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Algebraic and combinatorial methods in the branching problems in representation theory

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Khanh Nguyen Duc

**Méthodes algébriques et combinatoires pour les
problèmes de branchement en théorie des
représentations**

Devant le jury composé de :

M. Jiang ZENG	Professeur, Université Claude Bernard Lyon 1	Président
M. Cédric LECOUCVEY	Professeur, Université François Rabelais, Tours	Rapporteur
M. Shrawan KUMAR	Professeur, University of North Carolina at Chapel Hill	Rapporteur
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Khanh Nguyen Duc

Thèse de doctorat

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Méthodes algébriques et combinatoires pour les problèmes de branchement en théorie des représentations

Thèse présentée en vue d'obtenir le diplôme de
Doctorat de l'Université de Lyon

soutenue publiquement le 25 septembre 2020 par
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Introduction

The objective of my thesis is to use algebraic and combinatorial methods to study questions around the branching problem in representation theory.

The branching problem is one of the most important in representation theory. It allows us to understand how an irreducible module of a Lie algebra \mathfrak{g} can be decomposed into irreducible modules of a subalgebra $\dot{\mathfrak{g}} \subset \mathfrak{g}$ and several properties around the decomposition. For example, let $\dot{\mathfrak{g}}$ be the generalized linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ ($n \in \mathbb{Z}_{\geq 0}$). Then the set of all irreducible polynomial $\mathfrak{gl}_n(\mathbb{C})$ -modules corresponds to the set of partitions of length at most n . If λ is such a partition, let V^λ be the corresponding irreducible polynomial $\mathfrak{gl}_n(\mathbb{C})$ -module. Then

$$\{V^\lambda \otimes V^\mu \mid \lambda, \mu \text{ are partitions of length at most } n\} \quad (1)$$

is the set of all irreducible polynomial modules of $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$. These irreducible modules can be considered as $\mathfrak{gl}_n(\mathbb{C})$ -modules. Hence we have decompositions

$$V^\lambda \otimes V^\mu = \bigoplus_{\nu} (V^\nu)^{\text{mult}_{\lambda, \mu}(\nu)}, \quad (2)$$

where $\text{mult}_{\lambda, \mu}(\nu)$ is the multiplicity of V^ν in $V^\lambda \otimes V^\mu$.

The branching problem consists in many questions around the multiplicity $\text{mult}_{\lambda, \mu}(\nu)$ and their generalizations to other pairs $\dot{\mathfrak{g}} \subset \mathfrak{g}$ than $\mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$. For example,

- what is the formula of the multiplicities?
- what are good algorithms to compute the multiplicities?
- can we describe explicitly the support of the decomposition? etc.

There are a lot of approaches to answer those questions from different domains : algebraic geometry, number theory, combinatorics, analysis, For example,

- by geometric methods, S. Kumar, N. Ressayre and P. Belkale obtained strong results on the support of the decomposition when $\dot{\mathfrak{g}}$ is a symmetrizable Kac-Moody algebra and $\mathfrak{g} = \dot{\mathfrak{g}} \oplus \dot{\mathfrak{g}}$ (see [BK06], [KR17], [Res17], [Res10], [BKR12] etc).
- by using theory of modular forms, V. Kac and M. Wakimoto obtained results about branching functions which help us to compute explicitly the decomposition in many particular cases (see [KW90]).

- by combinatorial methods, P. Littelman obtained the generalized Littlewood-Richardson rule for any symmetrizable Kac-Moody algebras. His model is well known and named path model (see [Lit94], [Lit95a], [Lit95b], [Lit97], [Lit98]). It has too many applications in representation theory, for example, a simple proof of PRV conjecture (see [Lit94]). There are also a lot of notable works using combinatorial methods. For example, [KT01], [KT99], [KTW04] for the proof of saturation conjecture using honeycomb model. By, e.g., [RW84], [Wor84], [Ste89], [Sag87], [Ser10], [Vak06] and so on for developing theory of (shifted) Littlewood-Richardson coefficients.

In this thesis, we study the branching rule on winding subalgebras of affine Kac-Moody algebras and investigate some non-trivial properties of shifted Littlewood-Richardson coefficients which appear in the projective representations of symmetric groups. The explicit results can be described as follows :

The first work is the results mentioned in our preprint [Kha19] : “Branching problem on winding subalgebras of affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$.” Let \mathfrak{g} be an affine Kac-Moody algebra and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $\mathfrak{g}[u]$ be a winding subalgebra of \mathfrak{g} for some given positive integer u . The winding subalgebra $\mathfrak{g}[u]$ is isomorphic to \mathfrak{g} but nontrivially embedded in \mathfrak{g} (see [KW90]). The integrable irreducible highest weight \mathfrak{g} -modules (resp. $\mathfrak{g}[u]$ -modules) are parametrized by the semigroup P_+ (resp. \dot{P}_+) of dominant integral weights. For $\Lambda \in P_+$ (resp. $\lambda \in \dot{P}_+$), let $L(\Lambda)$ (resp. $\dot{L}(\lambda)$) be the integrable irreducible highest weight \mathfrak{g} -module (resp. $\mathfrak{g}[u]$ -module) with highest weight Λ (resp. λ). Under the action of \mathfrak{h} , the \mathfrak{g} -module $L(\Lambda)$ decomposes as

$$L(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} L(\Lambda)_\lambda. \quad (3)$$

where

$$L(\Lambda)_\lambda = \{v \in L(\Lambda) \mid hv = \lambda(h)v \ \forall h \in \mathfrak{h}\}. \quad (4)$$

Let $\text{mult}_{\Lambda, \mathfrak{h}}(\lambda)$ be the dimension of $L(\Lambda)_\lambda$. Set

$$\Gamma(\mathfrak{g}, \mathfrak{h}) = \{(\Lambda, \lambda) \in P_+ \times \mathfrak{h}^* \mid \text{mult}_{\Lambda, \mathfrak{h}}(\lambda) \neq 0\}.$$

Now, the \mathfrak{g} -module $L(\Lambda)$ is integrable as an $\mathfrak{g}[u]$ -module. In particular, under the action of $\mathfrak{g}[u]$, it decomposes as

$$L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+} \dot{L}(\lambda)^{\text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda)}, \quad (5)$$

where $\text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda)$ is the multiplicity of $\dot{L}(\lambda)$ in $L(\Lambda)$. Set

$$\Gamma(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda) \neq 0\}.$$

Our first main result is the following theorem.

Theorem 0.1. *As a subset of $\mathfrak{h}^* \times \mathfrak{h}^*$, the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is a semigroup.*

Let $(\Lambda, \lambda) \in P_+ \times \dot{P}_+$. Let δ be the basis imaginary root of \mathfrak{g} . To understand the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$, we study the set

$$\mathcal{B}(\Lambda, \lambda) = \{b \in \mathbb{C} \mid \dot{L}(\lambda + b\delta) \subset L(\Lambda)\}. \quad (6)$$

Let $P_u(\Lambda)$ is the set of all λ such that $(\Lambda, \lambda) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. The set $\mathcal{B}(\Lambda, \lambda)$ is nonempty if and only if $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. In this case, we define $b_{\Lambda, \lambda, u}$ be number $b \in \mathcal{B}(\Lambda, \lambda)$ such that $b + n \notin \mathcal{B}(\Lambda, \lambda)$ for any $n \in \mathbb{Z}_{>0}$. We also define a number $h_{\Lambda, \lambda}^{[u]}$ as in (2.118). We describe the set $\mathcal{B}(\Lambda, \lambda)$ as in the following theorem.

Theorem 0.2. *Let \mathfrak{g} be an affine Kac-Moody algebra. Let $\Lambda \in P_+$ and $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. We have*

1. $b_{\Lambda, \lambda, u} - (\mathbb{Z}_{\geq 0} \setminus \{1\}) \subset \mathcal{B}(\Lambda, \lambda) \subset b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.
2. *If in addition $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $\mathcal{B}(\Lambda, \lambda) = b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.*

Let $P(\Lambda)$ be the set of all $\lambda \in \mathfrak{h}^*$ such that $(\Lambda, \lambda) \in \Gamma(\mathfrak{g}, \mathfrak{h})$. For each $\lambda \in P(\Lambda) + \mathbb{C}\delta$, we set $b_{\Lambda, \lambda}$ the complex number b such that $\lambda + b\delta \in P(\Lambda)$ and $\lambda + (b+n)\delta \notin P(\Lambda)$ for any $n \in \mathbb{Z}_{>0}$. For \mathfrak{g} of type $A_1^{(1)}$ and $A_2^{(2)}$, we can compute explicitly the number $b_{\Lambda, \lambda}$ in Propositions 2.32 and 2.37. Let Λ_0 be the 0-th fundamental weight and α be the simple root α_1 . Let $m \in \mathbb{Z}_{>0}$. For each $m \in \mathbb{Z}$, we denote by P^m the set of all integral weights of level m . The results below help us to understand the branching rule on $\mathfrak{g}[u]$ in these two particular cases.

Theorem 0.3. *Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_1^{(1)}$. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in P_+^m$ and let $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \dot{P}_+$.*

1. *If there exists $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta)$ belongs to $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then $j' - j \in 2\mathbb{Z}$ and $m' = m$.*
2. *If moreover*
 - a. *$j \leq j' \leq um - j$ and u is even; or*
 - b. *$j \leq j' \leq um - (m - j)$ and u is odd,*

then $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

Theorem 0.4. *Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_2^{(2)}$. Let $\lambda = m\Lambda_0 + \frac{j\alpha}{2} \in P_+^m$ and let $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \dot{P}_+$.*

1. *If there exists $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta)$ belongs to $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then $m' = m$.*
2. *If moreover*
 - a. *$j \leq j'$; and*
 - b. *$j' \in \frac{m(u-1)}{2} - j + (2\mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{<0})$,*

then $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

For Λ like in Theorems 0.3, 0.4, we denote by $\mathcal{A}_u(\Lambda)$ the set of λ such that the theorems apply. The saturated setting of $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is defined by

$$\tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \lambda \in \Lambda + Q, \dot{L}(N\lambda) \subset L(N\Lambda) \text{ for some integer } N > 1\}. \quad (7)$$

Let Q be the root lattice of \mathfrak{g} . The following result is a corollary of Theorems 0.2, 0.3, 0.4.

Corollary 0.5. *Let \mathfrak{g} be affine Kac-Moody algebra of type $A_1^{(1)}$ or $A_2^{(2)}$. Fix $u \in \mathbb{Z}_{>1}$ (u is an odd number in the case $A_2^{(2)}$). Let $\Lambda \in P_+$ and let $\lambda \in \mathcal{A}_u(\Lambda) \cap (\Lambda + Q)$. For all $b \in \mathbb{C}$, we have*

1. $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow d(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$ for all $d \in \mathbb{Z}_{\geq 2}$.
2. *If in addition $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow (\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$.*

The second work is the results mentioned in our second preprint [Kha20] : “On the shifted Littlewood-Richardson coefficients and Littlewood-Richardson coefficients.” Namely, let λ, μ, ν be partitions. Let $l(\lambda)$ be the length of λ , and s_λ be the Schur function associated to the partition λ . The Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ appear in the expansion (see [Ful97] or [Mac15])

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu. \quad (8)$$

If now λ, μ, ν are strict partitions, let Q_λ be the shifted Schur Q -function associated to λ . The shifted Littlewood-Richardson coefficients appear in the expansion (see [Ste89])

$$Q_\lambda Q_\mu = \sum_{\nu} 2^{l(\lambda)+l(\mu)-l(\nu)} f_{\lambda\mu}^\nu Q_\nu. \quad (9)$$

For any strict partition λ , and a partition μ of the same integer, the coefficients $g_{\lambda\mu}$ appear in the decomposition

$$Q_\lambda = 2^{l(\lambda)} \sum_{\mu} g_{\lambda\mu} s_\mu. \quad (10)$$

The coefficients $g_{\lambda\mu}$ can be considered as shifted Littlewood-Richardson coefficients by the identity

$$g_{\lambda\mu} = f_{\lambda\delta}^{\mu+\delta}, \quad (11)$$

where $\delta = (l, l-1, \dots, 1)$ with $l = l(\mu)$.

Our first result is a new interpretation for the shifted Littlewood-Richardson coefficients. More precisely, given a skew shifted shape ν/μ , we construct the set $\tilde{\mathcal{O}}(\nu/\mu)$ of tableaux from the shifted reverse filling of the skew shifted shape ν/μ .

Theorem 0.6. *Let λ, μ, ν be strict partitions. Then the coefficient $f_{\lambda\mu}^\nu$ is the number of the tableaux T in $\tilde{\mathcal{O}}(\nu/\mu)$ of shape λ .*

The coefficient $g_{\lambda\mu}$ can be considered as particular shifted Littlewood-Richardson coefficient, we obtain a new model for $g_{\lambda\mu}$.

Theorem 0.7. *Let λ be a strict partition and μ be a partition. Then the coefficient $g_{\lambda\mu}$ is the number of the tableaux T in $\tilde{\mathcal{O}}(\mu + \delta/\delta)$ of shape λ .*

Our second result is also a new combinatorial interpretation of the coefficients $g_{\lambda\mu}$. More precisely, let $\tilde{\lambda}$ be the partition such that its Young diagram is the union of shifted diagram corresponding to λ and its reflection through the main diagonal. Let μ^t be the conjugate partition of μ . We prove that $g_{\lambda\mu}$ is the cardinal of a subset of a set that counts the coefficients $c_{\mu^t\mu}^{\tilde{\lambda}}$. As a corollary, we prove that

Theorem 0.8. *Let λ be a strict partition and μ be a partition. Then $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$.*

We also conjecture a stronger inequality

Conjecture 0.9. *Let λ be a strict partition and μ be a partition. Then $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\tilde{\lambda}}$.*

We formulate a conjecture on our combinatorial models whose validity implies the conjecture above. An evidence of the second conjecture is that it implies easily a well known equality $g_{\lambda\mu} = g_{\lambda\mu^t}$.

The thesis contains four chapters. In the first chapter, we introduce the fundamental background about Kac-Moody algebras and their representations (in particular, affine Kac-Moody algebras). In the second chapter, we present our results about branching rule on winding subalgebras of affine Kac-Moody algebras of type $A_1^{(1)}$ and $A_2^{(2)}$. In the third chapter, we introduce the shifted Littlewood-Richardson coefficients and explain the motivation from projective representation theory of the symmetric groups. In the last chapter, we present our results about new interpretations for shifted Littlewood-Richardson coefficients. We prove an inequality, and then conjecture a stronger inequality with some combinatorial structures which help us to understand better the relations between the coefficients.

Introduction en Français

L'objectif de ma thèse est d'utiliser des méthodes algébriques et combinatoires pour étudier le problème de branchement en théorie des représentations.

Le problème de branchement est l'un des plus importants de la théorie des représentations. Il demete de comprendre comment un module irréductible d'une algèbre \mathfrak{g} se décompose en modules irréductibles d'une sous-algèbre $\mathfrak{h} \subset \mathfrak{g}$ et étudier les propriétés de cette décomposition. Bien entendu, des hypothèses sur les algèbres de Lie à considérer doivent être faites pour donner un cadre précis à cette question. Par exemple, considérons l'algèbre de Lie linéaire $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{C})$ ($n \in \mathbb{Z}_{\geq 0}$). Alors, l'ensemble des représentations polynomiales irréductibles de $\mathfrak{gl}_n(\mathbb{C})$ correspond à l'ensemble des partitions de longueur au plus n . Si λ est une telle partition, notons V^λ le $\mathfrak{gl}_n(\mathbb{C})$ -module polynomiales irréductible correspondant à λ . Alors,

$$\{V^\lambda \otimes V^\mu \mid \lambda, \mu \text{ sont partitions de longueur au plus } n\} \quad (12)$$

est l'ensemble de tous les modules polynomiales irréductibles de $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$. Considérant ces modules comme $\mathfrak{gl}_n(\mathbb{C})$ -modules, on obtient les décompositions

$$V^\lambda \otimes V^\mu = \bigoplus_{\nu} (V^\nu)^{\text{mult}_{\lambda, \mu}(\nu)}, \quad (13)$$

où $\text{mult}_{\lambda, \mu}(\nu)$ est la multiplicité de V^ν dans $V^\lambda \otimes V^\mu$.

Le problème de branchement consiste en de nombreuses questions sur les multiplicités $\text{mult}_{\lambda, \mu}(\nu)$ et leurs généralisations à d'autres paires $\mathfrak{h} \subset \mathfrak{g}$ que $\mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$. Par exemple,

- quelle est la formule des multiplicités ?
- quels sont les bons algorithmes pour calculer les multiplicités ?
- peut-on décrire explicitement le support de la décomposition ? etc.

Des approches variées pour aborder ces questions relèvent de domaines différents : géométrie algébrique, théorie des nombres, combinatoire, analyse, ... Par exemple,

- par des méthodes géométriques, S. Kumar, N. Ressayre et P. Belkale ont obtenu des résultats précis sur le support de la décomposition lorsque \mathfrak{h} est une algèbre de Kac-Moody symétrisable et $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ (voir [BK06], [KR17], [Res17], [Res10], [BKR12] etc).
- en utilisant la théorie des formes modulaires, V. Kac et M. Wakimoto ont obtenu des résultats sur les fonctions de branchement qui nous aident à calculer explicitement la décomposition dans de nombreux cas particuliers (voir [KW90]). Ici les algèbres \mathfrak{h} et \mathfrak{g} sont des algèbres de Lie affines.

- par des méthodes combinatoires, P. Littelmann a obtenu une règle de Littlewood-Richardson pour toute algèbre de Kac-Moody symétrisable. Son modèle, connu sous le nom de modèle des chemins de Littelmann décrit les multiplicités de la décomposition du produit tensoriel (voir [Lit94], [Lit95a], [Lit95b], [Lit97], [Lit98]). Il a de nombreuses applications en théorie des représentations. Par exemple, une preuve simple de la conjecture PRV (voir [Lit94]). Il existe également de nombreux travaux notables utilisant des méthodes combinatoires. Par exemple, [KT01], [KT99], [KTW04] pour la preuve de la conjecture de saturation en utilisant le modèle en nid d'abeille. Ou [RW84], [Wor84], [Ste89], [Sag87], [Ser10], [Vak06] et ainsi de suite pour développer la théorie des coefficients (décalés) de Littlewood-Richardson, etc.

Ici, nous étudions la règle de branchement pour les sous-algèbres d'enroulement des algèbres Kac-Moody affines et étudions certaines propriétés non triviales des coefficients Littlewood-Richardson qui apparaissent dans la théorie de représentation projective des groupes symétriques. Les résultats explicites peuvent être décrits comme suit :

Le premier travail constitue la pré-publication [Kha19] : “ Branching problem on winding subalgebras of affine Kac-Moody algebras $A_1^{(1)}$ et $A_2^{(2)}$.” À savoir, soit \mathfrak{g} une algèbre de Kac-Moody affine et soit \mathfrak{h} une sous-algèbre Cartan de \mathfrak{g} . Soit $\mathfrak{g}[u]$ une sous-algèbre d'enroulement de \mathfrak{g} pour un entier positif donné u . La sous-algèbre d'enroulement $\mathfrak{g}[u]$ est isomorphe à \mathfrak{g} mais est plongé dans \mathfrak{g} de manière non triviale (voir [KW90]). Les \mathfrak{g} -modules irréductibles (resp. $\mathfrak{g}[u]$ -modules) sont paramétrés par le semigroupe P_+ (resp. \dot{P}_+) de poids entiers dominants. Pour $\Lambda \in P_+$ (resp. $\lambda \in \dot{P}_+$), soit $L(\Lambda)$ (resp. $\dot{L}(\lambda)$) le \mathfrak{g} -module (resp. $\mathfrak{g}[u]$ -module) irréductible intégrable de plus haut poids Λ (resp. λ). Sous l'action de \mathfrak{h} , le \mathfrak{g} -module $L(\Lambda)$ se décompose comme

$$L(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} L(\Lambda)_\lambda. \quad (14)$$

où

$$L(\Lambda)_\lambda = \{v \in L(\Lambda) \mid hv = \lambda(h)v \ \forall h \in \mathfrak{h}\}. \quad (15)$$

Soit $\text{mult}_{\Lambda, \mathfrak{h}}(\lambda)$ la dimension de $L(\Lambda)_\lambda$. Soit

$$\Gamma(\mathfrak{g}, \mathfrak{h}) = \{(\Lambda, \lambda) \in P_+ \times \mathfrak{h}^* \mid \text{mult}_{\Lambda, \mathfrak{h}}(\lambda) \neq 0\}.$$

Maintenant, le \mathfrak{g} -module $L(\Lambda)$ est intégrable en tant que $\mathfrak{g}[u]$ -module. En particulier, sous l'action de $\mathfrak{g}[u]$, il se décompose comme

$$L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+} \dot{L}(\lambda)^{\text{mult}_{\Lambda, \mathfrak{g}}(\lambda)}, \quad (16)$$

où $\text{mult}_{\Lambda, \mathfrak{g}}(\lambda)$ est la multiplicité de $\dot{L}(\lambda)$ dans $L(\Lambda)$. Posons

$$\Gamma(\mathfrak{g}, \mathfrak{g}) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \text{mult}_{\Lambda, \mathfrak{g}}(\lambda) \neq 0\}.$$

Notre premier résultat principal est le théorème suivant.

Theorem 0.10. *En tant que sous-ensemble de $\mathfrak{h}^* \times \mathfrak{h}^*$, l'ensemble $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ est un semigroupe.*

Soit $(\Lambda, \lambda) \in P_+ \times \dot{P}_+$. Soit δ la racine imaginaire de base de \mathfrak{g} . Pour comprendre l'ensemble $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$, nous étudions l'ensemble

$$\mathcal{B}(\Lambda, \lambda) = \{b \in \mathbb{C} \mid \dot{L}(\lambda + b\delta) \subset L(\Lambda)\}. \quad (17)$$

Soit $P_u(\Lambda)$ l'ensemble de λ tel que $(\Lambda, \lambda) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. L'ensemble $\mathcal{B}(\Lambda, \lambda)$ n'est pas vide si et seulement si $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. Dans ce cas, nous définissons $b_{\Lambda, \lambda, u}$ le nombre $b \in \mathcal{B}(\Lambda, \lambda)$ tel que $b + n \notin \mathcal{B}(\Lambda, \lambda)$ pour tout $n \in \mathbb{Z}_{>0}$. Nous définissons un nombre $h_{\Lambda, \lambda}^{[u]}$ dans (2.118). Nous décrivons l'ensemble $\mathcal{B}(\Lambda, \lambda)$ dans le théorème suivant.

Theorem 0.11. *Soit \mathfrak{g} une algèbre de Kac-Moody affine. Soit $\Lambda \in P_+$ et $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. Nous avons*

1. $b_{\Lambda, \lambda, u} - (\mathbb{Z}_{\geq 0} \setminus \{1\}) \subset \mathcal{B}(\Lambda, \lambda) \subset b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.
2. Si en plus $h_{\Lambda, \lambda}^{[u]} \neq 0$ alors $\mathcal{B}(\Lambda, \lambda) = b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.

Soit $P(\Lambda)$ l'ensemble de tous les $\lambda \in \mathfrak{h}^*$ tel que $(\Lambda, \lambda) \in \Gamma(\mathfrak{g}, \mathfrak{h})$. Pour chaque $\lambda \in P(\Lambda) + \mathbb{C}\delta$, nous désignons par $b_{\Lambda, \lambda}$ le nombre complexe b tel que $\lambda + b\delta \in P(\Lambda)$ et $\lambda + (b+n)\delta \notin P(\Lambda)$ pour tout $n \in \mathbb{Z}_{>0}$. Pour \mathfrak{g} de type $A_1^{(1)}$ et $A_2^{(2)}$, nous pouvons calculer explicitement le nombre $b_{\Lambda, \lambda}$ dans Propositions 2.32 and 2.37. Soit Λ_0 le 0-e poids fondamental et α la racine simple α_1 . Soit $m \in \mathbb{Z}_{>0}$. Pour chaque $m \in \mathbb{Z}$, on note P^m l'ensemble de tous les poids intégraux de niveau m . Les résultats ci-dessous nous aident à comprendre la règle de branchement sur $\mathfrak{g}[u]$ dans ces deux cas particuliers.

Theorem 0.12. *Soit \mathfrak{g} l'algèbre affine de Kac-Moody de type $A_1^{(1)}$. Soit $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in P_+^m$ et soit $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \dot{P}_+$.*

1. S'il existe $b \in \mathbb{C}$ tel que $(\Lambda, \lambda + b\delta)$ appartient à $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ puis $j' - j \in 2\mathbb{Z}$ et $m' = m$.
2. Si en plus
 - a. $j \leq j' \leq um - j$ et u est pair ; ou
 - b. $j \leq j' \leq um - (m - j)$ et u est impair,
alors $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

Theorem 0.13. *Soit \mathfrak{g} l'algèbre affine de Kac-Moody de type $A_2^{(2)}$. Soit $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in P_+^m$ et soit $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \dot{P}_+$.*

1. S'il existe $b \in \mathbb{C}$ tel que $(\Lambda, \lambda + b\delta)$ appartient à $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ puis $m' = m$.
2. Si en plus
 - a. $j \leq j'$; et
 - b. $j' \in \frac{m(u-1)}{2} - j + (2\mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{< 0})$,
alors $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

Pour Λ comme dans les théorèmes 0.12, 0.13, nous notons $\mathcal{A}_u(\Lambda)$ l'ensemble de λ tel que les Théorèmes s'appliquent. Le paramètre saturé de $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ est défini par

$$\tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \lambda \in \Lambda + Q, \dot{L}(N\lambda) \subset L(N\Lambda) \text{ pour un entier } N > 1\}. \quad (18)$$

Soit Q le réseau racine de \mathfrak{g} . Le résultat suivant est un corollaire des Théorèmes 0.11, 0.12, 0.13.

Corollary 0.14. *Soit \mathfrak{g} une algèbre de Kac-Moody affine de type $A_1^{(1)}$ ou $A_2^{(2)}$. Fixe $u \in \mathbb{Z}_{>1}$ (u est un nombre impair dans le cas $A_2^{(2)}$). Soit $\Lambda \in P_+$ et soit $\lambda \in \mathcal{A}_u(\Lambda) \cap (\Lambda + Q)$. Pour tous $b \in \mathbb{C}$, nous avons*

1. $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow d(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$ pour tous $d \in \mathbb{Z}_{\geq 2}$.
2. Si en plus $h_{\Lambda, \lambda}^{[u]} \neq 0$ alors $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow (\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$.

Notre deuxième travail constitue ma deuxième pré-publication [Kha20] : “On the shifted Littlewood-Richardson coefficients et Littlewood-Richardson coefficients.” Soient λ, μ, ν trois partitions. Soit $l(\lambda)$ la longueur de λ et soit s_λ la fonction de Schur associée à la partition λ . Les coefficients de Littlewood-Richardson $c_{\lambda\mu}^\nu$ apparaissent dans l’extension (voir [Ful97] ou [Mac15])

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu. \quad (19)$$

Si maintenant λ, μ, ν sont des partitions strictes, soit Q_λ la fonction de Schur Q décalée associée à la partition λ . Les coefficients décalés de Littlewood-Richardson apparaissent dans l’expansion (voir [Ste89])

$$Q_\lambda Q_\mu = \sum_{\nu} 2^{l(\lambda)+l(\mu)-l(\nu)} f_{\lambda\mu}^\nu Q_\nu. \quad (20)$$

Pour toute partition stricte λ , et une partition μ du même entier, les coefficients $g_{\lambda\mu}$ apparaissent dans la décomposition

$$Q_\lambda = 2^{l(\lambda)} \sum_{\mu} g_{\lambda\mu} s_\mu. \quad (21)$$

Les coefficients $g_{\lambda\mu}$ peuvent être considérés comme des coefficients de Littlewood-Richardson décalés par l’identité

$$g_{\lambda\mu} = f_{\lambda\delta}^{\mu+\delta}, \quad (22)$$

où $\delta = (l, l-1, \dots, 1)$ avec $l = l(\mu)$.

Notre premier résultat est une nouvelle interprétation des coefficients décalés de Littlewood-Richardson. Plus précisément, étant donné une forme décalée ν/μ , nous construisons un ensemble $\tilde{\mathcal{O}}(\nu/\mu)$ de tableaux à partir du remplissage inversé décalé de la forme décalée symétrique ν/μ .

Theorem 0.15. *Soit λ, μ, ν des partitions strictes. Alors le coefficient $f_{\lambda\mu}^\nu$ est le nombre des tableaux T dans $\tilde{\mathcal{O}}(\nu/\mu)$ de forme λ .*

Le coefficient $g_{\lambda\mu}$ peut être considéré comme un coefficient décalé de Littlewood-Richardson, nous obtenons un nouveau modèle pour $g_{\lambda\mu}$.

Theorem 0.16. *Soit λ une partition stricte et soit μ une partition. Alors le coefficient $g_{\lambda\mu}$ est le nombre des tableaux T dans $\tilde{\mathcal{O}}(\mu + \delta/\delta)$ de forme λ .*

Notre deuxième résultat est également une nouvelle interprétation combinatoire des coefficients $g_{\lambda\mu}$. Plus précisément, soit $\tilde{\lambda}$ la partition telle que son diagramme de Young soit l’union du diagramme décalé correspondant à λ et sa réflexion à travers la diagonale principale. Soit μ^t la partition conjuguée de μ . Nous prouvons que $g_{\lambda\mu}$ est le cardinal d’un sous-ensemble d’un ensemble qui compte les coefficients $c_{\mu^t\mu}^{\tilde{\lambda}}$. En corollaire, nous prouvons que

Theorem 0.17. *Soit λ une partition stricte et soit μ une partition. Alors $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$.*

Nous conjecturons également une inégalité plus forte

Conjecture 0.18. *Soit λ une partition stricte et soit μ une partition. Alors $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\tilde{\lambda}}$.*

Nous formulons une conjecture sur nos modèles combinatoires dont la validité implique la conjecture ci-dessus. L'évidence de la deuxième conjecture est qu'elle implique facilement une égalité bien connue $g_{\lambda\mu} = g_{\lambda\mu^t}$.

La thèse contient quatre chapitres. Dans le premier chapitre, nous introduisons les algèbres de Kac-Moody et leurs représentations (en particulier, les algèbres de Kac-Moody affines). Dans le deuxième chapitre, nous présentons nos résultats sur la règle de branchement pour les sous-algèbres d'enroulement des algèbres de Kac-Moody affines de type $A_1^{(1)}$ et $A_2^{(2)}$. Dans le troisième chapitre, nous introduisons les coefficients décalés de Littlewood-Richardson et expliquons la motivation de la théorie de représentation projective des groupes symétriques. Dans le dernier chapitre, nous présentons nos résultats sur les nouvelles interprétations des coefficients décalés de Littlewood-Richardson. Nous prouvons une inégalité, et puis conjecturons une inégalité plus forte avec quelques structures combinatoires qui nous aident à mieux comprendre les relations entre les coefficients.

Contents

Acknowledgements	i
Introduction	v
Introduction en Français	xi
Contents	xvi
1 Kac-Moody algebras and their representations	1
1.1 Generalized Cartan matrix	2
1.2 Realization of a complex square matrix	6
1.3 Kac-Moody algebras	6
1.4 Symmetrizable Kac-Moody algebras	8
1.4.1 Standard invariant form	8
1.4.2 The Weyl group	9
1.4.3 Representations of symmetrizable Kac-Moody algebras	10
1.4.4 Contravariant Hermitian forms on Verma modules	12
1.4.5 Affine Kac-Moody algebras	13
2 Branching problem on winding subalgebras of affine Kac-Moody algebras	
$A_1^{(1)}$ and $A_2^{(2)}$	17
2.1 Introduction	18
2.2 Preliminaries	21
2.2.1 Generalized Cartan matrix of affine type	21
2.2.2 Realization of a generalized Cartan matrix	21
2.2.3 Affine Kac-Moody algebras	22
2.2.4 Weyl group	22
2.2.5 Realization of affine Kac-Moody algebras	23
2.2.6 Dominant integral weights	24
2.2.7 Highest weight representations of affine Kac-Moody algebras	24
2.3 Branching on Cartan subalgebras	25
2.3.1 About the character ch_Λ	25
2.3.2 Semigroup structure	26
2.4 Branching on winding subalgebras: the general case	26
2.4.1 Winding subalgebras of an affine Kac-Moody algebra	26
2.4.2 Formulas for characteristic elements	27
2.4.3 The set of weights $P_u(\Lambda)$	28
2.4.4 Character method	28

2.4.4.1	The Virasoro algebra	28
2.4.4.2	Sugawara construction of Virasoro operators	29
2.4.4.3	Coset construction of Virasoro operators for winding subalgebras	32
2.4.4.4	Unitarizability	35
2.4.4.5	An identity of characters	35
2.4.5	Semigroup structure	35
2.4.6	The description of the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$	37
2.5	The cases $A_1^{(1)}$ and $A_2^{(2)}$	38
2.5.0.1	Computation of $b_{\Lambda, \lambda}$ for the case $A_1^{(1)}$	39
2.5.0.2	The conditions for $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$ in the case $A_1^{(1)}$	40
2.5.0.3	Computation of $b_{\Lambda, \lambda}$ for the case $A_2^{(2)}$	44
2.5.0.4	The conditions for $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$ in the case $A_2^{(2)}$	46
2.5.0.5	Relation between $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ and its saturated setting	50
3	Projective representations of symmetric groups	53
3.1	Shifted Littlewood-Richardson coefficients	54
3.1.1	Young tableaux	54
3.1.2	Ring of symmetric functions	54
3.1.3	Shifted tableaux	55
3.1.4	Shifted Littlewood-Richardson coefficients	56
3.2	Projective representations of symmetric groups	56
3.2.1	Projective representations of S_n	57
3.2.2	Conjugacy classes	58
3.2.2.1	Conjugacy classes of \widetilde{S}_n	58
3.2.2.2	Conjugacy classes of parabolic subgroups of \widetilde{S}_n	58
3.2.3	The space of class functions of \widetilde{S}_n	59
3.2.3.1	Basic spin characters of \widetilde{S}_n	60
3.2.3.2	Spin characters of parabolic subgroups of \widetilde{S}_n	61
3.2.3.3	Irreducible spin characters of \widetilde{S}_n	62
3.2.4	Relation with shifted Littlewood-Richardson coefficients	63
4	On the shifted Littlewood-Richardson coefficients and the Littlewood-Richardson coefficients	65
4.1	Introduction	66
4.1.1	Young tableaux	68
4.1.2	Row-insertion and product tableau	69
4.1.3	Sliding and jeu de taquin	69
4.1.4	The Robinson-Schensted-Knuth correspondence	70
4.1.5	Littlewood-Richardson rule	71
4.1.6	Tableau switching	74
4.1.7	The symmetry of Littlewood-Richardson coefficients	76
4.2	The shifted Littlewood-Richardson coefficients	76
4.2.1	Shifted tableaux	77
4.2.2	Shifted jeu de taquin	77
4.2.3	Shifted Littlewood-Richardson rule	78
4.2.4	Geometric interpretation of the coefficients $f_{\lambda\mu}^\nu$ and $g_{\lambda\mu}$	80
4.2.5	Application to the identity $g_{\lambda\mu} = g_{\lambda\mu^t}$	80
4.3	A new combinatorial model for the coefficients $f_{\lambda\mu}^\nu$	82

4.4	On the coefficients $g_{\lambda\mu}$	86
4.4.1	A new interpretation of the coefficients $g_{\lambda\mu}$	86
4.4.2	Inequality $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$	88
4.4.3	Inequality $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\tilde{\lambda}}$	89
	Bibliography	95

Chapter 1

Kac-Moody algebras and their representations

In this chapter, we collect some fundamental facts about the Kac-Moody algebras and their representations, which are necessary to understand the next chapter, namely, the first preprint [\[Kha19\]](#). One of the key notions are maximal weights, branching functions, string functions, Weyl-Kac's character formula. For more details, see [\[Kac90\]](#), [\[Car05\]](#).

Contents

1.1	Generalized Cartan matrix	2
1.2	Realization of a complex square matrix	6
1.3	Kac-Moody algebras	6
1.4	Symmetrizable Kac-Moody algebras	8
1.4.1	Standard invariant form	8
1.4.2	The Weyl group	9
1.4.3	Representations of symmetrizable Kac-Moody algebras	10
1.4.4	Contravariant Hermitian forms on Verma modules	12
1.4.5	Affine Kac-Moody algebras	13

1.1 Generalized Cartan matrix

Let $I = \{1, \dots, n\}$. A matrix $A = (a_{ij})_{i,j \in I}$ is called a **generalized Cartan matrix** if it satisfies the conditions below:

1. $a_{ii} = 2$,
2. $-a_{ij} \in \mathbb{Z}_{\geq 0}$ for all $i \neq j$,
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Two generalized Cartan matrices A and A' are called **equivalent** if they have the same size, say $n \times n$, and there exists a permutation $\sigma \in S_n$ such that $a'_{ij} = a_{\sigma(i)\sigma(j)}$ for all $i, j = 1, \dots, n$.

A generalized Cartan matrix A is called **indecomposable** if it is not equivalent to a block diagonal of nonzero generalized Cartan matrices.

Lemma 1.1. *If A is a generalized Cartan matrix, then the transpose tA is also a generalized Cartan matrix. Moreover, tA is indecomposable if and only if A is.*

We can classify indecomposable generalized Cartan matrices into three types according to its corank. Namely, A is of **finite type** if $\text{corank}(A) = 0$, of **affine type** if $\text{corank}(A) = 1$, and of **indefinite type** otherwise.

Let $u = {}^t(u_1, \dots, u_n)$ be a column vector in \mathbb{R}^n . We write $u > 0$ if $u_i > 0$ for all $i = 1, \dots, n$. We write $u \geq 0$ if $u_i \geq 0$ for all $i = 1, \dots, n$.

Theorem 1.2. *Let A be an indecomposable generalized Cartan matrix. Then tA and A have the same type, and*

1. A is of finite type iff there exists a vector $u \in \mathbb{R}^n$ such that $u > 0$, $Au > 0$,
2. A is of affine type iff there exists a vector $u \in \mathbb{R}^n$ such that $u > 0$, $Au = 0$,
3. A is of indefinite type iff there exists a vector $u \in \mathbb{R}^n$ such that $u > 0$, $Au < 0$.

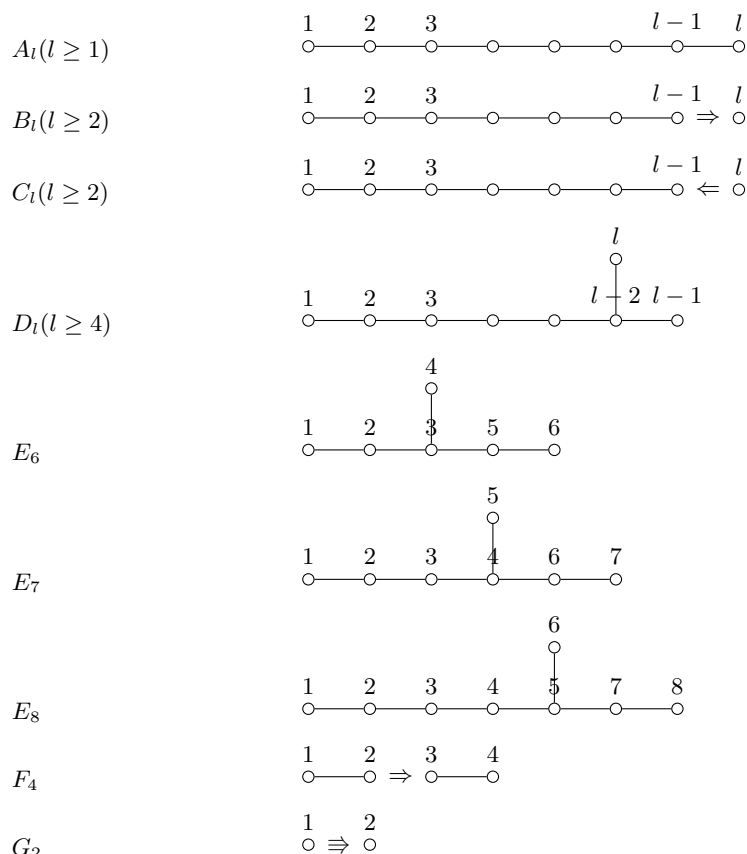
A generalized Cartan matrix A is called **symmetrizable** if $A = DB$ for some non-singular diagonal matrix D and symmetric matrix B .

Theorem 1.3. *Let A be an indecomposable generalized Cartan matrix of finite type or affine type. Then A is symmetrizable.*

A generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ is determined by its **Dynkin diagram** $\mathcal{D}(A)$, which is defined as follows:

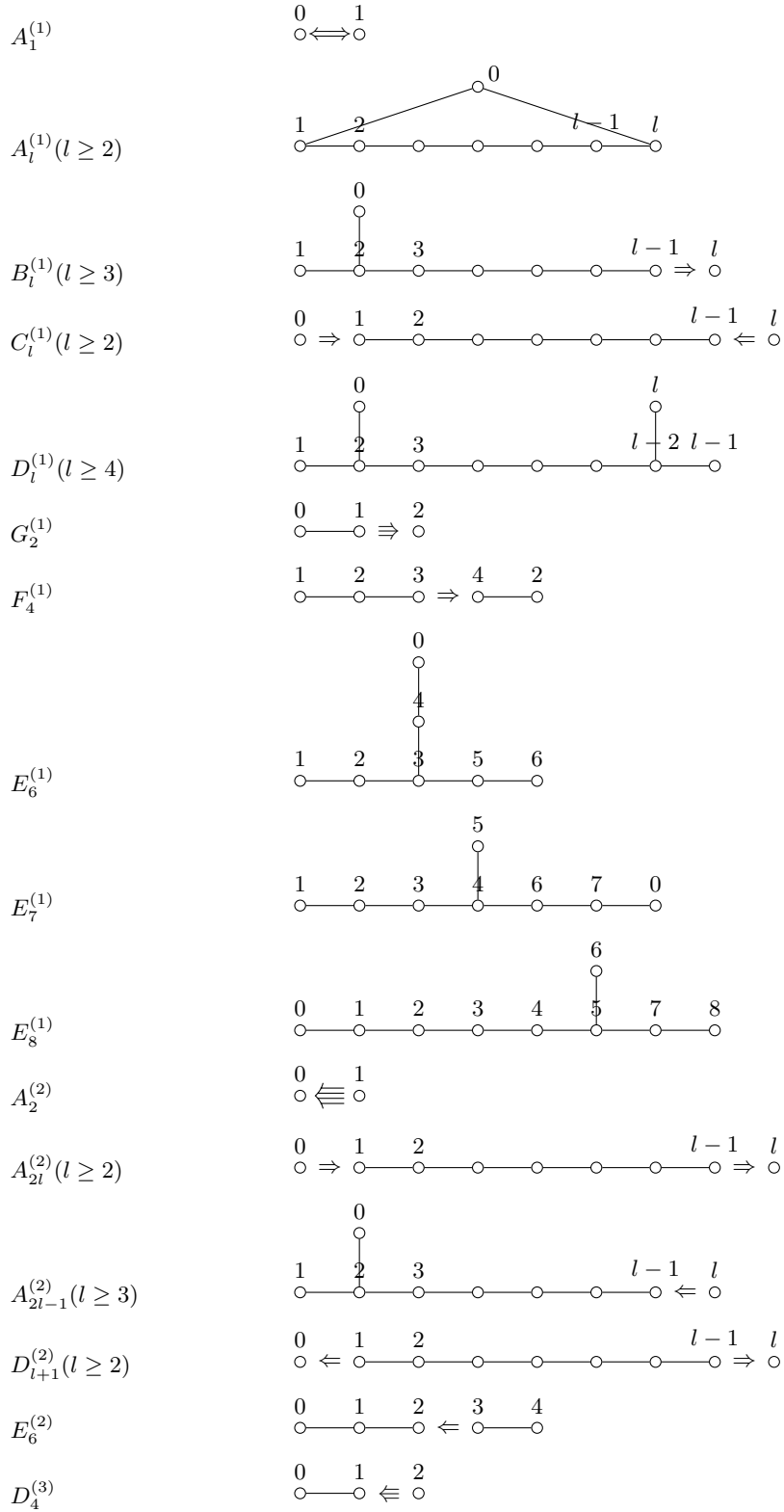
1. The vertices of $\mathcal{D}(A)$ are labelled by $i \in I$.
2. Two distinct vertices i, j of $\mathcal{D}(A)$ are joined by the rules:
 - If $a_{ij}a_{ji} = 0$, then vertices i, j are joined by no edge.
 - If $a_{ij}a_{ji} = 1$, then vertices i, j are joined by a single edge.
 - If $a_{ij}=-1, a_{ji} = -2$, then vertices i, j are joined by a double edge with an arrow pointing towards j .
 - If $a_{ij}=-1, a_{ji} = -3$, then vertices i, j are joined by a triple edge with an arrow pointing towards j .
 - If $a_{ij}=-1, a_{ji} = -4$, then vertices i, j are joined by a quadruple edge with an arrow pointing towards j .
 - If $a_{ij}=-2, a_{ji} = -2$, then vertices i, j are joined by a double edge with two arrows pointing away from i, j .
 - If $a_{ij}a_{ji} \geq 5$, then vertices i, j are joined by an edge with numbers $|a_{ij}|, |a_{ji}|$ shown on it.

A generalized Cartan matrix A is indecomposable if and only if its Dynkin diagram $\mathcal{D}(A)$ is connected. Moreover, the list of all connected Dynkin diagrams corresponding to generalized Cartan matrices of finite type is given below.



The list of all connected Dynkin diagrams corresponding to generalized Cartan matrices of affine

type is given below. We use the index set $I = \{0, \dots, l\}$ and standard notations as in [Kac90].



1.2 Realization of a complex square matrix

A **realization** of a generalized Cartan matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that

1. \mathfrak{h} is a \mathbb{C} -vector space of finite dimension,
2. $\Pi^\vee = \{h_1, \dots, h_n\}$ is a linearly independent subset of \mathfrak{h} ,
3. $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a linearly independent subset of \mathfrak{h}^* ,
4. $\alpha_i(h_j) = a_{ji}$ for all $i, j \in I$.

Let $(\mathfrak{h}', \Pi', (\Pi')^\vee)$ be another realization of A with $(\Pi')^\vee = \{h'_1, \dots, h'_n\} \subset \mathfrak{h}'$ and $\Pi = \{\alpha'_1, \dots, \alpha'_n\} \subset (\mathfrak{h}')^*$ such that $\alpha'_i(h'_j) = a_{ji}$. We say that $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}', \Pi', (\Pi')^\vee)$ are **isomorphic** if there is an isomorphism $\phi: \mathfrak{h} \rightarrow \mathfrak{h}'$ such that

$$\phi(h_i) = h'_i, \quad {}^t\phi(\alpha'_i) = \alpha_i, \quad (1.1)$$

where ${}^t\phi: (\mathfrak{h}')^* \rightarrow \mathfrak{h}^*$ is the induced map of ϕ on dual spaces.

By [Car05], we have

$$\dim(\mathfrak{h}) \geq 2n - \text{rank}(A). \quad (1.2)$$

In the case the equality happens, we say that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a **minimal realization** of A . Moreover, there exists a unique, up to isomorphism, minimal realization of a complex matrix.

1.3 Kac-Moody algebras

We define the **Kac-Moody algebra** $\mathfrak{g}(A)$ associated with A as follows:

Step 1. Let $\tilde{\mathfrak{g}}(A)$ be the Lie algebra with Lie bracket $[\cdot, \cdot]$ generated by elements

$$e_1, \dots, e_n, f_1, \dots, f_n, \tilde{h} \text{ for all } h \in \mathfrak{h}, \quad (1.3)$$

satisfying the relations

1. $\lambda \widetilde{h_1} + \mu \widetilde{h_2} = \widetilde{\lambda h_1 + \mu h_2}$ for all $\lambda, \mu \in \mathbb{C}$, $h_1, h_2 \in \mathfrak{h}$,
2. $[\widetilde{h_1}, \widetilde{h_2}] = 0$ for all $h_1, h_2 \in \mathfrak{h}$,
3. $[e_i, f_i] - \tilde{h}_i = 0$ for all $i = 1, \dots, n$,
4. $[e_i, f_j] = 0$ for all $i \neq j$,
5. $[\tilde{h}, e_i] - \alpha_i(h)e_i = 0$ for all $i = 1, \dots, n$ and $h \in \mathfrak{h}$,
6. $[\tilde{h}, f_i] + \alpha_i(h)f_i = 0$ for all $i = 1, \dots, n$ and $h \in \mathfrak{h}$.

Step 2. Let $\tilde{\mathfrak{h}}$ be the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by elements \tilde{h} for $h \in \mathfrak{h}$. The Lie algebra $\tilde{\mathfrak{g}}(A)$ contains a unique maximal ideal \mathfrak{I} with $\mathfrak{I} \cap \tilde{\mathfrak{h}} = 0$ (Proposition 14.13 in [Car05]). We define the Kac-Moody algebra associated to A by

$$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{I}. \quad (1.4)$$

Let $\tilde{\mathfrak{n}}_+$ be the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by elements e_1, \dots, e_n . Let $\tilde{\mathfrak{n}}_-$ be the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by elements f_1, \dots, f_n . Let ψ be the natural homomorphism $\tilde{\mathfrak{g}}(A) \rightarrow \mathfrak{g}(A)$, $x \mapsto x + \mathfrak{I}$. We define

$$\mathfrak{n}_+ = \psi(\tilde{\mathfrak{n}}_+) \text{ and } \mathfrak{n}_- = \psi(\tilde{\mathfrak{n}}_-). \quad (1.5)$$

From the fact that the map $\mathfrak{h} \rightarrow \tilde{\mathfrak{h}}, h \mapsto \tilde{h}$ is an isomorphism of vector spaces (Corollary 14.7 in [Car05]) and the restriction of ψ on $\tilde{\mathfrak{h}}$ is an isomorphism onto $\psi(\tilde{\mathfrak{h}})$ (Proposition 14.14 in [Car05]), we can identify \mathfrak{h} with $\psi(\tilde{\mathfrak{h}})$. The decomposition of $\tilde{\mathfrak{g}}(A)$

$$\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+ \quad (1.6)$$

induces the **triangular decomposition** of $\mathfrak{g}(A)$

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \quad (1.7)$$

We call \mathfrak{h} the **Cartan subalgebra**, \mathfrak{n}_+ the **positive subalgebra**, \mathfrak{n}_- the **negative subalgebra** of $\mathfrak{g}(A)$.

Let Q be the subgroup of \mathfrak{h}^* generated by $\{\alpha_1, \dots, \alpha_n\}$. Let Q_+ and Q_- be the subsets of Q defined by

$$Q_+ = \{k_1\alpha_1 + \dots + k_n\alpha_n \mid k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}\}, \quad (1.8)$$

$$Q_- = \{k_1\alpha_1 + \dots + k_n\alpha_n \mid k_1, \dots, k_n \in \mathbb{Z}_{\leq 0}\}. \quad (1.9)$$

For each $\alpha \in Q$, we define

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}. \quad (1.10)$$

Proposition 1.4. *We have*

1. $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$,
2. $\dim(\mathfrak{g}_\alpha)$ is finite for all $\alpha \in Q$,
3. $\mathfrak{g}_0 = \mathfrak{h}$,
4. If $\alpha \neq 0$, then $\mathfrak{g}_\alpha = 0$ unless $\alpha \in Q_+$ or $\alpha \in Q_-$,
5. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in Q$.

We call an element $\alpha \in \mathfrak{h}^*$ a **root** if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The set of all roots of $\mathfrak{g}(A)$ is denoted by Φ and call it the **root system**. The set Q is called the **root lattice** of $\mathfrak{g}(A)$. Every root belongs to either Q_+ or Q_- . The roots in Q_+ are called **positive roots**. The roots in Q_- are called **negative roots**. When α is a root, we call \mathfrak{g}_α the **root space** of α , and $\dim(\mathfrak{g}_\alpha)$ the **multiplicity** of α .

Proposition 1.5. *We have*

$$\dim(\mathfrak{g}_{\pm k\alpha_i}) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \quad (1.11)$$

We call $\alpha_1, \dots, \alpha_n$ the **simple roots** of $\mathfrak{g}(A)$ and h_1, \dots, h_n the **simple coroots** of $\mathfrak{g}(A)$.

1.4 Symmetrizable Kac-Moody algebras

In this section, assuming that A is symmetrizable, we define the standard invariant form, the Weyl group and the representations of the Kac-Moody algebra $\mathfrak{g}(A)$. Finally, we describe those objects in the particular case when A is of affine type.

1.4.1 Standard invariant form

Suppose that $A = DB$ for some non-singular diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ and symmetric matrix B . Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a minimal realization of A . For $\alpha = k_1\alpha_1 + \dots + k_n\alpha_n \in Q$, we define the **height** of α by

$$ht(\alpha) = k_1 + \dots + k_n. \quad (1.12)$$

For each $i \in \mathbb{Z}$, set

$$\mathfrak{g}_i = \bigoplus_{\alpha \in Q, ht(\alpha)=i} \mathfrak{g}_\alpha. \quad (1.13)$$

We have

$$\mathfrak{g}(A) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i. \quad (1.14)$$

From the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Hence, $\mathfrak{g}(A)$ can be considered as a \mathbb{Z} -graded Lie algebra. For each $r \in \mathbb{Z}_{\geq 0}$, we define

$$\mathfrak{g}(r) = \bigoplus_{-r \leq i \leq r} \mathfrak{g}_i. \quad (1.15)$$

Observe that $\mathfrak{g}(r)$ is finite dimensional but not necessarily a Lie algebra. Moreover,

$$\mathfrak{h} = \mathfrak{g}(0) \subset \mathfrak{g}(1) \subset \dots \subset \bigcup_{r \geq 0} \mathfrak{g}(r) = \mathfrak{g}(A). \quad (1.16)$$

We define the **standard invariant form** on $\mathfrak{g}(A)$ by induction as follows (as in the proof of Theorem 16.2 in [Car05]):

A. We define a bilinear form on \mathfrak{h} . Namely, let \mathfrak{h}' be the subspace of \mathfrak{h} generated by the elements h_1, \dots, h_n and let \mathfrak{h}'' be a complementary subspace of \mathfrak{h}' in \mathfrak{h} . We define a bilinear form $(|)$ on \mathfrak{h} by:

1. $(h_i|h_j) = d_i d_j B_{ij}$ for all $i, j \in I$,
2. $(h_i|h'') = (h''|h_i) = d_i \alpha_i(h'')$ for all $h'' \in \mathfrak{h}''$,
3. $(h''_1|h''_2) = 0$ for all $h''_1, h''_2 \in \mathfrak{h}''$.

We can prove that the bilinear form $(|)$ we have constructed on \mathfrak{h} is symmetric, non-degenerate (Proposition 16.1 in [Car05]) and invariant, i.e.,

$$([x, y]|z) = (x|[y, z]) \quad (1.17)$$

for all $x, y, z \in \mathfrak{h}$.

B. We first extend the bilinear form $(|)$ on

$$\mathfrak{g}(1) = \bigoplus_{i=1}^n \mathbb{C}f_i \oplus \mathfrak{h} \oplus \bigoplus_{i=1}^n \mathbb{C}e_i \quad (1.18)$$

by setting more conditions:

1. $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ unless $i + j = 0$.
2. $(e_i|f_j) = (f_j|e_i) = d_i\delta_{i,j}$.

This form is bilinear, symmetric, invariant.

C. Now, we consider the case $r \geq 2$. Suppose that we have extended the bilinear form to $\mathfrak{g}(r-1)$, we then extend it to $\mathfrak{g}(r)$ by setting more conditions:

1. $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ unless $i + j = 0, |i|, |j| \leq r$.
2. For $x \in \mathfrak{g}_r, y \in \mathfrak{g}_{-r}$, we can always write

$$y = \sum_j [c_j, d_j] \quad (1.19)$$

for some $c_j \in \mathfrak{g}_{-u_j}, d_j \in \mathfrak{g}_{-v_j}$ with $u_j, v_j > 0, u_j + v_j = r$. From the fact that $[x, c_j], d_j$ are all in $\mathfrak{g}(r-1)$ and the sum $\sum_j ([x, c_j]|d_j)$ does not depend on the way to express y as in the formula (1.19) (see [Car05]), we define

$$(x|y) = \sum_j ([x, c_j]|d_j) \quad (1.20)$$

for all $x \in \mathfrak{g}_r, y \in \mathfrak{g}_{-r}$.

We can check that the bilinear form (|) we have extended to $\mathfrak{g}(r)$ is bilinear, symmetric, non-degenerate and invariant, i.e.,

$$([x, y]|z) = (x|[y, z]) \quad (1.21)$$

for all $x, y, z \in \mathfrak{g}(r)$.

Finally, the bilinear form (|) is extended to $\mathfrak{g}(A)$.

1.4.2 The Weyl group

For $i = 1, \dots, n$, we define the map $s_i : \mathfrak{h} \rightarrow \mathfrak{h}$ by

$$s_i(h) = h - \alpha_i(h)h_i \quad (1.22)$$

for all $h \in \mathfrak{h}$.

Proposition 1.6. *We have $s_i^2 = 1, s_i(h_i) = -h_i$.*

We call the maps $s_i : \mathfrak{h} \rightarrow \mathfrak{h}$ the **fundamental reflections**. The group W of non-singular linear transformations of \mathfrak{h} generated by s_1, \dots, s_n is called the **Weyl group** of $\mathfrak{g}(A)$.

Let (|) be the standard invariant form on $\mathfrak{g}(A)$.

Proposition 1.7. *We have $(h|h') = (w(h)|w(h'))$ for all $h, h' \in \mathfrak{h}$ and $w \in W$.*

We can define the action of the Weyl group on \mathfrak{h}^* by

$$(w(\lambda))(h) = \lambda(w^{-1}(h)) \quad (1.23)$$

for all $w \in W, \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}$. Then for any $\lambda \in \mathfrak{h}^*$, we have

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i. \quad (1.24)$$

Moreover, the Weyl group acts on the root system Φ of $\mathfrak{g}(A)$. And the multiplicity of a root is invariant under the action.

Proposition 1.8. *If $\alpha \in \Phi$, then $w(\alpha) \in \Phi$. Moreover, $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{w(\alpha)})$.*

Consider the decomposition

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right). \quad (1.25)$$

We say that a root $\alpha \in \Phi$ is **real** if there exists $w \in W$ and $\alpha_i \in \Pi$ such that $\alpha = w(\alpha_i)$, and $\alpha \in \Phi$ is **imaginary** otherwise.

Theorem 1.9. *Let A be a symmetrizable indecomposable generalized Cartan matrix. Then*

1. *If A is of finite type, then $\mathfrak{g}(A)$ has no imaginary root.*
2. *If A is of affine type, then there exists a unique vector $a = {}^t(a_1, \dots, a_n)$ of relatively prime integers such that $a_i > 0$, $Aa = {}^t(0, \dots, 0)$. Set $\delta = a_1\alpha_1 + \dots + a_n\alpha_n$. Then the imaginary roots of $\mathfrak{g}(A)$ have form $k\delta$, where $k \in \mathbb{Z} \setminus \{0\}$.*

In the case A is of affine type, we call δ the **basic imaginary root** of $\mathfrak{g}(A)$.

1.4.3 Representations of symmetrizable Kac-Moody algebras

We define an order in \mathfrak{h}^* by

$$\lambda \geq \mu \text{ if and only if } \lambda - \mu \in Q_+. \quad (1.26)$$

The **category** \mathcal{O} is defined as follows:

1. The objects in the category \mathcal{O} are $\mathfrak{g}(A)$ -modules V satisfying the conditions:

- $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}. \quad (1.27)$$

- $\dim(V_\lambda)$ is finite for each $\lambda \in \mathfrak{h}^*$.
- There exists a finite set $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$ such that if $V_\lambda \neq 0$, then $\lambda \leq \lambda_i$ for some $i \in \{1, \dots, s\}$

2. The morphisms in the category \mathcal{O} are the homomorphisms of $\mathfrak{g}(A)$ -modules.

For each $V \in \mathcal{O}$, we define the **character** $ch(V)$ as a function from \mathfrak{h}^* to \mathbb{Z} by

$$ch(V)(\lambda) = \dim(V_\lambda). \quad (1.28)$$

For each $\lambda \in \mathfrak{h}^*$, we define the function $e^\lambda : \mathfrak{h}^* \rightarrow \mathbb{Z}$ by

$$e^\lambda(\mu) = \delta_{\lambda\mu}. \quad (1.29)$$

Then we have

$$ch(V) = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_\lambda) e^\lambda. \quad (1.30)$$

Lemma 1.10. *We have*

1. *If $V \in \mathcal{O}$ and U is a submodule of V , then $U \in \mathcal{O}$ and $V/U \in \mathcal{O}$,*

2. If $V_1, V_2 \in \mathcal{O}$, then $V_1 \oplus V_2 \in \mathcal{O}$ and $V_1 \otimes V_2 \in \mathcal{O}$.

For each $\lambda \in \mathfrak{h}^*$, we define the **Verma module** $M(\lambda)$ with highest weight λ by

$$M(\lambda) = \mathcal{U}(\mathfrak{g}(A))/K_\lambda, \quad (1.31)$$

where $\mathcal{U}(\mathfrak{g}(A))$ is the universal enveloping algebra of $\mathfrak{g}(A)$ and

$$K_\lambda = \mathcal{U}(\mathfrak{g}(A))\mathfrak{n}_+ + \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g}(A))(h - \lambda(h)) \quad (1.32)$$

for all $h \in \mathfrak{h}$.

The Verma module $M(\lambda)$ has a unique maximal submodule $J(\lambda)$. We define

$$L(\lambda) = M(\lambda)/J(\lambda). \quad (1.33)$$

Proposition 1.11. *For each $\lambda \in \mathfrak{h}^*$, we have $M(\lambda) \in \mathcal{O}$. Hence, $L(\lambda) \in \mathcal{O}$. Moreover, the modules $L(\lambda)$ are the only irreducible modules in the category \mathcal{O} .*

A $\mathfrak{g}(A)$ -module V is called **integrable** if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \quad (1.34)$$

and $e_i : V \rightarrow V$, $f_i : V \rightarrow V$ are locally nilpotent for all $i = 1, \dots, n$.

Proposition 1.12. *Let V be an integrable $\mathfrak{g}(A)$ -module. Then*

$$\dim(V_\lambda) = \dim(V_{w(\lambda)}) \quad (1.35)$$

for each $\lambda \in \mathfrak{h}^*$, $w \in W$.

We define the set of all **dominant integral weights**

$$P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \dots, n\}. \quad (1.36)$$

Proposition 1.13. *Let $\mathfrak{g}(A)$ be a symmetrizable Kac-Moody algebra. The module $L(\lambda)$ is integrable if and only if $\lambda \in P_+$.*

Since h_1, \dots, h_n are linearly independent, we can choose an element $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i = 1, \dots, n$.

Theorem 1.14. *(Weyl-Kac's character formula) Let $\mathfrak{g}(A)$ be a symmetrizable Kac-Moody algebra. For each $\lambda \in P_+$, we have*

$$ch(L(\lambda)) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e^{w(\rho)}}. \quad (1.37)$$

1.4.4 Contravariant Hermitian forms on Verma modules

Let $e_1, \dots, e_n, f_1, \dots, f_n, h_1, \dots, h_n, d$ be the Chevalley generators of \mathfrak{g} . Let ω_0 be the \mathbb{C} -**anti-linear anti-involution on \mathfrak{g}** defined by

$$\omega_0(e_i) = f_i, \quad \omega_0(f_i) = e_i, \quad \omega_0(h_i) = h_i, \quad \omega_0(d) = d. \quad (1.38)$$

Here anti-involution means we have $\omega_0([x, y]) = [\omega_0(y), \omega_0(x)]$ for all $x, y \in \mathfrak{g}$ and $\omega_0^2 = \text{Id}$. The map ω_0 extends uniquely to an anti-linear anti-involution on $\mathcal{U}(\mathfrak{g})$, which we still denote by ω_0 . The triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with the Poincaré-Birkhoff-Witt theorem imply that

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}_- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g}) \mathfrak{n}_+). \quad (1.39)$$

This splitting is stable by ω_0 since ω_0 pointwise fixes $\mathcal{U}(\mathfrak{h})$ and it interchanges $\mathfrak{n}_- \mathcal{U}(\mathfrak{g})$ with $\mathcal{U}(\mathfrak{g}) \mathfrak{n}_+$. In the decomposition (1.39), $\mathcal{U}(\mathfrak{h})$ is the symmetric algebra $\mathcal{S}(\mathfrak{h})$ since \mathfrak{h} is abelian. Let $\pi : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{S}(\mathfrak{h})$ be the projection to the first component of the decomposition. It is a $\mathcal{S}(\mathfrak{h})$ -bimodule map, i.e, $\pi(xgy) = x\pi(g)y$ for all $g \in \mathcal{U}(\mathfrak{g}), x, y \in \mathcal{S}(\mathfrak{h})$. Let $\langle \cdot, \cdot \rangle : \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{S}(\mathfrak{h})$ be the map defined by

$$\langle x, y \rangle = \pi(\omega_0(x)y). \quad (1.40)$$

Proposition 1.15. *We have*

1. $\langle \cdot, \cdot \rangle$ is linear in the second variable, anti-linear in the first variable.
2. $\langle gx, y \rangle = \langle x, \omega_0(g)y \rangle$ for all $x, y, g \in \mathcal{U}(\mathfrak{g})$.
3. $\langle x, y \rangle = 0$ if $x \in \mathcal{U}(\mathfrak{g})\mathfrak{n}_+$ or $y \in \mathcal{U}(\mathfrak{g})\mathfrak{n}_+$.

Let M be a \mathfrak{g} -module, a **Hermitian contravariant form on M with respect to the anti-involution ω_0** is a map $H : M \times M \rightarrow \mathbb{C}$ satisfying conditions below

1. $H(ax + y, z) = \bar{a}H(x, z) + H(y, z)$ for all $x, y, z \in M, a \in \mathbb{C}$,
2. $H(z, ax + y) = aH(z, x) + H(z, y)$ for all $x, y, z \in M, a \in \mathbb{C}$,
3. $H(gx, y) = H(x, \omega_0(g)y)$ for all $x, y \in M, g \in \mathfrak{g}$.

For each $\Lambda \in \mathfrak{h}^*$, let $ev_\Lambda : \mathcal{S}(\mathfrak{h}) \rightarrow \mathbb{C}$ be the homomorphism defined by

$$ev_\Lambda(x_1 \dots x_r) = \Lambda(x_1) \dots \Lambda(x_r) \text{ for all } x_1, \dots, x_r \in \mathfrak{h}. \quad (1.41)$$

The kernel of ev_Λ is generated by all elements $h - \Lambda(h)$ with $h \in \mathfrak{h}^*$. Set $\langle \cdot, \cdot \rangle_\Lambda = ev_\Lambda \circ \langle \cdot, \cdot \rangle$. Let $\mathfrak{h}_\mathbb{R}^*$ be the dual space of $\mathbb{R}d \oplus \bigoplus_{i=1}^n \mathbb{R}h_i$. With the help of Proposition 1.15, we obtain the following statement.

Proposition 1.16. *We have $\langle x, y \rangle_\Lambda = 0$ for all x or y in $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+ + \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(h - \Lambda(h))$ if and only if $\Lambda \in \mathfrak{h}_\mathbb{R}^*$.*

By the Proposition 1.16, with the assumption that $\Lambda \in \mathfrak{h}_\mathbb{R}^*$, the map $\langle \cdot, \cdot \rangle_\Lambda$ induces a Hermitian form on the Verma module $M(\Lambda)$, which we still denote by $\langle \cdot, \cdot \rangle_\Lambda$.

Proposition 1.17. *Let $\Lambda \in \mathfrak{h}_\mathbb{R}^*$, we have*

1. $\langle \cdot, \cdot \rangle_\Lambda$ is a Hermitian contravariant form on $M(\Lambda)$.

2. $\langle x, y \rangle_\Lambda = 0$ for all $x \in M(\Lambda)_{\Lambda-\alpha}, y \in M(\Lambda)_{\Lambda-\beta}$ with $\alpha \neq \beta$ in Q_+ .
3. The radical of \langle, \rangle_Λ is the maximal submodule $J(\Lambda)$ of $M(\Lambda)$.

A \mathfrak{g} -module M is said to be **unitarizable** if it has a positive definite Hermitian form which is contravariant with respect to the anti-involution ω_0 . Let $\Lambda \in \mathfrak{h}_\mathbb{R}^*$. By 3. in the Proposition 1.17, \langle, \rangle_Λ induces such a Hermitian contravariant form on $L(\Lambda)$. The result below is mentioned in Theorem 11.7 in [Kac90].

Theorem 1.18. *Let $\Lambda \in \mathfrak{h}_\mathbb{R}^*$. The \mathfrak{g} -module $L(\Lambda)$ is unitarizable if and only if $\Lambda \in P_+$.*

Remark 1.19.

1. It can be proved that \langle, \rangle_Λ is the only candidate to be a Hermitian contravariant form with respect to ω_0 .
2. Assume that $\mathfrak{g}(A)$ is finite dimensional and G is the simply connected group associated to $\mathfrak{g}(A)$ and $L(\Lambda)$ is finite dimensional of dimension d . Then G is not compact whereas $U_d(\mathbb{C})$ is. So we can not hope an invariant positive definite Hermitian form on $L(\Lambda)$. The role of ω_0 is here to solve this paradox since G^{ω_0} is a compact form of G .

1.4.5 Affine Kac-Moody algebras

Assume A is of affine type and number the index set as $I = \{0, \dots, l\}$. There exists a unique pair of vectors $a = {}^t(a_0, \dots, a_l)$ and $c = (c_0, \dots, c_l)$ of relatively prime integers such that $a_i, c_i > 0$ and $Aa = {}^t(0, \dots, 0), cA = (0, \dots, 0)$.

Proposition 1.20. *We have*

1. $c_0 = 1$,
2. $a_0 = \begin{cases} 2 & \text{if } A \text{ is of type } A_{2l}^{(2)}, \\ 1 & \text{otherwise.} \end{cases}$

The **Coxeter number** h and **dual Coxeter number** h^\vee of A are defined by

$$h = \sum_{i \in I} a_i, \quad h^\vee = \sum_{i \in I} c_i. \quad (1.42)$$

Recall that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a minimal realization of A then $\dim(\mathfrak{h}) = l + 2$. Fix an element $d \in \mathfrak{h}$ such that

$$\alpha_0(d) = 1, \quad \alpha_i(d) = 0 \text{ for all } i > 0. \quad (1.43)$$

We call d a **scaling element**. The **0-th fundamental weight** Λ_0 is defined by

$$\Lambda_0(h_j) = \delta_{0j}, \quad \Lambda_0(d) = 0 \text{ for all } i \in I. \quad (1.44)$$

Proposition 1.21. *We have*

1. $\{\alpha_0, \dots, \alpha_l, \Lambda_0\}$ is a basis of \mathfrak{h}^* ,
2. $\{h_0, \dots, h_l, d\}$ is a basis of \mathfrak{h} .

Recall that A is symmetrizable. More precisely, we have the following proposition.

Proposition 1.22. *We have $A = DB$, where $D = \text{diag}(d_0, \dots, d_l)$ with $d_i = a_i/c_i$ and B is a symmetric matrix.*

Let $(|)$ be the standard invariant form on $\mathfrak{g}(A)$ constructed as in Section 1.4.1. We have

1. $(h_i|h_j) = d_i d_j B_{ij} = \frac{a_j}{c_j} a_{ij}$ for all $i, j \in I$,
2. $(h_i|d) = a_0 \delta_{i0}$ for all $i \in I$,
3. $(d|d) = 0$.

There is a bijection $\mathfrak{h}^* \rightarrow \mathfrak{h}$ defined by

$$\lambda \mapsto h'_\lambda, \quad (1.45)$$

where $\lambda(h) = (h'_\lambda|h)$ for all $h \in \mathfrak{h}$. Set

$$K = \sum_{i=0}^l c_i h_i, \quad \delta = \sum_{i=0}^l a_i \alpha_i. \quad (1.46)$$

Lemma 1.23. *Under the bijection $\mathfrak{h}^* \rightarrow \mathfrak{h}, \lambda \mapsto h'_\lambda$, we have*

1. $\alpha_i \mapsto \frac{c_i}{a_i} h_i$,
2. $\Lambda_0 \mapsto \frac{1}{a_0} d$,
3. $\delta \mapsto K$.

From the fact that the centre of $\mathfrak{g}(A)$ is the 1-dimensional vector space spanned by K , we call K the **canonical central element** of $\mathfrak{g}(A)$. The element δ is called the **basic imaginary root** as in Section 1.4.2.

The bijection $\mathfrak{h}^* \rightarrow \mathfrak{h}, \lambda \mapsto h'_\lambda$ allows us to consider $(|)$ as a standard invariant form on \mathfrak{h}^* by

1. $(\alpha_i|\alpha_j) = \frac{c_i}{a_i} a_{ji}$ for all $i, j \in I$,
2. $(\alpha_i|\Lambda_0) = \frac{1}{a_0} \delta_{i0}$ for all $i \in I$,
3. $(\Lambda_0|\Lambda_0) = 0$.

The Weyl group W of $\mathfrak{g}(A)$ is generated by the **fundamental reflections** s_0, \dots, s_l , where s_i is given by

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad (1.47)$$

for all $\lambda \in \mathfrak{h}^*$.

For each $\alpha \in \mathfrak{h}^*$, we define $t_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ by

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - \left(\lambda + \frac{\lambda(K)\alpha}{2} \middle| \alpha \right) \delta. \quad (1.48)$$

Proposition 1.24. *We have*

1. $t_\alpha t_\beta = t_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{h}^*$.

2. $wt_\alpha w^{-1} = t_{w(\alpha)}$ for all $\alpha \in \mathfrak{h}^*$, $w \in W$.

For each $i = 0, \dots, l$, set

$$\alpha_i^\vee = \frac{a_i}{c_i} \alpha_i. \quad (1.49)$$

Suppose that A is of type $X_N^{(r)}$. We denote the type of ${}^t A$ by $X_{N^\vee}^{(r^\vee)}$. Let M be a sublattice of Q defined by

$$M = \begin{cases} \bigoplus_{i=1}^l \mathbb{Z} \alpha_i & \text{if } r^\vee = 1, \\ \bigoplus_{i=1}^l \mathbb{Z} \alpha_i^\vee & \text{if } r^\vee > 1. \end{cases} \quad (1.50)$$

Set $t_M = \{t_\alpha \mid \alpha \in M\}$. Let \overline{W} be the subgroup of W generated by $\{s_1, \dots, s_l\}$.

Theorem 1.25. *We have*

$$W \cong t_M \rtimes \overline{W}. \quad (1.51)$$

For each $i \in I$, we define the i -th fundamental weight Λ_i by

$$\Lambda_i(h_j) = \delta_{ij} \text{ for all } j \in I, \quad \Lambda_i(d) = 0. \quad (1.52)$$

We remark that the 0-th fundamental weight Λ_0 is already defined in (1.44). Since

$$\delta(h_i) = 0 \text{ for all } i \in I, \quad \delta(d) = 1, \quad (1.53)$$

then $\{\Lambda_0, \dots, \Lambda_l, \delta\}$ is a basis of \mathfrak{h}^* .

The set of dominant integral weights is given by

$$P_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i + \mathbb{C} \delta. \quad (1.54)$$

For each $\Lambda \in P_+$, let $L(\Lambda)$ be the integrable irreducible highest weight $\mathfrak{g}(A)$ -module with highest weight Λ . Let $P(\Lambda)$ be the set of all weights of $L(\Lambda)$, i.e.,

$$P(\Lambda) = \{\mu \in \mathfrak{h}^* \mid L(\Lambda)_\mu \neq 0\}. \quad (1.55)$$

Theorem 1.26. *Let $\Lambda \in P_+$. We have $\mu \in P(\Lambda)$ if and only if there exists $w \in W$ such that $w(\mu) \in P_+$ and $w(\mu) \leq \Lambda$.*

Corollary 1.27. *If $\mu \in P(\Lambda)$, then $\mu - \delta \in P(\Lambda)$.*

The corollary implies that if $\mu \in P(\Lambda)$, then $\mu - n\delta \in P(\Lambda)$ for any $n \in \mathbb{Z}_{\geq 0}$. Since there exist finitely many positive integers n such that $\mu + n\delta \leq \Lambda$, we say that μ is **maximal weight** of $L(\Lambda)$ if and only if $\mu \in P(\Lambda)$ but $\mu + \delta \notin P(\Lambda)$. We denote the set of all maximal weights of $L(\Lambda)$ by $\max(\Lambda)$. In summary,

$$P(\Lambda) = W((\Lambda - Q_+) \cap P_+) = \max(\Lambda) - \mathbb{Z}_{\geq 0} \delta. \quad (1.56)$$

Since δ is W -invariant, we have

Proposition 1.28. *Let $\Lambda \in P_+$. The set $\max(\Lambda)$ is invariant under the action of the Weyl group. In particular,*

$$\max(\Lambda) = W(\max(\Lambda) \cap P_+). \quad (1.57)$$

Set $e^{-\delta} = q$. For each $\lambda \in \mathfrak{h}^*$, we define the **string function** $c_\lambda^\Lambda \in \mathbb{C}((q))$ associated to λ by

$$c_\lambda^\Lambda = \sum_{n \in \mathbb{Z}} \dim(L(\Lambda)_{\lambda - n\delta}) q^n. \quad (1.58)$$

By Proposition 1.12, for all $w \in W$, we have

$$c_\lambda^\Lambda = c_{w(\lambda)}^\Lambda. \quad (1.59)$$

Denote by ch_Λ the character $ch(L(\Lambda))$. Choose $\rho = \sum_{i \in I} \Lambda_i$, we see that $\rho(h_i) = 1$ for all $i \in I$.

We have formulas for character of $L(\Lambda)$ as follows

$$ch_\Lambda = \sum_{\lambda \in \max(\Lambda)} c_\lambda^\Lambda e^\lambda. \quad (1.60)$$

$$ch_\Lambda = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e^{w(\rho)}}. \quad (1.61)$$

Chapter 2

Branching problem on winding subalgebras of affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$

The chapter is a reproduction of our first preprint [Kha19].

Abstract

We consider an affine Kac-Moody algebra \mathfrak{g} . Since \mathfrak{g} is infinite dimensional, it could occur that a strict subalgebra is isomorphic to \mathfrak{g} . It is the situation for the winding subalgebra $\mathfrak{g}[u]$ depending on a positive integer u . We are interested in the branching problem for such a pair $\mathfrak{g}[u] \subset \mathfrak{g}$ and integrable highest weights \mathfrak{g} -modules. Fix such a \mathfrak{g} -module $L(\Lambda)$ depending on a dominant integral weight Λ . Under the action of $\mathfrak{g}[u]$, $L(\Lambda)$ is still integrable with finite multiplicities. Hence, it decomposes in direct sum of $\mathfrak{g}[u]$ -modules $\dot{L}(\lambda)$ as $L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+} \dot{L}(\lambda)^{\text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda)}$. Here P_+ (resp. \dot{P}_+) denote the set of dominant weights and $L(\Lambda)$ (resp. $\dot{L}(\lambda)$) the irreducible integral \mathfrak{g} -module (resp. $\mathfrak{g}[u]$ -module) associated to $\Lambda \in P_+$ (resp. $\lambda \in \dot{P}_+$). Consider the support of the decomposition $\Gamma(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda) \neq 0\}$. Our first result is qualitative and extends to our context a well know Brion-Knop's theorem for finite dimensional reductive algebras: $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is a semigroup. Note that by lack of geometry, our proof is not a straightforward generalization to known proofs. Our second result is a partial description of $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$. When \mathfrak{g} is of type $A_1^{(1)}$ or $A_2^{(2)}$, we describe explicit sub-semigroups of this semigroup.

Contents

2.1	Introduction	18
2.2	Preliminaries	21
2.2.1	Generalized Cartan matrix of affine type	21
2.2.2	Realization of a generalized Cartan matrix	21
2.2.3	Affine Kac-Moody algebras	22
2.2.4	Weyl group	22
2.2.5	Realization of affine Kac-Moody algebras	23
2.2.6	Dominant integral weights	24
2.2.7	Highest weight representations of affine Kac-Moody algebras	24
2.3	Branching on Cartan subalgebras	25
2.3.1	About the character ch_Λ	25
2.3.2	Semigroup structure	26
2.4	Branching on winding subalgebras: the general case	26
2.4.1	Winding subalgebras of an affine Kac-Moody algebra	26
2.4.2	Formulas for characteristic elements	27
2.4.3	The set of weights $P_u(\Lambda)$	28
2.4.4	Character method	28
2.4.4.1	The Virasoro algebra	28
2.4.4.2	Sugawara construction of Virasoro operators	29
2.4.4.3	Coset construction of Virasoro operators for winding subalgebras	32
2.4.4.4	Unitarizability	35
2.4.4.5	An identity of characters	35
2.4.5	Semigroup structure	35
2.4.6	The description of the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$	37
2.5	The cases $A_1^{(1)}$ and $A_2^{(2)}$	38
2.5.0.1	Computation of $b_{\Lambda, \lambda}$ for the case $A_1^{(1)}$	39
2.5.0.2	The conditions for $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$ in the case $A_1^{(1)}$	40
2.5.0.3	Computation of $b_{\Lambda, \lambda}$ for the case $A_2^{(2)}$	44
2.5.0.4	The conditions for $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$ in the case $A_2^{(2)}$	46
2.5.0.5	Relation between $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ and its saturated setting	50

2.1 Introduction

One of the most important question in representation theory is how an irreducible module of a Lie algebra \mathfrak{g} can be decomposed when we consider it as a representation of some given Lie subalgebra $\dot{\mathfrak{g}}$. Assume first that \mathfrak{g} and $\dot{\mathfrak{g}}$ are finite dimensional and semi-simple. Then the finite dimensional irreducible \mathfrak{g} -modules (resp. $\dot{\mathfrak{g}}$ -modules) are parametrized by the semigroup P_+ (resp. \dot{P}_+) of dominant integral weights. Given $\Lambda \in P_+$, under the action of $\dot{\mathfrak{g}}$, the irreducible \mathfrak{g} -module $L(\Lambda)$ of highest weight Λ decomposes as

$$L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+} \dot{L}(\lambda)^{\text{mult}_{\Lambda, \dot{\mathfrak{g}}}(\lambda)}, \tag{2.1}$$

where $\text{mult}_{\Lambda, \dot{\mathfrak{g}}}(\lambda)$ is the multiplicity of $\dot{L}(\lambda)$ in $L(\Lambda)$. Understanding the number $\text{mult}_{\Lambda, \dot{\mathfrak{g}}}(\lambda)$ is referred as the branching problem. For example, for $\dot{\mathfrak{g}}$ diagonally embedded in $\mathfrak{g} = \dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$, the coefficients $\text{mult}_{\Lambda, \dot{\mathfrak{g}}}(\lambda)$ are the multiplicities of the tensor product decomposition of two

irreducible representations of $\dot{\mathfrak{g}}$. If $\dot{\mathfrak{g}} = \mathfrak{gl}_n(\mathbb{C})$, then \dot{P}_+ may be identified with the set of non-increasing sequences of n integers and the coefficients are the Littlewood-Richardson coefficients. If $\dot{\mathfrak{g}}$ is a Cartan subalgebra of \mathfrak{g} , the multiplicities $\text{mult}_{\Lambda, \dot{\mathfrak{g}}}(\lambda)$ are the Kostka coefficients. The support

$$\Gamma(\mathfrak{g}, \dot{\mathfrak{g}}) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ : \text{mult}_{\Lambda, \dot{\mathfrak{g}}}(\lambda) \neq 0\}$$

of these multiplicities is also a fascinating object. Actually, it is a finitely generated semigroup that generates a polyhedral convex cone (see [Bri13]). For $\dot{\mathfrak{g}}$ diagonally embedded in $\mathfrak{g} = \dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$ this cone is the famous Horn cone. Its description has a very long and rich story (see [Bri15], [Bri13], [Kum14], [BK06], [Res10]).

In this chapter, we are interested in similar questions for affine Kac-Moody algebras. Assume now that \mathfrak{g} is an affine Kac-Moody algebra and consider the integrable highest weight \mathfrak{g} -module $L(\Lambda)$ as module over some subalgebra $\dot{\mathfrak{g}}$. In the following three cases, we have decompositions similar to (2.1) with finite multiplicities:

1. $\dot{\mathfrak{g}} = \mathfrak{h}$ is a Cartan subalgebra of \mathfrak{g} ,
2. $\dot{\mathfrak{g}}$ is diagonally embedded in $\mathfrak{g} = \dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$,
3. $\dot{\mathfrak{g}}$ is a winding subalgebra of \mathfrak{g} introduced by V. G. Kac and M. Wakimoto in [KW90].

Recently, several authors studied $\Gamma(\mathfrak{g}, \dot{\mathfrak{g}})$ in the case of the tensor product decomposition of affine (or symmetrizable) Kac-Moody Lie algebras (see [Res17], [KR17], [Lit94], [BK14]).

Here, we begin a study of $\Gamma(\mathfrak{g}, \dot{\mathfrak{g}})$ in the winding case. This case is important and interesting since its relation with the tensor product decomposition (see [KW90]) and a solution to Frenkel's conjecture (see [KW90], [Fre82]). The winding subalgebras $\mathfrak{g}[u]$ are isomorphic to \mathfrak{g} but nontrivially embedded in \mathfrak{g} (the embedding depends on a fixed u , see Section 2.4 or [KW90] for details).

Let P_+ (resp. \dot{P}_+) be the set of all dominant integral weights of \mathfrak{g} (resp. $\mathfrak{g}[u]$). For $\Lambda \in P_+$ (resp. $\lambda \in \dot{P}_+$), let $L(\Lambda)$ (resp. $\dot{L}(\lambda)$) be the irreducible highest weight \mathfrak{g} -module (resp. $\mathfrak{g}[u]$ -module) with highest weight Λ (resp. λ). By [KW90], under the action of $\mathfrak{g}[u]$, $L(\Lambda)$ is still integrable and decomposes as

$$L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+} \dot{L}(\lambda)^{\text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda)}. \quad (2.2)$$

We define $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ to be the support of this decomposition:

$$\Gamma(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda) \neq 0\}. \quad (2.3)$$

Our first main result is the following theorem.

Theorem 2.1. *The set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is a sub-semigroup of $\mathfrak{h}^* \times \mathfrak{h}^*$.*

Let δ be the basic imaginary root of \mathfrak{g} . The description of the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is equivalent to describe the sets

$$\mathcal{B}(\Lambda, \lambda) = \{b \in \mathbb{C} \mid \dot{L}(\lambda + b\delta) \subset L(\Lambda)\},$$

for any $(\Lambda, \lambda) \in P_+ \times \dot{P}_+$. Let $P_u(\Lambda)$ is the set of all λ such that $(\Lambda, \lambda) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. The set $\mathcal{B}(\Lambda, \lambda)$ is nonempty if and only if $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. In this case, we define $b_{\Lambda, \lambda, u}$ to be number $b \in \mathcal{B}(\Lambda, \lambda)$ such that $b + n \notin \mathcal{B}(\Lambda, \lambda)$ for any $n \in \mathbb{Z}_{>0}$. We also define a number $h_{\Lambda, \lambda}^{[u]}$ as in (2.118). We describe the set $\mathcal{B}(\Lambda, \lambda)$ as in the following theorem.

Theorem 2.2. Let \mathfrak{g} be an affine Kac-Moody algebra of type $X_N^{(r)}$. Fix $u \in \mathbb{Z}_{>1}$ such that $u = 1 \pmod r$. Let $\Lambda \in P_+$ and $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. We have

1. $b_{\Lambda, \lambda, u} - (\mathbb{Z}_{\geq 0} \setminus \{1\}) \subset \mathcal{B}(\Lambda, \lambda) \subset b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.
2. If in addition $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $\mathcal{B}(\Lambda, \lambda) = b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.

Since \mathfrak{h} is contained in $\mathfrak{g}[u]$, if $(\Lambda, \lambda) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then λ is a weight of $L(\Lambda)$. Set

$$L(\Lambda)_\lambda = \{v \in L(\Lambda) \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}. \quad (2.4)$$

Set

$$\Gamma(\mathfrak{g}, \mathfrak{h}) = \{(\Lambda, \lambda) \in P_+ \times \mathfrak{h}^* \mid L(\Lambda)_\lambda \neq \{0\}\}. \quad (2.5)$$

Let us recall how $\Gamma(\mathfrak{g}, \mathfrak{h})$ can be described. Let $P(\Lambda)$ be the set of weights of $L(\Lambda)$. Fix $\lambda \in P(\Lambda) + \mathbb{C}\delta$, then the set of $b \in \mathbb{C}$ such that $\lambda + b\delta$ is a weight of $L(\Lambda)$ is of the form $b_{\Lambda, \lambda} - \mathbb{Z}_{\geq 0}$ for some well defined $b_{\Lambda, \lambda}$. In particular, describing $\Gamma(\mathfrak{g}, \mathfrak{h})$ is equivalent to compute the numbers $b_{\Lambda, \lambda}$. For \mathfrak{g} of type $A_1^{(1)}$ and $A_2^{(2)}$, we determine this number in Proposition 2.32 and Proposition 2.37, respectively. Let us introduce more notation to describe our results on $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ for these two Lie algebras. Let Λ_0 be the 0-th fundamental weight and let α be the simple root α_1 . Up to tensoring by some one dimensional \mathfrak{g} -module any integrable highest weight \mathfrak{g} -module is $L(\Lambda)$ with $\Lambda = m\Lambda_0 + \frac{j\alpha}{2}$ for some $j, m \in \mathbb{Z}_{\geq 0}$ and $j \leq m$ (for the case $A_1^{(1)}$) or $j \leq \frac{m}{2}$ (for the case $A_2^{(2)}$).

Theorem 2.3. Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_1^{(1)}$. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in P_+$ and let $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \dot{P}_+$.

1. If there exists $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta)$ belongs to $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then $j' - j \in 2\mathbb{Z}$ and $m' = m$.
2. If moreover
 - a. $j \leq j' \leq um - j$ and u is even; or
 - b. $j \leq j' \leq um - (m - j)$ and u is odd,

then $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

For Λ like in Theorem 2.3, we denote by $\mathcal{A}_u(\Lambda)$ the set of λ such that the theorem applies. Namely

$$\mathcal{A}_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in [j, um - j^*] \cap (j + 2\mathbb{Z}) \right\} \quad (2.6)$$

where $j^* = j$ if u is even and $j^* = m - j$ if u is odd.

Theorem 2.4. Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_2^{(2)}$. Let $\lambda = m\Lambda_0 + \frac{j\alpha}{2} \in P_+$ and let $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \dot{P}_+$.

1. If there exists $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta)$ belongs to $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then $m' = m$.
2. If moreover
 - a. $j \leq j'$; and
 - b. $j' \in \frac{m(u-1)}{2} - j + (2\mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{< 0})$,

then $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

For Λ like in Theorem 2.4, we denote by $\mathcal{A}_u(\Lambda)$ the set of λ such that the theorem applies. Namely

$$\mathcal{A}_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \left[j, \frac{um}{2} \right] \cap \mathbb{Z} \cap \left(\frac{m(u-1)}{2} - j + (2\mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{< 0}) \right) \right\}. \quad (2.7)$$

The saturated set of the support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is defined by

$$\tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \lambda \in \Lambda + Q, \dot{L}(N\lambda) \subset L(N\Lambda) \text{ for some integer } N > 1\}. \quad (2.8)$$

The following result is a corollary of Theorems 2.2, 2.3, 2.4.

Corollary 2.5. *Let \mathfrak{g} be an affine Kac-Moody algebra of type $A_1^{(1)}$ or $A_2^{(2)}$. Fix $u \in \mathbb{Z}_{>1}$ (u is an odd number in the case $A_2^{(2)}$). Let $\Lambda \in P_+$ and let $\lambda \in \mathcal{A}_u(\Lambda) \cap (\Lambda + Q)$. For all $b \in \mathbb{C}$, we have*

1. $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow d(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$ for all $d \in \mathbb{Z}_{\geq 2}$.
2. If in addition $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow (\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$.

This chapter is organized as follows. In Section 2.2, we prepare fundamental knowledge of affine Kac-Moody algebras. In Section 2.3, we present results around branching on Cartan subalgebras. We prove that the set $\Gamma(\mathfrak{g}, \mathfrak{h})$ is a semigroup. In Section 2.4, we introduce the notion of winding subalgebras and study the branching problem for general affine Kac-Moody algebras. The main results in this section are Theorems 2.1, 2.2. In Section 2.5, we present the results in the cases $A_1^{(1)}$ and $A_2^{(2)}$. The main results in this section are Theorems 2.3, 2.4, and Corollary 2.5.

2.2 Preliminaries

In this section, we recall basic results about affine Kac-Moody algebras in [Car05], [Kac90], [KW90].

2.2.1 Generalized Cartan matrix of affine type

Set $I = \{0, \dots, l\}$. Let $A = (a_{ij})_{i, j \in I}$ be a generalized Cartan matrix of affine type, i.e., A is indecomposable of corank 1, $a_{ii} = 2, -a_{ij} \in \mathbb{Z}_{\geq 0}$ for $i \neq j, a_{ij} = 0$ iff $a_{ji} = 0$ and there exists a column vector u with positive integer entries such that $Au = 0$.

Let $a = {}^t(a_0, \dots, a_l)$ and $c = (c_0, \dots, c_l)$ be the vectors of relatively prime integers such that $a_i, c_i > 0$ and $Aa = {}^t(0, \dots, 0), cA = (0, \dots, 0)$. The Coxeter number and dual Coxeter number of A are defined by $h = \sum_{i \in I} a_i$ and $h^\vee = \sum_{i \in I} c_i$.

2.2.2 Realization of a generalized Cartan matrix

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A where \mathfrak{h} is a \mathbb{C} -vector space of dimension $l + 2$, $\Pi^\vee = \{h_0, \dots, h_l\}$ is a linearly independent subset in \mathfrak{h} and $\Pi = \{\alpha_0, \dots, \alpha_l\}$ is a linearly independent subset in \mathfrak{h}^* (the dual space of \mathfrak{h}) such that $\alpha_i(h_j) = a_{ji}$.

Let $K = \sum_{i \in I} c_i h_i$ be the canonical central element and $\delta = \sum_{i \in I} a_i \alpha_i$ be the basic imaginary root.

Let $d \in \mathfrak{h}$ be the scaling element, i.e., $\alpha_0(d) = 1, \alpha_i(d) = 0$ for $i > 0$. Let Λ_i ($i \in I$) be the i -th fundamental weights, i.e., $\Lambda_i(h_j) = \delta_{ij}, \Lambda_i(d) = 0$ for all $j \in I$. Set $\rho = \sum_{i \in I} \Lambda_i$. Then

$\{\alpha_0, \dots, \alpha_l, \Lambda_0\}$ is a basis of \mathfrak{h}^* and $\{h_0, \dots, h_l, d\}$ is a basis of \mathfrak{h} .

2.2.3 Affine Kac-Moody algebras

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Kac-Moody algebra corresponding to the matrix A . We call \mathfrak{h} a Cartan subalgebra of \mathfrak{g} and Π, Π^\vee the set of simple roots, simple coroots of \mathfrak{g} , respectively. We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad (2.9)$$

where \mathfrak{n}_- is the negative subalgebra of \mathfrak{g} and \mathfrak{n}_+ is the positive subalgebra of \mathfrak{g} .

An affine Kac-Moody algebra has type $X_N^{(r)}$ with $r = 1, 2, 3$ (here we use the standard notation in [Kac90] to label the type of affine Kac-Moody algebras). In particular, the untwisted affine Kac-Moody of type $A_1^{(1)}$ is defined by the generalized Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (2.10)$$

and the twisted affine Kac-Moody algebra of type $A_2^{(2)}$ is defined by the generalized Cartan matrix

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}. \quad (2.11)$$

It is known that the matrix ${}^t A$ is also a generalized Cartan matrix of affine type. We denote $X_N^{(r^\vee)}$ the type of the algebra $\mathfrak{g}({}^t A)$.

2.2.4 Weyl group

Let $Q = \mathbb{Z}\Pi$ be the root lattice and let $Q_+ = \mathbb{Z}_{\geq 0}\Pi$. We define an order on \mathfrak{h}^* by $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. For each $i \in I$, set $\alpha_i^\vee = \frac{a_i}{c_i} \alpha_i$. We define a sublattice M of Q by

$$M = \begin{cases} \bigoplus_{i=1}^l \mathbb{Z} \alpha_i & \text{if } r^\vee = 1, \\ \bigoplus_{i=1}^l \mathbb{Z} \alpha_i^\vee & \text{if } r^\vee > 1. \end{cases} \quad (2.12)$$

Let $(\cdot | \cdot)$ be the standard invariant form on \mathfrak{h}^* which is defined by

$$(\alpha_i | \alpha_j) = \frac{c_i}{a_i} a_{ij}, \quad (\alpha_i | \Lambda_0) = \frac{1}{a_0} \alpha_i(d), \quad (\Lambda_0 | \Lambda_0) = 0 \quad \forall i, j \in I. \quad (2.13)$$

For each $\alpha \in \mathfrak{h}^*$, we define $t_\alpha \in GL(\mathfrak{h}^*)$ by

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - \left(\lambda + \frac{\lambda(K)\alpha}{2} \middle| \alpha \right) \delta. \quad (2.14)$$

Let W be the Weyl group of \mathfrak{g} , i.e., the group generated by fundamental reflections $\{s_0, \dots, s_l\}$, where $s_i \in GL(\mathfrak{h}^*)$ is defined by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. We have

$$W \cong t_M \rtimes \overline{W}, \quad (2.15)$$

where $t_M = \{t_\alpha \mid \alpha \in M\}$ and \overline{W} is the subgroup of W generated by $\{s_1, \dots, s_l\}$.

2.2.5 Realization of affine Kac-Moody algebras

Let $\bar{\mathfrak{g}}$ be a simple Lie algebra with Lie bracket $[\cdot, \cdot]_0$, and normalized invariant form $(\cdot | \cdot)_0$ (Here we use the notion of normalized invariant form for a simple Lie algebra in the book of Carter [Car05]). We extend $\bar{\mathfrak{g}}$ to a new Lie algebra

$$\hat{\bar{\mathfrak{g}}} = \mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}d \quad (2.16)$$

with new Lie bracket

$$\begin{aligned} & [t^i \otimes x + \lambda K + \mu d, t^j \otimes y + \lambda' K + \mu' d] \\ &= t^{i+j} \otimes [x, y]_0 + \mu j t^j \otimes y - \mu' i t^i \otimes x + i \delta_{i+j,0} (x|y)_0 K. \end{aligned} \quad (2.17)$$

for all $i, j \in \mathbb{Z}, x, y \in \bar{\mathfrak{g}}$ and $\lambda, \lambda', \mu, \mu' \in \mathbb{C}$.

Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$ and let $\bar{\Phi}$ be the root system of $\bar{\mathfrak{g}}$. For each $\alpha \in \bar{\Phi}$, let

$$\bar{\mathfrak{g}}_\alpha = \{x \in \bar{\mathfrak{g}} \text{ such that } [h, x]_0 = \alpha(h)x \text{ for all } h \in \bar{\mathfrak{h}}\}. \quad (2.18)$$

Let $\bar{\Pi} = \{\bar{\alpha}_1, \dots, \bar{\alpha}_l\}$ be the set of simple roots and $\bar{\Pi}^\vee = \{\bar{h}_1, \dots, \bar{h}_l\}$ be the set of simple coroots of $\bar{\mathfrak{g}}$. It is known that the dimension of $\bar{\mathfrak{g}}_\alpha$ is one for each $\alpha \in \bar{\Phi}$. For each $i \in \{1, \dots, l\}$, let \bar{e}_i be a basis vector of $\bar{\mathfrak{g}}_{-\bar{\alpha}_i}$ and \bar{f}_i be a basis vector of $\bar{\mathfrak{g}}_{\bar{\alpha}_i}$. Then $\bar{\mathfrak{g}}$ is a Lie algebra with generators $\{\bar{h}_1, \dots, \bar{h}_l, \bar{e}_1, \dots, \bar{e}_l, \bar{f}_1, \dots, \bar{f}_l\}$.

Suppose that $\bar{\mathfrak{g}}$ is associated to a generalized Cartan matrix \bar{A} of finite type. Each $\sigma \in S_l$ such that $\bar{a}_{ij} = \bar{a}_{\sigma(i)\sigma(j)}$ for all $i, j \in \{1, \dots, l\}$ can be considered as an automorphism of $\bar{\mathfrak{g}}$ by sending

$$\bar{e}_i \mapsto \bar{e}_{\sigma(i)}, \quad \bar{f}_i \mapsto \bar{f}_{\sigma(i)}, \quad \bar{h}_i \mapsto \bar{h}_{\sigma(i)}. \quad (2.19)$$

Let m be the order of σ and set $\eta = e^{2i\pi/m}$. We define the automorphism τ of $\hat{\bar{\mathfrak{g}}}$ by

$$\tau(t^j \otimes x) = \eta^{-j} t^j \otimes \sigma(x), \quad \tau(K) = K, \quad \tau(d) = d \quad (2.20)$$

for all $j \in \mathbb{Z}, x \in \bar{\mathfrak{g}}$. This map is called a twisted automorphism of $\hat{\bar{\mathfrak{g}}}$.

Now, let \mathfrak{g} be an affine Kac-Moody algebra of type $X_N^{(r)}$. Let $\bar{\mathfrak{g}}$ be the simple Lie algebra of type X_N . If \mathfrak{g} is an untwisted affine Kac-Moody algebra, i.e., type $X_N^{(1)}$, we have

$$\mathfrak{g} \simeq \hat{\bar{\mathfrak{g}}}. \quad (2.21)$$

If \mathfrak{g} is a twisted affine Kac-Moody algebra, i.e., type $X_N^{(r)}$ with $r = 2, 3$, then the order of nontrivial σ mentioned above is r . In this case, we have

$$\mathfrak{g} \simeq \hat{\bar{\mathfrak{g}}}^{(\tau)}. \quad (2.22)$$

The simple coroots h_1, \dots, h_l of \mathfrak{g} have the property

$$h_i \in 1 \otimes \bar{\mathfrak{h}} \quad (2.23)$$

for each $i \in \{1, \dots, l\}$, where $\bar{\mathfrak{h}}$ is the Cartan subalgebra of $\bar{\mathfrak{g}}$. For the details, we refer the reader to the proofs of Theorem 18.5, Theorem 18.9 and Theorem 18.14 in [Car05].

2.2.6 Dominant integral weights

Let

$$P = \sum_{i \in I} \mathbb{Z}\Lambda_i + \mathbb{C}\delta \quad (2.24)$$

be the set of integral weights. For any set $S \subset \mathfrak{h}^*$, we denote by \bar{S} the subset of all $\lambda \in S + \mathbb{C}\delta$ such that $\lambda(d) = 0$. We have

$$\bar{P} = \sum_{i \in I} \mathbb{Z}\Lambda_i. \quad (2.25)$$

For each $\lambda \in P$, the number $\lambda(K)$ is an integer and we call it the level of λ . For each $m \in \mathbb{Z}$, we denote by P^m the set of all integral weights of level m . Then we have

$$P^m = \left\{ \sum_{i \in I} m_i \Lambda_i \mid \sum_{i \in I} m_i c_i = m, m_i \in \mathbb{Z} \right\} + \mathbb{C}\delta. \quad (2.26)$$

Let

$$P_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i + \mathbb{C}\delta \quad (2.27)$$

be the set of dominant integral weights. Let \bar{P}_+, \bar{P}_+^m be the intersection of \bar{P}, \bar{P}^m with P_+ , respectively.

2.2.7 Highest weight representations of affine Kac-Moody algebras

Let \mathfrak{g} be an affine Kac-Moody algebra. Let \mathcal{O} be the category which is defined as follows:

1. The objects in the category \mathcal{O} are \mathfrak{g} -modules V satisfying the conditions:

- $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$.
- $\dim(V_\lambda)$ is finite for each $\lambda \in \mathfrak{h}^*$.
- There exists a finite set $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$ such that if $V_\lambda \neq 0$, then $\lambda \leq \lambda_i$ for some $i \in \{1, \dots, s\}$

2. The morphisms in the category \mathcal{O} are the homomorphisms of \mathfrak{g} -modules.

For each $\Lambda \in \mathfrak{h}^*$, we define the Verma module $M(\Lambda)$ with highest weight Λ by

$$M(\Lambda) = \mathcal{U}(\mathfrak{g})/K_\Lambda, \quad (2.28)$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} and

$$K_\Lambda = \mathcal{U}(\mathfrak{g})\mathfrak{n}_+ + \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(h - \lambda(h)) \quad (2.29)$$

for all $h \in \mathfrak{h}$.

The Verma module $M(\Lambda)$ has a unique maximal submodule $J(\Lambda)$. We define

$$L(\Lambda) = M(\Lambda)/J(\Lambda). \quad (2.30)$$

Proposition 2.6. *For each $\Lambda \in \mathfrak{h}^*$, we have $M(\Lambda) \in \mathcal{O}$. Hence, $L(\Lambda) \in \mathcal{O}$. Moreover, the modules $L(\Lambda)$ are the only irreducible modules in the category \mathcal{O} .*

A \mathfrak{g} -module V is called integrable if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ and $e_i : V \rightarrow V$, $f_i : V \rightarrow V$ are locally nilpotent for all $i = 1, \dots, n$.

Proposition 2.7. *Let \mathfrak{g} be an affine Kac-Moody algebra. The module $L(\Lambda)$ is integrable if and only if $\Lambda \in P_+$.*

2.3 Branching on Cartan subalgebras

In this section, we recall some facts about branching on Cartan subalgebras of affine Kac-Moody algebras.

Let $\Lambda \in P_+$ be a dominant integral weight of \mathfrak{g} , and let \mathfrak{h} be Cartan subalgebra of \mathfrak{g} . Regarding the \mathfrak{g} -module $L(\Lambda)$ as an \mathfrak{h} -module, it can be decomposed into direct sum of weights spaces

$$L(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} L(\Lambda)_\lambda, \quad (2.31)$$

where

$$L(\Lambda)_\lambda = \{v \in L(\Lambda) \mid hv = \lambda(h)v \ \forall h \in \mathfrak{h}\}. \quad (2.32)$$

Let $\text{mult}_{\Lambda, \mathfrak{h}}(\lambda)$ be the dimension of the space $L(\Lambda)_\lambda$. Then the decomposition (2.31) is encoded by a formal series ch_Λ on \mathfrak{h}^* as follows

$$ch_\Lambda = \sum_{\lambda \in \mathfrak{h}^*} \text{mult}_{\Lambda, \mathfrak{h}}(\lambda) e^\lambda, \quad (2.33)$$

called the character of $L(\Lambda)$, where $e^\lambda(\mu) = \delta_{\lambda, \mu}$ for $\mu \in \mathfrak{h}^*$. The set of weights of $L(\Lambda)$ is defined by

$$P(\Lambda) = \{\lambda \in \mathfrak{h}^* \mid \text{mult}_{\Lambda, \mathfrak{h}}(\lambda) \neq 0\}. \quad (2.34)$$

For each $\lambda \in \mathfrak{h}^*$, we say that λ is a maximal weight of $L(\Lambda)$ if $\lambda \in P(\Lambda)$ but $\lambda + n\delta \notin P(\Lambda)$ for any $n > 0$. Let $\max(\Lambda)$ be the set of all maximal weights of $L(\Lambda)$. We have

$$\max(\Lambda) = W(\max(\Lambda) \cap P_+) \quad (2.35)$$

and

$$P(\Lambda) = W((\Lambda - Q_+) \cap P_+) = \max(\Lambda) - \mathbb{Z}_{\geq 0}\delta. \quad (2.36)$$

For each $\lambda \in P(\Lambda) + \mathbb{C}\delta$, let $b_{\Lambda, \lambda}$ be the complex number such that $\lambda + b_{\Lambda, \lambda}\delta \in \max(\Lambda)$. It is an integer if $\lambda \in \Lambda + Q$ because $b_{\Lambda, \lambda}\delta \in Q$. For each $b \in \mathbb{C}$, $\lambda + b\delta \in P(\Lambda + b\delta)$ if and only if $\lambda \in P(\Lambda)$. Hence

$$b_{\Lambda + b_2\delta, \lambda + b_1\delta} = b_{\Lambda, \lambda} + b_2 - b_1 \quad (2.37)$$

for any $b_1, b_2 \in \mathbb{C}$. Let $\Gamma(\mathfrak{g}, \mathfrak{h})$ be the set of all $(\Lambda, \lambda) \in P_+ \times \mathfrak{h}^*$ such that $\lambda \in P(\Lambda)$.

2.3.1 About the character ch_Λ

We now recall some facts about the character ch_Λ for any affine Kac-Moody algebra.

Denote $e^{-\delta}$ by q . For each $\lambda \in \mathfrak{h}^*$, we define the string function $c_\lambda^\Lambda \in \mathbb{C}((q))$ of $L(\Lambda)$ associated to λ by

$$c_\lambda^\Lambda = \sum_{n \in \mathbb{Z}} \text{mult}_{\Lambda, \mathfrak{h}}(\lambda - n\delta) q^n. \quad (2.38)$$

Then for any $w \in W$ we have

$$c_\lambda^\Lambda = c_{w\lambda}^\Lambda \quad (2.39)$$

and

$$ch_\Lambda = \sum_{\lambda \in \max(\Lambda)} c_\lambda^\Lambda e^\lambda. \quad (2.40)$$

For each $w \in W$, let $l(w)$ be the minimal length of w as a product of fundamental reflections s_0, \dots, s_l . Set $\epsilon(w) = (-1)^{l(w)}$. The character ch_Λ can be written in terms of Weyl group, with the Weyl-Kac character formula (see Corollary 19.18 in [Car05]):

$$ch_\Lambda = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e^{w(\rho)}}. \quad (2.41)$$

2.3.2 Semigroup structure

In this part, we study the set $\Gamma(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^* \times \mathfrak{h}^*$.

Theorem 2.8. *The set $\Gamma(\mathfrak{g}, \mathfrak{h})$ is a sub-semigroup of $\mathfrak{h}^* \times \mathfrak{h}^*$.*

Proof. Let (Λ, λ) and $(\bar{\Lambda}, \bar{\lambda})$ be elements in the set $\Gamma(\mathfrak{g}, \mathfrak{h})$. We will show that $(\Lambda + \bar{\Lambda}, \lambda + \bar{\lambda})$ is still in this set. Indeed, $\lambda + \bar{\lambda}$ is a weight of $L(\Lambda) \otimes L(\bar{\Lambda})$. Hence $\lambda + \bar{\lambda}$ is a weight of $L(\Lambda'')$ for some $\Lambda'' \in ((\Lambda + \bar{\Lambda}) - Q_+) \cap P_+$. By (2.36), we have

$$P(\Lambda'') \subset P(\Lambda + \bar{\Lambda}). \quad (2.42)$$

It means $\lambda + \bar{\lambda} \in P(\Lambda + \bar{\Lambda})$ and then $\Gamma(\mathfrak{g}, \mathfrak{h})$ is a semigroup. \square

Remark 2.9. We can prove that $\Gamma(\mathfrak{g}, \mathfrak{h})$ is a semigroup for any symmetrizable Kac-Moody algebra \mathfrak{g} by the same argument.

2.4 Branching on winding subalgebras: the general case

In this section, we study the branching problem on winding subalgebras.

2.4.1 Winding subalgebras of an affine Kac-Moody algebra

In this subsection, we recall the notation of winding subalgebras of an affine Kac-Moody algebra in [KW90]. The studying of winding subalgebras is important and interesting because of its relation with the tensor product decomposition (see [KW90]) and a solution to Frenkel's conjecture (see [KW90], [Fre82]).

Let \mathfrak{g} be an affine Kac-Moody algebra of type $X_N^{(r)}$ which is defined by (2.21), (2.22). Fix $u \in \mathbb{Z}_{>1}$ such that $u \equiv 1 \pmod{r}$. It is easy to check that

$$t^j \otimes x \mapsto t^{uj} \otimes x, \quad K \mapsto uK, \quad d \mapsto \frac{d}{u}, \quad (2.43)$$

where $j \in \mathbb{Z}$, $x \in \bar{\mathfrak{g}}$ extend to an injective homomorphism ψ_u of Lie algebras and the image of ψ_u is stable by $\langle \tau \rangle$ in the case $r > 1$. Let $\mathfrak{g}[u]$ be the subalgebra of \mathfrak{g} which is defined as follows.

- If $r = 1$, set $\mathfrak{g}[u] = \psi_u(\hat{\mathfrak{g}})$.

- If $r > 1$, set $\mathfrak{g}[u] = \psi_u(\widehat{\mathfrak{g}})^{\langle \tau \rangle}$.

Then $\mathfrak{g}[u]$ is a subalgebra of \mathfrak{g} and it is isomorphic to \mathfrak{g} . We call $\mathfrak{g}[u]$ the winding subalgebra of \mathfrak{g} associated to u .

2.4.2 Formulas for characteristic elements

Set $\dot{K} = \psi_u(K) = uK$. Let $\tilde{\psi}_u : \mathfrak{h} \rightarrow \mathfrak{h}$ be the restriction of ψ_u to the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . For each $i \in I$, set $\dot{h}_i = \tilde{\psi}_u(h_i)$. Then by (2.23), we see that

$$\dot{h}_i = h_i \text{ for all } i > 0 \text{ and } \dot{h}_0 = \frac{u-1}{c_0}K + h_0. \quad (2.44)$$

Let ${}^t\tilde{\psi}_u : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ the dual map of $\tilde{\psi}_u$. Namely, for each $\lambda \in \mathfrak{h}^*$ we define ${}^t\tilde{\psi}_u$ by

$${}^t\tilde{\psi}_u(\lambda)(h) = \lambda(\tilde{\psi}_u(h)) \quad (2.45)$$

for all $h \in \mathfrak{h}$. For each $i \in I$, set $\dot{\alpha}_i = {}^t\tilde{\psi}_u(\alpha_i)$. Then by (2.44), (2.45) we have

$$\dot{\alpha}_i = \alpha_i \text{ for all } i > 0 \text{ and } \dot{\alpha}_0 = \frac{u-1}{a_0}\delta + \alpha_0. \quad (2.46)$$

For each $i \in I$, set $\dot{\Lambda}_i = {}^t\tilde{\psi}_u(\Lambda_i)$ and $\dot{\rho} = {}^t\tilde{\psi}_u(\rho)$. By (2.44), (2.45) we have

$$\dot{\Lambda}_i = \Lambda_i + \left(\frac{1}{u} - 1\right) \frac{c_i}{c_0} \Lambda_0 \quad (2.47)$$

$$\dot{\rho} = \sum_I \dot{\Lambda}_i = \rho + \left(\frac{1}{u} - 1\right) \frac{h^\vee}{c_0} \Lambda_0. \quad (2.48)$$

The map ${}^t\tilde{\psi}_u$ induces simple reflections $\dot{s}_i \in \text{Aut}(\mathfrak{h}^*)$, which are defined by

$$\dot{s}_i(\lambda) = \lambda - \lambda(\dot{h}_i)\dot{\alpha}_i. \quad (2.49)$$

The Weyl group \dot{W} of $\mathfrak{g}[u]$ is generated by the simple reflections \dot{s}_i ($i \in I$) turns out to be

$$\dot{W} \cong t_{uM} \rtimes \overline{W}. \quad (2.50)$$

In particular, by (2.15), it is a subgroup of the Weyl group W .

Let

$$\dot{P}_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \dot{\Lambda}_i + \mathbb{C}\delta \quad (2.51)$$

be the set of dominant integral weights of $\mathfrak{g}[u]$. For each $m \in \mathbb{Z}_{\geq 0}$, let \dot{P}_+^m be the set of dominant integral weights of $\mathfrak{g}[u]$ of level m , i.e.,

$$\dot{P}_+^m = \left\{ \sum_{i \in I} m_i \dot{\Lambda}_i \mid \sum_{i \in I} m_i c_i = m, m_i \in \mathbb{Z}_{\geq 0} \right\} + \mathbb{C}\delta. \quad (2.52)$$

Let $\lambda \in \dot{P}_+$, we denote the irreducible integrable $\mathfrak{g}[u]$ -module of highest weight λ by $\dot{L}(\lambda)$. The winding subalgebra $\mathfrak{g}[u]$ has a triangular decomposition

$$\mathfrak{g}[u] = \dot{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \dot{\mathfrak{n}}_+, \quad (2.53)$$

where $\dot{\mathfrak{n}}_-$ is the negative subalgebra of $\mathfrak{g}[u]$ and $\dot{\mathfrak{n}}_+$ is the positive subalgebra of $\mathfrak{g}[u]$.

2.4.3 The set of weights $P_u(\Lambda)$

For each $\Lambda \in P_+^m$ ($m \in \mathbb{Z}_{\geq 0}$), the \mathfrak{g} -module $L(\Lambda)$ can be regarded as a $\mathfrak{g}[u]$ -module of level um . Then it can be decomposed into direct sum of integrable irreducible $\mathfrak{g}[u]$ -module of level um

$$L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+^{um}} \dot{L}(\lambda)^{\text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda)}. \quad (2.54)$$

Set

$$P_u(\Lambda) = \{\lambda \in \dot{P}_+ \mid \text{mult}_{\Lambda, \mathfrak{g}[u]}(\lambda) \neq 0\}. \quad (2.55)$$

It is easy to see that $P_u(\Lambda) \subset P(\Lambda)$ (by part 0.5 page 5 in [KW90]). Hence $P_u(\Lambda)$ is a subset of $\max(\Lambda) - \mathbb{Z}_{\geq 0}\delta$. We say $\lambda \in P_u(\Lambda)$ a $\mathfrak{g}[u]$ -maximal weight of Λ if there does not exist $n \in \mathbb{Z}_{>0}$ such that $\lambda + n\delta \in P_u(\Lambda)$. We denote the set of all $\mathfrak{g}[u]$ -maximal weights of Λ by $\max_u(\Lambda)$. For each $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$, there exists a unique complex number $b_{\Lambda, \lambda, u}$ such that $\lambda + b_{\Lambda, \lambda, u}\delta \in \max_u(\Lambda)$. By definition, we have

$$b_{\Lambda, \lambda} - b_{\Lambda, \lambda, u} \in \mathbb{Z}_{\geq 0}. \quad (2.56)$$

Let $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ be the set of all $(\Lambda, \lambda) \in P_+ \times \dot{P}_+$ such that $\lambda \in P_u(\Lambda)$.

2.4.4 Character method

In this subsection, we prepare fundamental background on the representation theory of the Virasoro algebras which help us to study the branching problem. The idea is analogous to the idea in [KW90], [BK14]. In general, we need a formula expressing the character of an irreducible highest weight \mathfrak{g} -module $L(\Lambda)$ ($\Lambda \in P_+$) in terms of the characters of irreducible highest weight $\mathfrak{g}[u]$ -modules $\dot{L}(\lambda)$ ($\lambda \in \dot{P}_+$). The Virasoro characters plays a role in this formula to help us understanding the branching rule.

2.4.4.1 The Virasoro algebra

The Virasoro algebra Vir is spanned by symbols $\{L_n (n \in \mathbb{Z}), Z\}$ over \mathbb{C} with the Lie bracket

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}Z \text{ and } [Vir, Z] = 0. \quad (2.57)$$

Set $Vir_0 = \mathbb{C}L_0 \oplus \mathbb{C}Z$. Let V be a Vir -module. For each $\lambda \in Vir_0^*$, the weight space V_λ of V is

$$V_\lambda = \{v \in V \mid Xv = \lambda(X)v \text{ for all } X \in Vir_0\}. \quad (2.58)$$

Let V be a Vir -module which admits a weight decomposition such that each weight subspace has finite dimension. Let ω_0^{Vir} be the \mathbb{C} -anti-linear anti-involution on Vir defined by

$$\omega_0^{Vir}(L_n) = L_{-n} \ (n \in \mathbb{Z}), \quad \omega_0(Z) = Z. \quad (2.59)$$

Here anti-involution means we have $\omega_0^{Vir}([X, Y]) = [\omega_0^{Vir}(Y), \omega_0^{Vir}(X)]$ for all $X, Y \in Vir$ and $(\omega_0^{Vir})^2 = \text{Id}$. A Hermitian form $\langle \cdot, \cdot \rangle$ on V is said to be contravariant with respect to ω_0^{Vir} if

$$\langle Xv, w \rangle = \langle v, \omega_0^{Vir}(X)w \rangle, \quad (2.60)$$

for all $v, w \in V$, $X \in Vir$. A Vir -module is said to be unitarizable if there exists a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ which is contravariant with respect to ω_0^{Vir} . It is known that any unitarizable Vir -module is completely reducible.

A Vir -module V is called a highest weight module if there exists a nonzero vector v_0 in V such that v_0 is a Vir_0 -eigenvector, $L_n v_0 = 0$ for all $n \in \mathbb{Z}_{>0}$ and $\mathcal{U}(\bigoplus_{n<0} \mathbb{C}L_n)v_0 = V$ (here $\mathcal{U}(\cdot)$ is the universal enveloping algebra). We say that such a module has highest weight $\lambda \in Vir_0^*$ if $Xv_0 = \lambda(X)v_0$ for all $X \in Vir_0$.

Let V be a unitarizable highest weight Vir -module with highest weight $\lambda \in Vir_0^*$. As in the case affine Kac-Moody algebras, we have $\lambda \in Vir_{\mathbb{R}}^* = \mathbb{R}(L_0)^* \oplus \mathbb{R}Z^*$. Moreover, $\lambda(L_0), \lambda(Z)$ are non-negative since

$$0 \leq \langle L_{-n}v_0, L_{-n}v_0 \rangle = \langle L_n L_{-n}v_0, v_0 \rangle = (2n\lambda(L_0) + \frac{1}{12}(n^3 - n)\lambda(Z))\langle v_0, v_0 \rangle. \quad (2.61)$$

Let $\{(L_0)^*, Z^*\}$ be the dual basis of $\{L_0, Z\}$. We now restate Lemma 4.1 in [BK14].

Lemma 2.10. *Let V be a unitarizable highest weight (irreducible) Vir -module with highest weight λ . Then*

1. *If $\lambda(L_0) \neq 0$ then $V_{\lambda+n(L_0)^*} \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$.*
2. *If $\lambda(L_0) = 0$ and $\lambda(Z) \neq 0$, then $V_{\lambda+n(L_0)^*} \neq 0$ for all $n \in \mathbb{Z}_{>1}$ and $V_{\lambda+(L_0)^*} = 0$.*
3. *If $\lambda(L_0) = \lambda(Z) = 0$, then V is one dimensional.*

We next present some facts about the Virasoro algebras.

2.4.4.2 Sugawara construction of Virasoro operators

We recall the Sugawara construction of the Virasoro operator for the reader's convenience. For details, see, e.g., [KW88], [Kac90].

Untwisted case

We first introduce the Sugawara construction of the Virasoro operators for the untwisted affine Kac-Moody algebras. Let \mathfrak{g} be an untwisted affine Kac-Moody algebra of type $X_N^{(1)}$. Let $\bar{\mathfrak{g}}$ be the simple Lie algebra of type X_N .

$$\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}d. \quad (2.62)$$

with Lie bracket

$$\begin{aligned} & [t^i \otimes x + \lambda K + \mu d, t^j \otimes y + \lambda' K + \mu' d] \\ &= t^{i+j} \otimes [x, y]_0 + \mu j t^j \otimes y - \mu' i t^i \otimes x + i \delta_{i+j,0} (x|y)_0 K. \end{aligned} \quad (2.63)$$

for all $i, j \in \mathbb{Z}, x, y \in \bar{\mathfrak{g}}$ and $\lambda, \lambda', \mu, \mu' \in \mathbb{C}$. We denote $x^{(m)}$ for $t^m \otimes x$ ($x \in \bar{\mathfrak{g}}, m \in \mathbb{Z}$). Let $\{u_i\}$ and $\{u^i\}$ be a dual basis of $\bar{\mathfrak{g}}$, i.e., $(u_i|u^j) = \delta_{ij}$. The Sugawara operators T_n ($n \in \mathbb{Z}$) are defined by:

$$T_0 = \sum_i u_i u^i + 2 \sum_{m>0} \sum_i u_i^{(-m)} u^{i(m)}, \quad (2.64)$$

$$T_n = \sum_{m \in \mathbb{Z}} \sum_i u_i^{(-m)} u^{i(m+n)} \text{ if } n \neq 0. \quad (2.65)$$

The operators T_n ($n \in \mathbb{Z}$) does not make sense as an element of $\mathcal{U}(\mathfrak{g})$ in general because they are infinite sum over (m, i) . However, they are well defined endomorphisms of V for any \mathfrak{g} -module V in the category \mathcal{O} because for all $v \in V$ there exists finitely many (m, i) such that

$u_i^{(-m)}u^{i(m+n)}(v) \neq 0$. Let V be a \mathfrak{g}' -module in the category \mathcal{O} such that K acts on V as the multiplication by scalar k , where $k + h^\vee \neq 0$. As endomorphisms of V , we have the following statement.

Proposition 2.11. *For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}$, we have*

1. $[x^{(m)}, T_n] = 2(k + h^\vee)mx^{(m+n)}$
2. $[T_m, T_n] = 2(k + h^\vee)((m - n)T_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{6} (\dim \bar{\mathfrak{g}})k\text{Id}_V)$.

Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \bar{\mathfrak{g}} \oplus \mathbb{C}K$. We define the Virasoro operators on V as follows:

$$L_n = \frac{1}{2(k + h^\vee)} T_n \quad (2.66)$$

for each $n \in \mathbb{Z}$, and Z is the multiplication by scalar $\frac{k(\dim \bar{\mathfrak{g}})}{k + h^\vee}$. The following proposition automatically follows Proposition 2.11.

Proposition 2.12. *For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}$, we have*

1. $[x^{(m)}, L_n] = mx^{(m+n)}$.
2. $[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} Z$.

Twisted case

In this part, we present the Sugawara construction of the Virasoro operators for the twisted affine Kac-Moody algebras. Let \mathfrak{g} be an twisted affine Kac-Moody algebra of type $X_N^{(r)}$ ($r = 2, 3$). Let $\bar{\mathfrak{g}}$ be the simple Lie algebra of type X_N and let σ be the diagram automorphism of $\bar{\mathfrak{g}}$. The $\mathbb{Z}/r\mathbb{Z}$ -graduation of $\bar{\mathfrak{g}}$ with respect to σ is $\bar{\mathfrak{g}} = \bigoplus_{\bar{j} \in \mathbb{Z}/r\mathbb{Z}} \bar{\mathfrak{g}}_{\bar{j}}$, where $\bar{\mathfrak{g}}_{\bar{j}} = \{x \in \bar{\mathfrak{g}} \mid \sigma(x) = e^{2\pi i \bar{j}/m} x\}$.

Then we have

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} t^j \otimes \bar{\mathfrak{g}}_{\bar{j}} \oplus \mathbb{C}K \oplus \mathbb{C}d', \quad (2.67)$$

with Lie bracket

$$\begin{aligned} & [t^i \otimes x + \lambda K + \mu d', t^j \otimes y + \lambda' K + \mu' d'] \\ &= t^{i+j} \otimes [x, y]_0 + \mu j t^j \otimes y - \mu' i t^i \otimes x + \frac{i}{r} \delta_{i+j,0} (x|y)_0 K. \end{aligned} \quad (2.68)$$

The normalized invariant form on \mathfrak{g} is given by

$$\begin{aligned} & (t^i \otimes x + \lambda K + \mu d' | t^j \otimes y + \lambda' K + \mu' d')' \\ &= \frac{1}{r} \delta_{i+j,0} (x|y)_0 + (\lambda' \mu + \lambda \mu'). \end{aligned} \quad (2.69)$$

Remark 2.13.

1. We have $(\alpha|\alpha)_0 = 2$ for all long roots α of $\bar{\mathfrak{g}}$. However, $(\alpha|\alpha)' = 2r$ for all long roots α of \mathfrak{g} .
2. The element d' is not the scaling element of \mathfrak{g} as in the untwisted case. The formula of the scaling element contains d' .

Let $\{u_{i,-\bar{j}}\}$ be a basis of $\bar{\mathfrak{g}}_{-\bar{j}}$ and let $\{u^{i,\bar{j}}\}$ be the dual basis of $\bar{\mathfrak{g}}_{\bar{j}}$. The Sugawara operators T_n ($n \in \mathbb{Z}$) are defined by:

$$T_0 = \sum_i u_{i,\bar{0}} u^{i,\bar{0}} + 2 \sum_{m>0} \sum_i u_{i,-\bar{m}}^{(-m)} u^{i,\bar{m}(m)} + \frac{r-1}{2r} (\dim \bar{\mathfrak{g}} - \dim \bar{\mathfrak{g}}_0) K, \quad (2.70)$$

$$T_n = \sum_{m \in \mathbb{Z}} \sum_i u_{i,-\bar{m}}^{(-m)} u^{i,\bar{m}(m+rn)} \text{ if } n \neq 0. \quad (2.71)$$

These operators are well defined endomorphisms of V for any \mathfrak{g} -module V in the category \mathcal{O} because for any $v \in V$ there exists finitely many (m, i) such that $u_{i,-\bar{m}}^{(-m)} u^{i,\bar{m}(m+rn)}(v) \neq 0$. Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \bigoplus_{j \in \mathbb{Z}} t^j \otimes \bar{\mathfrak{g}}_{\bar{j}} \oplus \mathbb{C}K$. Let V be a \mathfrak{g}' -module in the category \mathcal{O} such that K acts on V as the multiplication by scalar k , where $k + \mathfrak{h}^\vee \neq 0$. As endomorphisms of V , we have the following statement.

Proposition 2.14. *For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}_{\bar{m}}$, we have*

1. $[x^{(m)}, T_n] = \frac{2(k+\mathfrak{h}^\vee)}{r} m x^{(m+rn)}$.
2. $[T_m, T_n] = 2(k + \mathfrak{h}^\vee)((m-n)T_{m+n} + \delta_{m+n,0} \frac{m^3-m}{6} r (\dim \bar{\mathfrak{g}}) k \text{Id}_V)$.

We define the Virasoro operators on V as follows:

$$L_n = \frac{1}{2(k + \mathfrak{h}^\vee)} T_n \quad (2.72)$$

for each $n \in \mathbb{Z}$ and Z is the multiplication by scalar $\frac{rk(\dim \bar{\mathfrak{g}})}{k+\mathfrak{h}^\vee}$. The following proposition automatically follows Proposition 2.14.

Proposition 2.15. *For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}_{\bar{m}}$, we have*

1. $[x^{(m)}, L_n] = \frac{m}{r} x^{(m+rn)}$.
2. $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} Z$.

Anti-involution ω_0 on Virasoro operators

The first conclusions of the Proposition Proposition 2.12 and Proposition 2.15 allow us to define a $\mathfrak{g}' \rtimes \text{Vir}$ -module structure on V . Here Vir is the Virasoro algebra with Virasoro operators L_n ($n \in \mathbb{Z}$) and Z . We recall the anti-linear anti-involution ω_0 on $\mathcal{U}(\mathfrak{g})$ in the part 1.4.4. Up to completion of $\mathcal{U}(\mathfrak{g})$ to get series in place of finite sums, we have the following statement.

Proposition 2.16. $\omega_0(L_n) = L_{-n}$ ($n \in \mathbb{Z}$) and $\omega_0(Z) = Z$.

Remark 2.17. The Proposition 2.16 implies

$$\langle L_n v, w \rangle_\Lambda = \langle v, \omega_0(L_n) w \rangle_\Lambda = \langle v, L_{-n} w \rangle_\Lambda, \quad (2.73)$$

$$\langle Z v, w \rangle_\Lambda = \langle v, \omega_0(Z) w \rangle_\Lambda = \langle v, Z w \rangle_\Lambda. \quad (2.74)$$

2.4.4.3 Coset construction of Virasoro operators for winding subalgebras

Let \mathfrak{g} be an affine Kac-Moody algebra of type $X_N^{(r)}$. Let $\bar{\mathfrak{g}}$ be the simple Lie algebra of type X_N . We now recall the coset construction of Virasoro operators for winding subalgebras $\mathfrak{g}[u]$, introduced in [KW90], with some details for the reader's convenience.

Untwisted case

In the case $r = 1$, we have

$$\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad (2.75)$$

$$\mathfrak{g}[u] = \mathbb{C}[t^u, t^{-u}] \otimes \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}d. \quad (2.76)$$

For each $\Lambda \in P_+^k$, the \mathfrak{g} -module $L(\Lambda)$ is considered as a $\mathfrak{g}[u]$ -module of level uk . Let $\{u_i\}$ and $\{v_i\}$ be a dual basis of $\bar{\mathfrak{g}}$. For each $x \in \bar{\mathfrak{g}}, n \in \mathbb{Z}$, we set $x^{(n)} = t^n \otimes x$. Let L_n ($n \in \mathbb{Z}$) and Z be the Virasoro operators given by the Sugawara construction for the \mathfrak{g} -module $L(\Lambda)$. Namely,

$$L_0 = \frac{1}{2(k + h^\vee)} \left(\sum_i u_i u^i + 2 \sum_{m>0} \sum_i u_i^{(-m)} u^{i(m)} \right), \quad (2.77)$$

$$L_n = \frac{1}{2(k + h^\vee)} \left(\sum_{m \in \mathbb{Z}} \sum_i u_i^{(-m)} u^{i(m+n)} \right) \text{ if } n \neq 0, \quad (2.78)$$

$$Z \text{ acts on } L(\Lambda) \text{ as the multiplication by scalar } c_k = \frac{k \dim(\bar{\mathfrak{g}})}{k + h^\vee}. \quad (2.79)$$

We construct the Virasoro operators \dot{L}_n ($n \in \mathbb{Z}$) and \dot{Z} on $\mathfrak{g}[u]$ -modules as follows.

$$\dot{L}_0 = \frac{1}{2(uk + h^\vee)} \left(\sum_i u_i u^i + 2 \sum_{m>0} \sum_i u_i^{(-um)} u^{i(um)} \right), \quad (2.80)$$

$$\dot{L}_n = \frac{1}{2(uk + h^\vee)} \left(\sum_{m \in \mathbb{Z}} \sum_i u_i^{(-um)} u^{i(um+un)} \right) \text{ if } n \neq 0, \quad (2.81)$$

$$\dot{Z} \text{ acts on } L(\Lambda) \text{ as the multiplication by scalar } c_{uk} = \frac{uk \dim(\bar{\mathfrak{g}})}{uk + h^\vee}. \quad (2.82)$$

Remark 2.18. The map ψ_u extends to $\psi_u : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$. Then we have $\psi_u(L_n) = \dot{L}_n$ and $\psi(Z) = \dot{Z}$ up to completion to get series in place of finite sums.

Proposition 2.19. For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}$, we have

1. $[x^{(um)}, \dot{L}_n] = mx^{(um+un)}$.
2. $[\dot{L}_m, \dot{L}_n] = (m - n)\dot{L}_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} \dot{Z}$.

The map $\zeta_u : \text{Vir} \rightarrow \text{Vir}$ defined by $\zeta_u(L_n) = u^{-1}L_{un} + \delta_{n,0} \frac{u-u^{-1}}{24} Z$ and $\zeta_u(Z) = uZ$ is a Lie algebra map. Set $\tilde{L}_n = \zeta_u(L_n)$ and $\tilde{Z} = \zeta_u(Z)$.

Proposition 2.20. For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}$, we have

1. $[x^{(um)}, \tilde{L}_n] = mx^{(um+un)}$.
 2. $[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} \tilde{Z}$.
- Set $L_n^{[u]} = \tilde{L}_n - \dot{L}_n$ and $Z^{[u]} = \tilde{Z} - \dot{Z}$.

Proposition 2.21. *We have*

1. $[\mathfrak{g}'[u], L_n^{[u]}] = 0$.
2. $[L_m^{[u]}, L_n^{[u]}] = (m-n)L_{m+n}^{[u]} + \delta_{m+n,0} \frac{m^3-m}{12} Z^{[u]}$.

Twisted case

In the case $r = 2, 3$, we have

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} t^j \otimes \bar{\mathfrak{g}}_j \oplus \mathbb{C}K \oplus \mathbb{C}d', \quad (2.83)$$

$$\mathfrak{g}[u] = \bigoplus_{j \in \mathbb{Z}} t^{uj} \otimes \bar{\mathfrak{g}}_{uj} \oplus \mathbb{C}K \oplus \mathbb{C}d', \quad (2.84)$$

where $\bar{\mathfrak{g}} = \bigoplus_{\bar{j} \in \mathbb{Z}/r\mathbb{Z}} \bar{\mathfrak{g}}_{\bar{j}}$ is the $\mathbb{Z}/r\mathbb{Z}$ -gradation of $\bar{\mathfrak{g}}$ with respect to the diagram automorphism σ of $\bar{\mathfrak{g}}$.

For each $\Lambda \in P_+^k$, the \mathfrak{g} -module $L(\Lambda)$ is considered as a $\mathfrak{g}[u]$ -module of level uk . Let $\{u_{i,-\bar{j}}\}$ be a basis of $\bar{\mathfrak{g}}_{-\bar{j}}$ and $\{u^{i,\bar{j}}\}$ be a dual basis of $\bar{\mathfrak{g}}_{\bar{j}}$. For each $n \in \mathbb{Z}$, $x \in \bar{\mathfrak{g}}_n$, we set $x^{(n)} = t^n \otimes x$. Let L_n ($n \in \mathbb{Z}$) and Z be the Virasoro operators given by the Sugawara construction for the \mathfrak{g} -module $L(\Lambda)$ in the twisted case. Namely,

$$L_0 = \frac{1}{2(k + \mathfrak{h}^\vee)} \left(\sum_i u_{i,\bar{0}} u^{i,\bar{0}} + 2 \sum_{m>0} \sum_i u_{i,-\bar{m}}^{(-m)} u^{i,\bar{m}(m)} + \frac{r-1}{2r} (\dim \bar{\mathfrak{g}} - \dim \bar{\mathfrak{g}}_{\bar{0}}) k \text{Id}_V \right), \quad (2.85)$$

$$L_n = \frac{1}{2(k + \mathfrak{h}^\vee)} \left(\sum_{m \in \mathbb{Z}} \sum_i u_{i,-\bar{m}}^{(-m)} u^{i,\bar{m}(m+rn)} \right) \text{ if } n \neq 0, \quad (2.86)$$

$$Z \text{ acts on } L(\Lambda) \text{ as the multiplication by scalar } c_k = \frac{rk \dim(\bar{\mathfrak{g}})}{k + \mathfrak{h}^\vee}. \quad (2.87)$$

Let \dot{L}_n ($n \in \mathbb{Z}$) and \dot{Z} be the Virasoro operators given by the Sugawara construction for the $\dot{\mathfrak{g}}$ -module $L(\Lambda)$. Namely,

$$\dot{L}_0 = \frac{1}{2(uk + \mathfrak{h}^\vee)} \left(\sum_i u_{i,\bar{0}} u^{i,\bar{0}} + 2 \sum_{m>0} \sum_i u_{i,-\bar{um}}^{(-um)} u^{i,\bar{um}(um)} + \frac{r-1}{2r} (\dim \bar{\mathfrak{g}} - \dim \bar{\mathfrak{g}}_{\bar{0}}) uk \text{Id}_V \right), \quad (2.88)$$

$$\dot{L}_n = \frac{1}{2(uk + \mathfrak{h}^\vee)} \left(\sum_{m \in \mathbb{Z}} \sum_i u_{i,-\bar{um}}^{(-um)} u^{i,\bar{um}(um+urn)} \right) \text{ if } n \neq 0, \quad (2.89)$$

$$\dot{Z} \text{ acts on } L(\Lambda) \text{ as the multiplication by scalar } c_{uk} = \frac{urk \dim(\bar{\mathfrak{g}})}{uk + \mathfrak{h}^\vee}. \quad (2.90)$$

Proposition 2.22. *For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}_m$, we have*

1. $[x^{(um)}, \dot{L}_n] = \frac{m}{r} x^{(um+urn)}$.
2. $[\dot{L}_m, \dot{L}_n] = (m-n)\dot{L}_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} \dot{Z}$.

The map $\zeta_u : Vir \rightarrow Vir$ defined by $\zeta_u(L_n) = u^{-1}L_{un} + \delta_{n,0} \frac{u-u^{-1}}{24}Z$ and $\zeta_u(Z) = uZ$ is a Lie algebra map. Set $\tilde{L}_n = \zeta_u(L_n)$ and $\tilde{Z} = \zeta_u(Z)$.

Proposition 2.23. *For any $n, m \in \mathbb{Z}$ and $x \in \bar{\mathfrak{g}}_{um}$, we have*

1. $[x^{(um)}, \tilde{L}_n] = mx^{(um+urn)}$.
2. $[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} \tilde{Z}$.

Set $L_n^{[u]} = \tilde{L}_n - r\dot{L}_n$ and $Z^{[u]} = \tilde{Z} - r\dot{Z}$.

Proposition 2.24. *We have*

1. $[\mathfrak{g}'[u], L_n^{[u]}] = 0$.
2. $[L_m^{[u]}, L_n^{[u]}] = (m-n)L_{m+n}^{[u]} + \delta_{m+n,0} \frac{m^3-m}{12} Z^{[u]}$.

Anti-involution ω_0 on Virasoro operators

The first conclusions of the Proposition 2.19 and Proposition 2.22 allow us to define a $\mathfrak{g}'[u] \rtimes Vir$ -module structure on $L(\Lambda)$. Here Vir is the Virasoro algebra with Virasoro operators \dot{L}_n ($n \in \mathbb{Z}$) and \dot{Z} . We recall the anti-linear anti-involution ω_0 on $\mathcal{U}(\mathfrak{g})$ in the part 1.4.4. Up to completion of $\mathcal{U}(\mathfrak{g})$ to get series in place of finite sums, we have the following statement.

Proposition 2.25. $\omega_0(\dot{L}_n) = \dot{L}_{-n}$ ($n \in \mathbb{Z}$) and $\omega_0(\dot{Z}) = \dot{Z}$.

Remark 2.26. The Proposition 2.25 implies

$$\langle \dot{L}_n v, w \rangle_\Lambda = \langle v, \omega_0(\dot{L}_n) w \rangle_\Lambda = \langle v, \dot{L}_{-n} w \rangle_\Lambda, \quad (2.91)$$

$$\langle \dot{Z} v, w \rangle_\Lambda = \langle v, \omega_0(\dot{Z}) w \rangle_\Lambda = \langle v, \dot{Z} w \rangle_\Lambda. \quad (2.92)$$

By Propositions 2.16, 2.25, we have the following statement.

Proposition 2.27. $\omega_0(L_n^{[u]}) = L_{-n}^{[u]}$ and $\omega_0(Z^{[u]}) = Z^{[u]}$.

Remark 2.28. The Proposition 2.27 implies

$$\langle L_n^{[u]} v, w \rangle_\Lambda = \langle v, \omega_0(L_n^{[u]}) w \rangle_\Lambda = \langle v, L_{-n}^{[u]} w \rangle_\Lambda, \quad (2.93)$$

$$\langle Z^{[u]} v, w \rangle_\Lambda = \langle v, \omega_0(Z^{[u]}) w \rangle_\Lambda = \langle v, Z^{[u]} w \rangle_\Lambda. \quad (2.94)$$

2.4.4.4 Unitarizability

For $\Lambda \in P_+$, $\lambda \in \dot{P}_+ + \mathbb{C}\delta$, set

$$\mathcal{U}(\Lambda, \lambda) = \{v \in L(\Lambda) \mid (\mathfrak{n}_+ \cap \mathfrak{g}'[u])v = 0 \text{ and } hv = \lambda(h) \text{ for all } h \in \mathfrak{h} \cap \mathfrak{g}'[u]\}. \quad (2.95)$$

The space $\mathcal{U}(\Lambda, \lambda)$ is stable by $L_n^{[u]}$ because of the commutativity of $L_n^{[u]}$ with $\mathfrak{g}'[u]$ (see Propositions 2.21, 2.24 in 2.4.4.3). Moreover, we have the decomposition of $L(\Lambda)$ with respect to the direct sum of $\mathfrak{g}'[u] \oplus Vir$ -modules (see [KW88], [KW90])

$$L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+ \pmod{\mathbb{C}\delta}} \dot{L}(\lambda) \otimes \mathcal{U}(\Lambda, \lambda). \quad (2.96)$$

Recall the positive definite contravariant Hermitian form $\langle \cdot, \cdot \rangle_\Lambda$ in Subsection 1.4.4. It induces a positive definite Hermitian form on $\mathcal{U}(\Lambda, \lambda)$ by restriction. By Proposition 2.27, this Hermitian form is contravariant with respect to the Virasoro algebra:

$$\langle L_n^{[u]}v, w \rangle_\Lambda = \langle v, L_{-n}^{[u]}w \rangle_\Lambda = \langle v, \omega_0^{Vir}(L_n)w \rangle_\Lambda, \quad (2.97)$$

$$\langle Z^{[u]}v, w \rangle_\Lambda = \langle v, Z^{[u]}w \rangle_\Lambda = \langle v, \omega_0^{Vir}(Z)w \rangle_\Lambda. \quad (2.98)$$

Thus, the Vir -module $\mathcal{U}(\Lambda, \lambda)$ is unitarizable.

2.4.4.5 An identity of characters

Let Λ be an element in P_+^m ($m \in \mathbb{Z}_{\geq 0}$). By (2.35), (2.39), (2.40) and a fact that \dot{W} is a subgroup of W , we have

$$\left(\sum_{w \in \dot{W}} \epsilon(w) e^{w(\dot{\rho})} \right) ch_\Lambda = \sum_{\lambda \in \max(\Lambda)} \left(\sum_{w \in \dot{W}} \epsilon(w) e^{w(\lambda + \dot{\rho})} \right) c_\lambda^\Lambda. \quad (2.99)$$

We may assume that $\lambda + \dot{\rho}$ in the above equality is regular with respect to \dot{W} . In this case, there exists unique $\sigma \in \dot{W}$ and $\lambda' \in \dot{P}_+$ such that $\sigma(\lambda + \dot{\rho}) = \lambda' + \dot{\rho}$. Let $p(\lambda)$ and $\{\lambda\}$ be $\epsilon(\sigma)$ and λ' in this case, respectively. In the case $\lambda + \dot{\rho}$ is nonregular, set $p(\lambda)$ and $\{\lambda\}$ to be 0. Since

$$\sum_{w \in \dot{W}} \epsilon(w) e^{w(\lambda + \dot{\rho})} = p(\lambda) \sum_{w \in \dot{W}} \epsilon(w) e^{w(\{\lambda\} + \dot{\rho})}, \quad (2.100)$$

it follows from the identities (2.41), (2.99) that:

Proposition 2.29.

$$ch_\Lambda = \sum_{\lambda \in \max(\Lambda)} p(\lambda) ch_{\{\lambda\}} c_\lambda^\Lambda \quad (2.101)$$

2.4.5 Semigroup structure

We state our first result about the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$.

Theorem 2.30. *The set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is a sub-semigroup of $\mathfrak{h}^* \times \mathfrak{h}^*$.*

Proof. Let (Λ, λ) and $(\bar{\Lambda}, \bar{\lambda})$ be elements in the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$. We need to show that $(\Lambda + \bar{\Lambda}, \lambda + \bar{\lambda}) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. The pair (Λ, λ) is an element of $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ if and only if $\dot{L}(\lambda) \subset L(\Lambda)$. The condition is equivalent to the existence of a nonzero vector $v \in L(\Lambda)$ such that

$$g(v) = 0, \forall g \in \dot{\mathfrak{n}}_+ \quad \text{and} \quad h(v) = \lambda(h)v, \forall h \in \mathfrak{h}. \quad (2.102)$$

Let \bar{v} be a nonzero vector in $L(\bar{\Lambda})$ satisfying the same conditions but for the pair $(\bar{\Lambda}, \bar{\lambda})$. In order to prove the semigroup structure of $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ we just need to show the existence of a nonzero vector \tilde{v} in $L(\Lambda + \bar{\Lambda})$ which satisfies the conditions (2.102) with the weight $\lambda + \bar{\lambda}$. We make the details in the two following steps.

Step 1. Construction of the vector \tilde{v} . By the fact that $L(\Lambda + \bar{\Lambda})$ is a \mathfrak{g} -submodule of $L(\Lambda) \otimes L(\bar{\Lambda})$ of multiplicity one, there exists a unique \mathfrak{g} -stable complementary subspace S such that

$$L(\Lambda) \otimes L(\bar{\Lambda}) = L(\Lambda + \bar{\Lambda}) \oplus S. \quad (2.103)$$

Let $\pi : L(\Lambda) \otimes L(\bar{\Lambda}) \rightarrow L(\Lambda + \bar{\Lambda})$ be the projection with kernel S . Set $\tilde{v} = \pi(v \otimes \bar{v})$. We show that \tilde{v} is nonzero and satisfies the conditions (2.102) in the next steps.

Step 2. \tilde{v} is nonzero. Let

$$L(\Lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} L(\Lambda)_\mu \quad (2.104)$$

be the weight spaces decomposition of $L(\Lambda)$. We define

$$L(\Lambda)^\vee = \bigoplus_{\mu \in \mathfrak{h}^*} (L(\Lambda)_\mu)^*. \quad (2.105)$$

There exists a nonzero vector $\psi \in L(\Lambda)^\vee$ such that

$$g(\psi) = 0, \forall g \in \mathfrak{n}_- \quad \text{and} \quad h(\psi) = -\Lambda(h)\psi, \forall h \in \mathfrak{h}. \quad (2.106)$$

Let G be the minimal Kac-Moody group corresponding to the Kac-Moody algebra \mathfrak{g} (see [Kum02]). To the vector $v \in L(\Lambda)$ defined above, we associate a function $f_v : G \rightarrow \mathbb{C}, g \mapsto \psi(g^{-1}(v))$. Since $L(\Lambda)$ is irreducible, the function f_v is nonzero ($f_v = 0$ implies $Gv \subset \ker \psi$). Let B^- be the negative Borel subgroup of G . We have

$$(1, b).f_v = \Lambda(b)^{-1}f_v \text{ for all } b \in B^-. \quad (2.107)$$

Similarly, for $L(\bar{\Lambda})$, we define $\bar{\psi} \in L(\bar{\Lambda})^\vee$ and $f_{\bar{v}} : G \rightarrow \mathbb{C}, g \mapsto \bar{\psi}(g^{-1}(\bar{v}))$. Then $f_{\bar{v}}$ is nonzero and

$$(1, b).f_{\bar{v}} = \bar{\Lambda}(b)^{-1}f_{\bar{v}} \text{ for all } b \in B^-. \quad (2.108)$$

Set $f = f_v f_{\bar{v}}$. Since G is irreducible as an indvariety, the function f is a well-defined nonzero function on G . And of course,

$$(1, b).f = (\Lambda + \bar{\Lambda})(b)^{-1}f \text{ for all } b \in B^-. \quad (2.109)$$

Moreover, we have

$$f(g) = (\psi \otimes \bar{\psi})(g^{-1}(v \otimes \bar{v})). \quad (2.110)$$

Indeed, by definition

$$\begin{aligned} f(g) &= f_v(g) f_{\bar{v}}(g) \\ &= \psi(g^{-1}(v)) \bar{\psi}(g^{-1}(\bar{v})) \\ &= (\psi \otimes \bar{\psi})(g^{-1}(v) \otimes g^{-1}(\bar{v})) \\ &= (\psi \otimes \bar{\psi})(g^{-1}(v \otimes \bar{v})). \end{aligned}$$

Now, $\psi \otimes \bar{\psi}$ is an element of

$$L(\Lambda)^\vee \otimes L(\bar{\Lambda})^\vee = (L(\Lambda) \otimes L(\bar{\Lambda}))^\vee = L(\Lambda + \bar{\Lambda})^\vee \oplus S^\vee. \quad (2.111)$$

By (2.109), (2.110) we have

$$\psi \otimes \bar{\psi} \in L(\Lambda + \bar{\Lambda})^\vee. \quad (2.112)$$

It implies that

$$\ker(\psi \otimes \bar{\psi}) \supset S. \quad (2.113)$$

Rewrite $v \otimes \bar{v} = \pi(v \otimes \bar{v}) + s$ for some $s \in S$. Then we have

$$\begin{aligned} (\psi \otimes \bar{\psi})(g^{-1}(v \otimes \bar{v})) &= (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v}) + s)) \\ &= (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v}))). \end{aligned}$$

It means $f(g) = (\psi \otimes \bar{\psi})(g^{-1}(\pi(v \otimes \bar{v})))$. Since $f \neq 0$, we have $\tilde{v} = \pi(v \otimes \bar{v}) \neq 0$.

Step 3. \tilde{v} satisfies the conditions (2.102). For any $g \in \dot{\mathfrak{n}}_+$ and $h \in \mathfrak{h}$, we have:

$$\begin{aligned} g(\pi(v \otimes \bar{v})) &= \pi(g(v \otimes \bar{v})) \\ &= \pi(g(v) \otimes \bar{v} + v \otimes g(\bar{v})) \\ &= \pi(0) = 0, \\ h(\pi(v \otimes \bar{v})) &= \pi(h(v \otimes \bar{v})) \\ &= \pi(h(v) \otimes \bar{v} + v \otimes h(\bar{v})) \\ &= \pi((\lambda + \bar{\lambda})(h)(v \otimes \bar{v})) \\ &= (\lambda + \bar{\lambda})(h)\pi(v \otimes \bar{v}). \end{aligned}$$

We conclude that the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is a semigroup. \square

2.4.6 The descripton of the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$

Let $(\Lambda, \lambda) \in P_+ \times \dot{P}_+$. The description of the set $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is equivalent to problem of describing the set

$$\mathcal{B}(\Lambda, \lambda) = \{b \in \mathbb{C} \mid \dot{L}(\lambda + b\delta) \subset L(\Lambda)\}. \quad (2.114)$$

We may assume that $\Lambda \in \overline{P}_+$ and $\lambda \in \overline{\dot{P}}_+$. The reason is that $\dot{L}(\lambda + b_1\delta) \subset L(\Lambda + b_2\delta)$ if and only if $\dot{L}(\lambda + (b_1 - b_2)\delta) \subset L(\Lambda)$. We have

$$\mathcal{B}(\Lambda + b_2\delta, \lambda + b_1\delta) = \mathcal{B}(\Lambda, \lambda) + b_2 - b_1. \quad (2.115)$$

Since the set $\mathcal{B}(\Lambda, \lambda)$ is nonempty if and only if $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$, we just need to suppose that $\lambda \in \overline{P}_u(\Lambda)$. Let $m \in \mathbb{Z}_{>0}$ be the level of Λ , we set

$$c_m = \frac{rm \dim \bar{\mathfrak{g}}}{m + \mathfrak{h}^\vee} \text{ and } c_m^{[u]} = uc_m - c_{um}, \quad (2.116)$$

$$m_\Lambda = \frac{|\Lambda + \rho|^2}{2(m + \mathfrak{h}^\vee)} - \frac{|\rho|^2}{2\mathfrak{h}^\vee} \text{ and } \dot{m}_\lambda = \frac{|\lambda + \dot{\rho}|^2}{2(um + \mathfrak{h}^\vee)} - \frac{|\dot{\rho}|^2}{2\mathfrak{h}^\vee}, \quad (2.117)$$

$$h_{\Lambda, \lambda}^{[u]} = u^{-1}m_\Lambda - \dot{m}_\lambda + c_m^{[u]}/24 - b_{\Lambda, \lambda, u}. \quad (2.118)$$

The following theorem gives us a description of the set $\mathcal{B}(\Lambda, \lambda)$.

Theorem 2.31. *Let \mathfrak{g} be an affine Kac-Moody algebra of type $X_N^{(r)}$. Fix $m \in \mathbb{Z}_{>0}$, $u \in \mathbb{Z}_{>1}$ such that $u \equiv 1 \pmod{r}$. Let $\Lambda \in P_+^m$ and $\lambda \in P_u(\Lambda) + \mathbb{C}\delta$. We have*

1. $b_{\Lambda, \lambda, u} - (\mathbb{Z}_{\geq 0} \setminus \{1\}) \subset \mathcal{B}(\Lambda, \lambda) \subset b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.

2. If in addition $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $\mathcal{B}(\Lambda, \lambda) = b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$.

Proof. The inclusion $\mathcal{B}(\Lambda, \lambda) \subset b_{\Lambda, \lambda, u} - \mathbb{Z}_{\geq 0}$ is trivial by definitions of $\mathcal{B}(\Lambda, \lambda)$, $b_{\Lambda, \lambda, u}$ and by the fact that $P_u(\Lambda) \subset P(\Lambda)$. By (2.115), we may assume that $\Lambda \in \overline{P_+^m}$ and $\lambda \in \overline{P_u(\Lambda)}$. We have the decomposition of $L(\Lambda)$ with respect to the direct sum of $\mathfrak{g}^{[u]} \oplus Vir$ -modules

$$L(\Lambda) = \bigoplus_{\lambda \in \overline{P_u(\Lambda)}} \dot{L}(\lambda) \otimes \mathcal{U}(\Lambda, \lambda). \quad (2.119)$$

Here the Virasoro operators are $L_n^{[u]}$ and $Z^{[u]}$. The operator $Z^{[u]}$ acts on $\mathcal{U}(\Lambda, \lambda)$ as the multiplication by non-zero scalar $c_m^{[u]}$. The lowest eigenvalue with respect to $L_0^{[u]}$ is $h_{\Lambda, \lambda}^{[u]}$ (see [KW90]). By [KW88], [KW90], we have

$$ch_{\Lambda} = \sum_{\lambda \in \overline{P_u(\Lambda)}} \dot{c}h_{\lambda} q^{m_{\lambda} - u^{-1} m_{\Lambda} tr_{\mathcal{U}(\Lambda, \lambda)}(q^{L_0^{[u]} - Z^{[u]}/24})}. \quad (2.120)$$

The representation of Vir on $\mathcal{U}(\Lambda, \lambda)$ is unitrizable. We rewrite (2.120) as

$$ch_{\Lambda} = \sum_{\lambda \in \overline{P_u(\Lambda)}} \dot{c}h_{\lambda} q^{m_{\lambda} - u^{-1} m_{\Lambda} - c_m^{[u]}/24 + h_{\Lambda, \lambda}^{[u]}} (\dim \mathcal{U}(\Lambda, \lambda)_{\lambda'} + \dim \mathcal{U}(\Lambda, \lambda)_{\lambda' + (L_0^{[u]})^* q} + \dots) \quad (2.121)$$

$$= \sum_{\lambda \in \overline{P_u(\Lambda)}} \dot{c}h_{\lambda} q^{-b_{\Lambda, \lambda, u}} (\dim \mathcal{U}(\Lambda, \lambda)_{\lambda'} + \dim \mathcal{U}(\Lambda, \lambda)_{\lambda' + (L_0^{[u]})^* q} + \dots) \quad (2.122)$$

$$= \sum_{\lambda \in \overline{P_u(\Lambda)}} \dot{c}h_{\lambda + b_{\Lambda, \lambda, u} \delta} (\dim \mathcal{U}(\Lambda, \lambda)_{\lambda'} + \dim \mathcal{U}(\Lambda, \lambda)_{\lambda' + (L_0^{[u]})^* q} + \dots). \quad (2.123)$$

Suppose that V is the highest weight Vir -module of highest weight $\lambda' \in Vir_0^*$ with $\lambda'(Z^{[u]}) = c_m^{[u]}$ and $\lambda'(L_0^{[u]}) = h_{\Lambda, \lambda}^{[u]}$, which is contained in $\mathcal{U}(\Lambda, \lambda)$. By Lemma 2.10, we have

- If $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $0 \neq \dim V_{\lambda' + n(L_0^{[u]})^*} \leq \dim \mathcal{U}(\Lambda, \lambda)_{\lambda' + n(L_0^{[u]})^*}$ for all $n \in \mathbb{Z}_{\geq 0}$. It implies $\dot{L}(\lambda + (b_{\Lambda, \lambda, u} - n)\delta) \subset L(\Lambda)$ for all $n \in \mathbb{Z}_{\geq 0}$.
- If $h_{\Lambda, \lambda}^{[u]} = 0$ then $0 \neq \dim V_{\lambda' + n(L_0^{[u]})^*} \leq \dim \mathcal{U}(\Lambda, \lambda)_{\lambda' + n(L_0^{[u]})^*}$ for all $n \in \mathbb{Z}_{\geq 0} \setminus \{1\}$. It implies $\dot{L}(\lambda + (b_{\Lambda, \lambda, u} - n)\delta) \subset L(\Lambda)$ for all $n \in \mathbb{Z}_{\geq 0} \setminus \{1\}$.

We have proven the theorem. □

2.5 The cases $A_1^{(1)}$ and $A_2^{(2)}$

In this part, we study two particular cases $A_1^{(1)}$ and $A_2^{(2)}$. We compute explicitly the number $b_{\Lambda, \lambda}$ and show the conditions where we know that $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$. Then we obtain relations between the support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ and its saturated setting.

2.5.0.1 Computation of $b_{\Lambda, \lambda}$ for the case $A_1^{(1)}$

We are going to compute explicitly the set $\max(\Lambda)$, hence the number $b_{\Lambda, \lambda}$ for the cases $A_1^{(1)}$. By (2.37), we may assume that $\Lambda \in \overline{P}_+$. The idea of computations is based on the work on S. Kumar and M. Brown in [BK14].

Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_1^{(1)}$. Fix $m \in \mathbb{Z}_{>0}$. Let α be the simple root α_1 of \mathfrak{g} . We have

$$\overline{P}_+^m = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \mid j \in [0, m] \cap \mathbb{Z} \right\}. \quad (2.124)$$

We can describe explicitly the set $\max(\Lambda)$ and the number $b_{\Lambda, \lambda}$ as below. It is the combination of Lemma 5.2 and Lemma 5.3 in [BK14]. We restate and re-explain in details the results.

Proposition 2.32. *Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}_+^m$. For each $k \in \mathbb{Z}$, let $\phi(m, j, k)$ be a number which is uniquely determined by k, m, j as follows*

1. Write $k = mq + r$ for some $q \in \mathbb{Z}, r \in [0, m)$.

2. Set

$$\phi(m, j, k) = -q(k + r + j) + \begin{cases} -r & \text{if } r \in [0, m - j], \\ m - j - 2r & \text{if } r \in [m - j, m). \end{cases} \quad (2.125)$$

Then we have

$$\max(\Lambda) = \{ \Lambda + k\alpha + \phi(m, j, k)\delta \mid k \in \mathbb{Z} \}. \quad (2.126)$$

Or equivalently, for each $\lambda = m\Lambda_0 + \frac{j'\alpha}{2}$ with $j' \in j + 2\mathbb{Z}$, we have

$$b_{\Lambda, \lambda} = \phi(m, j, \frac{j' - j}{2}). \quad (2.127)$$

In order to prove above proposition, we need the following lemma. In fact, in Proposition 4.4 of [BK14], the authors give the same statement for only untwisted affine Kac-Moody algebras. But the statement and arguments in the proof still work well for any affine Kac-Moody algebras.

Lemma 2.33. *For any affine Kac-Moody algebra, let $\Lambda \in P_+$ of positive level, then*

$$\max(\Lambda) \cap P_+ = \left\{ \Lambda - \sum_{i \in I} m_i \alpha_i \mid m_i \in \mathbb{Z}_{\geq 0} \text{ for all } i, m_i < a_i \text{ for some } i \in I \right\} \cap P_+. \quad (2.128)$$

With the aid of Lemma 2.33, we can prove Proposition 2.32 as follows.

Proof. We have $\max(\Lambda) = W(\max(\Lambda) \cap P_+)$. By Lemma 2.33,

$$\max(\Lambda) \cap P_+ = \left\{ \Lambda - m_0(\delta - \alpha), \Lambda - m_1\alpha \mid m_i \in \mathbb{Z}_{\geq 0}, m_0 \leq \frac{m-j}{2}, m_1 \leq \frac{j}{2} \right\}. \quad (2.129)$$

Recall that $W = \{t_{n\alpha}, t_{n\alpha}s_1 \mid n \in \mathbb{Z}\}$. We have

$$t_{n\alpha}(\Lambda - m_0(\delta - \alpha)) = \Lambda + (m_0 + mn)\alpha - ((j + 2m_0 + mn)n + m_0)\delta, \quad (2.130)$$

$$t_{n\alpha}s_1(\Lambda - m_0(\delta - \alpha)) = \Lambda + (-j - m_0 + mn)\alpha - ((-j - 2m_0 + mn)n + m_0)\delta, \quad (2.131)$$

$$t_{n\alpha}(\Lambda - m_1\alpha) = \Lambda + (-m_1 + mn)\alpha - (j - 2m_1 + mn)n\delta, \quad (2.132)$$

$$t_{n\alpha}s_1(\Lambda - m_1\alpha) = \Lambda + (-j + m_1 + mn)\alpha - (-j + 2m_1 + mn)n\delta. \quad (2.133)$$

So an element λ in $\max(\Lambda)$ has form $\Lambda + r\alpha + n'\delta$ for some $n' \in \mathbb{Z}$, $r \in \mathbb{Z}$. Fix such a λ . Then, for any $q \in \mathbb{Z}$

$$t_{q\alpha}(\Lambda + r\alpha + n'\delta) = \Lambda + (mq + r)\alpha + (n' - (j + 2r + mq)q)\delta \quad (2.134)$$

is still in $\max(\Lambda)$. Set $k = mq + r$ for some $q \in \mathbb{Z}$, then

$$\Lambda + k\alpha + n''\delta \in \max(\Lambda) \quad \text{where } n'' = n' - q(k + r + j). \quad (2.135)$$

Assume now that $0 \leq r < m$, then the expression $k = mq + r$ is the Euclidean division. By (2.130), (2.131), (2.132), (2.133) we get

$$n' = \begin{cases} -r & \text{if } r \in [0, m - j], \\ m - j - 2r & \text{if } r \in [m - j, m). \end{cases} \quad (2.136)$$

Hence we obtain $n'' = \phi(m, j, k)$ as in (2.125). It means

$$\max(\Lambda) = \{\Lambda + k\alpha + \phi(m, j, k)\delta \mid k \in \mathbb{Z}\}. \quad (2.137)$$

□

2.5.0.2 The conditions for $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$ in the case $A_1^{(1)}$

We have seen in the previous part, we know explicetely the number $b_{\Lambda, \lambda}$ in the case $A_1^{(1)}$. By Theorem 2.31, the number $b_{\Lambda, \lambda, u}$ is the key to understand the support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$. The concrete formula for the number $b_{\Lambda, \lambda, u}$ is not known in general, even for the case $A_1^{(1)}$. However, in this part, we will give some conditions where we know that $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

Theorem 2.34. *Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_1^{(1)}$. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}_+^m$ and let $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \overline{P}_+$.*

1. *If there exists $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta)$ belongs to $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then $j' - j \in 2\mathbb{Z}$ and $m' = m$.*

2. *If moreover*

- a. *$j \leq j' \leq um - j$ and u is even; or*
- b. *$j \leq j' \leq um - (m - j)$ and u is odd,*

then $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

For Λ like in Theorem 2.34, we denote by $\mathcal{A}_u(\Lambda)$ the set of λ such that the theorem applies. Namely

$$\mathcal{A}_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in [j, um - j^*] \cap (j + 2\mathbb{Z}) \right\} \quad (2.138)$$

where $j^* = j$ if u is even and $j^* = m - j$ if u is odd.

Before going to proof of the theorem, we need to state some propositions.

Proposition 2.35. *Let \mathfrak{g} be affine Kac-Moody algebra of type $A_1^{(1)}$. Fix $m \in \mathbb{Z}_{>0}$, $u \in \mathbb{Z}_{>1}$. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}_+^m$.*

1. We parametrize $\lambda \in \max(\Lambda)$ such that $\lambda + \dot{\rho}$ is regular with respect to \dot{W} by $\lambda_k = \Lambda + k\alpha + \phi(m, j, k)\delta$. Then the only possible values of k are

$$k = \frac{j' - j}{2} - n(um + 2) \text{ and } k = -\frac{j' + j}{2} - 1 + n(um + 2) \quad (2.139)$$

where $j' \in [0, um] \cap (j + 2\mathbb{Z})$ and $n \in \mathbb{Z}$.

2. Let

$$N_k = -\phi(m, j, k) + un(j' + 1 - num - 2n). \quad (2.140)$$

Then

- i. If $k = \frac{j' - j}{2} - n(um + 2)$, then $p(\lambda_k) = 1$ and $\{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta$.
- ii. If $k = -\frac{j' + j}{2} - 1 + n(um + 2)$, then $p(\lambda_k) = -1$ and $\{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta$.
- iii. The function N_k is considered as a function on n and it attains the minimum at $n = 0$ in the first case and at $n = 0$ or $n = 1$ in the second case.

Proof. We need the following data

$$|\alpha|^2 = 2, \quad \dot{\rho} = \frac{2}{u}\Lambda_0 + \frac{1}{2}\alpha, \quad \dot{W} = \{t_{un\alpha}, t_{un\alpha}s_1 \mid n \in \mathbb{Z}\}. \quad (2.141)$$

Since $\lambda_k + \dot{\rho}$ is regular with respect to \dot{W} , there exists unique $\sigma \in \dot{W}$ and $\mu = m\Lambda_0 + \frac{j'\alpha}{2} + b'\delta \in \dot{P}_+^{um}$ such that $\sigma(\lambda_k + \dot{\rho}) = \mu + \dot{\rho}$.

- a. (Proof of 1. and 2.i.) If $\sigma = t_{un\alpha}$ for some $n \in \mathbb{Z}$, then $\sigma(\lambda_k + \dot{\rho}) - \dot{\rho}$ equals

$$m\Lambda_0 + \left(num + 2n + k + \frac{j}{2} \right) \alpha + (\phi(m, j, k) - un(2k + j + 1 + num + 2n))\delta. \quad (2.142)$$

Hence

$$j' \in [0, um] \cap (j + 2\mathbb{Z}) \text{ and } k = \frac{j' - j}{2} - n(um + 2). \quad (2.143)$$

In this case, we have

$$p(\lambda_k) = 1 \text{ and } \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} + (\phi(m, j, k) - un(j' + 1 - num - 2n))\delta. \quad (2.144)$$

- b. (Proof of 1. and 2.ii.) If $\sigma = t_{un\alpha}s_1$ for some $n \in \mathbb{Z}$, then $\sigma(\lambda_k + \dot{\rho}) - \dot{\rho}$ equals

$$m\Lambda_0 + \left(num + 2n - k - \frac{j}{2} - 1 \right) \alpha + (\phi(m, j, k) - un(-2k - j - 1 + num + 2n))\delta. \quad (2.145)$$

Hence

$$j' \in [0, um] \cap (j + 2\mathbb{Z}) \text{ and } k = -\frac{j' + j}{2} - 1 + n(um + 2). \quad (2.146)$$

In this case, we have

$$p(\lambda_k) = -1 \text{ and } \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} + (\phi(m, j, k) - un(j' + 1 - num - 2n))\delta. \quad (2.147)$$

- c. (Proof of 2.iii.) Put $M = um + 2$. We consider the first case when $k = \frac{j'-j}{2} - nM$. Write $k = qm + r$ for some $q \in \mathbb{Z}, 0 \leq r < m$, then

$$-N_k = \phi(m, j, r) - q(k + r + j) - un(j' + 1 - nM) \quad (2.148)$$

$$= \phi(m, j, r) - \frac{(\frac{j'-j}{2} - nM - r)(\frac{j+j'}{2} - nM + r)}{m} - un(j' + 1 - nM) \quad (2.149)$$

$$= n^2 M \left(u - \frac{M}{m} \right) - n \left(u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{4m} + \left(\frac{r^2}{m} + \frac{rj}{m} + \phi(m, j, r) \right). \quad (2.150)$$

We have

$$\frac{r^2}{m} + \frac{rj}{m} + \phi(m, j, r) = \begin{cases} \frac{1}{m} r(r + j - m) & \text{if } 0 \leq r \leq m - j, \\ \frac{1}{m} (r - m)(r + j - m) & \text{if } m - j \leq r < m. \end{cases} \quad (2.151)$$

The condition $0 \leq r \leq m - j$ can be rewritten as

$$\frac{j - m}{2} \leq -nM + \frac{j'}{2} - m \left(q + \frac{1}{2} \right) \leq \frac{m - j}{2} \quad (2.152)$$

and $m - j \leq r < m$ can be rewritten as

$$-\frac{j}{2} \leq -nM + \frac{j'}{2} - m(q + 1) < \frac{j}{2}. \quad (2.153)$$

It implies that $\frac{r^2}{m} + \frac{rj}{m} + \phi(m, j, r)$ equals

$$\begin{cases} \frac{|-nM + \frac{j'}{2} - \frac{m}{2}p|^2 - \frac{(m-j)^2}{4}}{m} & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that} \\ & | -nM + \frac{j'}{2} - \frac{m}{2}p | \leq \frac{m-j}{2}, \\ \frac{|-nM + \frac{j'}{2} - \frac{m}{2}p|^2 - \frac{j^2}{4}}{m} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that} \\ & | -nM + \frac{j'}{2} - \frac{m}{2}p | \leq \frac{j}{2}. \end{cases} \quad (2.154)$$

Let $P_{j,j'} : \mathbb{Z} \rightarrow \mathbb{R}$ be the function that maps $-n$ to (2.154). Let $F_{j,j'} : \mathbb{Z} \rightarrow \mathbb{R}$ be the function defined by

$$F_{j,j'}(t) = t^2 M \left(u - \frac{M}{m} \right) + t \left(u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{4m} + P_{j,j'}(t). \quad (2.155)$$

So $-N_k = F_{j,j'}(-n)$.

We will show that the maximum of $F_{j,j'}(-n)$ appears when $n = 0$, i.e., $k = \frac{j'-j}{2}$. Let $P : \mathbb{R} \times [0, m] \times [0, um] \rightarrow \mathbb{R}$ be the function defined by $P(t, j, j') = P_{j,j'}(t)$. Let $F : \mathbb{R} \times [0, m] \times [0, um] \rightarrow \mathbb{R}$ be the function defined by $F(t, j, j') = F_{j,j'}(t)$. The function F is a continuous, piecewise smooth function. Set $\Delta(t, j, j') = F(t + 1, j, j') - F(t, j, j')$. We will prove that it is nonincreasing in t and $\Delta(-1, j, j') > 0 > \Delta(0, j, j')$. Indeed:

$$\Delta(t, j, j') = 2tM \left(u - \frac{M}{m} \right) + (M + j') \left(u - \frac{M}{m} \right) + u + P(t + 1, j, j') - P(t, j, j'). \quad (2.156)$$

We denote the numbers p defined on $P_{j,j'}(t+1)$ and $P_{j,j'}(t)$ by p_1, p_0 , respectively. By definition, we have $\frac{p_1-p_0}{2} \geq u$. Hence the partial derivatives of Δ where they exist are

$$\partial_t \Delta(t, j, j') = 2M \left(u - \frac{p_1 - p_0}{2} \right) \leq 0, \quad (2.157)$$

$$\partial_{j'} \Delta(t, j, j') = u - \frac{p_1 - p_0}{2} \leq 0. \quad (2.158)$$

It implies that Δ is nonincreasing in t and j' . So $\Delta(0, j, j') \leq \Delta(0, j, 0)$ and $\Delta(-1, j, um) \leq \Delta(-1, j, j')$. We can easily check that $\Delta(0, j, 0) < 0 < \Delta(-1, j, um)$.

It implies that $F(0, j, j') > F(t, j, j')$ for any $t \in \mathbb{Z}, t \neq 0$, i.e., $F_{i,j}(-n)$ attains its maximum when $n = 0$. Hence, in the case $k = \frac{j'-j}{2} - nM$, the minimum of N_k occurs when $n = 0$.

For the case $k = -\frac{j+j'}{2} - 1 + nM$. Since $k = \frac{j'-j}{2} + \left(n - \frac{j'+1}{M}\right)M$, we have

$$-N_k = F\left(n - \frac{j'+1}{M}, j, j'\right). \quad (2.159)$$

Then N_k attains its minimum when $n = 0$ or 1 .

□

Here is the next proposition we will use in the proof of Theorems 2.34.

Proposition 2.36. *With $\phi(m, j, k)$ is defined as in (2.125), for each $j \in [0, m]$ and $j' \in [0, um] \cap (j + 2\mathbb{Z})$, we have*

$$-\phi\left(m, j, \frac{j'-j}{2}\right) = \min\left(-\phi\left(m, j, -\frac{j'+j}{2} - 1\right), u - \phi\left(m, j, -\frac{j'+j}{2} + 1\right)\right) \quad (2.160)$$

if and only if one of the following three conditions below is satisfied:

(A1) $m > 1$ and $j' \leq j - 2$.

(B1) $m > 1, u$ is even, $j' \geq um - j + 1$.

(C1) $m > 1, u$ is odd, $j' \geq m(u-1) + j + 2$.

Proof. We use a fact that

$$\phi(m, j, -(j+k)) = \phi(m, j, k). \quad (2.161)$$

Indeed, if $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} + b\delta \in P_+^m$ and $\lambda = \Lambda + k\alpha + \phi(m, j, k)\delta \in \max(\Lambda)$, then $s_1(\lambda) = \Lambda - (j+k)\alpha + \phi(m, j, k)\delta \in \max(\Lambda)$. We use the equality (2.161) to rewrite

$$\phi\left(m, j, -\frac{j'+j}{2} - 1\right) = \phi\left(m, j, \frac{j'-j}{2}\right). \quad (2.162)$$

as $\phi(m, j, x) = \phi(m, j, x+1)$, where $x = -\frac{j'+j}{2} - 1$. Use (2.125) for $\phi(m, j, x)$ we check that it happens if and only if (A1) happens. Similarly, use (2.161) to rewrite

$$-u + \phi\left(m, j, -\frac{j'+j}{2} + 1\right) \leq \phi\left(m, j, \frac{j'-j}{2}\right) \quad (2.163)$$

as $\phi(m, j, x+1) - u \leq \phi(m, j, x)$, where $x = -\frac{j'+j}{2}$. Use (2.125) for $\phi(m, j, x)$ we can check that it happens if and only if (B1) or (C1) happens. □

We are now going to the proof of Theorem 2.34.

Proof. The first step is writing explicitly ch_Λ in Proposition 2.29. It can be done by substituting values of $p(\lambda_k)$ and $\{\lambda_k\}$ in Proposition 2.35 to the formula (2.101). We can rewrite ch_Λ as follows

$$\sum_{j' \in [0, um] \cap (j+2\mathbb{Z})} \dot{c}h_{m\Lambda_0 + \frac{j'\alpha}{2}} \left(\sum_{\substack{n \in \mathbb{Z}, \\ k = \frac{j'-j}{2} - n(um+2)}} q^{N_k} c_{\lambda_k}^\Lambda - \sum_{\substack{n \in \mathbb{Z}, \\ k = -\frac{j'+j}{2} - 1 + n(um+2)}} q^{N_k} c_{\lambda_k}^\Lambda \right). \quad (2.164)$$

The identity (2.164) implies the condition $m' = m$ and $j' \in [0, um] \cap (j+2\mathbb{Z})$ that we need to prove for Theorem 2.34. The coefficients of $c_{\lambda_k}^\Lambda$ in the formula (2.164) are always positive integers since $\lambda_k \in \max(\Lambda)$. Proposition 2.35 says that N_k attains minimums at $n = 0$ for those on the left of (2.164), and at $n = 0$ or $n = 1$ for those on the right of (2.164). The corresponding minimums of N_k are

$$-\phi(m, j, \frac{j'-j}{2}) \text{ and } \min \left(-\phi(m, j, -\frac{j'+j}{2} - 1), u - \phi(m, j, -\frac{j'+j}{2} + 1) \right). \quad (2.165)$$

By (2.127), (2.56), (2.164), (2.165), we can imply that

$$-\phi(m, j, \frac{j'-j}{2}) \leq \min \left(-\phi(m, j, -\frac{j'+j}{2} - 1), u - \phi(m, j, -\frac{j'+j}{2} + 1) \right). \quad (2.166)$$

Moreover, the equality in (2.166) happens if and only if one of the three conditions (A1), (B1), (C1) in Proposition 2.36 is satisfied. So, for any $\lambda = m\Lambda_0 + \frac{j'\alpha}{2} \in \mathcal{A}_u(\Lambda)$, we have strict inequality in (2.166). By (2.164), in this case we have $\lambda + \phi(m, j, \frac{j'-j}{2})\delta \in \max_u(\Lambda)$. That means $b_{\Lambda, \lambda, u} = \phi(m, j, \frac{j'-j}{2}) = b_{\Lambda, \lambda}$. \square

2.5.0.3 Computation of $b_{\Lambda, \lambda}$ for the case $A_2^{(2)}$

For the case $A_2^{(2)}$, the computation is similar as we did for the case $A_1^{(1)}$. Let Λ_0 be the 0-th fundamental weight and α be the simple root α_1 of \mathfrak{g} . Fix $m \in \mathbb{Z}_{>0}$, we

$$\overline{P}_+^m = \left\{ m\Lambda_0 + \frac{j\alpha}{2} \mid j \in \left[0, \frac{m}{2}\right] \cap \mathbb{Z} \right\}. \quad (2.167)$$

We can describe explicitly the set $\max(\Lambda)$ and the number $b_{\Lambda, \lambda}$ for the case $A_2^{(2)}$ as follows.

Proposition 2.37. *Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}_+^m$. For each $k \in \frac{1}{2}\mathbb{Z}$, let $\phi(m, j, k)$ be a number which is uniquely determined by k, m, j as follows*

1. Write $k = \frac{m}{2}q + r$ for some $q \in \mathbb{Z}, r \in [0, \frac{m}{2})$.

2. Set

$$\phi(m, j, k) = -q(k+r+j) + \begin{cases} -r & \text{if } r \in [0, \frac{m}{2} - j], \\ \frac{m}{2} - j - 2r & \text{if } r \in [\frac{m}{2} - j, \frac{m}{2}) \cap (\frac{m}{2} + \mathbb{Z}), \\ \frac{m-1}{2} - j - 2r & \text{if } r \in [\frac{m-1}{2} - j, \frac{m}{2}) \cap (\frac{m-1}{2} + \mathbb{Z}). \end{cases} \quad (2.168)$$

Then we have

$$\max(\Lambda) = \left\{ \Lambda + k\alpha + \phi(m, j, k)\delta \mid k \in \frac{1}{2}\mathbb{Z} \right\}. \quad (2.169)$$

Or equivalently, for each $\lambda = m\Lambda_0 + \frac{j'\alpha}{2}$ with $j \in \mathbb{Z}$, we have

$$b_{\Lambda, \lambda} = \phi(m, j, \frac{j' - j}{2}). \quad (2.170)$$

Proof. We have $\max(\Lambda) = W(\max(\Lambda) \cap P_+)$. By Lemma 2.33, $\max(\Lambda) \cap P_+$ contains exactly elements

$$\Lambda - m_0\alpha_0, \Lambda - m_1\alpha, \Lambda - \alpha_0 - m_2\alpha \quad (2.171)$$

such that

$$m_i \in \mathbb{Z}_{\geq 0}, m_0 \leq \frac{m}{2} - j, m_1 \leq \frac{j}{2}, \frac{j+1}{2} - \frac{m}{4} \leq m_2 \leq \frac{j+1}{2}. \quad (2.172)$$

Recall that $W = \{t_{\frac{n\alpha}{2}}, t_{\frac{n\alpha}{2}}s_1 \mid n \in \mathbb{Z}\}$. We have

$$t_{\frac{n\alpha}{2}}(\Lambda - m_0\alpha_0) = \Lambda + \frac{mn + m_0}{2}\alpha - \frac{(mn + 2j + 2m_0)n + m_0}{2}\delta, \quad (2.173)$$

$$t_{\frac{n\alpha}{2}}s_1(\Lambda - m_0\alpha_0) = \Lambda + \frac{mn - 2j - m_0}{2}\alpha - \frac{(mn - 2j - 2m_0)n + m_0}{2}\delta, \quad (2.174)$$

$$t_{\frac{n\alpha}{2}}(\Lambda - m_1\alpha) = \Lambda + \frac{mn - 2m_1}{2}\alpha - \frac{(mn + 2j - 4m_1)n}{2}\delta, \quad (2.175)$$

$$t_{\frac{n\alpha}{2}}s_1(\Lambda - m_1\alpha) = \Lambda + \frac{mn - 2j + 2m_1}{2}\alpha - \frac{(mn - 2j + 4m_1)n}{2}\delta, \quad (2.176)$$

$$t_{\frac{n\alpha}{2}}(\Lambda - \alpha_0 - m_2\alpha) = \Lambda + \frac{mn + 1 - 2m_2}{2}\alpha - \frac{(mn + 2j + 2 - 4m_2)n + 1}{2}\delta, \quad (2.177)$$

$$t_{\frac{n\alpha}{2}}s_1(\Lambda - \alpha_0 - m_2\alpha) = \Lambda + \frac{mn - 1 - 2j + 2m_2}{2}\alpha - \frac{(mn - 2j - 2 + 4m_2)n + 1}{2}\delta. \quad (2.178)$$

So an element λ in $\max(\Lambda)$ has form $\Lambda + r\alpha + n'\delta$ for some $n' \in \frac{1}{2}\mathbb{Z}$, $r \in \frac{1}{2}\mathbb{Z}$. Fix such a λ . Then for any $q \in \mathbb{Z}$

$$t_{\frac{q\alpha}{2}}(\Lambda + r\alpha + n'\delta) = \Lambda + \left(\frac{m}{2}q + r\right)\alpha + \left(n' - \left(j + 2r + \frac{m}{2}q\right)q\right)\delta \quad (2.179)$$

is still in $\max(\Lambda)$. Set $k = \frac{m}{2}q + r$ for some $q \in \mathbb{Z}$, then

$$\Lambda + k\alpha + n''\delta \in \max(\Lambda) \quad \text{where } n'' = n' - q(k + r + j). \quad (2.180)$$

Assume now that $0 \leq r < \frac{m}{2}$, then the expression $k = \frac{m}{2}q + r$ is the Euclidean division. By (2.173), (2.174), (2.175), (2.176), (2.177), (2.178) we get

$$n' = \begin{cases} -r & \text{if } r \in [0, \frac{m}{2} - j], \\ \frac{m}{2} - j - 2r & \text{if } r \in [\frac{m}{2} - j, \frac{m}{2}) \cap (\frac{m}{2} + \mathbb{Z}), \\ \frac{m-1}{2} - j - 2r & \text{if } r \in [\frac{m-1}{2} - j, \frac{m}{2}) \cap (\frac{m-1}{2} + \mathbb{Z}). \end{cases} \quad (2.181)$$

Hence we obtain $n'' = \phi(m, j, k)$ as in (2.168). It means

$$\max(\Lambda) = \left\{ \Lambda + k\alpha + \phi(m, k, j)\delta \mid k \in \frac{1}{2}\mathbb{Z} \right\}. \quad (2.182)$$

□

2.5.0.4 The conditions for $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$ in the case $A_2^{(2)}$

In this part, as we did for the case $A_1^{(1)}$, we give conditions where we know that $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$. This fact with Theorem 2.31 help us to understand the support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ for the case $A_2^{(2)}$.

Theorem 2.38. *Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_2^{(2)}$. Let $\lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}_+^m$ and let $\lambda = m'\Lambda_0 + \frac{j'\alpha}{2} \in \overline{P}_+$.*

1. *If there exists $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta)$ belongs to $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ then $m' = m$.*

2. *If moreover*

a. *$j \leq j'$; and*

b. *$j' \in \frac{m(u-1)}{2} - j + (2\mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{< 0})$,*

then $b_{\Lambda, \lambda, u} = b_{\Lambda, \lambda}$.

For Λ like in Theorem 2.38, we denote by $\mathcal{A}_u(\Lambda)$ the set of λ such that the theorem applies. Namely

$$\mathcal{A}_u(\Lambda) = \left\{ m\Lambda_0 + \frac{j'\alpha}{2} \mid j' \in \left[j, \frac{um}{2} \right] \cap \mathbb{Z} \cap \left(\frac{m(u-1)}{2} - j + (2\mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{< 0}) \right) \right\}. \quad (2.183)$$

Before going to proof of the theorem, we need to state some propositions.

Proposition 2.39. *Let \mathfrak{g} be the affine Kac-Moody algebra of type $A_2^{(2)}$. Fix $m \in \mathbb{Z}_{> 0}$, $u \in \mathbb{Z}_{> 1}$ relatively prime to 2. Let $\Lambda = m\Lambda_0 + \frac{j\alpha}{2} \in \overline{P}_+^m$.*

1. *We parametrize $\lambda \in \max(\Lambda)$ such that $\lambda + \rho$ is regular with respect to \dot{W} by $\lambda_k = \Lambda + k\alpha + \phi(m, j, k)\delta$. Then the only possible values of k are*

$$k = \frac{j' - j}{2} - n \frac{um + 3}{2} \text{ and } k = -\frac{j' + j}{2} - 1 + n \frac{um + 3}{2} \quad (2.184)$$

where $j' \in [0, \frac{um}{2}] \cap \mathbb{Z}$ and $n \in \mathbb{Z}$.

2. *Let*

$$N_k = -\phi(m, j, k) + un \left(j' + 1 - n \frac{um + 3}{2} \right). \quad (2.185)$$

Then

i. *If $k = \frac{j' - j}{2} - n \frac{um + 3}{2}$, then $p(\lambda_k) = 1$ and $\{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta$.*

ii. *If $k = -\frac{j' + j}{2} - 1 + n \frac{um + 3}{2}$, then $p(\lambda_k) = -1$ and $\{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} - N_k\delta$.*

iii. *The function N_k is considered as a function on n and it attains the minimum at $n = 0$ in the first case and at $n = 0$ or $n = 1$ in the second case.*

Proof. We need the following data

$$|\alpha|^2 = 4, \quad \dot{\rho} = \frac{3}{u}\Lambda_0 + \frac{1}{2}\alpha, \quad \dot{W} = \{t \frac{un\alpha}{2}, t \frac{un\alpha}{2} s_1 \mid n \in \mathbb{Z}\}. \quad (2.186)$$

Since $\lambda_k + \rho$ is regular with respect to \dot{W} , there exists unique $\sigma \in \dot{W}$ and $\mu = m\Lambda_0 + \frac{j'\alpha}{2} + b'\delta \in \dot{P}_+^{um}$ such that $\sigma(\lambda_k + \rho) = \mu + \dot{\rho}$.

a. (Proof of 1. and 2.i.) If $\sigma = t \frac{un\alpha}{2}$ for some $n \in \mathbb{Z}$, then $\sigma(\lambda_k + \rho) - \rho$ equals

$$m\Lambda_0 + \left(k + \frac{j}{2} + n \frac{um+3}{2}\right) \alpha + \left(\phi(m, j, k) - un \left(2k + j + 1 + n \frac{um+3}{2}\right)\right) \delta. \quad (2.187)$$

Hence

$$j' \in \left[0, \frac{um}{2}\right] \cap \mathbb{Z} \text{ and } k = \frac{j' - j}{2} - n \frac{um+3}{2}. \quad (2.188)$$

In this case, we have

$$p(\lambda_k) = 1 \text{ and } \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} + \left(\phi(m, j, k) - un \left(j' + 1 - n \frac{um+3}{2}\right)\right) \delta. \quad (2.189)$$

b. (Proof of 1. and 2.ii.) If $\sigma = t \frac{un\alpha}{2} s_1$ for some $n \in \mathbb{Z}$, then $\sigma(\lambda_k + \rho) - \rho$ equals

$$m\Lambda_0 + \left(-k - \frac{j}{2} - 1 + n \frac{um+3}{2}\right) \alpha + \left(\phi(m, j, k) - un \left(-2k - j - 1 + n \frac{um+3}{2}\right)\right) \delta. \quad (2.190)$$

Hence

$$j' \in \left[0, \frac{um}{2}\right] \cap \mathbb{Z} \text{ and } k = -\frac{j' + j}{2} - 1 + n \frac{um+3}{2}. \quad (2.191)$$

In this case, we have

$$p(\lambda_k) = -1 \text{ and } \{\lambda_k\} = m\Lambda_0 + \frac{j'\alpha}{2} + \left(\phi(m, j, k) - un \left(j' + 1 - n \frac{um+3}{2}\right)\right) \delta. \quad (2.192)$$

c. (Proof of 2.iii.) Put $M = um + 3$. We consider the first case $k = \frac{j'-j}{2} - \frac{nM}{2}$. Write $k = \frac{m}{2}q + r$ for some $q \in \mathbb{Z}, 0 \leq r < \frac{m}{2}$, then

$$-N_k = \phi(m, j, r) - q(k + r + j) - un \left(j' + 1 - \frac{nM}{2}\right) \quad (2.193)$$

$$= \phi(m, j, r) - \frac{2\left(\frac{j'-j}{2} - \frac{nM}{2} - r\right)\left(\frac{j'+j}{2} - \frac{nM}{2} + r\right)}{m} - un \left(j' + 1 - \frac{nM}{2}\right) \quad (2.194)$$

$$= n^2 \frac{M}{2} \left(u - \frac{M}{m}\right) - n \left(u + uj' - \frac{M}{m}j'\right) + \frac{j^2 - j'^2}{2m} + \left(\frac{2r^2}{m} + \frac{2rj}{m} + \phi(m, j, r)\right). \quad (2.195)$$

We have $\frac{2r^2}{m} + \frac{2rj}{m} + \phi(m, j, r)$ equals

$$\begin{cases} \frac{2}{m}r\left(r + j - \frac{m}{2}\right) & \text{if } 0 \leq r \leq \frac{m}{2} - j, \\ \frac{2}{m}\left(r - \frac{m}{2}\right)\left(r + j - \frac{m}{2}\right) & \text{if } \frac{m}{2} - j \leq r < \frac{m}{2}, r \in \frac{m}{2} + \mathbb{Z}, \\ \frac{2}{m}\left(r - \frac{m}{2}\right)\left(r + j - \frac{m}{2}\right) - \frac{1}{2} & \text{if } \frac{m-1}{2} - j \leq r < \frac{m}{2}, r \in \frac{m+1}{2} + \mathbb{Z}. \end{cases} \quad (2.196)$$

The condition $0 \leq r \leq \frac{m}{2} - j$ can be rewritten as

$$\frac{2j - m}{4} \leq \frac{-nM + j'}{2} - \frac{m}{2} \left(q + \frac{1}{2}\right) \leq \frac{m - 2j}{4} \quad (2.197)$$

and $\frac{m}{2} - j \leq r < \frac{m}{2}$ can be rewritten as

$$\frac{-j}{2} \leq \frac{-nM + j'}{2} - \frac{m}{2}(q + 1) < \frac{j}{2}. \quad (2.198)$$

It implies that $\frac{2r^2}{m} + \frac{2rj}{m} + \phi(m, j, r)$ equals

$$\left\{ \begin{array}{ll} \frac{2}{m} (|\frac{-nM+j'}{2} - \frac{m}{4}p|^2 - |\frac{m-2j}{4}|^2) & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that} \\ & |\frac{-nM+j'}{2} - \frac{m}{4}p| \leq \frac{m-2j}{4}, \\ \frac{2}{m} (|\frac{-nM+j'}{2} - \frac{m}{4}p|^2 - \frac{j^2}{4}) & \text{if } \exists p \in 2\mathbb{Z} \text{ such that} \\ & |\frac{-nM+j'}{2} - \frac{m}{4}p| \leq \frac{j}{2}, \\ & \frac{j'-j}{2} - \frac{nM}{2} - \frac{m}{2}(\frac{p}{2} - 1) \in \frac{m}{2} + \mathbb{Z}, \\ \frac{2}{m} (|\frac{-nM+j'}{2} - \frac{m}{4}p|^2 - \frac{j^2}{4}) - \frac{1}{2} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that} \\ & \frac{-nM+j'}{2} - \frac{m}{4}p \in [-\frac{1+j}{2}, \frac{j}{2}], \\ & \frac{j'-j}{2} - \frac{nM}{2} - \frac{m}{2}(\frac{p}{2} - 1) \in \frac{m+1}{2} + \mathbb{Z}. \end{array} \right. \quad (2.199)$$

Let $P_{j,j'} : \mathbb{Z} \rightarrow \mathbb{R}$ be the function that maps $-n$ to (2.199). Let $F_{j,j'} : \mathbb{Z} \rightarrow \mathbb{R}$ be the function defined by

$$F_{j,j'}(t) = t^2 \frac{M}{2} \left(u - \frac{M}{m} \right) + t \left(u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{2m} + P_{j,j'}(t). \quad (2.200)$$

So $-N_k = F_{j,j'}(-n)$.

We will show that the maximum of $F_{j,j'}(n)$ appears when $n = 0$, i.e., $k = \frac{j'-j}{2}$. To do that, we show that the upper bound function $F^+ : \mathbb{R} \times [0, \frac{m}{2}] \times [0, \frac{um}{2}] \rightarrow \mathbb{R}$ of $F_{j,j'}(t)$ and lower bound function $F^- : \mathbb{R} \times [0, \frac{m}{2}] \times [0, \frac{um}{2}] \rightarrow \mathbb{R}$ of $F_{j,j'}(t)$ given below attains their maximum along $t \in \mathbb{Z}$ when $t = 0$.

$$F^+(t, j, j') = t^2 \frac{M}{2} \left(u - \frac{M}{m} \right) + t \left(u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{2m} + P^+(t, j, j'), \quad (2.201)$$

$$F^-(t, j, j') = t^2 \frac{M}{2} \left(u - \frac{M}{m} \right) + t \left(u + uj' - \frac{M}{m} j' \right) + \frac{j^2 - j'^2}{2m} + P^-(t, j, j'), \quad (2.202)$$

where $P^+(t, j, j')$ is

$$\left\{ \begin{array}{ll} \frac{2}{m} (|\frac{tM+j'}{2} - \frac{m}{4}p|^2 - |\frac{m-2j}{4}|^2) & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that} \\ & |\frac{tM+j'}{2} - \frac{m}{4}p| \leq \frac{m-2j}{4}, \\ \frac{2}{m} (|\frac{tM+j'}{2} - \frac{m}{4}p|^2 - \frac{j^2}{4}) & \text{if } \exists p \in 2\mathbb{Z} \text{ such that} \\ & |\frac{tM+j'}{2} - \frac{m}{4}p| \leq \frac{j}{2}. \end{array} \right. \quad (2.203)$$

and $P^-(t, j, j')$ is

$$\left\{ \begin{array}{ll} \frac{2}{m} (|\frac{tM+j'}{2} - \frac{m}{4}p|^2 - |\frac{m-2j}{4}|^2) & \text{if } \exists p \in 2\mathbb{Z} + 1 \text{ such that} \\ & \frac{tM+j'}{2} - \frac{m}{4}p \in [\frac{2j-m}{4}, \frac{m-2j}{4} - \frac{1}{2}], \\ \frac{2}{m} (|\frac{tM+j'}{2} - \frac{m}{4}p|^2 - \frac{j^2}{4}) - \frac{1}{2} & \text{if } \exists p \in 2\mathbb{Z} \text{ such that} \\ & \frac{tM+j'}{2} - \frac{m}{4}p \in [-\frac{1+j}{2}, \frac{j}{2}]. \end{array} \right. \quad (2.204)$$

We just consider the function F^+ and apply similar arguments for F^- . The function F^+ is a continuous, piecewise smooth function. Let $\Delta^+(t, j, j') = F^+(t+1, j, j') - F^+(t, j, j')$, then it is nonincreasing in t and $\Delta^+(-1, j, j') > 0 > \Delta^+(0, j, j')$. Indeed: $\Delta^+(t, j, j')$ equals

$$tM \left(u - \frac{M}{m} \right) + \left(j' + \frac{M}{2} \right) \left(u - \frac{M}{m} \right) + u + P^+(t+1, j, j') - P^+(t, j, j'). \quad (2.205)$$

We denote the numbers p defining $P^+(t+1, j, j')$ and $P^+(t, j, j')$ by p_1, p_0 , respectively. Use definition, we have $\frac{p_1 - p_0}{2} \geq u$. Hence the partial derivatives of Δ^+ , where they exist are

$$\partial_t \Delta^+(t, j, j') = M \left(u - \frac{p_1 - p_0}{2} \right) \leq 0, \quad (2.206)$$

$$\partial_{j'} \Delta^+(t, j, j') = u - \frac{p_1 - p_0}{2} \leq 0. \quad (2.207)$$

It implies that Δ^+ is nonincreasing in t and j' . So $\Delta^+(0, j, j') \leq \Delta^+(0, j, 0)$ and $\Delta^+(-1, j, \frac{um}{2}) \leq \Delta^+(-1, j, j')$. We can check that $\Delta^+(0, j, 0) < 0 < \Delta^+(-1, j, \frac{um}{2})$.

It implies that $F^+(0, j, j') > F^+(t, j, j')$ for any $t \in \mathbb{Z}, t \neq 0$. The same results is true for F^- . Hence $F_{i,j}(-n)$ attains its maximum when $n = 0$. Hence, in the case $k = \frac{j'-j}{2} - \frac{nM}{2}$, the minimum of N_k occurs when $n = 0$.

For the case $k = -\frac{j'+j}{2} - 1 + \frac{nM}{2}$. Since $k = \frac{j'-j}{2} + \left(n - 2\frac{j'+1}{M} \right) \frac{M}{2}$, we have

$$-N_k = F \left(n - 2\frac{j'+1}{M}, j, j' \right). \quad (2.208)$$

Then N_k attains its minimum when $n = 0$ or 1 .

□

Here is the next proposition we will use in the proof of Theorems 2.38.

Proposition 2.40. *With $\phi(m, j, k)$ is defined as in (2.168), for each $j \in [0, \frac{m}{2}]$ and $j' \in [0, \frac{um}{2}] \cap \mathbb{Z}$, we have*

$$-\phi(m, j, \frac{j'-j}{2}) = \min \left(-\phi(m, j, -\frac{j'+j}{2} - 1), \frac{u}{2} - \phi(m, j, -\frac{j'+j}{2} + \frac{1}{2}) \right). \quad (2.209)$$

if and only if one of the following two conditions is satisfied:

(A2) $m > 2$ and $j' \leq j - 1$.

(B2) $m \geq 2, j' \in \frac{m(u-1)}{2} + 1 - j + 2\mathbb{Z}_{\geq 0}$.

Proof. We again use the equality (2.161) to rewrite

$$\phi(m, j, -\frac{j'+j}{2} - 1) = \phi(m, j, \frac{j'-j}{2}) \text{ and } -\frac{u}{2} + \phi(m, j, -\frac{j'+j}{2} + \frac{1}{2}) = \phi(m, j, \frac{j'-j}{2}) \quad (2.210)$$

as $\phi(m, j, x) = \phi(m, j, x-1)$ and $\phi(m, j, x) = -\frac{u}{2} + \phi(m, j, x + \frac{1}{2})$, where $x = -\frac{j'+j}{2}$. Use (2.168), we can check that it happens if and only if condition (A2) or (B2) is satisfied. □

We are now going to the proof of Theorem 2.38.

Proof. The strategy is the same as in the case $A_1^{(1)}$. The first step is writing explicitly ch_Λ in Proposition 2.29. By substituting values of $p(\lambda_k)$ and $\{\lambda_k\}$ in Proposition 2.39 to the formula (2.101), we can rewrite ch_Λ as follows

$$\sum_{j' \in [0, \frac{um}{2}] \cap \mathbb{Z}} \dot{c}h_{m\Lambda_0 + \frac{j'\alpha}{2}} \left(\sum_{\substack{n \in \mathbb{Z}, \\ k = \frac{j'-j}{2} - n \frac{um+3}{2}}} q^{N_k} c_{\lambda_k}^\Lambda - \sum_{\substack{n \in \mathbb{Z}, \\ k = -\frac{j'+j}{2} - 1 + n \frac{um+3}{2}}} q^{N_k} c_{\lambda_k}^\Lambda \right). \quad (2.211)$$

The identity (2.211) implies the condition $j' \in [0, \frac{um}{2}] \cap \mathbb{Z}$ that we need to prove for Theorem 2.38. The coefficients of $c_{\lambda_k}^\Lambda$ in the formula (2.211) are always positive integers since $\lambda_k \in \max(\Lambda)$. Proposition 2.39 says that the number N_k attains minimums at $n = 0$ for those on the left hand side of (2.211), and at $n = 0$ or $n = 1$ for those on the right hand side of (2.211). The corresponding minimums of N_k are

$$-\phi(m, j, \frac{j'-j}{2}) \text{ and } \min \left(-\phi(m, j, -\frac{j'+j}{2} - 1), \frac{u}{2} - \phi(m, j, -\frac{j'+j}{2} + \frac{1}{2}) \right). \quad (2.212)$$

By (2.170), (2.56), (2.211), (2.212), we can imply that

$$-\phi(m, j, \frac{j'-j}{2}) \leq \min \left(-\phi(m, j, -\frac{j'+j}{2} - 1), \frac{u}{2} - \phi(m, j, -\frac{j'+j}{2} + \frac{1}{2}) \right). \quad (2.213)$$

Moreover, the equality happens if and only if condition (A2) or (B2) in Proposition 2.40 is satisfied. So, for any $\lambda = m\Lambda_0 + \frac{j'\alpha}{2} \in \mathcal{A}_u(\Lambda)$, we have strict inequality in (2.213). By (2.211), in this case we have $\lambda + \phi(m, j, \frac{j'-j}{2})\delta \in \max_u(\Lambda)$. That means $b_{\Lambda, \lambda, u} = \phi(m, j, \frac{j'-j}{2}) = b_{\Lambda, \lambda}$. \square

2.5.0.5 Relation between $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ and its saturated setting

The saturated setting of the support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ is defined by

$$\tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) = \{(\Lambda, \lambda) \in P_+ \times \dot{P}_+ \mid \lambda \in \Lambda + Q, \dot{L}(N\lambda) \subset L(N\Lambda) \text{ for some integer } N > 1\}. \quad (2.214)$$

Corollary 2.41. *Let \mathfrak{g} be affine Kac-Moody algebra of type $A_1^{(1)}$ or $A_2^{(2)}$. Fix $m \in \mathbb{Z}_{>0}$, $u \in \mathbb{Z}_{>1}$ (u is an odd number in the case $A_2^{(2)}$). Let $\Lambda \in \overline{P}_+^m$ and let $\lambda \in \mathcal{A}_u(\Lambda) \cap (\Lambda + Q)$. For all $b \in \mathbb{C}$, we have*

1. $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow d(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$ for all $d \in \mathbb{Z}_{\geq 2}$.
2. If in addition $h_{\Lambda, \lambda}^{[u]} \neq 0$ then $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u]) \Leftrightarrow (\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$.

Before going to the proof, we first restate Lemma 6.3 and Lemma 8.5 in [BK14].

Lemma 2.42. *Let \mathfrak{g} be an affine Kac-Moody algebra of type $A_1^{(1)}$ or $A_2^{(2)}$. Let $\Lambda \in P_+$ and $\lambda \in (\Lambda + Q)$. Fix a positive integer N . Then $\lambda \in \max(\Lambda)$ if and only if $N\lambda \in \max(N\Lambda)$.*

Now we are going to the proof of Corollary 2.41.

Proof. Fix a positive integer d . Since $\lambda \in \mathcal{A}_u(\Lambda) \cap (\Lambda + Q)$, we have $d\lambda \in \mathcal{A}_u(d\Lambda) \cap (d\Lambda + Q)$. By Theorems 2.34, 2.38 and Lemma 2.42, we have

$$b_{d\Lambda, d\lambda, u} = b_{d\Lambda, d\lambda} = db_{\Lambda, \lambda}. \quad (2.215)$$

Suppose that $(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u])$, then $\lambda + b\delta \in \Lambda + Q$ and $\dot{L}(N\lambda + Nb\delta) \subset L(N\Lambda)$ for some integer $N > 1$. Hence $b \in \mathbb{Z}$ (since $\lambda \in \Lambda + Q$) and $Nb \leq b_{N\Lambda, N\lambda, u} = Nb_{\Lambda, \lambda}$ for some $N > 1$ (by (2.215)). So $b \leq b_{\Lambda, \lambda}$ and they are integers. Hence for any $d \in \mathbb{Z}_{\geq 2}$, we have $b_{d\Lambda, d\lambda} - db = d(b_{\Lambda, \lambda} - b) \in \mathbb{Z}_{\geq 0} \setminus \{1\}$. Then by Theorem 2.31 1., $d(\Lambda, \lambda + b\delta) \in \tilde{\Gamma}(\mathfrak{g}, \mathfrak{g}[u])$. If $h_{\Lambda, \lambda}^{[u]} \neq 0$, then by Theorems 2.31 2., $(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. \square

Remark 2.43. In the case $A_1^{(1)}$, we have $\mathcal{A}_u(\Lambda) \cap (\Lambda + Q) = \mathcal{A}_u(\Lambda)$. In the case $A_2^{(2)}$, we have $\mathcal{A}_u(\Lambda) \cap (\Lambda + Q)$ is the subset of elements in $\mathcal{A}_u(\Lambda)$ with strict condition $j' \in j + 2\mathbb{Z}$ compare to (2.183).

Chapter 3

Projective representations of symmetric groups

In this chapter, we present the projective representation theory of the symmetric groups. It explains a motivation for studying the shifted Littlewood-Richardson coefficients in our second preprint [\[Kha20\]](#).

Contents

3.1	Shifted Littlewood-Richardson coefficients	54
3.1.1	Young tableaux	54
3.1.2	Ring of symmetric functions	54
3.1.3	Shifted tableaux	55
3.1.4	Shifted Littlewood-Richardson coefficients	56
3.2	Projective representations of symmetric groups	56
3.2.1	Projective representations of S_n	57
3.2.2	Conjugacy classes	58
3.2.2.1	Conjugacy classes of \widetilde{S}_n	58
3.2.2.2	Conjugacy classes of parabolic subgroups of \widetilde{S}_n	58
3.2.3	The space of class functions of \widetilde{S}_n	59
3.2.3.1	Basic spin characters of \widetilde{S}_n	60
3.2.3.2	Spin characters of parabolic subgroups of \widetilde{S}_n	61
3.2.3.3	Irreducible spin characters of \widetilde{S}_n	62
3.2.4	Relation with shifted Littlewood-Richardson coefficients	63

3.1 Shifted Littlewood-Richardson coefficients

3.1.1 Young tableaux

For a **partition** λ , we write $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with $\lambda_l > 0$ and $\sum_{i=1}^l \lambda_i = n$, we write $l(\lambda) = l$, $|\lambda| = n$.

Each partition λ is presented by a **Young diagram** $Y(\lambda)$ that is a collection of boxes such that:

- (D1) The leftmost boxes of each row are in a column.
- (D2) The number of boxes from top row to bottom row are $\lambda_1, \lambda_2, \dots$, respectively.

A **semistandard Young tableau** of shape λ is a filling of the Young diagram $Y(\lambda)$ by the ordered alphabet $\{1 < 2 < \dots\}$ such that:

- (Y1) The entries in each column are strictly increasing.
- (Y2) The entries in each row are weakly increasing.

A Young tableau T is said to have **content** $\gamma = (\gamma_1, \gamma_2, \dots)$ if γ_i is the number of entries i in the tableau T . We write

$$x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \dots \quad (3.1)$$

3.1.2 Ring of symmetric functions

For each partition λ , the **Schur function** s_λ in variables x_1, x_2, \dots is defined as the sum of x^T , where T runs over the semistandard Young tableaux of shape λ .

The **power-sum symmetric function** p_r with $r \geq 1$ is defined by

$$p_r = x_1^r + x_2^r + \dots \quad (3.2)$$

For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \tag{3.3}$$

Let $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$ be the **graded ring of symmetric functions** in the variables x_1, x_2, \dots with coefficients in \mathbb{Z} , where Λ^n is the \mathbb{Z} -submodule of elements of degree n . The following sets are \mathbb{Z} -basis of Λ^n (see [Ful97] or [Mac15])

$$\{s_\lambda \mid \lambda \text{ is a partition of } n\},$$

$$\{p_\lambda \mid \lambda \text{ is a partition of } n\}.$$

3.1.3 Shifted tableaux

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is said to be **strict** if $\lambda_1 > \lambda_2 > \dots$.

Each strict partition λ is presented by a **shifted diagram** $sY(\lambda)$ that is a collection of boxes such that:

- (SD1) The leftmost boxes of each row are in the main diagonal.
- (SD2) The number of boxes from top row to bottom row are $\lambda_1, \lambda_2, \dots$, respectively.

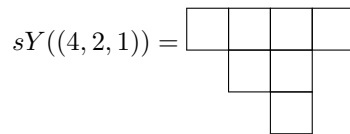
A **shifted tableau** T of shifted shape λ is a result of filling the shifted diagram $sY(\lambda)$ by the ordered alphabet $\{1' < 1 < 2' < 2 < \dots\}$ such that

- (T1) The entries in each column and in each row are weakly increasing.
- (T2) The entries k' in each row are strictly increasing.
- (T3) The entries k in each column are strictly increasing.

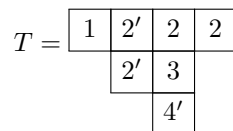
The shifted tableau T is said to have **content** $\gamma = (\gamma_1, \gamma_2, \dots)$ if γ_i is the number of i or i' in T . We write

$$x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \dots \tag{3.4}$$

Example 3.1. Let $\lambda = (4, 2, 1)$. Then the shifted diagram $sY(\lambda)$ is



And



is a shifted tableau of shifted shape $(4, 2, 1)$. The content of T is $(1, 4, 1, 1)$.

3.1.4 Shifted Littlewood-Richardson coefficients

For each strict partition λ , the **Schur Q -function** $Q_\lambda = Q_\lambda(x)$ in variables x_1, x_2, \dots is defined as the sum of x^T where T runs over the shifted tableaux of shape λ . Since every coefficient in Q_λ is divisible by $2^{l(\lambda)}$, we can define a formal power series with integer coefficients

$$P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x). \quad (3.5)$$

Let $\Omega_{\mathbb{Q}} = \bigoplus_{n \geq 0} \Omega_{\mathbb{Q}}^n$ be the graded subalgebra of $\Lambda_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ generated by $1, p_1, p_3, p_5, \dots$. Let $\Omega = \Omega_{\mathbb{Q}} \cap \Lambda$ be the \mathbb{Z} -hyper subring of $\Omega_{\mathbb{Q}}$. We write $\Omega = \bigoplus_{n \geq 0} (\Omega_{\mathbb{Q}}^n \cap \Lambda)$ as a graded ring. A fundamental result in the theory of strict partitions is that

$$\{P_\lambda \mid \lambda \text{ is a strict partition of } n\}$$

is a \mathbb{Z} -basis of $\Omega_{\mathbb{Q}}^n \cap \Lambda$ (see [Ste89]). We can define integers $f_{\lambda\mu}^\nu$ for each strict partitions λ, μ, ν by

$$P_\lambda P_\mu = \sum_{\nu} f_{\lambda\mu}^\nu P_\nu. \quad (3.6)$$

The integers $f_{\lambda\mu}^\nu$ are called the **shifted Littlewood-Richardson coefficients**.

Moreover, one can think about P_λ as an element of Λ^n endowed with its Schur basis. Let $g_{\lambda\mu}$ be the integers defined by

$$P_\lambda = \sum_{|\mu|=n} g_{\lambda\mu} s_\mu. \quad (3.7)$$

Note that in the notation of $g_{\lambda\mu}$, λ is a strict partition and μ is a partition. It turns out that the coefficients $g_{\lambda\mu}$ are related with the shifted Littlewood-Richardson coefficients. More precisely (see [Ste89]),

$$g_{\lambda\mu} = f_{\lambda\delta}^{\mu+\delta}, \quad (3.8)$$

where

$$\mu = (\mu_1, \mu_2, \dots, \mu_l) \text{ with } l = l(\mu), \quad (3.9)$$

$$\delta = (l, l-1, \dots, 1), \quad (3.10)$$

$$\mu + \delta = (\mu_1 + l, \mu_2 + l - 1, \dots, \mu_l + 1). \quad (3.11)$$

We have two motivations to study the coefficients $f_{\lambda\mu}^\nu$ and $g_{\lambda\mu}$. The first one explained in the following section, interprets these coefficients in terms of projective representations of the symmetric groups. The second one, see subsection ??, interprets them in terms of Schubert calculus.

3.2 Projective representations of symmetric groups

In this subsection, we present fundamental background about projective representations of the symmetric groups and the relation with shifted Littlewood-Richardson coefficients. The main reference are [Ste89] and [HH92].

3.2.1 Projective representations of S_n

Let G be a finite group and V be a finite dimensional complex vector space. A **projective representation** of the group G on V is a group homomorphism of G in $PGL(V) = GL(V)/\mathbb{C}^\times 1_V$. Such an homomorphism may be viewed as a map $P : G \rightarrow GL(V)$ such that

1. $P(1)$ is the identity linear map of V .
2. $P(x)P(y) = c(x, y)P(xy)$ ($x, y \in G$) for some $c(x, y) \in \mathbb{C}^\times$.

The map $c : G \times G \rightarrow \mathbb{C}^\times$, $(x, y) \mapsto c(x, y)$ is called the **factor set** of P . It satisfies

$$c(x, 1) = c(1, x) = 1 \text{ for all } x \in G. \quad (3.12)$$

By the associativity of $GL(V)$, we have

$$c(x, y)c(xy, z) = c(x, yz)c(y, z) \text{ for all } x, y, z \in G. \quad (3.13)$$

Remark 3.2. An ordinary representation of G is a projective representation with $c(x, y) = 1$ for all $x, y \in G$.

Let $P : G \rightarrow GL(V)$ and $Q : G \rightarrow GL(W)$ be two projective representations of G . We say that P and Q are **equivalent** if there exists an invertible linear map $S : V \rightarrow W$ and a map $b : G \rightarrow \mathbb{C}^\times$ such that

$$b(1) = 1 \text{ and } b(x)SP(x)S^{-1} = Q(x) \text{ for all } x \in G. \quad (3.14)$$

Let c and c' be factor sets of P and Q , respectively. In this case, we also say that c and c' are **equivalent**. More precisely, the equivalence of factor sets is given by

$$\frac{c'(x, y)}{c(x, y)} = \frac{b(x)b(y)}{b(xy)} \text{ for all } x, y \in G. \quad (3.15)$$

The set of all factor sets modulo equivalence is called the **Schur multiplier** of G . It is isomorphic to the second cohomology group $H^2(G, \mathbb{C}^\times)$.

We associate to each factor set c a **twisted group algebra** $\mathbb{C}G^c$ with a basis $\{\alpha_x \mid x \in G\}$ and the multiplication defined by $\alpha_x \alpha_y = c(x, y)\alpha_{xy}$. The twisted group algebras associated to equivalent factor sets are isomorphic. A projective representation of G with factor set c is identified with a $\mathbb{C}G^c$ -module.

Now, suppose that $G = S_n$. Let $\mathbb{C}S_n$ be the group algebra generated by s_1, \dots, s_{n-1} with the relations

$$(s_j)^2 = 1, \quad (s_j s_k)^2 = 1 \text{ for all } |j - k| \geq 2, \quad (s_j s_{j+1})^3 = 1. \quad (3.16)$$

Let $\mathbb{C}S'_n$ be the group algebra generated by $\alpha_1, \dots, \alpha_{n-1}$ with the relations

$$(\alpha_j)^2 = 1, \quad (\alpha_j \alpha_k)^2 = -1 \text{ for all } |j - k| \geq 2, \quad (\alpha_j \alpha_{j+1})^3 = 1. \quad (3.17)$$

Proposition 3.3. *The algebras $\mathbb{C}S_n$ and $\mathbb{C}S'_n$ are the only possible twisted group algebras structures for S_n . They are isomorphic only for $n \leq 3$. In particular, $H^2(S_n, \mathbb{C}^\times)$ is of order 2 only for $n \geq 4$.*

The above proposition says that any projective representation of S_n is either a $\mathbb{C}S_n$ -module or a $\mathbb{C}S'_n$ -module. In the case $n \geq 4$, set $\sigma_j = \sqrt{-1}\alpha_j$ for each $j = 1, \dots, n-1$. By relations (3.17), we have

$$(\sigma_j)^2 = -1, \quad (\sigma_j\sigma_k)^2 = -1 \text{ for all } |j-k| \geq 2, \quad (\sigma_j\sigma_{j+1})^3 = -1. \quad (3.18)$$

Let \widetilde{S}_n be the group generated by $-1, \sigma_1, \dots, \sigma_{n-1}$ with relations (3.18). A $\mathbb{C}S'_n$ -module V is identified with an ordinary representation of \widetilde{S}_n with -1 represented faithfully in $GL(V)$, and such a representation is called a **spin representation** of \widetilde{S}_n . A $\mathbb{C}S_n$ -module V is identified with an ordinary representation of \widetilde{S}_n where the image of -1 in $GL(V)$ is 1.

3.2.2 Conjugacy classes

Let $\lambda = (\lambda_1, \dots, \lambda_l)$ ($\lambda_l > 0$) be a partition of n . We say that λ is **even (odd)** if $n - l(\lambda)$ is even (odd). The conjugacy classes of S_n are indexed by the partitions of n . Namely, an element $\sigma \in S_n$ belongs to the λ^{th} conjugacy class if its cycle length are $\lambda_1, \dots, \lambda_l$. In this case, we write $\text{type}(\sigma) = \lambda$. Moreover, we say that σ is **even (odd)** if λ is even (odd).

3.2.2.1 Conjugacy classes of \widetilde{S}_n

Let $|\cdot|$ be the canonical homomorphism from \widetilde{S}_n to S_n , which is defined by $-1 \mapsto 1, \sigma_j \mapsto s_j$. Set

$$C_\lambda = \{\sigma \in \widetilde{S}_n \text{ such that } \text{type}(|\sigma|) = \lambda\}. \quad (3.19)$$

Let

$$\sigma^\lambda = \pi_1 \dots \pi_l \quad (l = l(\lambda)), \quad (3.20)$$

where

$$\pi_j = \sigma_{a+1} \dots \sigma_{a+\lambda_j-1} \quad (a = \lambda_1 + \dots + \lambda_{j-1}). \quad (3.21)$$

Set

$$C_\lambda^+ = \{\tau\sigma^\lambda\tau^{-1} \mid \tau \in \widetilde{S}_n\}, \quad (3.22)$$

$$C_\lambda^- = \{-\tau\sigma^\lambda\tau^{-1} \mid \tau \in \widetilde{S}_n\}. \quad (3.23)$$

The conjugacy classes of \widetilde{S}_n are described as follows:

- If any $\sigma \in \widetilde{S}_n$ is conjugate to $-\sigma$, then C_λ is an conjugacy class of \widetilde{S}_n .
- If there is no $\sigma \in \widetilde{S}_n$ conjugate to $-\sigma$, then C_λ is the disjoint union of two \widetilde{S}_n conjugacy classes C_λ^+ and C_λ^- . We know that (see [Ste89]) this case happens if and only if λ has only odd parts or it is a strict odd partition.

3.2.2.2 Conjugacy classes of parabolic subgroups of \widetilde{S}_n

Let J be a subset of the set $\{1, \dots, n-1\}$. There exists a unique sequence of nonnegative integers $\beta = (\beta_1, \dots, \beta_l)$ such that

$$\beta_1 + \dots + \beta_l = n \text{ and } J = \{\beta_1 + \dots + \beta_j \mid 1 \leq j < l\}. \quad (3.24)$$

The subgroup S_n^J of S_n generated by all elements s_j ($j \in J$) is called a **parabolic subgroup** of S_n . Moreover,

$$S_n^J \cong S_{\beta_1} \times \dots \times S_{\beta_l}. \quad (3.25)$$

The subgroup \widetilde{S}_n^J of \widetilde{S}_n generated by elements -1 and all σ_j ($j \in J$) is called a **parabolic subgroup** of \widetilde{S}_n . We write \widetilde{S}_β (resp. S_β) for the group \widetilde{S}_n^J (resp. S_n^J) and regard \widetilde{S}_{β_j} (resp. S_{β_j}) as subgroups of \widetilde{S}_β (resp. S_β). We also let $\mathbb{C}S'_\beta$ (resp. $\mathbb{C}S_\beta$) be the corresponding subalgebra of $\mathbb{C}S'_n$ (resp. $\mathbb{C}S_n$). The conjugacy classes of S_β are indexed by l -tuple $\lambda = (\lambda^1, \dots, \lambda^l)$ where λ^j is a partition of β_j . For such an l -tuple λ , set

$$C_\lambda = \{\pi_1 \dots \pi_l \text{ such that } \pi_j \in \widetilde{S}_{\beta_j} \text{ and } \text{type}(|\pi_j|) = \lambda^j\}. \quad (3.26)$$

Set $\sigma^\lambda = \sigma^{\lambda^1} \dots \sigma^{\lambda^l}$ where $\sigma^{\lambda^j} \in \widetilde{S}_{\beta_j}$ is defined by (3.20) and

$$C_\lambda^+ = \{\tau \sigma^\lambda \tau^{-1} \mid \tau \in \widetilde{S}_\beta\}, \quad (3.27)$$

$$C_\lambda^- = \{-\tau \sigma^\lambda \tau^{-1} \mid \tau \in \widetilde{S}_\beta\}. \quad (3.28)$$

The conjugacy classes of \widetilde{S}_β are described as follows:

- If any $\sigma \in \widetilde{S}_\beta$ is conjugate to $-\sigma$ then C_λ is a conjugacy class of \widetilde{S}_β .
- If there is no $\sigma \in \widetilde{S}_\beta$ conjugate to $-\sigma$ then C_λ is the disjoint union of two \widetilde{S}_β classes C_λ^+ and C_λ^- . Let $\lambda^* = \lambda^1 \cup \dots \cup \lambda^l$ be the partition whose parts are union of those of $\lambda^1, \dots, \lambda^l$ (we call λ^* the **multiset partition union** of $\lambda^1, \dots, \lambda^l$). Then this case happens if and only if λ^* has only odd parts or λ^* is odd and λ^j has distinct parts.

3.2.3 The space of class functions of \widetilde{S}_n

The **space of all class functions** of \widetilde{S}_n is of the form $Z_n \oplus Z'_n$ where Z_n is the space spanned by ordinary characters and Z'_n is the space spanned by spin characters.

For each partition λ of n , we define the **indicator function** 1_λ by

$$1_\lambda(\sigma) = \begin{cases} 1 & \text{if } \text{type}(|\sigma|) = \lambda, \\ 0 & \text{if } \text{type}(|\sigma|) \neq \lambda, \end{cases} \quad (3.29)$$

and the **spin-indicator function** $1'_\lambda$ by

$$1'_\lambda(\sigma) = \begin{cases} 1 & \text{if } \sigma \in C_\lambda^+, \\ -1 & \text{if } \sigma \in C_\lambda^-, \\ 0 & \text{if } \text{type}(|\sigma|) \neq \lambda. \end{cases} \quad (3.30)$$

Then

$$\{1_\lambda \mid \lambda \text{ is a partition of } n\}$$

is a basis of Z_n , and

$$\{1'_\lambda \mid \lambda \text{ is a partition of } n \text{ with only odd parts or a strict odd partition of } n\}$$

is a basis of Z'_n . The **product of characters** in the space $Z_n \oplus Z'_n$ is given by

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda\mu} 1_\lambda, \\ 1'_\lambda 1'_\mu &= \delta_{\lambda\mu} 1'_\lambda, \\ 1'_\lambda 1'_\mu &= \delta_{\lambda\mu} 1_\lambda. \end{aligned}$$

We denote the **inner product of characters** of the group \widetilde{S}_n by $\langle \cdot, \cdot \rangle$.

3.2.3.1 Basic spin characters of \widetilde{S}_n

The **Clifford algebra** \mathfrak{C}_n is the algebra generated by ζ_1, \dots, ζ_n with relations

$$\zeta_j^2 = 1, \quad \zeta_j \zeta_k + \zeta_k \zeta_j = 0 \text{ for all } k \neq j. \quad (3.31)$$

As a vector space, it has a basis

$$\{\zeta_A \mid A = \{a_1, \dots, a_r \text{ such that } 1 \leq a_1 < \dots < a_r \leq n\}\}, \quad (3.32)$$

where $\zeta_A = \zeta_{a_1} \dots \zeta_{a_r}$.

Let M_m be the matrix algebra of degree m .

If n is an even number, write $n = 2k$, then \mathfrak{C}_{2k} is isomorphic to M_{2^k} . In particular, the map $\rho : \mathfrak{C}_{2k} \rightarrow M_{2^k}$ defined by

$$\begin{aligned} \zeta_{2j-1} &\mapsto \epsilon \otimes \dots \otimes \epsilon \otimes x \otimes 1 \otimes \dots \otimes 1, \\ \zeta_{2j} &\mapsto \epsilon \otimes \dots \otimes \epsilon \otimes y \otimes 1 \otimes \dots \otimes 1, \end{aligned}$$

where x and y are in the j^{th} ($1 \leq j \leq k$) position, and

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad (3.33)$$

is an isomorphism. Then character of ρ is given by

$$\text{tr} \rho \left(\sum_A c_A \zeta_A \right) = 2^k c_\emptyset, \quad (3.34)$$

where c_A is the coefficients of ζ_A . Hence, ρ modulo equivalence is the unique irreducible representation of \mathfrak{C}_{2k} .

If n is an odd number, write $n = 2k + 1$, then

$$\mathfrak{C}_{2k+1} = \mathfrak{C}_{2k} \oplus \zeta \mathfrak{C}_{2k}, \quad (3.35)$$

where $\zeta = \zeta_1 \dots \zeta_{2k+1}$. The representation ρ of \mathfrak{C}_{2k} defined above is extended to two representations ρ_\pm of \mathfrak{C}_{2k+1} by

$$\rho_\pm(\zeta) = \pm(\sqrt{-1})^k. \quad (3.36)$$

The characters of ρ_\pm are given by

$$\text{tr} \rho_\pm \left(\sum_A c_A \zeta_A \right) = 2^k c_\emptyset \pm (2\sqrt{-1})^k c_\zeta, \quad (3.37)$$

where c_ζ is the coefficient of ζ . Hence, ρ_\pm modulo equivalence are the unique irreducible representation of \mathfrak{C}_{2k+1} .

The group \widetilde{S}_n is embedded into $\mathfrak{C}_{n-1}^\times$ by a map $\psi : \widetilde{S}_n \rightarrow \mathfrak{C}_{n-1}^\times$ given by

$$\sigma_j \mapsto a_j \zeta_j + b_j \zeta_{j+1}, \quad (3.38)$$

where $a_j, b_j \in \mathbb{C}^\times$ ($1 \leq j \leq n-1$) are such that

$$a_j^2 + b_j^2 = -1, \quad a_{j+1} b_j = \frac{1}{2}, \quad b_{n-1} = 0. \quad (3.39)$$

The **basic spin representations** of \widetilde{S}_n with respect to the embedding ψ is defined as follows:

- If $n = 2k + 1$, it is the composition

$$\widetilde{S}_n \xrightarrow{\psi} \mathfrak{C}_{n-1}^\times \xrightarrow{\rho} GL(2^k), \quad (3.40)$$

denoted by φ^n .

- If $n = 2k$, it is both of the compositions

$$\widetilde{S}_n \xrightarrow{\psi} \mathfrak{C}_{n-1}^\times \xrightarrow{\rho_\pm} GL(2^{k-1}), \quad (3.41)$$

denoted by φ_\pm^n , respectively. We use the notation φ^n for either φ_+^n or φ_-^n .

These representations modulo equivalence are independent of ψ . The characters of basic spin representations will be denoted by the same symbols φ^n, φ_\pm^n , and we call them the **basic spin characters**.

Theorem 3.4. *The basic spin representations of \widetilde{S}_n are irreducible. Moreover, we have*

$$\varphi^{2k+1}(\sigma^\lambda) = \begin{cases} 2^{(l(\lambda)-1)/2} & \text{if } \lambda \text{ is a partition of } 2k+1 \text{ with only odd parts,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.42)$$

$$\varphi_\pm^{2k}(\sigma^\lambda) = \begin{cases} 2^{(l(\lambda)-2)/2} & \text{if } \lambda \text{ is a partition of } 2k \text{ with only odd parts,} \\ \pm(\sqrt{-1})^k \sqrt{k} & \text{if } \lambda = (2k), \\ 0 & \text{otherwise.} \end{cases} \quad (3.43)$$

3.2.3.2 Spin characters of parabolic subgroups of \widetilde{S}_n

Let J be a subset of $\{1, \dots, n-1\}$. Let β be the sequence of nonnegative integers defined as in (3.24). For each $1 \leq k \leq l$, set

$$J_k = \{j \text{ such that } \beta_1 + \dots + \beta_{k-1} < j < \beta_1 + \dots + \beta_k\}. \quad (3.44)$$

Then $J = \bigcup_{k=1}^l J_k$ and \widetilde{S}_{β_k} is the subgroup of \widetilde{S}_β generated by -1 and all elements σ_j ($j \in J_k$).

In this subsection, we construct spin representations of \widetilde{S}_β (or $\mathbb{C}S'_\beta$ -modules).

First, let V and V' be $\mathbb{C}S'_n$ -modules. We say that V and V' are **associates** if $V' \cong \text{sgn} \otimes V$. An $\mathbb{C}S'_n$ -module V is called **self-associate** if $V \cong \text{sgn} \otimes V$. In this case, there exists $S \in GL(V)$ such that

$$S\sigma_j|_V = -\sigma_j S|_V \text{ for all } 1 \leq j \leq n-1. \quad (3.45)$$

We call S an **associator** for V .

Now, let V_j ($1 \leq j \leq l$) be $\mathbb{C}S'_{\beta_j}$ -modules among which exactly r are self-associate. We assume that V_1, \dots, V_r are the self-associate modules. Let V be an irreducible \mathfrak{C}_{l-r} -module where \mathfrak{C}_{l-r} is the Clifford algebra generated by $\{\zeta_{j+1}, \dots, \zeta_l\}$ and let S_j be an associator of V_j for each $1 \leq j \leq r$. The **reduced Clifford product** of V_1, \dots, V_l with respect to V and S_1, \dots, S_r is the tensor product $V \otimes V_1 \otimes \dots \otimes V_l$ with $\mathbb{C}S'_\beta$ -module structure defined by

$$\sigma_j(v \otimes v_1 \otimes \dots \otimes v_l) = \begin{cases} v \otimes A_1 v_1 \otimes \dots \otimes A_l v_l & \text{if } j \in J_k, 1 \leq k \leq r, \\ \zeta_k v \otimes B_1 v_1 \otimes \dots \otimes B_l v_l & \text{if } j \in J_k, r < k \leq l, \end{cases} \quad (3.46)$$

where

$$(A_1, \dots, A_l) = (S_1, \dots, S_{k-1}, \sigma_j, 1, \dots, 1), \quad (3.47)$$

$$(B_1, \dots, B_l) = (S_1, \dots, S_r, 1, \dots, \sigma_j, \dots, 1). \quad (3.48)$$

Here, σ_j in (3.48) is in k^{th} position.

Let $\varphi, \varphi_1, \dots, \varphi_l$ be spin characters of V, V_1, \dots, V_l , respectively. We denote $\varphi \times_c \varphi_1 \times_c \dots \times_c \varphi_l$ the spin character of $V \otimes V_1 \otimes \dots \otimes V_l$, and call it the **reduced Clifford product of spin characters** $\varphi, \varphi_1, \dots, \varphi_l$.

Theorem 3.5. *A reduced Clifford product of the $\mathbb{C}S'_{\beta_j}$ -modules is irreducible. Conversely, every irreducible $\mathbb{C}S'_{\beta}$ -module is of this form.*

3.2.3.3 Irreducible spin characters of \widetilde{S}_n

For each partition λ , let z_λ be the cardinal of the centralizer of any permutation σ of type λ . Let m_j ($j = 1, 2, 3, \dots$) be the number of the parts of λ of size j . Then we have

$$z_\lambda = (m_1!1^{m_1})(m_2!2^{m_2}) \dots \quad (3.49)$$

The **characteristic map** $ch : Z_n \rightarrow \Lambda_{\mathbb{C}}^n$ is the linear isomorphism given by

$$ch(1_\lambda) = p_\lambda / z_\lambda. \quad (3.50)$$

Set $\chi^\lambda = ch^{-1}(s_\lambda)$.

Theorem 3.6. *The irreducible characters of S_n are χ^λ (λ is a partition of n).*

Define an inner product $[,]$ on $\Omega_{\mathbb{C}}^n$ by

$$[p_\lambda, p_\mu] = \frac{1}{2^{l(\lambda)}} z_\lambda \delta_{\lambda\mu}. \quad (3.51)$$

For each strict partition λ , set $Q_\lambda^* = 2^{-l(\lambda)/2} Q_\lambda$. Then

$$\{Q_\lambda^* \mid \lambda \text{ is a strict partition}\}$$

is an orthonormal basis of $\Omega_{\mathbb{R}}$. The irreducible spin characters of \widetilde{S}_n associated to λ are defined as follows:

- If λ is an even partition of n with distinct parts, we let $\varphi^\lambda \in Z'_n$ be the self-associate class function defined by

$$\varphi^\lambda(\sigma^\mu) = \begin{cases} [Q_\lambda^*, 2^{l(\mu)/2} p_\mu] & \text{if } \mu \text{ is a partition of } n \text{ with only odd parts,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.52)$$

- If λ is a strict odd partition of n , we let $\varphi_\pm^\lambda \in Z'_n$ be the pair of associate class functions defined by

$$\varphi_\pm^\lambda(\sigma^\mu) = \begin{cases} \frac{1}{\sqrt{2}} [Q_\lambda^*, 2^{l(\mu)/2} p_\mu] & \text{if } \mu \text{ is a partition of } n \text{ with only odd parts,} \\ \pm (\sqrt{-1})^{(n-l(\lambda)+1)/2} \sqrt{\frac{z_\lambda}{2}} & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases} \quad (3.53)$$

In this case, we use the notation φ^λ for either φ_+^λ or φ_-^λ .

Theorem 3.7. *The irreducible spin characters of \widetilde{S}_n are φ^λ (λ is an even partition of n with distinct parts) and φ_\pm^λ (λ is a strict odd partition of n).*

3.2.4 Relation with shifted Littlewood-Richardson coefficients

This subsection explains how the shifted Littlewood-Richardson coefficients appear in the projective representations of symmetric groups.

For each partition λ , set

$$\epsilon_\lambda = \begin{cases} \sqrt{2} & \text{if } \lambda \text{ is odd,} \\ 1 & \text{if } \lambda \text{ is even.} \end{cases} \quad (3.54)$$

Theorem 3.8. *Let λ be a strict partition of k , μ be a strict partition of $n - k$, and ν be a strict partition of n . Then*

$$\langle (\varphi^\lambda \times_c \varphi^\mu) \uparrow \widetilde{S}_n, \varphi^\nu \rangle = \frac{1}{\epsilon_\nu \epsilon_{\lambda \cup \mu}} 2^{(l(\lambda) + l(\mu) - l(\nu))/2} f_{\lambda\mu}^\nu, \quad (3.55)$$

unless ν is odd and $\nu = \lambda \cup \mu$. In that case, the multiplicity of φ_\pm^ν is 0 or 1 according to the choice of associates.

Theorem 3.9. *Let λ be a strict partition of n and μ be a partition of n . Then*

$$\langle \varphi^n \chi^\mu, \varphi^\lambda \rangle = \frac{1}{\epsilon_\lambda \epsilon_{(n)}} 2^{(l(\lambda) - 1)/2} g_{\lambda\mu}, \quad (3.56)$$

unless $\lambda = (n)$, n is even, and μ is a hook-partition. In that case, the multiplicity of φ_\pm^λ is 0 or 1 according to the choice of associates.

Corollary 3.10. *The coefficients $f_{\lambda\mu}^\nu$ and $g_{\lambda\mu}$ are non-negative integers.*

Chapter 4

On the shifted Littlewood-Richardson coefficients and the Littlewood-Richardson coefficients

The chapter is a reproduction of our second preprint [\[Kha20\]](#).

Abstract

In relation with projective representations of the symmetric group, Stembridge introduced in 1989, the shifted Schur Q -functions Q_λ associated to any strict partition λ . The shifted Littlewood-Richardson coefficients $f_{\lambda\mu}^\nu \in \mathbb{N}$ (where λ, μ, ν are strict partitions) is defined by the relation $Q_\lambda Q_\mu = \sum_\nu 2^{l(\lambda)+l(\mu)-l(\nu)} f_{\lambda\mu}^\nu Q_\nu$. In this note, we give a new combinatorial model for these coefficients. The coefficients $g_{\lambda\mu}$ appear in the decomposition of Schur Q -function Q_λ into the sum of Schur functions: $Q_\lambda = 2^{l(\lambda)} \sum_\mu g_{\lambda\mu} s_\mu$. Actually, $g_{\lambda\mu}$ is equal to some explicit shifted Littlewood-Richardson coefficient and hence, we have a combinatorial model to express it. We also give another description for $g_{\lambda\mu}$ as the cardinal of a subset of a set that counts Littlewood-Richardson coefficients $c_{\mu^t\mu}^{\tilde{\lambda}}$. This new point of view allows us to establish connections between $g_{\lambda\mu}$ and $c_{\mu^t\mu}^{\tilde{\lambda}}$. More precisely, we prove that $g_{\lambda\mu} = g_{\lambda\mu^t}$, and $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$. We conjecture that $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\tilde{\lambda}}$ and formulate some conjectures on our combinatorial models which would imply this inequality if it is valid.

Contents

4.1	Introduction	66
4.1.1	Young tableaux	68
4.1.2	Row-insertion and product tableau	69
4.1.3	Sliding and jeu de taquin	69
4.1.4	The Robinson-Schensted-Knuth correspondence	70
4.1.5	Littlewood-Richardson rule	71
4.1.6	Tableau switching	74
4.1.7	The symmetry of Littlewood-Richardson coefficients	76
4.2	The shifted Littlewood-Richardson coefficients	76
4.2.1	Shifted tableaux	77
4.2.2	Shifted jeu de taquin	77
4.2.3	Shifted Littlewood-Richardson rule	78
4.2.4	Geometric interpretation of the coefficients $f_{\lambda\mu}^\nu$ and $g_{\lambda\mu}$	80
4.2.5	Application to the identity $g_{\lambda\mu} = g_{\lambda\mu^t}$	80
4.3	A new combinatorial model for the coefficients $f_{\lambda\mu}^\nu$	82
4.4	On the coefficients $g_{\lambda\mu}$	86
4.4.1	A new interpretation of the coefficients $g_{\lambda\mu}$	86
4.4.2	Inequality $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$	88
4.4.3	Inequality $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\tilde{\lambda}}$	89

4.1 Introduction

Let λ, μ, ν be partitions. Let $l(\lambda)$ be the length of λ , and s_λ be the Schur function associated to the partition λ . The Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ appear in the expansion (see [Ful97])

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu. \tag{4.1}$$

If now λ, μ, ν are strict partitions, let Q_λ be the shifted Schur Q -function associated to the strict partition λ . The shifted Littlewood-Richardson coefficients appear in the expansion (see [Ste89])

$$Q_\lambda Q_\mu = \sum_{\nu} 2^{l(\lambda)+l(\mu)-l(\nu)} f_{\lambda\mu}^\nu Q_\nu. \tag{4.2}$$

For any strict partition λ , and a partition μ of the same integer, the coefficients $g_{\lambda\mu}$ appear in the decomposition (see [Ste89])

$$Q_\lambda = 2^{l(\lambda)} \sum_{\mu} g_{\lambda\mu} s_\mu. \tag{4.3}$$

The coefficients $g_{\lambda\mu}$ can be considered as shifted Littlewood-Richardson coefficients by the identity (see [Ste89])

$$g_{\lambda\mu} = f_{\lambda\delta}^{\mu+\delta}, \tag{4.4}$$

where $\delta = (l, l-1, \dots, 1)$ with $l = l(\mu)$.

There were several developments beyond the Littlewood-Richardson rule. For example,

- Zelevinsky [Zel81] expressed the coefficients $c_{\lambda\mu}^\nu$ as the number of pictures between μ and ν/λ .

- Remmel and Whitney [RW84] described $c_{\lambda\mu}^{\nu}$ as the number of standard tableaux of shape λ satisfying some rules depending on the skew shape ν/μ . There is also a similar version by Chen, Garsia, Remmel [CGR84] where they replace λ with ν and ν/μ with $\lambda * \mu$.
- White [Whi81] showed that the set of tableaux in the construction of Remmel and Whitney [RW84] can be understood from a different point of view. It arises from Robinson-Schensted insertion of reading words of column-strict tableaux of a fixed skew shape.

There are new approaches that come from geometry: the algorithm by Liu [Liu10] and the rule of Ravi Vakil [Vak06] etc.

The theory and methods for shifted Littlewood-Richardson coefficients are also developed parallelly with the theory of Littlewood-Richardson coefficients. Based on the work of Worley [Wor84], Sagan [Sag87], Stembridge [Ste89], there are several versions of the shifted Littlewood-Richardson rule for $f_{\lambda\mu}^{\nu}$, for example, the works of Serrano [Ser10] and Shimozono [Shi99] and so on. The shifted Littlewood-Richardson rule given by Stembridge [Ste89] is also re-obtained by using the theory of crystal bases for the quantum queer superalgebra (see [GJK⁺14] and [CK18]). In [CNO14], the authors established the bijections between three models for shifted Littlewood-Richardson coefficients in [Ste89], [Ser10] and [GJK⁺14].

In this article, we use Stembridge's rule [Ste89] to obtain a new combinatorial models for the coefficients $f_{\lambda\mu}^{\nu}$ and $g_{\lambda\mu}$. The advantage of our rules is that they let appear connections with ordinary Littlewood-Richardson coefficients. The motivation of our work comes from the work of P. Belkale, S. Kumar and N. Ressayre [BKR12]. The main results in the article [BKR12] raised up some first clues about relations between shifted Littlewood-Richardson coefficients with Littlewood-Richardson coefficients. N. Ressayre conjectures an inequality between them in [Res19]. We do not use the approach from geometry as in [BKR12], but we try to develop the combinatorial model of Stembridge [Ste89] to discover bridges between these coefficients. To be more precise, we describe the results as follows.

Our first result, Theorem 4.22 is a new combinatorial model for the shifted Littlewood-Richardson coefficients. This is analogous to Remmel and Whitney's work [RW84]. The combinatorial model proposed by Shimozono in [Shi99] is analogous to White's model [Whi81], arising from Sagan's shifted insertion [Sag87]. Despite the case of Littlewood-Richardson coefficients where Remmel and Whitney's construction is identified with White's construction, our construction and Shimozono's construction do not produce the same model.

Since $g_{\lambda\mu}$ can be considered as a shifted Littlewood-Richardson coefficient, we obtain a new model for $g_{\lambda\mu}$ in Theorem 4.23.

Our second result, Theorem 4.28 is another combinatorial interpretation of the coefficients $g_{\lambda\mu}$. More precisely, let $\tilde{\lambda}$ be the partition such that its Young diagram is the union of shifted diagram corresponding to λ and its reflection through the main diagonal. Let μ^t be the conjugate partition of μ . We prove that $g_{\lambda\mu}$ is the cardinality of a subset of a set that counts the coefficients $c_{\mu^t\mu}^{\tilde{\lambda}}$. This implies Theorem 4.30 that

$$g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}. \quad (4.5)$$

We conjecture a stronger inequality (see Conjecture 4.31)

$$g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\tilde{\lambda}}. \quad (4.6)$$

Using a computer program, we checked this conjecture on a lot of examples. Based on our combinatorial model for the coefficients $g_{\lambda\mu}$, we formulate Conjecture 4.32 whose validity implies Conjecture 4.31. An evidence for Conjecture 4.32 is that it implies easily the equality

$$g_{\lambda\mu} = g_{\lambda\mu^t}. \quad (4.7)$$

The equality (4.7) might be well known among experts, nevertheless we include a geometric proof in Proposition 4.19.

The article contains four sections. In the first section, we collect some basic background about the theory of Young tableaux, and related models for Littlewood-Richardson coefficients. In the second section, we present the theory of shifted tableaux, and related models, some interpretations for shifted Littlewood-Richardson coefficients. The last two sections present our two main results on the coefficients $f_{\lambda\mu}^\nu$ and $g_{\lambda\mu}$.

In this section, we present Young tableaux, and related models for Littlewood-Richardson coefficients.

4.1.1 Young tableaux

For a **partition** λ , we write $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with $\lambda_l > 0$ and $\sum_{i=1}^l \lambda_i = n$, we write $l(\lambda) = l$, $|\lambda| = n$.

Each partition λ is presented by a **Young diagram** $Y(\lambda)$.

Example 4.1.

$$Y((3, 2)) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array}$$

The reflection $\sigma(Y)$ through the main diagonal of a Young diagram Y is also a Young diagram. The **conjugate partition** λ^t of λ is defined by $\sigma(Y(\lambda)) = Y(\lambda^t)$.

A **semistandard Young tableau** of shape λ is a filling of the Young diagram $Y(\lambda)$ by the ordered alphabet $\{1 < 2 < \dots\}$ such that:

- (Y1) The entries in each column are strictly increasing.
- (Y2) The entries in each row are weakly increasing.

Let $\nu = (\nu_1, \nu_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be two partitions. We say that ν is bigger than μ if and only if $\nu_i \geq \mu_i$ for all i , and we write $\nu \geq \mu$. In this case, we define the **skew Young diagram** $Y(\nu/\mu)$ as the result of removing boxes in the Young diagram $Y(\mu)$ from the Young diagram $Y(\nu)$. We write $|\nu/\mu| = |\nu| - |\mu|$. A **skew Young tableau** T of skew shape ν/μ is a result of filling the skew Young diagram $Y(\nu/\mu)$ by the ordered alphabet $\{1 < 2 < \dots\}$ satisfying the rules (Y1) and (Y2).

The **word** $w(T)$ of a Young tableau T is defined to be the sequence obtained by reading the rows of T from left to right, starting from bottom to top. A Young tableau of skew shape ν/μ is said to be a **standard skew Young tableau** if its word is a permutation of the word $12 \dots |\nu/\mu|$. The **transpose** of a standard skew Young tableau T is also a standard skew Young tableau and it is denoted by T^t .

4.1.2 Row-insertion and product tableau

For a Young tableau T and a positive integer x , we recall **row-insertion** x to T from [Ful97]. The result of row-insertion x to T is a Young tableau, is denoted by $T \leftarrow x$.

Example 4.2.

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \hline \end{array} \leftarrow 1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}$$

Let T and U be Young tableaux, the **product tableau** $T.U$ is defined by

$$T.U := (\dots((T \leftarrow x_1) \leftarrow x_2) \leftarrow \dots \leftarrow x_{n-1}) \leftarrow x_n, \tag{4.8}$$

where $w(U) = x_1x_2 \dots x_n$.

Example 4.3. Let

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \hline \end{array} \quad \text{and} \quad U = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 & \\ \hline \end{array}$$

then $w(U) = 24123$ and

$$T.U = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 4 & 4 & & & \\ \hline 3 & & & & & \\ \hline \end{array}$$

4.1.3 Sliding and jeu de taquin

For the skew Young diagram $Y(\nu/\mu)$, an **inner corner** of $Y(\nu/\mu)$ is a box in the Young diagram $Y(\mu)$ such that the boxes below and to the right are not in $Y(\mu)$. An **outside corner** is a box in the Young diagram $Y(\nu)$ such that the boxes below and to the right are not in $Y(\nu)$.

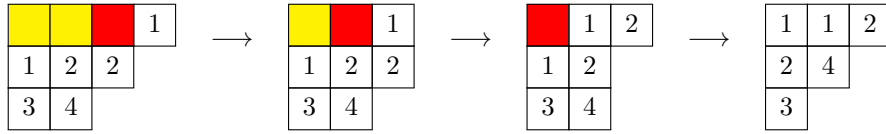
Let T be a skew Young tableau of skew shape ν/μ . Let b be an inner corner of ν/μ . We recall **sliding** b out of T from [Ful97]. The result of applying sliding b out of T gives us a new skew Young tableau T' of skew shape ν'/μ' such that $|\nu'| = |\nu| - 1$, $|\mu'| = |\mu| - 1$. Choose a random inner corner b' of T' and do sliding b' out of T' as before, we get a new skew Young tableau T'' of skew shape ν''/μ'' such that $|\nu''| = |\nu| - 2$, $|\mu''| = |\mu| - 2$. So repeat the process as many times as possible, we finally get a Young tableau and the process will terminate. There is a fact that the Young tableau we get does not depend on the choice of random inner corners in each step. The final tableau we have obtained is called the **rectification** of T and it is denoted by $Rect(T)$. The whole process we apply on T to get $Rect(T)$ is called the **jeu de taquin**.

Lemma 4.4. Let T and U be skew Young tableaux. If $w(T) = w(U)$ then $Rect(T) = Rect(U)$.

Example 4.5. Let

$$T = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 2 & 2 & \\ \hline 3 & 4 & & \\ \hline \end{array} \quad \text{and} \quad U = \begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline & & 1 & 2 & 2 \\ \hline & & 4 & & \\ \hline 3 & & & & \\ \hline \end{array}$$

The process of applying the jeu de taquin on T can be visualized as follows:

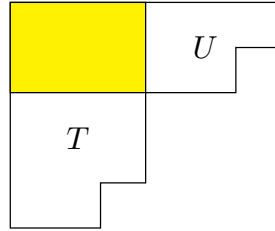


where the boxes in red are chosen to be slid. Hence,

$$\text{Rect}(T) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}$$

One can easily check that $\text{Rect}(U) = \text{Rect}(T)$.

Let T and U be Young tableaux. We denote $T * U$ the new skew Young tableau which is defined as follows:



We have another point of view about the product tableau $T.U$.

Lemma 4.6. *Let T and U be Young tableaux. We have $T.U = \text{Rect}(T * U)$.*

4.1.4 The Robinson-Schensted-Knuth correspondence

A two-rowed array is defined by

$$w = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix}, \quad (4.9)$$

with u_i 's and v_i 's are in two independent alphabets. We say that w is in **lexicographic order** if

1. $u_1 \leq u_2 \leq \cdots \leq u_n$.
2. If $u_{k-1} = u_k$ for some k , then $v_{k-1} \leq v_k$.

The **Robinson-Schensted-Knuth correspondence** is mentioned in [Ful97], which set up an one-to-one correspondence between a two-rowed array in lexicographic order $\begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix}$

and a pair of tableaux of the same shape $\begin{pmatrix} Q \\ P \end{pmatrix}$. We write

$$\begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix} \xleftrightarrow{RSK} \begin{pmatrix} Q \\ P \end{pmatrix}. \quad (4.10)$$

Example 4.7. Let w be the two-rowed array

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 6 & 7 & 4 & 5 & 1 \end{pmatrix}. \tag{4.11}$$

The tableaux P and Q are

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 7 & & \\ \hline 6 & & & \\ \hline \end{array} \text{ and } Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \hline 7 & & & \\ \hline \end{array}$$

4.1.5 Littlewood-Richardson rule

A Young tableau T is said to have **content** $\gamma = (\gamma_1, \gamma_2, \dots)$ if γ_i is the number of entries i in the tableau T . We write

$$x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \dots \tag{4.12}$$

For each partition λ , the **Schur function** s_λ in variables x_1, x_2, \dots is defined as the sum of x^T , where T runs over the semistandard Young tableaux of shape λ . Let $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$ be the **graded ring of symmetric functions** in the variables x_1, x_2, \dots with coefficients in \mathbb{Z} . The following set is a \mathbb{Z} -basis of Λ^n

$$\{s_\lambda \mid \lambda \text{ is a partition of } n\}.$$

The integers $c_{\lambda\mu}^\nu$ for each partitions λ, μ, ν defined by

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu, \tag{4.13}$$

are called **Littlewood-Richardson coefficients**.

For any Young tableaux V_0 of shape ν , let $\mathcal{T}(\lambda, \mu, V_0)$ be the set

$$\{(\Lambda, U) \mid \Lambda, U \text{ are Young tableaux of shapes } \lambda, \mu, \text{ respectively and } \Lambda.U = V_0\}. \tag{4.14}$$

For any tableau U_0 of shape μ , let $\mathcal{S}(\nu/\lambda, U_0)$ be the set

$$\{\text{Skew tableaux } S \text{ of skew shape } \nu/\lambda \text{ such that } \text{Rect}(S) = U_0\}. \tag{4.15}$$

We can describe a one-to-one correspondence between $\mathcal{T}(\lambda, \mu, V_0)$ and $\mathcal{S}(\nu/\lambda, U_0)$ as follows:

1. Let (Λ, U) be an element of the set $\mathcal{T}(\lambda, \mu, V_0)$. Suppose that

$$\begin{pmatrix} U \\ U_0 \end{pmatrix} \xleftrightarrow{RSK} \begin{pmatrix} u_1 & \dots & u_m \\ v_1 & \dots & v_m \end{pmatrix}, \tag{4.16}$$

where $m = |\mu|$. Let S be the new skew tableau obtained by placing u_1, \dots, u_m into the new boxes while doing row-insertion v_1, \dots, v_m into Λ . Then S is an element of $\mathcal{S}(\nu/\lambda, U_0)$.

2. Conversely, let S be an element of $\mathcal{S}(\nu/\lambda, U_0)$. Let Λ' be an arbitrary Young tableau of shape λ . Put an order on the letters in Λ' and S in such a way that all letters in Λ' are smaller than those in S . Now, suppose that

$$\begin{pmatrix} V_0 \\ \Lambda' \cup S \end{pmatrix} \xleftrightarrow{RSK} \begin{pmatrix} t_1 & \dots & t_n & u_1 & \dots & u_m \\ x_1 & \dots & x_n & v_1 & \dots & v_m \end{pmatrix}, \tag{4.17}$$

where $n = |\lambda|$. Then we can construct a tableau Λ such that $(\Lambda, U) \in \mathcal{T}(\lambda, \mu, V_0)$ by

$$\begin{pmatrix} t_1 & \cdots & t_n \\ x_1 & \cdots & x_n \end{pmatrix} \xleftrightarrow{RSK} \begin{pmatrix} \Lambda \\ \Lambda' \end{pmatrix}. \quad (4.18)$$

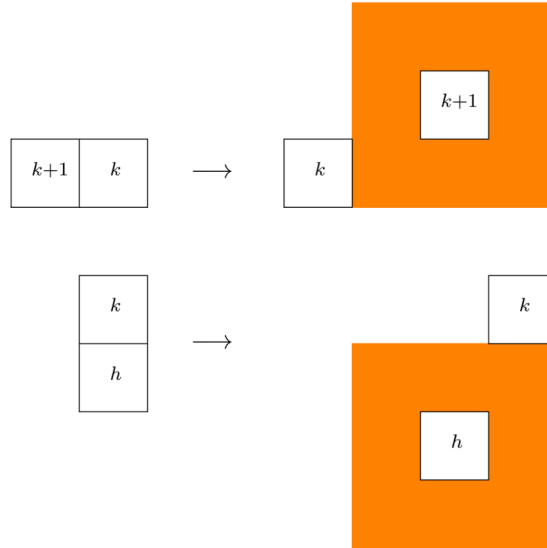
Denote by $\mathcal{F}_{\nu/\lambda, U_0}^{\lambda, \mu, V_0}$ the map that sends (Λ, U) in $\mathcal{T}(\lambda, \mu, V_0)$ to S in $\mathcal{S}(\nu/\lambda, U_0)$. This map is a bijection.

A method to compute the set $\mathcal{S}(\nu/\lambda, U_0)$ is explained in subsection 4.1.7. When U_0 is the Young tableau \mathcal{U}_μ of shape μ whose all entries in the k^{th} row are equal to k , one can compute explicitly the set $\mathcal{S}(\nu/\lambda, \mathcal{U}_\mu)$ by the model of Remmel and Whitney [RW84] as follows:

1. Number the boxes of the skew shape ν/λ from top to bottom and right to left in each row by $1, 2, \dots, |\nu/\lambda|$, respectively. The result is called the **reverse filling of the skew shape** ν/λ . We denote it by $T_{\nu/\lambda}$.
2. Define $\mathcal{O}(\nu/\lambda)$ to be the set of Young tableaux T of size $|\nu/\lambda|$, constructed from $T_{\nu/\lambda}$ satisfying the following conditions:

(R1) If k and $k + 1$ appear in the same row of $T_{\nu/\lambda}$, then $k + 1$ appears weakly above and strictly right of k in T .

(R2) If h appears in the box directly below k in $T_{\nu/\lambda}$, then h appears strictly below and weakly left of k in T .



3. Let $\mathcal{O}_\mu(\nu/\lambda)$ be the set of all tableaux T in $\mathcal{O}(\nu/\lambda)$ of shape μ . For each T in $\mathcal{O}_\mu(\nu/\lambda)$, we construct a word $x_{|\mu|} \dots x_1$, where x_k is the row where k belongs to in the tableau T . There exists a unique skew Young tableau T^* of skew shape ν/λ such that $w(T^*)$ is the word we have created.
4. It is proved that the set $\mathcal{S}(\nu/\lambda, \mathcal{U}_\mu)$ is the set of all skew tableaux T^* where T runs over the set of all tableaux in $\mathcal{O}_\mu(\nu/\lambda)$.

Theorem 4.8. Let λ, μ, ν be partitions. Let T_0 be a Young tableau of shape ν and U_0 be a Young tableau of shape μ . We have $c_{\lambda\mu}^\nu = \#\mathcal{O}_\mu(\nu/\lambda) = \#\mathcal{S}(\nu/\lambda, U_0) = \#\mathcal{T}(\lambda, \mu, V_0)$.

Let us illustrate the three models presented in Theorem 4.8 by an example.

Example 4.9. Set $\lambda = (3, 2, 1, 1)$, $\mu = (4, 2, 1)$ and $\nu = (6, 4, 2, 1, 1)$. Then

$$T_{\nu/\lambda} = \begin{array}{cccccc} & & & 3 & 2 & 1 \\ & & & 5 & 4 & \\ & & 6 & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ 7 & & & & & \end{array}$$

The tableaux of the set $\mathcal{O}_\mu(\nu/\lambda)$ are

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & & \\ \hline 6 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \hline 7 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & & \\ \hline 7 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 7 & & \\ \hline 6 & & & \\ \hline \end{array}$$

Hence, $c_{\lambda\mu}^\nu = 4$. We have

$$U_\mu = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array}$$

The tableaux of the set $\mathcal{S}(\nu/\lambda, U_\mu)$ are

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 2 & 2 & \\ \hline & & 3 & & & \\ \hline & & & & & \\ \hline 1 & & & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 2 & 2 & \\ \hline & & 1 & & & \\ \hline & & & & & \\ \hline 3 & & & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 1 & 2 & \\ \hline & & 2 & & & \\ \hline & & & & & \\ \hline 3 & & & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 1 & 2 & \\ \hline & & 3 & & & \\ \hline & & & & & \\ \hline 2 & & & & & \\ \hline \end{array}$$

Set

$$V_0 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 6 & 6 & 7 & & \\ \hline 3 & 7 & & & & \\ \hline 4 & & & & & \\ \hline 5 & & & & & \\ \hline \end{array} \quad \text{and} \quad \Lambda' = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array}$$

with order that $1 < 2 < 3 < 4 < 1 < 2 < 3$. Set

$$S = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 2 & 2 & \\ \hline & & 3 & & & \\ \hline & & & & & \\ \hline 1 & & & & & \\ \hline \end{array} \in \mathcal{S}(\nu/\lambda, U_\mu)$$

The two-rowed array corresponding to the pair $\left(\begin{smallmatrix} V_0 \\ \Lambda' \cup S \end{smallmatrix} \right)$ is

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 1 & 5 & 6 & 2 & 4 & 3 & 2 \end{array} \quad \begin{array}{cccccc} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 3 & 7 & 7 & 4 & 6 & 5 \end{array} \right). \quad (4.19)$$

The tableaux Λ and U such that $(\Lambda, U) \in \mathcal{T}(\lambda, \mu, V_0)$ corresponding to S are

$$\Lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} \quad \text{and} \quad U = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 6 & 7 & & \\ \hline 7 & & & \\ \hline \end{array}$$

4.1.6 Tableau switching

In this subsection, we recall the definition and basic properties of the switching procedure. The main reference is the article [BSS96] by G. Benkart, F. Sottile, J. Stroomer.

For each skew shape γ , we define a **perforated tableau** T of shape γ to be a result of filling some boxes in $Y(\gamma)$ with integers such that:

(PT1) The entries in each column are strictly increasing.

(PT2) The entries in the northwest of each entry t of T are less than or equal to t .

Let S, T be perforated tableaux of shape γ . We say that S, T **completely fill** γ if all boxes in $Y(\gamma)$ are filled by entries of S or T , and no box is filled twice. We then call $S \cup T$ a **perforated pair** of shape γ .

Let $S \cup T$ be a perforated pair of shape γ . Let s in S and t in T be adjacent integers, t is below or in the right of s . We define **switching** $s \leftrightarrow t$ by interchanging s and t such that after the action, both perforated tableau of shape γ filled by entries t , and perforated tableau of shape γ filled by entries s satisfy the conditions (PT1) and (PT2).

Choose a random pair (s, t) in $S \cup T$ such that we can do the switching $s \leftrightarrow t$. Repeat this process until there are no more pair (s, t) in $S \cup T$ that can be switched $s \leftrightarrow t$. The result is a new perforated pair $T' \cup S'$ of shape γ , where S' is the perforated tableau filled by entries s and T' is the perforated tableau filled by entries t . The point is that the resulting pair $T' \cup S'$ does not depend on the choices, it is denoted by ${}^S T \cup S_T$ (see [BSS96]). The process we have done to produce ${}^S T \cup S_T$ from $S \cup T$ is called the **switching procedure**. The map that sends $S \cup T$ to ${}^S T \cup S_T$ is called the **switching map**.

The example below visualizes switching procedure.

Example 4.10. Let $\gamma = (4, 3, 3, 2)/(2, 1)$. The tableau S with red entries and the tableau T with blue entries below are perforated tableaux of shape γ .

$$S = \begin{array}{|c|c|c|c|} \hline & & 1 & \\ \hline & & & \\ \hline 1 & & 2 & \\ \hline & 3 & & \\ \hline \end{array} \quad T = \begin{array}{|c|c|c|c|} \hline & & & -1 \\ \hline & -2 & -2 & \\ \hline & 1 & & \\ \hline 2 & & & \\ \hline \end{array} \quad S \cup T = \begin{array}{|c|c|c|c|} \hline & & 1 & -1 \\ \hline & -2 & -2 & \\ \hline 1 & 1 & 2 & \\ \hline 2 & 3 & & \\ \hline \end{array}$$

Look at the entries inside the circles below

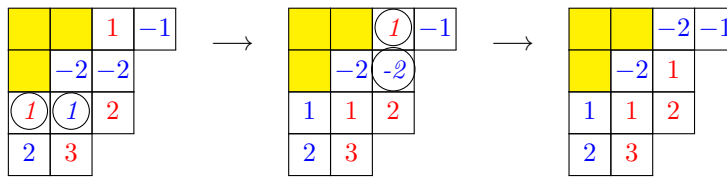
		1	-1
	-2	-2	
1	1	2	
2	3		

We see that we can just switch $1 \leftrightarrow 1$, but we cannot switch $1 \leftrightarrow 2$. Indeed, after switching $1 \leftrightarrow 1$, we get

		1	-1
	-2	-2	
1	1	2	
2	3		

The new tableau formed by the red entries and the new tableau formed by blue entries satisfy the conditions (PT1), (PT2). But after switching $1 \leftrightarrow 2$, the new tableau formed by the blue entries does not satisfy the condition (PT2).

Here is the visualization of switching procedure with starting point $S \cup T$ (we choose pairs in circles to switch).



Hence,

$${}^S T = \begin{array}{|c|c|c|c|} \hline & & -2 & -1 \\ \hline & -2 & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array} \quad \text{and} \quad S_T = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 1 & \\ \hline & 1 & 2 & \\ \hline & 3 & & \\ \hline \end{array}$$

Let S, T be skew tableaux. We say that T extends S if T has skew shape ν/λ and S has shape λ/μ for some partitions $\nu \geq \lambda \geq \mu$. The following theorem is a collection of some important properties in Theorem 2.2 and Theorem 3.1 in the article [BSS96].

Theorem 4.11. *Let S, T be skew Young tableaux such that T extends S . Then*

1. S_T and ${}^S T$ are skew Young tableaux, S_T extends ${}^S T$.
2. ${}^S T \cup S_T$ has the same shape as $S \cup T$.
3. $\text{Rect}(S) = \text{Rect}(S_T)$.
4. $\text{Rect}(T) = \text{Rect}({}^S T)$.
5. The switching map $S \cup T \mapsto {}^S T \cup S_T$ is an involution.

Example 4.12. *Let*

$$S = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} \quad \text{and} \quad T = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & & 2 & 2 & \\ \hline & & 3 & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline 1 & & & & & \\ \hline \end{array}$$

Then T extends S and

$$S_T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \text{and} \quad S_T = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 1 & 1 \\ \hline & & & 1 & 2 & \\ \hline & & 2 & & & \\ \hline & & & & & \\ \hline & & & 3 & & \\ \hline & & & 4 & & \\ \hline \end{array}$$

4.1.7 The symmetry of Littlewood-Richardson coefficients

The tableau switching provides a bijective proof of the symmetry of Littlewood-Richardson coefficients

$$c_{\lambda\mu}^\nu = c_{\mu\lambda}^\nu. \tag{4.20}$$

Indeed, let Λ_0 be a Young tableau of shape λ and U_0 be a Young tableau of shape μ . We can describe a one-to-one correspondence between $\mathcal{S}(\nu/\mu, \Lambda_0)$ and $\mathcal{S}(\nu/\lambda, U_0)$ by tableau switching as follows:

1. Let S be an element of $\mathcal{S}(\nu/\lambda, U_0)$. The switching map sends $\Lambda_0 \cup S$ to ${}^{\Lambda_0}S \cup (\Lambda_0)_S$. By Theorem 4.11, we have ${}^{\Lambda_0}S = U_0$ and $\text{Rect}((\Lambda_0)_S) = \Lambda_0$. Hence, $(\Lambda_0)_S \in \mathcal{S}(\nu/\mu, \Lambda_0)$.
2. By Theorem 4.11, the switching map is an involution. Hence, the map that sends S to $(\Lambda_0)_S$ is a bijection between $\mathcal{S}(\nu/\lambda, U_0)$ and $\mathcal{S}(\nu/\mu, \Lambda_0)$. We denote this map by $\mathcal{B}_{\nu/\mu, \Lambda_0}^{\nu/\lambda, U_0}$.

Let V_0 and W_0 be Young tableaux of shape ν . The composition of the bijections below

$$\mathcal{T}(\lambda, \mu, V_0) \xrightarrow{\mathcal{F}_{\nu/\lambda, U_0}^{\lambda, \mu, V_0}} \mathcal{S}(\nu/\lambda, U_0) \xrightarrow{\mathcal{B}_{\nu/\mu, \Lambda_0}^{\nu/\lambda, U_0}} \mathcal{S}(\nu/\mu, \Lambda_0) \xrightarrow{(\mathcal{F}_{\nu/\mu, \Lambda_0}^{\mu, \lambda, W_0})^{-1}} \mathcal{T}(\mu, \lambda, W_0) \tag{4.21}$$

gives us a bijection between the set $\mathcal{T}(\lambda, \mu, V_0)$ and the set $\mathcal{T}(\mu, \lambda, W_0)$. We denote this map by $\mathcal{S}_{V_0, U_0, \Lambda_0, W_0}^{\lambda, \mu, \nu}$.

Remark 4.13. Subsection 4.1.5 provides an algorithm to determine the set $\mathcal{S}(\nu/\lambda, \mathcal{U}_\mu)$. Applying then $\mathcal{B}_{\nu/\mu, \Lambda_0}^{\nu/\lambda, \mathcal{U}_\mu}$, we get an algorithm to compute $\mathcal{S}(\nu/\mu, \Lambda_0)$ for any Λ_0 .

4.2 The shifted Littlewood-Richardson coefficients

In this section, we present the definition and Stembridge's models, geometric points of view for shifted Littlewood-Richardson coefficients.

4.2.1 Shifted tableaux

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is said to be **strict** if $\lambda_1 > \lambda_2 > \dots$.

Each strict partition λ is presented by a **shifted diagram** $sY(\lambda)$ that is a collection of boxes such that:

(SD1) The leftmost boxes of each row are in the main diagonal.

(SD2) The number of boxes from top row to bottom row are $\lambda_1, \lambda_2, \dots$, respectively.

A **shifted tableau** T of shifted shape λ is a result of filling the shifted diagram $sY(\lambda)$ by the ordered alphabet $\{1' < 1 < 2' < 2 < \dots\}$ such that

(T1) The entries in each column and in each row are weakly increasing.

(T2) The entries k' in each row are strictly increasing.

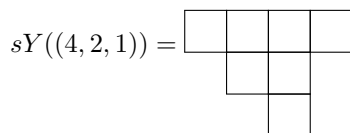
(T3) The entries k in each column are strictly increasing.

The shifted tableau T is said to have **content** $\gamma = (\gamma_1, \gamma_2, \dots)$ if γ_i is the number of i or i' in T . We write

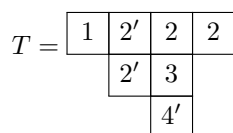
$$x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \dots \tag{4.22}$$

Let $\nu = (\nu_1, \nu_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be two strict partitions with $\nu \geq \mu$. We define the **skew shifted diagram** $sY(\nu/\mu)$ as the result of removing boxes in shifted diagram $sY(\mu)$ from shifted diagram $sY(\nu)$. A **skew shifted tableau** T of skew shifted shape ν/μ is a result of filling the shifted diagram $sY(\nu/\mu)$ by the ordered alphabet $\{1' < 1 < 2' < 2 < \dots\}$ satisfying the rules (T1), (T2) and (T3). The **content** of a skew shifted tableau T is defined by the same way as for a shifted tableau.

Example 4.14. Let $\lambda = (4, 2, 1)$. Then the shifted diagram $sY(\lambda)$ is



And



is a shifted tableau of shifted shape $(4, 2, 1)$. The content of T is $(1, 4, 1, 1)$.

4.2.2 Shifted jeu de taquin

For the skew shifted diagram $sY(\nu/\mu)$, we also define **inner corners** and **outside corners** by the same way as for the case of skew Young diagrams. Let T be a skew shifted tableau of skew shifted shape ν/μ without entries k' . Let b be an inner corner of skew shifted diagram $sY(\nu/\mu)$, we define **shifted sliding** b out of T , and **shifted jeu de taquin** on T , **shifted rectification** of T which we denote by $sRect(T)$, by the same way as for the case of skew Young tableaux.

Here is an example of shifted jeu de taquin.

Example 4.15. Set

$$T = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline & 4 & 5 & \\ \hline \end{array}$$

The process of applying the shifted jeu de taquin on T can be visualized as follows:

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline & 4 & 5 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline & 4 & 5 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & 1 & 5 \\ \hline & 2 & 3 & \\ \hline & 4 & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & 1 & 3 & 5 \\ \hline & 2 & 4 & \\ \hline & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline & & 4 & \\ \hline & & & \\ \hline \end{array}$$

where the boxes in red are chosen to be slid. Hence,

$$sRect(T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline & & 4 & \\ \hline & & & \\ \hline \end{array}$$

4.2.3 Shifted Littlewood-Richardson rule

The **Schur Q -function** $Q_\lambda = Q_\lambda(x)$ in variables x_1, x_2, \dots is defined as the sum of x^T where T runs over the shifted tableaux of shape λ . Since every coefficient in Q_λ is divisible by $2^{l(\lambda)}$, we can define a formal power series with integer coefficients

$$P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x). \quad (4.23)$$

We define the **power-sum symmetric function** p_r with $r \geq 1$ by

$$p_r = x_1^r + x_2^r + \dots \quad (4.24)$$

For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \quad (4.25)$$

The following set is a \mathbb{Z} -basis of Λ^n

$$\{p_\lambda \mid \lambda \text{ is a partition of } n\}.$$

Let $\Omega_{\mathbb{Q}} = \bigoplus_{n \geq 0} \Omega_{\mathbb{Q}}^n$ be the graded subalgebra of $\Lambda_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ generated by $1, p_1, p_3, p_5, \dots$. Let $\Omega = \Omega_{\mathbb{Q}} \cap \Lambda$ be the \mathbb{Z} -hyper subring of $\Omega_{\mathbb{Q}}$. We write $\Omega = \bigoplus_{n \geq 0} (\Omega_{\mathbb{Q}}^n \cap \Lambda)$ as a graded ring. Since

$$\{P_\lambda \mid \lambda \text{ is a strict partition of } n\}$$

is a \mathbb{Z} -basis of $\Omega_{\mathbb{Q}}^n \cap \Lambda$, we can define integers $f_{\lambda\mu}^\nu$ for each strict partitions λ, μ, ν by

$$P_\lambda P_\mu = \sum_{\nu} f_{\lambda\mu}^\nu P_\nu. \quad (4.26)$$

The integers $f_{\lambda\mu}^\nu$ are called the **shifted Littlewood-Richardson coefficients**.

For any (skew) shifted tableau T , we define the **word** $w(T)$ to be the sequence obtained by reading the rows of T from left to right, starting from bottom to top.

Given a word $w = w_1 w_2 \dots w_n$ over the alphabet $\{1' < 1 < 2' < 2 < \dots\}$, we define a sequence of statistics $m_i(j)$ ($0 \leq j \leq 2n, i \geq 1$) as follows:

$$\begin{aligned} m_i(j) &= \text{multiplicity of } i \text{ among } w_n \dots w_{n-j+1} && (0 \leq j \leq n), \\ m_i(j) &= \text{multiplicity of } i' \text{ among } w_1 \dots w_{j-n} \\ &\quad + \text{multiplicity of } i \text{ among } w_n \dots w_1 && (n < j \leq 2n). \end{aligned}$$

We say that the word w is a **shifted lattice word** if, whenever $m_i(j) = m_{i-1}(j)$, we have

$$\begin{aligned} w_{n-j} &\neq i, i' \text{ if } 0 \leq j < n, \\ w_{j-n+1} &\neq i - 1, i' \text{ if } n \leq j < 2n. \end{aligned}$$

Stembridge in [Ste89] obtained a shifted analogue of the Littlewood-Richardson rule as follows.

Theorem 4.16. *Let λ, μ, ν be strict partitions. Then the coefficient $f_{\lambda\mu}^\nu$ is the number of skew shifted tableaux T of skew shifted shape ν/μ and content λ satisfying*

- (F1) *The leftmost letter of $\{i, i' \text{ in } w(T)\}$ is unmarked ($1 \leq i \leq l(\lambda)$).*
- (F2) *The word $w(T)$ is a shifted lattice word.*

For each strict partition λ and partition μ of the same integer n , let $g_{\lambda\mu}$ be the integer defined by

$$P_\lambda = \sum_{|\mu|=n} g_{\lambda\mu} s_\mu. \tag{4.27}$$

In the proof of Theorem 9.3 in [Ste89], Stembridge used the fact that

$$g_{\lambda\mu} = f_{\lambda\delta}^{\mu+\delta}, \tag{4.28}$$

where

$$\mu = (\mu_1, \mu_2, \dots, \mu_l) \text{ with } l = l(\mu), \tag{4.29}$$

$$\delta = (l, l - 1, \dots, 1), \tag{4.30}$$

$$\mu + \delta = (\mu_1 + l, \mu_2 + l - 1, \dots, \mu_l + 1). \tag{4.31}$$

With the identity (4.28), he obtained an explicit interpretation of $g_{\lambda\mu}$ as in the following theorem.

Theorem 4.17. *Let λ be a strict partition and let μ be a partition. Then the coefficient $g_{\lambda\mu}$ is the number of skew shifted tableaux T of shape μ and content λ satisfying*

- (G1) *The leftmost letter of $\{i, i' \text{ in } w(T)\}$ is unmarked ($1 \leq i \leq l(\lambda)$).*
- (G2) *The word $w(T)$ is a shifted lattice word.*

A skew shifted tableau of skew shifted shape ν/μ is said to be **standard** if its word is a permutation of the word $12 \dots |\nu/\mu|$. The following result can be translated equivalently from Lemma 8.4 in the article [Ste89] of J. Stembridge.

Theorem 4.18. *Let λ, μ, ν be strict partitions. Choose a standard shifted tableau \mathcal{T}_λ of shifted shape λ . Then the coefficient $f_{\lambda\mu}^\nu$ is the number of standard skew shifted tableaux S of skew shifted shape ν/μ such that $sRect(S) = \mathcal{T}_\lambda$.*

4.2.4 Geometric interpretation of the coefficients $f_{\lambda\mu}^\nu$ and $g_{\lambda\mu}$

Let V be a complex vector space of dimension $m+n$. The set $Gr(m, V)$ of linear subspaces of dimension m in V is called a **complex Grassmannian**. Fix a complete flag of V

$$\mathcal{F} : 0 = V_0 \subset \cdots \subset V_i \subset \cdots \subset V_{m+n} = V, \quad (4.32)$$

where each V_i is a vector subspace of V of dimension i . To each partition $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_m \geq 0$, contained in the $m \times n$ rectangle, we associate the Schubert variety

$$X_\lambda(\mathcal{F}) = \{W \in Gr(m, V) \mid \dim(W \cap V_{n+i-\lambda_i}) \geq i \ (1 \leq i \leq m)\}. \quad (4.33)$$

The Poincaré dual class of $X_\lambda(\mathcal{F})$ is denoted by σ_λ and called a Schubert class. Then σ_λ is an element of $H^{2|\lambda|}(Gr(m, V))$. We have (see [Ful97])

$$H^*(Gr(m, V)) = \bigoplus_{\lambda \text{ is a partition contained in the } m \times n \text{ rectangle}} \mathbb{Z}\sigma_\lambda. \quad (4.34)$$

Now, let V be a complex vector space of dimension $2n$, endowed with a nondegenerate skew-symmetric bilinear form ω . A subspace W of V is isotropic if the form ω vanishes on it, i.e., $\omega(v, w) = 0$ for all $v, w \in W$. A maximal isotropic subspace of V is called **Lagrangian**. The set $LG(n, V)$ of Lagrangian subspaces in V is called the **Lagrangian Grassmannian**. Fix a complete isotropic flag of V

$$\mathcal{L} : 0 = V_0 \subset \cdots \subset V_i \subset \cdots \subset V_n \subset V, \quad (4.35)$$

where each V_i is a vector subspace of V , of dimension i and V_n is Lagrangian. To each strict partition $\lambda = (\lambda_1, \dots, \lambda_l)$ with $\lambda_l > 0$, contained in $(n, n-1, \dots, 1)$, we associate the Schubert variety

$$Y_\lambda(\mathcal{L}) = \{W \in LG(n, V) \mid \dim(W \cap V_{n+1-\lambda_i}) \geq i \ (1 \leq i \leq l)\}. \quad (4.36)$$

The Poincaré dual class of $Y_\lambda(\mathcal{L})$ is denoted by θ_λ and called a Schubert class. Then θ_λ is an element of $H^{2|\lambda|}(LG(n, V))$. We have (see [Pra91])

$$H^*(LG(n, V)) = \bigoplus_{\lambda \text{ is a strict partition contained in } (n, n-1, \dots, 1)} \mathbb{Z}\theta_\lambda, \quad (4.37)$$

and

$$\theta_\lambda \theta_\mu = \sum_{\nu} 2^{l(\lambda)+l(\mu)-l(\nu)} f_{\lambda\mu}^\nu \theta_\nu. \quad (4.38)$$

There is a canonical embedding $\iota : LG(n, V) \rightarrow Gr(n, V)$. The map ι induces the ring homomorphism $\iota^* : H^*(Gr(n, V)) \rightarrow H^*(LG(n, V))$. For each partition μ contained in the $n \times n$ rectangle, we have (see [Pra00])

$$\iota^*(\sigma_\mu) = \sum_{\lambda \text{ is a strict partition contained in } (n, n-1, \dots, 1)} g_{\lambda\mu} \theta_\lambda. \quad (4.39)$$

4.2.5 Application to the identity $g_{\lambda\mu} = g_{\lambda\mu^t}$

Proposition 4.19. *Let λ be a strict partition and let μ be a partition. Then $g_{\lambda\mu} = g_{\lambda\mu^t}$.*

Proof. Let V be a complex vector space of dimension $2n$, endowed with a nondegenerate skew-symmetric bilinear form ω . For each subspace W of V , set

$$W^{\perp\omega} = \{v' \in V \text{ such that } \omega(v', v) = 0 \text{ for all } v \in W\}, \quad (4.40)$$

$$W^\perp = \{f \in V^* \text{ such that } f(v) = 0 \text{ for all } v \in W\}. \quad (4.41)$$

Fix a complete isotropic flag of V

$$\mathcal{L} : 0 = V_0 \subset \cdots \subset V_i \subset \cdots \subset V_n \subset V. \quad (4.42)$$

Then we can extend \mathcal{L} to a complete flag \mathcal{F} of V as follow

$$\mathcal{F} : 0 = V_0 \subset \cdots \subset V_i \subset \cdots \subset V_{2n} = V, \quad (4.43)$$

where $V_{n+i} = (V_{n-i})^{\perp\omega}$ for each $i = 1, \dots, n$. Moreover, the flag \mathcal{F}^\perp defined below is a complete flag of V^*

$$\mathcal{F}^\perp : 0 = (V_{2n})^\perp \subset \cdots \subset (V_{2n-i})^\perp \subset \cdots \subset (V_0)^\perp = V^*. \quad (4.44)$$

Then flag \mathcal{L}^\perp defined below is a complete isotropic flag of V^*

$$\mathcal{L}^\perp : 0 = (V_{2n})^\perp \subset \cdots \subset (V_{2n-i})^\perp \subset \cdots \subset (V_n)^\perp \subset V^*. \quad (4.45)$$

We define an isomorphism $\eta : Gr(n, V) \rightarrow Gr(n, V^*)$ by $W \mapsto W^\perp$. By [GH78], we know that

$$\eta(X_\mu(\mathcal{F})) = X_{\mu^t}(\mathcal{F}^\perp). \quad (4.46)$$

Hence, the map η induces the ring isomorphism $\eta^* : H^*(Gr(n, V^*)) \rightarrow H^*(Gr(n, V))$ with

$$\eta^*(\sigma_\mu) = \sigma_{\mu^t}. \quad (4.47)$$

The restriction of η on $LG(n, V)$ is also an isomorphism and we still denote it by η . We have

$$\eta(Y_\lambda(\mathcal{L})) = Y_\lambda(\mathcal{L}^\perp). \quad (4.48)$$

Indeed,

1. For each $W \in Y_\lambda(\mathcal{L})$, we have $W^{\perp\omega} = W$, and

$$\dim(W \cap V_{n+1-\lambda_i})^{\perp\omega} = 2n - 1 + \lambda_i - \dim(W \cap V_{n-1+\lambda_i}). \quad (4.49)$$

Then we can rewrite (4.36) as

$$Y_\lambda(\mathcal{L}) = \{W \in LG(n, V) \mid \dim(W \cap V_{n-1+\lambda_i}) \geq i + \lambda_i - 1 \ (1 \leq i \leq l)\}. \quad (4.50)$$

2. Now, for any $W \in Y_\lambda(\mathcal{L})$ given by (4.50), we have

$$\dim(W \cap V_{n-1+\lambda_i})^\perp = 2n + 1 - \lambda_i - \dim(W^\perp \cap V_{n-1+\lambda_i}^\perp). \quad (4.51)$$

Then $W^\perp \in Y_\lambda(\mathcal{L}^\perp)$ given by (4.36).

Hence, the map η induces the ring isomorphism $\eta^* : H^*(LG(n, V^*)) \rightarrow H^*(LG(n, V))$ with

$$\eta^*(\theta_\lambda) = \theta_\lambda. \quad (4.52)$$

We have

$$\eta^* \iota^* = \iota^* \eta^*. \quad (4.53)$$

Apply η^* on both sides of the equality (4.39), with the help of (4.47), (4.52), (4.53), we get

$$\eta^*(\iota^*(\sigma_\mu)) = \sum_{\lambda \text{ is a strict partition contained in } (n, n-1, \dots, 1)} g_{\lambda\mu} \theta_\lambda \quad (4.54)$$

$$= \sum_{\lambda \text{ is a strict partition contained in } (n, n-1, \dots, 1)} g_{\lambda\mu^t} \theta_\lambda. \quad (4.55)$$

It implies $g_{\lambda\mu} = g_{\lambda\mu^t}$. □

4.3 A new combinatorial model for the coefficients $f_{\lambda\mu}^\nu$

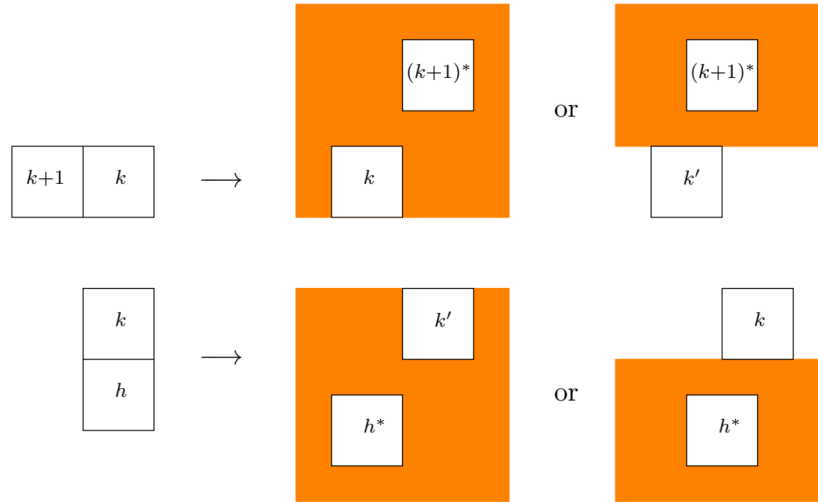
Given a skew shifted shape ν/μ , we number the boxes from top to bottom and right to left in each row by $1, 2, \dots, |\nu/\mu|$, respectively. The result is called the **shifted reverse filling of the skew shifted shape ν/μ** . We denote it by $\tilde{T}_{\nu/\mu}$.

For each $k = 1, 2, \dots, |\nu/\mu|$, let k^* to be meant k or k' .

We now let $\tilde{\mathcal{O}}(\nu/\mu)$ be the set of all tableaux T of size $|\nu/\mu|$, unshifted shape, labelled by the alphabet $\{1' < 1 < 2' < 2 < \dots < |\nu/\mu|' < |\nu/\mu|\}$, satisfying the following conditions:

(C1) If k and $k+1$ appear in the same row of $\tilde{T}_{\nu/\mu}$, then $(k+1)^*$ appears weakly above k or $(k+1)^*$ appears strictly above k' in T .

(C2) If h appears in the box directly below k in $\tilde{T}_{\nu/\mu}$, then h^* appears weakly below k' or h^* appears strictly below k in T .



(C3) The rightmost letter in each row of T is unmarked.

(C4) Let \hat{T} be the result of reordering each row of T relatively to the order $1 < 2 < \dots < |\nu/\mu| < |\nu/\mu|' < \dots < 2' < 1'$. Then

- The entries in each column of \hat{T} are increasing.
- If the entry j' belong to the i -th ($i > 1$) row of \hat{T} , and the number of entries less than j in the $(i-1)$ -th, i -th row of \hat{T} are τ_{i-1} , τ_i , respectively, then $\tau_{i-1} > \tau_i$.
- If the entry j belong to the i -th row of \hat{T} , and the number of entries less than j' in the i -th, $(i+1)$ -th row of \hat{T} are τ_i , τ_{i+1} , respectively, then $\tau_i > \tau_{i+1}$.

Remark 4.20.

- Since the size of T is $|\nu/\mu|$, the conditions (C1) and (C2) imply that only one of k or k' appears in T for each $k = 1, 2, \dots, |\nu/\mu|$.
- In the condition (C1) for the set $\tilde{\mathcal{O}}(\nu/\mu)$, $(k+1)^*$ must appear strictly right of k^* in T . It is similar to the condition (R1) for the set $\mathcal{O}(\nu/\mu)$. However, in the condition (C2) for the set $\tilde{\mathcal{O}}(\nu/\mu)$, it is not necessary that h^* appears weakly left of k^* in T . It is not similar as the condition (R2) for the set $\mathcal{O}(\nu/\mu)$. Indeed, for $\nu = (3, 1)$, $\mu = (1)$, in $\tilde{T}_{\nu/\mu}$, the entry 2 is directly above the entry 3. But $\tilde{\mathcal{O}}(\nu/\mu)$ contains

$$T = \begin{array}{|c|c|c|} \hline 1 & 2' & 3 \\ \hline \end{array}$$

with 3 is on the right of $2'$.

Example 4.21. We illustrate how the condition (C4) works. Let T be the following tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5' & 8' & 10 \\ \hline 6 & 7' & 9 & & & & \\ \hline 11 & 12 & & & & & \\ \hline \end{array}$$

We have

$$\hat{T} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 10 & 8' & 5' \\ \hline 6 & 9 & 7' & & & & \\ \hline 11 & 12 & & & & & \\ \hline \end{array}$$

We can see that the first and the second condition in (C4) are satisfied. However, the third condition in (C4) is not satisfied. Indeed, the entry 9 belongs to the second row of \hat{T} . The number of entries less than $9'$ in the second row and the third row of \hat{T} are equal two.

Theorem 4.22. Let λ, μ, ν be strict partitions. Then the coefficient $f_{\lambda\mu}^\nu$ is the number of the tableaux T in $\tilde{\mathcal{O}}(\nu/\mu)$ of shape λ .

Proof. Let $\tilde{\mathcal{S}}_\lambda(\nu/\mu)$ be the set of tableaux in Theorem 4.16. Let $\tilde{\mathcal{O}}_\lambda(\nu/\mu)$ be the set of tableaux in the set $\tilde{\mathcal{O}}(\nu/\mu)$ of shape λ .

Let $T \in \tilde{\mathcal{S}}_\lambda(\nu/\mu)$ with $w(T) = w_1 w_2 \dots w_{|\nu/\mu|}$. We associate T with a unique tableau T' of unshifted shape by the rules: For each $i = |\nu/\mu|, \dots, 2, 1$, we have

- If $w_i = k$, then $|\nu/\mu| + 1 - i$ appears in the k^{th} row of T' .
- If $w_i = k'$, then $(|\nu/\mu| + 1 - i)'$ appears in the k^{th} row of T' .

We can easily check that $T' \in \tilde{\mathcal{O}}_\lambda(\nu/\mu)$. Indeed,

- T has content λ if and only if T' has shape λ .
- The conditions (T1), (T2) and (T3) of T are equivalent to the conditions (C1) and (C2) of T' . Indeed, suppose that $(k+1)^*$ belongs to the x -th row of T' and k^* belongs to the y -th row of T' , then $k+1$ and k appear in the same row of $\tilde{T}_{\nu/\mu}$ if and only if $x \leq y$ and $x < y$ if $k^* = k'$ in T' . This is the statement of the condition (C1). For the condition (C2), suppose that k^* belongs to the x -th row of T' and h^* belongs to the y -th row of T' , then k is above h in $\tilde{T}_{\nu/\mu}$ if and only if $x \leq y$ and $x < y$ if $k^* = k$ in T' .
- The condition (F1) of T is equivalent to the condition (C3) of T' .
- The condition (F2) of T is equivalent to the conditions (C4) of T' . Indeed, by the definition of the shifted lattice words, for each $j = 1, \dots, n-1$, if $m_i(j) = m_{i-1}(j)$ then $w_{n-j} \neq i, i'$. Equivalently, if the shape of T' after deleting entries k' and $k < j$ is $\tau = (\tau_1, \tau_2, \dots)$, then τ must be a partition and if $\tau_{i-1} = \tau_i$ for some i then $(j+1)'$ does not depend on the t -th row of T . Now, for each $j = n, \dots, 2n-1$, if $m_i(j) = m_{i-1}(j)$ then $w_{j-n+1} \neq i-1, i'$. This condition equivalent to say that for each $j = n, \dots, 2$, if the shape of T' after deleting entries k' for $k < j$ is $\tau = (\tau_1, \tau_2, \dots)$, then τ must be a partition and if $\tau_{i-1} = \tau_i$ then $j-1$ does not belong to the $(i-1)$ -th row of T' . The conditions τ is a partition in both cases is rewritten shortly by \hat{T}' having strictly increasing entries in rows and columns. Of course, the entries in each row of \hat{T}' are strictly increasing. Then, we just need conditions on columns. The last conditions of \hat{T}' are the trivial translation of remaining conditions on T' .

Hence, we can define an injection $\phi : \tilde{\mathcal{S}}_\lambda(\nu/\mu) \rightarrow \tilde{\mathcal{O}}_\lambda(\nu/\mu), T \mapsto T'$.

Moreover, for each $T' \in \tilde{\mathcal{O}}_\lambda(\nu/\mu)$, we associate T' with a unique tableau T of skew shifted shape ν/μ and word $w(T) = w_1 w_2 \dots w_{|\nu/\mu|}$ by the rule: for each $j = |\nu/\mu|, \dots, 2, 1$, we have

- If j appears in the k^{th} row of T' , then $w_{|\nu/\mu|+1-j} = k$.
- If j' appears in the k^{th} row of T' , then $w_{|\nu/\mu|+1-j} = k'$.

The equivalence of the conditions we have already shown implies that $T \in \tilde{\mathcal{S}}_\lambda(\nu/\mu)$. So we can define an injection $\psi : \tilde{\mathcal{O}}_\lambda(\nu/\mu) \rightarrow \tilde{\mathcal{S}}_\lambda(\nu/\mu), T' \mapsto T$. Moreover, $\phi\psi = Id$. Hence, ϕ is a bijection and $f_{\lambda\mu}^\nu = \#\tilde{\mathcal{S}}_\lambda(\nu/\mu) = \#\tilde{\mathcal{O}}_\lambda(\nu/\mu)$. \square

Theorem 4.23. *Let λ be a strict partition and let μ be a partition. Then the coefficient $g_{\lambda\mu}$ is the number of the tableaux T in $\tilde{\mathcal{O}}(\mu + \delta/\delta)$ of shape λ .*

Proof. This follows from Theorem 4.22 and identity (4.28). \square

We illustrate the method to compute the coefficients $f_{\lambda\mu}^\nu$ through an example.

Example 4.24. *Set $\lambda = (3, 2), \mu = (3, 2), \nu = (5, 3, 2)$.*

(1) *The shifted reverse filling of the skew shifted shape ν/μ is*

$$\tilde{T}_{\nu/\mu} = \begin{array}{|c|c|c|c|c|} \hline & & & 2 & 1 \\ \hline & & & 3 & \\ \hline & & 5 & 4 & \\ \hline \end{array}$$

(2) *To construct the tableaux T' in $\tilde{\mathcal{O}}_\lambda(\nu/\mu)$, we first use three conditions (C1), (C2) and (C3). Then check the results if they satisfy the condition (C4) or not.*

1. We start with 1^* , there are two possibilities, they are $1'$ and 1 . But if $1'$ appears in the tableau T' then the next position of 2^* will be in the row above the first row by the condition (C1). It is impossible. Hence, just only one case that 1 appears in T' . Then the next two possibilities by the condition (C1) are

1	2	1	2'
---	---	---	----

2. For the second case, by the condition (C2), there are four possibilities below

1	2'	1	2'	1	2'	3	1	2'	3'
3'		3							

- The last one cannot happen since the tableau T' has shape $\lambda = (3, 2)$. Then we consider $3'$ as the rightmost letter in the first row of T' and it should be 3 to satisfy the condition (C3).
- The second one also cannot happen because the next position of 4^* will be in the row below the row of 3 by the condition (C2). It cannot produce a tableau of shape $\lambda = (3, 2)$ later.
- For the third one, the next position of 4^* is based on the condition (C2). To produce the shape $\lambda = (3, 2)$ later, it will be as follows:

1	2'	3	1	2'	3
4			4'		

- For the first one, the next position of 4^* is based on the conditions (C2) and (C3). To produce the shape $\lambda = (3, 2)$ later, will be as follows:

1	2'
3'	4

Continue until the end on the remaining cases by similar arguments, we finally can find the tableaux of shape $\lambda = (3, 2)$ satisfying all conditions (C1), (C2) and (C3) as follows:

1	2	5	1	2'	5	1	2'	3
3'	4		3'	4		4	5	

We can check that only the first two tableaux above satisfy the condition (C4). Hence,

$$f_{\lambda\mu}^\nu = 2.$$

We automatically find out the set of skew shifted tableaux described in Theorem 4.16 by using the bijection we mentioned in the proof of Theorem 4.22. Here they are

			1	1			1'	1
			2'				2'	
		1	2				1	2

Remark 4.25. The model of Remmel and Whitney for Littlewood-Richardson coefficients [RW84], i.e., the set $\mathcal{O}_\mu(\nu/\lambda)$ in Theorem 4.8, has another interpretation by White [Whi81]. Namely, each tableau in the set $\mathcal{O}_\mu(\nu/\lambda)$ can be considered as the recording tableau of the word rewritten in inverse order of a tableau in the set $\mathcal{S}(\nu/\lambda, \mathcal{U}_\mu)$. Our model, i.e., the set $\tilde{\mathcal{O}}_\lambda(\nu/\mu)$, is analogous to Remmel and Whitney's model but for shifted Littlewood-Richardson coefficients. In [Shi99], M. Shimozono gave an analogous model to the White's model in [Whi81] but for shifted Littlewood-Richardson coefficients. Our model and Shimozono's model are totally different. For example, with $\lambda = (3, 2)$, $\mu = (3, 2)$ and $\nu = (5, 3, 2)$, our model $\tilde{\mathcal{O}}_\lambda(\nu/\mu)$ consists of the elements below

1	2	5
3'	4	

1	2'	5
3'	4	

However, Shimozono's model consists of the elements below

1	2	4'
	3	5

1	2	4'
	3	5'

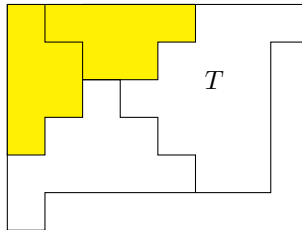
4.4 On the coefficients $g_{\lambda\mu}$

In this section, we present our second result. Namely, we present a new interpretation of the coefficients $g_{\lambda\mu}$ as a subset of a set that counts Littlewood-Richardson coefficients. As corollaries, we can compute the coefficients $g_{\lambda\mu}$ by models for Littlewood-Richardson coefficients. We will prove and conjecture inequalities between the coefficients and also state some conjectures that explain the hidden structure behind them.

4.4.1 A new interpretation of the coefficients $g_{\lambda\mu}$

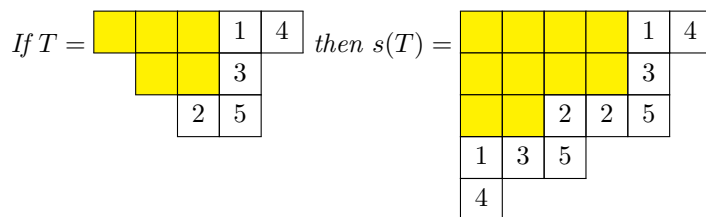
For any (skew) shifted tableau T without entries k' , let $s(T)$ be the new (skew) tableau which is defined as follow:

1. Create an image of T by the symmetry through its main diagonal.
2. Combine the image we have created with T by gluing them along the main diagonal as the image below.



Let ν/μ be the skew shifted shape of T , then we denote the shape of $s(T)$ by $\widetilde{\nu/\mu}$.

Example 4.26.



The following result is a restatement of Proposition 5.4 in the article [Hai89] of M. D. Haiman.

Proposition 4.27. *Let T be a skew shifted tableau without entries k' . Then we have $s(sRect(T)) = Rect(s(T))$.*

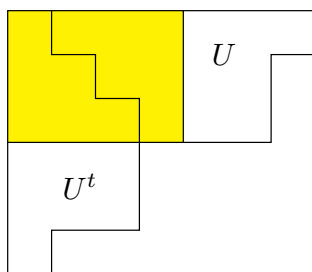
For any strict partition λ of n , let \mathcal{T}_λ be the shifted tableau of shifted shape λ , obtained by putting numbers $1, 2, \dots, n$ in the boxes of shifted diagram $sY(\lambda)$ from left to right, starting from top to bottom. Let $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ be the subset of $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$ of all pairs (T, U) such that $T = U^t$.

Theorem 4.28. *Let λ be a strict partition and let μ be a partition. Then $g_{\lambda\mu} = \#\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$.*

Proof. We have $g_{\lambda\mu} = f_{\lambda\delta}^{\mu+\delta}$. By Theorem 4.18, it is the number of standard skew shifted tableaux S of skew shifted shape $(\mu + \delta)/\delta$ such that $sRect(S) = \mathcal{T}_\lambda$. The condition $sRect(S) = \mathcal{T}_\lambda$, by Proposition 4.27 is equivalent to the condition

$$s(\mathcal{T}_\lambda) = Rect(s(S)). \tag{4.56}$$

The tableau $s(S)$ has form



where U is a standard Young tableau of shape μ . Since $s(S)$ and $U^t * U$ have the same word, then by Lemma 4.4 and Lemma 4.6, we have

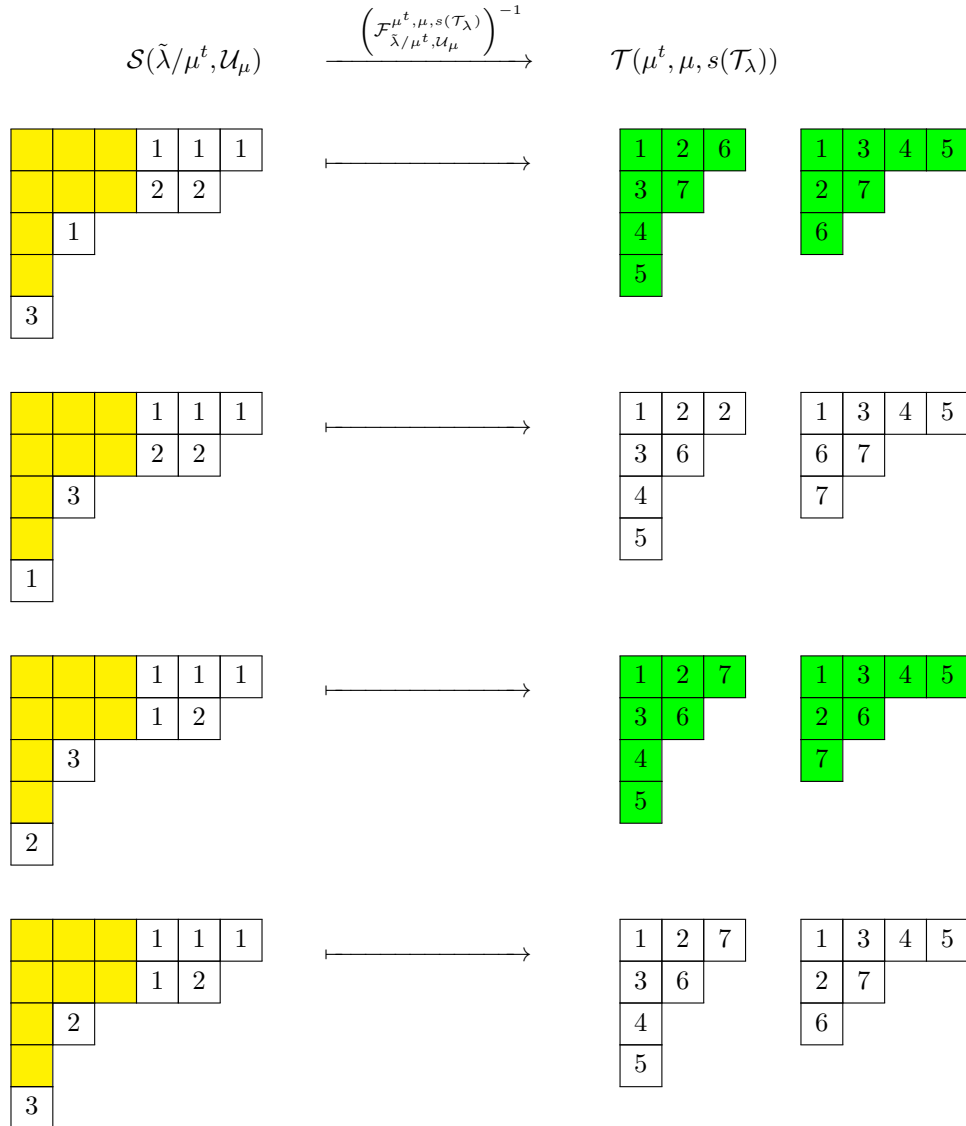
$$s(\mathcal{T}_\lambda) = Rect(U^t * U) = U^t \cdot U. \tag{4.57}$$

It is clear that U is uniquely determined by S . Hence, $g_{\lambda\mu}$ is the number of pairs (T, U) in the set $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$ such that $T = U^t$. \square

Theorem 4.28 gives a way to compute the coefficients $g_{\lambda\mu}$.

Example 4.29. *Let $\lambda = (5, 2)$ and $\mu = (4, 2, 1)$. Since $\mu^t = (3, 2, 1, 1)$, we can re-use the computation in Example 4.9. The elements in the set $\mathcal{S}(\tilde{\lambda}/\mu^t, \mathcal{U}_\mu)$ with the corresponding elements*

in the set $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$ are (the elements in the subsets $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ are marked by coloring all boxes in green).



Hence, $g_{\lambda\mu} = 2$.

4.4.2 Inequality $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$

In this subsection, we prove a conjecture of N. Ressayre [Res19]: $g_{\lambda\mu} \leq c_{\mu^t\mu}^{\tilde{\lambda}}$. The conjecture was based on the facts below

$$g_{\lambda\mu} = 1 \text{ implies } c_{\mu^t\mu}^{\tilde{\lambda}} = 1, \quad (4.58)$$

$$g_{\lambda\mu} \neq 0 \text{ implies } c_{\mu^t\mu}^{\tilde{\lambda}} \neq 0. \quad (4.59)$$

The conclusions (4.58), (4.59) are versions of Theorem 1.4 and Proposition 1.6 in the article [BKR12] for the case $G = Sp(2n, \mathbb{C})$. Indeed, with the notations in the article [BKR12], by

[Res12], in the case $G = Sp(2n, \mathbb{C})$, G/P is the Lagrangian Grassmannian $LG(n, \mathbb{C}^{2n})$, the corresponding Levi group is $GL(n)$, and W^P is parametrized by strict partitions. Suppose that $w \in W^P$ corresponds to strict partition λ , then χ_w corresponds to the partition $\tilde{\lambda}$. Let λ^\vee be the strict partition corresponding to the completion of $sY(\lambda)$ in $sY((n, n-1, \dots, 1))$. Let $\lambda_1, \lambda_2, \lambda_3$ be the strict partitions corresponding to $w_1, w_2, w_3 \in W^P$, respectively in Theorem 1.4 in [BKR12]. Since the structure constants for the singular cohomology and the deformed cohomology \odot_0 in this case are the same, the theorem says that for any positive integer k , we have

$$f_{\lambda_1 \lambda_2}^{\lambda_3^\vee} = 1 \text{ implies } c_{k\lambda_1 k\lambda_2}^{\widetilde{k\lambda_3^\vee}} = 1. \quad (4.60)$$

In particular, when $\lambda_1 = \lambda$, $\lambda_2 = \delta$, $\lambda_3^\vee = \mu + \delta$, the left-hand side of (4.60) becomes

$$1 = f_{\lambda \delta}^{\mu + \delta} = g_{\lambda\mu}.$$

With $k = 1$, the right-hand side of (4.60) becomes

$$\begin{aligned} 1 &= c_{\lambda \delta}^{\widetilde{\mu + \delta}} = \#\mathcal{S}(\widetilde{\mu + \delta / \delta}, s(\mathcal{T}_\lambda)) \\ &= \#\mathcal{S}(\mu^t * \mu, s(\mathcal{T}_\lambda)) \\ &= \#\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda)) \\ &= c_{\mu^t \mu}^{\tilde{\lambda}}. \end{aligned}$$

Hence, we get the conclusion (4.58). Similarly, the conclusion (4.59) follows Proposition 1.6 in [BKR12].

Theorem 4.30. *Let λ be a strict partition and let μ be a partition. Then $g_{\lambda\mu} \leq c_{\mu^t \mu}^{\tilde{\lambda}}$.*

Proof. By Theorem 4.28 and Theorem 4.8, we have

$$g_{\lambda\mu} = \#\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))} \leq \#\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda)) = c_{\mu^t \mu}^{\tilde{\lambda}}. \quad (4.61)$$

□

4.4.3 Inequality $g_{\lambda\mu}^2 \leq c_{\mu^t \mu}^{\tilde{\lambda}}$

In this subsection, we propose a stronger conjectural inequality than Theorem 4.30. We provide some examples to support this conjecture. Indeed, we formulate a conjecture on combinatorial models whose validity implies the first conjecture.

Conjecture 4.31. *Let λ be a strict partition and let μ be a partition. Then $g_{\lambda\mu}^2 \leq c_{\mu^t \mu}^{\tilde{\lambda}}$.*

To compute the decomposition of P_λ into Schur functions s_μ by computer, we use the code below. For example, with $\lambda = (4, 2)$, we have

$$P_{(4,2)} = s_{(2,2,1,1)} + s_{(2,2,2)} + s_{(3,1,1)} + 2s_{(3,2,1)} + s_{(3,3)} + s_{(4,1,1)} + s_{(4,2)}. \quad (4.62)$$

The code in SageMath (online version: <https://cocalc.com/>) is

```
Sym = SymmetricFunctions(FractionField(QQ['t']))
SP = Sym.hall_littlewood(t=-1).P();
s = Sym.schur();
s(SP([4,2]))
```

The result appears in computer is

$$s[2, 2, 1, 1] + s[2, 2, 2] + s[3, 1, 1, 1] + 2*s[3, 2, 1] + s[3, 3] + s[4, 1, 1] + s[4, 2].$$

To compute the Littlewood-Richardson coefficients by computer, we use the code below. For example, for $\lambda = (4, 2)$, $\mu = \mu^t = (3, 2, 1)$, we have

$$c_{\mu^t \mu}^{\lambda} = 4. \tag{4.63}$$

The code is

```
import sage.libs.lrcalc.lrcalc as lrcalc
A = [5,4,2,1]
B = [3,2,1]
C = [3,2,1]
lrcalc.lrccoef(A,B,C)
```

The result appears in computer is

4.

We check the conjecture for all strict partitions λ such that $|\lambda| \leq 11$. By Theorem 4.30, we just need to check the cases $g_{\lambda\mu} > 1$. Here is the data of computations on computer.

$ \lambda $	strict partitions λ such that there exists $g_{\lambda\mu} > 1$	μ such that $g_{\lambda\mu} > 1$	$g_{\lambda\mu}$	$c_{\mu^t \mu}^{\lambda}$
11	(9,2)	(3, 2, 1, 1, 1, 1, 1)	2	4
11	(9,2)	(4, 2, 1, 1, 1, 1, 1)	2	4
11	(9,2)	(5, 2, 1, 1, 1, 1)	2	4
11	(9,2)	(6, 2, 1, 1, 1)	2	4
11	(9,2)	(7, 2, 1, 1)	2	4
11	(9,2)	(8, 2, 1)	2	4
11	(8,3)	(3, 2, 2, 1, 1, 1)	2	4
11	(8,3)	(4, 2, 1, 1, 1, 1)	2	4
11	(8,3)	(4, 2, 2, 1, 1, 1)	2	4
11	(8,3)	(4, 3, 1, 1, 1, 1)	2	4
11	(8,3)	(5, 2, 1, 1, 1, 1)	2	4
11	(8,3)	(5, 2, 2, 1, 1)	2	4
11	(8,3)	(5, 3, 1, 1, 1)	2	4
11	(8,3)	(6, 2, 1, 1, 1)	2	4
11	(8,3)	(6, 2, 2, 1)	2	4
11	(8,3)	(6, 3, 1, 1)	2	4
11	(8,3)	(7, 2, 1, 1)	2	4
11	(8,3)	(7, 3, 1)	2	4
11	(7,4)	(3, 2, 2, 2, 1, 1)	2	4
11	(7,4)	(4, 2, 2, 1, 1, 1)	2	4
11	(7,4)	(4, 2, 2, 2, 1)	2	4
11	(7,4)	(4, 3, 2, 1, 1)	2	4
11	(7,4)	(5, 2, 1, 1, 1, 1)	2	4
11	(7,4)	(5, 2, 2, 1, 1)	2	4
11	(7,4)	(5, 3, 1, 1, 1)	2	4
11	(7,4)	(5, 3, 2, 1)	2	4
11	(7,4)	(5, 4, 1, 1)	2	4
11	(7,4)	(6, 2, 1, 1, 1)	2	4
11	(7,4)	(6, 3, 1, 1)	2	4
11	(7,4)	(6, 4, 1)	2	4
11	(7,3,1)	(3, 3, 2, 1, 1, 1)	2	6
11	(7,3,1)	(4, 2, 2, 1, 1, 1)	2	5
11	(7,3,1)	(4, 3, 1, 1, 1, 1)	2	5
11	(7,3,1)	(4, 3, 2, 1, 1)	3	13
11	(7,3,1)	(5, 2, 2, 1, 1)	2	5
11	(7,3,1)	(5, 3, 1, 1, 1)	2	5
11	(7,3,1)	(5, 3, 2, 1)	3	13
11	(7,3,1)	(6, 2, 2, 1)	2	5

11	(7,3,1)	(6, 3, 1, 1)	2	5
11	(7,3,1)	(6, 3, 2)	2	6
11	(6,4,1)	(3, 3, 2, 2, 1)	2	6
11	(6,4,1)	(4, 2, 2, 2, 1)	2	5
11	(6,4,1)	(4, 3, 2, 1, 1)	3	14
11	(6,4,1)	(4, 3, 2, 2)	2	7
11	(6,4,1)	(4, 3, 3, 1)	2	4
11	(6,4,1)	(4, 4, 2, 1)	2	7
11	(6,4,1)	(5, 2, 2, 1, 1)	2	5
11	(6,4,1)	(5, 3, 1, 1, 1)	2	5
11	(6,4,1)	(5, 3, 2, 1)	3	14
11	(6,4,1)	(5, 4, 1, 1)	2	5
11	(6,4,1)	(5, 4, 2)	2	6
11	(6,3,2)	(4, 3, 2, 1, 1)	3	10
11	(6,3,2)	(4, 3, 2, 2)	2	4
11	(6,3,2)	(4, 3, 3, 1)	2	5
11	(6,3,2)	(4, 4, 2, 1)	2	4
11	(6,3,2)	(5, 3, 2, 1)	3	10
11	(5,4,2)	(4, 3, 2, 1, 1)	2	4
11	(5,4,2)	(4, 3, 2, 2)	2	5
11	(5,4,2)	(4, 3, 3, 1)	2	5
11	(5,4,2)	(4, 4, 2, 1)	2	5
11	(5,4,2)	(5, 3, 2, 1)	2	4
10	(8,2)	(3, 2, 1, 1, 1, 1, 1)	2	4
10	(8,2)	(4, 2, 1, 1, 1, 1)	2	4
10	(8,2)	(5, 2, 1, 1, 1)	2	4
10	(8,2)	(6, 2, 1, 1)	2	4
10	(8,2)	(7, 2, 1)	2	4
10	(7,3)	(3, 2, 2, 1, 1, 1)	2	4
10	(7,3)	(4, 2, 1, 1, 1, 1)	2	4
10	(7,3)	(4, 2, 2, 1, 1)	2	4
10	(7,3)	(4, 3, 1, 1, 1)	2	4
10	(7,3)	(5, 2, 1, 1, 1)	2	4
10	(7,3)	(5, 2, 2, 1)	2	4
10	(7,3)	(5, 3, 1, 1)	2	4
10	(7,3)	(6, 2, 1, 1)	2	4
10	(7,3)	(6, 3, 1)	2	4
10	(6,4)	(3, 2, 2, 2, 1)	2	4
10	(6,4)	(4, 2, 2, 1, 1)	2	4
10	(6,4)	(4, 3, 2, 1)	2	4
10	(6,4)	(5, 2, 1, 1, 1)	2	4
10	(6,4)	(5, 3, 1, 1)	2	4
10	(6,4)	(5, 4, 1)	2	4
10	(6,3,1)	(3, 3, 2, 1, 1)	2	6
10	(6,3,1)	(4, 2, 2, 1, 1)	2	5
10	(6,3,1)	(4, 3, 1, 1, 1)	2	5
10	(6,3,1)	(4, 3, 2, 1)	3	13
10	(6,3,1)	(5, 2, 2, 1)	2	5
10	(6,3,1)	(5, 3, 1, 1)	2	5
10	(6,3,1)	(5, 3, 2)	2	6
10	(5,4,1)	(4, 3, 2, 1)	2	7
10	(5,3,2)	(4, 3, 2, 1)	3	9
9	(7,2)	(3, 2, 1, 1, 1, 1)	2	4
9	(7,2)	(4, 2, 1, 1, 1)	2	4
9	(7,2)	(5, 2, 1, 1)	2	4
9	(7,2)	(6, 2, 1)	2	4
9	(6,3)	(3, 2, 2, 1, 1)	2	4
9	(6,3)	(4, 2, 1, 1, 1)	2	4
9	(6,3)	(4, 2, 2, 1)	2	4
9	(6,3)	(4, 3, 1, 1)	2	4
9	(6,3)	(5, 2, 1, 1)	2	4

9	(6,3)	(5, 3, 1)	2	4
9	(5,3,1)	(3, 3, 2, 1)	2	6
9	(5,3,1)	(4, 2, 2, 1)	2	5
9	(5,3,1)	(4, 3, 1, 1)	2	5
9	(5,3,1)	(4, 3, 2)	2	6
8	(6,2)	(3, 2, 1, 1, 1)	2	4
8	(6,2)	(4, 2, 1, 1)	2	4
8	(6,2)	(5, 2, 1)	2	4
8	(5,3)	(3, 2, 2, 1)	2	4
8	(5,3)	(4, 2, 1, 1)	2	4
8	(5,3)	(4, 3, 1)	2	4
7	(5,2)	(3, 2, 1, 1)	2	4
7	(5,2)	(4, 2, 1)	2	4
6	(4,2)	(3,2,1)	2	4
< 6	\emptyset	\emptyset		

Conjecture 4.32. *We have*

1. *The restriction of the map $\mathcal{S}_{s(\mathcal{T}_\lambda), \mathcal{U}_\mu, \mathcal{U}_{\mu^t}, s(\mathcal{T}_\lambda)}^{\mu^t, \mu, \bar{\lambda}}$ on the set $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ is a bijection onto the set $\overline{\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))}$.*
2. *The elements of the set $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ have the form (U_α^t, U_α) , with index α . Let (V_α, V_α^t) be the image of (U_α^t, U_α) through the bijection $\mathcal{S}_{s(\mathcal{T}_\lambda), \mathcal{U}_\mu, \mathcal{U}_{\mu^t}, s(\mathcal{T}_\lambda)}^{\mu^t, \mu, \bar{\lambda}}$. Let (U_α^t, U_α) and (U_β^t, U_β) be elements of the set $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$. If (U_α^t, U_β) is not in the set $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$, then (V_α, V_β^t) is in the set $\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))$.*

Remark 4.33. Thanks to Theorem 4.28, the validity of Conjecture 4.32 1. implies the equality $g_{\lambda\mu} = g_{\lambda\mu^t}$, which was proved in Proposition 4.19.

Proposition 4.34. *Suppose that Conjecture 4.32 holds. Then we have $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\bar{\lambda}}$.*

Proof. We suppose that (U_α^t, U_α) and (U_β^t, U_β) are elements in $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$. We construct an element $(\widetilde{U_\alpha^t}, \widetilde{U_\beta})$ is still in the set as follows:

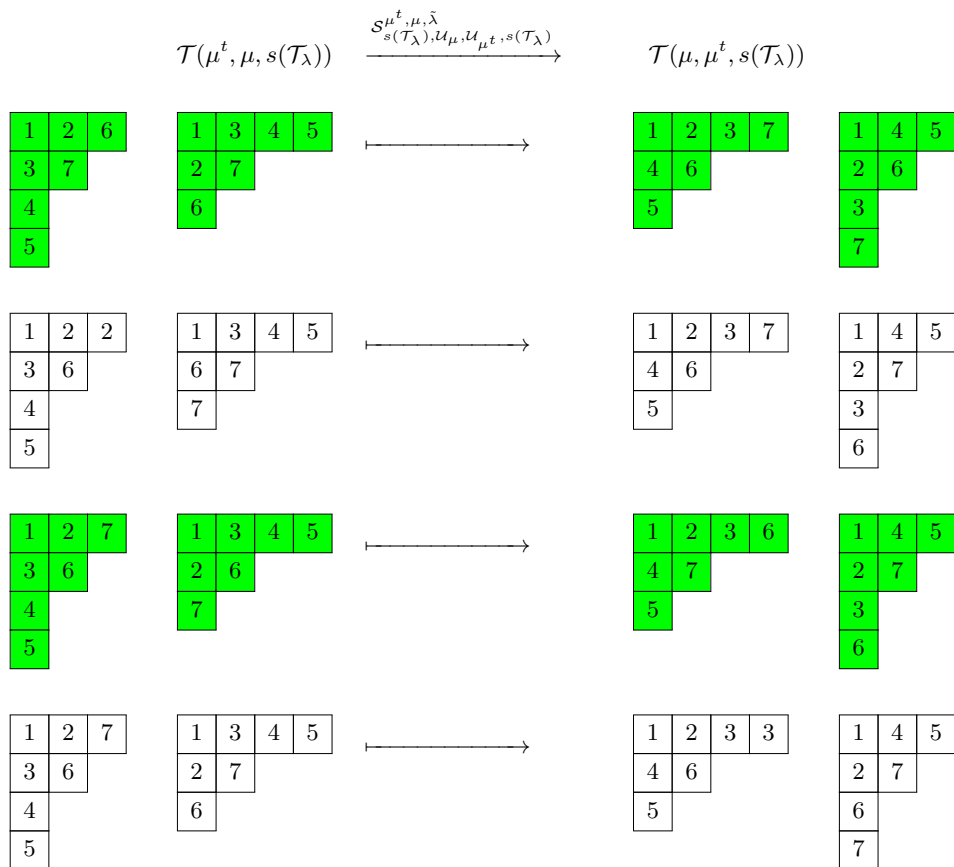
1. If (U_α^t, U_β) belongs to $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$, then we set $(\widetilde{U_\alpha^t}, \widetilde{U_\beta}) = (U_\alpha^t, U_\beta)$.
2. If (U_α^t, U_β) does not belong to $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$, then by Conjecture 4.32, (V_α, V_β^t) belongs to $\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))$. Set $(\widetilde{U_\alpha^t}, \widetilde{U_\beta})$ is the image of (V_α, V_β^t) through the bijection $\left(\mathcal{S}_{s(\mathcal{T}_\lambda), \mathcal{U}_\mu, \mathcal{U}_{\mu^t}, s(\mathcal{T}_\lambda)}^{\mu^t, \mu, \bar{\lambda}}\right)^{-1}$.

The set of all pairs $(\widetilde{U_\alpha^t}, \widetilde{U_\beta})$ we have constructed is a subset of $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$. Since its cardinal is $g_{\lambda\mu}^2$, we have $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^{\bar{\lambda}}$. \square

We can see the conjecture through following example.

Example 4.35. *Let $\lambda = (5, 2)$ and $\mu = (4, 2, 1)$. The correspondence between elements in $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$ and elements in $\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))$ is showed below (the elements in the subsets $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ and $\overline{\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))}$)*

are marked by coloring all boxes in green).



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Méthodes algébriques et combinatoires pour les problèmes de branchement en théorie des représentations

Résumé. Le but de cette thèse est d'étudier les questions entourant le problème de branchement en théorie des représentations à l'aide de méthodes algébriques et combinatoires. Sur la base des modèles et des idées des autres construits précédemment, nous développons et créons de nouvelles techniques, des modèles pour obtenir des résultats plus profonds. Concrètement, nous nous concentrons sur deux projets principaux :

Dans le premier projet, nous étudions le problème de branchement des algèbres Kac-Moody affines \mathfrak{g} sur ses sous-algèbres sinieuses $\mathfrak{g}[u]$ par des méthodes algébriques. Soit \mathfrak{h} la sous-algèbre Cartan de \mathfrak{g} . Nous prouvons que le support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ et $\Gamma(\mathfrak{g}, \mathfrak{h})$ de la décomposition de \mathfrak{g} -modules en tant que $\mathfrak{g}[u]$ -modules et \mathfrak{h} -modules sont des semi-groupes. Soit Λ, λ des poids entiers dominants de $\mathfrak{g}, \mathfrak{g}[u]$, respectivement. Soit δ la racine imaginaire de base de \mathfrak{g} . Dans les cas où \mathfrak{g} est de type $A_1^{(1)}$ et $A_2^{(2)}$, avec une certaine condition sur λ , nous pouvons décrire l'ensemble des nombres $b \in \mathbb{C}$ tels que $(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. Le résultat nous aide à réaliser la relation entre $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ et son paramètre saturé.

Dans le deuxième projet, nous étudions les coefficients apparaissant dans la théorie de représentation projective des groupes de symétrie S_n : les coefficients décalés de Littlewood-Richardson $f_{\lambda\mu}^\nu$ (λ, μ, ν sont des partitions strictes) et $g_{\lambda\mu}$ (λ est une partition stricte et μ est une partition) qui peuvent être considérés comme des cas particuliers de coefficients Littlewood-Richardson décalés. Nous obtenons une nouvelle interprétation pour les coefficients $f_{\lambda\mu}^\nu$. De plus, pour les coefficients $g_{\lambda\mu}$, nous obtenons également une autre description combinatoire, qui nous permet de voir les relations entre $g_{\lambda\mu}$ avec les coefficients Littlewood-Richardson $c_{\mu^t\mu}^\lambda$. Plus précisément, nous prouvons que $g_{\lambda\mu} = g_{\lambda\mu^t}$, et $g_{\lambda\mu} \leq c_{\mu^t\mu}^\lambda$. Nous conjecturons que $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^\lambda$ et formulons quelques conjectures sur nos modèles combinatoires qui impliquent cette inégalité si cela est valable.

Mots-clés : Algèbres affines de Kac-Moody, sous-algèbres sinieuses, règle de branchement, tableaux de Young, fonctions Schur, coefficients Littlewood-Richardson, changement de tableau, Grassmanniens, variétés Schubert, tableaux décalés, fonctions Schur Q , coefficients décalés Littlewood-Richardson, Grassmanniens Lagrangiens.

Algebraic and combinatorial methods in the branching problems in representation theory

Abstract. The purpose of this thesis is to study the questions surrounding the branching problem in representation theory using algebraic and combinatorial methods. Based on previously built models and ideas of the others, we develop and create new techniques, models to achieve deeper results. Concretely, we focus on two main projects:

In the first project, we study the branching problem of affine Kac-Moody algebras \mathfrak{g} on its winding subalgebras $\mathfrak{g}[u]$ by algebraic methods. Let \mathfrak{h} be the Cartan subalgebra of \mathfrak{g} . We prove that the support $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ and $\Gamma(\mathfrak{g}, \mathfrak{h})$ of the decomposition of \mathfrak{g} -modules as $\mathfrak{g}[u]$ -modules and \mathfrak{h} -modules are semigroups. Let Λ, λ be dominant integral weights of $\mathfrak{g}, \mathfrak{g}[u]$, respectively. Let δ be the basis imaginary root of \mathfrak{g} .

In the cases where \mathfrak{g} are types $A_1^{(1)}$ and $A_2^{(2)}$, with a certain condition on λ , we can describe the set of numbers $b \in \mathbb{C}$ such that $(\Lambda, \lambda + b\delta) \in \Gamma(\mathfrak{g}, \mathfrak{g}[u])$. The result helps us to realize the relation between $\Gamma(\mathfrak{g}, \mathfrak{g}[u])$ and its saturated setting.

In the second project, we study the coefficients appear in projective representation theory of symmetric groups S_n : the shifted Littlewood-Richardson coefficients $f_{\lambda\mu}^\nu$ (λ, μ, ν are strict partitions) and $g_{\lambda\mu}$ (λ is a strict partition and μ is partition) which can be considered as special cases of shifted Littlewood-Richardson coefficients. We obtain a new interpretation for the coefficients $f_{\lambda\mu}^\nu$. Moreover, for the coefficients $g_{\lambda\mu}$, we also obtain another combinatorial description, which allows us to see the relations between $g_{\lambda\mu}$ with Littlewood-Richardson coefficients $c_{\mu^t\mu}^\lambda$. Specifically, we prove that $g_{\lambda\mu} = g_{\lambda\mu^t}$, and $g_{\lambda\mu} \leq c_{\mu^t\mu}^\lambda$. We conjecture that $g_{\lambda\mu}^2 \leq c_{\mu^t\mu}^\lambda$ and formulate some conjectures on our combinatorial models which would imply this inequality if it is valid.

Keywords: Affine Kac-Moody algebras, winding subalgebras, branching rule, Young tableaux, Schur functions, Littlewood-Richardson coefficients, tableau switching, Grassmannians, Schubert varieties, shifted tableaux, Schur Q -functions, shifted Littlewood-Richardson coefficients, Lagrangian Grassmannians.

Image de couverture : Forêt de fleurs de lavande.

