pi_R^{-1}(R) \subset \operatorname{Bl}_R \mathbb{P}^2 \setminus \pi_R^* \mathscr{A}$ and its restriction to $f^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ is isomorphic to $\mathbb{C}$. The meridian γ_{D_∞} can be chosen to lie inside this restriction and therefore $\gamma_{D_\infty} = 1$. We obtain that $\pi_1(f^{-1}(\mathbb{P}^1 \setminus \{\infty\})) \cong \pi_1(M(\mathscr{A}))$.

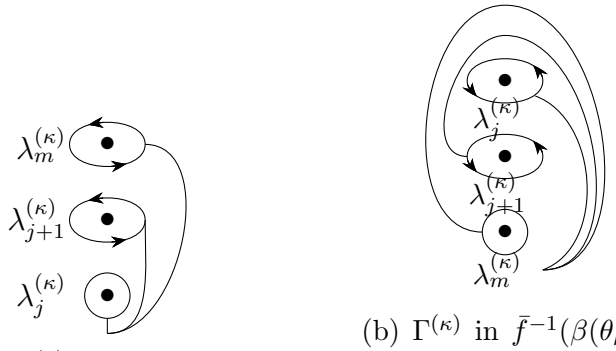
Observe that $\cup_{\kappa=1}^{\nu+s} M_\kappa(\mathscr{A})$ has the same homotopy as $(\operatorname{Bl}_R \mathbb{P}^2 \setminus \mathscr{A}) \setminus f^{-1}(\infty)$. We conclude by successive applications of the van-Kampen Theorem: by construction $B_1 \cap B_{\nu+s} = \varnothing$, we obtain that $\pi_1(\cup_{\kappa=1}^{\nu+s} M_\kappa(\mathscr{A}))$ is isomorphic to

$$\pi_1(M_1(\mathscr{A})) \underset{\pi_1(M_1(\mathscr{A}) \cap M_2(\mathscr{A}))}{*} \cdots \underset{\pi_1(M_{\nu+s-1}(\mathscr{A}) \cap M_{\nu+s}(\mathscr{A}))}{*} \pi_1(M_{\nu+s}(\mathscr{A}))$$

□

We want to compute now $\pi_1(M_\kappa(\mathscr{A}))$ for $\kappa = 1, \dots, \nu+s$ and the morphisms of amalgamation $\pi_1(M_\kappa(\mathscr{A})) \leftarrow \pi_1(M_\kappa(\mathscr{A}) \cap M_{\kappa+1}(\mathscr{A})) \to \pi_1(M_{\kappa+1}(\mathscr{A}))$. In fact, if no point of $\operatorname{Sing} \mathscr{A}$ lies in $M_\kappa(\mathscr{A})$ we will have that $\pi_1(M_\kappa(\mathscr{A})) \cong F_n$. However, some conjugations may arise in the meridians due to braiding of the wires in $\mathcal{W}$.

We have to distinguish 3 cases depending in the nature of M_κ: M_κ contains a point of $\operatorname{Sing} \mathscr{A}$, it contains a *positive braiding* of $\mathcal{W}$ or it contains a *negative braiding*.



(a) $\Gamma^{(\kappa)}$ in $\bar{f}^{-1}(\beta(\theta_\kappa))$

(b) $\Gamma^{(\kappa)}$ in $\bar{f}^{-1}(\beta(\theta_{\kappa+1}))$.

Figure 3.3: Geometric generating set in different fibers

Let $\theta_\kappa < t_\kappa$ be sufficiently close so that $\beta(\theta_\kappa) \in B_\kappa \cap B_{\kappa-1}$ and denote by $x_1^{(\kappa)}, \ldots, x_{n+1}^{(\kappa)}$ the set of points in the planar representation $p(\beta^*(\mathcal{W}))$ of the wiring diagram $\mathcal{W}$ labeled from bottom to top corresponding to the points in $\bar{f}^{-1}(\beta(\theta_\kappa)) \cap \mathscr{A}$.

Definition 3.2.3. A *geometric generating set* $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$ of the group $\pi_1(\bar{f}^{-1}(\beta(\theta_\kappa)) \setminus (\bar{f}^{-1}(\beta(\theta_\kappa)) \cap \mathscr{A}), q_\kappa)$ with $q_\kappa = \pi_R^{-1}(R) \cap \bar{f}^{-1}(\beta(\theta_\kappa))$ is the datum of $\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}$ meridians around $x_1^{(\kappa)}, \ldots, x_{n+1}^{(\kappa)}$ respectively, all of them based at q_κ such that $\lambda_{n+1}^{(\kappa)} \cdots \lambda_1^{(\kappa)}$ is nullhomotopic in $\bar{f}^{-1}(\beta(\theta_\kappa)) \setminus \{x_1^{(\kappa)}, \ldots, x_{n+1}^{(\kappa)}\} \cong \mathbb{P}^1 \setminus \{(n+1) - \text{points}\}$.

Remark 3.2.7. A geometric generating set $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$ induces a geometric base $\Gamma^{(\kappa)\prime} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_n^{(\kappa)}\}$ of $\pi_1(\mathbb{C} \setminus \{x_1^{(\kappa)}, \ldots, x_n^{(\kappa)}\}, q_\kappa)$ as in 1.2.1.

We consider here the geometric generating set $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$ as in figure 3.3a. As $\pi_1(\pi_R^{-1}(R))$ is trivial, we can fix a point $q \in \pi_R^{-1}(R)$ as a global base point for all the geometric generating set $\Gamma^{(\kappa)}$ with $\kappa = 1, \ldots, \nu + s$ by joining q_κ to q by a simple path in $\pi_R^{-1}(R)$.

We describe how the meridians change when we move the generators of $\Gamma^{(\kappa)}$ to the fiber $\bar{f}^{-1}(\beta(\theta_{\kappa+1}))$ and express them in the generators $\Gamma^{(\kappa+1)}$, see figure 3.3b. We record as well the relations arising in between.

Suppose that $p \in \operatorname{Sing} \mathscr{A} \cap M_\kappa$, and let $\Gamma^{(\kappa)}$ be as above. Denote by j the first index of the meridians of $\Gamma^{(\kappa)}$ corresponding to a line passing through p, and by m the last. We have that $\lambda_k^{(\kappa+1)} = \lambda_k^{(\kappa)}$ for $k < j$ and $k > m$ as we can deform continuously $\lambda_k^{(\kappa)}$ to $\lambda_k^{(i+1)}$ having the same homotopy type in $\pi_1(M_\kappa(\mathscr{A}))$.

Let $R_\kappa = [\lambda_m^{(\kappa)}, \lambda_{m-1}^{(\kappa)}, \ldots, \lambda_j^{(\kappa)}]$ denote the set of equations of the form $\lambda_m^{(\kappa)} \cdot \lambda_{m-1}^{(\kappa)} \cdots \lambda_j^{(\kappa)} = \lambda_{\sigma(m)}^{(\kappa)} \lambda_{\sigma(m-1)}^{(\kappa)} \cdots \lambda_{\sigma(j)}^{(\kappa)}$ where σ varies in the set of cyclic permutations in $m - j + 1$ elements.

Lemma 3.2.8. *Let $p \in \operatorname{Sing} \mathscr{A} \cap M_\kappa$. Then $\pi_1(M_\kappa(\mathscr{A}), q_\kappa)$ is generated by the elements of $\Gamma^{(\kappa)}$ and $\Gamma^{(\kappa+1)}$ together with the relations R_κ, $\lambda_k^{(\kappa+1)} = \lambda_k^{(\kappa)}$*

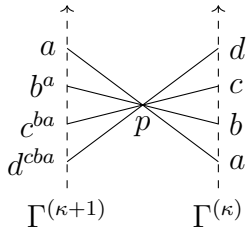


Figure 3.4: Actual vertex

for $k < j$ *or* $m < k$, $\lambda_{n+1}^{(\kappa)} \cdots \lambda_1^{(\kappa)} = 1$ *and*

$$\begin{aligned}
\lambda_m^{(\kappa+1)} &= \lambda_j^{(\kappa)}, \\
\lambda_{m-1}^{(\kappa+1)} &= {\lambda_{j+1}^{(\kappa)}}^{\lambda_j{}^{(\kappa)}}, \\
\lambda_{m-2}^{(\kappa+1)} &= {\lambda_{j+2}^{(\kappa)}}^{\lambda_{j+1}{}^{(\kappa)}\lambda_j{}^{(\kappa)}}, \\
&\vdots \\
\lambda_j^{(\kappa+1)} &= {\lambda_m^{(\kappa)}}^{\lambda_{m-1}^{(\kappa)}\cdots\lambda_j^{(\kappa)}}.
\end{aligned}$$

(see figures 3.3 and 3.4.)

Proof. The proof follows closely that of Proposition 1.2.7.

Let V_p be a neighborhood around p homeomorphic to a product $B_\kappa \times D$ with D a disk not intersecting $L_k \in \mathscr{A}$ with $k < j$ or $k > m$. The local fundamental $\pi_1(V_p \setminus \mathscr{A})$ equals the fundamental group of the link associated to the singularity p which is a Hopf link of $m-j+1$ circles (see [OT92, Lemma 5.75]).

For the complement $M_\kappa(\mathscr{A}) \setminus V_p$ we have $\pi_1(M_\kappa(\mathscr{A}) \setminus V_p) \cong F_{n-(m-j)}$ and if V' is a small neighborhood of V_p we have that as $V_p \setminus \mathscr{A}$ retracts to $\partial V_p \setminus \mathscr{A}$ then $\pi_1((M_\kappa(\mathscr{A}) \setminus V_p) \cap V') \cong \pi_1(V_p \setminus \mathscr{A})$. By van-Kampen we obtain the relation $\lambda_{n+1}^{(\kappa)} \cdots \lambda_1^{(\kappa)} = 1$. □

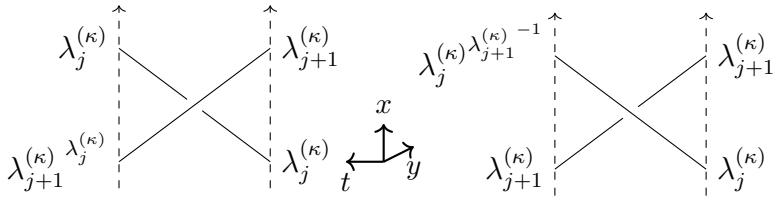
Lemma 3.2.9. *Suppose that there is a* positive braiding *of the wires* j *and* $j+1$ *in* $M_\kappa(\mathscr{A})$. *Then the group* $\pi_1(M_\kappa(\mathscr{A}), q_\kappa)$ *admits the presentation*

$$\left\langle \lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}, \lambda_j^{(\kappa+1)}, \lambda_{j+1}^{(\kappa+1)} \mid \lambda_{j+1}^{(\kappa+1)} = \lambda_j^{(\kappa)}, \lambda_j^{(\kappa+1)} = {\lambda_{j+1}^{(\kappa)}}^{\lambda_j^{(\kappa)}} \right\rangle$$

(See fig. 3.5a.)

Proof. As in lemma 3.2.8, we have that we can deform $\lambda_k^{(\kappa+1)}$ to $\lambda_k^{(\kappa)}$ for $k < j$ or $j+1 < k$ without changing the homotopy type.

The result follows from the Wirtinger presentation of a braid interchanging the j and the $j+1$ wire: consider the meridians $\lambda_j^{(\kappa)}, \lambda_{j+1}^{(\kappa)}$ in $\bar{f}^{-1}(\beta(\theta_{\kappa+1}))$ as



(a) Positive braiding (b) Negative braiding

Figure 3.5: Braiding in $\mathcal{W}$

in figure 3.3b. Note that in $\pi_1(\bar{f}^{-1}(\beta(\theta_{\kappa+1})), q_{\kappa+1})$ these meridians satisfy the relations:

$$\lambda_{j+1}^{(\kappa+1)} = \lambda_j^{(\kappa)}, \quad \lambda_j^{(\kappa+1)} = \lambda_{j+1}^{(\kappa)\ \lambda_{j+1}^{(\kappa+1)}} = \lambda_{j+1}^{(\kappa)\ \lambda_j^{(\kappa)}}.$$

This can be seen directly from Figure 3.3b. (C.f. [OT92, Lemmas 5.73, 5.74].) □

Lemma 3.2.10. *Suppose that there is a* negative braiding *in $M_\kappa(\mathscr{A})$, then the group $\pi_1(M_\kappa(\mathscr{A}), q_\kappa)$ admits the presentation*

$$\left\langle \lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}, \lambda_j^{(\kappa+1)}, \lambda_{j+1}^{(\kappa+1)} \mid \lambda_{j+1}^{(\kappa+1)} = {\lambda_j^{(\kappa)}}^{{\lambda_{j+1}^{(\kappa)}}^{-1}}, \lambda_j^{(\kappa+1)} = \lambda_{j+1}^{(\kappa)} \right\rangle$$

(See fig. 3.5b.)

We can summarize the information carried by a wiring diagram $\mathcal{W}$ and the changes in the geometric sets $\Gamma^{(\kappa)}$ as they cross a vertex in $\mathcal{W}$ as follows.

For every $t_\kappa \in \{t_1, \ldots, t_{\nu+s}\}$ there exists a crossing p_κ in the planar representation of $\mathcal{W}$, let $\Pi^{(\kappa)} = \{\sigma^{(\kappa)}(1) < \ldots < \sigma^{(\kappa)}(n+1)\}$ be an ordered set, with $\sigma^{(\kappa)}$ a permutation of $\{1, \ldots, n+1\}$ such that the k-th element $\sigma^{(\kappa)}(k)$ records the position of the wire $W_{\sigma^{(\kappa)}(k)}$ in the fiber $\bar{f}^{-1}(\beta(\theta_\kappa))$, when $\mathcal{W}$ is read from bottom to top, with θ_κ as in Definition 3.2.3. This is, $x_k^{(\kappa)} \in W_{\sigma^{(\kappa)}(k)}$ for $k = 1, \ldots, n+1$. Note that $\sigma^{(1)} = id$.

The order in $\Pi^{(\kappa)}$ records the *local position* of the wires of $\mathcal{W}$ in $\bar{f}^{-1}(\beta(\theta_\kappa))$, while the order $\{1, \ldots, n+1\}$ induced from the order of the lines in $\mathscr{A}$ is a *global order*. For a wire W_k of $\mathcal{W}$, we write ${\sigma^{(\kappa)}}^{-1}(k)$ to indicate that the wire W_k is in the ${\sigma^{(\kappa)}}^{-1}(k)$ position in the fiber $\bar{f}(\beta(\theta_\kappa))$.

Consider the free group $F_{n+1}^{(\kappa)}$ generated by the meridians in $\Gamma^{(\kappa)}$ and let $\tau^{(\kappa)} : \{1, \ldots, n+1\} \to F_{n+1}^{(\kappa)}$ defined as follows:

Suppose that the crossing p_κ corresponding to t_κ satisfies $p_\kappa = W_{\sigma^{(\kappa)}(j)} \cap W_{\sigma^{(\kappa)}(j+1)} \cap \ldots \cap W_{\sigma^{(\kappa)}(m)}$, then

$$\tau^{(\kappa)}(k) = \begin{cases} e & \text{for } k = 1, \ldots, j, m+1, \ldots, n+1, \\ \lambda_{k-1}^{(\kappa)} \cdots \lambda_j^{(\kappa)} & \text{for } j < k \leq m, \end{cases}$$

if t_κ is an actual vertex,

$$\tau^{(\kappa)}(k) = \begin{cases} e & \text{for } k = 1, \dots, j, j+2, \dots, n+1, \\ \lambda_j^{(\kappa)} & \text{for } k = j+1, \end{cases}$$

if t_κ is a positive virtual vertex, and

$$\tau^{(\kappa)}(k) = \begin{cases} e & \text{for } k = 1, \dots, j-1, j+1, \dots, n+1, \\ (\lambda_{j+1}^{(\kappa)})^{-1} & \text{for } k = j, \end{cases}$$

if t_κ is a negative virtual vertex.

The Lemmas 3.2.8, 3.2.9 and 3.2.10 imply the following proposition.

Proposition 3.2.11. *Let $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \dots, \lambda_{n+1}^{(\kappa)}\}, \Gamma^{(\kappa+1)} = \{\lambda_1^{(\kappa+1)}, \dots, \lambda_{n+1}^{(\kappa+1)}\}$ be geometric generating set as in 3.2.2 and suppose that $p_\kappa \in M_\kappa$. Then we have that in $\pi_1(M_\kappa(\mathscr{A}), q_\kappa)$:*

$$\lambda^{(\kappa+1)}_{\sigma^{(\kappa+1)^{-1}}(\sigma^{(\kappa)}(k))} = (\lambda_k^{(\kappa)})^{\tau^{(\kappa)}(k)} \quad \text{for } k = 1, \dots, n+1,$$

or equivalently,

$$\lambda_k^{(\kappa+1)} = (\lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(\sigma^{(\kappa)}(k))})^{\tau^{(\kappa)}\left(\sigma^{(\kappa)^{-1}}(\sigma^{(\kappa+1)}(k))\right)} \quad \text{for } k = 1, \dots, n+1.$$

Note that if $p_\kappa = W_{\sigma^{(\kappa)}(j)} \cap \ldots \cap W_{\sigma^{(\kappa)}(m)}$ we have that

$$\sigma^{(\kappa+1)^{-1}}(\sigma^{(\kappa)}(k)) = \begin{cases} k & \text{for } k = 1, \dots, j-1, m+1, \dots, n+1, \\ m - \iota & \text{for } k = j + \iota \text{ and } \iota = 0, \dots, m-j. \end{cases}$$

As the fundamental group of $M(\mathscr{A})$ is generated by the meridians around each line, we fix the geometric generating set $\Gamma^{(1)} = \{\lambda_1^{(1)}, \dots, \lambda_{n+1}^{(1)}\} = \{\lambda_1, \dots, \lambda_{n+1}\} \subset M_1(\mathscr{A})$.

Theorem 3.2.12. *Let $\mathscr{A} = \{L_1, \dots, L_{n+1}\}$ be a complex arrangement of lines in $\mathbb{P}^2$ and let $\Gamma^{(1)}$ be a geometric generating set as above. A presentation for the fundamental group of $M(\mathscr{A})$ is given by*

$$\pi_1(M(\mathscr{A}), q) = \left\langle \lambda_1, \dots, \lambda_{n+1} \mid \bigcup_\kappa R_\kappa, \lambda_{n+1} \cdots \lambda_1 \right\rangle$$

with R_κ as Lemma 3.2.8 and each κ corresponding to a point $p_\kappa \in \operatorname{Sing} \mathscr{A}$.

Remark 3.2.13. The relations R_κ are expressed in terms of the geometric generating set $\Gamma^{(1)}$ by substituting $\lambda_k^{(\kappa)}$ by a conjugate of $\lambda^{(1)}_{\sigma^{(\kappa)}(k)}$ by elements of $\Gamma^{(1)}$ by repeated applications of Proposition 3.2.11.

Proof. From Proposition 3.2.6 we know that $\pi_1(M(\mathscr{A})) = \pi_1(M_1(\mathscr{A})) *_{F_n} \cdots *_{F_n} \pi_1(M_{\nu+s}(\mathscr{A}))$. Now, the groups $\pi_1(M_\kappa(\mathscr{A}))$ are presented in generators $\Gamma^{(\kappa)}$ and $\Gamma^{(\kappa+1)}$, and relations which are words in these letters (see Lemmas 3.2.8, 3.2.9, 3.2.10). The geometric generating set $\Gamma^{(\kappa)}$ is chosen in such a way that it lies in a fiber over a point of $B_{\kappa-1} \cap B_\kappa$, and therefore, we can assume that the amalgamation $\pi_1(M_{\kappa-1}(\mathscr{A})) *_{F_n} \pi_1(M_\kappa(\mathscr{A}))$ permits to see $\Gamma^{(\kappa)}$ in $M_\kappa(\mathscr{A})$ and $M_{\kappa-1}(\mathscr{A})$ simultaneously.

Note that $\lambda_m^{(\kappa+1)} \cdots \lambda_j^{(\kappa+1)} = \lambda_m^{(\kappa)} \cdots \lambda_j^{(\kappa)}$, hence $\lambda_{n+1}^{(\kappa+1)} \ldots \lambda_1^{(\kappa+1)} = \lambda_{n+1}^{(\kappa)} \cdots \lambda_1^{(\kappa)}$ for every $\kappa = 1, \ldots, \nu + s - 1$.

The relations in $\pi_1(M_\kappa(\mathscr{A}))$ when there is a positive or virtual vertex in $M_\kappa(\mathscr{A})$, can be omitted in the presentation of $\pi_1(M(\mathscr{A}))$ by writing every meridian of $\Gamma^{(\kappa+1)}$ in terms of $\Gamma^{(\kappa)}$ as in Lemmas 3.2.9, 3.2.10.

When there is an actual vertex in $M_\kappa(\mathscr{A})$, the relation $R_\kappa = [\lambda_m^{(\kappa)}, \lambda_{m-1}^{(\kappa)}, \ldots, \lambda_j^{(\kappa)}]$ will appear in the presentation of $\pi_1(M(\mathscr{A}))$. This relation can be expressed in terms of $\Gamma^{(1)}$ in a recursive way, by expressing $\Gamma^{(\kappa)}$ in terms of $\Gamma^{(\kappa-1)}$ by using the amalgamation of $\pi_1(M_{\kappa-1}(\mathscr{A}))$ and $\pi_1(M_\kappa(\mathscr{A}))$ over $B_{\kappa-1} \cap B_\kappa$ and the presentation of $M_{\kappa-1}(\mathscr{A})$ given by proposition 3.2.11. More precisely, we have that

$$\lambda_k^{(\kappa)} = \lambda_{\sigma^{(\kappa)}(k)}^{\tau^{(1)}\left(\sigma^{(\kappa)}(k)\right) \cdot \tau^{(2)}\left({\sigma^{(2)}}^{-1}(\sigma^{(\kappa)}(k)\right) \cdots \tau^{(\kappa-1)}\left({\sigma^{(\kappa-1)}}^{-1}(\sigma^{(\kappa)}(k))\right)} \text{ for } k = 1, \ldots, n+1,$$

and every $\tau^{(r)}\left({\sigma^{(r)}}^{-1}(\sigma^{(\kappa)}(k))\right)$ can be expressed in terms of $\Gamma^{(1)}$ in a recursive way for $r = 1, \ldots, i-1$. □

Algorithm for determining the presentation for a partial compactification $M(\mathscr{A}, I, P)$

Let $\mathcal{W}$ be a wiring diagram and $\langle \lambda_1, \ldots, \lambda_{n+1} \mid \cup_k R_k, \lambda_{n+1} \cdots \lambda_1 \rangle$ a presentation of $\pi_1(M(\mathscr{A}))$ as in Theorem 3.2.12.

Consider a partial compactification $M(\mathscr{A}, I, P)$ of $\mathbb{P}^2 \setminus \mathscr{A}$ as in 3.1.3. Here, we let $P_0 = \{p_1, \ldots, p_{s_0}\} \subset \operatorname{Sing} \mathscr{A}$ denote the points of multiplicity strictly bigger than two, consider $\pi : \bar{X} = \operatorname{Bl}_{P_0} \mathbb{P}^2 \to \mathbb{P}^2$ and denote by $D = \sum_{i=1}^{n+1+s_0} D_i = \pi^* \mathscr{A}$. Select $I \subset \{1, \ldots, n+1+s_0\}$ and $P = \{p'_1, \ldots, p'_{s_1}\} \subset \operatorname{Sing} \sum_{i \notin I} D_i$. Consider another blow-up $\pi' : \operatorname{Bl}_P \bar{X} \to \bar{X}$ and write $\pi'^* D = \sum_{i=1}^{n+1+s_0+s_1} D'_i$. Define $D' = \pi'^*(D) - \sum_{i \in I} D'_i - \sum_{i > n+1+s_0} D'_i$ and $M(\mathscr{A}, I, P) = \operatorname{Bl}_{P_1} \bar{X} \setminus D'$.

From Proposition 3.1.1, we have that a presentation for the fundamental group $\pi_1(M(\mathscr{A}, I, P))$ can be obtained from $\langle \lambda_1, \ldots, \lambda_{n+1} \mid \cup_k R_k, \lambda_{n+1} \cdots \lambda_1 \rangle$ by adding as relations certain words $\lambda(D'_i)$ representing some meridians around the irreducible components D'_i with either $i \in I$ or $n + s_0 + 1 < i$. In order to do so, we have to distinguish four cases for these irreducible components D'_i of $\pi'^* D$:

1. D'_i is the strict transform of a line in $\mathscr{A}$. In this case $i \leq n+1$.

2. D'_i is the strict transform of an exceptional divisor D_i in $\bar{X}$. In this case $n+1 < i \leq n+1+s_0$,

3. D'_p is an exceptional divisor coming from a double point p in $\operatorname{Sing}\mathscr{A}$.

4. D'_p is an exceptional divisor obtained by blowing-up a point $p = D_r \cap D_k$ with $r \leq n+1$ and $n+1 < k \leq n+1+s_0$.

For the lines as in (1) we let $\lambda(D'_i) = \lambda_i$.

For the lines as in (2), suppose that $D_i \subset \bar{X}$ is an exceptional divisor coming from a point $p \in \operatorname{Sing}\mathscr{A}$ and suppose that $p \in M_\kappa$, this is, t_p is the κ-element in the ordered set of vertices $t_1, \ldots, t_{\nu+s}$ of a planar representation of $\mathcal{W}$ as in 3.2.2. In other words $t_p = t_\kappa \in [0,1]$ satisfies $\beta(t_\kappa) = \bar{f}(p)$ and consider $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$ the geometric generating set of $\bar{f}^{-1}(\beta(\theta_\kappa)) \setminus (x_1^{(\kappa)}, \ldots, x_{n+1}^{(\kappa)}) \subset M_\kappa(\mathscr{A})$ and suppose that $p = W_{\sigma^{(\kappa)}(j)} \cap W_{\sigma^{(\kappa)}(j+1)} \cap \ldots \cap W_{\sigma^{(\kappa)}(m)}$ with the local index $\Pi^{(\kappa)} = \{\sigma^{(\kappa)}(1) < \ldots < \sigma^{(\kappa)}(n+1)\}$ as in 3.2.2. Associate to D_i and to its strict transform D'_i, the word $\lambda(D_i) = \lambda(D'_i) = \lambda_m^{(\kappa)} \cdot \lambda_{m-1}^{(\kappa)} \cdots \lambda_{j+1}^{(\kappa)} \cdot \lambda_j^{(\kappa)}$.

Lemma 3.2.14. *Let D'_i be a line as in (2). Then $\lambda(D'_i) = \lambda_m^{(\kappa)} \cdot \lambda_{m-1}^{(\kappa)} \cdots \lambda_{j+1}^{(\kappa)} \cdot \lambda_j^{(\kappa)}$ represents a meridian around D_i, and by pull-back, also around D'_i.*

Proof. Let $\psi : U \to D$ and $\psi_i : U_i \to D_i$ be the boundary manifolds of D and D_i in $\bar{X}$ respectively. Note that we can use the meridians $\lambda_j^{(\kappa)}, \ldots, \lambda_m^{(\kappa)}$ to give a presentation of $\pi_1(\partial U^*)$, with $\partial U_i^* = \partial U \cap \partial U_i$ as in 3.1.4, as follows: the projection $\pi(\partial U_i)$ to $\mathbb{P}^2$ can be seen as the boundary of a 4-real ball B_p centered at p. There exists $R_p \in \partial B_p$ such that for each $j \leq k \leq m$ the loop $\alpha_k := \lambda_k^{(\kappa)}$ is homotopic to a product $\alpha_{k_1}^{\alpha_{k_2}}$ with

- The loop α_{k_1} starting at R_p, lying completely in ∂B_p and surrounding the line $L_{\sigma^{(\kappa)}(k)}$.

- The loop α_{k_2} is a simple path connecting R_p and the point $R \in \mathbb{P}^2 \setminus \mathscr{A}$.

By pulling-back the meridians $\alpha_{j_1}, \ldots, \alpha_{m_1}$ to $\bar{X}$ we can see them as lying in ∂U. By construction of the geometric generating set $\Gamma^{(\kappa)}$, the product $\alpha_{m_1} \cdots \alpha_{j_1}$ is homotopic to a path encircling the lines $L_{\sigma^{(\kappa)}(j)}, \ldots, L_{\sigma^{(\kappa)}(m)}$ and therefore the projection $\psi_{i*}(\alpha_{m_1} \cdots \alpha_{j_1}) = e$ in $\pi_1(D_i^*, \psi_i(R_p))$. We can construct a continuous map $h_i : \cup_{k=j}^m \psi_i(\alpha_{k_1}) \to \cup_{k=j}^m \alpha_{k_1}$ such that $h_i(\psi(\alpha_{k_1})) = \alpha_{k_1}$ and therefore the loops $\alpha_{j_1}, \ldots, \alpha_{m_1}$ together with a fiber α_i of ∂U_i generate the group $\pi_1(\partial U_i^*)$ as in Lemma 3.1.4. Moreover, as $D_i \cdot D_i = -1$, we have the relation $\alpha_i = \alpha_{m_1} \cdots \alpha_{j_1}$ in $\pi_1(\partial U_i^*, R_p)$.

By construction of $\Gamma^{(\kappa)}$, we have that every two α_{k_2} and $\alpha_{k'_2}$ with $j \leq k, k' \leq m$ are homotopic. Therefore, by connecting α_i to R via α_{j_2}, we obtain the relation $\alpha_i^{\alpha_{j_2}} = \lambda_m^{(\kappa)} \cdots \lambda_j^{(\kappa)}$ in $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$.

By pulling-back $\alpha_i^{\alpha_{j_2}}$ to Bl_{P_1} we obtain that it is homotopic to a meridian around D'_i. $\square$

For the lines D'_p as in (3), suppose that $p = D_r \cap D_k$ with $r, k \leq n+1$. Consider the unique index $1 \leq \kappa \leq \nu + s$ such that $p \in M_\kappa$ and let $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$ be a geometric generating set of $\pi_1(M_\kappa(\mathscr{A}))$. We denote

$$\lambda(D'_p) := \lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(r)} \lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(k)}.$$

Recall that $\sigma^{(\kappa)^{-1}}(r)$ and $\sigma^{(\kappa)^{-1}}(k)$ record the local position of the wires W_r, W_k respectively, in the local order of the wires of $\mathcal{W}$ in $\bar{f}^{-1}(\beta(\theta_\kappa))$ given by $\Pi^{(\kappa)} = \{\sigma^{(\kappa)}(1) < \ldots < \sigma^{(\kappa)}(n+1)\}$.

Finally, let D'_p be as in (4) with $p \in P$. We have that $p = D_r \cap D_k$ with $r \leq n+1$ and D_k an exceptional divisor coming from a point $p(k) \in P_0$. Let us suppose that $p(k) \in M_\kappa$. Denote by $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\} \subset M_\kappa(\mathscr{A})$ the geometric generating set as above. We can suppose that $p(k) = W_{\sigma^{(\kappa)}(j)} \cap \ldots \cap W_{\sigma^{(\kappa)}(m)}$. As $n+1 \leq k \leq n+1+s_0$, we can consider the word $\lambda(D_k) = \lambda_m^{(\kappa)} \cdots \lambda_j^{(\kappa)}$ as in Lemma 3.2.14 above.

Lemma 3.2.15. *A meridian of D'_p is given by $\lambda(D'_p) = \lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(r)} \lambda(D_k)$. Moreover, $\lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(r)}$ commutes with $\lambda(D_k)$.*

Proof. Recall that by construction, $\lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(r)}$ is the meridian of L_r lying in the geometric generating set $\Gamma^{(\kappa)}$.

Let $\psi_{D_k} : \partial U_{D_k} \to D_k$ be the boundary manifold of D_k in $\bar{X}$. For $k' = j, \ldots, m$, let us decompose the loops $\alpha_{k'} = \lambda_{k'}^{(\kappa)}$ in two parts $\alpha_{k'_1}, \alpha_{k'_2}$, as in the proof of the Lemma 3.2.14, such that $\alpha_{k'}$ is homotopic to $\alpha_{k'_1}^{\alpha_{k'_2}}$. The proof of the same Lemma and 3.1.4 give us that

$$\pi_1(\partial U^*_{D_k}, R_k) = \langle \alpha_{j_1}, \ldots, \alpha_{m_1}, \alpha_k \mid [\alpha_k, \alpha_{k'_1}], \alpha_k = \alpha_{m_1} \cdots \alpha_{j_1} \rangle.$$

for a point $R_k \in \partial U^*_{D_k}$ and α_k a fiber of $\partial U^*_{D_k}$. We can globalize the relations in this presentation by considering $\alpha_{k'_1}^{\alpha_{k'_2}}$ and obtain that $\lambda(D_k)$ commutes with $\lambda_{k'}^{(\kappa)}$ for $k' = j, \ldots, m$, in particular as D_r intersect D_k, we have that $\lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(r)}$ commutes with $\lambda(D_k)$.

Furthermore, the point R_k can be chosen to lie in the boundary ∂B_p of a ball $B_p \subset \bar{X}$ around p. Let $\psi_{D'_p} : \partial U_{D'_p} \to D'_p$ be the boundary manifold of D'_p in $\mathrm{Bl}_P \bar{X}$ and Δ_1, Δ_2 a pair of disks about the points $D'_p \cap D'_r$ and $D'_p \cap D'_k$ respectively. Denote $\partial U^*_{D'_p} = \psi^{-1}_{D'_p}(D'_p \setminus (\Delta_1 \cup \Delta_2))$. By working in local coordinates, it can be seen that $\alpha_k, \alpha_{\sigma^{(\kappa)^{-1}}(r)_1}$ and a fiber α_p of $\partial U_{D'_p}$ at R_k generate the group $\pi_1(\partial U^*_{D'_p})$ and that

$$\pi_1(\partial U^*_{D'_p}) = \left\langle \alpha_k, \alpha_{\sigma^{(\kappa)^{-1}}(r)_1}, \alpha_p \;\middle|\; \begin{array}{l} [\alpha_k, \alpha_p],\ [\alpha_{\sigma^{(\kappa)^{-1}}(r)_1}, \alpha_p], \\ \alpha_p = \alpha_k \cdot \alpha_{\sigma^{(\kappa)^{-1}}(r)_1} \end{array} \right\rangle$$

by Lemma 3.1.4 and because $D'_p \cdot D'_p = -1$. □

Theorem 3.2.16. *Let $\mathscr{A} \subset \mathbb{P}^2$ be an arrangement of lines, $\mathcal{W}$ a wiring diagram and $M(\mathscr{A}, I, P)$ a partial compactification. Then*

$$\pi_1(M(\mathscr{A}, I, P), q) = \left\langle \lambda_1, \dots, \lambda_{n+1} \mid \bigcup_k R_k, \lambda_{n+1} \cdots \lambda_1, \bigcup_{i \in I} \lambda(D'_i), \bigcup_{p \in P} \lambda(D'_p), \right\rangle$$

is a presentation for the fundamental group of the partial compactification.

Proof. We only have to justify the expression for those meridians around lines as in (1) and (3). For the meridians of lines as in (2) and (4), the expression $\lambda(D'_i)$ and $\lambda(D'_p)$ is explained by Lemmas 3.2.14 and 3.2.15 respectively. We will conclude by Proposition 3.1.1.

For the meridians around lines as in (1), it is immediate by the biholomorphism property of the blow-up outside the exceptional divisor.

Consider a line D_i as in (3) and suppose that it comes from a point $p = D_r \cap D_k$ with $r, k \leq n+1$. Note that there is essentially no difference with a line as in (2) besides the change of local indexation to a global one, and therefore, we can proceed as in the proof of Lemma 3.2.14 to obtain that $\lambda^{(\kappa)}_{\sigma^{(\kappa)-1}(r)} \lambda^{(\kappa)}_{\sigma^{(\kappa)-1}(k)}$ is homotopic to a fiber of ∂U_i^* connected to the global base point R.

□

As $D = \sum_{i=1}^N D_i$ is a simple normal crossing divisor with $N = n+1+s_0$, we can consider an orbifold structure in $(\mathrm{Bl}_{P_0} \mathbb{P}^2, D)$ (see [Eys17] for the notation) by choosing weights $r = (r_1, \dots, r_N) \in (\mathbb{N}^* \cup \{+\infty\})^N$.

Theorem 3.2.17. *Let $\mathscr{A}$ be a complex arrangement of lines, $\mathcal{W}$ a wiring diagram and consider the weights r of D as above. The fundamental group $\pi_1(\mathcal{X}(\mathrm{Bl}_{P_0} \mathbb{P}^2, D, r))$ of the orbifold $\mathcal{X}(\mathrm{Bl}_{P_0} \mathbb{P}^2, D, r)$ admits the following presentation:*

$$\left\langle \lambda_1, \dots, \lambda_{n+1} \mid \bigcup_k R_k, \lambda_{n+1} \cdots \lambda_1, \bigcup_{i=1}^N \lambda(D_i)^{r_i} \right\rangle$$

where the relation $\lambda(D_i)^{r_i}$ is omitted if $r_i = +\infty$.

3.3 Boundary Manifolds methods

In this Section we use the results of Mumford as stated in 3.1.4 in order to study the fundamental group of the boundary manifold ∂U of an arrangement of lines $\mathscr{A}$.

The notion of wiring diagram defined in the previous section will play an important role, a presentation of $\pi_1(M(\mathscr{A}, I, P))$ will be obtained as a quotient of the presentation of $\pi_1(\partial U)$ and compared with Theorem 3.2.16.

3.3.1 Boundary manifold of an arrangement of lines

Fundamental group of the boundary manifold of an arrangement of lines

Let $\mathscr{A} = \{L_1, \ldots, L_{n+1}\} \subset \mathbb{P}^2$ be an arrangement of lines and denote by $\pi : \bar{X} \to \mathbb{P}^2$ the blow-up of the projective plane at the s_0 points of $\operatorname{Sing} \mathscr{A}$ of multiplicity equal or higher than 3 as in 3.1.3. Recall that $D = |\pi^* D| = \sum_{i=1}^{n+s_0+1} D_i$ is the reduced total transform of $\mathscr{A}$ in $\bar{X}$ and let $\psi : \partial U \to D$ be its boundary manifold.

Using the description of Mumford (Theorem 3.1.6) and that of a weighted graph, Westlund gave a presentation of the fundamental group $\pi_1(\partial U)$ of ∂U [Wes67] (see also [CS08]). Let us describe this presentation.

Denote by Δ the dual graph of D and by $\mathcal{E}$ the set of edges of Δ as in 3.1.3 above. Associate to each vertex v_i a weight w_i corresponding with the self-intersection number of the associated line D_i in $\bar{X}$.

Let $\mathcal{T}$ be a maximal tree of Δ (a subgraph of Δ containing no cycles and all the vertices of Δ) and denote by $\mathcal{C} = \Delta \setminus \mathcal{T}$. Note that $g = |\mathcal{C}| = b_1(\Delta)$ equals the number of independent cycles in Δ.

The edges in $\mathcal{C}$ correspond to g points $\{p_1, \ldots, p_g\}$ in $\operatorname{Sing} D$. Let us denote by $\pi^{(1)} : \operatorname{Bl}_{p_1,\ldots,p_g} \bar{X} \to \bar{X}$ the blow-up at these points. Denote by $D' = \sum_{i=1}^{n+s_0+1} D'_i$ the strict transform of D in $\operatorname{Bl}_{p_1,\ldots,p_g} \bar{X}$ and let $\psi' : \partial U' \to D'$ be the boundary manifold of D'. Note that the dual graph of D' is a tree that can be identified with $\mathcal{T}$ by removing from Δ the edges in $\mathcal{C}$. In particular, D' and $\partial U'$ are connected. Let ${\pi^{(1)}}^*(D) = D' + \sum_{k=1}^{g} E_k$ be the total transform of D with $E_1, \ldots, E_k$ exceptional divisors.

Now, if $(i, j) \in \mathcal{C}$ corresponds to the point p_k for some $1 \leq k \leq g$, there exists an exceptional divisor $E_k \in \operatorname{Bl}_{p_1,\ldots,p_g} \bar{X}$ and D'_i, D'_j strict transforms of irreducible components D_i, D_j of D respectively such that $E_k \cap D'_i \neq \varnothing$, $E_k \cap D'_j \neq \varnothing$ and $D_i \cap D_j = p_k$. Denote its boundary manifold by $\psi_{E_k} : \partial U_{E_k} \to E_k, \psi'_i : \partial U'_i \to D'_i, \psi'_j : \partial U'_j \to D'_j$.

Select a base point $Q_i \in D'_i \setminus (\cup_{j \neq i} D'_j \bigcup \cup_{k=1}^{g} E_k)$ as in 3.1.4 and a simple curve $l_i \subset D'_i$ containing Q_i and every intersection of the form:

1. $D'_i \cap D'_j$, with (i, j) an edge in $\mathcal{T}$,
2. $D'_i \cap E_k$, with E_k coming from a point p_k corresponding to an edge (i, j) in $\mathcal{C}$.

Let us label these points by the order they intersect l_i as $P_{i1}, \ldots, P_{ik'_i}$. Note that for every P_{im} there corresponds a unique edge $(i, j_\Delta(i, m))$ in Δ. This defines an injective function $m \mapsto j_\Delta(i, m)$ from $\{1, \ldots, k'_i\}$ to $\{1, \ldots, n + s_0 + 1\}$.

We also label only the points as in (1) by the order they intersect l_i as $P'_{i1}, \ldots, P'_{ik_i}$ and define a function $m \mapsto j_\mathcal{T}(i, m)$ from $\{1, \ldots, k_i\}$ to $\{1, \ldots, n + s_0 + 1\}$ as in 3.1.4.

Let $l = \cup l_k \subset D'$ and $h' : l \to \partial U'$ be a continuous function such that $\psi' \circ h' = id_l$. For an exceptional divisor E_k corresponding to an edge (i, j)

in $\mathcal{C}$, we let $l_{E_k} \subset E_k$ be a simple path connecting $E_k \cap D'_i$ to $E_k \cap D'_j$ and $h_{E_k} : l_{E_k} \to \partial U_{E_k}$ such that $\psi_{E_k} \circ h_{E_k} = id_{l_{E_k}}$, $h_{E_k}(l_{E_k}) \cap h'(l_i) \neq \varnothing$ and $h_{E_k}(l_{E_k}) \cap h'(l_j) \neq \varnothing$. This create a cycle $c_k = c_{ij}$ in the boundary manifold of $\pi^{(1)^*}(D)$, which we orient passing first by $h'(l_i)$, following $h_{E_k}(l_{E_k})$ and coming back by $h'(l_j)$. We denote by $\gamma_1, \ldots, \gamma_{n+1+s_0}$ the meridians around $D'_1, \ldots, D'_{n+1+s_0}$ obtained as in 3.1.4 using $h'(l)$.

Theorem 3.3.1 (Westlund)**.** *A presentation for $\pi_1(\partial U)$ is given by*

$$\pi_1(\partial U) = \left\langle \begin{array}{l|ll} \gamma_1, \ldots, \gamma_{n+s_0+1} & [\gamma_i, \gamma_j^{s_{ij}}], & (i,j) \in \mathcal{E} \\ c_1, \ldots, c_g & \gamma_i^{-w_i} = \prod_{m=1}^{k'_i} \gamma_{j_\Delta(i,m)}^{s_{ij_\Delta(i,m)}} & 1 \leq i \leq n+s_0+1 \end{array} \right\rangle$$

where

$$s_{ij} = \begin{cases} c_k^{-1} & \text{if } (i,j) \text{ equals the } k\text{-th element in } \mathcal{C}, \\ c_k & \text{if } (j,i) \text{ equals the } k\text{-th element in } \mathcal{C}, \\ 1 & \text{if } (i,j) \text{ is an edge of } \mathcal{T}. \end{cases}$$

Proof. From Theorem 3.1.6 we know that

$$\pi_1(\partial U') = \left\langle \gamma_1, \ldots \gamma_{n+s_0+1} \middle| [\gamma_i, \gamma_{j_{\mathcal{T}(i,m)}}] \ m = 1, \ldots, k_i, \gamma_i^{-w'_i} = \prod_{m=1}^{k_i} \gamma_{j_{\mathcal{T}(i,m)}} \right\rangle$$

where w'_i is the intersection number of the strict transform D'_i of D_i in $\mathrm{Bl}_{p_1,\ldots,p_g} \bar{X}$. Note that (i,l) is an edge of $\mathcal{T}$ if and only if $l = j_{\mathcal{T}}(i,m)$ for some $m \in \{1, \ldots, k_i\}$ and therefore the set of relations $A = \{[\gamma_i, \gamma_{j_{\mathcal{T}}(i,m)}] \mid m = 1, \ldots, k_i, i = 1, \ldots, n+s_0+1\}$ is the same as $B = \{[\gamma_i, \gamma_l] \mid (i,l) \text{ an edge of } \mathcal{T}\}$.

Let E_k be an exceptional divisor corresponding to an edge (i,j) in $\mathcal{C}$ as above. We can remove two disks $\Delta'_1 \subset D'_i, \Delta'_2 \subset D'_j$ in D' around the points $E_k \cap D'_i$ and $E_k \cap D'_j$ respectively, and obtain a pair of torus T'_i, T'_j as boundary from $\partial U'^{\circ} = \psi'^{-1}(D' \setminus \Delta'_1 \cup \Delta'_2)$. Let $\gamma(E_k)'_i, \gamma_i$ and $\gamma(E_k)'_j, \gamma_j$ be generators of $\pi_1(T'_i)$ and $\pi_1(T'_j)$ with $\gamma(E_k)'_i, \gamma(E_k)'_j$ constructed from $\partial\Delta'_1$, $\partial\Delta'_2$ as in 3.1.4. We obtain the following presentation for $\pi_1(\partial U'^{\circ})$:

$$\left\langle \gamma_1, \ldots \gamma_{n+s_0+1}, \gamma(E_k)'_i, \gamma(E_k)'_j \middle| \begin{array}{l} A, [\gamma_i, \gamma(E_k)'_i], [\gamma_j, \gamma(E_k)'_j] \\ \gamma_l^{-w'_l} = \prod_{m=1}^{k_l} \gamma_{j_{\mathcal{T}}(l,m)} \text{ for } l \neq i,j, \\ \gamma_i^{-w'_i} = \gamma_{j_{\mathcal{T}}(i,1)} \cdots \gamma(E_k)'_i \cdots \gamma_{j_{\mathcal{T}}(i,k_i)}, \\ \gamma_j^{-w'_j} = \gamma_{j_{\mathcal{T}}(j,1)} \cdots \gamma(E_k)'_j \cdots \gamma_{j_{\mathcal{T}}(i,k_j)} \end{array} \right\rangle$$

where the products in the lowest row of the relations are taken in such a way that $1 = \psi_i(\gamma'_{i1}) \cdots \psi_i(\gamma(E_k)'_i) \cdots \psi_i(\gamma'_{ik_i})$ holds in $\pi_1(D'^*_i \setminus \Delta_1)$ and similarly $1 = \psi_j(\gamma'_{j1}) \cdots \psi_j(\gamma(E_k)'_j) \cdots \psi_j(\gamma'_{jk_j})$ in $\pi_1(D'^*_j \setminus \Delta_2)$ for generators $\gamma_r, \gamma'_{r1}, \ldots, \gamma'_{rk_r}$ generators of $\pi_1(\partial U'^*_r)$ for $r = i,j$ as in 3.1.4.

Let E^*_k denote the submanifold of E_k obtained by removing another pair of disks Δ_1, Δ_2 of E_k about the points $E_k \cap D'_i$ and $E_k \cap D'_j$ as in 3.1.4. Write $\partial U^*_{E_k}$ for $\psi^{-1}_{E_k}(E^*_k)$. Note that the boundary of $\partial U^*_{E_k}$ consists also of a pair

of torus T_i, T_j corresponding to Δ_1 and Δ_2 respectively. Let $\gamma_i', \gamma(E_k)$ and $\gamma_j', \gamma(E_k)$ be generators of $\pi_1(T_i)$ and $\pi_1(T_j)$ respectively. By (3.1), we have that $\pi_1(\partial U^*_{E_k}) = \langle \gamma(E_k), \gamma_i', \gamma_j' \mid [\gamma(E_k), \gamma_i'], [\gamma(E_k), \gamma_j'], \gamma(E_k) = \gamma_i'\gamma_j' \rangle \cong \langle \gamma_i', \gamma_j' \mid [\gamma_i', \gamma_j'] \rangle \cong \mathbb{Z}^2$ because $E_k \cdot E_k = -1$.

We can glue $\partial U'^{\circ}$ to $\partial U^*_{E_k}$ by first gluing T_i to T_i' by a longitude-to-meridian orientation-preserving attaching map f, and similarly T_j to T_j' by a map g.

First, by the van Kampen Theorem we obtain that $\gamma_i = \gamma_i'$ and $\gamma(E_k) = \gamma(E_k)_i'$. Then, from HNN extension we get $\gamma_j = c_k^{-1}\gamma_j' c_k$ and $\gamma(E_k)_j' = c_k^{-1}\gamma(E_k)c_k$.

We obtain the following presentation of $\pi_1(\partial U'^{\circ} \cup_{f,g} \partial U^*_{E_k})$ by replacing $\gamma_i' = \gamma_i, \gamma_j' = \gamma_j^{c_k^{-1}}, \gamma(E_k) = \gamma_i\gamma_j^{c_k^{-1}}, \gamma(E_k)_i' = \gamma_i\gamma_j^{c_k^{-1}}, \gamma(E_k)_j' = \gamma_i^{c_k}\gamma_j$ in terms of γ_i, γ_j, c_k

$$\left\langle \gamma_1, \dots \gamma_{n+s_0+1}, c_k \;\middle|\; \begin{array}{l} [\gamma_i, \gamma_j^{c_k^{-1}}], [\gamma_i, \gamma_l] \text{ with } (i,l) \in \mathcal{T} \\ \gamma_l^{-w_l'} = \prod_{m=1}^{k_l} \gamma_{j_{\mathcal{T}}(l,m)} \text{ for } l \neq i, j, \\ \gamma_i^{-w_i'} = \gamma_{j_{\mathcal{T}}(i,1)} \cdots \gamma_i \gamma_j^{c_k^{-1}} \cdots \gamma_{j_{\mathcal{T}}(i,k_i)}, \\ \gamma_j^{-w_j'} = \gamma_{j_{\mathcal{T}}(j,1)} \cdots \gamma_i^{c_k} \gamma_j \cdots \gamma_{j_{\mathcal{T}}(j,k_j)} \end{array} \right\rangle$$

Note that the row of the relations corresponding to i can be simplified to

$$\gamma_i^{-(w_i'+1)} = \gamma_{j_{\mathcal{T}}(i,1)} \cdots \gamma_j^{c_k^{-1}} \cdots \gamma_{j_{\mathcal{T}}(i,k_i)}, \tag{3.2}$$

as γ_i commutes with every $\gamma_{j_{\mathcal{T}}(i,m)}$. A similar simplification can be made for the relation corresponding to j.

We repeat the above process for every E_k with $k = 1, \dots, g$. After this, the order for the product as in (3.2), is given by the function $m \mapsto j_\Delta(i,m)$ and the conjugations s_{ij} as in the statement of the Theorem. We get that $\gamma_i^{-(w_i'+(k_i'-k_i))} = \prod_{m=1}^{k_i'} \gamma_{j_\Delta(i,m)}^{s_{ij_\Delta(i,m)}}$. Note that $k_i' - k_i$ equals the number of points in $\{p_1, \dots, p_g\} \cap D_i$, and therefore $w_i' + (k_i' - k_i) = w_i$.

This gives a presentation for the fundamental group of the boundary manifold of the total transform of D, which is homeomorphic to ∂U. □

A central computation in our work is the expression of the meridians around the exceptional divisors $D_{n+2}, \dots, D_{n+s_0+1}$ in $D = \sum_{i=1}^{n+s_0+1} D_i$ in terms of meridians of the lines in $\mathscr{A}$. As a partial result we obtain an expression in the following corollary. The cycles c_k will be expressed in terms of meridians of the lines in 3.3.2.

Corollary 3.3.2. *For $r = n+2, \dots, n+s_0+1$, we have that in $\pi_1(\partial U)$,*

$$\gamma_r = \prod_{m=1}^{k_r'} \gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}} \text{ with } s_{rj} = \begin{cases} c_k^{-1} & \text{if } (r,j) \text{ equals the } k\text{-th element in } \mathcal{C}, \\ c_k & \text{if } (j,r) \text{ equals the } k\text{-th element in } \mathcal{C}, \\ 1 & \text{if } (r,j) \text{ is an edge in } \mathcal{T}. \end{cases}$$

Proof. It follows from the relation $\gamma_r^{-w_r} = \prod_{m=1}^{k_r'} \gamma_{j_\Delta(i,m)}^{s_{rj_\Delta(r,m)}}$ in the presentation of $\pi_1(\partial U)$ in Theorem 3.3.1, the fact that $w_r = -1$ because D_r is an exceptional divisor and hence $j_\Delta(r,m) \in \{1, \dots, n+1\}$. □

Choice of a maximal tree

In what follows, we will define a maximal tree $\mathcal{T}'$ of the dual graph Δ of D as defined in [CS08, Section 3.3].

In the arrangement $\mathscr{A} = \{L_1, \ldots, L_{n+1}\}$, we will fix the line L_{n+1} as the line at infinity, recall that we denote by D_i the strict transform by L_i for $i \leq n+1$ in $D = \sum D_i \subset \bar{X}$.

Consider the following subset of edges $\mathcal{E}' \subset \mathcal{E}$ which defines a maximal tree $\mathcal{T}' \subset \Delta$ of the dual graph Δ of D:

1. Let $(j, n+1), (n+1, j) \in \mathcal{E}'$ if $D_{n+1} \cap D_j \neq \varnothing$. This is, all the edges having as an endpoint the vertex corresponding to D_{n+1}.

2. Let $(i, j) \in \mathcal{E}'$ if $n+1 < j$ (D_j is an exceptional divisor) with either

   - $D_j \cap D_{n+1} = \varnothing$ and $i = \min\{l \mid D_l \cap D_j \neq \varnothing\}$. Note that D_j comes from a point in $\operatorname{Sing} \mathscr{A} \setminus L_{n+1}$.
   - or $D_j \cap D_{n+1} \neq \varnothing$ and $D_i \cap D_j \neq \varnothing$. The line D_i corresponds then to a line L_i touching L_{n+1} in a point of multiplicity > 2.

   Note that $\mathcal{E} \setminus \mathcal{E}'$ consists either:

   - of edges corresponding to double points $L_i \cap L_j$ with $i, j < n+1$,
   - or, if $p = L_{i_1} \cap \ldots \cap L_{i_l}$ with $i_1 < \ldots < i_l < n+1$, $2 < l$, and E_j denotes the exceptional divisor obtained by blowing up at p, of edges of the form (i_r, j) with $r = 2, \ldots, l$.

Let us consider the presentation of $\pi_1(\partial U)$ as in Theorem 3.3.1. If (i, j) equals the k-th element in $\Delta \setminus \mathcal{T}'$ as in the first point above, we denote the cycle c_k by $c_{i,j}$. Recall that if $i < j$, we pass first through $h'(l_i)$ and then through $h'(l_j)$.

For the cycles created by the edges in the second point, let us suppose that the irreducible component of D are ordered in such a way that $D_{n+1} \cap D_j \neq \varnothing$ for $j = n+2, \ldots, s'$ and $D_{n+1} \cap D_k = \varnothing$ for $k > s'$.

For $s' < \iota \leq n+1+s_0$, we have that, as D_ι is an exceptional divisor, $1 \leq j_\Delta(\iota, m) \leq n$ for $1 \leq m \leq k'_i$, and $\gamma_\iota^{-w_\iota} = \prod_{m=1}^{k'_\iota} \gamma_{j_\Delta(\iota,m)}^{s_{\iota j_\Delta(\iota,m)}}$ holds as in Theorem 3.3.1. Note that if $(j_\Delta(\iota, m), \iota)$ equals the k-th element in $\Delta \setminus \mathcal{T}'$, we have that $s_{\iota j_\Delta(\iota,m)} = c_k$. In this case, we denote c_k by $c_{j_\Delta(\iota,m),\iota}$. As $\mathcal{T}'$ is a maximal tree, the edges corresponding to $(j_\Delta(\iota, m), \iota)$ for $1 < m \leq k'_i$, give rise to $k'_i - 1$ independent cycles $c_{j_\Delta(\iota,m),\iota}$ in Δ.

Using the tree $\mathcal{T}'$ and corollary 3.3.2, we can express the meridian around an exceptional divisor in terms of the meridians of the lines and the cycles $c_{s''_t, s''}$:

$$\gamma_\iota = \gamma_{j_\Delta(\iota,1)}^{c_{j_\Delta(\iota,1),\iota}} \gamma_{j_\Delta(\iota,2)}^{c_{j_\Delta(\iota,2),\iota}} \cdots \gamma_{j_\Delta(\iota,k'_\iota)}^{c_{j_\Delta(\iota,k'_\iota),\iota}} \quad \text{for } s' < \iota \leq n+1+s_0 \tag{3.3}$$

with $c_{j_\Delta(\iota,r),\iota} = 1$ if $r = \min\{j_\Delta(\iota, m) \mid m = 1, \ldots, k'_i\}$.

3.3.2 From a presentation for the boundary manifold of an arrangement of lines to a presentation of its complement

Let $\mathscr{A} \subset \mathbb{P}^2$ be an arrangement of lines and ∂U its boundary manifold. We identify ∂U with the boundary manifold of the total transform D of $\mathscr{A}$ in $\pi : \bar{X} \to \mathbb{P}^2$, the blow-up of $\mathbb{P}^2$ at the points of $\operatorname{Sing}\mathscr{A}$ of multiplicity higher than two. Denote by $i : \partial U \hookrightarrow \mathbb{P}^2 \setminus \mathscr{A}$ the inclusion map and by $i_* : \pi_1(\partial U) \to \pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ the induced homomorphism.

Consider the presentations $\langle \gamma_1, \ldots, \gamma_{n+1}, c_1, \ldots, c_g \mid R' \rangle$ of $\pi_1(\partial U)$ with R' the set of relations as in Theorem 3.3.1 and $\langle \lambda_1, \ldots, \lambda_{n+1} \mid \cup R_k, \lambda_{n+1} \cdots \lambda_1 \rangle$ of $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ as in Theorem 3.2.12.

Recall that the construction of the meridian γ_k around the irreducible component D_k of $D = \sum_{k=1}^{n+1+s_0} D_k$ depends on a choice of a maximal tree $\mathcal{T}$ of the dual graph Δ of D, contractible paths $l_k \subset D_k$, and a section $h : l = \cup l_k \to \partial U$, see 3.1.4. We choose the maximal tree $\mathcal{T}'$ constructed at 3.3.1. For $p = L_{\eta_1} \cap \ldots \cap L_{\eta_r} \in \operatorname{Sing}\mathscr{A} \setminus L_{n+1}$, we have a unique cycle c_{η_1,η_2} if $r = 2$ and $r-1$ cycles if $r > 2$, in this case let us denote by D_ι the corresponding exceptional divisor in $\bar{X}$, therefore we have the cycles $c_{\eta_2,\iota}, \ldots, c_{\eta_r,\iota}$. See 3.3.1.

Consider a wiring diagram $\mathcal{W}$ of $\mathscr{A}$ as in 3.2.1. There exists $\kappa \in \mathbb{N}^*$ such that $p \in M_\kappa$. Consider the geometric generating set $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$. Recall that, as in remark 3.2.13, there exists a word $\xi_j^{(\kappa)}$ in $\lambda_1^{(1)}, \ldots, \lambda_{n+1}^{(1)}$ (see also 3.3.2), such that

$$\lambda^{(\kappa)}_{\sigma^{(\kappa)^{-1}}(j)} = \lambda_j^{(1)^{\xi_j^{(\kappa)}}} \quad \text{for } j = 1, \ldots, n+1.$$

The main objective of this subsection is to prove the following Theorem.

Theorem 3.3.3. *The paths $l_1, \ldots, l_{n+1+s_0}$, the map $h : l \to \partial U$ and the wiring diagram $\mathcal{W}$ of $\mathscr{A}$ can be chosen in such a way that*

1. *The generator γ_k of $\pi_1(\partial U)$ lies in the same homotopy class as λ_k in $\mathbb{P}^2 \setminus \mathscr{A}$ for $k = 1, \ldots, n+1$.*

2. *If $p = L_{\eta_1} \cap \ldots \cap L_{\eta_r} \in \operatorname{Sing}\mathscr{A} \setminus L_{n+1}$ and $p \in M_\kappa$ as above, then*

   - *if $r = 2$, the cycle c_{η_1,η_2} is homotopic in $\mathbb{P}^2 \setminus \mathscr{A}$ to $\xi_{\eta_1}^{(\kappa)}(\xi_{\eta_2}^{(\kappa)})^{-1}$ and*
   - *if $r > 2$, the cycle $c_{\eta_a,\iota}$ is homotopic to $\xi_{\eta_a}^{(\kappa)}{\xi_{\eta_1}^{(\kappa)}}^{-1}$, for $a = 2, \ldots, r$.*

   By the point (1), we can also consider each $\xi_{\eta_a}^{(\kappa)}$ as a word in $\gamma_1, \ldots, \gamma_{n+1}$.

3. *If $p = L_{\eta_1} \cap \ldots \cap L_{\eta_r} \in \operatorname{Sing}\mathscr{A} \setminus L_{n+1}$, denote by $R'(p)$ the set of relations:*

   - $\{c^{-1}_{\eta_1,\eta_2}\xi_{\eta_1}^{(\kappa)}(\xi_{\eta_2}^{(\kappa)})^{-1}\}$ *if $r = 2$, or*

- $\{c_{\eta_a,\iota}^{-1}\xi_{\eta_a}^{(\kappa)}(\xi_{\eta_1}^{(\kappa)})^{-1} \mid a = 2,\dots,r\}$ *if* $r > 2$.

We have that $\langle \gamma_1,\dots,\gamma_{n+1},c_1,\dots,c_g \mid R', \cup_{p\in \mathrm{Sing}\,\mathscr{A}\setminus L_{n+1}} R'(p)\rangle$ *and* $\langle \lambda_1,\dots,\lambda_{n+1} \mid \cup R_k, \lambda_{n+1}\cdots\lambda_1\rangle$ *are Tietze-equivalent presentations of* $\pi_1(\mathbb{P}^2\setminus\mathscr{A})$.

By using a different presentation of $\pi_1(\partial U)$ and different techniques, the image of the generators of $\pi_1(\partial U)$ under the map i_* was computed in [FGBMB15] (See Proposition 2.13 and Theorem 4.5 of loc. cit.). The proof of Theorem 3.3.3 is inspired by the ideas of [FGBMB15].

Constructing equivalent generators

Let us choose the point $R \in \mathbb{P}^2\setminus\mathscr{A}$ close to L_{n+1}, consider the blow-up $\pi_R : \mathrm{Bl}_R\,\mathbb{P}^2 \to \mathbb{P}^2$ and denote by $\bar{f} : Bl_R\mathbb{P}^2 \to \mathbb{P}^1$ the associated pencil as in 3.2.1.

Let $\beta : [0,1] \to \mathbb{P}^1$ be as in 3.2.1 such that it passes first through the projection of the points $\mathrm{Sing}\,\mathscr{A}\cap L_{n+1}$ to $\mathbb{P}^1$ via $\bar{f}$. Take its associated wiring diagram $\mathcal{W}$ corresponding to the arrangement $\mathscr{A}$ and fix a planar representation $p(\beta^*\mathcal{W})$ as in 3.2.1.

Let us order the representation of all the singular points $\mathrm{Sing}\,\mathscr{A} = \{p_1,\dots,p_s\}$ in $p(\beta^*\mathcal{W})$ together with the virtual vertices $\{p'_1,\dots,p'_\nu\} \in p(\beta^*\mathcal{W})$, by the order they are crossed by the fiber $p(\beta^*\mathcal{W})|_t$ with t increasing in $[0,1]$, and let $t_1,\dots,t_{s+\nu} \in (0,1)$ be such that either an actual or a virtual vertex lies in $p(\beta^*(\mathcal{W}))|_{t_\kappa}$, for all $\kappa = 1,\dots,s+\nu$. By abuse of notation we will also denote by t_κ the crossings in $p(\beta^*\mathcal{W})$ at the fiber $p(\beta^*(\mathcal{W}))|_{t_\kappa}$ and we will write $\mathcal{W}$ for $p(\beta^*\mathcal{W})$. Let $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)},\dots,\lambda_{n+1}^{(\kappa)}\}$ be the geometric generating set defined in 3.2.2, for $\kappa = 1,\dots,s+\nu$.

Recall that we have assumed that the order of the lines $L_1,\dots,L_n$ is such that, at the very right of the planar representation of $\mathcal{W}$, the wire W_1 is at the bottom of $\mathcal{W}$, above it is the wire W_2 and then W_3, continuing in this way until W_n.

For an irreducible component D_k of $D \subset \bar{X}$, denote its boundary manifold by $\psi_k : \partial U_k \to D_k$ and recall that we can consider $\psi_k|_{\partial U_k^*} : \partial U_k^* = \partial U_k \cap \partial U \to D_k^*$ (see 3.1.4). A set of generators for $\pi_1(\partial U_k^*)$ was constructed by fixing a base point $Q_k \in D_k^*$, simple paths $l_k \subset D_k$ from which we obtain paths $l'_k \subset D_k^*$ (see figure 3.1) and $h : \cup l_k \to \partial U$ as in 3.1.4. The generators $\gamma_1,\dots,\gamma_{n+1+s}$ were constructed by joining the different generators of $\pi_1(\partial U_k^*)$ to a common base point Q via the contractible path $h(\cup l_k)$ in ∂U.

Recall that the first $n+1$ irreducible components $D_1,\dots,D_{n+1}$ of D correspond to the lines $L_1,\dots,L_{n+1}$ respectively and that, as in the end of 3.3.1, there exists s' such that for $j = n+2,\dots,s'$, we have that $D_{n+1}\cap D_j \neq \varnothing$ and for $j > s'$, we have $D_{n+1}\cap D_j = \varnothing$.

Lemma 3.3.4. *For* $k = n+1,\dots,s'$, *we can choose* $l'_k \subset D_k^*$, *a continuous map* $h'_k : l'_k \to \partial U_k^*$ *and a base point* $Q \in h_{n+1}(l'_{n+1})$ *for the fundamental group*

$\pi_1(\partial U)$ *in such a way that* $i_*(\gamma_r)$ *lies in the same homotopy class as* $\lambda_r^{(1)}$ *for* $r = 1, \ldots, n+1$.

Proof. We begin by defining those l'_k for $k = n+2, \ldots, s'$. Essentially, we arrange the choices in an appropriate way to obtain the stated in the lemma.

More precisely, let D_k be an exceptional divisor corresponding to a point $p = p(k) \in \operatorname{Sing} \mathscr{A} \cap L_{n+1}$ with multiplicity higher or equal to three. Suppose that $p = L_j \cap L_{j+1} \cap \ldots \cap L_{m-1} \cap L_m \cap L_{n+1}$ (which can be written in this way by the order of the lines chosen above). Consider the boundary manifold $\psi_k : \partial U_k \to D_k$ in $\bar{X}$. We will also write ∂U_p for the image $\pi(\partial U_k) \subset \mathbb{P}^2$ under the map $\pi : \bar{X} \to \mathbb{P}^2$. For $r = j, \ldots, m$, each meridian $\lambda_r^{(1)}$ (see figure 3.3a) is homotopic to a meridian ${\lambda'_r}^{(1)}$ (see figure 3.6) that can be decomposed in the following way: ${\lambda'_r}^{(1)} = \lambda_p \lambda_{r_1} \lambda_p^{-1}$ with $\lambda_{r_1} \subset \partial U_p$ a meridian of L_r based at a point $q_p \in \partial U_p$ and λ_p a path connecting R and q_p.

We can further decompose each λ_{r_1} as the boundary of a disk Δ around a point in L_r and a path λ_{r_2} connecting the point q_p to $\partial\Delta$. Define the path l'_k in D_k^* as the projection $\psi_k(\cup_{r=j}^m \lambda_{r_2})$. We define $h|_{l'_k}$ such that $h|_{l'_k} \circ \psi_k|_{\cup_{r=j}^m \lambda_{r_2}} = id|_{\cup_{r=j}^m \lambda_{r_2}}$.

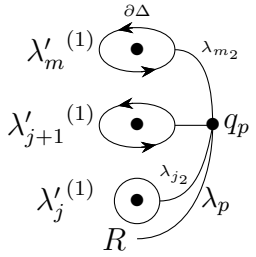


Figure 3.6: Decomposing a meridian

Note that, up to a slight change in R, the paths λ_p are homotopic to paths λ'_p lying in ∂U_{n+1}^*.

Now, let $p = L_j \cap L_{n+1}$ be a double point in $\operatorname{Sing} \mathscr{A} \cap L_{n+1}$. The meridian $\lambda_j^{(1)}$ can be decomposed as $\lambda'_j \cdot \partial\Delta_j \cdot \lambda_j'^{-1}$, with $\partial\Delta_j$ a fiber of ∂U_j^* and $\lambda'_j \subset \partial U_{n+1}^*$ a path starting at R and finishing at point $q_j \in \partial U_j^* \cap \partial U_{n+1}^*$.

Finally, for $k = n+1$ we define $l_k \in D_k$ as the image of $\beta([0,1])$ under the section of the pencil $\bar{f} : \operatorname{Bl}_R \mathbb{P}^2 \to \mathbb{P}^1$ with range D_k. By construction l_k passes over all the points in $\operatorname{Sing} \mathscr{A} \cap L_{n+1}$. We let $h|_{l_{n+1}}$ be a continuous function such that $\psi_{n+1}|_{h|_{l_{n+1}}(l_{n+1})} \circ h|_{l_{n+1}} = id_{l_{n+1}}$, $h|_{l_{n+1}}(l_{n+1})$ is a simple path passing through each q_j with $L_j \cap L_{n+1}$ a double point, touching each $\lambda_{(n+1)_2}(p)$ for each point $p \in \operatorname{Sing} \mathscr{A} \cap L_{n+1}$ of multiplicity greater or equal to two, and such that each $\lambda_p \cdot \lambda_{j_2}$ is homotopic to a segment of $h|_{l_{n+1}}(l_{n+1})$.

By the construction of the maximal tree $\mathcal{T}'$, these paths are sufficient to construct γ_j for $j = 1, \ldots, n+1$ and by construction, they lie in the same homotopy class as $\lambda_j^{(1)}$. □

Corollary 3.3.5. *The morphism $i_* : \pi_1(\partial U) \to \pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ is surjective.*

Proof. The group $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ is generated by the elements $\Gamma^{(1)} = \{\lambda_1^{(1)}, \ldots, \lambda_{n+1}^{(1)}\}$, as $i_*(\gamma_k) = \lambda_k^{(1)}$ for $k = 1, \ldots, n+1$ the result follows. □

Suppose that $p \in M_\kappa \cap (\operatorname{Sing} \mathscr{A} \setminus L_{n+1})$ is of multiplicity higher or equal to three and that $p = L_{\sigma^{(\kappa)}(j)} \cap L_{\sigma^{(\kappa)}(j+1)} \cap \ldots \cap L_{\sigma^{(\kappa)}(m)}$. Denote by $\psi_p : \partial U_p \to E_p$ the boundary manifold of the exceptional divisor $E_p \subset \bar{X}$ obtained by blowing-up p. We select $l'_p \subset E_p^*$ in a similar way as in the proof of the precedent Lemma for a point of multiplicity higher than two lying in L_{n+1}: decompose each $\gamma_j^{(\kappa)}, \ldots, \gamma_m^{(\kappa)}$ into a path $\lambda_p^{(\kappa)}$ connecting $R \subset \mathbb{P}^2$ and a point $q_p \in \partial U_p$, and ${\lambda_{r_1}}^{(\kappa)}$ with $r \in \{j, \ldots, m\}$ based at q_p and generating $\pi_1(\partial U_p^*)$ as in figure 3.6. Decompose further ${\lambda_{r_1}}^{(\kappa)}$ into a boundary of a disk $\partial\Delta_r$ around a point of the line $L_{\sigma^{(\kappa)}(r)}$ and a path ${\lambda_{r_2}}^{(\kappa)}$ connecting q_p and $\partial\Delta_r$. We take $l'_p = \psi_p(\cup {\lambda_{r_2}}^{(\kappa)})$ and define $h|_{l'_p}$ such that $h|_{l'_p} \circ \psi_p|_{\cup {\lambda_{r_2}}^{(\kappa)}} = id|_{\cup {\lambda_{r_2}}^{(\kappa)}}$.

For every $k = 1, \ldots, n$, we define $l_k \subset D_k$ as the image of $\beta([0,1])$ under the section of $\bar{f} : \operatorname{Bl}_R \mathbb{P}^2 \to \mathbb{P}^1$ that has as range D_k. We define $h|_{l_k}$ such that it is continuous, $\psi_k|_{h|_{l_k}(l_k)} \circ h_{l_k} = id_{l_k}$, $h_{l_k}(l_k)$ intersects $\cup \lambda_{r_2}^{(\kappa)}$ in a point if $p \in M_\kappa \cap (\operatorname{Sing} \mathscr{A} \cap L_k)$ with the notations as in the paragraph above, $h_{l_k}(l_k) \cap h_{l_{k'}}(l_{k'}) \neq \varnothing$ if $L_k \cap L_{k'} \neq \varnothing$ is a double point and $h_{l_k}(l_k)$ is not homotopic to a multiple of a fiber S^1 of ∂U_k.

Expressing the cycles in terms of the meridians

Let $t_\eta \in \mathcal{W}$ be an actual vertex and suppose that $t_\eta = W_{\eta_1} \cap W_{\eta_2} \cap \ldots \cap W_{\eta_r}$ with the global order of the wires of $\mathcal{W}$ such that $\eta_1 < \eta_2 < \ldots < \eta_r < n+1$. By definition of the maximal tree $\mathcal{T}'$, to each η_a, with $a > 1$, corresponds a cycle c_{η_a, t_η} which is a generator of $\pi_1(\partial U)$, see 3.3.1. This cycle is constructed by connecting $h|_{\eta_a}(l_{\eta_a}) \cdot h|_{t_\eta}(l'_{t_\eta}) \cdot h|_{\eta_1}(l_{\eta_1})$ to R if $r > 2$ and by connecting $h|_{\eta_1}(l_{\eta_1}) \cdot h|_{\eta_a}(l_{\eta_a})$ to R if $r = 2$.

For every $\kappa \leq \eta$, consider the geometric generating set $\Gamma^{(\kappa)} = \{\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)}\}$ as in 3.2.2 and recall the construction of the functions $\tau^{(\kappa)} : \{1, \ldots, n+1\} \to F_{n+1}^{(\kappa)}$ as defined before Proposition 3.2.11. For $1 \leq a \leq r$, denote by

$$\xi_{\eta_a}^{(\kappa)} = \tau^{(1)}(\eta_a) \cdot \tau^{(2)}\left(\sigma^{(2)^{-1}}(\eta_a)\right) \ldots \tau^{(\kappa-1)}\left(\sigma^{(\kappa-1)^{-1}}(\eta_a)\right). \tag{3.4}$$

Proposition 3.3.6. *Let $1 < a \leq r$. The image of the cycle c_{t_η, η_a} under the map i_* equals $\xi_{\eta_1}^{(\eta)}(\xi_{\eta_a}^{(\eta)})^{-1}$ if $r = 2$ or $\xi_{\eta_a}^{(\eta)}(\xi_{\eta_1}^{(\eta)})^{-1}$ if $r > 2$.*

We consider the points $\theta_\kappa < t_\kappa$ very close to t_κ as before Definition 3.2.3.

Lemma 3.3.7. *Let $\Gamma^{(\kappa)} = (\lambda_1^{(\kappa)}, \ldots, \lambda_{n+1}^{(\kappa)})$ be a generating set as above. Then, for $\iota = 1, \ldots, n+1$ we have that $(\lambda_\iota^{(\kappa)})^{\xi_{\eta_a}^{(\kappa)^{-1}}}$ is homotopic to a meridian of $L_{\sigma^{(\kappa)}(\iota)}$ at the point $x_\iota^{(\kappa)} = \bar{f}^{-1}(\beta(\theta_\kappa)) \cap L_{\sigma^{(\kappa)}(\iota)}$ constructed by:*

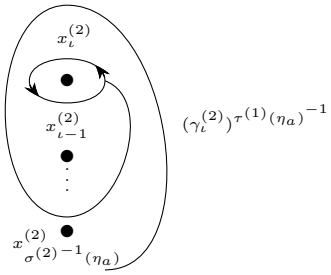


Figure 3.7: A meridian follows another boundary manifold

1. *following* $h|_{\eta_a}(l_{\eta_a})$ *until* $\bar{f}^{-1}(\beta(\theta_\kappa))$,
2. *then joining it to a circle in* $\bar{f}^{-1}(\beta(\theta_\kappa))$ *about* $x_\iota^{(\kappa)}$ *and,*
3. *coming back via* $h|_{\eta_a}(l_{\eta_a})$.

See figure 3.7.

Proof. Note that if $\sigma^{(\kappa)}(\iota) = \eta_a$ and as $\lambda_\iota^{(\kappa)} = \lambda_{\eta_a}^{\xi_{\eta_a}^{(\kappa)}}$, by successive applications of Proposition 3.2.11, we can choose a meridian in the homotopy class of $(\lambda_\iota^{(\kappa)})^{\xi_{\eta_a}^{(\kappa)^{-1}}} = \lambda_{\eta_a}$ that satisfies the properties stated in the Lemma (see figure 3.3b).

Now, if $\sigma^{(\kappa)}(\iota) \neq \eta_a$, we proceed by induction. Let $t_1 = W_j \cap W_{j+1} \cap \ldots \cap W_m$ and consider

$$\xi_{\eta_a}^{(2)} = \tau^{(1)}(\eta_a) = \begin{cases} \lambda_{\eta_a - 1}^{(1)} \cdots \lambda_j^{(1)} & \text{if } n_a \in \{j+1, \ldots, m\}, \\ 1 & \text{if } \eta_a \notin \{j+1, \ldots, m\}. \end{cases}$$

By construction, for $k \in \{j+1, \ldots, m\}$ the product $\lambda_k^{(1)} \ldots \lambda_j^{(1)}$ is freely homotopic to a circle containing the points $x_j^{(1)}, \ldots, x_k^{(1)}$.

Now, note that the paths in $\Gamma^{(1)}$ are homotopic to paths in the fiber $\bar{f}^{-1}(\beta(\theta_2))$ as in Figure 3.3b. Such representative of the homotopy class of $\lambda_\iota^{(1)}$ can be seen as lying in the boundary manifold ∂U_ι^*.

By considering $(\lambda_\iota^{(2)})^{\xi_{\eta_a}^{(2)^{-1}}}$ we obtain a path as in figure 3.7 if $\tau^{(1)}(\eta_a) \neq 1$. This meridian can be decomposed as stated.

For a general $\Gamma^{(\kappa+1)}$, note that as $\xi_{\eta_a}^{(\kappa+1)} = \xi_{\eta_a}^{(\kappa)} \cdot \tau^{(\kappa)}(\sigma^{(\kappa)^{-1}}(\eta_a))$ and by repeating the above procedure, we can decompose $(\lambda_\iota^{(\kappa+1)})^{\tau^{(\kappa)^{-1}}(\sigma^{(\kappa)^{-1}}(\eta_a))}$ as a meridian of $L_{\sigma^{(\kappa+1)}(\iota)}$ that follows l'_{η_a} between $\bar{f}^{-1}(\beta(\theta_\kappa))$ and $\bar{f}^{-1}(\beta(\theta_{\kappa+1}))$ (see figure 3.7). By applying induction, we obtain that $(\lambda_\iota^{(\kappa+1)})^{\xi_{\eta_a}^{(\kappa+1)^{-1}}}$ can be decomposed as stated in the lemma. □

Proof of Proposition 3.3.6. Note that we have that

$$\begin{aligned} \xi_{\eta_a}^{(\eta)} &= \tau^{(1)}(\eta_a) \cdots \tau^{(\eta-1)}(\sigma^{(\eta-1)^{-1}}(\eta_a)) \\ &= \tau^{(\eta-1)}\left(\sigma^{(\eta-1)^{-1}}(\eta_a)\right)^{\xi_{\eta_a}^{(\eta-1)^{-1}}} \cdots \tau^{(2)}\left(\sigma^{(2)^{-1}}(\eta_a)\right)^{\xi_{\eta_a}^{(2)^{-1}}} \tau^{(1)}(\eta_a). \end{aligned}$$

by using $\xi_{\eta_a}^{(\kappa)} = \xi_{\eta_a}^{(\kappa-1)} \cdot \tau^{(\kappa-1)}\left(\sigma^{(\kappa-1)^{-1}}(\eta_a)\right)$ for $\kappa = 2, \ldots, \eta$.

Now, if $\tau^{(\kappa)}(\sigma^{(\kappa)^{-1}}(\eta_a)) \neq 1$, it is homotopic to a path in $\bar{f}^{-1}(\beta(\theta_\kappa))$ encircling the points $x_j^{(\kappa)}, \ldots, x_{\sigma^{(\kappa)^{-1}}(\eta_a)}^{(\kappa)}$ and by applying Lemma 3.3.7 to each factor of $\tau^{(\kappa)}(\sigma^{(\kappa)^{-1}}(\eta_a)) = \lambda_{\sigma^{(\kappa)^{-1}}(\eta_a)}^{(\kappa)} \ldots \lambda_j^{(\kappa)}$ we obtain that $\tau^{(\kappa)}(\sigma^{(\kappa)^{-1}}(\eta_a))$ can be decomposed in three parts as in Lemma 3.3.7.

Recall that we have constructed the $l_{\eta_a} \subset D_{\eta_a}$ from a section of the map $\bar{f}$. By the choice of $h|_{\eta_a}$ we can suppose that $h_{\eta_a} \subset \bar{f}^{-1}(\beta[0,1]) \cap \partial U_{\eta_a}$.

By considering $Y = \bar{f}^{-1}(\beta([0,1])) \setminus \pi_R^{-1}(R) \subset \mathrm{Bl}_R \mathbb{P}^2$, we can see the cycles $c_{t_\eta,\eta_a} \subset Y \subset \mathbb{R}^3$. Moreover, we can choose coordinates in $\mathbb{R}^3$ and define that $h|_{\eta_a}(l_{\eta_a})$ passes above $h|_{l_k}(l_k)$ (or $h|_{l_k}(l_k)$ passes below $h|_{l_{\eta_a}}(l_{\eta_a})$) in some fiber $\bar{f}^{-1}(\beta(\theta'))$ with $\theta' \in [t_\kappa - \varepsilon, t_\kappa + \varepsilon]$ with $\varepsilon > 0$ sufficiently small, if the wires $W_{\eta_a} \cap W_k \neq \emptyset$ in a planar representation of the fiber $\bar{f}^{-1}(\beta(t_\kappa))$ and $\sigma^{(\kappa)^{-1}}(\eta_a) < \sigma^{(\kappa)^{-1}}(k)$.

We can see then $(\tau^{(\kappa)}(\sigma^{(\kappa)}(\eta_a)))^{\xi_{\eta_a}^{(\kappa)^{-1}}}$ as a path encircling the lines L_k corresponding to those $h|_{l_k}(l_k)$ passing below $h|_{l_{\eta a}}(l_{\eta_a})$ in some fiber $\bar{f}^{-1}(\beta'(\theta'))$ with $\theta' \in [t_\kappa - \varepsilon, t_\kappa + \varepsilon]$. By construction, $\xi_{\eta_a}^{(\eta)}$ is homotopic to a path encircling all the lines L_k such that $h|_{l_k}(l_k)$ lies below $h_{l_{\eta a}}(l_{\eta_a})$ at some point in $\beta([0, t_\eta])$.

Therefore, we can decompose $\xi_{\eta_a}^{(\eta)}$ in three parts:

1. The first path starting at $Q \in h(l_{n+1})$ and following $h(l_{\eta_a})$ until $\bar{f}^{-1}(\beta(\theta_\eta))$. Then,

2. a simple path starting at $h(l_{\eta_a}) \cap \bar{f}^{-1}(\beta(\theta_\eta))$, lying completely in $\bar{f}^{-1}(\beta(\theta_\eta))$ and finishing at $\bar{f}^{-1}(\beta(\theta_\eta)) \cap \pi_R^{-1}(R)$, and

3. a path connecting $\pi_R^{-1}(R) \cap \bar{f}^{-1}(\beta(\theta_\eta))$ to $Q \subset \pi_R^{-1}(R)$.

By decomposing in a similar fashion $\xi_{\eta_1}^{(\eta)}$, it follows that the cycle c_{t_η,η_a} is homotopic in $\mathbb{P}^2 \setminus \mathscr{A}$ to $\xi_{\eta_1}^{(\eta)}(\xi_{\eta_a}^{(\eta)})^{-1}$ if $r = 2$ and to $\xi_{\eta_a}^{(\eta)}(\xi_{\eta_1}^{(\eta)})^{-1}$ if $r > 2$. □

Expressing the relations in terms of the generators

For every $r = n+2, \ldots, n+s_0+1$, we let R'_r be the subset of the set of relations R' of the presentation of $\pi_1(\partial U)$ as in Theorem 3.3.1 such that

$$R'_r = \{[\gamma_k, \gamma_r^{s_{kr}}], \gamma_r^{-w_r} = \prod_{m=1}^{k'_r} \gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}} \mid (k,r) \in \mathcal{E}\}$$

with

$$s_{kr} = \begin{cases} c_\iota^{-1} & \text{if } (k,r) \text{ equals the } \iota\text{-th element in } \mathcal{C}, \\ c_\iota & \text{if } (r,k) \text{ equals the } \iota\text{-th element in } \mathcal{C}, \\ 1 & \text{if } (k,r) \text{ is an edge of } \mathcal{T}'. \end{cases}$$

Proposition 3.3.8. *Consider an exceptional divisor $E_\kappa = D_r \subset \bar{X}$ coming from a singular point $t_\kappa \in \operatorname{Sing}\mathscr{A} \cap M_\kappa$ of multiplicity higher or equal to 3. The image of the set of relations R'_r as above, under the map i_*, equals the set of relations $R_\kappa = [\lambda_m^{(\kappa)}, \ldots, \lambda_j^{(\kappa)}]$ as in Lemma 3.2.8.*

Proof. Let $t_\kappa = W_{\sigma^{(\kappa)}(j)} \cap W_{\sigma^{(\kappa)}(j+1)} \cap \ldots \cap W_{\sigma^{(\kappa)}(m)}$ with the local order given by $\Pi^{(\kappa)} = \{\sigma^{(\kappa)}(1) < \ldots < \sigma^{(\kappa)}(n+1)\}$. As $w_r = -1$ and by the local order of the wires we have that $\gamma_r = \gamma_{\sigma^{(\kappa)}(m)}^{c_{\sigma^{(\kappa)}(m),t_\kappa}} \cdots \gamma_{\sigma^{(\kappa)}(j)}^{c_{\sigma^{(\kappa)}(j),t_\kappa}}$, that $[\gamma_r, \gamma_{\sigma(\kappa)}^{c_{\sigma(\kappa),t_\kappa}}]$ and that $c_{\sigma^{(\kappa)}(k),t_\kappa} = c_{t_\kappa,\sigma^{(\kappa)}(k)}^{-1}$.

Let us omit the superscript $\lambda_\kappa = \lambda_\kappa^{(1)}$ for the elements in $\Gamma^{(1)}$.

By considering the image under i_* of the elements in R'_r, we have by Lemma 3.3.4 and by Proposition 3.3.6 that

$$i_*(\gamma_r) = \lambda_{\sigma^{(\kappa)}(m)}^{\xi_{\sigma^{(\kappa)}(m)}^{(\kappa)}\xi_\iota^{(\kappa)-1}} \cdots \lambda_{\sigma^{(\kappa)}(j+1)}^{\xi_{\sigma^{(\kappa)}(j+1)}^{(\kappa)}\xi_\iota^{(\kappa)-1}} \lambda_{\sigma^{(\kappa)}(j)}^{\xi_{\sigma^{(\kappa)}(j)}^{(\kappa)}\xi_\iota^{(\kappa)-1}}, \quad [i_*(\gamma_r), \lambda_{\sigma^{(\kappa)}(k)}^{\xi_{\sigma^{(\kappa)}(k)}^{(\kappa)}\xi_\iota^{(\kappa)-1}}].$$

with $\iota = \min\{\sigma^{(\kappa)}(j), \sigma^{(\kappa)}(j+1), \ldots, \sigma^{(\kappa)}(m)\}$.

The commutators can also be written as $[i_*(\gamma_r)^{\xi_\iota^{(\kappa)}}, \lambda_{\sigma^{(\kappa)}(k)}^{\xi_{\sigma^{(\kappa)}(k)}^{(\kappa)}}]$. But as $i_*(\gamma_r)^{\xi_\iota^{(\kappa)}} = \lambda_{\sigma^{(\kappa)}(m)}^{\xi_{\sigma^{(\kappa)}(m)}^{(\kappa)}} \cdots \lambda_{\sigma^{(\kappa)}(j+1)}^{\xi_{\sigma^{(\kappa)}(j+1)}^{(\kappa)}} \lambda_{\sigma^{(\kappa)}(j)}^{\xi_{\sigma^{(\kappa)}(j)}^{(\kappa)}}$, we have that the relations $[i_*(\gamma_r)^{\xi_\iota^{(\kappa)}}, \lambda_{\sigma^{(\kappa)}(k)}^{\xi_{\sigma^{(\kappa)}(k)}^{(\kappa)}}]$ can be condensed as $[\lambda_{\sigma^{(\kappa)}(m)}^{\xi_{\sigma^{(\kappa)}(m)}^{(\kappa)}}, \cdots, \lambda_{\sigma^{(\kappa)}(j+1)}^{\xi_{\sigma^{(\kappa)}(j+1)}^{(\kappa)}}, \lambda_{\sigma^{(\kappa)}(j)}^{\xi_{\sigma^{(\kappa)}(j)}^{(\kappa)}}]$. Now, if $R_\kappa = [\lambda_m^{(\kappa)}, \ldots, \lambda_{j+1}^{(\kappa)}, \lambda_j^{(\kappa)}]$ denotes the relation given in Theorem 3.2.12 for the point t_κ as in Lemma 3.2.8, recall that we have the equality $\lambda_k^{(\kappa)} = \lambda_{\sigma^{(\kappa)}(k)}^{\xi_{\sigma^{(\kappa)}(k)}}$. By replacing it in the commutators above, the result follows. □

Proposition 3.3.9. *For $r = 1, \ldots, n+1$, we have the equality*

$$i_*(\gamma_r^{w_r} \prod_{m=1}^{k'_r} \gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}) = \gamma_{n+1} \cdots \gamma_1$$

in $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ with s_{rj} as above.

Proof. Fix $L_r \in \mathscr{A}$. Let $\{p_1, \ldots, p_b\} \subset \operatorname{Sing}\mathscr{A} \cap L_r$ be the singular points of the arrangement lying in L_r. Note that $b = k'_r$. Indeed, we can find a partition $A \cup B = \{1, \ldots, k'_r\}$, with A a set indexing the double points of $\operatorname{Sing}\mathscr{A} \cap L_r$, and B indexing the points of multiplicity strictly bigger than two. Let $\pi : \bar{X} \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points of $\operatorname{Sing}\mathscr{A}$ of multiplicity strictly bigger than two and let D_r denote the strict transform of L_r in $\bar{X}$. We have that A also indexes all the strict transforms of lines in $\mathscr{A}$ which have no empty intersection with D_r, and B the exceptional divisors of $\bar{X}$ crossing D_r. It is clear then that $b = k'_r$.

It follows that $\gamma_{j_\Delta(r,m)}$ is a meridian of an irreducible component $D_{j_\Delta(r,m)}$ of $D = \pi^* D$ for $m = 1, \ldots, k'_r$. Recall that γ_r commutes with $\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}$ and note that the self-intersection number w_r of D_r is $1 - |B|$.

Let us study the geometric meaning of the product $\gamma_r^{-1}\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}$ with $m \in B$. Let us write $\iota = j_\Delta(r,m)$, denote by $D_\iota = D_{j_\Delta(r,m)}$ the exceptional divisor that $\gamma_{j_\Delta(r,m)}$ surrounds, and let $D_{\iota_1}, \ldots, D_{\iota_k}$ be the irreducibles components of $D = \pi^*\mathscr{A}$ that intersect D_ι ordered in such a way that, if we denote by γ_{ι_j} the meridians around D_{ι_j} used for the presentation of $\pi_1(\partial U)$, $\gamma_\iota^{-w_\iota} = \gamma_{\iota_1}^{s_{\iota\iota_1}} \cdots \gamma_{\iota_k}^{s_{\iota\iota_k}}$ holds. As D_ι is an exceptional divisor, we have that $w_\iota = -1$. By Theorem 3.3.1, we have that $[\gamma_\iota, \gamma_{\iota_j}^{s_{\iota\iota_j}}]$ for $j = 1, \ldots, k = k(\iota)$.

Replacing the expression γ_ι as above in $[\gamma_\iota, \gamma_{\iota_j}^{s_{\iota\iota_j}}]$, we can show that these commutators relations are equivalent to

$$\gamma_{\iota_1}^{s_{\iota\iota_1}}\gamma_{\iota_2}^{s_{\iota\iota_2}} \cdots \gamma_{\iota_k}^{s_{\iota\iota_k}} = \gamma_{\sigma(\iota_1)}^{s_{\iota\sigma(\iota_1)}}\gamma_{\sigma(\iota_2)}^{s_{\iota\sigma(\iota_2)}} \cdots \gamma_{\sigma(\iota_k)}^{s_{\iota\sigma(\iota_k)}}$$

where σ runs over the cyclic permutations of the elements $\{\iota_1, \ldots, \iota_k\}$. Hence there exists some cyclic permutation σ' such that $\sigma'(\iota_1) = r$ because D_r intersects $D_{j_\Delta(r,m)} = D_\iota$. Note that $s_{r\iota} = s_{\iota r}^{-1} = s_{\iota\sigma'(\iota_1)}^{-1}$ and hence $\gamma_r^{-1}\gamma_\iota^{s_{r\iota}} = (\gamma_{\sigma'(\iota_2)}^{s_{\iota\sigma'(\iota_2)}} \cdots \gamma_{\sigma'(\iota_k)}^{s_{\iota\sigma'(\iota_k)}})^{s_{r\iota}}$ represents a loop which surrounds the lines $L_{\sigma'(\iota_2)}, \ldots, L_{\sigma'(\iota_k)}$ following l'_r by construction of the cycle $s_{r\iota}$.

Now, the product $\gamma_r^{w_r}\prod_{m=1}^{k'_r}\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}$ can be written as $\gamma_r\prod_{m=1}^{k'_r}\Upsilon_m$ with

$$\Upsilon_m = \begin{cases} \gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}} & \text{if } m \in A, \\ \gamma^{-1}\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}} & \text{if } m \in B. \end{cases}$$

by commuting γ_r with $\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}$. Note that, for Υ_m with $m \in A$, the path Υ_m is a meridian around the other line that intersects D_r in the double point corresponding to $m \in A$. Hence, by the precedent paragraph, $\gamma_r\prod_{m=1}^{k'_r}\Upsilon_m$ is a product of the meridians of all the lines in $\mathscr{A}$ ordered in the way they intersect L_r.

Now, by choosing a line L sufficiently close to L_r we have that the product $\gamma_r^{w_r}\prod_{m=1}^{k'_r}\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}$ is a path encircling $L\setminus(L\cap\mathscr{A})$ and therefore it is equivalent to $\lambda_{n+1}\cdots\lambda_1$ in $\pi_1(\mathbb{P}^2\setminus\mathscr{A})$. □

End of proof of the Theorem 3.3.3

The point (1) of the Theorem is obtained by Lemma 3.3.4.

The point (2) follows from Proposition 3.3.8.

For the point (3), recall that R' denotes the set of relations for the presentation of $\pi_1(\partial U)$ as in Theorem 3.3.1. Using the notation of 3.3.2 we have that

$$R'\setminus\cup R'_k = \{\gamma_r^{w_r}\prod_{m=1}^{k'_r}\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}, [\gamma_k, \gamma_r^{s_{kr}}] \mid r = 1, \ldots, n+1, D_k\cap D_r \in \operatorname{Sing}\mathscr{A}\setminus P_0\}$$

this is, $D_k \cap D_r$ is a double point.

By Proposition 3.3.9, we have that $i_*(\gamma_r^{w_r}\prod_{m=1}^{k'_r}\gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(r,m)}}) = \lambda_{n+1}\ldots\lambda_1$.

By proceeding as in Proposition 3.3.8, it can be seen that for a double point $p_\kappa = D_k \cap D_r$, the relation $[\gamma_k, \gamma_r^{s_{kr}}]$ correspond to the relation R_κ as in Theorem 3.2.12.

Hence, in $\langle \gamma_1, \dots, \gamma_{n+1}, c_1, \dots, c_g \mid R', \cup_{p \in \operatorname{Sing} \mathscr{A} \setminus L_{n+1}} R'(p) \rangle$ the set of relations R' is equivalent to the set of relations $\cup R_\kappa \cup \{\lambda_{n+1} \cdots \lambda_1\}$.

This concludes the proof of Theorem 3.3.3.

Independence of the maximal tree

Let $D = \sum_{k=1}^{n+1+s_0} D_k$ the total transform of the arrangement $\mathscr{A}$ in $\bar{X}$ and denote by Δ the dual graph of D as above.

Let $\mathcal{T} \subset \Delta$ be an arbitrary maximal tree and denote by $G(\mathcal{T}) = \{\gamma_1(\mathcal{T}), \dots, \gamma_{n+1+s_0}(\mathcal{T}), c_1(\mathcal{T}), \cdots, c_g(\mathcal{T})\}$, the set of generators of $\pi_1(\partial U)$ as in Theorem 3.3.1. Recall that these are constructed using $\mathcal{T}$. Denote by $R(\mathcal{T})$ the set of relations given in the same Theorem.

Consider also the maximal tree $\mathcal{T}'$ defined as in 3.3.1 and denote by $\gamma_1, \dots, \gamma_{n+1+s_0}$, $c_1, \dots, c_g$ the generators of $\pi_1(\partial U)$ as in Theorem 3.3.3 and by R the set of relations.

Consider the inclusion $i : \partial U \hookrightarrow \mathbb{P}^2 \setminus \mathscr{A}$ and fix $i_*(\gamma_1) = \lambda_1, \dots, i_*(\gamma_{n+1+s_0}) = \lambda_{n+1+s_0}$ as a set of generators for $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ with $\Gamma^{(1)} = \{\lambda_1, \dots, \lambda_{n+1}\}$ as in Theorem 3.3.3. For $\iota = 1, \dots, n+1+s_0$, we have that $i_*(\gamma_\iota(\mathcal{T}))$ and λ_ι are meridians of the same smooth curve D_ι, therefore, we can express $i_*(\gamma_\iota(\mathcal{T}))$ as a conjugate of λ_ι by elements in $\lambda_1, \dots, \lambda_{n+1}$. We let δ_ι denote the word in $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ representing $i_*(c_\iota)$ in the letters $\lambda_1, \dots, \lambda_{n+1+s_0}$ and by δ'_ι the same word in the letters $\gamma_1, \dots, \gamma_{n+1+s_0}$ as in Theorem 3.3.3.

Reciprocally, by fixing $i_*(\gamma_1(\mathcal{T})), \dots, i_*(\gamma_{n+1+s_0}(\mathcal{T}))$ as generators of $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$, we can express λ_ι as a conjugate of $i_*(\gamma_\iota(\mathcal{T}))$ by elements in $i_*(\gamma_1(\mathcal{T})), \dots, i_*(\gamma_{n+1+s_0}(\mathcal{T}))$ for $\iota = 1, \dots, n+1$. The image $i_*(c_\iota(\mathcal{T}))$ of the cycle $c_\iota(\mathcal{T})$ can be expressed in terms of $i_*(\gamma_1(\mathcal{T})), \dots, i_*(\gamma_{n+1+s_0}(\mathcal{T}))$ for $\iota = 1, \dots, g$. We let $\delta_\iota(\mathcal{T})$ be this expression when it is written in terms of $\gamma_1(\mathcal{T}), \dots, \gamma_{n+1+s_0}(\mathcal{T})$ such that $\delta_\iota(\mathcal{T}) \in \langle G(\mathcal{T}) \mid R(\mathcal{T}) \rangle$.

Proposition 3.3.10. *A presentation of $\pi_1(\mathbb{P}^2 \setminus \mathscr{A})$ can be obtained as follows*

$$\pi_1(\mathbb{P}^2 \setminus \mathscr{A}) \cong \langle G(\mathcal{T}) \mid R(\mathcal{T}), c_1(\mathcal{T}) \cdot \delta_1(\mathcal{T})^{-1}, \cdots, c_g(\mathcal{T}) \cdot \delta_g(\mathcal{T}))^{-1} \rangle,$$

Proof. The presentations $\langle G(\mathcal{T}) \mid R(\mathcal{T}) \rangle$ and $\langle \gamma_1, \dots, \gamma_{n+1+s_0}, c_1, \dots, c_g \mid R \rangle$ of $\pi_1(\partial U)$ as in Theorem 3.3.1 can also be obtained as graphs of groups (see [Hir00]). These graphs of groups are constructed over Δ as follows: the vertices groups are given as in Lemma 3.1.4, the edges groups are $\mathbb{Z}^2$. To each tree of Δ there correspond a presentation and the presentations are Tietze-equivalent.

Let us fix v_{n+1}, the vertex corresponding to D_{n+1} as a base point for $\pi_1(\Delta)$ and $c_1, \dots, c_g$ a generating set. Every cycle $c_\iota(\mathcal{T}) \in \pi_1(\Delta, v_{n+1})$ can be expressed as $c_\iota(\mathcal{T}) = c_{\iota 1} \cdots c_{\iota r_\iota}$ where $c_{\iota m} \in \{c_1, \dots, c_g\}$ with $m = 1, \dots, r_\iota$ and $\iota = 1, \dots, g$. Therefore $i_*(c_\iota(\mathcal{T})) = i_*(c_{\iota 1}) \cdots i_*(c_{\iota r_\iota}) = \delta_{\iota 1} \cdots \delta_{\iota r_\iota}$. Let us show

that

$$c_\iota(\mathcal{T})\cdot\delta_\iota(\mathcal{T})^{-1} = c_{\iota 1}\cdots c_{\iota r_\iota}{\delta'_{\iota r_\iota}}^{-1}\cdots{\delta'_{\iota 1}}^{-1} \in \langle\langle c_1\cdot{\delta'_{\iota 1}}^{-1},\ldots,c_g\cdot{\delta'_g}^{-1}\rangle\rangle.$$

Note that

$$\begin{aligned}(c_{\iota 1}{\delta'_{\iota 1}}^{-1})(c_{\iota 2}{\delta'_{\iota 2}}^{-1})^{\delta_{\iota 1}^{-1}} &= c_{\iota 1}c_{\iota 2}{\delta'_{\iota 2}}^{-1}{\delta'_{\iota 1}}^{-1}\\ &\vdots\\ (c_{\iota 1}{\delta'_{\iota 1}}^{-1})\cdots(c_{\iota r_\iota}{\delta'_{\iota r_\iota}}^{-1})^{{\delta'_{\iota r_\iota}}^{-1}\cdots{\delta'_{\iota 1}}^{-1}} &= c_{\iota 1}\cdots c_{\iota\iota_r}{\delta'_{\iota r_\iota}}^{-1}\cdots{\delta'_{\iota 1}}^{-1}\end{aligned}$$

In a similar way we can prove that $c_\iota\cdot{\delta'_\iota}^{-1} \in \langle\langle c_1(\mathcal{T})\cdot\delta_1(\mathcal{T})^{-1},\ldots,c_g(\mathcal{T})\cdot\delta_g(\mathcal{T})^{-1}\rangle\rangle$. This proves that the presentations $\langle G(\mathcal{T}) \mid R(\mathcal{T}), c_1(\mathcal{T})\delta_1(\mathcal{T})^{-1},\cdots,c_g(\mathcal{T})\delta_g(\mathcal{T}))^{-1}\rangle$, and $\langle\gamma_1,\ldots,\gamma_{n+1+s_0},c_1,\ldots,c_g \mid R, c_1{\delta'_1}^{-1},\ldots,c_g{\delta'_g}^{-1}\rangle$ are equivalent. We conclude by Theorem 3.3.3. □

3.3.3 Boundary manifold of a partial compactification

Here we will present another presentation for the fundamental group of certain partial compactifications $M(\mathscr{A},I,P)$, where $M(\mathscr{A},I,P)$ is as in 3.1.3, but the lines of D indexed by I correspond only to exceptional divisors, this is, $I\subset\{n+2,\ldots,n+1+s_0\}$

Inclusion of the boundary of a partial compactification

Let us recall the notation of section 3.1.3.

Let $\mathscr{A}\subset\mathbb{P}^2$ be an arrangement of lines and $\bar{X}$ the blow-up at the points $P_0=\{p_1,\ldots,p_{s_0}\}$ of $\operatorname{Sing}\mathscr{A}$ with multiplicity strictly higher than two and let $D=\sum_{k=1}^{n+1+s_0}D_k$ be the reduced total transform of $\mathscr{A}$ in $\bar{X}$.

Here, *we suppose that* $I\subset\{n+2,\ldots,n+1+s_0\}$ and let $P=\{p'_1,\ldots,p'_{s_1}\}\subset\operatorname{Sing}\sum_{k\notin I}D_k$. Denote by $\pi':\operatorname{Bl}_P\bar{X}\to\bar{X}$ the blow-up map and the dual graph of $|\pi'^*D|$ by Δ. Note that in the previous section Δ denoted instead the dual graph of D. Consider the divisor $D'\subset\operatorname{Bl}_P\bar{X}$ as in 3.1.3 and denote by Δ' the dual graph of D'. Recall that Δ' is obtained from Δ by removing some vertices and the corresponding adjacent edges.

In 3.1.3 we defined the partial compactification $M(\mathscr{A},I,P)$ of $M(\mathscr{A})$ as $\operatorname{Bl}_P\bar{X}\setminus D'$.

Let us assume that D' *is connected*, which is equivalent to Δ' being connected. Therefore, there exists a maximal tree $\mathcal{T}_{\Delta'}\subset\Delta'$. Note that every cycle in Δ' can be seen as a cycle in Δ.

Lemma 3.3.11. *Any maximal tree $\mathcal{T}_{\Delta'}$ can be completed to a maximal tree $\mathcal{T}_{\Delta',\Delta}$ in Δ.*

Proof. Let $\{v_1,\ldots,v_k\}$ be the vertices of Δ which are to be removed along with its adjacent edges in order to obtain Δ'.

As $I \subset \{n+2, \ldots, n+1+s_0\}$ and $P \subset \operatorname{Sing} \sum_{\iota \notin I} D_\iota$, we have that all the vertices in $\{v_1, \ldots, v_k\}$ correspond to exceptional divisors in $\operatorname{Bl}_P \bar{X}$, therefore there is no edge connecting v_ι and v_j for $\iota \neq j$ and to complete $\mathcal{T}_{\Delta'}$ to a maximal tree of Δ it suffices to take no matter what edge connecting a vertex in $\mathcal{T}_{\Delta'}$ and v_ι for $\iota = 1, \ldots, k$ because no cycle will be created in this way. $\square$

Corollary 3.3.12. *Let g denote the number of independent cycles in Δ. Let $c_1(\mathcal{T}_{\Delta'}), \ldots, c_{g'}(\mathcal{T}_{\Delta'})$ be independent cycles in Δ' each one formed by adjoining one edge in Δ' to the maximal tree $\mathcal{T}_{\Delta'}$. There exists $c_{g'+1}(\mathcal{T}_{\Delta',\Delta}), \ldots, c_g(\mathcal{T}_{\Delta',\Delta})$ cycles in Δ that together with $c_1(\mathcal{T}_{\Delta',\Delta}) = c_1(\mathcal{T}_{\Delta'}), \ldots, c_{g'}(\mathcal{T}_{\Delta',\Delta}) = c_{g'}(\mathcal{T}_{\Delta'})$ complete a generating set of $\pi_1(\Delta, v_{n+1})$.*

Let us denote by ∂U the boundary manifold of the total transform of D in $\operatorname{Bl}_P \bar{X}$. By proceeding as in the proof of Theorem 3.3.1, we have that a presentation for $\pi_1(\partial U)$, by using the maximal tree $\mathcal{T}_{\Delta',\Delta}$, has generators $\gamma_1 = \gamma_1(\mathcal{T}_{\Delta',\Delta}), \ldots, \gamma_{n+1+s_0+s_1} = \gamma_{n+1+s_0+s_1}(\mathcal{T}_{\Delta',\Delta}), c_1 = c_1(\mathcal{T}_{\Delta',\Delta}), \ldots, c_g = c_g(\mathcal{T}_{\Delta',\Delta})$ and a set of relations

$$R = \left\{ \begin{array}{ll} [\gamma_r, \gamma_j^{s_{rj}}], & (r,j) \in \mathcal{E}(\Delta) \\ \gamma_r^{-w'_r} = \prod_{m=1}^{k'_r} \gamma_{j_\Delta(r,m)}^{s_{rj_\Delta(i,m)}} & 1 \leq r \leq n+1+s_0+s_1 \end{array} \right\} \tag{3.5}$$

where $w'_r = D'_r \cdot D'_r$, for an irreducible component D'_r of $\pi'^* D$, we denoted by k'_r the number of points in $\operatorname{Sing} \pi'^* D \cap D'_r$ (see the proof of Proposition 3.3.9), and

$$s_{rj} = \left\{ \begin{array}{ll} c_k^{-1} & \text{if } (r,j) \text{ equals the } k\text{-th element in } \Delta \setminus \mathcal{T}_{\Delta',\Delta}, \\ c_k & \text{if } (j,r) \text{ equals the } k\text{-th element in } \Delta \setminus \mathcal{T}_{\Delta',\Delta}, \\ 1 & \text{if } (r,j) \text{ is an edge of } \mathcal{T}_{\Delta',\Delta}. \end{array} \right.$$

Moreover, let $\partial U'$ denote the boundary manifold of $D' \subset \operatorname{Bl}_P \bar{X}$. Here, if $r \notin I$ let us denote by k''_r the number of points in $(D_k \cap \sum_{\iota \notin I} D_\iota) \setminus P$ or equivalently, in $D'_k \cap D'$. By using the maximal tree $\mathcal{T}_{\Delta'}$ of Δ' and proceeding as in the proof of Theorem 3.3.1, we obtain the following Proposition.

Proposition 3.3.13. *A presentation for $\pi_1(\partial U')$ is given by*

$$\left\langle \begin{array}{ll} \gamma_\iota, \quad \iota \in J & [\gamma_r, \gamma_j^{s_{rj}}], \qquad (r,j) \in \mathcal{E}(\Delta') \\ c_1, \ldots, c_{g'} & \gamma_r^{-w'_r} = \prod_{m=1}^{k''_r} \gamma_{j_{\Delta'}(r,m)}^{s_{rj_{\Delta'}(r,m)}} \quad r \in J \end{array} \right\rangle$$

where $J = \{1, \ldots, n+1+s_0\} \setminus I$, $\mathcal{E}(\Delta')$ denotes the set of edges of Δ', w'_r the self-intersection number of the strict transform D'_r of D_r in $\operatorname{Bl}_P \bar{X}$ and

$$s_{rj} = \left\{ \begin{array}{ll} c_k^{-1} & \textit{if } (r,j) \textit{ equals the } k\textit{-th element in } \Delta' \setminus \mathcal{T}_{\Delta'}, \\ c_k & \textit{if } (j,r) \textit{ equals the } k\textit{-th element in } \Delta' \setminus \mathcal{T}_{\Delta'}, \\ 1 & \textit{if } (r,j) \textit{ is an edge of } \mathcal{T}_{\Delta'}. \end{array} \right.$$

For every $\iota \in I$, we have that, as D'_ι is an exceptional divisor, the following relation is in R:

$$\gamma_\iota = \prod_{m=1}^{k'_\iota} \gamma_{j_{\Delta',\Delta}(\iota,m)}^{s_{\iota j_{\Delta',\Delta}(\iota,m)}}. \tag{3.6}$$

Analogously, if $p = p_\iota \in P$, by abuse of notation we will write $p = n + 1 + s_0 + \iota$. We have that if $p = D_r \cap D_j$:

$$\gamma_p = \gamma_r^{s_{pr}} \gamma_j^{s_{pj}} \tag{3.7}$$

By using the map $i : \partial U \to M(\mathscr{A}) = \mathbb{P}^2 \setminus \mathscr{A}$ as in 3.3.2, we can express the image of the cycles $i_*(c_r)$ as a word in the letters $i_*(\gamma_1), \ldots, i_*(\gamma_{n+1})$, for $r = 1, \ldots, g$. Let us denote by δ_r the word obtained by replacing the letters $i_*(\gamma_1), \ldots, i_*(\gamma_{n+1})$ by $\gamma_1, \ldots, \gamma_{n+1}$ in this precedent word associated to $i_*(c_r)$.

By using $\delta_1, \ldots, \delta_g$ and replacing $i_*(\gamma_1), \ldots, i_*(\gamma_{n+1})$ by $\gamma_1, \ldots, \gamma_{n+1}$, we can express the words $i_*(\prod_{m=1}^{k'_\iota} \gamma_{j_{\Delta',\Delta}(\iota,m)}^{s_{\iota j_{\Delta',\Delta}(\iota,m)}})$ and $i_*(\gamma_r^{s_{pr}} \gamma_j^{s_{pj}})$ with $\iota \in I$ and $p \in P$ as words $\gamma(\iota), \gamma(p) \in \pi_1(\partial U')$ respectively.

Let us denote by R' the set of relations in the presentation given by Proposition 3.3.13.

Theorem 3.3.14. *A presentation of $\pi_1(M(\mathscr{A}, I, P))$ is given by*

$$\langle c_1, \ldots, c_{g'}, \gamma_\iota \quad \iota \in J \mid R', c_1\delta_1^{-1}, \ldots, c_{g'}\delta_{g'}^{-1}, \cup_{\iota \in I}\gamma(\iota), \cup_{p \in P}\gamma(p) \rangle$$

with $J = \{1, \ldots, n + 1 + s_0\} \setminus I$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \pi_1(\partial U) & \twoheadrightarrow & \pi_1(M(\mathscr{A})) & \underset{\cong}{\longleftarrow} & \pi_1(\partial U)/\langle\langle c_1\delta_l^{-1}, \ldots, c_g\delta_g^{-1}\rangle\rangle \\ & & \downarrow & & \downarrow \\ \pi_1(\partial U') & \twoheadrightarrow & \pi_1(M(\mathscr{A}, I, P)) & \underset{\cong}{\longleftarrow} & \pi_1(M(\mathscr{A}))/\langle\langle i_*(\gamma_\iota), i_*(\gamma_p)\rangle\rangle \end{array}$$

Where the isomorphism in the right of the first row comes from Theorem 3.3.3 and Proposition 3.3.10.

From the rightest column we obtain that

$$\pi_1(\partial U)/\langle\langle c_1\delta_1^{-1}, \ldots, c_g\delta_g^{-1}, \gamma(\iota), \gamma(p)\rangle\rangle \cong \pi_1(M(\mathscr{A}, I, P)). \tag{3.8}$$

We will see that this presentation is equivalent to

$$\pi_1(\partial U')/\langle\langle \gamma(\iota), \gamma(p), c_1 \cdot \delta_1^{-1}, \ldots, c_{g'}\delta_{g'}^{-1})\rangle\rangle \tag{3.9}$$

Indeed, by the choice of the maximal tree $\mathcal{T}_{\Delta',\Delta}$, the are only four types of relations in R of the presentation of $\pi_1(\partial U)$ involving the cycles $c_{g'}, \ldots, c_g$:

- commutators $[\gamma_r, \gamma_{r'}^{s_{rr'}}]$ with $r' = \iota, p$,

- those relations as in (3.6),
- those relations as in (3.7), and
- relations $\gamma_r^{-w'_r} = \prod_{m=1}^{k'_r} \gamma_{j_{\Delta',\Delta}(r,m)}^{s_{rj_{\Delta',\Delta}(r,m)}}$ with (r,ι) or (r,p) an edge in Δ.

By adding the relations $c_1 = \delta_1, \ldots, c_g = \delta_g$, we can see these relations as expressed in terms of $\gamma_1, \ldots, \gamma_{n+1+s_0}$.

Note that the commutator-relation as in the first point above becomes trivial in $\pi_1(\partial U)/\langle\langle c_1\delta_1^{-1}, \ldots, c_g\delta_g^{-1}, \gamma(\iota), \gamma(p)\rangle\rangle$.

The relations in the points two and three above, are by construction, equivalent to the words $\gamma(\iota), \gamma(p)$.

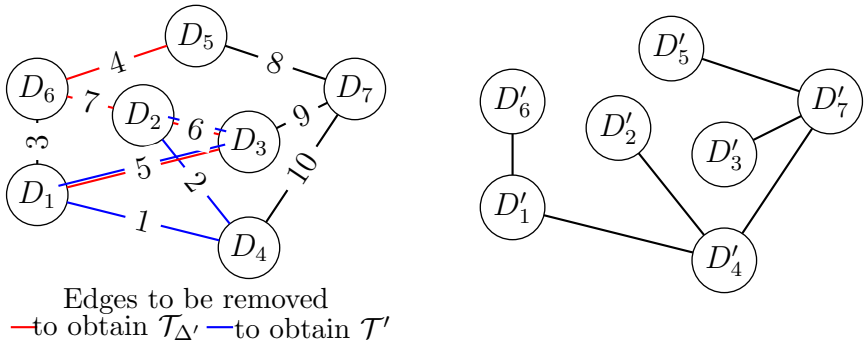
For the relations as in the fourth point, note that $k''_r = k'_r - |P \cap D_r| - \{\iota \in I \mid D_\iota \cap D_r \neq \varnothing\}$.

□

An example

The following example of a partial compactification $M(\mathscr{A}, I, P)$ of the complement $M(\mathscr{A})$ of an arrangement of lines $\mathscr{A}$ in $\mathbb{P}^2$ is such that $M(\mathscr{A}, I, P)$ is a $\mathbb{Q}$-homology plane. In fact, it is the arrangement $\mathscr{A}$ with the fewest number of lines such that the fundamental group $\pi_1(M(\mathscr{A}))$ is non-abelian and $M(\mathscr{A})$ has partial compactifications that are $\mathbb{Q}$-homology planes, see [tDP93]. [1]

Let $\mathscr{A} = \{L_1, \ldots, L_5\} \subset \mathbb{P}^2$ be an arrangement of 5 lines with 2 triple points and 4 double points. Projective equations are obtained by homogenizing the equation of example 3.2.1 and adding the line at infinity $z_3 = 0$.



(a) Dual Graph (b) Dual Tree $\mathcal{T}_{\Delta'}$

Figure 3.8: Dual graphs

Let $P_0 = \{p_1, p_2\} \subset \operatorname{Sing}\mathscr{A}$ be the triple points in the arrangement $\mathscr{A}$, $\pi : \bar{X} := \operatorname{Bl}_{P_0}\mathbb{P}^2 \to \mathbb{P}^2$ be the blow up at P_0 of the projective plane and $D = \pi^*\mathscr{A} = \sum_{i=1}^7 D_i$ the reduced total transform of the arrangement $\mathscr{A}$ seen

[1] We plan to investigate the fundamental group of the $\mathbb{Q}$-homology planes arising as partial compactifications of such arrangements in future work.

as divisor, we obtain a dual graph Δ of D as in figure 3.8a with the labeling of the edges as indicated on it.

We will delete the edges labeled as $[4,5,6,7]$ to obtain a tree as in Fig 3.8b, we let $P_1 := \{p'_1, p'_2, p'_3, p'_4\} \subset \operatorname{Sing} D$ be the points in D corresponding to the required edges, with the numeration as in figure 3.8a. Consider $\pi' : \operatorname{Bl}_{P_1} \bar{X} \to \bar{X}$ and let $D' = \pi'^*(D) - \sum_{7<i} D'_i$. If we set $I = \varnothing$, we have defined $M(\mathscr{A}, \varnothing, P_1) = \operatorname{Bl}_{P_1} \bar{X} \setminus D'$ in 3.1.3.

Denote by Δ' the dual graph of D', note that $\Delta' = \mathcal{T}_{\Delta'}$ it is already a tree. Let $\langle \gamma'_1, \ldots, \gamma'_7 \mid R' \rangle$ be the presentation of $\pi_1(\partial U')$ of the boundary manifold $\partial U'$ of the divisor D' as in Theorem 3.1.6 using the tree Δ'.

The following Lemma follows the computations as in [Wag71].

Lemma 3.3.15. *The group $\langle \gamma'_1, \ldots, \gamma'_7 \mid R' \rangle$ is infinite solvable.*

Proof. Recall that $D_1, \ldots, D_5$ are strict transform of the lines L_k for $k = 1, \ldots, 5$ and that D_6, D_7 are exceptional divisors. The lines $D'_1, \ldots, D'_7$ are strict transforms of these lines in $\operatorname{Bl}_P \bar{X}$.

Denote by $\partial U'_k$ the boundary manifold of D'_k in $\operatorname{Bl}_P \bar{X}$, by $\partial U'$ the boundary manifold of D', and by $\partial U'^*_k = \partial U'_k \cap \partial U'$. By abuse of notation, we will denote also by γ'_k the fiber S^1 in the group $\pi_1(\partial U'^*_k)$.

As $D'_6 \cdot D'_6 = -3$ in $\operatorname{Bl}_P \bar{X}$ we obtain by Lemma 3.1.4 that $\pi_1(\partial U'^*_6) = \langle \gamma'_6, \gamma'_1 \mid {\gamma'_6}^3 = \gamma'_1, [\gamma'_6, \gamma'_1] \rangle$. Note that $D'_1 \cdot D'_1 = -1$ in $\operatorname{Bl}_{P_1} \bar{X}$ and therefore $\pi_1(\partial U'^*_1) = \langle \gamma'_1, \gamma'_6, \gamma'_4 \mid \gamma'_1 = \gamma'_6\gamma'_4, [\gamma'_1, \gamma'_6], [\gamma'_1, \gamma'_4] \rangle$. By replacing the relation $\gamma'_1 = \gamma'_6\gamma'_4$ inside $[\gamma'_1, \gamma'_6], [\gamma'_1, \gamma'_4]$, and ${\gamma'_6}^3 = \gamma'_1$ in $\gamma'_1 = \gamma'_6\gamma'_4$, we obtain that $\pi_1(\partial U'^*_1) = \langle \gamma'_6, \gamma'_4 \mid {\gamma'_6}^2 = \gamma'_4, [\gamma'_6, \gamma'_4] \rangle$. We obtain that $\pi_1(\partial U')$ is isomorphic to the boundary manifold of a divisor such that its dual graph is as in Figure 3.9.

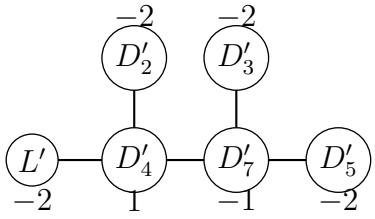


Figure 3.9: Dual tree with weights

We will prove that the subgroup $N := \langle \gamma'_4, \gamma'_7 \rangle$ is in fact normal in $\pi_1(\partial U')$. By Theorem 3.1.6, and using the reduction above of $\pi_1(\partial U'^*_6)$, we have the following presentation for $\pi_1(\partial U')$

$$\left\langle \gamma'_2, \gamma'_3, \gamma'_4, \gamma'_5, \gamma'_6, \gamma'_7 \;\middle|\; \begin{array}{l} {\gamma'_6}^2 = \gamma'_4, {\gamma'_2}^2 = \gamma'_4, {\gamma'_3}^2 = \gamma'_7, {\gamma'_5}^2 = \gamma'_7, [\gamma'_4, \gamma'_6], [\gamma'_4, \gamma'_2] \\ {\gamma'_4}^{-1} = \gamma'_6\gamma'_2\gamma'_7, \gamma'_7 = \gamma'_5\gamma'_4\gamma'_3, [\gamma'_7, \gamma'_4], [\gamma'_7, \gamma'_3], [\gamma'_7, \gamma'_5] \end{array} \right\rangle. \tag{3.10}$$

Note that N is abelian and that $\pi_1(\partial U')/N \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. The element γ'_7 commutes with γ'_4, γ'_3, and γ'_5, we will show that ${\gamma'_7}^{\gamma'_2}, {\gamma'_7}^{\gamma'_6} \in N$. To

emphasize the symmetry let us write $n=-1$, therefore ${\gamma'_4}^n = {\gamma'_2}^{2n} = \gamma'_6\gamma'_2\gamma'_7$. This implies $\gamma'_6 = {\gamma'_7}^{-1}{\gamma'_2}^{2n-1}$ by using $[{\gamma'_2}^2, \gamma'_7]$, then we use ${\gamma'_6}^2 = \gamma'_4$ to obtain $({\gamma'_7}^{-1}{\gamma'_2}^{2n-1})({\gamma'_7}^{-1}{\gamma'_2}^{2n-1}) = \gamma'_4 = {\gamma'_2}^2$, which implies ${\gamma'_7}^{-1}{\gamma'_2}^{-1}{\gamma'_7}^{-1}{\gamma'_2}^{4n-3} = 1$, and therefore ${\gamma'_7}^{-1}{\gamma'_2}^{4n-4} = \gamma'_2\gamma'_7{\gamma'_2}^{-1}$. Finally ${\gamma'_7}^{-1}{\gamma'_4}^{2n-2} = {\gamma'_7}^{\gamma'_2{}^{-1}}$. The rest are obtained in a similar way by symmetry (see figure 3.9) and therefore N is normal. □

Proposition 3.3.16. *The group $\pi_1(M(\mathscr{A}, \varnothing, P_1))$ is finite, cyclic of order four.*

Proof. The maximal tree $\mathcal{T}'$ defined in 3.3.1 is obtained from Δ by deleting the edges $[1,2,5,6]$ in figure 3.8a. Let us denote by $\gamma_1, \ldots, \gamma_7$ the meridians around the lines $D'_1, \ldots, D'_7$ constructed using the tree $\mathcal{T}'$ as in Theorem 3.3.3. We will denote by $c_{r,j}$ the cycle created following $\mathcal{T}'$ by joining the vertex D'_r to D'_j, we have: $c_{1,3}, c_{1,4}, c_{2,3}, c_{2,4}$. By removing redundant relations, we obtain the following presentation of the fundamental group of the boundary manifold ∂U of $\mathscr{A}$ from Theorem 3.3.1 (see also the wiring diagram of Figure 3.2):

$$\left\langle \begin{array}{r} \gamma_1, \gamma_2, \gamma_3, \gamma_4 \\ c_{1,3}, c_{1,4}, c_{2,3}, c_{2,4} \end{array} \middle| \begin{array}{l} \gamma_1^0 = 1 = \gamma_3{}^{c_{1,3}^{-1}}\gamma_4{}^{c_{1,4}^{-1}}\gamma_6, \gamma_2^0 = 1 = \gamma_3{}^{c_{2,3}^{-1}}\gamma_4{}^{c_{2,4}^{-1}}\gamma_6, \\ \gamma_3^0 = 1 = \gamma_2{}^{c_{2,3}}\gamma_1{}^{c_{1,3}}\gamma_7, \gamma_4^0 = 1 = \gamma_2{}^{c_{2,4}}\gamma_1{}^{c_{1,4}}\gamma_7, \\ [\gamma_1, \gamma_4{}^{c_{1,4}^{-1}}], [\gamma_1, \gamma_3{}^{c_{1,3}^{-1}}], [\gamma_2, \gamma_4{}^{c_{2,4}^{-1}}], [\gamma_2, \gamma_3{}^{c_{2,3}^{-1}}] \end{array} \right\rangle \tag{3.11}$$

We have that $\gamma_6 = \gamma_5\gamma_2\gamma_1$, $\gamma_7 = \gamma_5\gamma_4\gamma_3$ and $\gamma_5 = \gamma_7\gamma_6$ hence $\gamma_5 = \gamma_5\gamma_4\gamma_3\gamma_5\gamma_2\gamma_1$ and therefore we have that $\gamma_5 = (\gamma_4\gamma_3)^{-1}(\gamma_2\gamma_1)^{-1}$.

We will use the wiring diagram in figure 3.2 of the arrangement $\mathscr{A}$ to express $i_*(c_{r,j})$ in terms of $\gamma_1, \ldots, \gamma_4$ in $M(\mathscr{A})$. Using Proposition 3.3.6 we obtain that

$$i_*(c_{1,4}) = \gamma_3^{-1},\ i_*(c_{1,3}) = 1,\ i_*(c_{2,4}) = \gamma_1(\gamma_3\gamma_1)^{-1} = \gamma_3^{-1},\ i_*(c_{2,3}) = \gamma_1\gamma_1^{-1} = 1. \tag{3.12}$$

The non-commutator relations of (3.11) are already trivial in $\pi_1(M(\mathscr{A}))$ by Proposition 3.3.9. In order to see this concretely, let us compute for example:

$$i_*(\gamma_3{}^{c_{1,3}^{-1}}\gamma_4{}^{c_{1,4}^{-1}}\gamma_6) = \gamma_3\gamma_4^{\gamma_3}\gamma_5\gamma_2\gamma_1 = \gamma_4\gamma_3(\gamma_4\gamma_3)^{-1}(\gamma_2\gamma_1)^{-1}\gamma_2\gamma_1 = 1,$$

by (3.12) and using the expressions for γ_6 and γ_5 given after (3.11).

From (3.11), (3.12), and Theorem 3.3.3, we obtain the following presentation for the fundamental group of $M(\mathscr{A})$:

$$\pi_1(M(\mathscr{A})) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid [\gamma_1, \gamma_4{}^{\gamma_3}], [\gamma_1, \gamma_3], [\gamma_2, \gamma_4{}^{\gamma_3}], [\gamma_2, \gamma_3] \rangle \tag{3.13}$$

By using the commutators $[\gamma_1, \gamma_3], [\gamma_2, \gamma_3]$, we can see that the conjugations inside the other commutators are redundant. We found that $\pi_1(M(\mathscr{A})) \cong \mathbb{F}_2 \times \mathbb{F}_2$, which can be seen directly by noticing that $M(\mathscr{A}) \cong \mathbb{C}\setminus\{2-\text{points}\} \times \mathbb{C} \setminus \{2-\text{points}\}$.

Let us come back to the tree $\mathcal{T}_{\Delta'} = \Delta'$ obtained from Δ by removing $[4,5,6,7]$, denote the meridians around $D'_1, \ldots, D'_{11}$ constructed following $\mathcal{T}_{\Delta'}$ by $\gamma'_1, \ldots, \gamma'_{11}$, and the cycles by $k_{5,6} = c_{5,6}(\mathcal{T}_{\Delta'}), k_{1,3} = c_{1,3}(\mathcal{T}_{\Delta'}), k_{2,3} = c_{2,3}(\mathcal{T}_{\Delta'}), k_{2,6} = c_{2,6}(\mathcal{T}_{\Delta'})$ obtained by joining the vertex D'_r to D'_j. From figures 3.8a and 3.8b it is easy to see that in $\pi_1(\partial U)$ they satisfy the following relations:

$$k_{5,6} = c_{1,4},\ k_{1,3} = c_{1,4}^{-1}c_{1,3},\ k_{2,3} = c_{2,4}^{-1}c_{2,3},\ k_{2,6} = c_{2,4}^{-1}c_{1,4}. \tag{3.14}$$

To every point $p'_\iota \in P = \{p'_1, \ldots, p'_4\}$ corresponds an irreducible component $D'_{7+\iota}$ of $\pi'^*(D)$ which is an exceptional divisor. Using the relations as in (3.5), we have that

$$\gamma'_8 = \gamma_6'^{k_{5,6}^{-1}}\gamma'_5,\ \gamma'_9 = \gamma_1'^{k_{1,3}}\gamma'_3,\ \gamma'_{10} = \gamma_2'^{k_{2,3}}\gamma'_3,\ \gamma'_{11} = \gamma_2'^{k_{2,6}}\gamma'_6, \tag{3.15}$$

we have that $\gamma_{7+\iota}$ is a meridian around $D'_{7+\iota}$ for $i = 1,2,3,4$.

The following meridians lie in the same homotopy class in the boundary manifold ∂U when constructed either with the tree $\mathcal{T}_{\Delta'}$ or $\mathcal{T}'$:

$$\gamma'_5 = \gamma_5,\ \gamma'_7 = \gamma_7,\ \gamma'_3 = \gamma_3,\ \gamma'_4 = \gamma_4,$$

the others satisfy the relations:

$$\gamma'_6 = \gamma_6^{c_{1,4}},\ \gamma'_1 = \gamma_1^{c_{1,4}},\ \gamma'_2 = \gamma_2^{c_{2,4}}.$$

Finally, as in the proof of Theorem 3.3.14, by using proposition 3.1.1 and the presentation in (3.13), we found that

$$\pi_1(M(\mathscr{A}, \varnothing, P_1)) = \pi_1(M(\mathscr{A}))/\langle\langle i_*(\gamma'_8), \ldots, i_*(\gamma'_{11})\rangle\rangle = \langle \gamma_3 \mid \gamma_3^4 = 1\rangle.$$

Indeed, this follows from

$$\begin{aligned} i_*(\gamma'_8) &= (\gamma_4\gamma_3)^{-2}(\gamma_2\gamma_1)^{-1}, & i_*(\gamma'_9) &= \gamma_1\gamma_3, \\ i_*(\gamma'_{10}) &= \gamma_2\gamma_3, & i_*(\gamma'_{11}) &= \gamma_2^{\gamma_3^{-1}}((\gamma_4\gamma_3)^{-1})^{\gamma_3^{-1}}, \end{aligned}$$

and $\pi_1(M(\mathscr{A}))/\langle\langle i_*(\gamma'_9), i_*(\gamma'_{10})\rangle\rangle \cong \mathbb{Z}^2$, where we have used (3.12), (3.14) and (3.15) to express $i_*(\gamma'_{7+\iota})$ in terms of γ_ι for $\iota = 1, \ldots, 4$. □

Remark 3.3.17. In the last part of the proof of Proposition 3.3.16, we could have used directly Theorem 3.3.14 to obtain a presentation of $\pi_1(M(\mathscr{A}, \varnothing, P_1))$ by using the presentation of $\pi_1(\partial U')$ as (3.10) in Lemma 3.3.15 and adding the relations $\{1 = \gamma_6'^{\gamma'_3}\gamma'_5, 1 = (\gamma'_6\gamma'_4)^{\gamma'_3}\gamma'_3, 1 = \gamma_2'^{\gamma'_3}\gamma'_3, 1 = \gamma'_2\gamma_6\}$ corresponding respectively to the relations $\gamma(p'_1), \ldots, \gamma_(p'_4)$ in Theorem 3.3.14.

Indeed, we can express the images $i_*(k_{r,j})$ of the cycles associated to $\mathcal{T}_{\Delta'}$ in terms of $\gamma'_1, \ldots, \gamma'_4$ by using (3.12), (3.14) and the correspondence between γ_r and γ'_r as above. We then substitute this in (3.15) obtaining the words $\gamma(p'_1), \ldots, \gamma(p'_4)$.

It can be shown that the presentation obtained for $\pi_1(M(\mathscr{A}, \varnothing, P_1))$ is the same as that of Proposition 3.3.16. However, the computations are longer so we have preferred to avoid them.

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