



**HAL**  
open science

# Identifiability of a class of SIR models and applications

Hoang Thanh Nguyen

► **To cite this version:**

Hoang Thanh Nguyen. Identifiability of a class of SIR models and applications. Modeling and Simulation. Université de Bordeaux, 2021. English. NNT : 2021BORD0164 . tel-03355411

**HAL Id: tel-03355411**

**<https://theses.hal.science/tel-03355411>**

Submitted on 27 Sep 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET  
D'INFORMATIQUE

par **Hoang Thanh NGUYEN**

POUR OBTENIR LE GRADE DE

**DOCTEUR**

SPÉCIALITÉ : MATHÉMATIQUES APPLIQUÉES

---

**IDENTIFIABILITY OF A CLASS OF SIR MODELS AND  
APPLICATIONS**

---

**Date de soutenance :** 22 Juin 2021

**Devant la commission d'examen composée de :**

Alain Miranville ... ..	Professeur, Université de Poitiers .....	Rapporteur
Rong Yuan ... ..	Professeur, Beijing Normal University .....	Rapporteur
Arnaud Ducrot ... ..	Professeur, Université Le Havre Normandie .....	Examineur
Jacques Demongeot ... ..	Professeur, Université Grenoble Alpes .....	Examineur
Jean-Pierre Françoise ... ..	Professeur, Sorbonne Université .....	Examineur
Pierre Magal ... ..	Professeur, Université de Bordeaux .....	Directeur de these



---

**Résumé** Cette thèse est consacrée à l'étude de l'identifiabilité et de l'estimation des paramètres d'une classe de modèles SIR et se divise en deux parties.

Dans la première partie (chapitre 1) nous tentons à estimer les paramètres de modèles SIR simples à partir des données réelles. Dans le chapitre 1, nous construisons un schéma qui permet de retrouver tous les paramètres des modèles SIR sous l'hypothèse que nous avons la connaissance des données des cas déclarés chaque semaine lors d'une épidémie. Ce schéma a été utilisé pendant des épidémies qui ont eu lieu en Amérique, en France et en Inde.

Dans la deuxième partie (chapitre 2, chapitre 3) nous étudions l'identifiabilité d'une classe de modèles SIR avec un taux d'incidence non linéaire. Dans le chapitre 2, nous fournissons quelques préliminaires d'algèbre différentielle, et une méthode pour trouver les paramètres identifiables d'un système dynamique. Dans le chapitre 3, nous appliquons la méthode du chapitre 2 pour certains modèles SIR à taux d'incidence non linéaire.

**Abstract** This thesis is devoted to the identifiability and parameter estimation of a class of SIR models and divided into two parts.

The first part (Chapter 1) aims to estimate the parameters of simple SIR models from the real data. In Chapter 1, we build a scheme that we can find all the parameters of SIR models if we know the weekly reported case data of an epidemic. This scheme is applied to some real epidemics in America, France, and India.

The second part (Chapter 2, Chapter 3) aims to study the identifiability of a class of SIR models with nonlinear incidence rate. Chapter 2 provides some preliminaries of differential algebra, and a method to find the identifiable parameters of a dynamical system. In Chapter 3, we applied the method in Chapter 2 for some SIR models with nonlinear incidence rate.

**Keywords** Identifiability, SIR models, Epidemic, SIR models with nonlinear incidence rate, Parameter Estimation.

**Mots-clés** Identifiabilité, Les modèles de SIR, Epidémie, Les modèles SIR avec un taux d'incident non linéaire, Estimation des paramètres.

**Laboratoire d'accueil** Bordeaux University, Institut de Mathématiques (IMB), CNRS UMR 5251. Address: 351, Cours de la Libération 33405 Talence, France.

---

# Acknowledgement

Firstly, I would like to express my sincere gratitude to my supervisors Professor Pierre Magal and Professor Arnaud Ducrot for their continuous support of my doctoral study and related research. Their patience, motivations, and immense knowledge inspired me to take a long journey into the research. Their suggestions also allowed me to improve the manuscript of this thesis. I could not have imagined having better supervisors and mentors for my doctoral study.

Besides my supervisors, I would like to thank Prof. Andreas Hartmann, and Prof. David Lannes not only for their insightful comments and encouragement but also for the hard question which encouraged me to widen my research from various perspectives.

My sincere thanks also go to Dr. Xiaoming Fu, Dr. Minh Phan Tran Duc, Dr. That Tu Ton, Dr. Duc Trung Hoang, Dr. Lara Abi Rizk who helped me a lot when I got some problems concerning the study. I also want to thank my doctoral student friends Tuan Thuong Dang, Abhinandan, Trung Hieu Vu as well as the other doctoral students, who give me a lot of valuable advice in the researches. Without their precious support, it would not be possible to conduct this research.

I also thank my fellow labmates Felipe Negreira, and Rolando Iii Perez for the stimulating discussions and all the fun we have had in the last four years.

I am also grateful to the project 911 of the Vietnamese government for granting me the scholarship so that I can pursue this Ph.D. program.

Last but not the least, I would like to thank my family: my parents, and my wife for supporting me spiritually throughout writing this thesis and my life in general. Their encouragement helped me to overcome the difficulties encountered during the doctoral study.



# Contents

<b>Contents</b>	<b>vii</b>
<b>Résumé</b>	<b>1</b>
<b>Introduction</b>	<b>7</b>
<b>1 Identifying the number of unreported cases in SIR epidemic models</b>	<b>13</b>
1.1 Introduction . . . . .	13
1.2 Identifiability of SIR model . . . . .	15
1.3 Computation the combinations of the parameters of SIR model . . . . .	17
1.3.1 System of equations to identify the parameters . . . . .	17
1.3.2 Derivation of the equation for the turning point . . . . .	18
1.3.3 Analysis of the turning point equation . . . . .	22
1.4 An identification method and applications to some outbreak epidemics	30
1.4.1 Description of the method . . . . .	30
1.4.2 Application to Hong-Kong influenza in New York City in 1968-1969 . . . . .	32
1.4.3 Application to the plague epidemic in Bombay, India in 1906	37
1.4.4 Application to the influenza epidemic in France in the consecutive six years 1988-1994 . . . . .	41
<b>2 Some preliminaries of differential algebra and application to system identifiability</b>	<b>61</b>
2.1 Some basic definitions of differential algebra . . . . .	61
2.2 Introduction to Rosenfeld-Gröbner algorithm implemented in Maple	64
2.3 Differential algebra approach for system identifiability . . . . .	66
2.4 Finding identifiable combinations of parameters of SIR model by differential approach . . . . .	70
<b>3 Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data</b>	<b>75</b>
3.1 The sufficient condition of identifiability of SIR model with nonlinear incidence rate . . . . .	75
3.2 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 1, \beta = 2$	77
3.3 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 1, \beta = 1$	84
3.4 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 2, \beta = 1$	87



3.5	SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 2, \beta = 2$	94
3.6	SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 1, \beta = 3$	112
<b>Conclusion</b>		<b>117</b>
<b>Bibliography</b>		<b>119</b>

# Résumé

Tout au long de l'histoire de l'humanité, on a été témoin de tant d'épidémies qui ont causé beaucoup de décès. Dans une petite maison datant de 5000 ans découverte dans un village préhistorique en Chine [77], se trouve les os des 97 corps étant entassés et ensuite brûlés et comprenant de tout âge : des adultes, des adolescents et des enfants. C'est très probablement une trace d'une épidémie, constatent les anthropologues. Autres exemples : la peste bubonique (connue au monde comme "Black Deaths"), entre 1346 et 1350, qui bouleversait l'Occident et retirait environ un tiers de la population; la pandémie grippale (également appelée "Grippe espagnole" [14]), pour la période 1918-1919, qui a causé plus de 50 millions de décès dans le monde entier; et plus récemment, la pandémie de coronavirus, le Covid-19 (SARS-CoV-2) d'origine Wuhan, en Chine (identifié pour la première fois en décembre 2019), répandus rapidement dans le monde entier et a causé 1 160 416 décès au total compté jusqu'au 25 octobre 2020 [76].

Pour comprendre la mécanique d'une épidémie et pour y faire face, l'épidémiologie mathématique a été née et devenue un domaine de recherche très actif, et typiquement, dans la situation sanitaire récente dans le monde. L'épidémiologie mathématique, par son nom, joue le rôle essentiel dans la mise en place des modèles mathématiques qui permettent de clarifier et expliquer les mécanismes de transmission et de propagation des maladies. Depuis les travaux de Daniel Bernoulli en 1760, en mettant en ouvre un modèle mathématique sous forme d'équations différentielles ordinaires [5], jusqu'à près d'un siècle et demi, en 1906 Hamer proposa un modèle à temps discret [32], qui peut être considéré comme l'idée pionnière des modèles compartimentaux, où il supposa que le nombre de nouveaux cas par unité de temps d'une épidémie dépend du nombre d'individus sensibles et du nombre d'individus infectés. Aujourd'hui, la plupart des modèles compartimentaux développés pour les systèmes déterministes ainsi que les systèmes stochastiques se basent au modèle SIR très connue de Kermack et McKendrick introduit en 1927 [36], [37]. Dans ce modèle, une population est divisée en trois compartiments (ceux qui donnent le nom SIR du modèle) : les individus sensibles, abrégés par le symbole S, les individus infectés, abrégés par le symbole I, les individus retirés, abrégés par le symbole R.

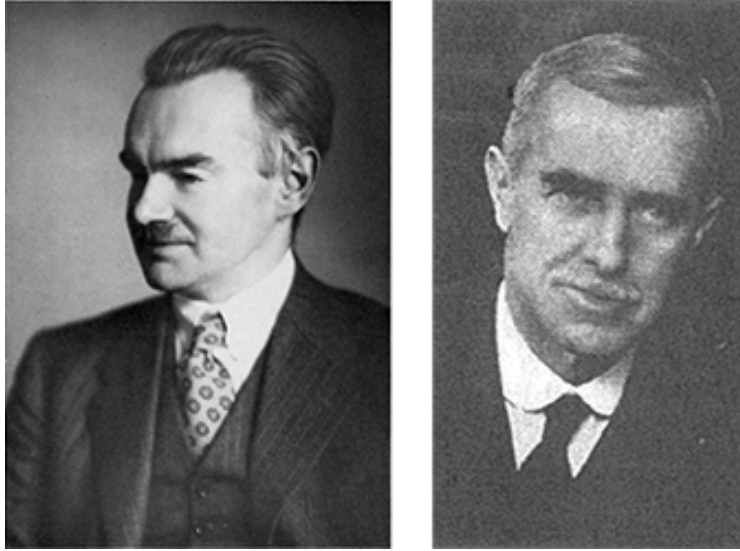


Figure 1: *William Kermack et Anderson McKendrick*

Dans cette thèse, nous étudions l'identifiabilité des modèles SIR et des modèles SIR à taux d'incidence *non linéaire* et leurs applications. Nous rappelons que le modèle SIR introduit par Kermack et McKendrick s'écrit :

$$\begin{cases} S'(t) = -\tau S(t)I(t), \\ I'(t) = \tau S(t)I(t) - \nu I(t), t \geq 0 \\ R'(t) = \nu I(t) \end{cases} \quad (0.0.1)$$

avec la condition initiale  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ ,  $R(0) = R_0 > 0$ .

Ici,  $S(t)$ ,  $I(t)$  et  $R(t)$  indiquent respectivement le nombre de personnes sensibles, infectées et retirées, au moment  $t > 0$ . Le paramètre  $\tau > 0$  correspond au taux de transmission de la maladie et le paramètre  $\nu > 0$  correspond au taux d'élimination des individus infectés. *Tous les paramètres et les conditions initiales considérées comme inconnues.*

En réalité, pour étudier une épidémie, nous ne disposons que des données fournies par un tableau contenant le nombre d'individus infectés déclarés. Pour comprendre les causes d'une épidémie et estimer sa taille, nous devons trouver un modèle approprié qui correspond aux données dans un certain sens. Cela se pose naturellement l'un des problèmes les plus importants de l'épidémiologie mathématique : l'ajustement des données au modèle.

Supposons que nous disposions des données sur les cas signalés chaque semaine lors d'une épidémie, comme le montre le tableau suivant

Semaine ( $t$ )	Les cas signalés
1	$N_1$
2	$N_2$
...	...
$k$	$N_k$

Table 1: *Données sur les cas signalés chaque semaine pour une épidémie de la semaine 1 à la semaine  $k$ .*

Afin d'ajuster les données, nous déterminons d'abord les paramètres  $\tau, \nu, S_0, I_0, R_0$  de (0.0.1) pour que nous puissions obtenir un modèle SIR approprié. Certains travaux tentent d'estimer ces paramètres par la méthode des moindres carrés, la méthode des vraisemblances,...tels que [38], [48], [16], [28], [18], [29]. L'inconvénient de ces méthodes est qu'elles ne peuvent pas fournir une vue générale des paramètres de (0.0.1). Cela signifie que l'on ne peut pas trouver tous les paramètres de (0.0.1) et qu'elles ne mentionnent pas l'identifiabilité des paramètres de (0.0.1). En 2018, Pierre Magal et Glenn Webb [46] ont fourni une nouvelle approche pour estimer les paramètres de (0.0.1) en utilisant *le point tournant*. Dans leur travail, ils considèrent les cas non signalés en ajoutant le nouveau paramètre  $\nu_1$ , le taux retiré des personnes infectées en raison des données sur les cas signalés. Le paramètre  $\nu$  est considéré comme le taux retiré de personnes infectées de tous cas, y compris les cas déclarés et les cas non déclarés, puis un autre nouveau paramètre apparaît  $\nu_2 = \nu - \nu_1$ , le taux retiré de personnes infectées en raison des cas non déclarés. La deuxième colonne du tableau de données ci-dessus contient maintenant les valeurs de la fonction  $\nu_1 I(t)$ , et en construisant la fonction  $CR(t) = \int_0^t \nu_1 I(s) ds$ , le nombre cumulé de cas signalés au moment  $t$ , Pierre Magal et et Glenn Webb ont fourni un algorithme pour calculer les paramètres numériquement. Cependant, le calcul numérique dans leurs travaux nécessite de connaître les valeurs de  $S_0$  et  $I_0$ , et une fois de plus, l'identifiabilité des paramètres est négligée.

L'identifiabilité des paramètres des systèmes dynamiques est un domaine actif étudié par de nombreux mathématiciens, par exemple [24], [25], [7], [43],[47],[49], [66], [67],[68], [69]. Afin de comprendre le rôle important de l'identifiabilité d'un système dynamique pour l'estimation des paramètres, nous présentons ici la définition pour (0.0.1).

**Definition 0.0.1.** Considérons le système (0.0.1) avec la fonction de sortie  $CR(t) = \int_0^t \nu_1 I(s) ds$ . Les paramètres  $\tau, \nu, \nu_1, S_0, I_0, R_0$  sont supposés être positifs et nous désignons  $p = (\tau, \nu, \nu_1, S_0, I_0, R_0)$ ,  $\bar{p} = (\bar{\tau}, \bar{\nu}, \bar{\nu}_1, \bar{S}_0, \bar{I}_0, \bar{R}_0)$ . Si nous avons  $p = \bar{p}$  de la condition  $CR(t, p) = CR(t, \bar{p})$  pour chaque  $t \geq 0$ , le (0.0.1) est appelé pour être identifiable à partir de la fonction de sortie  $CR(t)$ .

À partir des données sur les cas déclarés chaque semaine, on considère que la fonction de sortie  $CR(t)$  est déterminée. Si le système (0.0.1) est identifiable, on

---

peut alors affirmer que les paramètres peuvent être déterminés de manière unique à partir de la fonction de sortie  $CR(t)$ . Malheureusement, d'après le résultat que nous présentons au Chapitre 1, (0.0.1) n'est pas identifiable. De plus, nous avons la relation

$$\nu = \bar{\nu}, \frac{\tau}{\nu_1} = \frac{\bar{\tau}}{\bar{\nu}_1}, \tau S_0 = \bar{\tau} \bar{S}_0, \tau I_0 = \bar{\tau} \bar{I}_0. \quad (0.0.2)$$

si  $CR(t, p) = CR(t, \bar{p})$  pour chaque  $t \geq 0$ . Cela signifie qu'à partir des données des cas déclarés chaque semaine, il y a tant de vecteurs de paramètres  $p$  donnant la même fonction de sortie  $CR(t)$ . Cette analyse nous montre l'importance de prendre en compte l'identifiabilité avant d'essayer d'estimer les paramètres à partir des données. Dans le Chapitre 1, du point de vue de [46] et en considérant l'identifiabilité de (0.0.1), nous construisons un schéma qui nous permet de trouver un modèle approprié à partir des données réelles d'une épidémie. Ce schéma est appliqué pour certaines épidémies réelles survenues à New York, Bombay et en France. La clé de la construction de ce schéma est *l'équation du tournant*, les valeurs du tournant  $t_p$ ,  $CR(t_p)$ ,  $\lim_{t \rightarrow \infty} CR(t)$ , et  $CR'(t_p)$ . Ces valeurs peuvent certainement être tirées des données sur les cas déclarés chaque semaine. Ensuite, nous pouvons déterminer toutes les valeurs des paramètres  $p$  si nous ne faisons varier que la valeur de  $S_0$ , et estimer la taille finale de l'épidémie en fonction de  $S_0$ .

Depuis que Kermack et McKendrick ont publié leurs travaux sur les modèles d'épidémie, un grand nombre de publications tentent d'étendre ce modèle pour étudier l'épidémiologie. En 1978, après avoir étudié la propagation de l'épidémie de choléra à Bari, Capasso et Serio [17] ont généralisé le modèle (0.0.1) comme suit

$$\begin{cases} S'(t) = -S(t)g(I(t)), \\ I'(t) = S(t)g(I(t)) - \nu I(t), \\ R'(t) = \nu I(t) \end{cases} \quad (0.0.3)$$

avec la même condition initiale de (0.0.1). La fonction  $g : [0, +\infty) \rightarrow [0, +\infty)$  est supposé remplir les conditions suivantes

- i.  $g(0) = 0$ .
- ii. Il existe  $c > 0$  tel que  $g(x) \leq c$  pour tous les  $x \in [0, +\infty)$ .
- iii. La dérivée de  $g$  existe et est bornée sur tout intervalle compact de  $[0, +\infty)$ , avec  $g'(0) > 0$ .

La fonction  $g(x)$  ci-dessus est considérée d'inclure le phénomène de saturation. Capasso et Serio [17] ont également mentionné  $g(x) = \frac{\tau x}{1 + \kappa x}$  comme un exemple de leurs recherches.

Afin d'étudier le comportement dynamique du modèle épidémique (0.0.3), de nombreux auteurs ont modifié (0.0.3) d'une certaine manière. En 1986, Liu *et al.* [42] a brièvement examiné l'un des modèles modifiés de (0.0.3) en utilisant la fonction spécifique

$$g(x) = \frac{\tau x^\alpha}{1 + \kappa x^\beta} \quad (0.0.4)$$

dans laquelle  $\alpha = \beta > 0$ .

D'autres études utilisant la fonction (0.0.4) sont Ruan *et al.* [60] avec  $\alpha = \beta = 2$  et Xiao *et al.* [71] avec  $\alpha = 1, \beta = 2$ .

Au Chapitre 3, nous considérons le système (0.0.3) discuté dans Magal *et al.*[45] avec la fonction (0.0.4) comme suit

$$\begin{cases} S'(t) = -\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}, \\ I'(t) = \frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} - \nu I(t), \quad t \geq 0 \\ R'(t) = \nu I(t) \end{cases} \quad (0.0.5)$$

avec la condition initiale  $S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = R_0 \geq 0$ ,  $\alpha, \beta$  sont donnés et  $\beta + 1 \geq \alpha \geq 1$ .

On voit que la troisième équation de (0.0.5) est juste une conséquence des deux premières équations de (0.0.5), alors (0.0.5) peut être lu comme suit

$$\begin{cases} S'(t) = -\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}, \\ I'(t) = \frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} - \nu I(t), \quad t \geq 0 \end{cases} \quad (0.0.6)$$

où  $\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}$  est appelé un taux d'incidence non linéaire,  $\nu$  est le taux d'élimination de tous les individus infectieux d'une maladie. En réalité, ce taux ne peut pas être connu car de nombreux individus retirent la maladie sans le signaler. Nous ne pouvons avoir que le nombre d'individus infectieux  $y(t) = \mu I(t)$  avec  $0 < \mu < \nu$  déclarés par les responsables de la santé publique (voir [46], [23]). Supposons que  $\tau, \kappa, \nu, \mu, S_0, I_0$  soient les paramètres inconnus de (0.0.6). L'un des problèmes les plus importants concernant le modèle (0.0.6) est le suivant

*Problem 0.0.2.* Considérons le système (0.0.6), si nous avons les données réelles des cas déclarés comme une fonction  $y(t) = \mu I(t)$ , peuvent-ils les paramètres de (0.0.6) être déterminés de manière unique à partir de celle-ci ?

Considérons le système (0.0.6), le nombre  $\mathcal{R}_0 = \frac{\tau I_0^{\alpha-1} S_0}{\nu(1 + \kappa I_0^\beta)}$  est appelé le nombre de reproduction de base de (0.0.6) [45]. Si le  $\mathcal{R}_0 < 1$ , alors  $I(t)$  tombe à zéro, et l'épidémie s'atténue. Désormais, nous considérerons (0.0.6) seulement lorsque l'épidémie se déclenche avec  $\mathcal{R}_0 > 1$ . Une autre question est la suivante :

*Problem 0.0.3.* Le nombre de reproduction de base de (0.0.6) est-il déterminé uniquement à partir de la fonction  $y(t) = \mu I(t)$  lorsque l'épidémie se déclenche ?

Les deux problèmes 0.0.2, et 0.0.3 peuvent être compris comme une sorte de problème concernant l'identifiabilité de (0.0.6). Comme nous l'avons mentionné ci-dessus, il est très important de comprendre l'identifiabilité d'un système dynamique pour l'estimation des paramètres à partir des données réelles. Il existe certaines approches pour tester l'identifiabilité d'un système dynamique, telles que l'approche des séries de Taylor, l'approche de la transformation de similitude, l'approche de l'algèbre

---

différentielle, ... [4], [7], [8], [10], [23], [24], [25], [43], [47], [49], [52], [56], [61], [64], [66], [67], [68], [69], [70], [72]. Dans chaque approche, nous pouvons obtenir certains avantages ou inconvénients pour certains systèmes dynamiques spécifiques. Afin de résoudre les problèmes 0.0.2, et 0.0.3, nous choisissons l'approche de l'algèbre différentielle. L'avantage de cette approche est que nous pouvons extraire une équation ne concernant aucun état du système mais seulement la fonction de sortie que nous connaissons dans la réalité. En combinant d'autres hypothèses du problème, nous pouvons obtenir la relation entre deux paramètres lorsque le système dynamique a la même sortie. Ainsi, nous pouvons tester l'identifiabilité du système à partir de la sortie et également obtenir la relation entre les paramètres. Cette relation nous aide à déterminer les combinaisons identifiables de paramètres, il est donc très important d'estimer les paramètres. Le Chapitre 2 de cette thèse fournit une méthode pour trouver les combinaisons identifiables de paramètres. Dans ce chapitre, nous présentons quelques préliminaires en algèbre différentielle et fournissons un schéma pour trouver des combinaisons identifiables de paramètres, puis nous appliquons ce schéma pour le modèle SIR. Le Chapitre 3 vise à trouver les combinaisons de paramètres de (0.0.6) qui peuvent être déterminées de manière unique à partir de la fonction  $y(t) = \mu I(t)$  dans laquelle  $\mu$  est le taux de retrait des personnes infectées déclarées. La première section du Chapitre 3 fournit la condition suffisante pour deviner que les combinaisons de paramètres peuvent être déterminées de manière unique à partir de la fonction  $y(t) = \mu I(t)$ . Les sections suivantes sont les réponses aux problèmes 0.0.2, 0.0.3 dans certains cas particuliers  $\{\alpha = 1, \beta = 2\}$ ,  $\{\alpha = 1, \beta = 1\}$ ,  $\{\alpha = 2, \beta = 2\}$ ,  $\{\alpha = 2, \beta = 1\}$ ,  $\{\alpha = 1, \beta = 3\}$ . Ces réponses sont réalisées en appliquant l'approche de l'algèbre différentielle présentée dans les Sections 2.2 et 2.3 du Chapitre 2.

# Introduction

An epidemic is understood as an infectious disease that affects a large number of people during a certain period of time. Along with the history of human beings, people witnessed so many epidemics that caused a lot of deaths. About 5000 years ago, an epidemic swept a prehistoric village in China. The bodies were piled up in a house and then burned down. The epidemic does not rule out anyone. The bones found included adults, teenagers and children [77]. Up to now, when mankind's knowledge of epidemic is increasing, there have been still new epidemics that spread worldwide. These epidemics are called pandemics, for instance, COVID-19 pandemic. The COVID-19 also called the coronavirus pandemic, is a pandemic by the transmission of severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) [76]. The COVID-19 was first identified in Wuhan, China in December 2019 and up to the 25th of October 2020, COVID-19 has caused 1,160,416 deaths all over the world [78]. In the past, the bubonic plague that is called Black Deaths caused about one-third of the population of Europe from 1346 to 1350. Another epidemic that caused more than 50,000,000 deaths all over the world in the period 1918-1919 is "Spanish" influenza [14].

Since epidemic is the cause of so many deaths, it becomes a problem that humans should concern carefully. Epidemics should be investigated in various ways, from the infectious mechanisms to their influences. One of the ways that mathematicians have been trying is modeling the epidemic by mathematical dynamical systems. This can explain why mathematical epidemiology has become a very active field of research. The goal of mathematical epidemiology is setting up mathematical models giving us an understanding of the mechanisms of disease transmission and spread. Mathematical models also help us determine the main factors of the disease transmission process, give us suggestions for effectively controlling and methods of prevention, and provide an estimation for the severity and the size of an epidemic. Roughly speaking, mathematical models can be a part of the toolbox of public health research and decision making [26]. One of the first mathematical models of epidemics is the work entitled *An attempt at a new analysis of the mortality caused by smallpox and of the advantages of inoculation to prevent it* of Daniel Bernoulli (1700-1782) [5]. In his work, Daniel Bernoulli tried to find a comparison between the benefit of inoculation and the immediate risk of dying by setting up a model under the form of ordinary differential equations. After that, he applied this model with *Halley's life table*. This life table is introduced in a book "*Natural and Political Observations Made upon the Bills of Mortality*", published in 1662 in London, and known as a kind of table of data. Nowadays, combining mathematical models and data for studying



---

epidemics is a common way in mathematical epidemiology [5].

From the work in 1760 of Daniel Bernoulli to the late nineteenth century, mathematical models for epidemics didn't have any significant development. In 1906, for studying the measles epidemics, Hamer suggested using the mass action law to formulate a discrete-time model [32]. Hamer supposes that the number of new cases per unit time of an epidemic depends on the number of susceptible individuals and the number of infected individuals. Hamer's model can be considered as the first idea of compartmental models, and then so many studies about mathematical models of epidemics also were carried out this way [5]. One of the most famous compartmental models describing the spread of the epidemic is the SIR model that is firstly introduced by Kermack and McKendrick in 1927 [36], [37]. Both Kermack and McKendrick were public health physicians but they spent a lot of time studying mathematics in epidemiology. The model introduced by them is now extended in different ways for the deterministic systems and stochastic systems. In Kermack and McKendrick's model, a population is divided into three compartments: susceptible individuals abbreviated by the symbol  $S$ , the infected individuals abbreviated by the symbol  $I$ , the removed individuals abbreviated by the symbol  $R$ . Accordingly, some assumptions should be made to set up this model [26].

- The number of population is always a constant. New births or deaths from other reasons can be neglected.
- All the individuals can be susceptible individuals equally. Every infected individuals can transmit disease to susceptible individuals.
- Removed individuals cannot become infected individuals and transmit the disease.



Figure 2: *Flow diagram of SIR model*

Denote by  $S(t)$ ,  $I(t)$ , and  $R(t)$  are the number of susceptible individuals, the number of infected individuals, and the removed individuals at time  $t$  respectively. Suppose that  $\tau$  is the transmission rate and  $\nu$  is the removed rate of reported infected individuals per unit of time. Kermack and McKendrick's model can be written as follows

$$\begin{cases} S'(t) = -\tau S(t)I(t), \\ I'(t) = \tau S(t)I(t) - \nu I(t), t \geq 0 \\ R'(t) = \nu I(t) \end{cases} \quad (0.0.7)$$

with the initial  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ ,  $R(0) = R_0 > 0$ .

Considering (0.0.7) as a dynamical system, there are so many problems from the qualitative properties to computations. In reality, for studying an epidemic, we have

only the data given by a table containing the number of reported infected individuals. To understand the causes of an epidemic and estimate its size, we have to find an appropriate model that matches the data in some sense. It is naturally derived one of the most important problems in mathematical epidemiology: fitting data with the model.

Suppose that we have the weekly reported case data from an epidemic as the following table

Week ( $t$ )	Reported Cases
1	$N_1$
2	$N_2$
...	...
$k$	$N_k$

Table 2: Weekly reported case data for an epidemic from the week 1 to the week  $k$ .

To fit the data, we first determine the parameters  $\tau, \nu, S_0, I_0, R_0$  of (0.0.7) so that we can have the appropriate SIR model. There are some works trying to estimate these parameters by least square method, likelihood method,...such as [38], [48], [16], [28], [18], [29]. The disadvantage of these methods is that they cannot provide a general view of the parameters of (0.0.7). It means that neither we can find all the parameters of (0.0.7) nor they mention the identifiability of parameters of (0.0.7). In 2018, Pierre Magal and Glenn Webb [46] provided a new approach to estimate the parameters of (0.0.7) by using *the turning point*. In their work, they consider the unreported cases by adding the new parameter  $\nu_1$ , the removed rate of infected individuals due to the reported case data. The parameter  $\nu$  is considered as a removed rate of infected individuals of all cases including reported cases and unreported cases, then another new parameter appears  $\nu_2 = \nu - \nu_1$ , the removed rate of infected individuals due to unreported cases. The second column of the table of data above is now the values of the function  $\nu_1 I(t)$ , and by building the function

$CR(t) = \int_0^t \nu_1 I(s) ds$ , the cumulative number of reported cases at time  $t$ , Pierre Magal

and Glenn Webb provided an algorithm to compute the parameters numerically. However, the numerical computation in their works requires to know the values of  $S_0$  and  $I_0$ , and once again the identifiability of the parameters is neglected.

The identifiability of parameters of dynamical systems is an active field studied by many mathematicians, for instance [24], [25], [7], [43],[47],[49], [66], [67],[68], [69]. For understanding the important role of identifiability of a dynamical system to the parameter estimation we present here the definition for (0.0.7).

**Definition 0.0.4.** Consider the system (0.0.7) with the output function  $CR(t) = \int_0^t \nu_1 I(s) ds$ . The parameters  $\tau, \nu, \nu_1, S_0, I_0, R_0$  are supposed to be positive and let  $p = (\tau, \nu, \nu_1, S_0, I_0, R_0)$ ,  $\bar{p} = (\bar{\tau}, \bar{\nu}, \bar{\nu}_1, \bar{S}_0, \bar{I}_0, \bar{R}_0)$ . If we have  $p = \bar{p}$  from the condition  $CR(t, p) = CR(t, \bar{p})$  for every  $t \geq 0$ , the (0.0.7) is called to be identifiable from the

---

output function  $CR(t)$ .

From the weekly reported case data, the output function  $CR(t)$  is considered to be determined. If the system (0.0.7) is identifiable, then we can affirm that the parameters can be uniquely determined from the the output function  $CR(t)$ . Unfortunately, by the result that we present in Chapter 1, (0.0.7) is not identifiable . Moreover, we have the relationship

$$\nu = \bar{\nu}, \frac{\tau}{\nu_1} = \frac{\bar{\tau}}{\bar{\nu}_1}, \tau S_0 = \bar{\tau} \bar{S}_0, \tau I_0 = \bar{\tau} \bar{I}_0. \quad (0.0.8)$$

if  $CR(t, p) = CR(t, \bar{p})$  for every  $t \geq 0$ . It means that from the weekly reported case data, there are so many parameter vectors  $p$  giving the same output function  $CR(t)$ . This analysis let us see the important role of considering the identifiability before trying to estimate the parameters from the data. In Chapter 1, by the view point of [46] and considering the identifiability of (0.0.7), we construct a scheme that we find an appropriate model from the real data of an epidemic. This scheme is applied for some real epidemics happened in New York, Bombay, and France. The key of constructing this scheme is *the turning point equation*, the values of the turning point  $t_p$ ,  $CR(t_p)$ ,  $\lim_{t \rightarrow \infty} CR(t)$ , and  $CR'(t_p)$ . Certainly, these values can be taken from the weekly reported case data. After that we can determine all the values of the parameters  $p$  if we vary just only the value of  $S_0$ , and estimate the final size of the epidemic depending on  $S_0$ .

Since Kermack and McKendrick published their works on epidemic models, there is a large number of publications trying to extend this model for studying epidemiology. In 1978, after studying cholera epidemic spread in Bari, Capasso and Serio [17] generalized the model (0.0.7) as follows

$$\begin{cases} S'(t) = -S(t)g(I(t)), \\ I'(t) = S(t)g(I(t)) - \nu I(t), \\ R'(t) = \nu I(t) \end{cases} \quad (0.0.9)$$

with the same initial condition of (0.0.7). The function  $g : [0, +\infty) \rightarrow [0, +\infty)$  is assumed to satisfy the following conditions

- i.  $g(0) = 0$ .
- ii. There exists  $c > 0$  such that  $g(x) \leq c$  for all  $x \in [0, +\infty)$ .
- iii. The derivative of  $g$  exists and is bounded on any compact interval of  $[0, +\infty)$ , with  $g'(0) > 0$ .

The function  $g(x)$  above is considered to include the saturation phenomenon. Capasso and Serio [17] also mentioned  $g(x) = \frac{\tau x}{1 + \kappa x}$  as an example of their researches.

To study the dynamical behavior of epidemic model (0.0.9), many authors modified (0.0.9) in some ways. In 1986, Liu *et al.* [42] briefly discussed one of the modified models of (0.0.9) using the specific function

$$g(x) = \frac{\tau x^\alpha}{1 + \kappa x^\beta} \quad (0.0.10)$$


---

in which  $\alpha = \beta > 0$ .

Some other studies using the function (0.0.11) are Ruan *et al.* [60] with  $\alpha = \beta = 2$  and Xiao *et al.* [71] with  $\alpha = 1, \beta = 2$ .

In Chapter 3, we consider the system (0.0.9) discussed in Magal *et al.* [45] with the function (0.0.10) as follows

$$\begin{cases} S'(t) = -\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}, \\ I'(t) = \frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} - \nu I(t), \\ R'(t) = \nu I(t) \end{cases} \quad t \geq 0 \quad (0.0.11)$$

with the initial condition  $S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = R_0 \geq 0$ ,  $\alpha, \beta$  are given and  $\beta + 1 \geq \alpha \geq 1$ .

It is seen that the third equation of (0.0.11) is just a consequence of the first two equations of (0.0.11), then (0.0.11) can be read as follows

$$\begin{cases} S'(t) = -\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}, \\ I'(t) = \frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} - \nu I(t), \end{cases} \quad t \geq 0 \quad (0.0.12)$$

where  $\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}$  is called a nonlinear incidence rate,  $\nu$  is the removed rate of all infectious individuals of a disease. In reality, this rate cannot be known because many individuals remove the disease without report. We can only have the number of infectious individuals  $y(t) = \mu I(t)$  with  $0 < \mu < \nu$  reported from the public health officials (see [46], [23]). Suppose that  $\tau, \kappa, \nu, \mu, S_0, I_0$  are the unknown parameters of (0.0.12). One of the most important problems concerning the model (0.0.12) is the following

*Problem 0.0.5.* Consider the system (0.0.12), if we know the real reported case data as a function  $y(t) = \mu I(t)$ , is the parameter of (0.0.12) uniquely determined?

Consider the system (0.0.12), the number  $\mathcal{R}_0 = \frac{\tau I_0^{\alpha-1} S_0}{\nu(1 + \kappa I_0^\beta)}$  is called the basic reproduction number of (0.0.12) [45]. If the  $\mathcal{R}_0 < 1$ , then  $I(t)$  decreases to zero, and the epidemic subsides. From now on, we will only consider (0.0.12) when the epidemic outbreaks with  $\mathcal{R}_0 > 1$ . Another question is the following

*Problem 0.0.6.* Is the basic reproduction number of (0.0.12) uniquely determined from the function  $y(t) = \mu I(t)$  when the epidemic outbreaks?

The two problems 0.0.5, and 0.0.6 can be understood as a kind of the problem about the identifiability of (0.0.12). As we mentioned above, understanding the identifiability of a dynamical system is very important for parameter estimation from the real data. There are some approaches in testing the identifiability of a dynamical system such as Taylor series approach, similarity transformation approach, differential algebra approach, ... [4], [7], [8], [10], [23], [24], [25], [43], [47],[49], [52], [56], [61], [64], [66], [67], [68], [69], [70], [72]. In each approach, we can obtain

---

some advantages or disadvantages for some specific dynamical systems. To solve the problems 0.0.5, and 0.0.6, we choose the differential algebra approach. The advantage of this approach is that we can extract one equation not concerning any state of the system but only the output function that we know in reality. Combining other assumptions from the problem, we can obtain the relationship between two parameters when the dynamical system has the same output. Hence, we can test the identifiability of the system from the output and also obtain the relationship between parameters. This relationship helps us determine identifiable combinations of parameters so it is very important to estimate parameters. Chapter 2 of this thesis is providing a method for finding the identifiable combinations of parameters. In this chapter, we present some preliminaries in differential algebra and provide a scheme to find identifiable combinations of parameters, then we apply this scheme for SIR model. Chapter 3 aims to find the combinations of the parameters of (0.0.12) that can be uniquely determined from the function  $y(t) = \mu I(t)$  in which  $\mu$  is the removed rate of reported infected individuals. The first section of Chapter 3 provides the sufficient condition to guess the combinations of parameters that can be uniquely determined from the function  $y(t) = \mu I(t)$ . The next sections are the answers for the problems 0.0.5, 0.0.6 in some particular cases  $\{\alpha = 1, \beta = 2\}$ ,  $\{\alpha = 1, \beta = 1\}$ ,  $\{\alpha = 2, \beta = 2\}$ ,  $\{\alpha = 2, \beta = 1\}$ ,  $\{\alpha = 1, \beta = 3\}$ . These answers are performed by applying the differential algebra approach presented in sections 2.2 and 2.3 of Chapter 2.

# Chapter 1

## Identifying the number of unreported cases in SIR epidemic models

In this chapter, we consider the classical SIR epidemic model with its parameters and initial values, based upon reported case data from public health sources. The objective of this chapter is to provide a method for numerical computing the parameters of the SIR model from the real reported case data. The method here is based on the fact that the unreported cases are counted in the model. In many epidemic diseases, the reported cases are a small fraction of the unreported cases. We compute this fraction numerically by determining identifiable combinations of parameters of the model from reported case data. The numerical calculations in this chapter are applied to the Hong Kong seasonal influenza epidemic in New York City in 1968-1969, the influenza epidemic in France for the six consecutive years from 1988-1989 to 1993-1994, and some other epidemics.

### 1.1 Introduction

Mathematical models of epidemics have a long history [1, 6, 12, 13, 15, 21, 30, 32, 35, 51, 63]. One of the most important considerations of epidemic models is the identification of parameters needed for applications. The parameter identification problem for the SIR model has been investigated by many researchers, including [2, 3, 18, 19, 20, 22, 25, 27, 28, 29, 31, 33, 34, 40, 41, 44, 50, 55, 57, 58, 65]. Our objective here is to continue the investigation in [46] of the parameter identification problem for the standard SIR ordinary differential equations model of an outbreak epidemic:

$$\begin{cases} S'(t) = -\tau S(t)I(t), \\ I'(t) = \tau S(t)I(t) - \nu I(t). \end{cases} \quad (1.1.1)$$

Here  $S(t)$  and  $I(t)$  denote the number of susceptible and infected individuals, respectively, at time  $t > 0$ . The parameter  $\tau > 0$  corresponds to the disease transmission rate and the parameter  $\nu > 0$  corresponds to the removal rate of infected individuals.

The initial conditions of the model are

$$S(0) = S_0 > 0 \text{ and } I(0) = I_0 > 0. \quad (1.1.2)$$

For specific applications, the parameters  $\tau$ ,  $\nu$ , as well as the initial conditions,  $S_0$ ,  $I_0$ , are usually unknown. Our objective here is to determine these values from specific time data of reported infected cases.

Typically, the reported cases are only a small fraction of the total number of cases, since only the most severe symptomatic cases are reported. Our approach is based on knowledge of the data of newly reported cases (typically weekly) over the time course of the epidemic. This known data consists of the cumulative reported cases at time  $t$ , denoted by  $CR(t)$ , that correspond to the total number of reported infected cases up to time  $t$ . To handle these data we assume that the removal rate  $\nu$  takes the following form  $\nu = \nu_1 + \nu_2$ , where  $\nu_1$  is the removal rate of reported infected individuals, and  $\nu_2$  is the removal rate of infected individuals due to all other causes, such as mortality, recovery, or other reasons. With this assumption and notation, the cumulative reported cases are related to the number of infected by the following formula

$$CR(t) = \nu_1 \int_0^t I(s) ds, \quad (1.1.3)$$

where  $\nu_1 > 0$  is an unknown parameter. We formulate our problem as follows

*Problem 1.1.1.* How can we identify the parameter set  $\Theta = \{(\tau, \nu, \nu_1, S_0, I_0)\} \subset (0, \infty)^5$  if we know the cumulative reported cases  $CR(t)$  for all time  $t > 0$ ?

We will show that the knowledge of the cumulative reported cases  $CR(t)$  is not sufficient to recover the parameter set  $\Theta$ . Roughly speaking, this set is defined up to one degree of freedom. More precisely, under suitable hypotheses on the cumulative reported case data  $CR(t)$ ,  $t \geq 0$ , only the following combination of parameters and initial values can be reconstructed

$$\frac{I_0}{S_0}, S_0\tau, S_0\nu_1 \text{ and } \nu. \quad (1.1.4)$$

As a consequence, the knowledge about the value of  $S_0$ , the number of susceptible people in the population before the epidemic outbreak, allows us to obtain precise information about the values  $I_0$ ,  $\tau$ , and  $\nu_1$ . Then the *basic reproduction number* of the epidemic

$$\mathcal{R}_0 = \frac{S_0\tau}{\nu}, \quad (1.1.5)$$

can be obtained from (1.1.4). The interpretation of  $\mathcal{R}_0$  is that if  $\mathcal{R}_0 < 1$ , then the epidemic subsides, and if  $\mathcal{R}_0 > 1$ , then the epidemic outbreaks. We will describe a simple method to compute the parameter set in (1.1.4), and apply this method to specific epidemic data.

## 1.2 Identifiability of SIR model

In this section we show that the parameter set  $\Theta$  is not identifiable from the reported case data  $CR(t), t \geq 0$ . We refer to Evans *et al.* [25] for more results on this topic. Here we perform a simple proof for the SIR model.

Consider the parameter  $p := (\tau, \nu, \nu_1, S_0, I_0) \in (0, +\infty)^5$ . Define  $(S(t, p), I(t, p))$ , as the unique solution of (1.1.1)-(1.1.2), and  $CR(t, p)$  as the output function (1.1.3), for a given value of the parameter set  $p \in (0, +\infty)^5$ . We have the following proposition

**Proposition 1.2.1.** *Suppose that  $(S(t, p), I(t, p))$  and  $(S(t, \bar{p}), I(t, \bar{p}))$  are the two solutions of (1.1.1)-(1.1.2) for the parameter  $p = (\tau, \nu, \nu_1, S_0, I_0)$  and the parameter  $\bar{p} = (\bar{\tau}, \bar{\nu}, \bar{\nu}_1, \bar{S}_0, \bar{I}_0)$ , respectively. Then*

$$CR(t, p) = CR(t, \bar{p}) \quad (1.2.1)$$

for every  $t \geq 0$ , if and only if

$$\nu = \bar{\nu}, \frac{\tau}{\nu_1} = \frac{\bar{\tau}}{\bar{\nu}_1}, \tau S_0 = \bar{\tau} \bar{S}_0, \tau I_0 = \bar{\tau} \bar{I}_0. \quad (1.2.2)$$

*Proof.* ( $\Rightarrow$ ) Assume first that (1.2.1) holds and differentiate both sides for  $t$  to obtain

$$CR'(t, p) = CR'(t, \bar{p}), \forall t \geq 0,$$

which is equivalent to

$$\nu_1 I(t, p) = \bar{\nu}_1 I(t, \bar{p}), \forall t \geq 0. \quad (1.2.3)$$

Differentiate both sides of (1.2.3) with respect to  $t$ , to obtain

$$\nu_1 I'(t, p) = \bar{\nu}_1 I'(t, \bar{p}), \forall t \geq 0.$$

Replacing  $I'(t)$  by its formula in (1.1.1) on both sides of the above equality, we obtain

$$\begin{aligned} \nu_1 I'(t, p) &= \bar{\nu}_1 I'(t, \bar{p}) \\ \Leftrightarrow \nu_1 \left( \tau S(t, p) I(t, p) - \nu I(t, p) \right) &= \bar{\nu}_1 \left( \bar{\tau} S(t, \bar{p}) I(t, \bar{p}) - \bar{\nu} I(t, \bar{p}) \right) \\ \Leftrightarrow \nu_1 I(t, p) \left( \tau S(t, p) - \nu \right) &= \bar{\nu}_1 I(t, \bar{p}) \left( \bar{\tau} S(t, \bar{p}) - \bar{\nu} \right), \end{aligned}$$

and (1.2.3) implies that

$$\tau S(t, p) - \nu = \bar{\tau} S(t, \bar{p}) - \bar{\nu}, \forall t \geq 0. \quad (1.2.4)$$

Differentiating both side of (1.2.4) with respect to  $t$ , we obtain

$$\begin{aligned} \tau S'(t, p) &= \bar{\tau} S'(t, \bar{p}) \\ \Leftrightarrow \tau \left( -\tau S(t, p) I(t, p) \right) &= \bar{\tau} \left( -\bar{\tau} S(t, \bar{p}) I(t, \bar{p}) \right) \\ \Leftrightarrow \tau^2 S(t, p) I(t, p) &= \bar{\tau}^2 S(t, \bar{p}) I(t, \bar{p}) \end{aligned}$$



and by using again (1.2.3), we obtain

$$\frac{\bar{\nu}_1}{\nu_1} \tau^2 S(t, p) = \bar{\tau}^2 S(t, \bar{p}), \forall t \geq 0. \quad (1.2.5)$$

Then, by using (1.2.4), and replacing  $S(t, \bar{p})$  in (2.4.1), we obtain

$$\frac{\bar{\nu}_1}{\nu_1} \tau^2 S(t, p) = \bar{\tau}^2 \left( \frac{\bar{\nu} - \nu + \tau S(t, p)}{\bar{\tau}} \right).$$

Therefore,

$$\left( \frac{\bar{\nu}_1}{\nu_1} \tau - \bar{\tau} \right) S(t, p) = \frac{\bar{\tau}}{\tau} (\bar{\nu} - \nu), \forall t \geq 0. \quad (1.2.6)$$

Differentiating both sides of equation (1.2.3) with respect to  $t$ , we obtain

$$\left( \frac{\bar{\nu}_1}{\nu_1} \tau - \bar{\tau} \right) S'(t, p) = 0,$$

which implies that

$$\left( \frac{\bar{\nu}_1}{\nu_1} \tau - \bar{\tau} \right) (-\tau S(t, p) I(t, p)) = 0, \forall t \geq 0. \quad (1.2.7)$$

Setting  $t = 0$  in equations (1.2.3), (1.2.4), (1.2.6) and (1.2.7) we obtain the following system of equations

$$\begin{cases} I_0 \nu_1 = \bar{I}_0 \bar{\nu}_1 \\ \tau S_0 - \nu = \bar{\tau} \bar{S}_0 - \bar{\nu} \\ \left( \frac{\bar{\nu}_1}{\nu_1} \tau - \bar{\tau} \right) S_0 = \frac{\bar{\tau}}{\tau} (\bar{\nu} - \nu) \\ -\tau S_0 I_0 \left( \frac{\bar{\nu}_1}{\nu_1} \tau - \bar{\tau} \right) = 0 \end{cases} \quad (1.2.8)$$

and (1.2.2) follows.

( $\Leftarrow$ ) To prove the converse implication, let  $(S(t), I(t))$  be a solution of (1.1.1)-(1.1.2). Let  $\bar{S}_0 > 0$ ,  $\bar{I}_0 > 0$ , and set

$$\bar{S}(t) := \frac{\bar{S}_0}{S_0} S(t) \text{ and } \bar{I}(t) := \frac{\bar{I}_0}{I_0} I(t).$$

Since  $(S(t), I(t))$  satisfies (1.1.1)-(1.1.2), we obtain by replacing  $S(t)$  and  $I(t)$  with the above formulas,

$$\begin{cases} \frac{S_0}{\bar{S}_0} \bar{S}'(t) = -\tau \frac{S_0}{\bar{S}_0} \bar{S}(t) \frac{I_0}{\bar{I}_0} \bar{I}(t) \\ \frac{I_0}{\bar{I}_0} \bar{I}'(t) = \tau \frac{S_0}{\bar{S}_0} \bar{S}(t) \frac{I_0}{\bar{I}_0} \bar{I}(t) - \nu \frac{I_0}{\bar{I}_0} \bar{I}(t) \\ \bar{S}(0) = \bar{S}_0 \\ \bar{I}(0) = \bar{I}_0. \end{cases}$$

After simplifying,

$$\begin{cases} \bar{S}'(t) = -\tau \frac{I_0}{\bar{I}_0} \bar{S}(t) \bar{I}(t) \\ \bar{I}'(t) = \tau \frac{S_0}{\bar{S}_0} \bar{S}(t) \bar{I}(t) - \nu \bar{I}(t) \\ \bar{S}(0) = \bar{S}_0 \\ \bar{I}(0) = \bar{I}_0 \end{cases} \quad (1.2.9)$$

and by using (1.2.2) we deduce that  $\tau \frac{I_0}{\bar{I}_0} = \tau \frac{S_0}{\bar{S}_0} = \bar{\tau}$  and  $\nu = \bar{\nu}$ . Therefore,  $(\bar{S}(t), \bar{I}(t))$  satisfies (1.1.1)-(1.1.2) with the new parameter set  $\bar{p}$ . By the uniqueness of the solution of (1.1.1)-(1.1.2) with the parameter set  $\bar{p}$ ,  $(\bar{S}(t), \bar{I}(t)) = (S(t, \bar{p}), I(t, \bar{p}))$ . It means that

$$CR(t, \bar{p}) := \bar{\nu}_1 \int_0^t I(s, \bar{p}) ds = \bar{\nu}_1 \int_0^t \bar{I}(s) ds = \bar{\nu}_1 \frac{\bar{I}_0}{I_0} \int_0^t I(s) ds, \forall t \geq 0,$$

and by using (1.2.2) we deduce that  $\frac{\tau}{\nu_1} = \frac{\bar{\tau}}{\bar{\nu}_1}$ ,  $\tau I_0 = \bar{\tau} \bar{I}_0$ . Therefore,

$$CR(t, \bar{p}) = CR(t, p), \forall t \geq 0.$$

□

*Remark 1.2.2.* Suppose that  $p = (\tau, \nu, \nu_1, S_0, I_0) \in (0, +\infty)^5$  is parameter of the system (1.1.1)-(1.1.3), then the cumulative reported cases function  $CR(t)$  in (1.1.3) retains the same value if we replace  $p$  by a new parameter as follows

$$q = \left( \frac{\tau}{a}, \nu, \frac{\nu_1}{a}, aS_0, aI_0 \right)$$

for every  $a > 0$ .

## 1.3 Computation the combinations of the parameters of SIR model

### 1.3.1 System of equations to identify the parameters

In this section, we consider the SIR model (1.1.1)-(1.1.2) when the epidemic outbreaks with the basic reproduction number  $\mathcal{R}_0 = \tau S_0 / \nu > 1$ . The cumulative reported case function  $CR(t) := \nu_1 \int_0^t I(s) ds$ ,  $t \geq 0$  is assumed to be known. We aim to provide a simple method to identify the parameters (1.1.4).

Recall that, since  $\mathcal{R}_0 > 1$ , the solutions of (1.1.1)-(1.1.2) have a typical outbreak behavior [46] as follows

- (i) The function  $t \mapsto S(t)$  is not increasing on  $[0, \infty)$  with  $S(0) = S_0$  and  $S(\infty) > 0$ .

- (ii) There exists a unique turning point  $t_p > 0$  such that  $I'(t_p) = 0$ , and  $I(t)$  is not decreasing on  $[0, t_p]$  and not increasing on  $[t_p, \infty)$ . Moreover  $I(\infty) = 0$  and  $t \mapsto I(t)$  is integrable on  $[0, \infty)$ .

In addition to the turning point  $t_p$ , the above properties allow us to define several important quantities related to the function  $CR$ :

$$CR(t_p), CR'(t_p) \text{ and } CR(\infty). \quad (1.3.1)$$

As it will be seen later, these quantities will be sufficient to compute the combinations of parameters in (1.1.4). To compute these four combined parameters we will provide four independent equations. Three of them is derived in [46]. Following the notations introduced in [46], we set

$$c := CR(\infty), r := \frac{CR(t_p)}{CR(\infty)}.$$

Next, by setting

$$X := c \frac{\tau}{\nu_1}, \quad (1.3.2)$$

and then by multiplying both sides by  $S_0\nu_1$  we deduce that

$$X \times (S_0\nu_1) = cS_0\tau. \quad (1.3.3)$$

Moreover, by using respectively, equations (3.3), (3.7) and (3.9) in [46], we derive the three following independent equations

$$e^{-X} + Xe^{-rX} = 1 + \frac{I_0}{S_0}, \quad (1.3.4)$$

$$(S_0\nu_1) \times \left[ 1 + \frac{I_0}{S_0} - e^{-rX} (1 + rX) \right] = CR'(t_p) \quad (1.3.5)$$

and

$$\nu = (S_0\tau) \times e^{-rX}. \quad (1.3.6)$$

We recall first, that by Proposition 3.1 in [46], equation (1.3.4) implies the following compatibility condition with the data:  $r \in (0, 1/2)$ . Thus, for the model (1.1.1)-(1.1.2) more than half of all cases occur after the turning point  $t_p$ . As noted in [46], some outbreak epidemics have more than half of all cases occurring before the turning point, and the model (1.1.1)-(1.1.2) does not apply these examples.

### 1.3.2 Derivation of the equation for the turning point

To define an equation for the turning point, we first introduce the function

$$F(X) := e^{-X} + Xe^{-rX} - 1.$$

**Lemma 1.3.1.** *Suppose that  $1 > r \geq \frac{1}{2} \Rightarrow F(X) < 0, \forall X > 0$ .*

*Proof.* We have  $F(0) = 0$  and  $F'(X) = e^{-rX}G(x)$ , with

$$G(X) := 1 - rX - e^{-(1-r)X}.$$

Then  $G(0) = 0$  and  $G'(X) = -r + (1-r)e^{-(1-r)X}$ . So if  $1 > r \geq 1/2$  we have

$$G'(X) < 0, \forall X > 0.$$

The result follows. □

**Lemma 1.3.2.** *Assume that  $r \in (0, \frac{1}{2})$ . There exists a unique strictly positive solution  $X(r) > 0$  of equation*

$$F(X) = 0 \Leftrightarrow e^{-X} + Xe^{-rX} - 1 = 0. \quad (1.3.7)$$

Moreover, there exists  $X_{\max} \in (0, X(r))$ , such that the function  $F(X)$  is strictly increasing on  $(0, X_{\max})$  and strictly decreasing on  $(X_{\max}, X(r))$ .

Furthermore,

$$\begin{cases} F(X) > 0, & \text{if } X \in (0, X(r)), \\ F(X) < 0, & \text{if } X \in (X(r), \infty). \end{cases} \quad (1.3.8)$$

*Proof.* We have  $F(0) = 0$  and  $F'(X) = e^{-rX}G(x)$ , with

$$G(X) := 1 - rX - e^{-(1-r)X}.$$

Then  $G(0) = 0$  and  $G'(X) = -r + (1-r)e^{-(1-r)X}$ . Moreover, we have

$$G'(X) = 0 \Leftrightarrow X = \frac{1}{1-r} \ln \left( \frac{1}{r} - 1 \right) := X^* > 0.$$

Thus,  $G'(X) > 0$  for  $X \in (0, X^*)$  and  $G'(X) < 0$  for  $X > X^*$ . We also have

$$\lim_{X \rightarrow \infty} G(X) = -\infty.$$

Let  $X_{\max} > X^*$  be the unique value in  $(0, +\infty)$  such that  $G(X_{\max}) = 0$ . Moreover,  $F'(X) > 0$  on  $(0, X_{\max})$ ,  $F'(X) < 0$  on  $(X_{\max}, \infty)$ , and  $F'(0) = F'(X_{\max}) = 0$ . Hence,  $F(X_{\max}) > 0$  is the maximum of  $F$ . Next, we observe that

$$\lim_{X \rightarrow \infty} F(X) = -1$$

Therefore there exists a unique  $X(r) \in (X_{\max}, \infty)$  such that  $F(X(r)) = 0$ . □

The above results are summarized in Figure 1.1.

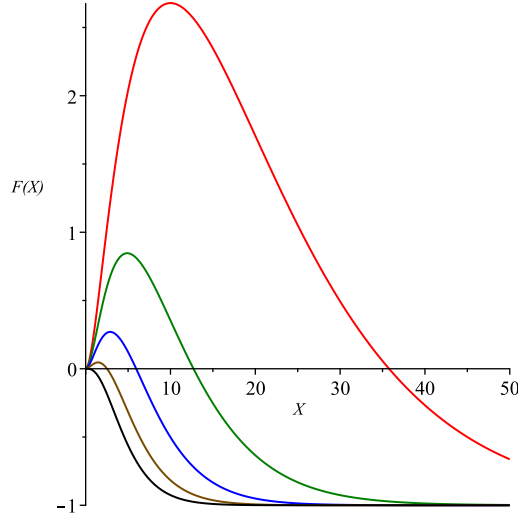


Figure 1.1: The function  $X \rightarrow F(X)$  when  $r = 0.1, 0.2, 0.3, 0.4$  and  $0.49$  respectively in red, green, blue, orange, black.

Next, we derive an additional independent equation involving the turning point  $t_p$ . To that aim, recall that

$$CR'(t) = \nu_1 I(t), \quad \forall t > 0 \text{ and } CR(0) = 0.$$

As a consequence, we obtain from (1.1.1)-(1.1.2),

$$\frac{d}{dt} \left( S(t) + I(t) + \frac{\nu}{\nu_1} CR(t) \right) = 0, \quad \forall t > 0,$$

so that

$$S(t) + I(t) + \frac{\nu}{\nu_1} CR(t) = S_0 + I_0, \quad \forall t \geq 0.$$

Hence, we obtain

$$CR'(t) = \nu_1 I(t) = \nu_1 \left[ S_0 + I_0 - S(t) - \frac{\nu}{\nu_1} CR(t) \right], \quad \forall t \geq 0.$$

However, by using (1.1.1), we may eliminate  $S(t)$ , since  $S(t) = S_0 e^{-\frac{\tau}{\nu_1} CR(t)}$ . As a consequence,  $CR(t)$  satisfies the equation

$$CR'(t) = \nu_1 I(t) = S_0 \nu_1 \left[ 1 + \frac{I_0}{S_0} - e^{-\frac{\tau}{\nu_1} CR(t)} - \frac{\nu}{S_0 \nu_1} CR(t) \right], \quad \forall t \geq 0.$$

Now note that since  $CR'(t) = \nu_1 I(t) > 0$ , the function  $t \rightarrow CR(t)$  must be increasing on  $(0, \infty)$ . We thus have

$$1 + \frac{I_0}{S_0} - e^{-\frac{\tau}{\nu_1} y} - \frac{\nu}{S_0 \nu_1} y > 0, \quad \forall y \in [0, CR(\infty)).$$

*Remark 1.3.3.* The above inequality gives a condition on the final number of cumulative reported cases  $c = CR(\infty)$ .

Integration of the differential equation for  $CR(t)$  above, from  $t = 0$  to  $t = t_p$ , yields

$$\int_0^{t_p} \frac{CR'(t)}{1 + \frac{I_0}{S_0} - e^{-\frac{\tau}{\nu_1} CR(t)} - \frac{\nu}{S_0 \nu_1} CR(t)} dt = (S_0 \nu_1) t_p.$$

Set  $s = CR(t)$  and we obtain

$$\int_0^{CR(t_p)} \frac{1}{1 + \frac{I_0}{S_0} - e^{-\frac{\tau}{\nu_1} s} - \frac{\nu}{S_0 \nu_1} s} ds = (S_0 \nu_1) t_p.$$

Now recalling that  $CR(t_p) = rc$ , the change of variable  $s = c\sigma$  yields

$$\int_0^r \frac{1}{1 + \frac{I_0}{S_0} - e^{-\frac{c\tau}{\nu_1} \sigma} - \frac{c\nu}{S_0 \nu_1} \sigma} d\sigma = (S_0 \nu_1) \frac{t_p}{c}.$$

By (1.3.2), that is  $X = \frac{c\tau}{\nu_1}$ , we deduce that

$$\int_0^r \frac{1}{1 + \frac{I_0}{S_0} - e^{-X\sigma} - \frac{\nu}{\tau S_0} X\sigma} d\sigma = (S_0 \nu_1) \frac{t_p}{c}.$$

By (1.3.6), that is  $\nu = S_0 \tau e^{-rX}$ , we have

$$\int_0^r \frac{d\sigma}{1 + \frac{I_0}{S_0} - e^{-X\sigma} - X e^{-rX} \sigma} = (S_0 \nu_1) \frac{t_p}{c}.$$

By (1.3.5), that is  $S_0 \nu_1 \left[ 1 + \frac{I_0}{S_0} - e^{-rX} (1 + rX) \right] = CR'(t_p)$ , we obtain

$$\left[ 1 + \frac{I_0}{S_0} - e^{-rX} (1 + rX) \right] \times \int_0^r \frac{d\sigma}{1 + \frac{I_0}{S_0} - e^{-X\sigma} - X e^{-rX} \sigma} = CR'(t_p) \frac{t_p}{c}.$$

Finally, by (1.3.4), that is  $e^{-X} + X e^{-rX} = 1 + \frac{I_0}{S_0}$  and Lemma 1.3.2, we deduce that

$$0 < X < X(r) \tag{1.3.9}$$

and that  $X$  must satisfy the *turning point equation*

$$T(X, r) = \frac{CR'(t_p) t_p}{c}, \tag{1.3.10}$$

where the mapping  $T$  is defined by

$$T(X, r) := \int_0^r \frac{H(X, r)}{H(X, \sigma)} d\sigma, \tag{1.3.11}$$

where the right hand side is an improper integral and the function  $H$  is defined by

$$H(X, \sigma) := e^{-X} + X e^{-rX} - e^{-X\sigma} - X e^{-rX} \sigma. \tag{1.3.12}$$

*Remark 1.3.4.* We observe that  $H(X, 0) = e^{-X} + X e^{-rX} - 1 = F(X)$ .

### 1.3.3 Analysis of the turning point equation

**Lemma 1.3.5.** *If  $r \in (0, \frac{1}{2})$ , and  $X \in (0, X(r))$ , then*

$$0 < F(X) = H(X, 0) \leq H(X, \sigma) \leq H(X, r)$$

for every  $\sigma \in [0, r]$ .

*Proof.* Suppose that  $r \in (0, \frac{1}{2})$ ,  $X \in (0, X(r))$ , and consider

$$\partial_\sigma H(X, \sigma) = X(e^{-\sigma X} - e^{-rX}) \geq 0, \forall \sigma \in [0, r],$$

which means that  $H(X, \sigma)$  is increasing with respect to  $\sigma$ . □

**Lemma 1.3.6.** *The function  $X \rightarrow T(X, r)$  is well-defined on the open interval  $(0, X(r))$ . Moreover,*

$$\lim_{X \rightarrow X(r)^-} T(X, r) = \infty, \quad (1.3.13)$$

and

$$\lim_{X \rightarrow 0^+} T(X, r) = \frac{r-1}{2} \ln(1-2r). \quad (1.3.14)$$

*Proof. Proof of (1.3.13):* By Lemma 1.3.5,  $T(X, r)$  is well-defined on the interval  $(0, X(r))$ . Since  $X(r)$  is the unique positive solution of the equation (1.3.7), we have  $H(X(r), \sigma) = 1 - e^{-\sigma X(r)} - \sigma(1 - e^{-\sigma X(r)})$ . Let  $k(x, \sigma) = 1 - e^{-\sigma x} - \sigma(1 - e^{-x})$  on  $[0, \infty)$ . Then,  $\partial_x k(x, \sigma) = \sigma(e^{-\sigma x} - e^{-x}) > 0$  for every  $x \in [0, \infty)$ , and  $\sigma \in (0, \frac{1}{2})$ . This means that  $k(x, \sigma) > k(0, \sigma) = 0$  for every  $x \in (0, \infty)$ , and  $\sigma \in (0, \frac{1}{2})$ .

Therefore,  $H(X(r), \sigma) = 1 - e^{-\sigma X(r)} - \sigma(1 - e^{-\sigma X(r)}) > 0$  for every  $\sigma \in (0, \frac{1}{2})$ .

Moreover,

$$\begin{aligned} \lim_{\sigma \rightarrow 0^+} \frac{H(X(r), \sigma)}{\sigma} &= \lim_{\sigma \rightarrow 0^+} \frac{1 - e^{-\sigma X(r)} - rX(r)e^{-rX(r)}\sigma}{\sigma} \\ &= \lim_{\sigma \rightarrow 0^+} \frac{1 - e^{-\sigma X(r)}}{\sigma} - rX(r)e^{-rX(r)} \\ &= X(r) - X(r)e^{-rX(r)} = X(r)(1 - e^{-rX(r)}) > 0. \end{aligned}$$

This means that  $\int_0^r \lim_{X \rightarrow X(r)^-} \frac{H(X, r)}{H(X, \sigma)} d\sigma = \int_0^r \frac{H(X(r), r)}{H(X(r), \sigma)} d\sigma = \infty$ , and by Fatou's

Lemma, we have  $\lim_{X \rightarrow X(r)^-} T(X, r) = \infty$ .

*Proof of (1.3.14):* Next, taking the Taylor's expansions of the functions  $e^{-X}$ ,  $e^{-rX}$ ,  $e^{-hX}$  at  $X = 0$ , and  $\sigma \in [0, r] \subset (0, \frac{1}{2})$ , we obtain

$$\begin{aligned} e^{-\sigma X} &= 1 - \sigma X + \frac{1}{2}\sigma^2 X^2 + o((\sigma X)^3) = 1 - \sigma X + \frac{1}{2}\sigma^2 X^2 + o(X^3), \\ e^{-rX} &= 1 - rX + \frac{1}{2}r^2 X^2 + o(X^3), \text{ and } e^{-X} = 1 - X + \frac{1}{2}X^2 + o(X^3), \end{aligned}$$

where  $o(X^3)$  does not depend on  $\sigma$ . Thus,

$$H(X, \sigma) = \frac{1 - 2r - \sigma^2 + 2r\sigma}{2} X^2 + o(X^3), \quad H(X, r) = \frac{(1 - r)^2}{2} X^2 + o(X^3).$$

Hence,

$$\frac{H(X, r)}{H(X, \sigma)} = \frac{\frac{(1 - r)^2}{2} X^2 + o(X^3)}{\frac{1 - 2r - \sigma^2 + 2r\sigma}{2} X^2 + o(X^3)} = \frac{(1 - r)^2 + o(X)}{1 - 2r - \sigma^2 + 2r\sigma + o(X)}.$$

Since  $o(X)$  does not depend on  $\sigma$ , when  $X$  tends to  $0^+$  the function  $\frac{H(X, r)}{H(X, \sigma)}$  is uniformly convergent to  $h(\sigma) = \frac{(1 - r)^2}{1 - 2r - \sigma^2 + 2r\sigma}$  on  $[0, r]$ . Thus,

$$\begin{aligned} \lim_{X \rightarrow 0^+} T(X, r) &= \int_0^r \frac{(1 - r)^2}{1 - 2r - \sigma^2 + 2r\sigma} d\sigma \\ &= (1 - r)^2 \int_0^r \frac{1}{(2r - 1 - \sigma)(\sigma - 1)} d\sigma = \frac{(1 - r)}{2} \ln\left(\frac{1}{1 - 2r}\right). \end{aligned}$$

□

In Figure 1.2 we plot the mapping  $r \rightarrow X(r)$ , where  $X = X(r)$  is the solution of (1.3.7), as  $r$  varies in  $(0, \frac{1}{2})$ . In Figure 1.3 we plot the mapping  $x \rightarrow T(xX(r), r)$ , where  $T$  is defined by (1.3.11), as  $x$  varies in  $(0, 1)$ , for different values of  $r$ . From Figure 1.3 we observe that numerically, the mappings  $X \rightarrow T(X, r)$  are all monotone increasing for each value of  $r$ . As a consequence we can conclude (numerically) that equation (1.3.10) has a unique solution  $X \in (0, X(r))$ .

*Remark 1.3.7.* From Figure 1.3 we can also visualize the minimum value for  $T(X, r)$ , which is given by (1.3.14). By using (1.3.10) we deduce that we must have

$$\frac{CR'(t_p)t_p}{CR(\infty)} > \frac{r - 1}{2} \ln(1 - 2r).$$

where  $r = \frac{CR(t_p)}{CR(\infty)}$ . Therefore, we obtain a new relationship between the values  $t_p, CR(t_p), CR'(t_p), CR(\infty)$ .



1.3. Computation the combinations of the parameters of SIR model

---

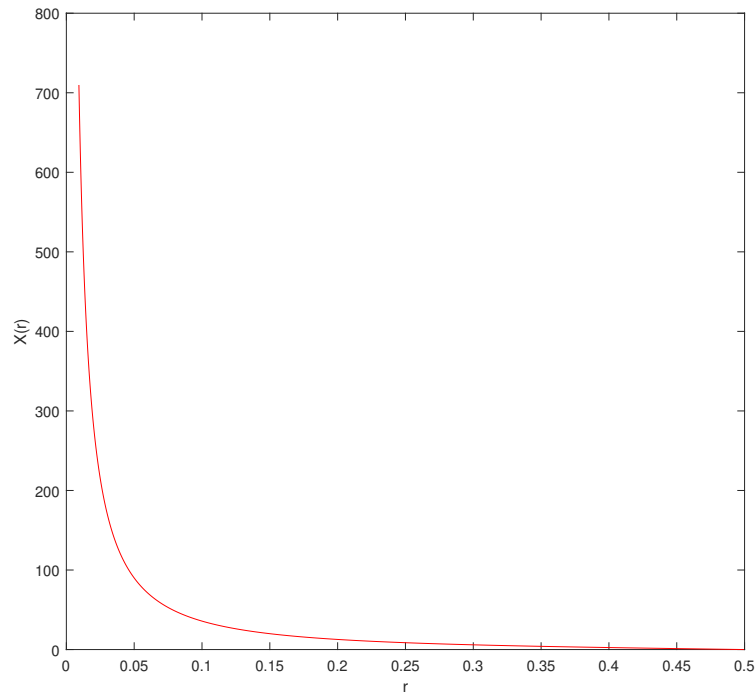


Figure 1.2: *The mapping  $r \rightarrow X(r)$  where  $r$  varies in  $(0, \frac{1}{2})$ .*

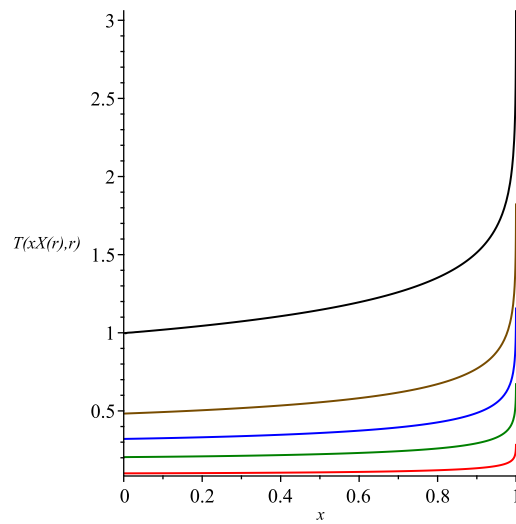


Figure 1.3: *The mapping  $x \rightarrow T(xX(r), r)$  where  $x$  varies in  $(0, 1)$ . The red, green, blue, orange, black curves correspond (from the bottom to the top) to  $r = 0.1$ ,  $r = 0.2$ ,  $r = 0.3$ ,  $r = 0.4$  and  $r = 0.49$ , respectively.*

As a consequence of Proposition 1.2.1, we have the following theorem.

**Theorem 1.3.8.** *Assume that the equation (1.3.10) has a unique solution  $X$  in  $(0, X(r))$ . Then the function  $t \rightarrow CR(t, p)$  is uniquely determined by the turning point  $t_p$ ,  $CR(t_p)$ ,  $CR'(t_p)$  and  $CR(\infty)$  in which  $p = (\tau, \nu, \nu_1, S_0, I_0)$  is the arbitrary parameter of (1.1.1)-(1.1.3).*

*Remark 1.3.9.* Assume that the equation (1.3.10) has a unique solution  $X$  in  $(0, X(r))$ . Then the basic reproduction number  $\mathcal{R}_0$  is uniquely determined by the turning point  $t_p$ ,  $CR(t_p)$ ,  $CR'(t_p)$  and  $CR(\infty)$ .

Note that, if the function  $T(X, r)$  is strictly decreasing on  $(0, X(r))$  then the equation (1.3.10) has a unique solution on  $(0, X(r))$ . About the monotony of the function  $T(X, r)$  on  $(0, X(r))$ , we have the following theorem

**Theorem 1.3.10.** *Suppose that  $r \in (0, \frac{1}{2})$ , and  $X(r)$  is the unique positive solution of the equation  $e^{-X} + Xe^{-rX} - 1 = 0$ , then there exists  $\frac{1}{2} > \alpha > 0$  such that for every  $r \in (\frac{1}{2} - \alpha, \frac{1}{2})$  the function  $T(X, r) = \int_0^r \frac{H(X, r)}{H(X, \sigma)} d\sigma$  is strictly increasing on  $(0, X(r))$ .*

Firstly, the derivative of  $T(X, r)$  is computed as follows

$$T'_X(X, r) = \int_0^r \frac{H'_X(X, r)H(X, \sigma) - H'_X(X, \sigma)H(X, r)}{H^2(X, \sigma)} d\sigma.$$

It is seen that  $T'_X(X, r)$  is a complicated expression, then we hope we can prove that  $H'_X(X, r)H(X, \sigma) - H'_X(X, \sigma)H(X, r) > 0$  for every  $X \in (0, X(r))$ ,  $\sigma \in (0, r)$  for concluding the monotony of  $T(X, r)$ . However, taking some computations by Maple, the result is not as our expectation. For instance, if  $r = \frac{25}{54}$ , and  $\sigma = 0.01$ , there is an interval so that  $T'_X(X, r)$  is negative. It is the reason why the study of the monotony of  $T(X, r)$  becomes difficult.

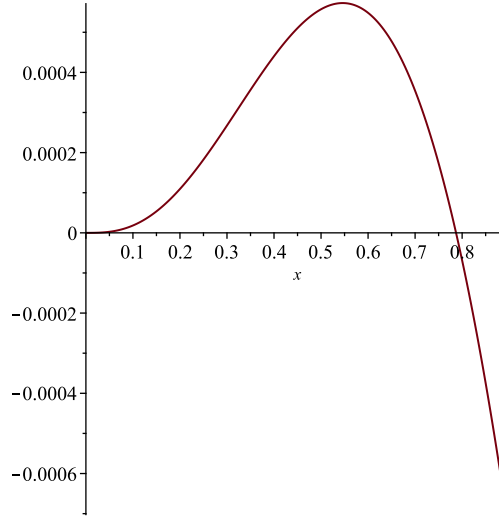


Figure 1.4: The graph of  $H'_X(X, r)H(X, \sigma) - H'_X(X, \sigma)H(X, r)$  where  $r = \frac{25}{54}$ , and  $\sigma = 0.01$  and  $X$  varies in  $(0, X(r))$ .

Now we study the monotony of  $T(X, r)$  when  $r$  is closed enough to  $\frac{1}{2}$ . Suppose that  $X(r)$  is the unique positive solution of the equation  $F(X) = e^{-X} + Xe^{-rX} - 1 = 0$  as Lemma 1.3.2, it is seen that  $X(r)$  is a positive function of the variable  $r$  on  $(0, \frac{1}{2})$ . We have the following lemma

**Lemma 1.3.11.**  $\lim_{r \rightarrow \frac{1}{2}^-} X(r) = 0$ .

*Proof.* Let  $f(X) = \frac{1}{X} \ln\left(\frac{Xe^X}{e^X - 1}\right)$ . It is seen that

$$X^2(e^X - 1)f'(X) = (1 - e^X)\left(\ln\left(\frac{X}{e^X - 1}\right) + x\right) + e^X - 1 - X.$$

We have  $1 - e^X = -X - \frac{X^2}{2} - \frac{X^3}{6} + o(X^4)$  and

$$\ln\left(\frac{X}{e^X - 1}\right) = -\frac{X}{2} - \frac{X^2}{24} + o(X^4).$$

Therefore

$$\begin{aligned} X^2(e^X - 1)f'(X) &= \left(-X - \frac{X^2}{2} - \frac{X^3}{6} + o(X^4)\right)\left(-\frac{X}{2} - \frac{X^2}{24} + o(X^4)\right) \\ &\quad + \left(\frac{X^2}{2} + \frac{X^3}{6} + o(X^4)\right) \\ &= -\frac{X^3}{24} + o(X^4) \end{aligned}$$

It means that  $\frac{e^X - 1}{X}f'(X) = -\frac{1}{24} + o(X)$  for  $X > 0$ . Since  $\lim_{X \rightarrow 0^+} o(X) = 0$ , there exists  $\eta > 0$  such that  $|o(X)| < \frac{1}{24}$  for every  $X \in (0, \eta)$ . It implies that  $X^2(e^X - 1)f'(X) < 0$  for every  $X \in (0, \eta)$ . Then  $f'(X) < 0$  for every  $X \in (0, \eta)$ , so  $f$  is strictly decreasing on  $X \in (0, \eta)$ .

It is clear that  $e^{-X} + Xe^{-rX} - 1 = 0 \Leftrightarrow f(X) = r$ . So  $X(r)$  is the unique positive solution of the equation  $f(X) = r$ . Since  $f$  is strictly decreasing on  $X \in (0, \eta)$ ,  $f$  is continuous and  $\lim_{r \rightarrow \frac{1}{2}^-} X(r) = 0$  then  $f$  is a bijection from  $(0, \eta)$  to  $(f(\eta), \frac{1}{2})$ . It can be rewritten  $X(r) = f^{-1}(r)$ , and  $X(r) = f^{-1}(r)$  is strictly decreasing on  $(f(\eta), \frac{1}{2})$ .

Moreover,  $X(r) > 0$  for every  $r \in (f(\eta), \frac{1}{2})$ , then there exists the limit  $\lim_{r \rightarrow \frac{1}{2}^-} X(r) = \tilde{X} \geq 0$ . Since  $X(r)$  is the unique positive solution of  $e^{-X} + Xe^{-rX} - 1 = 0$ , we have  $\lim_{r \rightarrow \frac{1}{2}^-} e^{-X(r)} + X(r)e^{-rX(r)} - 1 = 0$ . It implies that  $e^{-\tilde{X}} + \tilde{X}e^{-\frac{1}{2}\tilde{X}} - 1 = 0$ .

Let  $p(X) = e^{-X} + Xe^{-\frac{1}{2}X} - 1$  for every  $X \in [0, \eta]$ . We have  $p'(X) = -e^{-X} + e^{-\frac{1}{2}X} - \frac{X}{2}e^{-\frac{1}{2}X} = e^{-\frac{1}{2}X}(1 - \frac{X}{2} - e^{-\frac{1}{2}X})$ . Define  $q(X) = 1 - \frac{X}{2} - e^{-\frac{1}{2}X}$ , then  $q'(X) = \frac{1}{2}(e^{-\frac{X}{2}} - 1) < 0$  for every  $X \in [0, \eta]$ . It means that  $q(X)$  is decreasing on  $[0, \eta]$ , so  $q(X) < 0$  for every  $X \in (0, \eta)$ . It follows that  $p'(X) < 0$ , therefore  $p$  is strictly decreasing on  $(0, \eta)$ . Moreover  $p(X) < 0$  for every  $X \in (0, \eta)$ , and  $p(0) = 0$ . It means that  $X = 0$  is the unique solution of the equation  $p(X) = 0$ , so  $\tilde{X} = 0$ , and  $\lim_{r \rightarrow \frac{1}{2}^-} X(r) = 0$ .

□

To conclude the monotony of  $T(X, r)$ . We need the following lemmas

**Lemma 1.3.12.** *Taylor's expansion of  $N(X) = H'_X(X, r)H(X, \sigma) - H'_X(X, \sigma)H(X, r)$  at  $X = 0$  is  $\frac{(1 - \sigma)(\sigma - r)^2(1 - r)^2}{12}X^4 + o(X^5)$ .*

*Proof.* We have

$$\begin{aligned} H(X, \sigma) &= e^{-X} + Xe^{-rX} - e^{-X\sigma} - Xe^{-rX}\sigma \\ \text{and } H(X, r) &= e^{-X} + Xe^{-rX} - e^{-rX} - Xe^{-rX}r. \\ \text{So } H'_X(X, \sigma) &= -e^{-X} + e^{-rX} - rXe^{-Xr} + \sigma e^{-\sigma X} - \sigma e^{-rX} + r\sigma Xe^{-rX} \\ \text{and } H'_X(X, r) &= -e^{-X} + e^{-rX} - rXe^{-Xr} + r^2Xe^{-rX}. \end{aligned}$$

We take the Taylor's expansion of the functions  $e^{-X}$ ,  $e^{-rX}$ ,  $e^{-\sigma X}$  at  $X = 0$

$$\begin{aligned} e^{-\sigma X} &= 1 - \sigma X + \frac{1}{2}\sigma^2 X^2 - \frac{1}{6}\sigma^3 X^3 + \frac{1}{24}\sigma^4 X^4 - \frac{1}{120}\sigma^5 X^5 + o(X^6) \\ \text{so } e^{-rX} &= 1 - rX + \frac{1}{2}r^2 X^2 - \frac{1}{6}r^3 X^3 + \frac{1}{24}r^4 X^4 - \frac{1}{120}r^5 X^5 + o(X^6) \\ \text{and } e^{-X} &= 1 - X + \frac{1}{2}X^2 - \frac{1}{6}X^3 + \frac{1}{24}X^4 - \frac{1}{120}X^5 + o(X^6) \end{aligned}$$

Replacing these above Taylor's expansion to the expression  $H(X, \sigma)$ , we have

$$\begin{aligned} H(X, \sigma) &= \frac{(1-\sigma)(\sigma-2r+1)}{2}X^2 + \left(\frac{(1-\sigma)}{2}r^2 + \frac{\sigma^3-1}{6}\right)X^3 \\ &\quad + \left(\frac{(\sigma-1)}{6}r^3 - \frac{\sigma^4-1}{24}\right)X^4 + \left(\frac{(1-\sigma)}{24}r^4 + \frac{\sigma^5-1}{120}\right)X^5 + o(X^6) \\ \text{and } H(X, r) &= \frac{(1-r)^2}{2}X^2 + \left(-\frac{1}{6} + \frac{1}{2}r^2 - \frac{1}{3}r^3\right)X^3 \\ &\quad + \left(\frac{1}{24} - \frac{1}{6}r^3 + \frac{1}{8}r^4\right)X^4 + \left(-\frac{1}{120} + \frac{1}{24}r^4 - \frac{1}{30}r^5\right)X^5 + o(X^6). \\ \text{So } H'_X(X, \sigma) &= (1-\sigma)(\sigma-2r+1)X + \left(\frac{3(1-\sigma)}{2}r^2 + \frac{\sigma^3-1}{2}\right)X^2 \\ &\quad + \left(\frac{2(\sigma-1)}{3}r^3 - \frac{\sigma^4-1}{6}\right)X^3 + \left(\frac{5(1-\sigma)}{24}r^4 + \frac{\sigma^5-1}{24}\right)X^4 + o(X^5) \\ \text{and } H'_X(X, r) &= (1-r)^2X + \left(-\frac{1}{2} + \frac{3}{2}r^2 - r^3\right)X^2 \\ &\quad + \left(\frac{1}{6} - \frac{2}{3}r^3 + \frac{1}{2}r^4\right)X^3 + \left(-\frac{1}{24} + \frac{5}{24}r^4 - \frac{1}{6}r^5\right)X^4 + o(X^5). \end{aligned}$$

Finally,

$$H'_X(X, r)H(X, \sigma) - H'_X(X, \sigma)H(X, r) = \frac{(1-\sigma)(\sigma-r)^2(1-r)^2}{12}X^4 + o(X^5).$$

□

**Lemma 1.3.13.**  $M(r) = \int_0^r (1-h)(h-r)^2(1-r)^2 dh > 0.0069$  for every  $r \in [\frac{2}{5}, \frac{1}{2}]$ .

*Proof.* We have

$$\begin{aligned}
 M(r) &= \int_0^r (r-1)^2(1-h)(h-r)^2 dh \\
 &= (r-1)^2 \int_0^r (1-h)(h-r)^2 dh \\
 &= (r-1)^2 \int_0^r (1-r+r-h)(h-r)^2 dh \\
 &= (r-1)^2 \left( (1-r) \int_0^r (h-r)^2 dh + \int_0^r (h-r)^3 dh \right) \\
 &= (r-1)^2 \left( -(1-r) \frac{(-r)^3}{3} + \frac{r^4}{4} \right) \\
 &= \frac{(r-1)^2 r^3 (4-r)}{12}.
 \end{aligned}$$

It implies that  $I'(r) = \frac{(1-r)r^2(r^2-4r+2)}{12}$ . Note that  $(r^2-4r+2)' = 2r-2 < 0$  for every  $r \in [\frac{2}{5}, \frac{1}{2}]$ , so  $r^2-4r+2 \geq \frac{1}{2} - 4\frac{1}{2} + 2 > 0$  for every  $r \in [\frac{2}{5}, \frac{1}{2}]$ . It shows that  $I'(r) > 0$  for every  $r \in [\frac{2}{5}, \frac{1}{2}]$ , and  $M(r) \geq I(\frac{2}{5}) > 0.0069$ . □

Now we show the *Proof of Theorem 1.3.10*:

*Proof.* (*Proof of Theorem 1.3.10*)

By Lemma 1.3.12 and Lemma 1.3.5, we have

$$\begin{aligned}
 T'_X(X, r) &= \int_0^r \frac{H'_X(X, r)H(X, \sigma) - H'_X(X, \sigma)H(X, r)}{H^2(X, \sigma)} d\sigma \\
 &= \int_0^r \frac{(1-\sigma)(\sigma-r)^2(1-r)^2}{12} X^4 + o(X^5)}{H^2(X, \sigma)} d\sigma \\
 &= \frac{X^4}{12} \int_0^r \frac{(1-\sigma)(\sigma-r)^2(1-r)^2 + o(X)}{H^2(X, \sigma)} d\sigma \\
 &\geq \frac{X^4}{12H^2(X, r)} \int_0^r \left( (1-\sigma)(\sigma-r)^2(1-r)^2 + o(X) \right) d\sigma
 \end{aligned}$$

Using Lemma 1.3.13, we also have

$$\begin{aligned} \int_0^r \left( (1-\sigma)(\sigma-r)^2(1-r)^2 + o(X) \right) d\sigma &= \int_0^r (1-\sigma)(\sigma-r)^2(1-r)^2 d\sigma + \int_0^r o(X) d\sigma \\ &> 0.0069 + \int_0^r o(X) d\sigma \end{aligned}$$

for every  $r \in [\frac{2}{5}, \frac{1}{2}]$ ,  $X \in (0, X(r))$ .

By Lemma 1.3.11, there exists  $\alpha > 0$  such that if  $r \in (\frac{1}{2} - \alpha, \frac{1}{2})$ , then  $|o(X)| < 0.12$  (note that, we can choose  $\alpha \leq \frac{1}{10}$  so that  $(\frac{1}{2} - \alpha, \frac{1}{2}) \subset [\frac{2}{5}, \frac{1}{2}]$ ). It implies that

$$\begin{aligned} \left| \int_0^r o(X) d\sigma \right| &\leq \int_0^r |o(X)| d\sigma \leq \int_0^r 0.12 d\sigma = 0.12r \leq 0.006. \text{ It follows that} \\ \int_0^r \left( (1-\sigma)(\sigma-r)^2(1-r)^2 + o(X) \right) d\sigma &\geq 0.0069 - 0.006 > 0. \end{aligned}$$

Therefore  $T'_X(X, r) > 0$  for every  $X \in (0, X(r))$ , and  $T(X, r) = \int_0^r \frac{H(X, r)}{H(X, \sigma)} d\sigma$  is strictly increasing on  $(0, X(r))$ . □

*Remark 1.3.14.* Suppose that  $r \in (0, \frac{1}{2})$ , and  $X(r)$  is the unique positive solution of the equation  $e^{-X} + Xe^{-rX} - 1 = 0$ , then there exists  $\frac{1}{2} > \alpha > 0$  such that for every  $r \in (\frac{1}{2} - \alpha, \frac{1}{2})$  the function  $T(X, r) = \int_0^r \frac{H(X, r)}{H(X, \sigma)} d\sigma$  is strictly increasing on  $(0, X(r))$ . It implies the uniqueness of solution of (1.3.10), and then the function  $CR(t, p)$  is uniquely determined by the turning point  $t_p$ ,  $CR(t_p)$ ,  $CR'(t_p)$  and  $CR(\infty)$  in which  $p = (\tau, \nu, \nu_1, S_0, I_0)$  is the arbitrary parameter of 1.1.1-1.1.3.

## 1.4 An identification method and applications to some outbreak epidemics

### 1.4.1 Description of the method

By Proposition 1.2.1, the combination of parameters  $\frac{I_0}{S_0}$ ,  $\nu_1 S_0$ ,  $S_0 \tau$ ,  $\nu_1 I_0$ , and  $\nu$  from the system (1.1.1)-(1.1.2) are uniquely determined by the cumulative reported cases function  $CR(t)$  for the parameter  $p = (\tau, \nu, \nu_1, S_0, I_0)$ . Moreover, the analysis in Section 1.3 allows us to derive a method to compute this combination by the following three steps:

Assume that the values of  $t_p$ ,  $CR(t_p)$ ,  $CR'(t_p)$ , and  $CR(\infty)$  are known, and set  $c = CR(\infty)$  and  $r = \frac{CR(t_p)}{CR(\infty)}$ .

**Step 1:** Solve the equation (1.3.7),  $e^{-X} + Xe^{-rX} - 1 = 0$ , to obtain the unique positive solution  $X(r)$ .

**Step 2:** Solve the turning point equation (1.3.10)

$$\int_0^r \frac{e^{-X} + Xe^{-rX} - e^{-\sigma X} - \sigma Xe^{-rX}}{e^{-X} + Xe^{-rX} - e^{-\sigma X} - \sigma Xe^{-rX}} d\sigma = \frac{CR'(t_p)t_p}{c},$$

with the condition (1.3.9),  $0 < X < X(r)$ , to obtain the value  $X = c \frac{\tau}{\nu_1}$ .

**Step 3:**

- i. Compute the value of  $a_1 := \frac{\tau}{\nu_1} = \frac{X}{c}$  by the formula (1.3.2);
- ii. Compute the value of  $a_2 := \frac{I_0}{S_0} = e^{-X} + Xe^{-rX} - 1$  by the formula (1.3.4);
- iii. Compute the value of  $a_3 := \nu_1 S_0 = \frac{CR'(t_p)}{1 + a_2 - e^{-rX}(1 + rX)}$  by the formula (1.3.5);
- iv. Compute the value of  $a_4 := \tau S_0 = a_1 a_3$ ;
- v. Compute the value of  $a_5 := \nu_1 I_0 = a_2 a_3$ ;
- vi. Compute the value of  $a_6 := \nu = a_4 e^{-rX}$  by the formula (1.3.6).

*Remark 1.4.1.* By the fact that  $\nu_1 < \nu$ , we obtain the following evaluations:

$$\begin{aligned} S_0 &> S_0 \frac{\nu_1}{\nu} = \frac{a_3}{a_6} \\ I_0 &> I_0 \frac{\nu_1}{\nu} = \frac{a_5}{a_6} \\ \tau &< \nu \frac{\tau}{\nu_1} = a_1 a_6. \end{aligned} \tag{1.4.1}$$

Moreover, the basic reproduction number is

$$\mathcal{R}_0 = \frac{\tau S_0}{\nu} = \frac{a_4}{a_6}. \tag{1.4.2}$$

*Remark 1.4.2.* Recall from [46] that

$$CU(t) = \nu_2 \int_0^t I(s) ds$$



is the cumulative unreported infected cases at time  $t > 0$ , and

$$C(t) = \nu \int_0^t I(s) ds$$

is the cumulative total cases at time  $t > 0$ . If  $S_0$  is known, the epidemic *final size* of the epidemic is written as follows:

$$\begin{aligned} C(\infty) &= CR(\infty) + CU(\infty) = CR(\infty) + \frac{\nu_2}{\nu_1} CR(\infty) \\ &= \frac{\nu}{\nu_1} CR(\infty) = \nu \frac{S_0}{\nu_1 S_0} CR(\infty) = a_6 \frac{c}{a_3} S_0. \end{aligned} \quad (1.4.3)$$

Denote by  $N = S_0 + I_0$ , the number of the individuals involved in the epidemic, which is typically smaller than the total number of the population, since some people have immunity. Then  $S_0$  satisfies

$$\frac{a_3}{a_6} < S_0 \leq N - I_0 = N - S_0 \frac{I_0}{S_0},$$

which implies

$$\frac{a_3}{a_6} < S_0 \leq \frac{N}{1 + \frac{I_0}{S_0}} = \frac{N}{1 + a_2}. \quad (1.4.4)$$

Moreover, the number of susceptible individuals at the end of epidemic can be computed by the following formula

$$S(\infty) = S_0 + I_0 - C(\infty) = S_0 \left( 1 + a_2 - a_6 \frac{c}{a_3} \right). \quad (1.4.5)$$

### 1.4.2 Application to Hong-Kong influenza in New York City in 1968-1969

To illustrate the method we performed in the previous section, we compute the parameters of SIR model for Hong-Kong influenza in New York City during 13 weeks of the winter of 1968-1969. The data of this epidemic given in the following table are weekly reported numbers of influenza deaths (see [62]).

1. Identifying the number of unreported cases in SIR epidemic models

---

Week ( $t$ )	Reported Cases ( $\nu_1 I(t)$ )
1	14
2	28
3	50
4	66
5	156
6	190
7	156
8	108
9	68
10	77
11	33
12	65
13	24

Table 1.1: Reported cases data for Hong-Kong influenza in New York in 1968-1969.

In this application, we have the values  $CR(\infty) = 1080$ ,  $CR(t_p) = 500$ ,  $t_p = 6.15$ ,  $CR'(t_p) = 190$  (see [46]). The total population of New York City in 1968 is 7,900,000. Consider the equation

$$\int_0^{\frac{500}{1080}} \frac{e^{-X} + X e^{-\frac{500}{1080}X} - e^{-\frac{500}{1080}X} + \frac{500}{1080} X e^{-\frac{500}{1080}X}}{e^{-X} + X e^{-\frac{500}{1080}X} - e^{-\sigma X} - \sigma X e^{-\frac{500}{1080}X}} d\sigma = \frac{11685}{10800}. \quad (1.4.6)$$

for  $X \in (0, X(r))$  where  $X(r) = 0.89478$  is the positive solution of the equation

$$e^{-X} + X e^{-\frac{500}{1080}X} - 1 = 0. \quad (1.4.7)$$

First, solve the equation (1.4.7), and obtain the value of  $X(r) = 0.89478$ . This value corresponds to the positive zero of the function in Figure 1.5 - left side.

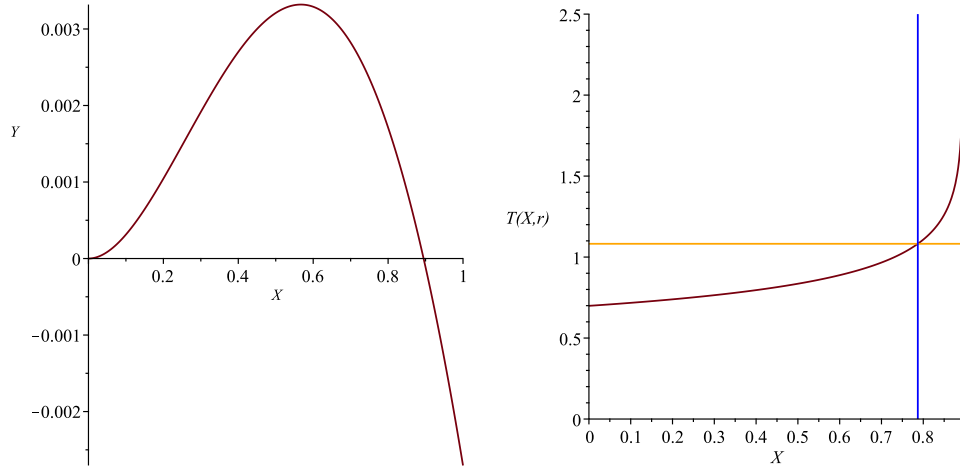


Figure 1.5: *Left side: The graph of  $Y = e^{-X} + Xe^{-\frac{500}{1080}X} - 1$ . Right side: The intersection of  $Y = T(X, \frac{500}{1080})$  and  $Y = \frac{t_p CR'(t_p)}{CR(\infty)}$ . When  $r = \frac{500}{1080}$  we obtain  $X(r) = 0.89478$ ,  $X = 0.7869$  and  $\frac{I_0}{S_0} = e^{-X} + Xe^{-\frac{500}{1080}X} - 1 = 0.0019$ .*

Finally, by applying Step 3 of the method described above we obtain the following table of values

Variable	Description	Estimated value
$X$	$c\tau/\nu_1$	0.79
$a_1$	$\tau/\nu_1$	$7.3 \times 10^{-4}$
$a_2$	$I_0/S_0$	0.002
$a_3$	$\nu_1 S_0$	3509.1
$a_4$	$\tau S_0$	2.56
$a_5$	$\nu_1 I_0$	6.65
$a_6$	$\nu$	1.78

Table 1.2: *List of a combination of parameters obtained for Hong-Kong influenza in New York in 1968-1969.*

*Remark 1.4.3.* From the reported case data for the Hong Kong influenza in New York City in 1968-1969, there are at least three infected individuals at the beginning of the epidemic.

If the value of the initial susceptible individuals  $S_0$  is given, then all the parameters can be obtained. The following table gives these values when  $S_0 = 1,976$ ,  $S_0 = 4,000,000$ , and  $S_0 = 7,885,047$

1. Identifying the number of unreported cases in SIR epidemic models

---

Variable	Estimated value 1	Estimated value 2	Estimated value 3
$S_0$	1976	4,000,000	7,885,047
$I_0$	3.7472	7,586	14,953
$\tau$	$1.3 \times 10^{-3}$	$6.4 \times 10^{-7}$	$3.2 \times 10^{-7}$
$\nu_1$	1.78	$0.88 \times 10^{-3}$	$4.5 \times 10^{-4}$
$\nu_2$	$3.1 \times 10^{-4}$	1.78	1.78
$C(\infty)$	1080	$2.19 \times 10^6$	$4.31 \times 10^6$
$S(\infty)$	899	$1.82 \times 10^6$	$3.59 \times 10^6$

---

Table 1.3: List of parameters obtained for Hong-Kong influenza in New York in 1968-1969. In this table we vary the value of  $S_0$  between the minimal value 1976 up to the maximal value 7,885,047 and compute the corresponding estimated parameters values.

In Figures 1.6, 1.7, and 1.8 we provide model (1.1.1)-(1.1.2) output for the Hong Kong influenza epidemic in New York City in 1968-1968 for the parameters in Table 1.4.2 and the values under  $S_0 = 4,000,000$  in Table 1.3 (see [46]). In Figure 1.6 we compare the model output to the reported case data (see [46]). In Figure 1.7 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.8 we illustrate the epidemic final size as a function of the turning point of the epidemic.

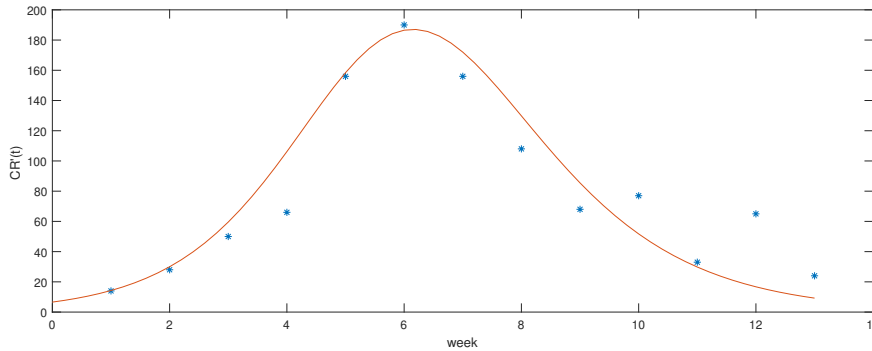


Figure 1.6: Hong Kong influenza epidemic in New York City in 1968-1969. The weekly reported mortality case data and cumulative reported case data, and the model output graph  $CR'(t)$ . To run this simulation we fix  $S_0 = 4,000,000$ .

Moreover, the basic reproduction number is

$$\mathcal{R}_0 = \frac{\tau S_0}{\nu} = \frac{a_4}{a_6} = 1.4. \quad (1.4.8)$$

From Remark 1.4.2 whenever  $S_0$  is known the final size of the epidemic  $C(\infty)$  is

expressed linearly in function of  $S_0$

$$C(\infty) = \frac{a_6 CR(\infty)}{a_3} S_0, \quad (1.4.9)$$

and we have the following upper and lower bounded for  $S_0$

$$\frac{a_3}{a_6} < S_0 \leq \frac{N}{1 + a_2}. \quad (1.4.10)$$

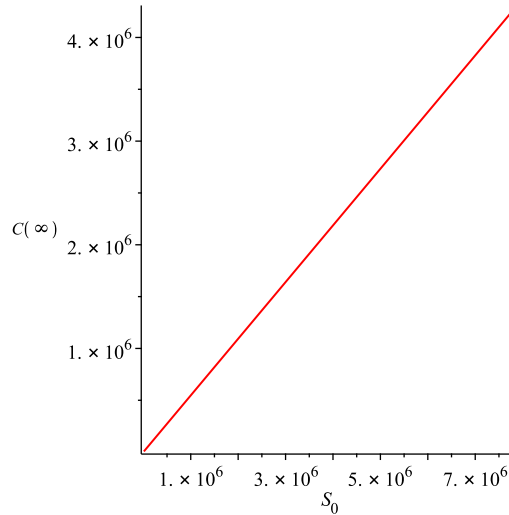


Figure 1.7: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$ . Here  $S_0$  varies from 1,976 (which is strictly larger than the minimal value  $\frac{a_3}{a_6} = 1,975$ ) up to the maximal value  $S_0 = \frac{N}{1+a_2} = 7,885,047$  which corresponds to  $I_0 = 14,953$ .*

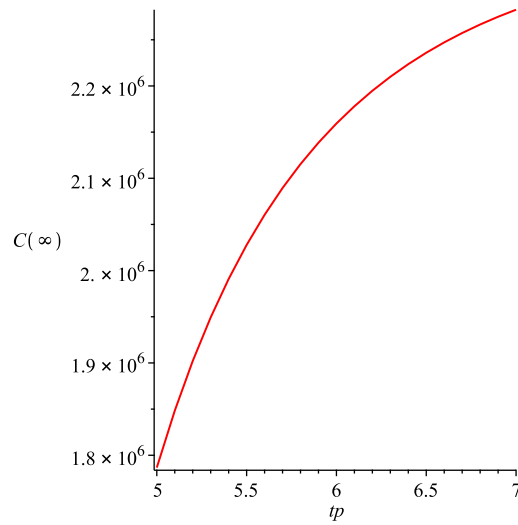


Figure 1.8: In this figure we fix  $S_0 = 4,000,000$ , vary the turning point  $t_p$  and plot the final size  $C(\infty)$  of the epidemic as a function of the turning point.

### 1.4.3 Application to the plague epidemic in Bombay, India in 1906

The weekly reported case data of the plague epidemic in Bombay, India in 1906 is obtained from the website [75]. The data is taken from the first 30 weeks of the year 1906 when the plague epidemic broke out in Bombay. We have the table of data as follows

Week ( $t$ )	Reported Cases ( $\nu_1 I(t)$ )
01	16
02	24
03	48
04	51
05	92
06	124
07	178
08	280
09	387
10	442
11	644
12	779
13	702
14	695
15	870
16	925
17	802
18	578
19	404
20	296
21	162
22	106
23	64
24	46
25	35
26	27
27	28
28	24
29	26
30	29

Table 1.4: *Reported cases data for the plague in Bombay, India in 1906.*

In this application, we get from the data the values (see [46])  $CR_\infty = 8840$ ,  $CR(tp) = 4330$ ,  $tp = 13.5$ ,  $CR'(tp) = 770$ . The total population of Bombay in 1906 is approximately 1,000,000.

Consider the equation

$$\int_0^{\frac{4330}{8840}} \frac{e^{-X} + X e^{-\frac{4330}{8840}X} - e^{-\frac{4330}{8840}X} + \frac{4330}{8840} X e^{-\frac{4330}{8840}X}}{e^{-X} + X e^{-\frac{4330}{8840}X} - e^{-\sigma X} - \sigma X e^{-\frac{4330}{8840}X}} d\sigma = 1.175904977. \quad (1.4.11)$$

where  $X \in (0, X(r))$  with  $X(r)$  is the unique positive solution of the equation

$$e^{-X} + X e^{-\frac{4330}{8840}X} - 1 = 0. \quad (1.4.12)$$

1. Identifying the number of unreported cases in SIR epidemic models

---

Solving the equation (1.4.12), we obtain the value of  $X(r) = 0.2444655268$ . This value corresponds to the positive zero of the function in the following figure 1.9 - left side.

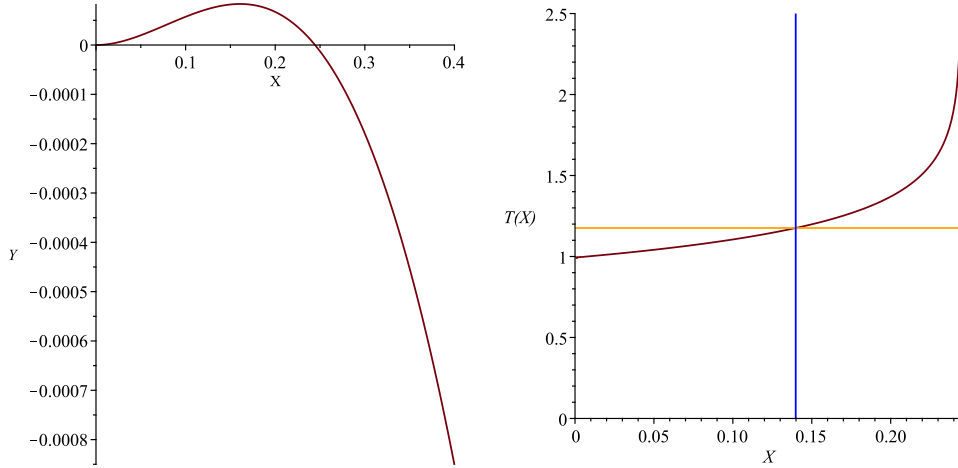


Figure 1.9: *Left side: The graph of  $Y = e^{-X} + Xe^{-\frac{4330}{8840}X} - 1$ . Right side: The intersection of  $Y = T(X, \frac{4330}{8840})$  and  $Y = \frac{t_p CR'(t_p)}{CR(\infty)}$  where  $X(r) = 0.2444655268$ ,  $X = 0.1398251831$ .*

Next, applying **Step 3** of the method described in Section 1.4.1 we obtain the following table of values

Variable	Description	Estimated value
$X$	$c\tau/\nu_1$	.1398251831
$a_1$	$\tau/\nu_1$	$0.1581732840 \times 10^{-4}$
$a_2$	$I_0/S_0$	$0.79501 \times 10^{-4}$
$a_3$	$\nu_1 S_0$	331826.8395
$a_4$	$\tau S_0$	5.248614094
$a_5$	$\nu_1 I_0$	26.38056557
$a_6$	$\nu$	4.901175271

Table 1.5: *List of a combination of parameters obtained for the plague epidemmic in Bombay in 1906.*

The value of the initial susceptible individuals  $S_0 = 100,000$  is taken as the paper [46], and then all the parameters can be obtained in the following table.



Variable	Estimated value 1
$S_0$	100,000
$I_0$	7.950100000
$\tau$	$0.5248614094 \times 10^{-4}$
$\nu_1$	3.318268395
$\nu_2$	1.582906876
$C(\infty)$	13056.92736
$S(\infty)$	86951.02274

Table 1.6: List of parameters obtained for the plague epidemic in Bombay in 1906. In this table, we take  $S_0 = 100,000$  and compute all other parameters' values.

In Figures 1.10, 1.11, and 1.12 we provide model (1.1.1)-(1.1.2) output for the plague epidemic in Bombay, India in 1906 for the parameters in Table 1.4.3 and the values under  $S_0 = 100,000$  in Table 1.6 (see [46]). In Figure 1.10 we compare the model output to the reported case data (see [46]). In Figure 1.11 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.12 we illustrate the epidemic final size as a function of the turning point of the epidemic.

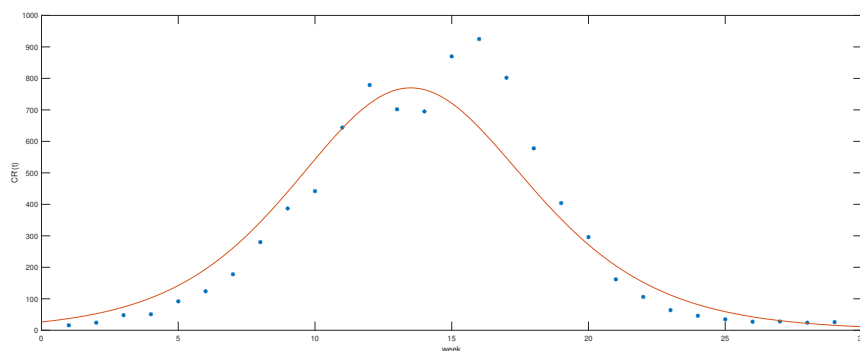


Figure 1.10: The plague epidemic in Bombay, India in 1906. The weekly reported mortality case data and cumulative reported case data, and the model output graph  $CR'(t)$ . To run this simulation we fix  $S_0 = 100,000$ .

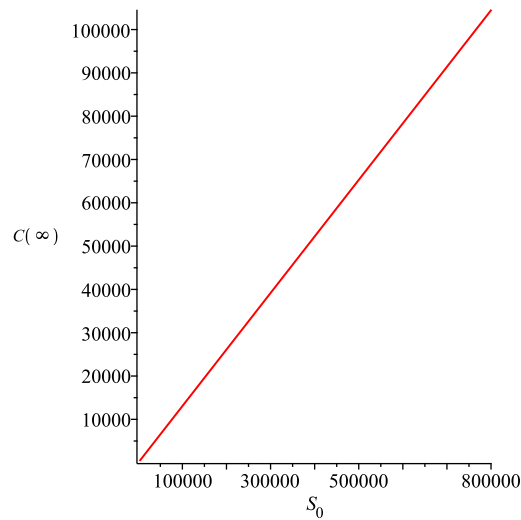


Figure 1.11: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$ . Here  $S_0$  varies from 1,976 (which is strictly larger than the minimal value  $\frac{a_3}{a_6} = 1,975$ ) up to the maximal value  $S_0 = \frac{N}{1+a_2} = 7,885,047$  which corresponds to  $I_0 = 14,953$ .*

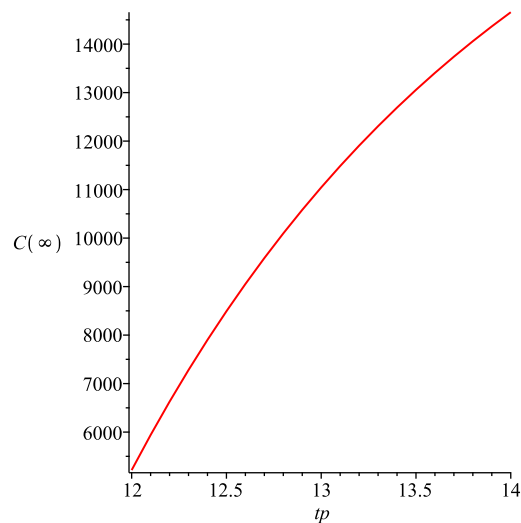


Figure 1.12: *In this figure we fix  $S_0 = 4,000,000$ , vary the turning point  $t_p$  and plot the final size  $C(\infty)$  of the epidemic as a function of the turning point.*

#### 1.4.4 Application to the influenza epidemic in France in the consecutive six years 1988-1994

In this part, we study the influenza epidemics in France in the consecutive six years from week 37 of the year 1988 to week 15 of the year 1994 with the data from

[73].

We have the following figure 1.13 from [73]

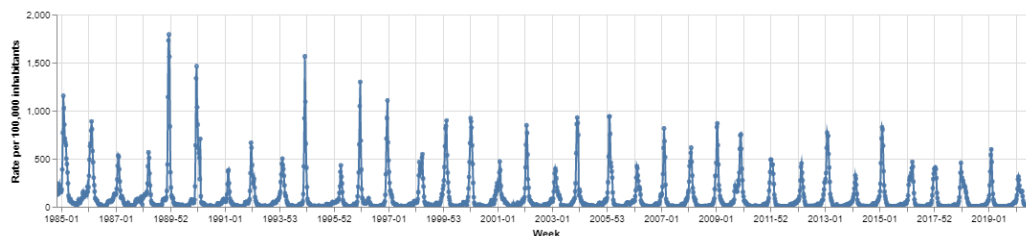


Figure 1.13: *This figure indicates the outbreak of the influenza epidemics from 1985 to 2020 (This figure is from [73])*

According to the figure 1.13, it is seen that in the period from 1988 to 1994 there are three times the reported infected cases over 1500 individuals over 100,000 inhabitants. It means that the three highest peaks of these influenza epidemics in France took place from 1988 to 1994. That is why we study the influenza epidemics in France in this period.

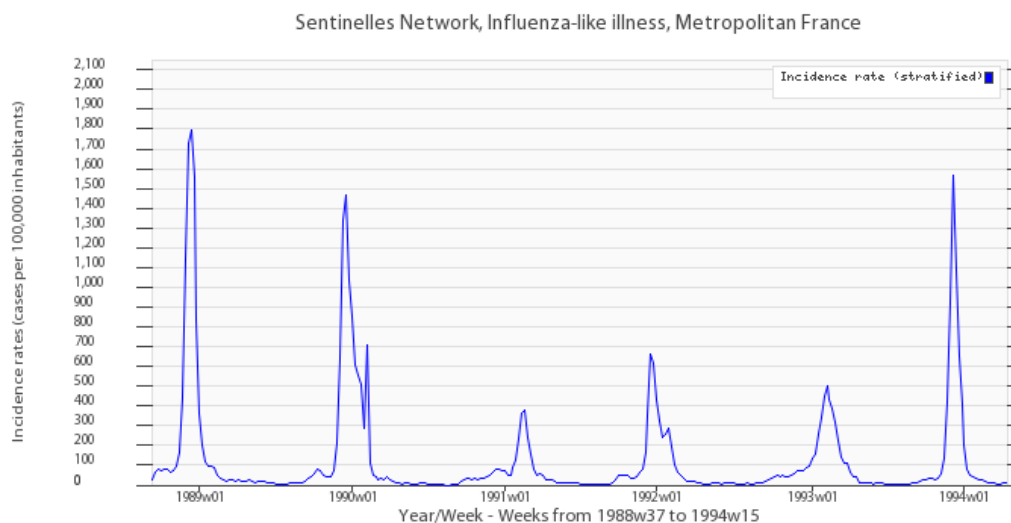


Figure 1.14: *This figure indicates the outbreak of the epidemics during the period 1988-1994 (This figure is from [73])*

What we do next is choosing the starting point and the endpoint of each influenza epidemic in France for every year according to the data from [73], and applying the method in section 1.4.1 to investigate the epidemics. Note that the incidence rate from the data is the reported case per 100,000 inhabitants for each epidemic.

#### I. Period from week 43 of the year 1988 to week 8 of the year 1989

From the data in [73](see Figure 1.15), we obtain the values  $CR_\infty = 9000$ ,  $CR(tp) = 4400$ ,  $tp = 7.8$ ,  $CR'(tp) = 1810$ .

## 1. Identifying the number of unreported cases in SIR epidemic models

---

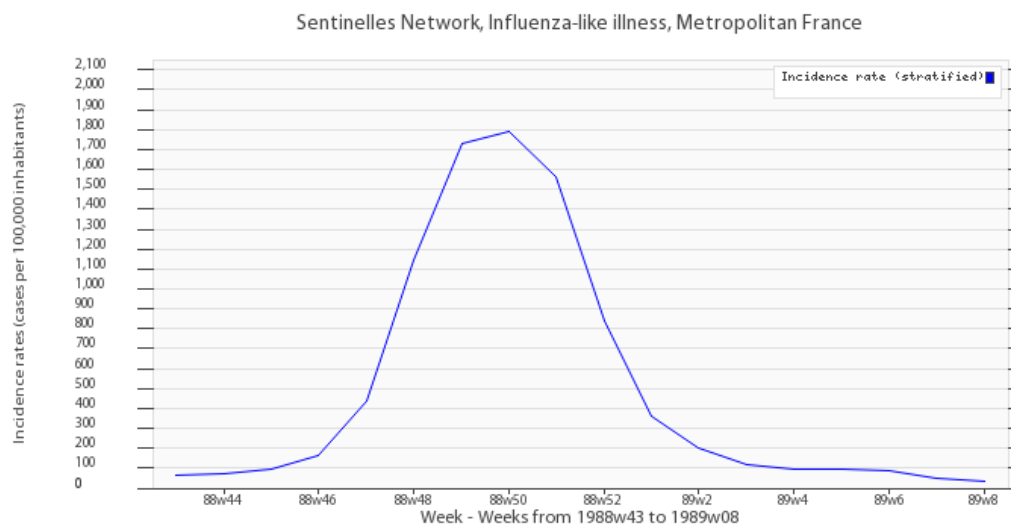


Figure 1.15: *This figure indicates the outbreak of the epidemics during the period from week 43 of the year 1988 to week 8 of the year 1989 (This figure is from [73])*

Next, applying the method described in section 1.4.1 we obtain the following table with  $S_0 = 89200$

Variable	Estimated value
$S_0$	89200
$I_0$	3.756390400
$\tau$	$0.8185035674 \times 10^{-4}$
$\nu_1$	2.967019577
$\nu_2$	3.499492216
$C_\infty$	19615.17429
$S_\infty$	69588.58210

Table 1.7: List of parameters obtained for the influenza-like illness in France in 1988w43-1989w8.

In Figures 1.16, 1.17, 1.18, and 1.19 we provide model (1.1.1)-(1.1.2) output for the influenza epidemic in France during the period from week 43 of the year 1988 to the week 8 of the year 1989 for the parameters in Table 1.4.4. In Figure 1.16, 1.17 we compare the model output to the reported case data. In Figure 1.18 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.19 we illustrate the epidemic final size as a function of the turning point of the epidemic.

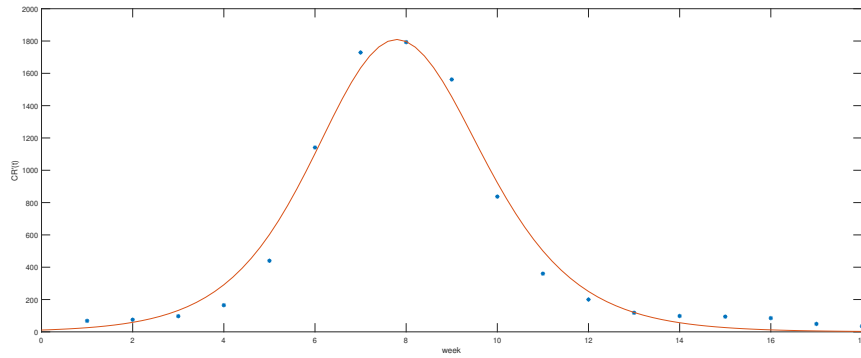


Figure 1.16: *The weekly reported case data(blue dots) and model output graph of  $\nu_1 I(t)$  for the influenza-like illness in France in 1988w43-1989w8*

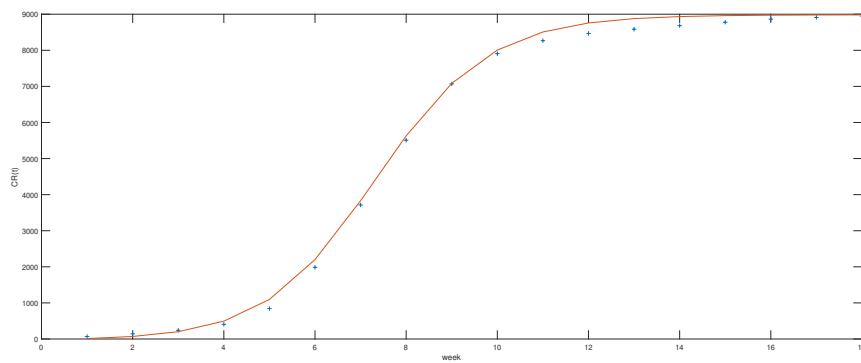


Figure 1.17: *The weekly reported cumulative case data(blue dots) and model output graph of  $CR(t)$  for the influenza-like illness in France in 1988w43-1989w8*

---

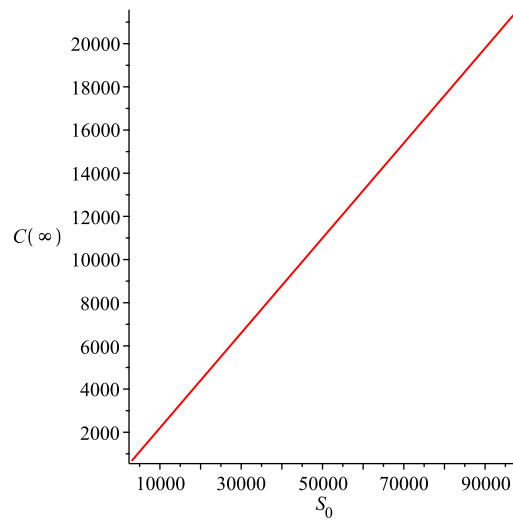


Figure 1.18: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$  for the influenza-like illness in France in 1988w43-1989w8*

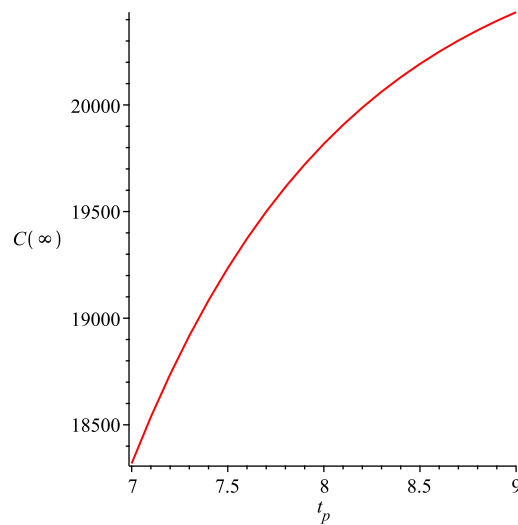


Figure 1.19: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and the turning point  $t_p$  for the influenza-like illness in France in 1988w43-1989w8*

## II. Period from week 45 of the year 1989 to week 10 of the year 1990

From the data in [73](see Figure 1.20), we obtain the values  $CR_\infty = 8585$ ,  $CR(tp) = 3802$ ,  $tp = 7$ ,  $CR'(tp) = 1463$ .

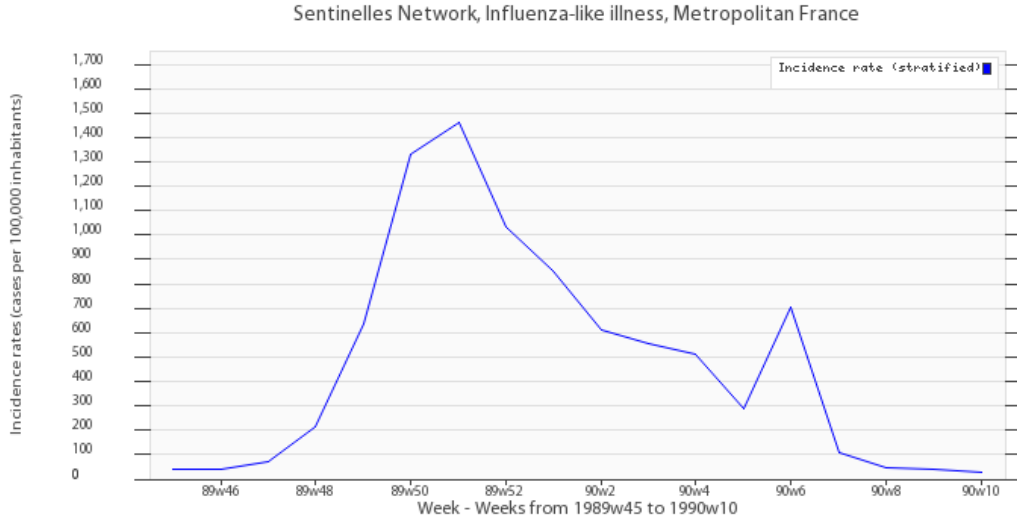


Figure 1.20: This figure indicates the outbreak of the epidemics during the period from week 45 of the year 1989 to week 10 of the year 1990 (This figure is from [73])

Next, applying the method described in section 1.4.1 we obtain the following table with  $S_0 = 89200$

Variable	Estimated value
$S_0$	89200
$I_0$	153.8498408
$\tau$	$0.2098327123 \times 10^{-4}$
$\nu_1$	0.1334583415
$\nu_2$	0.8960421725
$C_\infty$	66224.87447
$S_\infty$	23128.97537

Table 1.8: List of parameters obtained for the influenza-like illness in France in 1989w45-1990w10.

In Figures 1.21, 1.22, 1.23, and 1.24 we provide model (1.1.1)-(1.1.2) output for the influenza epidemic in France during the period from week 45 of the year 1989 to week 10 of the year 1990 for the parameters in Table 1.4.4. In Figure 1.21, 1.22 we compare the model output to the reported case data. In Figure 1.23 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.24 we illustrate the epidemic final size as a function of the turning point of the epidemic.

1. Identifying the number of unreported cases in SIR epidemic models

---

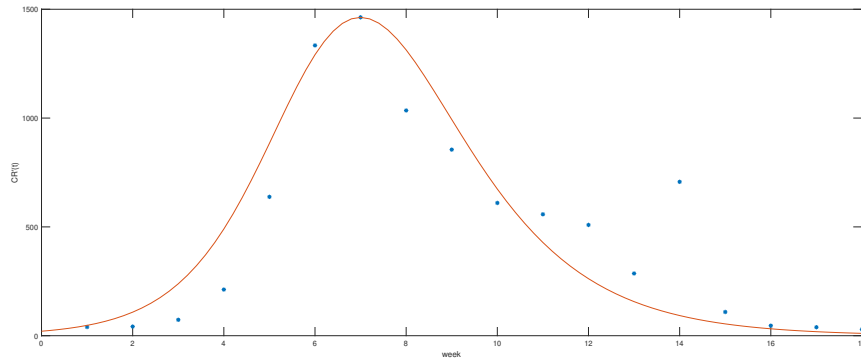


Figure 1.21: The weekly reported case data(blue dots) and model output graph of  $\nu_1 I(t)$  for the influenza-like illness in France in 1989w45-1990w10

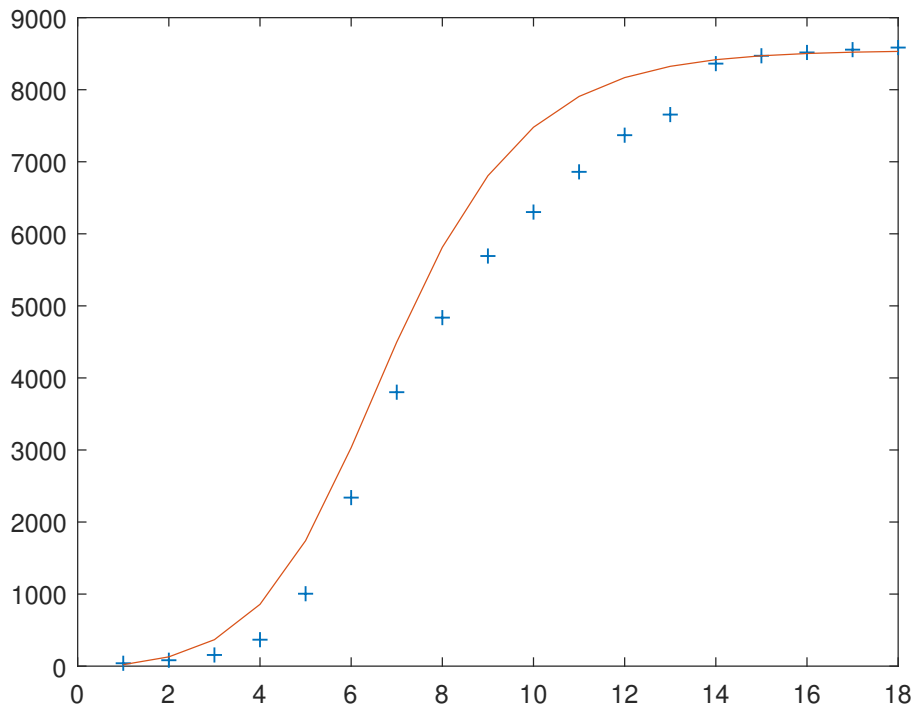


Figure 1.22: The weekly reported cumulative case data(blue dots) and model output graph of  $CR(t)$  for the influenza-like illness in France in 1989w45-1990w10



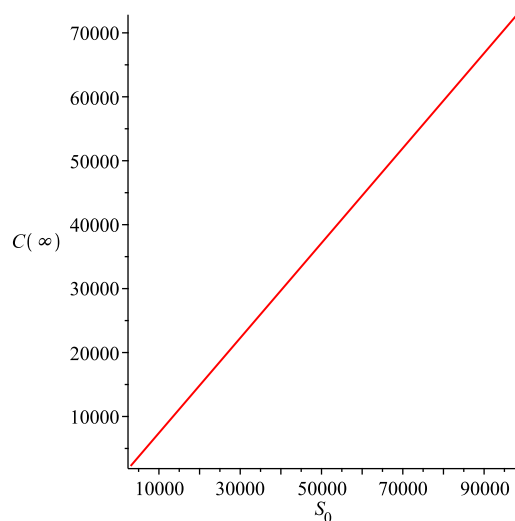


Figure 1.23: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$  for the influenza-like illness in France in 1989w45-1990w10*

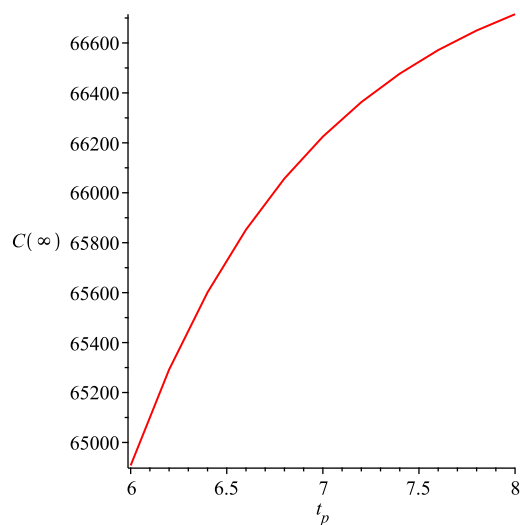


Figure 1.24: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and the turning point  $t_p$  for the influenza-like illness in France in 1989w45-1990w10*

### III. Period from week 2 of the year 1991 to week 15 of the year 1991

From the data in [73](see Figure 1.25), we obtain the values  $CR_\infty = 1970$ ,  $CR(tp) = 970$ ,  $tp = 6.15$ ,  $CR'(tp) = 385$ .

## 1. Identifying the number of unreported cases in SIR epidemic models

---

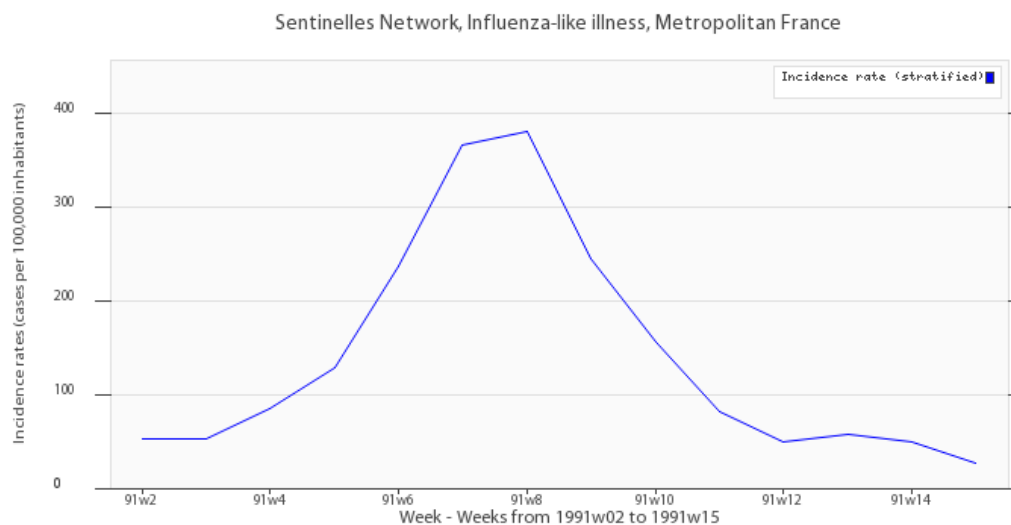


Figure 1.25: *This figure indicates the outbreak of the epidemics during the period from week 2 of the year 1991 to week 15 of the year 1991 (This figure is from [73])*

Next, applying the method described in section 1.4.1 we obtain the following table with  $S_0 = 89200$

Variable	Estimated value
$S_0$	89200
$I_0$	2.545054400
$\tau$	$0.2095686786 \times 10^{-3}$
$\nu_1$	4.806980959
$\nu_2$	13.11249984
$C_\infty$	7343.773041
$S_\infty$	81858.77201

Table 1.9: List of parameters obtained for the influenza-like illness in France in 1991w2-1991w15.

In Figures 1.26, 1.27, 1.28, and 1.29 we provide model (1.1.1)-(1.1.2) output for the influenza epidemic in France during the period from week 2 of the year 1991 to week 15 of the year 1991 for the parameters in Table 1.4.4. In Figure 1.26, 1.27 we compare the model output to the reported case data. In Figure 1.28 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.29 we illustrate the epidemic final size as a function of the turning point of the epidemic.

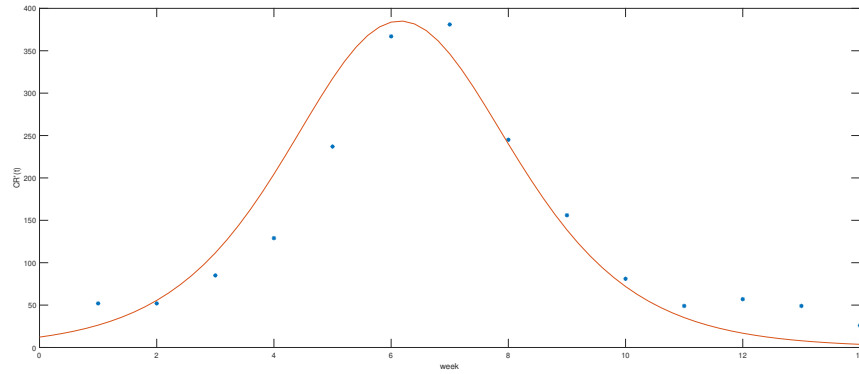


Figure 1.26: *The weekly reported case data(blue dots) and model output graph of  $\nu_1 I(t)$  for the influenza-like illness in France in 1991w2-1991w15*

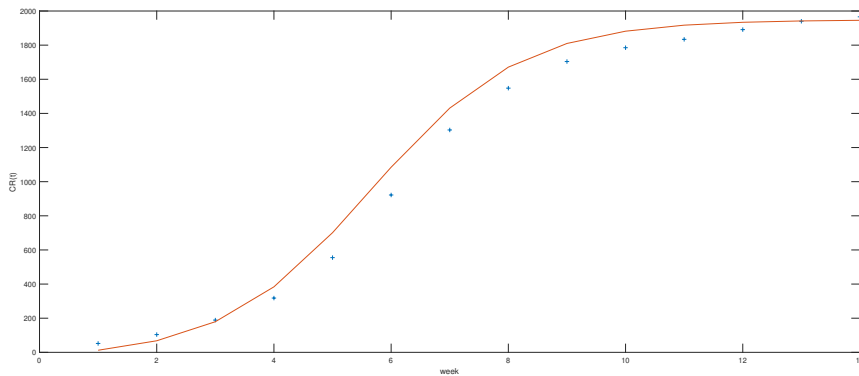


Figure 1.27: *The weekly reported cumulative case data(blue dots) and model output graph of  $CR(t)$  for the influenza-like illness in France in 1991w2-1991w15*

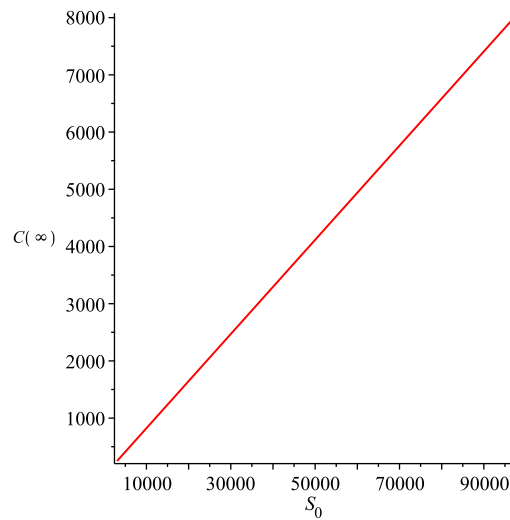


Figure 1.28: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$  for the influenza-like illness in France in 1991w2-1991w15*

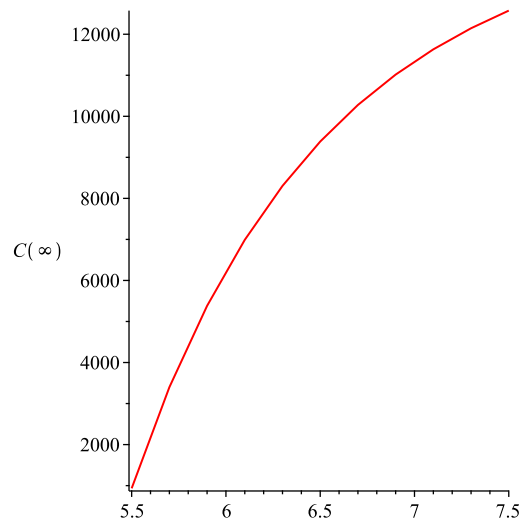


Figure 1.29: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and the turning point  $t_p$  for the influenza-like illness in France in 1991w2-1991w15*

#### IV. Period from week 45 of the year 1991 to week 10 of the year 1992

From the data in [73](see Figure 1.30), we obtain the values  $CR_\infty = 4007$ ,  $CR(tp) = 1745$ ,  $tp = 7.6$ ,  $CR'(tp) = 680$ .

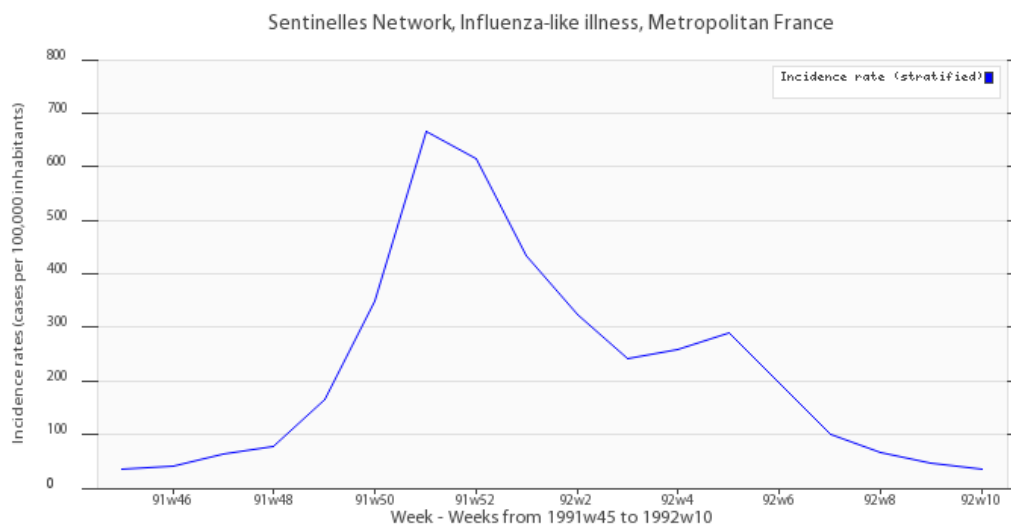


Figure 1.30: This figure indicates the outbreak of the epidemics during the period from week 45 of the year 1991 to week 10 of the year 1992 (This figure is from [73])

Next, applying the method described in section 1.4.1 we obtain the following table with  $S_0 = 89200$

Variable	Estimated value
$S_0$	89200
$I_0$	96.62144000
$\tau$	$0.1983750434 \times 10^{-4}$
$\nu_1$	0.05103676158
$\nu_2$	0.8469815172
$C_\infty$	70505.24235
$S_\infty$	18791.37909

Table 1.10: List of parameters obtained for the influenza-like illness in France in 1991w45-1992w10.

In Figures 1.31, 1.32, 1.33, and 1.34 we provide model (1.1.1)-(1.1.2) output for the influenza epidemic in France during the period from week 45 of the year 1991 to week 10 of the year 1992 for the parameters in Table 1.4.4. In Figure 1.31, 1.32 we compare the model output to the reported case data. In Figure 1.33 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.34 we illustrate the epidemic final size as a function of the turning point of the epidemic.

1. Identifying the number of unreported cases in SIR epidemic models

---

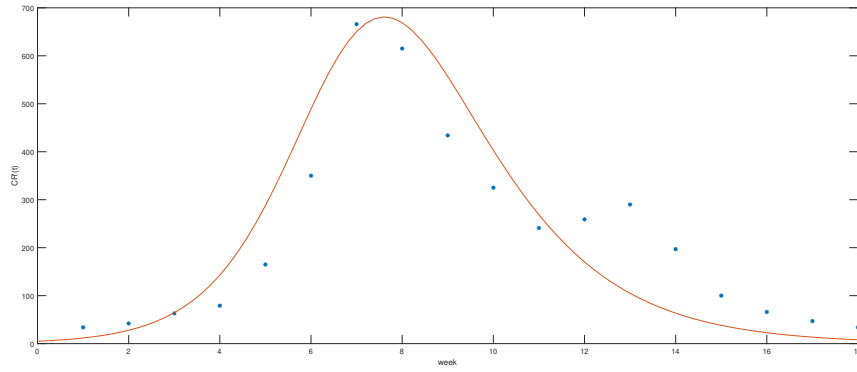


Figure 1.31: *The weekly reported case data(blue dots) and model output graph of  $\nu_1 I(t)$  for the influenza-like illness in France in 1991w45-1992w10*

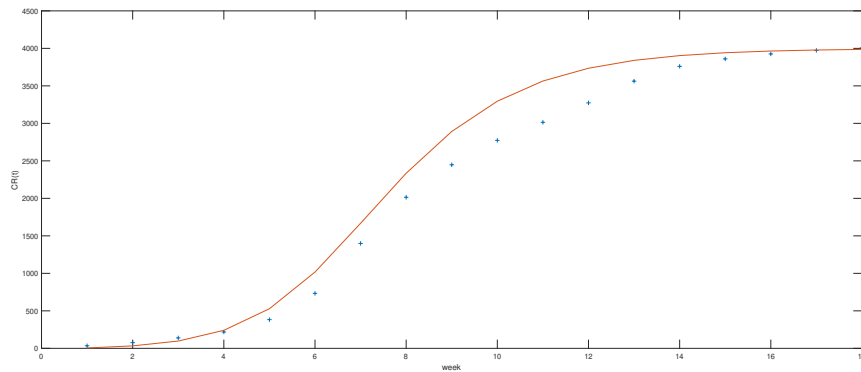


Figure 1.32: *The weekly reported cumulative case data(blue dots) and model output graph of  $CR(t)$  for the influenza-like illness in France in 1991w45-1992w10*

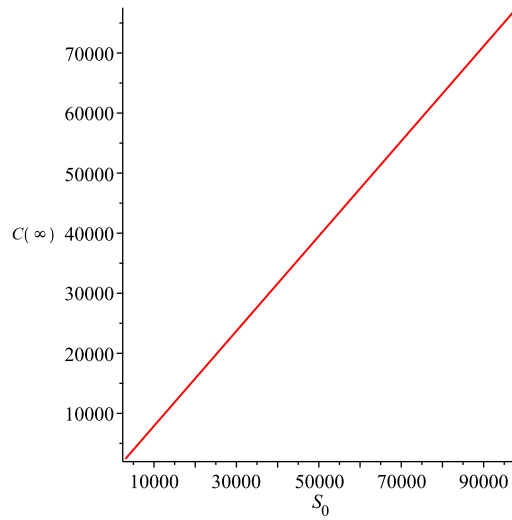


Figure 1.33: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$  for the influenza-like illness in France in 1991w45-1992w10*

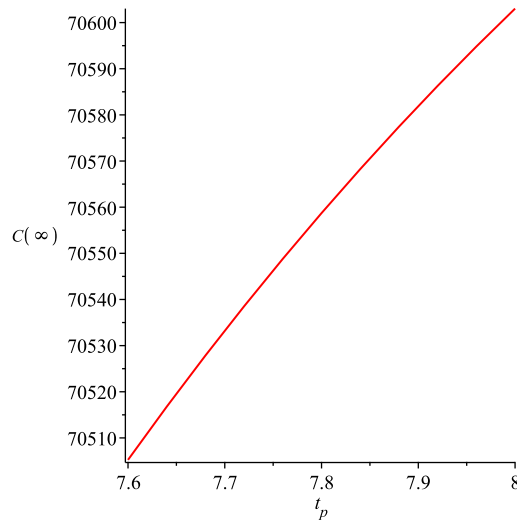


Figure 1.34: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and the turning point  $t_p$  for the influenza-like illness in France in 1991w45-1992w10*

#### V. Period from week 51 of the year 1992 to week 17 of the year 1993

From the data in [73](see Figure 1.35), we obtain the values  $CR_\infty = 3980$ ,  $CR(tp) = 1920$ ,  $tp = 8.7$ ,  $CR'(tp) = 502$ .

## 1. Identifying the number of unreported cases in SIR epidemic models

---

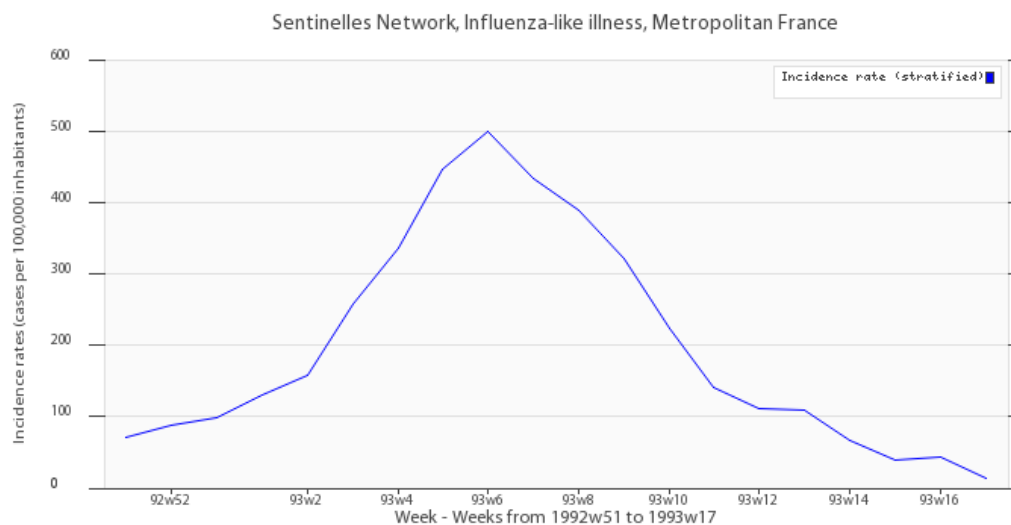


Figure 1.35: *This figure indicates the outbreak of the epidemics during the period from week 51 of the year 1992 to week 17 of the year 1993 (This figure is from [73])*

Next, applying the method described in section 1.4.1 we obtain the following table with  $S_0 = 89200$

Variable	Estimated value
$S_0$	89200
$I_0$	35.99505440
$\tau$	$0.4420270591 \times 10^{-4}$
$\nu_1$	0.6102895730
$\nu_2$	2.820697355
$C_\infty$	22375.16185
$S_\infty$	66860.83320

Table 1.11: List of parameters obtained for the influenza-like illness in France in 1992w51-1993w17.

In Figures 1.36, 1.37, 1.38, and 1.39 we provide model (1.1.1)-(1.1.2) output for the influenza epidemic in France during the period from week 51 of the year 1992 to week 17 of the year 1993 for the parameters in Table 1.4.4. In Figure 1.36, 1.37 we compare the model output to the reported case data. In Figure 1.38 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.39 we illustrate the epidemic final size as a function of the turning point of the epidemic.



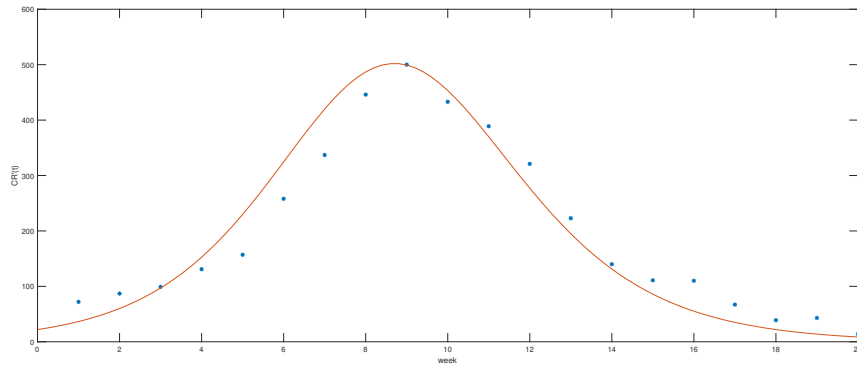


Figure 1.36: *The weekly reported case data(blue dots) and model output graph of  $\nu_1 I(t)$  for the influenza-like illness in France in 1992w51-1993w17*

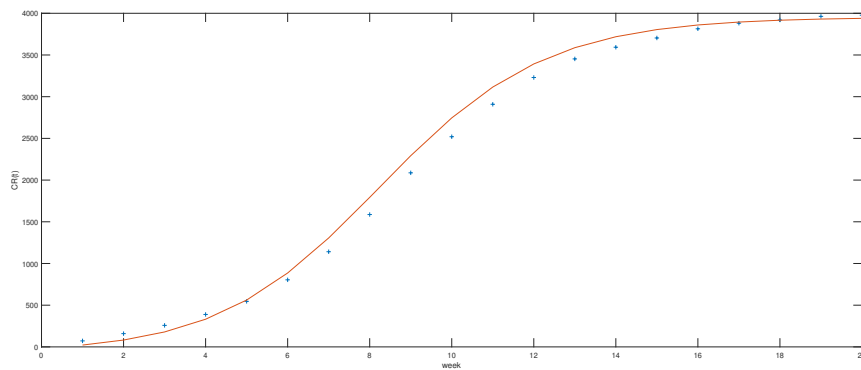


Figure 1.37: *The weekly reported cumulative case data(blue dots) and model output graph of  $CR(t)$  for the influenza-like illness in France in 1992w51-1993w17*

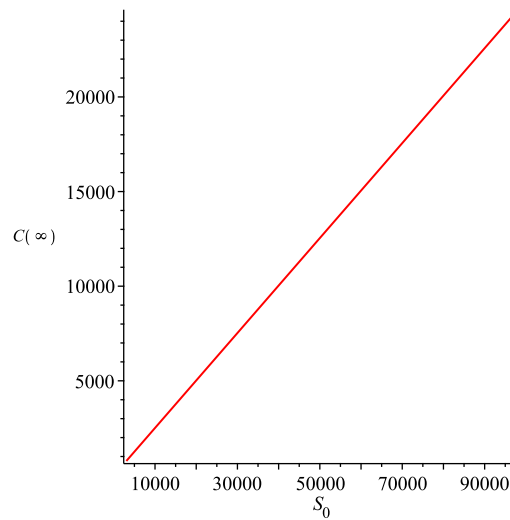


Figure 1.38: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$  for the influenza-like illness in France in 1992w51-1993w17*

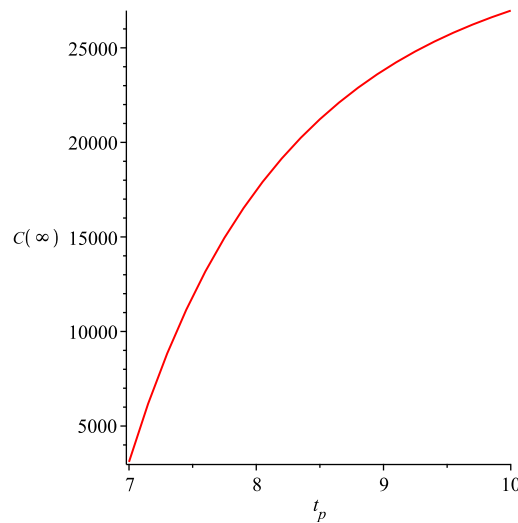


Figure 1.39: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and the turning point  $t_p$  for the influenza-like illness in France in 1992w51-1993w17*

#### VI. Period from week 45 of the year 1993 to week 5 of the year 1994

From the data in [73](see Figure 1.40), we obtain the values  $CR_\infty = 5760$ ,  $CR(tp) = 2730$ ,  $tp = 5.2$ ,  $CR'(tp) = 1568$ .

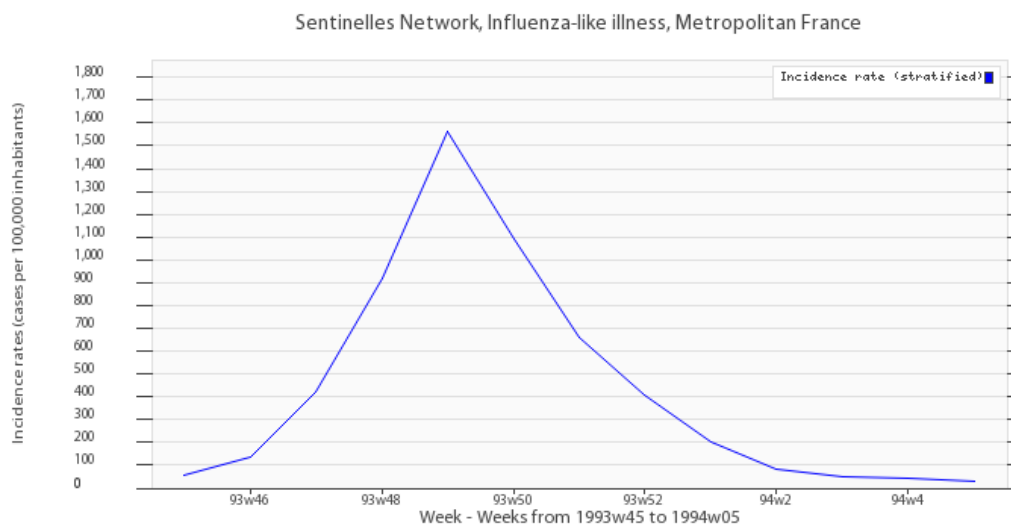


Figure 1.40: *This figure indicates the outbreak of the epidemics during the period from week 45 of the year 1993 to week 5 of the year 1994 (This figure is from [73])*

Next, applying the method described in section 1.4.1 we obtain the following table with  $S_0 = 89200$

Variable	Estimated value
$S_0$	89200
$I_0$	14.80889480
$\tau$	$0.7255641309 \times 10^{-4}$
$\nu_1$	0.9864734585
$\nu_2$	4.308166928
$C_\infty$	30432.25414
$S_\infty$	58782.55475

Table 1.12: List of parameters obtained for the influenza-like illness in France in 1993w45-1994w5.

In Figures 1.41, 1.42, 1.43, and 1.44 we provide model (1.1.1)-(1.1.2) output for the influenza epidemic in France during the period from week 45 of the year 1993 to week 5 of the year 1994 for the parameters in Table 1.4.4. In Figure 1.41, 1.42 we compare the model output to the reported case data. In Figure 1.43 we illustrate the epidemic final size as a function of the initial number of susceptibles  $S_0$ . In Figure 1.44 we illustrate the epidemic final size as a function of the turning point of the epidemic.

1. Identifying the number of unreported cases in SIR epidemic models

---

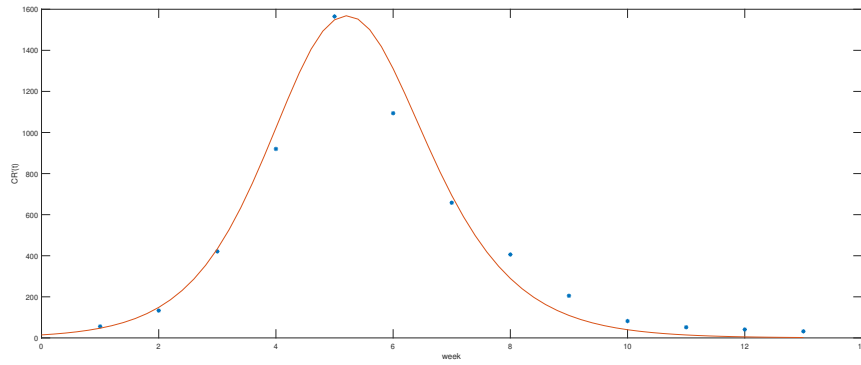


Figure 1.41: *The weekly reported case data(blue dots) and model output graph of  $\nu_1 I(t)$  for the influenza-like illness in France in 1993w45-1994w5*

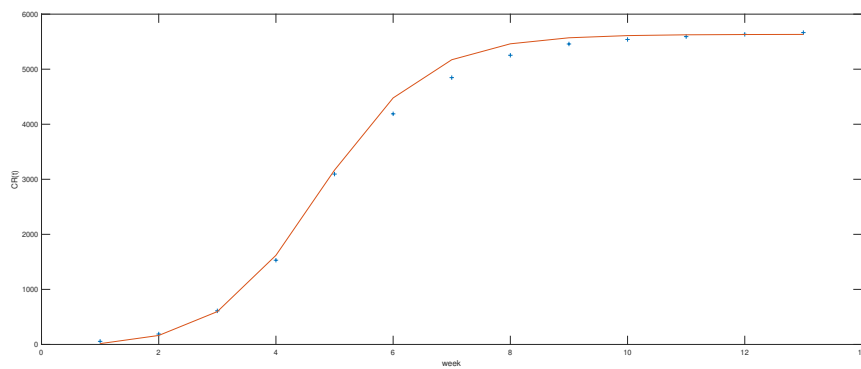


Figure 1.42: *The weekly reported cumulative case data(blue dots) and model output graph of  $CR(t)$  for the influenza-like illness in France in 1993w45-1994w5*

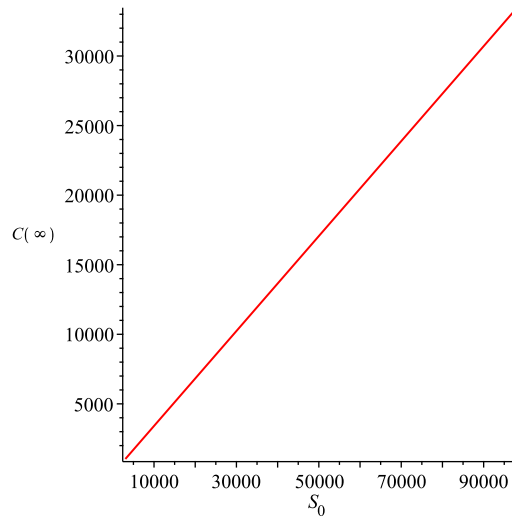


Figure 1.43: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and  $S_0$  for the influenza-like illness in France in 1993w45-1994w5*

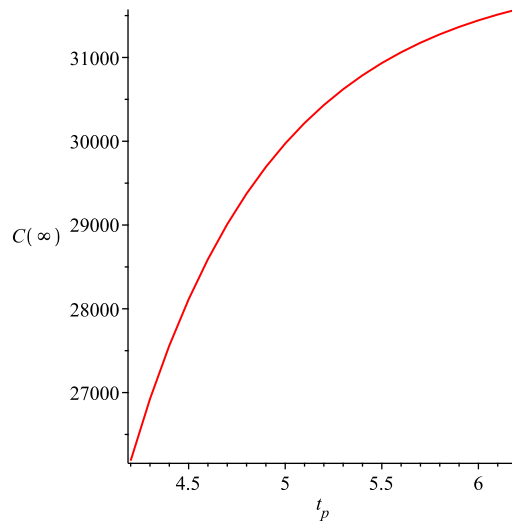


Figure 1.44: *The relationship between the total case number at the end of epidemic  $C(\infty)$  and the turning point  $t_p$  for the influenza-like illness in France in 1993w45-1994w5*

## Chapter 2

# Some preliminaries of differential algebra and application to system identifiability

In this chapter we present some necessary preliminaries of differential algebra and differential elimination for our work. What we present here is based on the input-output approach from the papers [43], [66], [67], [7], [47], [8] [9], [11], [10], [49],[53], [61], [69], [54].

### 2.1 Some basic definitions of differential algebra

The definitions below are based on the papers [9], [11], [10], the books [59], [39], the lecture [54], and Maple Help [74].

**Definition 2.1.1.** Let  $R$  be a commutative ring with identity.  $R$  is called a differential ring if there exists a finite set of maps  $\{\partial_1, \dots, \partial_n\}$  in which  $\partial_i : R \rightarrow R$ ,  $i \in \{1, 2, \dots, n\}$ , satisfied

- i.  $\partial_i(a + b) = \partial_i(a) + \partial_i(b)$  for all  $a, b \in R$ ,
- ii.  $\partial_i(ab) = b\partial_i(a) + a\partial_i(b)$  for all  $a, b \in R$ ,
- iii.  $\partial_i\partial_j(a) = \partial_j\partial_i(a)$  for all  $a \in R$ ,  $1 \leq i, j \leq n$ .

The map  $\partial_i$  here is called a derivation of the ring  $R$ . If a set of all derivations of a differential ring  $R$  has only one element, we call  $R$  an ordinary differential ring.

From now on, for simplicity the term ring  $R$  is always understood as a commutative ring with identity.

**Definition 2.1.2.** If a differential ring  $R$  is a field, it is called a differential field.

Let  $K$  be a field of characteristic zero, we denote by  $K[X]$  the ring of all polynomials that  $X$  is the set of all variables and the coefficients are in  $K$ .

**Definition 2.1.3.** Suppose that  $\{\partial_1, \dots, \partial_n\}$  is the set of all the derivations of the differential field  $R$ , and  $X := \{x_1, x_2, \dots, x_k\}$ ,  $K$  is a field of characteristic zero. Let  $\mathcal{D} = \{\partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n} : i_1, \dots, i_n \in \mathbb{N}\}$ , then  $K[\mathcal{D}(X)]$  is called the ring of differential polynomials with coefficients in  $K$  where  $\mathcal{D}(X) = \{\theta x_i | \theta \in \mathcal{D}, x_i \in X, i = 1, 2, \dots, k\}$ . We denote  $K[\mathcal{D}(X)]$  by  $K\{X\}$ . Every element of  $K\{X\}$  is called a differential polynomial and  $x_1, x_2, \dots, x_k$  are called the indeterminates of  $K\{X\}$ .

*Example 2.1.4.* Consider  $C^\infty[0, \infty)$  with the derivation  $\partial(x) := x'$  for every  $x \in C^\infty[0, \infty)$ . Suppose that  $x_1, x_2, \dots, x_n$  are the elements of  $C^\infty[0, \infty)$  and  $K$  is a subfield of  $\mathbb{R}$ . The ring  $K[x_1, x_2, \dots, x_n, x_1', \dots, x_n', \dots, x_1'', \dots]$  is a ring of differential polynomial in indeterminates  $x_1, \dots, x_n$  with coefficients in  $K$  and denoted by  $K\{x_1, x_2, \dots, x_n\}$ . Every element of  $K\{x_1, x_2, \dots, x_n\}$  is a differential polynomial. For instance,  $a_3 x_1' + a_1 x_1' x_2' x_3' - a_2 x_1^3 (x_2')^2 x_3''$  is a differential polynomial of  $K\{x_1, x_2, \dots, x_n\}$  with  $a_1, a_2, a_3 \in K$ .

**Definition 2.1.5.** Given a differential ring  $R$ , an ideal  $I$  of  $R$  is called the differential ideal of  $R$  if  $\partial(a) \in I$  for all the derivations  $\partial$  of  $R$  and  $a \in I$ .

Let  $R$  be a commutative ring and  $S \subset R$ , the smallest ideal containing  $S$  denoted by  $(S)$  is called an ideal generated by  $S$ . The smallest differential ideal containing  $S$  denoted by  $[S]$  is called a differential ideal generated by  $S$ .

*Remark 2.1.6.* Suppose that  $\{\partial_1, \dots, \partial_n\}$  is the set of all the derivations of the ring  $R$ . Let  $\mathcal{D} = \{\partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n} : i_1, \dots, i_n \in \mathbb{N}\}$ , then  $[S]$  is the ideal generated by  $\theta(S)$  with  $\theta \in \mathcal{D}$ .

**Definition 2.1.7.** Let  $R$  be a commutative ring and  $I$  is an ideal of  $R$ . The radical of  $I$  denoted by  $\sqrt{I}$  is the set  $\{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$ .

An ideal  $I$  of  $R$  is said to be radical if it coincides with its radical.

An ideal  $I$  of a differential ring  $R$  is called a radical differential ideal if  $I$  is a differential ideal and is also radical.

Let  $R$  be a differential ring, and  $S \subset R$ , the smallest radical differential ideal containing  $S$  denoted by  $\{S\}$  is called a radical differential ideal generated by  $S$ .

**Definition 2.1.8.** Let  $R$  be a differential ring, the differential ideal  $P$  of  $R$  is called a prime differential ideal if  $ab \in P$  then  $a \in P$  or  $b \in P$ .

A relation  $\leq$  over a set  $A$  is called a total order over  $A$  if the following conditions hold

- i.  $a \leq a$  for every  $a \in A$ .
- ii. If  $a \leq b$  and  $b \leq a$  then  $a = b$  for every  $a, b \in A$ .
- iii. If  $a \leq b$  and  $b \leq c$  then  $a \leq c$  for every  $a, b, c \in A$ .
- iv.  $a \leq b$  or  $b \leq a$  for every  $a, b \in A$ .

Suppose that  $\leq$  is a total order over  $A$ . A strict total order  $<$  over  $A$  is a relation over  $A$  such that for every  $a, b \in A$ ,  $a < b$  if  $a \leq b$  and  $a \neq b$ .

**Definition 2.1.9.** Suppose that  $\{\partial_1, \dots, \partial_n\}$  is the set of all the derivations of the differential field  $R$ , and  $X := \{x_1, x_2, \dots, x_k\}$ . Let  $\mathcal{D} = \{\partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n} : i_1, \dots, i_n \in \mathbb{N}\}$ . A strict total order  $<$  over the set of the derivatives  $\mathcal{D}(X) = \{\theta x_i | \theta \in \mathcal{D}, x_i \in X, i = 1, 2, \dots, k\}$  is called to be a ranking if for every  $\delta \in \{\partial_1, \dots, \partial_n\}$

- i.  $u < \delta u$ ,
- ii.  $u < v$  then  $\delta u < \delta v$ ,

for all  $u, v \in \mathcal{D}(X)$ .

A ranking such that  $u < v$  implies  $\delta u < \phi v$  for all  $\delta, \phi \in \mathcal{D}$  is said to be an elimination ranking and written  $[u] < [v]$ .

**Definition 2.1.10.** Let  $q$  be a polynomial of  $K\{x_1, \dots, x_k\}$  and  $<$  is a ranking on the  $\mathcal{D}(X)$ . The leader  $u$  of  $q$  is the largest derivative  $\partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n} x_j$  of  $\mathcal{D}(X)$  with respect to the ranking  $<$  which appear in  $q$ .

Let  $d$  be the degree of  $u$  in  $q$ , then the coefficient of  $u^d$  is called the initial of  $q$ , denoted by  $I_q$ . The derivative  $S_q = \frac{\partial q}{\partial u}$  is called the separant of  $q$ , and  $u^d$  is called the rank of  $q$ .

*Example 2.1.11.* We consider again the example 2.1.4, suppose that  $<$  is the elimination ranking on  $K\{x_1, x_2, \dots, x_n\}$  such that  $x_1 < x_2 < \dots < x_n$ . It means that  $x_1 < x_1' < x_1'' < \dots < x_2 < x_2' < \dots < x_n < x_n' < \dots$ . It is clear that  $x_1''$  is the leader of the differential polynomial  $a_1 x_1 x_2^2 x_3^3 - a_2 (x_1'')^2 + a_3 x_1 x_2'$ . The initial of this polynomial is  $-a_2$  and the separant of this polynomial is  $-2a_2 x_1''$  where the rank is  $(x_1'')^2$ .

**Definition 2.1.12.** Let  $q_1$  and  $q_2$  be two polynomials of  $K\{u_1, \dots, u_k\}$ , and  $u^d$  is the rank of  $q_1$ .  $q_2$  is said to be partially reduced w.r.t (with respect to)  $q_1$  if  $q_2$  does not contain any proper derivative of  $u$ .

$q_2$  is said to be reduced w.r.t  $q_1$  if it is partially reduced w.r.t  $q_1$  and its degree in  $u$  is less than  $d$ .

A set of polynomials  $A$  is called an autoreduced set if its elements are pairwise reduced.

**Definition 2.1.13.** An autoreduced set  $C$  of a differential ideal  $I$  is said to be a characteristic set if  $I$  does not contain any non-zero element reduced w.r.t all the elements of  $C$ .

**Definition 2.1.14.** A set of polynomials  $A$  is called an orthonomic set if the initial and separants of its elements are in  $K$ .

For an ordinary differential ring  $K\{u_1, \dots, u_k\}$ , an ideal of  $K\{u_1, \dots, u_k\}$  generated by an autoreduced and orthonomic set is a prime ideal (see **Boulier et al.** [9]).

*Example 2.1.15.* Let  $\mathbb{Q}(p)$  be the field of the rational expressions of  $p_1, \dots, p_4 \in \mathbb{R}$ , consider the ring  $\mathbb{Q}(p)\{x_1, x_2, x_3\}$  of differential polynomials in indeterminates  $x_1, x_2, x_3$  with the coefficients in  $\mathbb{Q}(p)$ . We equip on  $\mathbb{Q}(p)\{x_1, x_2, x_3\}$  the elimination ranking such that

$$x_1 < x_1' < \dots x_1^{(n)} < \dots < x_2 < x_2' < \dots < x_3 < x_3' < \dots$$



Let  $A$  be the set of three differential polynomials  $q_1 = p_4x'_1 + p_1x_1x_2x_3$ ,  $q_2 = x'_2 - p_1x_1x_2x_3 + p_3x_2$ ,  $q_3 = p_4^2x'_3 + 2p_1p_2x_1x_2^2x_3^3 - 2p_2p_3x_2^2x_3^2$  of  $\mathbb{Q}(p)\{x_1, x_2, x_3\}$ . The leaders of  $q_1, q_2, q_3$  are respectively  $x'_1, x'_2, x'_3$ .  $q_1$  does not contain any derivative of  $x'_2, x'_3$  so that  $q_1$  is partial reduced w.r.t  $q_2, q_3$ , and then  $q_1$  is reduced w.r.t  $q_2, q_3$ . The initials and separants of  $q_1, q_2, q_3$  are (respectively  $p_4, 1, p_4$ ) in  $\mathbb{Q}(p)$ , then  $A$  is an autoreduced and orthonomic set.

## 2.2 Introduction to Rosenfeld-Gröbner algorithm implemented in Maple

What we present in this section is based on the papers [9], [11], [10], and [74]

Rosenfeld-Gröbner algorithm works with a list of differential polynomials and a ranking on a differential ring of differential polynomials that is understood as input. Suppose that we have a system of differential equations in which every left-hand side of each equation can be considered as a differential polynomial when the right-hand side is zero. Rosenfeld-Gröbner algorithm splits the system into cases and in each case, there is no one equation that is the consequence of the others. To understand precisely, we present here the Ritt's reduction with *pseudodivision* of the differential polynomials that is a generalization of the Euclidean division.

Let  $f$  and  $g$  be two polynomials in one variable  $x$  with coefficients in a ring. Denote by  $\deg(f, x)$  the degree of  $f$  in  $x$ . The *pseudodivision* of  $f$  by  $g$  is given by the formula  $c^d f = gq + r$  in which  $c$  is the coefficient of the leader  $x$  of  $g$ ,  $d = \deg(f, x) - \deg(g, x) + 1$ , and  $\deg(r, x) < \deg(g, x)$ . We call the polynomial  $r$  *pseudoremainder* of  $f$  by  $g$ , and denote  $r := \text{prem}(f, g, x)$ .

Let  $A = \{f_1, f_2, \dots, f_n\}$  be the set of differential polynomials of  $K\{X\} \setminus K$  in which  $K$  is a field and  $X$  is a set of indeterminates, and  $g$  be a differential polynomial of  $K\{X\} \setminus K$ . Suppose that  $x_i$  is the leader of  $f_i$  for  $i \in \{1, 2, \dots, n\}$ . Now we perform a procedure reducing  $g$  w.r.t every element of  $A$  that we denote by  $\text{Rittreduction}(g, A)$ . This procedure generate a sequence  $g_0, g_1, \dots, g_s$  as follows

- First step, we set  $g_0 = g$ .
- Next steps, we compute  $g_{k+1}$  from  $g_k$  in the following cases

**Case 1.** If  $g_k$  does not depend on any proper derivative of  $x_i$  and the  $\deg(g_k, x_i) < \deg(f_i, x_i)$  for every  $i \in \{1, 2, \dots, n\}$  then we stop the procedure and assign  $g_s := g_k$ .

**Case 2.** If  $\deg(g_k, x_i) \geq \deg(f_i, x_i)$  for some index  $i \in \{1, 2, \dots, n\}$  then we assign  $g_{k+1} := \text{prem}(g_k, f_i, x_i)$ .

**Case 3.** If there is some index  $i \in 1, 2, \dots, n$  such that  $g_k$  depends on some proper derivative  $x_i^{(m)}$  of  $x_i$  then we assign  $g_{k+1} := \text{prem}(g_k, f_i^{(m)}, x_i^{(m)})$ .

- Last step, when the procedure finishes we obtain the differential polynomial  $g_s$  and we define  $\text{Rittreduction}(g, A)$  to be  $g_s$ .

### The Rosenfeld-Gröbner algorithm

- The input of the algorithm is a list  $P$  of differential polynomials and a ranking.

-The output of the algorithm is a finite family  $A_1, A_2, \dots, A_s$  of finite subsets of  $K\{X\} \setminus K$  in which  $K$  is a field and  $X$  is a set of indeterminates. Denote by  $\mathcal{A}_i$  the differential radical ideal for every  $i \in \{1, 2, \dots, s\}$  such that for any differential polynomial  $p \in K\{X\}$ ,

$$p \in \mathcal{A}_i \text{ if and only if } \text{Rittreduction}(p, A_i) = 0.$$

Denote by  $\mathcal{P}$  the radical ideal generated by  $P$ , Rosenfeld-Gröbner algorithm gives us the relationship between  $\mathcal{P}$  and  $\mathcal{A}_i$  as follows

$$\mathcal{P} = \bigcap_{i=1}^s \mathcal{A}_i.$$

We have  $\mathcal{P} = K\{X\}$  when  $s = 0$ . Moreover, an algorithm for testing membership problem in  $\mathcal{P}$  is obtained as follows:

Given a differential polynomial  $p \in K\{X\}$ , we have  $p \in \mathcal{P}$  if and only if  $\text{Rittreduction}(p, A_i) = 0$  for every  $i \in \{1, 2, \dots, s\}$ .

*Example 2.2.1.* Let  $\mathbb{Q}(p)$  be the field of the rational expressions of  $p_1, p_2, p_3 \in \mathbb{R}$ , consider the ring  $\mathbb{Q}(p)\{x_1, y\}$  of differential polynomials in indeterminates  $x_1, y$  with the coefficients in  $\mathbb{Q}(p)$ . We equip on  $\mathbb{Q}(p)\{x_1, y\}$  the elimination ranking such that

$$y < y' < \dots < y^{(n)} < \dots < x_1 < x_1' < \dots < x_1^{(n)} < \dots$$

Let  $P$  be the list of two differential polynomials  $p_3x_1' + p_1x_1y, y' - p_1x_1y + p_2y$ . Applying Rosenfeld-Gröbner algorithm with the elimination ranking above, we obtain a family of two lists  $A_1 = \{x_1', y\}$  and  $A_2 = \{p_1x_1y - y' - p_2y, p_3yy'' - p_3(y')^2 + p_1y^2y' + p_1p_2y^3\}$ . We conduct the procedure  $\text{Rittreduction}(y + 1, A_1)$  as follows:

- Firstly, we assign  $g_0 := y + 1$ .
- Secondly, we see that the leaders of  $f_1 := x_1'$  and  $f_2 := y$  are respectively  $x_1'$  and  $y$ . Since  $\deg(g_0, y) = \deg(f_2, y)$  then we assign  $g_1 := \text{prem}(g_0, f_2, y) = 1$ .
- Finally,  $\text{Rittreduction}(y + 1, A_1) := g_1 = 1$ .

Since  $\text{Rittreduction}(y + 1, A_1) \neq 0$  then  $y + 1 \notin \mathcal{A}_1$ , then  $y + 1 \notin \mathcal{P}$  where  $\mathcal{P}$  is the differential radical ideal generated by  $P$ .

The Rosenfeld-Gröbner algorithm is implemented in Maple. To use Rosenfeld-Gröbner algorithm we have to call the package "*DifferentialAlgebra*" in Maple. Suppose that we have a list of differential polynomials and a ranking on them.

Firstly, we have to build the differential ring of polynomials specifying the variables and a ranking for these polynomials by calling the function `DifferentialRing` (`derivations = ...`, `blocks = ...`, `arbitrary = ...`) in which "`derivations = ...`" is a list of independent variables specifying the possible dependency of the dependent variables, "`blocks = ...`" is a list of the dependent variables, possibly including sublists of them; only one level of nested sublists is allowed, and "`arbitrary = [...]`" is a list of algebraically independent arbitrary objects; these can be dependent variables (functions of one or many variables specified in `derivations`) or other variables (not those specified in `derivations`) ( see [74]).

Secondly, after we have  $R = \text{DifferentialRing}(\text{derivations} = \dots, \text{blocks} = \dots, \text{arbitrary} = \dots)$ , we can call Rosenfeld-Gröbner algorithm by the function `RosenfeldGroebner(sys, R)` where "`sys`" is a list or a set of differential equations or inequations; all rational in the independent and dependent variables and their derivatives. `RosenfeldGroebner(sys, R)` splits the given system `sys` into cases, each one specified by their

Equations and Inequations. One and only one of these cases is the general case, and the rest are singular cases that may exist only when  $\text{sys}$  is nonlinear. In each of the cases returned, each equation is not the consequence of the other equations and all the integrability conditions are taken into account. The relation between the system  $\text{sys}$  and the systems of each of these cases is the solutions, general and singular, of  $\text{sys}$ , are given by the union of the general solutions of each of the returned systems (see [74]).

Finally, after calling the function  $\text{I}=\text{RosenfeldGroebner}(\text{sys}, R)$ , we can test if a differential polynomial  $p$  of the differential ring  $R$  belongs to the radical ideals generated by the lists of  $\text{I}$  or not by calling the function  $\text{BelongsTo}(p, \text{I})$ . This function returns true if  $p$  belongs to the differential ideals represented by  $\text{I}$ , else it returns false ( see [74]).

## 2.3 Differential algebra approach for system identifiability

Consider the system as follows

$$\begin{cases} x'(t, p) = f(x(t, p), p), \\ x(0, p) = x_0(p), \quad t \geq 0 \\ y(t, p) = h(x(t, p), p). \end{cases} \quad (2.3.1)$$

where  $p \in \Omega$ , with  $\Omega$  is open subset of  $\mathbb{R}^s$ , is a parameter vector,  $x(t, p) \in \mathbb{R}^n$  is a state variable and  $y(t, p) \in \mathbb{R}^r$  is the output function.

Assuming that, for each  $p \in \Omega$ , (2.3.1) has a unique solution, and there exists an open connected subset  $M(p)$  of  $\mathbb{R}^n$  such that  $x(t, p) \in M(p)$  for all  $t \geq 0$  and  $f(\bullet, p)$ ,  $h(\bullet, p)$  are analytic on  $M(p)$ .

The problem that we want to solve is the following

*Problem 2.3.1.* Consider the system (2.3.1), then what is the relationship between  $p, \bar{p} \in \Omega$  from the condition  $y(t, p) = y(t, \bar{p})$ ?

The following proposition gives us an easy way to predict the relation between  $p, \bar{p} \in \Omega$  from the condition  $y(t, p) = y(t, \bar{p})$ , and we can have similar results from the papers of **Evans et al.** [24], [25].

**Proposition 2.3.2.** *Given a system (2.3.1), and parameters  $p, \bar{p} \in \Omega$ . Suppose that there exists  $\epsilon \neq 0$  such that*

$$\epsilon x \in M(p), \quad (2.3.2)$$

$$\epsilon x_0(\bar{p}) = x_0(p), \quad (2.3.3)$$

$$f(\epsilon x, p) = \epsilon f(x, \bar{p}), \quad (2.3.4)$$

$$h(\epsilon x, p) = h(x, \bar{p}), \quad (2.3.5)$$

for all  $x \in M(\bar{p})$ , then  $y(t, p) = y(t, \bar{p})$  for all  $t \geq 0$ .

## 2. Some preliminaries of differential algebra and application to system identifiability

*Proof.* Define that  $z(t) = \epsilon x(t, \bar{p})$  for all  $t \geq 0$ . Since  $x(t, \bar{p}) \in M(\bar{p})$  then  $z(t) = \epsilon x(t, \bar{p}) \in M(p)$  for all  $t \geq 0$ .

$$\begin{aligned} z'(t) &= \epsilon x'(t, \bar{p}) \\ &= \epsilon f(x(t, \bar{p}), \bar{p}) \quad (\text{since (2.3.1)}) \\ &= f(\epsilon x(t, \bar{p}), p) \quad (\text{since (2.3.4)}) \\ &= f(z(t), p). \end{aligned}$$

Moreover,  $z(0) = \epsilon x_0(\bar{p}) = x_0(p)$  (since (2.3.3)). It means that  $z(t)$  is a solution of (2.3.1) with respect to  $p$ . By the uniqueness of solution of (2.3.1),  $z(t) = x(t, p)$  for all  $t \geq 0$ . Therefore,

$$\begin{aligned} y(t, p) &= h(x(t, p), p) \\ &= h(z(t), p) \\ &= h(\epsilon x(t, \bar{p}), p) \\ &= h(x(t, \bar{p}), \bar{p}) \quad (\text{since (2.3.5)}) \\ &= y(t, \bar{p}). \end{aligned}$$

□

Consider the system (2.3.1) in which  $f_i$  is a rational function for all  $i = 1, \dots, n$  and  $h_j$  is a polynomial function for all  $j = 1, \dots, r$ . Now we consider Problem 2.3.1 with  $r = 1$ .

If there is some  $i \in \{1, \dots, n\}$  such that  $f_i(x, p) = \frac{q_i(x, p)}{q(x, p)}$  where  $q_i(x, p), q(x, p)$  are polynomials, by introducing new variable  $x_{n+1}(t, p) = \frac{1}{q(x, p)}$ , we have  $x'_{n+1}(t, p) = -\frac{1}{q^2(x, p)} \sum_{i=1}^n \frac{\partial q}{\partial x_i} x'_i(t)$ . It derives that

$$x'_{n+1}(t, p) = -x_{n+1}^3(x, p) \sum_{i=1}^n \frac{\partial q}{\partial x_i} q_i(x(t, p), p)$$

in which the right hand side is a polynomial function in  $x_1, \dots, x_n, x_{n+1}$  (see [47])

Hence we can assume that from (2.3.1), by introducing a new state variable  $z = (x_1, \dots, x_n, x_{n+1})$  we can obtain a new system

$$\begin{cases} z'(t, p) = F(z(t, p), p), \\ z(0, p) = z_0(p), \quad t \geq 0 \\ y(t, p) = H(z(t, p), p). \end{cases} \quad (2.3.6)$$

where  $F_i, H$  are polynomial functions for all  $i = 1, \dots, n, n + 1$ .

Next, we present an approach by using differential algebra and differential elimination for finding identifiable parameters for the system (2.3.1). This part is based on the input-output approach from the papers [43], [66], [67], [7], [47], [8] [9], [11], [10], [49], [53], [61], [69].

**Step 1.** The system (2.3.1) is rewritten as follows

$$\begin{cases} z'(t, p) - F(z(t, p), p) = 0, & t \geq 0 \\ y(t, p) - H(z(t, p), p) = 0. \end{cases} \quad (2.3.7)$$

Here we ignore the initial condition for simplicity. However, the initial condition will be used later for another analysis of the method.

Consider the differential ring  $\mathbb{Q}(p)\{z_1, \dots, z_{n+1}, y\}$  where  $\mathbb{Q}(p)$  is the field of the rational expressions of  $p_1, \dots, p_s$ . From (2.3.7), we make some arrangements on the polynomials of the left hand side of (2.3.7) to obtain new polynomials such that they form an autoreduced and orthonomic set, and we call  $\mathcal{P}$  the prime differential ideal generated by this set. We use the ranking in  $\mathbb{Q}(p)\{z_1, \dots, z_{n+1}, y\}$  to eliminate the state variable  $z$  that is described as follows

$$y < y' < \dots < y^{(n)} < \dots < z_1 < \dots < z_1^{(n)} < \dots < z_2 < \dots$$

or simply

$$[y] < [z_1] < \dots < [z_{n+1}].$$

**Step 2.** Using the Rosenfeld-Gröbner algorithm implemented in Maple (see [9], [11]) to decompose the prime ideal  $\mathcal{P}$ . Rosenfeld-Gröbner algorithm with the elimination ranking above decomposes  $\mathcal{P}$  as an intersection of radical differential ideals. Concretely, Rosenfeld-Gröbner algorithm splits (2.3.7) into cases, each one specified by their equations and inequations. In each of the cases returned, no one equation is the consequence of the others (see [74]). Among these cases, only one is the general solution and others are singular solutions of (2.3.7). Consider the initial condition given in (2.3.6) for analyzing the singular cases. If these cases cannot describe the complete system, the system (2.3.7) with the initial condition, then the general case is corresponding to (2.3.7).

Assume that (2.3.7) is corresponding to the general case. It means that in the decomposition of  $\mathcal{P}$ , all the singular cases are redundant. Note that the Rosenfeld-Gröbner algorithm is implemented in Maple, and the general case after using Rosenfeld-Gröbner algorithm is just a list of the generators of a differential ideal. This set of generators is called a characteristic presentation of  $\mathcal{P}_G$  (see [11]). One of the generators of this characteristic presentation is a differential polynomial concerning only the indeterminate  $y$  with the ranking we used above, and it can be written by the form  $\phi_{\gamma+1}(y) + \sum_{i=1}^{\gamma} c_i(p)\phi_i(y)$  where  $\phi_i(y)$  is the differential polynomial w.r.t the indeterminate  $y$  for all  $i = 1, \dots, \gamma, \gamma + 1$ , and  $c_i(p)$  is the rational function of  $p_1, \dots, p_s$  for all  $i = 1, \dots, \gamma$ . We call this polynomial an output polynomial of (2.3.7). We have

$$\phi_{\gamma+1}(y(t, p)) + \sum_{i=1}^{\gamma} c_i(p)\phi_i(y(t, p)) = 0. \quad (2.3.8)$$

It is called an output equation of (2.3.7) ( see [67], [7],[49], [61] [69]).

Consider the Wronskian of the polynomials  $\phi_i(y), \dots, \phi_\gamma(y)$  as follows

$$\begin{vmatrix} \phi_1(y) & \phi_2(y) & \dots & \phi_\gamma(y) \\ \phi_1'(y) & \phi_2'(y) & \dots & \phi_\gamma'(y) \\ \dots & \dots & \dots & \dots \\ \phi_1^{(\gamma-1)}(y) & \phi_2^{(\gamma-1)}(y) & \dots & \phi_\gamma^{(\gamma-1)}(y) \end{vmatrix}. \quad (2.3.9)$$

We have the following proposition that mentioned in [67], [7], [69]

**Proposition 2.3.3.** *Assume that  $p, \bar{p}$  are the parameters of the system (2.3.7). If the Wronskian (2.3.9) does not belong to the ideal  $\mathcal{P}$ , then from the condition  $y(t, p) = y(t, \bar{p})$  we have  $c_i(p) = c_i(\bar{p})$  for all  $i = 1, \dots, \gamma$ .*

*Proof.* The Wronskian does not belong to the ideal  $\mathcal{P}$  implies that this determinant is not identically zero. It means that if  $y(t)$  is the solution of (2.3.7) such that the Wronskian at this solution is not identically zero. It derives the linear independence of the functions  $\phi_1(y(t, p)), \dots, \phi_\gamma(y(t, p))$ .

If we have  $y(t, p) = y(t, \bar{p})$  for all  $t \geq 0$ , then  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$  and for all  $t \geq 0$ . It implies that  $\phi_j(y(t, p)) = \phi_j(y(t, \bar{p}))$  since  $\phi_j$  is a differential polynomial w.r.t the indeterminate  $y$  for all  $j = 1, \dots, \gamma, \gamma + 1$ .

From (2.3.8), substituting  $p$  by  $\bar{p}$  we have

$$\phi_{\gamma+1}(y(t, \bar{p})) + \sum_{i=1}^{\gamma} c_i(\bar{p})\phi_i(y(t, \bar{p})) = 0.$$

Then, substituting  $\phi(y(t, \bar{p}))$  by  $\phi(y(t, p))$ , we have

$$\phi_{\gamma+1}(y(t, p)) + \sum_{i=1}^{\gamma} c_i(\bar{p})\phi_i(y(t, p)) = 0.$$

Subtracting this equation by (2.3.8), we obtain

$$\sum_{i=1}^{\gamma} (c_i(p) - c_i(\bar{p}))\phi_i(y(t, p)) = 0.$$

By the linear independence of  $\phi_1(y(t, p)), \dots, \phi_\gamma(y(t, p))$ , we obtain  $c_i(p) = c_i(\bar{p})$  for all  $i = 1, \dots, \gamma$ .  $\square$

Now the set  $\{c_1(p), c_2(p), \dots, c_\gamma(p)\}$  is called an exhaustive summary of the output equation (2.3.8). (Ollivier [53]).

**Step 3.** Solving the equations  $c_i(p) = c_i(\bar{p})$  for  $i = 1, \dots, \gamma$ , we obtain the relationships between  $p$  and  $\bar{p}$ . Note that the initial values are concerning the parameters so that we take the initial conditions and other generators of the general case to obtain the remaining relations between  $p$  and  $\bar{p}$ .

## 2.4 Finding identifiable combinations of parameters of SIR model by differential approach

Consider the system (1.1.1) including the output function as mentioned in Chapter 1

$$\begin{cases} S'(t) = -\tau S(t)I(t), \\ I'(t) = \tau S(t)I(t) - \nu I(t), t \geq 0 \\ y(t) = \nu_1 I(t) \end{cases}$$

with the initials  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and the basic production number  $\mathcal{R}_0 = \frac{\tau S_0}{\nu} > 1$ .

Denote by  $p = (\tau, \nu, \nu_1, S_0, I_0)$ ,  $\bar{p} = (\bar{\tau}, \bar{\nu}, \bar{\nu}_1, \bar{S}_0, \bar{I}_0)$  the unknown parameter vectors, and the set of possible  $p$  is given by  $\Omega = \{p \in \mathbb{R}^5 : p_i > 0, i = 1..5\}$ .

Let  $x(t, p) = \begin{pmatrix} x_1(t, p) \\ x_2(t, p) \end{pmatrix} = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix}$  then

$$x'(t, p) = \begin{pmatrix} -\tau S(t)I(t) \\ \tau S(t)I(t) - \nu I(t) \end{pmatrix} = \begin{pmatrix} -\tau x_1(t)x_2(t) \\ \tau x_1(t)x_2(t) - \nu x_2(t) \end{pmatrix} = f(x(t, p), p),$$

$x_0(p) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} S_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} p_4 \\ p_5 \end{pmatrix}$ , and  $y(t, p) = \nu_1 I(t) = p_3 x_2(t) = h(x(t, p), p)$ .

As we know from Chapter 1 and [46],  $S(t), I(t) > 0$  for all  $t \geq 0$ , so we choose  $M(p) = \mathbb{R}_+^2$  for every  $p \in \Omega$ . It means that  $x(t, p) \in M(p)$  for all  $p \in \Omega$ .

**Definition 2.4.1.** Consider the system (1.1.1) with the parameters

$$p = (\tau, \nu, \nu_1, S_0, I_0)$$

and

$$\bar{p} = (\bar{\tau}, \bar{\nu}, \bar{\nu}_1, \bar{S}_0, \bar{I}_0).$$

If the condition  $y(t, p) = y(t, \bar{p})$  for all  $t \geq 0$  implies  $p = \bar{p}$  then (1.1.1) is called to be identifiable.

By Proposition 2.3.2 we obtain the following proposition

**Proposition 2.4.2.** *The SIR model (1.1.1) is not identifiable from the output function  $y(t) = \nu_1 I(t)$ , and if*

$$\nu = \bar{\nu}, \frac{\tau}{\nu_1} = \frac{\bar{\tau}}{\bar{\nu}_1}, \tau S_0 = \bar{\tau} \bar{S}_0, \tau I_0 = \bar{\tau} \bar{I}_0, \quad (2.4.1)$$

then  $y(t, p) = y(t, \bar{p})$  for all  $t \geq 0$ .

*Proof.* For every  $x \in M(p) = \mathbb{R}_+^2$ , let  $f(x) = \begin{pmatrix} -\tau x_1 x_2 \\ \tau x_1 x_2 - \nu x_2 \end{pmatrix}$ , and  $h(x) = p_3 x_2$ .

Choose  $\epsilon = \frac{\bar{S}_0}{S_0} > 0$ , by the condition (2.4.1) we easily check that (2.3.2)-(2.3.5) from Proposition 2.3.2 hold for all  $x \in M(p) = \mathbb{R}_+^2$ . It implies that  $y(t, p) = y(t, \bar{p})$  for all  $t \geq 0$ . It means that SIR model (1.1.1) is not identifiable.  $\square$

## 2. Some preliminaries of differential algebra and application to system identifiability

To show the necessary condition of the above proposition we will apply the tools of differential algebra approach presented in the previous section.

The system (1.1.1) now can be rewritten as follows

$$\begin{cases} x_1'(t) = -p_1x_1(t)x_2(t), \\ x_2'(t) = p_1x_1(t)x_2(t) - p_2x_2(t), \\ y(t) = p_3x_2(t) \end{cases} \quad t \geq 0 \quad (2.4.2)$$

with  $x_1(0) = p_4$ ,  $x_2(0) = p_5$ .

Substituting  $x_2(t) = \frac{y(t)}{p_3}$  into (2.4.1) we obtain

$$\begin{cases} p_3x_1'(t) + p_1x_1(t)y(t) = 0, \\ y'(t) - p_1x_1(t)y(t) + p_2y(t) = 0, \end{cases} \quad t \geq 0, \quad (2.4.3)$$

with  $x_1(0) = p_4$ ,  $x_2(0) = p_5$ ,  $y(0) = p_3p_5$ .

Consider the differential polynomials on the left hand sides of (2.4.3)  $f_1 = p_3x_1' + p_1x_1y$  and  $f_2 = y' - p_1x_1y + p_2y$  of differential ring  $\mathbb{Q}(p)\{x_1, y\}$  with the elimination ranking

$$y < y' < \dots < y^{(n)} < \dots < x_1 < x_1' < \dots < x_1^{(n)} < \dots$$

The leaders of  $f_1, f_2$  are respectively  $x_1', y'$ . Since  $f_1, f_2$  do not contain any derivatives of the leaders of each other then  $f_1$  is reduced w.r.t  $f_2$ . It means the set  $A = \{f_1, f_2\}$  is an autoreduced set in the differential ring  $\mathbb{Q}(p)\{x_1, y\}$ . Moreover, the initials and separants of these polynomials are respectively  $p_3, 1$  in  $\mathbb{Q}(p)$ , then  $A$  is an orthonomic set of  $\mathbb{Q}(p)\{x_1, y\}$ . Let  $\mathcal{P}$  is a differential ideal generated these polynomials.

Next, we decompose the ideal  $\mathcal{P}$  by using Rosenfeld-Gröbner algorithm implemented in Maple. This decomposition returns two lists of differential polynomials. Each list forms the set of generators of a radical ideal corresponding to either the singular solutions or general solution of (2.4.3). We show here the lists

$$[x_1'(t), y(t)],$$

$$[p_1x_1(t)y(t) - y'(t) - p_2y(t), p_3y(t)y''(t) - p_3(y'(t))^2 + p_1y^2(t)y'(t) + p_1p_2y^3(t)].$$

The meaning of the lists is that (2.4.3) is equivalent to

$$\begin{cases} x_1'(t) = 0, \\ y(t) = 0, \end{cases} \quad t \geq 0,$$

or

$$\begin{cases} p_1x_1(t)y(t) - y'(t) - p_2y(t) = 0, \\ p_3y(t)y''(t) - p_3(y'(t))^2 + p_1y^2(t)y'(t) + p_1p_2y^3(t) = 0, \end{cases} \quad t \geq 0,$$

It can be seen that the first of the lists above are not corresponding to the complete system including (2.4.3) and the initial condition since the first list requires  $y(t) = 0$  for all  $t \geq 0$ , contradicting to  $y(0) = p_3p_5 > 0$ .

In the second list (the general case), there is an expression concerning only the indeterminate  $y$ , and it is called the output polynomial of (2.4.3)



$$p_3 y(t) y''(t) - p_3 (y'(t))^2 + p_1 y^2(t) y'(t) + p_1 p_2 y^3(t)$$

We rewrite this differential polynomial, then the output equation is obtained as follows

$$p_3 (y(t) y''(t) - (y'(t))^2) + p_1 y^2(t) y'(t) + p_1 p_2 y^3(t) = 0, \quad t \geq 0, \quad (2.4.4)$$

Dividing (2.4.4) by  $p_3$  we obtain

$$(y(t) y''(t) - (y'(t))^2) + \frac{p_1}{p_3} y^2(t) y'(t) + \frac{p_1 p_2}{p_3} y^3(t) = 0, \quad t \geq 0, \quad (2.4.5)$$

Now we check the linear independence of two polynomials

$$q_1 = y^2(t) y'(t),$$

$$q_2 = y^3(t).$$

Next, we compute the Wronskian  $W$  of  $q_1, q_2$  and the result is that

$$W = -y^5(t) y''(t) - y^4(t) (y'(t))^2.$$

By proposition 2.3.3 we have to justify that  $W$  is not in the ideal  $\mathcal{P}$ . This is a testing membership problem of an ideal in *Differential Algebra* and we can conduct to check if  $W$  is not in the ideal  $\mathcal{P}$  by the function called *BelongsTo* in the package *DifferentialAlgebra* of Maple. The result in Maple returned is "**false**". It means that  $W \notin \mathcal{P}$ .

## 2. Some preliminaries of differential algebra and application to system identifiability

---

```

with(DifferentialAlgebra);
params := [p1, p2, p3]
[BelongsTo, DifferentialRing, Equations, Get, Inequations, Is, NormalForm, PowerSeriesSolution, ReducedForm, RosenfeldGroebner,
Tools]
                                params := [p1, p2, p3]
R := DifferentialRing(derivations = [t], blocks = [x1, y], arbitrary = params)
                                R := differential_ring
edo1 := p3 * (d/dt x1(t)) + p1 * x1(t) * y(t)
                                edo1 := p3 * (d/dt x1(t)) + p1 * x1(t) * y(t)
edo2 := d/dt y(t) - p1 * x1(t) * y(t) + p2 * y(t)
                                edo2 := d/dt y(t) - p1 * x1(t) * y(t) + p2 * y(t)
ideal := RosenfeldGroebner([edo1, edo2], R)
                                ideal := [regular_differential_chain, regular_differential_chain]
Get(attributes, ideal)
[[differential, prime, autoreduced, primitive, squarefree, normalized], [differential, prime, autoreduced, primitive, squarefree,
normalized]]
Equations(ideal)
[[[p1 * x1(t) * y(t) - d/dt y(t) - p2 * y(t), (d^2/dt^2 y(t)) * y(t) * p3 - (d/dt y(t))^2 * p3 + (d/dt y(t)) * y(t)^2 * p1 + y(t)^3 * p1 * p2], [d/dt x1(t), y(t)]]]
BelongsTo((d/dt y(t))^2 * y(t)^4 - y(t)^5 * (d^2/dt^2 y(t)), ideal)
                                false

```

Figure 2.1: Maple code for decomposing the ideal generated by the left hand side of (2.4.3) and checking if  $W$  is not in the ideal  $\mathcal{I}$

By proposition 2.3.3 and (2.4.5), if  $y(t, p) = y(t, \bar{p})$ , then

$$\begin{cases} \frac{p_1 p_2}{p_3} = \frac{\bar{p}_1 \bar{p}_2}{\bar{p}_3}, \\ \frac{p_1}{p_3} = \frac{\bar{p}_1}{\bar{p}_3} \end{cases} \quad (2.4.6)$$

Now, consider the initial condition of (2.4.3), we have  $y(0, p) = y(0, \bar{p})$  then

$$p_3 p_5 = \bar{p}_3 \bar{p}_5. \quad (2.4.7)$$

Next, we consider another equation of the second list

$p_1 x_1(t) y(t) - y'(t) - p_2 y(t) = 0$ . We show that  $y(t)$  are not generically zeros. Indeed, by the result in [23](Chapter 1) and the condition  $\mathcal{R}_0 > 1$ , the function  $x_2(t)$  is firstly increasing and then decreasing on the interval  $[0, +\infty)$ . It implies that  $y(t)$  are not generically zeros. Therefore, we can rewrite the above equation as follows

$$p_1 x_1(t, p) = \frac{y'(t, p)}{y(t, p)} + p_2$$

By (2.4.6) and  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$ , we have

$$p_1 x_1(0, p) = \bar{p}_1 x_1(0, \bar{p}).$$

It implies that

$$p_1 p_4 = \bar{p}_1 \bar{p}_4. \quad (2.4.8)$$

Solving algebraic system (2.4.6)-(2.4.8), we have the relationships between  $p$  and  $\bar{p}$  as follows  $p_2 = \bar{p}_2$ ,  $\frac{\bar{p}_1}{\bar{p}_3} = \frac{p_1}{p_3}$ ,  $p_1 p_4 = \bar{p}_1 \bar{p}_4$ ,  $p_1 p_5 = \bar{p}_1 \bar{p}_5$ . Together with Proposition 2.4.2, we obtain Proposition 1.2.1 as a consequence. It means that  $\nu$ ,  $\frac{\tau}{\nu_1}$ ,  $\tau S_0$ ,  $\tau I_0$  are the identifiable combinations of parameters of (1.1.1) from the output function  $y(t) = \nu_1 I(t)$ .

The following is a consequence of proposition 1.2.1

**Proposition 2.4.3.** *The basic production number of the SIR model (1.1.1) is uniquely determined from the output function  $y(t) = \nu_1 I(t)$ .*

*Proof.* From the output function  $y(t) = \nu_1 I(t)$  we have  $\nu$ ,  $\tau S_0$  are uniquely determined. It implies that  $\mathcal{R}_0 = \frac{\tau S_0}{\nu}$  is uniquely determined.  $\square$

## Chapter 3

# Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

In parameter estimation of a given system model from real experimental data, the identifiability analysis is the most important prerequisite. There are so many papers mentioning the criterion for testing the identifiability, however, when a given system is not identifiable there is not a general way for showing the combinations of parameters that are identifiable. Thank to differential algebra and differential elimination, the output equation of a system model can be given easier for approaching the identifiable combinations of parameters. This chapter uses the tools of differential algebra for studying the identifiability of the SIR model with a nonlinear incidence rate. The approach here can be easily applied for some other biological models because some of the tools of differential algebra were implemented in Maple.

### 3.1 The sufficient condition of identifiability of SIR model with nonlinear incidence rate

Consider the SIR model with nonlinear incidence rate as follows

$$\begin{cases} S'(t) = -\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)}, \\ I'(t) = \frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} - \nu I(t), \\ y(t) = \mu I(t) \end{cases} \quad t \geq 0 \quad (3.1.1)$$

with  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and  $\beta + 1 \geq \alpha \geq 1$ .

Denote by  $p = (\tau, \kappa, \nu, \mu, S_0, I_0)$ ,  $\bar{p} = (\bar{\tau}, \bar{\kappa}, \bar{\nu}, \bar{\mu}, \bar{S}_0, \bar{I}_0)$  the unknown parameter vectors, and the set of possible  $p$  is given by  $\Omega = \{p \in \mathbb{R}^6 : p_i > 0, i = 1..6\}$ .

$$\text{Let } x(t, p) = \begin{pmatrix} x_1(t, p) \\ x_2(t, p) \end{pmatrix} = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix}, f(x(t, p), p) = \begin{pmatrix} -\frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} \\ \frac{\tau S(t)I^\alpha(t)}{1 + \kappa I^\beta(t)} - \nu I(t) \end{pmatrix}, x_0(p) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} S_0 \\ I_0 \end{pmatrix}, \text{ and } y(t, p) = h(x(t, p), p) = \mu I(t).$$

By the **Theorem 1** of the paper [45], we have  $S(t), I(t) > 0$  for all  $t \geq 0$ .

For every  $p \in \Omega$ , we choose  $M(p) = \mathbb{R}_+^2$ , then  $x(t, p) = \begin{pmatrix} x_1(t, p) \\ x_2(t, p) \end{pmatrix} = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix} \in M(p)$ .

we will look for  $\epsilon > 0$  satisfied (2.3.2)-(2.3.5) of Proposition 2.3.2 of Chapter 2. Since  $\epsilon > 0$ , it can be seen that (2.3.2) holds.

Then (2.3.3) is read as follows

$$\epsilon x_0(\bar{p}) = x_0(p)$$

$$\Leftrightarrow \begin{pmatrix} \epsilon \bar{S}_0 \\ \epsilon \bar{I}_0 \end{pmatrix} = \begin{pmatrix} S_0 \\ I_0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \epsilon \bar{S}_0 = S_0 & (3.1.a) \\ \epsilon \bar{I}_0 = I_0 & (3.1.b) \end{cases}$$

Next, (2.3.4) is read as follows

$$f(\epsilon x, p) = \epsilon f(x, \bar{p}) \text{ for all } x \in M(\bar{p}) = \mathbb{R}_+^2$$

$$\Leftrightarrow \begin{pmatrix} -\frac{\tau \epsilon^{\alpha+1} x_1 x_2^\alpha}{1 + \epsilon^\beta \kappa x_2^\beta} \\ \frac{\tau \epsilon^{\alpha+1} x_1 x_2^\alpha}{1 + \epsilon^\beta \kappa x_2^\beta} - \epsilon \nu x_2 \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon \bar{\tau} x_1 x_2^\alpha}{1 + \bar{\kappa} x_2^\beta} \\ \frac{\epsilon \bar{\tau} x_1 x_2^\alpha}{1 + \bar{\kappa} x_2^\beta} - \epsilon \bar{\nu} x_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \frac{-\tau \epsilon^{\alpha+1} x_1 x_2^\alpha}{1 + \epsilon^\beta \kappa x_2^\beta} = \frac{-\epsilon \bar{\tau} x_1 x_2^\alpha}{1 + \bar{\kappa} x_2^\beta} \\ \frac{\tau \epsilon^{\alpha+1} x_1 x_2^\alpha}{1 + \epsilon^\beta \kappa x_2^\beta} - \epsilon \nu x_2 = \frac{\epsilon \bar{\tau} x_1 x_2^\alpha}{1 + \bar{\kappa} x_2^\beta} - \epsilon \bar{\nu} x_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \tau \epsilon^\alpha + \tau \epsilon^\alpha \bar{\kappa} x_2^\beta = \bar{\tau} + \bar{\tau} \epsilon^\beta \kappa x_2^\beta & (3.1.c) \\ \epsilon \nu x_2 = \epsilon \bar{\nu} x_2 & (3.1.d) \end{cases}$$

Since (3.1.c) – (3.1.d) hold for all  $x \in M(\bar{p}) = \mathbb{R}_+^2$ , then (3.1.c) – (3.1.d) is equivalent to

$$\begin{cases} \tau \epsilon^\alpha = \bar{\tau} & (3.1.e) \\ \tau \epsilon^\alpha \bar{\kappa} = \bar{\tau} \epsilon^\beta \kappa & (3.1.f) \\ \epsilon \nu = \epsilon \bar{\nu} & (3.1.g) \end{cases}$$

Finally, the equation (2.3.5) is equivalent to  $h(\epsilon x, p) = h(x, \bar{p})$  for all  $x \in M(\bar{p}) = \mathbb{R}_+^2$ . It implies that (2.3.5) is equivalent to

$$\mu \epsilon x_2 = \bar{\mu} x_2$$

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\Leftrightarrow \epsilon\mu = \bar{\mu} \quad (3.1.h)$$

Since (3.1.a) – (3.1.b), and (3.1.e) – (3.1.h), we obtain  $\bar{\nu} = \nu$ ,  $\left(\frac{\bar{\tau}}{\tau}\right)^\beta = \left(\frac{\bar{\kappa}}{\kappa}\right)^\alpha$ ,  $\frac{\bar{\tau}}{\tau} = \left(\frac{\bar{\mu}}{\mu}\right)^\alpha$ ,  $\frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0} = \epsilon$ .

Hence, for each  $p \in \Omega$ , there are infinite parameters  $\bar{p} \in \Omega$  such that  $y(t, p) = y(t, \bar{p})$  for all  $t \geq 0$ .

As a consequence of Proposition 2.3.2, we obtain the following proposition

**Proposition 3.1.1.** *The SIR model with nonlinear incidence rate (3.1.1) is not identifiable, and if  $\bar{\nu} = \nu$ ,  $\left(\frac{\bar{\tau}}{\tau}\right)^\beta = \left(\frac{\bar{\kappa}}{\kappa}\right)^\alpha$ ,  $\frac{\bar{\tau}}{\tau} = \left(\frac{\bar{\mu}}{\mu}\right)^\alpha$ ,  $\frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}$ , we have  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$ .*

### 3.2 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 1, \beta = 2$

Consider the system

$$\begin{cases} S'(t) = -\frac{\tau S(t)I(t)}{1 + \kappa I^2(t)}, \\ I'(t) = \frac{\tau S(t)I(t)}{1 + \kappa I^2(t)} - \nu I(t), \\ y(t) = \mu I(t) \end{cases} \quad t \geq 0 \quad (3.2.1)$$

with  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0^2)} > 1$ .

Denote by  $p = (\tau, \kappa, \nu, \mu, S_0, I_0)$ ,  $\bar{p} = (\bar{\tau}, \bar{\kappa}, \bar{\nu}, \bar{\mu}, \bar{S}_0, \bar{I}_0, \bar{\mu})$  the unknown parameter vectors, and the set of possible  $p$  is given by  $\Omega = \{p \in \mathbb{R}^6 : p_i > 0, i = 1..6\}$ .

$$\text{Let } x(t, p) = \begin{pmatrix} x_1(t, p) \\ x_2(t, p) \end{pmatrix} = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix}, \quad f(x(t, p), p) = \begin{pmatrix} -\frac{\tau S(t)I(t)}{1 + \kappa I^2(t)} \\ \frac{\tau S(t)I(t)}{1 + \kappa I^2(t)} - \nu I(t) \end{pmatrix},$$

$$x_0(p) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} S_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} p_4 \\ p_5 \end{pmatrix}, \text{ and } y(t, p) = h(x(t, p), p) = \mu I(t) = p_6 x_2(t).$$

the system (3.2.1) can be rewritten as follows

$$\begin{cases} x_1'(t) = -\frac{p_1 x_1(t)x_2(t)}{1 + p_2 x_2^2(t)}, \\ x_2'(t) = \frac{p_1 x_1(t)x_2(t)}{1 + p_2 x_2^2(t)} - p_3 x_2(t), \\ y(t) = p_4 x_2(t) \end{cases} \quad t \geq 0 \quad (3.2.2)$$

with  $x_1(0) = p_5$ ,  $x_2(0) = p_6$ .

We introduce here a new state variable  $x_3(t) = \frac{1}{1 + p_2 x_2^2(t)}$ , then  $x_3'(t) = -\frac{2p_2 x_2(t)x_2'(t)}{(1 + p_2 x_2^2(t))^2} = -2p_2 x_2(t)x_3^2(t)(p_1 x_1(t)x_2(t)x_3(t) - p_3 x_2(t))$ , and (3.2.2) can be

written as follows

$$\begin{cases} x_1'(t) = -p_1x_1(t)x_2(t)x_3(t), \\ x_2'(t) = p_1x_1(t)x_2(t)x_3(t) - p_3x_2(t), & t \geq 0 \\ x_3'(t) = -2p_1p_2x_1(t)x_2^2(t)x_3^3(t) + 2p_2p_3x_2^2(t)x_3^2(t), \\ y(t) = p_4x_2(t) \end{cases} \quad (3.2.3)$$

with  $x_1(0) = p_5, x_2(0) = p_6, x_3(0) = \frac{1}{1 + p_2p_6^2}$ .

Substituting  $x_2(t) = \frac{y(t)}{p_4}$  into (3.2.3) we obtain

$$\begin{cases} p_4x_1'(t) + p_1x_1(t)y(t)x_3(t) = 0, \\ y'(t) - p_1x_1(t)y(t)x_3(t) + p_3y(t) = 0, & t \geq 0 \\ p_4^2x_3'(t) + 2p_1p_2x_1(t)y(t)^2x_3^3(t) - 2p_2p_3y^2(t)x_3^2(t) = 0 \end{cases} \quad (3.2.4)$$

with  $x_1(0) = p_5, y(0) = p_4p_6, x_3(0) = \frac{1}{1 + p_2p_6^2}$ .

Consider the differential polynomials on the left-hand sides of (3.2.4). These polynomials form an autoreduced set in the differential ring  $\mathbb{Q}(p)\{x_1, x_3, y\}$  with the elimination ranking (that we mentioned in Chapter 2)  $[y] < [x_1] < [x_3]$ . Moreover, the initials and separants of these polynomials are in  $\mathbb{Q}(p)$ , then these polynomials form an orthonomic set of  $\mathbb{Q}(p)\{x_1, x_3, y\}$ . Let  $\mathcal{P}$  is a prime differential ideal generated by these polynomials.

Next, the ideal  $\mathcal{P}$  is decomposed by using Rosenfeld-Gröbner algorithm implemented in Maple. The result is the four lists of differential polynomials. Each list forms the set of generators of a radical ideal corresponding to either the singular solutions or general solution of (3.2.4). We show here the lists

$$\begin{aligned} & [x_3(t), x_1'(t), y'(t) + p_3y(t)], \\ & [x_3'(t), x_1'(t), y(t)], \\ & [p_4^2x_3'(t) - 2p_2p_3y^2(t)x_3^2(t), x_1(t), y'(t) + p_3y(t)], \\ & [2x_3(t)(y')^2(t)p_2 + 2x_3(t)y'(t)y^3(t)p_2p_3 + x_3(t)y'(t)y^2(t)p_1p_4 \\ & \quad + x_3(t)y^3(t)p_1p_3p_4 + y''(t)y(t)p_4^2 - (y')^2(t)p_4^2, \\ & x_1(t)y''(t)y(t)p_1p_4^2 - x_1(t)(y')^2(t)p_1p_4^2 - x_1(t)(y')^2(t)p_1p_4^2 + 2(y')^3(t)y(t)p_2 \\ & \quad + 4(y')^2(t)y^2(t)p_2p_3 + 2y'(t)y^3(t)p_2p_3^2 + 2y'(t)y^2(t)p_1p_3p_4 + y^3(t)p_1p_3^2p_4, \\ & 2y^{(3)}(t)(y')^2(t)y^2(t)p_2 + 2y^{(3)}(t)y'(t)y^3(t)p_2p_3 + y^{(3)}(t)y'(t)y^2(t)p_1p_4 \\ & \quad + y^{(3)}(t)y^3(t)p_1p_3p_4 - 6y''(t)(y')^3(t)y^2(t)p_2 - 2(y'')^2(t)y^3(t)p_2p_3 \\ & \quad - (y'')^2(t)y^2(t)p_1p_4 + 2y''(t)(y')^3(t)y(t)p_2 - 6y''(t)(y')^2(t)y^2(t)p_2p_3 \\ & \quad - 2y''(t)(y')^2(t)y(t)p_1p_4 - 4y''(t)y'(t)y^2(t)p_1p_3p_4 + 2(y')^5(t)p_2 + 6(y')^4(t)y(t)p_2p_3 \\ & \quad + 2(y')^4(t)p_1p_4 + 3(y')^3(t)y(t)p_1p_3p_4]. \end{aligned}$$

### 3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

```

params := [p1, p2, p3, p4]:
R := DifferentialRing(derivations = [t], blocks = [x3, x1, y], arbitrary = params):

R := differential_ring

a := 1:
b := 2:

edo1 := p4^a * (d/dt x[1](t) + p1 * x[1](t) * (y(t))^a * x[3](t)):
edo2 := p4^a - 1 * (d/dt y(t) - p1 * x[1](t) * (y(t))^a * x[3](t) + p3 * p4^a - 1 * y(t)):
edo3 := p4^a + b - 1 * (d/dt x[3](t) + b * p1 * p2 * x[1](t) * (y(t))^(a+b-1) * (x[3](t))^3 - b * p2 * p3 * p4^a - 1 * (y(t))^b * (x[3](t))^2):
ideal := RosenfeldGroebner([edo1, edo2, edo3], R):
Get(attributes, ideal):
Equations(ideal)
[[2*x3(t) * (d/dt y(t))^2 * y(t)^2 * p2 + 2*x3(t) * (d/dt y(t)) * y(t)^3 * p2 * p3 + x3(t) * (d/dt y(t)) * y(t)^2 * p1 * p4
+ x3(t) * y(t)^3 * p1 * p3 * p4 + (d^2/dt^2 y(t)) * y(t) * p4^2 - (d/dt y(t))^2 * p4^2, x1(t) * (d^2/dt^2 y(t)) * y(t) * p1 * p4^2
- x1(t) * (d/dt y(t))^2 * p1 * p4^2 + 2 * (d/dt y(t))^3 * y(t) * p2 + 4 * (d/dt y(t))^2 * y(t)^2 * p2 * p3 + (d/dt y(t))^2 * y(t) * p1 * p4
+ 2 * (d/dt y(t)) * y(t)^3 * p2 * p3^2 + 2 * (d/dt y(t)) * y(t)^2 * p1 * p3 * p4 + y(t)^3 * p1 * p3^2 * p4, 2 * (d^3/dt^3 y(t)) * (d/dt y(t)) * y(t)^2 * p2 + 2 * (d^3/dt^3 y(t)) * (d/dt y(t)) * y(t)^2 * p1 * p4 + (d^3/dt^3 y(t)) * y(t)^3 * p1 * p3 * p4 - 6 * (d^2/dt^2 y(t)) * (d/dt y(t)) * y(t)^2 * p2 - 2 * (d^2/dt^2 y(t))^2 * y(t)^3 * p2 * p3 - (d^2/dt^2 y(t)) * y(t)^2 * p1 * p4 + 2 * (d^2/dt^2 y(t)) * (d/dt y(t))^3 * y(t) * p2 - 6 * (d^2/dt^2 y(t)) * (d/dt y(t))^2 * y(t)^2 * p2 * p3 - 2 * (d^2/dt^2 y(t)) * (d/dt y(t))^2 * y(t) * p1 * p4 - 4 * (d^2/dt^2 y(t)) * (d/dt y(t)) * y(t)^2 * p1 * p3 * p4 + 2 * (d/dt y(t))^5 * p2 + 6 * (d/dt y(t))^4 * y(t) * p2 * p3 + 2 * (d/dt y(t))^4 * p1 * p4 + 3 * (d/dt y(t))^3 * y(t) * p1 * p3 * p4], [x3(t), d/dt x1(t), d/dt y(t)
+ p3 * y(t)], [d/dt x3(t), d/dt x1(t), y(t)], [p4^2 * (d/dt x3(t)) - 2 * p2 * p3 * y(t)^2 * x3(t)^2, x1(t), d/dt y(t) + p3 * y(t)]]

```

Figure 3.1: Maple code for decomposing the ideal generated by the left hand side of (3.2.4)

It can be seen that the first three lists above are not corresponding to the complete system including (3.2.3) and the initial condition. For instance, the third list requires  $x_1(t) = 0$  for all  $t \geq 0$ , contradicting to  $x_1(0) = p_5 > 0$ .

In the fourth list, there is an expression that concerning only the indeterminate  $y$ , and it is written as follows



$$\begin{aligned}
& 2y^{(3)}(t)(y')^2(t)y^2(t)p_2 + 2y^{(3)}(t)y'(t)y^3(t)p_2p_3 + y^{(3)}(t)y'(t)y^2(t)p_1p_4 \\
& + y^{(3)}(t)y^3(t)p_1p_3p_4 - 6y''(t)(y')^3(t)y^2(t)p_2 - 2(y'')^2(t)y^3(t)p_2p_3 - (y'')^2(t)y^2(t)p_1p_4 \\
& + 2y''(t)(y')^3(t)y(t)p_2 - 6y''(t)(y')^2(t)y^2(t)p_2p_3 - 2y''(t)(y')^2(t)y(t)p_1p_4 \\
& - 4y''(t)y'(t)y^2(t)p_1p_3p_4 + 2(y')^5(t)p_2 + 6(y')^4(t)y(t)p_2p_3 \\
& + 2(y')^4(t)p_1p_4 + 3(y')^3(t)y(t)p_1p_3p_4.
\end{aligned}$$

Then the output equation is obtained as follows

$$\begin{aligned}
& p_2 \left( 2y^{(3)}(t)(y')^2(t)y^2(t) - 6y''(t)(y')^3(t)y^2(t) + 2y''(t)(y')^3(t)y(t) + 2(y')^5(t) \right) \\
& - 2p_2p_3 \left( 3(y')^4(t)y(t) - 3y''(t)(y')^2(t)y^2(t) - y^{(3)}(t)y'(t)y^3(t) + (y'')^2(t)y^3(t) \right) \\
& + p_1p_4 \left( y^{(3)}(t)y'(t)y^2(t) - (y'')^2(t)y^2(t) - 2y''(t)(y')^2(t)y(t) + 2(y')^4(t) \right) \\
& - p_1p_3p_4 \left( 4y''(t)y'(t)y^2(t) - y^{(3)}(t)y^3(t) - 3(y')^3(t)y(t) \right) = 0
\end{aligned} \tag{3.2.5}$$

Dividing both side of (3.2.5) by  $p_2$  we obtain

$$\begin{aligned}
& \left( 2y^{(3)}(t)(y')^2(t)y^2(t) - 6y''(t)(y')^3(t)y^2(t) + 2y''(t)(y')^3(t)y(t) + 2(y')^5(t) \right) \\
& - 2p_3 \left( 3(y')^4(t)y(t) - 3y''(t)(y')^2(t)y^2(t) - y^{(3)}(t)y'(t)y^3(t) + (y'')^2(t)y^3(t) \right) \\
& + \frac{p_1p_4}{p_2} \left( y^{(3)}(t)y'(t)y^2(t) - (y'')^2(t)y^2(t) - 2y''(t)(y')^2(t)y(t) + 2(y')^4(t) \right) \\
& - \frac{p_1p_3p_4}{p_2} \left( 4y''(t)y'(t)y^2(t) - y^{(3)}(t)y^3(t) - 3(y')^3(t)y(t) \right) = 0
\end{aligned} \tag{3.2.6}$$

Now we check the linear independence of three polynomials

$$\begin{aligned}
q_1 &= 2y(t) \left( 3(y')^4(t) - 3y''(t)(y')^2(t)y(t) - y^{(3)}(t)y'(t)y^2(t) + (y'')^2(t)y^2(t) \right), \\
q_2 &= -y^{(3)}(t)y'(t)y^2(t) + (y'')^2(t)y^2(t) + 2y''(t)(y')^2(t)y(t) - 2(y')^4(t), \\
q_3 &= - \left( 4y''(t)y'(t)y^2(t) - y^{(3)}(t)y^3(t) - 3(y')^3(t)y(t) \right)
\end{aligned}$$

In order to check that, we compute the Wronskian  $W$  of  $q_1, q_2, q_3$ . It is easy to obtain this evaluation by the package *VectorCalculus* in Maple( see figures 3.2, 3.3).

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

```

with(VectorCalculus) :
Wronskian(
  [
    - 2y(t) ( (d^2/dt^2 y(t))^2 y(t)^2 + 3 (d^2/dt^2 y(t)) (d/dt y(t))^2 y(t) - (d^3/dt^3 y(t)) (d/dt y(t)) y(t)^2
    - 3 (d/dt y(t))^4 ),
    (d^2/dt^2 y(t))^2 y(t)^2 + 2 (d^2/dt^2 y(t)) (d/dt y(t))^2 y(t) - (d^3/dt^3 y(t)) (d/dt y(t)) y(t)^2
    - 2 (d/dt y(t))^4 ,
    - y(t) ( 4 (d^2/dt^2 y(t)) (d/dt y(t)) y(t) - (d^3/dt^3 y(t)) y(t)^2 - 3 (d/dt y(t))^3 )
  ], t,
  determinant
)

```

Figure 3.2: Maple code for computing the Wronskian  $W$  of  $q_1, q_2, q_3$ .

$$\begin{aligned}
 & \text{BelongsTo} \left( 6y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) + 2y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^4}{dr^4} y(t) \right)^2 \right. \\
 & + 8y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^3 - 40y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) \\
 & + 4y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^4}{dr^4} y(t) \right) - 16y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^5}{dr^5} y(t) \right) \\
 & + 8y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^2 - 4y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^4}{dr^4} y(t) \right)^2 \\
 & - 16y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^3 + 64y(t)^5 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right) \\
 & + 38y(t)^5 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^4}{dr^4} y(t) \right) + 14y(t)^5 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^5}{dr^5} y(t) \right) \\
 & - 76y(t)^5 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 - 2y(t)^5 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) \\
 & + 148y(t)^4 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right) - 76y(t)^4 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^4}{dr^4} y(t) \right) \\
 & - 4y(t)^4 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^5}{dr^5} y(t) \right) + 104y(t)^4 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^2 \\
 & + 8y(t)^4 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^4}{dr^4} y(t) \right) - 412y(t)^3 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right) \\
 & + 52y(t)^3 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^4}{dr^4} y(t) \right) + 324y(t)^2 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right) \\
 & - 14y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^4}{dr^4} y(t) \right) - 2y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^3}{dr^3} \right. \\
 & y(t) \left. \right) \left( \frac{d^5}{dr^5} y(t) \right) + 36y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^4}{dr^4} y(t) \right) + 4y(t)^6 \left( \frac{d^2}{dr^2} \right. \\
 & y(t) \left. \right) \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) - 30y(t)^5 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^4}{dr^4} y(t) \right) \\
 & + 24y(t)^6 \left( \frac{d^2}{dr^2} y(t) \right)^7 - 96 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d}{dr} y(t) \right)^{12} + 540y(t) \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d}{dr} y(t) \right)^{10} \\
 & - 84y(t) \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^3}{dr^3} y(t) \right) - 6y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^4}{dr^4} y(t) \right) + 6y(t)^7 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 \\
 & + 54y(t)^5 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d}{dr} y(t) \right)^2 + 2y(t)^5 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^4}{dr^4} y(t) \right)^2 + 8y(t)^5 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^3 \\
 & - 564y(t)^4 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d}{dr} y(t) \right)^4 + 1218y(t)^3 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d}{dr} y(t) \right)^6 - 42y(t)^3 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^2 - 1176y(t)^2 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d}{dr} y(t) \right)^8 - 12y(t)^2 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^4}{dr^4} y(t) \right), \text{ideal} \left. \right) \\
 & \text{false}
 \end{aligned}$$

Figure 3.3: Maple code for justifying that  $W$  is not in the ideal  $\mathcal{P}$ . This code must be added at the end of the code of Figure 3.1.

By Proposition 2.3.3 we have to justify that  $W$  is not in the ideal  $\mathcal{P}$ . This is a testing membership problem of an ideal in *Differential Algebra* and we can conduct

### 3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

to check if  $W$  is not in the ideal  $\mathcal{P}$  by the function called *BelongsTo* in the package *DifferentialAlgebra* of Maple. The result in Maple returned is "**false**". It means that  $W \notin \mathcal{P}$ . By Proposition 2.3.3 and (3.2.6), if  $y(t, p) = y(t, \bar{p})$ , then

$$\begin{cases} -2p_3 = -2\bar{p}_3, \\ \frac{p_1 p_4}{p_2} = \frac{\bar{p}_1 \bar{p}_4}{\bar{p}_2}, \\ \frac{-p_1 p_3 p_4}{p_2} = \frac{-\bar{p}_1 \bar{p}_3 \bar{p}_4}{\bar{p}_2} \end{cases} \quad (3.2.7)$$

Now, consider the initial condition of (3.2.3), (3.2.4), we have  $y(0, p) = y(0, \bar{p})$  then

$$p_4 p_6 = \bar{p}_4 \bar{p}_6 \quad (3.2.8)$$

Next, we consider the two other equations of the fourth list

$$y^2(t) (2p_2 y'(t) + p_1 p_4) (y'(t) + p_3 y(t)) x_3(t) = p_4^2 (y(t) y''(t) - (y')^2(t)), \quad (3.2.9)$$

and

$$p_1 p_4^2 (y(t) y''(t) - (y')^2(t)) x_1(t) = y(t) (2p_2 y'(t) + p_1 p_4) (y'(t) + p_3 y(t))^2. \quad (3.2.10)$$

We show that  $y(t)$ ,  $2p_2 y'(t) + p_1 p_4$ ,  $y'(t) + p_3 y(t)$ , and  $y(t) y''(t) - (y')^2(t)$  are not generically zeros. Indeed, by the **Theorem 1** in [45] and the condition  $\mathcal{R}_0 > 1$ , the function  $x_2(t)$  is firstly increasing and then decreasing on the interval  $[0, +\infty)$ . It implies that  $y(t)$ ,  $2p_2 y'(t) + p_1 p_4$ ,  $y'(t) + p_3 y(t)$  are not generically zeros. If  $y(t) y''(t) - (y')^2(t)$  is generically zero then  $x_3(t)$  is generically zero. This contradicts the fact  $x_3(t) = \frac{1}{1 + p_2 x_2^2(t)}$ . Therefore, we can rewrite the two above equations as follows

$$\frac{p_2}{p_4^2} x_3(t, p) = \frac{(y(t, p) y''(t, p) - (y')^2(t, p))}{y^2(t, p) \left( 2y'(t, p) + \frac{p_1 p_4}{p_2} \right) (y'(t, p) + p_3 y(t, p))}, \quad (3.2.11)$$

and

$$p_4 x_1(t, p) = \frac{y(t, p) \left( 2y'(t, p) + \frac{p_1 p_4}{p_2} \right) (y'(t, p) + p_3 y(t, p))}{\frac{p_1 p_4}{p_2} (y(t, p) y''(t, p) - (y')^2(t, p))}. \quad (3.2.12)$$

By (3.2.7) and  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$ , we have

$$\begin{cases} p_4 x_1(0, p) = \bar{p}_4 x_1(0, \bar{p}), \\ \frac{p_2}{p_4^2} x_3(0, p) = \frac{\bar{p}_2}{\bar{p}_4^2} x_3(0, \bar{p}), \end{cases} \quad (3.2.13)$$

It implies that

$$\begin{cases} p_4 p_5 = \bar{p}_4 \bar{p}_5, \\ \frac{p_2}{p_4^2} \frac{1}{1 + p_2 p_6^2} = \frac{\bar{p}_2}{\bar{p}_4^2} \frac{1}{1 + \bar{p}_2 \bar{p}_6^2}, \end{cases} \quad (3.2.14)$$

Solving the algebraic system (3.2.7), (3.2.8), and (3.2.14) we have the relationships between  $p$  and  $\bar{p}$  as follows  $p_3 = \bar{p}_3$ ,  $\frac{\bar{p}_4}{p_4} = \frac{p_5}{\bar{p}_5} = \frac{p_6}{\bar{p}_6}$ ,  $\frac{\bar{p}_4}{p_4} = \frac{\bar{p}_1}{p_1}$ ,  $\left(\frac{\bar{p}_4}{p_4}\right)^2 = \frac{\bar{p}_2}{p_2}$ . Together with Proposition 3.1.1, we have the following theorem

**Theorem 3.2.1.** *The SIR model with nonlinear incidence rate (3.2.1) is not identifiable, and we have  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$  if and only if  $\bar{\nu} = \nu$ ,  $\left(\frac{\bar{\tau}}{\tau}\right)^2 = \frac{\bar{\kappa}}{\kappa}$ ,  $\frac{\bar{\tau}}{\tau} = \frac{\bar{\mu}}{\mu}$ ,  $\frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}$ .*

The following is the consequence of Theorem 3.2.1

**Theorem 3.2.2.** *The basic production number of the SIR model with nonlinear incidence rate (3.2.1) is uniquely determined from the output function  $y(t) = \mu I(t)$ .*

*Proof.* By theorem 3.2.1, from the output function  $y(t) = \mu I(t)$  we have  $\nu, \tau S_0, \kappa I_0^2$  are uniquely determined. It derives that  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0^2)}$  is uniquely determined.  $\square$

### 3.3 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 1, \beta = 1$

The model system (3.1.1) in which  $\alpha = 1, \beta = 1$  can be read as follows

$$\begin{cases} S'(t) = -\frac{\tau S(t)I(t)}{1 + \kappa I(t)}, \\ I'(t) = \frac{\tau S(t)I(t)}{1 + \kappa I(t)} - \nu I(t), \\ y(t) = \mu I(t) \end{cases} \quad t \geq 0 \quad (3.3.1)$$

with  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0)} > 1$ .

With the same notations as in Section 3.2, (3.3.1) can be read as follows

$$\begin{cases} x'_1(t) = -\frac{p_1 x_1(t) x_2(t)}{1 + p_2 x_2(t)}, \\ x'_2(t) = \frac{p_1 x_1(t) x_2(t)}{1 + p_2 x_2(t)} - p_3 x_2(t), \\ y(t) = p_4 x_2(t) \end{cases} \quad t \geq 0 \quad (3.3.2)$$

with  $x_1(0) = p_5$ ,  $x_2(0) = p_6$ .

Let  $x_3(t) = \frac{1}{1 + p_2 x_2(t)}$ , then

$$\begin{aligned} x'_3(t) &= -\frac{p_2 x'_2(t)}{(1 + p_2 x_2(t))^2} \\ &= -p_2 x_3^2(t) (p_1 x_1(t) x_2(t) x_3(t) - p_3 x_2(t)). \end{aligned}$$

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

We write (3.3.2) as follows

$$\begin{cases} p_4x_1'(t) + p_1x_1(t)y(t)x_3(t) = 0, \\ y'(t) - p_1x_1(t)y(t)x_3(t) + p_3y(t) = 0, \quad t \geq 0 \\ p_4x_3'(t) + p_1p_2x_1(t)y(t)x_3^3(t) - p_2p_3y(t)x_3^2(t) = 0 \end{cases} \quad (3.3.3)$$

with  $x_1(0) = p_5$ ,  $y(0) = p_4p_6$ ,  $x_3(0) = \frac{1}{1 + p_2p_6}$ .

The differential polynomials on the left hand sides of (3.3.3) form an autoreduced set in the differential ring  $\mathbb{Q}(p)\{x_1, x_3, y\}$  with the elimination ranking (that we mentioned in Chapter 2)  $[y] < [x_1] < [x_3]$ . Moreover, the initials and separants of these polynomials are in  $\mathbb{Q}(p)$ , then these polynomials form an orthonomic set of  $\mathbb{Q}(p)\{x_1, x_3, y\}$ . Let  $\mathcal{P}$  is a prime differential ideal generated by these polynomials.

Next, using Rosenfeld-Gröbner algorithm implemented in Maple to decompose the ideal  $\mathcal{P}$ . This decomposition returns five lists of differential polynomials. Each list forms the set of generators of a radical ideal corresponding to either the singular solutions or general solution of (3.3.3). We show here the first four lists

$$[x_3(t), x_1'(t), y'(t) + p_3y(t)],$$

$$[x_3'(t), x_1'(t), y(t)],$$

$$[p_4x_3'(t) - p_2p_3y(t)x_3^2(t), x_1(t), y'(t) + p_3y(t)],$$

$$[p_1p_2x_1(t)x_3(t) + p_1 - p_2p_3, p_2p_4x_1'(t) - p_1y(t) + p_2p_3y(t), p_2y'(t) + p_1y(t)],$$

It is clear that the first three lists above are not corresponding to the complete system including (3.3.3) and the initial condition. For instance, the third list requires  $x_1(t) = 0$  for all  $t \geq 0$ , contradicting to  $x_1(0) = p_5 > 0$ .

The fourth list requires  $p_2y'(t) + p_1y(t) = 0$  for all  $t \geq 0$ . It means that  $y'(t) = -\frac{p_1}{p_2}y(t)$  or  $x_2(t) = \frac{y(t)}{p_4}$  is decreasing for all  $t \geq 0$ . However, by the **Theorem 1**

from the paper of **Pierre Magal et al.** [45] and the condition  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0)} > 1$ , the function  $x_2(t)$  is firstly increasing and then decreasing on the interval  $[0, +\infty)$ . It derives that the fourth list is not corresponding to the complete system (3.3.3).

The fifth list is now corresponding to the general case, and we obtain the output equation of (3.3.3) as follows

$$\begin{aligned} & 2(y')^3(t)y''(t) + y(t)(y')^2(t)y'''(t) - 3y(t)y'(t)(y'')^2(t) \\ & + \left(\frac{p_1}{p_2} + p_3\right)(y^2(t)y'(t)y'''(t) + 2(y')^4(t) - 2y(t)(y')^2(t)y''(t) - y^2(t)(y'')^2(t)) \\ & + \frac{p_1p_3}{p_2}(y^3(t)y'''(t) + 3y(t)(y')^3(t) - 4y^2(t)y'(t)y''(t)) = 0 \end{aligned} \quad (3.3.4)$$

Now we check the linear independence of two polynomials

$$q_1 = y^2(t)y'(t)y'''(t) + 2(y')^4(t) - 2y(t)(y')^2(t)y''(t) - y^2(t)(y'')^2(t), \quad (3.3.5)$$

and

$$q_2 = y^3(t)y'''(t) + 3y(t)(y')^3(t) - 4y^2(t)y'(t)y''(t). \quad (3.3.6)$$

The Wronskian  $W$  of  $q_1, q_2$  is computed by the package *VectorCalculus* in Maple. Next, the function *BelongsTo* in the package *DifferentialAlgebra* of Maple shows that  $W \notin \mathcal{P}$ . (see Figure 3.4).

The screenshot shows the following Maple output:

```

BelongsTo(4*(d^2/dt^2 y(t))^4 y(t)^4 - 17*(d^2/dt^2 y(t))^3 (d/dt y(t))^2 y(t)^3 + 29*(d^2/dt^2 y(t))^2 (d/dt y(t))^4 y(t)^2
- (d^2/dt^2 y(t))^2 (d/dt y(t)) y(t)^4 (d^3/dt^3 y(t)) - (d^2/dt^2 y(t))^2 y(t)^5 (d^4/dt^4 y(t)) - 22*(d^2/dt^2 y(t)) (d/dt
y(t))^6 y(t) + 2*(d^2/dt^2 y(t)) (d/dt y(t))^2 y(t)^4 (d^4/dt^4 y(t)) + (d^2/dt^2 y(t)) y(t)^5 (d^3/dt^3 y(t))^2 + 6*(d/dt
y(t))^8 + (d/dt y(t))^5 y(t)^2 (d^3/dt^3 y(t)) - (d/dt y(t))^4 y(t)^3 (d^4/dt^4 y(t)) - (d/dt y(t))^2 y(t)^4 (d^3/dt^3 y(t))^2,
ideal)
false

```

Figure 3.4: The Wronskian  $W$  of  $q_1, q_2$ .

By Proposition 2.3.3 and (3.3.4), if  $y(t, p) = y(t, \bar{p})$ , then

$$\begin{cases} \frac{p_1 p_3}{p_2} = \frac{\bar{p}_1 \bar{p}_3}{\bar{p}_2}, \\ \frac{p_1}{p_2} + p_3 = \frac{\bar{p}_1}{\bar{p}_2} + \bar{p}_3 \end{cases} \quad (3.3.7)$$

(3.3.7) is equivalent to

$$\begin{cases} \frac{p_1}{p_2} = \frac{\bar{p}_1}{\bar{p}_2}, \\ p_3 = \bar{p}_3 \end{cases} \quad (3.3.8)$$

or

$$\begin{cases} \frac{p_1}{p_2} = \frac{\bar{p}_3}{\bar{p}_2}, \\ p_3 = \frac{\bar{p}_1}{\bar{p}_2} \end{cases} \quad (3.3.9)$$

By the initial condition of (3.3.3),  $y(0, p) = y(0, \bar{p})$  if

$$p_4 p_6 = \bar{p}_4 \bar{p}_6 \quad (3.3.10)$$

Next, by the similar arguments as in the section 3.2, we obtain the two other equations from the fifth list

$$\frac{p_2}{p_4} x_3(t, p) = \frac{y(t)y''(t) - (y')^2(t)}{y(t)(y'(t) + p_3 y(t)) \left( y'(t) + \frac{p_1}{p_2} y(t) \right)}, \quad (3.3.11)$$

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$p_1 x_1(t, p) x_3(t, p) + p_3 = \frac{y'(t)}{y(t)} \quad (3.3.12)$$

By (3.3.8), (3.3.9) and  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$ , we have

$$\begin{cases} \frac{p_2}{p_4} x_3(0, p) = \frac{\bar{p}_2}{\bar{p}_4} x_3(0, \bar{p}), \\ p_1 x_1(0, p) x_3(0, p) - p_3 = \bar{p}_1 x_1(0, \bar{p}) x_3(0, \bar{p}) - \bar{p}_3, \end{cases} \quad (3.3.13)$$

It implies that

$$\begin{cases} \frac{p_2}{p_4} \frac{1}{1 + p_2 p_6} = \frac{\bar{p}_2}{\bar{p}_4} \frac{1}{1 + \bar{p}_2 \bar{p}_6}, \\ \frac{p_1 p_5}{1 + p_2 p_6} - p_3 = \frac{\bar{p}_1 \bar{p}_5}{1 + \bar{p}_2 \bar{p}_6} - \bar{p}_3 \end{cases} \quad (3.3.14)$$

(3.3.7), (3.3.10), (3.3.14) give us the relationships between  $p$  and  $\bar{p}$  as follows  $p_3 = \bar{p}_3$ ,  $\frac{\bar{p}_1}{p_1} = \frac{\bar{p}_2}{p_2} = \frac{\bar{p}_4}{p_4} = \frac{p_5}{\bar{p}_5} = \frac{p_6}{\bar{p}_6}$ . Together with Proposition 3.1.1, we have the following

**Theorem 3.3.1.** *The SIR model with nonlinear incidence rate (3.3.1) is not identifiable, and we have  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$  if and only if  $\bar{\nu} = \nu$ ,  $\frac{\bar{\tau}}{\tau} = \frac{\bar{\kappa}}{\kappa} = \frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}$ .*

The following is the consequence of Theorem 3.3.1

**Theorem 3.3.2.** *The basic production number  $\mathcal{R}_0$  of the SIR model with nonlinear incidence rate (3.3.1) is uniquely determined from the output function  $y(t) = \mu I(t)$ .*

*Proof.* By Theorem 3.3.1, from the output function  $y(t) = \mu I(t)$  we have  $\nu$ ,  $\tau S_0$ ,  $\kappa I_0$  are uniquely determined. It derives that  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0)}$  is uniquely determined.  $\square$

### 3.4 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 2, \beta = 1$

In the case  $\alpha = 2, \beta = 1$  we rewrite (3.1.1) as follows

$$\begin{cases} S'(t) = -\frac{\tau S(t) I^2(t)}{1 + \kappa I(t)}, \\ I'(t) = \frac{\tau S(t) I^2(t)}{1 + \kappa I(t)} - \nu I(t), \quad t \geq 0 \\ y(t) = \mu I(t) \end{cases} \quad (3.4.1)$$

with  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and  $\mathcal{R}_0 = \frac{\tau S_0 I_0}{\nu(1 + \kappa I_0)} > 1$ .

Using the same notations as in Section 3.2, (3.4.1) can be rewritten as follows

$$\begin{cases} x'_1(t) = -\frac{p_1 x_1(t) x_2^2(t)}{1 + p_2 x_2(t)}, \\ x'_2(t) = \frac{p_1 x_1(t) x_2^2(t)}{1 + p_2 x_2(t)} - p_3 x_2(t), \quad t \geq 0 \\ y(t) = p_4 x_2(t) \end{cases} \quad (3.4.2)$$



with  $x_1(0) = p_5, x_2(0) = p_6$ .

Introducing a new variable  $x_3(t) = \frac{1}{1 + p_2 x_2(t)}$ , then  $x_3'(t) = -\frac{p_2 x_2'(t)}{(1 + p_2 x_2(t))^2} = -p_2 x_3^2(t) (p_1 x_1(t) x_2^2(t) x_3(t) - p_3 x_2(t))$ , and (3.4.2) can be written as follows

$$\begin{cases} p_4^2 x_1'(t) + p_1 x_1(t) y^2(t) x_3(t) = 0, \\ p_4 y'(t) - p_1 x_1(t) y^2(t) x_3(t) + p_3 p_4 y(t) = 0, & t \geq 0 \\ p_4^2 x_3'(t) + p_1 p_2 x_1(t) y^2(t) x_3^3(t) - p_2 p_3 p_4 y(t) x_3^2(t) = 0 \end{cases} \quad (3.4.3)$$

with  $x_1(0) = p_5, y(0) = p_4 p_6, x_3(0) = \frac{1}{1 + p_2 p_6}$ .

Consider the differential polynomials on the left-hand sides of (3.4.3). It is seen that the set of these polynomials is an autoreduced set in the differential ring  $\mathbb{Q}(p)\{x_1, x_3, y\}$  with the elimination ranking  $[y] < [x_1] < [x_3]$ . Moreover, the initials and separants of these polynomials are in  $\mathbb{Q}(p)$ , then these polynomials form an orthonomic set of  $\mathbb{Q}(p)\{x_1, x_3, y\}$ . Let  $\mathcal{P}$  is a prime differential ideal generated by these polynomials.

Next, we use Rosenfeld-Gröbner algorithm implemented in Maple for decomposing the ideal  $\mathcal{P}$ , and obtain four lists of differential polynomials. Each list is the set of generators of a radical ideal corresponding to the solution of (3.4.3). The lists are shown as follows

$$\begin{aligned} & [x_3(t), x_1'(t), y'(t) + p_3 y(t)], \\ & [x_3'(t), x_1'(t), y(t)], \\ & [p_4 x_3'(t) - p_2 p_3 y(t) x_3^2(t), x_1(t), y'(t) + p_3 y(t)], \\ & [p_2 p_4 x_3(t) y(t) (y')^2(t) + p_1 x_3(t) y^3(t) y'(t) + p_2 p_3 p_4 x_3(t) y^2(t) y'(t) \\ & \quad + p_1 p_3 x_3(t) y^4(t) + p_4^2 y(t) y'(t) - 2p_4^2 (y'')^2(t) - p_3 p_4^2 y(t) y'(t), \\ & p_1 p_4 x_1(t) y^2(t) y''(t) - 2p_1 p_4 x_1(t) y(t) (y')^2(t) - p_1 p_3 p_4 x_1(t) y^2(t) y'(t) + p_2 p_4 (y')^3(t) \\ & \quad + 2p_2 p_3 p_4 y(t) (y')^2(t) + 2p_1 p_3 y^3(t) y'(t) + p_2 p_3^2 p_4 y^2(t) y'(t) + p_1 p_3^2 y^4(t), \\ & p_2 p_4 y^2(t) (y')^2(t) y^{(3)}(t) + p_1 y^4(t) y'(t) y^{(3)}(t) + p_2 p_3 p_4 y^3(t) y'(t) y^{(3)}(t) \\ & \quad + p_1 p_3 y^5(t) y^{(3)}(t) - 3p_2 p_4 y^2(t) y'(t) (y'')^2(t) - p_1 y^4(t) (y'')^2(t) \\ & \quad - p_2 p_3 p_4 y^3(t) (y'')^2(t) + 4p_2 p_4 y(t) (y')^3(t) y''(t) - 4p_1 y^3(t) (y')^2(t) y''(t) \\ & \quad - 7p_1 p_3 y^4(t) y'(t) y''(t) - p_1 p_3^2 y^5(t) y''(t) - 2p_2 p_4 (y')^5(t) + 6p_1 y^2(t) (y')^4(t) \\ & \quad + 10p_1 p_3 y^3(t) (y')^3(t) + 3p_1 p_3^2 p_4 y^4(t) (y')^2(t)]. \end{aligned}$$

The first three lists above are not corresponding to the complete system including (3.4.3) and the initial condition. For instance, the third list requires  $x_1(t) = 0$  for all  $t \geq 0$ , contradicting to  $x_1(0) = p_5 > 0$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

The fourth list is now corresponding to the general case, and the last polynomial in the fourth list forms the output equation of (3.4.3) as follows

$$\begin{aligned}
& y^2(t)(y')^2(t)y^{(3)}(t) - 3y^2(t)y'(t)(y'')^2(t) + 4y(t)(y')^3(t)y''(t) - 2(y')^5(t) \\
& + \frac{p_1}{p_2p_4}(y^4(t)y'(t)y^{(3)}(t) + 6y^2(t)(y')^4(t) - 4y^3(t)(y')^2(t)y''(t) - y^4(t)(y'')^2(t)) \\
& + p_3(y^3(t)y'(t)y^{(3)}(t) - y^3(t)(y'')^2(t)) \\
& + \frac{p_1p_3}{p_2p_4}(y^5(t)y^{(3)}(t) + 10y^3(t)(y')^3(t) - 7y^4(t)y'(t)y''(t)) \\
& + \frac{p_1p_3^2}{p_2p_4}(3y^4(t)(y')^2(t) - y^5(t)y''(t)) = 0
\end{aligned} \tag{3.4.4}$$

Next, we check the linear independence of four polynomials

$$q_1 = y^4(t)y'(t)y^{(3)}(t) + 6y^2(t)(y')^4(t) - 4y^3(t)(y')^2(t)y''(t) - y^4(t)(y'')^2(t), \tag{3.4.5}$$

$$q_2 = y^3(t)y'(t)y^{(3)}(t) - y^3(t)(y'')^2(t) \tag{3.4.6}$$

$$q_3 = y^5(t)y^{(3)}(t) + 10y^3(t)(y')^3(t) - 7y^4(t)y'(t)y''(t), \tag{3.4.7}$$

$$q_4 = 3y^4(t)(y')^2(t) - y^5(t)y''(t). \tag{3.4.8}$$

The Wronskian  $W$  of  $q_1, q_2, q_3, q_4$  is computed by the package *VectorCalculus* in Maple. Next, the the function *BelongsTo* in the package *DifferentialAlgebra* of Maple shows that  $W \notin \mathcal{P}$ . ( see figures 3.5, 3.6, 3.7, 3.8)

$$\begin{aligned}
 \text{BelongsTo} & \left( -2y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^4 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 2y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 122y(t)^{16} \left( \frac{d^3}{dt^3} \right. \right. \\
 & y(t) \left. \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 18y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^5}{dt^5} y(t) \right) + 70y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^2}{dt^2} \right. \\
 & y(t) \left. \right)^4 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 18y(t)^{16} \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^5}{dt^5} y(t) \right)^2 + 29y(t)^{16} \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right)^3 \\
 & - 18y(t)^{16} \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 290y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^5 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right) + 123y(t)^{15} \left( \frac{d^3}{dt^3} \right. \\
 & y(t) \left. \right)^4 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) - 1214y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^4 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^3 - 3y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} \right. \\
 & y(t) \left. \right)^4 \left( \frac{d^6}{dt^6} y(t) \right) - 608y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^4}{dt^4} y(t) \right) - 88y(t)^{15} \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right)^2 \\
 & - 96y(t)^{15} \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right)^3 + 63y(t)^{15} \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^6}{dt^6} y(t) \right) - 463y(t)^{15} \left( \frac{d}{dt} \right. \\
 & y(t) \left. \right) \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 3292y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^4 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 - 66528y(t)^9 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} \right. \\
 & y(t) \left. \right)^{10} \left( \frac{d^2}{dt^2} y(t) \right)^3 - 6840y(t)^9 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 26064y(t)^9 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \\
 & + 13176y(t)^8 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right)^2 - 4680y(t)^8 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) \\
 & + 118y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^5}{dt^5} y(t) \right) + 5862y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \\
 & + 67y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 5452y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^6 + 2y(t)^{14} \left( \frac{d^3}{dt^3} \right. \\
 & y(t) \left. \right) \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^5}{dt^5} y(t) \right)^2 - y(t)^{14} \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 180y(t)^{14} \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} \right. \\
 & y(t) \left. \right)^2 \left( \frac{d^5}{dt^5} y(t) \right)^2 + 95y(t)^{14} \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^3 - 155y(t)^{14} \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^6}{dt^6} y(t) \right) \\
 & + 2207y(t)^{14} \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 958y(t)^{14} \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^5}{dt^5} y(t) \right) + 2666y(t)^{14} \left( \frac{d}{dt} \right. \\
 & y(t) \left. \right) \left( \frac{d^2}{dt^2} y(t) \right)^7 \left( \frac{d^4}{dt^4} y(t) \right) + 528y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^4 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) + 194y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} \right. \\
 & y(t) \left. \right)^6 \left( \frac{d^4}{dt^4} y(t) \right) - 1440y(t)^{10} \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 15048y(t)^{10} \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) \\
 & - 47736y(t)^{10} \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 16992y(t)^9 \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right) \\
 & - 1512y(t)^9 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^4}{dt^4} y(t) \right) + 368y(t)^{13} \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 3910y(t)^{13} \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 1162y(t)^{13} \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^5}{dt^5} y(t) \right)
 \end{aligned}$$

Figure 3.5: The first part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& -9950y(t)^{13} \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^4}{dt^4} y(t) \right) + 66y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^2 \\
& + 402y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^5}{dt^5} y(t) \right) + 58212y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 + 252y(t)^{12} \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right) \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 39104y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^6 - 12y(t)^{12} \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^4}{dt^4} \right. \\
& y(t) \left. \right) \left( \frac{d^6}{dt^6} y(t) \right) - 1244y(t)^{12} \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^6}{dt^6} y(t) \right) + 3352y(t)^{12} \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
& + 4544y(t)^{12} \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^5}{dt^5} y(t) \right) + 2042y(t)^{12} \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^4}{dt^4} y(t) \right) \\
& + 10872y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) + 420y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^4}{dt^4} y(t) \right) \\
& - 47328y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^3 + 252y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^6}{dt^6} y(t) \right) \\
& - 25356y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^5 - 96y(t)^{11} \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
& + 2052y(t)^{11} \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) - 1776y(t)^{11} \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
& - 14196y(t)^{11} \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) + 31728y(t)^{11} \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^4}{dt^4} y(t) \right) \\
& - 7632y(t)^{10} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) + 1656y(t)^{10} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^5}{dt^5} y(t) \right) \\
& + 96984y(t)^{10} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^4 - 15936y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \\
& + 30y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^6}{dt^6} y(t) \right) - 9104y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 + 18788y(t)^{13} \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right) \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^7 - 164y(t)^{13} \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2 + 5y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} \right. \\
& y(t) \left. \right) \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 77y(t)^{16} \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right) \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 252y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} \right. \\
& y(t) \left. \right) - 309y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 327y(t)^{15} \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^4}{dt^4} \right. \\
& y(t) \left. \right) \left( \frac{d^5}{dt^5} y(t) \right) - 1246y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) - 34y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} \right. \\
& y(t) \left. \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 73y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^4 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) - y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} \right. \\
& y(t) \left. \right)^2 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 41y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 9y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} \right.
\end{aligned}$$

Figure 3.6: The second part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4$ .

$$\begin{aligned}
 & y(t) \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^6}{dt^6} y(t) \right) + 36y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 2y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right) \\
 & y(t) \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right)^2 - 4y(t)^{16} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^3 + y(t)^{16} \left( \frac{d}{dt} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) - 9y(t)^{16} \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 127y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 894y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^3 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \\
 & y(t) + 3y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 7y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) \\
 & y(t) - 109y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^2 - 228y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^3 \\
 & y(t)^3 \left( \frac{d^5}{dt^5} y(t) \right) + 1951y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^4 - 73y(t)^{15} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^6}{dt^6} y(t) \right) + 118y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 444y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right) \\
 & y(t) \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) - 1324y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
 & - 6698y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) + 19542y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^4}{dt^4} y(t) \right) \\
 & y(t) + 46y(t)^{13} \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 1100y(t)^{13} \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
 & y(t) - 4598y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) - 648y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \\
 & y(t) \left( \frac{d^6}{dt^6} y(t) \right) + 9552y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) - 39154y(t)^{12} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^6 \\
 & y(t)^6 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) - 262y(t)^{12} \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 6444y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right) \\
 & y(t) \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 28236y(t)^{11} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \\
 & - 5256y(t)^{10} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) - 1090y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
 & y(t) + 11802y(t)^{13} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) + 1650y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 2367y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^5}{dt^5} y(t) \right) - 1684y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d}{dt} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^4}{dt^4} y(t) \right) - 63y(t)^{14} \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 1043y(t)^{14} \left( \frac{d}{dt} y(t) \right)^4 \\
 & y(t)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 9043y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^2 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) - y(t)^{14} \left( \frac{d^3}{dt^3} y(t) \right)
 \end{aligned}$$

Figure 3.7: The third part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4$ .



and

$$\frac{p_1}{p_4}x_1(t, p)x_3(t, p) = \frac{y'(t) + p_3y(t)}{y^2(t)} \quad (3.4.13)$$

By (3.4.10) and  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$ , we have

$$\begin{cases} \frac{p_2}{p_4}x_3(0, p) = \frac{\bar{p}_2}{\bar{p}_4}x_3(0, \bar{p}), \\ \frac{p_1}{p_4}x_1(0, p)x_3(0, p) = \frac{\bar{p}_1}{\bar{p}_4}x_1(0, \bar{p})x_3(0, \bar{p}), \end{cases} \quad (3.4.14)$$

It implies that

$$\begin{cases} \frac{p_2}{p_4} \frac{1}{1 + p_2p_6} = \frac{\bar{p}_2}{\bar{p}_4} \frac{1}{1 + \bar{p}_2\bar{p}_6}, \\ \frac{p_1}{p_4} \frac{p_5}{1 + p_2p_6} = \frac{\bar{p}_1}{\bar{p}_4} \frac{\bar{p}_5}{1 + \bar{p}_2\bar{p}_6} \end{cases} \quad (3.4.15)$$

The algebraic system (3.4.10), (3.4.11), (3.4.15) derives the relationships between  $p$  and  $\bar{p}$  as follows  $p_3 = \bar{p}_3$ ,  $\frac{\bar{p}_1}{p_1} = \left(\frac{\bar{p}_4}{p_4}\right)^2$ ,  $\frac{\bar{p}_2}{p_2} = \frac{\bar{p}_4}{p_4} = \frac{p_5}{p_5} = \frac{p_6}{p_6}$ . By Proposition 3.1.1, we have the following

**Theorem 3.4.1.** *The SIR model with nonlinear incidence rate (3.4.1) is not identifiable, and we have  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$  if and only if  $\bar{\nu} = \nu$ ,  $\frac{\bar{\tau}}{\tau} = \frac{\bar{\mu}^2}{\mu^2}$ ,  $\frac{\bar{\kappa}}{\kappa} = \frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}$ .*

The following is the corollary of the above theorem

**Theorem 3.4.2.** *The basic production number  $\mathcal{R}_0$  of the SIR model with nonlinear incidence rate (3.4.1) is uniquely determined from the output function  $y(t) = \mu I(t)$ .*

*Proof.* By Theorem 3.4.1, from the output function  $y(t) = \mu I(t)$  we have  $\nu$ ,  $\tau S_0 I_0$ ,  $\kappa I_0$  are uniquely determined. It derives that  $\mathcal{R}_0 = \frac{\tau S_0 I_0}{\nu(1 + \kappa I_0)}$  is uniquely determined.  $\square$

### 3.5 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 2, \beta = 2$

Consider the system

$$\begin{cases} S'(t) = -\frac{\tau S(t)I^2(t)}{1 + \kappa I^2(t)}, \\ I'(t) = \frac{\tau S(t)I^2(t)}{1 + \kappa I^2(t)} - \nu I(t), \\ y(t) = \mu I(t) \end{cases} \quad t \geq 0 \quad (3.5.1)$$

with  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0^2)} > 1$ .

### 3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

While the same notations are using as in Section 3.2, (3.5.1) can be rewritten as follows

$$\begin{cases} x_1'(t) = -\frac{p_1x_1(t)x_2^2(t)}{1+p_2x_2^2(t)}, \\ x_2'(t) = \frac{p_1x_1(t)x_2^2(t)}{1+p_2x_2^2(t)} - p_3x_2(t), \\ y(t) = p_4x_2(t) \end{cases} \quad t \geq 0 \quad (3.5.2)$$

with  $x_1(0) = p_5$ ,  $x_2(0) = p_6$ .

As mentioned in Chapter 2, we let  $x_3(t) = \frac{1}{1+p_2x_2^2(t)}$ , then

$$\begin{aligned} x_3'(t) &= -\frac{2p_2x_2(t)x_2'(t)}{(1+p_2x_2^2(t))^2} \\ &= -2p_2x_2(t)x_3^2(t)(p_1x_1(t)x_2^2(t)x_3(t) - p_3x_2(t)). \end{aligned}$$

Therefore, (3.5.2) can be written as follows

$$\begin{cases} p_4^2x_1'(t) + p_1x_1(t)y^2(t)x_3(t) = 0, \\ p_4y'(t) - p_1x_1(t)y^2(t)x_3(t) + p_3p_4y(t) = 0, \\ p_4^3x_3'(t) + 2p_1p_2x_1(t)y^3(t)x_3^3(t) - 2p_2p_3p_4y^2(t)x_3^2(t) = 0 \end{cases} \quad t \geq 0 \quad (3.5.3)$$

with  $x_1(0) = p_5$ ,  $y(0) = p_4p_6$ ,  $x_3(0) = \frac{1}{1+p_2p_6^2}$ .

The set of all differential polynomials on the left hand sides of (3.5.3) is an autoreduced set in the differential ring  $\mathbb{Q}(p)\{x_1, x_3, y\}$  with the elimination ranking  $[y] < [x_1] < [x_3]$ . Moreover, the initials and separants of these polynomials are in  $\mathbb{Q}(p)$ , then these polynomials form an orthonomic set of  $\mathbb{Q}(p)\{x_1, x_3, y\}$ . Let  $\mathcal{P}$  is a prime differential ideal generated by these polynomials.

Next, we decompose the ideal  $\mathcal{P}$  by using Rosenfeld-Gröbner algorithm implemented in Maple. The four lists of differential polynomials is returned by this decomposition. Each list forms the set of generators of a radical ideal corresponding to either the singular solutions or general solution of (3.5.3). We show here the lists

$$\begin{aligned} &[x_3(t), x_1'(t), y'(t) + p_3y(t)], \\ &[x_3'(t), x_1'(t), y(t)], \\ &[p_4^2x_3'(t) - 2p_2p_3y^2(t)x_3^2(t), x_1(t), y'(t) + p_3y(t)], \\ &[2p_3x_3(t)y^2(t)(y')^2(t) + p_1x_3(t)y^3(t)y'(t) + 2p_2p_3x_3(t)y^3(t)y'(t) \\ &\quad + p_1p_3x_3(t)y^4(t) + p_4^2y(t)(y'')^2(t) - 2p_4^2(y')^2(t) - p_3p_4^2y(t)y'(t), \\ &\quad p_1p_4x_1(t)y(t)y''(t) - 2p_1p_4x_1(t)(y')^2(t) - p_1p_3p_4x_1(t)y(t)y'(t) + 2p_2(y')^3(t) \\ &\quad + p_1y(t)(y')^2(t) + 4p_2p_3y(t)(y')^2(t) + 2p_1p_3y^2(t)y'(t) + 2p_2p_3^2y^2(t)y'(t) + p_1p_3^2y^3(t), \\ &\quad 2p_2y(t)(y')^2(t)y^{(3)}(t) + p_1y^2(t)y'(t)y^{(3)}(t) + 2p_2p_3y^2(t)y'(t)y^{(3)}(t) \\ &\quad + p_1p_3y^3(t)y^{(3)}(t) - 6p_2y(t)y'(t)(y'')^2(t) - p_1y^2(t)(y'')^2(t) - 2p_2p_3y^2(t)(y'')^2(t) \end{aligned}$$



$$\begin{aligned}
& +6p_2(y')^3(t)y''(t) - 4p_1y(t)(y')^2(t)y''(t) - 2p_2p_3y(t)(y')^2(t)y''(t) \\
& -7p_1p_3y^2(t)y'(t)y''(t) - p_1p_3^2y^3(t)y''(t) + 6p_1(y')^4(t) + 6p_2p_3(y')^4(t) \\
& +10p_1p_3y(t)(y')^3(t) + 2p_2p_3^2y(t)(y')^3(t) + 3p_1p_3^2y^2(t)(y')^2(t)].
\end{aligned}$$

It is seen that the first three lists above are not corresponding to the complete system including (3.5.3) and the initial condition. For instance, the third list requires  $x_1(t) = 0$  for all  $t \geq 0$ , contradicting to  $x_1(0) = p_5 > 0$ .

The fourth list is now corresponding to the general case, and the last polynomial in the fourth list forms the output equation of (3.5.3) as follows

$$\begin{aligned}
& 6(y')^3(t)y''(t) + 2y(t)(y')^2(t)y^{(3)}(t) - 6y(t)y'(t)(y'')^2(t) \\
& + \frac{p_1}{p_2} \left( 6(y')^4(t) - 4y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t) \right) \\
& + 2p_3 \left( 3(y')^4(t) - y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t) \right) \\
& + \frac{p_1p_3}{p_2} \left( 10y(t)(y')^3(t) + y^3(t)y^{(3)}(t) - 7y^2(t)y'(t)y''(t) \right) \\
& + \frac{p_1p_3^2}{p_2} \left( 3y^2(t)(y')^2(t) - y^3(t)y''(t) \right) + 2p_3^2y(t)(y')^3(t) = 0
\end{aligned} \tag{3.5.4}$$

Next, we check the linear independence of five polynomials

$$q_1 = 6(y')^4(t) - 4y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t), \tag{3.5.5}$$

$$q_2 = 3(y')^4(t) - y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t), \tag{3.5.6}$$

$$q_3 = 10y(t)(y')^3(t) + y^3(t)y^{(3)}(t) - 7y^2(t)y'(t)y''(t), \tag{3.5.7}$$

$$q_4 = 3y^2(t)(y')^2(t) - y^3(t)y''(t), \tag{3.5.8}$$

$$q_5 = y(t)(y')^3(t). \tag{3.5.9}$$

We compute the Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$  by the package *VectorCalculus* in Maple. Next, the function *Belongs To* in the package *DifferentialAlgebra* of Maple is used to show that  $W \notin \mathcal{P}$ . ( see figures from Figure 3.9 to Figure 3.23)

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
 \text{BelongsTo} & \left( -113244 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) + 13380 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} \right. \right. \\
 & y(t) \left. \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 133236 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} \right. \right. \\
 & y(t) \left. \left. + 2544 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) - 2736 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} \right. \right. \\
 & y(t) \left. \right) y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 11316 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) \\
 & - 70440 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) + 240 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} \right. \\
 & y(t) \left. \right) y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 96 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) \\
 & + 25068 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) - 2580 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} \right. \\
 & y(t) \left. \right) y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 2640 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) \\
 & + 2340 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 84 \left( \frac{d}{dr} y(t) \right)^8 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^3 \left( \frac{d^6}{dr^6} y(t) \right) \\
 & - 60 \left( \frac{d}{dr} y(t) \right)^8 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right)^2 + 2303640 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^4 - 253152 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^5 \left( \frac{d^5}{dr^5} y(t) \right) - 7722 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^6 \left( \frac{d^7}{dr^7} y(t) \right) + 1983840 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} \right. \\
 & y(t) \left. \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^5 + 322092 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^6 - 366 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} \right. \\
 & y(t) \left. \right)^2 y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right)^3 + 13008 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) - 126 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^8 \left( \frac{d^7}{dr^7} \right. \\
 & y(t) \left. \right) + 534 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right)^2 - 210 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^4 \\
 & + 1242972 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) - 176928 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \\
 & + 48816 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^6 \left( \frac{d^6}{dr^6} y(t) \right) - 527100 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right)^2 + 33198 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^5 y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right)^2 - 679758 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^6 - 282 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} \right. \\
 & y(t) \left. \right)^4 y(t)^8 \left( \frac{d^6}{dr^6} y(t) \right)^2 - 426 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^3 + 4704 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dr^4} \right. \\
 & y(t) \left. \right)^4 - 86172 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^6 y(t)^7 - 2670 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right)
 \end{aligned}$$

Figure 3.9: The first part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

$$\begin{aligned}
 & + 3576 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 852120 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^2}{dr^2} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \\
 & + 180018 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) + 5724 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^7 \left( \frac{d^7}{dr^7} y(t) \right) - 16416 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^{17} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) \left( \frac{d^4}{dr^4} y(t) \right) - 6480 \left( \frac{d}{dr} y(t) \right)^{17} \left( \frac{d^3}{dr^3} y(t) \right) y(t) \left( \frac{d^6}{dr^6} y(t) \right) + 4320 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^{17} y(t) \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) + 5040 \left( \frac{d}{dr} y(t) \right)^{16} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t) \left( \frac{d^6}{dr^6} y(t) \right) - 74304 \left( \frac{d}{dr} y(t) \right)^{16} \left( \frac{d^2}{dr^2} \right. \\
 & y(t) \left. \right) y(t) \left( \frac{d^4}{dr^4} y(t) \right)^2 + 18576 \left( \frac{d}{dr} y(t) \right)^{16} \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t) \left( \frac{d^4}{dr^4} y(t) \right) - 648 \left( \frac{d}{dr} y(t) \right)^{16} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^2 \left( \frac{d^7}{dr^7} y(t) \right) \\
 & - 4320 \left( \frac{d}{dr} y(t) \right)^{16} y(t)^2 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 65232 \left( \frac{d}{dr} y(t) \right)^{15} \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t) \left( \frac{d^5}{dr^5} y(t) \right) + 504 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^{15} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^2 \left( \frac{d^7}{dr^7} y(t) \right) - 101664 \left( \frac{d}{dr} y(t) \right)^{15} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t) - 34488 \left( \frac{d}{dr} y(t) \right)^{15} \left( \frac{d^3}{dr^3} \right. \\
 & y(t) \left. \right)^2 y(t)^2 \left( \frac{d^5}{dr^5} y(t) \right) + 52056 \left( \frac{d}{dr} y(t) \right)^{15} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^2 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 648 \left( \frac{d}{dr} y(t) \right)^{15} y(t)^3 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} \right. \\
 & y(t) \left. \right) + 360 \left( \frac{d}{dr} y(t) \right)^{15} y(t)^3 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 73728 \left( \frac{d}{dr} y(t) \right)^{14} \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t) \left( \frac{d^4}{dr^4} y(t) \right) \\
 & + 229248 \left( \frac{d}{dr} y(t) \right)^{14} \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t) - 29808 \left( \frac{d}{dr} y(t) \right)^{14} \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^2 \left( \frac{d^6}{dr^6} y(t) \right) \\
 & + 95976 \left( \frac{d}{dr} y(t) \right)^{14} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^2 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 14952 \left( \frac{d}{dr} y(t) \right)^{14} \left( \frac{d^2}{dr^2} y(t) \right) y(t)^3 \left( \frac{d^5}{dr^5} y(t) \right)^2 \\
 & - 15840 \left( \frac{d}{dr} y(t) \right)^{14} \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^3 \left( \frac{d^6}{dr^6} y(t) \right) - 108 \left( \frac{d}{dr} y(t) \right)^{14} y(t)^4 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) - 240912 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^{13} \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right) y(t) + 165960 \left( \frac{d}{dr} y(t) \right)^{13} \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^2 \left( \frac{d^5}{dr^5} y(t) \right) - 2736 \left( \frac{d}{dr} y(t) \right)^{13} \left( \frac{d^2}{dr^2} \right. \\
 & y(t) \left. \right)^3 y(t)^3 \left( \frac{d^7}{dr^7} y(t) \right) + 311688 \left( \frac{d}{dr} y(t) \right)^{13} \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^2 + 42192 \left( \frac{d}{dr} y(t) \right)^{13} \left( \frac{d^3}{dr^3} \right. \\
 & y(t) \left. \right)^3 y(t)^3 \left( \frac{d^4}{dr^4} y(t) \right) - 1404 \left( \frac{d}{dr} y(t) \right)^{13} \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^4 \left( \frac{d^7}{dr^7} y(t) \right) + 7644 \left( \frac{d}{dr} y(t) \right)^{13} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^4 \left( \frac{d^5}{dr^5} \right. \\
 & y(t) \left. \right)^2 - 1548 \left( \frac{d}{dr} y(t) \right)^{13} y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right) - 452520 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right)^5 y(t)^2 \left( \frac{d^4}{dr^4} y(t) \right) \\
 & - 1125216 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^2 + 56520 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^3 \left( \frac{d^6}{dr^6} y(t) \right) \\
 & + 232080 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^3 \left( \frac{d^4}{dr^4} y(t) \right)^2 + 12102 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^4 \left( \frac{d^5}{dr^5} y(t) \right)^2 \\
 & - 254208 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^3 - 282 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right) y(t)^5 \left( \frac{d^6}{dr^6} y(t) \right)^2 + 27324 \left( \frac{d}{dr} \right. \\
 & y(t) \left. \right)^{12} \left( \frac{d^2}{dr^2} y(t) \right) y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right)^3 - 36348 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^4 \left( \frac{d^5}{dr^5} y(t) \right) + 62838 \left( \frac{d}{dr} y(t) \right)^{12} \left( \frac{d^3}{dr^3} \right.
 \end{aligned}$$

Figure 3.10: The second part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .



$$\begin{aligned}
 & y(t)^6 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) + 5034 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \\
 & y(t) \left( \frac{d^6}{dt^6} y(t) \right) + 20730 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 60 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \\
 & y(t) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 612 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 240 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 444 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 \\
 & y(t)^2 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 72 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 462 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2 + 360 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 \\
 & y(t)^3 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 270 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) \\
 & + 39060 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 24 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^4 \\
 & y(t) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 246 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 480 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 2844 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^3 \\
 & y(t) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 6 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) \\
 & + 2934 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 30 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 \\
 & y(t)^2 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 12 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) \\
 & + 30 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 42 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 \\
 & y(t)^3 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 72 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) \\
 & + 72 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2 + 60 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 \\
 & y(t) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) - 1800 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 15396 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 60 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 \\
 & y(t) y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 228 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
 & - 24 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 48 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^3
 \end{aligned}$$

Figure 3.12: The fourth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& y(t)^2 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 78 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) \\
& - 372 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 432 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) - 414 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
& + 6660 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 36 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 36 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
& + 792 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 270 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 1152 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
& - 144 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 1188 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 10080 \left( \frac{d}{dt} y(t) \right)^{17} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) - 3168 \left( \frac{d}{dt} y(t) \right)^{16} \left( \frac{d^2}{dt^2} \right. \\
& y(t) \left. \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) - 16128 \left( \frac{d}{dt} y(t) \right)^{16} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 + 3600 \left( \frac{d}{dt} y(t) \right)^{16} y(t)^2 \left( \frac{d^5}{dt^5} y(t) \right)^2 \\
& + 18720 \left( \frac{d}{dt} y(t) \right)^{15} \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) + 41472 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^2 + 90 \left( \frac{d}{dt} y(t) \right)^{14} y(t)^4 \left( \frac{d^6}{dt^6} \right. \\
& y(t) \left. \right)^2 - 5832 \left( \frac{d}{dt} y(t) \right)^{14} y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right)^3 + 69552 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right)^7 y(t) + 49872 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^3}{dt^3} \right. \\
& y(t) \left. \right)^5 y(t)^4 + 48 \left( \frac{d}{dt} y(t) \right)^{11} y(t)^6 \left( \frac{d^5}{dt^5} y(t) \right)^3 - 296352 \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^2}{dt^2} y(t) \right)^8 y(t)^2 + 1062 \left( \frac{d}{dt} \right. \\
& y(t) \left. \right)^{10} y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right)^4 + 597240 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^9 y(t)^3 + 26700 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^3}{dt^3} y(t) \right)^6 y(t)^6 \\
& - 580230 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^{10} y(t)^4 + 60 \left( \frac{d}{dt} y(t) \right)^6 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^5 + 6504 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^7 y(t)^8 \\
& + 237006 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^{11} y(t)^5 - 33264 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^{12} y(t)^6 + 324 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^8 y(t)^{10} \\
& - 14364 \left( \frac{d^2}{dt^2} y(t) \right)^{11} y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) + 14868 \left( \frac{d^2}{dt^2} y(t) \right)^{10} \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 - 756 \left( \frac{d^2}{dt^2} y(t) \right)^{10} y(t)^9 \left( \frac{d^6}{dt^6} y(t) \right) \\
& + 4860 \left( \frac{d^2}{dt^2} y(t) \right)^9 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 7200 \left( \frac{d^2}{dt^2} y(t) \right)^7 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 - 432 \left( \frac{d^2}{dt^2} y(t) \right)^7 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 \\
& + 756 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^6 y(t)^{10} - 12960 \left( \frac{d}{dt} y(t) \right)^{18} \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 157020 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} \right. \\
& y(t) \left. \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^8 - 144 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^3 - 8460 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} \right.
\end{aligned}$$

Figure 3.13: The fifth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

$$\begin{aligned}
& y(t)^7 y(t)^9 + 72 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^2 + 57042 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^{10} y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \\
& + 94914 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 - 2898 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^9 y(t)^8 \left( \frac{d^6}{dr^6} y(t) \right) - 19494 \left( \frac{d}{dr} y(t) \right)^2 \\
& \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 1908 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right)^2 + 112140 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^6 \\
& y(t)^6 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^8 + 54 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^{10} \left( \frac{d^6}{dr^6} y(t) \right)^2 + 90 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right)^3 \\
& + 648 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^4 + 21690 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^6 y(t)^9 - 59472 \left( \frac{d}{dr} y(t) \right) \\
& y(t) \left( \frac{d^2}{dr^2} y(t) \right)^{11} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 - 4788 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^{10} y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right) - 252 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^9 \\
& y(t)^9 y(t)^9 \left( \frac{d^7}{dr^7} y(t) \right) - 63180 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^8 - 20880 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^5 \\
& y(t)^5 y(t)^9 - 324 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^7 y(t)^{10} + 3276 \left( \frac{d^2}{dr^2} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right) \\
& - 14580 \left( \frac{d^2}{dr^2} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) + 108 \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 108 \left( \frac{d^2}{dr^2} y(t) \right)^7 \\
& \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^6}{dr^6} y(t) \right) + 468 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right) + 1980 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^2 \\
& y(t)^2 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^2 - 2304 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right) + 174 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^2 \\
& y(t)^2 y(t)^6 \left( \frac{d^6}{dr^6} y(t) \right)^2 - 44808 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right)^3 + 36204 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^3}{dr^3} y(t) \right)^4 \\
& y(t)^4 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) - 888 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^6 \left( \frac{d^7}{dr^7} y(t) \right) + 4062 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right)^2 \\
& y(t)^2 + 24 \left( \frac{d}{dr} y(t) \right)^{10} y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right)^2 - 12 \left( \frac{d}{dr} y(t) \right)^{10} y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right)^2 \left( \frac{d^6}{dr^6} y(t) \right) \\
& - 2352240 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^3 + 245052 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^4 \left( \frac{d^5}{dr^5} y(t) \right) \\
& + 1260 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^5 y(t)^5 \left( \frac{d^7}{dr^7} y(t) \right) - 1065960 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^4 \\
& - 233640 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^5 + 36 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right)^3 - 15636 \left( \frac{d}{dr} y(t) \right)^9 \\
& \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) + 27390 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right)^2 + 198 \left( \frac{d}{dr} y(t) \right)^9 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^3 \\
& y(t)^3 \left( \frac{d^5}{dr^5} y(t) \right) - 1752840 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right) - 1788300 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^2 \\
& y(t)^2 y(t)^4 - 50220 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^5 \left( \frac{d^6}{dr^6} y(t) \right) + 990192 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^5 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right)^2
\end{aligned}$$

Figure 3.14: The sixth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& -55848 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right)^2 - 248844 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^5 + 252 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^7 \left( \frac{d^6}{dr^6} y(t) \right)^2 + 26418 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right)^3 - 3450 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^4 - 3816 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^7 \left( \frac{d^6}{dr^6} y(t) \right) + 60 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^8 \left( \frac{d^6}{dr^6} y(t) \right)^2 - 3030 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^3 + 60 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right)^3 - 661584 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^6 + 35544 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^7 + 426 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right)^3 - 456 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^9 \left( \frac{d^6}{dr^6} y(t) \right) - 630 \left( \frac{d}{dr} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right)^3 - 400122 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^9 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) + 400638 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^6 - 6570 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^7 \left( \frac{d^6}{dr^6} y(t) \right) + 123894 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 3354 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right)^2 + 438726 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^7 - 6 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^5 y(t)^9 \left( \frac{d^6}{dr^6} y(t) \right)^2 - 2202 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^3 - 2970 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right)^4 + 43230 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^6 y(t)^8 - 48 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^5 + 840 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^6 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) - 18 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^5 y(t)^{10} \left( \frac{d^7}{dr^7} y(t) \right) + 18 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right)^2 + 84 \left( \frac{d}{dr} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^4 + 81144 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^2}{dr^2} y(t) \right)^{10} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 - 47466 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^2}{dr^2} y(t) \right)^9 y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right) - 882 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^8 \left( \frac{d^7}{dr^7} y(t) \right) - 204132 \left( \frac{d}{dr} y(t) \right)^3 \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^7 + 372 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 66 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^6}{dr^6} y(t) \right) - 252 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right)^2 + 753684 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^5}{dr^5} y(t) \right) - 1550268 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) + 12708 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^7}{dr^7} y(t) \right) - 19356 \left( \frac{d}{dr} y(t) \right)^8 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 150 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) - 97032 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right) + 32640 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^2
\end{aligned}$$

Figure 3.15: The seventh part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .



$$\begin{aligned}
& -1524 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^6}{dr^6} y(t) \right) + 1554 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right)^2 + 215880 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) - 1836 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^8 \left( \frac{d^7}{dr^7} y(t) \right) + 315192 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 3828 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) - 660 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^5 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) \\
& - 357780 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right) + 6792 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \left( \frac{d^7}{dr^7} y(t) \right) - 25584 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^5}{dr^5} y(t) \right)^2 + 7848 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^2}{dr^2} y(t) \right) y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right) + 26196 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) - 84 \left( \frac{d}{dr} y(t) \right)^{11} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) - 120 \left( \frac{d}{dr} y(t) \right)^{11} y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) \\
& - 501960 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^4 \left( \frac{d^5}{dr^5} y(t) \right) - 213132 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right) - 852 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^7}{dr^7} y(t) \right) + 37392 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 99708 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \left( \frac{d^6}{dr^6} y(t) \right) - 372 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 89304 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^5 \left( \frac{d^5}{dr^5} y(t) \right) \\
& - 208566 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 2136 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right) y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^6}{dr^6} y(t) \right) + 2640 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^2}{dr^2} y(t) \right) y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right)^2 - 672 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 3738 \left( \frac{d}{dr} y(t) \right)^{10} \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right) - 12 \left( \frac{d}{dr} y(t) \right)^{10} y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 3468780 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^4 \left( \frac{d^4}{dr^4} y(t) \right) \\
& + 67116 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^6}{dr^6} y(t) \right) - 462588 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^4 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) - 110652 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^5 \left( \frac{d^5}{dr^5} y(t) \right) + 123732 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right)^2 + 684 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 5520 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 746268 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^5 \left( \frac{d^4}{dr^4} y(t) \right) - 9084 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) - 9084 \left( \frac{d}{dr} y(t) \right)^9 \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right)
\end{aligned}$$

Figure 3.16: The eighth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& y(t)^9 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^6 \left( \frac{d^7}{dt^7} y(t) \right) + 22158 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^6 \left( \frac{d^5}{dt^5} y(t) \right)^2 \\
& - 9750 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 39504 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^6 \left( \frac{d^6}{dt^6} y(t) \right) \\
& y(t) - 312 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^7 \left( \frac{d^6}{dt^6} y(t) \right)^2 + 23130 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) \\
& y(t) y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right)^3 - 60 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) - 180 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^3}{dt^3} y(t) \right) \\
& y(t)^2 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 552 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^7}{dt^7} y(t) \right) - 3900 \left( \frac{d}{dt} y(t) \right)^4 \\
& y(t)^4 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 10452 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^6}{dt^6} y(t) \right) \\
& + 6 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 28656 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right) \\
& y(t) - 66120 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 1086 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \\
& y(t)^2 \left( \frac{d^6}{dt^6} y(t) \right) + 3768 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^6 \left( \frac{d^6}{dt^6} y(t) \right) - 12 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \\
& y(t)^3 y(t)^7 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 9072 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^6 \left( \frac{d^5}{dt^5} y(t) \right) + 173550 \left( \frac{d}{dt} y(t) \right)^8 \\
& y(t)^8 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 2142 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) \\
& - 2760 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2 - 188436 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right) \\
& y(t) + 2364 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^7 \left( \frac{d^7}{dt^7} y(t) \right) - 10938 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) \\
& y(t)^2 y(t)^7 \left( \frac{d^5}{dt^5} y(t) \right)^2 - 60 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right)^2 + 30 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \\
& y(t) y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) + 7092 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 36 \left( \frac{d}{dt} y(t) \right)^8 \\
& y(t)^8 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 6 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) \\
& - 2178816 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^5 \left( \frac{d^4}{dt^4} y(t) \right) - 35904 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) \\
& y(t) y(t)^6 \left( \frac{d^6}{dt^6} y(t) \right) + 392652 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 37812 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right) \\
& y(t)^4 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^6 \left( \frac{d^5}{dt^5} y(t) \right) - 513282 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 2316 \left( \frac{d}{dt} y(t) \right)^7 \\
& y(t)^7 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 7068 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^7 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right)
\end{aligned}$$

Figure 3.17: The ninth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

$$\begin{aligned}
 & -232824 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) + 2256 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 \left( \frac{d^7}{dr^7} y(t) \right) + 8628 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right)^2 - 102 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^3 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right) - 28464 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^7 \left( \frac{d^6}{dr^6} y(t) \right) + 204 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^8 \left( \frac{d^6}{dr^6} y(t) \right)^2 - 6762 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^3 \\
 & - 90 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^7}{dr^7} y(t) \right) + 29832 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^7 \left( \frac{d^5}{dr^5} y(t) \right) - 59388 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right)^2 - 516 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^3 \left( \frac{d^5}{dr^5} y(t) \right) - 246 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 240 \left( \frac{d}{dr} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right) - 569754 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^6 \left( \frac{d^5}{dr^5} y(t) \right) + 1870320 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^6 \left( \frac{d^4}{dr^4} y(t) \right) - 9192 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^7 \left( \frac{d^7}{dr^7} y(t) \right) \\
 & + 8376 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^5 y(t)^7 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) + 55212 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^7 \left( \frac{d^6}{dr^6} y(t) \right) + 32958 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right) - 117666 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right) + 756 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^9 \left( \frac{d^7}{dr^7} y(t) \right) + 792 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 8388 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^9 \left( \frac{d^6}{dr^6} y(t) \right) - 36 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^6 y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 19728 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^5 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right) \\
 & + 8388 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^8 \left( \frac{d^5}{dr^5} y(t) \right)^2 + 48 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) - 24 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right)^2 y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right)^2 \left( \frac{d^6}{dr^6} y(t) \right) + 7230 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^8 \left( \frac{d^6}{dr^6} y(t) \right) - 108 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^9 \left( \frac{d^6}{dr^6} y(t) \right)^2 + 4176 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right)^3 - 114 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} y(t) \right) y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right)^3 - 144 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right)^3 \left( \frac{d^6}{dr^6} y(t) \right) + 102 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^2}{dr^2} y(t) \right) y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right)^2 \left( \frac{d^5}{dr^5} y(t) \right)^2 - 36 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) + 42 \left( \frac{d}{dr} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right)
 \end{aligned}$$

Figure 3.18: The tenth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& + 90 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) - 66 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2 \\
& - 90 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) + 330222 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^7 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^6 \left( \frac{d^4}{dt^4} y(t) \right) \\
& y(t) - 33090 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^7 \left( \frac{d^6}{dt^6} y(t) \right) - 143850 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) \\
& y(t) \left( \frac{d^5}{dt^5} y(t) \right) + 179988 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^7 \left( \frac{d^5}{dt^5} y(t) \right) + 299952 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^5 \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 1614 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) + 3312 \left( \frac{d}{dt} y(t) \right)^5 \\
& y(t) \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 465300 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) \\
& + 2526 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 \left( \frac{d^7}{dt^7} y(t) \right) - 19026 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) \\
& y(t) y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right)^2 + 6786 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) - 7362 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^8 \left( \frac{d^6}{dt^6} y(t) \right) + 96 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^6}{dt^6} y(t) \right)^2 - 18894 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \\
& y(t) \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^3 + 150 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) \\
& - 2724 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right) + 35472 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 \\
& y(t) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 522 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) - 31248 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) + 186 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^7}{dt^7} y(t) \right) - 720 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right)^2 + 60 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^4 + 1278 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 111324 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^7 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^7 \left( \frac{d^5}{dt^5} y(t) \right) \\
& - 452226 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) + 324 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) \\
& y(t) + 252 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^6}{dt^6} y(t) \right) - 216 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) \\
& y(t) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 + 36 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right) + 396 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^4 \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 + 504 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) - 468 \left( \frac{d^2}{dt^2} y(t) \right)^4 \\
& y(t) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 732 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2
\end{aligned}$$

Figure 3.19: The fourth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

$$\begin{aligned}
 & -22950 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) + 234 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^7}{dt^7} y(t) \right) \\
 & - 816 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right)^2 - 12 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \\
 & \left( \frac{d^6}{dt^6} y(t) \right)^2 + 6 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) - 2040 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^4 \\
 & y(t)^9 \left( \frac{d^6}{dt^6} y(t) \right) + 48 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^6}{dt^6} y(t) \right)^2 + 2070 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^3 + 54 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^3 + 60 \left( \frac{d}{dt} y(t) \right)^4 \\
 & y(t)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^6}{dt^6} y(t) \right) - 42 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right)^2 \\
 & + 1770 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) - 4470 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
 & + 78 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 162 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 22446 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^8 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) + 14790 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^7 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \\
 & \left( \frac{d^6}{dt^6} y(t) \right) + 15966 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^7 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 93876 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^6 \\
 & \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right) - 20442 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 216 \left( \frac{d}{dt} y(t) \right)^3 \\
 & y(t)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 264 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) \\
 & + 236424 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) - 960 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^7}{dt^7} y(t) \right) \\
 & y(t) + 5688 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right)^2 - 3570 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
 & y(t)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 8160 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^6}{dt^6} y(t) \right) - 90 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \\
 & \left( \frac{d^6}{dt^6} y(t) \right)^2 + 11430 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^3 - 54 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^4 \\
 & y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) - 8274 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) - 5880 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \\
 & \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 174 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) \\
 & + 19110 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) + 18 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^7}{dt^7} y(t) \right) \\
 & y(t) + 126 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^2 + 12 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)
 \end{aligned}$$

Figure 3.20: The eleventh part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& -22950 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) + 234 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^7}{dt^7} y(t) \right) - 816 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right)^2 - 12 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right)^2 + 6 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^2 \left( \frac{d^6}{dt^6} y(t) \right) - 2040 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^6}{dt^6} y(t) \right) + 48 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^6}{dt^6} y(t) \right)^2 + 2070 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^3 + 54 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^3 + 60 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^6}{dt^6} y(t) \right) - 42 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right)^2 + 1770 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) - 4470 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 78 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 162 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 22446 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^8 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^7 \left( \frac{d^4}{dt^4} y(t) \right) + 14790 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^7 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^6}{dt^6} y(t) \right) + 15966 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^7 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 93876 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^8 \left( \frac{d^5}{dt^5} y(t) \right) - 20442 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right)^2 + 216 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^7}{dt^7} y(t) \right) - 264 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^6 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 236424 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^8 \left( \frac{d^4}{dt^4} y(t) \right) - 960 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^9 \left( \frac{d^7}{dt^7} y(t) \right) + 5688 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right)^2 - 3570 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 8160 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^6}{dt^6} y(t) \right) - 90 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^6}{dt^6} y(t) \right)^2 + 11430 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^3 - 54 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^7}{dt^7} y(t) \right) - 8274 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^5}{dt^5} y(t) \right) - 5880 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 174 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) + 19110 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) + 18 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^7}{dt^7} y(t) \right) + 126 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 y(t)^{10} \left( \frac{d^5}{dt^5} y(t) \right)^2 + 12 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)
\end{aligned}$$

Figure 3.21: The twelfth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

$$\begin{aligned}
 & y(t) \Big) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^4 + 198 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^5 y(t)^{10} \left( \frac{d^6}{dt^6} y(t) \right) + 12960 \left( \frac{d}{dt} y(t) \right)^{18} \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
 & + 12960 \left( \frac{d}{dt} y(t) \right)^{17} \left( \frac{d^3}{dt^3} y(t) \right)^3 - 6048 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^2}{dt^2} y(t) \right)^6 + 10584 \left( \frac{d^2}{dt^2} y(t) \right)^{13} y(t)^7 + 216 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 102 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 18144 \left( \frac{d}{dt} y(t) \right)^{15} \left( \frac{d^2}{dt^2} y(t) \right) y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 206928 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) - 230184 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 4248 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 21672 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^2}{dt^2} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \left( \frac{d^6}{dt^6} y(t) \right) + 25416 \left( \frac{d}{dt} y(t) \right)^{14} \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 572976 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) - 91224 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) - 23880 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 131568 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) - 245088 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 2844 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right) y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 1092 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^2}{dt^2} y(t) \right) y(t)^4 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 4860 \left( \frac{d}{dt} y(t) \right)^{13} \left( \frac{d^3}{dt^3} y(t) \right) y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 304392 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 676488 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) - 7596 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) - 40776 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right)^2 y(t)^2 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 75864 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) + 372 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right) y(t)^5 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) - 864 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^3}{dt^3} y(t) \right) y(t)^5 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 912 \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^3}{dt^3} y(t) \right) y(t)^5 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 2261712 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 36960 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) + 235776 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right)^3 y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 93540 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 y(t)^4 \left( \frac{d^4}{dt^4} y(t) \right) + 4590 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 378 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right) y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) + 252 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^5 y(t)^{10} \left( \frac{d^4}{dt^4} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right)^2 - 44010 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^4 y(t)^9 \left( \frac{d^4}{dt^4} y(t) \right)
 \end{aligned}$$

Figure 3.22: The thirteenth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
& -630 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^4 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right)^2 - 486 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^6}{dr^6} \right. \\
& y(t) \left. \right) - 2034 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^3 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^3 - 54 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} \right. \\
& y(t) \left. \right) y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right) + 1908 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right)^4 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right)^2 - 1242 \left( \frac{d}{dr} y(t) \right)^2 \left( \frac{d^2}{dr^2} \right. \\
& y(t) \left. \right) \left( \frac{d^3}{dr^3} y(t) \right)^6 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right) + 65448 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^9 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^8 \left( \frac{d^4}{dr^4} y(t) \right) + 4032 \left( \frac{d}{dr} \right. \\
& y(t) \left. \right) \left( \frac{d^2}{dr^2} y(t) \right)^8 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^9 \left( \frac{d^6}{dr^6} y(t) \right) - 180 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^8 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^5}{dr^5} y(t) \right) - 13860 \left( \frac{d}{dr} \right. \\
& y(t) \left. \right) \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^9 \left( \frac{d^5}{dr^5} y(t) \right) - 12240 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^7 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right)^2 + 36 \left( \frac{d}{dr} \right. \\
& y(t) \left. \right) \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^{10} \left( \frac{d^4}{dr^4} y(t) \right) \left( \frac{d^7}{dr^7} y(t) \right) - 108 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^7 y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right) \left( \frac{d^6}{dr^6} y(t) \right) \\
& + 43380 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^3 y(t)^9 \left( \frac{d^4}{dr^4} y(t) \right) - 36 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right)^2 y(t)^{10} \left( \frac{d^7}{dr^7} \right. \\
& y(t) \left. \right) + 468 \left( \frac{d}{dr} y(t) \right) \left( \frac{d^2}{dr^2} y(t) \right)^6 \left( \frac{d^3}{dr^3} y(t) \right) y(t)^{10} \left( \frac{d^5}{dr^5} y(t) \right) + 84816 \left( \frac{d}{dr} y(t) \right)^{16} \left( \frac{d^2}{dr^2} y(t) \right) \left( \frac{d^3}{dr^3} \right. \\
& y(t) \left. \right) y(t) \left( \frac{d^5}{dr^5} y(t) \right) + 3312 \left( \frac{d}{dr} y(t) \right)^{15} \left( \frac{d^2}{dr^2} y(t) \right)^2 \left( \frac{d^3}{dr^3} y(t) \right) y(t) \left( \frac{d^4}{dr^4} y(t) \right) + 43920 \left( \frac{d}{dr} y(t) \right)^{15} \left( \frac{d^2}{dr^2} \right. \\
& y(t) \left. \right) \left( \frac{d^3}{dr^3} y(t) \right) y(t)^2 \left( \frac{d^6}{dr^6} y(t) \right), \textit{ideal}
\end{aligned}$$

*false*

Figure 3.23: The fourteenth part of Wronskian  $W$  of  $q_1, q_2, q_3, q_4, q_5$ .

By Proposition 2.3.3 and (3.5.4), if  $y(t, p) = y(t, \bar{p})$ , then

$$\begin{cases} \frac{p_1}{p_2} = \frac{\bar{p}_1}{\bar{p}_2}, \\ 2p_3 = 2\bar{p}_3, \\ \frac{p_1 p_3}{p_2} = \frac{\bar{p}_1 \bar{p}_3}{\bar{p}_2}, \\ \frac{p_1 p_3^2}{p_2} = \frac{\bar{p}_1 \bar{p}_3^2}{\bar{p}_2}, \\ \frac{p_2}{2p_3^2} = \frac{\bar{p}_2}{2\bar{p}_3^2}. \end{cases} \quad (3.5.10)$$

(3.5.10) is equivalent to

$$\begin{cases} \frac{p_1}{p_2} = \frac{\bar{p}_1}{\bar{p}_2}, \\ p_3 = \bar{p}_3 \end{cases} \quad (3.5.11)$$

By the initial condition of (3.5.3),  $y(0, p) = y(0, \bar{p})$  if

$$p_4 p_6 = \bar{p}_4 \bar{p}_6. \quad (3.5.12)$$

By the similar arguments from Section 3.2, we obtain the two other equations of the fourth list

$$\frac{p_2}{p_4^2} x_3(t, p) = \frac{p_3 y(t) y'(t) - y(t) y''(t) + 2(y')^2(t)}{y(t) (y'(t) + p_3 y(t)) \left( \frac{p_1}{p_2} y(t) + 2y'(t) \right)}, \quad (3.5.13)$$



and

$$\frac{p_1}{p_4}x_1(t, p)x_3(t, p) = \frac{y'(t) + p_3y(t)}{y^2(t)}. \quad (3.5.14)$$

By (3.5.11) and  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$ , we have

$$\begin{cases} \frac{p_2}{p_4^2}x_3(0, p) = \frac{\bar{p}_2}{\bar{p}_4^2}x_3(0, \bar{p}), \\ \frac{p_1}{p_4}x_1(0, p)x_3(0, p) = \frac{\bar{p}_1}{\bar{p}_4}x_1(0, \bar{p})x_3(0, \bar{p}), \end{cases} \quad (3.5.15)$$

It implies that

$$\begin{cases} \frac{p_2}{p_4^2} \frac{1}{1 + p_2p_6^2} = \frac{\bar{p}_2}{\bar{p}_4^2} \frac{1}{1 + \bar{p}_2\bar{p}_6^2}, \\ \frac{p_1}{p_4} \frac{p_5}{1 + p_2p_6^2} = \frac{\bar{p}_1}{\bar{p}_4} \frac{\bar{p}_5}{1 + \bar{p}_2\bar{p}_6^2} \end{cases} \quad (3.5.16)$$

we solve the algebraic system (3.5.11), (3.5.12), (3.5.16), and obtain the relationships between  $p$  and  $\bar{p}$  as follows  $p_3 = \bar{p}_3$ ,  $\frac{\bar{p}_1}{p_1} = \frac{\bar{p}_2}{p_2} = \left(\frac{\bar{p}_4}{p_4}\right)^2$ ,  $\frac{\bar{p}_4}{p_4} = \frac{p_5}{\bar{p}_5} = \frac{p_6}{\bar{p}_6}$ . Together with Proposition 3.1.1, we have the following

**Theorem 3.5.1.** *The SIR model with nonlinear incidence rate (3.5.1) is not identifiable, and we have  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$  if and only if  $\bar{\nu} = \nu$ ,  $\frac{\bar{\tau}}{\tau} = \frac{\bar{\kappa}}{\kappa} = \frac{\bar{\mu}^2}{\mu^2}$ ,  $\frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}$ .*

Moreover, we also have the following theorem

**Theorem 3.5.2.** *The basic production number  $\mathcal{R}_0$  of the SIR model with nonlinear incidence rate (3.5.1) is uniquely determined from the output function  $y(t) = \mu I(t)$ .*

*Proof.* By Theorem 3.5.1, from the output function  $y(t) = \mu I(t)$  we have  $\nu$ ,  $\tau S_0 I_0$ ,  $\kappa I_0^2$  are uniquely determined. It derives that  $\mathcal{R}_0 = \frac{\tau S_0 I_0}{\nu(1 + \kappa I_0^2)}$  is uniquely determined.  $\square$

### 3.6 SIR model with nonlinear incidence rate (3.1.1) in the case $\alpha = 1, \beta = 3$

Consider the system

$$\begin{cases} S'(t) = -\frac{\tau S(t)I(t)}{1 + \kappa I^3(t)}, \\ I'(t) = \frac{\tau S(t)I(t)}{1 + \kappa I^3(t)} - \nu I(t), \\ y(t) = \mu I(t) \end{cases} \quad t \geq 0 \quad (3.6.1)$$

with  $S(0) = S_0 > 0$ ,  $I(0) = I_0 > 0$ , and  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0^3)} > 1$ .

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

Using the same notation as in Section 3.2, then (3.5.1) can be rewritten as follows

$$\begin{cases} x_1'(t) = -\frac{p_1 x_1(t) x_2(t)}{1 + p_2 x_2^3(t)}, \\ x_2'(t) = \frac{p_1 x_1(t) x_2(t)}{1 + p_2 x_2^3(t)} - p_3 x_2(t), \\ y(t) = p_4 x_2(t) \end{cases} \quad t \geq 0 \quad (3.6.2)$$

with  $x_1(0) = p_5$ ,  $x_2(0) = p_6$ .

Let  $x_3(t) = \frac{1}{1 + p_2 x_2^3(t)}$ , then

$$\begin{aligned} x_3'(t) &= -\frac{3p_2 x_2^2(t) x_2'(t)}{(1 + p_2 x_2^3(t))^2} \\ &= -2p_2 x_2^2(t) x_3^2(t) (p_1 x_1(t) x_2(t) x_3(t) - p_3 x_2(t)). \end{aligned}$$

(3.6.2) can be written as follows

$$\begin{cases} p_4^2 x_1'(t) + p_1 x_1(t) y^2(t) x_3(t) = 0, \\ p_4 y'(t) - p_1 x_1(t) y^2(t) x_3(t) + p_3 p_4 y(t) = 0, \\ p_4^3 x_3'(t) + 2p_1 p_2 x_1(t) y^3(t) x_3^3(t) - 2p_2 p_3 p_4 y^2(t) x_3^2(t) = 0 \end{cases} \quad t \geq 0 \quad (3.6.3)$$

with  $x_1(0) = p_5$ ,  $y(0) = p_4 p_6$ ,  $x_3(0) = \frac{1}{1 + p_2 p_6^3}$ .

Consider the differential polynomials on the left-hand sides of (3.6.3). These polynomials form an autoreduced set in the differential ring  $\mathbb{Q}(p)\{x_1, x_3, y\}$  with the elimination ranking  $[y] < [x_1] < [x_3]$ . Moreover, the initials and separants of these polynomials are in  $\mathbb{Q}(p)$ , then these polynomials form an orthonomic set of  $\mathbb{Q}(p)\{x_1, x_3, y\}$ . Let  $\mathcal{P}$  is a prime differential ideal generated by these polynomials.

Next, using Rosenfeld-Gröbner algorithm implemented in Maple to decompose the ideal  $\mathcal{P}$ . This decomposition returns four lists of differential polynomials. Each list forms the set of generators of a radical ideal corresponding to either the singular solutions or general solution of (3.6.3). We show here the lists

$$[x_3(t), x_1'(t), y'(t) + p_3 y(t)],$$

$$[x_3'(t), x_1'(t), y(t)],$$

$$[p_4^3 x_3'(t) - 3p_2 p_3 y^3(t) x_3^2(t), x_1(t), y'(t) + p_3 y(t)],$$

$$\begin{aligned} &[3p_2 x_3(t) y^3(t) (y')^2(t) + 3p_2 p_3 x_3(t) y^4(t) y'(t) + p_1 p_4^2 x_3(t) y^2(t) y'(t) \\ &\quad + p_1 p_3 p_4^2 x_3(t) y^3(t) + p_4^3 y(t) y''(t) - p_4^3 (y')^2(t), \\ &p_1 p_4^3 x_1(t) y(t) y''(t) - p_1 p_4^3 x_1(t) (y')^2(t) + 3p_2 y^2(t) (y')^3(t) + 6p_2 p_3 y^3(t) (y')^2(t) \\ &\quad + p_1 p_4^2 y(t) (y')^2(t) + 3p_2 p_3^2 y^4(t) y'(t) + 2p_1 p_3 p_4^2 y^2(t) y'(t) + p_1 p_3^2 p_4^2 y^3(t), \end{aligned}$$

$$\begin{aligned}
& 3p_2y^3(t)(y')^2(t)y^{(3)}(t) + 3p_2p_3y^4(t)y'(t)y^{(3)}(t) + p_1p_4^2y^2(t)y'(t)y^{(3)}(t) \\
& + p_1p_3p_4^2y^3(t)y^{(3)}(t) - 9p_2y^3(t)y'(t)(y'')^2(t) - 3p_2p_3y^4(t)(y'')^2(t) - p_1p_4^2y^2(t)(y'')^2(t) \\
& - 12p_2p_3y^3(t)(y')^2(t)y''(t) - 2p_1p_4^2y(t)(y')^2(t)y''(t) - 4p_1p_3p_4^2y^2(t)y'(t)y''(t) \\
& + 6p_2y(t)(y')^5(t) + 12p_2p_3y^2(t)(y')^4(t) + 2p_1p_4^2(y')^4(t) + 3p_1p_3p_4^2y(t)(y')^3(t)].
\end{aligned}$$

It can be seen that the first three lists above are not corresponding to the complete system including (3.6.3) and the initial condition. For instance, the third list requires  $x_1(t) = 0$  for all  $t \geq 0$ , contradicting to  $x_1(0) = p_5 > 0$ .

The fourth list is now corresponding to the general case, and the last polynomial in the fourth list forms the output equation of (3.6.3) as follows

$$\begin{aligned}
& 3y(t)y'(t) \left( 2(y')^4(t) + y^2(t)y'(t)y^{(3)}(t) - 3y^2(t)(y'')^2(t) \right) \\
& + 3p_3y^2(t) \left( 4(y')^4(t) - 4y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t) \right) \\
& + \frac{p_1p_4^2}{p_2} \left( 2(y')^4(t) - 2y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t) \right) \\
& + \frac{p_1p_3p_4^2}{p_2} y(t) \left( 3(y')^3(t) - 4y(t)y'(t)y''(t) + y^2(t)y^{(3)}(t) \right) = 0
\end{aligned} \tag{3.6.4}$$

Next, we check the linear independence of three polynomials

$$q_1 = 3y^2(t) \left( 4(y')^4(t) - 4y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t) \right), \tag{3.6.5}$$

$$q_2 = 2(y')^4(t) - 2y(t)(y')^2(t)y''(t) + y^2(t)y'(t)y^{(3)}(t) - y^2(t)(y'')^2(t), \tag{3.6.6}$$

$$q_3 = y(t) \left( 3(y')^3(t) - 4y(t)y'(t)y''(t) + y^2(t)y^{(3)}(t) \right). \tag{3.6.7}$$

Using the package *VectorCalculus* in Maple to compute the Wronskian  $W$  of  $q_1, q_2, q_3$  and the function *BelongsTo* in the package *DifferentialAlgebra* of Maple shows that  $W \notin \mathcal{P}$ . (see Figure 3.24)

3. Determining the identifiable combinations of parameters of SIR model with nonlinear incidence rate from the reported case data

---

$$\begin{aligned}
 \text{BelongsTo} & \left( -72y(t)^7 \left( \frac{d^2}{dt^2} y(t) \right)^7 + 6y(t)^8 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) + 42y(t)^8 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) - 12y(t)^7 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) - 102y(t)^7 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) + 78y(t)^6 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) + 144 \left( \frac{d}{dt} y(t) \right)^{14} \\
 & - 6y(t)^8 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right)^2 - 18y(t)^8 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^5}{dt^5} y(t) \right) - 24y(t)^8 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^3 + 12y(t)^7 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right)^2 + 42y(t)^7 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^5}{dt^5} y(t) \right) \\
 & + 48y(t)^7 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^3 + 24y(t)^7 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^4}{dt^4} y(t) \right) - 60y(t)^7 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right)^2 + 120y(t)^7 \left( \frac{d}{dt} y(t) \right) \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right) + 6y(t)^6 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
 & - 24y(t)^6 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^5}{dt^5} y(t) \right) - 216y(t)^6 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^4}{dt^4} y(t) \right) + 294y(t)^6 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right)^2 - 90y(t)^6 \left( \frac{d}{dt} y(t) \right)^3 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right) - 6y(t)^5 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^5}{dt^5} y(t) \right) \\
 & - 18y(t)^5 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^3}{dt^3} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) + 312y(t)^5 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^4}{dt^4} y(t) \right) - 336y(t)^5 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right)^2 - 780y(t)^5 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^2}{dt^2} y(t) \right)^3 \left( \frac{d^3}{dt^3} y(t) \right) - 162y(t)^4 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^4}{dt^4} y(t) \right) \\
 & + 1710y(t)^4 \left( \frac{d}{dt} y(t) \right)^7 \left( \frac{d^2}{dt^2} y(t) \right)^2 \left( \frac{d^3}{dt^3} y(t) \right) - 1320y(t)^3 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^2}{dt^2} y(t) \right) \left( \frac{d^3}{dt^3} y(t) \right) \\
 & + 18y(t)^8 \left( \frac{d^2}{dt^2} y(t) \right)^5 \left( \frac{d^4}{dt^4} y(t) \right) - 18y(t)^8 \left( \frac{d^2}{dt^2} y(t) \right)^4 \left( \frac{d^3}{dt^3} y(t) \right)^2 - 6y(t)^6 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^4}{dt^4} y(t) \right)^2 \\
 & - 24y(t)^6 \left( \frac{d}{dt} y(t) \right)^5 \left( \frac{d^3}{dt^3} y(t) \right)^3 - 306y(t)^6 \left( \frac{d}{dt} y(t) \right)^2 \left( \frac{d^2}{dt^2} y(t) \right)^6 + 2268y(t)^5 \left( \frac{d}{dt} y(t) \right)^4 \left( \frac{d^2}{dt^2} y(t) \right)^5 \\
 & + 6y(t)^4 \left( \frac{d}{dt} y(t) \right)^9 \left( \frac{d^5}{dt^5} y(t) \right) + 120y(t)^4 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^3}{dt^3} y(t) \right)^2 - 4410y(t)^4 \left( \frac{d}{dt} y(t) \right)^6 \left( \frac{d^2}{dt^2} y(t) \right)^4 \\
 & + 24y(t)^3 \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^4}{dt^4} y(t) \right) + 3600y(t)^3 \left( \frac{d}{dt} y(t) \right)^8 \left( \frac{d^2}{dt^2} y(t) \right)^3 + 360y(t)^2 \left( \frac{d}{dt} y(t) \right)^{11} \left( \frac{d^3}{dt^3} y(t) \right) \\
 & - 936y(t)^2 \left( \frac{d}{dt} y(t) \right)^{10} \left( \frac{d^2}{dt^2} y(t) \right)^2 - 288y(t) \left( \frac{d}{dt} y(t) \right)^{12} \left( \frac{d^2}{dt^2} y(t) \right), \text{ideal}
 \end{aligned}$$

false

Figure 3.24: The Wronskian  $W$  of  $q_1, q_2, q_3$ .

By Proposition 2.3.3 and (3.6.4), if  $y(t, p) = y(t, \bar{p})$ , then

$$\begin{cases} 3p_3 = 3\bar{p}_3, \\ \frac{p_1 p_4^2}{p_2} = \frac{\bar{p}_1 \bar{p}_4^2}{\bar{p}_2}, \\ \frac{p_1 p_3 p_4^2}{p_2} = \frac{\bar{p}_1 \bar{p}_3 \bar{p}_4^2}{\bar{p}_2}. \end{cases} \quad (3.6.8)$$

(3.6.8) is equivalent to

$$\begin{cases} \frac{p_1 p_4^2}{p_2} = \frac{\bar{p}_1 \bar{p}_4^2}{\bar{p}_2}, \\ p_3 = \bar{p}_3 \end{cases} \quad (3.6.9)$$

By the initial condition of (3.6.3),  $y(0, p) = y(0, \bar{p})$  if

$$p_4 p_6 = \bar{p}_4 \bar{p}_6. \quad (3.6.10)$$

By the similar arguments from Section 3.2, we obtain the two other equations of the fourth list

$$\frac{p_2}{p_4^3} x_3(t, p) = \frac{y(t) y''(t) - (y'(t))^2(t)}{y^2(t) (y'(t) + p_3 y(t)) \left( 3y(t) y'(t) + \frac{p_1 p_4^2}{p_2} \right)}, \quad (3.6.11)$$

and

$$\frac{p_1 p_4^3}{p_2} x_1(t, p) = \frac{y(t) (y'(t) + p_3 y(t))^2 \left( 3y(t) y'(t) + \frac{p_1 p_4^2}{p_2} \right)}{y(t) y''(t) - (y'(t))^2(t)}. \quad (3.6.12)$$

Multiply (3.6.11) and (3.6.12), we obtain

$$p_1 x_1(t, p) x_3(t, p) = \frac{y'(t) + p_3 y(t)}{y(t)}. \quad (3.6.13)$$

By (3.6.11), (3.6.13) and  $y^{(i)}(t, p) = y^{(i)}(t, \bar{p})$  for all  $i \in \mathbb{N}$ , we have

$$\begin{cases} \frac{p_2}{p_4^3} x_3(0, p) = \frac{\bar{p}_2}{\bar{p}_4^3} x_3(0, \bar{p}), \\ p_1 x_1(0, p) x_3(0, p) = \bar{p}_1 x_1(0, \bar{p}) x_3(0, \bar{p}), \end{cases} \quad (3.6.14)$$

It implies that

$$\begin{cases} \frac{p_2}{p_4^3} \frac{1}{1 + p_2 p_6^3} = \frac{\bar{p}_2}{\bar{p}_4^3} \frac{1}{1 + \bar{p}_2 \bar{p}_6^3}, \\ \frac{p_1 p_5}{1 + p_2 p_6^3} = \frac{\bar{p}_1 \bar{p}_5}{1 + \bar{p}_2 \bar{p}_6^3} \end{cases} \quad (3.6.15)$$

Solving the algebraic system (3.6.9), (3.6.10), (3.6.15), we have the relationships between  $p$  and  $\bar{p}$  as follows  $p_3 = \bar{p}_3$ ,  $\frac{\bar{p}_2}{p_2} = \left( \frac{\bar{p}_4}{p_4} \right)^3$ ,  $\frac{\bar{p}_1}{p_1} = \frac{\bar{p}_4}{p_4} = \frac{p_5}{\bar{p}_5} = \frac{p_6}{\bar{p}_6}$ . Together with Proposition 3.1.1, we have the following

**Theorem 3.6.1.** *The SIR model with nonlinear incidence rate (3.6.1) is not identifiable, and we have  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$  if and only if  $\bar{\nu} = \nu$ ,  $\frac{\bar{\kappa}}{\kappa} = \frac{\bar{\mu}^3}{\mu^3}$ ,  $\frac{\bar{\tau}}{\tau} = \frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}$ .*

Moreover, we also have the following theorem

**Theorem 3.6.2.** *The basic production number  $\mathcal{R}_0$  of the SIR model with nonlinear incidence rate (3.6.1) is uniquely determined from the output function  $y(t) = \mu I(t)$ .*

*Proof.* By Theorem 3.6.1, from the output function  $y(t) = \mu I(t)$  we have  $\nu$ ,  $\tau S_0$ ,  $\kappa I_0^3$  are uniquely determined. It derives that  $\mathcal{R}_0 = \frac{\tau S_0}{\nu(1 + \kappa I_0^3)}$  is uniquely determined.  $\square$

# Conclusion

In this thesis, we consider the identifiability of SIR model and SIR model with nonlinear incidence rate. There are so many problems concerning these models and one of the most important problems is the study of identifiability and its applications. The identifiability analysis of these systems is the most prerequisite for parameter estimation from the real data of an epidemic. However, parameter estimation from the real data of an epidemic is one kind of converse problem and there is not a complete answer. This work aims to continue the ideas about the turning point of Pierre Magal and Glenn Webb in the paper [46], and find the identifiable combinations of parameters of SIR model with nonlinear incidence rate.

In chapter one, we use the ideas about the turning point of the model from the real data of an epidemic and the identifiability of SIR model for providing a scheme to estimate the parameters of SIR model. Using this scheme we apply for some real epidemics in New York, Bombay, and France. The results that we obtain can be divided into two parts. The first part includes the analysis of SIR models, the identifiability analysis of SIR model, and the scheme to compute the parameters of SIR model. The second part is the applications of the first part for the epidemics in New York, Bombay, and France. The first part and the application to Hong Kong influenza in New York in 1968-1969 is published in [23]. Moreover, while we analyze the SIR model with the turning point for building the above scheme we obtain the result about the uniqueness of the solution of the turning point equation (1.3.10) (Theorem 1.3.10). It indicates that when  $r$  is closed to  $\frac{1}{2}$  enough, the identifiable combination of parameters is uniquely determined from the value  $t_p$ ,  $CR(t_p)$ ,  $CR'(t_p)$  and  $CR(\infty)$ . It helps us ensure the uniqueness of the combinations of parameters that we compute in the applications. What we obtain in this chapter is providing the complete scheme to recover the parameters of SIR models from the weekly reported case data by finding the identifiable combinations of parameters and parameters estimation.

In chapter two and chapter three, we continue the idea about the scheme in chapter one by studying SIR model with nonlinear incidence rate. The problem becomes more complicated and difficult with the nonlinear incidence rate. Firstly, we provide the method for finding identifiable combinations of parameters for general nonlinear models in chapter two and apply for the simple SIR model. The method in chapter two is based on the results in differential algebra and the package implemented in Maple. Some parts of the computations for finding the identifiable combinations of parameters of SIR model with nonlinear incidence rate in chapter three that we conduct are based on the codes we write in Maple. The next part is finding the

---

method to calculate numerically the identifiable combinations of parameters from the real data but it is a very difficult part, and we do not have any answer until now after a lot of experiments. Maybe, it is the next direction for this study. In chapter three, the following problem holds for the cases  $\{\alpha = 1, \beta = 2\}$ ,  $\{\alpha = 1, \beta = 1\}$ ,  $\{\alpha = 2, \beta = 2\}$ ,  $\{\alpha = 2, \beta = 1\}$ ,  $\{\alpha = 1, \beta = 3\}$ . We can even compute for the case  $\{\alpha = 2, \beta = 3\}$  but the general case is still not answered.

**Problem.** Consider the SIR model with nonlinear incidence rate (3.1.1) with the condition  $\mu I(t, p) = \bar{\mu} I(t, \bar{p})$ . Can we have  $\bar{\nu} = \nu$ ,  $\left(\frac{\bar{\tau}}{\tau}\right)^\beta = \left(\frac{\bar{\kappa}}{\kappa}\right)^\alpha$ ,  $\frac{\bar{\tau}}{\tau} = \left(\frac{\bar{\mu}}{\mu}\right)^\alpha$ ,

$$\frac{\bar{\mu}}{\mu} = \frac{S_0}{\bar{S}_0} = \frac{I_0}{\bar{I}_0}?$$

The answer for the above problem for the general case  $\alpha, \beta \in \mathbb{N}$  such that  $\beta + 1 \geq \alpha \geq 1$  is also another work of this study.

# Bibliography

- [1] R.M. Anderson and R.M. May, *Infective Diseases of Humans: Dynamics and Control*, Oxford University Press, Oxford, 1991.
- [2] V. Andreasen, The final size of an epidemic and its relation to the basic reproduction number, *Bull. Math. Biol.*, **73(10)** (2011), 2305-2321.
- [3] J. Arino, F. Brauer, P. Van Den Driessche, J. Watmough and J. Wu, A final size relation for epidemic models, *Math. Biosci. Eng.*, **4(2)** (2007), 159-175.
- [4] S. Audoly, G. Bellu, L. D'Angiò, M. Saccomani, C. Cobelli, Global identifiability of nonlinear models of biological systems, *IEEE Transactions on Biomedical Engineering*, **48** (2001) 55–65.
- [5] Nicolas Bacaër, *A short history of population dynamics*, Springer-Verlag, London, 2011.
- [6] N.T.J. Bailey, *The Mathematical Theory of Epidemics*, Charles Griffin, London, 1957.
- [7] Daniel J. Bearup, Neil D. Evans, Micheal J. Chappell, The input-output relationship approach to structural identifiability analysis, *Computer Methods and Programs in Biomedicine*, **109** (2013), 171-181.
- [8] Giuseppina Bellu, Maria Pia Saccomani, Stefania Audoly, Leontina D'Angiò, DAISY: A new software tool to test global identifiability of biological and physiological systems , *Computer Methods and Programs in Biomedicine*, **88** (2007), 52-61.
- [9] François Boulier, Daniel Lazard, François Ollivier, and Michel Petitot, Representation for the radical of a finitely generated differential ideal, *ISSAC '95 Proceedings of the 1995 international symposium on Symbolic and algebraic computation*, (1995), 158-166.
- [10] François Boulier, Differential Elimination and Biological Modelling, *Radon Series Comp. Appl. Math* , **2**(2007), 111-139.
- [11] François Boulier, Daniel Lazard, François Ollivier, and Michel Petitot, Computing representations for radicals of a finitely generated differential ideals, *Applicable Algebra in Engineering, Communication, and Computing*, (2009), 73-121.



- [12] F. Brauer, P. van den Driessche and J. Wu (eds.), *Mathematical epidemiology*, Springer, Berlin, 2008.
- [13] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, New York, 2000.
- [14] Fred Brauer, Mathematical epidemiology: Past, present, and future, *Infectious Disease Modelling* **2** (2017), 113-127.
- [15] S. Busenberg and K. Cooke, *Vertically Transmitted Diseases: Models and Dynamics*, Lecture Notes in Biomathematics, **23**, Springer-Verlag, Berlin, 1993. Alex Capaldi, Samuel Behrend, Benjamin Berman, Jason Smith, Justin Wright, Alun Lloyd,
- [16] Alex Capaldi, Samuel Behrend, Benjamin Berman, Jason Smith, Justin Wright, Alun Lloyd, Parameter Estimation And Uncertainty Quantification For An Epidemic Model, *Mathematical Biosciences and Engineering*, **9** (2012), 553–576.
- [17] V. Capasso, G.Serio, A generalization of the Kermack-McKendrick Deterministic Epidemic Model, *Mathematical Biosciences*, **42** (1978), 43-61.
- [18] M. Capistran, M. Moreles, and B. Lara, Parameter estimation of some epidemic models. The case of recurrent epidemics caused by respiratory syncytial virus, *Bull. Math Biol.*, **71** (2009), 1890-1901.
- [19] G. Chowell, E. Shim, F. Brauer, P. Diaz-Dueñas, J.M. Hyman, C. Castillo-Chavez, Modelling the transmission dynamics of acute haemorrhagic conjunctivitis: Application to the 2003 outbreak in Mexico, *Stat. Med.*, **25**, (2006), 1840-1857.
- [20] G. Chowell, P. Diaz-Dueñas, J.C. Miller, A. Alcazar-Velazco, J.M. Hyman, P.W. Fenimore, and C. Castillo-Chavez, Estimation of the reproduction number of dengue fever from spatial epidemic data, *Math. Biosci.*, **208**, (2007), 571-589.
- [21] O. Diekmann, H. Heesterbeek and T. Britton, *Mathematical Tools for Understanding Infectious Disease Dynamics*, Princeton University Press, Princeton, 2013.
- [22] O. Diekmann, J.A.P. Heesterbeek, and J.A.J. Metz, On the definition and the computation of the basic reproduction ratio  $R_0$  in models for infectious diseases in heterogeneous populations, *J. Math. Biol.* **28** (1990), 365-382.
- [23] A. Ducrot, P. Magal, T. Nguyen, G. Webb, Identifying the Number of Unreported Cases in SIR Epidemic Models, *Mathematical Medicine and Biology: A Journal of the IMA*, **37** 2020, 243–261.
- [24] N.D. Evans, M.J. Chapman, M. Chappell, and K.R. Godfrey, Identifiability of uncontrolled nonlinear rational systems, *Automatica*, **38**(2) (2002), 1799-1805.

## BIBLIOGRAPHY

---

- [25] N.D. Evans, L.J. White, M.J. Chapman, K.R. Godfrey, and M. Chappell, The structural identifiability of the susceptible infected recovered model with seasonal forcing, *Math. Biosci.*, **194** (2005), 175-197.
- [26] Gauthier Sallet, *Mathematical Epidemiology*, Université de Lorraine, 2018.
- [27] N. Grassly and C. Fraser, Seasonal infectious disease epidemiology, *Proc. Roy. Soc. Lond. B: Biol. Sci.*, **273** (2006), 2541-2550.
- [28] K.P. Hadeler, Parameter identification in epidemic models, *Math. Biosci.*, **229** (2011), 185-189.
- [29] K.P. Hadeler, Parameter estimation in epidemic models: simplified formulas, *Can. Appl. Math. Q.*, **19** (2011), 343-356.
- [30] H.W. Hethcote, Qualitative analyses of communicable disease models, *Math. Biosci.*, **28** (1976), 335-356.
- [31] H. Hethcote, Modeling heterogeneous mixing in infectious disease dynamics, in V. Isham and G. Medley (eds.), *Models for infectious human diseases: their structure and relation to data*, Cambridge University Press, Cambridge, 1996.
- [32] H.W. Hethcote, The mathematics of infectious diseases, *SIAM Review*, **42(4)** (2000), 599-653.
- [33] G. Hooker, S.P. Ellner, L. De Vargas Roditi, and D.J.D. Earn, Parameterizing state space models for infectious disease dynamics by generalized profiling: measles in Ontario, *J. Roy. Soc. Interface* **8**, (2011), 961-974.
- [34] Y-H. Hsieh, D. Fisma, and J. Wu, On epidemic modeling in real time: An application to the 2009 Novel A (H1N1)influenza outbreak in Canada, *BMC Research Notes*, **3** (2010), 283.
- [35] M. Keeling and P. Rohani, *Modeling infectious diseases in humans and animals*, Princeton University Press, Princeton, 2007.
- [36] W.O. Kermack and A.G. McKendrick, Contributions to the mathematical theory of epidemics, part I, *Proc. Roy. Soc. Ser. A*, **115** (1927), 700-721.
- [37] W.O. Kermack and A.G. McKendrick, Contributions to the mathematical theory of epidemics, part II, *Proc. Roy. Soc. Ser. A*, **138** (1932), 55-83.
- [38] Aaron A. King, Introduction to inference: parameter estimation, *ICTP Workshop on Mathematical Models of Climate Variability, Environmental Change and Infectious Diseases* **10** (2017). <https://kingaa.github.io/clim-dis/parest/parest.html#introduction>
- [39] E. R. Kolchin, *Differential Algebra and Algebra Groups*, Academic Press, New York 1973.
- [40] A. Lange, Reconstruction of disease transmission rates: Applications to measles, dengue, and influenza, *J. Theor. Biol.* **400** (2016), 138-153.

- [41] J. Li and Y. Lou, Characteristics of an epidemic outbreak with a large initial infection size, *J. Biol. Dyn.* **10** (2016), 366-378.
- [42] Wei-min Liu , Simon A. Levin , and Yoh Iwasa, Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models, *J. Math. Biology*, **23** (1986), 187-204.
- [43] Ljung, L. and S. T. Glad, On global identifiability for arbitrary model parametrizations, *Automatica*, **30**(2) (1994), 265-276.
- [44] J. Ma and D.J.D. Earn, Generality of the final size formula for an epidemic of a newly invading infectious disease, *Bull. Math. Biol.*, **68** (2006), 679-702.
- [45] P. Magal, G. Webb, and Yixiang Wu, Spatial Spread of Epidemic Diseases in Geographical Settings: Seasonal Influenza Epidemics in Puerto Rico, (2018). <https://arxiv.org/pdf/1801.01856.pdf>
- [46] P. Magal, G. Webb, The Parameter Identification Problem for SIR Epidemic Models: Identifying Unreported Cases, (2018). *Journal of Mathematical Biology*, **77**(6-7) (2018), 1629-1648.
- [47] Gabriella Margaria, Eva Riccomagno, Micheal J. Chappel, Henry P. Wynn, Differential algebra methods for the study of the structural identifiability of rational function state-space models in the biosciences, *Mathematical Biosciences*, **174** (2001), 1-26.
- [48] Tchavdar T. Marinov, Rossitza S. Marinova, Joe Omojola, Michael Jackson, Inverse problem for coefficient identification in SIR epidemic models , *Computers and Mathematics with Applications*, **67** (2014) 2218–2227.
- [49] Nicolette Meshkat, Marisa Eisenberg, Joseph J. DiStefano III, An algorithm for finding globally identifiable parameter combinations of nonlinear ODE models using Gröbner Bases, *Mathematical Biosciences*, **222** (2009), 61-72.
- [50] A. Mummert, Studying the recovery procedure for the time-dependent transmission rate(s) in epidemic models, *J. Math. Biol.*, **67** (2013), 483-507.
- [51] J.D. Murray, *Mathematical Biology*, Springer, Berlin, 1993.
- [52] J. Nemcova, Structural identifiability of polynomial and rational systems, *Mathematical Biosciences*, **223** (2010) 83–96.
- [53] François Ollivier, Le problème de l'identifiabilité structurelle globale: étude théorique, méthodes effectives et bornes de complexité, *Phd thesis*, Ecole Polytechnique (1990).
- [54] Alexey Ovchinnikov, Differential Algebra(Lecture) <http://qcpages.qc.cuny.edu/~aovchinnikov/MATH87800/notes.pdf>
- [55] L. Pellis, N.M. Ferguson, and C. Fraser, Threshold parameters for a model of epidemic spread among households and workplaces, *J. Roy. Soc. Interface* **6**, (2009), 979-987.

## BIBLIOGRAPHY

---

- [56] H. Pohjanpalo, System identifiability based on the power series expansion of the solution, *Mathematical Biosciences* **41**, (1978) 21–33.
- [57] M. Pollicott, H. Wang H, and H. Weiss, Extracting the time-dependent transmission rate from infection data via solution of an inverse ODE problem, *J. Biol. Dyn.* **6** (2012), 509-523.
- [58] L.I.W. Roeger, Z. Feng and C. Castillo-Chavez, Modeling TB and HIV co-infections, *Math. Biosci. Eng.*, **6(4)** (2009), 815-837.
- [59] J. F. Ritt, *Differential Algebra*, Amer. Math. Soc, New York, 1950.
- [60] Shigui Ruan, Wendi Wang. Dynamical behavior of an epidemic model with a nonlinear incidence rate, *Journal of Differential Equations*, **188** (2003), 135-163.
- [61] Maria Pia Saccomani, Stefania Audoly, Leontina D’Angiò. Parameter identifiability of nonlinear systems: the role of initial conditions, *Automatica*, **39** (2003), 619-632.
- [62] D. Smith and L. Moore, The SIR model for spread of disease - background: Hong Kong flu, *J. Online Math. Appl.*, **Dec.** (2004).
- [63] H.R. Thieme, *Mathematics in Population Biology*, Princeton University Press, Princeton, 2003.
- [64] S. Vajda, K. Godfrey, H. Rabitz, Similarity transformation approach to identifiability analysis of nonlinear compartmental models, *Mathematical Biosciences* **93**(1989) 217–248.
- [65] P. Van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.* **180** (2002) 29-48.
- [66] Lilianne Denis-Vidal, Ghislaine Joly-Blanchard, Céline Noiret, Some effective approaches to check the identifiability of non-linear systems, In *Proceedings of 5th IFAC NOLCOS*, St Petersburg, Russia, volume **7** (2001), 174-178.
- [67] Lilianne Denis-Vidal, Ghislaine Joly-Blanchard, Céline Noiret, and M. Petitot, An algorithm to test identifiability of uncontrolled nonlinear systems, *Mathematics and Computer Methods in Simulation*, **57** (2001), 35-44.
- [68] Lilianne Denis-Vidal, Ghislaine Joly-Blanchard, Céline Noiret, System identifiability (symbolic computation) and parameter estimation (numerical computation), *Numerical algorithm*, **34** (2003), 283-292.
- [69] N. Verdière, S. Orange, A strategic algorithmic tool for doing an a priori identifiability study of dynamical nonlinear models, *Mathematical Biosciences*, **308** 2018, 105-113.
- [70] E. Walter, *Identifiability of State Space Models, with Applications to Transformation Systems*, Springer, Berlin, Heidelberg and New York, 1982.

- [71] Dongmei Xiao, Shigui Ruan, Global analysis of an epidemic model with non-monotone incidence rate, *Mathematical Biosciences*, **208** (2007), 419-429.
- [72] J.W.T. Yates, N.D. Evans, M.J. Chappell, Structural identifiability analysis via symmetries of differential equations, *Automatica*, **45** (2009) 2585–2591.
- [73] <https://websenti.u707.jussieu.fr/sentiweb/?page=table>
- [74] <https://www.maplesoft.com/support/help/Maple/view.aspx?path=DifferentialAlgebra>
- [75] <https://www.math.psu.edu/treluga/misc.html>
- [76] [https://en.wikipedia.org/wiki/COVID-19\\_pandemic](https://en.wikipedia.org/wiki/COVID-19_pandemic)
- [77] <https://www.livescience.com/51662-100-bodies-found-prehistoric-house.html>
- [78] <https://www.worldometers.info/coronavirus/>