Langlands Functoriality Conjecture for SO* (2n) in positive characteristic

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Conjecture de Fonctorialité de Langlands pour SO(2n) en caractéristique positive.*

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Summary

The Langlands program plays an important role in Number theory and Representation Theory. A crucial aspect of this program is the functoriality conjecture, expressed in a letter of Langlands to Weil in 1967. Let F be a global field with ring of adèles $\mathbb{A}_F$ and let

$$\rho\colon {}^L G \to {}^L H,$$

be a given L-homomorphism between the L-groups of two connected (quasi-split) reductive groups **G** and **H** over F. Then, according to this conjecture, for every cuspidal automorphic representation $\pi = \bigotimes_x \pi_x$ of $\mathbf{G}(\mathbb{A}_F)$, there exists an automorphic representation $\Pi = \bigotimes_x \Pi_x$ of $\mathbf{H}(\mathbb{A}_F)$ such that, at almost all places x where π_x is unramified, Π_x is unramified and its Satake parameter corresponds to the image under ρ of the Satake parameter of π_x. Such representation will be called a weak lift or transfer of π. Furthermore that transfer process should respect arithmetic information coming from γ-factors, L-functions and ε-factors, and lead to a local version of functoriality at the ramified places as well.

When **G** is a classical group, ${}^L G$ has a natural representation into ${}^L H$ for a specific general linear group **H**, and that case has been studied by many people. When F is a number field, two main tools have been used: converse theorem and trace formulas. The former was used by Cogdell, Kim, Piatetski-Shapiro and Shahidi in combination with the Langlands-Shahidi method to prove the conjecture for a globally generic automorphic representation π when **G** is a quasi-split symplectic, unitary or special orthogonal group. For the latter, Arthur and his continuators used trace formulas to get more complete results, not restricted to quasi-split groups in characteristic zero.

Lomelí extended the converse theorem method to global function fields, getting functoriality for globally generic automorphic representations of split classical groups and unitary groups. The present thesis further extends the converse theorem method, over a function field F, to establish the functoriality conjecture when **G** is a quasi-split non-split even special orthogonal group, and π a globally generic representation.

Theorem. *Let F be a global function field and π be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Then, π transfers to an irreducible automorphic*

representation Π of $\mathbf{GL}_{2n}(\mathbb{A}_F)$. Furthermore, Π can be expressed as an isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary self-dual cuspidal automorphic representation of $\mathbf{GL}_{N_i}(\mathbb{A}_F)$ for some N_i, and where $\Pi_i \ncong \Pi_j$ for $i \neq j$. Moreover if we write $\Pi = \otimes_x \Pi_x$, then for τ_x an irreducible generic unitary representation of $\mathbf{GL}_m(F_x)$

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x)$$

where the γ-factors on the right are obtained by the Rankin-Selberg method and those on the left by the Langlands-Sahidi method, as extended by Lomelí to positive characteristic.

As in Cogdell, Kim, Piatetski-Shapiro and Shahidi and Lomelí, the method of proof uses the converse theorem and L-functions to construct an automorphic representation of $\mathbf{GL}_n(\mathbb{A}_F)$: we provide a proof of a twisted version in positive characteristic of the converse theorem of Cogdell and Piatetski-Shapiro. To apply the converse theorem, one needs analytic properties of the Langlands-Shahidi L-functions, and to establish them we adapt Lomelí's arguments to our new case. We first obtain a weak lift which has the desired properties at almost all places. Then further properties of partial L-functions give that there is a lift which is an isobaric sum of unitary cuspidal automorphic representations. We prove the compatibility between the gamma local factors of π and the lift Π at all places.

As an application of the functoriality and the validity of the Ramanujan conjecture for general linear groups established by L. Lafforgue, we prove the unramified Ramanujan conjecture for globally generic cuspidal automorphic representations of our classical group in positive characteristic.

Theorem. *Let $\pi = \otimes_x \pi_x$ be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Then, if π_x is unramified, its Satake parameters have absolute value 1.*

Résumé

Le programme de Langlands joue un rôle important en théorie des nombres et en théorie des représentations. Un aspect crucial de ce programme est la conjecture de fonctorialité, formulée dans une lettre de Langlands à Weil en 1967. Soit F un corps global avec son anneau d'adèles $\mathbb{A}_F$ et soit

$$\rho\colon {}^L G \to {}^L H,$$

un L-homomorphisme donné entre les L-groupes de deux groupes réductifs quasi-déployés connexes **G** et **H** sur F. Alors, la conjecture dit que pour toute représentation automorphe cuspidale $\pi = \otimes_x \pi_x$ de $\mathbf{G}(\mathbb{A}_F)$, il existe une représentation automorphe cuspidale $\Pi = \otimes \Pi_x$ de $\mathbf{H}(\mathbb{A}_F)$ telle que, pour presque toute place x où π_x est non-ramifiée, Π_x est non ramifiée et son paramètre de Satake corresponds à l'image par ρ du paramètre de Satake de π_x. Une telle représentation sera appelée le transfert ou relèvement faible de π. Encore plus, elle doit respecter l'information arithmétique provenant des facteurs γ, fonctions L et facteurs ε, et donner aussi une version locale de la fonctorialité en les places ramifiées.

Quand **G** est un groupe classique, ${}^L G$ a une représentation naturelle dans un L-groupe ${}^L H$ d'un groupe linéaire général spécifique, et dans ce cas, la conjecture a été étudiée par diverses personnes. Quand F est un corps de nombres, deux techniques principales ont été utilisées : la méthode du théorème réciproque et la formule des traces. La première a été utilisée pour démontrer la conjecture pour les représentations automorphes cuspidales génériques π quand **G** est un groupe symplectique, un groupe unitaire, ou un groupe orthogonal spécial, tous quasi-déployés. La deuxième a été utilisée par Arthur et ses continuateurs pour obtenir de résultats plus complets, et non restreints aux groupes quasi-déployés en caractéristique nulle.

Lomelí étend la méthode du théorème réciproque au cas des corps de fonctions pour obtenir ces résultats en caractéristique positive pour les groupes classiques déployés et groupes unitaires. Cette thèse étend la méthode du théorème réciproque, sur un corps de fonctions F, pour établir la conjecture de fonctorialité quand $\mathbf{G}(\mathbb{A}_F)$ est un groupe spécial orthogonal pair quasi-déployé non-déployé, et π une représentation globalement générique.

Théorème. *Soit F un corps de fonctions et π une représentation automorphe cuspi-*

dale générique de $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$*. Alors,* π *se transfère à une représentation automorphe irréductible* Π *de* $\mathbf{GL}_{2n}(\mathbb{A}_F)$*. De plus,* Π *peut-être représentée par une somme isobare*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

où chaque Π_i *est une représentation automorphe cuspidale unitaire et auto-duale de* $\mathbf{GL}_{N_i}(\mathbb{A}_F)$*, pour quelque* N_i*, et où* $\Pi_i \ncong \Pi_j$ *pour* $i \neq j$*. Encore plus, si on écrit* $\Pi = \otimes_x \Pi_x$*, alors pour* τ_x *une représentation générique unitaire de* $\mathbf{GL}_m(F_x)$*, on a*

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x)$$

où les facteurs γ*à droite sont obtenus par la méthode de Rankin-Selberg et ceux à gauche par la méthode de Langlands-Shahidi, étendue par Lomelí à caractéristique positive.*

Comme chez Cogdell, Kim, Piatetski-Shapiro, et Shahidi, et Lomelí, la méthode de la preuve utilise le théorème réciproque de Cogdell et Piatetski-Shapiro et les fonctions L pour construire une représentation automorphe de $\mathbf{GL}_n(\mathbb{A}_F)$: On donne une version tordue du théorème réciproque de Cogdell et Piatetski-Shapiro. Pour appliquer le théorème réciproque, on a besoin de propriétés analytiques des fonctions L de Langlands-Shahidi, et pour les obtenir nous adaptons les arguments de Lomelí à notre nouveau cas. Ainsi, on obtient un relèvement faible qui satisfait les propriétés attendues en presque toute place. Des propriétés complémentaires des fonctions L partielles nous donnent l'existence d'un relèvement qui est une somme isobare de représentations automorphes cuspidales unitaires. Nous démontrons la compatibilité entre facteurs gamma locaux de π avec celles du transfert Π.

Comme application de la fonctorialité et de la conjecture de Ramanujan pour les groupes généraux linéaires, démontrée par L. Lafforgue, on démontre la conjecture de Ramanujan non-ramifiée pour les représentations automorphes cuspidales génériques de notre groupe classique quasi-déployé en caractéristique positive.

Théorème. *Soit* π *une représentation automorphe cuspidale générique de* $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$*. Alors, si* π_x *est non-ramifiée, ses paramètres ont valeur absolue* 1*.*

Resumen

El programa de Langlands juega un rol importante en teoría de números y en teoría de representaciones. Un aspecto crucial de este programa es la conjetura de funtorialidad, formulada en una carta de Langlands a Weil en 1967. Sea F un cuerpo global con anillo de adèles $\mathbb{A}_F$ y sea

$$\rho\colon {}^L G \to {}^L H,$$

un L-homomorfismo dado entre los L-grupos de dos grupos reductivos (cuasi escindidos) **G** y **H** sobre F. Entonces, según esta conjetura, para toda representación automorfa cuspidal π de $\mathbf{G}(\mathbb{A}_F)$, existe una representación automorfa Π de $\mathbf{H}(\mathbb{A}_F)$ tal que, para casi todo lugar x donde π_x es no ramificada, Π_x es no ramificada y su parámetro de Satake corresponde al de la imagen por ρ del parámetro de Satake de π_x. Una tal representación será llamada levantamiento débil o transferencia. Más aún, esta transferencia deberá respetar la información aritmética proveniente de los factores γ, funciones L y factores ε y conducir también a una versión local de la funtorialidad en los lugares ramificados.

Cuando **G** es un grupo clásico, ${}^L G$ tiene una representación natural en ${}^L H$ para un grupo general lineal específico **H**, y en ese caso ha sido estudiado por varias personas. Cuando F es un cuerpo de números, dos técnicas principales han sido usadas: teorema del recíproco y la fórmula de trazas. La primera técnica fue usada junto con el método de Langlands-Shahidi por Cogdell, Kim, Piatetski-Shapiro, y Shahidi para demostrar la conjetura de funtorialidad de Langlands para representaciones automorfas cuspidales globlamente genéricas de grupos simplécticos, unitarios y ortogonales cuasi escindidos. Con la segunda técnica, Arthur y sus continuadores lograron obtener resultados más completos, no restringiéndose a grupos cuasi escindidos en característica cero.

Lomelí extiende el método del teorema del recíproco a cuerpos de funciones, obteniendo así la funtorialidad para representaciones autormofas cuspidales de grupos clásicos escindidos y grupos unitarios. Esta tesis extiende el método del teorema del recíproco, sobre un cuerpo de funciones F, para obtener la conjetura de funtorialidad cuando **G** es un grupo ortogonal especial par cuasi escindido no escindido y π una representación globalmente genérica.

Teorema. *Sea F un cuerpo de funciones global y π una representación automorfa cuspidal globalmente genérica de $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Entonces, π se transfiere a una representación automorfa irreducible Π de $\mathbf{GL}_{2n}(\mathbb{A}_F)$. Además, Π puede ser expresada como una suma isobárica*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

donde cada Π_i es una representación globalmente genéricas automorfa cuspidal unitaria auto dual de $\mathbf{GL}_{N_i}(\mathbb{A}_F)$ para algún N_i, y donde $\Pi_i \ncong \Pi_j$ para $i \neq j$. Más aún, si escribimos $\Pi = \otimes \Pi_x$, entonces para τ_x una representación unitaria genérica de $\mathbf{GL}_m(F_x)$

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x)$$

donde los factores γ del lado izquierdo son obtenidos usando el método de Rankin-Selberg y lo del lado derecho es obtenido usando el método de Langlands-Shahidi, extendido por Lomelí a característica positiva.

Como en Cogdell, Kim, Piatetski-Shapiro y Shahidi y Lomelí, el método de la demostración usa el teorema del recíproco y funciones L para construir una representación automorfa de $\mathbf{GL}_n(\mathbb{A}_F)$: proveemos una versión alabeada del teorema del recíproco de Cogdell y Piatetski-Shapiro. Para aplicar el teorema del recíproco, uno necesita propiedades analíticas de las funciones L de Langlands-Shahidi, y para establecerlas adoptamos los argumentos de Lomelí en nuestro nuevo caso. Así, uno obtiene una transferencia débil, la cual tiene las propiedades deseadas en casi todo lugar. Más propiedades de las funciones L parciales, nos da una transferencia que es una suma isobárica de representaciones automorfas cuspidales unitarias. Finalmente, demostramos la compatibilidad entre los factores gamma locales de π y los de la transferencia Π.

Como aplicación de la funtorialidad y usando la conjetura de Ramanujan para los grupos generales lineales demostrada por L. Lafforgue, probamos la conjetura de Ramanujan no ramificada para representaciones automofas cuspidaldes globamente genéricas de nuestro grupo clásico cuasi escindido en característica positiva.

Teorema. *Sea $\pi = \otimes \pi_x$ una representación automorfa cuspidal globalmente genérica de $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Entonces, si π_x es no ramificada, sus parámetros de Satake tienen valor absoluto 1.*

Introduction

The Langlands program plays an important role in Number Theory and Representation Theory. A crucial aspect of this program is the Langlands functoriality conjecture. In [10, 11] Cogdell, Kim, Piatetski-Shapiro and Shahidi prove Langlands functoriality for globally generic representations from split classical groups, unitary groups or even quasi-split special orthogonal groups to general linear groups. They have done this in the context of characteristic zero. Later, Lomelí in [37, 38] extends this result to split classical groups and unitary groups in positive characteristic. Now the main objective of this thesis is to establish the functoriality conjecture in the case of the quasi-split (non-split) special orthogonal group for globally generic representations in positive characteristic (Theorem 7.2.1).

As in Cogdell, Kim, Piatetski-Shapiro and Shahidi over number fields and Lomelí over function fields, the method of proof uses the converse theorem, which allows us to use properties of L-functions to construct an automorphic representation of general linear groups; we provide a proof of a twisted version (Theorem 3.2.1) in positive characteristic of the converse theorem of Cogdell and Piatetski-Shapiro [9]. The L-functions, related local factors and their basic properties, necessary to apply the converse theorem, are provided by the Langlands-Shahidi method, as extended by Lomelí to positive characteristic. One first obtains a so-called "weak lift" which has the desired properties at almost all places, and further properties of partial L-functions give that there is a lift which is an isobaric sum of unitary cuspidal automorphic representations. As an application, we prove the Ramanujan conjecture for globally generic cuspidal representations of our quasi-split classical group (in positive characteristic).

Now we proceed to present the main results obtained in this thesis, by giving a small exposition of the ideas and tools used in this work to prove them.

Functoriality Conjecture

The functoriality conjecture first appeared in a letter from Langlands to André Weil in 1967. To formulate this conjecture, let F be a global field with ring of adèles $\mathbb{A}_F$ and let

$$\rho\colon {}^{L}G \to {}^{L}H,$$

be a given L-homomorphism between the L-groups of two connected (quasi-split) reductive groups **G** and **H** over F. Then, according to this conjecture, for every cuspidal automorphic representation $\pi = \otimes_x \pi_x$ of $\mathbf{G}(\mathbb{A}_F)$, there exists an automorphic representation $\Pi = \otimes \Pi_x$ of $\mathbf{H}(\mathbb{A}_F)$ such that, at almost all places x where π_x is unramified, Π_x is unramified and its Satake parameter corresponds to the image under ρ of the Satake parameter of π_x. Furthermore it respects arithmetic information coming from γ-factors, L-functions and ε-factors. This representation Π will be called lift or transfer.

This general problem has been central in the work of many people. Now, the main goal of this thesis is to find a transfer in the case where ρ is a certain embedding of the L-group of an even quasi-split special orthogonal group into the L-group of an appropriate general linear group, and π is globally generic.

Theorem A (7.2.1 & 7.3.1)**.** *Let F be a global function field and let π be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Then, π transfers to an irreducible automorphic representation Π of $\mathbf{GL}_{2n}(\mathbb{A}_F)$ (Section 6.4). Its central character is given by (6.1.3) and Π can be expressed as an isobaric sum*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary self-dual cuspidal automorphic representation of $\mathbf{GL}_{N_i}(\mathbb{A}_F)$ for some N_i, and where $\Pi_i \ncong \Pi_j$ for $i \neq j$. Moreover if we write $\Pi = \otimes \Pi_x$, then for τ_x an irreducible generic unitary representation of $\mathbf{GL}_m(F_x)$

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x)$$

where on the left hand side we use Langlands-Shahidi γ-factors.

Next, we explain the different elements of this result and provide the ideas and tools to prove it.

Special Orthogonal Groups

The group that we will be interested in is the non-split quasi-split even special orthogonal group. We are going to define this group in all characteristics, including two (Section 1.4). In order to define it, we follow the general construction of the special orthogonal group of even dimensional quadratic forms given in [12].

Having this general construction as a base, we then specialize it to the following case. Let E/F be a separable quadratic extension of a field F and consider the quadratic form

$$q_{E,n}(x_1, \ldots, x_{n-1}, x, x_{n+2}, \ldots, x_{2n}) = x_1 x_{2n} + \cdots + x_{n-1} x_{n+2} + \mathrm{N}_{E/F}(x),$$

where each $x_i \in F$ and $x \in E$. By $\mathbf{SO}^*_{2n}$, we refer to the non-split quasi-split even special orthogonal group $\mathbf{SO}(q_{E,n})$.

Following the general algebraic properties of these groups, we have that the connected component of the L-group of $\mathbf{SO}^*_{2n}$ is $\mathbf{SO}_{2n}(\mathbb{C})$ and we can construct the L-homomorphism for which we will answer the functoriality conjecture (Section 5.6):

$$\rho^*_{2n} : \mathbf{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_F \to \mathbf{GL}_{2n}(\mathbb{C}) \times \Gamma_F.$$
$$(g, \tau) \mapsto \begin{cases} (gw, \tau) & \text{if } \tau \notin \Gamma_E \\ (g, \tau) & \text{if } \tau \in \Gamma_E \end{cases}$$

where

$$w = \begin{pmatrix} 1_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1_{n-1} \end{pmatrix}.$$

To finish our discussion on the algebraic properties of our group, we mention two cases in Section 1.5, that will be relevant in calculating the central character of the lift, and also the γ-factor in a special case: when $n = 1$, $\mathbf{SO}^*_2$ is isomorphic to the "norm one F-torus" associated to E/F; when $n = 2$, the simply connected cover of $\mathbf{SO}^*_4$ is isomorphic to

$$\mathrm{Res}_{E/F} \mathbf{SL}_2 .$$

Langlands Parameters

Now that we have our groups and the L-homomorphism, we recall the notion of Langlands parameters. They will be useful in the construction of the candidate lift. First, if F is a non-archimedean local field, we fix a geometric Frobenius element $\mathrm{Fr} \in W_F$ (Section 5.1). For a connected quasi-split reductive group $\mathbf{G}$ over F, we denote by $\Phi(G)$ the set consisting of group homomorphism

$$\phi \colon W'_F = W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$

such that $\phi(\mathrm{Fr})$ is semi simple, $\phi|_{I_F}$ is continuous, $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic and ϕ satisfies certain relevance condition, modulo ${}^L G^\circ$-conjugacy (see Section 5.2). Moreover, in the case that $\phi|_{I_F}$ and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ are trivial, ϕ will be called unramified.

With this notion, the local unramified condition in the functoriality conjecture can be expressed (up to notation) as the commutativity of the following diagram (Section

5.4):

$$\begin{array}{ccc} \mathbf{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_F & \xrightarrow{\rho^*_{2n}} & \mathbf{GL}_{2n}(\mathbb{C}) \times \Gamma_F, \\ & {\scriptstyle \phi_\pi} \nwarrow \quad \nearrow {\scriptstyle \phi_{\Pi'}} & \\ & W'_F & \end{array}$$

where ϕ_π is the unramified parameter associated to the semisimple conjugacy class of an unramified representation π of $\mathbf{SO}^*_{2n}(F)$ and Π' is the unramified representation of $\mathbf{GL}_{2n}(F)$ associated to the semisimple conjugacy class $\phi_{\Pi'}(\mathrm{Fr}) = (\rho^*_{2n} \circ \phi_\pi)(\mathrm{Fr})$.

Furthermore, this setup also tells us how to find the transfer for $n = 1$ (relative rank 0). Indeed, using a combination of the Langlands correspondence for tori and $\mathbf{GL}_2$, we give a description of it. First, let σ be the non-trivial element of $\mathrm{Gal}(E/F)$ and $\phi_{E^\times}$ the Langlands parameter of the extension of scalar of $\mathbf{SO}^*_2$ along $E \supset F$, obtained from a restriction procedure for ϕ (5.3.3). The description of the transfer is thus obtained using the commutative diagram

$$\begin{array}{ccc} \mathbf{SO}^*_2(E) & \xrightarrow{\mathrm{Norm}} & \mathbf{SO}^*_2(F), \\ & {\scriptstyle \pi_{\phi_{E^\times}}} \searrow \quad \swarrow {\scriptstyle \pi_\phi} & \\ & \mathbb{C}^\times & \end{array}$$

given by naturality of the local Langlands correspondence $(\phi \mapsto \pi_\phi)$ for tori and the isomorphism between $\mathbf{SO}^*_2$ and the "norm one F-torus" associated to E/F mentioned before (thus for example $\mathbf{SO}^*_2(E) = E^\times$ and $\mathbf{SO}^*_2(F) = E^1$, and with these identifications the Norm sends x to $x\sigma(x)^{-1}$). This allows us to obtain that, for the character $\pi_\phi = \chi \colon E^1 \to \mathbb{C}^\times$, associated to the parameter ϕ, the representation Π' is given by (6.1.1)

$$\begin{cases} i^{GL_2}_{B_2}(\nu \otimes \varkappa\nu) & \text{if } \mu = \nu \circ \mathrm{N}_{E/F}, \text{ for some smooth character } \nu \text{ of } F^\times, \\ \pi_\mu & \text{otherwise,} \end{cases}$$

where $\mu \colon E^\times \to \mathbb{C}^\times$ is given by $[x \mapsto \chi(x\sigma(x)^{-1})]$, $\varkappa$ is the determinant of the representation of W_F induced by the trivial representation of W_E, viewed as character $F^\times$ via the Artin Reciprocity map (Section 5.1), and π_μ is a special case of the local Langlands correspondence for $\mathbf{GL}_2$ appearing in [8]. Moreover, we note that by construction the central character of Π' is

$$\varkappa\chi|_{F^\times} = \varkappa.$$

Finally, we also use this construction to produce a principal series with good enough data at the missing places to complete our unramified data (Section 6.1).

Up to now, we can think of forming a global representation from the local pieces. However, we need tools to check that these constructions form together an automorphic form for $\mathbf{GL}_{2n}$. For that, two ingredients are needed: the converse Theorem and the Langlands-Shahidi method.

Converse theorem

The converse theorem gives us a way to prove that an irreducible admissible representation of $\mathbf{GL}_n(\mathbb{A}_F)$ is automorphic, whenever we can prove certain properties of appropriately twisted families of Rankin-Selberg L-functions and ε-factors. This theorem is a vast generalization of the classical converse theorem of Hecke for modular forms. In our setup, we will prove and use an adapted version of the converse theorem in positive characteristic, already stated in [37].

Our proof of the converse theorem will closely follow the arguments appearing in the version by L. Lafforgue [33]. The differences between his version and ours are that we will allow a twist and that the representation of $\mathbf{GL}_n(\mathbb{A}_F)$ is irreducible and admissible.

Let F be a global field of positive characteristic with field of constants $\mathbb{F}_q$, $|F|$ its set of places and let ψ be a non-trivial character of $\mathbb{A}_F/F$. For a finite subset S of $|F|$ and a continuous character η of $\mathbb{A}_F^\times/F^\times$, we let $\mathcal{T}(S;\eta)$ be the set consisting of $\tau = \tau_0 \otimes \eta$, where τ_0 is a cuspidal automorphic representation of $\mathbf{GL}_m(\mathbb{A}_F)$, such that $\tau_{0,x}$ is unramified for every $x \in S$, and m is an integer ranging from $1 \leq m \leq n-1$. Given a smooth representation π, we let $\tilde{\pi}$ denote its smooth contragredient representation.

Theorem B (3.2.1)**.** *Let $n \geq 2$ be an integer and $\pi = \bigotimes_{x\in|F|} \pi_x$ an irreducible admissible representation of $\mathbf{GL}_n(\mathbb{A}_F)$. We suppose that, for a finite set S of places of F, π satisfies the following properties:*

i) *The central character $\chi_\pi = \bigotimes_{x\in|F|} \chi_{\pi_x}$ of π is invariant by the discrete subgroup $F^\times$ of $\mathbb{A}_F^\times$.*

ii) *For all $\pi' \in \mathcal{T}(S;\eta)$, the formal series in q^{-s}*

$$L(s, \pi \times \pi') \text{ and } L(s, \tilde{\pi} \times \tilde{\pi}')$$

are polynomials in q^{-s} and they satisfy the functional equation

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi', \psi) L(1-s, \tilde{\pi} \times \tilde{\pi}').$$

Then there exists an irreducible automorphic representation of $\mathbf{GL}_n(\mathbb{A}_F)$, where the factor at each place $x \notin S$ such that π_x is unramified, is unramified and it corresponds to the factor π_x of π, via the Satake parametrization. This representation is cuspidal if $S = \emptyset$.

Let Ξ_x be the representation of $\mathbf{GL}_n$, for which π_x is the Langlands quotient (Section 2.2). The representation Ξ_x has the following form

$$\Xi_x = i_{\mathbf{Q}(F_x)}^{\mathbf{GL}_n(F_x)}(\rho_{1,x}|\det|^{u_{1,x}} \otimes \cdots \otimes \rho_{m_x,x}|\det|^{u_{m_x,x}}),$$

where **Q** is the parabolic subgroup containing the Borel subgroup of upper triangular matrices associated to the ordered partition $(r_{m_x,x},\dots,r_{1,x})$ of n, $\rho_{i,x}$ is an irreducible tempered representation of $\mathbf{GL}_{r_i,x}(F_x)$ and the $u_{i,x}$ are real numbers satisfying $0 < u_{1,x} \leq \cdots \leq u_{m_x,x}$. The representation Ξ_x is induced of Whittaker type (Section 3.1).

The first step is to reduce the problem to $\eta = 1$. This can be done thanks to the following general identity: for every pair τ_x, τ'_x of induced representations of Whittaker type of $\mathbf{GL}_n(F_x)$ and $\mathrm{GL}_m(F_x)$, respectively, we have that

$$L(s,\tau_x \times (\tau'_x \otimes \eta_x)) = L(s,(\tau_x \otimes \eta_x) \times \tau'_x).$$

Once we are in the case where $\eta = 1$, we can follow the constructions in [33]. That is, using the Whittaker models, we construct in (3.2.5) a non-zero equivariant homomorphism from the admissible representation $\otimes_{x\notin S}\Xi_x$ to $\prod'_{x\notin S}\mathbf{GL}_n(F_x)$ into the space of functions on $\mathbf{GL}_n(F)\backslash\mathbf{GL}_n(\mathbb{A}_F)$ that are smooth in the sense of being invariant under some open compact subgroup of $\mathbf{GL}_n(\mathbb{A}_F)$ of $\mathbf{GL}_{n(\mathbb{A}_F)}$ acting via the right regular action. We then proceed to construct the desired automorphic representation. In this way we can obtain the desired version of the converse theorem. Thus after we locally construct the lift and form a global representation of $\mathbf{GL}_{2n}$ from the quasi-split Special Orthogonal Group $\mathbf{SO}^*_{2n}$ using the parameter formalism, we need to check the conditions of the converse theorem. For that we need to use the following central tool in our work.

Langlands-Shahidi Method

Let $\mathbf{P} = \mathbf{MN}$ be a (standard, i.e. containing a Borel subgroup) maximal parabolic subgroup of a connected quasi-split group **G** over a non-archimedean local field F of positive characteristic, where **M** and **N** are the Levi and unipotent subgroups, respectively. Let also ${}^L\mathfrak{n}$ be the Lie algebra of the corresponding unipotent subgroup of the L-group of **G**, on which ${}^LM$ acts via the adjoint action. Fix ψ a non-trivial additive character of F. The adjoint representation has the following decomposition into irreducible representations,

$$r = \bigoplus_{i=1}^{m_r} r_i : {}^LM \to \mathrm{GL}({}^L\mathfrak{n}).$$

The Langlands-Shahidi method associates to a generic representation σ of $\mathbf{M}(F)$ a rational polynomial in $\mathbb{C}(q_F^{-s})$, called γ-factor

$$\gamma(s,\sigma,r_i,\psi).$$

This construction was first developed by Shahidi [46] in characteristic zero. Later Lomelí [38] developed this in positive characteristic and it will be in this context that we will work.

In our case when $\mathbf{G} = \mathbf{SO}^*_{2(m+n)}$, and the maximal parabolic has Levi subgroup $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}^*_{2n}$, the adjoint representation we consider has the following decomposition

$$r = r_1 \oplus r_2,$$

where $r_1 = \rho_m \otimes \tilde{\rho}^*_{2n}$ and $r_2 = \wedge^2 \rho_m$ (Section 2.5). Here, ρ_m is the standard representation of $\mathbf{GL}_m(\mathbb{C})$ and $\tilde{\rho}^*_{2n}$ is the contragredient the representation ρ^*_{2n}, introduced before in the section titled Special Orthogonal Groups . Now using this general construction we can specify the first factor appearing in Theorem A. Let π be a generic representation of $\mathbf{SO}^*_{2n}(F)$ and τ a generic representation of $\mathbf{GL}_m(F)$. Then $\sigma = \tau \otimes \tilde{\pi}$ is a generic representation of $\mathbf{M}(F)$ and we denote

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \tau \otimes \tilde{\pi}, r_1, \psi).$$

An important feature of this system of γ-factors is the multiplicativity property and its relation with Tate local factors. If $m = 1$, π is a generic constituent of $i_{P_0}^{SO^*_{2n}}(\chi_1, ..., \chi_{n-1}, \eta)$ and τ is a representation of $F^\times$, these two properties give us the following form for the γ-factors (see (2.9.1)):

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \eta \times \tau, \psi) \prod_{i=1}^{n-1} \gamma(s, \chi_i \tau, \psi)\gamma(s, \chi_i^{-1}\tau, \psi),$$

where $\gamma(s, \chi_i\tau, \psi)$ are Tate factors. The factor $\gamma(s, \eta \times \tau, \psi)$ can be characterised using the construction of γ-factors and the determination of the simply connected cover of $\mathbf{SO}^*_2$ (see Proposition 2.9.2).

Another essential property of γ-factors, is their stability under twists by sufficiently highly ramified characters (Theorem 2.8.2). This important input comes from the work of Gan-Lomelí in [16], where they establish this powerful result in positive characteristic.

The Langlands-Shahidi method provides the means to construct local L-functions and ε-factors

$$\varepsilon(s, \pi, r_i, \psi) \quad \& \quad L(s, \pi, r_i)$$

that complete the ones defined in the unramified case and satisfy the expected properties. For that, a property is needed, whose validity amounts to the Shahidi tempered L-function conjecture. It is known for arbitrary quasi-split groups in characteristic zero [21] and for split groups in positive characteristic [39]. In this thesis we prove it in a special case (Remark 2.10.3), to use it in the construction of the lift. Later, in order to obtain the full compatibility between the local components of the cuspidal automorphic representation of $\mathbf{SO}^*_{2n}$ and those of its transfer, we will assume the property.

Now let us go back to the global situation. Let F be a global field of positive characteristic. Then the Langlands-Shahidi method allows us to define for a globally generic cuspidal automorphic representation $\pi = \otimes_x \pi_x$ of $\mathbf{M}(\mathbb{A}_F) \subset \mathbf{G}(\mathbb{A}_F)$, the global Langlands-Shahidi L-functions and ε-factors

$$L(s, \pi, r_i) := \prod_{x \in |F|} L(s, \pi_x, r_{i,x}) \quad \& \quad \varepsilon(s, \pi, r_i) := \prod_{x \in |F|} \varepsilon(s, \pi_x, r_{i,x}, \psi_x).$$

They satisfy the crucial functional equation

$$L(s, \pi, r_i) = \varepsilon(s, \pi, r_i)L(1 - s, \tilde{\pi}, r_i).$$

Global L-functions

Before we continue, we need to study global L-functions in more detail. For this, we use the constructions in the theory of Eisenstein Series in positive characteristic. They allow us to establish two main properties we need for the global L-functions. The first one is the holomorphy for each L-function associated to a generic cuspidal automorphic representation π of $\mathbf{M}(\mathbb{A}_F)$ that satisfies $\tilde{w}_0\pi \ncong \pi$ (Corollary 4.2.1), where $\tilde{w}_0$ is a representative of $w_0 = w_{l,\mathbf{G}}w_{l,\mathbf{M}}$ (Section 1.1). The holomorphy of the intertwining operator and the Eisenstein coefficient, for representations that satisfy $\tilde{w}_0\pi \ncong \pi_0$, gives us a partial result. We complete it by using a local property on normalized intertwining operators (2.11.2). We then obtain that

$$\prod_{i=1}^{m_r} \frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)} \quad (\text{resp. } \prod_{i=1}^{m_r} L(1+is,\pi,r_i))$$

is holomorphic for $\operatorname{Re} s \geq 1/2$ (resp. holomorphic and non-zero for $\operatorname{Re} s \geq 0$). We conclude by using induction on m_r. This discussion is valid for a general quasi-split reductive group, that satisfies the property on the normalized intertwining operators. This last property is known as the Kim's assumption. We will prove it under the assumption that the representations that we are considering, satisfy the standard module conjecture in a special case (Section 2.11).

The following property is the holomorphy (without twist) of partial L-functions (Theorem 4.2.6) for a cuspidal representation of

$$\mathbf{GL}_l \times \mathbf{SO}^*_{2m} \subset \mathbf{SO}^*_{2(l+m)}\,.$$

We study the intertwining operator as before. Using that $\mathbf{SO}(q_{F_x\otimes_F E,n})$ is split for a subset of density $1/2$ of the unramified places x of E/F and the relation of the intertwining operator with the residual spectrum, we show that it is holomorphic for $\operatorname{Re} s > 1$ (Proposition 4.1.5). We finish by using the fact that $m_r = 2$, which enables us to arrive at a Siegel Levi case studied in general in [39], and doing a similar analysis as above we arrive at the following

Theorem C (4.2.6)**.** *Let* $\mathbf{G} = \mathbf{SO}^*_{2n}$*, and let* $\mathbf{P} = \mathbf{M}\,\mathbf{N}$ *be a parabolic subgroup of with Levi subgroup* $\mathbf{M}$ *of the form* $\mathbf{GL}_m \times \mathbf{SO}^*_{2n}$*. Let* $\sigma = \tau \otimes \tilde{\pi}$ *be a generic cuspidal automorphic representation of* $\mathbf{M}(\mathbb{A}_F)$ *unramified outside of a finite set* S *of places. Then* $L^S(s,\sigma,r_1)$ *is holomorphic and non-vanishing for* $\operatorname{Re}(s) > 1$ *and has at most a simple pole at* $s = 1$*.*

This result will play a main role in the study of the image of functoriality.

Functoriality for $\mathbf{SO}^*_{2n}$

We now have all the ingredients ready to apply them to our problem. Let $\pi = \otimes_x \pi_x$ be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$, such that π_x is

unramified outside for $x \notin S$. We fix a maximal torus $\mathbf{T}$ of $\mathbf{SO}^*_{2n}$, $\mathbf{P}_0$ a Borel subgroup of $\mathbf{SO}^*_{2n}$ containing $\mathbf{T}$ and $\psi = \otimes\psi_x$ a nontrivial character of $\mathbb{A}_F/F$, such that ψ_x is unramified for $x \notin S$.

For every place x such that π_x is unramified, we choose a character λ_x of $\mathbf{T}(F_x)$ related to π_x via the Satake parametrization (5.4.1) and for x ramified, the induced representation $i_{P_0}^{SO^*_{2n}}(\lambda_x)$ has an (irreducible) generic subquotient $\pi'_{\lambda_x} = \pi'_x$ with the same central character of π_x. From λ_x, we get $\phi_{\lambda_x} : W'_{F_x} \to {}^L T_x$. Now let $\mathbf{T}_n = \{t = \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})\}$ be a maximal split torus of $\mathbf{SO}_{2n}$. Then, after composing these parameters with the inclusion $i_x : \mathbf{T}_n(\mathbb{C}) \rtimes \Gamma_{F_x} \hookrightarrow \mathbf{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{F_x}$ and applying local Langlands correspondence to

$$\rho_x \circ i_x \circ \phi_{\lambda_x} \colon W'_{F_x} \to \mathrm{GL}_{2n}(\mathbb{C}) \times \Gamma_{F_x},$$

we find an admissible irreducible representation Π'_x of $\mathbf{GL}_{2n}(F_x)$. Finally we put $\Pi' = \otimes_x \Pi'_x$, which is an irreducible admissible representation of $\mathbf{GL}_{2n}(\mathbb{A}_F)$. Using the construction (6.1.1) mentioned in the section titled Langlands Parameters, we have a description of these representations and their central characters.

We then use the properties of the Langlands-Shahidi γ-factors, L-functions and ε-factors, to prove that we have the following compatibility between the Langlands-Shahidi and Rankin-Selberg L-functions and ε-factors of the lift (Corollary 6.2.8). Namely, there is a character η of $\mathbb{A}_F^\times/F^\times$, such that for every cuspidal automorphic representation $\tau \in \mathcal{T}(S;\eta)$, we have that

$$L(s, \pi \times \tau) = L(s, \Pi' \times \tau),$$
$$\varepsilon(s, \pi \times \tau, \psi) = \varepsilon(s, \Pi' \times \tau, \psi).$$

With these relations, we are in position to check the hypotheses of the converse theorem, under the assumption that a special case of the standard module conjecture is valid (2.11.1). Indeed, using a combination of the holomorphicity of the L-functions after a twist (in this case by η) and the functional equation, we get the polynomial condition of the global Rankin-Selberg L-function. Thus after applying the converse theorem, we are able to find a (weak) automorphic lift Π (Section 6.4) as a subquotient of a representation

$$\mathrm{Ind}(\Pi_1, \cdots, \Pi_d),$$

where $\Pi_1, \dots, \Pi_d$ are cuspidal automorphic representations of smaller general linear groups and Ind is the (global) normalized induction functor from a certain parabolic subgroup of $\mathbf{GL}_{2n}$ to $\mathbf{GL}_{2n}$. This settles the existence of the desired lift, but we go further and we verify additional important properties, that will be useful for further applications.

Theorem D (7.2.1)**.** *Let π be a globally unitary generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Then, π transfers to an irreducible automorphic representation Π of*

$\mathbf{GL}_{2n}(\mathbb{A}_F)$. *Its central character is given by the quadratic character of* $\mathbb{A}_F/F^\times$ *defining* E *(6.1.3) and* Π *can be expressed as an isobaric sum (see Section 6.4)*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i *is a unitary self-dual cuspidal automorphic representation of* $\mathbf{GL}_{N_i}(\mathbb{A}_F)$*, and* $\Pi_i \ncong \Pi_j$ *for* $i \neq j$*.*

In fact, combining holomorphy properties of Rankin-Selberg partial L-functions, Langlands-Shahidi partial L-functions (without twist) and the relation

$$L^S(s, \pi \times \tau) = L^S(s, \Pi \times \tau),$$

we can prove that the cuspidal factors Π_i are distinct, unitary and self dual in positive characteristic. As Π_i is unitary for every i, $\mathrm{Ind}(\Pi_1, \cdots, \Pi_d)$ is irreducible and thus Π coincides with the isobaric sum.

Next, we prove the following compatibility between the γ-factors of π and the lift Π.

Theorem E (7.3.1)**.** *Let* $\pi = \otimes \pi_x$ *be a globally generic cuspidal automorphic representation of* $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$ *and* Π *its transfer to* $\mathbf{GL}_{2n}(\mathbb{A}_F)$*. Let* $x \in |F|$ *and* m *a positive integer. Then for* τ_x *an irreducible generic unitary representation of* $\mathbf{GL}_m(F_x)$

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x).$$

In order to prove this for the γ-factors, we use the global functional equation of the L-functions. The combination of Theorem D and E yield the main result of this thesis (Theorem A).

Ramanujan Conjecture

Finally, thanks to the validity of the Ramanujan conjecture for $\mathbf{GL}_{2n}(\mathbb{A}_F)$ established by L. Lafforgue in [33], we prove the unramified Ramanujan conjecture for $\mathbf{SO}^*_{2n}$.

Theorem F (7.4.1)**.** *Let* π *be a globally generic cuspidal representation of* $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$*. Then, if* π_x *is unramified, its Satake parameters have absolute value* 1*.*

Perspectives

Our work relies on the tempered L-function property, where the general case was settled in characteristic 0 by Heiermann and Opdam [21] and by Lomelí for split groups in characteristic p [39]. However, all of the machinery in positive characteristic is now available to prove the property for quasi-split groups. Thus, we aim to bridge this gap

in future work. We observe that arguing as in [20], the tempered L-function property implies the standard module conjecture and also Kim's assumption. In particular, we provide a proof of Kim's assumption in Section 2.11 under the assumption of a weak version of the standard module conjecture.

However, we take special care throughout the thesis to give complete proofs that do not involve the full use of the tempered L-function conjecture, property (T). This results in the fact that Theorems A, E and F do not use property (T), but do use the weak version of the standard module conjecture, if the compatibility at all local places is changed to all unramified places. All other results are unconditional. For example, we prove Theorem D using partial L-functions, hence property (T) at ramified places is not required.

The construction of the transfer was made in the context of globally generic cuspidal representations of the even quasi-split non-split special orthogonal group. But now with the work in positive characteristic of V. Lafforgue on the global Langlands parametrisation and that of the Lafforgue-Genestier on the local Langlands parametrisation up to semisimplification, there are new paths to approach the functoriality conjecture for non-generic representations.

While we do not use the results of V. Lafforgue and Genestier-Lafforgue in this thesis, we do make use of the landmark result of L. Lafforgue on the global Langlands correspondence for general linear groups [33]. In particular, combining our functorial lift from $\mathbf{SO}^*_{2n}$ to $\mathbf{GL}_{2n}$ with the results of L. Lafforgue, we establish the Ramanujan conjecture for $\mathbf{SO}^*_{2n}$, at least at every unramified place.

Chapter 1

Special orthogonal groups

In this chapter we set up, as in [12], the general facts and properties of algebraic groups that will be useful throughout the thesis. In particular we will define the quasi-split non-split special orthogonal group. Let k be a field and fix a separable closure k_s and denote $\Gamma_k = \mathrm{Gal}(k_s/k)$. Let us denote by $\mathbf{G}_{k_s}$, the base-change to k_s of a group $\mathbf{G}$ over k.

1.1 Weights and parabolic subgroups. Let $\mathbf{H}$ be a smooth connected group over a field k. We denote

$$X(\mathbf{H}) = \mathrm{Hom}(\mathbf{H}, \mathbb{G}_m) \quad \& \quad X_*(\mathbf{H}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{H}).$$

the set of characters and co-characters of $\mathbf{H}$.

Assume that $\mathbf{H}$ is abelian. Then these sets are abelian groups and since $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ we have that the composition

$$\begin{aligned} \mathrm{Hom}(\mathbb{G}_m, \mathbf{H}) \times \mathrm{Hom}(\mathbf{H}, \mathbb{G}_m) &\to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \\ (\mu, \nu) &\mapsto \nu \circ \mu \end{aligned}$$

induces a natural pairing of abelian groups

$$\begin{aligned} X(\mathbf{H}) \times X_*(\mathbf{H}) &\to \mathbb{Z} \\ (\nu, \mu) &\mapsto \langle \nu, \mu \rangle. \end{aligned}$$

Let $\mathbf{S}$ be a k-split subtorus of $\mathbf{H}$. As $\mathbf{H}$ acts functorially on $\mathfrak{h} := \mathrm{Lie}(\mathbf{H})$ via the adjoint action so does $\mathbf{S}$. In this case we define

$$\Phi(\mathbf{H}, \mathbf{S}) := \{a \in X(\mathbf{S}) - \{0\} : \mathfrak{h}_a \neq 0\} \subset X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R},$$

where $\mathfrak{h}_a$ is the unique subspace of $\mathfrak{h}$ characterized be the property that $(\mathfrak{h}_a)_R = (\mathfrak{h}_a \otimes_k R) := \{v \in \mathfrak{h}_R \colon t \cdot v = a(t)v,\ t \in \mathbf{S}(R)\}$, for every k-algebra R.

Let $\mathbf{G}$ be a connected reductive group over k, $\mathbf{S}$ a maximal k-split torus in $\mathbf{G}$ and $\mathbf{T}$ a maximal k-torus in $\mathbf{G}$ containing $\mathbf{S}$. With these choices, we denote

$$\Phi = \Phi(\mathbf{G}_{k_s}, \mathbf{T}_{k_s}) \quad \& \quad \Sigma = {}_k\Phi = \Phi(\mathbf{G}, \mathbf{S}).$$

In this case, these are moreover root systems, which are also called absolute and relative root system respectively. The dual root system is denoted by $\Phi^\vee \subset X_*(\mathbf{T}_{k_s}) \otimes \mathbb{R}$ and $\Sigma^\vee \subset X_*(\mathbf{S}) \otimes \mathbb{R}$.

We can relate these informations with each other. Pick a minimal parabolic k-subgroup $\mathbf{P}_0$ such that $\mathbf{S} \subset \mathbf{T} \subset \mathbf{P}_0$, a Borel k_s-subgroup $\mathbf{T}_{k_s} \subset \mathbf{B} \subset \mathbf{P}_{0,k_s}$, and let

$$\Phi^+ = \Phi(\mathbf{B}, \mathbf{T}_{k_s}) \quad \& \quad \Sigma^+ = {}_k\Phi^+ = \Phi(\mathbf{P}_0, \mathbf{S}).$$

We also denote by Δ a basis of simple roots of Φ^+. This set carries an action of $\mathrm{Gal}(k_s/k) = \Gamma_k$ called the $*$-action [13, p.607] and [12, Remark 7.1.2]. The Γ_k-action on Δ does not come from the one on $X(\mathbf{T}_{k_s})$, though it does if $\mathbf{G}$ is quasi-split, i.e has a Borel subgroup $\mathbf{B}$ defined over k. Now thanks to the inclusion $\mathbf{S}_{k_s} \subset \mathbf{T}_{k_s}$ we have a surjective (restriction) homomorphism

$$X(\mathbf{T}_{k_s}) \to X(\mathbf{S}_{k_s}) = X(\mathbf{S})$$

which caries Φ into $\Sigma \cup \{0\}$, hitting all of Σ. Also it sends Φ^+ into $\Sigma^+ \cup \{0\}$, as $\mathbf{B} \subset \mathbf{P}_{k_s}$. Denote by Δ_0 the set $\{a \in \Delta \colon \mathrm{Res}^{T_{k_s}}_{S_{k_s}} a = 1\}$ and by ${}_k\Delta$ the image of $\Delta - \Delta_0$ in Σ. It can be proven that it is a basis of Σ [13, Theorem C.2.15]. The restriction map does give rise to a Γ_k-invariant map

$$\Delta \to {}_k\Delta \cup \{0\}.$$

(*Parabolic subgroup*). We can parametrize parabolic subgroups containing a fixed minimal parabolic subgroup $\mathbf{P}_0$ that contains $\mathbf{S}$ in the following way [13, Proposition 3.5.1]:

$$\begin{aligned} \{\text{Parabolic subgroups } \mathbf{P} \supset \mathbf{P}_0\} &\to \{\text{Parabolic subsets } \Psi \subset \Sigma \text{ containing } \Sigma^+\} \\ \mathbf{P} &\mapsto \Phi(\mathbf{P}, \mathbf{S}). \end{aligned}$$

Moreover, the sets on the right hand side are exactly the subsets ${}_k\Phi^+ \cup [\theta]$, where $\theta \subset {}_k\Delta$ and $[\theta] \coloneqq (\mathbb{Z} \cdot \theta) \cap {}_k\Phi$ [6, Ch. VI, §1, n° 7. Proposition 20]. Thus we obtain an inclusion preserving bijection (in both directions)

$$\begin{aligned} \{\text{Parabolic subgroups } \mathbf{P} \supset \mathbf{P}_0\} &\leftrightarrow \{\text{Subsets } \theta \subset {}_k\Delta\} \\ \mathbf{P}_\theta &\hookleftarrow \theta. \end{aligned}$$

(*Weyl Group*). The Weyl group of $\mathbf{G}$ is

$$W^{\mathbf{G}} = W(\mathbf{G}, \mathbf{S}) = N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k) \subset \mathrm{GL}(X(\mathbf{S}) \otimes \mathbb{R}).$$

It can be identified with the Weyl group $W({}_k\Phi)$ of the root system ${}_k\Phi$ [13, Theorem C.2.15]. We denote by $w_l = w_{l,{}_k\Delta} = w_{\mathbf{G}}$ the longest element of $W^{\mathbf{G}}$. Finally, for $w \in W^{\mathbf{G}}$, we denote by $\tilde{w}$ a lift in $N_{\mathbf{G}}(\mathbf{S})(k) \subset \mathbf{G}(k)$.

1.2 Root groups and Levi subgroups. To know more about the structure of reductive groups and its subgroups, we use the construction in [13, Proposition 3.3.6] that we now summarize. Let $\mathbf{G}$ be a smooth connected affine k-group equipped with an action by a split k-torus $\mathbf{S}$ and let $A \subset X(\mathbf{S})$ be a semigroup not containing 0. There exists a unique $\mathbf{S}$-stable smooth connected k-subgroup

$$\mathbf{U}_A(\mathbf{G})$$

such that $\mathrm{Lie}(\mathbf{U}_A(\mathbf{G}))$ is the span of the a-weight spaces for all $a \in A \cap \Phi(\mathbf{G}, \mathbf{S})$. This k-group is unipotent and contains any S-stable smooth connected k-subgroup $\mathbf{H} \subset \mathbf{G}$ such that all weights of $\mathbf{S}$ on $\mathrm{Lie}(\mathbf{H})$ belong to A.

Now this construction allows us to consider the (relative) root groups as follows. For any $a \in X(\mathbf{S}) - \{0\}$, let $\langle a \rangle$ be the semigroup consisting of positive integral multiples of a. For $a \in \Sigma$, we define the (relative) root group associated to $a \in \Sigma$ as

$$\mathbf{U}_a \coloneqq \mathbf{U}_{\langle a \rangle}(\mathbf{G}).$$

(*Levi subgroups*). These groups can also be used to describe the parabolic subgroups. Indeed for any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{S}$, if we denote by $\Psi_{\mathbf{P}}$ the subset of Σ that consists of all the $a \in \Sigma$ such that $-a \notin \Phi(\mathbf{P}, \mathbf{S})$, then [13, Proposition C.2.26]

$$R_{u,k}(\mathbf{P}) = \mathbf{U}_{\Psi_{\mathbf{P}}},$$

where $R_{u,k}(\mathbf{P})$ is the k-unipotent radical of $\mathbf{P}$. In fact this description enables us to not only characterize the radical uniponent subgroups of a parabolic subgroup, but also to find an explicit expression of their Levi subgroups. Let $\mathbf{P}_0$ be a minimal parabolic subgroup containing $\mathbf{S}$. Let $\mathbf{P}_\theta \supset \mathbf{P}_0$, where $\theta \subset {}_k\Delta$. Let's consider

$$\mathbf{A}_{\mathbf{P}_\theta} = \mathbf{S}_\theta \coloneqq \Big(\bigcap \ker a\Big)^0_{red} \subset \mathbf{S} \quad \& \quad \mathbf{M}_{\mathbf{P}_\theta} \coloneqq \mathrm{Z}_{\mathbf{G}}(\mathbf{S}_\theta).$$

Then, we have the following Levi decomposition

$$\mathbf{P}_\theta = \mathrm{Z}_{\mathbf{G}}(\mathbf{S}_\theta) \ltimes \mathbf{U}_{\Psi_{P_\theta}}.$$

As a consequence of this explicit description we get that every parabolic $\mathbf{P}$ subgroup containing $\mathbf{S}$ has a unique Levi subgroup $\mathbf{M}_P \supset \mathbf{S}$. Moreover, $\mathbf{M}_{\mathbf{P}}$ is the centralizer of a split subtorus $\mathbf{A}_{\mathbf{P}} = \mathbf{A}_{\mathbf{M}_P} \subset \mathbf{S}$. For a fixed $\mathbf{S}$, these are often called semi-standard parabolic subgroup and semi-standard Levi subgroup.

1.3 L-group. Let $R = (X(\mathbf{T}_{k_s}), \Phi, X_*(\mathbf{T}_{k_s}), \Phi^\vee)$ be the absolute root datum associated to $\mathbf{G}$. Consider a pinning $(\Phi^+, \{X_a\}_{a \in \Delta})$ on $(\mathbf{G}_{k_s}, \mathbf{T}_{k_s})$ i.e. a choice of basis of $\mathfrak{g}_a$ for each

$a \in \Delta$. The Isomorphism Theorem [12, Theorem 6.1.17] gives that the natural map

$$\mathrm{Aut}_{k_s}(\mathbf{G}_{k_s}, \mathbf{T}_{k_s}, \{X_a\}_{a\in\Delta}) \to \mathrm{Aut}(R, \Delta)$$

is bijective.

Let $\widehat{\mathbf{G}}$ be the unique pinned connected reductive group (scheme) over $\mathbb{C}$ whose based root datum is equipped with an identification with $(R^\vee, \Delta^\vee)$, where $R^\vee = (X_*(\mathbf{T}_{k_s}), \Phi^\vee, X(\mathbf{T}_{k_s}), \Phi)$, obtained from the Isomorphism theorem [12, Theorem 6.1.17]. Moreover, as this latter root datum carries the $*$-action, it defines a composite homomorphism

$$\rho\colon \Gamma_k \to \mathrm{Aut}(R^\vee, \Delta^\vee) \hookrightarrow \mathrm{Aut}(\widehat{\mathbf{G}}). \tag{1.3.1}$$

The embedding

$$\mathrm{Aut}(R^\vee, \Delta^\vee) \hookrightarrow \mathrm{Aut}(\widehat{\mathbf{G}})$$

depends on the choice of pinning of $\widehat{\mathbf{G}}$ [12, p 243, (7.1.3)], but the $\widehat{\mathbf{G}}(\mathbb{C})$-conjugacy class of this homomorphism is independent of this choice. The Langlands dual of $\mathbf{G}$ is the disconnected locally algebraic group

$$^L\mathbf{G}_k := \Gamma_k \ltimes \widehat{\mathbf{G}},$$

well-defined up to $\widehat{\mathbf{G}}(\mathbb{C})$-conjugation. We write $^LG_k := {}^L\mathbf{G}_k(\mathbb{C})$ and $^LG_k^0 := \widehat{\mathbf{G}}_k(\mathbb{C}) \subset {}^LG_k$. If the context allows us, we just write ^LG.

If $\mathbf{H}$ is another reductive group over k with L-group $^L\mathbf{H}_k$, then an L-homomorphism from ^LG_k to ^LH_k is a continuous group homomorphism ρ whose restriction to $^LG_k^0$ is a complex analytic homomorphism of $^LG_k^0$ into $^LH_k^0$, such that the following diagram is commutative

$$\begin{array}{ccc} {}^LG_k & \xrightarrow{\quad\rho\quad} & {}^LH_k \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

1.4 Special orthogonal groups. We start with the general construction of $\mathbf{SO}(q)$, for an even dimensional non-degenerate quadratic space $Q = (V, q)$. First we recall some definitions: for a k-algebra R, a R-quadratic space is a pair (V, q) of a finite free R-module V and a quadratic form $q : V \to R$ i.e.

i) $q(rv) = r^2q(v)$, for all $r \in R$ and $v \in V$,

ii) the map $B_q\colon V \times V \to R$, defined by $B_q(x, y) = q(x + y) - q(x) - q(y)$, is R-bilinear.

The orthogonal group $\mathbf{O}(q)$ for a general k-quadratic space (V, q) over k, is a closed subscheme of $\mathrm{GL}(V)$, which represents the functor

$$R \mapsto \{g \in \mathrm{GL}(V_R)\colon q_R(gx) = q_R(x) \text{ for all } x \in V_R\}.$$

Now, suppose that V has even dimension n. In this case non-degenerate means that the bilinear form $B_q(x, y)$ is non-degenerate.

Now to define the special orthogonal group, we need for every k-algbera R, the Clifford algebra given by

$$C(V_R, q_R) = \mathrm{Sym}(V_R)/\langle q_R(x) - x \otimes x \colon x \in V_R \rangle,$$

where $\mathrm{SymV_R}$ denotes the symmetric algebra. The algebra $C(V_R, q_R)$ has a $\mathbb{Z}/2\mathbb{Z}$-grading, induced by the $\mathbb{Z}$-grading of $\mathrm{Sym}\, V_R$. This algebra satisfies the following structural property

Proposition 1.4.1. *[12, Corollary C.2.2] The R-algebra $C(V_R, q_R)$ and its even part $C_0(V_R, q_R)$ are respectively isomorphic, fppf-locally on R, to $\mathrm{M}_{2^{n/2}}(R)$ and a product of two copies of $\mathrm{M}_{2^{(n/2)-1}}(R)$, with the left $C_0(V_R, q_R)$-module $C_1(V_R, q_R)$ free of rank 1.*

Furthermore, $C(V_R, q_R)$ and $C_j(V_R, q_R)$ $(j = 1, 2)$ are R-free and the centers of $C(V_R, q_R)$ and $C_0(V_R, q_R)$ are respectively equal to R and a rank-2 finite étale R-algebra Z_q. $C_0(V_R, q_R)$ is the centralizer of Z_q in $C(V_R, q_R)$.

The induced action of $\mathbf{O}(q)(R)$ on $C(V_R, q_R)$ preserves the grading and, hence induces an action on $C_0(V_R, q_R)$, so finally we obtain an action on Z_{q_R}. Now the automorphism scheme $\mathrm{Aut}_{Z_q/k}$ is uniquely isomorphic to the constant group $\mathbb{Z}/2\mathbb{Z}$. Thus we get a homomorphism [12, C.2.2]

$$D_q \colon \mathbf{O}(q) \to \mathbb{Z}/2\mathbb{Z},$$

that is moreover compatible with isomorphisms of quadratic spaces (V, q), i.e. k-linear isomorphism preserving the quadratic forms. Its formation commutes with extension of scalars. This allows us to define

$$\mathbf{SO}(q) \coloneqq \ker D_q.$$

Theorem 1.4.2. *[12, Section C.2] The group $\mathbf{SO}(q)$ is connected, smooth and reductive of dimension $n(n-1)/2$. Its center is the 2-root of unity μ_2 as a group scheme. The Dickson morphism D_q is a smooth surjection, identifying $\mathbb{Z}/2\mathbb{Z}$ with $\mathbf{O}(q)/\mathbf{SO}(q)$.*

On the other hand, we also have the determinant homomorphism, that, thanks to the non-degeneracy property of B_q, factors through μ_2

$$\det \colon \mathbf{O}(q) \to \mu_2 \subset \mathbb{G}_m.$$

Furthermore one can prove [12, Corollary C.3.2],

$$\mathbf{SO}(q) \subset \ker(\det)$$

and that it is an equality if and only if $\mathrm{char}(k) \neq 2$.

Remark 1.4.3. There is a similar construction for $\mathbf{SO}(q)$ for odd dimensional quadratic space (V, q) [12, C.2].

We are going to consider two families of these groups: for $n \geq 1$ consider the quadratic space $Q_n = (k^{2n}, q_n)$, where

$$q_n(x_1, \ldots, x_{2n}) \coloneqq x_1 x_{2n} + \cdots + x_n x_{n+1}.$$

We have an orthogonal decomposition $Q_n = H_1 \perp \cdots \perp H_n$, for n hyperbolic planes $H_i = (ke_i \oplus ke_{2n-i+1}, x_i x_{2n-i+1})$. Indeed, we note that $B_{q_n}(x, y) = \sum(x_i y_{2n-i+1} + x_{2n-i+1} y_i)$, which it is non-degenerate. To simplify the notation, when the base field k is clear from context, we denote

$$\mathbf{SO}_{2n} \coloneqq \mathbf{SO}(q_{k,n}).$$

Let l a separable quadratic extension of k, $\mathrm{N}_{l/k}$ the norm, $\mathrm{Tr}_{l/k}$ the trace and σ the non-trivial element of $\mathrm{Gal}(l/k)$. For $n \geq 1$ consider the quadratic space $Q_{l,n} = (k^{n-1} \oplus l \oplus k^{n-1}, q_{l,n})$, where

$$q_{l,n}(x_1, \ldots, x_{n-1}, x, x_{n+2}, \ldots, x_{2n}) = x_1 x_{2n} + \cdots + x_{n-1} x_{n+2} + \mathrm{N}_{l/k}(x).$$

We have an orthogonal decomposition $Q_{l,n} = H_1 \perp \cdots \perp H_{n-1} \perp (l, \mathrm{N}_{l/k})$, for $n-1$ hyperbolic planes $H_i = (ke_i \oplus ke_{2n-i+1}, x_i x_{2n-i+1})$ and an (anisotropic) non-degenerated quadratic space $(l, \mathrm{N}_{l/k})$. Indeed, we note that $B_{q_{l,n}}(x, y) = \sum(x_i y_{2n-i+1} + x_{2n-i+1} y_i) + \mathrm{Tr}_{l/k}(x\sigma(y))$, which it is non-degenerate. To simplify the notation, when the extension l over k is clear from context, we write

$$\mathbf{SO}^*_{2n} \coloneqq \mathbf{SO}(q_{l,n}).$$

1.5 Properties and calculations for $\mathbf{SO}(q_{l,n})$. The group $\mathbf{SO}(q_{l,n})$ is a k-form, splitting over l, of the split group $\mathbf{SO}_{2n}$, whose absolute root system is D_n.

We let

$$\mathbf{S} = \prod_{i=1}^{n-1} \mathbf{SO}(H_i) \subset \mathbf{SO}(q_{l,n}) \quad \& \quad \mathbf{T} = \mathbf{S} \times \mathbf{SO}(l, \mathrm{N}_{l/k}) \subset \mathbf{SO}(q_{l,n}).$$

Since the dimension of $\mathbf{S}$ is $n-1$ and the dimension of $\mathbf{T}$ is n, we get that $\mathbf{S} \subset \mathbf{SO}(q_{l,n})$ is a maximal split k-torus and $\mathbf{T}$ is a maximal k-torus. Let $W = H_1 \perp \cdots \perp H_{n-1}$. Now, if we write the elements of $\mathrm{End}(Q_{l,n})$ as block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathrm{End}(W)$, $B \in \mathrm{Hom}(W, l)$, $C \in \mathrm{Hom}(l, W)$, and $D \in \mathrm{End}(l)$, then $\mathbf{S}$ can be identified with the set of block matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & \mathrm{Id}_l \end{pmatrix},$$

with A consisting of matrices

$$\mathrm{diag}(t_1, \cdots t_{n-1}, t_{n-1}^{-1}, \cdots, t_1^{-1}),$$

with respect to the ordered basis of W given by $\{e_1, \ldots, e_{n-1}, e_{n+2}, \ldots, e_{2n}\}$.

On the other hand, by noticing that $q_{l,n}$ restricted to $W \perp k \cdot 1$ is the split form of $2(n-1)+1$ variables

$$x_1 x_{2n} + \cdots + x_{n-1} x_{n+2} + x^2,$$

we get a copy of $\mathbf{SO}_{2n+1} \cong \mathbf{SO}(W \perp k \cdot 1)$ inside $\mathbf{SO}(q_{l,n})$. Finally, putting these together we can show that the (relative) root system $\Phi(\mathbf{SO}(q_{l,n}), \mathbf{S}) = \Phi(\mathbf{SO}(W \perp k \cdot 1), \mathbf{S}) = \Sigma = B_{n-1}$ and moreover

$$Z_{\mathbf{G}}(\mathbf{S}) = \mathbf{T}\,.$$

Let us choose $\alpha \in l$ such that $l = k(\alpha)$. Let $\mathbf{P}_0$ be the minimal parabolic subgroup of $\mathbf{SO}(q_{l,n})$ containing $\mathbf{S}$ consisting of upper triangular block matrices with respect to the ordered basis of $Q_{l,n}$ given by $\{e_1, \ldots, e_{n-1}, 1, \alpha, e_{n+2}, \ldots, e_{2n}\}$. Then, as $Z_{\mathbf{G}}(\mathbf{S}) = \mathbf{T}$, we have

$$\mathbf{P}_0 = \mathbf{T} \ltimes R_{u,k}(\mathbf{P}_0),$$

and that $\mathbf{SO}(q_{l,n})$ is quasi-split. As it will be used later, we study the following low relative rank cases.

(*Relative rank 0*). [12, Example C.6.1] Let us choose $\alpha \in l$ such that $l = k(\alpha)$. We look at $Q_{1,l} = (l, \mathrm{N}_{l/k})$. As $\dim_k(l) = 2$ we can check by definition that,

$$C_0 = C_0(l, \mathrm{N}_{l/k}) = Z_q = k[1 \otimes \alpha] = l \quad \& \quad C_1 = C_1(l, \mathrm{N}_{l/k}) = l.$$

The algebra C_0 acts on C_1, as l acts on l, i.e. by multiplication. Furthermore as the action of $\mathbf{SO}(\mathrm{N}_{l/k})$ on C_1 corresponds to the natural one and since its action on C_0 is trivial by definition, we have that the functor of points of $\mathbf{SO}(\mathrm{N}_{l/k}) \subset \mathbf{GL}_k(l)$ consists in l-linear maps that preserve $\mathrm{N}_{l/k}$. Now if we consider the morphism induced by the action of $l^\times$ on $\mathbf{GL}_k(l)$, this leads to an immersion of the norm one $N^1(\mathrm{Res}_{l/k}\,\mathbb{G}_m)$ elements of $\mathrm{Res}_{l/k}(\mathbb{G}_m)$, in $\mathbf{GL}_k(l)$. By definition this corresponds to the l-linear automorphisms that preserve $\mathrm{N}_{l/k}$, thus we have that

$$\mathbf{SO}(\mathrm{N}_{l/k}) \subset N^1(\mathrm{Res}_{l/k}\,\mathbb{G}_m).$$

Finally as both tori (smooth and connected) have the same dimension, the inclusion must be an equality.

(*Relative rank 1*). We relate $\mathrm{Res}_{l/k}\,\mathbf{SL}_2$ to the simply connected cover of $\mathbf{SO}(q_{l,2})$. First we note that $Q_{l,2}$ is isomorphic to the quadratic space (E, q) of Hermitian 2×2 matrices, with quadratic form $q = -\det$, via

$$(x_1, x_2, x) \mapsto \begin{pmatrix} -x_1 & x \\ \sigma(x) & x_2 \end{pmatrix}.$$

Furthermore, we have an action of $\mathrm{Res}_{l/k}\,\mathbf{SL}_2$ on E, via

$$a \mapsto gag^*,$$

where $g^* = {}^t\sigma(g)$. We observe that $q(a) = q(gag^*)$ and $\det(a \mapsto gag^*) = 1$. Thus the action gives us a morphism

$$\mathrm{Res}_{l/k}\,\mathbf{SL}_2 \to \mathbf{SO}'(q) = \ker(\det|_{\mathbf{O}(q)}).$$

As $\mathrm{Res}_{l/k}\,\mathbf{SL}_2$ is connected and $\mathbf{O}(q)/\,\mathbf{SO}(q) = \mathbb{Z}/2\mathbb{Z}$, this morphism factors through $\mathbf{SO}(q) \cong \mathbf{SO}(q_{l,2})$. Finally as the kernel of this morphism is μ_2 and the dimensions of the groups $\mathbf{SO}(q_{l,2})$ and $\mathrm{Res}_{l/k}\,\mathbf{SL}_2$ are the same, we have

$$1 \to \mu_2 \to \mathrm{Res}_{l/k}\,\mathbf{SL}_2 \to \mathbf{SO}(q_{2,l}) \to 1. \tag{1.5.1}$$

We can also obtain this description using the general theory of orthogonal groups. Recall that $\mathbf{Spin}(q)$ [12, p. 336] is the simply connected central cover of $\mathbf{SO}(q)$ [12, Lemma C.4.1]

$$1 \to \mu_2 \to \mathbf{Spin}(q) \to \mathbf{SO}(q) \to 1.$$

By definition we have that

$$\mathbf{Spin}(q) \subset \mathscr{D}(\mathrm{Res}_{Z_q/k}\,C_0(V,q)^\times),$$

where $\mathscr{D}(\mathrm{Res}_{Z_q/k}\,C_0(V,q)^\times)$ is the derived k-group of $\mathrm{Res}_{Z_q/k}\,C_0(V,q)^\times$. Now suppose V has dimension 4. We claim that in fact this is an equality. Indeed, as we know, $C_0(V,q)$ is a quaternion algebra over Z_q (Proposition 1.4.1), thus $\mathrm{Res}_{Z_q/k}\,C_0(V,q)^\times$ is an inner form of $\mathbf{GL}_2(Z_q)$, so $\mathscr{D}(\mathrm{Res}_{Z_q/k}\,C_0(V,q)^\times)$ is of dimension 6. On the other hand we have that $\mathbf{Spin}(q)$ also has dimension 6, and as both are smooth and connected, the inclusion is an equality.

In the case $(V,q) = Q_{l,2} = H_1 \perp (l, N_{l/k})$, we have, from the definition, a graded isomorphism [31, IV, (1.3.1)]

$$C(V,q) \xrightarrow{\sim} C(\mathbb{H})\widehat{\otimes}C(l, N_{l/k}).$$

Furthermore, as $C(\mathbb{H}) \cong \mathbf{M}_2(k)$ [31, V, (2.1.6)] and $C(l, N_{l/k}) \cong l \oplus ul$ [31, V, (2.2.1)] with the multiplication rules $xu = u\sigma(x)$ and $u^2 = 1$, we also have a graded isomorphism

$$\begin{aligned} \mathbf{M}_2(k)\widehat{\otimes}(l \oplus ul) &\to \mathbf{M}_2(l \oplus ul) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes (x + uy) &\mapsto \begin{pmatrix} a(x+uy) & b(x+uy) \\ c(x-uy) & d(x-uy) \end{pmatrix}, \end{aligned}$$

in particular we get that

$$C_0(V,q) \cong \begin{pmatrix} l & ul \\ ul & l \end{pmatrix} \subset \mathbf{M}_2(l \oplus ul).$$

Finally, the map

$$\begin{pmatrix} a \otimes x_1 & b \otimes ux_2 \\ c \otimes ux_3 & d \otimes x_4 \end{pmatrix} \mapsto \begin{pmatrix} ax_1 & b\sigma(x_2) \\ cx_3 & d\sigma(x_4) \end{pmatrix},$$

induces an isomorphism

$$C_0(V, q) \cong \mathbf{M}_2(l).$$

This implies that in fact $C_0(V, q)^\times \cong \mathbf{GL}_2(l)$ and $Z_q = l$, thus

$$\mathbf{Spin}(q) \cong \mathrm{Res}_{l/k} \mathbf{SL}_2 .$$

Chapter 2

The Langlands-Shahidi Method

In this chapter, we recall some local and global notions. We mainly follow [38], in order to present the Langlands-Shahidi γ-factors, L-functions and ε-factors. We apply this to the case $\mathbf{SO}^*_{2n}$. We finish by discussing three foundational conjectures in this theory, Shahidi's tempered L-function conjecture, the standard module conjecture and Kim's assumption.

Let $\mathbf{G}$ be a (connected) reductive group over a non-archimedean local or a global field F of positive characteristic. Let us fix a separable closure F_s of F. Finally, let us fix in $\mathbf{G}$ a maximal F-split subtorus $\mathbf{S}$ and $\mathbf{P}_0$ a minimal parabolic subgroup containing $\mathbf{S}$.

From now on, in order to reduce the size of the indices, we sometimes use the notation $M = \mathbf{M}(F)$, for the rational points of an algebraic group scheme $\mathbf{M}$ over F.

2.1 Basic structures. For a Levi subgroup $\mathbf{M}$ containing $\mathbf{S}$ (i.e. semi-standard, see Section 1.2) of $\mathbf{G}$, we let

$$a^*_{\mathbf{M}} = X(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{R} \quad \& \quad a^*_{\mathbf{M},\mathbb{C}} = X(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

In the case where $\mathbf{M}_\theta$ is the Levi subgroup of $\mathbf{G}$ associated to $\theta \subset {}_F\Delta$ (Section 1.2), we let $a^*_\theta = a^*_{\mathbf{M}_\theta}$ and $a^*_{\theta,\mathbb{C}} = a^*_{\mathbf{M}_\theta,\mathbb{C}}$.

Let us recall that, as $\mathbf{G}$ is connected reductive, the restriction homomorphism to $\mathbf{A}_\mathbf{G}$ (Section 1.2):

$$\mathrm{Res}^{\mathbf{G}}_{\mathbf{A}_\mathbf{G}} : X(\mathbf{G}) \to X(\mathbf{A}_\mathbf{G})$$

has finite cokernel and is injective; in particular it induces an isomorphism

$$a^*_\mathbf{G} \to X(\mathbf{A}_\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Now, let us consider $\mathbf{L}$ be a semi-standard Levi subgroup of $\mathbf{G}$ containing $\mathbf{M}$. By definition we have

$$\mathbf{A}_\mathbf{L} \subset \mathbf{A}_\mathbf{M} \subset \mathbf{M} \subset \mathbf{L}\,.$$

This chain induces the following commutative diagram by the restrictions

$$\begin{array}{ccc} X(\mathbf{L}) & \xrightarrow{\mathrm{Res}^{\mathbf{L}}_{\mathbf{A_L}}} & X(\mathbf{A_L}) \\ \mathrm{Res}^{\mathbf{L}}_{\mathbf{M}} \downarrow & & \uparrow \mathrm{Res}^{\mathbf{A_M}}_{\mathbf{A_L}} \\ X(\mathbf{M}) & \xrightarrow[\mathrm{Res}^{\mathbf{M}}_{A_{\mathbf{M}}}]{} & X(\mathbf{A_M}). \end{array}$$

Since the horizontal arrows are injections with finite cokernels between torsion free groups, they become isomorphisms after tensoring them by $\mathbb{R}$. Moreover, we obtain a split exact sequence

$$0 \longrightarrow a^*_{\mathbf{L}} \xleftarrow{\longrightarrow} a^*_{\mathbf{M}} \longrightarrow (a^{\mathbf{L}}_{\mathbf{M}})^* \longrightarrow 0,$$

where $(a^{\mathbf{L}}_{\mathbf{M}})^* \coloneqq \mathrm{coker}(X(\mathbf{L}) \to X(\mathbf{M})) \otimes_{\mathbb{Z}} \mathbb{R}$. In other words, we have a decomposition,

$$a^*_{\mathbf{M}} = a^*_{\mathbf{L}} \oplus (a^{\mathbf{L}}_{\mathbf{M}})^*.$$

Analogously, we have for $a_{\mathbf{M}} = (X(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{R})^*$ the following exact sequence

$$0 \longrightarrow a_{\mathbf{L}} \longrightarrow a_{\mathbf{M}} \xleftarrow{\longrightarrow} (a^{\mathbf{L}}_{\mathbf{M}}) \longrightarrow 0,$$

and thus a decomposition

$$a_{\mathbf{M}} = a_{\mathbf{L}} \oplus a^{\mathbf{L}}_{\mathbf{M}}.$$

By definition $\Phi(\mathbf{P}, \mathbf{A_M})$ (Section 1.1) is a subset of $X(\mathbf{A_M})$, hence of $a^*_{\mathbf{M}}$; in fact it lies in $(a^{\mathbf{G}}_{\mathbf{M}})^*$ and it spans it. In particular ${}_F\Delta \subset \Phi(\mathbf{P}_0, \mathbf{S})$ spans $(a^{\mathbf{G}}_0)^*$. Moreover, for $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}_0$, the restriction $\mathbf{A_M} \subset \mathbf{A}_{\mathbf{M}_0} = \mathbf{S}$ induces a map

$$\Phi(\mathbf{P}, \mathbf{S}) \to \Phi(\mathbf{P}, \mathbf{A_M}) \cup \{0\},$$

such that $\Phi(\mathbf{M}, \mathbf{A}_{\mathbf{P}_0})$ maps to 0. Denote by $\Delta_{\mathbf{P}} \subset a^*_{\mathbf{M}}$ the image of $\Delta \backslash \Phi(\mathbf{M}, \mathbf{A}_0)$ in $\Phi(\mathbf{P}, \mathbf{A_M})$.

Finally, let

$$a^{*,+}_{\mathbf{M}} = \{\nu \in a^*_{M_{\theta_0}} : \langle \nu, \alpha^\vee \rangle > 0, \alpha \in \Delta_{\mathbf{P}}\}$$
$$\overline{a}^{*,+}_{\mathbf{M}} = \{\nu \in a^*_{M_{\theta_0}} : \langle \nu, \alpha^\vee \rangle \geq 0, \alpha \in \Delta_{\mathbf{P}}\}.$$

(Unramified character). Now suppose that F is a non-archimedean local field, with residue field $\mathbb{F}_q$. For every $\chi \in X^*(\mathbf{G})$, we define

$$|\chi|_F \coloneqq |\cdot|_F \circ \chi(F) \colon \mathbf{G}(F) \xrightarrow{\chi(F)} F^\times \xrightarrow{|\cdot|_F} q^{\mathbb{Z}} \subset \mathbb{C}^\times,$$

where $\chi(F)$ is group homomorphism between the F-points of G and $\mathbb{G}_m$ induced from χ .

Let us consider the map

$$\log_G\colon\ \mathbf{G}(F) \to a_G = \mathrm{Hom}_{\mathbb{Z}}(X(\mathbf{G}), \mathbb{R}).$$
$$g \mapsto [\chi \mapsto -\log|\chi|_F(g)]$$

Again, from the restriction map of $\mathbf{A_G} \subset \mathbf{G}$, we have that $\mathrm{Im}(\log_G)$ is contained in the lattice $\mathrm{Hom}_{\mathbb{Z}}(X(\mathbf{G}), \frac{1}{\log q}\mathbb{Z})$ of $a_{\mathbf{G}}$ and has finite index there. We also consider a variant of this map, where we use $\log_q$ instead of $\log$, we denote it by

$$H_G\colon\ \mathbf{G}(F) \to X_*(\mathbf{G}) = \mathrm{Hom}_{\mathbb{Z}}(X(\mathbf{G}), \mathbb{Z}).$$

We put

$$G^1 \coloneqq \ker\log_G = \bigcap_{\chi\in X^*(\mathbf{G})} \ker|\chi|_F, \quad \mathcal{X}(G) \coloneqq \mathrm{Hom}_{\mathbb{Z}}(\mathbf{G}(F)/G^1, \mathbb{C}^\times).$$

We have a surjection,

$$\begin{aligned}\kappa\colon a^*_{\mathbf{G},\mathbb{C}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(a_{\mathbf{G},\mathbb{C}}, \mathbb{C}) &= \mathrm{Hom}_{\mathbb{C}}(\mathrm{Im}(\log_G)\otimes\mathbb{C}, \mathbb{C})\\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Im}(\log_G), \mathbb{C}) \xrightarrow{\exp\circ -} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Im}(\log_G), \mathbb{C}^\times)\\ &\xrightarrow{-\circ\log_G} \mathrm{Hom}_{\mathbb{Z}}(\mathbf{G}(F)/G^1, \mathbb{C}^\times) = \mathcal{X}(G),\end{aligned}$$

which can be also be written as

$$\chi\otimes s \mapsto [g \mapsto |\chi|^s_F(g)],$$

with kernel equal to $R = \frac{2i\pi}{\log q}\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Im}(\log_G), \mathbb{Z})$.

We put

$$\mathrm{Re}\,\mathcal{X}(G) = \kappa(a^*_{\mathbf{G}}) \quad \& \quad \mathrm{Im}\,\mathcal{X}(G) = \kappa(ia^*_{\mathbf{G}}).$$

We note that κ restricts to the following isomorphisms

$$a^*_{\mathbf{G}} \to \mathrm{Re}\,\mathcal{X}(G),$$

and

$$ia^*_{\mathbf{G}}/R \to \mathrm{Im}\,\mathcal{X}(G).$$

Let $\mathbb{C}[\mathbf{G}(F)/G^1]$ the group algebra of $\mathbf{G}(F)/G^1$ over $\mathbb{C}$. The universal property of this algebra gives us the following natural indentification

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbf{G}(F)/G^1, \mathbb{C}^\times) = \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[\mathbf{G}(F)/G^1], \mathbb{C}).$$

In other words, $\mathcal{X}(G)$ is the set of rational points of $\mathrm{Spec}(\mathbb{C}[\mathbf{G}(F)/G^1])$. As $\mathbf{G}(F)/G^1$ is free of some finite rank r, we have a (non-canonical) isomorphism from $\mathrm{Spec}(\mathbb{C}[\mathbf{G}(F)/G^1])$ to $\mathbb{G}^r_m$, giving us an algebraic structure on $\mathcal{X}(G)$, such that it is isomorphic to $(\mathbb{C}^\times)^r$.

We can also give a real manifold structure on $\mathrm{Re}\,\mathcal{X}(G)$ and $\mathrm{Im}\,\mathcal{X}(G)$, given by the isomorphism above. In the second case it is a compact torus.

(Inertia classes). The group $\mathcal{X}(G)$ acts on the set of equivalences classes of the smooth irreducible representations of $\mathbf{G}(F)$, $\mathrm{Irr}(G)$:

$$(\chi, [(\pi, V)]) \mapsto [(\pi \otimes \chi, V)], \quad \chi \in \mathcal{X}(G), \pi \in \mathrm{Irr}(G).$$

We denote the induced action of $a^*_{G,\mathbb{C}}$ via κ by $\pi_\nu = \pi \otimes \kappa(\nu)$ and

$$(\nu, [(\pi, V)]) \mapsto [(\pi_\nu, V_\nu)].$$

Using that $\mathcal{X}(G)$ is an algebraic complex torus and that the stabilizers of this action are finite we have that its orbits also have an algebraic complex structure. We denote the orbit of π as $\mathcal{O}_{\pi,\mathbb{C}}$, and let $R_{\pi,\mathbb{C}} \subset a^*_{\mathbf{G},\mathbb{C}}$ be the lattice such that

$$a^*_{\mathbf{G},\mathbb{C}}/R_{\pi,\mathbb{C}} \to \mathcal{O}_{\pi,\mathbb{C}}$$

is a bijection.

Similarly $\mathrm{Im}\,\mathcal{X}(G)$ acts on the set of equivalences classes of smooth unitary irreducible representations and this allows us to think of its orbits as real manifolds that are diffeomorphic to real compact tori. We denote the orbit of π as $\mathcal{O}_\pi$, and let $R_\pi \subset ia^*_{\mathbf{G}}$ be the lattice such that

$$a^*_{\mathbf{G}}/R_\pi \to \mathcal{O}_\pi$$

is a bijection.

(*Induced representation*). For a totally disconnected topological group H, we denote by $\mathrm{Rep}(H)$ the (abelian) category of smooth representations of H.

Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$, with $\mathbf{M}$ a Levi subgroup and $\mathbf{N}$ its unipotent radical. Then we denote the normalized parabolic induction functor by

$$i_P^G\colon \mathrm{Rep}(M) \to \mathrm{Rep}(G).$$

If $(\pi, V) \in \mathrm{Rep}(M)$ then we let

$$i_P^G(\pi, V) = (i_P^G\pi, i_P^G V).$$

Finally we denote, for a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ with $\mathbf{M} \supset \mathbf{S}$ and $\nu \in a^*_{\mathbf{M},\mathbb{C}}$

$$i_P^G(\nu, (\pi, V)) \coloneqq (i_P^G(\pi_\nu), i_P^G V_\nu).$$

2.2 The Langlands quotient. Suppose that F is a non-archimedean local field. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$ containing a fixed minimal parabolic subgroup

$\mathbf{P}_0$, σ an irreducible tempered representation of $\mathbf{M}(F)$, and $\nu \in a^*_{\mathbf{M},\mathbb{C}}$ such that $\mathrm{Re}(\nu) \in a^{*,+}_{\mathbf{M}}$. Then $i_P^G(\nu,\pi)$ has a unique irreducible quotient [45, Théorème VII.4.2] called the Langlands quotient and it is denoted by $J(\mathbf{P},\nu,\pi)$. Moreover, if $J(\mathbf{P}',\nu',\pi') \cong J(\mathbf{P},\nu,\pi)$ then $\mathbf{M}' = \mathbf{M}$, $\mathrm{Re}(\nu-\nu') = 0$ and $\pi_\nu \cong \pi'_{\nu'}$. Every irreducible smooth representation of $\mathrm{Rep}(\mathbf{G}(F))$ is isomorphic to a Langlands' quotient.

In the case of $\mathbf{M} = \mathbf{GL}_{n_b} \times \cdots \times \mathbf{GL}_{n_1} \times \mathbf{G}_{n_0} \subset \mathbf{G}_n$, where $\mathbf{G}_n = \mathbf{SO}^*_{2n}$ (we recall that we have already fixed a minimal parabolic subgroup in Section 1.5) or $\mathbf{SO}_{2n}$ (we will also fix a minimal parabolic subgroup in Section 5.6). Notice that the center of $\mathbf{G}_n$ is finite, and Langlands quotients are indeed quotients of representations of the form

$$i_P^{\mathbf{G}_n(F_x)}(\pi_b|\det|^{r_b} \otimes \cdots \otimes \pi_1|\det|^{r_1} \otimes \pi_0),$$

where $0 < r_1 \leq \cdots \leq r_b$, π_i is an irreducible tempered representation of $\mathbf{GL}_{n_i}(F)$ for $1 \leq i \leq b$ and π_0 is an irreducible tempered representation of $\mathbf{G}_{n_0}(F)$.

2.3 Local Intertwining operator. Suppose that F is a non-archimedean local field. Consider two parabolic subgroups $\mathbf{P} = \mathbf{M}\,\mathbf{U}$ and $\mathbf{P}' = \mathbf{M}\,\mathbf{U}'$ of $\mathbf{G}$, with a common Levi subgroup $\mathbf{M}$, containing $\mathbf{S}$, and let π be a smooth representation of $\mathbf{M}(F)$. We denote by $\overline{\mathbf{P}}'$ the unique parabolic subgroup $\mathbf{G}$ that is opposite to $\mathbf{P}'$ and contains $\mathbf{M}$. We first define formally the G-invariant operator,

$$J_{P'|P}(\pi)\colon i_P^G V \to i_{P'}^G V$$

characterized by

$$\langle (J_{P'|P}(\pi)f)(g), \check{v}\rangle = \int_{U\cap U'\backslash U'} \langle f(u'g),\check{v}\rangle du' \quad \text{for all } \check{v} \in \check{V}.$$

If for all $\check{v} \in \check{V}$ this integral converges absolutely and there exits $v \in V$ such that this integral is equal to $\langle v,\check{v}\rangle$ for all $\check{v} \in \check{V}$, we will say that $J_{P'|P}(\pi)$ is *defined by convergent integrals*. The vector obtained will be denoted by

$$(J_{P'|P}(\pi)f)(g) = \int_{(U\cap U')\backslash U'} f(u'g)du'.$$

Theorem 2.3.1. *[56, Théorème I V.1.1] Suppose that π is of finite length. Then there exists $R \in \mathbb{R}$ such that if $\langle \mathrm{Re}(\chi), \alpha^\vee\rangle > R$ for all $\alpha \in \Phi(\mathbf{P},\mathbf{A}_\mathbf{M}) \cap \Phi(\overline{\mathbf{P}}',\mathbf{A}_M)$, then $J_{P'|P}(\pi\otimes\chi)$ is defined by convergent integrals. Moreover, the operator $J_{P'|P}$ defined in the region just described of $\mathcal{O}_{\pi,\mathbb{C}}$ is rational.*

We also introduce a variant of that operator, used by Shahidi. We recall that we have a parametrization of parabolic subgroups containing $\mathbf{P}_0$, by subsets of ${}_F\Delta$ (Section 1.1). First, for $\theta,\theta' \subset {}_F\Delta$, let

$$W(\theta,\theta') = \{w \in W^G\colon w(\theta) = \theta'\}.$$

For a representative $\tilde{w} \in N_{\mathbf{G}}(\mathbf{S})(F)$ of $w \in W(\theta, \theta')$ and $\nu \in a^*_{M_\theta}$, as Shahidi, we can define

$$A(\nu, \pi, \tilde{w}) \coloneqq J_{P_{\theta'}|wP_\theta}(\pi_\nu) \circ t(\tilde{w})\colon i^G_{P_\theta}(\nu, \pi) \to i^G_{wP_\theta}(\tilde{w}(\nu), \tilde{w}(\pi)) \to i^G_{P_{\theta'}}(\tilde{w}(\nu), \tilde{w}(\pi)),$$

where $(t(\tilde{w})f)(g) = f(\tilde{w}^{-1}g)$, $\tilde{w}(\pi)(g) = \pi(\tilde{w}^{-1}g\tilde{w})$ and π_ν is an in Section 2.1. This operator is defined in a suitable cone as in the statement of Theorem 2.3.1 by convergent integrals as follows

$$A(\nu, \pi, \tilde{w})f(g) = \int_{wU_\theta w^{-1} \cap U_{\theta'} \backslash U_{\theta'}} f(\tilde{w}^{-1}u'g)du'. \tag{2.3.2}$$

2.4 Places and restricted products. Suppose F is a global field of positive characteristic. In this case we set up some global notations. We denote by $|F|$ the set of places of F. We denote by F_x the local field at x, $\mathcal{O}_x$ the ring of integers of F_x and $\mathbb{A}_F = \prod'_x F_x$ the ring of adèles. Let $\mathbf{M}$ be an algebraic group over F (finite type group scheme over F). The group $\mathbf{M}(\mathbb{A}_F)$ is the restricted product $\prod'_{K_x} \mathbf{M}(F_x)$ with respect to certain compact open subgroups K_x of $\mathbf{M}(F_x)$ [53, Section 4.10].

For each $x \in |F|$, let (π_x, V_x) be an admissible representation of $\mathbf{M}(F_x)$. Assume that for some finite subset S_0 of $|F|$, we are given a non-zero vector $v_x \in V_x$ for each $x \notin S_0$. These data define a representation of $\mathbf{M}(\mathbb{A}_F)$ [15, Section 2],

$$(\bigotimes_{v_x} \pi_x, \bigotimes_{v_x} V_x),$$

called the restricted tensor product of the collection $\{(\pi_x, V_x)\colon x \in |F|\}$ with respect to the collection $\{v_x \in V_x\colon x \notin S_0\}$.

If we assume that these representations satisfy $\dim V_x^{K_x} = 1$ and $v_x \in V_x^{K_x}$, then their isomorphism classes as $\mathbf{M}(\mathbb{A}_F)$-modules do not depend on the choice of the set $\{v_x\}$, and thus we just write $\bigotimes_x \pi_x$. The elements of these isomorphism classes will be called factorizable. Finally, we recall that an irreducible admissible representation of $\mathbf{M}(\mathbb{A}_F)$ is factorizable [15, Theorem 3].

2.5 Adjoint representation. Suppose that $\mathbf{G}$ is quasi-split. Let $\mathbf{P} = \mathbf{M}\,\mathbf{N}$ be a maximal parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}_0$, associated to ${}_F\Delta - \{\alpha\}$. We denote

$$\rho = \frac{1}{2} \sum_{b \in \Phi(\mathbf{P}, \mathbf{A}_\mathbf{M})} b \in a^*_\mathbf{M} \quad \& \quad \tilde{\alpha} \coloneqq \langle \rho, \alpha^\vee \rangle^{-1} \rho \in a^*_\mathbf{M}.$$

For $\beta \in \Sigma = \Phi(\mathbf{G}, \mathbf{S})$ we denote by $\underline{\beta} \in \Phi = \Phi(\mathbf{G}_{F_s}, \mathbf{T}_{F_s})$ a root such that $\mathrm{Res}^{T_{F_s}}_{S_{F_s}} \underline{\beta} = \beta$.

We consider the L-group ${}^L G$ (Section 1.3). Moreover we extend the pinning $\{X_b\}_{b \in \Delta^\vee}$, used to define $\hat{\mathbf{G}}$, to a Chevalley basis $\{X_b\colon b \in \Phi^\vee\}$. Now let us consider the adjoint

representation r of ${}^LM$ on ${}^L\mathfrak{n} = \mathrm{Lie}({}^LN)$. For every positive integer i, we consider the subrepresentation r_i acting on

$$V_i = \mathrm{Span}\{X_{\underline{\beta}^\vee} \in {}^L\mathfrak{n} \colon \langle \tilde{\alpha}, \beta^\vee|_{\mathbf{A_M}}\rangle = i\}.$$

Thanks to the algebraic arguments in [47, Section 2] valid in all characteristics, there exists a positive integer m_r such that

$$\{\langle \tilde{\alpha}, \beta^\vee|_{\mathbf{A_M}}\rangle \colon \beta \in \Sigma^+, X_{\underline{\beta}^\vee} \in {}^L\mathfrak{n}\} = \{1, \dots, m_r\}.$$

In fact these subspaces V_i correspond to the space generated by the weight spaces $\mathfrak{g}_{\underline{\gamma}^\vee}$ such that

$$\underline{\gamma}^\vee|_{\mathbf{A}_{L_M}} = i\underline{\alpha}^\vee|_{\mathbf{A}_{L_M}},$$

and they form an irreducible decomposition of r [47, Proposition 4.11]. Thus we can write

$$r = \oplus_{i=1}^{m_r} r_i : {}^LM \to \mathrm{GL}({}^L\mathfrak{n}). \tag{2.5.1}$$

Let E/F be a quadratic extension in F_s. In the case when $\mathbf{P} = \mathbf{M}\,\mathbf{N} \subset \mathbf{SO}(q_{E,m+n}) = \mathbf{SO}^*_{2(m+n)}$, with $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}^*_{2n}$ (See Section 1.5, for notations), we are in the case of $({}^2D_n - 1)$ [47, p. 565]. We obtain the following decomposition of the adjoint representation

$$r = r_1 \oplus r_2, \tag{2.5.2}$$

where $r_1 = \rho_m \otimes \tilde{\rho}^*_{2n}$ and $r_2 = \wedge^2\rho_m \otimes 1_{\mathbf{SO}^*_{2n}}$. Here $\rho_m = \mathrm{Id}_{\mathbf{GL}_m}$ is the standard representation of ${}^L\mathbf{GL}_m(\mathbb{C})$, ρ^*_{2n} the representation of ${}^L\mathbf{SO}^*_{2n}(\mathbb{C})$ constructed in (5.6.1) below and $1_{\mathbf{SO}^*_{2n}}$ is the trivial representation of ${}^L\mathbf{SO}^*_{2n}(\mathbb{C})$.

(*Restriction*). Now we recall some properties of the restriction of the adjoint representation to smaller Levi subgroups. These properties will play a role in the general multiplicativity formula of γ-factor below (Section 2.7).

Let $w_0 = w_{l,{}_F\Delta}w_{l,{}_F\Delta-\{\alpha\}}$. We study the restriction of the representation r_i to smaller Levi subgroups. Let $\theta \subset {}_F\Delta - \{\alpha\}$, fix a reduced decomposition $w_0 = w_{n-1}\cdots w_1$ [46, Lemma 2.1.2] and denote $\theta' = w_0(\theta) \subset {}_F\Delta$. For each j, $2 \leq j \leq n-1$, let $\overline{w_j} = w_{j-1}\cdots w_1$. Set $\overline{w}_1 = 1$. Also, let $\Omega_j = \theta_j \cup \{\alpha_j\}$, where $\theta_1 = \theta$, $\theta_n = \theta'$, and $\theta_{j+1} = w_j(\theta_j)$, $1 \leq j \leq n-1$. Then the group $\mathbf{M}_{\Omega_j}$ contains $\mathbf{M}_{\theta_j}\,\mathbf{N}_{\theta_j}$ as a maximal parabolic subgroup.

The L-group ${}^LM_\theta$ acts on V_i. Given an irreducible subrepresentation of this action, there exists a unique j, $1 \leq j \leq n-1$, which under w_j is equivalent to an irreducible subrepresentation of the action of ${}^LM_{\theta_j}$ on the Lie algebra of ${}^LN_{\theta_j}$. We denote $i(j)$ the index of this subspace of the Lie algebra of ${}^LN_{\theta_j}$. Finally, let S_i denote the set of all such i's where S_i, in general, is a proper subset of $1 \leq i \leq n-1$.

To shorten some indices we use the following notation. If σ be a representation of $\mathbf{M}(F)$ and $\nu \in a^*_{\mathbf{M}}$, then we denote by σ_j the representation of M_{θ_j} given by $\overline{w}_j(\sigma)$ and $\nu_j = \overline{w_j}(\nu) \in a^*_{\theta_j}$.

2.6 Generic representations. Suppose that $\mathbf{G}$ is quasi-split, so $\mathbf{P}_0 = \mathbf{B} = \mathbf{T}\mathbf{U}_0$ is a Borel subgroup and F is a non-archimedean local field. Let $\chi : \mathbf{U}_0(F) \to \mathbb{C}^\times$ be a character that is generic in the sense that it is non-trivial on $\mathbf{U}_a(F)$ for every $a \in {}_F\Delta$ and that is trivial on the normal subgroup $\prod_{a\in\Sigma^+-\Delta} \mathbf{U}_a(F)$. An irreducible admissible representation (π, V) is said to be χ-generic if there exists a non-zero χ-generic Whittaker functional for π, i.e a functional $\lambda_\chi \colon V \to \mathbb{C}$ such that

$$\lambda_\chi(\pi(u)v) = \chi(u)\lambda_\chi(v),$$

for all $u \in \mathbf{U}_0(F)$ and $v \in V$. The space of such Whittaker functionals, for an irreducible admissible representation has dimension at most one [50]. For a generic character χ and a χ-generic representation (π, V), there is a unique non-zero Whittaker functional V, up to multiplication by a non-zero constant.

2.7 The Langlands-Shahidi Method. Suppose F is a local field of positive characteristic, with residue field k_F, and $\mathbf{G}$ quasi-split. Denote $q_F = \#k_F$. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a maximal parabolic subgroup of $\mathbf{G}$, containing $\mathbf{P}_0$. The adjoint representation r decomposes intro irreducibile representations (Section 2.5)

$$r = \oplus_{i=1}^{m_r} r_i : {}^L M \to \mathrm{GL}({}^L\mathfrak{n}).$$

Let also (π, V) be a (χ-)generic representation of $\mathbf{M}(F)$ and $\psi \colon F \to \mathbb{C}^\times$ a smooth non-trivial character. Then the Langlands-Shahidi method [38, Section 5] constructs rational functions $\mathbb{C}(q^{-s})$ called γ-factors that are uniquely determined by the following properties

i) (Naturality). Let $\eta : F \to F'$ be an isomorphism of non-archimedean local fields and let π' be the representation of $\mathbf{M}(F')$ and ψ' the character of F', obtained via η. Then
$$\gamma(s, \pi, r_i, \psi) = \gamma(s, \pi', r_i, \psi').$$

ii) (Isomorphism) Let π_1 and π_2 be two isomorphic generic representations of $\mathbf{M}(F)$. Then
$$\gamma(s, \pi_1, r_i, \psi) = \gamma(s, \pi_2, r_i, \psi).$$

iii) (Compatibility with Artin factors) Let π be a generic unramified representation of $\mathbf{M}(F)$. Let $\sigma : W'_F \to {}^L M$ be the Langlands parameter corresponding to π (Section 5.4). Then
$$\gamma(s, \pi, r_i, \psi) = \gamma(s, r_i \circ \sigma, \psi).$$

iv) (Multiplicativity) Let π be the generic subquotient of the representation of $\mathbf{M}(F)$ given as

$$i^M_{P_{\theta_0}\cap M}\pi_0,$$

where π_0 is a generic representation $\mathbf{M}_{\theta_0}(F)$, with $\theta_0 \subset \theta$. Then

$$\gamma(s,\pi,r_i,\psi) = \prod_{j\in S_i}\gamma(s,\pi_{0,j},r_{i(j)},\psi),$$

where S_i, $\pi_{0,j}$ and $r_{i(j)}$ are as in the last paragraph Section 2.5.

v) (Dependence on ψ) For $a \in F^\times$, let $\psi^a\colon F \to \mathbb{C}^\times$ be a character given by $\psi^a(x) = \psi(ax)$. Then, there is a real number h_i and a rational character $t : F^\times \to Z(G)$ such that

$$\gamma(s,\pi,r_i,\psi^a) = \omega_\pi(t(a))^{h_i}|a|_F^{n_i(s-\frac{1}{2})}\cdot\gamma(s,\pi,r_i,\psi),$$

where $n_i = \dim V_i$ and ω_π is the central character of π.

vi) (Functional equation) Let k be a global field of positive characteristic, with field of constants $\mathbb{F}_q$ and $\mathbf{G}'$ a quasi-split reductive k-group. Let $\psi = \otimes_x\psi_x$ be a non-trivial character of $\mathbb{A}_k/k$ and $\pi = \bigotimes_x \pi_x$ be a globally generic cuspidal automorphic representation of $\mathbf{M}'(\mathbb{A}_k)$ (see Section (4.1.1)) such that both π and ψ are unramified outside a finite subset S of places of k, where $\mathbf{M}'$ is the Levi k-subgroup of a maximal parabolic subgroup of $\mathbf{G}'$, containing a Borel subgroup $\mathbf{P}'_0$. Let

$$L^S(s,\pi,r_i) = \prod_{x\notin S} L(s,\pi_x,r_{i,x}),$$

with $L(s,\pi_x,r_{i,x}) = \det(\mathrm{Id} - q_{F_x}^{-s}r_{i,x}(a_x))^{-1}$, $[a_x]$ the semisimple conjugacy class in ${}^LG_x$ associated to π_x (Section 5.4). It is a rational function on q^{-s} and

$$L^S(s,\pi,r_i) = \prod_{x\in S}\gamma(s,\pi_x,r_{i,x},\psi_x)\cdot L^S(1-s,\tilde{\pi},r_i).$$

2.8 Stability of γ-factors for $\mathbf{SO}^*_{2n}$.

We apply the construction of the Langlands-Shahidi γ-factors to the Levi subgroup $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}^*_{2n}$ of $\mathbf{SO}^*_{2(m+n)}$. As we mentioned (Section 2.5), the adjoint representation gives us two irreducible components r_1 and r_2. Thus we obtain two γ factors.

The one associated to r_1 will play a main role in functoriality (Section 6.2). For that reason, we will use the following notation. Let $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}^*_{2n} \subset \mathbf{SO}^*_{2(n+m)}$, π be a generic representation of $\mathbf{SO}^*_{2n}(F)$ and τ a generic representation of $\mathbf{GL}_m(F)$. We let $\tilde{\pi}$ denote the contragredient representation of π. Then $\tau\otimes\tilde{\pi}$ is a generic representation of $\mathbf{M}(F)$ and we write

$$\gamma(s,\tau\otimes\tilde{\pi},r_1,\psi) = \gamma(s,\pi\times\tau,\psi).$$

Among the properties of this factor, we would like to highlight two forms of the multiplicativity property. Let $\mathbf{M}_1 = \mathbf{GL}_m \times \mathbf{GL}_{n_b} \times \cdots \times \mathbf{GL}_{n_1} \times \mathbf{SO}^*_{2n_0} \subset \mathbf{M}$, and suppose that

π is the generic subquotient of

$$i_{P_1}^{\mathbf{SO}^*_{2n}(F)}(\pi_b \otimes \cdots \otimes \pi_1 \otimes \pi_0),$$

where $\mathbf{P}_1 = \mathbf{M}_1\mathbf{N}_1$ is the parabolic subgroup of $\mathbf{SO}^*_{2n}$, containing $\mathbf{P}_0$, π_i is a generic representation of $\mathbf{GL}_{n_i}(F)$ for $1 \leq i \leq b$ and π_0 is a generic representation of $\mathbf{SO}^*_{2n_0}(F)$. Then the multiplicative property gives us

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi_0 \times \tau, \psi) \prod_{i=1}^{b} \gamma(s, \pi_i \times \tau, \psi)\gamma(s, \tilde{\pi}_i \times \tau, \psi),$$

where $\gamma(s, \pi_i \times \tau, \psi)$ is Rankin-Selberg γ-function (Section 3.1). For the other case, let $\mathbf{M}_2 = \mathbf{GL}_{m_b} \times \cdots \times \mathbf{GL}_{m_1} \times \mathbf{SO}^*_{2n} \subset \mathbf{M}$ and suppose that τ is the generic subquotient of

$$i_{Q}^{\mathbf{GL}_m(F)}(\tau_b \otimes \cdots \otimes \tau_1),$$

where $\mathbf{Q} = \mathbf{M}_2\mathbf{N}_2$ is the parabolic subgroup of $\mathbf{GL}_m$ containing the upper triangular matrices, τ_i is a generic representation of $\mathbf{GL}_{m_i}(F)$ for $1 \leq i \leq b$. Then

$$\gamma(s, \pi \times \tau, \psi) = \prod_{i=1}^{b} \gamma(s, \pi \times \tau_i, \psi).$$

For r_2, we obtain the exterior square representation, that will play a role in the study of partial L-functions (Section 4.2),

$$\gamma(s, \tau \otimes \pi, r_2, \psi) = \gamma(s, \tau, \wedge^2\rho_m, \psi).$$

These factors have already been studied in detail [22]. The multiplicative property in this case has the following form. Let $\mathbf{M}_1 = \mathbf{GL}_{m_b} \times \cdots \times \mathbf{GL}_{m_1} \times \mathbf{SO}^*_{2n} \subset \mathbf{M}$ and suppose that τ is the generic subquotient of

$$i_{P}^{\mathbf{GL}_m(F)}(\tau_b \otimes \cdots \otimes \tau_1),$$

where τ_i is a generic representation of $\mathbf{GL}_{m_i}(F)$ for $1 \leq i \leq b$. Then

$$\gamma(s, \tau, \wedge^2\rho_m, \psi) = \prod_{i=1}^{b} \gamma(s, \tau_i, \wedge^2\rho_{m_i}, \psi) \prod_{i<j} \gamma(s, \tau_i \times \tau_j, \psi).$$

We now present two stability results for the γ-functions.

Lemma 2.8.1. *[49, Main Lemma 1] Let π be a generic representation of $\mathbf{SO}^*_{2n}(F)$ and τ a generic representation of $\mathbf{GL}_m(F)$. Then there exists a character χ of $F^\times$ so that $\gamma(s, \pi \times (\tau \cdot \chi), \psi)$ is a monomial in q_F^{-s}, for $1 \leq i \leq m$. Moreover χ can be replaced by any character of $F^\times$ whose conductor is larger than that of χ.*

We also have the following important result.

Theorem 2.8.2. *[16, Corollary 6.5] Let π_1 and π_2 be two irreducible generic representations of $\mathbf{SO}^*_{2n}(F)$ having the same central character, and let τ be an irreducible generic representation of $\mathbf{GL}_m(F)$. Then for a sufficiently highly ramified character χ of $F^\times$, we have*

$$\gamma(s, \pi_1 \times (\tau \cdot \chi), \psi) = \gamma(s, \pi_2 \times (\tau \cdot \chi), \psi).$$

2.9 Principal Series and Relative Rank one. Let E be a separable quadratic extension of F (local field of positive characteristic) contained in F_s, with Galois group $\mathrm{Gal}(E/F) = \{1, \sigma\}$. Let $(\chi_1, ..., \chi_{n-1}, \chi)$ be a character of the maximal subtorus $\mathbf{T}(F)$ of $\mathrm{SO}^*_{2n} = \mathbf{SO}(q_{E,n})(F)$ (See Section 1.5, for notations), where χ_i is a character of $F^\times$ for each $1 \leq i \leq n-1$ and χ is a character of E^1. Then if π is the generic subquotient of

$$i_{P_0}^{SO^*_{2n}}(\chi_1, ..., \chi_{n-1}, \chi)$$

and ξ a character of $F^\times$, the multiplicativity formula gives us

$$\gamma(s, \pi \times \xi, \psi) = \gamma(s, \chi \times \xi, \psi) \prod_{i=1}^{n-1} \gamma(s, \chi_i \xi, \psi)\gamma(s, \chi_i^{-1}\xi, \psi) \tag{2.9.1}$$

where the $\gamma(s, \chi_i\xi, \psi)$ are Tate factors.

Let us study the rank one case. First write $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ and let $\lambda(E/F, \psi)$ be the Langlands constant [8, Section 30.4]. Now let us recall that we constructed the simply connected cover of $\mathbf{SO}(q_{E,2})$ (1.5.1):

$$\mathrm{Res}_{E/F}\, \mathbf{SL}_2 \to \mathbf{SO}(q_{E,2}).$$

This morphism restricts to

$$\mathrm{diag}(t, t^{-1}) \mapsto [(x_1, x_2, x) \mapsto (N_{E/F}(t)x_1, N_{E/F}(t)^{-1}x_2, t\sigma^{-1}(t)x)].$$

Thus we have the following [38, Proposition 1.3]

Proposition 2.9.2. *Let (χ, ξ) be a smooth character of $\mathbf{T}(F)$, and μ the character of $E^\times$ defined by $[t \mapsto (\chi \circ N_{E/F})(t) \cdot \xi(t\sigma^{-1}(t))]$. Then*

$$\gamma(s, \chi \times \xi, \psi) = \lambda(E/F, \psi)\gamma(s, \mu, \psi_E).$$

2.10 L-Functions & ε-factors. The construction of γ-factors leads us to the construction of L-functions and ε-factors, for a general generic representation.

Let us start with the following definition: Let π be a tempered generic representation. Now let P_{π,r_i} be the unique polynomial with $P_{\pi,r_i}(0) = 1$ and such that $P_{\pi,r_i}(q_F^{-s})$ is the numerator of $\gamma(s, \pi, r_i, \psi)$. Then we define

$$L(s, \pi, r_i) = \frac{1}{P_{\pi,r_i}(q_F^{-s})}.$$

Starting from this definition, we can extend it to general generic representation (see the properties viii-ix below). In order that this construction has good properties (see for example properties vii,x), the following property (T), known as Shahidi's tempered L-function conjecture, must be valid. It was proved in most cases by Kim, see [30] for a detailed account, and Heiermann and Opdam in characteristic 0 for arbitrary quasi-split groups [21], using harmonic analytic tools developed in [19] and [51]. Now, we have the Langlands-Shahidi method available in positive characteristic [38], and observing that these harmonic analysis tools are also valid in this situation, it is natural to expect the validity of property (T) in all characteristic. Furthermore, a proof in positive characteristic for split reductive groups using the Kazhdan transfer is found in [39].

(T) Let π be a generic tempered representation of $\mathbf{M}(F)$, then $L(s,\pi,r_i)$ is holomorphic on $\mathrm{Re}(s)>0$.

Under the assumption that this property (T) holds, we have

vii) (Tempered ε-factors). Let π be a tempered generic representation of $\mathbf{M}(F)$. Then

$$\varepsilon(s,\pi,r_i,\psi)=\gamma(s,\pi,r_i,\psi)\frac{L(s,\pi,r_i)}{L(1-s,\tilde{\pi},r_i)},$$

is a monomial in q_F^{-s}.

Moreover, we can extend the definitions of L-functions and ε-factors for any generic representation in such a way that [38, Section 5]

viii) (Twists by unramified characters). Let π be a generic representation of $\mathbf{M}(F)$. Then

$$\begin{aligned}L(s+s_0,\pi,r_i)&=L(s,q_F^{-\langle s_0\tilde{\alpha},H_M(\cdot)\rangle}\otimes\pi,r_i),\\ \varepsilon(s+s_0,\pi,r_i,\psi)&=\varepsilon(s,q_F^{-\langle s_0\tilde{\alpha},H_M(\cdot)\rangle}\otimes\pi,r_i,\psi).\end{aligned}$$

ix) (Langlands' classification). Let π be a generic representation of $\mathbf{M}(F)$. Let π_0 be a tempered generic representation of $\mathbf{M}_{\theta_0}(F)$ and χ an unramified character of $\mathbf{M}_{\theta_0}(F)$ as in (Section 2.2). Suppose that π is the Langlands quotient of

$$i^M_{P_{\theta_0}\cap M}(\pi_{0,\chi}),$$

with $\pi_{0,\chi}=\pi_0\cdot\chi$. Note that each $\pi_{0,\chi,j}$ is quasi-tempered i.e. a tempered representation twisted by an unramified character. Then

$$\begin{aligned}L(s,\pi,r_i)&=\prod_{j\in S_i}L(s,\pi_{0,\chi,j},r_{i(j)}),\\ \varepsilon(s,\pi,r_i,\psi)&=\prod_{j\in S_i}\varepsilon(s,\pi_{0,\chi,j},r_{i(j)},\psi).\end{aligned}\tag{2.10.1}$$

x) (Global functional Equation). Let k be a global field of positive characteristic, with field of constants $\mathbb{F}_q$ and $\mathbf{G}'$ a quasi-split reductive k-group. Let $\pi=\otimes_x\pi_x$ be a

globally generic cuspidal automorphic representation of $\mathbf{M}'(\mathbb{A}_k)$ where $\mathbf{M}'$ is the Levi k-subgroup of a maximal parabolic subgroup of $\mathbf{G}'$, containing a Borel subgroup $\mathbf{P}'_0$. We define

$$L(s,\pi,r_i) \coloneqq \prod_{x\in|k|} L(s,\pi_x,r_{i,x}).$$

It is a rational function in q^{-s} and we have the functional equation

$$L(s,\pi,r_i) = \varepsilon(s,\pi,r_i)L(1-s,\tilde{\pi},r_i). \tag{2.10.2}$$

As before, we apply this construction to $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}^*_{2n} \subset \mathbf{SO}^*_{2(m+n)}$. Let π be a generic representation of $\mathbf{SO}^*_{2n}(F)$ and τ a generic representation of $\mathbf{GL}_m(F)$. Then $\tau\otimes\tilde{\pi}$ is a generic representation of $\mathbf{M}(F)$ and we denote by

$$L(s,\tau\otimes\tilde{\pi},r_1,\psi) = L(s,\pi\times\tau)$$
$$\varepsilon(s,\tau\otimes\tilde{\pi},r_1,\psi) = \varepsilon(s,\pi\times\tau,\psi).$$

We also note that these factors satisfy similar multiplicativity formulas, as in the case of γ-factors (Section 2.8).

Remark 2.10.3. Let us consider the case of a maximal Levi subgroup $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}^*_{2n} \subset \mathbf{SO}^*_{2(n+m)}$ and a tempered representation $\tau\otimes\pi$ of $\mathbf{M}(F)$, where π is an unramified representation of $\mathbf{SO}^*_{2n}(F)$. As each Satake parameter of π has absolute value 1, property (T) is true for general linear groups and by property iv) of Section 2.7, we get that $L(s,\tau\otimes\pi,r_i)$ satisfies property (T).

To finish, we note that it follows from the construction of the L-functions and Lemma 2.8.1 that given any generic representation π of $\mathbf{M}(F)$and τ a generic representation of $\mathbf{GL}_m(F)$, there exists a character χ of $F^\times$ so that

$$L(s,\pi\times(\tau\cdot\chi)) \equiv 1. \tag{2.10.4}$$

2.11 Kim's assumption. In this section we discuss Kim's assumption. In general its proof relies on property (T) and the standard module conjecture. This last conjecture, follows formally from property (T), as can be seen from [20], the machinery for its proof being in place.

We provide a proof of a version of Kim's assumption in Proposition 2.11.8, under the assumption of a weak version of the standard module conjecture: let π be a generic non-tempered unramified representation of $\mathbf{SO}^*_{2n}(F)$. Then it is a full induced representation of the form

$$i_P^{\mathbf{SO}^*_{2n}(F)}(\pi_b|\det|^{r_b}\otimes\cdots\otimes\pi_1|\det|^{r_1}\otimes\pi_0), \tag{2.11.1}$$

where $0<r_1\leq\cdots\leq r_b$, π_i is an irreducible tempered representation of $\mathbf{GL}_{n_i}(F)$ for $1\leq i\leq b$ and π_0 is an irreducible tempered representation of $\mathbf{SO}^*_{2n_0}(F)$.

In general the standard module conjecture was proven in characteristic zero [20] for quasi-split groups using the harmonic analytic tools in [19], which are valid in all characteristics. Having the Langlands-Shahidi method in positive characteristic, we plan to look these three interrelated properties, namely, Shahidi's tempered L-function conjecture, the standard module conjecture and Kim's assumption, in future work. For the moment, we will assume the version of the standard module conjecture in positive characteristic given above to prove Kim's assumption in the special case we need.

Let $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}_0$, associated to ${}_F\Delta \setminus \{\alpha\}$ and $\tilde{w}_0 \in \mathbf{G}(F)$ a representative of $w_0 = w_{l,\mathbf{G}} w_{l,\mathbf{M}} \in W^{\mathbf{G}}$ (Section 1.1). For π a generic representation of $\mathbf{M}(F)$, we define

$$r(s,\pi) = \prod_{i=1}^{m_r} \frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)\varepsilon(is,\pi,r_i,\psi)},$$

and the normalized intertwining operator $N(s,\pi,\tilde{w}_0)$ is defined to be such that

$$A(s,\pi,\tilde{w}_0) = r(s,\pi)N(s,\pi,\tilde{w}_0), \tag{2.11.2}$$

as a rational operator in s (See Section 2.3, for definition of $A(s,\pi,\tilde{w}_0)$). Kim's assumption then asserts that

(A) Let k be a global field. Assume that $\pi = \otimes_{x\in|k|}\pi_x$ is a globally generic unitary cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_k)$. Then $N(s,\pi_x,\tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}(s) \geq 1/2$ for all $x \in |k|$.

This assumption has been proven in characteristic 0 in [30] for quasi-split groups, using the tempered L-function conjecture and the standard module conjecture. We will prove a version (Proposition 2.11.8) of this assumption, but that will be enough for our purposes (Section 6.3).

Lemma 2.11.3. *Let ρ be a generic unitary cuspidal representation of $\mathbf{M}(F)$. Then $N(s,\rho,\tilde{w}_0)$ is holomorphic and non-zero outside of $\mathrm{Re}(s) = -1/2, -1$.*

Proof. Indeed from [38, Section 5], $\prod_{i=1}^{m} L(is,\rho,r_i)^{-1}A(s,\rho,\tilde{w}_0)$ is entire and non-zero. Therefore the poles of $N(s,\rho,\tilde{w}_0)$ are zeros of $\prod_{i=1}^{m} L(1+is,\rho,r_i)^{-1}$, but this does not have a zero outside of $\mathrm{Re}(s) = -\frac{1}{2}, -1$ [38, Section 5]. □

Lemma 2.11.4. *Let π be a generic tempered representation, for which property (T) is true. Then $N(s,\pi,\tilde{w}_0)$ is holomorphic on $\mathrm{Re}(s) \geq 0$ and non-zero on $\mathrm{Re}(s) > 0$.*

Proof. In fact, in this case $A(s,\pi,\tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}(s) > 0$ [56, Section IV, Proposition 2.1 & Equation (10)]. Now thanks to property (T), we conclude that $N(s,\pi,\tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}(s) > 0$. For $\mathrm{Re}(s) = 0$ we use [58, Lemma 2], which is also valid in positive characteristic. □

Let $\mathbf{M}_{\theta_0}$ be a non-maximal Levi subgroup $\mathbf{G}$, with $\theta_0 \subset {}_F\Delta \setminus \{\alpha\}$. In this case, we recall that we also have spaces of parameters denoted by $a^*_{\theta_0,\mathbb{C}} = a^*_{\mathbf{M}_{\theta_0},\mathbb{C}}$ and $a^*_{\theta_0} = a^*_{\mathbf{M}_{\theta_0}}$ (Section 2.1). We will need them to state the following non-vanishing result concerning the normalized intertwining operator.

Proposition 2.11.5. *Let π be a generic representation of $\mathbf{M}(F)$. Suppose that*

$$\pi = i^M_{P_{\theta_0}\cap M}(\Lambda, \pi_0),$$

where $\theta_0 \subset {}_F\Delta \setminus \{\alpha\}$, $\Lambda \in a^*_{\theta_0,\mathbb{C}}$ with $\mathrm{Re}(\Lambda) \in a^{*,+}_{\theta_0}$, π_0 is a tempered generic representation of $\mathbf{M}_{\theta_0}(F)$, and assume that $L(s, \pi_{0,j}, r_{i(j)})$ satisfies (T) for every i and $j \in S_i$. Then if $N(s, \pi, \tilde{w}_0)$ is holomorphic at $s_0 \in \mathbb{C}$, it is also non-zero at s_0.

This result is [58, Theorem 3] in characteristic zero. We mostly follow his proof with almost no changes to prove the result in positive characteristic.

Proof. As in [58], we extend the definition of the normalization $r(s, \pi)$, to deal with the non-maximal parabolic subgroup $\mathbf{P}_{\theta_0}$, by

$$r(\nu, \pi_0) = \prod_{l=1}^{n-1} r(\nu_i, \pi_{0,l}),$$

for $\nu \in a^*_{\theta_0,\mathbb{C}}$. This allows us to define

$$N(\nu, \pi_0, \tilde{w}_0) = r(\nu, \pi_0)^{-1} A(\nu, \pi_0, \tilde{w}_0).$$

From the condition that $L(s, \pi_{0,j}, r_{i(j)})$ satisfies (T) for every i and $j \in S_i$ and Lemma 2.11.4, we obtain that $N(\nu, \pi, \tilde{w}_0)$ is non-zero and holomorphic on $\mathrm{Re}(\nu) \in a^{*,+}_{\theta_0}$ and holomorphic on $\mathrm{Re}(\nu) \in \overline{a}^{*,+}_{\theta_0}$. Then arguing as in [58, Theorem 3, p. 393] we conclude that it is also non-vanishing in the closure. Finally, we can find $g \in W^G$ such that $g(\theta_0) \subset \Delta$, and $w_0(s_0\tilde{\alpha}+\Lambda)$ is in $\overline{a}^{*,+}_{g(\theta_0)}$. Then again thanks to the proof of [58, Theorem 3, pp. 393-394], we obtain that $N(\nu, \pi_0, \tilde{w}_0)$ is non-vanishing at $s_0\tilde{\alpha}+\Lambda$, as it is holomorphic there. Finally as $N(s_0, \pi, \tilde{w}_0) = N(s_0\tilde{\alpha} + \Lambda, \pi_0, \tilde{w}_0)|_{i^G_P(s,\pi)}$ [29, Proof of Lemma 4.3], we obtain the non-zeroness. □

Lemma 2.11.6. *[40, Proposition I.10] Let τ, τ' be two tempered representations of $\mathbf{GL}_n(F)$ and $\mathbf{GL}_l(F)$, respectively. Then $N(s, \tau \otimes \tau', \tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}(s) > -1$.*

We adapt the arguments in [28, Lemma 3.3] to our case.

Proposition 2.11.7. *Let π be a generic unramified representation of $\mathbf{SO}^*_{2n}(F)$. Suppose that $L(s, \pi \times \tau)$ is holomorphic on $\mathrm{Re}(s) > 1$, for every unitary generic representation τ of $\mathbf{GL}_m(F)$. Then π is the Langlands quotient of*

$$i_P^{\mathbf{SO}^*_{2n}(F)}(\pi_b |\det|^{r_b} \otimes \cdots \otimes \pi_1 |\det|^{r_1} \otimes \pi_0),$$

where $0 < r_1 \leq \cdots \leq r_b$, with $r_b < 1$, π_i is an irreducible tempered representation of $\mathbf{GL}_{n_i}(F)$ for $1 \leq i \leq b$ and π_0 is an irreducible tempered representation of $\mathbf{SO}^*_{2n_0}(F)$.

Proof. We can suppose that each π_i is a discrete series representation of $\mathbf{GL}_{n_i}(F)$, for $1 \leq i \leq b$. By property ix) of Section 2.10, $L(s - r_b, \pi_b \times \tilde{\pi}_b)$ is a factor of $L(s, \pi \times \tilde{\pi}_b)$. As $L(s - r_b, \pi_b \times \tilde{\pi}_b)$ is a Rankin-Selberg L-functions (Remark 3.1.1), it has a pole for $s = r_b$ and $L(s, \pi \times \tilde{\pi}_b)$ is holomorphic on $\mathrm{Re}(s) > 1$, we have that $r_b < 1$. □

The L-function condition in the previous proposition will be studied in Proposition 4.2.4, for a local component of a cuspidal automorphic representation. The next proposition is a weak version of Kim's assumption that we will prove under the assumption of the standard module conjecture in the case (2.11.1). To prove it we adapt the arguments in [28, Proposition 3.4] to our case.

Proposition 2.11.8. *Let π be generic unramified representation of $\mathbf{SO}^*_{2n}(F)$ such that it is the full induced representation*

$$i_P^{\mathbf{SO}^*_{2n}(F)}(\pi_b |\det|^{r_b} \otimes \cdots \otimes \pi_1 |\det|^{r_1} \otimes \pi_0),$$

where $0 < r_1 \leq \cdots \leq r_b$, with $r_b < 1$, π_i is an irreducible tempered representation of $\mathbf{GL}_{n_i}(F)$ for $1 \leq i \leq b$ and π_0 is an irreducible tempered representation of $\mathbf{SO}^*_{2n_0}(F)$. Then $N(s, \tau \otimes \tilde{\pi}, \tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}(s) \geq 1/2$ for every generic unitary representation τ of $\mathbf{GL}_m(F)$.

Proof. We can write τ as the full induced representation [54, Section 7]

$$i_Q^{\mathrm{GL}_m}(\xi_1 |\det|^{t_1} \otimes \cdots \otimes \xi_d |\det|^{t_d} \otimes \xi_{d+1} \otimes \xi_d |\det|^{-t_d} \otimes \cdots \otimes \xi_1 |\det|^{-t_1}),$$

where $\mathbf{Q}$ is a parabolic subgroup containing the Borel subgroup of $\mathbf{GL}_m$ consisting of upper triangular matrices, the ξ_i's are tempered representations of $\mathbf{GL}_{m_i}(F)$ and $0 < t_1 \leq \cdots \leq t_d < 1/2$.

Combining the description for π and τ as induced representations, we obtain that $\tau \otimes \tilde{\pi}$ is full induced from quasi-tempered datum. This allows us to use multiplicativity of the normalized intertwining operators (See [30, Proposition 4.6]), in order to reduce to the following rank one cases $\mathbf{GL}_k \times \mathbf{GL}_l \subset \mathbf{GL}_{l+k}$, $\mathbf{SO}^*_{2l} \times \mathbf{GL}_k \subset \mathbf{SO}^*_{2(l+k)}$ and $\mathbf{GL}_{l-1} \subset \mathbf{SO}^*_{2l}$ ($l \geq 3$):

i) For the case $\mathbf{GL}_k \times \mathbf{GL}_l \subset \mathbf{GL}_{l+k}$, we obtain from $\mathrm{Re}(s \pm r_i \pm t_j) > -1$, for $\mathrm{Re}(s) \geq 1/2$ the condition, thanks to Lemma 2.11.6.

ii) For the case $\mathbf{SO}^*_{2l} \times \mathbf{GL}_k \subset \mathbf{SO}^*_{2(l+k)}$, we note that $\mathrm{Re}(s \pm t_d) \geq 0$ for $\mathrm{Re}(s) \geq 1/2$. As π_0 is tempered and unramified it satisfies property (T) (Remark 2.10.3) and thus thanks to Lemma 2.11.4, we get our condition.

iii) Finally for the case $\mathbf{GL}_{l-1} \subset \mathbf{SO}^*_{2l}$ ($l \geq 3$), we use Lemma 2.11.3 as in [27, Lemma 3.3, Proposition 3.4] to conclude.

□

Chapter 3

Converse theorem

In this chapter, based on [9] and [33], we provide a proof of the twisted version of the converse theorem found in [11, Section 2] for an admissible irreducible representation in positive characteristic. This result is stated in [37, Theorem 8.1], and we now take the opportunity to provide a proof.

Let F be a global field of positive characteristic with field of constants $\mathbb{F}_q$. We keep the notations introduced in Section 2.4.

We denote by $\mathbf{U}_n$ the radical unipotent subgroup of the Borel F-subgroup $\mathbf{B}_n$ of upper triangular matrices of $\mathbf{GL}_n$ and by $\mathbf{Z}_n$ the center of $\mathbf{GL}_n$.

Let us also write

$$K = \prod_x K_x = \prod_x \mathbf{GL}_n(\mathcal{O}_x).$$

It is a maximal open compact subgroup of $\mathbf{GL}_n(\mathbb{A}_F)$, and $\mathbf{GL}_n(\mathbb{A}_F)$ is the restricted product of $\mathbf{GL}_n(F_x)$ with respect to the $K_x = \mathbf{GL}_n(\mathcal{O}_x)$.

3.1 Rankin-Selberg L-functions.

(*Local factors*). Let ψ_x be a non-trivial character of F_x and use it to also denote the (generic) character of the unipotent group $\mathbf{U}_n(F_x)$, that associates to $u = (u_{i,j})$ the complex number

$$\psi_x(u) = \sum_{i=1}^{n-1} \psi_x(u_{i,i+1}).$$

Let us consider a pair of irreducible smooth representations or induced smooth representations of Whittaker type π_1 and π_2 of $\mathbf{GL}_n(F_x)$ and $\mathbf{GL}_m(F_x)$, respectively. We define as in [26] the local Rankin-Selberg L-functions, ε-factors and γ-factors,

$$L(s, \pi_1 \times \pi_2), \quad \varepsilon(s, \pi_1 \times \pi_2, \psi_x) \quad \& \quad \gamma(s, \pi_1 \times \pi_2, \psi_x).$$

They are rational polynomials in $\mathbb{C}(q^{-s})$.

Let us recall the definition of L-functions for induced representations of Whittaker type, i.e. representations of the form

$$i_{\mathbf{Q}(F_x)}^{\mathbf{GL}_n(F_x)}(\rho_{1,x}|\det|^{u_{1,x}} \otimes \cdots \otimes \rho_{m_x,x}|\det|^{u_{m_x,x}}),$$

where $\mathbf{Q}$ is a parabolic subgroup containing $\mathbf{B}_n$ associated to a partition $(r_{1,x}, \ldots, r_{m_x,x})$ of n, $\rho_{i,x}$ is an irreducible square-integrable representation of $\mathbf{GL}_{r_{i,x}}(F_x)$ and the $u_{i,x}$ are real numbers satisfying $u_{1,x} \leq \cdots \leq u_{m_x,x}$. Every induced representation of Whittaker type π admits a (ψ_x)-Whittaker model, i.e the space $\mathcal{W}(\pi, \psi_x)$ spanned by functions on $\mathbf{GL}_n(F_x)$ of the form

$$g \mapsto \lambda_{\psi_x}(\pi(g)\xi_x),$$

where λ_{ψ_x} is a non-zero Whittaker functional, and ξ_x is a vector of π (See Section 2.6). Once the non-zero Whittaker functional is fixed, we denote such function by W_{ξ_x}.

Now, consider a pair of induced smooth representations τ of $\mathbf{GL}_n(F_x)$ and τ' of $\mathbf{GL}_m(F_x)$ of Whittaker type. We define for any $W \in \mathcal{W}(\tau, \psi_x)$, $W' \in \mathcal{W}(\tau', \psi_x)$, and any compactly supported locally constant function $\Phi\colon F_x^n \to \mathbb{C}$, the following local integrals, which define rational functions in $\mathbb{C}(q_{F_x}^{-s})$ [26, Theorem 2.7]. In the case where $m < n$, for $0 \leq j \leq n-m-1$, we denote

$$\Psi_j(s; W, W') = \int_{\mathbf{U}_m(F_x)\backslash \mathbf{GL}_m(F_x)} \left(\int_{\mathbf{M}_{j,m}(F_x)} W \begin{pmatrix} h & & \\ y & I_j & \\ & & I_{n-m-j} \end{pmatrix} dy \right) \cdot W'(h)|\det(h)|^{s-(n-m)/2} dh.$$

In the case where $m = n$, we put

$$\Psi(s; W, W', \Phi) = \int_{\mathbf{U}_m(F_x)\backslash \mathbf{GL}_m(F_x)} W(g)W'(g)\Phi(e_n g)|\det(g)|^s dg.$$

These integrals form $\mathbb{C}[q_{F_x}^s, q_{F_x}^{-s}]$-fractional ideals $I(\tau, \tau')$ in the case where $n = m$, and $I_j(\tau, \tau')$, in the case where $m < n$, for $0 \leq j \leq n-m-1$, in $\mathbb{C}(q_{F_x}^{-s})$. The unique generator of these ideals has the form

$$L(s, \tau \times \tau') = \frac{1}{P(q^{-s})}$$

with $P(X) \in \mathbb{C}[X]$ a polynomial with $P(0) = 1$. This is the Rankin-Selberg L-function of $\tau \times \tau'$.

Remark 3.1.1. We remark that from [23, Corollary 3.8], in the where case where π_1, π_2 are generic representations of $\mathbf{GL}_n(F_x)$ and $\mathbf{GL}_m(F_x)$ respectively, the polynomials $L(s, \pi_1 \times \pi_2)$ and $\varepsilon(s, \pi_1 \times \pi_2, \psi)$ coincide with $L(s, \pi_1 \otimes \tilde{\pi}_2, r)$ and $\varepsilon(s, \pi_1 \otimes \tilde{\pi}_2, \psi, r)$, defined in Section 2.10, where the maximal parabolic subgroup considered contains $\mathbf{B}_{n+m}$ and its Levi subgroup is isomorphic to $\mathbf{GL}_n \times \mathbf{GL}_m$.

(*Global Factors*). Let $\psi = \otimes \psi_x$ be a non-trivial character of $F\backslash\mathbb{A}_F$ and use it to also denote the character of the unipotent group $\mathbf{U}_n(\mathbb{A}_F)$, that associates to $u = (u_{i,j})$ the complex number

$$\psi(u) = \sum_{i=1}^{n} \psi(u_{i,i+1}).$$

Let $\pi = \otimes \pi_x$ be a factorizable representation of $\mathbf{GL}_r(\mathbb{A}_F)$ and $\pi' = \otimes \pi'_x$ be a factorizable representation of $\mathbf{GL}_{r'}(\mathbb{A}_F)$. We assume that π_x and π'_x are irreducible or induced of Whitaker type. By definition $L(s, \pi_x \times \pi'_x)$ is a rational function such that $L(s, \pi_x \times \pi'_x)^{-1}$ is a polynomial in the variable $q^{-\deg(x)s}$, with constant term equal to 1. Furthermore, $\varepsilon_x(s, \pi_x \times \pi'_x, \psi_x)$ is a monomial in the variable $q^{-\deg(x)s}$ and is equal to 1 for almost all x. We can define

$$L(s, \pi \times \pi') = \prod_x L(s, \pi_x \times \pi'_x), \text{ as a formal power series in } q^{-s},$$
$$\varepsilon(s, \pi \times \pi', \psi) = \prod_x \varepsilon(s, \pi \times \pi', \psi_x) \text{ as a monomial in } q^{-s},$$

because there are only a finite number of places x with given q_{F_x}.

3.2 Converse Theorem. Let S be a finite subset of $|F|$. For each integer m, let

$$\mathcal{A}_0(m) = \{\tau \mid \tau \text{ is a cuspidal representation of } \mathbf{GL}_m(\mathbb{A}_F)\},$$

and

$$\mathcal{A}_0^S(m) = \{\tau \in \mathcal{A}_0(m) \mid \tau_v \text{ is unramified for all } v \in S\}.$$

For $n \geq 2$, we set

$$\mathcal{T}(n-1) = \coprod_{m=1}^{n-1} \mathcal{A}_0(m) \quad \text{and} \quad \mathcal{T}^S(n-1) = \coprod_{m=1}^{n-1} \mathcal{A}_0^S(m).$$

If η is a continuous character of $F^\times \setminus \mathbb{A}_F^\times$, set

$$\mathcal{T}(S;\eta) = \mathcal{T}^S(n-1) \otimes \eta = \{\tau = \tau' \otimes \eta \colon \tau' \in \mathcal{T}^S(n-1)\}.$$

Theorem 3.2.1. *Let $n \geq 2$ be an integer, let $\pi = \bigotimes_{x\in|F|} \pi_x$ be an irreducible admissible representation of $\mathbf{GL}_n(\mathbb{A}_F)$ and let η be a continuous character of $\mathbb{A}_F^\times$ trivial on $F^\times$.*

We suppose that, for a finite set S of places $|F|$, π satisfies the following properties:

i) *The central character $\chi_\pi = \bigotimes_{x\in|F|} \chi_{\pi_x}$ of π is invariant by the discrete subgroup $F^\times$ of $\mathbb{A}_F^\times$.*

ii) *For all $\pi' \in \mathcal{T}(S;\eta)$, the formal series*

$$L(s, \pi \times \pi') \text{ and } L(s, \tilde{\pi} \times \tilde{\pi}')$$

are polynomials and they satisfy the functional equation

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi', \psi) L(1-s, \tilde{\pi} \times \tilde{\pi}').$$

Then there exists an irreducible automorphic representation ρ of $\mathbf{GL}_n(\mathbb{A}_F)$ such that, for each place $x \notin S$ such that π_x is unramified, ρ_x is unramified and $\pi_x \cong \rho_x$. Moreover, ρ is cuspidal if $S = \emptyset$.

In order to prove this, we will review some notions.

(Subgroups and Compact subgroups of $\mathbf{GL}_n$). We fix a normal and proper curve X_F over $\mathbb{F}_q$ (unique up to isomorphism) with field of fractions F.

Denote by $\mathbf{P}_n \subset \mathbf{GL}_n$ the subgroup of matrices of the form

$$\begin{pmatrix} * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For every closed subscheme N of X_F supported on S with the ring of global sections denoted by $\mathcal{O}_N$, we consider the finite index subgroup $K'_S(N)$ of $K_S = \prod_{x \in S} \mathbf{GL}_n(\mathcal{O}_x)$ of matrices with image in $\mathbf{GL}_n(\mathcal{O}_N)$ of the form

$$\begin{pmatrix} * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We denote $\mathbf{GL}_n(\mathbb{A}_F)'_S(N)$ the open subgroup of $\mathbf{GL}_n(\mathbb{A}_F)$ given by the inverse image of $K'_S(N)$ under $\mathbf{GL}_n(\mathbb{A}_F) \to \prod_{x \in S} \mathbf{GL}_n(F_x)$.

(Whittaker models). Every induced representation of (ψ_x)-Whittaker type π_x of $\mathbf{GL}_n(F_x)$ with a fixed Whittaker functional, admits a Whittaker model $\mathcal{W}(\pi_x, \psi_x)$, which we recall is a non-zero space spanned by functions $W_{\xi_x} \colon \mathbf{GL}_n(F_x) \to \mathbb{C}$, indexed by vector in the space of π_x. Note that each such function W_{ξ_x} is right-invariant under some open subgroups of $\mathbf{GL}_n(F_x)$ and the collection of these functions satisfies the following relation:

$$W_{\xi_x}(u_x g_x) = \psi_x(u_x) W_{\xi_x}(g_x), \text{ for every } g_x \in \mathbf{GL}_n(F_x), u_x \in \mathbf{U}_n(F_x).$$

Globally, let $\pi = \otimes_x \pi_x$ be an admissible representation of $\mathbf{GL}_n(\mathbb{A}_F)$, where π_x is induced of Whittaker type with fixed Whittaker functional. We can choose K_x-fixed vectors ξ_x°, for x outside some finite subset T of $|F|$, such that $W_{\xi_x^\circ} \in \mathcal{W}(\pi_x, \psi_x)$ is invariant under right multiplication by the compact open subgroup $\mathbf{GL}_n(\mathcal{O}_x)$ and it is equal to 1 at the identity. Now, for every vector $\xi = (\xi_x)_{x \in |F|}$ of π, such that $\xi_x = \xi_x^\circ$ for almost all x, we consider the complex valued function on $\mathbf{GL}_n(\mathbb{A}_F)$ given by

$$W_\xi : g = (g_x)_x \mapsto \prod_x W_{\xi_x}(g_x). \tag{3.2.2}$$

Each such W_ξ is right-invariant under a compact open subgroup of $\mathbf{GL}_n(\mathbb{A}_F)$ and satisfies

$$W_\xi(ug) = \psi(u)W_\xi(g), \text{ for every } g \in \mathbf{GL}_n(\mathbb{A}_F), u \in \mathbf{U}_n(\mathbb{A}_F).$$

This function will be our main ingredient for constructing a non-zero equivariant homomorphism to the space of automorphic forms.

(Twist). We have the following relation between Rankin-Selberg L-functions

Proposition 3.2.3. *Let E a non-archimedean local field, $\eta : E^\times \to \mathbb{C}^\times$ a character and $(\tau, V), (\tau', V')$ two induced representations of Whittaker type of $\mathbf{GL}_n(E)$ and $\mathbf{GL}_m(E)$, respectively. Then*

$$L(s, \tau \times (\tau' \otimes \eta)) = L(s, (\tau \otimes \eta) \times \tau').$$

Proof. By definition (Section 3.1), we notice that, after choosing a Whittaker functional $\Lambda : V \to \mathbb{C}$ of τ, we can compute the function $W_\xi \in \mathcal{W}(\tau \otimes \eta, \psi)$ as follows

$$W_\xi(g) = \eta(\det(g))\Lambda(\tau(g)\xi) = \Lambda(\tau \otimes \eta(g)\xi). \tag{3.2.4}$$

Now, let $\Lambda\colon V \to \mathbb{C}$ and $\Lambda'\colon V' \to \mathbb{C}$ be the respective Whittaker functionals of τ and τ', and $W_\xi \in \mathcal{W}(\tau, \psi)$ and $W'_{\xi'} \in \mathcal{W}(\tau' \otimes \eta, \psi)$. Then using the identity (3.2.4) we get that

$$\Psi(s; W_\xi, W'_{\xi'}) = \Psi(s; \Lambda(\tau(\cdot)\xi), \eta(\det(\cdot))\Lambda'(\tau'(\cdot)\xi')),$$

if $n = m$, and

$$\Psi_j(s; W_\xi, W'_{\xi'}) = \Psi_j(s; \Lambda(\tau(\cdot)\xi), \eta(\det(\cdot))\Lambda'(\tau'(\cdot)\xi')),$$

if $m < n$ and $0 \leq j \leq n - m - 1$. As these relations imply by definition the equality of the ideals, we have proved our desired relations. □

Now we go back to the proof of the converse theorem.

Proof of Theorem 3.2.1. For every $x \in |F|$ such that π_x unramified, we fix a vector $v_x \in V_x^{K_x}$. For every x, let Ξ_x be the representation of $\mathbf{GL}_n(F_x)$ that has π_x as its unique Langlands' quotient. Every Ξ_x is of the form

$$\Xi_x = i_{\mathbf{Q}(F_x)}^{\mathbf{GL}_n(F_x)}(\rho_{1,x}|\det|^{u_{1,x}} \otimes \cdots \otimes \rho_{m_x,x}|\det|^{u_{m_x,x}}),$$

where $\mathbf{Q}$ is a parabolic subgroup containing $\mathbf{B}_n$ associated to a partition $(r_{1,x}, \ldots, r_{m_x,x})$ of n, $\rho_{i,x}$ is an irreducible tempered representation of $\mathbf{GL}_{r_i,x}(F_x)$ and the $u_{i,x}$ are real numbers satisfying $u_{1,x} > \cdots > u_{m_x,x}$.

We can reduce the theorem to the case $\eta = 1$. Indeed, by definition of Rankin-Selberg L-function (Section 3.1) and using Proposition 3.2.3 we have

$$\begin{aligned} L(s, \pi_x \times (\pi'_x \otimes \eta_x)) &= L(s, \Xi_x \times (\Xi'_x \otimes \eta_x)) \\ &= L(s, (\Xi_x \otimes \eta_x) \times \Xi'_x) = L(s, (\pi_x \otimes \eta_x) \times \pi'_x), \end{aligned}$$

we can apply the Theorem 3.2.1, with trivial character, to $\pi \otimes \eta$. Therefore we have that there exists an automorphic representation Π' such that $\Pi'_x \cong \pi_x \otimes \eta$ for $x \notin S$ such that π_x is unramified. Then $\Pi := \Pi' \otimes \eta^{-1}$ is automorphic and satisfies that $\Pi_x \cong \pi_x$ for $x \notin S$ such that π_x is uramified.

Suppose that S is not empty. For every $x \notin S$ for which π_x is unramified, Ξ_x must have a unique K_x-fixed vector ξ_x° which projects to the fixed K_x-fixed v_x vector of π_x. From these choices, we can consider for every $\xi = (\xi_x)_x$ such that $\xi_x = \xi_x^\circ$ for almost all $x \notin S$, the global Whittaker function W_ξ (3.2.2).

Now for every $x \in S$ such that π_x is ramified, we can choose ξ_x° such that $(\xi_x^\circ)_{x\in S}$ is $K'_S(N)$-invariant for some subscheme N of X_F, supported on S, and ([9, Section 8 & p. 203])

$$W_{\xi_x^\circ}(1) = 1.$$

Thus, for every $x \in S$, ξ_x° is invariant under right multiplication by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$, with $h \in \mathbf{GL}_{n-1}(\mathcal{O}_x)$.
Finally we consider as in [33, Corollaire B.15], the well defined function on $\mathbf{GL}_n(\mathbb{A}_F)$

$$U_\xi(g) = \sum_{\gamma \in \mathbf{U}_n(F)\backslash \mathbf{P}_n(F)} W_\xi(\gamma g). \tag{3.2.5}$$

Putting these together we are able to consider, for every $\xi^S = (\xi_x)_{x\notin S}$ completed by $\xi = (\xi^S, (\xi_x^\circ)_{x\in S})$, the function U_{ξ^S} on $\mathbf{GL}_n(\mathbb{A}_F)$ defined by

$$U_{\xi^S}(g) = U_\xi(g'),$$

if g can be written as $g = \gamma g'$ with $\gamma \in \mathbf{GL}_n(F)$ and $g' \in \mathbf{GL}_n(\mathbb{A}_F)'_S(N)$ and, if not, by

$$U_{\xi^S}(g) = 0.$$

The map $\xi^S \mapsto U_{\xi^S}$ defines a non-zero [9, Lemma 6.3] equivariant homomorphism of the smooth admissible representation $\Xi^S = \otimes_{x\notin S} \Xi_x$ of $\prod'_{x\notin S} \mathbf{GL}_n(F_x)$ to the space of functions on $\mathbf{GL}_n(F)\backslash \mathbf{GL}_n(\mathbb{A}_F)$ that are invariant under right multiplication by open compact subgroups of $\mathbf{GL}_n(\mathbb{A}_F)$ [33, p. 237]. The action of the center $\mathbf{Z}_n(\mathbb{A}_F)$ of $\mathbf{GL}_n(\mathbb{A}_F)$ on the span of these functions is according to the central character χ_π of π.

Since Ξ^S has Π^S as its unique irreducible quotient, if we take a vector ξ^S which has a non-zero projection to Π^S, then ξ^S is a cyclic generator of Ξ^S. Thus the representation V of $\mathbf{GL}_n(\mathbb{A}_F)$ generated by the space of U_{ξ^S} is admissible [5, Section 5] and cyclic, generated by some element f_0. Let U be a maximal $\mathbf{GL}_n(\mathbb{A}_F)$-invariant subspace of V not containing f_0. Then $\Pi' = V/U$ is a non-zero subquotient of the space of automorphic forms; Π' is automorphic and at every place $x \notin S$ where π_x is unramified, its Satake parameter equal that to the one of π_x [9, Theorem A].

In the case where S is empty, we just consider $(\xi \mapsto U_\xi)$. As U_ξ is cuspidal [25, Proposition 12.3], we can conclude as before. □

Chapter 4

Global L-functions

In this chapter, we start by recalling the global theory of intertwinining operators and Eisenstein series, following [41], to have access to certain properties of global L-functions. More precisely, these properties are the holomorphicity of L-functions after a twist, that will be essential in the application of the converse theorem construction to a candidate lift (Section 6.3), and the holomorphicity of partial L-functions without a twist, that will be used to describe the image of the functoriality (Theorem 7.2.1). These two properties are obtained following the ideas in [39], in the context of the Langlands-Shahidi method in positive characteristic.

Let $\mathbf{G}$ be a quasi-split (connected) reductive group over a positive characteristic global field F with field of constants $\mathbb{F}_q$ and fix a separable closure F_s of F. Let us fix a maximal F-split subtorus $\mathbf{S}$ and $\mathbf{P}_0 = \mathbf{T}\mathbf{U}_0$ a minimal parabolic subgroup containing $\mathbf{S}$.

4.1 Intertwining operator & Eisenstein Series. We first introduce the intertwining operator and Eisenstein Series. Then we realize a connection between them using partial L-functions. We fix a maximal compact open subgroup $K = \prod_x K_x$ of $G = \mathbf{G}(\mathbb{A}_F)$, as in [41, Section I.1.4].

Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a maximal parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}_0$ associated to $\theta = {}_F\Delta - \{\alpha\} \subset {}_F\Delta$ and let $w_0 = w_{l,\mathbf{G}} w_{l,\mathbf{M}} \in W^{\mathbf{G}}$ (see Section 1.1). Let $\sigma = \otimes_x \sigma_x$ a unitary cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_F)$, where the restricted product is taken with respect to functions $\{\varphi_x^\circ\}$. We write

$$i_P^G(s,\sigma) = \bigotimes_x i_P^G(s,\sigma_x) = \bigotimes_x i_P^G(\sigma_x \otimes q^{\langle s\tilde{\alpha}, H_{\mathbf{P}_x}(\cdot)\rangle}),$$

where the restricted product is taken with respect to the functions $f_{x,s}^\circ \in i_P^G(s,\pi_x)$ such that $f_{x,s}^\circ(k_x) = \varphi_x^\circ$ for all $k_x \in K_x$. For $\tilde{w}_0$ a representative of w_0, we define the global

intertwining operator for $\mathrm{Re}(s)$ big enough, as in [39, Section 1.2], by

$$M(s,\sigma,\tilde{w}_0)\colon i_P^G(s,\sigma)\to i_{P'}^G(w_0(s),w_0(\sigma))$$
$$f\mapsto\left(g\mapsto\int_{\mathbf{N}'(\mathbb{A}_F)}f(\tilde{w}_0^{-1}ng)dn\right),$$

where $\mathbf{N}'$ is the radical of $\mathbf{P}'=\mathbf{P}_{w_0(\theta)}$, and the Eisenstein series, for an automorphic form $\Phi\colon\ \mathbf{M}(F)\,\mathbf{N}(\mathbb{A}_F)\backslash\mathbf{G}(\mathbb{A}_F)\to\mathbb{C}$

$$E(s,\Phi,g,\mathbf{P})=\sum_{\gamma\in\mathbf{P}(k)\backslash\mathbf{G}(k)}\Phi_s(\gamma g),$$

where $\Phi_s=\Phi\cdot q^{\langle s\tilde{\alpha}+\rho_{\mathbf{P}},H_{\mathbf{P}}(\cdot)\rangle}$. These two are moreover rational in the variable q^{-s} [41, Proposition IV.1.12].

Assume that σ is globally generic, i.e. that there exist a cusp form φ in σ such that, for some generic character χ of $(\mathbf{M}\cap\mathbf{U}_0)(F)\backslash(\mathbf{M}\cap\mathbf{U}_0)(\mathbb{A}_F)$ (i.e. that is non-trivial on the root subgroups of $\mathbf{M}$ associated to the simple roots and trivial on the other root subgroups, see Section 2.6), φ has a non-vanishing χ-Fourier coefficient

$$W_\varphi(g)=\int_{(\mathbf{M}\cap\mathbf{U}_0)(F)\backslash(\mathbf{M}\cap\mathbf{U}_0)(\mathbb{A}_F)}\varphi(ug)\overline{\chi}(u)du\neq 0, \tag{4.1.1}$$

and unramified outside a finite subset S of places. Then we have the following connections between the intertwining operator and the partial L-functions. Let $f_s=f_{S,s}\otimes f_s^S\in i_P^G(s,\pi)$, where $f_s^S=\otimes_{x\notin S}f_{x,s}^\circ$. Then [39, Eq. (3.2)]

$$M(s,\sigma,\tilde{w}_0)f_s=\otimes_{x\in S}A(s,\sigma_x,\tilde{w}_0)f_{x,s}\cdot\prod_{i=1}^{m_r}\frac{L^S(is,\sigma,r_i)}{L^S(1+is,\sigma,r_i)}\otimes_{x\notin S}\check{f}_{x,s}^\circ, \tag{4.1.2}$$

where $\check{f}_{x,s}^\circ(k_x)=\varphi_x^\circ$ for all $k_x\in K_x$ and

$$L^S(s,\sigma,r_i)=\prod_{x\notin S}L(s,\sigma_x,r_{i,x}),$$

with $L(s,\pi_x,r_{i,x})=\det(\mathrm{Id}-q_{F_x}^{-s}r_{i,x}(a_x))^{-1}$, where $[a_x]$ the semisimple conjugacy class in ${}^LG_x$ associated to π_x and $r_{i,x}$, the restriction via $\Gamma_{F_x}\hookrightarrow\Gamma_F$ of r_i (See Section 5.5 for the restriction notations).

On the other hand, we also have a connection between the Eisenstein series and the partial L-functions. Let $\psi=\otimes_x\psi_x$ be non-trivial character of $\mathbb{A}_F/F$, unramified outside of S, that we extend to $\mathbf{U}_0(F)\backslash\mathbf{U}_0(\mathbb{A}_F)$ as in [32, Section 5.3]. We define as in [39, Section 1.5]

$$E_\psi(s,\Phi,g,\mathbf{P})=\int_{\mathbf{U}_0(F)\backslash\mathbf{U}_0(\mathbb{A}_F)}E(s,\Phi,ug,\mathbf{P})\overline{\psi}(u)du.$$

The Fourier coefficients of these Eisenstein Series are also rational functions on q^{-s} [18, Section 1.6].

Starting from an automorphic form $\varphi = \otimes\varphi_x \colon \mathbf{M}(F)\backslash\mathbf{M}(\mathbb{A}_F) \to \mathbb{C}$ in σ, we construct an automorphic form $\Phi_\varphi \colon \mathbf{M}(F)\,\mathbf{N}(\mathbb{A}_F)\backslash\mathbf{G}(\mathbb{A}_F) \to \mathbb{C}$ and an element $f_{\varphi,s} = \otimes_x f_{x,\varphi,s} \in i_P^G(s,\pi)$ as in [39, Section 1.2 & 1.5]. Then we have the following formula [39, Eq. (1.3)]

$$E_\psi(s,\Phi_\varphi,g,\mathbf{P}) = \prod_{x\in S}\lambda_{\psi_x}(s,\pi_x)(i_P^G(s,\pi_x)(g_x)f_{x,\varphi,s})\prod_{i=1}^{m_r}L^S(1+is,\pi,r_i)^{-1}, \tag{4.1.3}$$

where λ_{ψ_x} is the (non-zero) Whittaker functional of $i_{\mathbf{P}(F_x)}^{\mathbf{G}(F_x)}(s,\pi_x)$ constructed in [39, Section 2.4].

Lemma 4.1.4. *[39, Lemma 3.3] Let $\mathbf{P} = \mathbf{M}\,\mathbf{N}$ be a parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}_0$ with Levi subgroup $\mathbf{M}$ and $\tilde{w}_0 \in \mathbf{G}(F)$ be a representative of $w_0 \in W^{\mathbf{G}}$. Let σ be a globally generic unitary cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_F)$, such that $\tilde{w}_0\sigma \ncong \sigma$. Then $M(s,\sigma,\tilde{w}_0)$ and $E(s,\Phi,g,\mathbf{P})$ are holomorphic on $\mathrm{Re}(s) \geq 0$.*

We go back to our case. Let E be a quadratic extension of F in F_s, and we denote $\mathbf{SO}_{2n}^* = \mathbf{SO}(q_{E,n})$, as usual. We also recall that we have fixed a minimal parabolic subgroup $\mathbf{P}_0$ (Section 1.5).

Proposition 4.1.5. *Let $\mathbf{G} = \mathbf{SO}_{2(m+n)}^*$, let $\mathbf{P} = \mathbf{M}\,\mathbf{N}$ be a parabolic subgroup containing $\mathbf{P}_0$ with Levi subgroup $\mathbf{M}$ isomorphic to $\mathbf{GL}_m \times \mathbf{SO}_{2n}^*$ and $\tilde{w}_0 \in \mathbf{G}(F)$ is a representative of $w_0 \in W^{\mathbf{G}}$. Let $\sigma = \tau \otimes \tilde{\pi}$ be a unitary generic cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_F)$. Then $M(s,\sigma,\tilde{w}_0)$ is holomorphic on $\mathrm{Re}(s) > 1$.*

Proof. Let S be a finite subset of $|F|$, such that σ_x is unramified for $x \notin S$.

Thanks to the work of L. Lafforgue [33, Théorème VI.10], we know that each local component of the globally generic cuspidal automorphic representation $\tau = \bigotimes_x \tau_x$ of $\mathbf{GL}_m(\mathbb{A}_F)$ is tempered. Then, for each $x \notin S$ we have that τ_x is of the form

$$i_{B_m}^{GL_m}(\chi_{1,x} \otimes \cdots \otimes \chi_{m,x}),$$

where $\chi_{j,x}$ is unitary unramified character of $F_x^\times$.

On the other hand, for each x such that $E_x = E \otimes_F F_x$ is a product of two fields (x is split in E), we have the classification of generic unitary representations of $\mathbf{SO}_{2n}(F_x) \cong \mathbf{SO}_{2n}^*(F_x) = \mathbf{SO}(q_{E_x,n})(F_x)$, given in [39, Theorem 5.6]. This gives that for split x, $\tilde{\pi}_x$ is of the form

$$i_{Q'}^{\mathbf{SO}_{2n}(F_x)}(\delta_{b,x}\nu^{r_b} \otimes \cdots \otimes \delta_{1,x}\nu^{r_1} \otimes \pi_{0,x}),$$

where $\mathbf{Q}'$ is a parabolic subgroup containing the Borel subgroup of upper triangular matrices in $\mathbf{SO}_{2n}$ and $\delta_{i,x}$ are unitary discrete series representation of $\mathbf{GL}_{n_i}(F_x)$; the constants $r_1,\ldots,r_b \in \mathbb{R}$ satisfy

$$1 > r_b \geq \cdots \geq r_1 > 0,$$

and $\pi_{0,x}$ is a tempered generic representation of $\mathbf{SO}_{2n_0}(F_x)$.

Now, if s_0 is a pole of $M(s,\sigma,\tilde{w}_0)$, then some subquotient of $i_P^G(s_0,\sigma)$ would be in the discrete residual spectrum [28, Section 1], thus unitary. Then for such s_0, we would have that for almost every $x \in |F|$, $i_{\mathbf{P}(F_x)}^{\mathbf{G}(F_x)}(s_0,\sigma_x)$ is unitary.

We argue by contradiction, and thus we assume that $\mathrm{Re}(s_0) > 1$. First, we can enlarge S, so that $i_{\mathbf{P}(F_x)}^{\mathbf{G}(F_x)}(s_0,\sigma_x)$ is unitary. As the set $\{x \in |F| \setminus S \colon x \text{ is split in } E\}$ has density $1/2$ by Chebotarev's theorem, we can always some find $x_0 \notin S$ split in E. But, thanks to Proposition A.2.1, we get that the unramified component of $i_{\mathbf{P}(F_x)}^{\mathbf{G}(F_x)}(s_0,\sigma_x)$ is not unitary, thus a contradiction. □

4.2 Global Langlands-Shahidi L-functions.

Using the intertwining operator, the Eisenstein Fourier coefficients and the normalized intertwining operator (Section 2.11) on ramified places, we prove the following result for global L-functions.

Corollary 4.2.1. *Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a parabolic subgroup of the general quasi-split reductive group $\mathbf{G}$ over F containing $\mathbf{P}_0$ with Levi subgroup $\mathbf{M}$ and $\tilde{w}_0 \in \mathbf{G}(F)$ be a representative of $w_0 \in W^{\mathbf{G}}$. Let σ be a generic cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_F)$ such that $w_0\sigma \ncong \sigma$. Suppose $T \subset S$ is a subset with the property that for $x \in T$, the normalized intertwining operator $N(s,\sigma_x,\tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}\, s \geq 1/2$. Then the partial L-function*

$$L^{S\setminus T}(s,\sigma,r_i) = \prod_{x\notin S\setminus T} L(s,\sigma_x,r_{i,x})$$

is holomorphic on $\mathrm{Re}\, s \geq 1/2$ and non-zero on $\mathrm{Re}\, s \geq 1$.

Proof. To shorten some notations, we denote, for the finite subset T of $|F|$,

$$L_T(s,\sigma,r_i) = \prod_{x\in T} L(s,\sigma_x,r_{i,x}) \quad \& \quad \varepsilon_T(s,\sigma,r_i) = \prod_{x\in T} \varepsilon(s,\sigma_x,r_{i,x}).$$

Now, putting the definition of the normalized intertwining operator in the right hand side of the formula (4.1.2), we get

$$\begin{aligned} M(s,\sigma,\tilde{w}_0)f = \bigotimes_{x\in S\setminus T} A(s\tilde{\alpha},\sigma_x,\tilde{w}_0)f_x \cdot \\ \prod_{i=1}^{m_r} \frac{L_T(is,\sigma,r_i)}{L_T(1+is,\sigma,r_i)\varepsilon_T(is,\sigma,r_i,\psi)} \bigotimes_{x\in T} N(s,\sigma_x,\tilde{w}_0)f_x \cdot \\ \prod_{i=1}^{m_r} \frac{L^S(is,\sigma,r_i)}{L^S(1+is,\sigma,r_i)} \bigotimes_{x\notin S} \check{f}_x^\circ. \end{aligned}$$

Using that $M(s,\sigma,\tilde{w}_0)$ is holomorphic on $\mathrm{Re}(s)\geq 0$ (Lemma 4.1.4), $N(s,\sigma_x,\tilde{w}_0)$ is holomorphic and non-zero on $\mathrm{Re}(s)\geq 1/2$ and that $A(s\tilde{\alpha},\sigma_x,\tilde{w}_0)$ [56, p. 283 Equation (10)] and $\varepsilon_T(s,\sigma,r_i,\psi)$ are non-vanishing, we have that

$$\prod_{i=1}^{m_r}\frac{L^{S\setminus T}(is,\sigma,r_i)}{L^{S\setminus T}(1+is,\sigma,r_i)}$$

is holomorphic on $\mathrm{Re}(s)\geq 1/2$. From the fact that L-functions are holomorphic on some $\mathrm{Re}(s)>N$ [4, Section 13.2], we get that

$$\prod_{i} L^{S\setminus T}(is,\sigma,r_i) \tag{4.2.2}$$

is holomorphic on $\mathrm{Re}(s)\geq 1/2$.

On the other hand, as $E(s,\Phi,g,\mathbf{P})$ is holomorphic on $\mathrm{Re}(s)\geq 0$ (Lemma 4.1.4) and the local L-functions are non-vanishing by definition, using the relation (4.1.3) we also get

$$\prod_{i=1}^{m_r} L_T(1+is,\sigma,r_i)\cdot\prod_{i=1}^{m_r} L^S(1+is,\sigma,r_i)=\prod_{i=1}^{m_r} L^{S\setminus T}(1+is,\sigma,r_i) \tag{4.2.3}$$

is non-zero on $\mathrm{Re}\, s\geq 0$.

Now we proceed by induction on m_r, as in [39, Lemma 2.4], to get that $L^{S\setminus T}(s,\sigma,r_i)$ is holomorphic on $\mathrm{Re}(s)\geq 1/2$ and non-zero on $\mathrm{Re}\, s\geq 1$. Indeed, for $m_r=1$ it follows directly from (4.2.2) and (4.2.3). Now, if we suppose that the statement is true for $L^{S\setminus T}(s,\sigma,r_i)$ with $i=2,\dots,m_r$, then $\prod_{i=2}^{m_r} L^{S\setminus T}(is,\sigma,r_i)$ is holomorphic and non-zero on $\mathrm{Re}\, s\geq 1/2$. Thus by (4.2.2), $L^{S\setminus T}(s,\sigma,r_1)$ is holomorphic on $\mathrm{Re}\, s\geq 1/2$. Similarly, but using (4.2.3), we get it is non-zero. □

We go back again to our case $\mathbf{G}=\mathbf{SO}^*_{2(m+n)}$ and a maximal Levi subgroup $\mathbf{M}=\mathbf{GL}_m\times\mathbf{SO}^*_{2n}$. The adjoint action decomposes as

$$r_1\oplus r_2,$$

where $r_1=\rho_m\otimes\tilde{\rho}^*_{2n}$ and $r_2=\wedge^2\rho_m$ (2.5.2). Using the Langlands-Shahidi method for $r_{1,x}$ and $r_{2,x}$ we obtain the Langlands-Shahidi L-functions

$$L(s,\pi_x\times\tau_x)=L(s,\tau_x\otimes\tilde{\pi}_x,r_{1,x})\quad\&\quad L(s,\tau,\wedge^2\rho_m)=L(s,,\tau_x\otimes\tilde{\pi}_x,r_{2,x}),$$

for π_x a generic representation of $\mathbf{SO}^*_{2n}(F_x)$ and τ_x a generic representation of $\mathbf{GL}_m(F_x)$ (See Section 2.8 for inert places i.e. when $E_x=E\otimes_F F_x$ is a field and [37, Section 6] for analogous description for split places, i.e, as in the case where $\mathbf{SO}^*_{2n}$ is replaced by the split group $\mathbf{SO}_{2n}$).

Proposition 4.2.4. *Let $\mathbf{G}=\mathbf{SO}^*_{2(m+n)}$ and let $\mathbf{P}=\mathbf{MN}$ be a parabolic subgroup with Levi subgroup $\mathbf{M}$ isomorphic to $\mathbf{GL}_m\times\mathbf{SO}^*_{2n}$. Let $\sigma=\tau\otimes\tilde{\pi}$ be a generic cuspidal*

automorphic representation of $\mathbf{M}(\mathbb{A}_F)$ *such that* $w_0\sigma \ncong \sigma$*. Suppose that for a fixed inert place* x_0*,* π_{x_0} *is unramified and*

$$\pi_{x_0} = i_{P_1}^{\mathbf{SO}^*_{2n}(F_{x_0})}(\pi_{b,x_0}|\det|^{r_b} \otimes \cdots \otimes \pi_{1,x_0}|\det|^{r_1} \otimes \pi_{0,x_0}), \tag{4.2.5}$$

where $0 < r_1 \leq \cdots \leq r_b$*,* π_{i,x_0} *is an irreducible tempered representation of* $\mathbf{GL}_{n_i}(F_{x_0})$ *for* $1 \leq i \leq b$ *and* π_{0,x_0} *is an irreducible tempered representation of* $\mathbf{SO}^*_{2n_0}(F_{x_0})$*. Then* $L(s, \pi_{x_0} \times \tau_{x_0})$ *is holomorphic on* $\operatorname{Re} s \geq 1$*.*

The following proof is an adaptation of [29, Proposition 4.9] to our case.

Proof. As before we input the definition of the normalized operator in the right hand side of the formula (4.1.2) to get

$$M(s,\sigma,\tilde{w}_0)f = \bigotimes_{x\in S\setminus\{x_0\}} A(s\tilde{\alpha},\sigma_x,\tilde{w}_0)f_x \cdot \prod_{i=1}^{2}\frac{L(is,\sigma_{x_0},r_{i,x_0})}{L(1+is,\sigma_{x_0},r_{i,x_0})} \bigotimes \frac{N(s,\sigma_{x_0},\tilde{w}_0)}{\varepsilon(is,\sigma_{x_0},r_{i,x_0},\psi)} f_{x_0} \cdot \prod_{i=1}^{2}\frac{L^S(is,\sigma,r_i)}{L^S(1+is,\sigma,r_i)} \bigotimes_{x\notin S} \check{f}^{\circ}_x.$$

Now let $N_0 \geq 2$ be big enough so that $L(1+s,\sigma_{x_0},r_1)$ has no poles on $\operatorname{Re}(s) \geq N_0$. This gives us that, if $\operatorname{Re}(s) \geq N_0 - 1$, then

$$\frac{L(is,\sigma_{x_0},r_{i,x_0})}{\varepsilon(is,\sigma_{x_0},r_{i,x_0},\psi)L(1+is,\sigma_{x_0},r_{i,x_0})}$$

is non-zero. Secondly, since τ_{x_0} is tempered, we have that $L(s,\sigma_{x_0},r_{2,x_0}) = L(s,\tau_{x_0},\wedge^2\rho_m)$ is holomorphic on $\operatorname{Re} s \geq 1$ and

$$\prod_{i=1}^{2}\frac{L(is,\sigma_{x_0},r_{i,x_0})}{\varepsilon(is,\sigma_{x_0},r_{i,x_0},\psi)L(1+s,\sigma_{x_0},r_{i,x_0})}$$

is non-zero on $\operatorname{Re}(s) \geq N_0 - 1$. Thirdly, using Corollary 4.2.1 for $T = \emptyset$, we get that

$$\prod_{i=1}^{2}\frac{L(is,\sigma_{x_0},r_{i,x_0})}{\varepsilon(is,\sigma_{x_0},r_{i,x_0},\psi)L(1+is,\sigma_{x_0},r_{i,x_0})}\prod_{i=1}^{2}\frac{L^S(is,\sigma,r_i)}{L^S(1+is,\sigma,r_i)}$$

is non-zero on $\operatorname{Re}(s) \geq N_0 - 1$. Since $\tilde{w}_0\sigma \ncong \sigma$, Lemma 4.1.4 gives us that $M(s,\sigma,\tilde{w}_0)$ is holomorphic on $\operatorname{Re}(s) \geq 0$. Lastly, using that $A(s\tilde{\alpha},\sigma_{x_0},\tilde{w}_0)$ is non-zero and the equality at the beginning of the proof, we have that $N(s,\sigma_{x_0},\tilde{w}_0)$ is holomorphic on $\operatorname{Re} s \geq N_0 - 1$. Now thanks to τ_{x_0} being tempered, π_{x_0} being unramified and of the form (4.2.5), and Remark 2.10.3, we have that $\sigma_{x_0} = \tau_{x_0} \otimes \tilde{\pi}_{x_0}$ satisfies the hypothesis of Proposition 2.11.5. Thus we have that $N(s,\sigma_{x_0},\tilde{w}_0)$ is also non-zero in on $\operatorname{Re}(s) \geq N_0 - 1$. Hence $L(s,\pi_{x_0}\times\tau_{x_0})$ has no poles on $\operatorname{Re}(s) \geq N_0 - 1$. Arguing inductively, we conclude that $L(s,\pi_{x_0}\times\tau_{x_0})$ is holomorphic on $\operatorname{Re}(s) \geq 1$. □

When we study the image of the functorial transfer of generic automorphic representations from the quasi-split special orthogonal group to general linear groups, it will be essential to have holomorphicity and non-vanishing results for L-functions in the case of Siegel Levi subgroups [39, Corollary 6.4], which relies on the work of L. Lafforgue on the Ramanujan conjecture in the $\mathbf{GL}_n$ case.

Theorem 4.2.6. *Suppose that* $\mathbf{G} = \mathbf{SO}^*_{2(m+n)}$, *and* $\mathbf{P} = \mathbf{MN}$ *parabolic subgroup with Levi subgroup* $\mathbf{M}$ *isomorphic to* $\mathbf{GL}_m \times \mathbf{SO}^*_{2n}$. *Let* $\sigma = \tau \otimes \tilde{\pi}$ *be a generic cuspidal automorphic representation of* $\mathbf{M}(\mathbb{A}_F)$. *Then* $L^S(s, \sigma, r_1)$ *is holomorphic and non-vanishing on* $\mathrm{Re}(s) > 1$ *and has at most a simple pole at* $s = 1$.

Proof. Let S be a finite subset of $|F|$, such that σ_x is unramified for $x \notin S$, as in the proof of Proposition 4.1.5. From (4.1.2) and Proposition 4.1.5, we have that

$$\prod_{i=1}^{m_r} \frac{L^S(is, \sigma, r_i)}{L^S(1 + is, \sigma, r_i)}$$

is holomorphic on $\mathrm{Re}(s) > 1$. As $L^S(s, \sigma, r_2) = L^S(s, \tau, \wedge^2 \rho_m)$ is holomorphic and non-zero on $\mathrm{Re}(s) > 1$ [39, Corollary 6.4], we can conclude that

$$\frac{L^S(s, \sigma, r_1)}{L^S(1 + s, \sigma, r_1)}$$

is holomorphic on $\mathrm{Re}(s) > 1$.

On the other hand, as [41, Proposition II.1.7]

$$E_{\mathbf{P}}(s, f, g, \mathbf{P}) = \int_{\mathbf{U}(F)\backslash \mathbf{U}(\mathbb{A}_F)} E(s, f, ug, \mathbf{P}) du = f(g) + M(s, \sigma, \tilde{w}_0) f(g)$$

and Proposition 4.1.5, we have that the poles of the constant terms $E_{\mathbf{P}}(s, f, g, \mathbf{P})$ are contained in $\mathrm{Re}\, s \leq 1$. Since $\mathbf{U}(F)\backslash \mathbf{U}(\mathbb{A}_F)$ is compact, the formula (4.1.3) and that $L^S(s, \sigma, r_2) = L^S(s, \tau, \wedge^2 \rho_m)$ is holomorphic and non-zero on $\mathrm{Re}(s) > 1$, we conclude that $L^S(1 + s, \sigma, r_1)^{-1}$ is holomorphic and non-vanishing on $\mathrm{Re}(s) > 1$. Thus $L^S(s, \sigma, r_1)$ is holomorphic on $\mathrm{Re}(s) > 1$.

Finally, we have that the poles of the global intertwining operator are all simple [41, Proposition IV.1.11, (c)]. Then, by (4.1.2) and the non-zeroness of $A(s, \pi_x, \tilde{w}_0)$, the quotient

$$\frac{L^S(s, \sigma, r_1) L^S(2s, \sigma, r_2)}{L^S(1 + s, \tilde{\sigma}, r_1) L^S(1 + 2s, \tilde{\sigma}, r_2)}$$

has at most a simple pole at $s = 1$. From [39, Corollary 6.4], $L^S(2, \sigma, r_2) = L^S(2, \tau, r_2)$ and $L^S(3, \tilde{\sigma}, r_2) = L^S(3, \tilde{\tau}, r_2)$ are non-zero. Thus, $L(s, \sigma, r_1)$ has at most a simple pole at $s = 1$. □

Chapter 5

Functoriality Conjecture

In this chapter, we introduce the main language and basic results to establish the main objective of this work. We introduce the notions of Langlands parameters [4] and Weil-Deligne representations [17]. Then we follow [36], to obtain a Langlands correspondence for tori. We introduce the notion of Satake parameter and the functoriality conjecture. We finish with the construction of the L-homomorphism, for which we will answer the generic functoriality conjecture.

Let **G** be a quasi-split (connected) reductive group over a field F and F_s a separable closure of F. As before $\Gamma_F = \mathrm{Gal}(F_s/F)$, and we let I_F be the intertia subgroup of Γ_F.

5.1 Weil group. Suppose that F is a non-archimedean local field. For each finite extension $F \subset E \subset F_s$, let k_E be the residue field of E and $q_E \coloneqq \#k_E$. Let $\overline{k} = \cup_E k_E$ be the residue field of F_s. We denote by W_F the dense subgroup of Γ_F consisting of the elements $\tau \in \Gamma_F$ which induce on $\overline{k}$ the map $x \mapsto x^{q_F^n}$ for some $n \in \mathbb{Z}$. By definition we have that the inertia subgroup I_F is a subgroup of W_F.

Let

$$r_E : E^\times \xrightarrow{\sim} W_E^{ab} \subset \Gamma_E^{ab} \quad \& \quad \varphi : W_F \to \Gamma_F$$

be the norm residue symbol [43, IV. (6.3)] and the inclusion, where W_F^c is the closure of the commutator group of W_F and $W_F^{ab} \coloneqq W_F/W_F^c$. We follow Deligne's convention [14, (2.3)], $r_E(a)$ induces $x \mapsto x^{|a|_E}$ in $\overline{k}$. The triple $(W_F, \varphi, \{r_E\}_E)$ is a Weil group [55, (1.4.1)]. In particular, for E a (finite) Galois extension, the extension $W_{E/F} = W_F/W_E^c$ given by

$$1 \to E^\times \xrightarrow{\overline{r}_E} W_{E/F} \xrightarrow{\overline{\varphi}} \mathrm{Gal}(E/F) \to 1,$$

where $\overline{r}_E$ and $\overline{\varphi}$ are the induced by r_E and φ. They correspond to the canonical class $\alpha_{E/F}$ in $H^2(\mathrm{Gal}(E/F), E^\times)$ [55, (1.2)].

5.2 Langlands parameters. Suppose that F is non-archimedean and fix a geometric

Frobenius element $\mathrm{Fr} \in W_F$. Let ${}^L G$ be the L-group of $\mathbf{G}$ and ${}^L G^0 = \widehat{\mathbf{G}}(\mathbb{C})$ its identity component (Section 1.3). We denote by $\Phi(G)$ the set of group morphisms [4, Section 8]

$$\phi\colon W_F' = W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$

such that $\phi(\mathrm{Fr})$ is semi simple, $\phi|_{I_F}$ is continuous, $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic and ϕ is relevant, i.e. if the image of ϕ is contained in a Levi subgroup of ${}^L G$ then it is the L-group of a Levi subgroup of $\mathbf{G}$, modulo ${}^L G^\circ$-conjugacy classes of parameters. Moreover, when $\phi|_{I_F}$ and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ are trivial, ϕ will be called unramified. We denote by $\Phi_{unr}(G)$ the set of these classes.

From the definition we have the following injection

$$\begin{aligned} \Phi(G) \hookrightarrow & H^1_{cts}(W_F', {}^L G^0), \\ \phi \mapsto & [g \mapsto \phi_0(g)] \end{aligned} \tag{5.2.1}$$

where ${}^L G^0$ has the action of W_F' induced by the action of Γ_F and ϕ_0 is the composition of ϕ with the projection of ${}^L G$ to ${}^L G^0 = \widehat{\mathbf{G}}(\mathbb{C})$ and $H^1_{cts}(W_F', {}^L G^0)$ is the set of continuous 1-cocycles modulo the continuous 1-coboundaries [4, Section 8.2].

We can present this data, in another way: A Weil-Deligne representation of ${}^L G$ is a pair (ρ, N), where $\rho\colon W_F \to {}^L G$, is continuous on I_F, $\rho(\mathrm{Fr})$ is semi simple and N is a nilpotent element of $\mathrm{Lie}({}^L G)$ such that $\mathrm{Ad}\,\rho(w)N = qN$ for all $w \in W_F$.

The map that sends ϕ to the pair (ρ, N) such that

$$\rho|_{I_F} = \phi|_{I_F}, \quad \rho(\mathrm{Fr}) = \phi(\mathrm{Fr}) \cdot \phi \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \quad N = d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.2.2}$$

induces a bijection between the equivalences classes (with the natural notion of equivalence) of Weil-Deligne representations and the ${}^L G^\circ$-conjugacy classes of parameters [17, Proposition 2.2].

5.3 Tori. Suppose that F is a non-archimedean local field. Let $\mathbf{T}$ be a torus that splits over a finite separable extension $K \subset F_s$ and write $T = \mathbf{T}(F)$ and $\mathbf{T}_K$ the base change to K from F of $\mathbf{T}$. The inflation morphism and (5.2.1) gives us a bijection

$$H^1_{cts}(W_{K/F}; \widehat{\mathbf{T}}(\mathbb{C})) \xrightarrow{\mathrm{Inf}} H^1_{cts}(W_F; \widehat{\mathbf{T}}(\mathbb{C})) = \Phi(T). \tag{5.3.1}$$

The evaluation map

$$\begin{aligned} H^1_{cts}(W_{K/F}; \widehat{\mathbf{T}}(\mathbb{C})) \times H_1(W_{K/F}; X_*(\mathbf{T})) &\to \mathbb{C}^\times, \\ (\overline{f}, \overline{x}) &\mapsto \sum_{w \in W_{K/F}} \langle \varphi(w), x(w) \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the pairing associated to

$$\mathrm{Hom}_{\mathbb{Z}}(X_*(\mathbf{T}_K), \mathbb{C}^\times) \times X_*(\mathbf{T}_K) \to \mathbb{C}^\times$$

is a perfect pairing and thus we get an isomorphism [36, p. 234]

$$H^1_{cts}(W_{K/F}; \widehat{\mathbf{T}}(\mathbb{C})) \to \mathrm{Hom}_{\mathrm{cts}}(H_1(W_{K/F}; X_*(\mathbf{T}_K)), \mathbb{C}^\times).$$

Now one can prove that the corestriction map

$$H_1(W_{K/F}; X_*(\mathbf{T}_K)) \to H_1(K^\times, X_*(\mathbf{T}_K))^{\mathrm{Gal}(K/F)} \tag{5.3.2}$$

is an isomorphism [36, p. 241]. Finally by definition of H_1 and T we have

$$\begin{aligned} H_1(K^\times; X_*(\mathbf{T}_K))^{\mathrm{Gal}(K/F)} = (X_*(\mathbf{T}_K) \otimes K^\times)^{\mathrm{Gal}(K/F)} &= \mathrm{Hom}(X(\mathbf{T}_K), K^\times)^{\mathrm{Gal}(K/F)} \\ &= \mathbf{T}(K)^{\mathrm{Gal}(K/F)} \\ &= \mathbf{T}(F) \end{aligned}$$

Thus we obtain a bijection [36, Theorem 2]

$$\begin{aligned} \Phi(T) &\cong \Pi(T) \\ \phi &\mapsto \pi_\phi \end{aligned}$$

where $\Pi(T)$ is the set of smooth characters $\chi : T \to \mathbb{C}^\times$.

(*Split case*). In the split case we have that $K = F$ and thus $W_{K/F} = F^\times$. Using the identifications

$$\mathbf{T}(F) = \mathrm{Hom}(X(\mathbf{T}), F^\times) \quad \& \quad \widehat{\mathbf{T}}(\mathbb{C}) = \mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times),$$

we can write this bijection as the following bijection between sets of continuous group homomorphisms,

$$\begin{aligned} \mathrm{Hom}_{grp-cts}(\mathrm{Hom}(X(\mathbf{T}), F^\times), \mathbb{C}^\times) &\to \mathrm{Hom}_{grp-cts}(F^\times, \mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times)) \\ \pi &\mapsto (w \mapsto [\widehat{\lambda} \mapsto \pi(\lambda \mapsto w^{\langle \lambda, \widehat{\lambda} \rangle})]). \end{aligned}$$

(*Restriction and Norm*). The restriction, the corestriction and the norm map N can be put in the following commutative diagram [36, p. 235]

$$\begin{array}{ccc} H_1(K^\times; X_*(\mathbf{T}_K)) & \xrightarrow{\mathrm{Cor}} & H_1(W_{K/F}; X_*(\mathbf{T}_K)) \\ \Big\downarrow{\scriptstyle N} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\ N(H_1(K^\times; X_*(\mathbf{T}_K))) & \hookrightarrow & H_1(K^\times; X_*(\mathbf{T}_K))^{\mathrm{Gal}(K/F)} \end{array}.$$

This gives us the commutative diagram

$$\begin{array}{ccc} \mathbf{T}(K) & \xrightarrow{\quad N \quad} & \mathbf{T}(F), \\ & {\scriptstyle \pi_{\phi_{K^\times}}} \searrow \quad \swarrow {\scriptstyle \pi_\phi} & \\ & \mathbb{C}^\times & \end{array} \tag{5.3.3}$$

where $\phi_{K^\times}$ is the parameter of $\mathbf{T}(K)$ obtained via the identification (5.3.1) and restriction to $K^\times$ via $\overline{r}_K : K^\times \to W_{K/F}$ (Section 5.1).

5.4 Satake parametrization & Parameters. In this section we follow [4]. We suppose that F is a non-archimedean local field and $\mathbf{G}$ is unramified, i.e. it is quasi-split and splits over an unramified extension of F.

Let $\mathbf{S}$ be a maximal split subtorus of $\mathbf{G}$, $\mathbf{T}$ its centralizer (as $\mathbf{G}$ quasi-split, is a maximal torus) and $W = N_{\mathbf{G}}(\mathbf{S})(F)/\,\mathbf{T}(F)$ the Weyl group. Let $\mathbf{P}_0$ be a minimal parabolic subgroup containing $\mathbf{S}$.

We note that W can be identified as the subgroup of $W(\mathbf{G}_{F_s}, \mathbf{T}_{F_s}) = N_{\mathbf{G}(F_s)}(\mathbf{T}(F_s))/\,\mathbf{T}(F_s)$ that fixes the subgroup $\mathbf{S}_{F_s}$ of $\mathbf{T}_{F_s}$ [4, Section 6]. We denote ${}_FN$ the subgroup of ${}^LG^0$, which is the pre-image of the subgroup W of $W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}}) = W(\mathbf{G}_{F_s}, \mathbf{T}_{F_s})$ under the natural projection homomorphism

$$N_{{}^LG^0}({}^LT^0) \to N_{{}^LG^0}({}^LT^0)/{}^LT^0 = W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}}).$$

In this section we consider the action of ${}_FN$ (resp. ${}^LG^0$) via conjugation on ${}^LT^0 \rtimes \mathrm{Fr} = \{(t, \mathrm{Fr}) : t \in {}^LT^0\}$ (resp. $({}^LG^0 \rtimes \mathrm{Fr})_{ss} = \{(g, \mathrm{Fr}) : g \in {}^LG^0, s \text{ semisimple}\}$). The set equivalences classes will be denoted by $({}^LT^0 \rtimes \mathrm{Fr})/\operatorname{Int}{}_FN$ (resp. $({}^LG^0 \rtimes \mathrm{Fr})_{ss}/\operatorname{Int}{}^LG^0$).

We have a natural bijection [4, Section 9.5]

$$\begin{aligned} \Phi_{unr}(G) &\to ({}^LT^0 \rtimes \mathrm{Fr})/\operatorname{Int}{}_FN. \\ (\mathrm{Fr} \mapsto (t, \mathrm{Fr})) &\mapsto [t, \mathrm{Fr}] \end{aligned}$$

We also have the following bijection, induced by restriction [4, Lemma 6.4]

$$\begin{aligned} \overline{\nu} : ({}^LT^0 \rtimes \mathrm{Fr})/\operatorname{Int}{}_FN &\to {}^LS^0/W. \\ [t, \mathrm{Fr}] &\mapsto [t] \end{aligned}$$

Furthermore, one can also show that the inclusion $({}^LT^0 \rtimes \mathrm{Fr}) \to ({}^LG^0 \rtimes \mathrm{Fr})_{ss}$ induces a bijection

$$({}^LT^0 \rtimes \mathrm{Fr})/\operatorname{Int}{}_FN \to ({}^LG^0 \rtimes \mathrm{Fr})_{ss}/\operatorname{Int}{}^LG^0.$$

On the other hand, choose a hyperspecial maximal compact subgroup K in good position of $\mathbf{G}(F)$ and let $\Pi_{K\text{-un}}(G)$ be the set of equivalence classes of irreducible representations of $\mathbf{G}(F)$ with a non-zero vector fixed by K. Now for a character χ of T trivial on $T \cap K = T^1 = \bigcap_{\chi \in X(\mathbf{T})} \ker |\chi|_F$ (i.e. in $\mathcal{X}(T)$ of Section 2.1), the representation $i_{P_0}^G \chi$ has a unique irreducible non-zero K-invariant factor. This builds a bijection ([2, Section 9.2] and [3, Section 4.9])

$$\mathcal{X}(T)/W \to \Pi_{K\text{-un}}(G).$$

As the inclusion $S \subset T$ induces an isomorphism [4, Section 9.5, (2)]

$$\mathcal{X}(T) = {}^L S^0,$$

we also have that

$$ {}^L S^0/W = \mathcal{X}(T)/W \to \Pi_{K\text{-un}}(G). \tag{5.4.1}$$

Thus we also obtain a bijection

$$\Pi_{K\text{-un}}(G) \to \Phi_{un}(G).$$
$$\pi \mapsto \phi_\pi$$

5.5 Conjecture. Suppose that F is a global field with set of places $|F|$. We put for $x \in |F|$, $\mathbf{G}_x := \mathbf{G}_{F_x}$ the group obtained from $\mathbf{G}$ by extending scalars along the inclusion $F \hookrightarrow F_x$. We choose a separable closure $F_{x,s}$ of F_x and an embedding $F_s \hookrightarrow F_{x,s}$ for each $x \in |F|$ that extends $F \hookrightarrow F_x$. We write $\Gamma_F = \mathrm{Gal}(F_s/F)$ and $\Gamma_{F_x} = \mathrm{Gal}(F_{x,s}/F_x)$ for each $x \in |F|$. These choices give us an injection $\Gamma_{F_x} \hookrightarrow \Gamma_F$ [43, (9.6)].

Now the restriction along $\Gamma_{F_x} \to \Gamma_F$ induces a (continuous) group homomorphism from ${}^L G_x$ to ${}^L G$, that fits into a commutative diagram

$$\begin{array}{ccc} {}^L G_x & \longrightarrow & {}^L G. \\ \downarrow & & \downarrow \\ \Gamma_{F_x} & \longrightarrow & \Gamma_F \end{array}$$

Now given an L-homomorphism $\rho : {}^L G \to {}^L H$ (Section 1.3), we can form, for every place x, an L-homomorphism $\rho_x : {}^L G_x \to {}^L H_x$.

(Functoriality Conjecture). Let

$$\rho \colon {}^L G \to {}^L H$$

be an L-homomorphism. Suppose that $\mathbf{H}$ and $\mathbf{G}$ are connected reductive quasi-split groups over F. For every cuspidal automorphic representation $\tau = \otimes_x \tau_x$ of $\mathbf{G}(\mathbb{A}_F)$, there exists an automorphic representation $\pi = \otimes \pi_x$ of $\mathbf{H}(\mathbb{A}_F)$, sometimes called a (weak) lift or transfer, such that for all places x such that $\mathbf{H}_x, \mathbf{G}_x, \tau_x$ and π_x are unramified, we have the following commutative diagram

$$\begin{array}{ccccc} {}^L G_x & & \xrightarrow{\quad\rho_x\quad} & & {}^L H_x \\ & {\scriptstyle \phi_{\tau_x}} \nwarrow & & \nearrow {\scriptstyle \phi_{\pi_x}} & \\ & & W'_{F_x} & & \end{array}$$

5.6 L-groups and L-homomorphisms. Let $E \subset F_s$ be a quadratic separable extension of F and $\mathrm{Gal}(E/F) = \{1, \sigma\}$. Let us consider the split special orthogonal group $\mathbf{SO}_{2n}$ over $\mathbb{C}$. We choose a split maximal torus $\mathbf{T}_n$ and a Borel subgroup $\mathbf{B}_n$ in the following way

$$\mathbf{T}_n(\mathbb{C}) = \{t = \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \colon t_i \in \mathbb{C}^\times, 1 \leq i \leq n\}$$

and

$$\mathbf{B}_n = \mathbf{T}_n \rtimes \{M(u)h(L) \colon v(L^t J)^t v = 0 \text{ for all } v \in \mathbb{C}^n\},$$

the subgroup of upper triangular matrices in $\mathbf{SO}_{2n}$, where J is the $n \times n$ matrix with 1's along the anti-diagonal and, for upper triangular unipotent $u \in \mathbf{GL}_n$ and $L \in \mathbf{Mat}_n$,

$$M(u) = \begin{pmatrix} u & 0_n \\ 0_n & J(u^t)^{-1}J \end{pmatrix} \quad \& \quad h(L) \coloneqq \begin{pmatrix} 1_n & L \\ 0_n & 1_n \end{pmatrix}.$$

We observe that the root datum associated to $(\mathbf{SO}_{2n}, \mathbf{T}_n)$ is isomorphic to the dual root datum $R^\vee$ of $(\mathbf{SO}(q_{E,n})_{F_s}, \mathbf{T}_{F_s})$. We choose a pinning of $(\mathbf{SO}_{2n}, \mathbf{T}_n, \mathbf{B}_n)$ corresponding to the based root datum $(R^\vee, \Delta^\vee)$ in the following manner,

$$X_{\alpha_i^\vee} = \begin{pmatrix} E_{i,i+1} & 0_n \\ 0_n & -E_{i,i+1} \end{pmatrix} \in \mathrm{Lie}(\mathbf{B}_n) \text{ for } 1 \leq i \leq n-1$$

where $E_{i_0,j_0} = (\delta_{i_0,j_0}(h,k))_{i,j} \in \mathbf{Mat}_n$, with δ the Kronecker's delta, and

$$X_{\alpha_n^\vee} = h \begin{pmatrix} & & 0_{n-2} \\ 1 & 0 & \\ 0 & -1 & \end{pmatrix} \in \mathrm{Lie}(\mathbf{B}_n).$$

Finally, following the equivalence of categories given in [12, Theorem 6.1.17], we can identify $\mathbf{SO}_{2n}$ to $\widehat{\mathbf{G}}$ and their Γ_F-actions (1.3.1), so also $\mathbf{SO}_{2n} \rtimes \Gamma_F$ and ${}^L G$. We thus fix this identification.

(*The $*$-action*). We denote

$$w = \begin{pmatrix} 1_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1_{n-1} \end{pmatrix}.$$

We note that $(g \mapsto wgw^{-1}) \in \mathrm{Aut}(\mathbf{SO}_{2n}, \mathbf{T}_n, \{X_a\}_{a \in \Delta^\vee})$ corresponds, via the equivalence of categories [12, Theorem 6.1.17], to the non-trivial automorphism in $\mathrm{Aut}(R^\vee, \Delta^\vee)$, obtaining thus an explicit expression of the construction (1.3.1)

$$\begin{aligned} \Gamma_F &\to \mathrm{Aut}(\mathbf{SO}_{2n}) \\ \tau &\mapsto \begin{cases} (g \mapsto wgw^{-1}) & \text{if } \tau \notin \mathrm{Gal}(F_s/E) \\ (g \mapsto g) & \text{if } \tau \in \mathrm{Gal}(F_s/E) \end{cases}. \end{aligned}$$

(*L-homomorphism*). Finally, we define the L-homomorphism needed to state our main result as follows

$$\rho_{2n}^{*} : \mathbf{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_F \to \mathbf{GL}_{2n}(\mathbb{C}) \times \Gamma_F$$
$$(g, \tau) \mapsto \begin{cases} (gw, \tau) & \text{if } \tau \notin \mathrm{Gal}(F_s/E) \\ (g, \tau) & \text{if } \tau \in \mathrm{Gal}(F_s/E) \end{cases}. \tag{5.6.1}$$

Chapter 6

Generic Functoriality

In this chapter, we establish all the main steps to prove the functoriality conjecture under consideration for globally generic cuspidal automorphic representations of adelic $\mathbf{SO}^*_{2n}$. The strategy is inspired from [11] and [38]. In particular, we construct and give a description of the lifts and prove the relations between its γ-factors, L-functions and ε-factors. Finally, assuming property (2.11.1) we verify all the hypotheses of the twisted version of the converse Theorem 3.2.1 and construct a (weak) lift.

Let F be a global field of positive characteristic, F_s a separable closure of F and $E \subset F_s$ a quadratic separable extension of F. After fixing E, we let

$$\mathbf{SO}^*_{2n} = \mathbf{SO}(q_{E,n}).$$

We let $\Gamma_F = \mathrm{Gal}(F_s/F)$. For every $x \in |F|$, we choose a separable closure $F_{x,s}$ of F_x and also an embedding $F_s \hookrightarrow F_{x,s}$ that extends $F \hookrightarrow F_x$. We let $\Gamma_{F_x} = \mathrm{Gal}(F_{x,s}/F_x)$ for every $x \in |F|$ and $\mathrm{Gal}(E/F) = \{1, \sigma\}$. These choices give us an injection $\Gamma_{F_x} \hookrightarrow \Gamma_F$.

We want to check the functoriality via ρ^*_{2n} (5.6.1) of a globally generic cuspidal automorphic representation $\pi = \otimes_x \pi_x$ of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$, unramified outside of a finite non-empty $S \subset |F|$. We also fix $\psi = \otimes_x \psi_x$ a non-trivial character of $\mathbb{A}_F/F$, unramified outside S.

6.1 Lift. For every place x we choose a character λ_x of $\mathbf{T}(F_x)$ such that for x unramified it is a character obtained by the Satake parametrization (5.4.1) and for x ramified, $i_{P_0}^{\mathbf{SO}^*_{2n}(F_x)}(\lambda_x)$ has an (irreducible) generic subquotient $\pi'_{\lambda_x} = \pi'_x$ with the same central character as π_x (see for example [11, Section 4.2]). Applying the local Langlands correspondence for tori (Section 5.3), from λ_x we get $\phi_{\lambda_x} : W'_{F_x} \to \mathbf{T}_n(\mathbb{C}) \rtimes \Gamma_{F_x}$.

Let $i_x : \mathbf{T}_n(\mathbb{C}) \rtimes \Gamma_{F_x} \hookrightarrow \mathbf{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{F_x}$ be the inclusion homomorphism. Then, applying the local Langlands correspondence for general linear groups to

$$\rho^*_{2n,x} \circ i_x \circ \phi_{\lambda_x} : W'_{F_x} \to \mathbf{GL}_{2n}(\mathbb{C}) \times \Gamma_{F_x},$$

we find an admissible representation Π_x of $\mathbf{GL}_{2n}(F_x)$. We put $\Pi = \otimes_x \Pi_x$, which is an irreducible admissible representation of $\mathbf{GL}_{2n}(\mathbb{A}_F)$. We call Π (resp. Π_x) a lift or transfer of π (resp. π_x).

(Expression for Π_x). First, let us note that by definition

$$\mathbf{SO}^*_{2n}(F_x) = \mathbf{SO}(q_{E_x,n})(F_x),$$

where $E_x \coloneqq E \otimes_F F_x$ is a degree two étale algebra over F_x. Thus, it is either a product of two (separable) fields extensions or a (separable) field extension over F_x. Let us concentrate on the case when E_x is a quadratic (separable) extension of F_x (i.e. x is an inert place), for which we have an embedding $E_x \hookrightarrow F_{x,s}$, coming from the one fixed in the beginning of this chapter $F_s \hookrightarrow F_{x,s}$.

Now let us consider $\mathbf{T}' = \mathbb{G}_m^{2n-2} \times \mathrm{Res}_{E_x/F_x}(\mathbb{G}_m)$. We have an isomorphism

$$E_x \otimes_{F_x} F_{x,s} \cong \prod_{\sigma \in \mathrm{Hom}(E_x, F_{x,s})} F_{x,s}.$$

This leads us to an isomorphism

$$X_*(\mathbf{T}'_{F_{s,x}}) = X_*(\mathbb{G}_m^{2n-2}) \times X_*(\mathrm{Res}_{E_x/F_x}(\mathbb{G}_m)_{F_{s,x}}) \cong \mathbb{Z}^{2n-2} \times \mathbb{Z}^2,$$

where the non-trivial action of the second factor, $\mathbb{Z}^2$, is given by

$$\begin{aligned} \Gamma_{F_x} &\to \mathrm{Aut}(\mathbb{Z}^2) \\ \tau &\mapsto \begin{cases} (a_1, a_2) \mapsto (a_2, a_1) & \text{if } \tau \notin \mathrm{Gal}(F_{x,s}/E_x) \\ (a_1, a_2) \mapsto (a_1, a_2) & \text{if } \tau \in \mathrm{Gal}(F_{x,s}/E_x) \end{cases}. \end{aligned}$$

If we denote by $\mathbf{D}_{2n} \subset \mathbf{GL}_{2n}$ the maximal diagonal torus, we can identify

$$\mathbf{D}_{2n}(\mathbb{C}) \rtimes \Gamma_{F_x} \cong {}^L T'_x,$$

where the action on the left hand side is given by conjugation by w.

Now thanks to this description we can construct the following embeddings

$$\begin{aligned} \iota_x \colon \mathbf{T}_n(\mathbb{C}) \rtimes \Gamma_{F_x} &\hookrightarrow \mathbf{D}_{2n}(\mathbb{C}) \rtimes \Gamma_{F_x} \cong {}^L T'_x \\ (t, \tau) &\mapsto ((t, t^{-1}), \tau) \end{aligned}$$

and

$$\begin{aligned} \mathbf{D}_{2n}(\mathbb{C}) \rtimes \Gamma_{F_x} &\hookrightarrow \mathbf{GL}_{2n}(\mathbb{C}) \times \Gamma_{F_x} \\ (d, \tau) &\mapsto \begin{cases} (dw, \tau) & \text{if } \tau \notin \mathrm{Gal}(F_{x,s}/E_x) \\ (d, \tau) & \text{if } \tau \in \mathrm{Gal}(F_{x,s}/E_x) \end{cases}. \end{aligned}$$

We can thus factor $\rho^*_{x,2n} \circ i_x$,

$$\begin{array}{ccc} {}^L T_x & \hookrightarrow & \mathbf{GL}_{2n}(\mathbb{C}) \times \Gamma_{F_x}. \\ & \searrow \quad \nearrow & \\ & {}^L T'_x & \end{array}$$

The definition of the lift and this factorization leads us to look at

$$\begin{array}{ccccc} \Phi(\mathbf{T}_x) & \longrightarrow & \Phi(\mathbf{T}'_x) & \longrightarrow & \Phi(\mathbf{GL}_{2n,x}) \\ \downarrow & ① & \downarrow & ② & \downarrow \\ \Pi(\mathbf{T}_x) & \longrightarrow & \Pi(\mathbf{T}'_x) & \longrightarrow & \Pi(\mathbf{GL}_{2n,x}), \end{array}$$

where first two vertical arrows are the ones given by Section 5.3, the third one is given by the local Langlands correspondence for the general linear group and the upper horizontal arrows are the ones obtained from composition with $\rho^*_{2n,x} \circ i_x$ (and its factorization).

Let $\lambda_x = (\chi_{1,x}, \dots, \chi_{n-1,x}, \chi_{n,x})$ be a character of $\mathbf{T}(F_x) = (F_x^\times)^{n-1} \times \mathbf{SO}_2^*(F_x)$. The image of ① is

$$\Lambda_x = (\chi_{1,x}, \dots, \chi_{n-1,x}, \mu_{n,x}, \chi_{n-1,x}^{-1}, \dots, \chi_{1,x}^{-1}),$$

where $\mu_{n,x} : E_x^\times \to \mathbb{C}^\times$ is the character obtained from $\chi_{n,x} : E_x^1 \to \mathbb{C}^\times$ via

$$\Phi(\mathbf{SO}_2^*) \to \Phi(\mathrm{Res}_{E_x/F_x} \mathbb{G}_m).$$

To specify $\mu_{n,x}$, let ϕ_x be the parameter of $\chi_{n,x} = \pi_{\phi_x}$. From (5.3.3), we have

$$\pi_{(\phi_x, E_x)} \colon \mathbf{SO}_2^*(E_x) \xrightarrow{\text{Norm}} \mathbf{SO}_2^*(F_x) \xrightarrow{\pi_{\phi_x}} \mathbb{C}^\times.$$

After the identification in Section 1.5 between $\mathbf{SO}_2^*$ and the norm one elements of $\mathrm{Res}_{E_x/F_x} \mathbb{G}_m$, we get

$$\begin{aligned} \pi_{(\phi_x, E_x)} \colon E_x^\times &\to \mathbb{C}^\times \\ x &\mapsto \chi_{n,x}(x\sigma(x)^{-1}). \end{aligned}$$

On the other hand, we recall that we have a natural isomorphism [4, Section 5] of Γ_{F_x}-groups

$$\begin{aligned} \widehat{\mathrm{Res}_{E_x/F_x} \mathbb{G}_m}(\mathbb{C}) &\cong I_{\Gamma_{E_x}}^{\Gamma_{F_x}} \widehat{\mathbb{G}_m}(\mathbb{C}) \\ &= \{f \colon \Gamma_{F_x} \to \widehat{\mathbb{G}_m}(\mathbb{C}) \colon f(a'a) = a' \cdot f(a),\ a' \in \Gamma_{E_x}, a \in \Gamma_{F_x}\}, \end{aligned}$$

where $I_{\Gamma_{E_x}}^{\Gamma_{F_x}}$ is the (non-smooth) induction functor from Γ_{F_x} to Γ_{E_x} and the action of Γ_{F_x} is given by right multiplication: $(a \cdot f)(x) = f(xa)$ for $x, a \in \Gamma_{F_x}$. Combining this isomorphism, along the restriction $W_E \subset W_F$ and the morphism

$$\begin{aligned} I_{\Gamma_{E_x}}^{\Gamma_{F_x}} \widehat{\mathbb{G}_m}(\mathbb{C}) &\to \widehat{\mathbb{G}_m}(\mathbb{C}) \\ f &\mapsto f(1_{\Gamma_{F_x}}), \end{aligned}$$

we get an isomorphism [4, Proposition 8.4 (Shapiro's Lemma)]

$$\Phi(\mathrm{Res}_{E_x/F_x}\mathbb{G}_m) \to \Phi(\mathbb{G}_{m,E_x})$$
$$(W_F \to I_{\Gamma_{E_x}}^{\Gamma_{F_x}}\widehat{\mathbb{G}_m}(\mathbb{C}) \rtimes \Gamma_{F_x}) \mapsto (W_E \to \widehat{\mathbb{G}_m}(\mathbb{C}) \times \Gamma_{E_x}).$$

The image of $\iota_x \circ \phi_x \in \Phi(\mathrm{Res}_{E_x/F_x}\mathbb{G}_m)$ through this isomoprhism is $\phi_{x,E_x} \in \Phi(\mathbb{G}_{m,E_x})$. Therefore, we have that $\mu_{n,x} = \pi_{(\phi_{x,E_x})}$.

Now for the second square, first we look at the image of the parameter corresponding to $\mu_{n,x}$ via

$$\Phi(\mathrm{Res}_{E_x/F_x}\mathbb{G}_m) \to \Phi(\mathbf{GL}_2).$$

First, using again the identification given above,

$$\Phi(\mathrm{Res}_{E_x/F_x}\mathbb{G}_m) \to \Phi(\mathbb{G}_{m,E_x}) \to \Pi(E^\times),$$

we have that via (5.2.2), the image of $\mu_{n,x}$ corresponds to the Weil-Deligne representation

$$(\mathrm{Ind}_{E_x/F_x}(\mu_n \circ \mathrm{ar}_{E_x}), 0),$$

where $\mathrm{ar}_{E_x} = r_{E_x}^{-1}$ is the reciprocity map [43, IV. (6.3)] and Ind_{E_x/F_x} is the smooth induction functor from Γ_{F_x} to Γ_{E_x}. Now using the (tame) local Langlands correspondence for $\mathbf{GL}_2$ ([8, Chapter 8]), we get our representation in $\Pi(\mathbf{GL}_2)$:

$$\Pi_{\mu_{n,x}} = \begin{cases} i_{B_2}^{GL_2}(\nu_{n,x}, \varkappa_x\nu_{n,x}) & \text{If } \mu_{n,x} = \nu_{n,x} \circ \mathrm{N}_{E_x/F_x}, \text{ for some } \nu_{n,x} \in \Pi(F_x^\times) \\ \pi_{\mu_{n,x}} & \text{otherwise,} \end{cases} \tag{6.1.1}$$

where $\varkappa_x = \det(\mathrm{Ind}_{E_x/F_x} 1_{E_x})$.

Putting all this together we get an expression for Π_x. In particular for $\lambda_x = (\chi_{1,x}, \dots, \chi_{n-1,x}, 1)$ unramified (Section 5.4) we have that Π_x is the one obtained in (5.4.1), i.e the constituent of

$$i_B^G(\chi_{1,x}, \dots, \chi_{n-1,x}, 1, \varkappa_x, \chi_{n-1,x}^{-1}, \dots, \chi_{1,x}^{-1}), \tag{6.1.2}$$

that has a nonzero vector fixed under the special maximal compact subgroup $\mathbf{GL}_{2n}(\mathcal{O}_x)$.

Finally, we note that this construction give us that the central character of Π_x is

$$\varkappa_x\mu_{n,x}|_{F_x^\times} = \varkappa_x.$$

Thus, the global character

$$\varkappa = \otimes_x \varkappa_x, \tag{6.1.3}$$

is trivial on $F^\times$.

6.2 L**-functions.** We recall that in the case where we consider a maximal Levi F-subgroup $\mathbf{M} = \mathbf{GL}_m \times \mathbf{SO}_{2n}^*$ of $\mathbf{SO}_{2(m+n)}^*$, the adjoint action decomposes as

$$r_1 \oplus r_2,$$

where $r_1 = \rho_m \otimes \tilde{\rho}^*_{2n}$ and $r_2 = \wedge^2 \rho_m$ (2.5.2). Using the Langlands-Shahidi method for $r_{1,x}$, when x is an inert place (i.e. $E \otimes_F F_x$ is a field), we obtained the Langlands-Shahidi L-functions

$$L(s, \pi_x \times \tau_x) = L(s, \tau_x \otimes \tilde{\pi}_x, r_{1,x}) \quad \& \quad L(s, \tau, \wedge^2 \rho_m) = L(s, , \tau_x \otimes \tilde{\pi}_x, r_{2,x}).$$

for π_x a generic representation of $\mathbf{SO}^*_{2n}(F_x)$ and τ_x a generic representation of $\mathbf{GL}_m(F_x)$. In what follows, we will only focus on the inert case, as for the split case analogous results are valid thanks to [37, Section 8].

(*Unramified case*). The definitions of π_x and Π_x in the unramified case give us that

$$L(s, \pi_x, \rho^*_{x,2n}) = L(s, \Pi_x)$$
$$\varepsilon(s, \pi_x, \rho^*_{x,2n}) = \varepsilon(s, \Pi_x).$$

Furthermore, we have

Proposition 6.2.1. *Let π_x be an unramified generic irreducible representation of $\mathbf{SO}^*_{2n}(F_x)$ and Π_x a lift as in Section 6.1. Then for a generic irreducible representation τ_x of $\mathbf{GL}_m(F_x)$ we have the following*

$$\begin{aligned} L(s, \pi_x \times \tau_x) &= L(s, \Pi_x \times \tau_x) \\ \varepsilon(s, \pi_x \times \tau_x, \psi_x) &= \varepsilon(s, \Pi_x \times \tau_x, \psi_x). \end{aligned} \tag{6.2.2}$$

(*Ramified case*).

Proposition 6.2.3. *Let π_x be an irreducible generic representation of $\mathbf{SO}^*_{2n}(F_x)$ and Π_x a lift as in Section 6.1. Then for any sufficiently ramified enough character η_x of $F_x^\times$, we have that*

$$\begin{aligned} L(s, \pi_x \times (\tau_x \cdot \eta_x)) &= L(s, \Pi_x \times (\tau_x \otimes \eta_x)) \\ \varepsilon(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) &= \varepsilon(s, \Pi_x \times (\tau_x \otimes \eta_x), \psi_x), \end{aligned} \tag{6.2.4}$$

for every unramified irreducible representation τ_x of $\mathbf{GL}_m(F_x)$.

Proof. First, making η_x sufficiently ramified to obtain (2.10.4)

$$L(s, \pi_x \times (\tau_x \cdot \eta_x)) \equiv 1 \equiv L(s, \Pi_x \times (\tau_x \otimes \eta_x)). \tag{6.2.5}$$

Using the relation between L-functions, ε-factors and γ-factors in the tempered case (Section 2.10, property vii) and the definition of the ε-factors, we are left to prove the corresponding identity for the γ-factors.

Now, as π'_x is generic (Section 6.1) we can use the stability of the gamma factors (Theorem 2.8.2). By this result, if we make η_x ramified enough, the following identity also holds

$$\gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \gamma(s, \pi'_x \times (\tau_x \cdot \eta_x), \psi_x).$$

On the other hand, using Section 2.9 we have

$$\begin{aligned}\gamma(s,\pi'_x\times\eta_x,\psi_x)&=\gamma(s,\chi_{n,x}\times\eta_x,\psi_x)\prod_{i=1}^{n-1}\gamma(s,\chi_{i,x}\eta_x,\psi)\gamma(s,\chi_{i,x}^{-1}\eta_x,\psi_x)\\&=\gamma(s,\Pi_{\mu_{n,x}}\times\eta_x,\psi_x)\prod_{i=1}^{n-1}\gamma(s,\chi_{i,x}\eta_x,\psi)\gamma(s,\chi_{i,x}^{-1}\eta_x,\psi_x)\\&=\gamma(s,\Pi_x\times\eta_x,\psi_x)\end{aligned}\tag{6.2.6}$$

Now, since τ_x is unramified, it is a subquotient of an induced representation of the form

$$i_{\mathbf{B}_m(F_x)}^{\mathbf{GL}_m(F_x)}(|\cdot|_{F_x}^{b_1}\otimes\cdots\otimes|\cdot|_{F_x}^{b_m}),$$

where $b_i\in\mathbb{R}$. Using multiplicativity of γ-factors, we have

$$\gamma(s,\pi_x\times(\tau_x\cdot\eta_x),\psi_x)=\prod_{i=1}^{m}\gamma(s-b_i,\pi_x\times\eta_x)$$

and

$$\begin{aligned}\gamma(s,\Pi_x\times(\tau_x\cdot\eta_x))&=\prod_{i=1}^{m}\gamma(s,\Pi_x\times(|\cdot|_x^{b_i}\cdot\eta_x))\\&=\prod_{i=1}^{m}\gamma(s-b_i,\Pi_x\times\eta_x).\end{aligned}$$

Comparing these two, using (6.2.6), we obtain the desired identity. □

Remark 6.2.7. We remark that using once more the stability property of γ-factors of $\mathbf{GL}_{2n}$, we can get the same result but with a representation Π'_x that has the same central character as Π_x, i.e. equal to $\varkappa_x$ (6.1.3).

(*Global case*). Using the equalities (6.2.2) twisted by any character and (6.2.4), we have the following result.

Corollary 6.2.8. *Let $\pi=\otimes_x\pi_x$ be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$, unramified outside of a non-empty $S\subset|F|$ and let Π a lift of π as in Section 6.1. Then, for a character $\eta=\otimes_x\eta_x$, sufficiently ramified in $x\in S$, (so as to satisfy (6.2.4)), we have*

$$\begin{aligned}L(s,\pi\times\tau)&=L(s,\Pi\times\tau)\\\varepsilon(s,\pi\times\tau)&=\varepsilon(s,\Pi\times\tau)\end{aligned}\tag{6.2.9}$$

for every $\tau\in\mathcal{T}(S;\eta)$ (as in 3.2).

6.3 Application of converse theorem. We know that a lift Π of π is irreducible and admissible and that its central character is trivial on $F^\times$ (6.1.3), but we do not necessarily

have that it is automorphic. For that we use the converse theorem (Theorem 3.2.1). Hence, we need to make sure that $L(s,\pi\times\tau)$ is a polynomial for $\tau\in\mathcal{T}(S;\eta)$, for some character η of $\mathbb{A}_F^\times$ trivial on $F^\times$.

(*Polynomial condition*). We can find as in [39, Proposition 4.1] a sufficiently ramified character η_{x_0}, with $x_0\in S$ (nonempty by definition), such that $\tilde{\omega}_0(\pi\otimes\tau\cdot\eta)\ncong\pi\otimes\tau\cdot\eta$. From Corollary 6.2.8 and (6.2.5)

$$L(s,\Pi\times\tau)=L(s,\pi\times\tau)=\prod_{x\notin S}L(s,\pi_x\times\tau_x).$$

Assume (2.11.1) for every inert $x_0\notin S$. Namely, the unramified representation π_{x_0} satisfies that it is the full induced representation

$$\pi_{x_0}=i_{P_1}^{\mathbf{SO}_{2l}^*(F_{x_0})}(\pi_{b,x_0}|\det|^{r_b}\otimes\cdots\otimes\pi_{1,x_0}|\det|^{r_1}\otimes\pi_{0,x_0}),$$

where $0<r_1\leq\cdots\leq r_b$, π_{i,x_0} is an irreducible tempered representation of $\mathbf{GL}_{n_i}(F_{x_0})$ for $1\leq i\leq b$ and π_{0,x_0} is a generic tempered representation of $\mathbf{SO}_{2n_0}^*(F_{x_0})$. Using Proposition 4.2.4 we obtain that $L(s,\pi_{x_0}\times\tau_{x_0})$ is holomorphic on $\operatorname{Re}s\geq 1$. This allows us to use Proposition 2.11.6 to obtain the condition needed for Proposition 2.11.8. Thus, after using Proposition 2.11.8, we obtain that the normalized operator

$$N(s,\tau_{x_0}\otimes\tilde{\pi}_{x_0},\tilde{w}_0)$$

is holomorphic and non-zero on $\operatorname{Re}s\geq 1/2$. Now, combining this and the known split case $\mathbf{SO}_{2n}$ [37, Section 7], we apply Proposition 4.2.1 to obtain that $L(s,\pi\times\tau)=\prod_{x\notin S}L(s,\pi_x\times\tau_x)$ is holomorphic on $\operatorname{Re}s\geq 1/2$. Finally using the Langlands-Shahidi functional equation (2.10.2), we get that $L(s,\pi\times\tau)$ is entire. In addition, using the rationality property of L-functions [39, Theorem 1.2] we see that $L(s,\Pi\times\tau)$ is a polynomial.

(Trivial on $F^\times$). Choosing characters ν_x for $x\in S$ sufficiently ramified as in the polynomial condition and in the ramified case of Section 6.2, we can find a character η of $\mathbb{A}_F^\times$ trivial on $F^\times$ [1, X, Theorem 5] and which satisfies $\eta_x=\nu_x$ for $x\in S$.

6.4 Isobaric sums and Transfer. As we have checked all the hypothesis of the converse Theorem 3.2.1 in the previous section, we find an irreducible automorphic representation Π' of $\mathbf{GL}_{2n}$ such that $\Pi'_x=\Pi_x$ for $x\notin S$.

We recall that from [35, Proposition 2], every automorphic representation Π of $\mathbf{GL}_{2n}$ arises as a subquotient of a representation induced from cuspidal representations,

$$i_{\mathbf{P}(\mathbb{A}_F)}^{\mathbf{GL}_{2n}(\mathbb{A}_F)}(\Pi_1,\cdots,\Pi_d),\tag{6.4.1}$$

where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{GL}_{2n}$ containing the Borel subgroup of $\mathbf{GL}_{2n}$ consisting of upper triangular matrices, and with every Π_i a cuspidal automorphic representation of $\mathbf{GL}_{N_i}$. Now, starting from the fact that Π_i is globally generic, the results on the

classification of automorphic representations for general linear groups [24, (4.3)] give us that there exists a unique generic subquotient of (6.4.1). We call this representation a transfer of π via ρ and we denote it $T_\rho(\pi)$. We observe that its central character is given by (6.1.3), because it coincides with it at almost every place. Finally, we also recall that Langlands' isobaric sum gives us another construction of a subquotient of (6.4.1) [34, Section 2], that we denote

$$\Pi_1 \boxplus \cdots \boxplus \Pi_d.$$

Chapter 7

Image of functoriality

In this chapter, we conclude with the main result of our work, Theorem 7.2.1. Inspired by [39] and [52], we prove that the cuspidal factors of the isobaric sum are distinct, unitary and self dual, in positive characteristic. We check that this lift respects the arithmetic information coming from γ-factors. We finish by proving, as an application, the unramified Ramanujan conjecture for globally generic cuspidal automorphic representations of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$.

7.1 Rankin-Selberg L-functions and poles. Let $\Pi = \bigotimes \Pi_x$ and $\Pi' = \bigotimes \Pi'_x$ be two unitary cuspidal automorphic representations of $\mathbf{GL}_r(\mathbb{A}_F)$ and $\mathbf{GL}_{r'}(\mathbb{A}_F)$ respectively. Assume that Π_x and Π'_x are unramified for all $x \notin S$, where S is a finite subset of $|F|$. Then [24, II; (3.3), (3.6) and (3.7)]

$$L^S(s, \Pi \times \Pi') = \prod_{x \notin S} L(s, \Pi_x \times \Pi'_x)$$

is holomorphic on $\operatorname{Re} s \geq 1$ if $r' < r$. If $r = r'$ it has at most simple poles, they occur if and only if there exists a real number h such that $\Pi \cong \tilde{\Pi}' \otimes |\det|^{ih}$ and, in this case, we get poles at $s = -ih$ and $s = 1 - ih$. Moreover, $L^S(s, \Pi \times \Pi')$ is non-vanishing on $\operatorname{Re}(s) \geq 1$.

7.2 Image.

Theorem 7.2.1. *Assume (2.11.1). Let π be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. Then, π transfers to the irreducible automorphic representation $\Pi \coloneqq T_\rho(\pi)$ of $\mathbf{GL}_{2n}(\mathbb{A}_F)$ constructed in Section 6.4. Its central character is given by (6.1.3) and Π can be expressed as an isobaric sum*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary self-dual cuspidal automorphic representation of $\mathbf{GL}_{N_i}(\mathbb{A}_F)$, and $\Pi_i \ncong \Pi_j$ for $i \neq j$.

Proof. As we are assuming (2.11.1), we can apply the converse theorem as in Section 6.3 to construct Π. Furthermore, by construction Π is a transfer of π and its central character is given by (6.1.3). We now show the properties of Π_i. Let S be a finite set of $|F|$ such that π is unramified outside of S.

(Unitarity). We write $\Pi_i = |\det|^{n_i}\Pi_i'$, where Π_i' is unitary for every $1 \leq i \leq d$ and $n_d \geq \cdots \geq n_1$. Given that the central character of Π is unitary, we have that $n_1 \leq 0$. By (6.2.2) and the multiplicativity property of Rankin-Selberg L-functions we have

$$\begin{aligned} L^S(s, \pi \times \tilde{\Pi}_1) = L^S(s, \Pi \times \tilde{\Pi}_1') &= \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_1') \\ &= \prod_j L^S(s + n_j, \Pi_j' \times \tilde{\Pi}_1'). \end{aligned}$$

Since the left hand side has at most a pole at $s = 1$ and it is holomorphic and non-vanishing for $\operatorname{Re} s > 1$ by Theorem 4.2.6, we must have that $n_1 = 0$. Recursively we can check that $n_i = 0$ for all i. Thus Π_i is unitary for all i.

As a consequence we have that Π is equal to the isobaric sum of the Π's, as each Π_i is unitary and thus Π is the full induced representation.

(Distinct). As before we consider

$$\begin{aligned} L^S(s, \pi \times \Pi_i) = L^S(s, \Pi \times \tilde{\Pi}_i) &= \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_i) \\ &= \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_i) \end{aligned}$$

Arguing as above, we must have $\Pi_i \ncong \Pi_j$ for $i \neq j$, because otherwise the right hand side would not have a simple pole by Section 7.1.

(Self-dual). First observe that linear map $\tilde{w}_0$ of $Q_{F,n+m}$ (see Section 1.5), given by $\tilde{w}_0 e_i = e_{2(n+m)-(n-i)}$ for $1 \leq i \leq n$, $\tilde{w}_0 e_i = e_i$ for $n+1 \leq i \leq 2m+n$, trivial on l and $\tilde{w}_0 e_i = e_{i-n-2m}$ for $n+2m+1 \leq i \leq 2n+2m$ is in $\mathbf{SO}^*_{2n+2m}(F)$ and is a representative of $w_0 = w_{l,\mathbf{G}} w_{l,\mathbf{M}} \in W^{\mathbf{G}}$, where $\mathbf{M} \cong \mathbf{GL}_m \times \mathbf{SO}^*_{2n}$. The action of $\tilde{w}_0$ on $(g_1, g_2) \in \mathbf{M}(\mathbb{A}_F)$ is $({}^t g_1^{-1}, g_2)$. Furthermore, $\tilde{w}_0(\sigma) = \tilde{\Pi}_i \otimes \tilde{\pi}$. Assume that Π_i not selfdual. In that we would have $\sigma \ncong \tilde{w}_0(\sigma)$. In that case, Corollary 4.2.1 implies that the left hand side

$$\begin{aligned} L^S(s, \pi \times \tilde{\Pi}_i) = L^S(s, \Pi \times \tilde{\Pi}_i) &= \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_i) \\ &= \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_i) \end{aligned}$$

is holomorphic on $\operatorname{Re}(s) > 1/2$. But the right hand side has a pole coming from $L(s, \Pi_i \times \tilde{\Pi}_i)$ (Section 7.1). A contradiction, thus the Π_i's are self-dual. □

Remark 7.2.2. Conjecturally the image is characterised by the condition in the theorem and the fact that $L^T(s, \Pi_i, \operatorname{Sym}^2)$ has a pole at $s = 1$ for any sufficiently large finite set

of places T containing all archimedean places. This is established in the work of J. Cogdell, I. Piatetski-Shapiro and F. Shahidi over number fields [11].

7.3 Local lift. In this section, we explore the local behavior of this transfer. We recall that we also have a non-trivial character $\psi = \otimes_x \psi_x$ of $\mathbb{A}_F/F$, unramified outside of S.

Theorem 7.3.1. *Let $\Pi_x := T_\rho(\pi)_x$ and τ_x be an irreducible generic unitary representation of $\mathbf{GL}_m(F_x)$. Then*

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x) \tag{7.3.2}$$

Proof. We first note that this is true when π_x is unramified, i.e when $x \notin S$ (6.2.2).

Let us fix $x_0 \in |F|$ and suppose first that τ_{x_0} is cuspidal. Then there is a cuspidal automorphic representation $\tau = \otimes \tau_x$ of $\mathbf{GL}_m(\mathbb{A}_F)$ that is τ_{x_0} at x_0 and such that τ_x is unramified for $x \notin S$ [39, Lemma 3.1]. Furthermore, thanks to the Grunwald-Wang theorem [1, X, Theorem 5] and Remark 6.2.7 (central character of Π_x is $\varkappa_x$), we can choose a character $\eta = \otimes \eta_x$ such that η_x is sufficiently ramified for $x \in S$ and $x \neq x_0$ so that

$$\gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \gamma(s, \Pi_x \times (\tau_x \otimes \eta_x), \psi_x)$$

and $\eta_{x_0} = 1$.

On the other hand the Langlands-Shahidi functional equation of Section 2.7, property vi) gives us that

$$L^S(s, \pi \times (\tau \cdot \eta)) = \gamma(s, \pi_{x_0} \times (\tau_{x_0} \cdot \eta_{x_0}), \psi_{x_0}) \prod_{x \in S - \{x_0\}} \gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) L^S(1 - s, \tilde{\pi} \times (\tilde{\tau} \cdot \tilde{\eta})).$$

Similarly for the Rankin-Selberg L-functions

$$L^S(s, \Pi \times (\tau \cdot \eta)) = \gamma(s, \Pi_{x_0} \times \tau_{x_0}, \psi_{x_0}) \prod_{x \in S - \{x_0\}} \gamma(s, \Pi_x \times (\tau_x \cdot \eta_x), \psi_x) L^S(1 - s, \tilde{\Pi} \times (\tilde{\tau} \cdot \tilde{\eta})).$$

Thus, after simplifying we get

$$\gamma(s, \pi_{x_0} \times \tau_{x_0}, \psi_{x_0}) = \gamma(s, \Pi_{x_0} \times \tau_{x_0}, \psi_{x_0}),$$

obtaining thus the relation for the cuspidal representation τ_x.

For a generic unitary representation we use that τ_x can be expressed as [54, Section 7]

$$i_{\mathbf{P}(F_x)}^{\mathbf{GL}_m(F_x)}(\delta_1 \nu^{t_1} \otimes \cdots \otimes \delta_d \nu^{t_d} \otimes \delta_{d+1} \otimes \cdots \otimes \delta_{d+k} \otimes \delta_d \nu^{-t_d} \otimes \cdots \otimes \delta_1 \nu^{-t_1}),$$

where $\mathbf{P}$ is a parabolic subgroup containing the Borel subgroup of $\mathbf{GL}_m$ consisting of upper triangular matrices, the δ_i's are generic unitary discrete series representation of

$\mathbf{GL}_{m_i}(F_x)$, $0 < t_1 \leq \cdots \leq t_d < 1/2$ and $\nu = |\det|$. Using the multiplicativity property of γ-factors (Section 2.7) we get

$$\begin{aligned}\gamma(s, \pi_x \times \tau_x, \psi_x) = &\prod_{i=1}^{k} \gamma(s, \pi_x \times \delta_{d+i}, \psi_x) \\ &\prod_{i=1}^{d} \gamma(s + t_i, \pi_x \times \delta_i, \psi_x)\gamma(s - t_i, \pi_x \times \delta_i, \psi_x).\end{aligned}$$

Similarly,

$$\begin{aligned}\gamma(s, \Pi_x \times \tau_x, \psi_x) = &\prod_{i=1}^{k} \gamma(s, \Pi_x \times \delta_{k+i}, \psi_x) \\ &\prod_{i=1}^{d} \gamma(s + t_i, \Pi_x \times \delta_i, \psi_x)\gamma(s - t_i, \Pi_x \times \delta_i, \psi_x).\end{aligned}$$

We thus have reduced the problem to the case of unitary generic discrete series representation δ_x of $\mathbf{GL}_m(F_x)$. But in this case we can write δ_x as the generic irreducible subquotient of [57, Section 9.3]

$$i_{\mathbf{Q}(F_x)}^{\mathbf{GL}_m(F_x)}(\rho\nu^{-\frac{t-1}{2}} \otimes \cdots \otimes \rho\nu^{\frac{t-1}{2}}),$$

where $\mathbf{Q}$ is a parabolic subgroup containing the Borel subgroup of $\mathbf{GL}_m$ consisting of upper triangular matrices, ρ is a cuspidal representation of $\mathbf{GL}_h(F_x)$, h divides m, and t is a positive integer. Then using once again the multiplicativity property of γ-factors and that we know the relation when the representation is cuspidal, we obtain

$$\begin{aligned}\gamma(s, \pi_x \times \delta_x, \psi_x) &= \prod_{l=0}^{t-1} \gamma(s - \frac{t-1}{2} + l, \pi_x \times \rho, \psi_x) \\ &= \prod_{l=0}^{t-1} \gamma(s - \frac{t-1}{2} + l, \Pi_x \times \rho, \psi_x) \\ &= \gamma(s, \Pi_x \times \delta_x, \psi_x).\end{aligned}$$

□

Now we would like to check the analogous relation, but for the L-functions and ε-factors, at least in the following situation.

Proposition 7.3.3. *Let $x \in |F|$. Assume Property (T) from Section 2.10 at x. Let $\Pi_x \coloneqq T_\rho(\pi)_x$ and τ_x be an irreducible generic unitary representation of $\mathbf{GL}_m(F_x)$. Then, if π_x is tempered, we have that*

$$\begin{aligned}L(s, \pi_x \times \tau_x, \psi_x) &= L(s, \Pi_x \times \tau_x) \\ \varepsilon(s, \pi_x \times \tau_x, \psi_x) &= \varepsilon(s, \Pi_x \times \tau_x, \psi_x).\end{aligned} \tag{7.3.4}$$

Proof. We can reduce to proving (7.3.4) for discrete series representations τ_x. In order to do this, we write τ_x as

$$i_{\mathbf{P}(F_x)}^{\mathbf{GL}_m(F_x)}(\delta_1\nu^{t_1}\otimes\cdots\otimes\delta_d\nu^{t_d}\otimes\delta_{d+1}\otimes\cdots\otimes\delta_{d+k}\otimes\delta_d\nu^{-t_d}\otimes\cdots\otimes\delta_1\nu^{-t_1})$$

where $\mathbf{P}$ is a parabolic subgroup containing the Borel subgroup of $\mathbf{GL}_m$ consisting of upper triangular matrices, the δ_i's are unitary discrete series representations of $\mathbf{GL}_{m_i}(F_x)$, $0<t_1\leq\cdots\leq t_d<1/2$ and $\nu=|\det|$. Let us write τ_0 the induced of

$$\delta_{d+1}\otimes\cdots\otimes\delta_{d+k}.$$

Using the multiplicativity properties viii) and ix) of Section 2.10, we obtain

$$L(s,\pi_x\times\tau_x)=L(s,\pi_x\times\tau_0)\prod_{i=1}^{d}L(s+t_i,\pi_x\times\delta_i)L(s-t_i,\pi_x\times\delta_i)$$

$$\varepsilon(s,\pi_x\times\tau_x)=\varepsilon(s,\pi_x\times\tau_0,\psi_x)\prod_{i=1}^{d}\varepsilon(s+t_i,\pi_x\times\delta_i,\psi_x)\varepsilon(s-t_i,\pi_x\times\delta_i,\psi_x).$$

On the other hand, we use the Langlands classification to express π_x as a Langlands quotient of

$$i_{\mathbf{Q}(F_x)}^{\mathbf{SO}_{2n}^*(F_x)}(\pi_1\otimes\cdots\otimes\pi_e\otimes\pi_0),$$

where $\mathbf{Q}$ is a parabolic subgroup containing $\mathbf{P}_0$ (See Section 1.5), π_i is a quasi-tempered representation (i.e. a tempered representation twisted by an unramified character) of $\mathbf{GL}_{n_i}(F_x)$, for $1\leq i\leq e$ and π_0 is a tempered representation of $\mathbf{SO}_{2n_0}^*(F_x)$.

Now, again using property ix) of Section 2.10, we have

$$L(s,\pi_x\times\tau_0)=L(s,\pi_0\times\tau_0)\prod_{i=1}^{e}L(s,\pi_i\times\tau_0)L(s,\tilde{\pi}_i\times\tau_0)$$

$$\varepsilon(s,\pi_x\times\tau_0,\psi_x)=\varepsilon(s,\pi_0\times\tau_0,\psi_x)\prod_{i=1}^{e}\varepsilon(s,\pi_i\times\tau_0,\psi_x)\varepsilon(s,\tilde{\pi}_i\times\tau_0,\psi_x)$$

All the representations involved in the previous two equations are quasi-tempered. From property vii) of Section 2.10, the multiplicativity of γ-factors (Section 2.7) and (7.3.2), we get

$$L(s,\pi_x\times\tau_0)=\prod_{l=1}^{k}L(s,\pi_x\times\delta_{d+l}),$$

$$\varepsilon(s,\pi_x\times\tau_0,\psi_x)=\prod_{l=1}^{k}\varepsilon(s,\pi_x\times\delta_{d+l},\psi_x).$$

Finally, using a similar analysis, but with the Rankin-Selberg L-functions, on the right hand side we get the reduction to discrete series.

Now that we have reduced to the case where τ_x is a discrete series, let us write Π_x as the parabolic induced representation of

$$i_{\mathbf{P}_1(F_x)}^{\mathbf{GL}_{2n}(F_x)}(\xi_1\nu^{t_1}\otimes\cdots\otimes\xi_d\nu^{t_d}\otimes\xi_{d+1}\otimes\cdots\otimes\xi_{d+k}\otimes\xi_d\nu^{-t_d}\otimes\cdots\otimes\xi_1\nu^{-t_1}),$$

where $\mathbf{P}_1$ is a parabolic subgroup containing the Borel subgroup of $\mathbf{GL}_{2n}$ consisting of upper triangular matrices, the ξ_i's are unitary discrete series representations of $\mathbf{GL}_{n_i}(F_x)$ and $0<t_1\leq\cdots\leq t_d<1/2$. Now thanks to (7.3.2) and the multiplicativity of Rankin-Selberg γ-factors, we have

$$\gamma(s,\pi_x\times\tau_x)=\prod_{i=1}^{k}\gamma(s,\xi_{f+i}\times\tau_x,\psi_x)\cdot \prod_{j=1}^{d}\gamma(s-t_j,\xi_j\times\tau_x,\psi_x)\gamma(s+t_j,\xi_j\times\tau_x,\psi_x).$$

As ξ_j and τ_x are discrete series representations, hence tempered, we can write the right hand side in the following formal rational expression

$$\varepsilon(q_{F_x}^{-s})\frac{P(q_{F_x}^{-s})}{Q(q_{F_x}^{-s})},$$

where

$$P(q_{F_x}^{-s})^{-1}=\prod_{i=1}^{k}L(s,\xi_{f+i}\times\tau_x)\prod_{j=1}^{d}L(s+t_j,\xi_j\times\tau_x)L(s-t_j,\xi_j\times\tau_x)$$

&

$$Q(q_{F_x}^{-s})^{-1}=\prod_{i=1}^{k}L(1-s,\tilde{\xi}_{f+i}\times\tilde{\tau}_x)\prod_{j=1}^{d}L(1-s-t_j,\tilde{\xi}_j\times\tilde{\tau}_x)L(1-s+t_j,\tilde{\xi}_j\times\tilde{\tau}_x),$$

and $\varepsilon(q_{F_x}^{-s})$ is a monomial in $q_{F_x}^{-s}$.

As each $L(s,\xi_j\times\tau_x)$ has no poles on $\mathrm{Re}(s)>0$, and since $t_j<1/2$ for each j, the regions where $P(q_{F_x}^{-s})$ and $Q(q_{F_x}^{-s})$ are zero do not intersect. Therefore, there are no cancellations involving the numerator and denominator of this formal expression. We can then conclude by the construction of Langlands-Shahidi L-functions [38, Section 5] and multiplicativity of the Rankin-Selberg L-functions that

$$L(s,\pi_x\times\tau_x)=\frac{1}{P(q_{F_x}^{-s})}=L(s,\Pi_x\times\tau_x).$$

□

7.4 Ramanujan Conjecture. We finish by using this study to prove the unramified Ramanujan conjecture for globally generic cuspidal representations of $\mathbf{SO}_{2n}^{*}$.

Theorem 7.4.1. *Assume (2.11.1). Let $\pi = \otimes_x \pi_x$ be a globally generic cuspidal representation of $\mathbf{SO}^*_{2n}(\mathbb{A}_F)$. If π_x is unramified, then its Satake parameter has absolute value 1.*

Proof. Let us fix $x \in |F|$ an inert place (the split case is obtained from [37, Theorem 9.14]). Using Theorem 7.2.1, we have that $T_\rho(\pi) = \Pi$ is the isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_e,$$

where each Π_i is a unitary self-dual cuspidal automorphic representation of $\mathbf{GL}_{N_i}(\mathbb{A}_F)$, and $\Pi_i \ncong \Pi_j$ for $i \neq j$. By [33, Théorème VI.10], each $\Pi_{i,x}$ is tempered.

If π_x is unramified, we may consider the semisimple conjugacy class

$$\mathrm{diag}(\alpha_1, \cdots, \alpha_{n-1}, 1) \rtimes \mathrm{Fr}_x\,.$$

Then, by definition, the semisimple conjugacy class of Π_x is given by

$$\mathrm{diag}(\alpha_1, \cdots, \alpha_{n-1}, 1, 1, \alpha_{n-1}^{-1}, \cdots, \alpha_1^{-1})$$

Each α_j or α_j^{-1} is the Satake parameter for one of the representations $\Pi_{i,x}$, which are unramified. But in the case $\mathbf{GL}_{n_i}$, we have that

$$|\alpha_i| = 1.$$

□

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Appendices

Appendix A

Appendix by Guy Henniart

In this appendix, we complete the proof of Proposition 4.1.5 which concerns the split group $\mathbf{SO}_{2n}$.

A.1 The result is local, but the setting comes form a global situation. Let us first recall that global setting. We have a global function field k, and a connected reductive group $\mathbf{G}$ over k, which is quasi-split but non-split, in fact a group of type $\mathbf{SO}^*_{2n}$ relative to a quadratic separable extension l/k. Such a group is unique up to isomorphism; for definiteness we consider on the k-vector space $V = k^{n-1} \oplus l \oplus k^{n-1}$ the quadratic form Q sending $(x_1, \dots, x_{n-1}, y, x_{n+2}, \dots, x_{2n})$ to $x_1x_{2n} + x_2x_{2n-1} + ... + x_{n-1}x_{n+2} + N_{l/k}(y)$, where the x_i's are in k, y in l, and $N_{l/k}$ is the norm map. We take $\mathbf{G}$ to be the connected component of the group of automorphisms of Q.

Remark A.1.1. If we allow for l the split k-algebra k^2, with coordinates x_n and x_{n+1}, and use the same recipe for Q and $\mathbf{G}$, we obtain a split group $\mathbf{SO}_{2n}$: indeed $N_{k^2/k}(x_n, x_{n+1}) = x_nx_{n+1}$. Consequently our group $\mathbf{G}$ over k splits if we base change to l, hence any extension of l.

For every place v of k, let k_v be the completion of k with respect to v and $\mathbf{G}_v$ be the base change to k_v from k of $\mathbf{G}$. Then $\mathbf{G}_v$ is the connected component of the group of automorphisms of the quadratic form Q_v on $V_v = k_v \otimes_k V$ given by the same formula, with N_{l_v/k_v} instead of $N_{l/k}$ and $l_v = l \otimes_k k_v$. The group $\mathbf{G}_v$ is split if and only if v splits in l. That group $\mathbf{G}_v$ is in fact defined, using the quadratic form Q_v, over the ring of integers $\mathcal{O}_v$ of k_v. When v is unramified in l, the group $\mathbf{G}_v$ is unramified, and conversely; the subgroup $\mathbf{G}_v(\mathcal{O}_v)$ is then a hyperspecial maximal compact subgroup of $\mathbf{G}_v(k_v)$.

From the global setting we also inherit Whittaker data. That involves a pinning of a root datum attached to $\mathbf{G}$, and a choice of non-trivial character Ψ of $\mathbb{A}_k$ trivial on k. We let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ preserving each line ke_i (where $(e_1, \dots, e_{n-1})$ is the canonical basis of the first copy of k^{n-1} in V, and $(e_{n+2}, \dots, e_{2n})$ that of the second

copy), and preserving l as well. It is the centralizer of the maximal split torus **S**, the subtorus of **T** acting trivially on l. We let **B** be the Borel subgroup of **G** preserving the flag of subspaces V_i of V, where V_i for $i = 1, \dots, n-1$ has the basis $(e_1, \dots, e_i)$; we write **U** for its unipotent radical. One identifies **T** to $\mathbb{G}_{m,k}^{n-1} \times \mathbf{T}'$, where $\mathbf{T}'$ is the kernel of the norm map from $\mathrm{Res}_{l/k}\mathbb{G}_{m,l}$ to $\mathbb{G}_{m,k}$, with $(x_1, \dots, x_{n-1})$ in $\mathbb{G}_{m,k}^{n-1}$ acting by x_i on ke_i, by its inverse on ke_{2n+1-i} for $i = 1, \dots, n-1$, and trivially on l, and $h \in \mathbf{T}'(k)$ acting by multiplication on l, and trivially on the e_i's. To get a matrix picture, one can choose a basis (e_n, e_{n+1}) of l over k, thus yielding a k-basis of V which we order as $e_1, \dots, e_{n-1}, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n}$, and then **B** appears as a group of upper triangular block diagonal matrices in $\mathbf{GL}_{2n}$, and **T** as its subgroup of block diagonal matrices, with blocks of size 1 except the central one of size 2; the group **S** is the subgroup of **T** with trivial central block. For $\mathbf{SO}_4^*$, the matrix picture of **B** is following: we write $l = k(\alpha)$ and we take the k-basis $e_2 = 1$ and $e_3 = \alpha$ of l. If $\mathrm{Tr}_{l/k}$ is the trace map, then

$$\mathbf{B}(k) = \left\{ \begin{pmatrix} t_1 & * & * & * \\ 0 & a & -bN_{l/k}(\alpha) & * \\ 0 & b & a + b\,\mathrm{Tr}_{l/k}(\alpha) & * \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} : t_1 \in k^\times, a, b \in k, N_{l/k}(a + \alpha b) = 1 \right\}.$$

Remark A.1.2. Over an extension k' of k splitting l, $k' \otimes_k l$ is isomorphic to k'^2, each component corresponding to a k-embedding ι of l into k', with $y \otimes u$ going to $y\iota(u)$. Then $\mathbf{G}_{k'}$ is split, indeed $\mathbf{T}'_{k'}$ is isomorphic to $\mathbb{G}_{m,k'}$ acting via x goes to (x, x^{-1}), and $\mathbf{T}_{k'}$ is isomorphic to $\mathbb{G}_{m,k'}^n$ using the action on $e_1, \dots, e_{n-1}, e_n$, where (e_n, e_{n+1}) is the canonical basis of k'^2.

The absolute root datum attached to **G** is of type D_n, the relative root datum of type B_{n-1}. More precisely let us choose a separable closure k^a of k, and call ι and ι' the two conjugate embeddings of l/k into k^a/k. Then as explained above $k^a \otimes_k l$ is isomorphic to $k^a \times k^a$ by sending $y \otimes u$ to $(y\iota(u), y\iota'(u))$; that gives a k^a-basis (e_n, e_{n+1}) of $k^a \otimes_k l$, and $\mathbf{T}_{k^a}$ is isomorphic to $(\mathbb{G}_{m,k^a})^n$ via its action on $e_1, \dots, e_n$. We use the corresponding coordinates $x_1, \dots, x_n$ on $\mathbf{T}_{k^a}$. The simple roots of $\mathbf{T}_{k^a}$ with respect to $\mathbf{B}_{k^a}$ are the characters $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$, where $\alpha_j(x_1, \dots, x_{n-1}, x_n) = x_j/x_{j+1}$ for $j < n$ and $\alpha_n(x_1, \dots, x_n) = x_{n-1}x_n$. Thus the Galois group of k^a/k fixes α_i for $i < n-1$ and exchanges α_{n-1} and α_n. The simple relative roots are then the restrictions β_i to **S** of those simple absolute roots, α_{n-1} and α_n having the same restriction $\beta_{n-1} = \beta_n$. For

$\mathbf{SO}_4^*$, we have the following matrix pictures:

$$\alpha_1 \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} = t_1/t_2, \quad \alpha_2 \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} = t_1 t_2,$$

$$\beta_1 \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} = t_1.$$

We can pin the root subgroup corresponding to α_i, for $i < n$, by sending t in k^a to the endomorphism $Id + tA_i$ of $k^a \otimes_k V$, where A_i acts trivially on all elements e_j except e_{i+1} and e_{2n+1-i}, and sends e_{i+1} to e_i, also e_{2n+1-i} to $-e_{2n-i}$. For $i < n-1$, the relative root subgroup corresponding to β_i is pinned (over k) by using the same formula. The root subgroup corresponding to α_n is pinned by sending t in k^a to the endomorphism $Id+tA_n$, where A_n acts trivially on the e_i's except e_{n+1} and e_{n+2}, and sends e_{n+1} to e_{n-1}, and e_{n+2} to $-e_n$. For $\mathbf{SO}_4$, we have the following matrix pictures:

$$Id + tA_1 = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Id + tA_2 = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Correspondingly, there is a kind of pinning, by $\mathrm{Res}_{l/k}\,\mathbb{G}_{a,l}$, of the relative root subgroup corresponding to $\beta_{n-1} = \beta_n$; that pinning sends u in l to $A(u)$, where $A(u)$ is the endomorphism of V acting trivially on the e_i's for $i < n$ and $i > n+2$, and sending y in l to $\mathrm{Tr}_{l/k}(uy)e_{n-1}$, also e_{n+2} to $(-u\overline{u}e_{n-1}, -\overline{u})$ in $ke_{n-1} \oplus l$, where bar indicates conjugation in l/k. Indeed, going over to k^a, and identifying $k^a \otimes_k l$ with $k^a \times k^a$ as above, β_n translates to sending (t, t') to $(Id + tA_{n-1})(Id + t'A_n)$. For $\mathbf{SO}_4^*$, we have the following matrix picture: if $\{e_2, e_3\}$ is a ordered basis of l, then

$$Id + A(u) = \begin{pmatrix} 1 & \mathrm{Tr}_{l/k}(e_2 u) & \mathrm{Tr}_{l/k}(e_3 u) & -N_{l/k}(u) \\ 0 & 1 & 0 & -\bar{u}_1 \\ 0 & 0 & 1 & -\bar{u}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\bar{u} = \bar{u}_1 e_2 + \bar{u}_2 e_3$.

We can use the additive character Ψ of $\mathbb{A}_k$ to get an additive character θ_Ψ of $\mathbf{U}(\mathbb{A}_k)$ sending $Id + tA_i$ to $\Psi(t)$ for $i < n-1$ and t in $\mathbb{A}_k$, and sending $Id + A(u)$ to $\Psi \circ \mathrm{tr}_{l/k}(u)$ for u in $\mathbb{A}_l$. Since Ψ is non-trivial, θ_Ψ is a non-degenerate character of $\mathbf{U}(\mathbb{A}_k)$. A general non-degenerate character θ of $\mathbf{U}(\mathbb{A}_k)$ sends $Id + tA_i$ to $\Psi(a_i t)$ for $i < n-1$ and t in $\mathbb{A}_k$,

and $Id + A(u)$ to $\Psi \circ \mathrm{Tr}_{l/k}(bu)$ for u in $\mathbb{A}_l$, for some a_i's in $k^\times$ and b in $l^\times$. We fix such a character θ.

Now if v is a place of k, the restriction θ_v of θ to $\mathbf{U}(k_v)$ is a non-degenerate character, and if π is a cuspidal θ-generic automorphic representation of $\mathbf{G}(\mathbb{A}_k)$ then at a place v its component π_v is generic with respect to θ_v. For all places v except finitely many, the conductor of Ψ_v is $\mathcal{O}_{k_v}$ (meaning that Ψ_v is trivial on $\mathcal{O}_{k_v}$ but not on larger fractional ideals), and the a_i's and b are units, so that θ_v is trivial on $\mathbf{U}(\mathcal{O}_v)$ but not on $\mathbf{U}(\mathcal{I})$ for any larger fractional ideal $\mathcal{I}$ of $\mathcal{O}_v$.

Furthermore, we have another positive integer m, and we consider the quadratic form Q' on $W = k^m \oplus V \oplus k^m$ given by $Q'(y_1, \dots, y_m, v, y_{m+1}, \dots, y_{2m}) = Q(v) + y_1 y_{2m} + \dots + y_m y_{m+1}$. The connected component $\mathbf{H}$ of the automorphism group of Q' is again a quasi-split group over k, of type $\mathbf{SO}^*_{2m+2n}$. We consider the parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$ stabilizing the first component k^m; it has the Levi subgroup stabilizing all three components, which is isomorphic to $\mathbf{GL}_m \times \mathbf{G}$, with $\mathbf{GL}_m$ acting on the first component and $\mathbf{G}$ on V. Again, at a place w split in l, $\mathbf{H}_w$ is split.

We are also given a unitary cuspidal automorphic representation τ of $\mathbf{GL}_m(\mathbb{A}_k)$ and a unitary θ-generic cuspidal automorphic representation π of $\mathbf{G}(\mathbb{A}_k)$. At a given place v of k, τ_v is tempered (by Laurent Lafforgue's result), and π_v is unitary and θ_v-generic.

A.2 We are interested in places v of k which are split in l (in particular unramified), such that Ψ_v has conductor $\mathcal{O}_{k_v}$, with the a_i's and b units at v, and such that furthermore τ_v and π_v are unramified. Such places form a set of density $1/2$, so there is a plethora of them. We want to show that if $\mathrm{Re}(s) > 1$ then the unramified component of the representation of $\mathbf{H}(k_v)$ parabolically induced from $\nu^s \tau_v \otimes \pi_v$ cannot be unitary, where unramified means with non-zero vectors fixed by $\mathbf{H}(\mathcal{O}_{k_v})$, and ν denotes the character of $\mathbf{GL}_m(k_v)$ given by the absolute value of the determinant character.

The proof requires a comparison with local fields of characteristic 0, so we change notation, and take as our base field a locally compact non-Archimedean field F, of arbitrary characteristic, and write $\mathcal{O}_F$ for its ring of integers, P_F for the maximal ideal of $\mathcal{O}_F$, val_F for the normalized valuation of F, and q_F for the cardinality of its residue field k_F. We consider the split groups $\mathbf{G} = \mathbf{SO}_{2n}$ and $\mathbf{H} = \mathbf{SO}_{2m+2n}$ with the standard Borel pairs as above, and $\mathbf{GL}_m \times \mathbf{G}$ seen, also as above, as a Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$. We call unramified a smooth irreducible representation of $\mathbf{G}(F)$ with non-zero vectors fixed by $\mathbf{G}(\mathcal{O}_F)$, and similarly for $\mathbf{H}$ and $\mathbf{GL}_m$. Genericity will not play a role in the following result, which implies what is needed for Proposition 4.1.5.

Proposition A.2.1. *Let τ be a tempered smooth irreducible unramified representation of $\mathbf{GL}_m(F)$ and π a unitary smooth irreducible unramified representation of $\mathbf{G}(F)$. Let s be a complex parameter with $\mathrm{Re}(s) > 1$. Then the unramified component of $\mathrm{Ind}_{P(F)}^{H(F)}(\nu^s \tau \otimes \pi)$ is not unitary.*

The proof relies on known classifications, for $\mathbf{GL}_m(F)$, $\mathbf{G}(F)$ and $\mathbf{H}(F)$: that of smooth irreducible unramified representations and that of unitary unramified ones. For $\mathbf{G}(F)$ and $\mathbf{H}(F)$, the first classification is established only when the characteristic of F is not 2, and the second one only for F of characteristic 0. Presumably the available proofs can be made to work in all positive characteristics as well, but that has not been done yet. So we proceed in two independent steps. On the one hand we use a comparison of local fields à la Kazhdan to show that if the result is true for F of characteristic 0, then it is true in general. On the other hand we prove the result when F has characteristic 0 using the known classifications.

Let us first recall the usual parametrization of smooth irreducible unramified representations of $\mathbf{G}(F)$ via unramified characters of $\mathbf{T}(F)$. That parametrization holds for unramified groups over F of any characteristic. If ξ is an unramified character of $\mathbf{T}(F)$, then $\mathrm{Ind}_{\mathbf{B}(F)}^{\mathbf{G}(F)}(\xi)$ has a unique unramified irreducible component $\pi(\xi)$, and all irreducible unramified smooth representations of $\mathbf{G}(F)$ are obtained in this way, with a character ξ which is unique up to the action of the Weyl group of $\mathbf{T}$ in $\mathbf{G}$. Concretely ξ sends $(x_1, \dots, x_n)$ in $\mathbf{T}(F)$ to the product of $z_i^{val_F(x_i)}$, i running from 1 to n, for some non-zero complex numbers z_i, and the tuple $(z_1, \dots, z_n)$ is determined by $\pi(\xi)$ up to permutation of the coordinates and sending z_i to its inverse for an even number of indices i; we write $\pi(\xi) = \pi(z_1, \dots, z_n)$. Such a description applies to $\mathbf{GL}_m(F)$, using unramified characters of its diagonal torus $\mathbf{A}$, and to $\mathbf{H}(F)$, using unramified characters of its maximal split torus $\mathbf{A} \times \mathbf{T}$. An unramified character of $\mathbf{A}(F)$ is given by an m-tuple of non-zero complex numbers $(y_1, \dots, y_m)$, and we write $\tau(y_1, \dots, y_m)$ for the corresponding unramified irreducible component; similarly an unramified character of $\mathbf{H}(F)$ is given by an $(m+n)$-tuple of non-zero complex numbers $(y_1, \dots, y_m, z_1, \dots, z_n)$, and gives the unramified irreducible component $\pi(y_1, \dots, y_m, z_1, \dots, z_n)$. Since the representation τ of $\mathbf{GL}_m(F)$ is supposed to be tempered, it has the form $\pi(y_1, \dots, y_m)$ where the y_j have modulus 1, and τ is the full parabolically induced representation.

The representation π is supposed to be unitary. To interpret that condition concretely, in terms of the parameters with $\pi = \pi(z_1, \dots, z_n)$, we now assume F of characteristic 0, to be able to use the results of Muic [3] and Muic-Tadic [4]. Note that in those papers the notion of unramified representation refers to the same choice of hyperspecial maximal compact subgroup as ours, viz. the group $\mathbf{SO}_{2n}(\mathcal{O}_F)$.

In [3] Muic gives a finer description of unramified smooth irreducible representations of $\mathbf{G}(F)$, which obviously also applies to $\mathbf{H}(F)$, in three stages: strongly negative representations, negative representations (which are unitary), general case. The classification of unitary unramified representations in [4] uses the classification of [3]. However we have to be careful in that both references consider the group $\mathbf{O}_{2n}$ instead of our group $\mathbf{SO}_{2n}$, and an irreducible smooth representation of $\mathbf{O}_{2n}(F)$ is called unramified if it has non-zero fixed vectors under $\mathbf{O}_{2n}(\mathcal{O}_F)$.

The major difference with $\mathbf{SO}_{2n}$ occurs already when $n = 1$, and concerns the re-

ducibility of (unramified) principal series. Indeed $\mathbf{SO}_2(F)$ is a split torus, and a principal series is simply a character ξ, in particular irreducible; but $\mathbf{O}_2(F) - \mathbf{SO}_2(F)$ acts on $\mathbf{SO}_2(F)$ by inversion, so ξ induces irreducibly to $\mathbf{O}_2(F)$, (to a unitary representation if and only if ξ is unitary), unless its square is trivial, in which case ξ induces to the direct sum of its two extensions to $\mathbf{O}_2(F)$, which are both unitary and can be distinguished by their value on the transposition matrix, which is a sign. But $\mathbf{O}_2(\mathcal{O}_F)$ contains that transposition matrix, so if ξ is unramified there is indeed a unique unramified component in the induced representation, where the transposition matrix acts trivially.

That phenomenon persists for all positive integers n. Indeed the normalizer of $\mathbf{T}$ in $\mathbf{O}_{2n}$ is twice bigger than the normalizer in $\mathbf{SO}_{2n}$; for example it contains the transposition matrix σ which exchanges e_n and e_{n+1}, and acts on an unramified character ξ of $\mathbf{T}(F)$ by changing z_n to its inverse in the parameter of ξ, yielding a character ξ^σ. We can inflate ξ to $\mathbf{B}(F)$, induce first to $\mathbf{SO}_{2n}(F)$, where the induced representation $I(\xi)$ has a line of $\mathbf{SO}_{2n}(\mathcal{O}_F)$-fixed vectors and a unique unramified irreducible component $\pi(\xi)$, and then induce further to a representation $I^+(\xi)$ of $\mathbf{O}_{2n}(F)$. It is clear that the restriction of $I^+(\xi)$ to $\mathbf{SO}_{2n}(F)$ is the direct sum of $I(\xi)$ and $I(\xi^\sigma)$, with the two lines of $\mathbf{SO}_{2n}(\mathcal{O}_F)$-fixed vectors, exchanged by σ. But $\mathbf{O}_{2n}(\mathcal{O}_F)$ is generated by $\mathbf{SO}_{2n}(\mathcal{O}_F)$ and σ, hence $I^+(\xi)$ has a unique line of $\mathbf{O}_{2n}(\mathcal{O}_F)$-fixed vectors. Also the direct sum of $\pi(\xi)$ and $\pi(\xi^\sigma)$ occurs as a subquotient of $I^+(\xi)$, with the two factors exchanged by σ. If those two factors are not isomorphic then the direct sum is an irreducible component $\pi^+(\xi)$ of $I^+(\xi)$, the unique unramified such. If they are isomorphic, then $\pi(\xi)$ extends to $\mathbf{O}_{2n}(F)$; there are two such extensions, one being the twist of the other by the nontrivial character of $\mathbf{O}_{2n}(F)$ trivial on $\mathbf{SO}_{2n}(F)$, but only one of them, which we call $\pi^+(\xi)$, is unramified – it is the only unramified irreducible component of $I^+(\xi)$. Note that if η is an unramifed character of $\mathbf{T}(F)$ then $\pi^+(\xi) = \pi^+(\eta)$ if and only if $\pi(\eta)$ is equal to $\pi(\xi)$ or $\pi(\xi^\sigma)$. It follows that ξ goes to $\pi^+(\xi)$ gives all unramified irreducible smooth representations of $\mathbf{O}_{2n}(F)$, and that $\pi^+(\xi)$ determines ξ up to the action of the normalizer of $\mathbf{T}(F)$ in $\mathbf{O}_{2n}(F)$, which acts on the parameters $(z_1, \ldots, z_n)$ by permutation of the indices and sending some of the z_i's to their inverses. It is clear that $\pi^+(\xi)$ can be unitary only if $\pi(\xi)$ is; if that is the case, then $\pi(\xi^\sigma)$ is also unitary, and so is $\pi^+(\xi)$. This means that a classification of unitary smooth irreducible unramified representations of $\mathbf{O}_{2n}(F)$ can be directly applied to $\mathbf{SO}_{2n}(F)$ instead. Moreover, a criterion of irreducibility of $\pi(\xi)$ in terms of ξ has to be unsensitive to replacing ξ with ξ^σ.

We now recall what we need of the classifications of [3] and [4], following the introduction of [4] specialized to our case of $\mathbf{O}_{2n}(F)$. The classification of general smooth irreducible unramified representations of $\mathbf{O}_{2n}(F)$ in terms of negative ones will be enough for us.

By [3] (see [4] definition 0-6 and surrounding comments) that general case is as follows: one considers a multiset E of triples (r, ξ, α) where r is a positive integer, ξ a unitary unramified character of $F^\times$ and α a positive real number, and a negative

representation ρ of $\mathbf{O}_{2a}(F)$, so that a plus the sum of the r's equals n. Then one attaches to those data the unique irreducible unramified component $\pi(E,\rho)$ of the representation of $\mathbf{O}_{2n}(F)$ parabolically induced from tensor (over triples) $(\nu^{\alpha}\xi)(r)\otimes\rho$, where $\eta(r)$ for a character η of $F^{\times}$ is the character $\eta\circ\det$ of $\mathbf{GL}_r(F)$. In that manner we get all irreducible unramified representations π of $\mathbf{O}_{2n}(F)$, up to isomorphism (the way we order the triples to construct the tensor product does not matter), and the representation π determines ρ (up to isomorphism) and the multiset of triples E.

Now if $\pi(E,\rho)$ is unitary, then ([4], second assertion of Theorem 0-8 and definition 0-7) for all triples (r,ξ,α) in E we have $\alpha<1$. ([4] gives necessary and sufficient conditions for $\pi(E,\rho)$ to be unitary, but we do not need them). Now we are in position to give the proof of the proposition when the characteristic of F is zero.

Proof of the Proposition when $\mathrm{char}\,F=0$*.* Our representation π of $\mathbf{SO}_{2n}(F)$ has the form $\pi(\xi)$ for some unramified character ξ of $\mathbf{T}(F)$ and the corresponding representation $\pi^{+}(\xi)$ of $\mathbf{O}_{2n}(F)$ (which is unramified unitary) has the preceding form $\pi(E,\rho)$. We have also the unramified tempered representation τ of $\mathbf{GL}_m(F)$ given by m unramified unitary characters, and we want to consider the unramified irreducible component π' of the representation of $\mathbf{O}_{2m+2n}$ parabolically induced by $\nu^{s}\tau\otimes\pi^{+}(\xi)$, where $t=\mathrm{Re}(s)>1$. We see $\nu^{s}\tau$ as given by $\nu^{t}\eta_1,\dots,\nu^{t}\eta_m$ for unitary unramified characters η_j of $F^{\times}$. Then π' is simply $\pi(E',\rho)$ where E' is obtained from E by adding the triples $(1,\eta_i,t)$ for $i=1$ to m. Indeed using parabolic induction in stages, we see that $\pi(E',\rho)$ is the unramified component of the induction of $\nu^{s}\tau\otimes\pi(E,\rho)$. It follows that π' does not satisfy the condition above, so cannot be unitary, and the components of the restriction of π' to $\mathbf{SO}_{2n}(F)$ cannot be unitary either, which is what we wanted to prove. □

A.3 Now that we have proved what we want for F of characteristic 0, we need to transfer it to the positive characteristic case, possibly 2. We operate that transfer using close local fields à la Kazhdan, and for convenience our reference is [2], although some earlier references could have been used in places.

We change notation a bit, and use $\mathbf{G}$ for a split connected group over $\mathbb{Z}$; that will be applied to $\mathbf{SO}_{2n}$, $\mathbf{SO}_{2m+2n}$ and $\mathbf{GL}_m$. We fix a Borel subgroup $\mathbf{B}=\mathbf{T}\,\mathbf{U}$. We have two non-Archimedean local fields of the same residue characteristic, one of them is our characteristic p local field F, the other one F' of characteristic 0, and we put a prime to indicate that we use over F' the notation over F. We have the maximal hyperspecial subgroup $K=\mathbf{G}(\mathcal{O}_F)$ of $\mathbf{G}(F)$, and the Iwahori subgroup I which is the inverse image in K of the subgroup $\mathbf{B}(k_F)$ of $\mathbf{G}(k_F)$ seen as a quotient of $\mathbf{G}(\mathcal{O}_F)$. We write $\mathcal{H}$ for the Hecke algebra (with complex coefficients) of I in $\mathbf{G}(F)$. It is known that taking fixed points under I yields an equivalence between the category of smooth representations of $\mathbf{G}(F)$ generated by their I-fixed vectors and the category of (right) modules over $\mathcal{H}$. Moreover a smooth irreducible representation π of $\mathbf{G}(F)$ is unitary if and only if the

corresponding $\mathcal{H}$-module π^I is unitary ([1, Corollary 1.3]; in fact the case of the Iwahori subgroup had been obtained previously by D. Barbasch and A. Moy). Since unramified principal series have K-fixed vectors, they can be detected by the corresponding $\mathcal{H}$-module, and their unitarity as well. But we also need compatibilities with parabolic induction.

We say that F and F' are r-close (for a positive integer r) if there is a ring isomorphism of $\mathcal{O}_F/P_F^r$ with $\mathcal{O}'_F/P_F'^r$. For $r = 1$, this means $q_F = q'_F$. If F and F' are 1-close, then there is an algebra isomorphism ι of $\mathcal{H}$ with $\mathcal{H}'$, sending the characteristic function of K to the characteristic function of K' (that is classical, see [2, Section 3], especially Theorem 3.13, for a generalization to the Hecke algebra relative to congruence subgroups of I). An irreducible smooth unramified representation π of $\mathbf{G}(F)$ gives a $\mathcal{H}$-module π^I with non-zero vectors fixed by the characteristic function of K, which yields via ι a $\mathcal{H}'$-module with non-zero vectors fixed by the characteristic function of K'; that module has the from $\pi'^{I'}$ for an irreducible smooth unramified representation π' of $\mathbf{G}(F')$ (unique up to isomorphism), and π is unitary if and only if π' is.

Finally let us deal with parabolic induction. Following [2, Section 4.3], we assume that F and F' are 4-closed, and choose a ring isomorphism of $\mathcal{O}_F/P_F^4$ onto $\mathcal{O}'_F/P_{F'}'^4$, and uniformizers t, t' of F, F' compatible with L. Let **P** be a parabolic subgroup of **G** containing **B**, and **M** its Levi subgroup containing **T**, **N** its unipotent radical (provisionally those letters do not stand for integers). Then $I_M = I \cap \mathbf{M}(F)$ is an Iwahori subgroup of $\mathbf{M}(F)$, with Hecke algebra $\mathcal{H}_M$ and $K_M = K \cap \mathbf{M}(F)$ a hyperspecial maximal compact subgroup of $\mathbf{M}(F)$. Again we put primes for the corresponding objects over F'. Let τ be a smooth irreducible representation of $\mathbf{M}(F)$ with I_M fixed vectors, and τ' the corresponding representation of $\mathbf{M}(F')$, obtained by the above process applied to $\mathbf{M}(F')$, via an isomorphism of τ^{I_M} onto $\tau'^{I'_M}$. Then there is an isomorphism of vector spaces of $(\mathrm{Ind}_{\mathbf{P}(F)}^{\mathbf{G}(F)} \tau)^I$ onto $(\mathrm{Ind}_{\mathbf{P}(F')}^{\mathbf{G}(F')} \tau')^{I'}$, which is moreover compatible with the natural actions of $\mathcal{H}$ and $\mathcal{H}'$ ([2, Lemma 4.10 & Theorem 4.14]). Now, we can prove the Proposition when the characteristic of F positive.

Proof of the Proposition when $\mathrm{char}\, F > 0$*.* We now choose F' of characteristic 0 such that F and F' are 4-close (that is possible), and apply those considerations to the ambient group $\mathbf{H} = \mathbf{SO}_{2m+2n}$ (instead of **G**) with the Levi subgroup $\mathbf{GL}_m(F) \times \mathbf{SO}_{2n}(F)$ (instead of the group **M**). We have the representation τ of $\mathbf{GL}_m(F)$ and the representation π of $\mathbf{SO}_{2n}(F)$, and corresponding representations τ' and π' obtained via the previous process. We also have the complex number s with $\mathrm{Re}(s) > 1$ and our goal is to show that the unramified irreducible component of the representation of $\mathbf{SO}_{2m+2n}(F)$ parabolically induced from $\nu^s\tau \otimes \pi$ is not unitary. By the first part of our proof, the result is true over the characteristic 0 field F', and it is enough to show that the hypotheses on τ and π transfer to the corresponding hypotheses on τ' and π', and that the result over F' transfers back to F. That we do presently. The representation τ is tempered,

parabolically induced from the unitary character $\xi = (\xi_1, \ldots, \xi_m)$ of $\mathbf{A}(F)$, and the compatibility with parabolic induction recalled above shows that τ' is induced from the unitary character $(\xi'_1, \ldots, \xi'_m)$ with ξ'_i taking the same value as ξ on uniformizers. Moreover multiplying τ by ν^s corresponds to multiplying τ' by ν'^s, since twisting by ν^s amounts to multiplying all ξ_i's by the character ν^s of $F^\times$. The representation π is smooth irreducible unitary unramified, and then π' is also smooth irreducible unitary unramified. Finally the representation I parabolically induced from $\nu^s\tau \otimes \pi$ corresponds to the representation I' parabolically induced from $\nu'^s\tau' \otimes \pi'$, and the unramified irreducible component of I corresponds to the unramified irreducible component of I'. By the result over F', that for I' is not unitary, hence that for I is not unitary either, which is what we wanted. $\square$

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Titre : Conjecture de fonctorialité de Langlands pour SO*(2n) en caractéristique positive.

Mots clés : Programme de Langlands, représentations automorphes, théorie des nombres.

Résumé : Dans cette thèse, nous nous intéressons à la conjecture de fonctorialité de Langlands pour les groupes spéciaux orthogonales pairs quasi-déployés non-déployés en caractéristique positive. Cogdell, Kim, Piatetski-Shapiro et Shahidi ont démontré la conjecture de fonctorialité dans le cas d'une représentation cuspidale automorphe globalement générique pour les groupes classiques déployés, les groupes unitaires, et les groupes orthogonales spéciaux paires quasi-déployés, en caractéristique nulle. Lomelí a étendu ces résultats aux groupes classiques déployés et unitaires en caractéristique positive.

On démontre dans cette thèse la conjecture dans le cas d'une représentation cuspidale automorphe globalement générique, pour les groupes spéciaux orthogonales pairs quasi-déployés non-déployés en caractéristique positive. Comme application de ce résultat, on démontre la compatibilité des facteurs locaux et la conjecture de Ramanujan non-ramifiée.

Title : Langlands Functoriality Conjecture for SO*(2n) in positive characteristic.

Keywords : Langlands Program, Automorphic Representation, Number Theory.

Abstract : In this thesis, we are concerned with the Langlands functioriality conjecture for the even quasi-split non-split special orthogonal groups in positive characteristic. Cogdell, Kim, Piatetski-Shapiro and Shahidi proved functioriality conjecture in the case of a globally generic cuspidal automorphic representation for the split classical groups, unitary groups or even quasi-split special orthogonal groups in characteristic zero. Lomelí extends this result to split classical groups and unitary groups in positive characteristic.

We prove in this thesis the conjecture in the case of a globally generic cuspidal automorphic representation for the even quasi-split non-split special orthogonal groups in positive characteristic. As an application of this result, we prove the compatibility of the local factors and the unramified Ramanujan conjecture.

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