Perverse Monodromic Sheaves

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Faisceaux Pervers Monodromiques

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Chapter 0

Introduction

0.1 Presentation

Let G be a connected reductive complex algebraic group, let B be a Borel subgroup and U be its unipotent radical. In this thesis, we study modular perverse monodromic sheaves over the basic affine space $\mathscr{X} := G/U$. Slightly more precisely, we obtain a "Soergel type" description of the category of monodromic perverse sheaves on $\mathscr{X}$, as an explicitly determined subcategory of the category of finitely generated modules over an explicitly determined algebra. This subcategory is given by a monodromic variation of the category of Soergel bimodules appearing in [So]

0.1.1 Context

Let us settle some notation: as above, let G be a complex connected reductive algebraic group, choose a Borel subgroup B and a decomposition $B = U \rtimes T$ with T a maximal torus and U the unipotent radical of B. We consider the Weyl group $W := N_G(T)/T$; this is a finite group and our choice of Borel subgroup gives us a family of simple reflections $S \subseteq W$ such that the pair (W, S) is a Coxeter system. The group W is then endowed with a partial order called the Bruhat order, denoted $\leq$. Let $\Bbbk$ be the algebraic closure of a finite field of characteristic $\ell > 0$; this will be our field of coefficients. Finally, we consider the associated root datum $(\mathrm{X}^*(T), \Phi, \mathrm{X}_*(T), \Phi^\vee)$; the cocharacter lattice $\mathrm{X}_*(T)$, that is the (finite rank) abelian group $\mathrm{Hom}(\mathbb{C}^*, T)$ will be of great importance; we set $\Bbbk[\mathrm{X}_*(T)]$ for the group algebra of $\mathrm{X}_*(T)$ over $\Bbbk$.

One can construct the category of B-constructible perverse sheaves on the flag variety of G (that is, on the quotient G/B). We denote it by $\mathscr{O}_{\mathrm{geom}} := \mathrm{P}_{(B)}(G/B, \Bbbk)$. This category is interesting because it has a representation theoretic interpretation: in case ℓ is sufficiently large, it is a geometric incarnation of Soergel's modular category $\mathscr{O}$ for representations of $G_{\Bbbk}$ (the reductive algebraic goup over $\Bbbk$ with root datum isomorphic to the one of G); in case $\ell = 0$ and G

semisimple, it is equivalent to the principal block $\mathscr{O}_0$ of the BGG-category $\mathscr{O}$ of the Lie algebra $\mathrm{Lie}(G)$.

In other words, $\mathscr{O}_{\mathrm{geom}}$ gives a geometric incarnation of representation-theoretic (i.e, of algebraic) categories, and as such, it has been extensively studied in the literature.

0.1.2 Work of Bezrukavnikov–Riche

In [BeR], Bezrukavnikov and Riche gave a completely general "Soergel-type" description of $\mathscr{O}_{\mathrm{geom}}$ as a full subcategory of the category of modules over an explicit ring. Slightly more precisely, it is well known that the category $\mathscr{O}_{\mathrm{geom}}$ admits a highest weight structure with $(W, \leq)$ as weight poset; as such, this category has enough projective objects, and indecomposable projective objects are parametrized by the Weyl group W. Understanding the whole category then reduces to understanding the subcategory of projective objects. However, as nice as projective objects can be, it is somehow easier to work with another class of objects, namely tilting objects. The existence of tilting objects is ensured by the highest weight structure; any tilting is a direct sum of indecomposable such objects, and the indecomposables are also parametrized by W. Moreover, the *geometric Ringel duality* states that the subcategories of projective and tilting objects are equivalent, thus to get a full understanding of the former, it suffices to consider the latter. The authors of *loc. cit.* then obtain a description of the category of tilting perverse objects in $\mathscr{O}_{\mathrm{geom}}$ as a full subcategory of the category of finitely generated modules over the endomorphism ring of the indecomposable tilting object associated to the longest element $w_\circ$ of W. Standard homological considerations then allow them to derive from this a complete description of $\mathscr{O}_{\mathrm{geom}}$.

The strategy they used is the starting point of the present work. Let us give a bit more details. Via the (pullback along) the quotient map $G/U \to G/B$, one can get a copy of $\mathscr{O}_{\mathrm{geom}}$ in the derived category $D^b_{(B)}(G/U, \Bbbk)$ of $\Bbbk$-sheaves constructible with respect to the B-orbits on $\mathscr{X}$. This means that we can see the objects in $\mathscr{O}_{\mathrm{geom}}$ as geometric objects on $\mathscr{X}$. In fact, one can get a description of the category $\mathscr{O}_{\mathrm{geom}}$ as a full subcategory of $D^b_{(B)}(\mathscr{X}, \Bbbk)$ considering actions of the cocharacter group $\mathrm{X}_*(T)$, which we explain now.

We note that the variety G/U is naturally a left and right T-variety for the actions induced by multiplication in G. Using a construction of Verdier [Ve], one can use these two actions to define canonical *monodromy actions* on $D^b_{(B)}(\mathscr{X}, \Bbbk)$. For any $\mathscr{F}$, this is defined as a pair of group morphism $\varphi_{\mathscr{F},l}, \varphi_{\mathscr{F},r} : \mathrm{X}_*(T) \to \mathrm{Aut}(\mathscr{F})$ where $\mathrm{Aut}(\mathscr{F})$ denotes the group of automorphisms of the object $\mathscr{F}$ in $D^b_{(B)}(\mathscr{X}, \Bbbk)$. We extend these maps to algebra morphisms $\varphi_{\mathscr{F},l}, \varphi_{\mathscr{F},r} : \Bbbk[\mathrm{X}_*(T)] \to \mathrm{End}(\mathscr{F})$. One then checks that the category $\mathscr{O}_{\mathrm{geom}}$ identifies in a natural way with the full subcategory of $D^b_{(B)}(G/U, \Bbbk)$ whose objects are those perverse sheaves $\mathscr{F}$ on G/U such that the right monodromy map $\varphi_{\mathscr{F},r}$ factors through the quotient of $\Bbbk[\mathrm{X}_*(T)]$ by the natural (max-

imal) augmentation ideal of $\Bbbk[\mathrm{X}_*(T)]$:

$$\varphi_{\mathscr{F},r} : \Bbbk[\mathrm{X}_*(T)] \twoheadrightarrow \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - 1 \mid \lambda \in \mathrm{X}_*(T)\rangle \to \mathrm{End}(\mathscr{F}). \tag{0.1.1}$$

The above considerations have the advantage of allowing the authors of *loc.cit.* to use Yun's completed category. This is defined as an appropriate subcategory of the category of pro-objects in $D^b_{(B)}(G/U, \Bbbk)$; since we will also consider (a variation of) Yun's completed category below, we do not extend on this topic right now. The point is, going to G/U allowed them to get new tools (namely, a monodromy action and a completed category formalism), crucial in order to obtain their description of the category $\mathscr{O}_{\mathrm{geom}}$.

0.1.3 Generalization

A natural question at this point is the following:

Question 1: What do we get if we replace the augmentation ideal in (0.1.1) by some other maximal ideal ?

The answer is: monodromic perverse sheaves.

One can show that the maximal ideals in $\Bbbk[\mathrm{X}_*(T)]$ are parametrized by the dual $\Bbbk$-torus $T^\vee_\Bbbk$ (for now, let us say that if we choose an identification $T \cong (\mathbb{C}^*)^r$, then we have an identification $T^\vee_\Bbbk \cong (\Bbbk^*)^r$). Then one can define the category $P_{[-,\underline{t}]}$ as the full subcategory of the category of B-constructible perverse sheaves on $\mathscr{X}$ whose right monodromy map factors through the quotient of $\Bbbk[\mathrm{X}_*(T)]$ by the maximal ideal corresponding to $t \in T^\vee_\Bbbk$. For $t = 1$, we get $P_{[-,\underline{1}]} = \mathscr{O}_{\mathrm{geom}}$; we can thus see the various $P_{[-,\underline{t}]}$ as deformations of $\mathscr{O}_{\mathrm{geom}}$ along a parameter varying in the dual $\Bbbk$-torus. By abuse, we will say that an object in $P_{[-,\underline{t}]}$ *has exact monodromy* t (there will be later a notion of *generalised* monodromy t).

0.1.4 Difficulties and strategy

Another reasonable question is then:

Question 2: What can we say about $P_{[-,\underline{t}]}$ in general ?

A slightly more precise question would be: does $P_{[-,\underline{t}]}$ for $t \neq 1$ share some of the known properties of $\mathscr{O}_{\mathrm{geom}}$? Can we obtain, in analogy with the work of Bezrukavnikov–Riche, a description of $P_{[-,\underline{t}]}$ as some category of modules over some appropriate ring? The answer to both questions is yes, and the strategy we use to prove this is quite similar in flavour to the one used for $\mathscr{O}_{\mathrm{geom}}$. However, there are, at least *a priori*, quite some differences between $P_{[-,\underline{t}]}$ and $\mathscr{O}_{\mathrm{geom}}$. Let us give two examples: the first one is that $P_{[-,\underline{t}]}$ does not identifies directly with the heart of a t-structure, thus common tools of homological algebra are no more available; in particular, the proof of the existence of a highest weight structure on $\mathscr{O}_{\mathrm{geom}}$ cannot be copied straightforwardly. Another convenient feature of $\mathscr{O}_{\mathrm{geom}}$ that we lack for $P_{[-,\underline{t}]}$ is the ability to project on, or pull back from, (partial) flag varieties: for example, if $\mathrm{IC}(s)_{\underline{\Bbbk}}$ denotes the simple perverse

sheaf of $\mathscr{O}_{\mathrm{geom}}$ associated to a simple reflection $s \in W$, then it is well known that $\mathrm{IC}(s)_{\Bbbk}$ can be obtained by pulling back the (perverse) skyscraper sheaf on stratum P_s/P_s in the partial flag variety G/P_s, where P_s is the standard minimal parabolic subgroup associated with s. For a monodromy $t \neq 1$, this is not available: the right action of T on any (partial) flag variety is trivial, hence there are no objects with nontrivial monodromy (i.e. whith monodromy $t \neq 1$) on any (partial) flag variety of G.

However, we can overcome these difficulties by adopting a slightly different point of view on $P_{[-,\underline{t}]}$; let us roughly explain this, we will detail below in this introduction. Once again, let us start with $\mathscr{O}_{\mathrm{geom}}$. By definition, the objects in this category (viewed here as a full subcategory of $D^b_{(B)}(\mathscr{X}, \Bbbk)$) are those perverse sheaves on $\mathscr{X}$ that "coms from" $G/B = (G/U)/T$, meaning, that are pullbacks of perverse sheaves on G/B along the morphism $\mathscr{X} \to G/B$. The latter morphism is the quotient morphism by the free (right) action of T on $\mathscr{X} = G/U$. Now, the category $D^b_{(B)}(G/B, \Bbbk)$ identifies naturally with the T-equivariant bounded derived category of B-constructible $\Bbbk$-sheaves on $\mathscr{X}$, and the subcategory of perverse sheaves in $D^b_{(B),T}(\mathscr{X}, \Bbbk)$ in turn identifies naturally with $\mathscr{O}_{\mathrm{geom}}$. Using this point of view, the functor of pullback along $\mathscr{X} \to G/B$ identifies with the natural forgetful functor $D^b_{(B),T}(\mathscr{X}, \Bbbk) \to D^b_{(B)}(\mathscr{X}, \Bbbk)$. To come back to a monodromy $t \neq 1$, we saw that we cannot identify $P_{[-,\underline{t}]}$ with the pullback from a (partial) flag variety of any appropriate category (again, because there is nothing that we can pull back from), but there may be a way to identify it with some category of "equivariant monodromic perverse sheaves". This is indeed possible, and the correct equivariant category was introduced in [LY].

0.1.5 Work of Lusztig–Yun

In their recent article [LY], Lusztig and Yun define monodromic Hecke categories, using a notion of equivariance "with respect to a local system" on T. The authors of *loc. cit.* consider varieties over a field of positive characteristic and $\overline{\mathbb{Q}}_\ell$ for field of coefficients. To any character sheaf $\mathscr{L}$ on T (see e.g. [Y, Appendix A] for a definition of character sheaves), they construct a derived category on $\mathscr{X}$, equivariant with respect to $\mathscr{L}$; in particular, if $\mathscr{L} = \underline{\Bbbk}_T$, the right *Lusztig–Yun equivariant category* associated to $\mathscr{L}$ is canonically equivalent to $D^b_{(B)}(G/B, \Bbbk)$. This equivariant category is exactly what we were looking for.

Copying the constructions in *loc.cit.* to our setting of complex varieties and coefficients of positive characteristic, we can define a Lusztig–Yun equivariant monodromic category, and a subcategory of perverse objects there. Our first key result is that the latter category is equivalent to $P_{[-,\underline{t}]}$; this notion of equivariance is the correct one to consider for our purposes, in particular, it gives natural solutions to the two problems mentioned above, as we will explain below.

In fact, numerous ideas from [LY] will be fundamental in the present work, and will be detailed below in this introduction.

0.1.6 Results

Let us roughly state our main results here. As suggested by the previous subsection, most of this thesis is written using the language of the Lusztig–Yun monodromic categories. With this at hand, we are able to derive almost all of the properties of $\mathscr{O}_{\mathrm{geom}}$ we were wishing for $P_{[-,\underline{t}]}$ to possess (or, in fact, its Lusztig–Yun-equivariant counterpart). The key point is always to find the correct translation to fit the monodromic (equivariant) setting; the results then follow from (often) standard considerations. In particular, we are able to show that monodromic perverse sheaves admit a natural highest weight structure; then we can define a tilting perverse monodromic subcategory, and we can relate it via a variant of the geometric Ringel duality to the subcategory of projective perverse monodromic sheaves. What remains to be done is then to describe the category of tilting objects as a category of modules over some explicitly determined ring. This is eventually achieved, using a "Soergel type" functor, providing an answer to our original question. Once the category of tilting objects is described explicitly, we can deduce a description of the whole category of perverse monodromic sheaves, and finally, following [LY], of the whole derived monodromic category: the latter is equivalent to the usual category of Borel-constructible $\Bbbk$-sheaves on the flag variety of some appropriate, endoscopic, group.

A last point: the description of the category of tilting objects as modules will be, as stated above, obtained via the use of an appropriately defined Soergel functor. Here, this means that we will consider a functor of the form $\mathrm{Hom}(\mathscr{T}, -)$ with $\mathscr{T}$ a nice tilting object. Such a functor takes its value in the category of right $\mathrm{End}(\mathscr{T})$-modules, thus one of our main missions will be (once we have identified a potential $\mathscr{T}$) to describe this endomorphism space. This can be seen as a geometric and monodromic version of Soergel's Endomorphismensatz in [So], and a direct variation of the analogous result in [BeR].

We review below the different chapters of this thesis, detailing for each one what is concretely done in order to obtain these results.

0.2 Overview of the chapters

This thesis is divided in two parts: the first one deals with general constructions on two different notions of monodromy, which we will detail below, and studies monodromic perverse sheaves following the pattern of [BGS, §3]; this is the content of the preprint [G1]. The second part focuses on the study of monodromic categories on the variety G/U, and will be published in [G2].

0.2.1 Topological monodromy

The first chapter is an exposition of the monodromy action evoked above, following Verdier's construction from [Ve]. Considering a complex algebraically stratified variety $(X, \mathcal{S})$ acted on by a complex algebraic torus A, we present the definition of the canonical monodromy action on the derived constructible

category $D^b_{\mathcal{S}}(X, \Bbbk)$. This chapter is essentially a gathering of definitions and elementary, although essential for the following, properties of monodromy.

0.2.2 Lusztig–Yun monodromy

The second chapter has for goal the presentation of another definition of monodromic categories. This is entirely based on [LY]; the considerations in *loc. cit.* are done on schemes defined over some finite field, endowed with the étale topology, and with coefficients a field of characteristic zero (more precisely, with $\overline{\mathbb{Q}}_\ell$-complexes of sheaves). Here, we consider a complex variety X acted on by a complex algebraic group H, and positive-characteristic coefficients, but the definitions and results of *loc. cit.* we will need for most part adapt quite straightforwardly.

The definition of the Lusztig–Yun monodromic category is roughly as follows: we consider a rank one multiplicative local system $\mathscr{L}$ on the algebraic group H (multiplicative meaning here that the pullback of $\mathscr{L}$ along the multiplication map $H \times H \to H$ is given by $\mathscr{L} \boxtimes \mathscr{L}$). Say that we have a surjective morphism of algebraic groups with finite kernel K whose order $|K|$ is prime to ℓ, central in $\widetilde{H}$ (that is, a finite central isogeny) $\widetilde{H} \xrightarrow{\nu} H$. The kernel K also acts on the pushforward $\nu_* \underline{\Bbbk}_{\widetilde{H}}$; assume that we have a character $\chi : K \to \Bbbk^*$ such that $\mathscr{L}$ identifies with the χ-isotypic component $\nu_* \underline{\Bbbk}_{\widetilde{H}}$.

One can consider the equivariant bounded derived category $D^b_{\widetilde{H}}(X, \Bbbk)$, where the action of $\widetilde{H}$ is via ν. The kernel K acts on this category, meaning that for any object $\mathscr{F}$ we have a group morphism $K \to \mathrm{Aut}(\mathscr{F})$ from K to the automorphisms group of $\mathscr{F}$ in the equivariant category, and this action is functorial. Then one can consider the full subcategory of $D^b_{\widetilde{H}}(X, \Bbbk)$ whose objects are those $\mathscr{F}$ for which the action of K is via the character χ. This yields the (equivariant) Lusztig–Yun monodromic category. A major part of the chapter is devoted to the following: the description just given requires choices, namely for the isogeny ν and the character χ. In order for this category to be manageable and convenient to work with, one wishes that in fact, it does not depend on these choices. This is indeed the case, as shown in sections 2.3 through 2.9; consequently, we obtain a "well defined" category denoted $\mathfrak{D}(X /\!\!/ H)_{\mathscr{L}}$.

0.2.3 Monodromic categories

Our third chapter studies the following situation: assume that we have an algebra R acting on a category $\mathcal{C}$. We show here that, under suitable assumptions, our category splits according to the generalised eigenvalues of the action. This is general and does not rely on the previous chapters, however, it applies to the categories that were introduced in these chapters: as we explain above, for $(X, \mathcal{S})$ a complex stratified variety acted on by a complex torus A, we have an action of $\Bbbk[\mathrm{X}_*(A)]$ on the category $D^b_{\mathcal{S}}(X, \Bbbk)$, given by monodromy. Our general

results imply that we have a direct sum decomposition

$$D^b_{\mathcal{S}}(X,\Bbbk) = \bigoplus_{t\in A^\vee_\Bbbk} D^b_{\mathcal{S}}(X,\Bbbk)_t,$$

where t is in the dual $\Bbbk$-torus and where we let $D^b_{\mathcal{S}}(X,\Bbbk)_t$ be the full subcategory of $D^b_{\mathcal{S}}(X,\Bbbk)$ whose objects are those $\mathscr{F}$ whose right monodromy map factors through the quotient $\Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t)\rangle^n$ for some $n > 0$ (note that for $t = 1$, this means that the monodromy is *unipotent*; this subcategory was the one studied in [BeR]). We say that an object in $D^b_{\mathcal{S}}(X,\Bbbk)_t$ *has generalized monodromy* t. In particular, if two objects $\mathscr{F},\mathscr{G}$ on $\mathscr{X}$ do not share a common generalized monodromy, then there are no nonzero morphism between them.

The category $D^b_{\mathcal{S}}(X,\Bbbk)_t$ admits a perverse t-structure, and we let $P_{\mathcal{S}}(X,\Bbbk)_t$ be its heart. It is clear that an object $\mathscr{F} \in D^b_{\mathcal{S}}(X,\Bbbk)$ that has exact monodromy t has in particular generalized monodromy t, in particular we have a full subcategory $P_{\mathcal{S}}(X,\Bbbk)_{\underline{t}} \subseteq P_{\mathcal{S}}(X,\Bbbk)_t$. Note that this subcategory is not stable by extensions (that was partly the cause of our preceding problems).

We then study monodromic categories in case $X = A$ and $\mathcal{S}$ has only one stratum given by the whole variety A. This reduces to the study of the monodromy of indecomposable local systems on A. We remark that a $\Bbbk$-representation of $\mathrm{X}_*(A)$ defines a $\Bbbk$-local system on A, moreover an irreducible local system always corresponds to a one dimensional representation of the form

$$\Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t)\rangle, \quad t \in A^\vee_\Bbbk; \tag{0.2.1}$$

in particular, isomorphism classes of irreducible $\Bbbk$-local systems on A are in bijection with the elements of $A^\vee_\Bbbk$. We then show that the monodromy of local systems is closely related to their structure of $\mathrm{X}_*(A)$-representation, which allows us to describe the monodromic categories on A.

Finally, we consider Lusztig–Yun categories, and show that a direct sum splitting holds again in this setting, as in the case of monodromic constructible sheaves.

0.2.4 Perverse monodromic sheaves

In the fourth chapter, we focus on algebraically stratified A-varieties $(X,\mathcal{S})$, where A is an algebraic (complex) torus. The fact that we restrict ourselves to the case of a torus has the following consequence: as above, we are able to classify the one-dimensional $\Bbbk$-local systems on A, and these are parametrized by elements t of the dual $\Bbbk$-torus $A^\vee_\Bbbk$, we let $\mathscr{L}^A_t$ be the local system on A corresponding to t, i.e. associated to the $\mathrm{X}_*(A)$-representation (0.2.1). We show in section 4.1 that for any t, this local system allows us to consider a Lusztig–Yun category on X (that is, we construct an isogeny $\widetilde{A} \to A$ and a character χ_t as in subsection 0.2.2). In this setting, we show that the perverse t-structure on the equivariant derived category $D^b_{\mathcal{S},\widetilde{A}}(X,\Bbbk)$ restricts to a t-structure on the Lusztig–Yun subcategory $\mathfrak{D}(X/\!\!/A)_{\mathscr{L}^A_t}$, thus defining a subcategory of equivariant Lusztig–Yun perverse sheaves, which we denote $\mathfrak{P}(X/\!\!/A)_{\mathscr{L}^A_t}$.

Now, our interest is in perverse monodromic sheaves on X with fixed monodromy t. The first step is to actually give a precise sense to "perverse monodromic sheaves": for now, we have two *a priori* distinct definitions, namely the category $P(X,\Bbbk)_{[t]}$ of perverse sheaves on $(X,\mathcal{S})$ with a fixed (topological) monodromy t (with t being an element in the dual $\Bbbk$-torus $A^\vee_\Bbbk$), and the subcategory $\mathfrak{P}(X/\!\!/A)_{\mathscr{L}^A_t}$ of perverse objects in a Lusztig–Yun equivariant category associated to the local system $\mathscr{L}^A_t$ on A. Considering the equivariant point of view, we can add two more reasonable definitions. Let us for a moment think about the (usual) constructible equivariant derived category on some stratified variety Y acted on by a connected algebraic group H, as defined e.g. in [BL]. The equivariant category is not a subcategory of the constructible category, however, it is well known that this is the case of the perverse subcategory. In fact, the subcategory of perverse objects in the equivariant category identifies with a full subcategory of the perverse subcategory in the constructible category; and an object $\mathscr{F}$ in the latter "is" equivariant if and only if it admits a descent datum with respect to the action of H on Y, meaning here that the pullbacks of $\mathscr{F}$ along the projection, resp. action map, $H\times Y\to Y$ are isomorphic.

Now in our monodromic setting, in analogy with this description of equivariant perverse sheaves, we can consider two new subcategories of the perverse subcategory in $D^b_{\mathcal{S}}(X,\Bbbk)$, whose objects are required to satisfy compatibility conditions on pullbacks analogous to those just evoked. This is done in section 4.2.

We then proceed to show our main results for this first part: first, all the definitions of monodromic perverse sheaves yield canonically equivalent categories. We thus have four equivalent categories, and we then show that under suitable assumptions on the strata of X, they share a natural highest weight structure, with finite weight poset; natural meaning here that this highest weight structure mimics precisely the case of $\mathscr{O}_{\mathrm{geom}}$: standard objects are shifted !-extension of appropriate local systems on the strata, whereas costandard objects are the $*$-extension of the same local systems; and the poset is given by the set of strata ordered by the closure inclusion.

In particular these categories have enough projective objects, as well as enough injective objects, and it makes sense to consider tilting objects. In the end of the thesis, we will work essentially with the Lusztig–Yun category $\mathfrak{P}(X/\!\!/A)_{\mathscr{L}^A_t}$ for $X=G/U$, as it is much more manageable than the category of perverse sheaves with fixed monodromy (though not forgetting the monodromic categories $D^b_{\mathcal{S}}(X,\Bbbk)_t$ and $P_{\mathcal{S}}(X,\Bbbk)_t$, which will still be quite useful). The subcategory $\mathfrak{T}(X/\!\!/A)_{\mathscr{L}^A_t}$ of tilting objects in $\mathfrak{P}(X/\!\!/A)_{\mathscr{L}^A_t}$ will become our primary interest in the rest of this thesis

0.2.5 Flag varieties

Chapter 5 is mainly a gathering of definitions and notation; we describe the constructions of part I in the particular case where $X=\mathscr{X}=G/U$ and the torus A is chosen to be the maximal torus T of G. There is a particularity however:

this variety, being both a left and right T-variety, allows us to define both left and right monodromy actions, as well as left and right Lusztig–Yun equivariant categories (now associated to a local system denoted $\mathscr{L}_t^T$ on T). In particular, from now on, our categories will be written with two indices: $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[?,?]}$, where ? can be either an element of the dual torus or the symbol $-$, the latter meaning that we make no assumption on the corresponding monodromy. For example, $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$ denotes the category whose objects are those $\mathscr{F}$ with right generalized monodromy t and no assumption on left monodromy. A similar notation will be used for Lusztig–Yun category, in this case we can even "mix monodromies": for $t, t' \in T^\vee_{\Bbbk}$, we set $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]}$ for the full subcategory of $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ (the usual Lusztig–Yun equivariant category for the right action of T) whose objects are those $\mathscr{F}$ for which, once viewed as constructible objects on $\mathscr{X}$ via an appropriate forgetful functor, the left generalized monodromy is given by t'.

The Bruhat stratification on G descends to a stratification on $\mathscr{X}$, indexed by the Weyl group W. The strata then admit a nice description in term of unipotent subgroups of U and lifts of elements of the Weyl group W of G. These descriptions allow us to see that the left and right monodromy actions on $\mathscr{X}$ are not independent but rather interconnected.

We also introduce what will be for us the replacement of the partial flag varieties G/P_s. As suggested earlier, a quotient in the usual (i.e. with monodromy 1) setting ought to correspond to an appropriate equivariance condition. For s a simple reflection in W, we consider the standard parabolic subgroup $B \subseteq P_s$; write $P_s = L_s \ltimes U^s$ where L_s is the Levi factor containing T, and U^s is the unipotent radical of P_s, and let $B_s = B \cap L_s$. We can consider the natural quotient $\mathscr{X}^s := G/U^s$; this is naturally a left and right B_s-variety, as well as a left and right L_s-variety. If $s = s_\alpha$ with $\alpha \in \Phi$ such that $\alpha^\vee(t) = 1$ (we will comeback to this condition later), the local system $\mathscr{L}_t^T$ extends to a local system $\mathscr{L}_t^{L_s}$ on L_s and hence to B_s as well. Then one can define L_s and B_s-Lusztig–Yun equivariant categories on $\mathscr{X}^s$. We will later consider various forgetful and averaging funcors between these categories, and these will serve as analogues of constructible categories on quotients of the form G/P_s.

Finally, our preceding results apply on $\mathscr{X}$; in particular, we obtain a highest weight category of right Lusztig–Yun equivariant perverse sheaves $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$, with weight poset given by W endowed with the Bruhat order. We then get standard and costandard objects, that we will denote $\Delta(w)_{\mathscr{L}_t^T}$ and $\nabla(w)_{\mathscr{L}_t^T}$ respectively, as well as tilting perverse objects denoted $\mathscr{T}(w)_{\mathscr{L}_t^T}$.

Let $\mathfrak{T}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ denote the full subcategory of tilting objects in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. This is the category we want to describe: in the end, this will allow us to give a description of the whole category of perverse sheaves. Following the strategy used for $\mathscr{O}_{\mathrm{geom}}$, our objective is to define a "Soergel functor" on this category, which will eventually lead to the wished-for description. This functor will be defined by evaluating the morphisms from a maximal tilting object (in a sense to be precised). The rest of the thesis is now a successive presentation of results which will allow us to define rigorously this functor and derive the properties we want.

0.2.6 Convolution

Chapter 6 is devoted to the construction of bifunctors on our monodromic categories, yielding a "convolution product". The thing is that G being a group, the action of G on variety $\mathscr{X} = G/U$ is induced by the natural multiplication on the left. This multiplication sort of upgrades to the derived categories to define a bifunctor (our convolution product) $(-)\star(-) : D^b_{(B)}(\mathscr{X}) \times D^b_{(B)}(\mathscr{X}) \to D^b_{(B)}(\mathscr{X})$. We then study the behavior of this convolution product with respect to left and right monodromy.

The next step is to define a similar convolution product on the Lusztig–Yun equivariant categories; the definition needs a slight adjustment in comparison to the constructible case; this is done in section 6.3.

0.2.7 Completed categories

In chapter 7, we present one of our main tools for the final understanding of $\mathfrak{T}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$, namely Yun's completed (monodromic) category. This construction is adapted from [BY, Appendix A].

Recall that for a category $\mathcal{C}$, a pro-object in $\mathcal{C}$ is a functor on $\mathcal{C}^{\mathrm{op}}$ isomorphic (as a functor) to a functor of the form $X \mapsto \varinjlim_n \mathrm{Hom}(X_n, X)$ for some projective system $(X_n)_n$ of objects in $\mathcal{C}$; such a functor is denoted "$\varprojlim$" X_n. Note that any object X of $\mathcal{C}$ can be viewed in a natural way as a pro-object, in particular, $\mathcal{C}$ identifies as a full subcategory of the category of pro-objects in $\mathcal{C}$.

The completed category we consider is a full subcategory of the category of pro-objects in $D^b_{(B)}(\mathscr{X}, \Bbbk)$. In the case of unipotent monodromy (i.e. $t = 1$), the definition of the completed category uses the pushforward functor along $\mathscr{X} \to G/B$. As we mentioned above, this is not available for non-unipotent monodromies; instead, we should consider a "projection functor" $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]} \to \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$. We define such a functor $\pi^t_\dagger$ using the convolution product defined in chapter 6, and study some of its elementary properties in 7.1.

Now, the functor $\pi^t_\dagger$ extends to a functor between the categories of pro-objects over $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$ and $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$. Let us, for the duration of this introduction, denote by $\widehat{\pi}^t_\dagger$ this extended functor. The completed category is then defined as the full subcategory of pro-objects "$\varprojlim$" $\mathscr{G}_n$ over $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$ such that the $\mathscr{G}_n$ have bounded cohomological amplitude and such that $\widehat{\pi}^t_\dagger(\text{"}\varprojlim\text{"}\,\mathscr{G}_n)$ is a representable functor on $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$. For the duration of this introduction, we denote it by $\widehat{D}_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$.

Once this is defined, we can start to derive properties of this completed category: first, the convolution product on $D^b_{(B)}(\mathscr{X}, \Bbbk)$ extends to a convolution bifunctor $(-)\widehat{\star}(-)$ at the completed level. Then the completed category is a triangulated category, and we can define a "perverse" t-structure on it, obtaining thus a category of completed perverse sheaves. We can then define *pro-standard* and *pro-costandard* perverse sheaves, denoted $\widehat{\Delta}_{w,t}$ and $\widehat{\nabla}_{w,t}$ for any $w \in W$, in analogy with the objects $\Delta(w)_{\mathscr{L}^T_t}$ and $\nabla(w)_{\mathscr{L}^T_t}$. In fact, the pro-standard and pro-costandard give lifts of the non-completed standard and costandard,

meaning that $\widehat{\pi}_!^t(\widehat{\Delta}_{w,t}) \cong \Delta(w)_{\mathscr{L}_t^T}$, and similarly for the ∇'s (note that here, the identification should rigorously be meant as an isomorphism of representable functors, but as explained above, we can and will directly identify $\widehat{\pi}_!^t$ of objects in the completed category as objects in the Lusztig–Yun category). We will see later that the objects in $\mathfrak{T}(\mathscr{X}/\!\!/T)_{\mathscr{L}_t^T}$ also lift to the completed category.

Importantly, these lifts admit *two non trivial* monodromy morphisms: more or less by definition, the objects of $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ have a fixed trivial right monodromy, meaning here that for such an object $\mathscr{F}$, the right monodromy morphism

$$\varphi_{r,\mathscr{F}} : \Bbbk[\mathrm{X}_*(T)] \to \mathrm{End}(\mathscr{F})$$

factors through the quotient $\Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t)\rangle \cong \Bbbk$. (We made an abuse of notation here, as monodromy is defined for constructible objects, and not really for equivariant objects, but for the purpose of this introduction, this seemed clearer). In contrast, completed objects can be thought as limits of objects $\mathscr{F}_n$ whose right and left monodromy factors through the quotient $\Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t)\rangle^m$, with m getting larger as n does so. In particular, for $\mathscr{F}$ in the completed category, the monodromy morphisms extends to a morphism from the completion $\widehat{R}_t := \varprojlim_n \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t)\rangle^n$ to the endomorphisms of $\mathscr{F}$.

In certain aspects, working in the completed setting is simpler that working with non-completed objects. For example, one can show that there are no nonzero morphisms between pro-standard associated to distinct elements v, w. We are also able to describe the morphisms between a pro-standard and a pro-costandard; usual arguments then allow us to see that the completed category is more manageable that one may a priori think: in particular, it is Krull–Schmidt and the morphism spaces are finitely generated $\widehat{R}_t$-modules (where the action is induced by monodromy).

In fact, the completed category will be particularly useful when we will study the endomorphism space of tilting objects: our version of the Endomorphismensatz will actually be a consequence of an analogous result for an appropriate object in the completed category.

0.2.8 Standard and costandard objects

Generalities

Chapter 8 deals with an extensive study of standard and costandard objects in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. In the case $t = 1$, that is to say, in the B-constructible derived category of sheaves on the flag variety G/B, the standard and costandard objects are given by $\Delta_w := (j_w)_! \underline{\Bbbk}_{BwB/B}[\dim(BwB/B)]$ and $\nabla_w := (j_w)_* \underline{\Bbbk}_{BwB/B}[\dim(BwB/B)]$ respectively; it is well known that they possess nice properties: for example, the convolution of Δ_w and Δ_v is isomorphic to Δ_{wv} if the lengths of w and v add up; we have $\Delta_w \star \nabla_{w^{-1}} = \Delta_e$; we know how to describe the socle of Δ_w and the cosocle of ∇_w (see [BBM] for an exposition of these results). Chapter 8 is devoted to the proofs that these facts are still valid in the category $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. For the flag-variety-case, a key tool is a bunch of exact sequences relating standard objects and simple objects. These

exact sequences are obtained via a push-pull to a partial flag variety operation, and once again, this is not available in our setting. However, we already found the replacement for such a process, via Lusztig–Yun equivariant categories for Levi factors of minimal parabolic subgroups. Then, a little work is needed to identify the objects involved, and we are able to derive exact sequences quite similar to those one is used to work with on flag varieties.

Convolution

The second step in the study of standards and costandards is the behavior of these objects with respect to convolution. Checking the analogue of the isomorphism $\Delta_w \star \Delta_v \cong \Delta_{wv}$ (and the similar one for costandards) is quite standard; the difficult part is to show that standard and costandard objects "cancel" each other. In order to show this fact, it suffices to consider a simple reflection s in W: the convolution of the standard associated to s with the costandard associated to the same s gives the standard object associated to $e \in W$. But, as briefly evoked above in subsection 0.2.5, the elements of W, in particular the simple reflections in this group, do not behave all in the same way with respect to the element t defining our monodromy (or, equivalently, with respect to the local system $\mathscr{L}_t^T$ defining our equivariance). For example, some elements may fix t, and some others may not. In fact the correct distinction to be made is the following (this is taken from [LY], and quickly appeared in subsection 0.2.5): let W_t° be the subgroup of W generated by the subset $S_t \subseteq S$ whose elements are those reflections $s = s_\alpha$ such that $\alpha^\vee(t) = 1$ (note that if $t = 1$, we have $W_t^\circ = W$). This subgroup will be fundamental in the following; let us say for now that the pair (W_t°, S_t) defines a Coxeter system, in particular, this group has its own Bruhat order $\leq_t$, its own length function ℓ_t and admits a unique maximal element $w_{t,\circ}$.

With a little work, we already see quite a difference: for s a simple reflection in W such that $s \notin W_t^\circ$, the standard object coincides with the costandard object and hence with the simple object associated to s; this is definitely not the case for $s \in W_t^\circ$. Thus from now on, we should always distinguish the cases $s \in W_t^\circ$ and $s \notin W_t^\circ$ (the latter being often quite simpler than the former). To come back to our Δ-∇ convolution problem, when $s \notin W_t^\circ$, the argument is topological, whereas the case $s \in W_t^\circ$ requires more care, and more involved (although quite standard) arguments. Once the case of objects is done, we can derive analogous results for pro-objects; all of this is done through sections 8.2 to 8.4.

Socle

We finally turn to the determination of the socle of Δ's. In the case $t = 1$, all the standard objects share the same socle, namely, the simple object $\mathrm{IC}(e)_{\Bbbk}$. If $t \neq 1$, however, it is clear that this will not be the case: for example, if $s \notin W_t^\circ$, then $\Delta(s)_{\mathscr{L}_t^T}$ is already simple. The thing is that "the Weyl group does not fix arbitrary element $t \in T_{\Bbbk}^\vee$ the way it fixes 1", i.e. the orbit of t

under W may contain more than just t. It is then plausible that we have to distinguish, in an appropriate way, a standard object from another one according to the elements of W that index them. Once again, the answer came from [LY]: as mentioned above, the left and right monodromy of an indecomposable object on $\mathscr{X}$ cannot be completely arbitrary. For example, an indecomposable object $\mathscr{F}$ concentrated on some stratum $\mathscr{X}_w$ with a fixed right monodromy t (e.g. $\Delta(w)_{\mathscr{L}_t^T}$) will automatically have a left monodromy given by $w(t)$. So if $w(t) \neq v(t)$, the standard associated to w and v are really different, and in particular, cannot share a common socle. Note however that if v is such that $v(t) = w(t)$, then it is reasonable to hope for our standards to be more "alike". We thus may be tempted to consider classes in some sort of quotient ${}_{t'}W_t/{}_tW_t$ where ${}_aW_b := \{w \in W \mid w(b) = a\}$ for $a, b \in T^\vee_{\Bbbk}$. It turns out that this is not exactly what we should consider, but instead the quotient ${}_{t'}\underline{W}_t := {}_{t'}W_t/W_t^\circ$ (note that $W_t^\circ \subseteq {}_tW_t$). Elements in this quotient are called *blocks*, and they are fundamental in what follows. Blocks possess numerous nice properties, for example, a block β in ${}_{t'}\underline{W}_t$ is a subset of W and as such it inherits the partial order of the Weyl group; one can show that there exists then a unique maximal element and a unique minimal element in β. For now, they give the distinction we were looking for: set

$$\mathfrak{D}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}_t^T]} = \langle \Delta(w)_{\mathscr{L}_t^T} \mid w \in \beta \rangle_{\text{triang.}},$$

we also call these subcategories "blocks". Adapting arguments from [LY], we are able to show that the category $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]}$ is a direct sum of its block subcategories:

$$\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]} = \bigoplus_{\beta \in {}_{t'}\underline{W}_t} \mathfrak{D}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}_t^T]}, \tag{0.2.2}$$

in particular, there are no nonzero morphisms from an object in one block to an object in another one.

Now it is really reasonable to hope for standard objects indexed by elements in a given block to behave well, for instance, to share a common socle. This is indeed the case, as shown in section 8.6, and this concludes our study of standard objects.

0.2.9 Tilting objects, Ringel duality and Soergel functors

The next chapter focuses on tilting objects, first those in the Lusztig–Yun categories, then their completed counterparts.

The block decomposition (0.2.2) induces a similar splitting of categories on the subcategories of perverse sheaves. We can thus consider categories of the type $\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}_t^T]}$; this is the Serre subcategory of $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]}$ generated by the simple objects $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ with $w \in \beta$. These categories are still highest weight, the corresponding projective and tilting objects are those of $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]}$ lying in our block $\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}_t^T]}$. This is the categories we are

really interested in. Our first goal is to show that the subcategories of projective and tilting perverse sheaves are related, (in fact, equivalent) via a monodromic version of Ringel duality. This is done in section 9.2, and it then follows from standard arguments that, as in $\mathscr{O}_{\mathrm{geom}}$, the projective cover of the minimal IC-sheaf (i.e. associated to the unique minimal element of β) in $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}^T_t]}$ is isomorphic to the maximal tilting object (i.e. associated to the unique maximal element of β).

The subcategory of perverse objects in the completed category also posses "tilting objects"; they can be characterized as those pro-objects $\widehat{\mathscr{T}}$ in the completed category whose projection $\pi^t_\dagger(\widehat{\mathscr{T}})$ is a tilting perverse object in the Lusztig–Yun category; let us set $\widehat{T}_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}$ this subcategory of tilting objects. We show that for any $w \in W$, there exists an indecomposable completed tilting object $\widehat{\mathscr{T}}_{w,t}$ such that $\pi^t_\dagger(\widehat{\mathscr{T}}_{w,t}) \cong \mathscr{T}(w)_{\mathscr{L}^T_t}$, and that any pro-tilting perverse object is a direct sum of these $\widehat{\mathscr{T}}_{w,t}$'s.

From now on, until the end of this introduction, we will focus on the class of the neutral element in ${}_t\underline{W}_t$, i.e. the subgroup W_t° (this is legitimate: it turns out that all blocks are equivalent, thus it suffices to consider this particular block). We call it the *neutral block*. In this case, we write $\mathfrak{D}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}^T_t]}$ for the corresponding block subcategory (and similarly for neutral blocks in some other subcategories).

We study in more details the (pro)-tilting objects associated to reflection $s \in W_t^\circ$, simple in this group. Quite standard features on convolution of tilting objects enable us to show that the convolution of tilting objects yields a tilting object, and that the categories of (pro)-tilting objects admit a "Bott–Samelson" type description, i.e. any indecomposable tilting object appears as a direct summand in an appropriate convolution product of objects of the form $\widehat{\mathscr{T}}_{s,t}$. This fact will later be put at use in order to describe our categories of tilting objects as modules.

The final section of this chapter is devoted to the introduction and first properties of another central players in our game, namely the functors $\widehat{\mathbb{V}}^\circ_t$ and $\mathbb{V}^\circ_t$. They are defined respectively as

$$\widehat{\mathbb{V}}^\circ_t(-) = \mathrm{Hom}(\widehat{\mathscr{T}}_{w_{t,\circ},t}, -) : \widehat{P}_{(B)}(\mathscr{X},\Bbbk)^\circ_{[t,t]} \to \mathrm{Mod}^{\mathrm{fg}}(\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}))$$

and

$$\mathbb{V}^\circ_t(-) = \mathrm{Hom}(\mathscr{T}(w_{t,\circ})_{\mathscr{L}^T_t}, -) : \mathfrak{P}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}^T_t]} \to \mathrm{Mod}^{\mathrm{fg}}(\mathrm{End}(\mathscr{T}(w_{t,\circ})_{\mathscr{L}^T_t})),$$

where for a ring A, we set $\mathrm{Mod}^{\mathrm{fg}}(A)$ for the category of finitely generated right-A-modules. The main result of this section is an adaptation of a result from [BY]: these functors, once restricted to the subcategories of tilting objects, are fully faithful. Thus, in order to obtain a full description of the latter categories, we must determine first $\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t})$ and $\mathrm{End}(\mathscr{T}(w_{t,\circ})_{\mathscr{L}^T_t})$, and then the essential images of $\widehat{\mathbb{V}}^\circ_t$ and $\mathbb{V}^\circ_t$. In fact, it suffices to consider the completed case:

one of the most convenient features of tilting objects in the completed category is the following: for any such objects $\widehat{\mathscr{T}}, \widehat{\mathscr{T}}'$, we have an isomorphism

$$\mathrm{Hom}(\pi_\dagger^t(\widehat{\mathscr{T}}), \pi_\dagger^t(\widehat{\mathscr{T}}')) \cong \mathrm{Hom}(\widehat{\mathscr{T}}, \widehat{\mathscr{T}}') \otimes_{\widehat{R}_t} \Bbbk\,. \tag{0.2.3}$$

In particular, by definition, we get

$$\mathrm{End}(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T}) \cong \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \otimes_{\widehat{R}_t} \Bbbk\,. \tag{0.2.4}$$

This partly justifies the importance of the completed category. Moreover, the pro-tilting objects are convenient objects to work with, notably because of the particularly simple description of morphisms between pro-standard objects in the completed category. We can then consider an associated graded functor

$$\mathrm{gr} : \widehat{T}_{(B)}(\mathscr{X}, \Bbbk)^\circ_{[t,t]} \to \widehat{D}_{(B)}(\mathscr{X}, \Bbbk)^\circ_{[t,t]},$$

and we show that this functor is faithful. Noticing that we have $\mathrm{gr}(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \cong \bigoplus_{w \in W_t^\circ} \widehat{\Delta}_{w,t}$, we obtain an injective map

$$\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \hookrightarrow \bigoplus_{w \in W_t^\circ} \mathrm{End}(\widehat{\Delta}_{w,t}).$$

It turns out that the right hand side is simply a direct sum of copies of the completion $\widehat{R}_t$. Thus, in order to determine $\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t})$, it suffices to determine the image of the above map; this will be done in the next chapter.

0.2.10 Completed Endomorphismensatz

Pittie–Steinberg theorem

Chapter 10 is devoted to the proof of one of our main result, namely, a variation of Soergel's Endomorphismensatz: we describe the endomorphism ring of the (pro)-tilting object associated to the element $w_{t,\circ}$. The strategy here is quite closely inspired by the arguments of [BeR, §8], themselves derived from results of [KK] and [AJS]. In the end, we will show that the monodromy morphism induces an isomorphism of algebras

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \xrightarrow{\sim} \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}). \tag{0.2.5}$$

Let us detail the strategy we will use; basically, the arguments are based on the Pittie–Steinberg theorem. Let H be a reductive semisimple simply-connected algebraic group over the algebraically closed field $\Bbbk$, and denote by T_H a maximal torus and W_H its Weyl group. This theorem (theorem 10.1.1 here) describes the $\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{X}^*(T_H)]$ of the abelian group $\mathrm{X}^*(T_H)$ as a module over the algebra $\mathbb{Z}[\mathrm{X}^*(T_H)]^{W_H}$ of fixed points under the natural W_H-action. Our first step is to obtain a variant of the Pittie–Steinberg theorem: the assumption that H is semisimple simply-connected is too restrictive for our purposes, but we

show, using verbatim the argument of [St], that the Pittie–Steinberg theorem still holds under the assumption that the quotient $\mathbb{Z}[\mathrm{X}_*(T_H)]/\Phi^\vee$ is free as a $\mathbb{Z}$-module, where $\Phi^\vee$ is the system of coroots of T_H (or, equivalently, that the derived subgroup of H is simply connected).

Completed version

Now, note that we want to prove the isomorphism (0.2.5): ultimately, we are interested in $\widehat{R}_t$, and not simply in $\mathbb{Z}[\mathrm{X}_*(T)]$; in fact our goal is not to obtain an exact analogue of the Pittie–Steinberg theorem, but rather a "completed version", using completions of the algebra $\Bbbk[\mathrm{X}^*(T_H)]$ along some maximal ideal associated to an element $t \in T_H$. But following [BeR], once the Pittie–Steinberg theorem is known, if t is central in H, one can derive such a completed version (thus here, maintaining our assumption on the derived subgroup of H). At this point, we should make a comment: the assumption that the derived subgroup of H is simply connected is still too restrictive: we would like to simply assume H to be any reductive group. Unfortunately, the price to pay for this generality is an assumption on the characteristic ℓ of $\Bbbk$: we assume that ℓ does not divide the order of the finite quotient $\mathbb{Z}[\mathrm{X}_*(T_{\mathscr{D}(H)})]/\mathbb{Z}\cdot\Phi^\vee$ (here $T_{\mathscr{D}(H)}$ denotes the maximal torus of the derived subgroup of H contained in T_H, and $\mathrm{X}_*(T_{\mathscr{D}(H)})$ its cocharacter lattice).

The fact that we want to consider only a completed version will actually be crucial in the proof of the isomorphism (0.2.5). The point is that for a general H reductive, there exists a reductive algebraic group $\widetilde{H}$ with simply connected derived subgroup and a morphism $\widetilde{H} \to H$ whose restriction to maximal tori $T_{\widetilde{H}} \to T_H$ is an étale map (with $T_{\widetilde{H}}$ being a maximal torus of $\widetilde{H}$ above T_H). Factually, the group $\widetilde{H}$ will be constructed by expliciting a root datum derived from the one of H, and the morphism $\widetilde{H} \to H$ will be obtain from a natural morphism of root data. Anyway, our variant of the Pittie–Steinberg theorem is true for $\widetilde{H}$, and we also obtain a completed version, for any element $\widetilde{t}$ central in $\widetilde{H}$. We now have to deduce from this fact for $\widetilde{H}$ the corresponding fact for H. The solution comes here from basic algebraic geometry: if $\widetilde{t} \in \widetilde{H}$ maps to $t \in H$, the map $T_{\widetilde{H}} \to T_H$, being an étale group morphism, induces an isomorphism between the completions of the local rings of the complex varieties $T_{\widetilde{H}}$ and T_H at $\widetilde{t}$ and t, respectively. But now, it is standard that in the case of algebraic complex tori, one can describe these local rings using the character lattices. Eventually, this allows us to derive the wished for result for H.

The endoscopic group

We come back to (0.2.5). In order to prove our isomorphism, we will apply the preceding considerations for a particular H, namely, the endoscopic group attached to t. Once again, we should make an assumption on the characteristic ℓ of $\Bbbk$: assume that ℓ is not a torsion prime for the Langlands dual group $G^\vee_\Bbbk$.

Let $\Phi_t \subseteq \Phi$ be the subset of roots α such that $s_\alpha \in S_t$; let $H_t^\vee$ be the connected reductive algebraic group over $\Bbbk$ defined as the connected component

of the neutral element in the centralizer in $G_{\Bbbk}^{\vee}$ of the element t. The complex Langlands dual H_t of $H_t^{\vee}$ is then the reductive group over $\mathbb{C}$ whose maximal torus identifies with T and whose root system is given by Φ_t; the Weyl group of H_t is then given by W_t°. Since t is central in $H_t^{\vee}$ by definition, we can apply our completed version of the Pittie–Steinberg theorem to this group. More or less by the definition of Langlands duals, this will imply that $\widehat{R}_t$ is a free $(\widehat{R}_t)^{W_t^{\circ}}$-module of rank $|W_t^{\circ}|$.

To deduce the isomorphism (0.2.5), we then adapt the strategy of [BeR], itself inspired by arguments from [KK] and [AJS]. The arguments are algebraic in nature.

0.2.11 Monoidality

Once we have obtained our monodromic version of the Endomorphismensatz, we need to describe the essential images of the functors $\widehat{\mathbb{V}}_t^{\circ}$ and $\mathbb{V}_t^{\circ}$. As was rapidly evoked above, we are able to obtain a "Bott–Samelson" description of (pro)-tilting block subcategories (recall that the convolution of two pro-tilting object is still a tilting object). In particular, in the completed case, the neutral block tilting subcategory is a monoidal category, and our description can be stated as an equivalence of categories (see corollary 9.9.2)

$$\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} = \langle \widehat{\mathscr{T}}_{s,t} \mid s \in S_t \rangle_{\widehat{\star},\oplus,\overset{\oplus}{\subseteq}}; \tag{0.2.6}$$

(recall that $\widehat{\star}$ denotes the extension of the convolution functor to the completed category). The Luztig–Yun case is deduced from the above equivalence, using the functor $\pi_{\dagger}^{t}$; we may thus once again focus on the completed category.

From (0.2.5), we deduce an isomorphism

$$\widehat{\mathbb{V}}_t^{\circ}(\widehat{\mathscr{T}}_{s,t}) \cong \widehat{R}_t \otimes_{(\widehat{R}_t)^s} \widehat{R}_t$$

for any $s \in S_t$ (where $(\widehat{R}_t)^s$ denotes the elements of $\widehat{R}_t$ invariant under the action of $s \in W_t^{\circ}$; see lemma 10.4.9). Combining this isomorphism and (0.2.6), one sees that it is possible to describe the essential image of $\widehat{\mathbb{V}}_t^{\circ}$ if one knows that our functor $\widehat{\mathbb{V}}_t^{\circ}$ respects the monoidal structure on both $(\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]},\widehat{\star})$ and $(\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^{\circ}}} \widehat{R}_t),\otimes_{\widehat{R}_t})$. Showing that there exists such a monoidal structure on $\widehat{\mathbb{V}}_t^{\circ}$ is the goal of Chapter 11 and Chapter 12.

Comparison results

Our proof will need a change of setting: we will consider schemes of finite type defined over an algebraically closed field $\mathbb{F}$ of positive characteristic endowed with the étale topology. Let us explain why this will be useful. The quotient $\mathscr{X} = G/U$ can be defined already over $\mathbb{Z}$: by general results of [SGA3], there exist a split reductive group scheme $\mathbf{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ whose root datum is the root datum of G, and such that (the $\mathbb{C}$-points of) $\mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathbf{G}_{\mathbb{Z}}$ identify with

G. We can moreover choose a maximal torus $\mathbf{T}_{\mathbb{Z}}$ and a Borel subgroup $\mathbf{B}_{\mathbb{Z}}$ that yield, after base change to $\mathbb{C}$, our torus T and Borel B. If $\mathbf{U}_{\mathbb{Z}}$ denotes the unipotent radical of $\mathbf{B}_{\mathbb{Z}}$, then $\mathscr{X}_{\mathbb{Z}} := \mathbf{G}_{\mathbb{Z}}/\mathbf{U}_{\mathbb{Z}}$ gives a $\mathbb{Z}$-version of $\mathscr{X}$. Now, we can consider the base change of $\mathscr{X}_{\mathbb{Z}}$ to any ring; in particular, we obtain a $\mathbb{F}$-version of our variety $\mathscr{X}$. The point of these considerations is that there are tools available only over schemes over fields of positive characteristic, in our case, the primary tool will be given by a Whittaker-category, on which we will return below in this introduction. In any case, our proof requires a transition step "over $\mathbb{F}$". Our main interest, however, is not étale categories, but rather the categories introduced before, defined over complex varieties endowed with their classical, "analytic", topology. Chapter 11 provides results that allow one to relate these two different settings. The general idea is to follows arguments of [BBD, §6] and [SGA4]. This will require some (somehow tedious) adaptations, or rather translations, of the constructions of our first chapters in the different setting of schemes over more general rings, and may involve quite heavy notation; but once these preliminaries are done, our results essentially follow from the big general machinery of [BBD] and [SGA4]. The main result is the following commutative diagram of categories and functors, where the horizontal arrows are equivalences of categories:

$$\begin{array}{ccc} D^b_{(B)}(\mathscr{X},\mathbb{K})_{[-,t]} & \xleftarrow{\sim} & D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})_{[-,t]} \\ \downarrow{\scriptstyle \pi^t_\dagger} & & \downarrow{\scriptstyle \pi^{t,\mathrm{et}}_\dagger} \\ \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]} & \xleftarrow{\sim} & \mathfrak{D}^{\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}_t]}. \end{array} \tag{0.2.7}$$

In this diagram, $\mathbb{K}$ is a finite field, and the objects in the right column, whose precise definition is irrelevant here, give analogues of the various objects appearing in the left column; analogues defined on schemes over an algebraically closed field of positive characteristic $\mathbb{F}$, endowed with the étale topology.

Étale constructions

Now that we have the comparison result of chapter 11 at hand, we can start to effectively work in the étale setting; this is done in chapter 12. We consider the scheme $\mathscr{X}_{\mathbb{F}} := \mathbf{G}_{\mathbb{F}}/\mathbf{U}_{\mathbb{F}}$ (the notation are as in the previous paragraph), endowed with the étale topology. We will consider the derived category of étale sheaves on $\mathscr{X}_{\mathbb{F}}$. In fact, we need a last piece of preliminary: the derived category of étale sheaves is well-behaved only for finite coefficients, but the complex-analytic-categories we are ultimately interested in are categories with algebraically closed coefficients. To solve this problem, we will "force" algebraically closed coefficients, by mimicking the construction of étale ℓ-adic categories. Once this is done, we obtain a version of diagram (0.2.7) with now algebraically closed coefficients. Then again, the beautiful machinery of Yun's completed category can now be performed on both sides of the diagram, and

we ultimately obtain a equivalence of categories (see corollary 12.5.4)

$$\widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})^{\circ}_{[-,t]} \xleftrightarrow{\sim} \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^{\circ}_{[-,t]}.$$

One more time, the precise definition of the right hand side is not necessary at this point; let us just state that it is an étale analogue of the left hand side. Both of these categories are monoidal, and this equivalence preserves the monoidal structure on both sides. In particular, we obtain an étale version $\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}$ of $\widehat{\mathscr{T}}_{w_{t,\circ},t}$ and an isomorphism of algebras

$$\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \cong \mathrm{End}(\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}).$$

The monoidality of $\widehat{\mathbb{V}}^{\circ}_t$ is then essentially equivalent to the monoidality of $\widehat{\mathbb{V}}^{\circ,\mathrm{et}}_t := \mathrm{Hom}(\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}, -)$. The proof of the latter property will be a consequence of an explicit construction of the object $\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}$. More precisely, as we stated above, the main point for us of working over $\mathbb{F}$ is the possibility to define an Artin–Schreier sheaf, and a Whittaker-equivariant category associated to it. We formally obtain a Whittaker-completed category, and we have averaging functors relating our (étale) completed monodromic category and this Whittaker completed version. Adapting arguments from [BY, §4] to our monodromic setting, we then show that $\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}$ appears as a direct summand in a "double-averaging" of the (étale version) of the minimal pro-standard, i.e. the pro-standard associated to the element $e \in W$. With this key result at hand, the monoidality of $\widehat{\mathbb{V}}^{\circ,\mathrm{et}}_t$ follows from (monodromic counterparts of) arguments from [BY]. We finally deduce the monoidality of the functor

$$\widehat{\mathbb{V}}^{\circ}_t : (\widehat{T}_{(B)}(\mathscr{X}, \Bbbk)^{\circ}_{[t,t]}, \widehat{\star}) \to (\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t), \otimes_{\widehat{R}_t}).$$

0.2.12 Main theorems

Tilting categories

At this point, we can obtain the wished for descriptions of our tilting categories: we identify the tilting category to a (variation of the usual) category of Soergel bimodules. The Lusztig–Yun case more or less follows from the description of the completed tilting category (using notably the identifications (0.2.3) and (0.2.4)). This description is a rather immediate consequence of the results obtained in previous chapters, and goes as follows:

Theorem 0.2.1. *There is a monoidal equivalence of categories*

$$\widehat{T}_{(B)}(\mathscr{X}, \Bbbk)^{\circ}_{[t,t]} \xrightarrow{\sim} \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t),$$

where $\mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t)$ *is the full subcategory of the category of finitely generated left* $\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t$*-modules generated under tensor product, direct sums*

and direct summand by the objects $\widehat{R}_t$ *and* $\widehat{R}_t \otimes_{(\widehat{R}_t)^s} \widehat{R}_t$ *for* $s \in S_t$ *(where* $(\widehat{R}_t)^s$ *denotes the* s*-invariants).*

There exists an equivalence of categories

$$\mathfrak{T}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)_+^{W_t^\circ}),$$

where $\mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)_+^{W_t^\circ})$ *is the full subcategory of the category of finitely generated left* $\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \Bbbk$*-modules generated under direct sums, direct summands and successive application of the functors* $\widehat{R}_t \otimes_{(\widehat{R}_t)^s} (-)$ *to the object* $\Bbbk$*, for* $s \in S_t$*.*

Once we have this result at hand, we can formally derive a description of the category $\mathfrak{P}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}_t^T]}$ as the category of left modules over the endomorphism algebra of an explicitly constructed projective generator of this category. Variants of these results still hold for any block.

Relation with the endoscopic group

Following one more time ideas from [LY], we can go further in our description of the neutral block. Between all the blocks, the neutral one W_t° has the particularity of admitting a group structure, in fact, we saw that W_t° identifies naturally with the Weyl group of the endoscopic group H_t. Moreover, the behavior of standard and tilting objects in $\mathfrak{P}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}_t^T]}$ suggests that this category is "governed by the group W_t°", in particular, it is reasonnable to hope that this category is somehow related to the category of Bruhat-constructible perverse sheaves on the flag variety of H_t. This is indeed the case: an exact analogue of theorem 0.2.1 holds for H_t, replacing the element t by the neutral element 1 and the local system $\mathscr{L}_t^T$ by the constant local sheaf $\underline{\Bbbk}_T$. The categories we obtain in this case are those studied in [BeR], in particular, the Lusztig–Yun equivariant category on the basic affine space of H_t is canonically equivalent to the usual derived category $D^b_{(B_t)}(H_t/B_t, \Bbbk)$ of Bruhat constructible $\Bbbk$-sheaves on a flag variety of H_t.

Then, basic homological algebra arguments allow us to obtain a perverse t-exact equivalence

$$\mathfrak{D}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}_t^T]} \xrightarrow{\sim} D^b_{(B_t)}(H_t/B_t, \Bbbk)$$

preserving standard, costandard, simple and tilting objects. This amounts to saying that, indeed, for an element $t \in T^\vee_\Bbbk$, the neutral block in the right Lusztig–Yun equivariant monodromic category is controlled by the Coxeter system (W_t°, S_t).

Description of maximal IC-sheaves

As a corollary of our preceding results, we obtain a description of monodromic IC-sheaves associated to elements w maximal in the block of ${}_{w(t)}\underline{W}_t$ they belong

to. In the case $w = w_{t,\circ}$ this description is as follows. Recall our local system $\mathscr{L}_t^T$ on T. For any $w \in W$, we derive local systems $\mathscr{L}_t^w$ on the strata $Bw_B/U := \mathscr{X}_w \subseteq \mathscr{X}$ associated to w. Set $\mathrm{IC}_{w_{t,\circ},t} := \mathrm{IC}(\mathscr{X}_{w_{t,\circ}}, \mathscr{L}_t^{w_{t,\circ}})$. Finally, recall that ℓ_t denotes the length function of the Coxeter system (W_t°, S_t). Then, for any $w \in W_t^\circ$, we have (see corollary 13.3.1)

$$j_w^* \mathrm{IC}_{w_{t,\circ},t} \cong \mathscr{L}_t^w[\dim(\mathscr{X}_w) + \ell_t(w_{t,\circ}) - \ell_t(w)],$$

$$j_w^! \mathrm{IC}_{w_{t,\circ},t} \cong \mathscr{L}_t^w[\dim(\mathscr{X}_w) - \ell_t(w_{t,\circ}) + \ell_t(w)].$$

0.3 Future Directions

Koszul duality The notion of Koszul duality was studied in [BGS], where it was described as an auto equivalence of the derived category of (a "graded version" of) the principal block of a semisimple complex Lie algebra. This result was later generalised by R. Bezrukavnikov and Z. Yun in [BY]. There, the authors worked with varieties over algebraic closure of finite fields and characteristic zero coefficients and constructed an equivalence of categories $D^b_{B,m}(G/B, \overline{\mathbb{Q}}_\ell) \cong \widehat{D}^b_{U^\vee,m}(G^\vee/U^\vee, \overline{\mathbb{Q}}_\ell)_{[1,1]}$, relating left B-equivariant mixed complexes on the flag variety of G to the completion along unipotent monodromy of $U^\vee$-equivariant mixed complexes on the basic affine space for the dual group (we will roughly explain what the term "mixed" means below). Moreover, the latter equivalence switches semisimple complexes and pro-tilting objects. Here, the term "mixed" refers to a feature due to Deligne, and requires to work on varieties over fields of positive characteristic, on which one can consider the Frobenius automorphism. The mixed condition is then given by technical assumptions on the eigenvalues of the Frobenius map on the stalks of a complex. The point is that with this framework, these categories are graded, in the sense that there exists on both sides an auto equivalence of categories (1), called Tate twist (coming from the Frobenius map); these auto equivalences actually reveal the hidden duality pattern stated above. One can rephrase this result with notation similar to those used above. We will consider the $\mathfrak{D}^m_{[\underline{\Bbbk},\underline{\Bbbk}]}$, where the left "$\underline{\Bbbk}$" means that we impose an equivariance condition on the left as well (before, we only considered a *right* Lusztig–Yun equivariance condition). The equivalence of [BY] is then a monoidal equivalence of categories

$$\mathfrak{D}^m_{[\underline{\Bbbk},\underline{\Bbbk}]} \xrightarrow{\sim} \widehat{D}^m_{[1,1]} \tag{0.3.1}$$

inducing an equivalence between the subcategory of mixed semisimple $(\underline{\Bbbk}_T, \underline{\Bbbk}_T)$-bi-equivariant complexes and the subcategory of mixed perverse tilting sheaves. In this setting (characteristic zero coefficients with schemes over positive characteristic fields), a huge step towards an analogous result with arbitrary monodromy was obtained in [LY].

One now wonders if such a duality may be obtained in our setting of complex varieties and coefficients of positive characteristic, and for general monodromy. In this case, it is known that the correct replacement for semisimple complexes

is given by the parity complexes of [JMW]. In fact, such an equivalence had already been obtained in the case of positive characteristic coefficients in [AR2], and even generalised (with a completely different proof) to the case of Kac–Moody groups in [AMRW], though only in the case of unipotent monodromy (that is, the exact analogue of (0.3.1)). The main problem here is that Deligne's theory of weights does not make sense anymore, so that we do not have clear auto-equivalences analogue to Tate twist; in other words, we do not have a mixed structure on our categories. This is an issue because parity complexes form a *graded* category, whereas tilting objects do not have any graded structure. Thus the two sides of the picture have different structure, and finding an equivalence is then hopeless. This difficulty was overcome in [AR2] where a new, more elementary, notion of mixed categories was introduced. This works well in the parity side, but it is not clear how to deal with the pro-tilting side. One may hope for an equivalence of monoidal categories of the form

$$(\widehat{T}^m_{[t',t]}, \widehat{\star}) \xrightarrow{\sim} (\mathrm{Parity}\mathfrak{D}^m_{[\mathscr{L}'_t, \mathscr{L}_t]}, \star)$$

for $t', t \in T^\vee_{\Bbbk}$. A first step in this direction would be to actually define the categories above: a mixed version of pro-tilting monodromic perverse sheaves (once again, this is already done in the case of unipotent monodromy), and Lusztig–Yun equivariant parity sheaves. Before that, one can hope to get a weaker result, namely obtain a degrading functor from the mixed parity side to the tilting category, as in [AR1]. These results would be an encouraging debut for proving the existence of a more general Koszul duality phenomenon.

Part I

Monodromy

Chapter 1

The topological monodromic category

In this chapter, we recall the monodromy constructions made by Verdier in [Ve], as well as some properties of the monodromy action.

1.1 Preliminaries

In all this document, unless stated otherwise, we will assume that $\Bbbk$ is the algebraic closure of a finite field. We let $\ell > 0$ denote the characteristic of $\Bbbk$.

We consider A a complex algebraic torus of rank $r \geq 1$. We will use several times the fact that we can choose a trivialisation $A \cong (\mathbb{C}^*)^r$ so we fix such a trivialization once and for all.

For any topological space Y, denote by $D^b(Y, \Bbbk)$ the bounded derived category of sheaves of $\Bbbk$-vector spaces on Y. Also, let $\mathrm{Loc}(Y, \Bbbk)$ be the abelian category of finite dimensional $\Bbbk$-local systems on Y (i.e. $\Bbbk$-local systems on Y with finite dimensional stalks).

For a morphism f between two algebraic varieties, the associated functors of pullback and (proper) pushforward will almost always be understood as derived functors. Thus, we will write f_* instead of Rf_* (and similarly for $f_!$ and f^*). In some places, we will have to consider usual, non-derived pushforward; the notation will then be f_*° and $f_!^\circ$ for the non-derived version.

Lemma 1.1.1. *Let Y be a connected, locally path-connected and locally simply connected topological space.*

Take $\mathscr{M}$, $\mathscr{N}$ in $\mathrm{Loc}(Y, \Bbbk)$. Then, the natural morphism

$$\mathrm{Ext}^1_{\mathrm{Loc}(Y,\Bbbk)}(\mathscr{M}, \mathscr{N}) \to \mathrm{Ext}^1_{D^b(Y,\Bbbk)}(\mathscr{M}, \mathscr{N})$$

is an isomorphism.

Proof. Recall that the left-hand side is defined to be the Hom-space in the derived category $D^b(Y, \Bbbk)$ between $\mathscr{M}$ and $\mathscr{N}[1]$. Denote by $\mathcal{C}$ the heart of the

natural t-structure on $D^b(Y,\Bbbk)$ ($\mathcal{C}$ is an abelian category). It is a known fact that $\mathcal{C}$ is equivalent to the category $\mathrm{Sh}(Y,\Bbbk)$ of sheaves of $\Bbbk$-modules on Y. It follows from general theory of t-structure (see for example [BBD, Remark 3.1.17 (ii)]) that the natural morphism

$$\cong \mathrm{Ext}^1_{\mathcal{C}}(\mathscr{M},\mathscr{N}) \to \mathrm{Ext}^1_{D^b(Y,\Bbbk)}(\mathscr{M},\mathscr{N})$$

is an isomorphism.

For $\mathscr{M},\mathscr{N}$ two local systems on Y, the inclusion $\mathrm{Loc}(Y,\Bbbk) \hookrightarrow \mathrm{Sh}(Y,\Bbbk)$ induces a map

$$\mathrm{Ext}^1_{\mathrm{Loc}(Y,\Bbbk)}(\mathscr{M},\mathscr{N}) \to \mathrm{Ext}^1_{\mathrm{Sh}(Y,\Bbbk)}(\mathscr{M},\mathscr{N}) \cong \mathrm{Ext}^1_{D^b(Y,\Bbbk)}(\mathscr{M},\mathscr{N}).$$

This map is injective: if an exact sequence of local systems splits in $\mathrm{Sh}(Y,\Bbbk)$, then it splits already in $\mathrm{Loc}(Y,\Bbbk)$. For the surjectivity, see [Ac, Lemma 1.8.6]. □

Lemma 1.1.2. *Let V be a finite dimensional $\Bbbk$-vector space. Then any $\phi \in GL_{\Bbbk}(V)$ is of finite order.*

Proof. Fixing a basis in V, we obtain an isomorphism $GL_{\Bbbk}(V) \cong GL_{\dim(V)}(\Bbbk)$. Now, $\Bbbk$ is the union of its finite subfields. This implies that $GL_{\dim(V)}(\Bbbk)$ is the union of its finite subgroups. The result follows. □

We recall below a well-known theorem that we will use many times. A space X is said semi-locally simply connected if each point $x \in X$ has a neighborhood U_x such that any loop in U_x is homotopic to a point when viewed as a path in X. Any complex algebraic variety is semi-locally simply connected, in fact it is even locally contractible (see for instance [Hi] and [Hat, Proposition A.4]).

Theorem 1.1.3. *Let Z be a connected, locally path connected, semi-locally simply connected topological space. Let $z_0 \in Z$ be any base-point. There is a canonical equivalence of categories*

$$\mathrm{Loc}(Z,\Bbbk) \cong \mathrm{Mod}^{\mathrm{fd}}(\Bbbk[\pi_1(Z,z_0)])$$

where the right-hand side denotes the category of finite-dimensional $\Bbbk$-representations of $\pi_1(Z,z_0)$. Consider (Y,y_0) another connected, locally path connected, semi-locally simply connected pointed topological space and $f:(Z,z_0)\to(Y,y_0)$ a continuous map of pointed spaces. The map f induces a group morphism

$$\pi_1(f):\pi_1(Z,z_0)\to\pi_1(Y,y_0)$$

and the functor $f^*:\mathrm{Loc}(Y,\Bbbk)\to\mathrm{Loc}(Z,\Bbbk)$ corresponds to the restriction of scalars along this map at the level of representations: we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Loc}(Y,\Bbbk) & \xrightarrow{\quad f^*\quad} & \mathrm{Loc}(Z,\Bbbk) \\ \wr\downarrow & & \wr\downarrow \\ \mathrm{Mod}^{\mathrm{fd}}(\Bbbk[\pi_1(Y,y_0)]) & \xrightarrow[\mathrm{Res}_{\pi_1(f)}]{} & \mathrm{Mod}^{\mathrm{fd}}(\Bbbk[\pi_1(Z,z_0)]). \end{array}$$

This theorem allows us to prove the following useful lemma. In this paper, unless stated otherwise, the fundamental group of any (topological) group will be considered with 1 as base point.

Lemma 1.1.4. *Let $\mathscr{L}$ be a $\Bbbk$-local system on A. Then there exists an integer n such that $e_n^*(\mathscr{L})$ is a constant sheaf.*

Proof. Thanks to theorem 1.1.3, the local system $\mathscr{L}$ corresponds to a $\Bbbk$-representation V of $\pi_1(A) \cong \mathbb{Z}^r$. This representation is entirely determined by the action of the elements $f_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^r$; this action is thus given by a family $\{g_i\}_{i=1,\dots,r}$ of commuting elements in $GL_{\Bbbk}(V)$. According to lemma 1.1.2, the g_i's are all of finite order; let n the least common multiple of these orders. For any $\Bbbk$-representation W of $\mathbb{Z}^r$ (equivalently, for any $\Bbbk[x_i, x_i^{-1} \mid i = 1, \dots, r]$-module) denote by W_n the representation defined in the following way: as $\Bbbk$-vector spaces, $W = W_n$, and the action " $\cdot^n$" is given by

$$x_i \cdot^n w = x_i^n \cdot w$$

for any i. The functor $e_n^* : \mathrm{Loc}(A, \Bbbk) \to \mathrm{Loc}(A, \Bbbk)$ corresponds to the functor $\mathrm{Mod}^{\mathrm{fd}}(\Bbbk[\pi_1(Y, y_0)]) \to \mathrm{Mod}^{\mathrm{fd}}(\Bbbk[\pi_1(Y, y_0)])$, $W \mapsto W_n$ via the equivalence of theorem 1.1.3. Thus the action of f_i on V_n is given by the matrix $g_i^n = 1 \in GL_{\Bbbk}(V)$. We get that V_n is trivial and so $e_n^*(\mathscr{L})$ is a constant local system. □

In this document, we will mostly be interested in several derived categories of constructible sheaves: let X be a complex algebraic variety endowed with an (algebraic) action of A. We fix a finite algebraic stratification $\mathcal{S}$ (see [CG, Definition 3.2.23]) such that each $S \in \mathcal{S}$ is A-stable.

Let $D^b_{\mathcal{S}}(X, \Bbbk)$ denote the $\mathcal{S}$-constructible bounded derived category of sheaves of $\Bbbk$-vector spaces on X.

We denote by $\mathrm{pr}_2 : A \times X \to X$ the projection and $a : A \times X \to X$ the action morphism. Let $e_n : A \to A$ be the morphism $t \mapsto t^n$. Finally, denote by $a(n)$ the composition $a \circ (e_n \times \mathrm{id})$.

Remark 1.1.5. Take $\mathscr{F}$ in $D^b_{\mathcal{S}}(X, \Bbbk)$. By definition, the restriction to a stratum S of any cohomology object of $\mathscr{F}$ is locally constant. By assumption, each A-orbit is contained in one of the strata ; each cohomology object of $\mathscr{F}$ is then locally constant on the A-orbits. This remark is in particular valid for a constructible sheaf.

Lemma 1.1.6. *Let $\mathscr{F}$ be an $\mathcal{S}$-constructible sheaf on X. There exists an integer n such that $a(n)^*(\mathscr{F})$ is constant on each fiber of* pr_2.

Proof. Since $\mathcal{S}$ is finite, it suffices to show the lemma for $\mathscr{F}_{|S}$ for any $S \in \mathcal{S}$. We can therefore assume that $\mathscr{F}$ is a local system on X. Then $a^*(\mathscr{F})$ is a local system on $A \times X$ and thus corresponds to a $\Bbbk$-representation V of $\pi_1(A \times X, (1, x_0)) \cong \mathbb{Z}^r \times \pi_1(X, x_0)$ (for any base-point $x_0 \in X$). Thanks to theorem 1.1.3, the sheaf $a(n)^*(\mathscr{F})$ corresponds under the equivalence of theorem 1.1.3 to the representation $\widetilde{V}$ of $\mathbb{Z}^r \times \pi_1(X, x_0)$, with $\widetilde{V} = V$ as $\Bbbk$-vector space

and $(m, [\alpha])$ acting as $(nm, [\alpha])$. If we view $\widetilde{V}$ as a representation of $\mathbb{Z}^r$ (via the canonical injection $\mathbb{Z}^r \hookrightarrow \mathbb{Z}^r \times \pi_1(X, x_0)$), we get the restriction of scalars of the representation associated to $\mathscr{F}$ along the group morphism $\pi_1(A) \to \pi_1(A) \times \pi_1(x_0, X)$ induced by the map

$$A \to A \times \{x_0\} \hookrightarrow A \times X,\ t \longmapsto (t^n, x_0).$$

We deduce from lemma 1.1.4 that for n sufficiently divisible, this corresponds to the trivial representation of $\mathbb{Z}^r$; the morphism defining the representation $\widetilde{V}$ of $\mathbb{Z}^r \times \pi_1(X, x_0)$ hence factors through the projection to $\pi_1(X, x_0)$. This says exactly that $a(n)^*(\mathscr{F})$ is of the form $\mathrm{pr}_2^*(\mathscr{G})$ for $\mathscr{G}$ a local system on X. □

Lemma 1.1.7. *Let $\mathscr{F}$ be a constructible sheaf on $A \times X$ that is constant on each fiber of pr_2. Then there exists a sheaf $\mathscr{G}$ on X such that*

$$\mathscr{F} \cong \mathrm{pr}_2^*(\mathscr{G}).$$

Proof. Consider the object $\mathrm{pr}_{2!}(\mathscr{F})$. For any $x \in X$, we have an isomorphism $\mathrm{pr}_{2!}(\mathscr{F})_x \cong R\Gamma_c^\bullet(\mathscr{F}_{|A\times\{x\}})$ in the bounded derived category of $\Bbbk$-vector spaces. Since $\mathscr{F}_{|A\times\{x\}}$ is constant by hypothesis, this cohomology lives in degrees $r, \ldots, 2r$. We consider the non-zero truncation morphism $\mathrm{pr}_{2!}(\mathscr{F}) \to \mathscr{H}^{2r}(\mathrm{pr}_{2!}(\mathscr{F}))[-2r]$. We use adjunction to obtain a (nonzero) morphism

$$\mathscr{F} \longrightarrow \mathrm{pr}_2^!\, \mathscr{H}^{2r}(\mathrm{pr}_{2!}(\mathscr{F}))[-2r] \cong \mathrm{pr}_2^*\, \mathscr{H}^{2r}(\mathrm{pr}_{2!}(\mathscr{F})). \tag{1.1.1}$$

(We used here the fact that pr_2 is a smooth morphism, so $\mathrm{pr}_2^! \cong \mathrm{pr}_2^*[2r]$.)

We check that this is an isomorphism by looking at the stalks. For a point $(z, x) \in A \times X$, we have a morphism

$$\mathscr{F}_{(z,x)} \to \mathscr{H}^{2r}(\mathrm{pr}_{2!}(\mathscr{F}))_x \cong \mathbf{H}_c^{2r}(A \times \{x\}, \mathscr{F}). \tag{1.1.2}$$

We can then assume that X is a one-point space and hence that $\mathscr{F}$ is a constant sheaf. It therefore suffices to check the isomorphism for $\mathscr{F}$ constant of rank 1; in this case, both sides are just $\Bbbk$ (thanks to Künneth's formula applied to $A \cong (\mathbb{C}^*)^r$). As our morphism (1.1.1) is nonzero, there exists a point (z, x) such that (1.1.2) is nonzero, and thus an isomorphism. Since $\mathscr{F}$ is assumed to be constant, the (global) morphism (1.1.1) is nonzero, whence an isomorphism. □

1.2 Verdier's proposition

Proposition 1.2.1. *Let $\mathscr{F}$ be an object in $D^b_{\mathcal{S}}(X, \Bbbk)$. There exists $n \in \mathbb{Z}_{>0}$ and a morphism*

$$\iota(n) : \mathrm{pr}_2^*(\mathscr{F}) \longrightarrow a(n)^*(\mathscr{F})$$

such that $\iota_{|\{1\}\times X}$ identifies with the identity of $\mathscr{F}$. Any such morphism is an isomorphism. If n_1, n_2 are two strictly positive integers and if

$$\iota(n_1) : \mathrm{pr}_2^*(\mathscr{F}) \longrightarrow a(n_1)^*(\mathscr{F}), \quad \iota(n_2) : \mathrm{pr}_2^*(\mathscr{F}) \longrightarrow a(n_2)^*(\mathscr{F})$$

are two such morphisms, then there exists a strictly positive integer n_3, multiple of both n_1 and n_2, such that

$$(e_{\frac{n_3}{n_1}} \times \mathrm{id})^*(\iota(n_1)) = (e_{\frac{n_3}{n_2}} \times \mathrm{id})^*(\iota(n_2)).$$

Finally, let $\mathscr{G}$ be another complex in $D^b_{\mathcal{S}}(X, \Bbbk)$ and $u : \mathscr{F} \to \mathscr{G}$ be any morphism. There exists $n \in \mathbb{Z}_{>0}$ and two morphisms $\iota_1(n) : \mathrm{pr}_2^*(\mathscr{F}) \to a(n)^*(\mathscr{F})$ and $\iota_2(n) : \mathrm{pr}_2^*(\mathscr{G}) \longrightarrow a(n)^*(\mathscr{G})$ as above such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{pr}_2^*(\mathscr{F}) & \xrightarrow[\sim]{\iota_1(n)} & a(n)^*(\mathscr{F}) \\ {\scriptstyle \mathrm{pr}_2^*(u)}\downarrow & & \downarrow{\scriptstyle a(n)^*(u)} \\ \mathrm{pr}_2^*(\mathscr{G}) & \xrightarrow[\sim]{\iota_2(n)} & a(n)^*(\mathscr{G}). \end{array} \tag{1.2.1}$$

Before giving the proof of the proposition, we need an preliminary result. We have

$$(e_n \times \mathrm{id})^* \,\mathrm{pr}_2^* = \mathrm{pr}_2^*,$$

so the inverse image functor of $(e_n \times \mathrm{id})$ induces an endomorphism

$$\rho_n : \mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X,\Bbbk)}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})) \longrightarrow \mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X,\Bbbk)}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})).$$

(Here, with a slight abuse of notation, we denote again by $\mathcal{S}$ the stratification of $A \times X$ whose strata are the $A \times S$ for $S \in \mathcal{S}$.) We clearly have $\rho_n \circ \rho_m = \rho_{nm}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}, \mathscr{G}) & & \\ {\scriptstyle \mathrm{pr}_2^*}\downarrow & \searrow^{\mathrm{pr}_2^*} & \\ \mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X,\Bbbk)}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})) & \xrightarrow[\rho_n]{} & \mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X,\Bbbk)}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})). \end{array}$$

Lemma 1.2.2. *For any $f \in \mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X,\Bbbk)}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G}))$, there exists an integer $n \gg 0$ such that $\rho_n(f) := e_n^* f = \mathrm{pr}_2^* g$ for some $g \in \mathrm{Hom}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}, \mathscr{G})$. Moreover, such a g is unique.*

Proof. Let us fix a f as in the statement. We begin by showing the existence of g. First, note that pr_2 is a smooth morphism since A is smooth. Thus we have $\mathrm{pr}_2^! \cong \mathrm{pr}_2^*[2\dim(A)] = \mathrm{pr}_2^*[2r]$. Now, thanks to [KS1, (2.6.4)], we can write

$$R\,\mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})) = R\Gamma(A \times X, R\,\mathcal{H}om(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})).$$

Using [KS1, Proposition 3.1.13], we get the following isomorphisms in the category $D^b_{\mathcal{S}}(A \times X, \Bbbk)$:

$$\begin{aligned} \mathrm{pr}_2^* \left(R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})\right) &\cong \mathrm{pr}_2^! \left(R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})\right)[-2r] \\ &\cong R\,\mathcal{H}om(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^!(\mathscr{G}))[-2r] \\ &\cong R\,\mathcal{H}om\left(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})[2r]\right)[-2r] \\ &\cong R\,\mathcal{H}om(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})). \end{aligned}$$

Using Künneth's formula, we obtain a chain of isomorphisms, in the derived category $D^b \operatorname{Vect}_{\Bbbk}$ of $\Bbbk$-vector spaces:

$$\begin{aligned} R\operatorname{Hom}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})) &\cong R\Gamma(A \times X, R\,\mathcal{H}om(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})) \\ &\cong R\Gamma(A \times X, \mathrm{pr}_2^*\, R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})) \\ &\cong R\Gamma(A \times X, \Bbbk \boxtimes R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})) \\ &\cong R\Gamma(A, \Bbbk) \otimes^L_{\Bbbk} R\operatorname{Hom}(\mathscr{F}, \mathscr{G}). \end{aligned}$$

Since we work over a field $\Bbbk$, we know that the tensor product functor over $\Bbbk$ is an exact functor, so taking 0-th cohomology on both sides, we get an isomorphism of vector spaces

$$\operatorname{Hom}_{D^b_{\mathcal{S}}(A\times X,\Bbbk)}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{G})) \cong \bigoplus_{i\in\mathbb{Z}} \mathbf{H}^i(A, \Bbbk) \otimes_{\Bbbk} \operatorname{Hom}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}, \mathscr{G}[-i]) \tag{1.2.2}$$

where $\mathbf{H}^i(A, \Bbbk)$ is the i-th cohomology of A. Under these isomorphisms, the morphism ρ_n identifies with $\rho_n(\Bbbk) \otimes \mathrm{id}$, where $\rho_n(\Bbbk)$ denote the morphism $\mathbf{H}^\bullet(A, \Bbbk) \to \mathbf{H}^\bullet(A, \Bbbk)$ induced by e_n^*.

Assume that A is of rank 1, i.e. that $A \cong \mathbb{G}_m = \mathbb{C}^*$. Recall that the cohomology of $\mathbb{G}_m$ is non-zero only in degree 0 and in degree 1.

We know that $\mathbf{H}^\bullet(\mathbb{G}_m, \Bbbk) = \operatorname{Ext}^\bullet_{D^b(\mathbb{G}_{\mathrm{m}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{G}_m}, \underline{\Bbbk}_{\mathbb{G}_m})$. Using lemma 1.1.1 and theorem 1.1.3, we have

$$\begin{aligned} \mathbf{H}^\bullet(\mathbb{G}_m, \Bbbk) &= \mathbf{H}^0(\mathbb{G}_{\mathrm{m}}, \Bbbk) \oplus \mathbf{H}^1(\mathbb{G}_{\mathrm{m}}, \Bbbk) \\ &\cong \operatorname{Hom}_{D^b(\mathbb{G}_{\mathrm{m}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{G}_m}, \underline{\Bbbk}_{\mathbb{G}_m}) \oplus \operatorname{Ext}^1_{D^b(\mathbb{G}_{\mathrm{m}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{G}_m}, \underline{\Bbbk}_{\mathbb{G}_m}) \\ &= \operatorname{Hom}_{\mathrm{Loc}(\mathbb{G}_m,\Bbbk)}(\underline{\Bbbk}_{\mathbb{G}_m}, \underline{\Bbbk}_{\mathbb{G}_m}) \oplus \operatorname{Ext}^1_{\mathrm{Loc}(\mathbb{G}_m,\Bbbk)}(\underline{\Bbbk}_{\mathbb{G}_m}, \underline{\Bbbk}_{\mathbb{G}_m}) \\ &\cong \operatorname{Hom}_{\mathrm{Rep}(\mathbb{Z},\Bbbk)}(\Bbbk_{\mathrm{triv}}, \Bbbk_{\mathrm{triv}}) \oplus \operatorname{Ext}^1_{\mathrm{Rep}(\mathbb{Z},\Bbbk)}(\Bbbk_{\mathrm{triv}}, \Bbbk_{\mathrm{triv}}). \end{aligned}$$

We can thus work in the category of $\Bbbk$-representations of $\mathbb{Z}$. We determine the action of $\rho_n(\Bbbk)$ on each of these summands.

It is obvious that $\Bbbk_{\mathrm{triv},n} = \Bbbk_{\mathrm{triv}}$ (see the proof of the lemma 1.1.4 for the notation), so the action of $\rho_n(\Bbbk)$ on $\operatorname{Ext}^0_{\mathrm{Rep}(\mathbb{Z},\Bbbk)}(\Bbbk_{\mathrm{triv}}, \Bbbk_{\mathrm{triv}}) = \operatorname{Hom}_{\mathrm{Rep}(\mathbb{Z},\Bbbk)}(\Bbbk_{\mathrm{triv}}, \Bbbk_{\mathrm{triv}})$ is simply the identity. Now, we know that the space $\operatorname{Ext}^1_{\mathrm{Rep}(\mathbb{Z},\Bbbk)}(\Bbbk_{\mathrm{triv}}, \Bbbk_{\mathrm{triv}})$ is one-dimensional; an isomorphism $\Bbbk \longrightarrow \operatorname{Ext}^1_{\mathrm{Rep}(\mathbb{Z},\Bbbk)}(\Bbbk_{\mathrm{triv}}, \Bbbk_{\mathrm{triv}})$ is given by

$$x \longmapsto V(x)$$

where $V(x) \cong \Bbbk \oplus \Bbbk$ as a vector space and the action of $1 \in \mathbb{Z}$ is given by the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. The action of $1 \in \mathbb{Z}$ on $V(x)_n$ is then given by the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nx \\ 0 & 1 \end{pmatrix}$. The endomorphism of $\Bbbk$ induced by $\rho_n(\Bbbk)$ is thus multiplication by n. We have determined the action of $\rho_n(\Bbbk)$ on each of the summands of $\mathbf{H}^\bullet(\mathbb{G}_{\mathrm{m}}, \Bbbk)$; since $\Bbbk$ is a finite field of characteristic ℓ, multiplication by n is zero as soon as ℓ divides n. Thus for $n >> 0$, the map $\rho_n(\Bbbk)$ kills all but the

zero-th degree component of $\mathbf{H}^\bullet(A,\Bbbk)$. In view of (1.2.2), this readily implies the existence of g as in the statement.

We now deal with the general case (still for the existence): we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad e_n \quad} & A \\ \downarrow\wr & & \downarrow\wr \\ (\mathbb{C}^*)^r & \xrightarrow[(e_n\times\cdots\times e_n)]{} & (\mathbb{C}^*)^r \end{array} \tag{1.2.3}$$

(we made a little abuse of notation, denoting by the same symbol "e_n" the maps $A \to A$ and $\mathbb{C}^* \to \mathbb{C}^*$ sending an element on its n-th power). Using Künneth's formula one more time, we get

$$\mathbf{H}^m(A,\Bbbk) \cong \bigoplus_{p_1+\cdots+p_r=m} \left(\bigotimes_{i=1}^{r} \mathbf{H}^{p_i}(\mathbb{C}^*,\Bbbk) \right). \tag{1.2.4}$$

Thanks to the diagram (1.2.3), the induced map $\rho_n(\Bbbk) : \mathbf{H}^n(A,\Bbbk) \to \mathbf{H}^n(A,\Bbbk)$ decomposes in a direct sum of tensor products of maps, each of which corresponding to the map already studied in the case $A = \mathbb{G}_{\mathrm{m}}$. Thanks to (1.2.4), the cohomology of A is in degrees $0,\dots,r$. From the case $A = \mathbb{G}_{\mathrm{m}}$, we deduce immediately that the induced map $\rho_n(\Bbbk)$ is the identity on $\mathbf{H}^0(A,\Bbbk) = \Bbbk$. Now for $\mathbf{H}^m(A,\Bbbk)$ with $m \geq 1$, there is in each summand at least one $p_i = 1$. On this summand, the map induced by $\rho_n(\Bbbk)$ is zero if n is divisible by ℓ. We can conclude as above.

We finally show the unicity: assume that we have two integers $n_1, n_2 \gg 0$ and two morphisms $g_1, g_2 : \mathscr{F} \to \mathscr{G}$ such that $\rho_{n_i}(f) = \mathrm{pr}_2^*\, g_i$.

Let $\{i,j\} = \{1,2\}$. Since $\rho_n(\mathrm{pr}_2^*\, g) = \mathrm{pr}_2^*\, g$ for any n and g, pulling back the equality $\rho_{n_i}(f) = \mathrm{pr}_2^*\, g_i$ along e_{n_j} for $i = 1, 2$ leads to

$$\mathrm{pr}_2^*\, g_1 = \mathrm{pr}_2^*\, g_2.$$

Restricting to $\{1\} \times X$, we find $g_1 = g_2$ and the proof is complete. □

1.3 Proof of Verdier's proposition

Proof of proposition 1.2.1. Let $\mathscr{F} \in D^b_{\mathcal{S}}(X,\Bbbk)$. We prove the existence of an isomorphism between $a(n)^*(\mathscr{F})$ and an object of the form $\mathrm{pr}_2^*(\mathscr{Q})$ with $\mathscr{Q} \in D^b_{\mathcal{S}}(X,\Bbbk)$ (for some $n \in \mathbb{Z}_{>0}$) by induction on the minimal length of an interval containing $\{i \in \mathbb{Z} \mid \mathscr{H}^i(\mathscr{F}) \neq 0\}$.

Assume that $\mathscr{F}$ has a unique non-zero cohomology object. We can use lemma 1.1.6 and lemma 1.1.7. We get an integer n such that $a(n)^*(\mathscr{F})$ is constant on the fibers of pr_2. Since $a(n)^*$ is an exact functor, we obtain an isomorphism $a(n)^*(\mathscr{F}) \cong \mathrm{pr}_2^*(\mathscr{Q})$ for a certain sheaf $\mathscr{Q}$ on X. This settles this case. Now for

a general $\mathscr{F}$, denote by N the largest integer such that $\mathscr{H}^N(\mathscr{F}) \neq 0$. We have a truncation triangle

$$\mathscr{H}^N(\mathscr{F})[-N-1] \xrightarrow{f} \tau_{\leq N-1}\mathscr{F} \longrightarrow \mathscr{F} \xrightarrow{[1]} .$$

Denote $\mathscr{G} := \tau_{\leq N-1}\mathscr{F}$ and $\mathscr{K} := \mathscr{H}^N(\mathscr{F})[-N-1]$. Both $\mathscr{G}$ and $\mathscr{K}$ have a smaller amplitude than $\mathscr{F}$ so by induction, there exists an n such that $a(n)^*(\mathscr{G})$ and $a(n)^*(\mathscr{K})$ are of the form $\mathrm{pr}_2^*(\mathscr{V})$ and $\mathrm{pr}_2^*(\mathscr{U})$.

The morphism $a(n)^*(f)$ defines a morphism $\mathrm{pr}_2^*(\mathscr{U}) \to \mathrm{pr}_2^*(\mathscr{V})$, and thus, by lemma 1.2.2, an element in $\varinjlim \mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{U}), \mathrm{pr}_2^*(\mathscr{V}))$ (say in the copy of $\mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{U}), \mathrm{pr}_2^*(\mathscr{V}))$ indexed by 1 in the inductive limit). With lemma 1.2.2, replacing n by a multiple if necessary, the morphism $a(n)^*(f)$ is of the form $\mathrm{pr}_2^*(g)$ for some $g : \mathscr{U} \to \mathscr{V}$; hence $a(n)^*(\mathscr{F})$ identifies with the cone of $\mathrm{pr}_2^*(g)$ and is of the form $\mathrm{pr}_2^*(\mathscr{Q})$.

So far, we have obtained an isomorphism

$$\mathrm{pr}_2^*(\mathscr{Q}) \xrightarrow{\sim} a(n)^*(\mathscr{F})$$

for a certain integer n and an object $\mathscr{Q} \in D^b_{\mathcal{S}}(X, \Bbbk)$. To identify $\mathscr{Q}$, we can restrict to the subset $\{1\} \times X$ to see that $\mathscr{F} \cong \mathscr{Q}$. We have thus obtained an isomorphism

$$\iota(n) : \mathrm{pr}_2^*(\mathscr{F}) \xrightarrow{\sim} a(n)^*(\mathscr{F}).$$

It is also clear that $\iota(n)$ restricts to the identity on $\{1\} \times X$.

We now prove the second statement. We know that there exists an isomorphism $\iota(n) : \mathrm{pr}_2^*(\mathscr{F}) \to a(n)^*(\mathscr{F})$ whose restriction to $\{1\} \times X$ is the identity of $\mathscr{F}$. Assume that there exists an integer $n' > 0$ and a morphism $\iota : \mathrm{pr}_2^*(\mathscr{F}) \to a(n')^*(\mathscr{F})$ is such that $\iota_{|\{1\}\times X} = \mathrm{id}_{\mathscr{F}}$; we want to show that ι is an isomorphism. We start by noticing the following: to check that a morphism of (complexes of) sheaves is an isomorphism, it suffices to verify that it is so on the stalk at any point. Thus if g is a surjective map and $g^*\iota$ is an isomorphism, ι is an isomorphism itself in the first place. Now consider a integer m which is a common multiple of n and n'. The composition

$$(e_{\frac{m}{n'}} \times \mathrm{id})^*\iota \circ (e_{\frac{m}{n}} \times \mathrm{id})^*\iota(n)^{-1}$$

is an endomorphism of $\mathrm{pr}_2^*(\mathscr{F})$. It then defines an element in

$$\varinjlim \mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{F})).$$

(Once again, we choose this element in the first copy of $\mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F}), \mathrm{pr}_2^*(\mathscr{F}))$ appearing in the limit.) With lemma 1.2.2, and replacing m by a multiple if necessary, we can assume that this element is of the form $\mathrm{pr}_2^*(f)$ for f an endomorphism of $\mathscr{F}$. Restricting again to $\{1\} \times X$, we obtain $f = \mathrm{id}_{\mathscr{F}}$. We deduce that, for m sufficiently divisible, $(e_{\frac{m}{n'}} \times \mathrm{id})^*(\iota) \circ (e_{\frac{m}{n'}} \times \mathrm{id})^*(\iota(n)^{-1}) = \mathrm{id}_{\mathrm{pr}_2^*\mathscr{F}}$; then

$$(e_{\frac{m}{n'}} \times \mathrm{id})^*\iota = (e_{\frac{m}{n}} \times \mathrm{id})^*\iota(n). \tag{1.3.1}$$

noticing that $e_{\frac{m}{n'}}$ is a surjective map, we get that ι is an isomorphism on the stalks and hence an isomorphism thanks to the comment above.

Now, assume that n_1 and n_2 are integers and consider morphisms $\iota(n_1)$ and $\iota(n_2)$ as in the proposition. We know from the second statement, just proved above, that $\iota(n_i)$ is an isomorphism for $i = 1, 2$. Then the exact same argument as before shows that there exists an integer n_3 sufficiently large, such that $(e_{\frac{n_3}{n_1}} \times \mathrm{id})^*\iota(n_1) = (e_{\frac{n_3}{n_2}} \times \mathrm{id})^*\iota(n_2)$.

Finally, consider a second complex $\mathscr{G}$ and a morphism $u : \mathscr{F} \to \mathscr{G}$. Choose n sufficiently divisible to have two isomorphisms

$$\iota_1(n) : \mathrm{pr}_2^*(\mathscr{F}) \longrightarrow a(n)^*(\mathscr{F}), \qquad \iota_2(n) : \mathrm{pr}_2^*(\mathscr{G}) \longrightarrow a(n)^*(\mathscr{G}).$$

The composition $\iota_2(n)^{-1} \circ a(n)^*(u) \circ \iota_1(n)$ is a morphism $\mathrm{pr}_2^*(\mathscr{F}) \to \mathrm{pr}_2^*(\mathscr{G})$. Using again lemma 1.2.2 and replacing n by a multiple if necessary, we can assume that this morphism is of the form $\mathrm{pr}_2^*(v)$ for $v : \mathscr{F} \to \mathscr{G}$. Restricting to $\{1\} \times X$, we obtain $v = u$ which finishes the proof. □

1.4 Definition

We define here the monodromy of a constructible complex. For any cocharacter $\lambda \in \mathrm{X}_*(A)$ and any integer n, we set $\lambda(e^{\frac{2i\pi}{n}}) = \lambda_n$. Let also

$$j_\sharp : \{\sharp\} \times X \hookrightarrow A \times X \quad \text{and} \quad \tau_\sharp : X \xrightarrow{\sim} \{\sharp\} \times X$$

denote the inclusion and the canonical map respectively, for any $\sharp \in A$.

Consider an object $\mathscr{F} \in D^b_{\mathcal{S}}(X, \Bbbk)$ and a cocharacter $\lambda \in \mathrm{X}_*(A)$. Thanks to proposition 1.2.1, there exists $n \in \mathbb{Z}_{>0}$ and an isomorphism

$$\iota(n) : \mathrm{pr}_2^*(\mathscr{F}) \to a(n)^*(\mathscr{F})$$

whose restriction to $\{1\} \times X$ identifies with the identity of $\mathscr{F}$. We set

$$\varphi_{\mathscr{F},n}(\lambda) := \tau_{\lambda_n}^* j_{\lambda_n}^* (\iota(n)) : \mathscr{F} \to \mathscr{F}\,.$$

Our first task is to show that in fact the isomorphism $\varphi_{\mathscr{F},n}(\lambda)$ does not depend on the integer n.

Proposition 1.4.1. *Consider two integers n_1, n_2 such that there exist two morphisms*

$$\iota_1 : \mathrm{pr}_2^*(\mathscr{F}) \to a(n_1)^*(\mathscr{F}) \quad \textit{and} \quad \iota_2 : \mathrm{pr}_2^*(\mathscr{F}) \to a(n_2)^*(\mathscr{F})$$

as in proposition 1.2.1. Then for any $\lambda \in \mathrm{X}_*(A)$, we have

$$\varphi_{\mathscr{F},n_1}(\lambda) = \varphi_{\mathscr{F},n_2}(\lambda).$$

Proof. We thus consider two morphisms ι_1 and ι_2 as in the statement of the lemma; $\iota_{1|\{1\}\times X}$ and $\iota_{2|\{1\}\times X}$ identify with $\mathrm{id}_{\mathscr{F}}$. We want to show that

$$\tau^*_{\lambda_{n_1}} j^*_{\lambda_{n_1}}(\iota_1) = \tau^*_{\lambda_{n_2}} j^*_{\lambda_{n_2}}(\iota_2).$$

We know that there exists a multiple m of both n_1 and n_2 such that $(e_{\frac{m}{n_1}} \times \mathrm{id})^*(\iota_1) = (e_{\frac{m}{n_2}} \times \mathrm{id})^*(\iota_2)$. Thus

$$j^*_{\lambda_m}(e_{\frac{m}{n_1}} \times \mathrm{id})^*(\iota_1) = j^*_{\lambda_m}(e_{\frac{m}{n_2}} \times \mathrm{id})^*(\iota_2).$$

In the following diagrams, we consider $i \in \{1, 2\}$. We will need some more notation: let $\overline{(e_{\frac{m}{n_i}} \times \mathrm{id})}$ be the map

$$\{\lambda_m\} \times X \longrightarrow \{\lambda_{n_i}\} \times X$$

induced by $(e_{\frac{m}{n_i}} \times \mathrm{id})$. The following diagram is clearly commutative:

$$\begin{array}{ccccc}
X & \xrightarrow[\sim]{\tau_{\lambda_m}} & \{\lambda_m\} \times X & \xrightarrow{j_{\lambda_m}} & T \times X \\
\| & & \downarrow{\scriptstyle \overline{(e_{\frac{m}{n_i}} \times \mathrm{id})}} & & \downarrow{\scriptstyle (e_{\frac{m}{n_i}} \times \mathrm{id})} \\
X & \xrightarrow[\sim]{\tau_{\lambda_n}} & \{\lambda_{n_i}\} \times X & \xrightarrow{j_{\lambda_n}} & A \times X.
\end{array}$$

We deduce that the following diagram is also commutative. (To gain some space, we omit the labels in the notation for Hom-spaces.)

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F}), a(n_i)^*(\mathscr{F})) & \xrightarrow[(e_{\frac{m}{n_i}} \times \mathrm{id})^*]{} & \mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F}), a(m)^*(\mathscr{F})) \\
\downarrow{\scriptstyle j^*_{\lambda_{n_i}}} & & \downarrow{\scriptstyle j^*_{\lambda_m}} \\
\mathrm{Hom}\left(j^*_{\lambda_{n_i}} \mathrm{pr}_2^*(\mathscr{F}), j^*_{\lambda_{n_i}} a(n_i)^*(\mathscr{F})\right) & \xrightarrow[(e_{\frac{m}{n_i}} \times \mathrm{id})^*]{} & \mathrm{Hom}\left(j^*_{\lambda_m} \mathrm{pr}_2^*(\mathscr{F}), j^*_{\lambda_m} a(m)^*(\mathscr{F})\right) \\
\wr \downarrow{\scriptstyle \tau^*_{\lambda_{n_i}}} & & \wr \downarrow{\scriptstyle \tau^*_{\lambda_m}} \\
\mathrm{Hom}(\mathscr{F}, \mathscr{F}) & = & \mathrm{Hom}(\mathscr{F}, \mathscr{F}).
\end{array}$$

As we have the equality $j^*_{\lambda_m}(e_{\frac{m}{n_1}} \times \mathrm{id})^*(\iota_1) = j^*_{\lambda_m}(e_{\frac{m}{n_2}} \times \mathrm{id})^*(\iota_2)$, we deduce that $\tau^*_{\lambda_{n_1}} j^*_{\lambda_{n_1}}(\iota_1)$ and $\tau^*_{\lambda_{n_2}} j^*_{\lambda_{n_2}}(\iota_2)$ are already equal in $\mathrm{Hom}(\mathscr{F}, \mathscr{F})$. □

Thanks to proposition 1.4.1, we now can get rid of the n in the notation $\varphi_{\mathscr{F},n}(\lambda)$. For any $\mathscr{F} \in D^b_{\mathcal{S}}(X, \Bbbk)$ have obtained an application

$$\varphi_{\mathscr{F}} : \mathrm{X}_*(A) \to \mathrm{Aut}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}), \quad \lambda \mapsto \varphi_{\mathscr{F}}(\lambda).$$

We state now a useful lemma.

Lemma 1.4.2. *Fix an object $\mathscr{F} \in D^b_{\mathcal{S}}(X, \Bbbk)$. The application $\varphi_{\mathscr{F}}$ is a group morphism $\mathrm{X}_*(A) \to \mathrm{Aut}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F})$.*

Proof. Consider $\lambda, \mu \in \mathrm{X}_*(A)$ and an integer n such that there exists a morphism

$$\iota : \mathrm{pr}_2^* \mathscr{F} \to a(n)^* \mathscr{F}$$

whose restriction to $\{1\} \times X$ identifies with $\mathrm{id}_{\mathscr{F}}$.

Let $f : A \times X \to A \times X$ denote the map

$$(t, x) \longmapsto (\mu_n t, x).$$

As we have $a(n) \circ f = a(n)$ and $\mathrm{pr}_2 \circ f = \mathrm{pr}_2$, the functor f^* induces a map

$$\mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X)}(\mathrm{pr}_2^* \mathscr{F}, a(n)^* \mathscr{F}) \to \mathrm{Hom}_{D^b_{\mathcal{S}}(A\times X)}(\mathrm{pr}_2^* \mathscr{F}, a(n)^* \mathscr{F}).$$

We consider $f^*(\iota)$: this is a morphism $a(n)^* \mathscr{F} \to \mathrm{pr}_2^* \mathscr{F}$ whose restriction to $\{1\} \times X$ has no reason to be $\mathrm{id}_{\mathscr{F}}$. So we cannot define $\varphi_{\mathscr{F}}(\lambda)$ from this map. To palliate that, we rather consider the composition

$$\mathrm{pr}_2^* \mathscr{F} \xrightarrow{\mathrm{pr}_2^*((\varphi_{\mathscr{F}}(\mu))^{-1})} \mathrm{pr}_2^* \mathscr{F} \xrightarrow{f^*(\iota)} a(n)^* \mathscr{F}\,. \tag{1.4.1}$$

We check that this morphism has the correct restriction to $\{1\} \times X$. What we consider is in fact

$$\tau_1^* j_1^* \left(f^*(\iota) \circ \mathrm{pr}_2^*((\varphi_{\mathscr{F}}(\mu)^{-1}) \right).$$

On the one hand, considering the following commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow[\sim]{\tau_1} & \{1\} \times X & \xhookrightarrow{j_1} & A \times X \\
\| & & \downarrow{\scriptstyle f_{|\{1\}\times X}} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow[\sim]{\tau_{\mu_n}} & \{\mu_n\} \times X & \xhookrightarrow{j_{\mu_n}} & A \times X
\end{array}$$

we have $\tau_1^* j_1^*(f^*(\iota)) = \varphi_{\mathscr{F}}(\mu)$. On the other hand, we have $\mathrm{pr}_2 \circ j_1 \circ \tau_1 = \mathrm{id}_X$ so $\tau_1^* j_1^*(\mathrm{pr}_2^*((\varphi_{\mathscr{F}}(\mu))^{-1})) = (\varphi_{\mathscr{F}}(\mu))^{-1}$. The restriction of the morphism (1.4.1) to $\{1\} \times X$ is then $\mathrm{id}_{\mathscr{F}}$. Thanks to proposition 1.4.1, we can define $\varphi_{\mathscr{F}}(\lambda)$ by restricting (1.4.1) to $\{\lambda_n\} \times X$ (and then applying $\tau_{\lambda_n}^*$). We do so:

$$\begin{aligned}
\varphi_{\mathscr{F}}(\lambda) &= \tau_{\lambda_n}^* j_{\lambda_n}^* \left(f^*(\iota) \circ \mathrm{pr}_2^*((\varphi_{\mathscr{F}}(\mu))^{-1}) \right) \\
&= \tau_{\lambda_n}^* j_{\lambda_n}^* (f^*(\iota)) \circ (\varphi_{\mathscr{F}}(\mu))^{-1}.
\end{aligned}$$

To conclude the proof, we must show that $\tau_{\lambda_n}^* j_{\lambda_n}^*(f^*(\iota)) = \varphi_{\mathscr{F}}(\lambda\mu)$. But this is clear from the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow[\sim]{\tau_{\lambda_n}} & \{\lambda_n\} \times X & \xhookrightarrow{j_{\lambda_n}} & A \times X \\
\| & & \downarrow{\scriptstyle f_{|\{\lambda_n\}\times X}} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow[\sim]{\tau_{\lambda_n\mu_n}} & \{\lambda_n\mu_n\} \times X & \xhookrightarrow{j_{\lambda_n\mu_n}} & A \times X.
\end{array}$$

The proof is now complete. □

Thanks to lemma 1.4.2, for any $\mathscr{F}$, we can extend the morphism $\varphi_{\mathscr{F}}$ to a $\Bbbk$-algebra morphism:

$$\varphi_{\mathscr{F}} : \Bbbk[\mathrm{X}_*(A)] \longrightarrow \mathrm{End}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}).$$

Definition 1.4.3. *The morphism of algebras $\varphi_{\mathscr{F}} : \Bbbk[\mathrm{X}_*(A)] \longrightarrow \mathrm{End}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F})$ is called the canonical monodromy morphism of $\mathscr{F}$. The action of $\mathrm{X}_*(A)$ on $\mathscr{F}$ defined by $\varphi_{\mathscr{F}}$ is the canonical monodromy action on $\mathscr{F}$.*

Remark 1.4.4. In our definition of monodromy, we considered, for any n, the simplest primitive root of unity $e^{\frac{2i\pi}{n}}$. Although convenient, this is not strictly necessary: if ξ is a primitive n-th root of unity and m divides n, the complex number $\xi^{\frac{n}{m}}$ is a primitive m-th root of unity. Then, the exact same definitions as above would still be valid replacing the family $\{e^{\frac{2i\pi}{n}}\}_{n>0}$ by an appropriate family of n-th roots of unity.

1.5 Properties

We study some basic properties of the monodromy morphism defined in the previous section.

The following result is a key feature:

Lemma 1.5.1. *Consider any $\lambda \in \mathrm{X}_*(A)$. For any $\mathscr{F}, \mathscr{G}$ in $D^b_{\mathcal{S}}(X, \Bbbk)$ and any morphism $f : \mathscr{F} \to \mathscr{G}$, we have $f \circ \varphi_{\mathscr{F}}(\lambda) = \varphi_{\mathscr{G}}(\lambda) \circ f$.*

Proof. This is an immediate consequence of the commutativity of diagram (1.2.1) of proposition 1.2.1: with the notation therein, the lemma follows from applying the functor $\tau_{\lambda_n}^* j_{\lambda_n}^*$ to the diagram. □

Consider any algebraic group morphism $\phi : A' \to A$ where A' is another torus. Via ϕ, the group A' acts on X. It is quite clear that each stratum of the stratification $\mathcal{S}$ is A'-stable. For any $\mathscr{F} \in D^b_{\mathcal{S}}(X, \Bbbk)$, we can thus define a monodromy application $\varphi'_{\mathscr{F}} : \mathrm{X}_*(A') \to \mathrm{Aut}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F})$. The following lemma relates the two applications $\varphi_{\mathscr{F}}$ and $\varphi'_{\mathscr{F}}$.

Lemma 1.5.2. *For any $\lambda' \in \mathrm{X}_*(A')$, we have the equality*

$$\varphi'_{\mathscr{F}}(\lambda') = \varphi_{\mathscr{F}}(\phi \circ \lambda').$$

Proof. Denote $a' : A' \times X \to X$ the application defining the action of A':

$$a'(t', x) = a(\phi(t'), x).$$

Then we define $a'(n)$ to be $a' \circ (e_n \times \mathrm{id})$. Since ϕ is a group morphism, we have

$$a'(n) = a(n) \circ (\phi \times \mathrm{id}).$$

Now consider a morphism

$$\iota : \mathrm{pr}_2^*(\mathscr{F}) \to a(n)^*(\mathscr{F})$$

as in proposition 1.2.1. We apply the functor $(\phi \times \mathrm{id})^*$ to this morphism to obtain a map

$$\iota' : (\mathrm{pr}'_2)^*(\mathscr{F}) \to a'(n)^*(\mathscr{F})$$

(where pr'_2 denotes the projection $A' \times X \to X$). We have $\iota'_{|\{1\}\times X} = \iota_{|\{1\}\times X} = \mathrm{id}_{\mathscr{F}}$. We set $\tau'_{\lambda'_n} : X \xrightarrow{\sim} \{\lambda'(e^{\frac{2i\pi}{n}})\} \times X = \{\lambda'_n\} \times X$ for the obvious isomorphism. We then get

$$\varphi'_{\mathscr{F}}(\lambda') = (\tau'_{\lambda'_n})^*(\iota'_{|\{\lambda'_n\}\times X}). \tag{1.5.1}$$

We remark that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xrightarrow{\tau'_{\lambda'_n}} & \{\lambda'_n\} \times X & \hookrightarrow & A' \times X \\ \downarrow{\scriptstyle \tau_{(\phi\circ\lambda')_n}} & & & & \downarrow{\scriptstyle (\phi\times\mathrm{id})} \\ \{\phi(\lambda'_n)\} \times X & & \xrightarrow[j_{(\phi\circ\lambda')_n}]{} & & A \times X. \end{array}$$

The vertical arrow on the left is in fact $\tau_{(\phi\circ\lambda')_n}$ i.e. is given by:

$$x \longmapsto (\phi \circ \lambda'(e^{\frac{2i\pi}{n}}), x).$$

Recalling that $\iota' = (\phi \times \mathrm{id})^*\iota$, we obtain the result from equality (1.5.1). □

We now consider the behaviour of monodromy with respect to functors constructed from a morphism. We begin with the following definition:

Definition 1.5.3. *Let $(X, \mathcal{S})$ and $(X', \mathcal{S}')$ be two varieties endowed with algebraic stratifications and $f : X \to X'$ a morphism. We say that f is stratified if for any stratum $S \in \mathcal{S}$ the image $f(S)$ is a union of strata of $\mathcal{S}'$. We say that f is locally trivial if it is stratified and if for any pair $(S', S) \in \mathcal{S}' \times \mathcal{S}$ such that $S' \subseteq f(S)$, the map $S \cap f^{-1}(S') \to S'$ induced by f is a Zariski locally trivial fibration.*

The point of this definition is that the pushforward and proper pushforward of an $\mathcal{S}$-constructible complex along a locally trivial f are $\mathcal{S}'$-constructible (see [BaR, Proposition 6.1] for the proper pushforward; a "Verdier dual" argument applies in fact also for the usual one). Before the next result, we make a remark on some morphisms of functors.

Remark 1.5.4. Consider any cartesian square of algebraic varieties and continuous maps:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[\delta]{} & D. \end{array}$$

We have natural morphisms of functors:

$$\gamma^*\delta^! \to \alpha^!\beta^* \qquad \text{and} \qquad \beta^*\delta_* \to \alpha_*\gamma^*. \tag{1.5.2}$$

Indeed, by adjunction, to give a morphism of functors $\gamma^*\delta^! \to \alpha^!\beta^*$ is the same as giving a morphism $\delta^! \to \gamma_*\alpha^!\beta^*$. With the base change theorem, this amounts to giving a morphism $\delta^! \to \delta^!\beta_*\beta^*$. We then take the image under $\delta^!$ of the adjunction morphism $\mathrm{id} \to \beta_*\beta^*$. Similar considerations give the second morphism of (1.5.2). Now assume that the diagram we consider is of the form

$$\begin{array}{ccc} E \times X & \xrightarrow{\mathrm{id}_E \times f} & E \times X' \\ {\scriptstyle j\times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle j\times \mathrm{id}_{X'}} \\ B \times X & \xrightarrow[\mathrm{id}_B \times f]{} & B \times X'. \end{array}$$

We have the same kind of morphisms as above. We consider an object in $D^b(B \times X', \Bbbk)$ of the form $\mathscr{F} \boxtimes \mathscr{G}$ (with $\mathscr{F} \in D^b(B, \Bbbk)$ and $\mathscr{G} \in D^b(X', \Bbbk)$). It is quite clear that the morphism

$$(j \times \mathrm{id}_X)^*(\mathrm{id}_B \times f)^! (\mathscr{F} \boxtimes \mathscr{G}) \to (\mathrm{id}_E \times f)^!(j \times \mathrm{id}_{X'})^* (\mathscr{F} \boxtimes \mathscr{G}) \tag{1.5.3}$$

is an isomorphism. Moreover, if $\mathscr{F}' \in D^b(B, \Bbbk)$ and $\mathscr{G} \in D^b(X', \Bbbk)$, and if

$$\varphi : \mathscr{F} \boxtimes \mathscr{G} \longrightarrow \mathscr{F}' \boxtimes \mathscr{G}'$$

is any morphism, the two morphisms

$$(\mathrm{id}_E \times f)^!(j \times \mathrm{id}_{X'})^*\varphi \qquad \text{and} \qquad (j \times \mathrm{id}_X)^*(\mathrm{id}_B \times f)^!\varphi$$

identify (under the isomorphism (1.5.3) for both objects $\mathscr{F} \boxtimes \mathscr{G}$ and $\mathscr{F}' \boxtimes \mathscr{G}'$). A similar statement can be made using the second morphism of (1.5.2), considering this time an object on $B \times X$. Finally, if $\mathscr{H} \cong \mathscr{F} \boxtimes \mathscr{G}$, then one still has the isomorphism (1.5.3), replacing $\mathscr{F} \boxtimes \mathscr{G}$ with $\mathscr{H}$.

Lemma 1.5.5. *Let $(X, \mathcal{S})$ and $(X', \mathcal{S}')$ be two stratified A-varieties. Consider $f : X \to X'$ an A-equivariant morphism of algebraic varieties. Take $\mathscr{F} \in D^b_{\mathcal{S}}(X, \Bbbk)$, $\mathscr{G} \in D^b_{\mathcal{S}'}(X', \Bbbk)$ and any $\lambda \in \mathrm{X}_*(A)$.*

(1) Assume that f is a locally trivial morphism of algebraic varieties (for $\mathcal{S}$ and $\mathcal{S}'$). We have:

$$f_!(\varphi_{\mathscr{F}}(\lambda)) = \varphi_{f_!(\mathscr{F})}(\lambda) \text{ and } f_*(\varphi_{\mathscr{F}}(\lambda)) = \varphi_{f_*(\mathscr{F})}(\lambda).$$

(2) Assume that for any stratum $S \in \mathcal{S}$, the image $f(S)$ is a stratum of $\mathcal{S}'$. Then

$$f^*(\varphi_{\mathscr{G}}(\lambda)) = \varphi_{f^*(\mathscr{G})}(\lambda) \text{ and } f^!(\varphi_{\mathscr{G}}(\lambda)) = \varphi_{f^!(\mathscr{G})}(\lambda).$$

Proof. We start by noticing that the lemma makes sense: $f^*\mathscr{G}$, $f^!\mathscr{G}$ and $f_!\mathscr{F}$, $f_*\mathscr{F}$ are respectively $\mathcal{S}$-constructible and $\mathcal{S}'$-constructible. We "prime" all the morphisms relative to X': a' denotes the action morphism $A \times X' \to X'$ for example; the same will apply for pr'_2, τ', ... We keep the notation of the proof of lemma 1.4.2.

We fix once and for all an $n \in \mathbb{Z}_{>0}$ such that there exist two morphisms

$$\iota : \mathrm{pr}_2^*(\mathscr{F}) \to a(n)^*(\mathscr{F})$$

and

$$\iota' : \mathrm{pr}_2'^*(\mathscr{G}) \to a'(n)^*(\mathscr{G})$$

whose restriction to $\{1\} \times X$ (resp. $\{1\} \times X'$) is $\mathrm{id}_{\mathscr{F}}$ (resp. $\mathrm{id}_{\mathscr{G}}$). Thanks to proposition 1.4.1, we can always find such an n, that is, an n that works for both $\mathscr{F}$ and $\mathscr{G}$. The arguments for both points are based on the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ {\scriptstyle \tau_{\lambda_n}}\downarrow\wr & & \wr\downarrow{\scriptstyle \tau'_{\lambda_n}} \\ \{\lambda_n\} \times X & \xrightarrow[\overline{(\mathrm{id} \times f)}]{} & \{\lambda_n\} \times X' \\ {\scriptstyle j_{\lambda_n}}\downarrow & & \downarrow{\scriptstyle j'_{\lambda_n}} \\ T \times X & \xrightarrow[(\mathrm{id} \times f)]{} & T \times X'. \end{array} \tag{1.5.4}$$

Note that both the upper and lower square are cartesian. Note also that the vertical arrows are all proper morphisms: the top ones because they are isomorphisms, the bottom ones because they are closed embeddings.

Now we prove the first equality of *(2)*. By definition, we have $\varphi_{\mathscr{G}}(\lambda) = (\tau'_{\lambda_n})^*(\iota'_{|\{\lambda_n\} \times X'})$. Using the top square in the diagram, we get

$$\begin{aligned} f^*(\varphi_{\mathscr{G}}(\lambda)) &= f^*(\tau_{\lambda_n}'^*(\iota'_{|\{\lambda_n\} \times X'})) \\ &= \tau_{\lambda_n}^* \overline{(\mathrm{id} \times f)}^*(\iota'_{|\{\lambda_n\} \times X'}). \end{aligned}$$

Now we use the bottom square to get

$$\overline{(\mathrm{id}\times f)}^{*}(\iota'_{|\{\lambda_n\}\times X'}) = ((\mathrm{id}\times f)^*\iota')_{|\{\lambda_n\}\times X}\,.$$

Using the fact that f is A-equivariant and the equality $\mathrm{pr}'_2\circ(\mathrm{id}\times f) = f\circ\mathrm{pr}_2$, we see that this identifies with a morphism

$$a(n)^*(f^*\mathscr{G}) \longrightarrow \mathrm{pr}_2^*(f^*\mathscr{G}).$$

Now, if we show that the restriction of $(\mathrm{id}\times f)^*\iota'$ to $\{1\}\times X$ identifies with the identity of $f^*\mathscr{G}$, we will be done. But we have:

$$\begin{aligned}\tau_1^* j_1^*((\mathrm{id}\times f)^*\iota') &= f^*\tau_1'^* j_1'^*(\iota')\\ &= f^*(\mathrm{id}_{\mathscr{G}})\\ &= \mathrm{id}_{f^*\mathscr{G}}\,.\end{aligned}$$

The first equality is thus proved.

We deal with the second equality of *(2)*. The beginning is similar to the preceding case, noticing first that τ and τ' are isomorphisms and so $\tau^*\cong\tau^!$ and $\tau'^*\cong\tau'^!$. Then, we remark that $a(n)$, $a'(n)$ and the projections pr_2, pr'_2 are smooth morphisms, thus $a(n)^*[2\dim(A)]\cong a(n)^!$ (and similarly for the other maps $a'(n)$, pr_2 and pr'_2). These remarks allow one to see that

$$\begin{aligned}f^!(\varphi_{\mathscr{G}}(\lambda)) &= f^!(\tau'^*_{\lambda_n}(\iota'_{|\{\lambda_n\}\times X'}))\\ &= \tau^*_{\lambda_n}\overline{(\mathrm{id}\times f)}^{!}(\iota'_{|\{\lambda_n\}\times X'})\end{aligned}$$

and that $(\mathrm{id}\times f)^!\iota'$ identifies with a morphism

$$\mathrm{pr}_2^*(f^!\mathscr{G}) \longrightarrow a(n)^*(f^!\mathscr{G}).$$

Let us evaluate $\overline{(\mathrm{id}\times f)}^{!}(\iota'_{|\{\lambda_n\}\times X'})$. To do so, one can apply remark 1.5.4 preceding the lemma to the bottom square of (1.5.4): we are precisely in the case described there. Indeed, all spaces involved are clearly products of spaces, and all the maps that appear can be written as the product of two maps, one of which is the identity (as an example, $j_{\lambda_n} = j_n\times\mathrm{id}_X$ where $j_n : \{\lambda_n\}\hookrightarrow A$ is the inclusion). We get that

$$\overline{(\mathrm{id}\times f)}^{!}(\iota'_{|\{\lambda_n\}\times X'}) \cong \big((\mathrm{id}\times f)^!\iota'\big)_{|\{\lambda_n\}\times X}\,.$$

Now, once again, if we can show that the restriction to $\{1\}\times X$ of $(\mathrm{id}\times f)^!\iota'$ identifies with the identity of $f^!\mathscr{F}$, we will be done. This follows from the next equalities:

$$\begin{aligned}\tau_1^* j_1^*(\mathrm{id}\times f)^!\iota' &= f^!\tau_1'^* j_1'^*\iota'\\ &= f^!(\mathrm{id}_{\mathscr{F}})\\ &= \mathrm{id}_{f^!(\mathscr{F})}\,.\end{aligned}$$

This settles this case.

Now consider the first equality in *(1)*. We have to evaluate

$$f_!(\tau^*(\iota_{|\{\lambda_n\}\times X})).$$

Using the base change theorem on the top square of diagram (1.5.4) first, and then on the bottom square, we get

$$f_!(\tau^*(\iota_{|\{\lambda_n\}\times X})) = \tau'^*\left((\mathrm{id}\times f)_!\iota\right)_{|\{\lambda_n\}\times X'}.$$

By definition, $(\mathrm{id}\times f)_!\iota$ is a morphism

$$(\mathrm{id}\times f)_!\,\mathrm{pr}_2^*(\mathscr{F}) \longrightarrow (\mathrm{id}\times f)_! a(n)^*(\mathscr{F}).$$

Another application of the base change theorem shows that $(\mathrm{id}\times f)_!\iota$ identifies with a morphism

$$\mathrm{pr}_2'^*\left(f_!(\mathscr{F})\right) \longrightarrow a'(n)^*\left(f_!(\mathscr{F})\right).$$

To evaluate its restriction to $\{1\}\times X'$, one must apply one more time the base change theorem; we see that this restriction is $\mathrm{id}_{f_!\mathscr{F}}$. Finally, we get

$$f_!(\tau^*(\iota_{|\{\lambda_n\}\times X})) = \tau'^*_{\lambda_n} j'^*_{\lambda_n}(\mathrm{id}\times f)_!\iota = \varphi_{f_!\mathscr{F}}(\lambda)$$

and the proof is complete. The case of f_* follows from similar considerations as above, using the second morphism in (1.5.2). □

Lemma 1.5.6. *Consider two stratified A-varieties $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ and $\mathscr{F},\mathscr{G}\in D^b_{\mathcal{S}_X}(X,\Bbbk)$, $\mathscr{H}\in D^b_{\mathcal{S}_Y}(Y,\Bbbk)$. We consider $X\times Y$ as an A-variety under the action*

$$t\cdot(x,y) = (t\cdot x, t\cdot y).$$

Then, we have $\varphi_{\mathscr{F}\otimes\mathscr{G}} = \varphi_{\mathscr{F}}\otimes\varphi_{\mathscr{G}}$ and $\varphi_{\mathscr{F}\boxtimes\mathscr{H}} = \varphi_{\mathscr{F}}\boxtimes\varphi_{\mathscr{H}}$, where these applications are defined by

$$\lambda\mapsto\varphi_{\mathscr{F}}(\lambda)\otimes\varphi_{\mathscr{G}}(\lambda) \quad \textit{and} \quad \lambda\mapsto\varphi_{\mathscr{F}}(\lambda)\boxtimes\varphi_{\mathscr{H}}(\lambda).$$

Proof. The equality $\varphi_{\mathscr{F}\otimes\mathscr{G}} = \varphi_{\mathscr{F}}\otimes\varphi_{\mathscr{G}}$ follows from the fact that we have

$$f^*(\mathscr{F}\otimes\mathscr{G}) \cong f^*\mathscr{F}\otimes f^*\mathscr{G}$$

for any map f whose codomain is X.

The second equality is clear if we look at $X\times Y$ as an $A\times A$ variety (under the action $((t,t')\cdot(x,y) = (t\cdot x, t'\cdot y)$). The reader should note that for the last sentence to make sense, we view $\varphi_{\mathscr{F}}\boxtimes\varphi_{\mathscr{H}}$ as a map from $\Bbbk[\mathrm{X}_*(A)]\otimes_\Bbbk\Bbbk[\mathrm{X}_*(A)]$. Now, what we consider in the lemma is this action pre-composed with the diagonal embedding $\Delta_A : A\to A\times A$. Thanks to lemma 1.5.2, we obtain the desired result. □

Chapter 2

The Lusztig–Yun monodromic category

2.1 Preliminaries

Consider a complex algebraic group H and a (complex) H-variety X. We denote the H-equivariant constructible bounded derived category of sheaves of $\Bbbk$-vector spaces on X by $D^b_H(X, \Bbbk)$.

Assume that there exists a finite subgroup K of H, contained in the center of H, which acts trivially on X. Recall that the center $Z(\mathcal{C})$ of an additive category $\mathcal{C}$ is the endomorphism ring of the identity functor $\mathrm{id}_{\mathcal{C}}$. If $\mathcal{C}$ is $\Bbbk$-linear, then $Z(\mathcal{C})$ is a commutative $\Bbbk$-algebra.

Lemma 2.1.1. *The finite abelian group K acts functorially on the identity functor of $D^b_H(X, \Bbbk)$. In other words, we have a $\Bbbk$-algebra morphism*

$$\Bbbk[K] \to Z(D^b_H(X, \Bbbk)).$$

The lemma means that for any $\mathscr{F} \in D^b_H(X, \Bbbk)$ and any $a \in K$, we have an automorphism $\vartheta_{\mathscr{F},a}$ of $\mathscr{F}$, and moreover these automorphisms are compatible with the group structure of K.

Proof. We recall one of the incarnations of the equivariant derived category as fibered category (see [BL, §2.4.3]). One can view an object $\mathscr{F} \in D^b_H(X, \Bbbk)$ as a collection of objects $\mathscr{F}(P) \in D^b(P/H)$ for every resolution $P \to X$, together with isomorphisms $\overline{\rho}^* \mathscr{F}(Q) \cong \mathscr{F}(P)$ for any morphism $P \xrightarrow{\rho} Q \to X$ of resolutions (where $\overline{\rho}$ is the induced morphism on quotients); this data is required to satisfy some usual compatibility conditions. To obtain such a collection from an object $\mathscr{F}$ in $D^b_H(X, \Bbbk)$, one proceeds in the following way: for any resolution $p : P \to X$, we have an object $p^* \mathscr{F}$ in $D^b_H(P, \Bbbk)$ since p is H-equivariant; this in turn defines an object $\mathscr{F}(P) \in D^b(P/H, \Bbbk)$ since P is a free H-space. The definition of the isomorphisms $\overline{\rho}^* \mathscr{F}(Q) \cong \mathscr{F}(P)$ is obvious.

Now, consider two resolutions $p : P \to X$ and $q : Q \to X$ and a morphism of resolutions $\rho : P \to Q$. We assume that we have a group L acting on P and Q, commuting to the H-action, making ρ an L-equivariant map and p, q two L-invariant maps. Then the objects $q^*\mathscr{F}$ and $p^*\mathscr{F}$ define objects in $D^b_{H\times L}(P, \Bbbk)$. Similarly, the isomorphism $\rho^* q^*\mathscr{F} \cong p^*\mathscr{F}$ lies in the category $D^b_{H\times L}(P, \Bbbk)$. As stated above, in order to get $\overline{\rho}^*\mathscr{F}(Q) \cong \mathscr{F}(P)$, we identify $D^b_H(P, \Bbbk)$ with $D^b(P/H, \Bbbk)$. Using the equivalence $D^b_{H\times L}(P, \Bbbk) \cong D^b_{H\times L/H\times\{1\}}(P/(H \times \{1\}), \Bbbk) = D^b_L(P/H, \Bbbk)$, we thus see that this is actually an isomorphism in $D^b_L(P/H, \Bbbk)$.

Consider $M = H \times X \xrightarrow{a} X$ and $M' = H \times H \times X \xrightarrow{a'} X$ with $a'(h, k, x) = hk \cdot x$. We have the following action of $H \times H$ on M' and M, respectively given by:

$$(h_1, h_2)\cdot(a, b, x) = (h_1 a h_2^{-1}, h_2 b h_2^{-1}, h_2 \cdot x), \qquad (h_1, h_2)\cdot(a, x) = (h_1 a h_2^{-1}, h_2 \cdot x).$$

The maps a and a' define two $H \cong H \times \{1\}$-resolutions of X. We also have two H-resolution morphisms $M' \xrightarrow{\varphi_1,\varphi_2} T$ given by $\varphi_1(h, k, x) = (hk, x)$ and $\varphi_2(h, k, x) = (h, k \cdot x)$. The induced maps on quotients are given respectively by pr_2 and $a : H \times X \to X$. Thus we obtain isomorphisms

$$\mathrm{pr}_2^*\mathscr{F}(M) \cong \mathscr{F}(M') \cong a^*\mathscr{F}(M).$$

By definition, this can be written as

$$\vartheta_{\mathscr{F}} : \mathrm{pr}_2^*(\mathrm{For}(\mathscr{F})) \xrightarrow{\sim} a^*(\mathrm{For}(\mathscr{F})). \tag{2.1.1}$$

It is clear by construction that this isomorphism is functorial and satisfies the usual cocycle condition.

The maps φ_i for $i = 1, 2$ are $H \times H$-equivariant for these actions, moreover, the maps a and a' are $\{1\} \times H$-invariant. We deduce from the discussion above that the isomorphisms $\overline{\varphi_i}^* \mathrm{For}(\mathscr{F}) \xrightarrow{\sim} \mathrm{For}(\mathscr{F})$ define isomorphisms in $D^b_H(H \times X, \Bbbk)$, where the latter category is defined with respect to the action of H on $H \times X$ given by $h \cdot (a, x) = (hah^{-1}, h \cdot x)$.

If k belongs to $K \subseteq Z(H)$, then the map $X \xrightarrow{\sim} \{k\} \times X \hookrightarrow H \times X$ is H-equivariant, thus the pullback $\vartheta_{\mathscr{F},k}$ of ϑ along this map gives an automorphism of $\mathscr{F}$ in $D^b_H(X, \Bbbk)$. The cocycle condition for $\vartheta_{\mathscr{F}}$, ensures that the $\vartheta_{\mathscr{F},k}$'s respect the group structure of K, i.e. that $\vartheta_{\mathscr{F},k} \circ \vartheta_{\mathscr{F},k'} = \vartheta_{\mathscr{F},kk'}$ for $k, k' \in K$. This concludes the proof of the lemma. □

2.2 Equivariant categories

Consider a finite central isogeny $\tilde{H} \xrightarrow{\nu} H$ with kernel K and $\chi : K \to \Bbbk^*$ any character of K. We can look at X as a $\tilde{H}$-variety via ν. We obtain in this way an equivariant bounded derived category $D^b_{\tilde{H}}(X, \Bbbk)$. Thanks to lemma 2.1.1, the finite abelian group K acts on the identity functor of this category. We can thus

consider the full subcategory consisting of objects $\mathscr{F}$ such that $\vartheta_{\mathscr{F},a} = \chi(a)\,\mathrm{id}_{\mathscr{F}}$ for all $a \in K$. We denote this subcategory $D^b_{\widetilde{H},\chi}(X,\Bbbk)$.

We consider now a (fixed) multiplicative rank-one $\Bbbk$-local system $\mathscr{L}$ on H. This means that the pullback of $\mathscr{L}$ along the multiplication map $H \times H \to H$ is $\mathscr{L} \boxtimes \mathscr{L}$. Assume that there exists a finite central isogeny ν with kernel K of cardinality prime to ℓ and a character χ of K such that

$$\mathscr{L} = (\nu_* \underline{\Bbbk}_{\widetilde{H}})[\chi].$$

Let us explain a bit what this equality means. The isogeny ν is a K-equivariant map (for H endowed with the trivial K-action). We can then define an action of K on the pushforward $\nu_* \underline{\Bbbk}_{\widetilde{H}}$; then $(\nu_* \underline{\Bbbk}_{\widetilde{H}})[\chi]$ denotes the χ-isotypic component. We will consider the category $D^b_{\widetilde{H},\chi}(X,\Bbbk)$. For now, the reader has probably noticed that the definition of $D^b_{\widetilde{H},\chi}(X,\Bbbk)$ requires some choices, for ν and χ. We will prove that under suitable assumptions on the characteristic of $\Bbbk$, this category in fact does not depend on the choices made.

To finish this section, let us remark the following facts. Lemma 2.1.1 states that the action of K on the equivariant derived category gives an algebra morphism $\Bbbk[K] \to Z(D^b_{\widetilde{H}}(X,\Bbbk))$. Since the cardinality of K is prime to ℓ, we have a decomposition $\Bbbk[K] = \bigoplus_\xi L_\xi$ indexed by the irreducible $\Bbbk$-representations of the finite group K; and since $\Bbbk$ is algebraically closed, these irreducible $\Bbbk$-representations are given by characters. Moreover, for each factor L_ξ, we have an idempotent e_ξ and a decomposition $\mathrm{id}_{\Bbbk[K]} = \sum_\xi e_\xi$. The identity I of $\mathrm{id}_{D^b_H(X,\Bbbk)}$ is the image of $\mathrm{id}_{\Bbbk[K]}$ so we obtain a decomposition (of functors) $I = \sum_\xi E_\xi$ with each E_ξ idempotent. Since $D^b_H(X,\Bbbk)$ is Karoubian (see e.g. [LC, §1, Theorem]), any object $\mathscr{F}$ decomposes as a direct sum $\mathscr{F} = \bigoplus_\xi \mathscr{F}_\xi$. The image of E_ξ is precisely $D^b_{\widetilde{H},\xi}(X,\Bbbk)$, and K acts via the character ξ on $\mathscr{F}_\xi$. This direct sum splitting is obviously functorial, thus $D^b_{\widetilde{H},\xi}(X,\Bbbk)$ is a direct factor subcategory in $D^b_{\widetilde{H}}(X,\Bbbk)$.

Remark 2.2.1. The above comment tells us that if there exists a nonzero morphism between two indecomposable objects of $D^b_{\widetilde{H}}(X,\Bbbk)$, then these two objects are in the same factor subcategory; in particular each factor inherits the structure of triangulated category from $D^b_{\widetilde{H}}(X,\Bbbk)$. This also implies the following important result: if we have a t-structure on $D^b_{\widetilde{H}}(X,\Bbbk)$ and if $\mathscr{F}$ belongs to the factor subcategory associated to a character χ of K, then the cohomology objects of $\mathscr{F}$ belong to the same factor.

2.3 Well-definiteness: first step

We start by making some general comments. Consider a connected algebraic group H acting on a variety X and a multiplicative rank-one local system $\mathscr{L}$ on H. Assume that we have two finite central isogenies $\widetilde{H}_i \xrightarrow{\nu_i} H$ with kernels K_i satisfying $\gcd(|K_i|,\ell) = 1$ for $i = 1,2$ and characters χ_1, χ_2 of K_1 and K_2

respectively such that $((\nu_i)_*(\underline{\Bbbk}_{\widetilde{H}_i}))[\chi_i] = \mathscr{L}$. We can then define the χ_1 and χ_2 equivariant categories

$$D_1 := D^b_{\widetilde{H}_1,\chi_1}(X,\Bbbk) \subseteq D^b_{\widetilde{H}_1}(X,\Bbbk) \quad \text{and} \quad D_2 := D^b_{\widetilde{H}_2,\chi_2}(X,\Bbbk) \subseteq D^b_{\widetilde{H}_2}(X,\Bbbk).$$

The question is: are these categories (canonically) equivalent? We begin by reducing to a somehow simpler case: the canonical map $\nu_1 \circ \mathrm{pr}_1 : \widetilde{H}_1 \times_H \widetilde{H}_2 \to H$ is again a finite central isogeny (here the fibered product has the group structure inherited from the one of the direct product $\widetilde{H}_1 \times \widetilde{H}_2$), and its kernel is $\ker(\nu_1) \times \ker(\nu_2)$ so its cardinality is prime to ℓ. If we can show that the "character-equivariant" category defined with respect to $(\nu_1 \circ \mathrm{pr}_1, \chi_1 \circ \mathrm{pr}_{1|\mathrm{pr}_1^{-1}(\ker(\nu_1))})$ is canonically equivalent to D_1 then we will get $D_1 \cong D_2$ (canonically). Indeed, by definition we have $\nu_1 \circ \mathrm{pr}_1 = \nu_2 \circ \mathrm{pr}_2$; one can check that $\chi_1 \circ \mathrm{pr}_{1|\mathrm{pr}_1^{-1}(\ker(\nu_1))}$ and $\chi_2 \circ \mathrm{pr}_{2|\mathrm{pr}_2^{-1}(\ker(\nu_2))}$ coincide because they define the same isotypic component $\mathscr{L}$ in $(\nu_1 \circ \mathrm{pr}_1)_* \underline{\Bbbk}$. Therefore we can reduce to the following situation:

$$\widetilde{H}_1 \xrightarrow{\nu} \widetilde{H}_2 \xrightarrow{\nu_2} H, \qquad \widetilde{H}_1 \xrightarrow{\nu_1} H$$

with ν and ν_2 two finite central isogenies and $\chi_1 = \chi_2 \circ \nu_{|\ker(\nu_1)}$ with kernel of cardinality coprime to ℓ. (Note that ν_1 is then itself a finite central isogeny, and that its kernel has cardinality coprime to ℓ.) The identity of X is a ν-map in the sense of [BL, §0.1]. We can thus consider the equivariant pullback and pushforward functors between the $\widetilde{H}_1$ and $\widetilde{H}_2$ equivariant derived categories on X (these functors are denoted Q^*_{id} and $Q_{\mathrm{id}*}$ in [BL, Part 6]). Set $(p_\nu)^* := Q^*_{\mathrm{id}}$. What we finally want to show is that $(p_\nu)^*$ induces an equivalence of categories $D_2 \to D_1$. For the proof, we will need an assumption on the characteristic ℓ of $\Bbbk$. The proof goes in two steps: first fully-faithfulness, then essential surjectivity; the essential surjectivity will be proved without hypothesis on ℓ but we will need an assumption for the fully-faithfulness.

2.4 Equivariant cohomology for algebraic groups

In the following lemma, we consider the notion of torsion primes for a reductive group; we refer to [SS, I.4.3, I.4.4] for the definition. Let us just recall that for a reductive group L, the torsion primes are the torsion primes of the simply-connected cover of its derived subgroup $\mathscr{D}(L)$, together with the primes dividing the order of the fundamental group of $\mathscr{D}(L)$.

Any connected algebraic group H over $\mathbb{C}$ can be written as a semidirect product $H = R_u(H) \rtimes L$ with $R_u(H)$ the unipotent radical of H and L a connected reductive subgroup (see e.g. [Ho, §VIII.1, Theorem 4.3]). We call L a Levi factor of H; we have an isomorphism $L \cong H/R_u(H)$.

Lemma 2.4.1. *Consider H a connected complex algebraic group, and fix a Levi decomposition $H = R_u(H) \rtimes L$. We fix a maximal torus T of L and we let W*

denote the Weyl group of (L,T). *Finally assume that the characteristic* ℓ *of* $\Bbbk$ *is not a torsion prime for* L. *Then the natural morphism* $\mathbf{H}_H^\bullet(\mathrm{pt},\Bbbk) \to \mathbf{H}_L^\bullet(\mathrm{pt},\Bbbk)$ *is an isomorphism, and the natural morphism*

$$\mathbf{H}_L^\bullet(\mathrm{pt},\Bbbk) \to \mathbf{H}_T^\bullet(\mathrm{pt},\Bbbk)$$

induces an isomorphism $\mathbf{H}_L^\bullet(\mathrm{pt},\Bbbk) \to \mathbf{H}_T^\bullet(\mathrm{pt},\Bbbk)^W$. *Moreover, we have an isomorphism* $\mathbf{H}_T^\bullet(\mathrm{pt}) \cong \mathrm{Sym}\,(\mathrm{X}^*(T) \otimes_{\mathbb{Z}} \Bbbk)$

Proof sketch. The H-equivariant cohomology of the point can be computed as the total cohomology of the space $BH = EH/H$ where EH is any contractible free H-space. Note that we have a split surjection

$$H \twoheadrightarrow L.$$

Thus the space EH can be viewed as a (contractible and free) L-space and then chosen as EL, i.e. we have $BL = EH/L$. We then obtain a locally trivial map

$$q : BL \twoheadrightarrow BH$$

whose fibers are isomorphic to the unipotent radical of H, so in particular, isomorphic to an affine space. Thus we have $\mathbf{H}_H^\bullet(\mathrm{pt},\Bbbk) \cong \mathbf{H}_L^\bullet(\mathrm{pt},\Bbbk)$. The last statement of the lemma is proved in [To, Theorem 1.3]; this is where the assumption on $\mathrm{char}(\Bbbk)$ is necessary. The last isomorphism is well-known. □

2.5 Isogenies and equivariant cohomology

We start by an easy general lemma:

Lemma 2.5.1. *Consider* $\widetilde{H}, H$ *two algebraic tori. Assume that* $\widetilde{H} \xrightarrow{\nu} H$ *is an isogeny with kernel* K *of order* k, *and let* $k = p_1^{a_1} \cdots p_r^{a_r}$ *be its decomposition as a product of prime numbers. Then the cokernel* A *of the induced map*

$$\nu^\# : \mathrm{X}^*(H) \to \mathrm{X}^*(\widetilde{H})$$

is finite and its order is of the form $p_1^{b_1} \cdots p_r^{b_r}$ *for some non-negative integers* b_i.

Proof. The map ν identifies with the quotient map $\widetilde{H} \to \widetilde{H}/K = H$. An element $\lambda \in \mathrm{X}^*(\widetilde{H})$ pulls back to an element of $\mathrm{X}^*(H)$ if and only if we have $\lambda(K) = \{1\}$. Now K is a finite group so any of its elements is of finite order, moreover this order must divide k. If we take the lowest common multiple of all the orders of the elements of K and denote it k', then we have $\lambda^{k'}(K) = \{1\}$ for any $\lambda \in \mathrm{X}^*(\widetilde{H})$, or in others terms $\lambda^{k'} \in \nu^\#(\mathrm{X}^*(\widetilde{H}))$. So any element in the cokernel of $\nu^\#$ is of finite order, and this order divides k'. It follows that A is a finitely generated abelian group (since it is an homomorphic image of such a group) and all its elements are of finite order, thus A is finite. Now assume that there is a prime p such that p divides the order of A. According to Cauchy's theorem, there must be an element in A whose order is p. So p must divides k' and thus be one of the p_i's. □

Consider now two complex algebraic groups $\widetilde{H}$ and H and a finite central isogeny $\nu : \widetilde{H} \to H$; denote its kernel by K. We can find a Levi decomposition $\widetilde{H} = R_u(\widetilde{H}) \rtimes \widetilde{L}$ (see the discussion before lemma 2.4.1).

Lemma 2.5.2. *Any finite subgroup of the center $Z(\widetilde{H})$ is contained in $\widetilde{L}$. In particular, $K \subseteq \widetilde{L}$.*

Proof. In fact, we will show that any element of finite order in $Z(\widetilde{H})$ is in $\widetilde{L}$. Consider such an x, and let n be its order. We can write $x = rs$ with $r \in R_u(\widetilde{H})$ and $s \in \widetilde{L}$. Since x is central, we have in particular $x = sxs^{-1} = sr$, so r and s commute. We then have $1 = x^n = r^n s^n$. These equalities imply that r^n is in $R_u(\widetilde{H}) \cap \widetilde{L}$, thus is trivial. So r is of finite order, and then semisimple. We then obtain $r = 1$ (as r is both unipotent and semisimple) and $x = s \in \widetilde{L}$. □

We deduce from lemma 2.5.2 that the map ν identifies with the natural projection

$$R_u(\widetilde{H}) \rtimes \widetilde{L} \longrightarrow R_u(\widetilde{H}) \rtimes \widetilde{L}/K.$$

We see in particular that two connected isogeneous groups over $\mathbb{C}$ have isomorphic unipotent radicals. We now would like to link the torsion primes for Levi factors of H and $\widetilde{H}$; in fact we show that the torsion primes for $\widetilde{L}$ are already torsion primes for L. We start by choosing a maximal torus $\widetilde{T}$ in $\widetilde{L}$; this torus contains the central subgroup K and we obtain in this way a maximal torus $T = \widetilde{T}/K$ in L. As we saw above, the isogeny ν induces an isogeny at the level of Levi factors. Moreover, the restriction $\nu_{|\mathscr{D}(\widetilde{L})}$ of ν to the derived (semi-simple) subgroup of $\widetilde{L}$ lands in $\mathscr{D}(L)$. We have an induced map on maximal tori $\widetilde{T} \to \widetilde{T}/K = T$; in turn this gives an injective group morphism $\mathrm{X}^*(T) \hookrightarrow \mathrm{X}^*(\widetilde{T})$. We finally obtain a surjection of groups $\pi_1(\mathscr{D}(L)) = A/\mathrm{X}^*(T) \twoheadrightarrow A/\mathrm{X}^*(\widetilde{T}) = \pi_1(\mathscr{D}(\widetilde{L}))$, where A is the abstract weight lattice of the root system of $\mathscr{D}(L)$ and $\mathscr{D}(\widetilde{L})$. Thus the prime numbers dividing the order of the fundamental group of $\pi_1(\mathscr{D}(\widetilde{L}))$ divide the order of $\pi_1(\mathscr{D}(L))$.

Lemma 2.5.3. *Take two algebraic groups $\widetilde{H}$ and H and a finite central isogeny $\nu : \widetilde{H} \to H$; denote its kernel by K. Assume*

- *the order of K is prime to* $\mathrm{char}(\Bbbk) = \ell$,
- *ℓ is not a torsion prime for the reductive group $H/R_u(H)$.*

Then the natural morphism

$$\mathbf{H}^\bullet_H(\mathrm{pt}, \Bbbk) \to \mathbf{H}^\bullet_{\widetilde{H}}(\mathrm{pt}, \Bbbk)$$

is an isomorphism.

Proof. According to lemmas 2.4.1 and 2.5.2, we can assume that $\widetilde{H}$ and H are reductive. As above we consider a maximal torus $\widetilde{T}$ in $\widetilde{H}$ and the maximal torus $T = \widetilde{T}/K$ under $\widetilde{T}$ in H. The isogeny ν induces an identification of the Weyl

groups $\widetilde{W}$ and W of $\widetilde{H}$ and H. The induced map $\widetilde{T} \to T$ is still an isogeny whose kernel has order prime to ℓ. The map ν induces an injective morphism of abelian group $\mathrm{X}^*(T) \to \mathrm{X}^*(\widetilde{T})$ whose cokernel is a finite group of order prime to ℓ according to lemma 2.5.1. Taking tensor product with $\Bbbk$, we obtain an isomorphism $\mathrm{X}^*(T) \otimes_{\mathbb{Z}} \Bbbk \xrightarrow{\sim} \mathrm{X}^*(\widetilde{T}) \otimes_{\mathbb{Z}} \Bbbk$ and then an isomorphism

$$\mathrm{Sym}\,(\mathrm{X}^*(T) \otimes_{\mathbb{Z}} \Bbbk) \xrightarrow{\sim} \mathrm{Sym}(\mathrm{X}^*(\widetilde{T}) \otimes_{\mathbb{Z}} \Bbbk). \tag{2.5.1}$$

Using the isomorphism $\mathbf{H}_T^\bullet(\mathrm{pt}) \cong \mathrm{Sym}\,(\mathrm{X}^*(T) \otimes_{\mathbb{Z}} \Bbbk)$ stated in lemma 2.4.1, we obtain

$$\mathbf{H}_T^\bullet(\mathrm{pt}) \cong \mathbf{H}_{\widetilde{T}}^\bullet(\mathrm{pt}). \tag{2.5.2}$$

The groups $\widetilde{W}$ and W act on the corresponding characters groups $\mathrm{X}^*(\widetilde{T})$ and $\mathrm{X}^*(T)$. Moreover, the map $\mathrm{X}^*(T) \to \mathrm{X}^*(\widetilde{T})$ induced by the isogeny commutes with the $\widetilde{W}$ and W actions. The action of W on $\mathrm{Sym}(\mathrm{X}^*(T) \otimes_{\mathbb{Z}} \Bbbk)$ induces the W-action on $\mathbf{H}_T^\bullet(\mathrm{pt}, \Bbbk)$ (and similarly for $\widetilde{T}$). As discussed above the statement of the lemma, since $\mathrm{char}(\Bbbk)$ is not a torsion prime H, it is not torsion for $\widetilde{H}$. Thanks to lemma 2.4.1 we have $\mathbf{H}_H^\bullet(\mathrm{pt}, \Bbbk) \cong \mathrm{Sym}(\mathrm{X}^*(T) \otimes_{\mathbb{Z}} \Bbbk)^W$ and $\mathbf{H}_{\widetilde{H}}^\bullet(\mathrm{pt}, \Bbbk) \cong \mathrm{Sym}(\mathrm{X}^*(\widetilde{T}) \otimes_{\mathbb{Z}} \Bbbk)^{\widetilde{W}}$. Thus the isomorphism (2.5.1) induces an isomorphism

$$\mathbf{H}_{\widetilde{H}}^\bullet(\mathrm{pt}, \Bbbk) \cong \mathbf{H}_H^\bullet(\mathrm{pt}, \Bbbk).$$

This concludes the proof of the lemma. □

2.6 A spectral sequence for equivariant cohomology

Lemma 2.6.1. *Consider a G-variety X, for G a connected algebraic group. For an object $\mathscr{F} \in D_G^b(X, \Bbbk)$ we have a converging spectral sequence*

$$E_2^{p,q} = \mathbf{H}_G^p(\mathrm{pt}, \Bbbk) \otimes_\Bbbk \mathbf{H}^q(X, \mathscr{F}) \Rightarrow \mathbf{H}_G^{p+q}(X, \mathscr{F}).$$

Proof. Fix a contractible free G-space EG. The projection map $p : EG \times X \to X$ gives an ∞-acyclic resolution of X in the sense of [BL, Definition 1.9.1] for the action $g \cdot (e, x) = (eg^{-1}, gx)$. The quotient is $BG := EG \times^G X$, the classifying space of G; since G is connected, BG is simply connected (this can be seen as follows: the natural map $EG \to BG$ is a locally trivial fibration with fibers G; one then uses the exact sequence of homotopy groups, since G is connected and EG simply-connected, the result follows). Denote by $q : EG \times X \to EG \times^G X$ the quotient map. Thanks to [BL, Lemma 2.3.2] we can consider our object in the equivariant derived category as an object $\mathscr{F}$ in $D^b(EG \times^G X, \Bbbk)$ such that there exists a $\mathscr{F}_X \in D^b(X, \Bbbk)$ and an isomorphism $p^*(\mathscr{F}_X) \cong q^*(\mathscr{F})$.

Consider the following diagram

$$EG \times^G X \xrightarrow{\mathrm{pr}} BG \xrightarrow{a} \{\mathrm{pt}\},$$

with $\mathrm{pr}([e,x]) = [e]$. We have a Grothendieck spectral sequence $R^q a_* \circ R^p(\mathrm{pr})_* \Rightarrow R^{p+q}(a \circ \mathrm{pr})_*$. We apply it to $\mathscr{F}$. The object $R^p(\mathrm{pr})_*(\mathscr{F})$ has constant stalks isomorphic to $\mathbf{H}^p(X, \mathscr{F}_X)$. We thus obtain a local system on BG; the latter space being simply connected, we have in fact a constant local system isomorphic to $\underline{\Bbbk}_{BG} \otimes_{\Bbbk} \mathbf{H}^p(X, \mathscr{F})$. When we apply $R^q a_*$ to this object, we obtain $\mathbf{H}^p_G(\mathrm{pt}, \Bbbk) \otimes_{\Bbbk} \mathbf{H}^q(X, \mathscr{F})$. Since $R^{p+q}(a \circ \mathrm{pr})_*(\mathscr{F}) = \mathbf{H}^{p+q}_G(X, \mathscr{F})$, we get the result. □

2.7 Full faithfulness of p_ν^*

We keep the notation of the beginning of section 2.3. We show here that under suitable assumptions, the functor p_ν^* is fully faithful.

Lemma 2.7.1. *With the notation of section 2.3, assume that ℓ is not a torsion prime for $H/R_u(H)$. Then the functor $(p_\nu)^*$ is fully faithful.*

Proof. Consider $\mathscr{F}, \mathscr{G}$ two objects in $D^b_{\widetilde{H}_i}(X, \Bbbk)$; applying lemma 2.6.1 to the derived internal Hom $R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})$ one obtains the following converging spectral sequence:

$$E_2^{p,q} = \mathbf{H}^p_{\widetilde{H}_i}(\mathrm{pt}, \Bbbk) \otimes_{\Bbbk} \mathrm{Hom}_{D^b(X,\Bbbk)}(\mathscr{F}, \mathscr{G}[q]) \Rightarrow \mathrm{Hom}_{D^b_{\widetilde{H}_i}(X)}(\mathscr{F}, \mathscr{G}[p+q]).$$

On the morphism spaces, the functor p_ν^* comes from a morphism of spectral sequences induced by the map

$$\mathbf{H}^\bullet_{\widetilde{H}_2}(\mathrm{pt}, \Bbbk) \to \mathbf{H}^\bullet_{\widetilde{H}_1}(\mathrm{pt}, \Bbbk). \tag{2.7.1}$$

According to lemma 2.5.3, the morphism (2.7.1) is an isomorphism (recall that the cardinality of the kernels of ν_1, ν_2 and ν are prime to ℓ). This allows us to conclude that p_ν^* is fully faithful, and thus lemma 2.7.1 is proved. □

2.8 Essential surjectivity of p_ν^*

We now show the essential surjectivity of p_ν^*. We keep the notation of §2.3. We also keep the assumptions of lemma 2.5.3 on ℓ; thanks to lemma 2.7.1, the functor $p_\nu^* : D_2 \to D_1$ is fully faithful. Since any object in D_1 is an extension of its perverse cohomology objects, using lemma 2.7.1, in order to show that p_ν^* is essentially surjective, we only need to check that the perverse sheaves in D_1 are in the essential image, thanks to the following well known fact:

"Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor between triangulated categories which is full, and that the essential image of F contains a family of objects generating $\mathcal{D}$ (as a triangulated category), then F is essentially surjective."

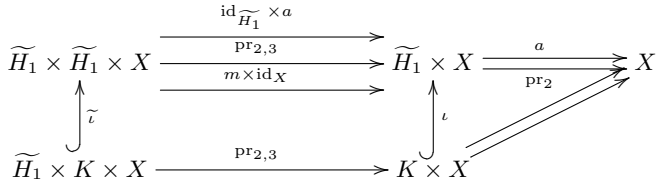
Set $K := \ker(\nu)$. In order for a perverse sheaf $\mathscr{F}$ on X to define a $\widetilde{H}_i$-equivariant object, we just need the existence of an isomorphism between the pullbacks

along the projection map and the action map (see [BaR, §A.1, 3rd "reasonable definition"]). Now if $\mathscr{F}$ is a perverse sheaf in D_1, then by definition it carries a $\widetilde{H}_1$-equivariant structure such that K acts trivially (via the map of lemma 2.1.1). Indeed, by definition, we have that the group $K_1 = \ker(\nu_1) = \nu^{-1}(\ker(\nu_2))$ acts on $\mathscr{F}$ via the character $\chi_1 = \chi_2 \circ \nu_{|\ker(\nu_1)}$. But $K \subseteq \ker(\chi_1)$, so K acts trivially on $\mathscr{F}$.

One then remarks that the isomorphism ϑ between the two pullbacks of $\mathscr{F}$ to $\widetilde{H}_1 \times X$ along the action map and the projection $\widetilde{H}_1 \times X \to X$ is the identity map of $(\mathrm{pr}_{2|K\times X})^* \mathscr{F}$ when restricted to $K \times X$, hence it descends to an isomorphism on $\widetilde{H}_2 \times X = \widetilde{H}_1/K \times X$. Let us elaborate on this fact: the perverse sheaves form a stack for the étale topology (see [BBD, §2.2.19]). Therefore to see that our isomorphism descends to an isomorphism on $\widetilde{H}_2 \times X$, we have to show that its pullbacks under the two projections $\pi_1, \pi_2 : (\widetilde{H_1} \times X) \times_{\widetilde{H_2}\times X} (\widetilde{H_1} \times X) \to \widetilde{H_1} \times X$ coincide.

Lemma 2.8.1. *We have* $\pi_1^* \vartheta = \pi_2^* \vartheta$.

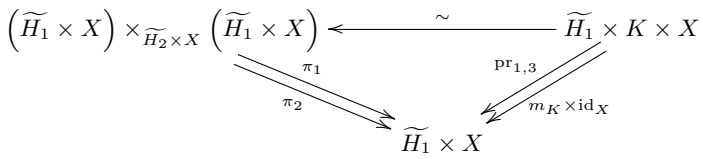
Proof. The different maps that we will consider are depicted in the following (non commutative) diagram.



(the two maps from $K \times X$ to X are given by $a \circ \iota$ and $\mathrm{pr}_2 \circ \iota$). The isomorphism ϑ satisfies the cocycle condition $(m \times \mathrm{id}_X)^* \vartheta = (\mathrm{id}_{\widetilde{H_1}} \times a)^* \vartheta \circ \mathrm{pr}_{2,3}^* \vartheta$. We want to show that the following diagram commutes:

$$\begin{array}{ccc} \pi_1^* a^*(\mathscr{F}) & = & \pi_2^* a^*(\mathscr{F}) \\ \downarrow{\scriptstyle \pi_1^*\vartheta} & & \downarrow{\scriptstyle \pi_2^*\vartheta} \\ \pi_1^* \mathrm{pr}_2^* \mathscr{F} & = & \pi_2^* \mathrm{pr}_2^* \mathscr{F}\,. \end{array}$$

We consider the diagram



(2.8.1)

where the horizontal map is given by $(p, k, x) \mapsto ((p, x), (pk, x))$.

Thanks to diagram (2.8.1), we are reduced to the commutativity of

$$\begin{array}{ccc} \mathrm{pr}_{1,3}^* a^*(\mathscr{F}) & = & (m_K \times \mathrm{id}_X)^* a^*(\mathscr{F}) \\ \downarrow{\scriptstyle \mathrm{pr}_{1,3}^*\vartheta} & & \downarrow{\scriptstyle (m_K\times \mathrm{id}_X)^*\vartheta} \\ \mathrm{pr}_{1,3}^* \mathrm{pr}_2^* \mathscr{F} & = & (m_K \times \mathrm{id}_X)^* \mathrm{pr}_2^* \mathscr{F}\,. \end{array} \tag{2.8.2}$$

But now, we use the isomorphism $(m_K \times \mathrm{id}_X)^* \cong \widetilde{\iota}^* \circ (m \times \mathrm{id}_X)^*$ and the cocycle condition to find that

$$\begin{aligned} (m_K \times \mathrm{id}_X)^*\vartheta &= \widetilde{\iota}^*(\mathrm{id}_{\widetilde{H_1}} \times a)^*\vartheta \circ \widetilde{\iota}^* \,\mathrm{pr}_{2,3}^*\,\vartheta \\ &= \mathrm{pr}_{1,3}^*\,\vartheta \circ \underbrace{\mathrm{pr}_{2,3}^*\,\iota^*\vartheta}_{\mathrm{id}} \end{aligned}$$

by the assumption that $\vartheta_{|K\times X} = \mathrm{id}$. Thus the diagram (2.8.2) commutes and we are done. □

We see that $\mathscr{F}$ is a $\widetilde{H}_2$-equivariant perverse sheaf, thus it defines an object in the $\widetilde{H}_2$-equivariant bounded derived category. Moreover, the isomorphism

$$a_2^*\,\mathscr{F} \to \mathrm{pr}_2^*\,\mathscr{F} \tag{2.8.3}$$

(where a_2 is the action map for $\widetilde{H}_2$ acting on X) on $\widetilde{H}_2 \times X$ comes from the one on $\widetilde{H}_1 \times X$ (i.e. is the descent of ϑ to $\widetilde{H}_2 \times X$). Since $\nu : \widetilde{H}_1 \to \widetilde{H}_2$ is an isogeny and hence is surjective, it is easy to see that the action of K_2 on $\mathscr{F}$ via (2.8.3) comes from the action of K_1 on $\mathscr{F}$. So in particular, one deduces that K_2 acts via χ_2 on $\mathscr{F}$. Thus our functor is essentially surjective.

2.9 Definition

Consider a finite central isogeny $\widetilde{H} \xrightarrow{\nu} H$ with kernels K satisfying $\gcd(|K|, \ell) = 1$ and characters $\chi_{\mathscr{L}}$ of K such that $((\nu)_*(\underline{\mathbb{k}}_{\widetilde{H}}))[\chi_{\mathscr{L}}] = \mathscr{L}$.

Definition 2.9.1. *In the situation of lemma 2.7.1, we set*

$$\mathfrak{D}(X/\!\!/H)_{\mathscr{L}} := D^b_{\widetilde{H},\chi_{\mathscr{L}}}(X, \mathbb{k})$$

and call this category the Lusztig–Yun monodromic equivariant category of monodromy $\mathscr{L}$. This category does not depend on the choice made for $\widetilde{H}$ and $\chi_{\mathscr{L}}$.

Remark 2.9.2. In the case where X admits a stratification $\mathcal{S}$, we can proceed all the preceding construction replacing the category $D^b_{\widetilde{H}}(X, \mathbb{k})$ by the $\mathcal{S}$-constructible derived equivariant category $D^b_{\widetilde{H},\mathcal{S}}(X, \mathbb{k})$. We obtain the $\mathcal{S}$-constructible Lusztig–Yun equivariant monodromic category $\mathfrak{D}_{\mathcal{S}}(X/\!\!/H)_{\mathscr{L}}$.

Let us fix an isogeny $\widetilde{H} \xrightarrow{\nu} H$ and a character χ of its kernel such that we can view $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$ as a full subcategory of $D^b_{\widetilde{H}}(X, \Bbbk)$. One can consider the restriction of the forgetful functor $\mathrm{For} : D^b_{\widetilde{H}}(X, \Bbbk) \to D^b(X, \Bbbk)$ to the subcategory $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$. Moreover, this functor does not depend of the choices made for ν and χ, in the following sense. With the notation of §2.3, we have an isomorphism $\mathrm{For}_1 \circ (p_\nu)^* \cong \mathrm{For}_2$ of functors $D_2 \to D^b(X, \Bbbk)$ (where For_i is the forgetful functor $D_i \to D^b(X, \Bbbk)$ for $i = 1, 2$).

Lemma 2.9.3. *For any $\mathscr{F}$ in $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$, there is an isomorphism $a^*(\mathrm{For}(\mathscr{F})) \cong \mathscr{L} \boxtimes \mathrm{For}(\mathscr{F})$ on $H \times X$.*

Proof. As $\mathscr{F}$ is $\widetilde{H}$-equivariant we have an isomorphism $\mathrm{pr}_2^* \mathrm{For}(\mathscr{F}) \cong a_\nu^* \mathrm{For}(\mathscr{F})$ (with $a_\nu = a_H \circ (\nu \times \mathrm{id}_X)$). Applying the functor $(\nu \times \mathrm{id}_X)_* = (\nu \times \mathrm{id}_X)_!$ to this isomorphism and using the projection formula, we get an isomorphism

$$a^*(\mathrm{For}(\mathscr{F})) \otimes^L_{\Bbbk} (\nu_* \underline{\Bbbk}_{\widetilde{H}} \boxtimes^L_{\Bbbk} \underline{\Bbbk}_X) \xrightarrow{\sim} \nu_* \underline{\Bbbk}_{\widetilde{H}} \boxtimes^L_{\Bbbk} \mathrm{For}(\mathscr{F}).$$

We have a decomposition in direct sum indexed by the irreducible representations ξ of K (see the end of section 2.2) $\nu_* \underline{\Bbbk}_{\widetilde{H}} = \bigoplus_\xi \mathscr{L}_\xi$. Here $\mathscr{L}_\xi$ is the irreducible local system on H corresponding to the representation ξ of K; note that by assumption $\mathscr{L}$ appears in this direct sum, associated to the one-dimensional representation $\xi = \chi_{\mathscr{L}}$ of K. Moreover, the action of K on $\mathscr{L}_\xi$ is induced by the action of K on ξ. We deduce a decomposition

$$\bigoplus_\xi a^*(\mathrm{For}(\mathscr{F})) \otimes^L_{\Bbbk} (\mathscr{L}_\xi \boxtimes^L_{\Bbbk} \underline{\Bbbk}_X) \xrightarrow{\sim} \bigoplus_\xi \mathscr{L}_\xi \boxtimes^L_{\Bbbk} \mathrm{For}(\mathscr{F}).$$

Moreover, the action of K on $\nu_* \underline{\Bbbk}_{\widetilde{H}} \boxtimes^L_{\Bbbk} \mathrm{For}(\mathscr{F})$ comes from the action of K on $\nu_* \underline{\Bbbk}_{\widetilde{H}}$. The direct summand $\mathscr{L} \boxtimes^L_{\Bbbk} \mathrm{For}(\mathscr{F})$ corresponds to the character $\chi_{\mathscr{L}}$; the group K acts via this character on $\mathscr{F}$. We have to identify the direct summand in $a^*(\mathrm{For}(\mathscr{F})) \otimes^L_{\Bbbk} (\nu_* \underline{\Bbbk}_{\widetilde{H}} \boxtimes^L_{\Bbbk} \underline{\Bbbk}_X)$ for which the action of K is via the same character $\chi_{\mathscr{L}}$. We investigate the action of K on $a^* \mathrm{For}(\mathscr{F})$. The map a is a $\widetilde{H}$-morphism for the action of $\widetilde{H}$ on $H \times X$ on the left factor uniquely. This implies that the action of K on $a^*(\mathrm{For}(\mathscr{F}))$ is given by the pullback along a of the action of K on $\mathscr{F}$; this action is then given by the character $\chi_{\mathscr{L}}$. We deduce that the direct summand that we are looking for is $a^*(\mathrm{For}(\mathscr{F})) \otimes^L_{\Bbbk} \underline{\Bbbk}_{H \times X} = a^*(\mathrm{For}(\mathscr{F}))$. The isomorphism of the statement then follows. □

2.10 Verdier duality

We conclude this chapter by general considerations about Verdier duality in the equivariant category. This section will not be used before part II. Let H be a connected algebraic group. Consider a H-variety X and two rank-one multiplicative $\Bbbk$-local systems $\mathscr{L}$ and $\mathscr{K}$ on H. Assume that there exists isogenies $H_{\mathscr{L}} \to H$ and $H_{\mathscr{K}} \to H$, with kernels $K_{\mathscr{L}}$ and $K_{\mathscr{K}}$ of order prime to

ℓ, and characters $\chi_{\mathscr{L}} : K_{\mathscr{L}} \to \Bbbk^*$ and $\chi_{\mathscr{K}} : K_{\mathscr{K}} \to \Bbbk^*$ that allow us to define the categories

$$\mathfrak{D}(X/\!\!/H)_{\mathscr{L}} \qquad \text{and} \qquad \mathfrak{D}(X/\!\!/H)_{\mathscr{K}}.$$

Note that the tensor product $\mathscr{L} \otimes \mathscr{K}$ is again a rank one multiplicative local system. Moreover, if we let $\chi_{\mathscr{L}} \cdot \chi_{\mathscr{K}}$ be the character of $K_{\mathscr{L}} \times K_{\mathscr{K}}$ given by $(x, x') \mapsto \chi_{\mathscr{L}}(x)\chi_{\mathscr{K}}(x')$, then $\mathscr{L} \otimes \mathscr{K}$ is the $\chi_{\mathscr{L}} \cdot \chi_{\mathscr{K}}$-isotypic component of the pushforward of the constant sheaf along the natural isogeny $H_{\mathscr{L}} \times_H H_{\mathscr{K}} \to H$. Note also that $\mathscr{L}$ and $\mathscr{K}$ appear naturally as direct summands in this pushforward. Finally, we will denote by $\mathscr{L}^{-1}$ the dual local system (i.e. the inverse local system under tensor product).

Consider the subcategory $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$ of $D^b_{H_{\mathscr{L}}}(X, \Bbbk)$. Via the natural isogeny $H_{\mathscr{L}} \times_H H_{\mathscr{K}} \to H_{\mathscr{L}}$, we can view $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$ as a full subcategory in the equivariant category $D^b_{H_{\mathscr{L}} \times_H H_{\mathscr{K}}}(X, \Bbbk)$. In fact, if we identify $\chi_{\mathscr{L}}$ with a character of $K_{\mathscr{L}} \times K_{\mathscr{K}}$ via the above map, we can view $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$ as the $\chi_{\mathscr{L}}$-equivariant direct summand subcategory in $D^b_{H_{\mathscr{L}} \times_H H_{\mathscr{K}}}(X, \Bbbk)$. A similar consideration allows us to see $\mathfrak{D}(X/\!\!/H)_{\mathscr{K}}$ as a full subcategory in $D^b_{H_{\mathscr{L}} \times_H H_{\mathscr{K}}}(X, \Bbbk)$. These facts will be useful for the next lemma.

Lemma 2.10.1. *Consider $\mathscr{F} \in \mathfrak{D}(X/\!\!/H)_{\mathscr{L}}$ and $\mathscr{G} \in \mathfrak{D}(X/\!\!/H)_{\mathscr{K}}$. Then we have*

$$R\,\mathcal{H}om(\mathscr{F}, \mathscr{G}) \in \mathfrak{D}(X/\!\!/H)_{\mathscr{L}^{-1} \otimes \mathscr{K}}.$$

Proof. Recall that the actions of the kernel $K_{\mathscr{L}}$ on $\mathscr{F}$ is defined via a natural isomorphism

$$\vartheta_{\mathscr{F}} : \mathrm{pr}^*\, \mathscr{F} \xrightarrow{\sim} a^*\, \mathscr{F},$$

see the proof of lemma 2.1.1.

Now, according to the discussion above, we can (and will) view the objects $\mathscr{F}, \mathscr{G}$ in the category $D^b_{H_{\mathscr{L}} \times_H H_{\mathscr{K}}}(X, \Bbbk)$. Using the same argument as the one used at the beginning of the proof of lemma 1.2.2, we see that the natural map

$$\mathrm{pr}^*\, R\,\mathcal{H}om(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} a^* R\,\mathcal{H}om(\mathscr{F}, \mathscr{G}) \tag{2.10.1}$$

is induced by the maps $\vartheta_{\mathscr{F}}$ and $\vartheta_{\mathscr{G}}$. More precisely, the smoothness of the action and projection maps provides isomorphisms

$$\mathrm{pr}^*\,(R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})) \cong R\,\mathcal{H}om(\mathrm{pr}^*\, \mathscr{F}, \mathrm{pr}^*\, \mathscr{G})$$

and

$$a^*\,(R\,\mathcal{H}om(\mathscr{F}, \mathscr{G})) \cong R\,\mathcal{H}om(a^*\, \mathscr{F}, a^*\, \mathscr{G}).$$

Then the map (2.10.1) is given by

$$R\,\mathcal{H}om(\vartheta_{\mathscr{F}}^{-1}, \vartheta_{\mathscr{G}}).$$

This fact allows us to conclude the proof of the lemma. □

We can consider the dualizing complex complex $\underline{\mathbb{D}}_X$ as an H-equivariant object (see [BL, §3.4.2 Example 1]). Said otherwise, we look at $\underline{\mathbb{D}}_X$ as an object of $\mathfrak{D}(X/\!\!/H)_{\underline{\Bbbk}_H}$. Lemma 2.10.1 then gives us a functor

$$\mathbb{D}_X : (\mathfrak{D}(X/\!\!/H)_{\mathscr{L}})^{\mathrm{opp}} \to \mathfrak{D}(X/\!\!/H)_{\mathscr{L}^{-1}}, \quad \mathscr{F} \mapsto R\,\mathcal{H}om(\mathscr{F}, \underline{\mathbb{D}}_X).$$

This is an equivalence of categories satisfying $\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id}$.

Chapter 3

Monodromic categories

3.1 General considerations

Lemma 3.1.1. *Let $\mathcal{C}$ be an additive Karoubian $\mathbb{K}$-linear category with $\mathbb{K}$ a field. Choose an object X of $\mathcal{C}$. Set $\mathrm{End}_{\mathcal{C}}(X) := \mathrm{Hom}_{\mathcal{C}}(X, X)$. Let $f \in \mathrm{End}_{\mathcal{C}}(X)$ and $\{P_i\}_{i=1,\dots,n}$ be a family of pairwise coprime polynomials in $\mathbb{K}[x]$ such that*

$$\prod_{i=1}^{n} P_i(f) = 0 \in \mathrm{End}_{\mathcal{C}}(X).$$

Then for each i the morphism $P_i(f)$ has a kernel X_i and we have a decomposition

$$X \cong \bigoplus_{i=1}^{n} X_i.$$

In particular, the morphism induced by $P_i(f)$ on X_i is zero.

Proof. We proceed by induction on n. The case $n = 1$ is clear; let us show the inductive step considering $n = 2$. We have two polynomials P_1 and P_2 such that $P_1(f)P_2(f) = 0$. By assumption, these P_i are coprime, so with Bézout's identity, we can find two polynomials A_1 and A_2 such that the equality

$$A_1P_1 + A_2P_2 = 1$$

holds in $\mathbb{K}[x]$. We deduce immediately that

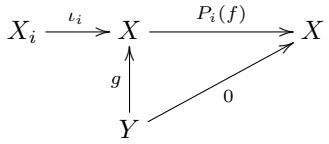
$$A_1(f)P_1(f) + A_2(f)P_2(f) = \mathrm{id}_X\,.$$

Moreover, for $\{i, j\} = \{1, 2\}$, we have

$$\left(A_i(f)P_i(f)\right)^2 = A_i(f)P_i(f)\left(\mathrm{id}_X - A_j(f)P_j(f)\right) = A_i(f)P_i(f)$$

since by assumption $P_i(f)P_j(f) = 0$. Both $A_1(f)P_1(f)$ and $A_2(f)P_2(f)$ are idempotents in $\mathrm{End}_{\mathcal{C}}(X)$; since $\mathcal{C}$ is Karoubian, each of these idempotents has

a kernel. In the following, we fix $\{i,j\} = \{1,2\}$. Let $\iota_i : X_i \to X$ be the kernel of $A_i(f)P_i(f)$. We show that in fact ι_i is the kernel of $P_i(f)$. This amounts to the following verification. First, we must show that $P_i(f) \circ \iota_i = 0$. Then, if we have a commutative diagram



we have to show that there exists a unique $\widetilde{g} : Y \to X_i$ such that $\iota_i \circ \widetilde{g} = g$. The first point is easy: as $\mathrm{id}_X = A_1(f)P_1(f) + A_2(f)P_2(f)$ and $A_i(f)P_i(f) \circ \iota_i = 0$ by definition, we have

$$\iota_i = A_j(f)P_j(f) \circ \iota_i.$$

Since $P_i(f) \circ P_j(f) = 0$, we are done. Now, assume that we have a commutative diagram as above. Since $P_i(f) \circ g = 0$, we have $A_i(f)P_i(f) \circ g = 0$. By definition of a kernel, there exists a unique map $\widetilde{g} : Y \to X_i$ satisfying $\iota_i \circ \widetilde{g} = g$, which proves the assertion. Taking $Y = X$ and $g = A_j(f)P_j(f)$ in the above diagram, we obtain a map $\pi_i : X \to X_i$ such that $\iota_i \circ \pi_i = A_j(f)P_j(f)$. Moreover, we have

$$\begin{cases} \pi_i \circ \iota_i = \mathrm{id}_{X_i}, \\ \pi_i \circ \iota_j = 0. \end{cases}$$

Indeed, as ι_i is a kernel, it is in particular a monomorphism (i.e. $\iota_i \circ f = \iota_i \circ g$ implies $f = g$). One the one hand, we have

$$\iota_i \circ \pi_i \circ \iota_j = A_j(f)P_j(f) \circ \iota_j = 0$$

which gives the second equality. On the other hand, as above we have

$$\iota_i = A_j(f)P_j(f) \circ \iota_i = \iota_i \circ \pi_i \circ \iota_i$$

and we deduce the first equality. In summary, we have

$$\pi_i : X \to X_i, \qquad \pi_j : X \to X_j, \qquad \iota_i : X_i \to X, \qquad \iota_j : X_j \to X$$

satisfying the following equalities:

1. $\pi_i \circ \iota_i = \mathrm{id}_{X_i}$,
2. $\pi_i \circ \iota_j = 0$,
3. $\iota_i \circ \pi_i = A_j(f)P_j(f)$.

Since we have

$$\mathrm{id}_X = \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2$$

we deduce that we have a decomposition $X \cong X_1 \oplus X_2$.

Now assume that the lemma is true for any family of n polynomials (satisfying the correct assumptions) and consider a family $\{P_i\}_{i=1,\dots,n+1}$ as in the statement of the lemma. Denote by Q the product of the first n elements of this family: $Q = \prod_{i=1}^{n} P_i$. Now Q and P_{n+1} are still coprime and we clearly have $Q(f)P_{n+1}(f) = 0$. With the case $n = 2$, we find a decomposition $X \cong X_Q \oplus X_{n+1}$ such that X_Q is the kernel of $Q(f)$ and X_{n+1} is the kernel of $P_{n+1}(f)$. In particular, (the restriction of) $Q(f) = \prod_{i=1}^{n} P_i(f)$ is the zero morphism of X_Q. We can thus apply the induction hypothesis to the data X_Q, $\{P_i\}_{i=1,\dots,n}$ to conclude the proof. □

Remark 3.1.2. The equality $\pi_i \circ \iota_i = \mathrm{id}_X$ implies easily that the π_i's are epimorphisms.

Recall that the center $Z(\mathcal{C})$ of an additive category $\mathcal{C}$ is defined to be the endomorphism ring of the identity functor $\mathrm{id}_{\mathcal{C}}$. If $\mathcal{C}$ is an additive $\mathbb{K}$-linear category, then $Z(\mathcal{C})$ is clearly a commutative $\mathbb{K}$-algebra. For any objects X, Y in $\mathcal{C}$ and any morphism $f : X \to Y$, an element $\varphi \in Z(\mathcal{C})$ satisfies

$$\varphi(Y) \circ f = f \circ \varphi(X).$$

Lemma 3.1.3. *Consider a family $\{f_i\}_{i=1,\dots,n}$ of pairwise commuting elements in a $\mathbb{K}$-algebra E. Assume that there exists elements $\{a_i\}_{i=1,\dots,n}$ in $\mathbb{K}$ such that*

$$(f_i - a_i) \text{ is nilpotent for any } i.$$

Then for any family of integers $\{m_i\}_{i=1,\dots,n} \subseteq \mathbb{Z}_{>0}$, the element

$$\prod_{i=1}^{n} f_i^{m_i} - \prod_{i=1}^{n} a_i^{m_i}$$

is nilpotent in E.

Proof. We proceed by induction on n. For the case $n = 1$, one remarks that for any $a, b \in E$ such that $ab = ba$ and $q > 0$ we have

$$(a^q - b^q) = (a - b)(a^{q-1} + \cdots + a^{q-k}b^{k-1} + \cdots + b^{q-1}).$$

If the statement of the lemma is true for an integer $n \geq 1$, we write

$$\begin{aligned}\prod_{i=1}^{n+1} f_i^{m_i} - \prod_{i=1}^{n+1} a_i^{m_i} &= \prod_{i=1}^{n+1} f_i^{m_i} - \left(\prod_{i=1}^{n} f_i^{m_i}\right) a_{n+1}^{m_{n+1}} + \left(\prod_{i=1}^{n} f_i^{m_i}\right) a_{n+1}^{m_{n+1}} - \prod_{i=1}^{n+1} a_i^{m_i} \\ &= \prod_{i=1}^{n} f_i^{m_i} \left(f_{n+1}^{m_{n+1}} - a_{n+1}^{m_{n+1}}\right) + \left(\prod_{i=1}^{n} f_i^{m_i} - \prod_{i=1}^{n} a_i^{m_i}\right) a_{n+1}^{m_{n+1}}\end{aligned}$$

and we apply the induction hypothesis on the last term and the case $n = 1$ to conclude the proof. □

We give a specific lemma that we will use in the proof of the proposition below.

Lemma 3.1.4. *Let R be a finitely generated, commutative $\mathbb{K}$-algebra with $\mathbb{K}$ a field. Let $\{r_1, \dots, r_n\}$ be a generating family of R. Assume that we are given the following data: a $\mathbb{K}$-algebra morphism $f : R \to E$ and scalars $\{a_1, \dots, a_n\}$ such that $f(r_i) - a_i$ is nilpotent on E for any $i \in \{1, \dots, n\}$. Then,*

$$R/\langle r_i - a_i \mid i = 1, \dots, n\rangle \cong \mathbb{K}$$

and $f(\langle r_i - a_i \mid i = 1, \dots, n\rangle)$ acts nipotently on E (i.e. multiplication by the image of any element of this ideal under f is a nilpotent endomorphism of E).

Proof. Let $J = \langle r_i - a_i \mid i = 1, \dots, n\rangle \subseteq R$. Note that the last assertion of the lemma is clear from the commutativity of R and the fact that a sum of commuting nilpotent elements is nilpotent. We have a surjective morphism $\mathbb{K}[x_1, \dots, x_n] \twoheadrightarrow R$ defined by $x_i \mapsto r_i$. Consider the ideal $I := \langle x_i - a_i \mid i = 1, \dots, n\rangle$ in $\mathbb{K}[x_1, \dots, x_n]$. The image of I in R is clearly J and R/J is a quotient of

$$\mathbb{K}[x_1, \dots, x_n]/I \cong \mathbb{K}\,.$$

Assume that $R/J = \{0\}$. This means that $R = J$, but then $1_E = f(1_R) \in f(J)$ would act in a nilpotent way on E. This is absurd, so $R/J \neq \{0\}$ and so is equal to $\mathbb{K}$. □

Remark 3.1.5. It is immediate that the assignment $r_i \mapsto a_i$ extends to an algebra morphism $R \to \mathbb{K}$, which is in fact the projection

$$R \twoheadrightarrow R/J \cong \mathbb{K}\,.$$

For a $\mathbb{K}$-algebra R, we denote $\mathrm{Specm}(R) := \{\mathfrak{m} \subseteq R \mid \mathfrak{m} \text{ maximal ideal}\}$. If $\mathbb{K}$ is algebraically closed and R finitely generated, the Nullstellensatz tells us that there is a bijection between $\mathrm{Specm}(R)$ and some subset of $\mathbb{K}^n$ (for a certain n). More precisely, if $R \cong \mathbb{K}[x_1, \dots, x_n]/I$ for some ideal I, we have a bijection

$$V(I) := \{x \in \mathbb{K}^n \mid \forall f \in I,\ f(x) = 0\} \longleftrightarrow \mathrm{Specm}(R).$$

In particular, we can identify a maximal ideal in R with an n-tuple of scalars.

Proposition 3.1.6. *Let $\mathcal{C}$ be an additive Karoubian $\mathbb{K}$-linear category with $\mathbb{K}$ an algebraically closed field and let R be a finitely generated commutative $\mathbb{K}$-algebra. Assume that $\mathrm{End}_{\mathcal{C}}(Y)$ is finite dimensional over $\mathbb{K}$ for any object Y of $\mathcal{C}$. Assume moreover that we are given a nonzero $\mathbb{K}$-algebra morphism*

$$\varphi : R \to Z(\mathcal{C}).$$

Consider an object X of $\mathcal{C}$. Then we have a canonical functorial decomposition $X \cong \bigoplus_{\mathfrak{m} \in \mathrm{Specm}(R)} X_{\mathfrak{m}}$ where the sum is finite and such that $\mathfrak{m}$ acts nilpotently on $X_{\mathfrak{m}}$ (i.e. $\varphi(\mathfrak{m}^n)(X_{\mathfrak{m}}) \subseteq \{0\}$ for a certain n) for all $\mathfrak{m} \in \mathrm{Specm}(R)$.

Before the proof, we introduce a notation: for any $\mathfrak{m} \in \mathrm{Specm}(R)$, let $\mathcal{C}_{\mathfrak{m}}$ be the full subcategory of $\mathcal{C}$ whose objects are those X such that $\varphi(\mathfrak{m})(X) \subseteq \mathrm{End}_{\mathcal{C}}(X)$ acts in a nilpotent way on X.

Proof. Consider a finite generating set $\{r_1, \dots, r_n\}$ of R. For any $i \in \{1, \dots, n\}$, set $\varphi_i := \varphi(r_i)(X)$. As R is a commutative algebra, we have $[\varphi_i, \varphi_j] := \varphi_i \circ \varphi_j - \varphi_j \circ \varphi_i = 0$ for all i, j. We treat in detail the case $n = 2$. We thus consider φ_1 and φ_2, two commuting endomorphisms of X.

Since φ_1 generates a finite dimensional algebra in $\mathrm{End}_{\mathcal{C}}(X)$, we can consider the minimal polynomial $P^1 \in \mathbb{K}[x]$ of φ_1 (with φ_1 viewed as an endomorphism of X). As $\mathbb{K}$ is algebraically closed, we can certainly write

$$P^1 = \prod_{i \in \Lambda^1} (x - a_i^1)^{m_i^1}$$

with Λ^1 a finite index set and $a_i^1 \neq a_j^1$ for $i \neq j$. The $P_i^1(x) := (x - a_i^1)^{m_i^1}$ are pairwise coprime, so we can apply lemma 3.1.1 to obtain a decomposition

$$X \cong \bigoplus_{i \in \Lambda^1} X_i^1 \tag{3.1.1}$$

where X_i^1 is the kernel of $P_i^1(\varphi_1)$. Note that this implies that $(\varphi_1 - a_i^1)$ acts nilpotently on X_i^1. A bit more precisely, for any $i, j \in \Lambda^1$, we have maps $\pi_i^1 : X \to X_i^1$ and $\iota_i^1 : X_i^1 \to X$ such that $\pi_i^1 \circ \iota_i^1 = \mathrm{id}_{X_i^1}$ and $\pi_j^1 \circ \iota_i^1 = 0$ if $i \neq j$. Now for $i \neq j$ in Λ^1 we have

$$\pi_j^1 \circ \varphi_2 \circ \iota_i^1 = \pi_j^1 \circ \iota_i^1 \circ \varphi(r_2)(X_i^1) = 0.$$

This tells us that φ_2 preserves the decomposition (3.1.1). Since the endomorphism $(\varphi_2)_{|X_i^1}$ induced by φ_2 on X_i^1 generates a finite dimensional $\mathbb{K}$-algebra in $\mathrm{End}_{\mathcal{C}}(X_i^1)$, we can apply the same reasoning to it to find a decomposition

$$X_i^1 \cong \bigoplus_{j \in \Lambda_i^2} X_{i,j}^{1,2} \tag{3.1.2}$$

and scalars $a_{i,j}^{1,2}$ such that $((\varphi_2)_{|X_i^1} - a_{i,j}^{1,2})$ acts nilpotently on $X_{i,j}^{1,2}$. We then glue the various decompositions (3.1.2) for all $i \in \Lambda^1$ to obtain a decomposition $X \cong \bigoplus_{i,j} X_{i,j}$ and scalars $\{a_{i,j}\}_{i,j}$ such that $(\varphi_1 - a_{i,j})$ and $(\varphi_2 - a_{i,j})$ act in a nilpotent way on $X_{i,j}$. Now, tanks to lemma 3.1.4 and remark 3.1.5, we see that the the assignment

$$r_1 \mapsto a_i, \qquad r_2 \mapsto a_j$$

for any (i, j) defines an algebra morphism $R \to \mathbb{K}$; moreover the maximal ideal $\langle r_1 - a_i,\ r_2 - a_j \rangle$ acts nilpotently on $X_{i,j}$ (via φ). Now for $n \geq 2$, the preceding reasoning extends in a straightforward way to give a finite decomposition

$$X \cong \bigoplus_{\mathfrak{m} \in \mathrm{Specm}(R)} X_{\mathfrak{m}}$$

such that each $\mathfrak{m}$ acts nilpotently on $X_{\mathfrak{m}}$.

We show that this decomposition is canonical and functorial, i.e. that it does not depend on the choice of the generating set $\{r_1, \dots, r_n\}$ and that it behaves

well with morphisms (in a sense to be precised). Note first that if $\mathfrak{m} \neq \mathfrak{m}'$, then $\mathfrak{m}'$ does **not** act in a nilpotent way on $X_\mathfrak{m}$. Indeed, we know that $\mathfrak{m}$ and $\mathfrak{m}'$ corresponds uniquely to some points $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of $\mathbb{K}^n$ (i.e. we have $\mathfrak{m} = \langle r_i - a_i \rangle$ and $\mathfrak{m}' = \langle r_i - b_i \rangle$). Since $\mathfrak{m} \neq \mathfrak{m}'$, we can assume that $a_1 \neq b_1$. For any $r \in \{r_1, \ldots, r_n\}$, we have $(\varphi(r)(X_\mathfrak{m}) - b_1) = (a_1 - b_1) + (\varphi(r)(X_\mathfrak{m}) - a_1)$. On $X_\mathfrak{m}$, the second term of the right-hand-side is nilpotent whereas the first is invertible. As they commute, we deduce that $(\varphi(r)(X_\mathfrak{m}) - b_1)$ is invertible on $X_\mathfrak{m}$ (and so, in particular, non-nilpotent).

Consider another object Y of $\mathcal{C}$ such that the vector space $\mathrm{End}_\mathcal{C}(Y)$ is finite dimensional and any map $f : X \to Y$. As above, we have a decomposition $Y \cong \bigoplus_\mathfrak{m} Y_\mathfrak{m}$, but we don't assume that this decomposition is obtained from the same generating set $\{r_1, \ldots, r_n\}$. Anyway, $\mathfrak{m}$ acts in a nilpotent way on $Y_\mathfrak{m}$. Let $\pi^X_\mathfrak{m} : X \to X_\mathfrak{m}$ and $\iota^X_\mathfrak{m} : X_\mathfrak{m} \to X$ be the epimorphisms and monomorphisms respectively that define the decomposition $X \cong \bigoplus X_\mathfrak{m}$; we will use the exact analogous (and obvious) notation for Y.

We show that $\pi^Y_{\mathfrak{m}'} \circ f \circ \iota^X_\mathfrak{m}$ is zero for $\mathfrak{m}' \neq \mathfrak{m}$. Set $\mathfrak{m} = \langle r_1 - a_1, \ldots, r_n - a_n \rangle$ and $\mathfrak{m}' = \langle r_1 - b_1, \ldots, r_n - b_n \rangle$. We can assume that $a_1 \neq b_1$. Since $X_\mathfrak{m}$ is in $\mathcal{C}_\mathfrak{m}$, there exists a $n_1 \geq 0$ such that $(\varphi(r_1)(X_\mathfrak{m}) - a_1)^{n_1}$ is the zero endomorphism of $X_\mathfrak{m}$. We get:

$$\pi^Y_{\mathfrak{m}'} \circ f \circ \iota^X_\mathfrak{m} \circ (\varphi(r_1)(X_\mathfrak{m}) - a_1)^{n_1} = (\varphi(r_1) Y_{\mathfrak{m}'} - a_1)^{n_1} \circ \pi^Y_{\mathfrak{m}'} \circ f \circ \iota^X_\mathfrak{m} = 0.$$

But $(\varphi(r_1)(Y_{\mathfrak{m}'}) - a_1)$ is an invertible endomorphism of $Y_{\mathfrak{m}'}$, as noticed above. We deduce that $\pi^Y_{\mathfrak{m}'} \circ f \circ \iota^X_\mathfrak{m} = 0$. This tells us that $f(X_\mathfrak{m}) \subseteq Y_\mathfrak{m}$. We also deduce that

$$\mathrm{Hom}_\mathcal{C}(X_\mathfrak{m}, Y_{\mathfrak{m}'}) = 0$$

if $\mathfrak{m} \neq \mathfrak{m}'$ (and more generally, that there is no morphism between an object of $\mathcal{C}_\mathfrak{m}$ and an object of $\mathcal{C}_{\mathfrak{m}'}$). This shows that our decomposition is indeed canonical (i.e. does not depend on the generating set of R chosen) and functorial. □

Corollary 3.1.7. *Consider the situation of proposition 3.1.6. Assume moreover that for any object X in $\mathcal{C}$, the $\mathbb{K}$-vector space $\mathrm{End}_\mathcal{C}(X)$ is finite dimensional. Then we have a direct sum of categories*

$$\mathcal{C} = \bigoplus_{\mathfrak{m} \in \mathrm{Specm}(R)} \mathcal{C}_\mathfrak{m}.$$

3.2 Decomposition in the topological monodromic category

In this section, we will use freely the notation of chapter 1; we apply the results of the preceding section to the category $D^b_\mathcal{S}(X, \Bbbk)$. Consider an object $\mathscr{F} \in D^b_\mathcal{S}(X, \Bbbk)$. We have the monodromy morphism of $\mathscr{F}$

$$\varphi_\mathscr{F} : \Bbbk[\mathrm{X}_*(A)] \longrightarrow \mathrm{End}_{D^b_\mathcal{S}(X,\Bbbk)}(\mathscr{F}).$$

This map defines an action of $\Bbbk[\mathrm{X}_*(A)]$ on $\mathscr{F}$. For any $\lambda \in \mathrm{X}_*(A)$, we denote the corresponding basis element of $\Bbbk[\mathrm{X}_*(A)]$ by e^λ. We consider the dual $\Bbbk$-torus $A^\vee_\Bbbk$ of A. By definition, this is the algebraic torus over $\Bbbk$ whose character lattice is the cocharacter lattice of A i.e. $\mathrm{X}_*(A) = \mathrm{X}^*(A^\vee_\Bbbk)$. It is a standard fact that the algebra of regular functions on a torus is isomorphic to the group algebra of the characters of the torus. We get

$$\Bbbk[\mathrm{X}_*(A)] = \Bbbk[\mathrm{X}^*(A^\vee_\Bbbk)] \cong \mathscr{O}(A^\vee_\Bbbk).$$

Thus we want to determine the algebra morphisms $\mathscr{O}(A^\vee_\Bbbk) \to \Bbbk$. It is well know that these morphisms correspond to closed points of the affine variety $A^\vee_\Bbbk$. For any $t \in A^\vee_\Bbbk$, let $\mathfrak{m}_t$ denote the ideal

$$\mathfrak{m}_t := \langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle.$$

We have an bijective map

$$A^\vee_\Bbbk \xleftrightarrow{\sim} \mathrm{Specm}(\Bbbk[\mathrm{X}_*(A)])$$

where $t \in A^\vee_\Bbbk$ is sent to $\mathfrak{m}_t$.

We then define $D^b_{\mathcal{S}}(X,\Bbbk)_{[t]}$ to be the full subcategory of $D^b_{\mathcal{S}}(X,\Bbbk)$ whose objects are given by

$$D^b_{\mathcal{S}}(X,\Bbbk)_{[t]} = \{\mathscr{F} \in D^b_{\mathcal{S}}(X,\Bbbk) \mid \mathfrak{m}_t \text{ acts in a nilpotent way via } \varphi_{\mathscr{F}}\}.$$

We also let $D^b_{\mathcal{S}}(X,\Bbbk)_{[\![t]\!]}$ be the full subcategory of $D^b_{\mathcal{S}}(X,\Bbbk)$ whose object are those $\mathscr{F}$ such that the monodromy morphism $\varphi_{\mathscr{F}} : \Bbbk[\mathrm{X}_*(A)] \to \mathrm{Aut}(\mathscr{F})$ factors through the quotient $\Bbbk[\mathrm{X}_*(A)]/\mathfrak{m}_t$. Note that this is not a triangulated subcategory.

Now thanks to [LC], the category $D^b_{\mathcal{S}}(X,\Bbbk)$ is Karoubian; it is a standard fact that the Hom-spaces are finite dimensional $\Bbbk$-vector spaces in this category. Finally, thanks to lemma 1.5.1 the monodromy defines an algebra map

$$\varphi : \Bbbk[\mathrm{X}_*(A)] \to Z(D^b_{\mathcal{S}}(X,\Bbbk)), \quad \lambda \mapsto (\varphi(\lambda) : \mathscr{F} \mapsto \varphi_{\mathscr{F}}(\lambda)).$$

We deduce from proposition 3.1.6 and the discussion above that we have a direct sum of subcategories

$$\bigoplus_{t \in A^\vee_\Bbbk} D^b_{\mathcal{S}}(X,\Bbbk)_{[t]} \cong D^b_{\mathcal{S}}(X,\Bbbk) \tag{3.2.1}$$

We can now give an auxiliary lemma that will be useful latter:

Lemma 3.2.1. *Let $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ be two stratified A-varieties and $f : X \to Y$ an A-equivariant morphism.*

1. *Assume that f is locally trivial (see definition 1.5.3). Then the functors f_* and $f_!$ map $D^b_{\mathcal{S}_X}(X,\Bbbk)_{[t]}$ to $D^b_{\mathcal{S}_Y}(Y,\Bbbk)_{[t]}$.*

2. *Assume that for any $S \in \mathcal{S}_X$, we have $f(S) \in \mathcal{S}_Y$. Then the functors f^* and $f^!$ map $D^b_{\mathcal{S}_Y}(Y,\Bbbk)_{[t]}$ to $D^b_{\mathcal{S}_X}(X,\Bbbk)_{[t]}$.*

In particular, assume that A acts trivially on Y and that f is locally trivial. If $\mathscr{F}$ is in $D^b_{\mathcal{S}_X}(X,\Bbbk)_{[t]}$ for $t \in A^\vee_\Bbbk \setminus \{1\}$, then $f_!(\mathscr{F}) \cong 0$.

Proof. The first two points follow directly from lemma 1.5.5 and the functorial decomposition 3.2.1.

To deduce the last statement, one should first remark that we have $D^b_{\mathcal{S}_Y}(Y,\Bbbk) = D^b_{\mathcal{S}_Y}(Y,\Bbbk)_{[1]}$, as follows immediately from the definition of monodromy, using proposition 1.2.1 and our hypothesis on the A-space Y (in fact, for any $\mathscr{G} \in D^b_{\mathcal{S}_Y}(Y,\Bbbk)$ and $r \in \Bbbk[\mathrm{X}_*(A)]$, we have $\varphi_{\mathscr{G}}(r) = \varepsilon(r)\,\mathrm{id}_{\mathscr{G}}$ where $\varepsilon : \Bbbk[\mathrm{X}_*(A)] \to \Bbbk$ is the augmentation morphism). Point (1) allows us to conlcude. □

3.3 Monodromic local systems on A

The decomposition obtained in the last section allows us to study local systems on A. If we let $D^b_{(A)}(A,\Bbbk)$ be the full category of $D^b(A,\Bbbk)$ whose objects are those $\mathscr{F}$ such that $\mathscr{H}^i(\mathscr{F})$ is a local system on A for any $i \in \mathbb{Z}$ (morally, the A-constructible complexes), we can naturally view the category $\mathrm{Loc}(A,\Bbbk)$ as a full subcategory of $D^b_{(A)}(A,\Bbbk)$. This allows us to define monodromy for local systems.

We need first some observation: denote by γ the application $[0,1] \to \mathbb{C}^*$ which maps t to $e^{2i\pi t}$. It is a classical fact that the application $\lambda \mapsto \lambda \circ \gamma$ gives a group isomorphism $\mathrm{X}_*(A) \to \pi_1(A)$. Using theorem 1.1.3, we obtain an equivalence $\mathrm{Loc}(A,\Bbbk) \cong \Bbbk[\mathrm{X}_*(A)]\text{-mod}$, where the latter category is the category of finite dimensional $\Bbbk[\mathrm{X}_*(A)]$-modules (or equivalently, finite dimensional $\Bbbk$-representations of the abelian group $\mathrm{X}_*(A)$).

Finally, the discussion at the beginning of section 3.2 tells us that to give an element of the dual torus $A^\vee_\Bbbk$ amounts to giving an irreducible finite dimensional representation of $\mathrm{X}_*(A)$ (equivalently an irreducible finite dimensional $\Bbbk[\mathrm{X}_*(A)]$-module): more precisely, we have a bijective map

$$A^\vee_\Bbbk \xleftrightarrow{\sim} \mathrm{Irr}\,(\Bbbk[\mathrm{X}_*(A)]\text{-mod})$$

where $\mathrm{Irr}\,(\Bbbk[\mathrm{X}_*(A)]\text{-mod})$ denotes the set of isomorphism classes of irreducible finite dimensional $\Bbbk[\mathrm{X}_*(A)]$-modules.

Remark 3.3.1. Considering a trivialisation $A^\vee_\Bbbk \cong (\Bbbk^*)^r$, an element of $A^\vee_\Bbbk$ can be viewed as an r-tuple $(t_1,\dots,t_r)$ of elements of $\Bbbk^*$. Since $\Bbbk$ is the union of its finite subfields, each of these t_i can be viewed as an element in a finite field, so each of these is of finite order in $\Bbbk^*$. We deduce easily that the element $t \in A^\vee_\Bbbk$ is of finite order in the torus. Note also that, $\Bbbk$ being the algebraic closure of a finite field, it is a perfect field of positive characteristic. Thus the morphism $x \mapsto x^\ell$ is a field automorphism and so the orders of x and x^ℓ are the same. Thus ℓ does not divide the order of any $t \in A^\vee_\Bbbk$.

For $t \in A^\vee_{\Bbbk}$, we denote by $\mathscr{L}^A_t$ the rank-one local system on A corresponding to the $\mathrm{X}_*(A)$-representation $L_t := \Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle$. More generally, denote by $\mathscr{L}^A_{t,n}$ the local system on A associated to the $\Bbbk[\mathrm{X}_*(A)]$-module

$$\Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(A)\rangle^n,$$

so that we have $\mathscr{L}^A_{t,1} = \mathscr{L}^A_t$.

Remark 3.3.2. If $m : A \times A \to A$ denotes the usual multiplication, we have that for any $t \in A^\vee_{\Bbbk}$ we have $m^*(\mathscr{L}^A_t) \cong \mathscr{L}^A_t \boxtimes \mathscr{L}^A_t$, that is to say, that $\mathscr{L}^A_t$ is a multiplicative local system. Indeed, the pullback of a local system is again a local system, and so we can rephrase the question in terms of representations. The map

$$\mathrm{X}_*(A) \times \mathrm{X}_*(A) \cong \pi_1(A,1) \times \pi_1(A,1) \to \pi_1(A,1) \cong \mathrm{X}_*(A)$$

induced by m is just the addition. What is left to do is to identify the following $\mathrm{X}_*(A) \times \mathrm{X}_*(A)$-representations: on the one hand,

$$\Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t)\rangle \text{ with action } (\lambda,\mu)\cdot x = e^{\lambda+\mu}x$$

and on the other hand

$$\big(\Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t)\rangle\big)^{\otimes 2} \text{ with action } (\lambda,\mu)\cdot x\otimes y = e^\lambda x \otimes e^\mu y.$$

The multiplication map $\big(\Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t)\rangle\big)^{\otimes 2} \to \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t)\rangle$ gives such an isomorphism.

Consider a local system $\mathscr{L}$ on A and denote by L its associated $\mathrm{X}_*(A)$-representation. The canonical monodromy action on $\mathscr{L}$ gives under the equivalence of theorem 1.1.3 an action of $\mathrm{X}_*(A)$ on L defined by a morphism

$$\varphi_{\mathscr{L}} : \mathrm{X}_*(A) \to \mathrm{End}_{\mathrm{X}_*(A)}(L).$$

Lemma 3.3.3. *With the above notation, we have $\varphi_{\mathscr{L}}(\lambda) = (e^\lambda)\cdot(-)$, the right-hand side being the action of λ on L given by the structure of $\mathrm{X}_*(A)$-module.*

Proof. The monodromy is defined by (the appropriate restriction of) an isomorphism $\iota : \mathrm{pr}_2^*\mathscr{L} \to m(n)^*\mathscr{L}$ satisfying $\iota_{|\{1\}\times A} = \mathrm{id}_{\mathscr{L}}$. In view of theorem 1.1.3, this amounts to $\iota_{(1,1)} = \mathrm{id}_L$. What we want to determine is $\iota_{|(\lambda_n,1)}$ for any $\lambda \in \mathrm{X}_*(A)$. In the following, we will denote by $\mathrm{pr}_2^* L$ and $m(n)^*L$ the $\mathrm{X}_*(A)$-representations associated to $\mathrm{pr}_2^*\mathscr{L}$ and $m(n)^*\mathscr{L}$ respectively. The action of a pair (λ,μ) on $\mathrm{pr}_2^* L$ (resp. $a(n)^*L$) is given by the action of μ (resp. $n\lambda+\mu$) on L. We start by some general considerations. Consider a topological space X satisfying the condition of theorem 1.1.3 and a loop $\gamma : [0,1] \to X$. The pullback of any local system $\mathscr{F}$ on X under γ gives a local system on $[0,1]$, which is trivial since $[0,1]$ is simply connected. We know that we then have a canonical identification $\Gamma([0,1],\gamma^*(\mathscr{F})) \cong (\gamma^*(\mathscr{F}))_x$ for any $x \in [0,1]$. Moreover, if $\mathscr{H}$

is another local system on X and $f : \mathscr{F} \to \mathscr{H}$ is an isomorphism, we have a commutative diagram (for any $x, y \in [0,1]$)

$$\begin{array}{ccccc} & & (\gamma^*(\mathscr{F}))_x & \xrightarrow[\sim]{(\gamma^*(f))_x} & (\gamma^*(\mathscr{H}))_x & & \\ & \overset{\sim}{\nearrow} & & & \overset{\sim}{\nwarrow} & & \\ \Gamma([0,1], \gamma^*(\mathscr{F})) & & & & & & \Gamma([0,1], \gamma^*(\mathscr{H})) \\ & \underset{\sim}{\searrow} & & & \underset{\sim}{\swarrow} & & \\ & & (\gamma^*(\mathscr{F}))_y & \xrightarrow[(\gamma^*(f))_y]{\sim} & (\gamma^*(\mathscr{H}))_y. & & \end{array} \tag{3.3.1}$$

Now, for any $n \geq 1$, let γ_n be the path $[0,1] \to \mathbb{C}^*, \quad t \mapsto e^{\frac{2i\pi t}{n}}$ and γ_n^λ the loop in A given by $\lambda \circ \gamma_n$. We consider the diagram (3.3.1) with $X = A \times A$, $\mathscr{F} = \mathrm{pr}_2^*(\mathscr{L})$, $\mathscr{H} = m(n)^*(\mathscr{L})$, the isomorphism $f = \iota$ and $\gamma = \gamma_n^\lambda \times \gamma_1^0$. Remark that γ_1^0 is just a fancy notation for the constant path at $1 \in A$. Also we choose $x = 0$ and $y = 1$. We get

$$\begin{array}{ccc} (\gamma^* \,\mathrm{pr}_2^*(\mathscr{L}))_0 & \xrightarrow{\mathrm{id}} & (\gamma^* m(n)^*(\mathscr{L}))_0 \\ \wr \downarrow & & \downarrow \wr \\ (\gamma^* \,\mathrm{pr}_2^*(\mathscr{L}))_1 & \xrightarrow[\sim]{} & (\gamma^* m(n)^*(\mathscr{L}))_1. \end{array}$$

We then remark that $\mathrm{pr}_2 = \mathrm{pr}_2 \circ (e_n \times \mathrm{id}_A)$ and that we have $(e_n \times \mathrm{id}_A) \circ \gamma = (\gamma_1^\lambda \times \gamma_1^0)$. The left (resp. right) vertical arrow is given (at the level of representations) by the action of $(\gamma_1^\lambda \times \gamma_1^0)$ on $\mathrm{pr}_2^* L$ (resp. $m(n)^* L$), which is by definition the action of 0 (resp. λ) on L. We can rewrite this diagram in the following form

$$\begin{array}{ccc} (\mathrm{pr}_2^*(\mathscr{L}))_{(1,1)} & \xrightarrow{\mathrm{id}=\iota_{(1,1)}} & (m(n)^*(\mathscr{L}))_{(1,1)} \\ \wr \downarrow & & \downarrow \wr \\ (\mathrm{pr}_2^*(\mathscr{L}))_{(\lambda_n,1)} & \xrightarrow[\sim]{\varphi_{\mathscr{L}}(\lambda)} & (m(n)^*(\mathscr{L}))_{(\lambda_n,1)}. \end{array}$$

and we can then conclude the proof.

□

We let $D^b_{\mathrm{Loc}}(A, \Bbbk)$ denote the full subcategory of the bounded derived category of sheaves of $\Bbbk$-modules on A whose objects are complexes $\mathscr{L}$ such that $\mathscr{H}^i(\mathscr{F}) \in \mathrm{Loc}(A, \Bbbk)$ for all $i \in \mathbb{Z}$. By definition, this category is the same as the category $D^b_{(A)}(A, \Bbbk)$ of A-constructible complexes of sheaves on A. We deduce immediately from lemma 3.3.3 the following result.

Corollary 3.3.4. *The object $\mathscr{L}_t^A$ is in $D^b_{\mathrm{Loc}}(A, \Bbbk)_{[\underline{t}]}$.*

In fact we can say a bit more. First, let us determine the full subcategory generated by $\mathscr{L}_t^A$ in $\mathrm{Loc}(A, \Bbbk)$. Let $\mathrm{Mod}^{\mathrm{nil},t}(\Bbbk[\mathrm{X}_*(A)])$ denote the category of

finite dimensional $\Bbbk[\mathrm{X}_*(A)]$-modules on which

$$\mathfrak{m}_t := \langle (e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(A) \rangle$$

acts nilpotently.

Lemma 3.3.5. *The full subcategory of* $\mathrm{Loc}(A, \Bbbk)$ *whose objects are local systems that are extension of copies of* $\mathscr{L}_t^A$ *is equivalent to the category* $\mathrm{Mod}^{\mathrm{nil},t}(\Bbbk[\mathrm{X}_*(A)])$.

Proof. The equivalence is given by the restriction of the equivalence of theorem 1.1.3. If a local system $\mathscr{L}$ is an extension of copies of $\mathscr{L}_t^A$, then the associated representation will clearly be annihilated by a power of $\mathfrak{m}_t$. In the other direction, start by noticing that if L is a finite dimensional $\Bbbk[\mathrm{X}_*(T)]$-module such that $\mathfrak{m}_t \cdot L = \{0\}$, then $L \cong \bigoplus L_t$ with $L_t := \Bbbk[\mathrm{X}_*(T)]/\mathfrak{m}_t$. Now if a power of $\mathfrak{m}_t$ annihilates L, we have a filtration $L \supseteq \mathfrak{m}_t \cdot L \supseteq \mathfrak{m}_t^2 \cdot L \cdots$ (as the action of $\mathfrak{m}_t$ is nilpotent, this sequence eventually ends). The module L is a successive extension of the $\mathfrak{m}_t^i \cdot L / \mathfrak{m}_t^{i+1} \cdot L$, and each of these satisfy $\mathfrak{m}_t \cdot (\mathfrak{m}_t^i \cdot L / \mathfrak{m}_t^{i+1} \cdot L) = \{0\}$. The equivalence is thus proved. □

The preceding result allows us to give a more precise description of the category $D^b_{(A)}(A, \Bbbk)_{[t]}$.

Lemma 3.3.6. *The full subcategory* $D^b_{(A)}(A, \Bbbk)_{[t]}$ *of* $D^b_{(A)}(A, \Bbbk)$ *is generated (as a triangulated category) by the object* $\mathscr{L}_t^A$.

Proof. We want to show that any $\mathscr{F} \in D^b_c(A, \Bbbk)_{[t]}$ can be obtained as successive extensions of $\mathscr{L}_t^A$. By a use of truncation triangles and a standard induction argument, it suffices to do so for $\mathscr{F} = \mathscr{L}[k]$ a shifted local system on A. It is harmless to assume that $k = 0$, and we do so. Any local system is of finite length; this means that we have a finite filtration

$$0 = \mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \cdots \subseteq \mathscr{L}_n = \mathscr{L}$$

of $\mathscr{L}$ with simple local systems as successive quotients. We show by induction on the length n of such a filtration that if $\mathscr{L}$ has monodromy t, then it is an successive extension of $\mathscr{L}_t^A$. If $n = 0$, there is nothing to prove; note that if $\mathscr{L}$ is simple (i.e. if $n = 1$) then it is of the form $\mathscr{L}_{t'}^A$ for a certain $t' \in A^\vee_{\Bbbk}$ (any simple object in $\mathrm{Loc}(A, \Bbbk)$ is of this type). By lemma 3.3.4 we have necessarily $t' = t$ and thus $\mathscr{L} \cong \mathscr{L}_t^A$ and we have the result. Now assume that the result is known for local systems of length n and take $\mathscr{L}$ of length $n+1$. We have an exact sequence

$$0 \to \mathscr{L}_n \to \mathscr{L} \to \mathscr{L}_{t'}^A \to 0.$$

Now the arrows in this sequence are obviously nonzero and $\mathscr{L}$ has monodromy t; proposition 3.1.7 tells us that the objects $\mathscr{L}_n$ and $\mathscr{L}_{t'}^A$ also have monodromy t and we can conclude by applying the induction hypothesis to $\mathscr{L}_n$ and the case $n = 1$ to $\mathscr{L}_{t'}^A = \mathscr{L}_t^A$. □

3.4 Decomposition in the equivariant category

We now consider the setting of section 2.1: H is an algebraic group acting on a variety X, and we assume that we have a finite central isogeny $\nu : \widetilde{H} \to H$ from a group $\widetilde{H}$ such that the kernel K is finite and of cardinality coprime to the characteristic of $\Bbbk$. We saw in 2.1 that these data allow us to define an algebra morphism

$$\vartheta : \Bbbk[K] \to Z(D^b_{\widetilde{H}}(X, \Bbbk)), \quad x \mapsto (\mathscr{F} \mapsto \vartheta_{\mathscr{F}}(x)).$$

The maximal ideals of $\Bbbk[K]$ correspond bijectively to characters χ of K. Moreover, we saw that we have a direct sum decomposition $\nu_* \underline{\Bbbk}_{\widetilde{H}} \cong \bigoplus_\chi \mathscr{L}_\chi$ where $\mathscr{L}_\chi$ is the χ-isotypic component of the pushforward for the natural action of K.

Thus we obtain a direct sum decomposition

$$D^b_{\widetilde{H}}(X, \Bbbk) = \bigoplus_{\chi \in \mathrm{Hom}(K, \Bbbk^*)} D^b_{\widetilde{H}}(X, \Bbbk)_\chi,$$

where $D^b_{\widetilde{H}}(X, \Bbbk)_\chi$ is the full subcategory whose objects are those $\mathscr{F}$ for which $\mathfrak{m}_\chi := \langle e^k - \chi(k) \mid k \in K \rangle$ acts nilpotently via $\vartheta_{\mathscr{F}}$.

Let us discuss this splitting of categories. In this particular case, thanks to our assumption on the cardinality of the finite group K, we get that the algebra $\Bbbk[K]$ is semisimple, hence we can write $\Bbbk[K] = \Bbbk^{|K|}$ (as algebras). The maximal ideals $\mathfrak{m}_\chi$ are now identified with subspaces of the form $\{(*, \dots, *, 0, *, \dots, *)\}$ and they are generated by idempotent elements, say e_χ. Since ϑ is a morphism of algebras, we see that for any $\mathscr{F} \in D^b_{\widetilde{H}}(X, \Bbbk)_\chi$, $\vartheta_{\mathscr{F}}(e_\chi)$ is both idempotent and nilpotent, thus is zero. We then immediately deduce that we have in fact $D^b_{\widetilde{H}}(X, \Bbbk)_\chi = \mathfrak{D}(X/\!\!/H)_{\mathscr{L}_\chi}$. We can now restate our result: we have a splitting of categories

$$D^b_{\widetilde{H}}(X, \Bbbk) = \bigoplus_{\chi \in \mathrm{Hom}(K, \Bbbk^*)} \mathfrak{D}(X/\!\!/H)_{\mathscr{L}_\chi},$$

and for any object $\mathscr{F}$ we have a canonical functorial decomposition $\mathscr{F} \cong \bigoplus_{\chi \in \widehat{K}} \mathscr{F}_\chi$ (where $\mathscr{F}_\chi$ lies in $\mathfrak{D}(X/\!\!/H)_{\mathscr{L}_\chi}$).

In the following corollary, we consider the action morphism $a_H : H \times X \to X$.

Corollary 3.4.1. *Assume that $\mathscr{F} \in D^b_{\widetilde{H}}(X, \Bbbk)$. Then $\mathscr{F}$ satisfies $a_H^*(\mathscr{F}) \cong \mathscr{L}_\chi \boxtimes \mathscr{F}$ if and only if we have $\mathscr{F} \in D^b(X, \Bbbk)_\chi = \mathfrak{D}(X/\!\!/H)_{\mathscr{L}_\chi}$.*

Proof. The "if" part has been done in lemma 2.9.3. We deal with the other direction. Thanks to the above proposition, we have a decomposition in direct sum $\mathscr{F} \cong \bigoplus_{\chi'} \mathscr{F}_{\chi'}$, with $\mathscr{F}_{\chi'} \in \mathfrak{D}(X/\!\!/H)_{\mathscr{L}_{\chi'}}$, the sum being taken over the characters of K. Now lemma 2.9.3 tells us that we have $a_H^*(\mathscr{F}_{\chi'}) \cong \mathscr{L}_{\chi'} \boxtimes \mathscr{F}_{\chi'}$. Our assumption on $\mathscr{F}$ thus implies that $\bigoplus_{\chi'} \mathscr{L}_{\chi'} \boxtimes \mathscr{F}_{\chi'} = \mathscr{L}_\chi \boxtimes \mathscr{F}_\chi$. Since $\mathscr{L}_{\chi'}$ is invertible, we must have $\mathscr{F}_{\chi'} = 0$ unless $\chi' = \chi$, which concludes the proof. □

Chapter 4

Perverse monodromic sheaves on stratified A-varieties

4.1 Perverse equivariant monodromic sheaves

According to §3.2, the isomorphism classes of irreducible local systems on A (which correspond to irreducible representations of $\mathrm{X}_*(A)$) are in bijection with elements of $A^\vee_{\Bbbk}$. For $t \in A^\vee_{\Bbbk}$, we denoted by $\mathscr{L}^A_t$ the rank-one local system on A corresponding to the $\mathrm{X}_*(A)$-representation $L_t := \Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(A)\rangle$. Recall that $\mathscr{L}^A_t$ is multiplicative thanks to remark 3.3.2.

Lemma 4.1.1. *For any simple local system $\mathscr{L}^A_t$ on A there exists a finite central isogeny $\nu : \widetilde{A} \to A$ and a character χ of $K := \ker(\nu)$ such that*

1. *the cardinality of K is prime to ℓ,*
2. *$\mathscr{L}^A_t = (\nu_*(\underline{\Bbbk}_A))_\chi$, the isotypic component of $\nu_*(\underline{\Bbbk}_A)$ on which K acts via χ.*

Proof. Denote by n the order of t (recall that t is of finite order thanks to remark 3.3.1, and that this order n is prime to the characteristic ℓ of $\Bbbk$). We consider the map $e_n : A \to A,\ x \mapsto x^n$; this is a finite central isogeny with kernel $K_n := \ker(e_n)$ of order $n^{\mathrm{rk}(A)}$, which is prime to ℓ. The pushforward of the constant local system along e_n is still a local system, thus we can make all our calculations in the representation-theoretic world; $(e_n)_*(\underline{\Bbbk}_A)$ identifies with $\Bbbk[\mathrm{X}_*(A)] \otimes_{\Bbbk[\mathrm{X}_*(A)]} \Bbbk$ where the structure of $\Bbbk[\mathrm{X}_*(A)]$-module on itself in the tensor product is given by $e^\lambda \cdot P = e^{n\lambda}P$. One easily checks that this tensor product identifies with

$$\Bbbk[\mathrm{X}_*(A)/e_n^\#(\mathrm{X}_*(A))]$$

$(e_n^{\#} : \mathrm{X}_*(A) \to \mathrm{X}_*(A), \lambda \mapsto n\lambda$ is the application induced by $e_n)$. Since $\mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A))$ is a finite abelian group of order $n^{\mathrm{rk}(A)}$ (so in particular prime to the characteristic ℓ of $\Bbbk$), its category of $\Bbbk$-representations is semisimple and we can decompose $\Bbbk[\mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A))]$ as a direct sum of simple, one-dimensional, representations, each one associated to a character of this quotient. Thus the associated local system decomposes as a direct sum of rank one local systems according to the characters of $\mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A))$.

We have a non-zero adjunction morphism $\mathscr{L}_t^A \to (e_n)_*(e_n)^* \mathscr{L}_t^A$. Using the diagram in the statement of theorem 1.1.3, it is easy to see that $e_n^*(\mathscr{L}_t^A) = \mathscr{L}_{t^n}^A \cong \underline{\Bbbk}_A$. With the preceding discussion, this adjunction morphism has to be the inclusion of a direct summand and this direct summand is obviously associated to the character of evaluation at t

$$ev(t) : \mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A)) \to \Bbbk^*, \quad [\lambda] \mapsto \lambda(t).$$

This character is well defined precisely because t is of order n.

One easily checks that evaluation at $e^{\frac{2i\pi}{n}}$ gives a (well-defined) isomorphism $\varphi : \mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A)) \xrightarrow{\sim} K_n$ (recall that K_n is the kernel of e_n). The morphism $ev(t) \circ \varphi^{-1}$ gives the wished-for character χ of K_n. □

According to the results of chapter 2 (using lemma 4.1.1), for any A-variety X, we can consider without any ambiguity the category $\mathfrak{D}(X/\!\!/A)_{\mathscr{L}_t^A}$ for any $t \in A_{\Bbbk}^{\vee}$.

Let us fix an isogeny $\nu : \widetilde{A} \to A$ and a character χ of $\ker(\nu)$ that allow us to define $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ (note that we impose constructibility with respect to $\mathcal{S}$ here) as a full direct factor in $D^b_{\widetilde{A},\mathcal{S}}(X, \Bbbk)$. The restriction of the perverse t-structure on $D^b_{\widetilde{A},\mathcal{S}}(X, \Bbbk)$ as defined in [BL, §5.1] to $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ gives a t-structure on this category. (This is true essentially because the category $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ is a full direct-factor-subcategory of $D^b_{\widetilde{A},\mathcal{S}}(X)$.) Note that, by the definition of the perverse t-structure on the equivariant bounded derived category and the discussion before lemma 2.9.3, this t-structure does not depend on the choices of ν and χ. We consider the t-structure obtained as the shift on the right by the rank of A of this perverse t-structure. We obtain in this way a new t-structure on $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$, called the perverse t-structure. Stated otherwise, we have the following definition:

Definition 4.1.2. *We say that an object $\mathscr{F} \in \mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ lies in ${}^p\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)^{\leq 0}_{\mathscr{L}_t^A}$ if and only if $\mathrm{For}(\mathscr{F})[r]$ lies in ${}^pD^b_{\mathcal{S}}(X, \Bbbk)^{\leq 0}$. The positive part of the t-structure is defined similarly; in particular, $\mathscr{F}$ is perverse if and only if the object $\mathrm{For}(\mathscr{F})[r]$ is perverse in $D^b_{\mathcal{S}}(X, \Bbbk)$. The category of perverse monodromic sheaves in the Lusztig–Yun category will be denoted $\mathfrak{P}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$. Its objects are called the Lusztig–Yun monodromic perverse sheaves.*

The restriction of $\mathrm{For}[r]$ to $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ then yields a t-exact functor (by definition) to the constructible category, which we will denote by For_t.

4.2 Perverse monodromic sheaves

In this section, we give four "reasonable definitions" of monodromic perverse sheaves, and show that they actually coincide.

Fix $t \in A^\vee_{\Bbbk}$. Here are the different possibilities for the category of monodromic perverse sheaves with "monodromy t":

1. the heart $\mathfrak{P}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}^A_t}$ of the perverse t-structure in the Lusztig–Yun equivariant monodromic category,

2. the full subcategory $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$ of $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ whose objects are those complexes in $D^b_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]} \cap \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$,

3. the category $\mathrm{P}^t_{\mathrm{co}}(X,\Bbbk)$ whose objects are pairs $(\mathscr{F}, \vartheta_{\mathscr{F}})$ with $\mathscr{F}$ a perverse sheaf in $D^b_{\mathcal{S}}(X,\Bbbk)$ and $\vartheta_{\mathscr{F}}$ an isomorphism

$$\mathscr{L}^A_t \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F})$$

satisfying

$$(m \times \mathrm{id}_X)^*(\vartheta_{\mathscr{F}}) = (\mathrm{id}_A \times a)^*(\vartheta_{\mathscr{F}}) \circ (\mathscr{L}^A_t \boxtimes \vartheta_{\mathscr{F}}) \quad \text{and} \quad (\vartheta_{\mathscr{F}})_{|\{1\}\times X} \cong \mathrm{id}_{\mathscr{F}}$$

and whose morphisms $f : (\mathscr{F}, \vartheta_{\mathscr{F}}) \to (\mathscr{G}, \vartheta_{\mathscr{G}})$ are given by morphisms $f : \mathscr{F} \to \mathscr{G}$ in $D^b_{\mathcal{S}}(X,\Bbbk)$ such that

$$\begin{array}{ccc} \mathscr{L}^A_t \boxtimes \mathscr{F} & \xrightarrow{\mathscr{L}^A_t \boxtimes f} & \mathscr{L}^A_t \boxtimes \mathscr{G} \\ {\scriptstyle \vartheta_{\mathscr{F}}}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle \vartheta_{\mathscr{G}}} \\ a^*(\mathscr{F}) & \xrightarrow[a^*(f)]{} & a^*(\mathscr{G}) \end{array}$$

is a commutative diagram,

4. the full subcategory $\mathrm{P}^t_{\mathrm{iso}}(X,\Bbbk)$ of perverse sheaves in $D^b_{\mathcal{S}}(X,\Bbbk)$ such that there exists an isomorphism $a^*(\mathscr{F}) \cong \mathscr{L}^A_t \boxtimes \mathscr{F}$.

Our aim is to show that these four definitions agree, in other words, that the four above categories are canonically equivalent. The proof of this fact will be given in a succession of lemmas; we will construct several natural functors between these categories. More precisely, we have the following natural functors:

1. the (restriction to perverse sheaves of the) functor $\mathrm{For}_t : \mathfrak{P}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}^A_t} \to \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$, defined in the end of subsection 4.1,

2. the functor $\mathrm{For}_{\mathrm{co}} : \mathrm{P}^t_{\mathrm{co}}(X,\Bbbk) \to \mathrm{P}^t_{\mathrm{iso}}(X,\Bbbk)$ that maps a pair $(\mathscr{F}, \vartheta_{\mathscr{F}})$ to $\mathscr{F}$ and a morphism f to itself,

3. the functor $\mathrm{For}_{\mathrm{mon}} : \mathrm{P}_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]} \to \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ that maps an object to itself and a morphism to itself.

Note that, according to the definition of $\mathfrak{P}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ (see definition 4.1.2), the functor For_t indeed maps perverse sheaves to perverse sheaves. We will show that these functors induce equivalences between our different monodromic perverse categories.

Remark that the stalk of the local system $\mathscr{L}_t^A$ at $\{1\}$ is $\Bbbk$. For m divisible by the order of t, we have a canonical isomorphism $\mathrm{can}_m : \underline{\Bbbk}_A \xrightarrow{\sim} e_m^*(\mathscr{L}_t^A)$. In the next proof we will consider $A = \mathbb{G}_{\mathrm{m}}$ and restriction of can to $e^{\frac{2i\pi}{m}}$. The reasoning of the proof of lemma 3.3.3 allows one to see that we have $\mathrm{can}_{|e^{\frac{2i\pi}{m}}} = t \cdot \mathrm{id}$. More generally, for a cocharacter $\lambda \in \mathrm{X}_*(A)$, one has $\mathrm{can}_{|\lambda_m} = \lambda(t) \cdot \mathrm{id}$.

Lemma 4.2.1. *Consider $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)$. Then $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]}$ if and only if there exists an isomorphism*

$$a^*(\mathscr{F}) \cong \mathscr{L}_t^A \boxtimes \mathscr{F}$$

whose restriction to $\{1\} \times X$ identifies with $\mathrm{id}_{\mathscr{F}}$.

Proof. To show the "only if" part, we assume that we have an object $\mathscr{F}$ in $\mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]}$ and we will construct locally an isomorphism as in the statement of the lemma (i.e. on an open covering). Now $a^*(\mathscr{F})$ is a shifted perverse sheaf (since a is smooth with connected fibers, cf. [BBD, Proposition 4.2.5]) and $\mathscr{L}_t^A \boxtimes \mathscr{F}$ as well (cf. [BBD, Proposition 4.2.8]), so according to [BBD, Corollaire 2.1.23], we will obtain a global isomorphism.

Consider the case $A = \mathbb{C}^*$. For $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]}$, there exists an integer $m \in \mathbb{Z}_{>0}$ such that we have an isomorphism $\iota : \mathrm{pr}_2^*(\mathscr{F}) \xrightarrow{\sim} a(m)^*(\mathscr{F})$ satisfying $\iota_{|\{1\}\times X} = \mathrm{id}_{\mathscr{F}}$. We can assume that m is divisible by the order of t, so that we have the isomorphism $\mathrm{can}_m : \underline{\Bbbk} \xrightarrow{\sim} e_m^*(\mathscr{L}_t^A)$. Thus we obtain an isomorphism

$$\mathrm{Hom}(e_m^*(\mathscr{L}_t^A) \boxtimes \mathscr{F}, a(m)^*(\mathscr{F})) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F}), a(m)^*(\mathscr{F})),$$

defined by $f \mapsto f \circ (\mathrm{can}_m \boxtimes \mathrm{id})$. Now since $\mathscr{F}$ is perverse, the isomorphism ι is unique. More precisely, for a given n, if there exists an isomorphism $\mathrm{pr}_2^*(\mathscr{F}) \xrightarrow{\sim} a(n)^*(\mathscr{F})$ whose restriction to $\{1\} \times X$ is the identity, then this isomorphism is unique (this is essentially because the functor pr_2^* is fully faithful on perverse sheaves, see [BBD, Proposition 4.2.5]). This implies that there exists a unique isomorphism $I := \iota \circ (\mathrm{can}_m^{-1} \boxtimes \mathrm{id})$ in $\mathrm{Hom}(e_m^*(\mathscr{L}_t^A) \boxtimes \mathscr{F}, a(m)^*(\mathscr{F}))$ whose restriction to $\{1\} \times X$ identifies with the identity of $\mathscr{F}$ (note that by definition the restriction of can to $\{1\} \subseteq A$ is id). Consider now the automorphism of $\mathbb{C}^* \times X$ given by

$$\xi : (z, x) \longmapsto (ze^{\frac{2i\pi}{m}}, x).$$

We clearly have $(e_m \times \mathrm{id}_X) \circ \xi = (e_m \times X)$ and similarly, $a(m) \circ \xi = a(m)$. The isomorphism $\xi^*(I) : e_m^*(\mathscr{L}_t^A) \boxtimes \mathscr{F} \xrightarrow{\sim} a(m)^*(\mathscr{F})$ satisfies

$$\begin{aligned}
\xi^*(I)_{|\{1\}\times \mathrm{id}_X} &= (\iota \circ (\mathrm{can}_m^{-1} \boxtimes \mathrm{id}))_{|\{e^{\frac{2i\pi}{m}}\}\times X} \\
&= (t \cdot \mathrm{id}_{\mathscr{F}}) \circ ((\mathrm{can}_m^{-1})_{e^{\frac{2i\pi}{m}}} \boxtimes \mathrm{id}) \\
&= (t \cdot \mathrm{id}_{\mathscr{F}}) \circ (t^{-1} \cdot \mathrm{id}_{\mathscr{F}}) \\
&= \mathrm{id}_{\mathscr{F}}.
\end{aligned}$$

(Note that we forgot the canonical isomorphism $e_m^*(\mathscr{L}_t^A)\boxtimes\mathscr{F} \xrightarrow{\sim} \xi^*((e_m^*(\mathscr{L}_t^A)\boxtimes \mathscr{F})$ in the computation above, because its restriction to $\{1\}\times X$ is the identity; similarly for the identification $\xi^* a(m)^*\mathscr{F} \cong a(m)^*\mathscr{F}$.) Thus we deduce that $\xi^*(I)$ coincides with $I = \iota\circ(\mathrm{can}\boxtimes \mathrm{id})$.

We will need a bit more of notation. Set

$$O_1 = \{z\in\mathbb{C}^* \mid z = re^{i\theta},\ \ r\in\mathbb{R}_{>0},\ \theta\in(0,\frac{2\pi}{m})\}$$

and

$$O_2 = \{z\in\mathbb{C}^* \mid z = re^{i\theta},\ \ r\in\mathbb{R}_{>0},\ \theta\in(-\frac{\pi}{m},\frac{\pi}{m})\}.$$

Then let $U_1 = \mathbb{C}^*\setminus\mathbb{R}_{>0}$ and $U_2 = \mathbb{C}^*\setminus\mathbb{R}_{<0}$. The application e_m induces homeomorphisms

$$O_1 \xrightarrow{\sim} U_1 \quad\text{and}\quad O_2 \xrightarrow{\sim} U_2.$$

We thus obtain isomorphisms

$$\begin{aligned}\mathrm{Hom}(\mathrm{pr}_2^*(\mathscr{F})_{|O_i\times X}, a(m)^*(\mathscr{F})_{|O_i\times X})&\\ \cong \mathrm{Hom}((e_m^*(\mathscr{L}_t^A)\boxtimes\mathscr{F})_{|O_i\times X}, a(m)^*(\mathscr{F})_{|O_i\times X})&\\ \cong \mathrm{Hom}((\mathscr{L}_t^A\boxtimes\mathscr{F})_{|U_i\times X}, a^*(\mathscr{F})_{|U_i\times X}).&\end{aligned}$$

The first isomorphism is given by $(-)\circ(\mathrm{can}_m^{-1}\boxtimes\mathrm{id})_{|O_i\times X}$ and the second one by $\big(((e_m\times\mathrm{id})_{|O_i\times X})^*\big)^{-1}$. We denote by I_i the image of $\iota_{|O_i\times X}$ obtained following these isomorphisms. By construction the restriction of the isomorphisms I_1 and I_2 to $\{z\in\mathbb{C}^*\mid \Im(z)>0\}$ coincide. (Note that this is a connected component of $U_1\cap U_2$.) For the other component

$$\{z\in\mathbb{C}^*\mid \Im(z)<0\}$$

we use the fact that $\xi^*(I)$ coincides with I to see that there again the restriction of I_1 and I_2 coincide. We thus have "constructed locally", i.e. on an open cover of $\mathbb{C}^*\times X$, an isomorphism between $a^*(\mathscr{F})$ and $\mathscr{L}_t^A\boxtimes\mathscr{F}$. Moreover these isomorphisms coincide on the intersection of the open subsets. We can then glue these "local isomorphisms" to obtain a global isomorphism

$$\mathscr{L}_t^A\boxtimes\mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F}). \tag{4.2.1}$$

To determine the restriction of this isomorphism to $\{1\}\times X$, one can first restrict to $U_2\times X$ (since $1\in U_2$). One then considers the pullback of (4.2.1) along the homeomorphism $O_2\times X\cong U_2\times X$. We obtain in this way the restriction of ι to $O_2\times X$. It is then easy to see that the restriction of (4.2.1) to $\{1\}\times X$ gives the identity of $\mathscr{F}$.

To deal with the general case, one chooses a trivialisation $A\cong(\mathbb{C}^*)^r$ and uses $\underline{\Bbbk}_A\cong\underline{\Bbbk}_{\mathbb{C}^*}\boxtimes\cdots\boxtimes\underline{\Bbbk}_{\mathbb{C}^*}$ and $\mathscr{L}_t^A\cong\mathscr{L}_{t_1}^{\mathbb{C}^*}\boxtimes\cdots\boxtimes\mathscr{L}_{t_r}^{\mathbb{C}^*}$ (for $t=(t_1,\dots,t_r)$ under the induced trivialisation $A_{\Bbbk}^\vee\cong(\Bbbk^*)^r$). We write $a_i:(\mathbb{C}^*)^i\times X\to(\mathbb{C}^*)^{i-1}\times X$ for $i=1,\dots,r$ with $a_i(t_1,\dots,t_i,x) = (t_1,\dots,t_{i-1},a((1,\dots,1,\underset{i}{t_i},1,\dots,1),x))$, so

that $a((t_1,\dots,t_r),x) = a_1 \circ \cdots \circ a_r(t_1,\dots,t_r,x)$. Applying the above reasoning successively to the a_i's, we can conclude.

Now we prove the converse. Assume that we have an isomorphism $a^*(\mathscr{F}) \cong \mathscr{L}_t^A \boxtimes \mathscr{F}$ whose restriction to $\{1\} \times X$ is the identity of $\mathscr{F}$ (under the identification $(\mathscr{L}_t^A)_1 \cong \Bbbk$). Then, for n the order of t, we obtain an isomorphism

$$\iota : e_n^*(\mathscr{L}_t^A) \boxtimes \mathscr{F} \xrightarrow{\sim} a(n)^*(\mathscr{F}).$$

We again have a canonical isomorphism $\mathrm{can}_n : \underline{\Bbbk} \xrightarrow{\sim} e_n^*(\mathscr{L}_t^A)$. We can consider

$$\tilde{\iota} : \underline{\Bbbk} \boxtimes \mathscr{F} \xrightarrow{\mathrm{can}_n \boxtimes \mathrm{id}_{\mathscr{F}}} e_n^*(\mathscr{L}_t^A) \boxtimes \mathscr{F} \xrightarrow{\iota} a(n)^*(\mathscr{F});$$

the restriction of this isomorphism to $\{1\} \times X$ identifies with the identity of $\mathscr{F}$. Thus the monodromy of $\mathscr{F}$ is given by the restriction of $\tilde{\iota}$ to various $\lambda_n = \lambda(e^{\frac{2i\pi}{n}})$ for $\lambda \in \mathrm{X}_*(A)$. From the argument above, one sees that this restriction coincides with the restriction of $\mathrm{can}_n \boxtimes \mathrm{id}_{\mathscr{F}}$ to $\{\lambda_n\} \times X$, which identifies with $\lambda(t) \cdot \mathrm{id}_{\mathscr{F}}$. This concludes the proof. □

In the following lemma we fix an isogeny $\widetilde{A} \xrightarrow{\nu} A$ in order to realize $\mathfrak{D}(X/\!\!/A)_{\mathscr{L}_t^A}$ as a full subcategory of $D^b_{\widetilde{A},\mathcal{S}}(X,\Bbbk)$. Namely, we choose the map $e_n : A \to A,\ x \mapsto x^n$, for n prime to ℓ and divisible by the order of t. Recall that we have a forgetful functor $\mathrm{For}_t : \mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A} \to D^b_{\mathcal{S}}(X,\Bbbk)$, as introduced at the end of section 2.7.

Lemma 4.2.2. *The functor* For_t *is fully faithful and has essential image included in* $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$.

Proof. The equivariant derived category is defined with respect to the action $t \cdot x = t^n x$ with $t \in A$ and $x \in X$. This action morphism $A \times X \to X$ is denoted $a(n) = a \circ (e_n \times \mathrm{id}_X)$. We will also consider the projection $\mathrm{pr}_2 : A \times X \to X$.

It is well know (see e.g. [BaR, Proposition A.2]) that the forgetful functor $\mathrm{For} : D^b_{\widetilde{A},\mathcal{S}}(X,\Bbbk) \to D^b_{\mathcal{S}}(X,\Bbbk)$ induces an equivalence between the category of equivariant perverse sheaves and the full subcategory of constructible perverse sheaves whose objects are those $\mathscr{F}$ such that there exists an isomorphism

$$\mathrm{pr}_2^*(\mathscr{F}) \xrightarrow{\sim} a(n)^*(\mathscr{F}).$$

The functor For_t is, up to a shift, the restriction to a direct summand subcategory of a fully faithful functor, we thus know that it is fully faithful. We need to show that its essential image is included in the category $P_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$. We will show that the restriction of For to $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ has essential image in $D^b_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$. As For maps perverse objects to perverse objects, this fact implies the lemma.

Consider $\mathscr{F} \in \mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$. As this is a $\widetilde{A}$-equivariant sheaf, we have a canonical isomorphism (in the constructible category)

$$\vartheta_{\mathscr{F}} : \mathrm{pr}_2^*(\mathrm{For}(\mathscr{F})) \xrightarrow{\sim} a(n)^*(\mathrm{For}(\mathscr{F}))$$

and this isomorphism satisfies the usual cocycle condition. We have two uses of this isomorphism: on the one hand, the condition $(\vartheta_{\mathscr{F}})_{|\{1\}\times X} = \mathrm{id}_{\mathscr{F}}$ allows us to define the canonical monodromy morphism of $\mathscr{F}$ from $\vartheta_{\mathscr{F}}$: using the notation of section 1.4, we have $\varphi^{\lambda}_{\mathscr{F}} = \vartheta_{\mathscr{F},\lambda_n}$.

On the other hand, the kernel $K = \ker(e_n)$ acts on $\mathscr{F}$ via the automorphisms $\vartheta_{\mathscr{F},a}$ for $a \in K$ (see the proof of lemma 2.1.1). One should remark that for any $\lambda \in \mathrm{X}_*(A)$, we have $\lambda_n \in K$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathrm{X}_*(A) & \xrightarrow{ev(e^{\frac{2i\pi}{n}})} & K \\ & {\scriptstyle \varphi_{\mathscr{F}}} \searrow \quad \swarrow {\scriptstyle \vartheta_{\mathscr{F},\cdot}} & \\ & \mathrm{Aut}(\mathscr{F}). & \end{array}$$

By definition of the category $\mathfrak{D}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A}$ the map $\vartheta_{\mathscr{F},(\cdot)}$ factors through the character $\chi_{\mathscr{L}_t^A} : K \to \Bbbk^*$, and it is clear that the horizontal map factors through the canonical projection $\mathrm{X}_*(A) \twoheadrightarrow \mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A))$. Using the proof of lemma 4.1.1, we get that the composition

$$\mathrm{X}_*(A) \twoheadrightarrow \mathrm{X}_*(A)/e_n^{\#}(\mathrm{X}_*(A)) \to K \to \Bbbk^*$$

coincides with the evaluation at $t \in A^{\vee}_{\Bbbk}$. This tells us that the canonical morphism of monodromy of $\mathscr{F}$ is given by

$$\varphi^{\lambda}_{\mathscr{F}} = \lambda(t)\,\mathrm{id}_{\mathscr{F}}$$

for any $\lambda \in \mathrm{X}_*(A)$. In others words, we have $\mathrm{For}_t(\mathscr{F}) \in D^b_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$. □

Corollary 4.2.3. *We keep the setting and notation of the proof of lemma 4.2.2. For any $\mathscr{F} \in D^b_{\widetilde{A},\mathcal{S}}(X,\Bbbk)$ and any $\lambda \in \mathrm{X}_*(A)$, we have $\vartheta_{\mathscr{F},\lambda_n} = \varphi^{\lambda}_{\mathrm{For}_t(\mathscr{F})}$.*

Lemma 4.2.4. *The functor $\mathrm{For}_t : \mathfrak{P}_{\mathcal{S}}(X/\!\!/A)_{\mathscr{L}_t^A} \to \mathrm{P}_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$ is an equivalence of categories.*

Proof. We consider once again the isogeny $\widetilde{A} = A \xrightarrow{e_n} A$ and view $\mathfrak{D}(X/\!\!/A)_{\mathscr{L}_t^A}$ as a full subcategory of $D^b_{\widetilde{A},\mathcal{S}}(X,\Bbbk)$. Thanks to lemma 4.2.2, we just have to show that the functor in the statement is essentially surjective. Consider an object $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)_{[\underline{t}]}$. By lemma 4.2.1, we have an isomorphism $\mathscr{L}_t^A \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F})$ whose restriction to $\{1\} \times X$ is the identity of $\mathscr{F}$. Pulling back along $(e_n \times \mathrm{id}_X)$ we obtain (as in the end of the proof of lemma 4.2.1) an isomorphism

$$\mathrm{pr}_2^*(\mathscr{F}) \xrightarrow{\sim} a(n)^*\,\mathscr{F}\,.$$

Thanks to [BaR, Proposition A.2], we know that there exists a perverse object $\mathscr{G}$ in $D^b_{\widetilde{A},\mathcal{S}}(X,\Bbbk)$ such that $\mathrm{For}(\mathscr{G}) = \mathscr{F}$. The main problem is to determine the action of $K = \ker(e_n)$ on $\mathscr{G}$. But thanks to corollary 4.2.3, the action of K on $\mathscr{G}$ gives (after applying For) the monodromy of $\mathscr{F}$. This tells us that K acts via the character of evaluation at t and concludes the proof. □

Lemma 4.2.5. *Consider $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)$ such that there exists an isomorphism*

$$\mathscr{L}_t^A \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F}).$$

Then there exists a unique isomorphism $\vartheta : \mathscr{L}_t^A \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F})$ such that $\vartheta_{|\{1\}\times X}$ identifies with the identity of $\mathscr{F}$.

Proof. We first show that $F = \mathscr{L}_t^A \boxtimes (\cdot) : D^b_{\mathcal{S}}(X, \Bbbk) \to D^b_{\tilde{\mathcal{S}}}(A \times X, \Bbbk)$ induces a fully faithful functor when restricted to perverse sheaves. (Here, $\tilde{\mathcal{S}}$ is the induced stratification $\{A \times S \mid S \in \mathcal{S}\}$ on $A \times X$.) Tensoring on the left with an invertible local system on $A \times X$ gives an auto-equivalence of category of $D^b_{\tilde{\mathcal{S}}}(A \times X, \Bbbk)$. Thus the functor $(\mathrm{pr}_1^* \mathscr{L}_{t^{-1}}^A \otimes_{\Bbbk} (-)) \circ F$ is fully faithful if and only if F is. This composition is clearly isomorphic to pr_2^*, and the latter functor is fully faithful on perverse sheaves thanks to [BBD, Proposition 4.2.5]. Thus the restriction of F to perverse sheaves is fully faithful. The inverse equivalence is induced by the functor i^* where $i : X \to A \times X$ is the map that sends x to $(1, x)$.

Now assume that $\beta : \mathscr{L}_t^A \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F})$ is any isomorphism, with $\mathscr{F}$ perverse. Then, it is easy to see that $\vartheta := F(i^*(\beta)^{-1}) \circ \beta$ is an isomorphism $\mathscr{L}_t^A \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F})$ such that $i^*(\vartheta) = \mathrm{id}_{\mathscr{F}}$. If ϑ_1, ϑ_2 are two such isomorphisms, then $\vartheta_1 \circ \vartheta_2^{-1}$ gives an automorphism of $\mathscr{L}_t^A \boxtimes \mathscr{F}$. Denote it by $F(g)$ (we can do that since F is full). We have $i^* F(g) = g = \mathrm{id}_{\mathscr{F}}$. This readily implies that $\vartheta_1 = \vartheta_2$ and concludes the proof of unicity, and hence the proof of the lemma. □

Lemma 4.2.6. *Consider $\mathscr{F} \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)$. Assume that there exists an isomorphism*

$$\vartheta_{\mathscr{F}} : \mathscr{L}_t^A \boxtimes \mathscr{F} \xrightarrow{\sim} a^*(\mathscr{F})$$

such that $(\vartheta_{\mathscr{F}})_{|\{1\}\times X}$ is $\mathrm{id}_{\mathscr{F}}$. Then we have

$$(m \times \mathrm{id}_X)^*(\vartheta_{\mathscr{F}}) = (\mathrm{id}_A \times a)^*(\vartheta_{\mathscr{F}}) \circ (\mathscr{L}_t^A \boxtimes \vartheta_{\mathscr{F}}). \tag{4.2.2}$$

Moreover for $(\mathscr{G}, \vartheta_{\mathscr{G}})$ another pair (perverse sheaf, isomorphism restricting to id*) and any map $f : \mathscr{F} \to \mathscr{G}$, we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{L}_t^A \boxtimes \mathscr{F} & \xrightarrow{\mathscr{L}_t^A \boxtimes f} & \mathscr{L}_t^A \boxtimes \mathscr{G} \\ {\scriptstyle \vartheta_{\mathscr{F}}}\downarrow \wr & & \wr \downarrow {\scriptstyle \vartheta_{\mathscr{G}}} \\ a^*(\mathscr{F}) & \xrightarrow[a^*(f)]{} & a^*(\mathscr{G}). \end{array}$$

Proof. Rewrite the equality (4.2.2) as $u = v$. We consider $u \circ v^{-1}$. This is an automorphism of $\mathscr{L}_t^A \boxtimes \mathscr{L}_t^A \boxtimes \mathscr{F}$. Denote by j the map $X \to A \times A \times X$ that sends x to $(1, 1, x)$. Using the same argument as in the proof of lemma 4.2.5, we see that since $\mathscr{F}$ is perverse, the maps

$$\mathrm{End}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}, \mathscr{F}) \leftrightarrow \mathrm{End}_{D^b_{\tilde{\mathcal{S}}}(A\times A\times X,\Bbbk)}((\mathscr{L}_t^A)^{\boxtimes 2} \boxtimes \mathscr{F}, (\mathscr{L}_t^A)^{\boxtimes 2} \boxtimes \mathscr{F})$$

induced by the functors

$$\mathscr{L}_t^A \boxtimes \mathscr{L}_t^A \boxtimes (\cdot) \quad \text{and} \quad j^*$$

are mutually inverse isomorphisms. This implies in particular that $u \circ v^{-1} = \mathrm{id}_{(\mathscr{L}_t^A)^{\boxtimes 2} \boxtimes \mathscr{F}}$. Thus we have $u = v$.

The proof of the statement about morphisms is similar to the end of the proof of lemma 4.2.5, and left to the reader. □

4.3 The equivalences

We can refine the definition of the functors For_t and $\mathrm{For}_{\mathrm{mon}}$. Lemma 4.2.1 tells us that $\mathrm{For}_{\mathrm{mon}}$ has essential image in $\mathrm{P}^t_{\mathrm{iso}}(X, \Bbbk)$. According to lemma 4.2.2, For_t induces a functor $\mathfrak{P}_{\mathcal{S}}(X /\!\!/ A)_{\mathscr{L}_t^A} \to \mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]}$. In fact, the results of section 4.2, give us a bit more: we have the following announced result.

Proposition 4.3.1. *The four categories $\mathfrak{P}_{\mathcal{S}}(X /\!\!/ A)_{\mathscr{L}_t^A}$, $\mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]}$, $\mathrm{P}^t_{\mathrm{co}}(X, \Bbbk)$ and $\mathrm{P}^t_{\mathrm{iso}}(X, \Bbbk)$ are canonically equivalent: the functors For_t, $\mathrm{For}_{\mathrm{co}}$ and $\mathrm{For}_{\mathrm{mon}}$ are equivalences of categories.*

Proof. All the work is already done: by lemma 4.2.1 and lemma 4.2.5, we have a canonical equivalence $\mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]} \cong \mathrm{P}^t_{\mathrm{iso}}(X, \Bbbk)$ (in fact these two categories are the same as full subcategories of $D^b_{\mathcal{S}}(X, \Bbbk)$). From lemma 4.2.6 and lemma 4.2.5, the categories $\mathrm{P}^t_{\mathrm{iso}}(X, \Bbbk)$ and $\mathrm{P}^t_{\mathrm{co}}(X, \Bbbk)$ are equivalent. Finally, lemma 4.2.4 gives the equivalence $\mathfrak{P}_{\mathcal{S}}(X /\!\!/ A)_{\mathscr{L}_t^A} \cong \mathrm{P}_{\mathcal{S}}(X, \Bbbk)_{[\underline{t}]}$. □

4.4 Standard, costandard and intersection cohomology complexes

We consider the following setting: $\mathscr{X} = \bigcup_{\alpha \in \Lambda} \mathscr{X}_\alpha$ is an algebraically stratified A-variety with A-stable strata. In particular, Λ is finite. We assume that there exist isomorphisms $\mathscr{X}_\alpha \cong A \times \mathbb{A}^{n_\alpha}$ for any α (and we fix such isomorphisms once and for all) and that the action of A on $\mathscr{X}_\alpha$ corresponds to the action of multiplication of A on itself. We let $d_\alpha := n_\alpha + r$ be the dimension of $\mathscr{X}_\alpha$. Let also $j_\alpha : \mathscr{X}_\alpha \hookrightarrow \mathscr{X}$ be the inclusion map.

The isomorphism classes of irreducible local systems on $\mathscr{X}_\alpha$ are in bijection with elements of $A^\vee_{\Bbbk}$: more precisely, the pullback along the map $\mathscr{X}_\alpha \xrightarrow{\sim} A \times \mathbb{A}^{n_\alpha} \xrightarrow{\mathrm{pr}} A$ gives an equivalence $\mathrm{Loc}(A, \Bbbk) \cong \mathrm{Loc}(\mathscr{X}_\alpha, \Bbbk)$. For any $t \in A^\vee_{\Bbbk}$, we obtain an irreducible local system $\mathscr{L}^\alpha_t$..

We also denote

$$\Delta^\alpha_t = (j_\alpha)_! \, \mathscr{L}^\alpha_t[d_\alpha] \quad \text{and} \quad \nabla^\alpha_t := (j_\alpha)_* \, \mathscr{L}^\alpha_t[d_\alpha].$$

Note that the j_α's are affine maps (indeed, the $\mathscr{X}_\alpha$'s are affine and $\mathscr{X}$ is a separated scheme), so that Δ^α_t and ∇^α_t are perverse. Note also that thanks

to lemma 1.5.5 and lemma 3.3.3, these objects are in $\mathrm{P}_\Lambda(\mathscr{X},\Bbbk)_{[\underline{t}]}$. Finally, let $\mathrm{IC}_t^\alpha := (j_\alpha)_{!*}\mathscr{L}_t^\alpha[d_\alpha]$ be the intermediate extension of $\mathscr{L}_t^\alpha$.

The Δ_t^α's and ∇_t^α's for various α and t are called standard and costandard objects respectively. (The justification for these names will be given in the next section.)

Note that the pullback under the isomorphism $\mathscr{X}_\alpha \cong A \times \mathbb{A}^{n_\alpha}$ of the local system $\mathscr{L}_t^\alpha$ on $\mathscr{X}_\alpha$ is the local system $\mathscr{L}_t^A \boxtimes \underline{\Bbbk}_{\mathbb{A}^{n_\alpha}}$. Then the pullback of $\mathscr{L}_t^\alpha$ under the action map $A \times \mathscr{X}_\alpha \to \mathscr{X}_\alpha$ is $\mathscr{L}_t^A \boxtimes \mathscr{L}_t^A \boxtimes \underline{\Bbbk}_{\mathbb{A}^{n_\alpha}}$. Since for n divisible by the order of t we have $e_n^*\mathscr{L}_t^A = \underline{\Bbbk}_A$, it follows easily that $\mathscr{L}_t^\alpha$ is A-equivariant for the action "twisted" by e_n. It is then immediate that $\mathscr{L}_t^\alpha$ defines an object in $\mathfrak{D}_\Lambda(\mathscr{X}_\alpha/\!\!/A)_{\mathscr{L}_t^A}$. It follows that, Δ_t^α, ∇_t^α and IC_t^α define objects in $\mathfrak{D}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}$. We set

$$\Delta(\alpha)_{\mathscr{L}_t^A} := \Delta_t^\alpha[-r], \quad \nabla(\alpha)_{\mathscr{L}_t^A} := \nabla_t^\alpha[-r], \quad \mathrm{IC}(\alpha)_{\mathscr{L}_t^A} := \mathrm{IC}_t^\alpha[-r]$$

when we consider these objects in $\mathfrak{D}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}$. The forgetful functor

$$\mathrm{For}_t : \mathfrak{P}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A} \to \mathrm{P}_\Lambda(\mathscr{X},\Bbbk)_{[\underline{t}]}$$

maps $\Delta(\alpha)_{\mathscr{L}_t^A}$ to Δ_t^α and similarly for the ∇'s and IC's.

4.5 The case of a single stratum

If we apply the equivalence of proposition 4.3.1 to the case where $\mathscr{X} = \mathscr{X}_\alpha$ (i.e. a stratified space with only one stratum), we obtain an equivalence

$$\mathfrak{P}_\Lambda(\mathscr{X}_\alpha/\!\!/A)_{\mathscr{L}_t^A} \longleftrightarrow \mathrm{P}_\Lambda(\mathscr{X}_\alpha,\Bbbk)_{[\underline{t}]}.$$

Once again, the shifted pullback along the map $\mathscr{X}_\alpha \xrightarrow{\sim} A \times \mathbb{A}^{n_\alpha} \to A$ gives an equivalence $\mathrm{P}_\Lambda(\mathscr{X}_\alpha,\Bbbk) \cong \mathrm{Loc}(A,\Bbbk)$ (since $\mathrm{Loc}(\mathscr{X}_\alpha,\Bbbk) \cong \mathrm{Loc}(A,\Bbbk)$) and obviously

$$\mathrm{P}_\Lambda(\mathscr{X}_\alpha,\Bbbk)_{[\underline{t}]} \cong \mathrm{Loc}(A,\Bbbk)_{[\underline{t}]}.$$

We showed previously (see lemma 3.3.3) that for local systems the canonical morphism of monodromy (which defines an action of $\Bbbk[\mathrm{X}_*(A)]$) coincides with the action of $\Bbbk[\mathrm{X}_*(A)]$ on the associated $\Bbbk[\mathrm{X}_*(A)]$-module. In particular, we have an equivalence

$$\mathrm{Loc}(A,\Bbbk)_{[\underline{t}]} \cong \Bbbk[\mathrm{X}_*(A)]/\langle e^\lambda - \lambda(t)\rangle\text{-mod} \cong \Bbbk\text{-mod}\,.$$

As $\Bbbk$ is a field, this last category is semi-simple, and we deduce that the category $\mathfrak{P}_\Lambda(\mathscr{X}_\alpha/\!\!/A)_{\mathscr{L}_t^A}$ is semi-simple as well.

4.6 Highest weight structure

This section and the arguments that follow are faithfully inspired from [BGS, §3.2 and 3.3]. For the results and definitions we will use about highest weight

categories, we refer to appendix A (and references mentioned there). The set Λ is a poset for the order $\leq$ defined by

$$(\alpha \leq \alpha') \Leftrightarrow (\overline{\mathscr{X}_\alpha} \subseteq \overline{\mathscr{X}_{\alpha'}}).$$

Theorem 4.6.1. *The category $\mathrm{P}_\Lambda(\mathscr{X}, \Bbbk)_{[\mathit{t}]}$ together with the natural morphisms $\Delta_t^\alpha \twoheadrightarrow \mathrm{IC}_t^\alpha \hookrightarrow \nabla_t^\alpha$ and with weight poset the pair $(\Lambda, \leq)$ is a highest-weight category.*

Proof. For the sake of simplicity (of notation) we will set, for the duration of this proof, $\mathrm{P}_\Lambda(\mathscr{X}, \Bbbk)_{[\mathit{t}]} = \mathcal{C}$. For I an ideal in Λ, we let $\mathcal{C}_I$ be the Serre subcategory generated by simple objects indexed by elements of I; this identifies with the image of the fully faithful pushforward functor along the closed inclusion $\bigcup_{\alpha \in I} \mathscr{X}_\alpha \hookrightarrow \mathscr{X}$.

According to definition A.0.1, we have 5 points to check. The first one is clear (since Λ itself is finite). Using the fact that $(j_\alpha)_{!*} : \mathrm{Loc}(\mathscr{X}_\alpha, \Bbbk) \to \mathrm{P}_\Lambda(\mathscr{X}, \Bbbk)$ is fully faithful and the equivalence $\mathrm{Loc}(\mathscr{X}_\alpha, \Bbbk) \cong \mathrm{Loc}(A, \Bbbk)$, one gets

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{IC}_t^\alpha, \mathrm{IC}_t^\alpha) = \mathrm{Hom}_{\mathrm{P}_\Lambda(\mathscr{X}, \Bbbk)}(\mathrm{IC}_t^\alpha, \mathrm{IC}_t^\alpha) = \mathrm{Hom}_{\mathrm{Loc}(A, \Bbbk)}(\mathscr{L}_t^A, \mathscr{L}_t^A) = \Bbbk\,.$$

Now consider an ideal $I \subset \Lambda$. Assume that α is a maximal element in I. We have to show that the canonical morphism $\Delta_t^\alpha \to \mathrm{IC}_t^\alpha$ is the projective cover of IC_t^α in the category $\mathcal{C}_I$. We must show that

$$\mathrm{Hom}_{\mathcal{C}_I}(\Delta_t^\alpha, \mathrm{IC}_t^\beta) = \begin{cases} \Bbbk & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \tag{4.6.1}$$

and that

$$\mathrm{Ext}^1_{\mathcal{C}_I}(\Delta_t^\alpha, \mathrm{IC}_t^\beta) = 0 \tag{4.6.2}$$

for any $\beta \in I$. We use the equivalence

$$\mathrm{P}_\Lambda(\mathscr{X}, \Bbbk)_{[\mathit{t}]} \cong \mathfrak{P}_\Lambda(\mathscr{X} /\!\!/ A)_{\mathscr{L}_t^A}$$

of lemma 4.2.4. In the latter category, we have

$$\mathrm{Ext}^1_{\mathfrak{P}_\Lambda(\mathscr{X} /\!\!/ A)_{\mathscr{L}_t^A}}(\Delta(\alpha)_{\mathscr{L}_t^A}, \mathrm{IC}(\beta)_{\mathscr{L}_t^A}) = \mathrm{Hom}_{\mathfrak{D}_\Lambda(\mathscr{X} /\!\!/ A)_{\mathscr{L}_t^A}}(\Delta(\alpha)_{\mathscr{L}_t^A}, \mathrm{IC}(\beta)_{\mathscr{L}_t^A}[1])$$

(see [BBD, Remarque 3.1.17 (ii)], this is a standard feature of t-structure). We can then use adjunction and we deduce (4.6.1) and (4.6.2) in the case $\alpha \neq \beta$. We treat the case $\alpha = \beta$. Using the equivalence above two times and adjunction, we obtain an isomorphism:

$$\mathrm{Ext}^1_{\mathrm{P}_\Lambda(\mathscr{X}_\alpha, \Bbbk)_{[\mathit{t}]}}(\mathscr{L}_t^\alpha[d_\alpha], \mathscr{L}_t^\alpha[d_\alpha]) \cong \mathrm{Ext}^1_{\mathrm{Loc}(A, \Bbbk)_{[\mathit{t}]}}(\mathscr{L}_t^A, \mathscr{L}_t^A).$$

Now the category $\mathrm{Loc}(A, \Bbbk)_{[\mathit{t}]}$ is semisimple as we saw in §4.5. so this Ext space vanishes. This concludes the verification of (4.6.2). We check similarly that $\mathrm{IC}_t^\alpha \to \nabla_t^\alpha$ is an injective hull in $\mathcal{C}_I$.

The fact that the kernel (resp. the cokernel) of $\Delta_t^\alpha \to \mathrm{IC}_t^\alpha$ (resp. of $\mathrm{IC}_t^\alpha \to \nabla_t^\alpha$) lies in $\mathcal{C}_{<\alpha}$ comes from the facts that these objects are supported on $\overline{\mathscr{X}_\alpha} = \sqcup_{\beta\leq\alpha}\mathscr{X}_\beta$ and that this morphism is an isomorphism once restricted to $\mathscr{X}_\alpha$.

We check that $\mathrm{Ext}^2_{\mathcal{C}}(\Delta_t^\alpha, \nabla_t^\beta) = 0$ for any $\alpha, \beta \in \Lambda$. We have

$$\mathrm{Ext}^2_{\mathrm{P}_\Lambda(\mathscr{X},\Bbbk)_{[t]}}(\Delta_t^\alpha, \nabla_t^\beta) \cong \mathrm{Ext}^2_{\mathfrak{P}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}}(\Delta(\alpha)_{\mathscr{L}_t^A}, \nabla(\beta)_{\mathscr{L}_t^A}).$$

Now we know $\mathfrak{P}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}$ is the heart of the perverse t-structure on the category $\mathfrak{D}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}$. Thanks to [BBD, Remarque 3.1.17 (ii)], we have an injection

$$\mathrm{Ext}^2_{\mathfrak{P}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}}(\Delta(\alpha)_{\mathscr{L}_t^A}, \nabla(\beta)_{\mathscr{L}_t^A}) \hookrightarrow \mathrm{Hom}_{\mathfrak{D}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}}(\Delta(\alpha)_{\mathscr{L}_t^A}, \nabla(\beta)_{\mathscr{L}_t^A}[2]).$$

We use adjunction; the only case that still needs work is the case $\alpha = \beta$. What we need to consider is now

$$\mathrm{Hom}_{\mathfrak{D}_\Lambda(\mathscr{X}_\alpha/\!\!/A)_{\mathscr{L}_t^A}}(\mathscr{L}_t^\alpha, \mathscr{L}_t^\alpha[2]).$$

Recall that we denoted K_n the kernel of the n-th power map $A \to A$. We have an isomorphism $A \cong \widetilde{A}/K_n \cong \widetilde{A} \times^{K_n} \{\mathrm{pt}\}$ where the last space is the quotient under the action of K_n by multiplication on $\widetilde{A}$. We also denote by $\mathrm{Rep}_{\Bbbk}(K_n)$ the category of finite dimensional $\Bbbk$-representations of K_n. Using [BL, §6.6, item 6] and [BL, §8.0 and §0.2, Remark], one has the following sequence of equivalences of categories:

$$D^b_{\tilde{A}}(A, \Bbbk) \cong D^b_{\tilde{A}}(\tilde{A} \times^{K_n} \{\mathrm{pt}\}, \Bbbk) \cong D^b_{K_n}(\{\mathrm{pt}\}, \Bbbk) \cong D^b(\mathrm{Rep}_{\Bbbk}(K_n)).$$

and this last category is the derived category of a semisimple abelian category (since the cardinality of K_n is prime to ℓ). Now $\mathfrak{D}_\Lambda(\mathscr{X}_\alpha/\!\!/A)_{\mathscr{L}_t^A}$ is a full subcategory of $D^b_{\tilde{A},\Lambda}(\mathscr{X}_\alpha, \Bbbk)$ and the pullback functor $D^b_{\tilde{A},\Lambda}(A, \Bbbk) \to D^b_{\tilde{A},\Lambda}(\mathscr{X}_\alpha, \Bbbk)$ is fully faithful, so we see that the above Hom-space vanishes and we get the result. □

Corollary 4.6.2. *The realization functor $D^b\mathfrak{P}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A} \to \mathfrak{D}_\Lambda(\mathscr{X}/\!\!/A)_{\mathscr{L}_t^A}$ is an equivalence of categories.*

Proof. This can be proved using the arguments of [BGS, Corollaries 3.2.2 and 3.3.2]. □

Part II

The case of flag varieties

Chapter 5

Flag varieties

From now on, we assume that $\mathbb{k}$ is the algebraic closure of a finite field of characteristic ℓ different from 2.

5.1 Setting and notation

5.1.1 G/U

Consider now a complex reductive algebraic group G. Fix a Borel subgroup B and a maximal torus $T \subseteq B$. Associated to these subgroups, we have a root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ where $X^*(T)$ denotes the character lattice, $X_*(T)$ the cocharacter lattice and Φ and $\Phi^\vee$ the roots and coroots respectively. The group B determines a subset of positive roots Φ^+ (our convention will be that the positive roots are those α such that $\mathrm{Lie}(B) = \oplus_{\alpha\in\Phi^+}\mathrm{Lie}(G)_\alpha$, with standard notation). For any root $\alpha \in \Phi$, we have the root subgroup U_α; this subgroup is isomorphic to $\mathbb{A}^1$ and we can choose such an isomorphism (non uniquely) $u_\alpha : \mathbb{A}^1 \xrightarrow{\sim} U_\alpha$ satisfying $tu_\alpha(a)t^{-1} = u_\alpha(\alpha(t)a)$ for any $a \in \mathbb{A}^1$ and $t \in T$ (see [J, §II. 1.2]). [J, §II. 1.3] tells us that for any α there exists a morphism $\varphi_\alpha : SL_2 \to G$ such that, under suitable normalization of the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} U_\alpha$, we have for any $a \in \mathbb{A}^1$ the image of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is given by $u_\alpha(a)$; similarly, the image of $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ is $u_{-\alpha}(a)$ and $\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ is mapped to $\alpha^\vee(b)$ for any $b \in \mathbb{A}^1 \smallsetminus \{0\}$.

We thus choose once and for all u_α and $u_{-\alpha}$ (for any positive root α) normalized so that we have the above relations with φ_α.

We denote by W the Weyl group $N_G(T)/T$. For all the rest of this document, we fix a section $W \to N_G(T)$, that is, we choose a lifting $\dot{w}$ in $N_G(T)$ for any $w \in W$ as follows. For a simple reflection s associated to a root α, we take $\dot{s} = \varphi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. According to [Sp, Proposition 9.3.2], the elements $\dot{s}$ (for

s simple) satisfy the braid relations; for $(s_1, \dots, s_t)$ a reduced expression of an element $w \in W$, [Sp, §9.3.3] ensures that we can set $\dot{w} = \dot{s_1} \cdots \dot{s_r}$ without any ambiguity, that is to say, the element $\dot{w}$ does not depends on the chosen reduced expression. Finally, note that the definition implies that $\dot{w}\dot{v}$ is equal to the lifting of the element wv for $\ell(wv) = \ell(w) + \ell(v)$.

We have the Bruhat stratification of the flag variety

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

Denote by $U \subseteq B$ the unipotent radical of B. We have a natural map

$$\pi : G/U \twoheadrightarrow G/B.$$

We obtain a stratification of G/U by pulling back the Bruhat stratification on G/B along π. We set $\mathscr{X} := G/U$, and let ϖ be the natural quotient map $G \twoheadrightarrow \mathscr{X}$. For any $w \in W$, let $\mathscr{X}_w = \pi^{-1}(BwB/B)$. Then $(\mathscr{X}, \{\mathscr{X}_w\}_{w \in W})$ is a stratified space; moreover, this stratification is algebraic is the sense of [CG, Definition 3.2.23]. Note also that these strata are $T \times T$-stable for the action

$$(t_1, t_2) \cdot gU = t_1 g t_2 U$$

(for g an arbitrary element in G). The bounded derived category of sheaves constructible with respect to this stratification will be denoted

$$D^b_{(B)}(\mathscr{X}, \Bbbk).$$

For any $w \in W$, let $j_w : \mathscr{X}_w \hookrightarrow \mathscr{X}$ be the inclusion. Let U_w be the product, in any fixed order, of the root subgroups U_α with $\alpha \in w(-\Phi^+) \cap \Phi^+$ (thanks to e.g. [Sp, 8.3.5 Lemma], the group we obtain does not depend on the chosen order). Then, the following map is an isomorphism (see e.g. [Sp, Lemma 8.3.6, (ii)])

$$U_w \times T \to \mathscr{X}_w, \qquad (u, t) \mapsto u\dot{w}tU.$$

We will consider the map

$$p_w : \mathscr{X}_w \xrightarrow{\sim} U_w \times T \xrightarrow{\mathrm{pr}_2} T$$

where the first map in the composition is the inverse of the above fixed isomorphism.

5.1.2 G/U^s

Consider a simple reflection $s \in W$ and the associated Levi subgroup L_s. We also denote $B_s = B \cap L_s$. We have the minimal parabolic subgroup $P_s = L_s \ltimes U^s$, with U^s the unipotent radical of P_s. We will denote the variety G/U^s by $\mathscr{X}^s$. It inherits a stratification

$$\mathscr{X}^s = \bigsqcup_{w \in W} \mathscr{X}^s_w$$

where $\mathscr{X}_w^s$ is the inverse image of $\mathscr{X}_w$ under the projection $G/U^s \to G/U$.

We fix $w \in W$. According to [Sp, §8.3.1 and 8.3.6], we have an isomorphism of varieties

$$U_w \times B_s \xrightarrow{\sim} \mathscr{X}_w^s, \quad (u, b) \mapsto u\dot{w}bU^s.$$

In particular, the composition of the inverse isomorphism with the projection $U_w \times B_s \to B_s$ gives a map $p_w^{B_s} : \mathscr{X}_w^s \to B_s$.

Now assume that w is such that $w < ws$. It is standard to show that the map

$$U_w \times L_s \to \mathscr{X}^s, \quad (u, l) \mapsto u\dot{w}lU^s$$

gives an isomorphism $U_w \times L_s \cong \mathscr{X}_w^s \sqcup \mathscr{X}_{ws}^s$.

In particular, we have a natural map

$$\mathscr{X}_w^s \sqcup \mathscr{X}_{ws}^s \xrightarrow{p_w^{L_s}} L_s$$

given by the composition of the inverse of the previous isomorphism with the projection on the last factor $U_w \times L_s \to L_s$.

5.1.3 Weyl group

We set several notation concerning roots and Weyl groups. Start by fixing an element $t \in T_{\Bbbk}^\vee$. The natural action of W on $T_{\Bbbk}^\vee$ translates in the world of local systems: if we set $w \cdot \mathscr{L} := (w^{-1})^* \mathscr{L}$ for any $w \in W$ and any local system $\mathscr{L}$ on T, it is easy to see that $w \cdot \mathscr{L}_t^T = \mathscr{L}_{w(t)}^T$. (Here, we denoted by w^{-1} the application $T \to T$ that maps t to $w^{-1}(t)$.) For $t, t' \in T_{\Bbbk}^\vee$, we set ${}_{t'}W_t := \{w \in W \mid w(t) = t'\}$. According to the above remark, we have ${}_{t'}W_t = \{w \in W \mid w \cdot \mathscr{L}_t^T = \mathscr{L}_{t'}^T\}$. If $t' = t$, we abbreviate ${}_tW_t = \mathrm{Stab}_W(t)$ by W_t.

Consider the subgroup W_t° of W defined as follows. Recall that, by definition, any coroot in $\Phi^\vee \subseteq \mathrm{X}_*(T)$ defines a character of $T_{\Bbbk}^\vee$. Set

$$\Phi_t^\vee := \{\alpha^\vee \in \Phi^\vee \mid \alpha^\vee(t) = 1\}.$$

Note that $\Phi_t^\vee = \{\alpha^\vee \in \Phi^\vee \mid (\alpha^\vee)^* \mathscr{L}_t^T \cong \underline{\Bbbk}_{\mathbb{C}^*}\}$. With the dual Φ_t of $\Phi_t^\vee$ (namely those roots α of G such that $\alpha^\vee \in \Phi_t^\vee$) we get a subroot system of Φ. We set

$$W_t^\circ := \langle s_\alpha \mid \alpha \in \Phi_t \rangle.$$

The subset $\Phi_t^+ := \Phi^+ \cap \Phi_t$ gives a positive subset in Φ_t. If we let S_t be the set $\{s_\alpha \mid \alpha \in \Phi_t,\ \alpha \text{ indecomposable in } \Phi_t^+\}$, then the pair (W_t°, S_t) is a Coxeter system. Since for a reflection s_α we have $s_\alpha(t) = t\alpha(\alpha^\vee(t))^{-1}$, we see that $W_t^\circ \subseteq W_t$. In fact W_t° is a normal subgroup of W_t.

Remark 5.1.1. In [LY], from which the previous definitions are borrowed, the authors consider ℓ-adic local systems instead of elements of the dual torus (their *definition* then being $\Phi_{\mathscr{L}}^\vee = \{\alpha^\vee \in \Phi^\vee \mid (\alpha^\vee)^* \mathscr{L} \cong \underline{\Bbbk}_{\mathbb{C}^*}\}$ for such a local system). Though our setting is slightly different, all the proofs of *loc. cit.* apply with the exact same arguments, thus we will often use [LY] as reference for facts concerning these groups.

Note that the restriction of the Bruhat order of W to W_t° endows the latter group with a poset structure. On the other hand, being a Coxeter group, W_t° has its own Bruhat order, denoted $\leq_t$, and its own length function denoted ℓ_t. Let us compare these orders:

By definition (see [H3, §5.9] and [H3, §5.7, Proposition]), this order is generated by the relations $w \leq_t ws$ for $w \in W_t^\circ$ and $s \in W_t^\circ$ a reflection associated to a root $\alpha \in \Phi_t^+$ such that $w(\alpha) \in \Phi_t^+$. Since $\Phi_t^+ \subseteq \Phi^+$, we see, using the analogous description of the Bruhat order in W, that

$$\forall v, w \in W_t^\circ, \quad (v \leq_t w) \Rightarrow (v \leq w). \tag{5.1.1}$$

Denote

$$ {}_{t'}\underline{W}_t := {}_{t'}W_t/W_t^\circ = W_{t'}^\circ \backslash {}_{t'}W_t.$$

Note that the converse is false: for two elements $v, w \in W_t^\circ$ such that $v \leq w$, it is not true that $v \leq_t w$. We will give an explicit example below.

Elements in ${}_{t'}\underline{W}_t$ are called blocks. Consider a block $\beta \in {}_{t'}\underline{W}_t$; the restriction of the Bruhat order of W to β allows us to view it as a poset; it can be shown as in [LY, Lemma 4.2] that for there exists a unique minimal element denoted $w_\beta^{\min}$ and a unique maximal element $w_\beta^{\max}$ in β (for the poset structure explicited above). This element can be characterized as follows (see [LY, Lemma 4.2]): $w_\beta^{\max}$ is the unique element sending all the positive roots in Φ_t to negative roots in Φ (in fact, it then maps Φ_t^+ to $-\Phi^+_{w_\beta^{\min}(t)}$). We have a similar characterisation of the minimal element $w_\beta^{\min}$: it is the only element in β such that $w_\beta^{\min}(\Phi_t^+) \subseteq \Phi^+$ (once again, this element in fact map Φ^+ to $\Phi^+_{w_\beta^{\min}(t)}$).

In the case where the block β is actually the neutral block W_t° we set

$$w_{t,\circ} := w_\beta^{\max}.$$

This element is the longest element of W_t° in the Coxeter system (W_t°, S_t) (this follows either from the above characterization of maximal elements or from (5.1.1)); in particular, we have

$$w_{t,\circ} \cdot w_{t,\circ} = e.$$

The following lemma gives some properties of blocks, it is a restatement of [LY, §4.1 to §4.4].

Lemma 5.1.2. *Consider three elements $t, t', t'' \in T_{\Bbbk}^\vee$ in the same W-orbit. Fix two blocks $\beta \in {}_{t'}\underline{W}_t$ and $\gamma \in {}_{t''}\underline{W}_{t'}$. Then the set*

$$\gamma\beta := \{w_1 w_2 \mid w_1 \in \gamma,\, w_2 \in \beta\}$$

is equal to $w_1 w_2 W_t^\circ = w_1 W_{t'}^\circ w_2 = W_{t''}^\circ w_1 w_2$ for any $w_1 \in \beta$ and $w_2 \in \gamma$. In particular, $\gamma\beta$ is a block in ${}_{t''}\underline{W}_t$. Moreover, we have

$$w_\gamma^{\min} w_\beta^{\min} = w_{\gamma\beta}^{\min}, \qquad w_\gamma^{\min} w_\beta^{\max} = w_{\gamma\beta}^{\max},$$

$$w_\gamma^{\max} w_\beta^{\min} = w_{\gamma\beta}^{\max}, \qquad w_\gamma^{\max} w_\beta^{\max} = w_{\gamma\beta}^{\min}.$$

Finally, multiplication on the left (resp. right) by $w_\gamma^{\min}$ (resp. by by $w_\beta^{\min}$) is an order-preserving map $(W_t^\circ, \leq_t) \to (W, \leq)$.

Proof. The first statement is obvious; for the first three equalities, see [LY, Corollary 4.3]. The last equality then follows: with the second and third equalities above, we can write $w_\gamma^{\max} = w_\gamma^{\min} w_{t',\circ}$ and $w_\beta^{\max} = w_{t',\circ} w_\beta^{\min}$ so that we get

$$w_\gamma^{\max} w_\beta^{\max} = w_\gamma^{\min} w_{t',\circ} w_{t',\circ} w_\beta^{\min} = w_\gamma^{\min} w_\beta^{\min} = w_{\gamma\beta}^{\min}.$$

This concludes the proof of the equalities. Let us prove the last statement, for left multiplication only; the other case being similar. We have to show that if $v, w \in W_t^\circ$, then $(v \leq_t w) \Rightarrow (w_\beta^{\min} v \leq w_\beta^{\min} w)$. To do so, we once again consider the definition of the orders $\leq_t$ and $\leq$. We can assume that $w = vt$ with t a reflection associated to a root $\alpha \in \Phi_t^+$ satisfying $v(\alpha) \in \Phi_t^+$. Then we have $w_\beta^{\min} w = (w_\beta^{\min} v)t$ and $w_\beta^{\min} v(\alpha) \in w_\beta^{\min}(\Phi_t^+) = \Phi^+$ thanks to the characterization of $w_\beta^{\min}$ stated above. This concludes the proof. □

Example 5.1.3. Assume that $\ell = \mathrm{char}(\Bbbk) \neq 2$. Let us give two examples to show that it may happen that $W_t^\circ \neq W_t$. First, consider $G = SL_2$, so that the dual $\Bbbk$-group is $PGL_{2,\Bbbk}$. Consider the element $t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the diagonal maximal torus of $PGL_{2,\Bbbk}$. The Weyl group has order 2 with nontrivial element given by the simple reflection s that exchanges the diagonal entries. The associated coroot $\alpha_s^\vee$ is given by $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto xy^{-1}$ (on the maximal torus of $PGL_{2,\Bbbk}$). Thus $\alpha_s^\vee(t) \neq 1$ and $W_t^\circ = \{e\}$. On the other hand, however, we have $W_t = \{e, s\} = W$.

We now give a slightly more interesting example. We place ourselves in the case $G = SL_4$; the dual is $PGL_{4,\Bbbk}$. The common Weyl group is now the symmetric group $W = \mathfrak{S}_4$. Let $s_1 := (1\,2)$, $s_2 := (2\,3)$ and $s_3 := (3\,4)$ so that

$$\langle s_1, s_2, s_3 \mid s_i^2 = 1 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 \rangle$$

is a Coxeter presentation of W.

Let t be the diagonal matrix given by $\mathrm{diag}(\xi, \xi, -\xi, -\xi)$ (with $\xi \neq 0$). Then obviously, we have $W_t^\circ = \langle s_1, s_3 \rangle$. Now simple computations shows that

$$W_t = \{e,\ s_1,\ s_3,\ s_1 s_3 = s_3 s_1,\ s_2 s_1 s_3 s_2,\ s_2 s_1 s_3 s_2 s_1,\ s_1 s_2 s_1 s_3 s_2,\ s_1 s_2 s_1 s_3 s_2 s_1\}.$$

We have two blocks in ${}_t\underline{W}_t$: there is the subgroup W_t° and the coset $s_2 s_1 s_3 s_2 W_t^\circ$.

Set $t' := s_2(t) = \mathrm{diag}(\xi, -\xi, \xi, -\xi)$; we have

$${}_{t'}W_t = \{s_2,\ s_2 s_1,\ s_2 s_3,\ s_1 s_3 s_2,\ s_2 s_1 s_3,\ s_1 s_3 s_2 s_1,\ s_3 s_1 s_2 s_3,\ s_1 s_2 s_3 s_2 s_1\}.$$

This time, the two blocks in ${}_{t'}\underline{W}_t$ are given respectively by

$$\beta = \{s_2,\ s_2 s_1,\ s_2 s_3,\ s_2 s_1 s_3\} \text{ and } \gamma = \{s_1 s_3 s_2,\ s_1 s_3 s_2 s_1,\ s_1 s_3 s_2 s_3,\ s_1 s_2 s_3 s_2 s_1\}.$$

Then one has $w_\beta^{\min} = s_2$ and $w_\beta^{\max} = s_2 s_1 s_3$; $w_\gamma^{\min} = s_1 s_3 s_2$ and $w_\gamma^{\max} = s_1 s_2 s_3 s_2 s_1$.

Let us give an example which shows that the converse to (5.1.1) is not true. We consider the group $\mathrm{Sp}_4(\mathbb{C})$ (whose root system is of type C_2) and its usual maximal torus; its $\Bbbk$-dual is $\mathrm{SO}_5(\Bbbk)$ and has root system of type B_2. Let W be the Weyl group of $\mathrm{Sp}_4(\mathbb{C})$. The element $t = \mathrm{diag}(-1,-1,1,-1,-1)$ (where $\mathrm{diag}(x_1,\dots,x_5)$ denotes the diagonal matrix whose terms are the x_i's) defines a semisimple element in $\mathrm{SO}_5(\Bbbk)$. The roots of $\mathrm{SO}_5(\Bbbk)$ mapping t to 1 are exactly the long roots. Consequently, the reflections of W lying in W_t° are those two reflections associated to the short roots of $\mathrm{Sp}_4(\mathbb{C})$, say s_1, s_2, and both are simple in W_t°. In particular, s_1 and s_2 cannot be compared in W_t°; on the other hand, they are comparable in W.

We conclude this subsection by two lemmas that describe the simple reflections in W_t°.

Lemma 5.1.4. *Consider an element $t \in T_\Bbbk^\vee$ and $x \in W_t^\circ$. Let w be any element of W. Then wxw^{-1} belongs to $W_{w(t)}^\circ$.*

Proof. Of course it suffices to show the lemma for $x = s$ a simple reflection in W_t°. By definition we have $s = s_{\alpha^\vee}$ with $\alpha^\vee$ a coroot such that $\alpha^\vee(t) = 1$. But then $wsw^{-1} = s_{w(\alpha^\vee)}$ and $w(\alpha^\vee)(w(t)) = \alpha^\vee(t) = 1$. Thus wsw^{-1} belongs to $W_{w(t)}^\circ$. □

Lemma 5.1.5. *Consider an element $t \in T_\Bbbk^\vee$ and a reflection $s \in W_t^\circ$, simple in this group. Then s can be written as a product wrw^{-1} with*

- *w a minimal element in the block $\beta \in {}_{w(t)}\underline{W}_t$ it belongs to (i.e. $w = w_\beta^{\min}$),*
- *r a reflection of W, simple in W and lying in $W_{w(t)}^\circ$.*

Proof. We note first that the element s is of length 1 in the Coxeter system (W_t°, S_t). Thanks to [LY, Lemma 4.6, (3) and (4)], if $r_N \cdots r_1$ is a reduced decomposition in W of s, then all but one of the r_i's are minimal in the block they belong to. More precisely, if we let $t_0 := t$ and $t_i := r_i r_{i-1} \cdots r_1(t)$ for $i \geq 1$, then there exists an i such that $r_j \notin W_{t_{j-1}}^\circ$ for $1 \leq j \leq N$, $j \neq i$ and $r_i \in W_{t_{i-1}}^\circ$. Note that this implies the equality $t_i = t_{i-1}$; we also have $t_0 = t_N$. Let us write $s = u r_i v$, so with $u = r_N \cdots r_{i+1}$ and $v = r_{i-1} \cdots r_1$. The preceding discussion implies that for $j \neq i$, the reflection r_j is minimal in the block of ${}_{t_j}\underline{W}_{t_{j-1}}$ it belongs to; thanks to lemma 5.1.2, this is also the case of the element u (resp. v) in the block of ${}_{t_N}\underline{W}_{t_i}$ (resp. ${}_{t_{i-1}}\underline{W}_{t_0}$) it belongs to; we denote this block by β (resp. γ). We thus have

$$s = w_\beta^{\min} \cdot r_i \cdot w_\gamma^{\min}.$$

Multiply both sides by $(w_\gamma^{\min,-1} r_i w_\gamma^{\min})$ on the right to obtain

$$s(w_\gamma^{\min,-1} r_i w_\gamma^{\min}) = w_\beta^{\min} w_\gamma^{\min}.$$

According to lemma 5.1.4, the element $(w_\gamma^{\min,-1} r_i w_\gamma^{\min})$ belongs to $W_{t_0}^\circ$, and so does s. Thus the product $w_\beta^{\min} w_\gamma^{\min}$ also belongs to $W_{t_0}^\circ$. Finally, lemma

5.1.2 tells us that the product $w_\beta^{\min} w_\gamma^{\min}$ is the minimal element in the block it belongs to, that is, the minimal element of $W_{t_0}^\circ$. We obtain $w_\beta^{\min} w_\gamma^{\min} = e$ and

$$s = w_\beta^{\min} r_i w_\beta^{\min,-1}.$$

The proof is now complete. □

5.1.4 Pro-objects and pro-complexes of sheaves

Later on, we will use pro-objects. We recall here some definitions and results from [KS1, §1.11]. We will only consider pro-objects indexed by the directed set $(\mathbb{Z}_{>0}, \leq)$, that we can view as a category in the usual way. A projective system in $\mathcal{C}$ indexed by this set is a functor

$$F : (\mathbb{Z}_{>0}, \leq)^{\mathrm{opp}} \to \mathcal{C}.$$

(Here, $(\mathbb{Z}_{>0}, \leq)^{\mathrm{opp}}$ denotes the opposite category of $(\mathbb{Z}_{>0}, \leq)$.) Thus a projective system is a diagram of the form

$$\cdots F_n \to F_{n-1} \to \cdots F_2 \to F_1.$$

For a projective system F in $\mathcal{C}$ indexed by $\mathbb{Z}_{>0}$, denote by "$\varprojlim$"$_n F$ the functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$ given by

$$X \longmapsto \varinjlim_{n>0} \mathrm{Hom}_{\mathcal{C}}(F_n, X).$$

If this functor is representable (in $\mathcal{C}$), one denotes "$\varprojlim$"$_n F_n$ a representative. Any functor from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$ isomorphic to a functor of the form "$\varprojlim$"$_{\mathbb{Z}_{>0}} F$ will be called a pro-object. We set $\mathrm{Pro}\,\mathcal{C}$ for the category of pro-objects in $\mathcal{C}$ (this is a subcategory of the category of functors from $\mathcal{C}$ to the category of sets). Note that any object in $\mathcal{C}$ defines a projective system, where $F_n = X$ for any n and all the applications $F_{n+k} \to F_n$ are the identity for $n, k > 0$. We obtain in this way a fully faithful embedding $\mathcal{C} \to \mathrm{Pro}\,\mathcal{C}$; one can in fact show that this coincides the Yoneda embedding.

For two pro-objects "$\varprojlim$" F_n and "$\varprojlim$" G_m the morphisms between these two objects can be described as

$$\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(\text{“}\varprojlim\text{”}\, F_n, \text{“}\varprojlim\text{”}\, G_m) = \varprojlim_m \varinjlim_n \mathrm{Hom}_{\mathcal{C}}(F_n, G_m).$$

If $\mathcal{D}$ is another category and $H : \mathcal{C} \to \mathcal{D}$ is a functor, we can extend H to the pro-categories in the "natural way". Namely there exists a unique functor H_{pro} such that the restriction of H_{pro} to $\mathcal{C}$ (under the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}\,\mathcal{C}$) is H, and it is given by what one may expect:

$$H_{\mathrm{pro}} : \mathrm{Pro}\,\mathcal{C} \to \mathrm{Pro}\,\mathcal{D}, \qquad \text{“}\varprojlim\text{”}\, F_n \mapsto \text{“}\varprojlim\text{”}\, H(F_n).$$

We will always forget the subscript "pro" in the following. We can apply this to the case of (complexes) of sheaves: if $\mathscr{F} =$ "$\varprojlim$" $\mathscr{F}_n$ is a pro-object in any of the usual categories $\mathrm{Loc}(X, \Bbbk)$, $D^b_{\mathcal{S}}(X, \Bbbk)$, ... , we can treat $\mathscr{F}$ (almost) like

a usual object. In particular, we can consider its restriction, its stalks or its cohomology. As an example, for a point $x \in X$, we have

$$\mathscr{F}_x := \text{“}\varprojlim\text{”}\,((\mathscr{F}_n)_x)\,.$$

We end this paragraph with the notion of monodromy for a pro-complex of sheaves (we keep the notation introduced in chapter 1). Consider as above such an object $\mathscr{F} = \text{“}\varprojlim\text{”}\,\mathscr{F}_n$. We have

$$\mathrm{End}_{\mathrm{Pro}\,D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}) = \varprojlim_m \varinjlim_n \mathrm{Hom}_{D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}_n, \mathscr{F}_m).$$

We define the monodromy of $\mathscr{F}$ to be the morphism given by

$$\varphi_{\mathscr{F}} : \Bbbk[\mathrm{X}_*(A)] \to \mathrm{End}_{\mathrm{Pro}\,D^b_{\mathcal{S}}(X,\Bbbk)}(\mathscr{F}), \qquad r \mapsto (\varphi_{\mathscr{F}_n}(r))_n\,.$$

Let us fix any element $t \in T^\vee_\Bbbk$. For any $n \in \mathbb{Z}_{\geq 0}$, we have a module $L_{t,n}$ over $\Bbbk[\mathrm{X}_*(T)]$, given by

$$L_{t,n} := \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle^n.$$

We then get an associated local system on T, denoted $\mathscr{L}^T_{t,n}$. The families $\{L_{t,n}\}_{n\in\mathbb{Z}_{n\geq 0}}$ and $\{\mathscr{L}^T_{t,n}\}_{n\in\mathbb{Z}_{n\geq 0}}$ naturally form projective systems. We can thus form the associated projective limit, resp. pro-object:

$$\widehat{R}_t := \varprojlim L_{t,n}$$

and

$$\widehat{\mathscr{L}^T_t} := \text{“}\varprojlim\text{”}\,\mathscr{L}^T_{t,n}\,.$$

The object $\widehat{R}_t$ is a $\Bbbk[\mathrm{X}_*(T)]$-module; whereas the pro-object $\widehat{\mathscr{L}^T_t}$ lies in the category of pro-local systems on T (i.e. the category $\mathrm{Pro}\,\mathrm{Loc}(T,\Bbbk)$).

5.2 Topological constructions

5.2.1 Right and left monodromy

As we said above, each $\mathscr{X}_w$ is stable under right and left multiplication by T. With the arguments of chapter 1 and section 1.4 (see in particular proposition 1.2.1 and definition 1.4.3), for any $\mathscr{F} \in D^b_{(B)}(\mathscr{X},\Bbbk)$ we can define two morphisms of monodromy

$$\varphi_{r,\mathscr{F}},\ \varphi_{l,\mathscr{F}} : \Bbbk[\mathrm{X}_*(T)] \longrightarrow \mathrm{End}_{D^b_{(B)}(\mathscr{X},\Bbbk)}(\mathscr{F}),$$

according to the action of T on $\mathscr{X}$ we consider (for the multiplication on the right, we obtain $\varphi_{r,\mathscr{F}}$; for multiplication on the left, we get $\varphi_{l,\mathscr{F}}$). Said otherwise, we can make $T \times T$ act on $\mathscr{X}$ as in subsection 5.1.1 and for any object

$\mathscr{F}$ in $D^b_{(B)}(\mathscr{X},\Bbbk)$ we can define a monodromy morphism $\varphi_{lr,\mathscr{F}}$ with respect to this action. One easily sees that this morphism identifies with

$$\varphi_{l,\mathscr{F}} \otimes \varphi_{r,\mathscr{F}} : \Bbbk[\mathrm{X}_*(T)] \otimes \Bbbk[\mathrm{X}_*(T)] \longrightarrow \mathrm{End}_{D^b_{(B)}(\mathscr{X},\Bbbk)}(\mathscr{F}).$$

In fact, by definition and with lemma 1.5.2, the left monodromy morphism can be described as $\varphi_{\mathscr{F}} \circ \iota_l$ where ι_l is the morphism

$$\Bbbk[\mathrm{X}_*(T)] \to \Bbbk[\mathrm{X}_*(T)] \otimes \Bbbk[\mathrm{X}_*(T)]$$

induced by $T \ni t \mapsto (t,1) \in T \times T$. Similarly, and with obvious notation, we have $\varphi_{r,\mathscr{F}} = \varphi_{\mathscr{F}} \circ \iota_r$. Notice also that the right and left monodromy commute with each other. Then we obtain a map

$$\begin{aligned} &\Bbbk[\mathrm{X}_*(T)] \otimes \Bbbk[\mathrm{X}_*(T)] \to \mathrm{End}_{D^b_{(B)}(\mathscr{X},\Bbbk)}(\mathscr{F}),\\ &c \otimes c' \mapsto \varphi_{l,\mathscr{F}}(c) \circ \varphi_{r,\mathscr{F}}(c') = \varphi_{r,\mathscr{F}}(c') \circ \varphi_{l,\mathscr{F}}(c), \end{aligned}$$

which is $\varphi_{lr,\mathscr{F}}$. As in section 3.2, one can define the full subcategories

$$\begin{aligned} D^b_{(B)}(\mathscr{X},\Bbbk)_{[t,t']} = \{\mathscr{F} \in D^b_{(B)}(\mathscr{X},\Bbbk);\\ \varphi_{r,\mathscr{F}}(\mathfrak{m}_{t'}) \text{ and } \varphi_{l,\mathscr{F}}(\mathfrak{m}_t) \text{ act nilpotently on } \mathscr{F}\}. \end{aligned}$$

For any $\mathscr{F} \in D^b_{(B)}(\mathscr{X},\Bbbk)$, we have a decomposition

$$\mathscr{F} \cong \bigoplus_{t,t' \in T^\vee_\Bbbk} \mathscr{F}_{t,t'}$$

with $\mathscr{F}_{t,t'} \in D^b_{(B)}(\mathscr{X},\Bbbk)_{[t,t']}$. Again as in section 3.2, we set

$$\begin{aligned} D^b_{(B)}(\mathscr{X},\Bbbk)_{[\underline{t},\underline{t'}]} = \{\mathscr{F} \in D^b_{(B)}(\mathscr{X},\Bbbk);\\ \varphi_{r,\mathscr{F}}(\mathfrak{m}_{t'}) \text{ and } \varphi_{l,\mathscr{F}}(\mathfrak{m}_t) \text{ act trivially on } \mathscr{F}\}. \end{aligned}$$

Quite often, we will just need a fixed right (or left) monodromy, but we won't need the monodromy on the other side to be fixed. In this case, we will denote

$$D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t']} = \{\mathscr{F} \in D^b_{(B)}(\mathscr{X},\Bbbk) \mid \varphi_{r,\mathscr{F}}(\mathfrak{m}_{t'}) \text{ acts nilpotently on } \mathscr{F}\},$$

and similarly for a fixed left monodromy. Similar (and obvious) notation holds for $\underline{t'}$ instead of t'. In order to simplify notation, and when no confusion is likely to occur, we will almost always write $D^b_{[-,t']}$ for $D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t']}$ and similarly for other monodromies. In particular, we will use these abbreviations only when we consider objects on $\mathscr{X}$. We can consider the subcategories of perverse sheaves in these various categories; we denote them as $\mathrm{P}_{[-,t']}$ (and accordingly for other monodromies). To conclude this section, we set the notation for standard and costandard objects in the above setting. For any $t \in T^\vee_\Bbbk$ and $w \in W$, set

$$\mathscr{L}^w_t := p_w^* \mathscr{L}^T_t .$$

We then have

$$\Delta_{w,t} = (j_w)_! \mathscr{L}^w_t[\dim(\mathscr{X}_w)] \quad \text{and} \quad \nabla_{w,t} = (j_w)_* \mathscr{L}^w_t[\dim(\mathscr{X}_w)].$$

5.2.2 Monodromic category over $\mathscr{X}$

For a topological space X, recall from section 3.3 that we denote by $D^b_{\mathrm{Loc}}(X,\Bbbk)$ the full triangulated subcategory of $D^b(X,\Bbbk)$ whose objects are those complexes $\mathscr{F}$ such that $\mathscr{H}^k(\mathscr{F})$ is a $\Bbbk$-local system on X for any $k\in\mathbb{Z}$. Thanks to [Ac, Theorem 1.9.7], remarking that the connected, locally contractible space $T\cong(\mathbb{C}^*)^r$ admits a contractible universal cover, we get an equivalence of categories $D^b\operatorname{Loc}(T,\Bbbk)\cong D^b_{(T)}(T,\Bbbk)$ induced by the inclusion functor $\operatorname{Loc}(T,\Bbbk)\hookrightarrow\operatorname{Sh}(T,\Bbbk)$ (where the last category is the category of sheaves of $\Bbbk$-modules on T). The arguments of subsection 3.3 then give a natural equivalence

$$D^b\operatorname{Loc}(T,\Bbbk)_{[t]}\xrightarrow{\sim}D^b_{(T)}(T,\Bbbk)_{[t]}. \tag{5.2.1}$$

Using the equivalence $\operatorname{Loc}(T,\Bbbk)\cong\operatorname{Loc}(\mathscr{X}_w,\Bbbk)$ given by the exact pullback functor p_w^*, the exact same argument as above gives an equivalence of triangulated categories

$$\Psi_w : D^b\operatorname{Loc}(T,\Bbbk)\xrightarrow{\sim}D^b_{\mathrm{Loc}}(\mathscr{X}_w,\Bbbk) \tag{5.2.2}$$

(given by $D^b\operatorname{Loc}(T,\Bbbk)\xrightarrow{p_w^*}D^b\operatorname{Loc}(\mathscr{X}_w,\Bbbk)\hookrightarrow D^b_{\mathrm{Loc}}(\mathscr{X}_w,\Bbbk)$).

Lemma 5.2.1. *Let $\mathscr{G}$ be in $D^b_{\mathrm{Loc}}(\mathscr{X}_w,\Bbbk)$. Then we have*

$$\varphi_{l,\mathscr{G}}=\varphi_{r,\mathscr{G}}\circ w^{-1}$$

where w^{-1} denotes here the $\Bbbk$-linear map $\Bbbk[\mathrm{X}_*(T)]\to\Bbbk[\mathrm{X}_*(T)]$ induced by $e^\lambda\mapsto e^{w^{-1}(\lambda)}$.

Proof. The decomposition (3.2.1) implies that monodromy is preserved under isomorphisms; we can assume that $\mathscr{G}=p_w^*\mathscr{F}$ for a certain $\mathscr{F}\in D^b_{\mathrm{Loc}}(T,\Bbbk)$. The map $p_w:\mathscr{X}_w\to T$ is T-equivariant for the actions on $\mathscr{X}_w$ and T given by:

$$t'\cdot gU=t'gU\qquad\text{and}\qquad t'\cdot t=w^{-1}(t')t.$$

It is also clearly equivariant for the action of T on $\mathscr{X}_w$ by multiplication on the right and the usual action of T on itself. For the duration of this proof, we will denote $\varphi^{\mathrm{reg}}_{\mathscr{F}}$ (resp. $\varphi^{\mathrm{twist}}_{\mathscr{F}}$) for the monodromy of $\mathscr{F}$ given by the usual (resp. twisted by w^{-1}) action of T on itself. For both actions on $\mathscr{X}_w$ described above, as p_w is T-equivariant, using lemma 1.5.5, we get

$$\varphi_{l,\mathscr{G}}=p_w^*(\varphi^{\mathrm{twist}}_{\mathscr{F}})\qquad\text{and}\qquad\varphi_{r,\mathscr{G}}=p_w^*(\varphi^{\mathrm{reg}}_{\mathscr{F}}).$$

With lemma 1.5.2, we see that the twisted monodromy of $\mathscr{F}$ is actually given by $\varphi^{\mathrm{reg}}_{\mathscr{F}}\circ w^{-1}$. We thus get

$$\varphi_{l,\mathscr{G}}=p_w^*(\varphi^{\mathrm{twist}}_{\mathscr{F}})=p_w^*(\varphi^{\mathrm{reg}}_{\mathscr{F}}\circ w^{-1})=\varphi_{r,p_w^*(\mathscr{F})}\circ w^{-1}=\varphi_{r,\mathscr{G}}\circ w^{-1}.$$

This concludes the proof. □

Assume moreover that $\mathscr{F}$ has a fixed monodromy, i.e. that $\mathscr{F} \in D^b_{(T)}(T, \Bbbk)_{[t]}$ for a certain $t \in T^\vee_\Bbbk$. We deduce that $p^*_w(\mathscr{F})$ has right monodromy given by t and left monodromy given by $w(t)$; said otherwise, we have $p^*_w(\mathscr{F}) \in D^b_{\mathrm{Loc}}(\mathscr{X}_w, \Bbbk)_{[w(t),t]}$. Indeed, by definition the ideal $\mathfrak{m}_t = \langle e^\lambda - \lambda(t) \mid \lambda \rangle$ acts in a nilpotent way on $\mathscr{F}$. Thanks to lemma 5.2.1 we get that the ideal $w(\mathfrak{m}_t)$ acts in a nilpotent way on $p^*_w(\mathscr{F})$ through $\varphi_{l,p^*_w(\mathscr{F})}$. But it is easily seen that $w(\mathfrak{m}_t) = \mathfrak{m}_{w(t)}$: since w acts as an algebra morphism on $\Bbbk[\mathrm{X}_*(T)]$, we get

$$w(\mathfrak{m}_t) = \langle e^{w(\lambda)} - \lambda(t) \mid \lambda \in \mathrm{X}_*(T) \rangle.$$

With the change of variable $\mu = w(\lambda)$ and by definition of the action on W on the character and cocharacter lattices, we obtain

$$w(\mathfrak{m}_t) = \langle e^\mu - \mu(w(t)) \mid \mu \in \mathrm{X}_*(T) \rangle$$

and so we deduce the result.

Lemma 5.2.2. *If the category $D^b_{[t,t']}$ is non-zero, then we have*

$$W \cdot t = W \cdot t'.$$

Moreover, we have

$$D^b_{[w(t),t]} = \langle \Delta_{v,t} \mid v \in wW_t \rangle$$

and

$$D^b_{[w(t),t]} = \langle \nabla_{v,t} \mid v \in wW_t \rangle$$

(as triangulated categories).

Proof. By definition, for any $\mathscr{F} \in D^b_{(B)}(\mathscr{X}, \Bbbk)$, we have

$$j^*_w(\mathscr{F}) \in D^b_{\mathrm{Loc}}(\mathscr{X}_w, \Bbbk).$$

Now consider a non-zero $\mathscr{F} \in D^b_{[t',t]}$; there exists $w \in W$ such that $j^*_w(\mathscr{F}) \neq 0$. From the discussion following lemma 5.2.1, we deduce that $t' = w(t)$. This implies in particular that $D^b_{[t',t]} = 0$ if t' is not in the W-orbit of t in $T^\vee_\Bbbk$.

We show the second statement of the lemma. Note that the essential image of $D^b \operatorname{Loc}(T, \Bbbk)_{[t]} = D^b_{\mathrm{Loc}}(T, \Bbbk)_{[t]}$ under Ψ_w is certainly equal to $D^b_{\mathrm{Loc}}(\mathscr{X}_w, \Bbbk)_{[w(t),t]}$. We know from lemma 3.3.6 that the former category is generated as a triangulated category by $\mathscr{L}^T_t$. We get that $D^b_{\mathrm{Loc}}(\mathscr{X}_w, \Bbbk)_{[w(t),t]}$ is generated (as a triangulated category) by the object $\Psi_w(\mathscr{L}^T_t) = p^*_w(\mathscr{L}^T_t)$. Take an object $\mathscr{F} \in D^b_{(B)}(\mathscr{X}, \Bbbk)_{[w(t),t]}$. We proceed by induction on the integer $N = \#\{v \in W \mid j^*_v(\mathscr{F}) \neq 0\}$. If $N = 0$, we have nothing to prove. Assume that $N = 1$; we have $\mathscr{F} = (j_v)_!(j_v)^* \mathscr{F}$ for a certain (unique) $v \in W$. Using the preceding argument, we see that v belongs to the coset wW_t. Since $(j_v)^*(\mathscr{F})$ is in $D^b_{\mathrm{Loc}}(\mathscr{X}_v, \Bbbk)_{[w(t),t]} = D^b_{\mathrm{Loc}}(\mathscr{X}_v, \Bbbk)_{[v(t),t]}$, we get $(j_v)_!(j_v)^* \mathscr{F} \in (j_v)_! \langle \Psi_v(\mathscr{L}^T_t)[n] \mid n \in Z \rangle$ and thus $\mathscr{F}$ can be written as a successive extension of shifts of $(j_v)_! p^*_v(\mathscr{L}^T_t) = \Delta_{v,t}$.

Assume now that $N > 1$. Consider a $v \in W$ such that $\mathscr{X}_v$ is open in the support $S_{\mathscr{F}}$ of $\mathscr{F}$. We have inclusions

$$\mathscr{X}_v \overset{j}{\hookrightarrow} S_{\mathscr{F}} \overset{i}{\hookleftarrow} S_{\mathscr{F}} \setminus \mathscr{X}_v$$

with j open and i closed. We can consider the extension by zero of the distinguished triangle

$$j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \overset{+1}{\to}$$

(which originally takes place in $D^b(S_{\mathscr{F}}, \Bbbk)$) to all of $\mathscr{X}$. We apply the case $N = 1$ to the left term and the induction hypothesis to the right one to conclude. The proof that the ∇'s generate the category is similar. □

5.3 Lusztig–Yun constructions

5.3.1 The case of G/U

We again consider $\mathscr{X} = G/U$ as a right and left stratified T-variety, with T-stable strata. According to section 4.1, for any $t \in T^\vee_{\Bbbk}$, we can consider the Lusztig–Yun category over $\mathscr{X}$ associated to the local system $\mathscr{L}^T_t$, for the right or the left action. These categories will be denoted respectively

$$\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]} \quad \text{and} \quad \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t,-]}.$$

We can consider perverse sheaves in these categories; one then obtains two categories $\mathfrak{P}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$ and $\mathfrak{P}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t,-]}$. The results of chapter 4 apply and thus these two categories are respectively equivalent to $\mathrm{P}_{[-,\underline{t}]}$ and $\mathrm{P}_{[\underline{t},-]}$; they both admit highest-weight structures.

We may occasionally have to consider the category $\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}^T_t,\mathscr{L}^T_t]}$ defined to be the category $\mathfrak{D}(T \times T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t \boxtimes \mathscr{L}^T_t]}$, where $T \times T$ acts via $(t_1, t_2) \cdot gU = t_1 g t_2 U$.

This category can be thought as "the full subcategory of $D^b_{(B),\widetilde{T}\times\widetilde{T}}(\mathscr{X}, \Bbbk)$ whose objects are those $\mathscr{F}$ that define objects in both $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$ and $\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t,-]}$". More precisely, choose a finite central isogeny $\widetilde{T} \to T$ as in section 4.1, so that $\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}^T_t,\mathscr{L}^T_t]}$ and $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[\mathscr{L}^T_t,\mathscr{L}^T_t]}$ are viewed respectively as full subcategory in $D^b_{(B),\widetilde{T}\times\widetilde{T}}(\mathscr{X}, \Bbbk)$ and $D^b_{(B),\widetilde{T}}(\mathscr{X}, \Bbbk)$. We view $\widetilde{T}$ as a subgroup in $\widetilde{T} \times \widetilde{T}$ via the natural inclusion

$$i : \widetilde{T} \cong \{1\} \times \widetilde{T} \hookrightarrow \widetilde{T} \times \widetilde{T}.$$

Then the identity of $\mathscr{X}$ is an i-map is the sense of [BL, I, §0.1], and we can consider the equivariant pullback $Q^*_{id_{\mathscr{X}}}$ (as defined in [BL, Definition 6.5]). We obviously have an analogous functor $\overline{Q}^*_{id_{\mathscr{X}}}$ corresponding to the inclusion $\overline{i} : \widetilde{T} \cong \widetilde{T} \times \{1\} \hookrightarrow \widetilde{T} \times \widetilde{T}$. If a $\widetilde{T} \times \widetilde{T}$-equivariant object $\mathscr{F}$ is in the bi-Lusztig–Yun equivariant category, then $Q^*_{\mathrm{id}_{\mathscr{X}}}(\mathscr{F})$ (resp. $\overline{Q}^*_{id_{\mathscr{X}}}(\mathscr{F})$) lies in the

right (resp. left) Lusztig–Yun equivariant category. The (restriction to the direct summand subcategory $\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T, \mathscr{L}_t^T]}$ of the) functor $Q_{id_{\mathscr{X}}}^*[r]$ defines a forgetful functor

$$\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T, \mathscr{L}_t^T]} \to \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-, \mathscr{L}_t^T]} \tag{5.3.1}$$

sending a complex $\mathscr{F}$ to itself (shifted by r) but forgetting the left equivariance (and similarly for $\overline{Q}_{id_{\mathscr{X}}}^*$).

We can obviously consider the subcategory of perverse objects in the bi-equivariant category which will be denoted by $\mathfrak{P}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T, \mathscr{L}_t^T]}$. The functor (5.3.1) is clearly t-exact for the perverse t-structure (this is the reason for the shift $[r]$), and induces a fully faithful functor

$$\mathfrak{P}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T, \mathscr{L}_t^T]} \to \mathfrak{P}(\mathscr{X} /\!\!/ T)_{[-, \mathscr{L}_t^T]}. \tag{5.3.2}$$

5.3.2 The case of G/U^s: settings

Let $s = s_\alpha$ be a simple reflection in W. By [J, §1.18] there exists a surjective map

$$Z^\circ(L_s) \times \mathscr{D}(L_s) \to L_s$$

whose kernel is isomorphic to a subgroup of $Z(\mathscr{D}(L_s))$, so is at most of order two since the derived group is either isomorphic to SL_2 or PGL_2. Now $\mathscr{D}(L_s)$ is a semisimple group of rank one; we have a surjective map $\varphi_\alpha : SL_2 \twoheadrightarrow \mathscr{D}(L_s)$, where φ_α is as defined in section 5.1.1, and the kernel is either of order one or two. The map $Z^\circ(L_s) \times SL_2 \to L_s$ we obtain defines a finite central isogeny whose kernel is either of order one (in which case we have an isomorphism) or of order two. We can precompose this map with the endomorphism $e_n \times \mathrm{id}_{SL_2}$ of $Z^\circ(L_s) \times SL_2$ (where, with a slight abuse of notation, we still denote by e_n the n-th power map on the torus $Z^\circ(L_s)$). In this way we obtain an isogeny

$$\nu_{L_s} : \widetilde{L_s} := Z^\circ(L_s) \times SL_2 \to L_s.$$

The kernel K of ν has then cardinality $n^{\mathrm{rk}(Z^\circ(L_s))}$ or $2n^{\mathrm{rk}(Z^\circ(L_s))}$. (This is one reason for which we assumed that the characteristic ℓ of $\Bbbk$ is not 2.)

We will also consider the following isogenies

$$\nu_{B_s} : \widetilde{B_s} = B_s \times_{L_s} \widetilde{L_s} \to B_s$$

and

$$\nu_T : \widetilde{T} = T \times_{L_s} \widetilde{L_s} \to T.$$

Note that we have embeddings $K \hookrightarrow \widetilde{T} \hookrightarrow \widetilde{B_s}$.

The above description tells us that $\widetilde{B_s}$ is the product $Z^\circ(L_s) \times B_{SL_2}$, with B_{SL_2} the Borel subgroup of upper triangular matrices in SL_2. Similarly, $\widetilde{T}$ is the product $Z^\circ(L_s) \times T_{SL_2}$ with T_{SL_2} the maximal torus of diagonal matrices in SL_2.

The groups B_s and L_s act naturally on the left and on the right on the variety $\mathscr{X}^s = G/U^s$ (but note that the strata $\mathscr{X}^s_w$ are not stable under the L_s-action). Thus we have actions of $\widetilde{B_s}$ and $\widetilde{L_s}$ on $\mathscr{X}^s$ via ν_{B_s} and ν_{L_s}, and we can consider the equivariant categories, either for the right or left action (the context should make clear what action we will actually consider)

$$D^b_{(B),\widetilde{B_s}}(\mathscr{X}^s, \Bbbk) \quad \text{and} \quad D^b_{(B),\widetilde{L_s}}(\mathscr{X}^s, \Bbbk).$$

Remark 5.3.1. As stated above, the strata $\mathscr{X}^s_w$ are not stable under the actions of L_s (and so of $\widetilde{L_s}$) on the right and the left. Thus the objects in $D^b_{(B),\widetilde{L_s}}(\mathscr{X}^s, \Bbbk)$ are actually constructible for the stratification on $\mathscr{X}^s$ obtained as the pullback of the Bruhat stratification on the quotient G/P_s via the natural map $\mathscr{X}^s \twoheadrightarrow G/P_s$. However, in analogy with the other categories we consider, we keep the subscript "(B)" in the notation.

Lemma 5.3.2. *Take $t \in T^\vee_\Bbbk$ and s a simple reflection of W that lies in W°_t. Then there exists a unique local system $\mathscr{L}^{L_s}_t$ on L_s such that $(\mathscr{L}^{L_s}_t)_{|T} \cong \mathscr{L}^T_t$.*

Proof. It is well known (see e.g. [BtD, Theorem (7.1)]) that $\pi_1(L_s) \cong \mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s^\vee$. More precisely, the inclusion $T \hookrightarrow L_s$ induces a quotient map $\mathrm{X}_*(T) \cong \pi_1(T) \twoheadrightarrow \pi_1(L_s) \cong \mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s^\vee$. Thus the isomorphism classes of simple local systems on L_s are in bijection with irreducible representations of $\mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s^\vee$, that is to say, with characters of this abelian group. Using the discussion at the beginning of section 3.2, these characters corresponds to elements $t \in T^\vee_\Bbbk$ such that $\alpha_s^\vee(t) = 1$. Now for such a t, let $\mathscr{L}^{L_s}_t$ be the associated local system on L_s. The restriction of this sheaf to T corresponds to the character of $\mathrm{X}_*(T)$ obtained by composition with the quotient map $\mathrm{X}_*(T) \twoheadrightarrow \mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s^\vee$ on fundamental groups. This is the same as the character of $\mathrm{X}_*(T)$ associated to t, thus we deduce the result. □

5.3.3 The case of G/U^s: Lusztig–Yun categories

In this section, we apply the Lusztig–Yun construction to the above setting: we consider the variety $\mathscr{X}^s$ and the groups L_s and B_s. Choose $t \in T^\vee_\Bbbk$ such that $s \in W^\circ_t$ and fix, for the rest of this section, n to be the smallest positive integer such that $t^n = 1$, that is, the order of t. Recall that this n is then coprime to ℓ, see remark 3.3.1. By lemma 5.3.2, we can extend $\mathscr{L}^T_t$ to L_s to obtain a local system $\mathscr{L}^{L_s}_t$. We will denote the restriction of $\mathscr{L}^{L_s}_t$ to B_s by $\mathscr{L}^{B_s}_t$. Note that $\mathscr{L}^{B_s}_t$ is also the pullback of $\mathscr{L}^T_t$ along the natural map $B_s \cong T \ltimes U_s \to T$. Our aim is to define Lusztig–Yun categories associated to these data. We must show that there exists a character χ_t of K such that $\mathscr{L}^{L_s}_t$ is the χ_t-isotypic component of $(\nu_{L_s})_* \underline{\Bbbk}_{\widetilde{L_s}}$. (Recall that we denoted K the kernel of ν_{L_s} and that the cardinality of K is prime to the characteristic ℓ of $\Bbbk$.)

The pushforward of $\underline{\Bbbk}_{\widetilde{L_s}}$ along the proper map ν_{L_s} identifies with the local system associated with the $\pi_1(L_s)$-representation

$$\Bbbk[\pi_1(L_s)/\pi_1(\widetilde{L_s})].$$

Now a finite central isogeny is a covering map. Note that the finite group K acts transitively and freely on the fibers of ν_{L_s}. This implies that we have an isomorphism

$$K \cong \mathrm{Aut}(\widetilde{L_s}/L_s),$$

where $\mathrm{Aut}(\widetilde{L_s}/L_s)$ denotes the deck tranformation group of the covering $\widetilde{L_s}/L_s$. In particular, the action of this automorphism group is transitive on the fibers of the covering, so this covering is Galois and we have an exact sequence of abelian groups

$$0 \to \pi_1(\widetilde{L_s}) \to \pi_1(L_s) \to K \to 0.$$

The pushforward of $\underline{\Bbbk}_{\widetilde{L_s}}$ then identifies with the local system associated to the $\pi_1(L_s)$-representation $\Bbbk[K]$. Since K has order prime to ℓ, this representation splits as a direct sum of simple subrepresentation of rank one according to the characters of K; the corresponding local system then splits as a direct sum of rank one local systems.

Lemma 5.3.3. *We have a nonzero morphism $\mathscr{L}_t^{L_s} \to (\nu_{L_s})_* \underline{\Bbbk}_{\widetilde{L_s}}$. This morphism corresponds to the inclusion of a direct summand associated to a character χ_t of K. Moreover, this morphism induces a morphism*

$$\mathscr{L}_t^{B_s} \to (\nu_{B_s})_* \underline{\Bbbk}_{\widetilde{B_s}}$$

corresponding to the inclusion of the $(\chi_t^{\widetilde{B_s}})$-isotypic component in the second term (where $\chi_t^{\widetilde{B_s}}$ is χ_t viewed as a character of $K \hookrightarrow \widetilde{B_s}$).

Proof. To prove the first part of the lemma, it suffices to show that $(\nu_{L_s})^* \mathscr{L}_t^{L_s} \cong \underline{\Bbbk}_{\widetilde{L_s}}$. We can then take our morphism to be the nonzero adjunction map

$$\mathscr{L}_t^{L_s} \to (\nu_{L_s})_* (\nu_{L_s})^* \mathscr{L}_t^{L_s}.$$

This map will automatically be the inclusion of a rank-one local system, corresponding to a certain character χ_t of K. Recall that we defined ν_{L_s} in two steps: first we consider $(e_n \times \mathrm{id}_{SL_2})$ and then we compose with the natural map $Z(L_s)^\circ \times SL_2 \to L_s$. If we pullback $\mathscr{L}_t^{L_s}$ along this last map, we obtain

$$(\mathscr{L}_t^{L_s})_{|Z(L_s)^\circ} \boxtimes \underline{\Bbbk}_{SL_2}$$

since SL_2 is simply connected (as a $\mathbb{C}$-variety). Now $(\mathscr{L}_t^{L_s})_{|Z(L_s)^\circ} = (\mathscr{L}_t^T)_{|Z(L_s)^\circ}$ and we know that the pullback of $\mathscr{L}_t^T$ along e_n is the trivial local system. We thus have $(\nu_{L_s})^* \mathscr{L}_t^{L_s} \cong \underline{\Bbbk}_{\widetilde{L_s}}$. This concludes the proof of the first statement.

The second part of the lemma then follows easily from the following facts: first we have an embedding $K \hookrightarrow \widetilde{B_s}$, then the following square is cartesian by construction:

$$\begin{array}{ccc} \widetilde{B_s} & \hookrightarrow & \widetilde{L_s} \\ {\scriptstyle \nu_{B_s}}\downarrow & & \downarrow{\scriptstyle \nu_{L_s}} \\ B_s & \hookrightarrow & L_s. \end{array}$$

A simple application of the base change theorem (recall that ν_{L_s} is proper) allows us to conclude the proof. □

Sitting in our equivariant derived categories defined above $D^b_{(B),\widetilde{B_s}}(G/U^s,\Bbbk)$ and $D^b_{(B),\widetilde{L_s}}(G/U^s,\Bbbk)$, we can thus define Lusztig–Yun categories

$$\mathfrak{D}(\mathscr{X}^s/\!\!/B_s)_{[-,\mathscr{L}_t^{B_s}]} \qquad \text{and} \qquad \mathfrak{D}(\mathscr{X}^s/\!\!/L_s)_{[-,\mathscr{L}_t^{L_s}]}.$$

(We could and will also consider the left monodromic categories.) Later on, we will use these categories in order to deduce informations on the structure of the category of perverse sheaves in $D^b_{[-,\underline{t}]}$ and $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$, thanks to the following discussion.

We have a natural map $p^s : G/U^s \to G/U$ given by the quotient by the free (right) action of U_s. According to the discussion following the proof of lemma 2.5.2, the unipotent radicals of the isogeneous groups B_s and $\widetilde{B_s}$ are isomorphic, so we have a copy of U_s inside $\widetilde{B_s}$ (in fact given by the upper triangular unipotent subgroup of SL_2 in the decomposition $\widetilde{B_s} \cong Z(L_s)^\circ \times B_{SL_2}$). We then easily get $\widetilde{B_s}/U_s \cong \widetilde{T}$. The map p^s is a $\widetilde{B_s}$-$\widetilde{T}$-map, and so we can consider the functor

$$Q^*_{p^s} : D^b_{(B),\widetilde{T}}(\mathscr{X},\Bbbk) \to D^b_{(B),\widetilde{B_s}}(\mathscr{X}^s,\Bbbk) \tag{5.3.3}$$

defined in [BL, §6]; according to [BL, §6.6, item 5], this functor is an equivalence of categories.

Lemma 5.3.4. *The functor $Q^*_{p^s}$ preserves the Lusztig–Yun categories, i.e. it induces an equivalence*

$$\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}^s/\!\!/B_s)_{[-,\mathscr{L}_t^{B_s}]}.$$

Proof. Recall that we set $\widetilde{T} = T \times_{L_s} \widetilde{L_s}$. It is clear that we have $K \hookrightarrow \widetilde{T}$ and that the character χ of K such that $\mathscr{L}_t^T \cong ((\nu_T)_* \underline{\Bbbk}_{\widetilde{T}})_\chi$ is the same that defines $\mathscr{L}_t^{B_s}$ as an isotypic component of $(\nu_{B_s})_* \underline{\Bbbk}_{\widetilde{B_s}}$. Moreover the projection $p : \widetilde{B_s} \twoheadrightarrow \widetilde{B_s}/U_s \cong \widetilde{T}$ does not affect K, that is to say, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{B_s} & \xrightarrow{\quad p \quad} & \widetilde{T} \\ & \nwarrow \quad \nearrow & \\ & K. & \end{array} \tag{5.3.4}$$

We consider $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ and $\mathfrak{D}(\mathscr{X}^s/\!\!/B_s)_{[-,\mathscr{L}_t^{B_s}]}$ as full subcategories of $D^b_{(B),\widetilde{T}}(\mathscr{X},\Bbbk)$ and $D^b_{(B),\widetilde{B_s}}(\mathscr{X}^s,\Bbbk)$ respectively. For an object $\mathscr{F}$ in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$, we have to determine the action of K on $Q^*_{p^s}(\mathscr{F})$. Note first that $\mathrm{For}_{\widetilde{B_s}} Q^*_{p^s} = (p^s)^* \mathrm{For}_{\widetilde{T}}$ (here $\mathrm{For}_?$ is the usual, non-shifted, forgetful functor

from the ?-equivariant category to the constructible category). We use the following commutative square:

$$\begin{array}{ccc} G/U^s \times \widetilde{B_s} & \xrightarrow{p^s \times p} & G/U \times \widetilde{T} \\ \downarrow{\scriptstyle a_s,\mathrm{pr}} & & \downarrow{\scriptstyle a,\mathrm{pr}} \\ G/U^s & \xrightarrow{p^s} & G/U \end{array}$$

where a_s stands for the action map. We deduce that the isomorphism

$$a_s^* \operatorname{For}_{\widetilde{B_s}} Q_{p^s}^*(\mathscr{F}) \xrightarrow{\sim} \mathrm{pr}^* \operatorname{For}_{\widetilde{B_s}} Q_{p^s}^*(\mathscr{F})$$

is the pullback under $p^s \times p$ of the isomorphism $a^* \operatorname{For}_{\widetilde{T}}(\mathscr{F}) \xrightarrow{\sim} \mathrm{pr}^* \operatorname{For}_{\widetilde{T}}(\mathscr{F})$. Using diagram (5.3.4), one deduces that the restriction of this isomorphism to $G/U^s \times \{k\} \cong G/U^s$ is the pullback along p^s of the corresponding isomorphism on G/U. In particular, K acts on $Q_{p^s}^*(\mathscr{F})$ via the same character as it does for $\mathscr{F}$.

Since $Q_{p^s}^*$ is an equivalence, it is automatically fully faithful; its restriction to a Lusztig–Yun factor induces an equivalence $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$ thanks to the above considerations. □

We end this section by an immediate, but nice property of the functor $Q_{p^s}^*$. First, a few definitions:

Definition 5.3.5. *For any $w \in W$, set*

$$\Delta(w)_{\mathscr{L}_t^T} := (j_w)_! p_w^* \mathscr{L}_t^T[\ell(w)], \quad \nabla(w)_{\mathscr{L}_t^T} := (j_w)_* p_w^* \mathscr{L}_t^T[\ell(w)].$$

Similarly, set

$$\Delta(w)_{\mathscr{L}_t^{B_s}} := (j_w^s)_! (p_w^{B_s})^* \mathscr{L}_t^{B_s}[\ell(w)], \quad \nabla(w)_{\mathscr{L}_t^{B_s}} := (j_w^s)_* (p_w^{B_s})^* \mathscr{L}_t^{B_s}[\ell(w)].$$

From the definition of section 4.2, it is clear that the image under the forgetful functor $\operatorname{For}_t : \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]} \to D^b_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$ of $\Delta(w)_{\mathscr{L}_t^T}$ (respectively $\nabla(w)_{\mathscr{L}_t^T}$) is $\Delta_{w,t}$ (respectively $\nabla_{w,t}$). In particular, the objects $\Delta(w)_{\mathscr{L}_t^T}$ and $\nabla(w)_{\mathscr{L}_t^T}$ are perverse. We define a perverse t-structure on the category $\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$ similarly as in definition 4.1.2. Recall from the proof of lemma 5.3.4 that we have a forgetful functor $\operatorname{For}_{\widetilde{B}_s}$ on $D^b_{(B),\widetilde{B}_s}(\mathscr{X}^s, \Bbbk)$. In the following definition, we consider the restriction of this functor to the direct factor subcategory $\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$; we keep the same notation for this restriction.

Definition 5.3.6. *Define the perverse t-structure on $\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$ to be the shift by $(r+1)$ of the usual perverse t-structure on $D^b_{(B)}(\mathscr{X}^s, \Bbbk)$. That is, $\mathscr{F} \in {}^p\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)^{\leq 0}_{[-,\mathscr{L}_t^{B_s}]}$ if and only if $\operatorname{For}_{\widetilde{B}_s}(\mathscr{F})[r+1]$ is in ${}^pD^{b,\leq 0}_{(B)}(\mathscr{X}^s, \Bbbk)$. Define ${}^p\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)^{\geq 0}_{[-,\mathscr{L}_t^{B_s}]}$ similarly. We set $\mathfrak{P}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$ for the heart of this t-structure, the subcategory of perverse sheaves.*

The objects $\Delta(w)_{\mathscr{L}_t^{B_s}}$ and $\nabla(w)_{\mathscr{L}_t^{B_s}}$ are then perverse, moreover, the negative subcategory ${}^p\mathfrak{D}(\mathscr{X}^s/\!\!/B_s)^{\leq 0}_{[-,\mathscr{L}_t^{B_s}]}$ is generated by the objects $\Delta(w)_{\mathscr{L}_t^{B_s}}[n]$ with $w \in W$ and $n \geq 0$, and ${}^p\mathfrak{D}(\mathscr{X}^s/\!\!/B_s)^{\geq 0}_{[-,\mathscr{L}_t^{B_s}]}$ by the objects $\nabla(w)_{\mathscr{L}_t^{B_s}}[n]$ with $w \in W$ and $n \leq 0$.

Lemma 5.3.7. *For any $w \in W$, one has*

$$Q_{p^s}^*(\Delta(w)_{\mathscr{L}_t^T}) = \Delta(w)_{\mathscr{L}_t^{B_s}}, \quad \text{and} \quad Q_{p^s}^*(\nabla(w)_{\mathscr{L}_t^T}) = \nabla(w)_{\mathscr{L}_t^{B_s}}.$$

In particular, $Q_{p^s}^*$ induces an equivalence

$$\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{P}(\mathscr{X}^s/\!\!/B_s)_{[-,\mathscr{L}_t^{B_s}]},$$

and maps $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ to $\mathrm{IC}(w)_{\mathscr{L}_t^{B_s}}$ for any w.

Proof. The isomorphisms are an immediate consequence of the smooth base change theorem. The second statement follows directly from these isomorphisms. □

5.3.4 Verdier duality

Recall that we defined a Verdier duality functor in section 2.10. We have such a functor on $\mathscr{X}$, but we will normalize it so that it preserves perverse objects.

Definition 5.3.8. *For any $t, t' \in T_{\Bbbk}^\vee$, we set*

$$\underline{\underline{\mathbb{D}}}_{\mathscr{X}} : \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X}/\!\!/T)_{[\mathscr{L}_{t'}^T, \mathscr{L}_t^T]} \to \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X}/\!\!/T)^{\mathrm{op}}_{[\mathscr{L}_{t'-1}^T, \mathscr{L}_{t-1}^T]},$$
$$\mathscr{F} \mapsto R\mathcal{H}om(\mathscr{F}, \underline{\mathbb{D}}_{\mathscr{X}}[-2r]).$$

This functor satisfies $\underline{\underline{\mathbb{D}}}_{\mathscr{X}} \circ \underline{\underline{\mathbb{D}}}_{\mathscr{X}} \cong \mathrm{id}$ and is t-exact for the perverse t-structures.

Similarly, we define $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}$ to be the functor $\mathscr{F} \mapsto R\mathcal{H}om(\mathscr{F}, \underline{\mathbb{D}}_{\mathscr{X}^s}[-2(r+1)])$ for any $\mathscr{F}$ in $\mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s/\!\!/B_s)_{[\mathscr{L}_{t'}^{B_s}, \mathscr{L}_t^{B_s}]}$ or $\mathfrak{D}(L_s \setminus\!\!\setminus \mathscr{X}^s/\!\!/L_s)_{[\mathscr{L}_{t'}^{L_s}, \mathscr{L}_t^{L_s}]}$.

The proof of the following lemma is omitted; the result follows easily from usual considerations. The reader should note that the normalization of $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}$ is the correct one in order for the lemma to be true.

Lemma 5.3.9. *We have $Q_{p^s}^* \circ \underline{\underline{\mathbb{D}}}_{\mathscr{X}} \cong \underline{\underline{\mathbb{D}}}_{\mathscr{X}^s} \circ Q_{p^s}^*$ (for $Q_{p^s}^*$ the functor of (5.3.3)). For any $w \in W$ and $t \in T_{\Bbbk}^\vee$, we have*

$$\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\Delta(w)_{\mathscr{L}_t^T}) \cong \nabla(w)_{\mathscr{L}_{t-1}^T} \quad \text{and} \quad \underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\nabla(w)_{\mathscr{L}_t^T}) \cong \Delta(w)_{\mathscr{L}_{t-1}^T}$$

and

$$\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\Delta(w)_{\mathscr{L}_t^{B_s}}) \cong \nabla(w)_{\mathscr{L}_{t-1}^{B_s}} \quad \text{and} \quad \underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\nabla(w)_{\mathscr{L}_t^{B_s}}) \cong \Delta(w)_{\mathscr{L}_{t-1}^{B_s}}.$$

Note that for $t, t' \in T_{\Bbbk}^{\vee}$, we have ${}_{t'}W_t = {}_{t'^{-1}}W_{t^{-1}}$. Similarly, we have $W_t^{\circ} = W_{t^{-1}}^{\circ}$, and thus ${}_{t'}\underline{W}_t = {}_{t'^{-1}}\underline{W}_{t^{-1}}$. The previous lemma then implies the following (obvious) corollary.

Corollary 5.3.10. *For any block $\beta \in {}_{t'}W_t$ the Verdier duality $\underline{\mathbb{D}}_{\mathscr{X}}$ restricts to a t-exact equivalence of categories*

$$\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)^{\beta}_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)^{\beta,\mathrm{op}}_{[\mathscr{L}^T_{t'^{-1}}, \mathscr{L}^T_{t^{-1}}]}.$$

Chapter 6

Convolution

6.1 Convolution in the topological case

Let $G \times^U \mathscr{X}$ be the coset space of $G \times \mathscr{X}$ under the action of U given by

$$u \cdot (g, hU) = (gu^{-1}, uhU)$$

(for $g, h \in G$ and $u \in U$). The quotient map is denoted $q_U : G \times \mathscr{X} \to G \times^U \mathscr{X}$. Let m_U be the natural multiplication map $G \times^U \mathscr{X} \to \mathscr{X}$

$$m_U([g; hU]) = ghU.$$

Also, recall that we let ϖ denotes the quotient map $G \to G/U$. Since the above antidiagonal action of U on $G \times \mathscr{X}$ is free, for any $\mathscr{F}, \mathscr{G} \in D^b_{(B)}(\mathscr{X}, \Bbbk)$, there exists a unique object $\mathscr{F} \boxtimes^U \mathscr{G}$ in $D^b(G \times^U \mathscr{X}, \Bbbk)$ such that

$$q_U^*(\mathscr{F} \boxtimes^U \mathscr{G}) = \varpi^*(\mathscr{F}) \boxtimes \mathscr{G}\,.$$

We then set

$$\mathscr{F} \star^U \mathscr{G} := (m_U)_!(\mathscr{F} \boxtimes^U \mathscr{G})[\dim(T)]$$

and call this object the convolution product of $\mathscr{F}$ and $\mathscr{G}$.

We have obtained in this way a bifunctor

$$(-) \star^U (-) : D^b_{(B)}(\mathscr{X}, \Bbbk) \times D^b_{(B)}(\mathscr{X}, \Bbbk) \longrightarrow D^b_{(B)}(\mathscr{X}, \Bbbk).$$

It is well known that convolution is associative i.e. that we have natural isomorphisms $(\mathscr{F} \star^U \mathscr{G}) \star^U \mathscr{H} \cong \mathscr{F} \star^U (\mathscr{G} \star^U \mathscr{H})$ for any $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ in $D^b_{(B)}(\mathscr{X}, \Bbbk)$.

6.2 Convolution and monodromy

We study the behavior of convolution and monodromy. Let $m_B : G \times^B \mathscr{X} \to \mathscr{X}$ be the map induced by the action of G on $\mathscr{X}$.

Lemma 6.2.1. *Let $\mathscr{F},\mathscr{G} \in D^b_{(B)}(\mathscr{X},\Bbbk)$. We have*

1. $\varphi_{l,\mathscr{F}\star^U\mathscr{G}} = \varphi_{l,\mathscr{F}} \star^U \mathrm{id}_{\mathscr{G}}$ *and* $\varphi_{r,\mathscr{F}\star\mathscr{G}} = \mathrm{id}_{\mathscr{F}} \star^U \varphi_{r,\mathscr{G}}$,

2. *if* $\mathscr{F} \in D^b_{[-,t]}$ *and* $\mathscr{G} \in D^b_{[t',-]}$ *with* $t \neq t'$, *then* $\mathscr{F}\star^U\mathscr{G} = 0$,

3. $\varphi_{r,\mathscr{F}} \star^U \mathrm{id}_{\mathscr{G}} = \mathrm{id}_{\mathscr{F}} \star^U \varphi_{l,\mathscr{G}}$.

Proof. (1) We prove the first equality, the second one being absolutely similar. We view $\mathscr{X}$, $G\times\mathscr{X}$ and $G\times^U\mathscr{X}$ as T-varieties under left multiplication (on the left factor uniquely for the two products). Note that both q_U and m_U are T-equivariant for these actions. By virtue of the definition of convolution product and lemma 1.5.5, if we show that the equality

$$\varphi_{l,\mathscr{F}\boxtimes^U\mathscr{G}} = \varphi_{l,\mathscr{F}} \boxtimes^U \mathrm{id}_{\mathscr{G}}$$

holds, we will be done. Since the functor q_U^* is fully faithful, it suffices to show that the pullback along q_U of these two morphisms coincide. But the functor

$$q_U^*((-)\boxtimes^U\mathscr{G}) : D^b_{(B)}(\mathscr{X},\Bbbk) \to D^b_{(B)}(G\times\mathscr{X},\Bbbk)$$

is by definition isomorphic to the functor $\varpi^*(-)\boxtimes\mathscr{G}$. Using twice lemma 1.5.5, we see that $q_U^*(\varphi_{l,\mathscr{F}}\boxtimes^U\mathrm{id}_{\mathscr{G}}) = \varphi_{l,\varpi^*(\mathscr{F})\boxtimes\mathscr{G}} = q_U^*(\varphi_{l,\mathscr{F}\boxtimes^U\mathscr{G}})$.

Assume now that we are in the situation of (2). We see $G\times\mathscr{X}$ and $G\times^U\mathscr{X}$ as T-varieties under the action

$$t\cdot(g,hU) = (gt^{-1},thU).$$

(Here (g,hU) could be an element of $G\times\mathscr{X}$ or an equivalence class in $G\times^U\mathscr{X}$.) We also view $\mathscr{X}$ as a T-variety for the trivial T-action. In this way, q_U and m_U are still T-equivariant morphisms. Using lemma 1.5.5 and lemma 1.5.6, we see that $\mathscr{F}\boxtimes^U\mathscr{G}$ is monodromic with monodromy given by $\varphi_{\mathscr{F}\boxtimes^U\mathscr{G}} = \varphi_{r,\mathscr{F}}^{-1}\boxtimes^U\varphi_{l,\mathscr{G}}$ (for the considered antidiagonal action of T). By hypothesis, this monodromy is non trivial; now an application of lemma 3.2.1 allows us to conclude that the convolution is zero.

The proof of (3) is borrowed from [BeR, Lemma 7.3]. We write

$$m_U : G\times^U\mathscr{X} \xrightarrow{m'} G\times^B\mathscr{X} \xrightarrow{m_B} \mathscr{X}.$$

Consider again the action of T on $G\times^U\mathscr{X}$ given by $t\cdot(g,hU) = (gt^{-1},thU)$. The object $\mathscr{F}\boxtimes^U\mathscr{G}$ is monodromic for this action and we have

$$\varphi_{\mathscr{F}\boxtimes^U\mathscr{G}} = \varphi_{r,\mathscr{F}}^{-1}\boxtimes^U\varphi_{l,\mathscr{G}}$$

i.e. we have $\varphi_{\mathscr{F}\boxtimes^U\mathscr{G}}(e^\lambda) = \varphi_{r,\mathscr{F}}(e^{-\lambda})\boxtimes^U\varphi_{l,\mathscr{G}}(e^\lambda)$ for any $\lambda\in\mathrm{X}_*(T)$. Now the map m' realizes the quotient $G\times^U\mathscr{X}\to(G\times^U\mathscr{X})/T$, in particular the action of T on $G\times^B\mathscr{X}$ is trivial and we have $(m')_!(\varphi_{\mathscr{F}\boxtimes^U\mathscr{G}}) = \mathrm{id}$ thanks to lemma 1.5.5. Since one has $\varphi_{r,\mathscr{F}}^{-1}\boxtimes^U\varphi_{l,\mathscr{G}} = (\varphi_{r,\mathscr{F}}^{-1}\boxtimes^U\mathrm{id}_{\mathscr{G}})\circ(\mathrm{id}_{\mathscr{F}}\boxtimes^U\varphi_{l,\mathscr{G}})$, this implies that

$$(m')_!(\varphi_{r,\mathscr{F}}\boxtimes^U\mathrm{id}_{\mathscr{G}}) = (m')_!(\mathrm{id}_{\mathscr{F}}\boxtimes^U\varphi_{l,\mathscr{G}}).$$

Applying $(m_B)_!$ to this equality, we get the desired result. □

6.3 Convolution in the equivariant case

6.3.1 Convolution over $\mathscr{X}$

Consider an element $t \in T^\vee_{\Bbbk}$. Recall the quotient maps $\varpi : G \to \mathscr{X}$ and $q_U : G \times \mathscr{X} \to G \times^U \mathscr{X}$.

Consider $\mathscr{F} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ and $\mathscr{G} \in \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t,-]}$. Consider the action of T on $G \times^U \mathscr{X}$ given by $t \cdot [g, hU] = [gt^{-1}, thU]$. There exists a unique object $\mathscr{F} \boxtimes^U \mathscr{G}$ on $G \times^U \mathscr{X}$ such that $q_U^*(\mathscr{F} \boxtimes^U \mathscr{G}) = \varpi^*(\mathscr{F}) \boxtimes \mathscr{G}$. Now choosing an isogeny $\widetilde{T} \to T$ allowing us to see the above Lusztig–Yun equivariant categories as full subcategories of $\widetilde{T}$-equivariant categories, the object $\mathscr{F} \boxtimes^U \mathscr{G}$ is in the category $D^b_{\widetilde{T}}(G \times^U \mathscr{X}, \Bbbk)$ for the action of T prescribed above. Since $\mathscr{F}$ is $\mathscr{L}^T_t$-equivariant for the action of T by multiplication on the right and $\mathscr{G}$ is $\mathscr{L}^T_t$-equivariant for the action of T by multiplication on the left, using lemma 2.9.3, one easily checks that the pullback of $\mathscr{F} \boxtimes^U \mathscr{G}$ along the action morphism $T \times G \times^U \mathscr{X} \to G \times^U \mathscr{X}$ is given by $\underline{\Bbbk}_T \boxtimes \mathscr{F} \boxtimes^U \mathscr{G}$. Thus, using corollary 3.4.1, one deduces that the object $\mathscr{F} \boxtimes^U \mathscr{G}$ lies in the category

$$\mathfrak{D}(G \times^U \mathscr{X}/\!\!/T, \Bbbk)_{[\Bbbk]} \cong D^b_{(B),T}(G \times^U \mathscr{X}, \Bbbk).$$

Thus it defines a unique object $\mathscr{F} \boxtimes^B \mathscr{G}$ on $(G \times^U \mathscr{X})/T = G \times^B \mathscr{X}$, that is to say, in the category $D^b_{(B)}(G \times^B \mathscr{X}, \Bbbk)$.

Definition 6.3.1. *Consider two objects $\mathscr{F} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ and $\mathscr{G} \in \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t,-]}$. We set*

$$\mathscr{F} \star^{\mathrm{LY}} \mathscr{G} := (m_B)_*(\mathscr{F} \boxtimes^B \mathscr{G}).$$

This object is called the convolution product of $\mathscr{F}$ and $\mathscr{G}$. This defines a bifunctor

$$\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]} \times \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t,-]} \to D^b_{(B)}(\mathscr{X}, \Bbbk).$$

Note that we could have defined convolution using the proper pushforward functor $(m_B)_!$ since the map m_B is proper.

Finally, consider t' in the W-orbit of t. It is clear that the exact same definition as above allows us to consider a convolution product

$$\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]} \times \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X}/\!\!/T)_{[\mathscr{L}^T_t,\mathscr{L}^T_{t'}]} \to \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_{t'}]}.$$

6.3.2 Convolution over $\mathscr{X}^s$: B_s-equivariance

We can consider the variety $\mathscr{X}^s$ as a right B_s-space as well as a left B_s-space; moreover these actions commute with each other. We can thus define the Lusztig–Yun categories $\mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}^{B_s}_t,-]}$ and $\mathfrak{D}(\mathscr{X}^s/\!\!/B_s)_{[-,\mathscr{L}^{B_s}_t]}$, respectively for the left and right action.

Take $\mathscr{F} \in \mathfrak{D}(\mathscr{X}^s/\!\!/B_s)_{[-,\mathscr{L}^{B_s}_t]}$ and $\mathscr{G} \in \mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}^{B_s}_t,-]}$. The object $\mathscr{G}$ lies in the U^s-equivariant derived category (for the left multiplication of U^s on $\mathscr{X}^s$) since it is left B-constructible. Similarly, $(\varpi^s)^*(\mathscr{F})$ is in the U^s-equivariant

derived category on G, for multiplication on the right. Thus $(\varpi^s)^*(\mathscr{F}) \boxtimes \mathscr{G}$ descends to an object $\mathscr{F} \boxtimes^{U^s} \mathscr{G}$ on $G \times^{U^s} \mathscr{X}^s$. Using arguments similar to those appearing befor definition 6.3.1, we see that the latter object defines an object in the category $D^b_{(B),B_s}(G \times^{U^s} \mathscr{X}^s, \Bbbk)$, for the antidiagonal action of B_s on $G \times^{U^s} \mathscr{X}^s$, and thus defines a object $\mathscr{F} \boxtimes^B_s \mathscr{G}$ in $D^b(G \times^B G/U^s, \Bbbk)$. We set

$$\mathscr{F} \star^{\mathrm{LY}}_{B_s} \mathscr{G} := (m^s_B)_!(\mathscr{F} \boxtimes^B_s \mathscr{G}).$$

(Here, m^s_B is the natural map $G \times^B \mathscr{X}^s \to \mathscr{X}^s$.) We define in this way a convolution product

$$(-) \star^{\mathrm{LY}}_{B_s} (-) : \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}^{B_s}_t]} \times \mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}^{B_s}_t,-]} \to D^b_{(B)}(\mathscr{X}^s, \Bbbk).$$

Here again, as in the very end of subsection 6.3.1, we have a variant of the convolution product where we impose a right equivariange for the object $\mathscr{G}$; in the following lemma, we consider such a variant, that is, the functor

$$(-) \star^{\mathrm{LY}}_{B_s} (-) : \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}^{B_s}_t]} \times \mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s /\!\!/ B_s)_{[\mathscr{L}^{B_s}_t,\mathscr{L}^{B_s}_{t'}]} \longrightarrow \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}^{B_s}_{t'}]}.$$

We consider the setting of lemma 5.3.4.

Lemma 6.3.2. *The equivalence* $Q^*_{p^s} : \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}^{B_s}_t]}$ *of* (5.3.3) *commutes with* $(-) \star^{\mathrm{LY}}_{B_s} (-)$*: for two objects* $\mathscr{F} \in \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$ *and* $\mathscr{G} \in \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}^T_t,\mathscr{L}^T_{t'}]}$ *we have* $Q^*_{p^s}(\mathscr{F} \star^{\mathrm{LY}} \mathscr{G}) \cong Q^*_{p^s}(\mathscr{F}) \star^{\mathrm{LY}}_{B_s} Q^*_{p^s}(\mathscr{G})$.

Proof. Consider $\mathscr{F}, \mathscr{G}$ as in the statement. Observe that the square

$$\begin{array}{ccc} G \times^B G/U^s & \xrightarrow{p'} & G \times^B G/U \\ \downarrow{\scriptstyle m^s_B} & & \downarrow{\scriptstyle m_B} \\ G/U^s & \xrightarrow{p} & G/U \end{array}$$

is cartesian. (Here, the horizontal maps are given by the quotient by the action of U_s on the right of the right factor). The maps m^s_B and m_B being smooth, [BL, Theorem 7.1] tells us that we have $Q^*_{p^s}(m_B)_! \cong (m^s_B)_! Q^*_{p'}$. Thus we must show that $Q^*_{p'}(\mathscr{F} \boxtimes^B \mathscr{G}) \cong Q^*_{p^s}(\mathscr{F}) \boxtimes^B_s Q^*_{p^s}(\mathscr{G})$. To do so, we use the characterisation of $Q^*_{p^s}(\mathscr{F}) \boxtimes^B_s Q^*_{p^s}(\mathscr{G})$: it is the only object such that the pullback under the quotient map $q^s_B : G \times G/U^s \to G \times^B G/U^s$ is $(\varpi^s)^* Q^*_{p^s}(\mathscr{F}) \boxtimes Q^*_{p^s}(\mathscr{G})$.

Consider the following commutative diagram:

$$\begin{array}{ccc} G \times G/U^s & \xrightarrow{\mathrm{id}_G \times p} & G \times G/U \\ \downarrow{\scriptstyle q^s_B} & & \downarrow{\scriptstyle q_B} \\ G \times^B G/U^s & \xrightarrow{p'} & G \times^B G/U. \end{array}$$

We deduce that (forgetting all the equivariance) we have $(q_B^s)^*(p')^*(\mathscr{F}\boxtimes^B\mathscr{G})\cong \varpi^*(\mathscr{F})\boxtimes p^*(\mathscr{G})$. Now $\varpi = p\circ\varpi^s$. Taking the equivariance into account, what we get is

$$(q_B^s)^* Q_{p'}^*(\mathscr{F}\boxtimes^B\mathscr{G})\cong(\varpi^s)^* Q_{p^s}^*(\mathscr{F})\boxtimes Q_{p^s}^*(\mathscr{G}).$$

This concludes the proof. □

Corollary 6.3.3. *For any $w, v\in W$, we have*

$$Q_{p^s}^*(\Delta(w)_{\mathscr{L}^T_{v(t)}}\star^{\mathrm{LY}}\nabla(v)_{\mathscr{L}^T_t})\cong \Delta(w)_{\mathscr{L}^{B_s}_{v(t)}}\star^{\mathrm{LY}}_{B_s}\nabla(v)_{\mathscr{L}^{B_s}_t}.$$

Proof. This follows directly from lemma 5.3.7 and lemma 6.3.2 above. □

Let us conclude by describing the effect of duality on convolution:

Lemma 6.3.4. *Consider $t\in T^\vee_{\Bbbk}$ and consider two objects $\mathscr{F}\in\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ and $\mathscr{G}\in\mathfrak{D}(T\setminus\!\!\setminus\mathscr{X})_{[\mathscr{L}^T_t,-]}$. Then one has $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F}\star^{\mathrm{LY}}\mathscr{G})\cong\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F})\star^{\mathrm{LY}}\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{G})$.*

Proof. In the following proof, a double-underline will always mean a shift by $-2r$ in a duality functor (this is the definition for $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}$ but we use this notation also for other varieties). Let us first note that the statement of the lemma makes sense: $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F})$ lies in $\mathfrak{D}(T\setminus\!\!\setminus\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_{t^{-1}}]}$ and $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F})$ lies in $\mathfrak{D}(T\setminus\!\!\setminus\mathscr{X}/\!\!/T)_{[\mathscr{L}^T_{t^{-1}},-]}$. Thus the convolution of $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F})$ and $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{G})$ indeed makes sense.

By definition, we have $\mathscr{F}\star^{\mathrm{LY}}\mathscr{G}=(m_B)_!(\mathscr{F}\boxtimes^B\mathscr{G})$. Since m_B is proper, we have $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}\circ(m_B)_!\cong(m_B)_!\circ\underline{\underline{\mathbb{D}}}_{G\times^B\mathscr{X}}$. We then have to show $\underline{\underline{\mathbb{D}}}_{G\times^B\mathscr{X}}(\mathscr{F}\boxtimes^B\mathscr{G})\cong\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F})\boxtimes^B\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{G})$. Apply the functor q_B^* to the object $\underline{\underline{\mathbb{D}}}_{G\times^B\mathscr{X}}(\mathscr{F}\boxtimes^B\mathscr{G})$, one then obtains:

$$\begin{aligned}
q_B^*\underline{\underline{\mathbb{D}}}_{G\times^B\mathscr{X}}(\mathscr{F}\boxtimes^B\mathscr{G}) &\cong \underline{\underline{\mathbb{D}}}_{G\times\mathscr{X}} q_B^!(\mathscr{F}\boxtimes^B\mathscr{G})\\
&\cong \underline{\underline{\mathbb{D}}}_{G\times\mathscr{X}}(q_B^*(\mathscr{F}\boxtimes^B\mathscr{G})[2\dim(B)])\\
&\cong \underline{\underline{\mathbb{D}}}_{G\times\mathscr{X}}(\varpi^*(\mathscr{F})\boxtimes\mathscr{G})[-2\dim(B)]\\
&\cong \mathbb{D}_G(\varpi^*(\mathscr{F}))\boxtimes\mathbb{D}_{\mathscr{X}}(\mathscr{G})[-2\dim(B)-2r]
\end{aligned}$$

(we used the smoothness of the maps q_B and ϖ)

Now we have $\mathbb{D}_G(\varpi^*(\mathscr{F}))\cong\varpi^!\,\mathbb{D}_G(\mathscr{F})\cong\varpi^*\,\mathbb{D}_G(\mathscr{F})[2\dim(U)]$, so we finally get

$$q_B^*\underline{\underline{\mathbb{D}}}_{G\times^B\mathscr{X}}(\mathscr{F}\boxtimes^B\mathscr{G})\cong\varpi^*\,\mathbb{D}_G(\mathscr{F})\boxtimes\mathbb{D}_{\mathscr{X}}(\mathscr{G})[-4r]\cong\varpi^*\underline{\underline{\mathbb{D}}}_{G}(\mathscr{F})\boxtimes\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{G}).$$

This concludes the proof. □

Remark 6.3.5. In the statement of lemma 6.3.4, one may want to impose a left equivariance on the object $\mathscr{F}$, or a right equivariance on the object $\mathscr{G}$, or both; obviously, Verdier duality still makes sense and "preserve" this equivariance. Say, for example, that $\mathscr{F}$ is in $\mathfrak{D}(T\setminus\!\!\setminus\mathscr{X}/\!\!/T)_{[\mathscr{L}^T_{t'},\mathscr{L}^T_t]}$ for some t'. Then the resulting object $\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F}\star^{\mathrm{LY}}\mathscr{G})\cong\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{F})\star^{\mathrm{LY}}\underline{\underline{\mathbb{D}}}_{\mathscr{X}}(\mathscr{G})$ lies in $\mathfrak{D}(T\setminus\!\!\setminus\mathscr{X})_{[\mathscr{L}^T_{t'^{-1}},-]}$

6.3.3 Convolution over $\mathscr{X}^s$: L_s-equivariance

We now proceed to define convolution over $\mathscr{X}^s$ in the case of L_s Lusztig–Yun equivariance. Namely, consider $\mathscr{F} \in \mathfrak{D}(\mathscr{X}^s /\!\!/ L_s)_{[-,\mathscr{L}_t^{L_s}]}$ and $\mathscr{G} \in \mathfrak{D}(L_s \setminus\!\setminus \mathscr{X}^s)_{[\mathscr{L}_t^{L_s},-]}$. As in subsection 6.3.2, the object $(\varpi^s)^* \mathscr{F} \boxtimes \mathscr{G}$ descends to an object $\mathscr{F} \boxtimes^{U^s} \mathscr{G}$ on $G \times^{U^s} \mathscr{X}^s$. Exactly as for the case of B_s-equivariance, but with L_s instead, this twisted external product defines an object in $D^b_{(B),L_s}(G \times^{U^s} \mathscr{X}^s, \Bbbk)$ for the antidiagonal action of L_s, and hence we obtain an object $\mathscr{F} \boxtimes_s^{P_s} \mathscr{G}$ in $D^b(G \times^{P_s} \mathscr{X}^s, \Bbbk)$. We then set

$$\mathscr{F} \star_{L_s}^{\mathrm{LY}} \mathscr{G} := (m^s_{P_s})_!(\mathscr{F} \boxtimes_s^{P_s} \mathscr{G}),$$

where $m^s_{P_s} : G \times^{P_s} \mathscr{X}^s \to \mathscr{X}^s$ is induced by the action map.

Chapter 7

Completed monodromic category

7.1 The functor $\pi_\dagger^t$

Throughout this section we consider any fixed $t \in T_{\Bbbk}^\vee$. We begin by showing that there is an action, given by a convolution product, of the category $D^b_{(B)}(\mathscr{X}, \Bbbk)$ on the category $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. We need, for the proof of the next result, a few observations. The following new notation will be used only in the next proof. Consider $\mathscr{X} \times \widetilde{T}$ as a G-variety for the action of G on the left of the first factor uniquely, i.e. $h \cdot (gU, t) = (hgU, t)$. An object on $\mathscr{X} \times \widetilde{T}$ constructible for the stratification $\{\mathscr{X}_w \times \widetilde{T}\}_{w \in W}$ is U-equivariant for the U-action induced by the one of G. For any objects $\mathscr{F} \in D^b_{(B)}(\mathscr{X}, \Bbbk)$ and $\mathscr{G} \in D^b(\mathscr{X} \times \widetilde{T}, \Bbbk)$, the external tensor product $\varpi^*(\mathscr{F}) \boxtimes \mathscr{G}$ on $G \times (\mathscr{X} \times \widetilde{T})$ is U-equivariant for the action

$$u \cdot (g, (hU, t)) = (gu^{-1}, (ugU, t)),$$

thus it defines an object $\mathscr{F} \boxtimes^U \mathscr{G}$ on the quotient $G \times^U (\mathscr{X} \times \widetilde{T})$. We can then define a convolution product

$$\mathscr{F} \star^{U,T} (\mathscr{G}) := (m_U \times \mathrm{id}_{\widetilde{T}})_! (\mathscr{F} \boxtimes^U \mathscr{G}).$$

Lemma 7.1.1. *For any $\mathscr{F} \in D^b_{(B)}(\mathscr{X}, \Bbbk)$, and any $\mathscr{G} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$, the convolution product $\mathscr{F} \star^U \mathscr{G}$ defines an object in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$.*

Proof. Consider an isogeny $\widetilde{T} \to T$ with kernel K (of cardinality coprime to ℓ) and a character χ of K such that we have $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} = D^b_{(B),\widetilde{T}}(\mathscr{X}, \Bbbk)_\chi$ as full subcategory of $D^b_{(B),\widetilde{T}}(\mathscr{X}, \Bbbk)$. The convolution product $\mathscr{F} \star^U \mathscr{G}$ lies obviously in the category $D_{(B),\widetilde{T}}(\mathscr{X}, \Bbbk)$ (for the action of $\widetilde{T}$ on the right). We must show that the action of K on this object is via the character χ. To determine

this action, what we have to consider is the canonical isomorphism

$$\mathrm{pr}_1^*(\mathscr{F} \star^U \mathscr{G}) \xrightarrow{\sim} a_{\widetilde{T}}^*(\mathscr{F} \star^U \mathscr{G}),$$

where $a_{\widetilde{T}}, \mathrm{pr}_1 : \mathscr{X} \times \widetilde{T} \to \mathscr{X}$ are the action map and the projection on the first factor.

Denote by $\vartheta_{\mathscr{G}}$ the analogous isomorphism $\mathrm{pr}_1^*(\mathscr{G}) \xrightarrow{\sim} a_{\widetilde{T}}(\mathscr{G})$ for $\mathscr{G}$. Now, consider the following commutative diagrams

$$\begin{array}{ccc} G \times^U \mathscr{X} \times \widetilde{T} & \xrightarrow{\eta} & G \times^U \mathscr{X} \\ {\scriptstyle m_U \times \mathrm{id}_{\widetilde{T}}}\downarrow & & \downarrow{\scriptstyle m} \\ \mathscr{X} \times \widetilde{T} & \xrightarrow[a_{\widetilde{T}}]{} & \mathscr{X} \end{array}$$

and

$$\begin{array}{ccc} G \times^U \mathscr{X} \times \widetilde{T} & \xrightarrow{\mathrm{pr}_1} & G \times^U \mathscr{X} \\ {\scriptstyle m_U \times \mathrm{id}_{\widetilde{T}}}\downarrow & & \downarrow{\scriptstyle m} \\ \mathscr{X} \times \widetilde{T} & \xrightarrow[\mathrm{pr}_1]{} & \mathscr{X} \end{array}$$

(with $\eta([g,(xU,t)]) = [g, xtU]$). Two applications of the base change theorem yield isomorphisms

$$a_{\widetilde{T}}^*(\mathscr{F} \star^U \mathscr{G}) \cong \mathscr{F} \star^{U,T} a_{\widetilde{T}}^*(\mathscr{G}) \quad \text{and} \quad \mathrm{pr}_1^*(\mathscr{F} \star^U \mathscr{G}) \cong \mathscr{F} \star^{U,T} \mathrm{pr}_1^*(\mathscr{G}).$$

We then deduce that the isomorphism that we are looking for is given by $\mathscr{F} \star^{U,T} \vartheta_{\mathscr{G}}$. The result then follows. □

Lemma 7.1.1 allows us to define a "projection functor" from the constructible derived category to the Lusztig–Yun equivariant category.

Definition 7.1.2. *For any $t \in T_{\Bbbk}^{\vee}$, we denote by $\pi_\dagger^t$ the functor*

$$\pi_\dagger^t := (\cdot) \star^U \Delta(e)_{\mathscr{L}_t^T} : D_{(B)}^b(\mathscr{X}, \Bbbk) \to \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]}.$$

Even though we defined the functor $\pi_\dagger^t$ on the category $D_{(B)}^b(\mathscr{X}, \Bbbk)$, we will mainly consider its restriction to the full subcategory $D_{(B)}^b(\mathscr{X}, \Bbbk)_{[-,t]}$; the notation will be the same however.

Lemma 7.1.3. *Let* For_t *be the forgetful functor*

$$\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]} \to D_{(B)}^b(\mathscr{X}, \Bbbk)_{[-,t]}$$

of the end of subsection 4.1. The pair $(\pi_\dagger^t, \mathrm{For}_t)$ is an adjoint pair.

Proof. Fix an isogeny $\nu : \widetilde{T} \to T$ with kernel K of cardinality prime to ℓ such that we can view $\mathfrak{D}_{(B\times T)}(\mathscr{X} \times T/\!\!/T)_{[-,\mathscr{L}_t^T]}$ as a full summand subcategory in $D^b_{(B\times T),\widetilde{T}}(\mathscr{X} \times T, \Bbbk)$ (the action of $\widetilde{T}$ being on the right factor uniquely, via ν) and similarly, $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ as a full summand subcategory in $D^b_{(B),\widetilde{T}}(\mathscr{X}, \Bbbk)$.

Let $e : \mathscr{X} \to \mathscr{X} \times T$ be the map $xU \mapsto (xU, 1)$. If we make $\widetilde{T}$ act trivially on $\mathscr{X}$, then e is a $\{1\}$-$\widetilde{T}$-equivariant map. We claim that the functor

$$Q_e^* : \mathfrak{D}_{(B\times T)}(\mathscr{X} \times T/\!\!/T)_{[-,\mathscr{L}_t^T]} \to D^b_{(B)}(\mathscr{X}, \Bbbk)$$

is an equivalence of categories and that its quasi-inverse is given by the functor av defined by

$$\mathrm{av}(\mathscr{F}) := \mathscr{F} \boxtimes \mathscr{L}_t^T .$$

To see that, we remark that taking the tensor product with the object $\underline{\Bbbk}_{\mathscr{X}} \boxtimes \mathscr{L}_{t^{-1}}^T$ induces an equivalence of categories

$$\mathfrak{D}_{(B\times T)}(\mathscr{X} \times T/\!\!/T)_{[-,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}_{(B\times T)}(\mathscr{X} \times T/\!\!/T)_{[-,\underline{\Bbbk}_T]} \cong D^b_{(B\times T),T}(\mathscr{X} \times T, \Bbbk).$$

Then the composition $(\underline{\Bbbk}_{\mathscr{X}} \boxtimes \mathscr{L}_{t^{-1}}^T) \otimes (-) \circ \mathrm{av}$ is isomorphic to the functor

$$Q^*_{\mathrm{pr}_2} : D^b_{(B)}(\mathscr{X}, \Bbbk) \to D^b_{(B\times T),T}(\mathscr{X} \times T, \Bbbk).$$

The latter functor is well known to be an equivalence of categories (see [BL, §6.6, item 5]). Thus

$$\mathrm{av} \cong (\underline{\Bbbk}_{\mathscr{X}} \boxtimes \mathscr{L}_t^T) \otimes (-) \circ Q^*_{\mathrm{pr}_2}$$

is isomorphic to a composition of equivalences, and so is itself an equivalence. Since we clearly have $e^* \circ \mathrm{av} \cong \mathrm{id}$, the functor e^* is indeed a quasi-inverse.

Now, we can identify $\pi_\dagger^t$ with the functor $a_!((-) \boxtimes \mathscr{L}_t^T)[r]$. From the discussion above, we deduce that this functor admits a right adjoint, given by

$$\mathscr{F} \mapsto e^* \circ a^!(\mathscr{F})[-r].$$

Since the action map a is smooth of relative dimension r, we have $a^! \cong a^*[2r]$. Moreover, the composition $e^* \circ a^*$ is isomorphic to $Q^*_{\mathrm{id}_{\mathscr{X}}}$ with $\mathrm{id}_{\mathscr{X}}$ viewed as a $\{1\}$-$\widetilde{T}$-map. This is just the (usual) forgetful functor, so finally, the right adjoint to $\pi_\dagger^t$ is given by (the restriction to $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ of) the functor

$$\mathrm{For}[r] : D^b_{(B),\widetilde{T}}(\mathscr{X}, \Bbbk) \to D^b_{(B)}(\mathscr{X}, \Bbbk).$$

This is indeed For_t and thus the proof of the lemma is complete. □

The following result is just a restatement of the fact that $\pi_\dagger^t$ lands in the category $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$; we make the monodromy of the objects in the image of this functor explicit. Denote by ε_t the natural map

$$\Bbbk[\mathrm{X}_*(T)] \twoheadrightarrow \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle.$$

Lemma 7.1.4. *Let $\mathscr{F}$ be an object of $D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}$ and let For_t be the forgetful functor $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} \to D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}$. Then, for any $r \in \Bbbk[\mathrm{X}_*(T)]$, we have*

$$\varphi_{\mathrm{For}_t(\pi_\dagger^t(\mathscr{F}))}(r) = \varepsilon_t(r)\cdot \mathrm{id}_{\mathrm{For}_t(\pi_\dagger^t(\mathscr{F}))}\,.$$

Proof. We know that the object $\pi_\dagger^t(\mathscr{F})$ lies in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. The statement of the lemma is then equivalent to saying that For_t actually lands in $D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,\underline{t}]}$. This was established during the proof of lemma 4.2.2. □

We will consider the following analogue of the functor For_t: we set $\mathrm{For}_{t'}^t$ for the natural forgetful functor

$$\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_{t'}^T]} \to D^b_{(B)}(\mathscr{X},\Bbbk)_{[\underline{t},\underline{t'}]}.$$

If we just consider equivariance on the left, the forgetful functor will be denoted just For^t.

Lemma 7.1.5. *For $\mathscr{F} \in D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}$, $\mathscr{G} \in \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}_t^T,-]}$ and $\mathscr{G}' \in \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_{t'}^T]}$ we have isomorphisms*

1. *$\mathscr{F} \star^U \mathrm{For}^t(\mathscr{G}) \cong \pi_\dagger^t(\mathscr{F}) \star^{\mathrm{LY}} \mathscr{G}$ in $D^b_{(B)}(\mathscr{X},\Bbbk)$,*

2. *$\mathscr{F} \star^U \mathrm{For}_{t'}^t(\mathscr{G}') \cong \mathrm{For}_{t'}(\pi_\dagger^t(\mathscr{F}) \star^{\mathrm{LY}} \mathscr{G})$ in $D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t']}$.*

Proof. We prove the isomorphism (1), the proof of the second case being similar. By definition, we have $\pi_\dagger^t(\mathscr{F}) \star^{\mathrm{LY}} \mathscr{G} = m_!^B(\pi_\dagger^t(\mathscr{F}) \boxtimes^B \mathscr{G})$ and $\mathscr{F} \star^U \mathrm{For}^t(\mathscr{G}) = m_!^U(\mathscr{F} \boxtimes \mathrm{For}^t(\mathscr{G}))[r]$ with

$$m_B : G \times^B \mathscr{X} \to \mathscr{X}, \qquad m_U : G \times^U \mathscr{X} \to \mathscr{X}.$$

If we denote by $\varrho : G \times^U \mathscr{X} \to G \times^B \mathscr{X}$ the natural quotient map, we have $m_U = m_B \circ \varrho$. We then have to show

$$\varrho_!(\mathscr{F} \boxtimes^U \mathrm{For}(\mathscr{G}))[r] \cong \pi_\dagger^t(\mathscr{F}) \boxtimes^B \mathscr{G}\,.$$

By definition, the right-hand side is the unique object on $G \times^B \mathscr{X}$ such that $q_B^*(\pi_\dagger^t(\mathscr{F}) \boxtimes^B \mathscr{G}) = \varpi^*(\pi_\dagger^t(\mathscr{F})) \boxtimes \mathscr{G}$. If we show that

$$q_B^*(\varrho_!(\mathscr{F} \boxtimes^U \mathrm{For}^t(\mathscr{G})))[r] \cong \varpi^*(\pi_\dagger^t(\mathscr{F})) \boxtimes \mathscr{G}$$

we will be done. Now, we use the identification $\pi_\dagger^t(\mathscr{F}) \cong a_!^T(\mathscr{F} \boxtimes \mathscr{L}_t^T)[r]$, where $a^T : \mathscr{X} \times T \to \mathscr{X}$ is the (right) action map. In order to prove the lemma, one has to show that there exists an isomorphism

$$q_B^*(\varrho_!(\mathscr{F} \boxtimes^U \mathrm{For}^t(\mathscr{G}))) \cong \varpi^*(a_!^T(\mathscr{F} \boxtimes \mathscr{L}_t^T)) \boxtimes \mathscr{G}\,. \tag{7.1.1}$$

We use the following cartesian squares

$$\begin{array}{ccc} G\times T\times \mathscr{X} & \xrightarrow{\gamma} & G\times^U \mathscr{X} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varrho} \\ G\times \mathscr{X} & \xrightarrow[q_B]{} & G\times^B \mathscr{X} \end{array}$$

and

$$\begin{array}{ccc} G\times T\times \mathscr{X} & \xrightarrow{\varpi\times \mathrm{id}\times \mathrm{id}} & \mathscr{X}\times T\times \mathscr{X} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle a^T\times \mathrm{id}} \\ G\times \mathscr{X} & \xrightarrow[\varpi\times \mathrm{id}]{} & \mathscr{X}\times \mathscr{X}. \end{array}$$

Here we have

- $\alpha(g,t,hU)=(gt,hU)$,
- $\gamma(g,t,hU)=[g,thU]$.

Using the base change theorem, (7.1.1) becomes

$$\alpha_!\gamma^*(\mathscr{F}\boxtimes^U \mathrm{For}^t(\mathscr{G}))\cong \alpha_!(\varpi^*(\mathscr{F})\boxtimes \mathscr{L}_t^T\boxtimes \mathscr{G}).$$

In order to conclude, it suffices to see that

$$\gamma^*(\mathscr{F}\boxtimes^U \mathrm{For}^t(\mathscr{G}))\cong \varpi^*(\mathscr{F})\boxtimes \mathscr{L}_t^T\boxtimes \mathscr{G}\,. \tag{7.1.2}$$

We then remark that $\gamma = q_U\circ(\mathrm{id}\times a_T)$ with $a_T: T\times\mathscr{X}\to\mathscr{X}$ the left action map. By definition, we obtain

$$\gamma^*(\mathscr{F}\boxtimes^U \mathrm{For}^t(\mathscr{G}))\cong \varpi^*(\mathscr{F})\boxtimes a_T^*\,\mathrm{For}^t(\mathscr{G}).$$

According to lemma 2.9.3, we have $a_T^*\,\mathrm{For}^t(\mathscr{G})\cong \mathscr{L}_t^T\boxtimes\mathscr{G}$, so we have an isomorphism as in (7.1.2) and the proof is complete. □

7.2 The completed monodromic category

In this section, we construct a "completed" monodromic category. All the constructions are copied from [BY, Appendix A]; we follow very closely the treatment of this appendix.

For the duration of this section, we consider a fixed locally closed union of strata Z of $\mathscr{X}$ (Z is then T-stable). For $\mathscr{F}\in D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}$, we have an identification of $\mathscr{F}\star^U\Delta(e)_{\mathscr{L}_t^T}$ with $a_!(\mathscr{F}\boxtimes\mathscr{L}_t^T)[r]$ where $a:\mathscr{X}\times T\to\mathscr{X}$ is the action map and $\mathscr{L}_t^T$ is viewed as a T-Lusztig–Yun-equivariant object on T. This has the following advantage: for our locally closed T-stable subset Z of $\mathscr{X}$, we can consider the restriction of $\pi_\dagger^t$ (which we will still denote by $\pi_\dagger^t$) to the category and $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$, with values in $\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$ (note that this last category indeed makes sense because Z is a T-variety).

Definition 7.2.1. *We denote by $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ the full subcategory of the category $\operatorname{Pro} D^b_{(B)}(Z,\Bbbk)_{[-,t]}$ of pro-objects on $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$ whose objects are those $\mathscr{F} = \text{“}\varprojlim\text{”}\, \mathscr{F}_n$ such that:*

1. *$\mathscr{F}$ is “$\pi^t_\dagger$-constant”: $\text{“}\varprojlim\text{”}\, \pi^t_\dagger(\mathscr{F}_n)$ is isomorphic to an object in the category $\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}^T_t]}$,*

2. *$\mathscr{F}$ is uniformly bounded: there exist $a, b \in \mathbb{Z}$ such that $\mathscr{F}$ is isomorphic to a $\text{“}\varprojlim\text{”}\, \mathscr{F}'_n$ with $\mathscr{F}'_n \in \mathfrak{D}^{[a,b]}(Z/\!\!/T)_{[-,\mathscr{L}^T_t]}$ for any $n \in \mathbb{Z}_{\geq 0}$.*

By definition, we obtain a functor $\pi^t_\dagger : \widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]} \to \mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}^T_t]}$.

We show that the category $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ admits a triangulated category structure. More precisely, we start by considering the family of triangles of objects of $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ of the form

$$\text{“}\varprojlim\text{”}\, \mathscr{F}_n \xrightarrow{\text{“}\varprojlim\text{”}\, f_n} \text{“}\varprojlim\text{”}\, \mathscr{G}_n \xrightarrow{\text{“}\varprojlim\text{”}\, g_n} \text{“}\varprojlim\text{”}\, \mathscr{H}_n \xrightarrow{\text{“}\varprojlim\text{”}\, h_n} \text{“}\varprojlim\text{”}\, \mathscr{F}_n[1]$$

with $(\mathscr{F}_n \xrightarrow{f_n} \mathscr{G}_n \xrightarrow{g_n} \mathscr{H}_n \xrightarrow{h_n} \mathscr{F}_n[1])_n$ a projective system of distinguished triangles in $D^b(Z,\Bbbk)_{[-,t]}$. Let $\delta(\operatorname{Pro} D^b_{(B)}(Z,\Bbbk)_{[-,t]})$ be the family of such “distinguished triangles”, and let $\mathbf{Tri}(\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]})$ be the family of such triangles with vertices in $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$. Also we denote by $[1]$ the obvious shift functor on the category $\operatorname{Pro} D^b_{(B)}(Z,\Bbbk)_{[-,t]}$.

Lemma 7.2.2. *The triple $(\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}, \mathbf{Tri}(\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}), [1])$ is a triangulated category.*

We will need the notion of a triangle-complete subcategory: consider a category D endowed with a family $\delta(D)$ of “distinguished” triangles and a shift functor $[1]$ (note that we do not impose that D is triangulated); a strictly full subcategory D' of D is said to be triangle complete if it is stable by $[1]$ and if for every triangle $X \to Y \to Z \xrightarrow{+1}$ in $\delta(D)$, if two vertices are in D', then so is the third one.

Proof. According to [BY, Theorem A.2.2], we must show:

1. $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ is triangle complete (with respect to the family of triangles $\delta(\operatorname{Pro} D^b_{(B)}(Z,\Bbbk)_{[-,t]})$ and $[1]$),

2. for any object $\text{“}\varprojlim\text{”}\, \mathscr{F}_m \in \widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ and $\mathscr{G} \in D^b(Z,\Bbbk)_{[-,t]}$, the $\Bbbk$-vector space
$$\varinjlim_m \operatorname{Hom}_{D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}}(\mathscr{F}_m, \mathscr{G})$$
is finite dimensional.

The proof is copied from the one of [BY, Theorem A.3.2]. To show the first point, we must show that if "$\varinjlim$" $\left(\mathscr{F}_n \xrightarrow{f_n} \mathscr{G}_n \xrightarrow{g_n} \mathscr{H}_n \xrightarrow{h_n} \mathscr{F}_n[1]\right)$ is a triangle with "$\varinjlim$" $\mathscr{F}_n$ and "$\varinjlim$" $\mathscr{G}_n$ in $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$, then "$\varinjlim$" $\mathscr{H}_n$ is also in $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$. First, since the $\mathscr{F}_n$ and $\mathscr{G}_n$ are uniformly bounded, so are the $\mathscr{H}_n$.

To show the $\pi_\dagger^t$-constancy, we fix $\mathscr{A},\mathscr{B} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ and isomorphisms $\alpha : \mathscr{A} \xrightarrow{\sim}$ "$\varinjlim$" $\pi_\dagger^t \mathscr{F}_n$ and $\beta : \mathscr{B} \xrightarrow{\sim}$ "$\varinjlim$" $\pi_\dagger^t \mathscr{G}_n$. These isomorphisms define map $\alpha_n : \mathscr{A} \to \pi_\dagger^t \mathscr{F}_n$ for any n (similarly for β). Also, let $a : \mathscr{A} \to \mathscr{B}$ be the map corresponding to "$\varinjlim$" f_n via the isomorphisms α and β. We denote by $\mathscr{C}$ the cone of a. We have diagrams

$$\begin{array}{ccccccc} \mathscr{A} & \xrightarrow{a} & \mathscr{B} & \xrightarrow{b} & \mathscr{C} & \xrightarrow{c} & \mathscr{A}[1] \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \vdots & & \downarrow{\scriptstyle \alpha_n[1]} \\ \pi_\dagger^t \mathscr{F}_n & \xrightarrow{\pi_\dagger^t(f_n)} & \pi_\dagger^t \mathscr{G}_n & \xrightarrow{\pi_\dagger^t(g_n)} & \pi_\dagger^t \mathscr{H}_n & \xrightarrow{\pi_\dagger^t(h_n)} & \pi_\dagger^t \mathscr{F}_n[1]. \end{array} \tag{7.2.1}$$

The subset M_n of maps in $\mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{C}, \pi_\dagger^t \mathscr{H}_n)$ making this diagram commutative is nonempty (this is one of the axioms in a triangulated category). We choose two such maps γ_n^1 and γ_n^2, the difference $\gamma_n^1 - \gamma_n^2$ fits in the diagram (7.2.1) with α_n and β_n replaced by the zero maps. Thus, if we denote by $\mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{C}, \pi_\dagger^t \mathscr{H}_n)^0$ the finite dimensional $\Bbbk$-subspace of

$$\mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{C}, \pi_\dagger^t \mathscr{H}_n)$$

whose elements are the maps that make the diagram (7.2.1) commute with α_n and β_n replaced by zero, we have

$$M_n = \gamma_n^1 + \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{C}, \pi_\dagger^t \mathscr{H}_n)^0.$$

In other words, M_n is a finite dimensional affine space lying in the vector space $\mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{C}, \pi_\dagger^t \mathscr{H}_n)$. This fact implies that the spaces M_n form a projective system satisying the Mittag–Leffler condition, thus the limit of this system is nonempty. Consequently, we have a morphism $\gamma : \mathscr{C} \to$ "$\varprojlim$" $\pi_\dagger^t \mathscr{H}_n$ that fits in diagram (7.2.1). We must show that this is in fact an isomorphism (since we do not know that "$\varinjlim$" $\pi_\dagger^t \mathscr{H}_n$ is representable, this is not automatic). Consider any object $\mathscr{Q} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. We apply the functor $\mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(-, \mathscr{Q})$ to the morphism of triangles (α, β, γ). The long exact sequences of Hom and the five lemma show that γ is indeed an isomorphism.

We now check the point (2). For "$\varinjlim$" $\mathscr{F}_m \in \widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$, the functor

$$\varinjlim \mathrm{Hom}_{D^b_{(B)}(Z,\Bbbk)_{[-,t]}}(\mathscr{F}_m, -)$$

is exact, thus we only need to verify (2) for $\mathscr{G}$ in some generating family of $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$. We can then assume that $\mathscr{G}$ is of the form $\mathrm{For}_t(\mathscr{H})$ for some

$\mathscr{H} \in \mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$. (Indeed, any object is a successive extension of its perverse cohomology objects, so it suffices to see that the objects $\mathrm{For}_t(\mathscr{H})$ generate the category of perverse sheaves, which in turn is clear since simple perverse sheaves lie in the image of For_t.) Using the adjunction of lemma 7.1.3, we then have

$$\begin{aligned}\varinjlim \mathrm{Hom}_{D^b_{(B)}(Z,\Bbbk)_{[-,t]}}(\mathscr{F}_m, \mathrm{For}_t(\mathscr{H})) &\cong \varinjlim \mathrm{Hom}_{\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\pi_\dagger^t(\mathscr{F}_m), \mathscr{H}) \\ &\cong \mathrm{Hom}_{\mathrm{Pro}\,\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\text{“}\varprojlim\text{”}\, \pi_\dagger^t(\mathscr{F}_m), \mathscr{H}).\end{aligned}$$

By assumption, “$\varprojlim$” $\pi_\dagger^t(\mathscr{F}_m)$ is isomorphic to an object of $\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$, so the last Hom-space above is actually an Hom-space in $\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$, hence finite dimensional. □

In the constructions above, we imposed a fixed monodromy on the right, i.e. the category $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ is a subcategory in the category of pro-objects on the right monodromic category $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$. We can however consider the exact same construction considering the category $D^b_{(B)}(Z,\Bbbk)_{[t',t]}$ (for some t' in the W-orbit of t) instead of $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$. More precisely, we set the following definition:

Definition 7.2.3. *Let t' be in the W-orbit of t. We denote by $\widehat{D}_{(B)}(Z,\Bbbk)_{[t',t]}$ the full subcategory of the category $\mathrm{Pro}\, D^b_{(B)}(Z,\Bbbk)_{[t',t]}$ of pro-objects on $D^b_{(B)}(Z,\Bbbk)_{[t',t]}$ whose objects are those $\mathscr{F} =$ “$\varprojlim$” $\mathscr{F}_n$ such that:*

1. *“$\varprojlim$” $\pi_\dagger^t(\mathscr{F}_n)$ is isomorphic to an object in the category $\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$,*
2. *there exist $a, b \in \mathbb{Z}$ such that $\mathscr{F}$ is isomorphic to a “$\varprojlim$” $\mathscr{F}'_n$ with $\mathscr{F}'_n \in \mathfrak{D}^{[a,b]}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$ for any $n \in \mathbb{Z}_{\geq 0}$.*

Remark 7.2.4.

1. It is clear from the definitions that $\widehat{D}_{(B)}(Z,\Bbbk)_{[t',t]}$ is a full triangulated subcategory of $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ (essentially because $D^b_{(B)}(Z,\Bbbk)_{[t',t]}$ is a triangulated subcategory of $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$.
2. It could have been possible to define a completed category using an “opposite picture”, that is to say, using the functor of convolution **on the left** with $\Delta(e)_{\mathscr{L}_t^T}$ and considering a subcategory of the category of pro-objects in $D^b_{(B)}(Z,\Bbbk)_{[t,-]}$. We then obtain a triangulated category “above” $\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}_t^T,-]}$. However, we emphasise that this is not what we did here, in particular, the definition of $\widehat{D}_{(B)}(Z,\Bbbk)_{[t',t]}$ is not symmetric.
3. In the case where $Z = \mathscr{X}$, we will denote

$$\widehat{D}_{[-,t]} := \widehat{D}_{(B)}(\mathscr{X},\Bbbk)_{[-,t]} \text{ and } \widehat{D}_{[t',t]} := \widehat{D}_{(B)}(\mathscr{X},\Bbbk)_{[t',t]}$$

7.3 Convolution in the completed category

We show that the convolution $(\cdot)\star^U(\cdot)$ extends to a bifunctor $\widehat{D}_{[t'',t']}\times\widehat{D}_{[t',t]}\to\widehat{D}_{[t'',t]}$.

Lemma 7.3.1. *Consider an object $\mathscr{F}=\text{“}\varprojlim\text{”}\,\mathscr{F}_n\in\widehat{D}_{[-,t']}$ and a $\mathscr{G}\in D^b_{(B)}(\mathscr{X},\Bbbk)_{[t',-]}$. Then $\text{“}\varprojlim\text{”}(\mathscr{F}_n\star^U\mathscr{G})$ is isomorphic to an object of $D^b_{(B)}(\mathscr{X},\Bbbk)$.*

Proof. Assume that the result is true when $\mathscr{G}$ is a perverse sheaf. We then proceed by induction on the length of the interval

$$I_{\mathscr{G}}:=\{i\in\mathbb{Z}\mid^p\mathscr{H}^i(\mathscr{G})\neq 0\}$$

(we will use similar notation for other complexes). Since any object is a successive extension of its perverse cohomology objects, for a general $\mathscr{G}$, we can write a distinguished triangle

$$\mathscr{G}'\to\mathscr{G}''\to\mathscr{G}\xrightarrow{+1}$$

with $\mathscr{G}'$ a (possibly shifted) perverse sheaf and $I_{\mathscr{G}''}$ has a strictly smaller length than $I_{\mathscr{G}}$. We obtain a projective system of distinguished triangles

$$(\mathscr{F}_n\star^U\mathscr{G}''\to\mathscr{F}_n\star^U\mathscr{G}\to\mathscr{F}_n\star^U\mathscr{G}'\xrightarrow{+1})_n.$$

By induction hypothesis, $\text{“}\varprojlim\text{”}(\mathscr{F}_n\star^U\mathscr{G}')$ and $\text{“}\varprojlim\text{”}(\mathscr{F}_n\star^U\mathscr{G}'')$ are representable; using arguments similar to those in the proof of point 1) of lemma 7.2.2, we see that this implies that $\text{“}\varprojlim\text{”}(\mathscr{F}_n\star^U\mathscr{G})$ is representable too.

We must treat the case $\mathscr{G}$ perverse. Since the objects of the form $\mathrm{For}^{t'}(\mathscr{G}')$ generate the perverse subcategory in $D^b_{(B)}(\mathscr{X},\Bbbk)_{[t',-]}$, the same kind of arguments as above tell us that it suffices to consider the case $\mathscr{G}=\mathrm{For}^{t'}(\mathscr{G}')$. In this case, using lemma 7.1.5, we have

$$\mathscr{F}_n\star^U\mathscr{G}=\mathscr{F}_n\star^U\mathrm{For}^{t'}(\mathscr{G}')\cong\pi^{t'}_\dagger(\mathscr{F}_n)\star^{\mathrm{LY}}\mathscr{G}'.$$

Thus $\text{“}\varprojlim\text{”}(\mathscr{F}_n\star^U\mathscr{G})\cong(\text{“}\varprojlim\text{”}\,\pi^{t'}_\dagger(\mathscr{F}_n))\star^{\mathrm{LY}}\mathscr{G}'$ is isomorphic to an object of $D^b_{(B)}(\mathscr{X},\Bbbk)$ since $\mathscr{F}$ is assumed to be $\pi^{t'}_\dagger$-constant. □

Before going on with the extension of convolution to the completed category, we derive from the preceding lemma an immediate corollary:

Corollary 7.3.2. *Consider $t,t'\in T^\vee_{\Bbbk}$ in the same W-orbit. Then the convolution product $\star^U$ extends to two bifunctors, triangulated in both entries*

$$\widehat{D}_{[-,t]}\times D^b_{(B)}(\mathscr{X},\Bbbk)_{[t,t']}\to D^b_{(B)}(\mathscr{X},\Bbbk)_{[-,t']}$$

$$\widehat{D}_{[-,t]}\times\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_{t'}]}\to\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_{t'}]}.$$

We denote these two functors by $\widehat{\star}$.

Consider now two pro-objects $\mathscr{F} = \text{“}\varprojlim\text{”}\,\mathscr{F}_n$ and $\mathscr{G} = \text{“}\varprojlim\text{”}\,\mathscr{G}_n$ respectively in $\widehat{D}_{[-,t']}$ and $\widehat{D}_{[t',t]}$. According to the lemma 7.3.1, for any m, the pro-object $\text{“}\varprojlim_n\text{”}(\mathscr{F}_n \star^U \mathscr{G}_m)$ is representable by an object of $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$. We can thus consider the pro-object

$$\text{“}\varprojlim_m\text{”}(\text{“}\varprojlim_n\text{”}(\mathscr{F}_n \star^U \mathscr{G}_m)).$$

Lemma 7.3.3. *The pro-object* $\text{“}\varprojlim_m\text{”}(\text{“}\varprojlim_n\text{”}(\mathscr{F}_n \star^U \mathscr{G}_m))$ *is in* $\widehat{D}_{[-,t]}$.

Proof. We must verify two things: first, this pro-object is uniformly bounded; then it is $\pi^t_\dagger$-constant.

The $\mathscr{F}_n$'s and $\mathscr{G}_n$'s are uniformly bounded by assumption; for any $m \in \mathbb{Z}$, the family $(\varpi^*(\mathscr{F}_n) \boxtimes \mathscr{G}_m)_{n\in\mathbb{Z}}$ is uniformly bounded, thus this is also the case for $(\mathscr{F}_n \boxtimes^U \mathscr{G}_m)_{n\in\mathbb{Z}}$. Now the spaces $\mathscr{X} = G/U$ and $G \times^U G/U$ are finite dimensional, thus, using [Iv, III, Definition 9.5 and X, Proposition 1.4], one sees that $(m_!(\mathscr{F}_n \boxtimes^U \mathscr{G}_m))_{n\in\mathbb{Z}}$ is a uniformly bounded family, which concludes the verification of this point.

Now we have to consider

$$\begin{aligned}
\text{“}\varprojlim_m\text{”}\left(\pi^t_\dagger(\text{“}\varprojlim_n\text{”}(\mathscr{F}_n \star^U \mathscr{G}_m))\right) &\cong \text{“}\varprojlim_m\text{”}(\text{“}\varprojlim_n\text{”}\pi^t_\dagger(\mathscr{F}_n \star^U \mathscr{G}_m)) \\
&\cong \text{“}\varprojlim_m\text{”}(\text{“}\varprojlim_n\text{”}(\mathscr{F}_n \star^U \pi^t_\dagger(\mathscr{G}_m))) \\
&\cong \text{“}\varprojlim_m\text{”}(\mathscr{F}\,\widehat{\star}\,\pi^t_\dagger(\mathscr{G}_m)) \\
&\cong \mathscr{F}\,\widehat{\star}\,(\text{“}\varprojlim_m\text{”}\pi^t_\dagger(\mathscr{G}_m)).
\end{aligned}$$

(To get the second line from the first one, we used the obvious fact that $\pi^t_\dagger(\mathscr{F}_n \star^U \mathscr{G}_m) \cong \mathscr{F}_n \star^U \pi^t_\dagger(\mathscr{G}_m)$, thanks to the associativity of convolution.) Since by definition $\text{“}\varprojlim_m\text{”}\pi^t_\dagger(\mathscr{G}_m)$ is representable by an object, lemma 7.3.1 tells us that $\mathscr{F}\,\widehat{\star}\,(\text{“}\varprojlim_m\text{”}\pi^t_\dagger(\mathscr{G}_m))$ is representable. This concludes the proof. □

We keep the notation used in lemma 7.3.3. Assume that $\mathscr{F}$ lies in $\widehat{D}_{[t'',t']}$; one can show that $\text{“}\varprojlim_n\text{”}(\text{“}\varprojlim_m\text{”}(\mathscr{F}_n \star^U \mathscr{G}_m))$ is in $\widehat{D}_{[-,t]}$ and, using [KS2, Proposition 2.1.7], that the latter is isomorphic to $\text{“}\varprojlim_m\text{”}(\text{“}\varprojlim_n\text{”}(\mathscr{F}_n \star^U \mathscr{G}_m))$ (as pro-objects). We deduce immediately the following proposition from lemma 6.2.1:

Proposition 7.3.4. *The convolution product* $(-) \star^U (-)$ *extends to a functor, triangulated in both entries,*

$$(-)\widehat{\star}(-) : \widehat{D}_{[t'',t']} \times \widehat{D}_{[t',t]} \to \widehat{D}_{[t'',t]}.$$

7.4 The single-stratum case

We are now interested in the case $D^b_{(T)}(T,\Bbbk)_{[t]}$: the construction of section 7.2 (and in particular definition 7.2.1) still make sense here, and we can consider a completed monodromic category on the torus, denoted $\widehat{D}_{(T)}(T,\Bbbk)_{[t]}$. Note that one can view T as B/U, and thus as a T-stable suvariety of $\mathscr{X}$. We obviously have $\widehat{D}_{(B)}(B/U,\Bbbk)_{[-,t]} = \widehat{D}_{(T)}(T,\Bbbk)_{[t]}$.

From now on, we set $\Bbbk_t = L_t := \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle$. We also let $\widehat{R}_t := \varprojlim_n \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle^n$ be the completion of R with respect to the ideal $\mathfrak{m}_t$. We then let $\widehat{\mathfrak{m}}_t = \mathfrak{m}_t\widehat{R}_t$. In particular we have $\widehat{R}_t/\widehat{\mathfrak{m}}_t \cong \Bbbk_t$.

We have an equivalence of categories between the finitely generated R-modules annihilated by a power of $\mathfrak{m}_t$ and the finitely generated $\widehat{R}_t$-modules annihilated by a power of $\widehat{\mathfrak{m}}_t$. This last category will be denoted $\mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t)$. For any $L \in \mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t)$ we can associate a local system $\mathscr{L}$ on T. According to corollary 3.3.4, we have $\mathscr{L} \in D^b_{(T)}(T,\Bbbk)_{[t]}$. We thus obtain a functor

$$\widetilde{\Phi}^t_T : D^b\,\mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t) \to D^b_{(T)}(T,\Bbbk)_{[t]}.$$

We denote Φ^t_T the composition $[r]\circ\widetilde{\Phi}^t_T$.

Lemma 7.4.1. *The functor Φ^t_T is a t-exact equivalence of categories for the natural t-structure on $D^b\,\mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t)$ and the perverse t-structure on $D^b(T,\Bbbk)_{[t]}$, and we have $\Phi^t_T(\Bbbk_t) = \mathscr{L}^T_t[r]$.*

Proof. The functor Φ^t_T is isomorphic to the following composition of equivalences

$$D^b\,\mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t) \xrightarrow{\sim} D^b\,\mathrm{Loc}(T,\Bbbk)_{[t]} \xrightarrow{\sim} D^b_{(T)}(T,\Bbbk)_{[t]},$$

where the second arrow is given by the equivalence (5.2.1). Moreover, we clearly have $\Phi^t_T(\Bbbk_t) = \mathscr{L}^T_t[r]$. The fact about the t-exactness is also clear. □

We once again fix a finite central isogeny $\widetilde{T} \xrightarrow{\nu} T$ whose kernel K is of order coprime to ℓ and a character χ of K such that we can view $\mathfrak{D}(T/\!\!/T)_{\mathscr{L}^T_t}$ as a full subcategory of $D^b_{(T),\widetilde{T}}(T,\Bbbk)$. We then have

$$D^b_{\widetilde{T}}(T,\Bbbk) \cong D^b_{\widetilde{T}}(\widetilde{T}\times^K \{\mathrm{pt}\}) \cong D^b(\mathrm{Rep}^{\mathrm{fd}}_{\Bbbk}(K)) = D^b(\mathrm{Mod}^{\mathrm{fg}}(\Bbbk[K])).$$

This last category is semisimple because the order of K is prime to ℓ. The full factor subcategory $\mathfrak{D}(T/\!\!/T)_{\mathscr{L}^T_t}$ identifies with the full subcategory

$$D^b(\Bbbk[K]/\langle e^k - \chi_t(k) \mid k \in K\rangle) \cong D^b(\mathrm{Vect}^{\mathrm{fd}}_{\Bbbk_t}).$$

We can consider the change of scalars functor (associated to the map $\widehat{R}_t \twoheadrightarrow \Bbbk_t$)

$$_{\widehat{R}_t}(-) : D^b(\mathrm{Vect}^{\mathrm{fd}}_{\Bbbk_t}) \to D^b\,\mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t).$$

This functor admits a left adjoint, namely

$$\Bbbk_t \otimes^L_{\widehat{R}_t} (-) : D^b \operatorname{Mod}^{\mathrm{nil}}(\widehat{R}_t) \to D^b(\operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t}).$$

Considering the equivalences $D^b_{(T)}(T,\Bbbk)_{[t]} \cong D^b \operatorname{Loc}(T,\Bbbk)_{[t]}$ and $\mathfrak{D}(T/\!\!/T)_{\mathscr{L}^T_t} \cong D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t}$ (respectively from (5.2.1) and justified above), we obtain an identification

$$\pi^t_\dagger \circ \Phi^t_T \;\cong\; \Bbbk_t \otimes^L_{\widehat{R}_t} (-). \tag{7.4.1}$$

Indeed the composition

$$\mathfrak{D}(T/\!\!/T)_{[\mathscr{L}^T_t]} \xrightarrow{\sim} D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t} \xrightarrow{\widehat{R}_t(-)} D^b \operatorname{Mod}^{\mathrm{nil}}(\widehat{R}_t) \xrightarrow{\Phi^t_T} D^b_{(T)}(T,\Bbbk)_{[t]}$$

is easily seen to correspond to the (shifted by r) forgetful functor $\mathfrak{D}(T/\!\!/T)_{[\mathscr{L}^T_t]} \to D^b_{(T)}(T,\Bbbk)_{[t]}$. Since the latter is right adjoint to $\pi^t_\dagger$ according to lemma 7.1.3, we deduce our identification from the adjunction $(\Bbbk_t \otimes^L_{\widehat{R}_t}(-),\ {}_{\widehat{R}_t}(-))$. Thus the category $\widehat{D}_{[-,t]}$ is equivalent to the full subcategory $\widehat{D}(\widehat{R}_t)$ of pro-objects "$\varprojlim$" M_n in $D^b \operatorname{Mod}^{\mathrm{nil}}(\widehat{R}_t)$ which are uniformly bounded and such that

$$\text{"}\varprojlim\text{"}\ \Bbbk_t \otimes^L_{\widehat{R}_t} M_n$$

is isomorphic to an object in $D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t}$.

We then have the following result, which is copied from [BeR, §4.2, 4.3] (though the authors of *loc. cit.* work with unipotent monodromy, that is, with $t = 1$, their arguments in §§4.2, 4.3 are entirely based on algebraic properties of the algebra $\widehat{R}_1$; since we have an algebra isomorphism $\widehat{R}_t \cong \widehat{R}_1$ for any $t \in T^\vee_{\Bbbk}$, their arguments apply in our setting, with the same proofs):

Proposition 7.4.2. *There exists a canonical equivalence of triangulated categories*

$$D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) \xrightarrow{\sim} \widehat{D}_{(T)}(T,\Bbbk)_{[t]}$$

sending $\widehat{R}_t$ *to* $\widehat{\mathscr{L}^T_t}[r]$. *In particular, we have*

$$\operatorname{Ext}^i_{\widehat{D}_{(T)}(T,\Bbbk)_{[t]}}(\widehat{\mathscr{L}^T_t}[r], \widehat{\mathscr{L}^T_t}[r]) = 0 \quad \textit{for } i > 0.$$

Remark 7.4.3. Let us try to (roughly) describe the equivalence of the preceding lemma. First, the natural extension to the categories of pro-objects of the functor Φ^t_T of lemma 7.4.1 induces an equivalence $\widehat{D}(\widehat{R}_t) \xrightarrow{\sim} \widehat{D}_{(T)}(T,\Bbbk)_{[t]}$ (that is, this equivalence maps a pro-object "$\varprojlim$" M_n to "$\varprojlim$" $\Phi^t_T(M_n)$. Then, since $\widehat{R}_t$ is local, noetherian and of finite global dimension, any object in $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ is isomorphic in this category to a bounded complex of free $\widehat{R}_t$-modules. For such a complex $(M^i)_{i\in\mathbb{Z}}$, the functor of proposition 7.4.2 is then given by

$$(M^i)_{i\in\mathbb{Z}} \mapsto \text{"}\varprojlim\text{"}(M^i/\mathfrak{m}_t^{n+1})_{i\in\mathbb{Z}}.$$

7.5 Perverse t-structure

In this section we define a "perverse t-structure" on the category $\widehat{D}_{[-,t]}$ following ideas of [BY] and [BeR]. Recall from the very begining of section 7.2 that the functor $\pi_\dagger^t$ "restricts" to locally closed union of strata, meaning that if Z is such a subspace, $\pi_\dagger^t$ induces functors from $\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]}$ and $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$ to $\mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$.

Lemma 7.5.1. *Consider $Z \subseteq \mathscr{X}$ any T-stable locally closed subset. Denote by $h : Z \hookrightarrow \mathscr{X}$ the inclusion map. Then we have isomorphisms of functors $D^b_{(B)}(X,\Bbbk)_{[-,t]} \to \mathfrak{D}(Z/\!\!/T)_{[-,\mathscr{L}_t^T]}$ (resp. $D^b_{(B)}(Z,\Bbbk)_{[-,t]} \to \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$):*

$$h^? \circ \pi_\dagger^t \cong \pi_\dagger^t \circ\ h^?, \tag{7.5.1}$$

$$h_? \circ \pi_\dagger^t \cong \pi_\dagger^t \circ\ h_? \tag{7.5.2}$$

for $? \in \{!, *\}$.

Proof. We look at $\pi_\dagger^t$ as the functor $a_!(- \boxtimes \mathscr{L}_t^T)[r]$. The isomorphisms (7.5.1) for $? = *$ and (7.5.2) for $? =!$ follow from standard arguments, using the commutative cartesian square

$$\begin{array}{ccc} Z \times T & \xrightarrow{h\times \mathrm{id}_T} & \mathscr{X} \times T \\ {\scriptstyle a_{|Z}}\downarrow & & \downarrow{\scriptstyle a} \\ Z & \xrightarrow[h]{} & \mathscr{X} \end{array}$$

and the fact that a and $a_{|Z}$ are smooth maps.

We prove (7.5.2) in the case $? = *$. Let us first show that we have a natural map

$$\pi_\dagger^t \circ h_* \longrightarrow h_* \circ \pi_\dagger^t. \tag{7.5.3}$$

It is clear that h_* commutes with the forgetful functor; recall from lemma 7.1.3 that $(\pi_\dagger^t, \mathrm{For}_t)$ forms an adjoint pair. First, we apply h_* to the adjunction morphism $\mathrm{id} \to \mathrm{For}_t \circ \pi_\dagger^t$. Then, use the isomorphism $h_* \circ \mathrm{For}_t \cong \mathrm{For}_t \circ h_*$ to get a map $h_* \to \mathrm{For}_t \circ h_* \circ \pi_\dagger^t$. We then apply the functor $\pi_\dagger^t$ to this map and use the adjunction $\pi_\dagger^t \circ \mathrm{For}_t \to \mathrm{id}$ to obtain the whished-for map.

Now it is enough to verify that this is an isomorphism of functors on a family of objects generating $D^b_{(B)}(Z,\Bbbk)_{[-,t]}$; we can thus consider an object of the form $\mathrm{For}_t(\mathscr{F})$ with $\mathscr{F} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. Lemma 2.9.3 tells us that there is an isomorphism $a_{|Z}^* \mathrm{For}_t(\mathscr{F}) \cong \mathrm{For}_t(\mathscr{F}) \boxtimes \mathscr{L}_t^T$. Thus we can identify $\pi_\dagger^t \circ \mathrm{For}_t(\mathscr{F})$ with $(a_{|Z})_! a_{|Z}^* \mathrm{For}_t(\mathscr{F})[r] \cong \mathrm{For}_t(\mathscr{F}) \otimes \mathbf{H}_c^{\bullet+r}(T)$, the last identification following from the projection formula. Thus, the morphism (7.5.3) applied to $\mathrm{For}_t(\mathscr{F})$ is given by the natural map

$$\mathrm{For}_t(h_*(\mathscr{F})) \otimes \mathbf{H}_c^{\bullet+r}(T) \to h_* \mathrm{For}_t(\mathscr{F}) \otimes \mathbf{H}_c^{\bullet+r}(T) \cong \mathrm{For}_t(h_*(\mathscr{F})) \otimes \mathbf{H}_c^{\bullet+r}(T).$$

This is an isomorphism, and settles the proof of the case h_*. The case $h^!$ is proved similarly. □

Remark 7.5.2. Lemma 7.5.1 allows us to apply [BY, Proposition A.3.3]: for $h : Z \to \mathscr{X}$ a locally closed embedding, we have a extensions to the completed categories of the usual functors $h^*, h^!, h_*$ and $h_!$. We will denote the functors in the completed case and the usual case the same way. Moreover, these functors still satisfy the usual adjunction properties, and lemma 7.5.1 still holds for these "extended functors".

If $i : Z \hookrightarrow \mathscr{X}$ is now a closed inclusion (with Z still a union of strata) and $j : U \hookrightarrow \mathscr{X}$ is its complement, we obtain a recollement diagram in the sense of [BBD, §1.4.3]:

$$\widehat{D}_{(B)}(Z,\Bbbk)_{[-,t]} \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{i_*}{\longrightarrow}}}} \widehat{D}_{[-,t]} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{j^*}{\longrightarrow}}}} \widehat{D}_{(B)}(U,\Bbbk)_{[-,t]}. \tag{7.5.4}$$

As we saw in corollary 7.4.2, we have an equivalence of categories

$$D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) \xrightarrow{\sim} \widehat{D}_{(T)}(T,\Bbbk)_{[t]}. \tag{7.5.5}$$

Now for a stratum $\mathscr{X}_w$ in $\mathscr{X}$, we have a map $p_w : \mathscr{X}_w \to T$. The pullback functor $p_w^*[\ell(w)] \cong p_w^![-\ell(w)]$ induces an equivalence of categories

$$\widehat{D}_{(T)}(T,\Bbbk)_{[t]} \xrightarrow{\sim} \widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)_{[-,t]}. \tag{7.5.6}$$

Composing the equivalences (7.5.5) and (7.5.6), we finally obtain an equivalence

$$D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) \xrightarrow{\sim} \widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)_{[-,t]} \tag{7.5.7}$$

for any $w \in W$. We can then transport the usual t-structure on $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ to the category $\widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)_{[-,t]}$. This new t-structure, denoted by

$$({}^p\widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)^{\leq 0}_{[-,t]},\ {}^p\widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)^{\geq 0}_{[-,t]}),$$

will be called the perverse t-structure. Using diagram (7.5.4), we can glue together the perverse t-structure on the $\mathscr{X}_w$'s to obtain a perverse t-structure $({}^p\widehat{D}^{\leq 0}_{[-,t]},\ {}^p\widehat{D}^{\geq 0}_{[-,t]})$ on $\widehat{D}_{[-,t]}$: we have

$$\left(\mathscr{F} \in {}^p\widehat{D}^{\leq 0}_{[-,t]}\right) \Longleftrightarrow \left(j_w^*\mathscr{F} \in {}^p\widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)^{\leq 0}_{[-,t]} \quad \text{for all } w \in W\right)$$

and

$$\left(\mathscr{F} \in {}^p\widehat{D}^{\geq 0}_{[-,t]}\right) \Longleftrightarrow \left(j_w^!\mathscr{F} \in {}^p\widehat{D}_{(B)}(\mathscr{X}_w,\Bbbk)^{\geq 0}_{[-,t]} \quad \text{for all } w \in W\right).$$

The heart of this t-structure will be denoted $\widehat{\mathrm{P}}_{(B)}(\mathscr{X},\Bbbk)_{[-,t]}$ or simply $\widehat{\mathrm{P}}_{[-,t]}$. The objects in $\widehat{\mathrm{P}}_{[-,t]}$ are called monodromic perverse pro-objects. The truncation functors for this t-structure are denoted by ${}^p\tau_{\leq n}$ and ${}^p\tau_{\geq n}$; the cohomology functors ${}^p\mathbf{H}^n(-)$.

Remark 7.5.3. In complete analogy with [BY, Remark A.4.8], one can describe the perverse t-structure $\widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]}$ is a more concrete way (already knowing that such a t-structure exists). More precisely, let $\mathscr{F}$ be an object in $\widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]}$. Then for $a \leq b$ in $\mathbb{Z}$ (and with obvious notation), $\mathscr{F} \in {}^p\widehat{D}^{[a,b]}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]}$ if and only if $\mathscr{F} \cong$ "$\varprojlim$" $\mathscr{F}_n$ with $\mathscr{F}_n \in {}^pD^{[a,b]}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]}$ for any n. This is a direct consequence of remark 7.4.3.

We can restate in our current setting [BY, Lemma A.6.2]:

Lemma 7.5.4. *The natural inclusions*

$$\mathrm{Pro}\,({}^pD_{(B)}(\mathscr{X}, \Bbbk)^{\leq 0}_{[-,t]}) \cap \widehat{D}_{[-,t]} \hookrightarrow {}^p\widehat{D}^{\leq 0}_{[-,t]},$$

$$\mathrm{Pro}\,({}^pD_{(B)}(\mathscr{X}, \Bbbk)^{\geq 0}_{[-,t]}) \cap \widehat{D}_{[-,t]} \hookrightarrow {}^p\widehat{D}^{\geq 0}_{[-,t]}$$

are equivalences of categories.

Proof sketch. We prove only the first equivalence, the second one being similar. The proof relies on the general [BY, Claim, Lemma A.6.2] (which we do not prove here, and restate in our current setting): if $\mathscr{F} \to \mathscr{G} \to \mathscr{H} \xrightarrow{+1}$ is a distinguished triangle in $\widehat{D}_{[-,t]}$, with $\mathscr{F}$ and $\mathscr{H}$ in $\mathrm{Pro}({}^pD_{(B)}(\mathscr{X}, \Bbbk)^{\leq 0})$, then $\mathscr{G}$ is isomorphic to an object in $\mathrm{Pro}({}^pD_{(B)}(\mathscr{X}, \Bbbk)^{\leq 0})$.

Once this is known, the proof of the lemma is by induction on the number N of strata in the support of $\mathscr{F}$. If $N = 0$, there is nothing to prove; the case $N = 1$ is a direct consequence of remark 7.5.3. Now assume that the result is true for some $N \geq 1$, and consider $\mathscr{F} \in \widehat{D}_{[-,t]}$ containing $N + 1$ strata in its support S. Let $\mathscr{X}_w$ be an open stratum in S, and $j : S \setminus \mathscr{X}_w \hookrightarrow \mathscr{X}$ be the inclusion map.

By induction hypothesis (resp. the case $N = 1$), we can find a pro-object "$\varprojlim$" $\mathscr{F}_n^{<w}$ (resp. "$\varprojlim$" $\mathscr{F}_n^{=w}$) in $\mathrm{Pro}\,({}^pD_{(B)}(\mathscr{X}, \Bbbk)^{\leq 0}_{[-,t]}) \cap \widehat{D}_{[-,t]}$ such that $j_*j^*\mathscr{F} \cong$ "$\varprojlim$" $\mathscr{F}_n^{<w}$ and $(j_w)_!(j_w)^*\mathscr{F} \cong$ "$\varprojlim$" $\mathscr{F}_n^{=w}$. We then get a distinguished triangle

$$(j_w)_!(j_w)^*\mathscr{F} \to \mathscr{F} \to j_*j^*\mathscr{F} \xrightarrow{+1}$$

which can be written as

$$\text{"}\varprojlim\text{"}\, \mathscr{F}_n^{=w} \to \mathscr{F} \to \text{"}\varprojlim\text{"}\, \mathscr{F}_n^{<w} \xrightarrow{+1}.$$

We then use the claim stated at the beginning of the proof in order to conclude that $\mathscr{F}$ is isomorphic to an object in $\mathrm{Pro}\,({}^pD_{(B)}(\mathscr{X}, \Bbbk)^{\geq 0}_{[-,t]}) \cap \widehat{D}_{[-,t]}$. □

This lemma has the following nice consequence: in order to show that a functor defined on the completed category is t-exact for the perverse t-structure, it suffices to check that its restriction to the non-completed category is exact.

Lemma 7.5.5. *Let M be in $D^b\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ and assume that $\Bbbk_t \otimes^L_{\widehat{R}_t} M$ is concentrated in non-negative degrees. Then M is isomorphic to a complex of free $\widehat{R}_t$-modules with nonzero terms in non-negative degrees only.*

Proof. See [BeR, Lemma 5.1]; once again, the authors consider only the case $t = 1$, but their argument is entirely based on agebraic properties of the algebra $\widehat{R}_1$. Since we have an algebra isomorphism $\widehat{R}_1 \cong \widehat{R}_t$, the same proof actually works in our current setting. □

Lemma 7.5.6. *Let $\mathscr{F}$ be in $\widehat{D}_{[-,t]}$.*

1. *If $\pi^t_\dagger(\mathscr{F})$ is perverse, then $\mathscr{F}$ is perverse itself.*
2. *If $\pi^t_\dagger(\mathscr{F}) = 0$ then $\mathscr{F} = 0$.*
3. *If $\mathscr{F}$ is perverse and ${}^p\mathbf{H}^0(\pi^t_\dagger(\mathscr{F})) = 0$, then $\mathscr{F} = 0$.*

Proof. We prove (1). By definition, $\mathscr{F}$ is perverse if and only if the restriction $j_w^* \mathscr{F} \in {}^p\widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)^{\leq 0}_{[-,t]}$ for all $w \in W$, and a similar condition for corestriction. According to lemma 7.5.1, we can focus on these restrictions (resp. corestrictions), and thus assume that $\mathscr{F} \in \widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]}$. More explicitly, we need to show that if $\pi^t_\dagger(\mathscr{F})$ lies in ${}^p\mathfrak{D}(\mathscr{X}_w /\!\!/ T)^{\leq 0}_{[-,\mathscr{L}^T_t]}$, then already $\mathscr{F}$ lies in ${}^p\widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)^{\leq 0}_{[-,t]}$ (and similarly for the ≥ 0 part of the t-structures).

Now we have the commutative diagram of categories

$$\begin{array}{ccc} D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) & \xrightarrow{\sim} & \widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]} \\ {\scriptstyle \Bbbk \otimes^L_{\widehat{R}_t} (-)} \downarrow & & \downarrow {\scriptstyle \pi^t_\dagger} \\ D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t} & \xrightarrow{\sim} & \mathfrak{D}(\mathscr{X}_w /\!\!/ T)_{[-,\mathscr{L}^T_t]} \end{array}$$

(i.e. the two compositions $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) \to \mathfrak{D}(\mathscr{X}_w /\!\!/ T)_{[-,\mathscr{L}^T_t]}$ are isomorphic, see (7.4.1)). Thus what we have to see is that if $M \in D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ satisfies

$$\mathbf{H}^i(\Bbbk_t \otimes^L_{\widehat{R}_t} M) = 0$$

for any $i > 0$ (resp. $i < 0$) then already $\mathbf{H}^i(M) = 0$ for any $i > 0$ (resp. $i < 0$). The case $i > 0$ follows from Nakayama's lemma: denote by d the largest integer such that $\mathbf{H}^n(M) \neq 0$. Then we have $\mathbf{H}^d(M) \otimes_{\widehat{R}_t} \Bbbk_t = \mathbf{H}^d(M \otimes^L_{\widehat{R}_t} \Bbbk_t)$. If the latter module is zero, then Nakayama's lemma tell us that $\mathbf{H}^d(M)$ is already zero. The case $i < 0$ is an immediate consequence of lemma 7.5.5.

The proof of (2) can be copied verbatim from [BY, Lemma A.3.5].

The proof of (3) is similar: it amounts to showing that if $M \in \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ and $\mathbf{H}^0(\Bbbk_t \otimes^L_{\widehat{R}_t} M) = 0$, then M is zero. But in this situation, we have

$$\mathbf{H}^0(\Bbbk_t \otimes^L_{\widehat{R}_t} M) = M \otimes_{\widehat{R}_t} \Bbbk_t,$$

and Nakayama's lemma allows one to conclude the proof. □

Remark 7.5.7. Assume that $\mathscr{X}'$ is a locally closed union of strata in $\mathscr{X}$, and consider a union of strata $Z \subseteq \mathscr{X}'$, closed in $\mathscr{X}'$; let $U := \mathscr{X}' \setminus Z$ be the complementary open in $\mathscr{X}'$. Then we can consider a recollement diagram similar to (7.5.4) but with $\widehat{D}_{(B)}(\mathscr{X}', \Bbbk)_{[-,t]}$ instead of $\widehat{D}_{[-,t]}$. This allows us to consider a perverse t-structure on $\widehat{D}_{(B)}(\mathscr{X}', \Bbbk)_{[-,t]}$, and the above results hold for $\widehat{D}_{[-,t]}$ replaced by $\widehat{D}_{(B)}(\mathscr{X}', \Bbbk)_{[-,t]}$.

7.6 Pro-standard and pro-costandard

We define two new objects of the category Pro $D^b_{(B)}(\mathscr{X}, \Bbbk)$. Recall that we have a pro-local system $\widehat{\mathscr{L}_t^T}$, defined in section 5.1.4. Set

$$\widehat{\mathscr{L}_t^w} := p_w^* \widehat{\mathscr{L}_t^T}[\dim(\mathscr{X}_w)];$$

then define

$$\widehat{\Delta}_{w,t} := (j_w)_! \widehat{\mathscr{L}_t^w} \quad \text{and} \quad \widehat{\nabla}_{w,t} := (j_w)_* \widehat{\mathscr{L}_t^w}.$$

Similarly, let $\mathscr{L}_{t,n}^T := p_w^* \mathscr{L}_{t,n}^T[\dim(\mathscr{X}_w)]$ and denote by $\Delta_{w,t}^n$ the object $(j_w)_! \mathscr{L}_{t,n}^T$ and by $\nabla_{w,t}^n$ the object $(j_w)_* \mathscr{L}_{t,n}^T$ (so that we have $\widehat{\mathscr{L}_t^w} = \text{“}\varprojlim\text{”} \mathscr{L}_{t,n}^T$ and $\widehat{\Delta}_{w,t} = \text{“}\varprojlim\text{”} \Delta_{w,t}^n$). For $n = 1$, we find the objects of the end of subsection 5.2.1

$$\nabla_{w,t} = \nabla_{w,t}^1 \qquad \text{and} \qquad \Delta_{w,t} = \Delta_{w,t}^1 .$$

The objects (resp. pro-objects) $\Delta_{w,t}$ and $\nabla_{w,t}$ (resp. $\widehat{\Delta}_{w,t}$ and $\widehat{\nabla}_{w,t}$) are called monodromic standard and costandard (resp. pro-standard and pro-costandard) sheaves.

Recall the notation W_t° from subsection 5.1.3.

Lemma 7.6.1. *Take $t \in T_\Bbbk^\vee$ and $s \in W$ a simple reflection. Assume that*

$$s \notin W_t^\circ.$$

Then, the natural morphism $\Delta_{s,t}^n \to \nabla_{s,t}^n$ is an isomorphism for any n. In particular, the natural morphism $\widehat{\Delta}_{s,t} \to \widehat{\nabla}_{s,t}$ obtained as the limit of the above morphisms is an isomorphism.

Proof. It suffices to show that the restriction to $\mathscr{X}_e$ of $(j_s)_* \mathscr{L}_{t,n}^s$ is zero. As everything takes place on $\overline{\mathscr{X}_s}$, we can replace G by its Levi subgroup that contains T as maximal torus and with roots $\pm\alpha$ (and thus assume that G is of semisimple rank 1). There exists a central isogeny

$$SL_2 \times Z \to G$$

with Z the identity component of the center of G (see the beginning of subsection 5.3.2). This induces a map

$$f : SL_2/U_{SL_2} \times Z \to G/U$$

that preserves the Bruhat stratification on both sides (here U_{SL_2} is the unipotent upper-triangular subgroup). Let O and C be the open and closed stratum respectively on SL_2/U_{SL_2}. We have a commutative diagram

$$\begin{array}{ccc} O\times Z & \xrightarrow{f_{|O\times Z}} & \mathscr{X}_s \\ {\scriptstyle j\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle j_s} \\ SL_2/U_{SL_2}\times Z & \xrightarrow{f} & \mathscr{X} \\ {\scriptstyle i\times\mathrm{id}}\uparrow & & \uparrow{\scriptstyle j_e} \\ C\times Z & \xrightarrow{f_{|C\times Z}} & \mathscr{X}_s \end{array} \tag{7.6.1}$$

where i and j are the inclusion maps. Since the horizontal maps are surjective, in order to show that $(j_e)^*(j_s)_*\mathscr{L}^s_{t,n}$ is zero, it suffices to show that $(f_{|C\times Z})^*(j_e)^*(j_s)_*\mathscr{L}^s_{t,n}$ is zero. But now we have isomorphisms

$$\begin{aligned}(f_{|C\times Z})^*(j_e)^*(j_s)_*\mathscr{L}^s_{t,n} &\cong (i\times\mathrm{id})^* f^*(j_s)_*\mathscr{L}^s_{t,n} \\ &\cong (i\times\mathrm{id})^*(j\times\mathrm{id})_*(f_{|O\times Z})^*\mathscr{L}^s_{t,n},\end{aligned}$$

where the last isomorphism comes from the smooth base change.

Thus we need to understand the object $(f_{|O\times Z})^*\mathscr{L}^s_{t,n}$. We identify SL_2/U_{SL_2} with $\mathbb{A}^2\setminus\{0\}$, so that we have $O=(\mathbb{A}^1\times\mathbb{A}^1\setminus\{0\})$ and $C=(\mathbb{A}^1\setminus\{0\})\times\{0\}$.

The pullback of $\mathscr{L}^s_{t,n}$ along $f_{|O\times Z}$ is thus of the form

$$\mathscr{L}^O_t\boxtimes(\mathscr{L}^T_{t,n_{|Z}}) \tag{7.6.2}$$

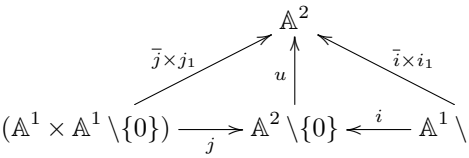
with $\mathscr{L}^O_t$ a local system on the open stratum O of SL_2/U_{SL_2}. In fact, we can identify $\mathscr{L}^O_t$ more precisely. Recall that (up to a shift by $r+1$, which we suppress for commodity of notation) we have $\mathscr{L}^s_{t,n}=p_s^*\mathscr{L}^T_{t,n}$ where $p_s:\mathscr{X}_s\to T$ is the map defined in the end of subsection 5.1.1. We can construct an analog of p_s: let $p_O: O\times Z\to\mathbb{A}^1\setminus\{0\}$ be defined as the projection $\mathbb{A}^1\times(\mathbb{A}^1\setminus\{0\})\times Z\to\mathbb{A}^1\setminus\{0\}$. Now we have a commutative diagram

$$\begin{array}{ccc} O\times Z & \xrightarrow{f} & \mathscr{X}_s \\ {\scriptstyle p_O}\downarrow & & \downarrow{\scriptstyle p_s} \\ \mathbb{A}^1\setminus\{0\} & \xrightarrow[\alpha^\vee]{} & T \end{array}$$

and one deduces that $\mathscr{L}^O_t$ is in fact $(p_O)^*(\alpha^\vee)^*\mathscr{L}^s_{t,n}$. We can identify the latter object with $\underline{\Bbbk}_{\mathbb{A}^1}\boxtimes(\alpha^\vee)^*\mathscr{L}^s_{t,n}\boxtimes(\mathscr{L}^T_{t,n})_{|Z}$ on $O\times Z=\mathbb{A}^1\times(\mathbb{A}^1\setminus\{0\})\times Z$. We can focus on what is happening on the left hand side of this product (that is to say we can "forget the factor that lives on Z"). In other terms, it suffices to see that

$$i^*j_*(\underline{\Bbbk}_{\mathbb{A}^1}\boxtimes(\alpha^\vee)^*\mathscr{L}^T_{t,n})=0.$$

Consider the following commutative diagram (where all arrows are inclusions)



Note that u is an open inclusion; the adjunction map $u^*u_* \to \mathrm{id}$ induces an isomorphism of functors $(\bar{i}\times i_1)^*(\bar{j}\times j_1)_* = i^*u^*u_*j_* \xrightarrow{\sim} i^*j_*$. Thus we see that it suffices to show that the stalk $i_1^*(j_1)_*((\alpha^\vee)^*\mathscr{L}^T_{t,n}))$ is zero (where $\mathbb{A}^1\setminus\{0\} \xrightarrow{j_1} \mathbb{A}^1 \xleftarrow{i_1} \{0\}$ are the inclusions). The hypothesis $s \notin W_t^\circ$ amounts to $\alpha^\vee(t) \neq 1$, and means that $(\alpha^\vee)^*\mathscr{L}^T_{t,n}$ is a non trivial local system; in particular, it has a non trivial monodromy on $\mathbb{A}^1\setminus\{0\}$ (for the natural action of $\mathbb{A}^1\setminus\{0\} = \mathbb{G}_\mathrm{m}$ on itself). But on the $\mathbb{A}^1\setminus\{0\}$-stable one-point space $\{0\}$, the action is of course trivial and we only have objects with trivial monodromy, so this stalk is zero (see also lemma 3.2.1). The proof is complete. □

Remark 7.6.2. Note that since $\Delta_{s,t}$ and $\nabla_{s,t}$ are perverse objects in the category $D^b_{(B)}(\mathscr{X},\Bbbk)_{[\underline{s(t)},\underline{t}]}$, using lemma 4.2.4, we see that under the hypothesis $s \notin W_t^\circ$, we also have

$$\Delta(s)_{\mathscr{L}^T_t} \cong \mathrm{IC}(s)_{\mathscr{L}^T_t} \cong \nabla(s)_{\mathscr{L}^T_t}.$$

7.7 Averaging by a pro-local system

Recall the local systems $\mathscr{L}^T_{t,n}$ defined in section 3.3: by definition, $\mathscr{L}^T_{t,n}$ is the local system on T associated with the $\pi_1(T) \cong \mathrm{X}_*(T)$-module

$$L_{t,n} := \Bbbk[\mathrm{X}_*(T)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle^n,$$

and we have $L_t := L_{t,1}$. For $n > m$, we have a natural morphism of local systems $\mathscr{L}^T_{t,n} \to \mathscr{L}^T_{t,m}$ induced by the natural quotient map between the associated $\mathrm{X}_*(T)$-representations.

Remark 7.7.1. Consider a trivialisation $T \cong (\mathbb{C}^*)^r$; this induces a trivialisation $T^\vee_\Bbbk \cong (\Bbbk^*)^r$ under which the element t corresponds to a r-tuple $(t_1,\dots,t_r)$. The group $\mathrm{X}_*(T)$ identifies with $\mathbb{Z}^r$; in turn, $\Bbbk[\mathrm{X}_*(T)]$ identifies with $\Bbbk[x_1^{\pm1},\dots,x_r^{\pm1}] \cong \Bbbk[x_1^{\pm1}]\otimes_\Bbbk\cdots\otimes_\Bbbk\Bbbk[x_r^{\pm1}]$, the indeterminate x_i corresponding to $(0,\dots,\underset{i}{1},\dots,0)$. Recall the ideal $\mathfrak{m}_t = \langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(T)\rangle$ of $\Bbbk[\mathrm{X}_*(T)]$; let $\mathfrak{m}_{t_i}^{\mathbb{C}^*} = \langle x_i - t_i\rangle$, an ideal of $\Bbbk[x_i^{\pm1}]$. With the above identifications, it is easy to see that we have

$$\mathfrak{m}_t^{nr} \subseteq ((\mathfrak{m}_{t_1}^{\mathbb{C}^*})^n \otimes_\Bbbk (\bigotimes_{i=2}^{r}\Bbbk[x_i^{\pm1}])) + (\Bbbk[x_1^{\pm1}]\otimes_\Bbbk(\mathfrak{m}_{t_2}^{\mathbb{C}^*})^n\otimes_\Bbbk(\bigotimes_{i=3}^{r}\Bbbk[x_i^{\pm1}])) + \cdots + ((\bigotimes_{i=1}^{r-1}\Bbbk[x_i^{\pm1}])\otimes_\Bbbk(\mathfrak{m}_{t_r}^{\mathbb{C}^*})^n) \subseteq \mathfrak{m}_t^n,$$

all the tensor products being taken over $\Bbbk$. One then deduces an isomorphism

$$\widehat{\mathscr{L}_t^T} \xrightarrow{\sim} \text{“}\varprojlim\text{”}_n(\mathscr{L}_{t_1,n}^{\mathbb{C}^*} \boxtimes \cdots \boxtimes \mathscr{L}_{t_r,n}^{\mathbb{C}^*}).$$

In the next lemma (whose proof is left to the reader), we assume that $T \cong \mathbb{G}_{\mathrm{m}}$; we then fix a generator λ of $\mathrm{X}_*(T)$. For L a $\widehat{R}_t$-module, set $L^{\lambda(t)} := \{v \in L \mid e^\lambda v = \lambda(t)v\}$ and $L_{\lambda(t)} := L/\langle e^\lambda - \lambda(t)\rangle \cdot L$.

Lemma 7.7.2. *For any $n > 0$, the natural map of $\mathrm{X}_*(T)$-representations $L_{t,n} \to L_t$ induces an isomorphism $(L_{t,n})_{\lambda(t)} \xrightarrow{\sim} (L_t)_{\lambda(t)}$; and $(L_{t,n})^{\lambda(t)} \to (L_t)^{\lambda(t)}$ is the zero map.*

Lemma 7.7.3. *We have $m_!(\widehat{\mathscr{L}_t^T} \boxtimes \mathscr{L}_t^T) \cong \mathscr{L}_t^T[-2r]$.*

Proof. We will freely use the notation of section 7.4 in the proof. We begin by assuming that T has rank one, that is, that we have $T \cong \mathbb{G}_{\mathrm{m}}$. In this case, we fix a generator λ of $\mathrm{X}_*(T)$. Also, for $n > m > 0$ two integers, we let f_n^m be the natural map of $\widehat{R}_t$-modules $L_{t,n} \to L_{t,m}$; remark that $L_{t,1} = L_t$.

Now note that (for any $n > 0$) the object $m_!(\mathscr{L}_{t,n}^T \boxtimes \mathscr{L}_t^T)$ naturally identifies with $\pi_\dagger^t(\mathscr{L}_{t,n}^T)[-1]$; in particular $m_!(\mathscr{L}_{t,n}^T \boxtimes \mathscr{L}_t^T)$ naturally defines an object in $\mathfrak{D}(T/\!\!/T)_{[\mathscr{L}_t^T]}$, (see lemma 7.1.1). Now, since by definition $\mathscr{L}_{t,n}^T = \Phi_T^t(L_{t,n})[-1]$, using the identification (7.4.1), we see that the object $m_!(\mathscr{L}_{t,n}^T \boxtimes \mathscr{L}_t^T)$ identifies with the object $L_{t,n} \otimes^L_{\widehat{R}_t} \Bbbk_t[-2]$ of $D^b\mathrm{Vect}^{\mathrm{fd}}_{\Bbbk_t}$.

We can compute this complex using the fact that we know a projective (in fact, a free) $\widehat{R}_t$-resolution of $\Bbbk_t$, namely

$$0 \to \widehat{R}_t \xrightarrow{1 \mapsto (e^\lambda - \lambda(t))} \widehat{R}_t \to \Bbbk_t \to 0. \tag{7.7.1}$$

Here the map on the right is the augmentation morphism $e^\lambda \mapsto \lambda(t)$. Thus $L_{t,n} \otimes^L_{\widehat{R}_t} \Bbbk_t[-2]$ is given by the complex

$$C_n := 0 \to L_{t,n} \underset{1}{\otimes}_{\widehat{R}_t} \widehat{R}_t \to L_{t,n} \underset{2}{\otimes}_{\widehat{R}_t} \widehat{R}_t \to 0,$$

the middle map being induced by the one in (7.7.1). One readily sees that the degree -1-cohomology of this complex computes $(L_{t,n})^{\lambda(t)}$, and that the degree -2-cohomology computes $(L_{t,n})_{\lambda(t)}$. Since any complex in $D^b\mathrm{Vect}^{\mathrm{fd}}_{\Bbbk_t}$ is isomorphic, in this category, to the direct sum of its shifted cohomology objects, we get that $C_n = (L_{t,n})^{\lambda(t)}[-1] \oplus (L_{t,n})_{\lambda(t)}[-2]$.

Now, for $n > m$, the map $m_!(\mathscr{L}_{t,n}^T \boxtimes \mathscr{L}_t^T) \to m_!(\mathscr{L}_{t,m}^T \boxtimes \mathscr{L}_t^T)$ identifies with

$$\begin{aligned}(L_{t,n})^{\lambda(t)}[-2r+1] \oplus (L_{t,n})_{\lambda(t)}[-2r] \\ \xrightarrow{(f_n^m)^{\lambda(t)}[-2r+1] \oplus (f_n^m)_{\lambda(t)}[-2r]} (L_{t,m})^{\lambda(t)}[-2r+1] \oplus (L_{t,m})_{\lambda(t)}[-2r],\end{aligned}$$

where $(f_n^m)_{\lambda(t)}$ and $(f_n^m)^{\lambda(t)}$ are the maps induced by f_n^m. Thanks to lemma 7.7.2, we get that $(f_n^1)_{\lambda(t)}$ (resp. $(f_n^1)^{\lambda(t)}$) is an isomorphism (resp. is zero) for

any $n > 0$. We readily deduce that the pro-object "$\varprojlim$" C_n is in fact an object (i.e. isomorphic to a constant pro-object), given by $(L_t)_{\lambda(t)}[-2r] = L_t[-2r]$.

Using the equivalence Φ_T^t once again, we get that the pro-object

$$\text{"}\varprojlim\text{"}\, m_!(\mathscr{L}_{t,n}^T \boxtimes \mathscr{L}_t^T) = m_!(\widehat{\mathscr{L}_t^T} \boxtimes \mathscr{L}_t^T)$$

is constant, with value $\mathscr{L}_t^T[-2]$. This concludes the proof in the case $T \cong \mathbb{G}_{\mathrm{m}}$. To deduce the general case from the former one, one uses remark 7.7.1 (we keep the notation used in the remark): considering a fixed trivialisation $T \cong (\mathbb{C}^*)^r$, we get isomorphisms $\widehat{\mathscr{L}_t^T} \xrightarrow{\sim}$ "$\varprojlim$"$_n(\mathscr{L}_{t_1,n}^{\mathbb{C}^*} \boxtimes \cdots \boxtimes \mathscr{L}_{t_r,n}^{\mathbb{C}^*})$ (as pro-objects) and $\mathscr{L}_t^T \cong (\mathscr{L}_{t_1}^{\mathbb{C}^*}) \boxtimes \cdots \boxtimes (\mathscr{L}_{t_r}^{\mathbb{C}^*})$. Moreover, the multiplication map m identifies with the componentwise multiplication on $(\mathbb{C}^*)^r$. Thus $m_!(\widehat{\mathscr{L}_t^T} \boxtimes \mathscr{L}_t^T)$ identifies with $(\mathscr{L}_{t_1}^{\mathbb{C}^*})[-2] \boxtimes \cdots \boxtimes (\mathscr{L}_{t_r}^{\mathbb{C}^*})[-2]$, which in turns, corresponds to $\mathscr{L}_t^T[-2r]$. We are done. □

We deduce the image of the pro-standard and pro-costandard objects under $\pi_\dagger^t$:

Corollary 7.7.4. *For any $w \in W$, we have $\pi_\dagger^t(\widehat{\Delta}_{w,t}) \cong \Delta(w)_{\mathscr{L}_t^T}$ and $\pi_\dagger^t(\widehat{\nabla}_{w,t}) \cong \nabla(w)_{\mathscr{L}_t^T}$.*

Proof. Thanks to lemma 7.5.1, we have $\pi_\dagger^t(\widehat{\Delta}_{w,t}) = (j_w)_! \pi_\dagger^t(\widehat{\mathscr{L}_t^w})$. It is easy to see that $\pi_\dagger^t(\widehat{\mathscr{L}_t^w}) \cong p_w^* m_!(\widehat{\mathscr{L}_t^T} \boxtimes \mathscr{L}_t^T)[r]$. Lemma 7.7.3 then directly implies the result. □

Lemma 7.7.5. *We have an isomorphism of functors*

$$a_!(\widehat{\mathscr{L}_t^T} \boxtimes -) \cong \mathrm{id}[-2r]$$

from $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$ to $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$.

Proof. We recall some ingredients that we will use in the proof. First, recall that if $\mathscr{G} \in D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t',-]}$ for $t' \neq t$, we have $\mathrm{Hom}_{D^b_{(B)}(\mathscr{X},\Bbbk)}(\mathscr{G}, \mathscr{F}) = 0$ (this is a consequence of (3.2.1)). Also recall from lemma 4.1.1 (and its proof) that for n the order of t in T, the local system $\mathscr{L}_t^T$ appears as a direct summand in $(e_n)_* \underline{\Bbbk}_T = (e_n)_! \underline{\Bbbk}_T$, ($e_n$ is proper, being finite); moreover, it is the only direct summand with monodromy t (that is, it is the only direct summand in $(e_n)_* \underline{\Bbbk}_T$ lying in $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$). Note that t is defined over a finite subfield of $\Bbbk$. Thus we can find an integer $s > 0$ such that $t^{\ell^s} = t$; we then have $t^{\ell^{bs}} = t$ for any $b > 0$. We deduce that for $b > 0$, we have $(e_{\ell^{bs}})_! \mathscr{L}_t^T \cong \mathscr{L}_{t,\ell^{bs}}^T$ and that the latter also lies in $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$. A final consideration: the map $e_{\ell^{bs}}$ is an isomorphism $T \to T$, in particular, the functor $(e_{\ell^{bs}})_*$ preserve monodromy (this follows for example from the decomposition). This has the following consequence: for any $b > 0$ and any direct summands $\mathscr{L} \neq \mathscr{L}_t^T$ appearing in $(e_n)_* \underline{\Bbbk}_T$, the object $(e_{\ell^{bs}})_* \mathscr{L}$ does not belong to $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$.

Consider $\mathscr{F} \in D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$; we construct a morphism $a_!(\widehat{\mathscr{L}^T_t} \boxtimes \mathscr{F}) \to \mathscr{F}[-2r]$. By adjunction, this is the same as a morphism $\widehat{\mathscr{L}^T_t} \boxtimes \mathscr{F} \to a^! \mathscr{F}[-2r]$. We know from proposition 1.2.1 that there exists an integer $m > 0$ such that we have an isomorphism $\iota(m) : \mathrm{pr}_2^* \mathscr{F} \xrightarrow{\sim} a(m)^* \mathscr{F}$ (the notation being the same as the ones used in proposition 1.2.1). Moreover, we showed that this isomorphism is essentially unique and functorial. Then we obtain an element $\iota(m)$ in

$$\begin{aligned}\mathrm{Hom}(\mathrm{pr}_2^* \mathscr{F}, a(m)^* \mathscr{F}) &\cong \mathrm{Hom}(\mathrm{pr}_2^* \mathscr{F}, a(m)^! \mathscr{F}[-2r]) \\ &\cong \mathrm{Hom}((e_m \times \mathrm{id})_! \mathrm{pr}_2^* \mathscr{F}, a^! \mathscr{F}[-2r]).\end{aligned}$$

Consider m of the form $a\ell^{bs}$ for $b > 0$ so that we have $\mathscr{L}^T_{t,\ell^{bs}}$ as a direct summand in $(e_m)_! \underline{\Bbbk}_T$; this is the only direct summand there that lies in $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t]}$. Since we assumed that $\mathscr{F}$ lies in $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t]}$, we have

$$\mathrm{Hom}((e_m \times \mathrm{id})_! \mathrm{pr}_2^* \mathscr{F}, a^! \mathscr{F}[-2r]) \cong \mathrm{Hom}(\mathscr{L}^T_{t,\ell^{bs}} \boxtimes \mathscr{F}, a^! \mathscr{F}[-2r])$$

and $\iota(m)$ defines an element in the latter space. The fact that these $\iota(m)$'s are essentially unique implies that they define an element in

$$\begin{aligned}\varinjlim_b \mathrm{Hom}(\mathscr{L}^T_{t,\ell^{bs}} \boxtimes \mathscr{F}, a^! \mathscr{F}[-2r]) &\cong \mathrm{Hom}(\text{“}\varprojlim_m\text{”}\, \mathscr{L}^T_{t,\ell^{bs}} \boxtimes \mathscr{F}, a^! \mathscr{F}[-2r]) \\ &\cong \mathrm{Hom}(\widehat{\mathscr{L}^T_t} \boxtimes \mathscr{F}, a^! \mathscr{F}[-2r]),\end{aligned}$$

(the limit being taken over $b \gg 0$). This gives us the morphism we were looking for.

We must check that this is an isomorphism. By dévissage, it suffices to do so on a generating family of $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,-]}$; thus we can assume that $\mathscr{F} = \mathrm{For}^t(\mathscr{G})$ for $\mathscr{G} \in \mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}^T_t, -]}$.

Remarking that $a_!(\widehat{\mathscr{L}^T_t} \boxtimes \mathscr{F}) = \widehat{\Delta}_{e,t} \mathbin{\widehat{\star}} \mathscr{F}[-r]$, the morphism $a_!(\widehat{\mathscr{L}^T_t} \boxtimes \mathscr{F}) \to \mathscr{F}[-2r]$ comes in the following way (and in particular, is an isomorphism):

$$\begin{aligned}a_!(\widehat{\mathscr{L}^T_t} \boxtimes \mathscr{F}) &\cong \widehat{\Delta}_{e,t} \mathbin{\widehat{\star}} \mathscr{F}[-r] \\ &\cong \widehat{\Delta}_{e,t} \mathbin{\widehat{\star}} \mathrm{For}^t(\mathscr{G})[-r] \\ &\cong \mathrm{For}^t(\pi^t_\dagger(\widehat{\Delta}_{e,t}) \star^{\mathrm{LY}} \mathscr{G})[-2r] \quad (1) \\ &\cong \mathrm{For}^t(\Delta(e)_{\mathscr{L}^T_t} \star^{\mathrm{LY}} \mathscr{G})[-2r] \quad (2) \\ &\cong \mathrm{For}^t(\mathscr{G}) = \mathscr{F}[-2r] \qquad (3).\end{aligned}$$

We used lemma 7.1.5 for the isomorphism (1), corollary 7.7.4 for (2) and lemma 8.3.1 for (3). This concludes the proof of the lemma. □

7.8 Consequence for the completed category

For any $w, v \in W$, using adjunction and proposition 7.4.2, we get

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\Delta}_{w,t}, \widehat{\nabla}_{v,t}[k]) = \begin{cases} \widehat{R}_t & \text{if } w = v \text{ and } k = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.8.1}$$

Lemma 7.8.1. *The triangulated category $\widehat{D}_{[-,t]}$ is generated as a triangulated category by the objects $\widehat{\Delta}_{w,t}$ for $w \in W$, as well as by the objects $\widehat{\nabla}_{w,t}$ for $w \in W$.*
For any $w \in W$, the monodromy morphism $\varphi_{\widehat{\Delta}_{w,t}}$ induces an isomorphism

$$\widehat{R}_t \xrightarrow{\sim} \mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\Delta}_{w,t}, \widehat{\Delta}_{w,t}), \tag{7.8.2}$$

and any nonzero element in this Hom*-space is injective.*

Proof. The fact that the $\widehat{\Delta}$'s generate the completed category can be proved as in the proof of lemma 5.2.2.

Now consider $\widehat{\mathscr{L}_t^w}$; this object corresponds to $\widehat{R}_t$ under the equivalence (7.5.7). Moreover the monodromy morphism $\varphi_{\widehat{\mathscr{L}_t^w}}$ is given by the $\widehat{R}_t$-action on the $\widehat{R}_t$-module $\widehat{R}_t$. Thus we obtain

$$\widehat{R}_t \cong \mathrm{Hom}_{\widehat{R}_t}(\widehat{R}_t, \widehat{R}_t) \xrightarrow[\sim]{\varphi} \mathrm{Hom}_{\widehat{D}_{(B)}(\mathscr{X}_w, \Bbbk)_{[-,t]}}(\widehat{\mathscr{L}_t^w}, \widehat{\mathscr{L}_t^w}).$$

The statement about $\widehat{\Delta}_{w,t}$ then follows because $(j_w)_!$ is fully faithful.

Finally we prove the last statement. According to what is above, we need to show that for any nonzero $x \in \widehat{R}_t$, the endomorphism $\varphi_{\widehat{\Delta}_{w,t}}(x)$ is injective. If C denotes the cone of this endomorphism, we need to show that C is concentrated in non-negative perverse degrees. Indeed, since $\widehat{\Delta}_{w,t}$ is perverse, the long exact sequence of perverse cohomology associated to the triangle

$$\widehat{\Delta}_{w,t} \xrightarrow{\varphi_{\widehat{\Delta}_{w,t}}(x)} \widehat{\Delta}_{w,t} \longrightarrow C \xrightarrow{+1}$$

will yield the result. By definition, this amounts to showing that

$$j_v^! C \in^p \widehat{D}_{(B)}(\mathscr{X}_v, \Bbbk)^{\geq 0}_{[-,t]}$$

for any $v \in W$. Consider $v \in W$. Let M_v be the object in $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ corresponding to $j_v^! \widehat{\Delta}_{w,t}$ under the equivalence (7.5.7). The inverse image of $j_v^! C$ under this equivalence is then the cone of the endomorphism of M_v given by the action of x. We have that $\pi_!^t(j_v^! \widehat{\Delta}_{w,t}) = j_v^! \Delta(w)_{\mathscr{L}_t^T}$ is concentrated in non-negative perverse degree, so $\Bbbk_t \otimes^L_{\widehat{R}_t} M_v$ is also concentrated in non-negative degrees. According to lemma 7.5.5, the complex M is then isomorphic in $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ to a complex N of free $\widehat{R}_t$-module such that $N^i = 0$ for $i < 0$. The action of x on M_v induces then an endomorphism of N, whose cone is obviously concentrated in non-negative degrees. We deduce that the cone of the action of x in M_v also has cohomology in non-negative degrees. This in turn implies that C lives in non-negative perverse degrees and concludes the proof. □

Remark 7.8.2. Let us discuss a bit point 2. of remark 7.2.4. Let us, for the duration of this remark only, denote by $\widehat{D}_{(B)}(Z, \Bbbk)^R_{[t',t]}$ the category $\widehat{D}_{(B)}(Z, \Bbbk)_{[t',t]}$ considered above (that is to say, the factor with fixed left monodromy t' in the

completed category $\widehat{D}_{(B)}(Z,\Bbbk)^R_{[-,t]}$ constructed as above, with respect to *right* convolution with $\Delta(e)_{\mathscr{L}_t^T}$). We also denote by $\widehat{D}_{(B)}(Z,\Bbbk)^L_{[t',t]}$ the "symmetric" construction, i.e. the same construction as above, but switching right convolution with $\Delta(e)_{\mathscr{L}_t^T}$ by left convolution by $\Delta(e)_{\mathscr{L}_{t'}^T}$, as well as the role of t and t'. The categories $\widehat{D}_{(B)}(Z,\Bbbk)^R_{[t',t]}$ and $\widehat{D}_{(B)}(Z,\Bbbk)^L_{[t',t]}$ are both full subcategory of the category of pro-objects above $D^b_{(B)}(\mathscr{X},\Bbbk)$.

Now, it is clear that for $w \in {}_{t'}W_t$, the object $\widehat{\Delta}_{w,t} \in \widehat{D}_{(B)}(Z,\Bbbk)^R_{[t',t]}$, define an object in $\widehat{D}_{(B)}(Z,\Bbbk)^L_{[t',t]}$, and it is easily seen that the family $\{\widehat{\Delta}_{w,t}\}_{w\in {}_{t'}W_t}$ generates the latter triangulated category. Since it also generates $\widehat{D}_{(B)}(Z,\Bbbk)^R_{[t',t]}$, we deduce that in fact these two categories coincide.

Corollary 7.8.3. *For any $\mathscr{F},\mathscr{G} \in \widehat{D}_{[-,t]}$ and $k \in Z$, the $\widehat{R}_t$-module*

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F},\mathscr{G}[k])$$

is finitely generated.

The category $\widehat{D}_{[-,t]}$ is Krull–Schmidt.

Proof. The fact on the Hom-space is a direct consequence of (7.8.1) and lemma 7.8.1. The Krull–Schmidt assertion can be proved exactly as in [BeR, Corollary 5.4]. □

Lemma 7.8.4. *The subcategory ${}^p\widehat{D}^{\leq 0}_{[-,t]}$ is generated under extensions by the objects of the form $\widehat{\Delta}_{w,t}[n]$ with $w \in W$ and $n \geq 0$.*

Proof. Consider $\mathscr{F} \in {}^p\widehat{D}^{\leq 0}_{[-,t]}$. Using recollement triangles and induction on the number r of strata contained in the support of $\mathscr{F}$, it suffices to treat the case $r = 1$. Denote by $\mathscr{X}_w$ the stratum on which $\mathscr{F}$ is nonzero. We then have $\mathscr{F} \cong (j_w)_! j_w^* \mathscr{F}$. Now $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)^{\leq 0}$ is generated under extensions by $\{\widehat{R}_t[n] \mid n \geq 0\}$. Indeed, any object M in $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)^{\leq 0}$ admits a finite projective, hence free, resolution (recall that $\widehat{R}_t$ is local of finite global dimension). The result then follows. □

Corollary 7.8.5. *The functor $\pi^t_\dagger$ is right t-exact for the perverse t-structures.*

Proof. Recall from corollary 7.7.4 that we have $\pi^t_\dagger(\widehat{\Delta}_{w,t}) \cong \Delta(w)_{\mathscr{L}_t^T}$ for any $w \in W$. Moreover, since $\Delta(w)_{\mathscr{L}_t^T}$ is perverse, $\Delta(w)_{\mathscr{L}_t^T}[n]$ lies ${}^p\mathfrak{D}(\mathscr{X}/\!\!/T)^{\leq 0}_{[-,\mathscr{L}_t^T]}$ for any $n \geq 0$. The result then follows from lemma 7.8.4. □

We conclude this section by two technical results that will be used for the study of tilting objects.

Definition 7.8.6. *1. Let $\mathscr{F}$ be an object in $\widehat{D}_{[-,t]}$. We say that $\mathscr{F}$ admits a $\widehat{\Delta}$-filtration, or a pro-standard filtration if it has a filtration with subquotients in $\{\widehat{\Delta}_{w,t} \mid w \in W\}$. Similarly, $\mathscr{F}$ has a $\widehat{\nabla}$-filtration, or pro-costandard filtration, if it admits a filtration with subquotients in $\{\widehat{\nabla}_{w,t} \mid w \in W\}$.*

2. *Let $\mathscr{F}$ be an object of $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. We say that $\mathscr{F}$ has a Δ-filtration or a standard filtration if it has a filtration with subquotients in $\{\Delta(w)_{\mathscr{L}_t^T} \mid w \in W\}$. Similarly, $\mathscr{F}$ has a ∇-filtration, or costandard filtration if it admits a filtration with subquotients in $\{\nabla(w)_{\mathscr{L}_t^T} \mid w \in W\}$.*

Lemma 7.8.7. *Consider $\mathscr{F} \in \widehat{D}_{[-,t]}$. Then*

$$(\mathscr{F} \text{ has a } \widehat{\Delta}\text{-filtration}) \Leftrightarrow (\text{for any } w \in W,\ j_w^* \mathscr{F} \text{ is a direct sum of } \widehat{\mathscr{L}_t^w}).$$

For $\mathscr{F} \in \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$, we have

$$(\mathscr{F} \text{ has a } \Delta\text{-filtration}) \Leftrightarrow (\text{for any } w \in W,\ j_w^* \mathscr{F} \text{ is a direct sums of } \mathscr{L}_{w,t}[-r]) \Leftrightarrow (\text{for any } w \in W,\ j_w^* \mathscr{F} \text{ is perverse}).$$

Proof. The proofs of the two statements are very similar, so we prove only the second one. It is clear that "$\mathscr{F}$ has a Δ-filtration" implies that the restrictions to each stratum are direct sums of $\mathscr{L}_{w,t}[-r]$. In turn this fact implies that these restrictions are perverse. Assume that $\mathscr{F}$ is such that for any $w \in W$, the object $j_w^* \mathscr{F}$ is perverse. We will prove that $\mathscr{F}$ has a Δ-filtration by induction of the number of strata in the support S of $\mathscr{F}$. Since the perverse sheaves on a single stratum are given by direct sums of $\mathscr{L}_{w,t}[-r]$, the initialisation is clear. Now choose a closed filtration $S \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq S_{n+1} = \varnothing$ of S such that $S_i \setminus S_{i+1}$ is a single stratum. Denote by $j : S \setminus S_n \hookrightarrow S$ the open inclusion map, and by i the closed inclusion $S_n \hookrightarrow S$. By induction, the object $j_! j^* \mathscr{F}$ is a successive extension of $\Delta(w)_{\mathscr{L}_t^T}$, for some $w \in W$. The case of a single stratum and the triangle

$$j_! j^* \mathscr{F} \to \mathscr{F} \to i_! i^* \mathscr{F} \xrightarrow{+1}$$

allow us to conclude that $\mathscr{F}$ is a successive extension of standard sheaves. □

Remark 7.8.8. Similarly, $\mathscr{F}$ admits a $\widehat{\nabla}$-filtration if and only if its corestriction to each stratum is a direct sum of $\widehat{\mathscr{L}_t^w}$. Similar statements hold for $\mathscr{F}$ in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$.

Assume that $\mathscr{F}$ in $\widehat{D}_{[-,t]}$ (resp. in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$) admits a $\widehat{\Delta}$ (resp. a Δ) filtration. Then, it is a standard fact that the number of occurrences of $\widehat{\Delta}_{w,t}$ (resp. of $\Delta(w)_{\mathscr{L}_t^T}$) as a subquotient does not depends on the chosen filtration. This number is then denoted $(\mathscr{F} : \widehat{\Delta}_{w,t})$ (resp. $(\mathscr{F} : \Delta(w)_{\mathscr{L}_t^T})$.)

Lemma 7.8.9. *Consider $\mathscr{F} \in \widehat{D}_{[-,t]}$ such that $\pi_\dagger^t(\mathscr{F})$ is perverse and admits a Δ-filtration. Then $\mathscr{F}$ is already perverse and admits a $\widehat{\Delta}$-filtration. More precisely, for the considered filtrations, we have*

$$(\mathscr{F} : \widehat{\Delta}_{w,t}) = (\pi_\dagger^t(\mathscr{F}) : \Delta(w)_{\mathscr{L}_t^T}).$$

Similar considerations apply to costandard filtrations.

Proof. We saw in lemma 7.8.7 that $\mathscr{F}$ has a $\widehat{\Delta}$-filtration if and only if the restrictions $j_w^* \mathscr{F}$ are direct sums of $\widehat{\mathscr{L}_t^w}$ for any $w \in W$. This is clearly equivalent to the fact that the inverse image of these restrictions under the equivalence (7.5.6) (say, denoted M_w^*) are free $\widehat{R}_t$-modules. In turn, this condition is equivalent to the fact that $\Bbbk_t \overset{L}{\otimes}_{\widehat{R}_t} M_w^*$ is concentrated in degree zero in $D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t}$. Thus $\mathscr{F}$ has a $\widehat{\Delta}$-filtration if and only if the $\pi_\dagger^t j_w^* \mathscr{F}$'s are perverse. Since $j_w^* \circ \pi_\dagger^t = \pi_\dagger^t \circ j_w^*$, using lemma 7.8.7, we see that $\mathscr{F}$ has a $\widehat{\Delta}$-filtration if and only if $\pi_\dagger^t(\mathscr{F})$ does.

The statement about multiplicities is obvious, and the proof in the costandard case is similar. □

Remark 7.8.10. Using remark 7.8.2, one can see that a "left-sided" analogue of lemma 7.8.9 holds: consider $t, t' \in T_{\Bbbk}^\vee$ in the same W-orbit and $\mathscr{F} \in \widehat{D}_{[t',t]}$. Then $\Delta(e)_{\mathscr{L}_{t'}^T} \widehat{\star} \mathscr{F}$ defines an object in $\mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}_{t'}^T, -]}$, and if it is perverse and admits a standard filtration in $\mathfrak{P}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}_{t'}^T, -]}$, then the same is true of $\mathscr{F}$ in $\widehat{D}_{[t',t]}$.

Chapter 8

Study of standard and costandard objects

The aim of this chapter is to study various properties of standard (and costandard) objects in the Lusztig–Yun categories previously introduced. We start by constructing forgetful and averaging functors between the categories $\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$ and $\mathfrak{D}(\mathscr{X}^s /\!\!/ L_s)_{[-,\mathscr{L}_t^{L_s}]}$, and we study the effect of these functors on standard objects. We continue by studying convolution between these objects. We then obtain the main result of this section: we get a description of the socle of standard objects and cosocle of costandard objects. Our proof is deeply inspired by ideas in [BBM, §2.1]. The obtained results will eventually allow us to study tilting objects, in particular to compare them to projective covers of minimal IC sheaves, as will be done in the next chapter.

8.1 Forgetful and averaging functors

For the duration of this section, we fix a complex algebraic group L and a closed subgroup B. Consider a right L-variety X. Let $\mathscr{L}_L$ be a rank one multiplicative $\Bbbk$-local system on L (recall that by multiplicative, we mean that the pullback of $\mathscr{L}_L$ under the multiplication map $L \times L \to L$ is given by $\mathscr{L}_L \boxtimes \mathscr{L}_L$). Let $\mathscr{L}_B := (\mathscr{L}_L)_{|B}$; this is again a rank one multiplicative $\Bbbk$-local system. In particular, we have a canonical trivialisation $(\mathscr{L}_L)_e \cong \Bbbk$ of the stalks at the neutral element e of L of the local system $\mathscr{L}_L$, and the same is true for the local system $\mathscr{L}_B$.

Assume that we have a finite central isogeny $\nu_L : \widetilde{L} \to L$ with kernel K_L of order prime to $\ell = \mathrm{char}(\Bbbk)$. Assume moreover that we have a $\Bbbk$-character χ_L of K_L such that $\mathscr{L}_L$ is the χ_L isotypic component of $(\nu_L)_* \underline{\Bbbk}_{\widetilde{L}}$. These data give us analogues for the group B: let $\nu_B : \widetilde{B} := B \times_L \widetilde{L} \to B$ and χ_B the character χ_L viewed as a character of $K \hookrightarrow B \times_L \widetilde{L}$. Then $\mathscr{L}_B$ identifies with the χ_B-isotypic

component of $(\nu_B)_* \underline{\Bbbk}_{\widetilde{B}}$. We can define the Lusztig–Yun categories

$$\mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B} \qquad \text{and} \qquad \mathfrak{D}(X/\!\!/L)_{\mathscr{L}_L}.$$

We warn the reader that the notation to appear in subsections 8.1.1 and 8.1.2 are only temporary: our main point of interest is the case $L = L_s$ and $B = B_s$, and we will settle more convenient notation later in subsection 8.1.3.

Remark 8.1.1. We consider here a right L-variety; this is arbitrary, the same arguments and results having exact analogues for a left action.

8.1.1 The functor For

We first define a natural "forgetful functor"

$$\mathrm{For} : \mathfrak{D}(X/\!\!/L)_{\mathscr{L}_L} \to \mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B}.$$

Consider an object $\mathscr{F}$ in $\mathfrak{D}(X/\!\!/L)_{\mathscr{L}_L}$, this is an $\widetilde{L}$-equivariant object on X, and the natural action of K (see lemma 2.1.1) is via a character χ_L. Now consider the object $\mathrm{For}_{\widetilde{L}}^{\widetilde{B}}(\mathscr{F})$, where $\mathrm{For}_{\widetilde{L}}^{\widetilde{B}}$ is the natural forgetful functor (which is denoted $Res_{\widetilde{B},\widetilde{L}}$ in [BL, §2.6.1]); this is a $\widetilde{B}$-equivariant object on X and we still have an action of K on this object. This action is once again via the character χ_L (but viewed as a character of $K \hookrightarrow B \times_L \widetilde{L}$, so in fact, χ_B) so we have $\mathrm{For}_{\widetilde{L}}^{\widetilde{B}}(\mathscr{F}) \in \mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B}$. In this way, we obtain the wished-for forgetful functor; we will denote it $\mathrm{For}_{L,B}^{\mathrm{LY}}$.

8.1.2 Averaging functors

Consider the variety $X \times^B L$. We obtain an L-variety, for the action of L on itself by multiplication on the right. We have a natural map μ, induced by the action of L on X, from $X \times^B L$ to X.

Lemma 8.1.2. *Denote by q_B the quotient map $X \times L \to X \times^B L$ for the antidiagonal action of B. For $\mathscr{F} \in \mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B}$, there exists a unique object $\mathscr{F} \boxtimes^B \mathscr{L}_L$ in $D^b(X \times^B L, \Bbbk)$ whose pullback under q_B is $\mathscr{F} \boxtimes \mathscr{L}_L$.*

Proof. Since the action of B on $X \times L$ is free, it suffices to show that $\mathscr{F} \boxtimes \mathscr{L}_L$ is in the B-equivariant derived category $D_B^b(X \times L, \Bbbk)$ for this action. This follows from arguments similar to those introduced for the definition of convolution in the equivariant case, see lemma 2.9.3 and section 6.3.2. □

We define two functors $\mathrm{Av}_{\mathscr{L}_L,*}$ and $\mathrm{Av}_{\mathscr{L}_L,!}$

$$\mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B} \to D^b(X, \Bbbk).$$

Definition 8.1.3. *Take $\mathscr{F} \in \mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B}$. Set*

$$\mathrm{Av}_{\mathscr{L}_L,*}(\mathscr{F}) = \mu_*(\mathscr{F} \boxtimes^B \mathscr{L}_L) \quad \textit{and} \quad \mathrm{Av}_{\mathscr{L}_L,!}(\mathscr{F}) = \mu_!(\mathscr{F} \boxtimes^B \mathscr{L}_L)[2\dim(L/B)].$$

The functors $\mathrm{Av}_{\mathscr{L}_L,*}$ and $\mathrm{Av}_{\mathscr{L}_L,!}$ are called averaging functors.

Lemma 8.1.4. *The averaging functors* $\mathrm{Av}_{\mathscr{L}_L,*}$ *and* $\mathrm{Av}_{\mathscr{L}_L,!}$ *canonically factor through* $\mathfrak{D}(X/\!\!/L)_{\mathscr{L}_L}$.

Proof. We show the lemma for the functor $\mathrm{Av}_{\mathscr{L}_L,*}$, the proof for $\mathrm{Av}_{\mathscr{L}_L,!}$ is similar. First, let us show that for any $\mathscr{F} \in D^b_{\widetilde{B}}(X,\Bbbk)_{\chi_B}$, we have $\mathscr{F} \boxtimes^B \mathscr{L}_L \in D^b_{\widetilde{L}}(X \times^B L, \Bbbk)_{\chi_L}$. The object $\mathscr{F} \boxtimes^B \mathscr{L}_L$ lies in the $\widetilde{L}$-equivariant derived category since $\mathscr{L}_L$ is $\widetilde{L}$-equivariant. We use the following commutative square

$$\begin{array}{ccc} X \times L \times L & \xrightarrow{q_B \times \mathrm{id}_L} & X \times^B L \times L \\ {\scriptstyle \mathrm{id}_X \times m_L}\downarrow & & \downarrow{\scriptstyle \widetilde{a}} \\ X \times L & \xrightarrow{q_B} & X \times^B L. \end{array}$$

Here, $\widetilde{a}$ is the action map given by $([x,l],l') \mapsto [x,ll']$. We investigate the object $\widetilde{a}^*(\mathscr{F} \boxtimes^B \mathscr{L}_L)$. We have a commutative square

$$\begin{array}{ccc} X \times L \times L & \xrightarrow{q_B \times \mathrm{id}_L} & X \times^B L \times L \\ {\scriptstyle \mathrm{id}_X \times m_L}\downarrow & & \downarrow{\scriptstyle \widetilde{a}} \\ X \times L & \xrightarrow{q_B} & X \times^B L. \end{array}$$

We deduce that $(q_B \times \mathrm{id}_L)^* \widetilde{a}^*(\mathscr{F} \boxtimes^B \mathscr{L}_L) \cong (\mathrm{id}_X \times m_L)^* q_B^*(\mathscr{F} \boxtimes^B \mathscr{L}_L)$. Since $\mathscr{L}_L$ is multiplicative, the latter object is isomorphic to $\mathscr{F} \boxtimes \mathscr{L}_L \boxtimes \mathscr{L}_L$. We then obtain that

$$\widetilde{a}^*(\mathscr{F} \boxtimes^B \mathscr{L}_L) \cong \mathscr{F} \boxtimes^B \mathscr{L}_L \boxtimes \mathscr{L}_L\,. \tag{8.1.1}$$

Thanks to corollary 3.4.1, we then see that $\mathscr{F} \boxtimes^B \mathscr{L}_L$ is indeed in the χ_L-equivariant subcategory.

Now note that the following square is cartesian:

$$\begin{array}{ccc} (X \times^B L) \times L & \xrightarrow{\widetilde{a}} & X \times^B L \\ {\scriptstyle \mu \times \mathrm{id}_L}\downarrow & & \downarrow{\scriptstyle \mu} \\ X \times L & \xrightarrow{a} & X, \end{array}$$

with $\widetilde{a}([x,l],l') = [x,ll']$ and $a(x,l) = x \cdot l$. The morphisms a and $\widetilde{a}$ are smooth, so an application of the smooth base-change theorem leads to

$$a^* \,\mathrm{Av}_{\mathscr{L}_L,*}(\mathscr{F}) \cong (\mu \times \mathrm{id})_* \widetilde{a}^*(\mathscr{F} \boxtimes^B \mathscr{L}_L). \tag{8.1.2}$$

Combining (8.1.1) and (8.1.2), we finally get

$$a^* \,\mathrm{Av}_{\mathscr{L}_L,*}(\mathscr{F}) \cong \mathrm{Av}_{\mathscr{L}_L,*}(\mathscr{F}) \boxtimes \mathscr{L}_L\,.$$

Since $\mathscr{F} \boxtimes^B \mathscr{L}_L$ is $\widetilde{L}$-equivariant and μ is $\widetilde{L}$-equivariant, the object $\mathrm{Av}_{\mathscr{L}_L,*}(\mathscr{F})$ is itself $\widetilde{L}$-equivariant. Thanks to corollary 3.4.1, this implies that $\mathrm{Av}_{\mathscr{L}_L,*}(\mathscr{F})$ belongs to $\mathfrak{D}(X/\!\!/L)_{\mathscr{L}_L}$. □

Consider the embedding $e : X \hookrightarrow X \times^B L$ given by $x \mapsto [x, e]$. This map makes the following into a commutative diagram:

$$\begin{array}{ccc} X \times B & \xrightarrow{e\times i} & X \times^B L \times L \\ {\scriptstyle ?}\downarrow & & \downarrow{\scriptstyle ?} \\ X & \xrightarrow{e} & X \times^B L. \end{array} \tag{8.1.3}$$

(Here i denotes the inclusion $B \hookrightarrow L$, and ? is either the action maps or the projection on the first factor.) This diagram implies that e is compatible with the $\widetilde{B}$-action on X and the $\widetilde{L}$-action on $X \times^B L$. Thus we obtain a functor $Q_e^* : D^b_{\widetilde{L}}(X \times^B L, \Bbbk) \to D^b_{\widetilde{B}}(X, \Bbbk)$. Using diagram (8.1.3), it is easy to see that the functor Q_e^* preserves the character-equivariant subcategories $D^b_{\widetilde{L}}(X \times^B L, \Bbbk)_{\chi_L}$ and $D^b_{\widetilde{B}}(X, \Bbbk)_{\chi_B} = \mathfrak{D}(X/\!\!/B)_{\mathscr{L}_B}$. In fact we can say a bit more:

Lemma 8.1.5. *The pullback functor Q_e^* induces an equivalence of categories $D^b_{\widetilde{L}}(X \times^B L, \Bbbk)_{\chi_L} \to D^b_{\widetilde{B}}(X, \Bbbk)_{\chi_B}$, with quasi-inverse $\mathscr{F} \mapsto \mathscr{F} \boxtimes^B \mathscr{L}_L$.*

Proof. According to the discussion above the statement of the lemma, the functor Q_e^* indeed maps the category $D^b_{\widetilde{L}}(X \times^B L, \Bbbk)_{\chi_L}$ to the category $D^b_{\widetilde{B}}(X, \Bbbk)_{\chi_B}$. Also we clearly have $Q_e^* \circ ((-) \boxtimes^B \mathscr{L}_L) \cong \mathrm{id}$. Now remark that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{e}} & X \times^{\widetilde{B}} \widetilde{L} \\ & {\scriptstyle e}\searrow & \downarrow{\scriptstyle \widetilde{\nu}} \\ & & X \times^B L. \end{array}$$

Here, the map $\widetilde{e}$ is given by $x \mapsto [x, e]$ and $\widetilde{\nu}$ maps $[x, l]$ to $[x, \nu_L(l)]$. It is easy to see that $\widetilde{\nu}$ is an isomorphism of $\widetilde{L}$-varieties. Moreover, the map $\widetilde{e}$ is a $\widetilde{B}$-$\widetilde{L}$ morphism of varieties. Then we clearly have $Q_e^* \cong Q_{\widetilde{e}}^* \circ Q_{\widetilde{\nu}}^*$. Now $Q_{\widetilde{e}}^*$ is an equivalence thanks to [BL, §6.6, item 6], and so is $Q_{\widetilde{\nu}}^*$ since $\widetilde{\nu}$ is an isomorphism. We deduce that Q_e^* is an equivalence $D^b_{(B),\widetilde{L}}(X \times^B L, \Bbbk) \to D^b_{(B),\widetilde{B}}(X, \Bbbk)$, and this fact readily implies that the restriction of Q_e^* to $D^b_{\widetilde{L}}(X \times^B L, \Bbbk)_{\chi_L}$ is fully faithful. It is moreover essentially surjective thanks to the begining of this proof. Thus, it is an equivalence, and has the prescribed quasi-inverse. □

Lemma 8.1.6. *We have adjoint pairs $(\mathrm{For}^{\mathrm{LY}}_{L,B}, \mathrm{Av}_{\mathscr{L}_L,*})$ and $(\mathrm{Av}_{\mathscr{L}_L,!}, \mathrm{For}^{\mathrm{LY}}_{L,B})$, where $\mathrm{For}^{\mathrm{LY}}_{L,B}$ is the functor defined in §8.1.1.*

Proof. We prove that $(\mathrm{For}^{\mathrm{LY}}_{L,B}, \mathrm{Av}_{\mathscr{L}_L,*})$ is an adjoint pair, the other case can be proved with similar arguments. From lemma 8.1.5, we know that $Q_e^* \circ \mu^*$ is right adjoint to $\mathrm{Av}_{\mathscr{L}_L,*}$. Indeed, Q_e^* is adjoint to $\mathscr{F} \mapsto \mathscr{F} \boxtimes^B \mathscr{L}_L$ and μ^* is right adjoint to $\mu_* : D^b_{\widetilde{L}}(X \times^B L, \Bbbk) \to D^b_{\widetilde{L}}(X, \Bbbk)$ (recall that the character equivariant categories, and in particular the Lusztig–Yun categories, are full subcategories of these $\widetilde{L}$-equivariant categories, so that the adjunction holds by standard considerations).

Now, the composition $\mu \circ e$ is the identity; one deduces that the composition $Q_e^* \circ \mu^*$ just forgets the $\widetilde{L}$-equivariant structure only to retain the $\widetilde{B}$-equivariant structure, hence is isomorphic to For. This concludes the proof. □

8.1.3 Forgetful and averaging functors over $\mathscr{X}^s$

This section has for only goal to fix the notation that we will use in future chapters.

Our main point of interest when we defined the forgetful and averaging functors in the previous subsections was to apply these constructions to the groups $L = L_s$ and $B = B_s$ and the variety $X = \mathscr{X}^s$. More precisely, let $t \in T^\vee_{\Bbbk}$ be an element and $s \in W$ a simple reflection such that $s \in W_t^\circ$. Then we can consider the local systems $\mathscr{L}_t^{L_s}$ and $\mathscr{L}_t^{B_s}$ and the isogenies ν_{L_s} and ν_{B_s} introduced in subsections 5.3.2 and 5.3.3. The constructions of subsection 8.1.1 and subsection 8.1.2 can be made in this setting. However, we can consider $\mathscr{X}^s$ as a right or left B_s- or L_s-variety. We introduce different notation for these different cases.

For the right action: we set

$$\mathrm{For}_t^s := \mathrm{For}^{\mathrm{LY}}_{L_s,B_s} : \mathfrak{D}(\mathscr{X}^s /\!\!/ L_s)_{[-,\mathscr{L}_t^{L_s}]} \to \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]},$$

and

$$\mathrm{Av}^s_{t,*} := \mathrm{Av}_{\mathscr{L}_t^{L_s},*},\ \mathrm{Av}^s_{t,!} := \mathrm{Av}_{\mathscr{L}_t^{L_s},!} : \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]} \to \mathfrak{D}(\mathscr{X}^s /\!\!/ L_s)_{[-,\mathscr{L}_t^{L_s}]}.$$

For the left action: we set

$$\mathrm{For}_s^t := \mathrm{For}^{\mathrm{LY}}_{L_s,B_s} : \mathfrak{D}(L_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}_t^{L_s},-]} \to \mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}_t^{B_s},-]},$$

and

$$\mathrm{Av}^t_{s,*} := \mathrm{Av}_{\mathscr{L}_t^{L_s},*},\ \mathrm{Av}^t_{s,!} := \mathrm{Av}_{\mathscr{L}_t^{L_s},!} : \mathfrak{D}(B_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}_t^{B_s},-]} \to \mathfrak{D}(L_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}_t^{L_s},-]}.$$

Note that in these cases, the (analogue of the) map μ is proper since it is a $L_s/B_s \cong \mathbb{P}^1$-fibration, so the $*$- and !-averaging functors just differ by a shift by $2\dim(L_s/B_s) = 2$.

The following result relates the averaging functors and Verdier duality introduced in subsection 5.3.4.

Lemma 8.1.7. *Consider $t \in T^\vee_{\Bbbk}$ and $s \in W$ a simple reflection with $s \in W_t^\circ$. For any $\mathscr{F}$ in $\mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$, we have*

$$\underline{\mathbb{D}}_{\mathscr{X}^s}(\mathrm{Av}^s_{t,*}(\mathscr{F})) \cong \mathrm{Av}^s_{t^{-1},!}(\underline{\mathbb{D}}_{\mathscr{X}^s}(\mathscr{F})) \text{ and } \underline{\mathbb{D}}_{\mathscr{X}^s}(\mathrm{Av}^s_{t,!}(\mathscr{F})) \cong \mathrm{Av}^s_{t^{-1},*}(\underline{\mathbb{D}}_{\mathscr{X}^s}(\mathscr{F})).$$

Proof. We clearly have $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s} \circ \mu_* \cong \mu_! \circ \mathbb{D}_{L_s \times^{B_s} \mathscr{X}^s}[-2(r+1)]$ and $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s} \circ \mu_! \cong \mu_* \circ \mathbb{D}_{L_s \times^{B_s} \mathscr{X}^s}[-2(r+1)]$. Moreover, as in the proof of lemma 6.3.4, one can easily see that

$$\mathbb{D}_{L_s \times^{B_s} \mathscr{X}^s}(\mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{L_s})[-2(r+1)] \cong \underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathscr{F}) \boxtimes^{B_s} \mathbb{D}_{L_s}(\mathscr{L}_t^{L_s})[-2(r+1)].$$

We must determine $\mathbb{D}_{L_s}(\mathscr{L}_t^{L_s})$. The group L_s is smooth and has dimension $r+2$. The duality functor applied to $\mathscr{L}_t^{L_s}$ yields $\mathbb{D}_{L_s}(\mathscr{L}_t^{L_s}) = \mathscr{L}_{t^{-1}}^{L_s}[2(r+2)]$ and so

$$\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathscr{F}) \boxtimes^{B_s} \mathbb{D}_{L_s}(\mathscr{L}_t^{L_s})[-2(r+1)] \cong \underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathscr{F}) \boxtimes^{B_s} \mathscr{L}_{t^{-1}}^{L_s}[2].$$

Thus we indeed have $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathrm{Av}_{t,*}^s(\mathscr{F})) \cong \mathrm{Av}_{t^{-1},!}^s(\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathscr{F}))$.

Now consider the preceding isomorphism for $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathscr{F})$ instead of $\mathscr{F}$ and t^{-1} instead of t. An application of $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}$ to the obtained isomorphism leads to $\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathrm{Av}_{t,!}^s(\mathscr{F})) \cong \mathrm{Av}_{t^{-1},*}^s(\underline{\underline{\mathbb{D}}}_{\mathscr{X}^s}(\mathscr{F}))$. □

Remark 8.1.8. An analogue of the previous lemma would hold in the generality of subsection 8.1.2, but we preferred to state it in the particular situation considered above: the main reason is that the Verdier duality functor on $\mathscr{X}^s$ is not the standard one (it is a shift of it), because we want this functor to preserve perverse sheaves in the Lusztig–Yun category over $\mathscr{X}^s$. Thus to avoid any confusion in the various shifts that may appear in the general case, we stick to the above version.

The exact arguments of lemma 7.1.5 allow one to prove the following lemma:

Lemma 8.1.9. *For any $\mathscr{F} \in \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$ and $\mathscr{G} \in \mathfrak{D}(L_s \setminus\!\!\setminus \mathscr{X}^s)_{[\mathscr{L}_t^{B_s},-]}$ we have an isomorphism*

$$\mathscr{F} \star_{B_s}^{\mathrm{LY}} \mathrm{For}_s^t(\mathscr{G}) \cong \mathrm{Av}_t^s(\mathscr{F}) \star_{L_s}^{\mathrm{LY}} \mathscr{G}\,.$$

8.2 Averaging of standards

In this section, unless otherwise stated, we consider $\mathscr{X}^s$ a right B_s or L_s-variety. To simplify slightly the notation, we set $\mathscr{X}_{w,ws}^s := \mathscr{X}_w^s \sqcup \mathscr{X}_{ws}^s$.

8.2.1 Averaging

We choose $w \in W$ such that $ws > w$. We have an induced map

$$\mathscr{X}_{w,ws}^s \times^{B_s} L_s \to \mathscr{X}_{w,ws}^s;$$

this is a $B_s \backslash L_s \cong \mathbb{P}^1$-fibration. It is a simple computation to see that the restriction

$$\mu_w : \mathscr{X}_w^s \times^{B_s} L_s \to \mathscr{X}_{w,ws}^s$$

of μ is an isomorphism.

We will also have to consider the restriction

$$\mu_{ws} : \mathscr{X}^s_{ws} \times^{B_s} L_s \to \mathscr{X}^s_{w,ws}.$$

We claim that μ_{ws} is an $\mathbb{A}^1$-fibration, thus a smooth map (with fibers all isomorphic to $\mathbb{A}^1$). To see this, we consider the identification $\mathscr{X}^s_{w,ws} = BwP_s/U^s$. With this point of view, we have

$$\mathscr{X}^s_w = BwB/U^s, \qquad \mathscr{X}^s_{ws} = BwsB/U^s = BwBsB/U^s.$$

If U_w is as in subsection 5.1.2, the we have an isomorphism $U_w \times P_s \xrightarrow{\sim} BwP_s$ given by $(u,p) \mapsto u\dot{w}p$. Now the twisted product $\mathscr{X}^s_{w,ws} \times^{B_s} L_s$ identifies with $U_w \times P_s \times^B P_s$, and its subvariety $\mathscr{X}^s_{ws} \times^{B_s} L_s$ identifies with $U_w \times BsB \times^B P_s$. These considerations show that it suffices to see that the map

$$BsB \times^B P_s \to P_s \tag{8.2.1}$$

induced by multiplication is an $\mathbb{A}^1$-fibration. In turn, this can be seen as follows: we have an isomorphism

$$\mathbb{A}^1 \times P_s \xrightarrow{\sim} BsB \times^B P_s,\ (x,p) \mapsto [u_s(x)\dot{s}; \dot{s}^{-1}u_s(-x)p]$$

where $\dot{s}$ and u_s are as in 5.1.1. Under this identification, the map (8.2.1) is given by the projection $\mathbb{A}^1 \times P_s \to P_s$; this proves our claim.

Let $d := \dim(\mathscr{X}^s_w)$; we have $\dim(\mathscr{X}^s_{ws}) = d+1$. The next commutative diagram gives a summary for the notation that we will use:

$$\begin{array}{ccccc}
\mathscr{X}^s_w & \xrightarrow{e_w} & \mathscr{X}^s_w \times^{B_s} L_s & \xrightarrow[\mu_w]{\sim} & \mathscr{X}^s_{w,ws} \\
\downarrow & & \downarrow & & \| \\
\mathscr{X}^s_{w,ws} & \xrightarrow[e_{w,ws}]{} & \mathscr{X}^s_{w,ws} \times^{B_s} L_s & \xrightarrow[\mu_{w,ws}]{\mathbb{P}^1 \text{ fibr.}} & \mathscr{X}^s_{w,ws} \\
\uparrow & & \uparrow & & \| \\
\mathscr{X}^s_{ws} & \xrightarrow[e_{ws}]{} & \mathscr{X}^s_{ws} \times^{B_s} L_s & \xrightarrow[\mu_{ws}]{\text{fibers} \cong \mathbb{A}^1} & \mathscr{X}^s_{w,ws}.
\end{array} \tag{8.2.2}$$

We want to determine the averaging of $\Delta(w)_{\mathscr{L}^{B_s}_t}$. Recall that this object is the !-extension of the (shifted) local system $\mathscr{L}(s)^w_t := (p^{B_s}_w)^* \mathscr{L}^{B_s}_t$ on $\mathscr{X}^s_w$ to the variety $\mathscr{X}^s$. We have the following diagram:

$$\begin{array}{ccccc}
\mathscr{X}^s_w & \xrightarrow{e_w} & \mathscr{X}^s_w \times^{B_s} L_s & \xrightarrow[\mu_w]{\sim} & \mathscr{X}^s_{w,ws} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{X}^s & \xrightarrow[e]{} & \mathscr{X}^s \times^{B_s} L_s & \xrightarrow[\mu]{} & \mathscr{X}^s.
\end{array} \tag{8.2.3}$$

It is easily seen that in order to determine $\mathrm{Av}^s_{t,*}(\Delta(w)_{\mathscr{L}_t^{L_s}})$, we can just stick to the top line of the diagram i.e. start with the shifted local system $\mathscr{L}(s)_t^w[\ell(w)]$, consider the local system $\mathscr{L}(s)_t^w \boxtimes^{B_s} \mathscr{L}_t^{L_s}$ on $\mathscr{X}_w^s \times^{B_s} L_s$, then apply $(\mu_w)_* = (\mu_w)_!$ and finally extend by zero via the inclusion given by the right vertical arrow. Doing so, we obtain $(j^s_{w,ws})_! \mathscr{L}_1[\ell(w)]$ for $\mathscr{L}_1$ a local system on $\mathscr{X}_w^s \sqcup \mathscr{X}_{ws}^s$.

Now we do the exact same thing for $\mathrm{Av}^s_{t,!}(\Delta(ws)_{\mathscr{L}_t^{B_s}})$. We have the same kind of diagram

$$\begin{array}{ccccc} \mathscr{X}_{ws}^s & \xhookrightarrow{e_{ws}} & \mathscr{X}_{ws}^s \times^{B_s} L_s & \xrightarrow[\mu_{ws}]{} & \mathscr{X}_{w,ws}^s \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{X}^s & \xhookrightarrow[e]{} & \mathscr{X}^s \times^{B_s} L_s & \xrightarrow[\mu]{} & \mathscr{X}^s. \end{array} \tag{8.2.4}$$

Let us show that $(\mu_{ws})_!(\mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s})$ on $\mathscr{X}_{w,ws}^s$ is a shifted local system; more precisely, that there exists a local system $\mathscr{L}_2$ on $\mathscr{X}_{w,ws}^s$ such that

$$(\mu_{ws})_!(\mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s}) \cong \mathscr{L}_2[-2]; \tag{8.2.5}$$

moreover, we have $\mu_{ws}^* \mathscr{L}_2 \cong \mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{B_s}$. We showed earlier that μ_{ws} is an $\mathbb{A}^1$-fibration. Thus, μ_{ws} is just a projection along an $\mathbb{A}^1$: $\mathscr{X}_{w,ws}^s \times \mathbb{A}^1 \to \mathscr{X}_{w,ws}^s$. Now, our local system $\mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s}$ identifies with a product $\mathscr{L}_2 \boxtimes \underline{\Bbbk}_{\mathbb{A}^1}$, for $\mathscr{L}_2$ a local system on $\mathscr{X}_{w,ws}^s$. But now the projection formula readily implies our result since the cohomology with compact support of $\mathbb{A}^1$ is given by $\Bbbk[-2]$. Now we have a nonzero adjunction map

$$\mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s} \to (\mu_{ws})^! (\mu_{ws})_! \mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s} \cong (\mu_{ws})^! \mathscr{L}_2[-2].$$

Since μ_{ws} is smooth with relative dimension 1, we have $(\mu_{ws})^! \cong (\mu_{ws})^*[2]$, so this adjunction map can be rewritten

$$\mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s} \to (\mu_{ws})^* \mathscr{L}_2 .$$

Now one can easily check on stalks that this is in fact an isomorphism (this again basically amounts to the calculation of the cohomology with compact support for $\mathbb{A}^1$). Thus we indeed have

$$(\mu_{ws})^* \mathscr{L}_2 \cong \mathscr{L}(s)_t^{ws} \boxtimes^{B_s} \mathscr{L}_t^{L_s} . \tag{8.2.6}$$

We readily deduce that the averaging $\mathrm{Av}_{t,!}(\Delta(ws)_{\mathscr{L}_t^{B_s}})$ is isomorphic to the !-extension of the shifted local system $\mathscr{L}_2[\ell(w)+1]$ on $\mathscr{X}_{w,ws}^s$ to $\mathscr{X}^s$.

We would like to see that $\mathscr{L}_1 \cong \mathscr{L}_2$; this is the subject of the following subsection.

8.2.2 Identification of the local systems

We start by showing that the restriction of $\mathscr{L}_1$ to $\mathscr{X}_w^s$ and the restriction of $\mathscr{L}_2$ to $\mathscr{X}_{ws}^s$ are what one expects:

Lemma 8.2.1. *The restriction of $\mathscr{L}_1$ to $\mathscr{X}_w^s$ is $\mathscr{L}(s)_t^w$. Similarly, the restriction of $\mathscr{L}_2$ to $\mathscr{X}_{ws}^s$ is $\mathscr{L}(s)_t^{ws}$ (i.e. the local systems we started with).*

Proof. To consider the first restriction, it suffices to pull back along the top line of diagram (8.2.3); said otherwise, to apply $e_w^* \circ \mu_w^*$. As μ_w is an isomorphism, and e_w^* is the inverse of the induction equivalence (see subsection 8.1.2, particularly lemma 8.1.2 and lemma 8.1.5), we clearly end up with $\mathscr{L}(s)_t^w$.

For the second restriction, we will use the exact same argument: once again to obtain this restriction, it suffices to pull back $\mathrm{Av}_{t,!}^s(\mathscr{L}(s)_t^{ws})$ along the top row of diagram (8.2.4). The isomorphism (8.2.6) and the above discussion allow us to conclude. □

We can consider the local system $\mathscr{L}(s)_t^{w,ws} := (p_w^{L_s})^* \mathscr{L}_t^{L_s}$ on the union of the two strata (the map $p_w^{L_s}$ has been defined in subsection 5.1.2). We will eventually show that this local system is the averaging of both local systems $\mathscr{L}(s)_t^w$ and $\mathscr{L}(s)_t^{ws}$, and that its restrictions to both of the strata are given by what one might expect, that is to say $\mathscr{L}(s)_t^w$ and $\mathscr{L}(s)_t^{ws}$.

Note that the pullback along the map $L_s \to \mathscr{X}_{w,ws}^s$ defined by $l \mapsto \dot{w} l U^s$ induces an equivalence

$$\mathrm{Loc}(\mathscr{X}_{w,ws}^s, \Bbbk) \xrightarrow{\sim} \mathrm{Loc}(L_s, \Bbbk).$$

Similarly, the map $B_s \to \mathscr{X}_w^s$ (resp. $B_s \to \mathscr{X}_{ws}^s$) given by $b \mapsto \dot{w} b U^s$ (resp. $b \mapsto \dot{w}\dot{s} b$) gives an equivalence $\mathrm{Loc}(\mathscr{X}_w^s) \xrightarrow{\sim} \mathrm{Loc}(B_s)$ (resp. $\mathrm{Loc}(\mathscr{X}_{ws}^s) \xrightarrow{\sim} \mathrm{Loc}(B_s)$).

We will need a few intermediary results. We start with general lemmas about fundamental groups of topological groups. For any path h (parametrized by $[0,1]$) in a topological space, denote by $\overline{h}$ the path defined by $\overline{h}(t) = h(1-t)$ (which is h "followed in the other direction").

Lemma 8.2.2. *Let H be a path-connected topological group. Let $g \in H$. Choose a path $\alpha : [0,1] \to H$ with $\alpha(0) = 1_H$ and $\alpha(1) = g$. We obtain a group isomorphism*

$$\begin{aligned} \varphi : \pi_1(H, g) &\longrightarrow \pi_1(H, 1), \\ [\gamma] &\mapsto [\overline{\alpha} \cdot \gamma \cdot \alpha]. \end{aligned}$$

This isomorphism does not depend on the path α.

Remark 8.2.3. Here, $\gamma \cdot \alpha$ denotes the usual composition of paths, with the following convention:

$$(\gamma \cdot \alpha)(t) = \begin{cases} \alpha(2t) & \text{if } t \in [0, 1/2] \\ \gamma(2t-1) & \text{if } t \in]1/2, 1]. \end{cases}$$

Proof. The proof is based on the known fact that the fundamental group of a topological group (at the identity element) is an abelian group (see e.g. [Hat, Chapter 3, Section 3.C, Exercise 5]). Choose another path α' with $\alpha'(0) = 1_H$ and $\alpha'(1) = g$. We obtain an isomorphism

$$\begin{aligned} \psi : \pi_1(H, 1) &\longrightarrow \pi_1(H, g), \\ [\gamma] &\mapsto [\alpha' \cdot \gamma \cdot \overline{\alpha'}]. \end{aligned}$$

The composition $\varphi \circ \psi$ is thus an automorphism of $\pi_1(H, 1)$ given by

$$[\gamma] \longmapsto [\alpha' \cdot \overline{\alpha} \cdot \gamma \cdot \alpha \cdot \overline{\alpha'}].$$

Now, $\alpha \cdot \overline{\alpha'}$ is a loop based in 1_H and $\varphi \circ \psi([\gamma]) = [\overline{\alpha \cdot \overline{\alpha'}}][\gamma][\alpha \cdot \overline{\alpha'}] = [\gamma]$ since the fundamental group is abelian. Finally, we get

$$[\overline{\alpha'} \cdot \gamma \cdot \alpha'] = [\overline{\alpha'} \cdot \overline{\alpha \cdot \overline{\alpha'}} \cdot \gamma \cdot \alpha \cdot \overline{\alpha'} \cdot \alpha'] = [\overline{\alpha} \cdot \gamma \cdot \alpha].$$

□

We keep the notation of the previous lemma: H is a path-connected topological group and g any element of H; consider any fixed path α in H satisfying $\alpha(0) = 1$ and $\alpha(1) = g$. We have a natural map $\pi_1(H, 1) \to \pi_1(H, g)$ that maps a class $[\beta]$ to the class of $[\beta_g]$, with

$$\beta_g : [0, 1] \to H, \quad t \mapsto g\beta(t).$$

(Here, $g\beta(t)$ denotes the multiplication in H of the elements g and $\beta(t)$.)

Lemma 8.2.4. *For any path γ in H based in 1, the paths γ and $\overline{\alpha} \cdot \gamma_g \cdot \alpha$ are homotopic. In particular, the functor of pullback along the map $h \mapsto gh$ induces the identity functor* $\mathrm{Loc}(H, \Bbbk) \to \mathrm{Loc}(H, \Bbbk)$.

Proof. For any $s \in [0, 1]$, let α_s be the path given by $t \mapsto \alpha(ts)$ and $\overline{\alpha}_s$ be the path given by $t \mapsto \overline{\alpha}((1-t)(1-s)+t)$. One can check that the following formula defines a homotopy between the two considered paths:

$$\eta(s, t) = (\overline{\alpha}_s \cdot (\alpha(s)\gamma) \cdot \alpha_s)(t).$$

The second point of the lemma is a consequence of the first, by virtue of theorem 1.1.3. Indeed, we have equivalences

$$\Bbbk[\pi_1(H, 1)]\text{-mod} \xleftarrow{\sim} \mathrm{Loc}(H, \Bbbk) \xrightarrow{\sim} \Bbbk[\pi_1(H, g)]\text{-mod}\,.$$

Thus we can reason entirely in terms of modules. The functor $\mathrm{Loc}(H, \Bbbk) \to \mathrm{Loc}(H, \Bbbk)$ given by pullback along the map $h \mapsto gh$ corresponds under the above equivalences to the functor $g^* : \Bbbk[\pi_1(H, g)]\text{-mod} \to \Bbbk[\pi_1(H, 1)]\text{-mod}$ that maps a $\Bbbk[\pi_1(H, g)]$-module V to the $\Bbbk[\pi_1(H, 1)]$-module V_g with action defined by

$$[\alpha] \cdot_g v = [\alpha_g] \cdot v.$$

Now for any path $\alpha : [0,1] \to H$ with $\alpha(0) = 1$ and $\alpha(1) = g$, we have a natural isomorphism $\pi_1(H,g) \xrightarrow{\sim} \pi_1(H,1)$ described in the statement of lemma 8.2.2; the same lemma tells us that this isomorphism is canonical in the sense that it does not depends on the chosen α. This gives us a canonical equivalence $\Phi : \Bbbk[\pi_1(H,g)]\text{-mod} \xrightarrow{\sim} \Bbbk[\pi_1(H,1)]\text{-mod}$. The first part of the lemma implies that the composition

$$\Bbbk[\pi_1(H,g)]\text{-mod} \xrightarrow[g^*]{\sim} \Bbbk[\pi_1(H,1)]\text{-mod} \xrightarrow[\Phi]{\sim} \Bbbk[\pi_1(H,g)]\text{-mod}$$

is the identity. This concludes the proof. □

Lemma 8.2.5. *1) A local system $\mathscr{L}$ on $\mathscr{X}^s_{w,ws}$ is uniquely determined by its restriction to $\mathscr{X}^s_w$, i.e. if $\mathscr{L}$ and $\mathscr{L}'$ are local systems on $\mathscr{X}^s_{w,ws}$, then*

$$\mathscr{L}_{|\mathscr{X}^s_w} \cong \mathscr{L}'_{|\mathscr{X}^s_w} \text{ in } \mathrm{Loc}(\mathscr{X}^s_w, \Bbbk) \Longrightarrow \mathscr{L} \cong \mathscr{L}' \text{ in } \mathrm{Loc}(\mathscr{X}^s_{w,ws}, \Bbbk).$$

2) A local system $\mathscr{L}$ on $\mathscr{X}^s_{w,ws}$ is uniquely determined by its restriction to $\mathscr{X}^s_{ws}$, i.e. if $\mathscr{L}$ and $\mathscr{L}'$ are local systems on $\mathscr{X}^s_{w,ws}$, then

$$\mathscr{L}_{|\mathscr{X}^s_{ws}} \cong \mathscr{L}'_{|\mathscr{X}^s_{ws}} \text{ in } \mathrm{Loc}(\mathscr{X}^s_{w,ws}, \Bbbk) \Longrightarrow \mathscr{L} \cong \mathscr{L}' \text{ in } \mathrm{Loc}(\mathscr{X}^s_{w,ws}, \Bbbk).$$

3) The local system $\mathscr{L}(s)^{w,ws}_t = (p^{L_s}_w)^*(\mathscr{L}^{L_s}_t)$ on $\mathscr{X}^s_{w,ws}$ satisfies

$$\mathscr{L}_{|\mathscr{X}^s_w} = \mathscr{L}(s)^w_t \quad \text{and} \quad \mathscr{L}_{|\mathscr{X}^s_{ws}} = \mathscr{L}(s)^{ws}_t.$$

Proof. 1) Let $i : B_s \hookrightarrow L_s$ be the natural inclusion. The morphism induced by this map on the fundamental groups is then given by the natural quotient map

$$\pi_1(B_s) \cong \pi_1(T) \cong \mathrm{X}_*(T) \twoheadrightarrow \mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s \cong \pi_1(L_s). \tag{8.2.7}$$

This quotient morphism in turn induces a natural functor $\Bbbk[\mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s]\text{-mod} \to \Bbbk[\mathrm{X}_*(T)]\text{-mod}$, the restriction of scalar, which corresponds under theorem 1.1.3 to the functor i^*. Now the following diagram is commutative

$$\begin{array}{ccc} \mathscr{X}^s_w & \hookrightarrow & \mathscr{X}^s_{w,ws} \\ {\scriptstyle \beta^{B_s}_w}\uparrow & & \uparrow{\scriptstyle \beta^{L_s}_w} \\ B_s & \xrightarrow{\ i\ } & L_s, \end{array} \tag{8.2.8}$$

where $\beta^{B_s}_w$ is defined to be the (right) action of B_s on the element $\dot{w}U^s$ and $\beta^{L_s}_w$ the action of L_s on the same element. The pullbacks along these maps give equivalences between the categories of local systems, thus it suffices to show that if two local systems $\mathscr{L}$ and $\mathscr{L}'$ on L_s have isomorphic restrictions to B_s then they are isomorphic in the first place. Using the discussion at the beginning of this proof, this amounts to showing that if two $\Bbbk[\mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s]$-modules coincide after restriction of scalars via the map (8.2.7), then these representations were isomorphic in the category of $\Bbbk[\mathrm{X}_*(T)/\mathbb{Z}\cdot\alpha_s]$-modules. This is obvious and concludes the proof of this point.

2) We use the same kind of arguments as in 1), but there is a slight difference. The analogue of diagram (8.2.8) with $\mathscr{X}^s_{ws}$ does not commute. Instead we must consider

$$\begin{array}{ccc} \mathscr{X}^s_{ws} & \hookrightarrow & \mathscr{X}^s_{w,ws} \\ {\scriptstyle \beta^{B_s}_{ws}}\uparrow & & \uparrow{\scriptstyle \beta^{L_s}_{ws}} \\ B_s & \xrightarrow[b\mapsto \dot{s}b]{i_s} & L_s. \end{array} \tag{8.2.9}$$

As for 1), it suffices to show that if two local systems $\mathscr{L}$ and $\mathscr{L}'$ on L_s satisfy $i_s^*\mathscr{L} \cong i_s^*\mathscr{L}'$, then we have $\mathscr{L} \cong \mathscr{L}'$. We note that the map $\widetilde{i}_{s^{-1}} : L_s \to L_s$, $l \mapsto \dot{s}^{-1}l$ is an isomorphism, so gives an auto-equivalence of category $\widetilde{i}^*_{s^{-1}} : \mathrm{Loc}(L_s, \Bbbk) \xrightarrow{\sim} \mathrm{Loc}(L_s, \Bbbk)$. Moreover, we have $\widetilde{i}_{s^{-1}} \circ i_s = i$. Now we can write $\mathscr{L} \cong \widetilde{i}^*_{s^{-1}}\widetilde{\mathscr{L}}$ and $\mathscr{L}' \cong \widetilde{i}^*_{s^{-1}}\widetilde{\mathscr{L}}'$ for unique (up to isomorphism) local systems $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{L}}'$. We thus obtain

$$i^*\widetilde{\mathscr{L}} \cong i_s^*\widetilde{i}^*_{s^{-1}}\widetilde{\mathscr{L}} \cong i_s^*\mathscr{L} \cong i_s^*\mathscr{L}' \cong i_s^*\widetilde{i}^*_{s^{-1}}\widetilde{\mathscr{L}}' \cong i^*\widetilde{\mathscr{L}}'.$$

The proof of 1) tells us that we must have $\widetilde{\mathscr{L}} \cong \widetilde{\mathscr{L}}'$, and so $\mathscr{L} \cong \mathscr{L}'$. Point 2) is now proved.

We prove 3). Firstly, we claim that the local system $(p^{L_s}_w)^*(\mathscr{L}^{L_s}_t)$ on $\mathscr{X}^s_{w,ws}$ coincides with the local system corresponding to $\mathscr{L}^{L_s}_t$ via the equivalence induced by the map $\beta^{L_s}_w$. This is a direct consequence of the commutativity of the following diagram

$$\begin{array}{ccc} L_s & \xrightarrow{\beta^{L_s}_w} & \mathscr{X}^s_{w,ws} \\ & {\scriptstyle \mathrm{id}_{L_s}}\searrow & \downarrow{\scriptstyle p^{L_s}_w} \\ & & L_s. \end{array}$$

To determine the restriction of this local system to $\mathscr{X}^s_{ws}$ or $\mathscr{X}^s_w$, we use the commutative diagrams (8.2.8) and (8.2.9). Using diagram (8.2.8), we see that the restriction of $\mathscr{L}(s)^{w,ws}_t$ to $\mathscr{X}^s_w$ is $\mathscr{L}(s)^w_t$. To determine the restriction of this local system to $\mathscr{X}^s_{ws}$, we remark that $i_s = \widetilde{i}_s \circ i$, where $\widetilde{i}_s$ denotes the map $L_s \to L_s$ of multiplication by $\dot{s}$. We invoke lemma 8.2.4 to see that pulling back along the bottom line of diagram (8.2.9) amounts to pulling back along the inclusion map $B_s \hookrightarrow L_s$ for a local system. The lemma is proved. □

Corollary 8.2.6. *We have isomorphisms*

$$\mathrm{Av}^s_{t,*}(\Delta(w)_{\mathscr{L}^{B_s}_t}[1]) \cong \mathrm{Av}^s_{t,!}(\Delta(ws)_{\mathscr{L}^{B_s}_t}) \cong (j_{w,ws})_!\,\mathscr{L}(s)^{w,ws}_t[\ell(w)+1].$$

In particular, these objects are perverse.

Proof. This is just putting lemma 8.2.1 and lemma 8.2.5 together. The assertion about perversity follows from the fact that $(j_{w,ws})_!\,\mathscr{L}(s)^{w,ws}_t[\ell(w)+1]$ is perverse ($\mathscr{X}^s_{w,ws}$ being smooth). □

Corollary 8.2.7. *We have isomorphisms*

$$\mathrm{Av}^s_{t,!}(\nabla(w)_{\mathscr{L}_t^{B_s}}[-1]) \cong \mathrm{Av}^s_{t,*}(\nabla(ws)_{\mathscr{L}_t^{B_s}}) \cong (j_{w,ws})_* \mathscr{L}(s)_t^{w,ws}[\ell(w)+1].$$

In particular, these objects are perverse.

Proof. These isomorphisms are obtained by an application of the Verdier duality functor (using lemma 8.1.7) to the isomorphisms of corollary 8.2.6 (with t^{-1} instead of t). □

Remark 8.2.8. Since $\mathrm{Av}^s_{t,*}$ and $\mathrm{Av}^s_{t,!}$ differ just by a shift by definition, one easily gets (for example) $\mathrm{Av}^s_{t,*}(\nabla(w)_{\mathscr{L}_t^{B_s}}) \cong (j_{w,ws})_*(\mathscr{L}(s)_t^{w,ws})[\ell(w)]$. The other possible cases can be determined in the same way.

8.2.3 Canonical triangle for standards

For the rest of this section, consider the following maps

$$\mathscr{X}^s_w \xhookrightarrow[\text{closed}]{i} \mathscr{X}^s_w \sqcup \mathscr{X}^s_{ws} \xhookleftarrow[\text{open}]{j} \mathscr{X}^s_{ws}.$$

Also let $j_{w,ws} : \mathscr{X}^s_w \sqcup \mathscr{X}^s_{ws} \hookrightarrow \mathscr{X}^s$ be the inclusion. According to corollary 8.2.6, we can identify the object $(j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}[\ell(w)+1]$ with $\mathrm{For}^s_t \mathrm{Av}^s_{t,!}(\Delta(ws)_{\mathscr{L}_t^{B_s}})$ and with $\mathrm{For}^s_t \mathrm{Av}^s_{t,*}(\Delta(w)_{\mathscr{L}_t^{B_s}}[1])$.

Lemma 8.2.9. *There exists a distinguished triangle*

$$\Delta(ws)_{\mathscr{L}_t^{B_s}} \to (j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}[\ell(w)+1] \to \Delta(w)_{\mathscr{L}_t^{B_s}}[1] \xrightarrow{+1} \qquad (8.2.10)$$

where the first arrow is given by the morphism

$$\Delta(ws)_{\mathscr{L}_t^{B_s}} \xrightarrow{\text{adj.}} \mathrm{For}^s_t \mathrm{Av}^s_{t,!}(\Delta(ws)_{\mathscr{L}_t^{B_s}}) \cong (j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}[\ell(w)+1]$$

and the second arrow by the morphism

$$(j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}[\ell(w)+1] \cong \mathrm{For}^s_t \mathrm{Av}^s_{t,*}(\Delta(w)_{\mathscr{L}_t^{B_s}}[1]) \xrightarrow{\text{adj.}} \Delta(w)_{\mathscr{L}_t^{B_s}}[1].$$

Proof. We apply the distinguished triangle $j_! j^* \to \mathrm{id} \to i_* i^* \xrightarrow{+1}$ to the local system $\mathscr{L}(s)_t^{w,ws}[\ell(w)+1]$. Using the third point of lemma 8.2.5 and considering its !-extension along the inclusion $j_{w,ws}$ we obtain a distinguished triangle

$$\Delta(ws)_{\mathscr{L}_t^{B_s}} \to (j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}[\ell(w)+1] \to \Delta(w)_{\mathscr{L}_t^{B_s}}[1] \xrightarrow{+1}.$$

If one applies the functor $j^* \circ j^*_{w,ws}$ to this triangle, the first morphism becomes an isomorphism, so it is in particular non-zero. The exact same argument with i^* ensures that the second morphism is non-zero too. In the following sequence

of isomorphisms, a subscript B_s (resp. L_s) on a Hom-space means that we are in the B_s-$\mathscr{L}_t^{B_s}$ (resp. L_s-$\mathscr{L}_t^{L_s}$) Lusztig–Yun category. We have

$$\begin{aligned}\mathrm{Hom}_{B_s}(\Delta(ws)_{\mathscr{L}_t^{B_s}}, \mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(ws)_{\mathscr{L}_t^{B_s}})) \\ \cong \mathrm{Hom}_{L_s}(\mathrm{Av}_{t,!}^s(\Delta(ws)_{\mathscr{L}_t^{B_s}}), \mathrm{Av}_{t,!}^s(\Delta(ws)_{\mathscr{L}_t^{B_s}})) \\ \cong \mathrm{Hom}_{L_s}((j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}, (j_{w,ws})_! \mathscr{L}(s)_t^{w,ws}).\end{aligned}$$

Consider the last Hom-space. The functor $(j_{w,ws})_!$ is fully faithful and $(p_w^{L_s})^*$ is an equivalence between the categories of local systems on L_s and $\mathscr{X}_{w,ws}^s$, so this Hom-space is actually isomorphic to the endomorphism space of $\mathscr{L}_t^{L_s}$ in the category of local systems on L_s, which is one dimensional. Thus we can assume that the first morphism in (8.2.10), which is non-zero, is given by the adjunction map $\Delta(ws)_{\mathscr{L}(s)_t^{ws}} \to \mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(ws)_{\mathscr{L}(s)_t^{B_s}})$. Similarly, one can assume that the second map in the triangle is given by the adjunction $\mathrm{For}_t^s \mathrm{Av}_{t,*}^s(\Delta(w)_{\mathscr{L}_t^{B_s}}[1]) \to \Delta(w)_{\mathscr{L}_t^{B_s}}[1]$. □

By Verdier duality, we obtain a distinguished triangle

$$\nabla(w)_{\mathscr{L}_t^{B_s}}[-1] \to (j_{w,ws})_* \mathscr{L}(s)_t^{w,ws}[\ell(w)+1] \to \nabla(ws)_{\mathscr{L}_t^{B_s}} \xrightarrow{+1}. \quad (8.2.11)$$

8.3 Convolution

8.3.1 Generalities

We study in this section the convolution, as defined in subsection 6.3.1, of standard and costandard objects in the Lusztig–Yun category.

Our first lemma in this study shows that the object $\Delta(e)_{\mathscr{L}_t^T} \cong \mathrm{IC}(e)_{\mathscr{L}_t^T} \cong \nabla(e)_{\mathscr{L}_t^T}$ is a unit for convolution.

Lemma 8.3.1. *For any $\mathscr{F} \in \mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}_t^T,-]}$, we have an isomorphism $\Delta(e)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \mathscr{F} \cong \mathscr{F}$ in the category $\mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}_t^T,-]}$. Similarly, for $\mathscr{F}' \in \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]}$, we have $\mathscr{F}' \star^{\mathrm{LY}} \Delta(e)_{\mathscr{L}_t^T} \cong \mathscr{F}'$ in $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]}$.*

Proof. We prove only the first isomorphism, the other one is very similar. It is easy to see that the object $\Delta(e)_{\mathscr{L}_t^T} \boxtimes^B \mathscr{F}$ on $G \times^B G/U$ identifies with the !-extension of the object $\mathscr{L}_t^e \boxtimes^B \mathscr{F}$ on $B \times^B \mathscr{X}$ to all of $G \times^B \mathscr{X}$. In turn, under the isomorphism $\mathscr{X} \cong B \times^B \mathscr{X}$ given by $xU \mapsto [e, xU]$, the latter object identifies simply with $\mathscr{F}$ and the action map $B \times^B \mathscr{X} \to \mathscr{X}$ identifies with the identity of $\mathscr{X}$. Thus indeed the objects $\Delta(e)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \mathscr{F}$ and $\mathscr{F}$ are isomorphic in $\mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}_t^T,-]}$ is clear. □

Lemma 8.3.2. *Assume that $w, v \in W$ are such that $\ell(wv) = \ell(w) + \ell(v)$. Consider $\mathscr{L}_t^T$ a simple local system on T. Then we have canonical isomorphisms*

$$\Delta(w)_{\mathscr{L}_{v(t)}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_t^T} \cong \Delta(wv)_{\mathscr{L}_t^T}$$

and

$$\nabla(w)_{\mathscr{L}^T_{v(t)}} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}^T_t} \cong \nabla(wv)_{\mathscr{L}^T_t}.$$

Proof. In this proof, we set $\mathscr{L} = \mathscr{L}^T_t$ and $\mathscr{L}' = \mathscr{L}^T_{v(t)}$. Let $\mathscr{L}^v$ denote $p_v^*\mathscr{L}[\ell(v)]$ with $p_v : BvB/U \cong U_v \times T \xrightarrow{\mathrm{pr}_2} T$ is the map defined in subsection 5.1.1. We will use a similar notation for other $w \in W$ or other local systems on T. By definition, we have $\Delta(w)_{\mathscr{L}'} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}} = (m_B)_!(\Delta(w)_{\mathscr{L}'} \boxtimes^B \Delta(v)_{\mathscr{L}})$ with $q_B^*(\Delta(w)_{\mathscr{L}'} \boxtimes^B \Delta(v)_{\mathscr{L}}) = \varpi^* \Delta(w)_{\mathscr{L}'} \boxtimes \Delta(v)_{\mathscr{L}}$. Using the base change theorem, one can show that we have in fact

$$\Delta(w)_{\mathscr{L}'} \boxtimes^B \Delta(v)_{\mathscr{L}} = (j_{w \times^B v})_!((\mathscr{L}')^w \boxtimes^B \mathscr{L}^v),$$

where $j_{w\times^B v} : BwB \times^B BvB/U \hookrightarrow G \times^B G/U$ is the inclusion map. Let $m_{w,v} : BwB \times^B BvB/U \to BwvB/U$ be the restriction of m_B; this map is an isomorphism. Using the commutative diagram

$$\begin{array}{ccc} BwB \times^B BvB/U & \xrightarrow[\sim]{m_{w,v}} & BwvB/U \\ \downarrow{\scriptstyle j_{w\times^B v}} & & \downarrow{\scriptstyle j_{wv}} \\ G \times^B G/U & \xrightarrow{m_B} & G/U, \end{array}$$

we need to determine $(m_{w,v})_!((\mathscr{L}')^w \boxtimes^B \mathscr{L}^v)$.

We use the following commutative diagram

$$\begin{array}{ccccc} BwB \times BvB/U & \longrightarrow & BwB \times^B BvB/U & \xrightarrow[\sim]{m_{w,v}} & BwvB/U \\ \wr\uparrow & & \wr\uparrow & & \wr\uparrow \\ U_w \times U \times T \times U_v \times T & \longrightarrow & U_w \times U_v \times T & \xrightarrow[\sim]{} & U_{wv} \times T. \end{array} \tag{8.3.1}$$

The vertical isomorphisms are given, from left to right, respectively by

- $(u, u', t, u'', t') \mapsto (u\dot{w}u't, u''\dot{v}t'U)$
- $(u, u', t) \mapsto [u\dot{w}, u'\dot{v}tU]$
- $(u, t) \mapsto u\dot{w}\dot{v}tU$.

The bottom horizontal arrows are given, from left to right, respectively by

- $(u, u', t, u'', t') \mapsto (u, \phi(u', t, u''), v^{-1}(t)t')$
- $(u, u', t) \mapsto (u\dot{w}u'\dot{w}^{-1}, t)$.

Here, $\phi : U \times T \times U_v \to U_v$ is a certain morphism of algebraic varieties. Let us give some more details: the product morphism $U \cong U_v \times U_{vw_\circ}$ gives an isomorphism (see [Sp, Lemma 8.3.5]). We would like to write that the considered map sends (u, u', t, u'', t') to $(u, u'tu''t^{-1}, v^{-1}(t)t')$, but $u'tu''t^{-1}$ has no reason

to be in U_v. Instead, this is an element of U, but thanks to our isomorphism $U \cong U_v \times U_{vw_\circ}$, we can write it uniquely as a product u^3u^4 with $u^3 \in U_v$ and $u^4 \in U_{vw_\circ}$. Then ϕ is given by the (algebraic) association $(u', t, u'') \mapsto u^3$.

Let $q_{w\times v}$ denote the restriction of q_B to $BwB \times BvB/U$. It is easy to see from the definitions that $(q_{w\times v})^*((\mathscr{L}')^w \boxtimes^B \mathscr{L}^v)$ identifies with the object $\underline{\Bbbk}_{U_w} \boxtimes \underline{\Bbbk}_U \boxtimes \mathscr{L}' \boxtimes \underline{\Bbbk}_{U_v} \boxtimes \mathscr{L}[\ell(wv)]$ on the bottom left product. We must then find the unique object on $U_w \times U_v \times T$ whose pullback under the left bottom horizontal map is this external product. But using remark 3.3.2 and the definition of $\mathscr{L}' = v(\mathscr{L}) = (v^{-1})^*\mathscr{L}$, we get that this object is $\underline{\Bbbk}_{U_w} \boxtimes \underline{\Bbbk}_{U_v} \boxtimes \mathscr{L}[\ell(wv)]$. Thus, using the right square in the diagram, we obtain that $(m_{w,v})_!((\mathscr{L}')^w \boxtimes^B \mathscr{L}^v)$ corresponds to $\underline{\Bbbk}_{U_{wv}} \boxtimes \mathscr{L}[\ell(wv)]$ under the right vertical isomorphism, i.e. is in fact $\mathscr{L}^{wv}$ (here we use our convention on liftings $\dot{w}$, see §5.1.1). This concludes the proof of the first isomorphism. The second isomorphism follows by similar arguments. □

The following is an immediate consequence:

Corollary 8.3.3. *Consider $w \in W$ and $(s_1, \dots, s_r)$ a reduced expression of w. For all $1 \le i < r$, let $\mathscr{L}_t^i$ be the local system $s_i s_{i+1} \cdots s_{r-1}(\mathscr{L}_t^T)$. Then one has*

$$\Delta(w)_{\mathscr{L}_t^T} \cong \Delta(s_1)_{\mathscr{L}_t^2} \star^{\mathrm{LY}} \cdots \star^{\mathrm{LY}} \Delta(s_r)_{\mathscr{L}_t^T};$$

and

$$\nabla(w)_{\mathscr{L}_t^T} \cong \nabla(s_1)_{\mathscr{L}_t^2} \star^{\mathrm{LY}} \cdots \star^{\mathrm{LY}} \nabla(s_r)_{\mathscr{L}_t^T}.$$

Another usual feature that we would like the Δ's and ∇'s to satisfy is the following:

$$\nabla(w^{-1})_{\mathscr{L}^T_{w(t)}} \star^{\mathrm{LY}} \Delta(w)_{\mathscr{L}_t^T} \cong \Delta(e)_{\mathscr{L}_t^T} \cong \Delta(w^{-1})_{\mathscr{L}^T_{w(t)}} \star^{\mathrm{LY}} \nabla(w)_{\mathscr{L}_t^T}.$$

Thanks to corollary 8.3.3, in order to prove this, we just need to treat the case $w = s$, a simple reflection in W. This is the object of the following sections. We will need to distinguish the cases $s \in W_t^\circ$ and $s \notin W_t^\circ$.

8.3.2 The case $s \notin W_t^\circ$

We prove the following result.

Lemma 8.3.4. *Assume that $s \notin W_t^\circ$. Then we have canonical ismorphisms*

$$\nabla(s)_{\mathscr{L}^T_{s(t)}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^T} \cong \Delta(e)_{\mathscr{L}_t^T} \cong \Delta(s)_{\mathscr{L}^T_{s(t)}} \star^{\mathrm{LY}} \nabla(s)_{\mathscr{L}_t^T}.$$

Proof. Some remarks first: we may replace G by its Levi subgroup L_s with roots $\pm\alpha_s$, thus we assume that G has semisimple rank one (from now on, we will just write α for α_s). In particular, we have $U = U_\alpha = U_s \cong \mathbb{A}^1$. We will use the following identifications: we have an isomorphism $\mathbb{A}^1 \times \mathbb{A}^1 \times T \times \mathbb{A}^1 \times T \cong BsB \times BsB/U$ given by

$$(x, a, t, y, t') \mapsto (u_\alpha(x)\dot{s}u_\alpha(a)t, u_\alpha(y)\dot{s}t'U). \tag{8.3.2}$$

Similarly, $\mathbb{A}^1 \times \mathbb{A}^1 \times T \cong BsB \times^B BsB/U$ via

$$(x, y, t) \mapsto [u_\alpha(x)\dot{s},\, u_\alpha(y)\dot{s}tU]. \tag{8.3.3}$$

The strategy is to determine the stalk of $\nabla(s)_{\mathscr{L}^T_{s(t)}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_t}$ at each point ξ of G/U. By definition, the stalk at ξ of this object is given by

$$\mathbf{H}^\bullet_c(m_B^{-1}(\xi); (\nabla(s)_{\mathscr{L}^T_{s(t)}} \boxtimes^B \Delta(s)_{\mathscr{L}^T_t})_{|m_B^{-1}(\xi)}).$$

We use the fact that under our assumption, we have $\nabla(s)_{\mathscr{L}^T_{s(t)}} \cong \Delta(s)_{\mathscr{L}^T_{s(t)}}$ thanks to remark 7.6.2 (and lemma 7.6.1); similarly to the proof of lemma 8.3.2, we find that

$$\nabla(s)_{\mathscr{L}^T_{s(t)}} \boxtimes^B \Delta(s)_{\mathscr{L}^T_t} = (j_{s\times^B s})_!(\mathscr{L}^s_{s(t)} \boxtimes^B \mathscr{L}^s_t),$$

where $j_{s\times^B s} : BsB \times^B BsB/U \hookrightarrow G \times^B G/U$ is the inclusion map. Thus we need only consider the fibers of $m_{s,s} := m_B \circ j_{s\times^B s}$.

Our first step is to determine the $m_{s,s}^{-1}(\xi)$'s. We consider the morphism $\varphi_\alpha : SL_2 \to G$ introduced in subsection 5.3.2. This allows us to make our computation in SL_2: using the Bruhat decomposition in this group, one can determine the action of the map $m_{s,s}$ on $[u_\alpha(x)\dot{s}; u_\alpha(y)\dot{s}tU]$. If $y \neq 0$, then we get

$$u_\alpha(x - y^{-1})\dot{s}\alpha^\vee(y)\dot{s}^2tU.$$

If $y = 0$, then we just get

$$u_\alpha(x)\dot{s}^2tU = \dot{s}^2tU.$$

We can now give $m_{s,s}^{-1}(\xi)$ for ξ a point in G/U. We distinguish two cases.

Assume first that $\xi \in B/U$. Consider an element $t' \in T$, representing ξ. We see then that under the isomorphism (8.3.3), we need y to be zero and $t = \dot{s}^{-2}t'$. There are no conditions on x however. Thus

$$m_{s,s}^{-1}(\xi) \cong \{(x, 0, \dot{s}^{-2}t') \mid x \in \mathbb{A}^1\} \cong \mathbb{A}^1\,. \tag{8.3.4}$$

Assume now that $\xi \in BsB/U$. We can write uniquely $\xi = u_\alpha(a)\dot{s}t'U$ with $a \in \mathbb{A}^1$ and $t' \in T$. Using the preceding discussion, we obtain

$$m_{s,s}^{-1}(\xi) \cong \{(a + y^{-1}, y, \alpha^\vee(y^{-1})\dot{s}^{-2}t') \mid y \in \mathbb{A}^1 \smallsetminus \{0\}\} \cong \mathbb{A}^1 \smallsetminus \{0\}. \tag{8.3.5}$$

The next step is to identify the sheaf $\mathscr{L}^s_{s(t)} \boxtimes^B \mathscr{L}^s_t$ that lives on $BsB \times^B BsB/U$ with something on $\mathbb{A}^1 \times \mathbb{A}^1 \times T$ (using the isomorphism (8.3.3)). To do so, we use the definition of $\mathscr{L}^s_{s(t)} \boxtimes^B \mathscr{L}^s_t$: it is the only object whose pullback along the projection $BsB \times BsB/U \to BsB \times^B BsB/U$ is $\varpi_s^* \mathscr{L}^s_{s(t)} \boxtimes \mathscr{L}^s_t$ (where ϖ_s is the restriction of $\varpi : G \to G/U$ to BsB). Now the latter object identifies clearly with $\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathscr{L}^T_{s(t)} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathscr{L}^T_t[2]$, by definition of $\mathscr{L}^s_t$. Thus we

need to determine two things: first, the map f that fits in the left column of the following diagram:

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{A}^1 \times T \times \mathbb{A}^1 \times T & \xrightarrow{\sim} & BsB \times BsB/U \\ \downarrow{\scriptstyle f} & & \downarrow \\ \mathbb{A}^1 \times \mathbb{A}^1 \times T & \xrightarrow{\sim} & BsB \times^B BsB/U; \end{array}$$

then, an object (in fact the unique object) on $\mathbb{A}^1 \times \mathbb{A}^1 \times T$ whose pullback along f is $\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathscr{L}^T_{s(t)} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathscr{L}^T_t[2]$. The map f is a particular case (using the isomorphism $\mathbb{A}^1 \cong U$) of the map given by the left bottom arrow of diagram (8.3.1), but it is this time explicitly computable; it maps (x, a, t, y, t') to $(x, a + \alpha(t)y, s(t)t')$. Consider the object $\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathscr{L}^T_t[2]$ on $\mathbb{A}^1 \times \mathbb{A}^1 \times T$. Its pullback under f is

$$\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes (s \times \mathrm{id}_T)^* m_T^* \mathscr{L}^T_t[2].$$

Using the fact that $\mathscr{L}^T_t$ is multiplicative (see remark 3.3.2), we finally get that this object is the one we were looking for.

We reduced our problem to the one of calculating cohomology with compact support of an explicit object on explicit subspaces of $\mathbb{A}^1 \times \mathbb{A}^1 \times T$.

We deal first with the case $\xi \in BsB/U$. If we pull back the restriction to $m_{s,s}^{-1}(\xi)$ of $\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathscr{L}^T_t[2]$ to $\mathbb{A}^1 \setminus \{0\}$ along the isomorphism (8.3.5), we obtain (up to a translation by $\dot{s}^{-2}t'$ and to a shift) the local system $(\alpha_s^\vee)^* \mathscr{L}^T_t = \mathscr{L}^T_{\alpha_s^\vee(t)}$. By hypothesis, this local system is nontrivial, so using lemma 3.2.1, we see that its cohomology with compact support vanishes.

Now using the same ideas for $\xi \in B/U$, what we get is clearly $\mathbf{H}^\bullet_c(\mathbb{A}^1, \underline{\Bbbk}_{\mathbb{A}^1}[2]) = \Bbbk$. We deduce that $\nabla(s)_{\mathscr{L}^T_{s(t)}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_t}$ is supported on B/U, in fact it is the extension by zero of some object $\mathscr{Q}$ in $\mathfrak{D}(\mathscr{X}_e /\!\!/ T)_{[-,\mathscr{L}^T_t]} = \mathfrak{D}(T /\!\!/ T)_{[\mathscr{L}^T_t]}$. Moreover, the preceding arguments tell us that $\mathscr{Q}$ is the image of $\Bbbk$ under the equivalence

$$\mathfrak{D}(T /\!\!/ T)_{[\mathscr{L}^T_t]} \cong D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk},$$

so we get $\mathscr{Q} = \mathscr{L}^T_t$. We finally proved that we have isomorphisms

$$\nabla(s)_{\mathscr{L}^T_{s(t)}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_t} \cong \Delta(e)_{\mathscr{L}^T_t} \cong \Delta(s)_{\mathscr{L}^T_{s(t)}} \star^{\mathrm{LY}} \nabla(s)_{\mathscr{L}^T_t}.$$

□

Corollary 8.3.5. *Fix $t \in T^\vee_{\Bbbk}$. Assume that $s \notin W^\circ_t$. Then:*

1) convolution on the left with $\nabla(s)_{\mathscr{L}^T_t} \cong \mathrm{IC}(s)_{\mathscr{L}^T_t} \cong \Delta(s)_{\mathscr{L}^T_t}$ induces an equivalence

$$\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_t, -]} \xrightarrow{\sim} \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}^T_{s(t)}, -]}$$

of triangulated categories, and is t-exact for the perverse t-structure. For any $t' \in W \cdot t$ and any $w \in {}_tW_{t'}$, this equivalence sends $\Delta(w)_{\mathscr{L}^T_{t'}}$ to $\Delta(sw)_{\mathscr{L}^T_{t'}}$, $\nabla(w)_{\mathscr{L}^T_{t'}}$ to $\nabla(sw)_{\mathscr{L}^T_{t'}}$ and $\mathrm{IC}(w)_{\mathscr{L}^T_{t'}}$ to $\mathrm{IC}(sw)_{\mathscr{L}^T_{t'}}$.

2) convolution on the right with $\nabla(s)_{\mathscr{L}_t^T} \cong \mathrm{IC}(s)_{\mathscr{L}_t^T} \cong \Delta(s)_{\mathscr{L}_t^T}$ *induces an equivalence*

$$\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_{s(t)}^T]}$$

of triangulated categories, and is t-exact for the perverse t-structure. For any $w \in W$*, this equivalence sends* $\Delta(w)_{\mathscr{L}_t^T}$ *to* $\Delta(ws)_{\mathscr{L}_{s(t)}^T}$*,* $\nabla(w)_{\mathscr{L}_t^T}$ *to* $\nabla(ws)_{\mathscr{L}_{s(t)}^T}$ *and* $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ *to* $\mathrm{IC}(ws)_{\mathscr{L}_{s(t)}^T}$*.*

Proof. We prove only 1); the proof of 2) is similar. The fact that our functor gives an equivalence $\mathfrak{D}(T\backslash\!\!\backslash \mathscr{X})_{[\mathscr{L}_t^T,-]} \xrightarrow{\sim} \mathfrak{D}(T\backslash\!\!\backslash \mathscr{X})_{[\mathscr{L}_{s(t)}^T,-]}$ follows directly from lemma 8.3.4 (and the fact that $\Delta(e)_{\mathscr{L}_t^T}$ is neutral for convolution). Corollary 8.3.3 tells us that the image of standard and costandard are the prescribed ones. It then follows that this equivalence is exact for the perverse t-structures. Finally, since this equivalence preserves the standard and costandard objects, it preserves the IC sheaves as in the statement of the lemma. □

Before giving and proving the analogue of lemma 8.3.4 in the case $s \in W_t^\circ$, we need some preliminaries, given in the next subsection.

8.3.3 The case $s \in W_t^\circ$

According to corollary 6.3.3, we can prove the case $s \in W_t^\circ$ of lemma 8.3.4 in the slightly different setting of the B_s-equivariant Lusztig–Yun category on $\mathscr{X}^s$.

We give a key-result:

Lemma 8.3.6. *Assume that* $s \in W_t^\circ$ *and take* $\mathscr{F} \in \mathfrak{D}(\mathscr{X}^s /\!\!/ B_s)_{[-,\mathscr{L}_t^{B_s}]}$*. Then* $\mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} = (j_{e,s})_* \mathscr{L}(s)_t^{e,s}[1]$ *and we have a functorial isomorphism*

$$\mathscr{F} \star_{B_s}^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} \cong \mathrm{For}_t^s(\mathrm{Av}_{t,*}^s(\mathscr{F}))[1].$$

Moreover, this isomorphisms fits in the following commutative diagram

$$\begin{array}{ccc} \mathscr{F} \star_{B_s}^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} & \xrightarrow{\mathscr{F} \star_{B_s}^{\mathrm{LY}} \mathrm{adj.}} & \mathscr{F} \star_{B_s}^{\mathrm{LY}} \mathrm{IC}(e)_{\mathscr{L}_t^{B_s}}[1] \\ \wr \downarrow & & \downarrow \wr \\ \mathrm{For}_t^s(\mathrm{Av}_{t,*}^s(\mathscr{F}))[1] & \xrightarrow{\mathrm{adj.}} & \mathscr{F}[1] \end{array} \tag{8.3.6}$$

where the right vertical isomorphism follows from the fact that $\mathrm{IC}(e)_{\mathscr{L}_t^{B_s}}$ *is the unit for convolution, and the top horizontal map comes from the adjunction* $\mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} \to (j_e)_*(j_e)^* \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} \cong \mathrm{IC}(e)_{\mathscr{L}_t^{B_s}}$*.*

Proof. The first statement is clear since $\mathscr{X}_{e,s}^s$ is a smooth variety.

By definition, we have $\mathscr{F} \star_s^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} = (m_B^s)_*(\mathscr{F} \boxtimes_s^B (j_{e,s})_* \mathscr{L}(s)_t^{e,s}[1])$ (note that m_B^s is a proper map, so that $(m_B^s)_* \cong (m_B^s)_!$). Using usual arguments, one can show that this is in fact equal to

$$(m_{B|G\times^B P_s/U^s}^s)_*(\mathscr{F} \boxtimes_s^B \mathscr{L}(s)_t^{e,s})[1],$$

where $\mathscr{F} \boxtimes_s^B \mathscr{L}(s)_t^{e,s}$ is viewed as an object on $G \times^B P_s/U^s$. We consider the natural isomorphism of varieties $\mathscr{X}^s \times^{B_s} L_s \xrightarrow{\sim} G \times^B P_s/U^s$ given by $[gU^s, l] \longmapsto [g, lU^s]$ (note that this is well defined since L_s normalizes U^s). The pullback of $\mathscr{F} \boxtimes_s^B \mathscr{L}(s)_t^{e,s}$ along this isomorphism is clearly given by $\mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{L_s}$. Thus, our convolution product identifies with the pushforward of $\mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{L_s}$ along the map induced by the action of L_s on the right of $\mathscr{X}^s$; this is exactly the definition of the averaging functor. If we only remember the $\widetilde{B_s}$-equivariant structure, we obtain the result.

We proceed to obtain the commutative square. Under the isomorphisms $\mathscr{X}^s \times^{B_s} L_s \xrightarrow{\sim} \mathscr{X} \times^B P_s$ and $\mathscr{X}^s \times^{B_s} B_s \xrightarrow{\sim} \mathscr{X}^s \times^B B$, the map

$$\mathscr{F} \boxtimes^{B_s} \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} \to \mathscr{F} \boxtimes^{B_s} \mathrm{IC}(e)_{\mathscr{L}_t^{B_s}}[1]$$

induced by adjunction identifies with (a shift of) the adjunction morphism

$$\mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{L_s} \to Q_{h*} Q_h^*(\mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{L_s}),$$

where h is the natural inclusion $\mathscr{X}^s \times^{B_s} B_s \hookrightarrow \mathscr{X}^s \times^{B_s} L_s$. Now,

$$Q_h^*(\mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{L_s}) \cong \mathscr{F} \boxtimes^{B_s} \mathscr{L}_t^{B_s},$$

and with arguments similar to those used in the proof of lemma 8.3.1, the pushforward of the latter object along the natural map $\mathscr{X}^s \times^{B_s} B_s \to \mathscr{X}^s$ identifies canonically with $\mathscr{F}$. This concludes the proof of the lemma. □

Lemma 8.3.7. *Assume that $s \in W_t^\circ$. Then we have canonical isomorphisms*

$$\nabla(s)_{\mathscr{L}_t^{B_s}} \star_{B_s}^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^{B_s}} \cong \Delta(e)_{\mathscr{L}_t^{B_s}} \cong \Delta(s)_{\mathscr{L}_t^{B_s}} \star_{B_s}^{\mathrm{LY}} \nabla(s)_{\mathscr{L}_t^{B_s}}.$$

Proof. We consider the triangle (8.2.10) for $w = e$:

$$\Delta(s)_{\mathscr{L}_t^{B_s}} \to \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} \to \Delta(e)_{\mathscr{L}_t^{B_s}}[1] \xrightarrow{+1}.$$

Note that the second morphism is induced by the adjunction $\mathrm{id} \to (j_e)_* j_e^*$. Apply the functor $\nabla(s)_{\mathscr{L}_t^{B_s}} \star_{B_s}^{\mathrm{LY}} (-)$ to this triangle to get, using lemma 8.3.6,

$$\nabla(s)_{\mathscr{L}_t^{B_s}} \star_{B_s}^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^{B_s}} \to \mathrm{For}_t^s(\mathrm{Av}_{t,*}^s(\nabla(s)_{\mathscr{L}_t^{B_s}}))[1] \to \nabla(s)_{\mathscr{L}_t^{B_s}}[1] \xrightarrow{+1}.$$

The second morphism is nonzero thanks to lemma 8.3.6 again and we can assume that it is induced by adjunction. Comparing this triangle with (8.2.11) (and using lemma 8.2.7), we get

$$\nabla(s)_{\mathscr{L}_t^{B_s}} \star_{B_s}^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^{B_s}} \cong \Delta(e)_{\mathscr{L}_t^{B_s}}.$$

The first isomorphism can be proven similarly, or can be obtained by an application of Verdier duality. □

Using the fact that the equivalence (5.3.3) is monoidal for convolution on both sides thanks to lemma 6.3.2, we finally get the following.

Corollary 8.3.8. *Assume that $s \in W_t^\circ$. Then we have canonical isomorphisms*

$$\nabla(s)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^T} \cong \Delta(e)_{\mathscr{L}_t^T} \cong \Delta(s)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \nabla(s)_{\mathscr{L}_t^T}.$$

Putting together lemma 8.3.2, lemma 8.3.4 and corollary 8.3.8, we get immediately the announced result:

Lemma 8.3.9. *Fix $t \in T^\vee_{\Bbbk}$ and $w \in W$. Then we have*

$$\nabla(w^{-1})_{\mathscr{L}^T_{w(t)}} \star^{\mathrm{LY}} \Delta(w)_{\mathscr{L}_t^T} \cong \Delta(e)_{\mathscr{L}_t^T} \cong \Delta(w^{-1})_{\mathscr{L}^T_{w(t)}} \star^{\mathrm{LY}} \nabla(w)_{\mathscr{L}_t^T}.$$

8.4 The case of pro-standard and pro-costandard objects

Lemma 8.4.1. *For any $\mathscr{F} \in D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t,\cdot]}$, we have*

$$\widehat{\Delta}_{e,t} \star^U \mathscr{F} \cong \mathscr{F}.$$

Proof. We easily obtain

$$\widehat{\Delta}_{e,t} \star^U \mathscr{F} \cong a_!(\widehat{\mathscr{L}_t^T}[r] \boxtimes \mathscr{F})[r]$$

where $a : T \times \mathscr{X} \to \mathscr{X}$ is the action morphism $(t, gU) \mapsto tgU$. Using arguments as in [BeR, Lemma 7.6] and lemma 7.7.5, we obtain an isomorphism

$$\widehat{\Delta}_{e,t} \star^U \mathscr{F} \cong \mathscr{F}.$$

□

Lemma 8.4.2. *Take w any element in W. Then we have*

$$\widehat{\Delta}_{w^{-1},w(t)} \mathbin{\widehat{\star}} \widehat{\nabla}_{w,t} \cong \widehat{\Delta}_{e,t} \cong \widehat{\nabla}_{w^{-1},w(t)} \mathbin{\widehat{\star}} \widehat{\Delta}_{w,t}.$$

If $v \in W$ is such that $\ell(wv) = \ell(w) + \ell(v)$, then

1. $\widehat{\Delta}_{w,v(t)} \mathbin{\widehat{\star}} \widehat{\Delta}_{v,t} \cong \widehat{\Delta}_{wv,t}$,
2. $\widehat{\nabla}_{w,v(t)} \mathbin{\widehat{\star}} \widehat{\nabla}_{v,t} \cong \widehat{\nabla}_{wv,t}$.

Proof. We prove (arbitrarily) only the isomorphism (1), the other ones being similar. According to lemma 7.1.5 and lemma 7.7.4, we have the following sequence of isomorphisms

$$\begin{aligned}
\pi^t_\dagger(\widehat{\Delta}_{w,v(t)} \mathbin{\widehat{\star}} \widehat{\Delta}_{v,t}) &\cong \widehat{\Delta}_{w,v(t)} \mathbin{\widehat{\star}} \pi^t_\dagger(\widehat{\Delta}_{v,t}) \\
&\cong \widehat{\Delta}_{w,v(t)} \mathbin{\widehat{\star}} \Delta(v)_{\mathscr{L}_t^T} \\
&\cong \pi^{v(t)}_\dagger(\widehat{\Delta}_{w,v(t)}) \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_t^T} \\
&\cong \Delta(w)_{\mathscr{L}^T_{v(t)}} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_t^T} \\
&\cong \Delta(wv)_{\mathscr{L}_t^T}.
\end{aligned}$$

We then use lemma 7.8.9 to conclude that $\widehat{\Delta}_{w,v(t)} \widehat{\star} \widehat{\Delta}_{v,t}$ is perverse, admits a $\widehat{\Delta}$-filtration with just one subquotient equal to $\widehat{\Delta}_{wv,t}$, so is isomorphic to the latter object. □

Lemma 8.4.3. *Consider $w, v \in W$ and an element $t \in T^\vee_{\Bbbk}$. Assume that we have $\ell(w) + \ell(v) = \ell(wv)$, and let t' be $v^{-1}(t)$. Then we have:*

1. $\widehat{\Delta}_{w,t} \widehat{\star} \Delta_{v,t'} \cong \Delta_{wv,t'}$ *and* $\widehat{\nabla}_{w,t} \widehat{\star} \nabla_{v,t'} \cong \nabla_{wv,t'}$,
2. $\widehat{\Delta}_{w,t} \widehat{\star} \nabla_{w^{-1},w(t)} \cong \Delta_{e,w(t)} \cong \widehat{\nabla}_{w,t} \widehat{\star} \Delta_{w^{-1},w(t)}$.

Proof. We prove the first isomorphism, the other ones can be proved similarly. We use the fact that for any $w \in W$ and $t \in T^\vee_{\Bbbk}$, we have $\Delta_{w,t} = \mathrm{For}(\Delta(w)_{\mathscr{L}^T_t})$, where For denotes the forgetful functor (see lemma 4.2.2)

$$\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}^T_{w(t)}, \mathscr{L}^T_t]} \to D^b_{(B)}(\mathscr{X}, \Bbbk)_{[\underline{w(t)}, \underline{t}]}.$$

Thus lemma 7.1.5 gives us

$$\begin{aligned}
\widehat{\Delta}_{w,t} \widehat{\star} \Delta_{v,t'} &= \mathrm{For}(\widehat{\Delta}_{w,t} \widehat{\star} \mathrm{For}(\Delta(v)_{\mathscr{L}^T_{t'}})) \\
&\cong \mathrm{For}(\pi^t_\dagger(\widehat{\Delta}_{w,t}) \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}^T_{t'}}) \\
&\cong \mathrm{For}(\Delta(w)_{\mathscr{L}^T_t} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}^T_{t'}}) \\
&\cong \mathrm{For}(\Delta(wv)_{\mathscr{L}^T_{t'}}) \\
&\cong \Delta_{wv,t'} .
\end{aligned}$$

□

8.5 Blocks

We use the notion of block from subsection 5.1.3; we keep the notation introduced there. Consider $\beta \in {}_{t'}\underline{W}_t$. We define a full subcategory of $D^b_{(B)}(\mathscr{X}, \Bbbk)_{[t',t]}$ by

$$D^\beta_{[t',t]} := D^b_{(B)}(\mathscr{X}, \Bbbk)^\beta_{[t',t]} := \langle \Delta_{w,t}[n] \mid n \in \mathbb{Z},\ w \in \beta \rangle.$$

Similarly, one sets

$$\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^\beta_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_t]} := \langle \Delta(w)_{\mathscr{L}^T_t}[n] \mid n \in \mathbb{Z},\ w \in \beta \rangle.$$

Recall from lemma 5.1.2 that we can consider the product of two blocks.

Lemma 8.5.1. *Take $t, t', t'' \in T^\vee_{\Bbbk}$ in the same W-orbit, and two blocks $\beta \in {}_t\underline{W}_{t'}$ and $\gamma \in {}_{t'}\underline{W}_{t''}$. Then for any objects $\mathscr{F} \in \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^\beta_{[\mathscr{L}^T_t, \mathscr{L}^T_{t'}]}$ and $\mathscr{G} \in \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^\gamma_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_{t''}]}$, we have*

$$\mathscr{F} \star \mathscr{G} \in \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^{\beta\gamma}_{[\mathscr{L}^T_t, \mathscr{L}^T_{t''}]}.$$

Proof. By definition of a block $\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)^{\beta}_{[\mathscr{L}^T_t, \mathscr{L}^T_{t'}]}$, it suffices to show that

$$\Delta(w_1)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_2)_{\mathscr{L}^T_{t''}} \in \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)^{\beta\gamma}_{[\mathscr{L}^T_t, \mathscr{L}^T_{t''}]}$$

for any $w_1 \in \beta$ and $w_2 \in \gamma$. As in [LY, Proposition 4.9], we proceed by induction on $\ell(w_1) + \ell(w_2)$ (thus allowing the elements t, t', t'' and the blocks β and γ to vary). If $\ell(w_1) + \ell(w_2) = 0$, then $w_1 = w_2 = e$ and the result is clear. Now assume that the lemma is true for all elements $w_1 \in \beta$ and $w_2 \in \gamma$ satisfying $\ell(w_1) + \ell(w_2) \leq n$. Consider such a pair of elements, but with $\ell(w_1) + \ell(w_2) = n + 1$. If $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ then lemma 8.3.2 tells us that $\Delta(w_1)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_2)_{\mathscr{L}^T_{t''}} \cong \Delta(w_1 w_2)_{\mathscr{L}^T_{t''}}$, and thus the result is clear. Now assume that $\ell(w_1) + \ell(w_2) > \ell(w_1 w_2)$. We can assume that there exists a simple reflection $s \in W$ such that $w_1 = w_1' s$ and $w_2 = s w_2'$, with $w_1' < w_1$ and $w_2' < w_2$. Indeed, consider a reduced expression $w_2 = s_1 \cdots s_r$; we can consider the largest $n < r$ such that $\ell(w_1 s_1 \cdots s_n) = \ell(w_1) + n$, then replace w_1 by $w_1 s_1 \cdots s_n$ and w_2 by $s_{n+1} \cdots s_r$. Thus the product $\Delta(w_1)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_2)_{\mathscr{L}^T_{t''}}$ is isomorphic to

$$\Delta(w_1')_{\mathscr{L}^T_{s(t')}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_{s(t')}} \star^{\mathrm{LY}} \Delta(w_2')_{\mathscr{L}^T_{t''}}.$$

If $s \notin W^\circ_{t'}$, lemma 8.3.4 and corollary 8.3.5 imply that $\Delta(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_{s(t')}} \cong \Delta(e)_{\mathscr{L}^T_{s(t')}}$, and this fact together with the induction hypothesis settle the problem, since $\ell(w_1') + \ell(w_2') = n - 1$. If $s \in W^\circ_{t'}$, then $\Delta(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_{s(t')}} = \Delta(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_{t'}}$ lies in the subcategory generated by $\Delta(e)_{\mathscr{L}^T_{t'}}$ and $\Delta(s)_{\mathscr{L}^T_{t'}}$. (Indeed, the distinguished triangle (8.2.10) and lemma 8.2.9 implies that the object $\mathrm{IC}(s)_{\mathscr{L}^T_{t'}}$ lies in this subcategory, now convolve the triangle (8.2.10) with $\Delta(s)_{\mathscr{L}^T_{t'}}$ and use lemma 8.2.9 again). Therefore, $\Delta(w_1)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_2)_{\mathscr{L}^T_{t''}}$ lies in the subcategory generated by the objects

$$\Delta(w_1')_{\mathscr{L}^T_{s(t')}} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_2')_{\mathscr{L}^T_{t''}} \cong \Delta(w_1' s)_{\mathscr{L}^T_{s(t')}} \star^{\mathrm{LY}} \Delta(w_2')_{\mathscr{L}^T_{t''}}$$

and

$$\Delta(w_1')_{\mathscr{L}^T_{s(t')}} \star^{\mathrm{LY}} \Delta(e)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_2')_{\mathscr{L}^T_{t''}} \cong \Delta(w_1')_{\mathscr{L}^T_{s(t')}} \star^{\mathrm{LY}} \Delta(w_2')_{\mathscr{L}^T_{t''}}$$

to which we can apply the induction hypothesis to conclude the proof. □

Proposition 8.5.2. *Take t, t' in the same W-orbit in $T^\vee_{\Bbbk}$ and choose $\beta \in {}_t\underline{W}_{t'}$. Then the canonical maps*

$$\Delta(w_\beta^{\min})_{\mathscr{L}^T_{t'}} \to \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}^T_{t'}} \to \nabla(w_\beta^{\min})_{\mathscr{L}^T_{t'}}$$

are isomorphisms. Moreover, for t'' another element in the W-orbit of t and $\gamma \in {}_{t'}\underline{W}_{t''}$ a bloc, $\Delta(w_\beta^{\min})_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} (-)$ defines a t-exact (for the perverse t-structure) equivalence of categories

$$\mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)^{\gamma}_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_{t''}]} \xrightarrow{\sim} \mathfrak{D}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)^{\beta\gamma}_{[\mathscr{L}^T_{t}, \mathscr{L}^T_{t''}]}.$$

This equivalence maps $\Delta(w)_{\mathscr{L}^T_{t''}}$ *to* $\Delta(w^{\min}_\beta w)_{\mathscr{L}^T_{t''}}$, $\nabla(w)_{\mathscr{L}^T_{t''}}$ *to* $\nabla(w^{\min}_\beta w)_{\mathscr{L}^T_{t''}}$ *and* $\mathrm{IC}(w)_{\mathscr{L}^T_{t''}}$ *to* $\mathrm{IC}(w^{\min}_\beta w)_{\mathscr{L}^T_{t''}}$, *for any* w.

Proof. To prove the first assertion, we proceed by induction on $\ell(w^{\min}_\beta)$ (thus allowing t, t' and β to vary). The case $\ell(w^{\min}_\beta) = 0$ is obvious and the case $\ell(w^{\min}_\beta) = 1$ is remark 7.6.2. Assume that the statement is true for $\ell(w^{\min}_\beta) = n$. For $w^{\min}_\beta$ of length $n+1$, consider $s \in W$ a simple reflection such that $\ell(sw^{\min}_\beta) = \ell(w^{\min}_\beta) - 1$. Note that we must have $s \notin W^\circ_t$. Indeed, otherwise, $sw^{\min}_\beta$ would be an element of β strictly smaller than $w^{\min}_\beta$, which by definition is not possible. Denote by η the block in ${}_{s(t)}W_{t'}$ to which $sw^{\min}_\beta$ belongs. According to lemma 5.1.2, we have $w^{\min}_\beta = sw^{\min}_\eta$.

Now we know that $\Delta(s)_{\mathscr{L}^T_t} \cong \mathrm{IC}(s)_{\mathscr{L}^T_t} \cong \nabla(s)_{\mathscr{L}^T_t}$ and corollary 8.3.5 tells us that convolution with this object is an equivalence, which preserves standard, costandard and IC-complexes. By induction hypothesis, we obtain the following sequence of isomorphisms:

$$\begin{aligned}\Delta(w^{\min}_\beta)_{\mathscr{L}^T_t} &\cong \mathrm{IC}(s)_{\mathscr{L}^T_t} \star^{\mathrm{LY}} \Delta(w^{\min}_\eta)_{\mathscr{L}^T_{t'}} \\ &\cong \mathrm{IC}(s)_{\mathscr{L}^T_t} \star^{\mathrm{LY}} \mathrm{IC}(w^{\min}_\eta)_{\mathscr{L}^T_{t'}} \\ &\cong \mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_{t'}}.\end{aligned}$$

Since we have $\dim \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_{t'}]}}(\Delta(w^{\min}_\beta)_{\mathscr{L}^T_{t'}}, \mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_{t'}}) = 1$, the above isomorphism identifies (up to a scalar) with the canonical morphism $\Delta(w^{\min}_\beta)_{\mathscr{L}^T_{t'}} \to \mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_{t'}}$. The proof that the morphism $\mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_{t'}} \to \nabla(w^{\min}_\beta)_{\mathscr{L}^T_{t'}}$ is an isomorphism is similar.

The fact that we have an equivalence follows from lemma 8.3.9, and the t-exactness will follow once we know that the images of the standard and costandard are the prescribed ones. This in turn can be proved again by induction on $\ell(w^{\min}_\beta)$, exactly as in [LY, Proposition 5.2]. □

Recall that for $t \in T^\vee_{\Bbbk}$, we denoted $w_{t,\circ}$ the longest element of the Coxeter system (W°_t, S_t) and that this element is also the maximal element in the neutral bloc of ${}_t\underline{W}_t$ (see subsection 5.1.3).

Corollary 8.5.3. *Consider* $t \in T^\vee_{\Bbbk}$. *For any* $v \in W^\circ_t$, *we have isomorphisms*

$$\Delta(w_{t,\circ})_{\mathscr{L}^T_t} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}^T_t} \cong \Delta(w_{t,\circ}v)_{\mathscr{L}^T_t},\ \nabla(v) \star^{\mathrm{LY}} \Delta(w_{t,\circ})_{\mathscr{L}^T_t} \cong \Delta(vw_{t,\circ})_{\mathscr{L}^T_t}.$$

Proof. We consider the longest element $w_\circ$ of the Coxeter system (W, S). Let $t' := w_\circ(t)$ and denote by β the block in ${}_{t'}\underline{W}_t$ to which $w_\circ$ belongs. Usual features of $w_\circ$ together with lemma 8.3.9 imply that for any $x \in W$, we have $\Delta(w_\circ)_{\mathscr{L}^T_t} \star^{\mathrm{LY}} \nabla(x)_{\mathscr{L}^T_{x^{-1}(t)}} \cong \Delta(w_\circ x)_{\mathscr{L}^T_{x^{-1}(t)}}$. We can write $w_\circ = w^{\min}_\beta w_{t,\circ}$ ac-

cording to lemma 5.1.2. Using the associativity of convolution, we have

$$\begin{aligned}\Delta(w_\beta^{\min})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} (\Delta(w_{t,\circ})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}_t^T}) &\\ \cong (\Delta(w_\beta^{\min})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \Delta(w_{t,\circ})_{\mathscr{L}_t^T}) \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}_t^T} &\\ \cong \Delta(w_\circ)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}_t^T} &\\ \cong \Delta(w_\circ v)_{\mathscr{L}_t^T} &\\ \cong \Delta(w_\beta^{\min} w_{t,\circ} v)_{\mathscr{L}_t^T} &\\ \cong \Delta(w_\beta^{\min})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \Delta(w_{t,\circ} v)_{\mathscr{L}_t^T}.&\end{aligned}$$

According to proposition 8.5.2, convolution with $\Delta(w_\beta^{\min})_{\mathscr{L}_t^T}$ gives an equivalence

$$\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^\circ_{[\mathscr{L}_t^T, \mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^\beta_{[\mathscr{L}_{t'}^T, \mathscr{L}_t^T]}.$$

Thus we obtain an isomorphism

$$\Delta(w_{t,\circ})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}_t^T} \cong \Delta(w_{t,\circ} v)_{\mathscr{L}_t^T}.$$

The proof of the other isomorphism is similar, and is left to the reader. □

Remark 8.5.4. Corollary 8.5.3 readily implies that we have an isomorphism

$$\Delta(w_{t,\circ} x^{-1})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \Delta(x)_{\mathscr{L}_t^T} \cong \Delta(w_{t,\circ})_{\mathscr{L}_t^T}$$

for any $x \in W_t^\circ$.

We can derive an analogous result for pro-objects:

Corollary 8.5.5. *Take t, t', t'' in the same W-orbit in $T^\vee_{\Bbbk}$ and choose $\beta \in {}_t\underline{W}_{t'}$ and $\gamma \in {}_{t'}\underline{W}_{t''}$. Then the canonical map*

$$\widehat{\Delta}_{w_\beta^{\min}, t'} \longrightarrow \widehat{\nabla}_{w_\beta^{\min}, t'}$$

is an isomorphism. Moreover, for any $w \in \gamma$, we have

$$\widehat{\Delta}_{w_\beta^{\min}, t'} \mathbin{\widehat{\star}} \widehat{\Delta}_{w, t''} \cong \widehat{\Delta}_{w_\beta^{\min} w, t''} \text{ and } \widehat{\Delta}_{w_\beta^{\min}, t'} \mathbin{\widehat{\star}} \widehat{\nabla}_{w, t''} \cong \widehat{\nabla}_{w_\beta^{\min} w, t''}.$$

Similar considerations hold for convolution on the right.

Proof. This is an immediate consequence of the preceding proposition. The isomorphism follows from the analogue result of proposition 8.5.2 in view of lemma 7.8.9. The statements about convolution follow from the exact same results. □

We have the following lemma, that finally gives a justification for the name "blocks" for elements of ${}_{t'}\underline{W}_t$. This is a direct adaptation of [LY, §4].

Lemma 8.5.6. *We have direct sums decomposition*

$$D^b_{[t',t]} = \bigoplus_{\beta \in {}_{t'}\underline{W}_t} D^\beta_{[t',t]},$$

and

$$\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_t]} = \bigoplus_{\beta \in {}_{t'}\underline{W}_t} \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^\beta_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_t]}.$$

Proof. We prove the first statement. By lemma 5.2.2, the subcategories

$$D^\beta_{[t',t]},\ \beta \in {}_{t'}\underline{W}_t$$

generate $D^b_{[t',t]}$. It remains to show that we have no nonzero morphism between two subcategories corresponding to different blocks. For this it suffices to show that

$$\mathrm{Hom}_{D^b_{(B)}(\mathscr{X},\Bbbk)}(\Delta_{w,t}, \Delta_{v,t}[r]) = 0$$

for w, v two elements of ${}_{t'}W_t$ that are in different blocks and any $r \in \mathbb{Z}$. We follow [LY, Proposition 4.7] and proceed by induction on $\ell(v)$. In the rest of this proof, we will omit the "$D^b_{(B)}(\mathscr{X}, \Bbbk)$" for the Hom-spaces, but the reader should keep in mind that we consider the Hom in this category.

Consider two distinct blocks $\beta, \gamma \in {}_{t'}\underline{W}_t$ and $w \in \beta$ and $v \in \gamma$. Assume that $\ell(v) = 0$ i.e. that $v = e$. Note that this implies that $w \neq e$. Using adjunction we have

$$\mathrm{Hom}(\Delta_{w,t}, \Delta_{e,t}[n]) = \mathrm{Hom}((j_e)^* \Delta_{w,t}, \mathscr{L}^T_t[r]),$$

which is zero as $w \neq e$. Assume that we have the wished-for vanishing for any v of length n. Consider a v of length $n+1$ and a simple reflection s such that $\ell(sv) < \ell(v)$. Lemmas 8.4.1 and 8.4.2 imply that convolution with $\widehat{\nabla}_{s,t'}$ gives an equivalence of categories, so we have

$$\begin{aligned}\mathrm{Hom}(\Delta_{w,t}, \Delta_{v,t}[r]) &\cong \mathrm{Hom}(\widehat{\nabla}_{s,t'} \mathbin{\widehat{\star}} \Delta_{w,t}, \widehat{\nabla}_{s,t'} \mathbin{\widehat{\star}} \Delta_{v,t}[r]) \\ &\cong \mathrm{Hom}(\widehat{\nabla}_{s,t'} \mathbin{\widehat{\star}} \Delta_{w,t}, \Delta_{sv,t}[r]).\end{aligned}$$

The second isomorphism here is a consequence of lemma 8.4.3; the same lemma allows us to conclude by induction hypothesis in the case $\ell(sw) < \ell(w)$. Using also corollary 8.5.5, we get the case $\ell(sw) > \ell(w)$ and $s \notin W^\circ_{t'}$ (remark that sw and sv do not belong to the same block; if $s \in W^\circ_{t'}$ this is obvious, otherwise this is a consequence of lemma 5.1.2). We deal with the case $\ell(sw) > \ell(w)$ and $s \in W^\circ_{t'}$. Using lemmas 7.1.5 and 7.7.4, one has

$$\begin{aligned}\widehat{\nabla}_{s,t'} \star^U \Delta_{w,t} &= \widehat{\nabla}_{s,t'} \star^U \mathrm{For}(\Delta(w)_{\mathscr{L}^T_t}) \\ &\cong \mathrm{For}(\pi^{t'}_\dagger(\widehat{\nabla}_{s,t'}) \star^{\mathrm{LY}} \Delta(w)_{\mathscr{L}^T_t}) \\ &\cong \mathrm{For}(\nabla(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w)_{\mathscr{L}^T_t}).\end{aligned}$$

Now $\nabla(s)_{\mathscr{L}^T_{t'}}$ lies in the subcategory generated by $\Delta(e)_{\mathscr{L}^T_{t'}}$ and $\Delta(s)_{\mathscr{L}^T_{t'}}$, so that $\widehat{\nabla}_{s,t'} \star^U \Delta_{w,t}$ lies in the subcategory generated by

$$\mathrm{For}_t(\Delta(w)_{\mathscr{L}^T_t}) \cong \Delta_{w,t} \text{ and } \mathrm{For}_t(\Delta(sw)_{\mathscr{L}^T_t}) \cong \Delta_{sw,t}\,.$$

Since w, sw are in the block β and sv is in γ, we can conclude by induction hypothesis.

The proof of the second statement of the lemma is very similar, in fact the original proof is in [LY, Proposition 4.7]. □

By adjunction we have a nonzero morphism $\Delta_{w,t} \to \nabla_{w,t}$. With the above lemma, we deduce that $\nabla_{w,t}$ lies in the same block as $\Delta_{w,t}$, and so does $\mathrm{IC}_{w,t}$. Similarly, $\Delta(w)_{\mathscr{L}^T_t}$, $\nabla(w)_{\mathscr{L}^T_t}$ and $\mathrm{IC}(w)_{\mathscr{L}^T_t}$ all belong to the same block. For any $t \in T^\vee_{\Bbbk}$, one deduces from the preceding lemma the following decomposition:

$$D^b_{[-,t]} = \bigoplus_{t' \in W\cdot t} \left(\bigoplus_{\beta \in {}_{t'}\underline{W}_t} D^\beta_{[t',t]} \right). \tag{8.5.1}$$

We can consider the (right) Lusztig–Yun category $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$; and we can in particular focus on the full subcategory $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}^T_t]}$ consisting of objects $\mathscr{F}$ that are in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ and such that $\mathrm{For}_t(\mathscr{F})$, which is just $\mathscr{F}$ viewed as an object in $D^b_{(B)}(\mathscr{X},\Bbbk)$, is in $D^b_{[t',-]}$. Of course, we can also consider the full subcategory of perverse sheaves

$$\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}^T_t]}.$$

An object $\mathscr{F}$ in the latter category is a perverse object $\mathscr{F} \in \mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ such that $\mathrm{For}_t(\mathscr{F}) \in \mathrm{P}_{[t',-]}$. We have a natural notion of blocks in these "hybrid" monodromic categories, and we have a decomposition

$$\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]} = \bigoplus_{t' \in W\cdot t} \left(\bigoplus_{\beta \in {}_{t'}\underline{W}_t} \mathfrak{D}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]} \right). \tag{8.5.2}$$

These decompositions induce decompositions on perverse sheaves:

$$\mathrm{P}_{[-,t]} = \bigoplus_{t' \in W\cdot t} \left(\bigoplus_{\beta \in {}_{t'}\underline{W}_t} \mathrm{P}^\beta_{[t',t]} \right), \tag{8.5.3}$$

and similarly

$$\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]} = \bigoplus_{t' \in W\cdot t} \left(\bigoplus_{\beta \in {}_{t'}\underline{W}_t} \mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]} \right). \tag{8.5.4}$$

Moreover, we can describe the simple objects in each block: for $\beta \in {}_{t'}\underline{W}_t$, the simple objects in $\mathrm{P}^\beta_{[t',t]}$ (resp. in $\mathfrak{P}(T \setminus\!\!\setminus \mathscr{X}/\!\!/T)^\beta_{[\mathscr{L}^T_{t'},\mathscr{L}^T_t]}$, $\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]}$) are the $\mathrm{IC}_{w,t}$ (resp. $\mathrm{IC}(w)_{\mathscr{L}^T_t}$) with $w \in \beta$.

8.6 Socle of standards and cosocle of costandards

Proposition 8.6.1. *Consider $t, t' \in T^\vee_{\Bbbk}$ and $\beta \in {}_{t'}\underline{W}_t$. Recall that there exists a unique minimal element $w^{\min}_\beta \in \beta$. For any $w \in \beta$:*

1. *the socle of $\Delta(w)_{\mathscr{L}^T_t}$ is $\mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_t}$. Moreover the cokernel of*

$$\mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_t} \hookrightarrow \Delta(w)_{\mathscr{L}^T_t}$$

   has composition factors indexed by $v \in \beta$ such that $v > w^{\min}_\beta$.

2. *the cosocle of $\nabla(w)_{\mathscr{L}^T_t}$ is $\mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_t}$. Moreover the kernel of*

$$\nabla(w)_{\mathscr{L}^T_t} \twoheadrightarrow \mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_t}$$

   has composition factors indexed by $v \in \beta$ such that $v > w^{\min}_\beta$.

Proof. We will show (1), then (2) follows by Verdier duality applied to

$$\mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_{t^{-1}}} \hookrightarrow \Delta(w)_{\mathscr{L}^T_{t^{-1}}}.$$

The proof is by induction on $\ell(w)$ (we will allow t, t' and β to vary). The case $\ell(w) = 0$ is clear since we have $\Delta(e)_{\mathscr{L}^T_t} = \mathrm{IC}(e)_{\mathscr{L}^T_t}$ for any t and e is clearly minimal in W°_t. Assume that the statement of the lemma is true for a $w \in W$ (for the block of ${}_{w(t)}\underline{W}_t$ to which it belongs). Consider $s \in W$ a simple reflection such that $\ell(w) > \ell(ws)$. We have to distinguish two cases. Assume first that $s \notin W^\circ_t$ and set $t'' = s(t)$. Then s is minimal in the block $\gamma \in {}_t\underline{W}_{t''}$ to which it belongs. By remark 7.6.2, we have

$$\Delta(s)_{\mathscr{L}^T_{t''}} \cong \mathrm{IC}(s)_{\mathscr{L}^T_{t''}} \cong \nabla(s)_{\mathscr{L}^T_{t''}}$$

and we know thanks to corollary 8.3.5 that convolution on the right with this object gives an equivalence of category

$$\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_{t''}]}.$$

Moreover this convolution sends $\Delta(w)_{\mathscr{L}^T_t}$ to $\Delta(ws)_{\mathscr{L}^T_{t''}}$ and $\mathrm{IC}(w^{\min}_\beta)_{\mathscr{L}^T_t}$ to $\mathrm{IC}(w^{\min}_\beta s)_{\mathscr{L}^T_{t''}}$. To end this case, we invoke lemma 5.1.2 to see that $w^{\min}_\beta s$ is minimal in its block. In both of these cases, the fact about the composition factors of the cokernel is clear.

Now we deal with the case $s \in W^\circ_t$. We use the equivalence of lemma 5.3.4 (and lemma 5.3.7) to go from the T-equivariant Lusztig–Yun category to the B_s-equivariant one. This will allow us to use the forgetful and averaging functors. According to lemma 8.2.9, we have an exact sequence

$$0 \to \Delta(ws)_{\mathscr{L}^{B_s}_t} \to \Delta(w)_{\mathscr{L}^{B_s}_t} \to \mathrm{For}^s_t \,\mathrm{Av}^s_{t,!}(\Delta(w)_{\mathscr{L}^{B_s}_t}) \to 0.$$

(This is indeed an exact sequence since $\mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ is a perverse object: For_t^s preserves the perverse objects and $\mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ is perverse thanks to corollary 8.2.6.) Consider an injective map $I \to \Delta(w)_{\mathscr{L}_t^{B_s}}$ with I a simple perverse object. Assume that the composition

$$I \hookrightarrow \Delta(w)_{\mathscr{L}_t^{B_s}} \to \mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$$

is nonzero. The object $\mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ is a $\widetilde{L_s}$-equivariant perverse object, thus all of its composition factors are $\widetilde{L_s}$-equivariant IC-complexes. The functor For_t^s preserves simple perverse sheaves, hence the images of the composition factors of $\mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ under For_t^s give all the composition factors of $\mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$. Consequently, I is of the form $\mathrm{For}_t^s(I')$ for I' a $\widetilde{L_s}$-equivariant simple perverse sheaf. We know that these objects are IC-complexes constructed from local systems on $\mathscr{X}_{v,vs}^s$, for $v \in W$ such that with $v < vs$. In particular, all composition factors of $\mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ are IC_v's with $v \in \beta$ such that $vs < v$, so different from $w_\beta^{\min}$. We can now write:

$$\begin{aligned}
\mathrm{Hom}_{B_s}(I, \Delta(w)_{\mathscr{L}_t^{B_s}}) &\cong \mathrm{Hom}_{B_s}\left(\mathrm{For}_t^s(I'), \Delta(w)_{\mathscr{L}_t^{B_s}}\right) \\
&\cong \mathrm{Hom}_{L_s}\left(I', \mathrm{Av}_{t,*}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})\right) \\
&\cong \mathrm{Hom}_{L_s}\left(I', \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})[-2]\right) \\
&= 0.
\end{aligned}$$

The last equality follows from the fact that $\mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ and I' are perverse. Thus the composition $I \to \Delta(w)_{\mathscr{L}_t^{B_s}} \to \mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$ is zero, and we have $I \hookrightarrow \Delta(ws)_{\mathscr{L}_t^{B_s}}$. We can conclude by induction that I is actually isomorphic to $\mathrm{IC}(w^\beta)_{\mathscr{L}_t^{B_s}}$. Finally, by induction hypothesis, the cokernel of

$$\mathrm{IC}(w^\beta)_{\mathscr{L}_t^{B_s}} \hookrightarrow \Delta(ws)_{\mathscr{L}_t^{B_s}}$$

has composition factors indexed by $v \in \beta$ such that $v > w_\beta^{\min}$; the previous argument tells us that this is also the case of the object $\mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})$. Since the inclusion $\mathrm{IC}(w^\beta)_{\mathscr{L}_t^{B_s}} \hookrightarrow \Delta(w)_{\mathscr{L}_t^{B_s}}$ is the composition

$$\mathrm{IC}(w^\beta)_{\mathscr{L}_t^{B_s}} \hookrightarrow \Delta(ws)_{\mathscr{L}_t^{B_s}} \hookrightarrow \Delta(w)_{\mathscr{L}_t^{B_s}},$$

these facts imply that the cokernel of $\mathrm{IC}(w^\beta)_{\mathscr{L}_t^{B_s}} \hookrightarrow \Delta(w)_{\mathscr{L}_t^{B_s}}$ has also composition factors indexed by elements $v \in \beta$ such that $v > w_\beta^{\min}$. □

Chapter 9

Tilting objects

9.1 Definition

Our main objective is the study of the various categories $\mathrm{P}_{[-,\underline{t}]}$. According to proposition 4.3.1, we have an equivalence

$$\mathrm{P}_{[-,\underline{t}]} \cong \mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} \tag{9.1.1}$$

and both of these categories are endowed with highest weight structures. Moreover, the equivalence (9.1.1) preserves these highest weight structures, i.e. it maps standards to standards, costandards to costandards and simples to simples. Thanks to [BGS, Theorem 3.2.1], our categories $\mathrm{P}_{[-,\underline{t}]}$ have enough projective objects and these projective objects admit standard filtrations. In particular, each simple object has a projective cover.

Definition 9.1.1. *Consider $t \in T_{\Bbbk}^{\vee}$ and $w \in W$. We denote by*

1. *$\mathscr{P}_{w,t}$ the projective cover of $\mathrm{IC}_{w,t}$ in the category $\mathrm{P}_{[-,\underline{t}]}$,*

2. *$\mathscr{P}(w)_{\mathscr{L}_t^T}$ the projective cover of $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$.*

Remark 9.1.2. We also have enough injective objects in these categories; the injective envelope of $\mathrm{IC}_{w,t}$ in $\mathrm{P}_{[-,\underline{t}]}$ (resp. $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ in $\mathfrak{P}(\mathscr{X}/T)_{[-,\mathscr{L}_t^T]}$) is denoted $\mathscr{I}_{w,t}$ (resp. $\mathscr{I}(w)_{\mathscr{L}_t^T}$). In fact, we will almost always work with projective objects rather than injective ones.

There is another kind of objects (in any general highest weight category) that is of interest, namely the tilting objects.

Definition 9.1.3. *Define an object in a highest weight category $(\mathcal{A}, (\mathcal{S}, \leq))$ to be tilting if it admits both a standard filtration and a costandard filtration.*

We restate below, and in our setting of perverse monodromic sheaves, a general statement about tilting objects, whose proof in the general setting can be found e.g. in [R, §7.5, Theorem 7.14].

Proposition 9.1.4. *For any $w \in W$ and $t \in T^\vee_{\Bbbk}$, there exists a unique (up to isomorphism) indecomposable tilting object $\mathscr{T}_{w,t}$ in the monodromic category $\mathrm{P}_{[-,\underline{t}]}$ such that*

$$[\mathscr{T}_{w,t} : \mathrm{IC}_{w,t}] = 1 \quad \textit{and} \quad ([\mathscr{T}_{w,t} : \mathrm{IC}_{v,t}] \neq 0) \Rightarrow v \leq w. \tag{9.1.2}$$

Moreover, there exists an embedding $\Delta_{w,t} \hookrightarrow \mathscr{T}_{w,t}$ whose cokernel admits a standard filtration, and a surjection $\mathscr{T}_{w,t} \twoheadrightarrow \nabla_{w,t}$ whose kernel admits a costandard filtration. Any indecomposable tilting in $\mathrm{P}_{[-,\underline{t}]}$ is isomorphic to one of the $\mathscr{T}_{w,t}$ for a unique $w \in W$, and any tilting object is a direct sum of indecomposable tilting objects.

The exact analogue of course holds in the category $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$:

Proposition 9.1.5. *For any $w \in W$ and $t \in T^\vee_{\Bbbk}$, there exists a unique (up to isomorphism) indecomposable tilting object $\mathscr{T}(w)_{\mathscr{L}^T_t}$ in the monodromic category $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ such that*

$$[\mathscr{T}(w)_{\mathscr{L}^T_t} : \mathrm{IC}(w)_{\mathscr{L}^T_t}] = 1 \quad \textit{and} \quad ([\mathscr{T}(w)_{\mathscr{L}^T_t} : \mathrm{IC}(v)_{\mathscr{L}^T_t}] \neq 0) \Rightarrow v \leq w. \tag{9.1.3}$$

Moreover, there exists an embedding $\Delta(w)_{\mathscr{L}^T_t} \hookrightarrow \mathscr{T}(w)_{\mathscr{L}^T_t}$ whose cokernel admits a standard filtration, and a surjection $\mathscr{T}(w)_{\mathscr{L}^T_t} \twoheadrightarrow \nabla(w)_{\mathscr{L}^T_t}$ whose kernel admits a costandard filtration. Any indecomposable tilting in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ is isomorphic to one of the $\mathscr{T}(w)_{\mathscr{L}^T_t}$ for a unique $w \in W$, and any tilting object is a direct sum of indecomposable tilting objects.

We set $\mathfrak{T}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$ for the subcategory of tilting objects in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}^T_t]}$. Since the equivalence (9.1.1) preserves standard, simple and costandard objects, it also preserves tilting objects, i.e. it maps $\mathscr{T}(w)_{\mathscr{L}^T_t}$ to $\mathscr{T}_{w,t}$.

9.2 Ringel duality

Fix $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit. We consider the hybrid categories $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}^T_t]}$ introduced after lemma 8.5.6, and its decomposition into blocks, stated as (8.5.4). Any indecomposable object, in particular any indecomposable projective or tilting object, belongs to some block. Fix $\beta \in {}_{t'}\underline{W}_t$ and a $w \in \beta$. The objects $\mathscr{T}(w)_{\mathscr{L}^T_t}$ and $\mathscr{P}(w)_{\mathscr{L}^T_t}$ are indecomposable and admit nonzero maps respectively to $\nabla(w)_{\mathscr{L}^T_t}$ and $\mathrm{IC}(w)_{\mathscr{L}^T_t}$. Thus these two objects are in

$$\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]}.$$

Now, the direct sum decomposition of lemma 8.5.6 implies that this block is itself a highest weight category. We then have enough projective objects in this subcategory, and a subcategory of tilting objects. Clearly, the objects in the subactegory $\mathfrak{T}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]}$ are those tilting objects $\mathscr{T}$ in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}^T_t]}$ such that the standards appearing as suquotients in a standard filtration of $\mathscr{T}$ are $\Delta(w)_{\mathscr{L}^T_t}$ with $w \in \beta$.

Lemma 9.2.1. *Let $\mathscr{P}$ to be an object in $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}_t^T]}$. If*

$$\mathrm{Ext}^i_{\mathfrak{D}(X/\!\!/T)_{[t',\mathscr{L}_t^T]}}(\mathscr{P}, \Delta(w)_{\mathscr{L}_t^T}) = 0$$

for any $i > 0$ and $w \in \beta$, then $\mathscr{P}$ is a projective object in $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}_t^T]}$.

Proof. Since ${}^p\mathfrak{D}(\mathscr{X}/\!\!/T)^{\leq 0}_{[t',\mathscr{L}_t^T]}$ is generated under extension by objects of the form $\Delta(w)_{\mathscr{L}_t^T}[n]$ with $w \in W$ and $n \geq 0$, any perverse object $\mathscr{F}$ is a successive extension (in the triangulated category $\mathfrak{D}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}_t^T]}$) of such objects. If $\mathscr{F}$ lies in $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}_t^T]}$, then the w that may occur in such a filtration by standard objects can be chosen in β (this is a direct consequence of the decomposition (8.5.2)).

Now, using corollary 4.6.2, to show that $\mathscr{P}$ is projective in $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}_t^T]}$, it suffices to show that $\mathrm{Ext}^i_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]}}(\mathscr{P}, \Delta(w)_{\mathscr{L}_t^T}[n]) = 0$ for any $w \in \beta$, $i > 0$ and $n \geq 0$. This concludes the proof. □

Proposition 9.2.2. *Fix $t, t', t'' \in T^\vee_{\Bbbk}$ in the same W-orbit and consider two blocks $\beta \in {}_{t'}\underline{W}_t$ and $\gamma \in {}_{t''}\underline{W}_{t'}$. Then we have a "Ringel duality"*

$$R^\gamma_\beta : \mathfrak{D}(\mathscr{X}/\!\!/T)^{\gamma}_{[t'',\mathscr{L}_{t'}^T]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}/\!\!/T)^{\gamma\beta}_{[t'',\mathscr{L}_t^T]}.$$

This is an equivalence of triangulated category and we have

$$R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}_{t'}^T}) \cong \mathscr{P}(ww^{\max}_\beta)_{\mathscr{L}_t^T}$$

and

$$R^\gamma_\beta(\nabla(w)_{\mathscr{L}_{t'}^T}) \cong \Delta(ww^{\max}_\beta)_{\mathscr{L}_t^T}$$

for any $w \in \gamma$. In particular, R^γ_β restrict to an equivalence $\mathfrak{T}(\mathscr{X}/\!\!/T)^{\gamma}_{[t'',\mathscr{L}_{t'}^T]} \cong \mathrm{Proj}\mathfrak{P}(\mathscr{X}/\!\!/T)^{\gamma\beta}_{[t'',\mathscr{L}_t^T]}$.

Similarly, we have an equivalence of triangulated categories

$$\widehat{R}^\beta_\gamma : \mathfrak{D}(X/\!\!/T)^{\beta}_{[t',\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}(X/\!\!/T)^{\gamma\beta}_{[t'',\mathscr{L}_t^T]},$$

such that

$$\widehat{R}^\beta_\gamma(\mathscr{T}(w)_{\mathscr{L}_t^T}) \cong \mathscr{P}(w^{\max}_\gamma w)_{\mathscr{L}_t^T}$$

and

$$\widehat{R}^\beta_\gamma(\nabla(w)_{\mathscr{L}_t^T}) \cong \Delta(w^{\max}_\gamma w)_{\mathscr{L}_t^T}$$

for any $w \in \beta$.

Proof. We copy the proof of [BBM, Proposition 2.3]. We abbreviate the notation $\mathfrak{D}(\mathscr{X}/\!\!/T)^{\gamma}_{[t'',\mathscr{L}_{t'}^T]}$ by $D^\gamma_{[t'',\underline{t'}]}$ and similarly for $\mathfrak{D}(\mathscr{X}/\!\!/T)^{\gamma\beta}_{[t'',\mathscr{L}_t^T]}$. Define R^γ_β to be the functor

$$R^\gamma_\beta := (-) \star^{\mathrm{LY}} \Delta(w^{\max}_\beta)_{\mathscr{L}_t^T} : D^\gamma_{[t'',\underline{t'}]} \to D^{\gamma\beta}_{[t'',\underline{t}]}.$$

Note that the target category is the right one thanks to lemma 8.5.1 (in fact, lemma 8.5.1 involves categories of sheaves equivariant on the right and the left, which is not what we consider here; the same proof works however, since $D^{\gamma}_{[t'',\underline{t'}]}$ is generated by standard objects $\Delta(w)_{\mathscr{L}^T_{t'}}$ with $w \in \gamma$). According to lemma 8.3.9, this is an equivalence with quasi-inverse given by

$$(-) \star^{\mathrm{LY}} \nabla((w_\beta^{\max})^{-1})_{\mathscr{L}^T_{t'}}.$$

Now fix a $w \in \gamma$ and consider $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$. This object admits a ∇-filtration by objects $\nabla(v)_{\mathscr{L}^T_{t'}}$ with $v \in \gamma$. We can write uniquely $v = w_\gamma^{\min} x$ with $x \in W^\circ_{t'}$ (recall that a γ is just a coset in ${}_{t''}W_{t'}/W^\circ_{t'}$). Similarly, we have $w_\beta^{\max} = w_{t',\circ} w_\beta^{\min}$, where $w_{t',\circ}$ is the maximal element in the neutral block in ${}_{t'}\underline{W}_{t'}$, and so $w_{t',\circ}$ is the longest element in the Coxeter group $W^\circ_{t'}$ (the discussion above lemma 5.1.2 ensures that these two notions are indeed the same). Now with proposition 8.5.2, we have

$$\Delta(w_\gamma^{\min})_{\mathscr{L}^T_{t'}} \cong \nabla(w_\gamma^{\min})_{\mathscr{L}^T_{t'}} \quad \text{and} \quad \nabla(w_\beta^{\min})_{\mathscr{L}^T_t} \cong \Delta(w_\beta^{\min})_{\mathscr{L}^T_t}.$$

The same proposition 8.5.2 allows us to deduce

$$\begin{aligned} R^\gamma_\beta(\nabla(v)_{\mathscr{L}^T_{t'}}) &= \nabla(v)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_\beta^{\max})_{\mathscr{L}^T_t} \\ &= \nabla(w_\gamma^{\min} x)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_{t',\circ} w_\beta^{\min})_{\mathscr{L}^T_t} \\ &\cong \nabla(w_\gamma^{\min})_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \nabla(x)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_{t',\circ})_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_\gamma^{\min})_{\mathscr{L}^T_t}. \end{aligned}$$

Thanks to corollary 8.5.3, we have $\nabla(x)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_{t',\circ})_{\mathscr{L}^T_{t'}} \cong \Delta(x w_{t',\circ})_{\mathscr{L}^T_{t'}}$. Hence

$$\begin{aligned} R^\gamma_\beta(\nabla(v)_{\mathscr{L}^T_{t'}}) &\cong \Delta(w_\gamma^{\min})_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(x w_{t',\circ})_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(w_\beta^{\min})_{\mathscr{L}^T_t} \\ &\cong \Delta(w_\gamma^{\min} x w_{t',\circ} w_\beta^{\min})_{\mathscr{L}^T_t}. \end{aligned}$$

We have finally obtained $R^\gamma_\beta(\nabla(v)_{\mathscr{L}^T_{t'}}) \cong \Delta(v w_\beta^{\max})_{\mathscr{L}^T_t}$. We deduce that the object $R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}})$ admits a Δ-filtration and thus is perverse. To check that this object is projective, using lemma 9.2.1, it suffices to show that $\mathrm{Ext}^i(R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}), \Delta(x)_{\mathscr{L}^T_t}) = 0$ for any $i > 0$ and $x \in \gamma\beta$ Note that such an x can be written as $x' w_\beta^{\max}$ with $x' \in \gamma$. We now write

$$\begin{aligned} \mathrm{Ext}^i(R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}), \Delta(x)_{\mathscr{L}^T_t}) &\cong \mathrm{Ext}^i(R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}), R^\gamma_\beta(\nabla(x')_{\mathscr{L}^T_{t'}})) \\ &\cong \mathrm{Ext}^i(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}, \nabla(x')_{\mathscr{L}^T_{t'}}) \\ &= 0. \end{aligned}$$

The last equality follows from the fact that $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$ has a Δ-filtration. We deduce that $R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}})$ is a projective perverse sheaf, it is moreover indecomposable since $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$ is.

We now have to determine this projective object. By proposition 9.1.5, there exists a surjective map from $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$ to $\nabla(w)_{\mathscr{L}^T_{t'}}$, whose kernel K admits a costandard filtration. If we apply R^γ_β to the associated distinguished triangle

$$K \to \mathscr{T}(w)_{\mathscr{L}^T_{t'}} \to \nabla(w)_{\mathscr{L}^T_{t'}} \xrightarrow{+1},$$

then we obtain a distinguished triangle

$$R^\gamma_\beta(K) \to R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}) \to \Delta(ww^{\max}_\beta)_{\mathscr{L}^T_t} \xrightarrow{+1}.$$

The three terms in this triangle are perverse (we use the costandard filtration of K here), and thus this triangle gives a short exact sequence in $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\gamma\beta}_{[t'',\mathscr{L}^T_t]}$. Consequently, we have a surjective map

$$R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}) \twoheadrightarrow \Delta(ww^{\max}_\beta)_{\mathscr{L}^T_t},$$

from which we get a surjective map $R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}) \twoheadrightarrow \mathrm{IC}(ww^{\max}_\beta)_{\mathscr{L}^T_t}$. Being projective and indecomposable, $R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}})$ has to be the projective cover of this IC-sheaf; said otherwise, we have

$$R^\gamma_\beta(\mathscr{T}(w)_{\mathscr{L}^T_{t'}}) \cong \mathscr{P}(ww^{\max}_\beta)_{\mathscr{L}^T_t}.$$

The proof of the existence and properties of $\widehat{R}^\beta_\gamma$ are similar: one sets

$$\widehat{R}^\beta_\gamma := \widehat{\Delta}_{w^{\max}_\gamma, t} \,\widehat{\star}(-).$$

The verifications are left to the reader. □

Corollary 9.2.3. *We keep the notation of proposition 9.2.2. For any $w \in \gamma$ and $v \in \gamma\beta$, we have*

$$(\mathscr{P}(ww^{\max}_\beta)_{\mathscr{L}^T_t} : \Delta(v)_{\mathscr{L}^T_t}) = (\mathscr{T}(w)_{\mathscr{L}^T_{t'}} : \nabla(v(w^{\max}_\beta)^{-1})_{\mathscr{L}^T_{t'}}).$$

In particular, we have

$$(\mathscr{P}(w^{\min}_{\gamma\beta})_{\mathscr{L}^T_t} : \Delta(v)_{\mathscr{L}^T_t}) = (\mathscr{T}(w^{\max}_\gamma)_{\mathscr{L}^T_{t'}} : \nabla(v(w^{\max}_\beta)^{-1})_{\mathscr{L}^T_{t'}}). \qquad (9.2.1)$$

Proof. The first statement is an immediate consequence of proposition 9.2.2. The equality (9.2.1) is just a particular case, using lemma 5.1.2. □

9.3 Comparison of tilting and projective objects

In this section we use the exact arguments as in [AR1, Section 5.11]. For the duration of this paragraph, we fix arbitrary $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit and β a block in ${}_{t'}\underline{W}_t$.

Corollary 9.3.1. *For any $w \in \beta$, we have $(\mathscr{P}(w_\beta^{\min})_{\mathscr{L}_t^T} : \Delta(w)_{\mathscr{L}_t^T}) = 1$.*

Proof. This follows immediately from the point (2) of proposition 8.6.1 and the reciprocity formula in a highest weight category. □

The proof of the following lemma can be copied from [AR1, Lemma 5.24]:

Lemma 9.3.2. *For any $w \in \beta$ we have*

1. $\dim \mathrm{Hom}(\Delta(w)_{\mathscr{L}_t^T}, \Delta(w_\beta^{\max})_{\mathscr{L}_t^T}) = 1$ *and every nonzero map*

$$\Delta(w)_{\mathscr{L}_t^T} \to \Delta(w_\beta^{\max})_{\mathscr{L}_t^T}$$

is injective;

2. $\dim \mathrm{Hom}(\nabla(w_\beta^{\max})_{\mathscr{L}_t^T}, \nabla(w)_{\mathscr{L}_t^T}) = 1$ *and every nonzero map*

$$\nabla(w_\beta^{\max})_{\mathscr{L}_t^T} \to \nabla(w)_{\mathscr{L}_t^T}$$

is surjective.

Proof. We prove only 1, the proof of 2 is similar. Write (uniquely) $w = w_\beta^{\min} v$ with $v \in W_t^\circ$. According to proposition 8.6.1, the space

$$\mathrm{Hom}(\Delta(e)_{\mathscr{L}_t^T}, \Delta(v^{-1} w_{t,\circ})_{\mathscr{L}_t^T})$$

is one dimensional since $\Delta(e)_{\mathscr{L}_t^T}$ is the socle of $\Delta(v^{-1}w_{t,\circ})_{\mathscr{L}_t^T}$, and any map there is injective. Since convolution on the left with $\Delta(v)_{\mathscr{L}_t^T}$ (in fact, with any standard or costandard object) yields an equivalence of categories; remark 8.5.4 tells us that the space

$$\mathrm{Hom}(\Delta(v)_{\mathscr{L}_t^T}, \Delta(w_{t,\circ})_{\mathscr{L}_t^T})$$

is also one dimensional. Finally, using proposition 8.5.2 and convolving with $\Delta(w_\beta^{\min})_{\mathscr{L}_t^T}$ we get that $\dim \mathrm{Hom}(\Delta(w)_{\mathscr{L}_t^T}, \Delta(w_\beta^{\max})_{\mathscr{L}_t^T}) = 1$. Now thanks to proposition 8.6.1, any map f in this space induces an isomorphism on socles; in particular, the kernel of f has a trivial socle and is zero, i.e. f is injective. □

Lemma 9.3.3. *For any $w \in \beta$, we have*

$$(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} : \nabla(w)_{\mathscr{L}_t^T}) = 1 = (\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} : \Delta(w)_{\mathscr{L}_t^T}).$$

Proof. The first equality is a direct consequence of corollaries 9.3.1 and 9.2.3. The second one follows from Verdier duality applied to $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t^{-1}}^T}$ (which makes sense thanks to the comment above corollary 5.3.10). □

The proof of the next proposition is exactly as in [AR1, Proposition 5.26].

Proposition 9.3.4. *We have $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \cong \mathscr{P}(w_\beta^{\min})_{\mathscr{L}_t^T}$.*

Proof. First, we have

$$\dim \mathrm{Hom}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}, \nabla(w)_{\mathscr{L}_t^T}) = 1 \tag{9.3.1}$$

for any $w \in \beta$. This follows from lemma 9.3.3. Then any nonzero map is the latter space is surjective: the case $w = w_\beta^{\max}$ is just a property of $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}$. For other w this follows from lemma 9.3.2 and the case $w = w_\beta^{\max}$. The next step is to see that

$$\mathrm{Hom}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}, \mathrm{IC}(w)) = 0 \text{ for } w \neq w_\beta^{\min}. \tag{9.3.2}$$

Indeed, if there were a nonzero map $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \to \mathrm{IC}(w)$ for $w \neq w_\beta^{\min}$, this would yield a nonzero, non-surjective map

$$\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \to \mathrm{IC}(w)_{\mathscr{L}_t^T} \hookrightarrow \nabla(w)_{\mathscr{L}_t^T},$$

contradicting what is stated above (note that the injection $\mathrm{IC}(w)_{\mathscr{L}_t^T} \hookrightarrow \nabla(w)_{\mathscr{L}_t^T}$ is not an isomorphism, e.g. because the cosocle of $\nabla(w)_{\mathscr{L}_t^T}$ is $\mathrm{IC}(w_\beta^{\min})$, and $w_\beta^{\min} \neq w$).

Now from (9.3.1) and proposition 8.5.2, we have

$$\dim \mathrm{Hom}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}, \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^T}) = 1.$$

Moreover, (9.3.2) implies that $\mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^T}$ is the unique simple quotient of $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}$. Thus the object $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}$ must be (isomorphic to) a quotient of $\mathscr{P}(w_\beta^{\min})_{\mathscr{L}_t^T}$. But corollary 9.3.1 and lemma 9.3.3 imply that these two objects have the same length, and thus they must be isomorphic. This concludes the proof. □

Remark 9.3.5. In complete analogy, and considering injective objects (see remark 9.1.2), one can show that $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \cong \mathscr{I}(w_\beta^{\min})_{\mathscr{L}_t^T}$.

9.4 Tilting perverse sheaves in the completed category

Definition 9.4.1. *We say that a perverse object $\mathscr{F} \in \widehat{D}_{[-,t]}$ is tilting if it admits a $\widehat{\Delta}$-filtration and a $\widehat{\nabla}$-filtration (that is, a filtration in $\widehat{\mathrm{P}}_{(B)}(\mathscr{X}, \Bbbk)_{[-,t]}$ with subquotients in $\{\widehat{\Delta}_{w,t} \mid w \in W\}$ and a filtration with subquotients in $\{\widehat{\nabla}_{w,t} \mid w \in W\}$).*

If $\mathscr{F}$ is tilting, one easily sees that the number of occurrence of $\widehat{\Delta}_{w,t}$ in a $\widehat{\Delta}$-filtration does not depend on the filtration; this is equal to the rank over $\widehat{R}_t$ of $\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F}, \widehat{\nabla}_{w,t})$. Similarly, the number of occurence of $\widehat{\nabla}_{w,t}$ in a $\widehat{\nabla}$-filtration is equal to the rank of $\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\Delta}_{w,t}, \mathscr{F})$ (in fact this is standard in highest weight category, but the same proof applies here, using (7.8.1)). We will denote these multiplicities respectively by $(\mathscr{F} : \widehat{\Delta}_{w,t})$ and $(\mathscr{F} : \widehat{\nabla}_{w,t})$.

Lemma 9.4.2. *(1) If $\mathscr{F} \in \widehat{D}_{[-,t]}$, then $\mathscr{F}$ is tilting if and only if $\pi_\dagger^t(\mathscr{F})$ is perverse and tilting in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$.*
(2) If $\mathscr{F}, \mathscr{G} \in \widehat{D}_{[-,t]}$ are tilting, then:

- *the functor $\pi_\dagger^t$ induces an isomorphism*

$$\Bbbk_t \otimes_{\widehat{R}_t} \mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\pi_\dagger^t(\mathscr{F}), \pi_\dagger^t(\mathscr{G})),$$

- $\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F}, \mathscr{G}[k]) = 0$ *if* $k \neq 0$,

- *the $\widehat{R}_t$-module $\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F}, \mathscr{G})$ is free of finite rank.*

Proof. (1) is just a particular case of lemma 7.8.9.

(2) Any tilting admits both a standard and costandard filtration, that is, is a successive finite extension of such objects in the perverse completed category. The fact that the Hom-space is free of finite rank is then a consequence of the equalities (7.8.1). The same equalities imply the second point above. Finally, according to lemma 7.1.4, the natural map

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\pi_\dagger^t(\mathscr{F}), \pi_\dagger^t(\mathscr{G}))$$

factors through the quotient $\Bbbk_t \otimes_{\widehat{R}_t} \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\pi_\dagger^t(\mathscr{F}), \pi_\dagger^t(\mathscr{G}))$. We use the fact that the $\widehat{R}_t$-module $\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\mathscr{F}, \mathscr{G})$ is free; thanks to the five lemma it then suffices to consider the case $\mathscr{F} = \widehat{\Delta}_{w,t}$ and $\mathscr{G} = \widehat{\nabla}_{v,t}$ for $w, v \in W$. But the first point is then obvious from the equalities (7.8.1). □

9.5 Classification of tilting perverse sheaves

Lemma 9.5.1. *Fix an element $w \in W$ and consider any open subset $O \subseteq \overline{\mathscr{X}_w}$ that is a union of strata. Then there exists a tilting sheaf $\mathscr{F} \in \widehat{D}_{(B)}(O, \Bbbk)_{[-,t]}$ whose restriction to $\mathscr{X}_w$ is $\widehat{\mathscr{L}_t^w}$.*

Proof. The proof is by induction on the number n of strata in $O \subseteq \overline{\mathscr{X}_w}$. We will have to consider standard objects in $\widehat{D}_{(B)}(O, \Bbbk)_{[t]}$, we will denote them with a superscript O.

If $n = 1$, then one takes $\widehat{\Delta}_{w,t} \cong \widehat{\nabla}_{w,t}$.

For a general O, consider a closed stratum $\mathscr{X}_v$ in O and let $V = O \setminus \mathscr{X}_v$; we denote by $j : V \hookrightarrow O$ the open embedding. By induction, we have a tilting sheaf $\widehat{\mathscr{T}}_V$ in $\widehat{D}_{(B)}(V, \Bbbk)_{[-,t]}$ whose restriction to $\mathscr{X}_w$ is $\widehat{\mathscr{L}_t^w}$. The object $j_! \widehat{\mathscr{T}}_V$ admits a $\widehat{\Delta}^O$-filtration in $\widehat{D}_{(B)}(O, \Bbbk)_{[t]}$, so it is perverse. Moreover, thanks to corollary 7.8.3, the space

$$E := \mathrm{Hom}_{\widehat{D}_{(B)}(O,\Bbbk)_{[-,t]}}(\widehat{\Delta}^O_{v,t}, j_! \widehat{\mathscr{T}}_V[1]) = \mathrm{Ext}^1_{\widehat{\mathrm{P}}_{(\mathrm{B})}(O,\Bbbk)_{[-,t]}}(\widehat{\Delta}^O_{v,t}, j_! \widehat{\mathscr{T}}_V)$$

is finitely generated as a $\widehat{R}_t$-module. We can choose a surjection $\widehat{R}_t^{\oplus m} \twoheadrightarrow E$. This map defines an element in

$$\begin{aligned}\mathrm{Hom}_{\widehat{R}_t\text{-mod}}(\widehat{R}_t^{\oplus m}, E) \cong \mathrm{Hom}_{\widehat{R}_t\text{-mod}}(\widehat{R}_t, E)^{\oplus m} &\cong E^{\oplus m} \\ &\cong \mathrm{Ext}^1_{\widehat{\mathrm{P}}_{(\mathrm{B})}(O,\Bbbk)_{[-,t]}}((\widehat{\Delta}^O_{v,t})^{\oplus m}, j_!\widehat{\mathscr{T}}_V). \quad (9.5.1)\end{aligned}$$

We thus obtain an extension in $\widehat{\mathrm{P}}_{(\mathrm{B})}(O,\Bbbk)_{[-,t]}$

$$0 \to j_!\widehat{\mathscr{T}}_V \to \widehat{\mathscr{T}} \to (\widehat{\Delta}^O_{v,t})^{\oplus m} \to 0. \quad (9.5.2)$$

The object $\widehat{\mathscr{T}}$ clearly admits a $\widehat{\Delta}^O$-filtration in $\widehat{D}_{(B)}(O,\Bbbk)_{[-,t]}$, so it is perverse. Applying the functor j_w^* to (9.5.2), we see that $\widehat{\mathscr{T}}$ has the correct restriction to $\mathscr{X}_w$. We now have to show that it admits a $\widehat{\nabla}^O$-filtration, or, according to lemma 7.8.9, that $\pi_\dagger^t(\widehat{\mathscr{T}})$ admits a ∇^O-filtration. According to [R, Proposition 7.10], this is equivalent to

$$\mathrm{Ext}^1_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(u)^O_{\mathscr{L}_t^T}, \pi_\dagger^t(\widehat{\mathscr{T}})) = 0$$

for any u with $\mathscr{X}_u \subseteq O$. We will consider the exact sequence

$$0 \to \pi_\dagger^t(j_!\widehat{\mathscr{T}}_V) \to \pi_\dagger^t(\widehat{\mathscr{T}}) \to (\Delta(v)^O_{\mathscr{L}_t^T})^{\oplus m} \to 0 \quad (9.5.3)$$

in $\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}$ obtained by an application of $\pi_\dagger^t$ to the exact sequence (9.5.2) (we indeed obtain an exact sequence since each term remains perverse after the application of $\pi_\dagger^t$). We begin with the case $u \neq v$. We will use remark 7.5.2: in what follows, we consider alternatively usual topological functors (such as $j_!$, $j^!$...) and their extension to the completed category; but we will not have different notation. It should be clear, however, what version is used in each case. Since $\pi_\dagger^t \circ\ j_! = j_! \circ \pi_\dagger^t$ and $\mathscr{X}_u \subseteq V$, we have $j_u^!\pi_\dagger^t(j_!\widehat{\mathscr{T}}_V) = j_u^!\pi_\dagger^t(\widehat{\mathscr{T}}_V)$. Since $\pi_\dagger^t(\widehat{\mathscr{T}}_V)$ is tilting, according to lemma 7.8.7 and remark 7.8.8, this corestriction perverse. Using the exact sequence (9.5.3), one obtains

$$\begin{aligned}\mathrm{Ext}^1_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(u)^O_{\mathscr{L}_t^T}, \pi_\dagger^t(\widehat{\mathscr{T}})) &= \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}_u/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{L}^u_t, j_u^!\pi_\dagger^t(\widehat{\mathscr{T}})[1]) \\ &= \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}_u/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{L}^u_t, j_u^!\pi_\dagger^t(j_!\widehat{\mathscr{T}}_V)[1]) \\ &= \mathrm{Ext}^1_{\mathfrak{P}(V/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(u)^U_{\mathscr{L}_t^T}, \pi_\dagger^t(\widehat{\mathscr{T}}_V)) \\ &= 0.\end{aligned}$$

The last equality follows from the fact that $\pi_\dagger^t(\widehat{\mathscr{T}}_V)$ is tilting in $\mathfrak{P}(V/\!\!/T)_{[-,\mathscr{L}_t^T]}$.

We deal with the case $u = v$. Applying the functor

$$\mathrm{Hom}_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, -)$$

to the exact sequence (9.5.3), we see that in order to show that

$$\mathrm{Ext}^1_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, \pi_\dagger^t(\widehat{\mathscr{T}})) = 0,$$

we have to show that the map

$$\Bbbk_t^{\oplus m} \cong \mathrm{Hom}_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, \Delta(v)_{\mathscr{L}_t^T}^{\oplus m}) \\ \to \mathrm{Ext}^1_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, \pi_\dagger^t(j_! \widehat{\mathscr{T}_V}))$$

is surjective. A first step in order to see that is to show that

$$\mathrm{Ext}^i_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, \pi_\dagger^t(j_! \widehat{\mathscr{T}}_V)) = 0 \text{ for any } i \geq 2.$$

Consider the following exact sequences in $\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}$

$$N \hookrightarrow j_! \pi_\dagger^t(\widehat{\mathscr{T}}_V) \twoheadrightarrow j_{!*} \pi_\dagger^t(\widehat{\mathscr{T}}_V) \quad \text{and} \quad j_{!*} \pi_\dagger^t(\widehat{\mathscr{T}}_V) \hookrightarrow j_* \pi_\dagger^t(\widehat{\mathscr{T}}_V) \twoheadrightarrow M \quad (9.5.4)$$

where the objects N and M are given by $N := \ker(\pi_\dagger^t(\widehat{\mathscr{T}}_V) \twoheadrightarrow j_{!*} \pi_\dagger^t(\widehat{\mathscr{T}}_V))$ and $M := \mathrm{coker}(j_{!*} \pi_\dagger^t(\widehat{\mathscr{T}}_V) \hookrightarrow j_* \pi_\dagger^t(\widehat{\mathscr{T}}_V))$. We note the following facts:

1. N and M are supported on $\mathscr{X}_v$ and are perverse, so they are direct sums of $\Delta(v)_{\mathscr{L}_t^T}$ in $\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}$,

2. the object $j_* \pi_\dagger^t(\widehat{\mathscr{T}}_V)$ admits a ∇^O-filtration in $\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}$ since $\pi_\dagger^t(\widehat{\mathscr{T}}_V)$ does,

3. we have $\mathrm{Ext}^i(\Delta(v)_{\mathscr{L}_t^T}, \Delta(v)_{\mathscr{L}_t^T}) = 0$ and $\mathrm{Ext}^i(\Delta(v)_{\mathscr{L}_t^T}, \nabla(x)_{\mathscr{L}_t^T}) = 0$ in $\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}$, for any $i > 0$ and any $x \in W$ such that $\mathscr{X}_x \subseteq V$.

We apply the functor $\mathrm{Hom}_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, -)$ to the second exact sequence in (9.5.4). According to the three points above, we obtain

$$\mathrm{Ext}^i_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, j_{!*} \pi_\dagger^t(\widehat{\mathscr{T}}_V)) = 0 \quad \text{for } i > 1.$$

Then apply the same functor to the first exact sequence in (9.5.4) to obtain the wished-for vanishing.

Using adjunction, we get $\mathrm{Hom}_{\mathfrak{D}(V/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{L}_t^v, j_v^! j_! \pi_\dagger^t(\widehat{\mathscr{T}}_V)[i]) = 0$ for $i > 1$. If we let $\mathcal{T}$ be the object of $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ corresponding to $j_v^! j_! \widehat{\mathscr{T}}_V$ under the equivalence (7.5.6) , then we obtain that

$$\mathrm{Hom}_{D^b \operatorname{Vect}^{\mathrm{fd}}_{\Bbbk_t}}(\Bbbk_t, \Bbbk_t \otimes^L_{\widehat{R}_t} \mathcal{T}[i]) = \mathbf{H}^i(\Bbbk_t \otimes^L_{\widehat{R}_t} \mathcal{T}) = 0$$

for $i > 1$. This in turn implies that $\mathcal{T}$ is concentrated in degrees ≤ 1, so we have an isomorphism $\Bbbk_t \otimes_{\widehat{R}_t} \mathbf{H}^1(\mathcal{T}) \cong \mathbf{H}^1(\Bbbk_t \otimes^L_{\widehat{R}_t} \mathcal{T})$. Using adjunction once again, this isomorphism can be rewritten as

$$\begin{aligned}\Bbbk_t \otimes_{\widehat{R}_t} E &\cong \mathrm{Hom}_{\mathfrak{D}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, j_! \pi_\dagger^t(\widehat{\mathscr{T}}_V)[1]) \\ &= \mathrm{Ext}^1_{\mathfrak{P}(V/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, j_! \pi_\dagger^t(\widehat{\mathscr{T}}_V)).\end{aligned}$$

This finally shows that our chosen surjection $\widehat{R}_t^{\oplus m} \twoheadrightarrow E$ induces a surjection $\Bbbk_t^{\oplus m} \twoheadrightarrow \mathrm{Ext}^1_{\mathfrak{P}(O/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(v)_{\mathscr{L}_t^T}, j_! \pi_\dagger^t(\widehat{\mathscr{T}}_V))$. This finishes the demonstration. □

Proposition 9.5.2. *For any $w \in W$, there exists a unique (up to isomorphism) indecomposable tilting object $\widehat{\mathscr{T}}_{w,t}$ in $\widehat{D}_{[-,t]}$ such that*

$$\pi_\dagger^t(\widehat{\mathscr{T}}_{w,t}) \cong \mathscr{T}(w)_{\mathscr{L}_t^T}.$$

Any indecomposable tilting object in $\widehat{D}_{[-,t]}$ is isomorphic to some $\widehat{\mathscr{T}}_{w,t}$ for a $w \in W$.

Proof. According to lemma 9.5.1, there exists a tilting $\widehat{\mathscr{T}}$ in $\widehat{D}_{[-,t]}$ whose support is $\overline{\mathscr{X}_w}$ and such that $j_w^* \widehat{\mathscr{T}} \cong \widehat{\mathscr{L}}_t^w$. We can obviously choose $\widehat{\mathscr{T}}$ to be indecomposable since each object in the completed category splits as a direct sum of indecomposable objects, as stated in corollary 7.8.3. According to lemma 9.4.2, the endomorphism ring of $\pi_\dagger^t(\widehat{\mathscr{T}})$ is a quotient of $\mathrm{End}(\widehat{\mathscr{T}})$, hence is local (a quotient of a local ring being local). We deduce that $\pi_\dagger^t(\widehat{\mathscr{T}})$ is an indecomposable tilting perverse sheaf with support $\overline{\mathscr{X}_w}$, so it must be isomorphic to $\mathscr{T}(w)_{\mathscr{L}_t^T}$. We set $\widehat{\mathscr{T}}_{w,t} := \widehat{\mathscr{T}}$.

It remains to show that any indecomposable tilting perverse sheaf $\widehat{\mathscr{T}}'$ in $\widehat{D}_{[-,t]}$ is isomorphic to one of the $\widehat{\mathscr{T}}_{w,t}$. The above arguments tell us that $\pi_\dagger^t(\widehat{\mathscr{T}}')$ is an indecomposable tilting perverse sheaf in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$ and thus is isomorphic to a $\mathscr{T}(v)_{\mathscr{L}_t^T}$ for a certain $v \in W$. Now according to lemma 9.4.2, the map induced by $\pi_\dagger^t$

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{T}}', \widehat{\mathscr{T}}_{v,t}) \otimes_{\widehat{R}_t} \Bbbk_t \to \mathrm{Hom}_{\mathfrak{D}(X/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\pi_\dagger^t(\widehat{\mathscr{T}}'), \mathscr{T}(v)_{\mathscr{L}_t^T})$$

is an isomorphism. Thus there exists a map $f : \widehat{\mathscr{T}}' \to \widehat{\mathscr{T}}_{v,t}$ such that $\pi_\dagger^t(f)$ is an isomorphism. If C denotes the cone of f, we then have $\pi_\dagger^t(C) = 0$. Lemma 7.5.6 allows us to conclude that $C = 0$ so f is already an isomorphism. □

Remark 9.5.3. Assume that w belongs to ${}_{t'}W_t$ for some $t' \in T^\vee_{\Bbbk}$. Echoing remarks 7.8.2 and 7.8.10, we see that $\widehat{\mathscr{T}}_{w,t}$ admits another characterisation: the category $\mathfrak{P}(T \setminus\!\!\setminus \mathscr{X})_{[\mathscr{L}_{t'}^T, -]}$ is a highest weight category, and thus we have

indecomposable tilting objects ${}_{\mathscr{L}^T_{t'}}\mathscr{T}(v)$ for $v \in W$ in this category. Then $\widehat{\mathscr{T}}_{w,t}$ is the unique (up to isomorphism) indecomposable tilting object in the completed category such that

$$\Delta(e)_{\mathscr{L}^T_{t'}} \widehat{\star} \widehat{\mathscr{T}}_{w,t} \cong {}_{\mathscr{L}^T_{t'}}\mathscr{T}(v).$$

Note that even if, for $v \in {}_{t'}W_t$ the standard and costandard objects $\Delta(v)_{\mathscr{L}^T_t}, \nabla(v)_{\mathscr{L}^T_t}$ define both left and right equivariant objects, this is not the case in general for tilting objects: there is no reason for ${}_{\mathscr{L}^T_{t'}}\mathscr{T}(v)$ to belong $\mathfrak{P}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}^T_t]}$.

9.6 Associated graded functor

We fix a total order $\preceq$ on W that refines the Bruhat order. Let

$$j_{\prec w} : \bigsqcup_{v \prec w} \mathscr{X}_v \hookrightarrow \mathscr{X}$$

be the natural (closed) inclusion. For any $w \in W$ and any tilting sheaf $\widehat{\mathscr{T}} \in \widehat{D}_{[-,t]}$, we have an adjunction morphism

$$\widehat{\mathscr{T}} \to (j_{\prec w})_*(j_{\prec w})^* \widehat{\mathscr{T}}. \tag{9.6.1}$$

This morphism fits in a functorial distinguished triangle whose terms are all perverse (we use the fact that $\widehat{\mathscr{T}}$ is tilting here), therefore it is actually an exact sequence in $\widehat{\mathrm{P}}_{[-,t]}$. The morphism (9.6.1) is thus surjective, and we let $\widehat{\mathscr{T}}_{\succeq w}$ be its kernel. Note that for $w \preceq v$, the map (9.6.1) factors as a composition

$$\widehat{\mathscr{T}} \to (j_{\prec v})_*(j_{\prec v})^* \widehat{\mathscr{T}} \to (j_{\prec v})_*(j_{\prec w})_*(j_{\prec w})^*(j_{\prec v})^* \widehat{\mathscr{T}} \cong (j_{\prec w})_*(j_{\prec w})^* \widehat{\mathscr{T}}.$$

Both maps in this diagram are induced by adjunction, and we abusively keep the notation $j_{\prec w}$ for the map $\bigsqcup_{x \prec w} \mathscr{X}_x \hookrightarrow \bigsqcup_{x \prec v} \mathscr{X}_x$. From the definition of $\widehat{\mathscr{T}}_{\succeq w}$ and $\widehat{\mathscr{T}}_{\succeq v}$ as kernels, we get that the natural inclusion $\widehat{\mathscr{T}}_{\succeq v} \hookrightarrow \widehat{\mathscr{T}}$ factors via the inclusion $\widehat{\mathscr{T}}_{\succeq w} \hookrightarrow \widehat{\mathscr{T}}$. Thus $(\widehat{\mathscr{T}}_{\succeq w})_{w \in W}$ gives a filtration of $\widehat{\mathscr{T}}$, indexed by W endowed with the order opposite to $\preceq$.

Definition 9.6.1. *For $\widehat{\mathscr{T}}$ a tilting in $\widehat{\mathrm{P}}_{[-,t]}$ and a pair (w, w') with w' the successor of w for $\preceq$ (i.e. the element of W such that $w \prec w'$ and $w \preceq v \prec w'$ implies $v = w$), we consider the quotient*

$$\mathrm{gr}_w(\widehat{\mathscr{T}}) := \widehat{\mathscr{T}}_{\succeq w} / \widehat{\mathscr{T}}_{\succeq w'}.$$

We then set

$$\mathrm{gr}(\widehat{\mathscr{T}}) := \bigoplus_{w \in W} \mathrm{gr}_w(\widehat{\mathscr{T}}).$$

Lemma 9.6.2. *The assignment $\widehat{\mathscr{T}} \mapsto \mathrm{gr}(\widehat{\mathscr{T}})$ defines an additive functor*

$$\widehat{\mathrm{T}}_{[-,t]} \to \widehat{\mathrm{P}}_{[-,t]}.$$

Proof. We must show that for any morphism $f : \widehat{\mathscr{T}} \to \widehat{\mathscr{T}}'$, we have $f(\widehat{\mathscr{T}}_{\succeq w}) \subseteq \widehat{\mathscr{T}}'_{\succeq w}$. This follows from the fact that the exact sequence

$$0 \to \widehat{\mathscr{T}}_{\succeq w} \to \widehat{\mathscr{T}} \to (j_{\prec w})_*(j_{\prec w})^*\widehat{\mathscr{T}} \to 0$$

comes from a *functorial* distinguished triangle in $\widehat{D}_{[-,t]}$. This readily implies that the composition $\widehat{\mathscr{T}}_{\succeq w} \hookrightarrow \widehat{\mathscr{T}} \xrightarrow{f} \widehat{\mathscr{T}}'$ factors as $\widehat{\mathscr{T}}_{\succeq w} \xrightarrow{f} \widehat{\mathscr{T}}'_{\succeq w} \hookrightarrow \widehat{\mathscr{T}}'$, which is exactly what we want. □

We show that the functor gr is faithful. To do so, we proceed exactly as in [BeR, §6.3].

Lemma 9.6.3. *For any $v, w \in W$, with $w \neq v$, we have*

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\Delta}_{w,t}, \widehat{\Delta}_{v,t}) = 0.$$

Proof. The proof is copied on [BeR, proof of lemma 6.2]. Let $f : \widehat{\Delta}_{w,t} \to \widehat{\Delta}_{v,t}$ be a nonzero morphism. We denote by $\mathscr{F}$ the image of f and we write f as a composition $\widehat{\Delta}_{w,t} \overset{f_2}{\twoheadrightarrow} \mathscr{F} \overset{f_1}{\hookrightarrow} \widehat{\Delta}_{v,t}$, with f_2 and f_1 the natural surjection and embedding respectively. According to lemma 1.5.1, for any $c \in \Bbbk[\mathrm{X}_*(T)]$, we have a commutative diagram

$$\begin{array}{ccccc}
\widehat{\Delta}_{w,t} & \xrightarrow{f_2} & \mathscr{F} & \xrightarrow{f_1} & \widehat{\Delta}_{v,t} \\
{\scriptstyle \varphi_{l,w}(c)}\downarrow\uparrow{\scriptstyle \varphi_{r,w}(c)} & & {\scriptstyle \varphi_{l,\mathscr{F}}(c)}\downarrow\uparrow{\scriptstyle \varphi_{r,\mathscr{F}}(c)} & & {\scriptstyle \varphi_{l,v}(c)}\downarrow\uparrow{\scriptstyle \varphi_{r,v}(c)} \\
\widehat{\Delta}_{w,t} & \xrightarrow{f_2} & \mathscr{F} & \xrightarrow{f_1} & \widehat{\Delta}_{v,t}
\end{array} \tag{9.6.2}$$

where we abbreviated $\varphi_{r,w}$ for $\varphi_{r,\widehat{\Delta}_{w,t}}$ (and similarly for $\varphi_{l,\widehat{\Delta}_{w,t}}$ and for the monodromy of $\widehat{\Delta}_{v,t}$). Thanks to lemma 7.8.1, if c is nonzero, the map $\varphi_{l,v}(c)$ is injective. Since f_1 is injective and $\varphi_{l,v}(c) \circ f_1 = f_1 \circ \varphi_{l,\mathscr{F}}(c)$, the map $\varphi_{l,\mathscr{F}}(c)$ is injective if $c \neq 0$, in particular it is nonzero. Now using lemma 5.2.1, we have $\varphi_{l,v}(c) = \varphi_{r,v}(v(c))$. We then get

$$f_1 \circ \varphi_{r,\mathscr{F}}(c) = \varphi_{r,v}(c) \circ f_1 = \varphi_{l,v}(v(c)) \circ f_1 = f_1 \circ \varphi_{l,\mathscr{F}}(v(c)).$$

Since f_1 is injective, one deduces that $\varphi_{l,\mathscr{F}}(v(c)) = \varphi_{r,\mathscr{F}}(c)$. Similarly, we have

$$\varphi_{r,\mathscr{F}}(c) \circ f_2 = f_2 \circ \varphi_{r,w}(c) = f_2 \circ \varphi_{l,w}(w(c)) = \varphi_{l,\mathscr{F}}(w(c)) \circ f_2,$$

which implies that $\varphi_{l,\mathscr{F}}(w(c)) = \varphi_{r,\mathscr{F}}(c)$ since f_2 is surjective. We then obtain

$$\varphi_{l,\mathscr{F}}(v(c)) = \varphi_{r,\mathscr{F}}(c) = \varphi_{l,\mathscr{F}}(w(c)).$$

These equalities give, replacing c by $w^{-1}(c)$, the equality

$$\varphi_{l,\mathscr{F}}(c - vw^{-1}(c)) = 0$$

for any $c \in \Bbbk[\mathrm{X}_*(T)]$. Since $\varphi_{l,\mathscr{F}}(d) \neq 0$ if $d \neq 0$, we get $c = vw^{-1}(c)$, and thus $v(c) = w(c)$ for any c, so that $w = v$. This concludes the proof. □

Corollary 9.6.4. *The functor* gr *is faithful.*

Proof. The proof can literally be copied from [BeR, Corollary 6.3], so we only sketch it. We want to show that if $f : \widehat{\mathscr{T}} \to \widehat{\mathscr{T}}'$ is a nonzero map between pro-tilting objects, then $\mathrm{gr}(f)$ is nonzero. Let $w \in W$ be the largest element such that $f(\widehat{\mathscr{T}}_{\succeq w}) \subseteq \widehat{\mathscr{T}}'_{\succeq w}$ is nonzero. The maximality assumption on w gives us a nonzero map $f_w : \mathrm{gr}_w(\widehat{\mathscr{T}}) \to \widehat{\mathscr{T}}'_{\succeq w}$. Since $\widehat{\mathscr{T}}'_{\succeq w}$ admits a pro-standard filtration, it follows from lemma 9.6.3 that the map $\mathrm{Hom}(\mathrm{gr}_w(\widehat{\mathscr{T}}), \widehat{\mathscr{T}}'_{\succeq w}) \to \mathrm{Hom}(\mathrm{gr}_w(\widehat{\mathscr{T}}), \mathrm{gr}_w(\widehat{\mathscr{T}}'))$ induced by the quotient $\widehat{\mathscr{T}}'_{\succeq w} \twoheadrightarrow \mathrm{gr}_w(\widehat{\mathscr{T}}')$ is injective. In particular, the image of f_w, which is $\mathrm{gr}_w(f)$, is nonzero, whence $\mathrm{gr}(f)$ is also nonzero. We are done. □

Recall now from subsection 5.2.1 that for any monodromic pro-object $\widehat{\mathscr{F}}$, we have a monodromy morphism $\varphi_{lr,\widehat{\varphi}}$, with respect to the natural left and right actions of T on G/U.

Lemma 9.6.5. *Consider $t, t' \in T^\vee_{\Bbbk}$, and $\beta \in {}_{t'}\underline{W}_t$. We make $\widehat{R}_{t'}$ a right $(\widehat{R}_t)^{W_t^\circ}$-module via $x \cdot r = xw_\beta^{\min}(r)$. For any $\widehat{\mathscr{T}} \in \widehat{\mathrm{T}}^\beta_{[t',t]}$, the morphism $\varphi_{lr,\widehat{\mathscr{T}}}$ factors through an algebra morphism*

$$\widehat{R}_{t'} \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \to \mathrm{End}(\widehat{\mathscr{T}}).$$

Proof. Let us remark the following: by definition of ${}_{t'}\underline{W}_t$, for any $w \in \beta$ there exists a unique $v \in W_t^\circ$ such that $w = w_\beta^{\min} v$. Now, we know that $\mathrm{gr}(\widehat{\mathscr{T}}) \cong \bigoplus_{w \in W} \widehat{\Delta}_{w,t}^{\oplus n_w}$ for certain $n_w \in \mathbb{Z}_{\geq 0}$. If we have $\widehat{\mathscr{T}} \in \widehat{\mathrm{T}}^\beta_{[t',t]}$, then $n_w = 0$ for $w \notin \beta$. Using lemma 5.2.1, we see that for any $r \in (\widehat{R}_t)^{W_t^\circ}$, we have $\mathrm{gr}(\varphi_{l,\widehat{\Delta}_{w,t}}(r)) = \mathrm{gr}(\varphi_{r,\widehat{\Delta}_{w,t}}(w_\beta^{\min}(r)))$. The result then follows from corollary 9.6.4. □

9.7 Convolution of tilting objects

We study in this subsection the convolution of tilting and pro-tilting objects. First we state a result about convolution with a standard object associated to an element minimal in its block (such a standard is isomorphic to the costandard object associated to this minimal element, and is thus tilting). To address the convolution of tilting objects in general, we then study a bit further the convolution of standard and costandard objects in the Lusztig–Yun category.

Lemma 9.7.1. *Consider $t, t', t'' \in T^\vee_{\Bbbk}$ in the same W-orbit and fix two blocks $\beta \in {}_{t'}\underline{W}_t$, $\gamma \in {}_{t''}\underline{W}_{t'}$.*

1. Convolution on the right with $\Delta(w_\beta^{\min})_{\mathscr{L}_t^T}$ induces an equivalence of categories

$$(-) \star^{\mathrm{LY}} \Delta(w_\beta^{\min})_{\mathscr{L}_t^T} : \mathfrak{T}(\mathscr{X} /\!\!/ T)^\gamma_{[t'',\mathscr{L}_{t'}^T]} \to \mathfrak{T}(\mathscr{X} /\!\!/ T)^{\gamma\beta}_{[t'',\mathscr{L}_t^T]}.$$

For $w \in \gamma$, this equivalence sends $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$ to $\mathscr{T}(ww^{\min}_\beta)_{\mathscr{L}^T_t}$.

2. Convolution on the left with the pro-object $\widehat{\Delta}_{w^{\min}_\gamma,t'}$ induces an equivalence of categories

$$\widehat{\Delta}_{w^{\min}_\gamma,t'} \widehat{\star} (-) : \mathfrak{T}(\mathscr{X}/\!\!/T)^{\beta}_{[t',\mathscr{L}^T_t]} \to \mathfrak{T}(\mathscr{X}/\!\!/T)^{\gamma\beta}_{[t'',\mathscr{L}^T_t]}.$$

For $w \in \beta$, this equivalence sends $\mathscr{T}(w)_{\mathscr{L}^T_t}$ to $\mathscr{T}(w^{\min}_\gamma w)_{\mathscr{L}^T_t}$.

3. Convolution on the left (resp. right) with the pro-object $\widehat{\Delta}_{w^{\min}_\gamma,t'}$ (resp. the pro-object $\widehat{\Delta}_{w^{\min}_\beta,t}$ induces an equivalence of categories

$$\widehat{\Delta}_{w^{\min}_\gamma,t'} \widehat{\star} (-) : \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\beta}_{[t',t]} \to \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\beta}_{[t'',t]},$$

(resp.

$$(-) \widehat{\star} \widehat{\Delta}_{w^{\min}_\beta,t} : \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\beta}_{[t'',t']} \to \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\beta}_{[t'',t]}).$$

Proof. The proofs of the three points are similar. Let us show the first one.

According to lemma 8.5.2, the functor $(-) \star^{\mathrm{LY}} \Delta(w^{\min}_\beta)_{\mathscr{L}^T_t}$ is an equivalence of categories

$$\mathfrak{P}(\mathscr{X}/\!\!/T)^{\circ}_{[-,\mathscr{L}^T_{t'}]} \to \mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta}_{[-,\mathscr{L}^T_t]}$$

which maps standard objects to standard objects and costandard objects to costandard objects. Thus, this functor preserves standard and costandard filtrations, so it indeed induces an equivalence on the subcategories of tilting objects.

The image of $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$ under this equivalence is an indecomposable tilting object. The standard filtration on $\mathscr{T}(w)_{\mathscr{L}^T_{t'}}$ induces a standard filtration on its image under $(-) \star^{\mathrm{LY}} \Delta(w^{\min}_\beta)_{\mathscr{L}^T_t}$, with quotients of the form $\Delta(vw^{\min}_\beta)_{\mathscr{L}^T_t}$ (with $v \in \gamma$). In particular, we deduce an injective map

$$\Delta(ww^{\min}_\beta)_{\mathscr{L}^T_t} \to \mathscr{T}(w) \star^{\mathrm{LY}} \Delta(w^{\min}_\beta)_{\mathscr{L}^T_t}$$

whose cokernel admits a Δ-filtration. Thanks to [R, Remark 7.16] (and proposition 9.1.5), this implies that the convolution $\mathscr{T}(w) \star^{\mathrm{LY}} \Delta(w^{\min}_\beta)_{\mathscr{L}^T_t}$ must be isomorphic to $\mathscr{T}(ww^{\min}_\beta)_{\mathscr{L}^T_t}$. □

Lemma 9.7.2. *Consider $t, t', t'' \in T^\vee_{\Bbbk}$ in the same W-orbit. For any $w \in {}_{t''}W_{t'}$ and $v \in {}_{t'}W_t$, the object $\Delta(w)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}^T_t}$ lies in the subcategory generated by the objects $\Delta(x)_{\mathscr{L}^T_t}[-n]$ with $x \in {}_{t''}W_t$ and $n \geq 0$. Similarly, the object $\nabla(w)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}^T_t}$ lies in the subcategory generated by the objects $\nabla(x)_{\mathscr{L}^T_t}[n]$ with $x \in {}_{t''}W_t$ and $n \geq 0$.*

Proof. We prove only the statement about the convolution of standard, the case of costandard being similar. First note that using lemma 8.4.2, it suffices to consider the case where v is a simple reflection s in W. If $ws > w$, then $\Delta(w)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(s')_{\mathscr{L}^T_t} \cong \Delta(ws')_{\mathscr{L}^T_t}$, which settles this case. If $s \notin W^\circ_t$, then the

convolution is again isomorphic to $\Delta(ws)_{\mathscr{L}_t^T}$ according to lemma 8.3.4. Thus it remains to consider $\Delta(w)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^T}$ with $s \in W_t^\circ$ and $ws < w$. Convolving the triangle

$$\Delta(e)_{\mathscr{L}_t^T} \to \Delta(s)_{\mathscr{L}_t^T} \to \mathrm{IC}(s)_{\mathscr{L}_t^T} \xrightarrow{+1}$$

on the left with $\Delta(s)_{\mathscr{L}_t^T}$ we see that $\Delta(w)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \Delta(s)_{\mathscr{L}_t^T}$ is in the category generated by $\Delta(w)_{\mathscr{L}_t^T}$ and $\Delta(w)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^T}$. We claim that the latter object belongs to the category generated by the objects $\Delta(w)_{\mathscr{L}_t^T}[-1]$ and $\Delta(ws)_{\mathscr{L}_t^T}[-1]$, which would conclude the proof. In order to show that, we use lemma to switch to the B_s-equivariant equivalent setting using the equivalence of lemma 5.3.4; namely, we consider the object $\Delta(w)_{\mathscr{L}_t^{B_s}} \star^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}}$ (lemma 5.3.7 and lemma 6.3.2 insure that this is indeed the correct object to consider). Now, we have isomorphisms

$$\Delta(w)_{\mathscr{L}_t^{B_s}} \star^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^{B_s}} \cong \mathrm{For}_t^s \mathrm{Av}_{t,*}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})[1] \cong \mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}})[-1].$$

Here, the first isomorphism comes from lemma 8.3.6, and the second one from the definition of the averaging functors (see subsection 8.1.3). With the triangle

$$\Delta(ws)_{\mathscr{L}_t^{B_s}} \to \Delta(w)_{\mathscr{L}_t^{B_s}} \to \mathrm{For}_t^s \mathrm{Av}_{t,!}^s(\Delta(w)_{\mathscr{L}_t^{B_s}}) \xrightarrow{+1}$$

from lemma 8.2.9, and going back to the T-equivariant setting using lemma 5.3.4 one more time, we see that $\Delta(w)_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \mathrm{IC}(s)_{\mathscr{L}_t^T}$ indeed lies in the category generated by $\Delta(w)_{\mathscr{L}_t^T}[-1]$ and $\Delta(ws)_{\mathscr{L}_t^T}[-1]$. This allows us to conclude the proof. □

Lemma 9.7.3. *Consider $t, t', t'' \in T_{\Bbbk}^\vee$ in the same W-orbit. For any $w \in {}_{t''}W_{t'}$ and $v \in {}_{t'}W_t$*

(1) the functors

$$\Delta(w)_{\mathscr{L}_{t'}^T} \star^{\mathrm{LY}} (-) : \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t'}^T, \mathscr{L}_t^T]} \to \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t''}^T, \mathscr{L}_t^T]}$$

and

$$(-) \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_t^T} : \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t''}^T, \mathscr{L}_{t'}^T]} \to \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t''}^T, \mathscr{L}_t^T]}$$

are left t-exact for the perverse t-structure,

(2) the functors

$$\nabla(w)_{\mathscr{L}_{t'}^T} \star^{\mathrm{LY}} (-) : \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t'}^T, \mathscr{L}_t^T]} \to \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t''}^T, \mathscr{L}_t^T]}$$

and

$$(-) \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}_t^T} : \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t''}^T, \mathscr{L}_{t'}^T]} \to \mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t''}^T, \mathscr{L}_t^T]}$$

are right t-exact for the perverse t-structure.

In particular, for any $w \in {}_{t''}W_{t'}$ and $v \in {}_{t'}W_t$, the objects $\Delta(w)_{\mathscr{L}_{t'}^T} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}_t^T}$ and $\nabla(w)_{\mathscr{L}_{t'}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_t^T}$ are perverse.

Proof. We show only the first case of the first point, the other proofs are similar.

We consider the functor $\Delta(w)_{\mathscr{L}^{T_s}_{t'}} \star^{\mathrm{LY}} (-)$. As a composition of left t-exact functors is t-exact it suffices to consider the case $w = s$ a simple reflection in W. Moreover, since

$$ {}^p\mathfrak{D}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)^{\geq 0}_{[\mathscr{L}^T_{t'}, \mathscr{L}^T_t]} $$

is generated by the objects $\nabla(x)_{\mathscr{L}^T_t}[-n]$ with $x \in {}_{t'}W_t$ and $n \geq 0$, we can restrict ourselves to the computation of $\Delta(s)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \nabla(x)_{\mathscr{L}^T_t}$. If $s \notin W^\circ_{t'}$ or $sx < x$ then this convolution is isomorphic to $\nabla(sx)_{\mathscr{L}^T_t}$, (using corollaries 8.3.5 and 8.3.8) so this settles these cases.

Assume now that $s \in W^\circ_{t'}$ and $sx > x$. We use the equivalence of lemma 5.3.4 and place ourselves in the monodromic categories $\mathfrak{D}(B_s \setminus\!\setminus \mathscr{X}^s /\!\!/ T)_{[\mathscr{L}^{B_s}_?, \mathscr{L}^T_t]}$ (where ? could be either t' or t'').

We start by applying the functor $(-) \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t}$ to the triangle

$$ \nabla(e)_{\mathscr{L}^{B_s}_{t'}}[-1] \to \mathrm{IC}(s)_{\mathscr{L}^{B_s}_{t'}} \to \nabla(s)_{\mathscr{L}^{B_s}_{t'}} \xrightarrow{+1} . $$

One obtains a distinguished triangle

$$ \nabla(x)_{\mathscr{L}^{B_s}_t}[-1] \to \mathrm{IC}(s)_{\mathscr{L}^{B_s}_{t'}} \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t} \to \nabla(sx)_{\mathscr{L}^{B_s}_t} \xrightarrow{+1}, $$

which implies that $\mathrm{IC}(s)_{\mathscr{L}^{B_s}_{t'}} \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t}$ lies in ${}^p\mathfrak{D}(B_s \setminus\!\setminus \mathscr{X}^s /\!\!/ B_s)^{\geq 0}_{[\mathscr{L}^{B_s}_{t'}, \mathscr{L}^{B_s}_t]}$. We finally apply $(-) \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t}$ to the distinguished triangle

$$ \Delta(e)_{\mathscr{L}^{B_s}_{t'}} \to \Delta(s)_{\mathscr{L}^{B_s}_{t'}} \to \mathrm{IC}(s)_{\mathscr{L}^{B_s}_{t'}} \xrightarrow{+1} $$

to get

$$ \nabla(x)_{\mathscr{L}^{B_s}_t} \to \Delta(s)_{\mathscr{L}^{B_s}_{t'}} \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t} \to \mathrm{IC}(s)_{\mathscr{L}^{B_s}_{t'}} \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t} \xrightarrow{+1} . $$

The above consideration on $\mathrm{IC}(s)_{\mathscr{L}^{B_s}_{t'}} \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t}$ tells us that the convolution $\Delta(s)_{\mathscr{L}^{B_s}_{t'}} \star^{\mathrm{LY}}_s \nabla(x)_{\mathscr{L}^{B_s}_t}$ is indeed in ${}^p\mathfrak{D}(B_s \setminus\!\setminus \mathscr{X}^s /\!\!/ B_s)^{\geq 0}_{[\mathscr{L}^{B_s}_{t'}, \mathscr{L}^{B_s}_t]}$, which concludes the proof.

The fact that the objects $\Delta(w)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \nabla(v)_{\mathscr{L}^T_t}$ and $\nabla(w)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}^T_t}$ are perverse is an immediate consequence of (1) and (2). □

Remark 9.7.4. The statements of lemma 9.7.3 are obviously still true if one forgets one side of the equivariance: for example, the functor $\Delta(w)_{\mathscr{L}^T_{t'}} \star^{\mathrm{LY}} (-) : \mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}^T_{t'}, -]} \to \mathfrak{D}(T \setminus\!\setminus \mathscr{X})_{[\mathscr{L}^T_{t''}, -]}$ is still left t-exact for the perverse t-structure.

The following result is copied from [BY, Proposition 4.3.4].

Lemma 9.7.5. *Consider $t, t', t'' \in T^\vee_{\Bbbk}$ in the same W-orbit. For $\mathscr{T} \in \widehat{D}_{[t'', t']}$ and $\mathscr{T}' \in \widehat{D}_{[t', t]}$ two tilting objects, the convolution $\mathscr{T} \widehat{\star} \mathscr{T}'$ is again a tilting object.*

Proof. First, thanks to lemma 9.4.2, it suffices to show that $\pi_\dagger^t(\mathscr{T}\,\widehat{\star}\,\mathscr{T}') \cong \mathscr{T}\,\widehat{\star}\pi_\dagger^t(\mathscr{T}')$ is tilting in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t'',\mathscr{L}_t^T]}$. The object $\pi_\dagger^t(\mathscr{T}')$ is tilting in $\mathfrak{D}(\mathscr{X}/\!\!/T)_{[t',\mathscr{L}_t^T]}$, so it admits a Δ-filtration. Similarly, $\mathscr{T}$ admits a $\widehat{\nabla}$-filtration. Consider x and y two elements of W such that $\Delta(x)_{\mathscr{L}_t^T}$ appears in a Δ-filtration of $\pi_\dagger^t(\mathscr{T}')$ and $\widehat{\Delta}_{y,t'}$ appears in a $\widehat{\Delta}$-filtration of $\mathscr{T}$. One has the following isomorphisms:

$$\begin{aligned}\widehat{\Delta}_{y,t'}\,\widehat{\star}\,\Delta(x)_{\mathscr{L}_t^T} &\cong \widehat{\Delta}_{y,t'}\,\widehat{\star}\,\mathrm{For}^t(\Delta(x)_{\mathscr{L}_t^T})\\ &\cong \pi_\dagger^{t'}(\widehat{\Delta}_{y,t'})\star^{\mathrm{LY}}\Delta(x)_{\mathscr{L}_t^T} \quad \text{(Lemma 7.1.5)}\\ &\cong \Delta(y)_{\mathscr{L}_{t'}^T}\star^{\mathrm{LY}}\Delta(x)_{\mathscr{L}_t^T}. \quad \text{(Lemma 7.7.4)}\end{aligned}$$

According to lemma 9.7.3, the object $\pi_\dagger^t(\mathscr{T}\,\widehat{\star}\,\mathscr{T}')$ lies in the subcategory generated by the objects $\Delta(w)_{\mathscr{L}_t^T}[-n]$ with $W \in {}_{t''}W_t$ and $n \geq 0$. Similar considerations for (pro-) costandard filtrations imply that $\pi_\dagger^t(\mathscr{T}\,\widehat{\star}\,\mathscr{T}')$ also lies in the subcategory generated by $\nabla(w)_{\mathscr{L}_t^T}[n]$ with $w \in_{t''} W_t$ and $n \geq 0$. This readily implies that $(j_w)^*\pi_\dagger^t(\mathscr{T}\,\widehat{\star}\,\mathscr{T})$ and $(j_w)^!\pi_\dagger^t(\mathscr{T}\,\widehat{\star}\,\mathscr{T})$ are perverse for any $w \in W$. Thanks to lemma 7.8.7, this tells us that $\mathscr{T}\,\widehat{\star}\,\mathscr{T}$ is tilting, and hence concludes the proof. □

Proposition 9.7.6. *Consider four elements $t_1, t_2, t_3, t_4 \in T_{\Bbbk}^\vee$ in the same W-orbit. Fix three blocks*

$$\delta \in {}_{t_4}\underline{W}_{t_3}, \quad \beta \in {}_{t_3}\underline{W}_{t_2}, \quad \gamma \in {}_{t_2}\underline{W}_{t_1}.$$

Finally, let α be the block $\delta\beta\gamma \in {}_{t_4}\underline{W}_{t_1}$. Then for any $w \in \delta$ and any $v \in \gamma$, we have an isomorphism

$$\widehat{\Delta}_{w,t_3}\,\widehat{\star}\,\widehat{\mathscr{T}}_{w_\beta^{\max},t_2}\,\widehat{\star}\,\widehat{\Delta}_{v,t_1} \cong \widehat{\mathscr{T}}_{w_\alpha^{\max},t_1}.$$

Proof. Of course it suffices to show that we have isomorphisms $\widehat{\mathscr{T}}_{w_\beta^{\max},t_2}\,\widehat{\star}\,\widehat{\Delta}_{v,t_1} \cong \widehat{\mathscr{T}}_{w_{\beta\gamma}^{\max},t_1}$ and $\widehat{\Delta}_{w,t_3}\,\widehat{\star}\,\widehat{\mathscr{T}}_{w_\beta^{\max},t_2} \cong \widehat{\mathscr{T}}_{w_{\delta\beta}^{\max},t_2}$. We start by showing the first isomorphism. It is enough to show that

$$\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}\star^{\mathrm{LY}}\Delta(v)_{\mathscr{L}_{t_1}^T} \cong \mathscr{T}(w_{\beta\gamma}^{\max})_{\mathscr{L}_{t_1}^T}. \tag{9.7.1}$$

Indeed, we have $\pi_\dagger^{t_1}(\widehat{\mathscr{T}}_{w_\beta^{\max},t_2}\,\widehat{\star}\,\widehat{\Delta}_{v,t_1}) \cong \mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}\star^{\mathrm{LY}}\Delta(v)_{\mathscr{L}_{t_1}^T}$. If we can show isomorphism (9.7.1), then lemma 9.4.2 will tell us that $\widehat{\mathscr{T}}_{w_\beta^{\max},t_2}\,\widehat{\star}\,\widehat{\Delta}_{v,t_1}$ is tilting, and proposition 9.5.2 will give us $\widehat{\mathscr{T}}_{w_\beta^{\max},t_2}\,\widehat{\star}\,\widehat{\Delta}_{v,t_1} \cong \widehat{\mathscr{T}}_{w_{\beta\gamma}^{\max},t_1}$.

We will show that $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}\star^{\mathrm{LY}}\Delta(v)_{\mathscr{L}_{t_1}^T}$ is the projective cover of the simple object $\mathrm{IC}(w_{\beta\gamma}^{\min})$ in the category $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}$. Thanks to lemma 9.3.4, this will prove the existence of the wished-for isomorphism. First, since $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}$ admits a ∇-filtration, lemma 9.7.3 tells us that the convolution

$\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_{t_1}^T}$ is a perverse object. Let us show that it is projective. As explained in lemma 9.2.1, it suffices to show that

$$\begin{aligned}\mathrm{Ext}^i_{\mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}}&(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_{t_1}^T}, \Delta(x)_{\mathscr{L}_{t_1}^T}) \\ &\cong \mathrm{Hom}_{\mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_{t_1}^T}, \Delta(x)_{\mathscr{L}_{t_1}^T}[i]) = 0\end{aligned}$$

for any $\Delta(x)_{\mathscr{L}_{t_1}^T}$ in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}$ and any $i > 0$. We use the fact that $(-) \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}$ gives an equivalence

$$\mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]} \xrightarrow{\sim} \mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_2}^T]},$$

with quasi-inverse $(-) \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_{t_1}^T}$ (see lemma 8.3.9). Thus we have to consider

$$\mathrm{Hom}_{\mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_2}^T]}}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}, \Delta(x)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}[i]).$$

The latter Hom-space is isomorphic to

$$\mathrm{Ext}^i_{\mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_2}^T]}}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}, \Delta(x)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}),$$

which is zero since $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}$ is projective in $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t_3,\mathscr{L}_{t_2}^T]}$. Finally, $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_{t_1}^T}$ is a projective perverse object.

We now proceed to show that this object is indeed the projective cover of the IC-sheaf with minimal support in the block $\mathfrak{P}(\mathscr{X}/\!\!/T)^{\beta\gamma}_{[t_3,\mathscr{L}_{t_1}^T]}$. Fix $x \in \beta\gamma$. We have to show the following:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}}&(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T} \star^{\mathrm{LY}} \Delta(v)_{\mathscr{L}_{t_1}^T}, \mathrm{IC}(x)_{\mathscr{L}_{t_1}^T}) \\ &= \begin{cases} \Bbbk & \text{if } x = w_{\beta\gamma}^{\min} \\ 0 & \text{otherwise.} \end{cases} \qquad (9.7.2)\end{aligned}$$

We will abbreviate $\mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}}$ by $\mathrm{Hom}_{\mathfrak{P}(t_3,\underline{t_1})}$ and $\mathrm{Hom}_{\mathfrak{D}(X/\!\!/T)_{[t_3,\mathscr{L}_{t_1}^T]}}$ by $\mathrm{Hom}_{\mathfrak{D}(t_3,\underline{t_1})}$ (and similarly for other monodromies). Once again we use the equivalence $(-) \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}$. We want to determine

$$\mathrm{Hom}_{\mathfrak{D}(t_3,\underline{t_1})}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_{t_2}^T}, \mathrm{IC}(x)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}), \qquad (9.7.3)$$

first in the case $x \neq w_{\beta\gamma}^{\min}$. We proceed by induction on $\ell(x)$. Precisely, we show by induction on $\ell(x)$ the following property:

"For any block $\beta \in {}_{t_3}\underline{W}_{t_2}$, any block $\gamma \in {}_{t_2}\underline{W}_{t_1}$ and any elements $v \in \gamma$, $x \in \beta\gamma$ with $x \neq w_{\beta\gamma}^{\min}$, the space (9.7.3) is zero."

We will allow the block β to vary during the induction. If $\ell(x) = 1$, then x must be a simple reflection s of W lying in $W_{t_1}^\circ$ (since x is not minimal in its block by assumption). In this case, taking into account the equivalence of lemma 5.3.4, the object $\mathrm{IC}(x)_{\mathscr{L}_{t_1}^T}$ is the image of $\mathrm{IC}(s)_{\mathscr{L}_{t_1}^{L_s}}$ under the forgetful functor

$$\mathrm{For}_s^{t_1} : \mathfrak{P}(L_s \setminus\!\setminus \mathscr{X}^s /\!\!/ T)_{[\mathscr{L}_{t_1}^{L_s}, \mathscr{L}_{t_1}^T]} \to \mathfrak{P}(B_s \setminus\!\setminus \mathscr{X}^s /\!\!/ T)_{[\mathscr{L}_{t_1}^{B_s}, \mathscr{L}_{t_1}^T]} \cong \mathfrak{P}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_{t_1}^T, \mathscr{L}_{t_1}^T]}.$$

In particular, this perverse sheaf is $\mathscr{L}_{t_1}^{L_s}$-equivariant on the left. Then, this is also the case for the composition factors of the perverse cohomology objects of $\mathrm{IC}(x)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}$ (since the above forgetful functor preserves simple perverse sheaves). In particular, these composition factors, which are indexed by elements lying in the block β, are constructible for the L_s-orbits, so in particular indexed by element y of W satisfying $sy < y$. This readily implies that none of these factors is isomorphic to $\mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_{t_2}^T}$: since $w_\beta^{\min}$ is minimal in β and $sw_\beta^{\min}$ belongs to β, we necessarily have $sw_\beta^{\min} > w_\beta^{\min}$.

Assume that (9.7.2) is true for any x of length $n \geq 1$, with $x \neq w_{\beta\gamma}^{\min}$ and consider an element x of length $n+1$ in $\beta\gamma$. There exists s a simple reflection in W satisfying $sx < x$. If $s \in W_{t_2}^\circ$, then the exact same argument as in the case $\ell(x) = 1$ allows us to conclude that (9.7.2) is true for our x.

If $s \notin W_{t_2}^\circ$, then s is minimal in the block it belongs to. Since x is not minimal in its block, the product sx cannot be minimal in its own block, by virtue of lemma 5.1.2. Convolution on the left with the object $\widehat{\Delta}_{s,t_3} \cong \widehat{\nabla}_{s,t_3}$ is an equivalence

$$\mathfrak{P}(\mathscr{X} /\!\!/ T)_{[t_3, \mathscr{L}_{t_1}^T]} \xrightarrow{\sim} \mathfrak{P}(\mathscr{X} /\!\!/ T)_{[s(t_3), \mathscr{L}_{t_1}^T]}.$$

Applying this functor to (9.7.3), we get the space

$$\mathrm{Hom}_{\mathfrak{P}(s(t_3), \underline{t_1})}(\mathscr{T}(sw_\beta^{\max})_{\mathscr{L}_{t_2}^T}, \mathrm{IC}(sx)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}).$$

The element $sw_\beta^{\max}$ is maximal in its block (again thanks to lemma 5.1.2) and so we can conclude that this Hom-space is zero by induction hypothesis. We proved the case "otherwise" of (9.7.2).

It remains to show the case $x = w_{\beta\gamma}^{\min}$. Since x is assumed to be minimal, convolution on the left with $\mathrm{IC}(x)_{\mathscr{L}_{t_1}^T}$ maps $\mathrm{IC}(y)_{\mathscr{L}_{t_2}^T}$ to $\mathrm{IC}(xy)_{\mathscr{L}_{t_2}^T}$ for any $y \in {}_{t_1}W_{t_2}$. Lemma 8.6.1 implies that there is only one composition factor of the perverse object $\mathrm{IC}(x)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \nabla(v^{-1})_{\mathscr{L}_{t_2}^T}$ which is isomorphic to $\mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_{t_2}^T}$ (namely, $\mathrm{IC}(x)_{\mathscr{L}_{t_1}^T} \star^{\mathrm{LY}} \mathrm{IC}(w_\gamma^{\min})_{\mathscr{L}_{t_2}^T}$). Lemma 9.3.4 implies that the Hom-space (9.7.3) is one dimensional, and thus we finally proved (9.7.2).

To prove the isomorphism $\widehat{\Delta}_{w,t_3} \widehat{\star} \widehat{\mathscr{T}}_{w_\beta^{\max}, t_2} \cong \widehat{\mathscr{T}}_{w_{\delta\beta}^{\max}, t_2}$, one can use the exact same argument, replacing $\pi_\dagger^t$ by the functor $\Delta(e)_{\mathscr{L}_{t_3}^T} \widehat{\star}(-)$ and using remark 9.5.3. The details are left to the reader. □

We deduce from the preceding proposition the following.

Corollary 9.7.7. *Consider three elements $t, t', t'' \in T^\vee_{\Bbbk}$ in the same W-orbit. Let $\gamma \in {}_{t'}\underline{W}_t$ and $\beta \in {}_{t''}\underline{W}_{t'}$ be any fixed blocks. Consider $\widehat{\mathscr{T}}$ a tilting object in $\widehat{\mathrm{P}}^\beta_{[t',t]}$ and $\widehat{\mathscr{T}}'$ a tilting object in $\widehat{\mathrm{P}}^\gamma_{[t'',t']}$. Then the convolutions*

$$\widehat{\mathscr{T}} \mathbin{\widehat{\star}} \widehat{\mathscr{T}}_{w_\gamma^{\max},t} \quad \text{and} \quad \widehat{\mathscr{T}}_{w_\beta^{\max},t'} \mathbin{\widehat{\star}} \widehat{\mathscr{T}}'$$

are isomorphic to direct sums of copies of $\widehat{\mathscr{T}}_{w_{\beta\gamma}^{\max},t}$.

Proof. Let us prove only the first isomorphism, the other one can be proved similarly. Being tilting, the object $\widehat{\mathscr{T}}$ admits a $\widehat{\Delta}$-filtration, that is, we have exact sequences of the form $\widehat{\mathscr{T}}_i \hookrightarrow \widehat{\mathscr{T}}_{i+1} \twoheadrightarrow \widehat{\Delta}_{w_i,t}$ for $\{\widehat{\mathscr{T}}\}_{i=1,\dots,n}$ a family of subobjects (and as such, tilting objects) of $\widehat{\mathscr{T}}$ and $w_i \in \beta$. Convolving these exact sequences on the right with $\widehat{\mathscr{T}}_{w_\gamma^{\max},t}$ and using lemmas 9.7.5 and 9.7.6, we obtain an exact sequence whose terms are all tilting objects, the third one being $\widehat{\mathscr{T}}_{w_{\beta\gamma}^{\max},t}$. Now one remark that any extension between tilting objects is split (using the equalities (7.8.1) for example). The lemma then follows by an easy induction argument. □

9.8 The objects $\widehat{\mathscr{T}}_{s,t}$ and $\mathscr{T}(s)_{\mathscr{L}^T_t}$

We finish this chapter with a study of (pro-) tilting objects associated to simple reflections of W_t°. In particular, we show that these objects are "constructed as in the classical case", that is to say, have the correct standard and costandard filtrations. We then proceed to describe the monodromy of these tilting objects. Finally, the results obtained allow us to describe quite explicitly the effect of convolution with such an object, and we obtain as a corollary a "Bott–Samelson" type description of tilting objects in the neutral block.

9.8.1 Standard and costandard filtrations

We begin by the description of the object $\mathscr{T}(s)_{\mathscr{L}^T_t}$ for s a reflection in W, simple in W_t°. We show that the objects $\Delta(s)_{\mathscr{L}^T_t}$ and $\Delta(e)_{\mathscr{L}^T_t}$ appear with multiplicity one (and are the only standards that appear) in a standard filtration of $\mathscr{T}(s)_{\mathscr{L}^T_t}$ and similarly for the objects $\nabla(e)_{\mathscr{L}^T_t}$ and $\nabla(s)_{\mathscr{L}^T_t}$ in a costandard filtration. Together with lemma 7.8.9, this will imply a similar claim for the object $\widehat{\mathscr{T}}_{s,t}$.

Assume first that s is simple in W (and so s is also simple in W_t°). From lemma 8.2.9, we get an exact sequence

$$\mathrm{IC}(e)_{\mathscr{L}^T_t} \hookrightarrow \Delta(s)_{\mathscr{L}^T_t} \twoheadrightarrow \mathrm{IC}(s)_{\mathscr{L}^T_t},$$

and this sequence is non split, because $\Delta(s)_{\mathscr{L}^T_t}$ does not admit any nonzero map to $\mathrm{IC}(e)_{\mathscr{L}^T_t}$. Similarly, we have a non split exact sequence

$$\mathrm{IC}(s)_{\mathscr{L}^T_t} \hookrightarrow \nabla(s)_{\mathscr{L}^T_t} \twoheadrightarrow \mathrm{IC}(e)_{\mathscr{L}^T_t}.$$

Lemma A.0.8 implies that $\mathscr{T}(s)_{\mathscr{L}_t^T}$ admits a standard filtration with subquotients given by $\Delta(s)_{\mathscr{L}_t^T}$ and $\Delta(e)_{\mathscr{L}_t^T}$, as well as a costandard filtration with subquotients given by $\nabla(e)_{\mathscr{L}_t^T}$ and $\nabla(s)_{\mathscr{L}_t^T}$.

The case s simple in W_t°

We now treat the case s simple in W_t° but not simple in W. According to lemma 5.1.5, we have

$$s = w_\beta^{\min} \cdot r \cdot w_\beta^{\min,-1},$$

with β a block in (say) ${}_t\underline{W}_{t'}$ and $r \in W_{t'}^\circ$ a reflection, simple in W. Now we use lemma 9.7.1 to get that the functor $\widehat{\Delta}_{w_\beta^{\min},t'} \widehat{\star}(-) \star^{\mathrm{LY}} \Delta(w_\beta^{\min,-1})_{\mathscr{L}_t^T}$ is an equivalence of categories

$$\mathfrak{T}(\mathscr{X} /\!\!/ T)_{[t',\mathscr{L}_{t'}^T]} \to \mathfrak{T}(\mathscr{X} /\!\!/ T)_{[t,\mathscr{L}_t^T]}.$$

We deduce (using proposition 8.5.2 and lemma 8.4.3 and its proof) from the case s simple in W that $\mathscr{T}(s)_{\mathscr{L}_t^T}$ admits a standard filtration with two subquotients, given by $\Delta(s)_{\mathscr{L}_t^T}$ and $\Delta(e)_{\mathscr{L}_t^T}$, as well as a costandard filtration with subquotients given by $\nabla(e)_{\mathscr{L}_t^T}$ and $\nabla(s)_{\mathscr{L}_t^T}$.

9.8.2 Monodromy

As explained in subsection 5.3.1, we have a fully faithful functor

$$\mathfrak{P}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]} \to \mathfrak{P}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]}.$$

In fact, lemma 9.8.1 easily implies that this functor induces an equivalence

$$\mathfrak{P}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{P}(\mathscr{X} /\!\!/ T)_{[\underline{t},\mathscr{L}_t^T]}. \tag{9.8.1}$$

Lemma 9.8.1. *Consider $t \in T_{\Bbbk}^\vee$ and s a simple reflection in W_t°. Then $\mathscr{T}(s)_{\mathscr{L}_t^T}$ does not belong to the category $\mathfrak{P}(T \setminus\!\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}$, i.e. this object does not belong to the essential image of the above functor.*

Before giving the proof, let us recall a few standard facts: first, on the equivariant cohomology $\mathbf{H}_H^\bullet(\{\mathrm{pt}\},\Bbbk)$, for H a complex algebraic torus: we have $\mathbf{H}_H^1(\{\mathrm{pt}\},\Bbbk) = 0$ and $\mathbf{H}_H^2(\{\mathrm{pt}\},\Bbbk) \cong \Bbbk \otimes_{\mathbb{Z}} \mathrm{X}^*(H)$. An isomorphism can be constructed in the following way. For a character $\lambda \in \mathrm{X}^*(H)$, consider the H-space $\mathbb{C}_\lambda$ where the action of $h \in H$ is given by multiplication by $\lambda(h)$. The closed subvariety $\{0\}$ is H-stable for this action; let $i_{\{0\}} : \{0\} \hookrightarrow \mathbb{C}_\lambda$ be the inclusion. The adjunction morphisms

$$(i_{\{0\}})_!(i_{\{0\}})^! \underline{\Bbbk}_{\mathbb{C}_\lambda} \to \underline{\Bbbk}_{\mathbb{C}_\lambda} \to (i_{\{0\}})_*(i_{\{0\}})^* \underline{\Bbbk}_{\mathbb{C}_\lambda}$$

define an element in $\mathrm{Hom}_{D_H^b(\{0\},\Bbbk)}(\underline{\Bbbk}_{\{0\}}, \underline{\Bbbk}_{\{0\}}[2]) = \mathbf{H}_H^2(\{0\},\Bbbk)$, which is defined to be the image of λ.

Then, for α the root associated to s as in the statement of the lemma, we have $\langle \alpha, \alpha^\vee \rangle = 2$. Since for any character $\lambda \in \mathrm{X}^*(T)$, we have $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$, this implies that $\alpha = p\lambda$ with λ a character and $p \in \mathbb{Z}$ a prime number is possible only for $p = 2$.

Proof. We know that the object $\mathscr{T}(s)_{\mathscr{L}_t^T}$ is a non-split extension of $\Delta(e)_{\mathscr{L}_t^T}$ by $\Delta(s)_{\mathscr{L}_t^T}$ in the category $\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}$. The proof consists in showing that

$$\mathrm{Ext}^1_{\mathfrak{P}(T\backslash\!\backslash\mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\Delta(e)_{\mathscr{L}_t^T},\Delta(s)_{\mathscr{L}_t^T})=0.$$

We use the isomorphism

$$\mathrm{Ext}^1_{\mathfrak{P}(T\backslash\!\backslash\mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\Delta(e)_{\mathscr{L}_t^T},\Delta(s)_{\mathscr{L}_t^T}) \cong \mathrm{Hom}_{\mathfrak{P}(T\backslash\!\backslash\mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\Delta(e)_{\mathscr{L}_t^T},\Delta(s)_{\mathscr{L}_t^T}[1]).$$

Let us first show that it suffices to consider the case where $s\in W_t^\circ$ is simple in W. In fact, as in the end of subsection 9.8.1, for s simple in W_t°, we can write $s=w_\beta^{\min}\cdot r\cdot w_\beta^{\min,-1}$ for some block $\beta\in {}_t\underline{W}_{t'}$ and $r\in W_{t'}^\circ$. The functor $\Delta(w_\beta^{\min})_{\mathscr{L}_{t'}^T}\star^{\mathrm{LY}}(-)\star^{\mathrm{LY}}\Delta(w_\beta^{\min,-1})_{\mathscr{L}_{t'}^T}$ then induces an isomorphism

$$\mathrm{Hom}_{\mathfrak{P}(T\backslash\!\backslash\mathscr{X}/\!\!/T)_{[\mathscr{L}_{t'}^T,\mathscr{L}_{t'}^T]}}(\Delta(e)_{\mathscr{L}_t^T},\Delta(r)_{\mathscr{L}_t^T}[1]) \cong \mathrm{Hom}_{\mathfrak{P}(T\backslash\!\backslash\mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\Delta(e)_{\mathscr{L}_t^T},\Delta(s)_{\mathscr{L}_t^T}[1]).$$

From now on, we assume that s is simple in W; let us abbreviate $\mathfrak{D}(T\setminus\!\!\setminus\mathscr{X}/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}=\mathfrak{D}(t,t)$. From the short exact sequence $\mathrm{IC}(e)_{\mathscr{L}_t^T}\hookrightarrow\Delta(s)_{\mathscr{L}_t^T}\twoheadrightarrow\mathrm{IC}(s)_{\mathscr{L}_t^T}$, we obtain the following long exact sequence:

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T},\mathrm{IC}(e)_{\mathscr{L}_t^T}[1])\to\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T},\Delta(s)_{\mathscr{L}_t^T}[1])\\&\quad\to\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T},\mathrm{IC}(s)_{\mathscr{L}_t^T}[1])\to\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T},\mathrm{IC}(e)_{\mathscr{L}_t^T}[2]).\end{aligned}$$

We will evaluate these four terms. We start with the first and fourth ones. Using adjunction, we have to evaluate

$$\mathrm{Hom}_{\mathfrak{D}(T\backslash\!\backslash\mathscr{X}_e/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\mathscr{L}_t^e,\mathscr{L}_t^e[i])$$

with $i=1,2$. With the isomorphism $T\cong\mathscr{X}_e$, this amounts to evaluating

$$\mathrm{Hom}_{\mathfrak{D}(T\backslash\!\backslash T/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\mathscr{L}_t^T,\mathscr{L}_t^T[i]).$$

Now let us fix a finite central isogeny $\nu:\widetilde{T}\to T$ as in §4.1.1. Recall the following facts: the pushforward $\nu_*\,\underline{\Bbbk}_{\widetilde{T}}$ splits as a direct sum of simple local systems, and $\mathscr{L}_t^T$ is a direct summand there; the category $\mathfrak{D}(T\setminus\!\!\setminus T/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}$ is a full subcategory in $D^b_{\widetilde{T}\times\widetilde{T}}(T,\Bbbk)$. Moreover, there is no nonzero morphism from $\mathscr{L}_t^T$ to any other direct summand in $\nu_*\,\underline{\Bbbk}_{\widetilde{T}}$. We thus have

$$\mathrm{Hom}_{\mathfrak{D}(T\backslash\!\backslash T/\!\!/T)_{[\mathscr{L}_t^T,\mathscr{L}_t^T]}}(\mathscr{L}_t^T,\mathscr{L}_t^T[i])\ \cong\ \mathrm{Hom}_{D^b_{\widetilde{T}\times\widetilde{T}}(T,\Bbbk)}(\mathscr{L}_t^T,\nu_*\,\underline{\Bbbk}_{\widetilde{T}}[i]).$$

Using adjunction and the isomorphism $\nu^*\,\mathscr{L}_t^T\cong\underline{\Bbbk}_{\widetilde{T}}$, we now have to evaluate

$$\mathrm{Hom}_{D^b_{\widetilde{T}\times\widetilde{T}}(\widetilde{T},\Bbbk)}(\underline{\Bbbk}_{\widetilde{T}},\underline{\Bbbk}_{\widetilde{T}}[i])\cong\mathbf{H}^i_{\widetilde{T}}(\{\mathrm{pt}\},\Bbbk).$$

This is well known to be zero if $i = 1$ and (isomorphic to) $\Bbbk \otimes_{\mathbb{Z}} \mathrm{X}^*(\widetilde{T})$ for $i = 2$, as stated before the proof of the lemma.

We now deal with the term $\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T}, \mathrm{IC}(s)_{\mathscr{L}_t^T}[1])$. Since $s \in W_t^\circ$, we have $\mathrm{IC}(s)_{\mathscr{L}_t^T} \cong (j_{e,s})_* \mathscr{L}_t^{e,s}[1]$ (see §8.2.2 for the notation). The Hom-space we wish to understand is then isomorphic to

$$\mathrm{Hom}_{D^b_{(B),\widetilde{T}\times\widetilde{T}}(\overline{\mathscr{X}_s},\Bbbk)}((j_e)_!\,\mathscr{L}_t^e, \mathscr{L}_t^{e,s}[2]).$$

(Here, we used the same considerations as above on equivariant categories.) Let $\mathscr{X}_s^\circ := \overline{\mathscr{X}_s} \smallsetminus \{\dot{s}tU_s \mid t \in T\}$; this variety is $\widetilde{T}\times\widetilde{T}$-stable. The map $U_s^- \times T \to \mathscr{X}_s^\circ$ given by $(u, \theta) \mapsto u\theta U$ is easily seen to be an isomorphism of varieties. It is moreover $\widetilde{T} \times \widetilde{T}$-equivariant if we let this group act via

$$(t_1, t_2) \cdot (u, \theta) = (\nu(t_1)u\nu(t_1)^{-1}, \nu(t_1)\theta\nu(t_2)).$$

Note that U_s^- is an affine space of dimension 1; considering the isomorphism $u_{-\alpha} : \mathbb{A}^1 \to U_s^-$ (see subsection 5.1.1), we get that the action of the element (t_1, t_2) on U_s^- is given by multiplication by the element $-\alpha(t_1)$.

Restriction along the map $i_s^\circ : \mathscr{X}_s^\circ \hookrightarrow \overline{\mathscr{X}_s}$ then gives an isomorphism

$$\begin{aligned}\mathrm{Hom}_{D^b_{(B),\widetilde{T}\times\widetilde{T}}(\overline{\mathscr{X}_s},\Bbbk)}&((j_e)_!\,\mathscr{L}_t^e, \mathscr{L}_t^{e,s}[2]) \\ &\cong \mathrm{Hom}_{D^b_{(B),\widetilde{T}\times\widetilde{T}}(\mathscr{X}_s^\circ,\Bbbk)}((i_s^\circ)^*(j_e)_!\,\mathscr{L}_t^e, (i_s^\circ)^*\,\mathscr{L}_t^{e,s}[2]).\end{aligned}$$

We have

$$\begin{aligned}\mathrm{Hom}_{D^b_{(B),\widetilde{T}\times\widetilde{T}}(\mathscr{X}_s^\circ,\Bbbk)}&((i_s^\circ)^*(j_e)_!\,\mathscr{L}_t^e, (i_s^\circ)^*\,\mathscr{L}_t^{e,s}[2]) \\ &\cong \mathrm{Hom}_{D^b_{\widetilde{T}\times\widetilde{T}}(U_s^-\times T,\Bbbk)}(\underline{\Bbbk}_{\{e\}} \boxtimes \mathscr{L}_t^T, \underline{\Bbbk}_{U_s^-} \boxtimes \mathscr{L}_t^T[2]) \\ &\cong \mathrm{Hom}_{D^b_{\widetilde{T}\times\widetilde{T}}(U_s^-\times T,\Bbbk)}(\underline{\Bbbk}_{\{e\}} \boxtimes \mathscr{L}_t^T, \underline{\Bbbk}_{U_s^-} \boxtimes \nu_*\,\underline{\Bbbk}_{\widetilde{T}}[2]) \\ &\cong \mathrm{Hom}_{D^b_{\widetilde{T}\times\widetilde{T}}(U_s^-\times \widetilde{T},\Bbbk)}(\underline{\Bbbk}_{\{e\}} \boxtimes \underline{\Bbbk}_{\widetilde{T}}, \underline{\Bbbk}_{U_s^-} \boxtimes \underline{\Bbbk}_{\widetilde{T}}[2]) \\ &\cong \mathrm{Hom}_{D^b_{\widetilde{T}}(U_s^-,\Bbbk)}(\underline{\Bbbk}_{\{e\}}, \underline{\Bbbk}_{U_s^-}[2]).\end{aligned}$$

(The last isomorphism follows from the induction equivalence). The latter space is one-dimensional over $\Bbbk$, a basis being given by the adjunction map induced by the inclusion $i_e : \{e\} \hookrightarrow U_s^-$.

Finally, the morphism

$$\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T}, \mathrm{IC}(s)_{\mathscr{L}_t^T}[1]) \to \mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T}, \mathrm{IC}(e)_{\mathscr{L}_t^T}[2])$$

considered above identifies with the map

$$\mathrm{Hom}_{D^b_{\widetilde{T}}(U_s^-,\Bbbk)}(\underline{\Bbbk}_{\{e\}}, \underline{\Bbbk}_{U_s^-}[2]) \to \mathrm{Hom}_{D^b_{\widetilde{T}}(\{e\},\Bbbk)}(\underline{\Bbbk}_{\{e\}}, \underline{\Bbbk}_{\{e\}}[2]) \tag{9.8.2}$$

induced by the adjunction morphism

$$\underline{\Bbbk}_{U_s^-} \to (i_e)_*(i_e)^*\,\underline{\Bbbk}_{U_s^-} = \underline{\Bbbk}_{\{e\}}\,.$$

From the fact recalled before the proof of the lemma, we see that the morphism (9.8.2) identifies with the morphism $\Bbbk \to \Bbbk \otimes_{\mathbb{Z}} \mathrm{X}^*(\widetilde{T})$ induced by the inclusion $\{\alpha\} \hookrightarrow \mathrm{X}^*(\widetilde{T})$ (α being viewed as a character of $\widetilde{T}$ via the isogeny ν). This map is injective (see the facts above the proof, and recall that the characteristic of $\Bbbk$ is assumed to be different from 2), so that finally the space $\mathrm{Hom}_{\mathfrak{D}(t,t)}(\mathrm{IC}(e)_{\mathscr{L}_t^T}, \Delta(s)_{\mathscr{L}_t^T}[1])$ is zero and the proof is complete. □

Proposition 9.8.2. *Consider $t \in T_{\Bbbk}^{\vee}$ and s a simple reflection in W_t°. Then the map*

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \to \mathrm{End}(\widehat{\mathscr{T}}_{s,t})$$

of lemma 9.6.5 is a surjective morphism of algebras.

Proof. According to lemma 9.4.2, item (2), we have an isomorphism

$$\mathrm{End}(\mathscr{T}(s)_{\mathscr{L}_t^T}) \cong \mathrm{End}(\widehat{\mathscr{T}}_{s,t}) \otimes_{\widehat{R}_t} \Bbbk_t \,.$$

Thanks to Nakayama's lemma, it suffices to show that the induced map

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \Bbbk_t \to \mathrm{End}(\mathscr{T}(s)_{\mathscr{L}_t^T})$$

is surjective. Now this map fits in a commutative diagram

$$\begin{array}{ccc} \widehat{R}_t & \xrightarrow{\varphi_{l,\mathscr{T}(s)_{\mathscr{L}_t^T}}} & \mathrm{End}(\mathscr{T}(s)_{\mathscr{L}_t^T}) \\ & \searrow \quad \nearrow & \\ & \widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \Bbbk_t & \end{array}$$

where the left map is the natural quotient morphism. Thus is suffices to see that $\varphi_{l,\mathscr{T}(s)_{\mathscr{L}_t^T}}$ is surjective. The description of $\mathscr{T}(s)_{\mathscr{L}_t^T}$ given in subsection 9.8.1 allows us to see that $\mathrm{End}(\mathscr{T}(s)_{\mathscr{L}_t^T})$ is two-dimensional over $\Bbbk$. Thus we must show that the image of $\varphi_{l,\mathscr{T}(s)_t}$ is not reduced to $\Bbbk \cdot \mathrm{id}$. But according to lemma 4.2.4, if this were the case, then the object $\mathscr{T}(s)_{\mathscr{L}_t^T}$ would lie in the category $\mathfrak{P}(T \setminus\!\setminus \mathscr{X} /\!\!/ T)_{[\mathscr{L}_t^T, \mathscr{L}_t^T]}$. This is not true according to lemma 9.8.1 (and (9.8.1)). This fact concludes our proof. □

9.9 Bott–Samelson description of tilting categories

We use here the results of the preceding subsection to derive useful descriptions of the categories of pro-tilting and tilting objects.

Lemma 9.9.1. *Let $t, t' \in T^\vee_{\Bbbk}$ be two elements in the same W-orbit and consider s a simple reflection of W°_t. For any tilting perverse sheaf $\widehat{\mathscr{T}} \in \widehat{T}_{[t,t']}$, the convolution $\widehat{\mathscr{T}}_{s,t} \widehat{\star} \widehat{\mathscr{T}}$ is a tilting perverse sheaf and we have*

$$(\widehat{\mathscr{T}}_{s,t} \widehat{\star} \widehat{\mathscr{T}} : \widehat{\Delta}_{w,t'}) = (\widehat{\mathscr{T}} : \widehat{\Delta}_{w,t'}) + (\widehat{\mathscr{T}} : \widehat{\Delta}_{sw,t'})$$

for any $w \in W_{t'}$.

Proof. The proof is (very closely) inspired by [BeR, Lemme 7.8]. There is a slight difference however, due to the fact that the length functions on $W^\circ_{t'}$ and W do not match. The strategy is nonetheless the same: we claim that for any $w \in {}_{t'}W_{t'}$, the object $\widehat{\mathscr{T}}_{s,t} \widehat{\star} \widehat{\Delta}_{w,t'}$ admits a standard filtration, in which the multiplicity of $\widehat{\Delta}_{v,t'}$ (for $v \in W^\circ_{t'}$) is one if $v \in \{w, sw\}$ and zero otherwise. According to lemma 5.1.5, there exists a t'' in the W-orbit of t such that we can write $s = w_\beta^{\min} r w_{\overline{\beta}}^{\min}$ with β a block of ${}_t\underline{W}_{t''}$ and $\overline{\beta}$ the "opposite" corresponding block in ${}_{t''}\underline{W}_t$ (that is, $\overline{\beta} = \{w \in {}_{t''}W_t \mid w^{-1} \in \beta\}$), and $r \in W^\circ_{t''}$ simple in W. (Of course, we have $w_{\overline{\beta}}^{\min} = w_\beta^{\min,-1}$, but we choose the former notation to emphasize the fact that this element is minimal in its block.) Using the results of subsection 9.8.1, we have two distinguished triangles

$$\widehat{\Delta}_{s,t} \to \widehat{\mathscr{T}}_{s,t} \to \widehat{\Delta}_{e,t} \xrightarrow{+1}$$

and

$$\widehat{\nabla}_{e,t} \to \widehat{\mathscr{T}}_{s,t} \to \widehat{\nabla}_{s,t} \xrightarrow{+1}.$$

Assume that $r w_{\overline{\beta}}^{\min} w > w_{\overline{\beta}}^{\min} w$ (for the Bruhat order on W). Then, using corollary 8.5.5, we have

$$\begin{aligned}
\widehat{\Delta}_{s,t} \widehat{\star} \widehat{\Delta}_{w,t'} &\cong \widehat{\Delta}_{w_\beta^{\min},t''} \widehat{\star} \widehat{\Delta}_{r,t''} \widehat{\star} \widehat{\Delta}_{w_{\overline{\beta}}^{\min},t} \widehat{\star} \widehat{\Delta}_{w,t'} \\
&\cong \widehat{\Delta}_{w_\beta^{\min} r w_{\overline{\beta}}^{\min} w,t'} \\
&\cong \widehat{\Delta}_{sw,t'}.
\end{aligned}$$

Convolving the first distinguished triangle above with $\widehat{\Delta}_{w,t'}$ on the right, we get

$$\widehat{\Delta}_{sw,t'} \to \widehat{\mathscr{T}}_{s,t} \widehat{\star} \widehat{\Delta}_{w,t'} \to \widehat{\Delta}_{w,t'} \xrightarrow{+1}.$$

We now consider the case $r w_{\overline{\beta}}^{\min} w < w_{\overline{\beta}}^{\min} w$. We apply $(-)\widehat{\star}\widehat{\Delta}_{w,t'}$ to the second distinguished triangle; the same kind of arguments as those used above allow us to get a distinguished triangle

$$\widehat{\Delta}_{w,t'} \to \widehat{\mathscr{T}}_{s,t} \widehat{\star} \widehat{\Delta}_{w,t'} \to \widehat{\Delta}_{sw,t'} \xrightarrow{+1}.$$

These triangles readily imply our claim; together with lemma 9.7.5 they give the lemma. (Alternatively, one could use a similar reasoning to show that $\widehat{\mathscr{T}}_{s,t} \widehat{\star} \widehat{\nabla}_{w,t'}$ admits a costandard filtration for any $w \in W^\circ_{t'}$.) □

We deduce from lemma 9.9.1 the following corollary, that provides a "Bott–Samelson" description of $\widehat{T}^{\circ}_{[t,t]}$:

Corollary 9.9.2. *Let $t, t' \in T^{\vee}_{\Bbbk}$ be two elements in the same W-orbit, and let $\beta \in {}_{t'}\underline{W}_t$ be any block.*

1. *The objects in $\widehat{T}^{\beta}_{[t',t]}$ are direct sums of direct summands of objects of the form $\widehat{\Delta}_{w^{\min}_{\beta},t}\widehat{\star}\widehat{\mathscr{T}}_{s_r,t}\widehat{\star}(\cdots)\widehat{\star}\widehat{\mathscr{T}}_{s_1,t}$ for any family $\{s_i\}_i$ of simple reflections in W°_t. In particular, for $\beta = W^{\circ}_t$, we have*

$$\widehat{T}^{\circ}_{[t,t]} = \langle \widehat{\mathscr{T}}_{s,t} \mid s \in S_t \rangle_{\widehat{\star},\oplus,\overset{\oplus}{\subseteq}}.$$

2. *The objects of $\mathfrak{T}(\mathscr{X} /\!\!/ T)^{\beta}_{[t',\mathscr{L}^T_t]}$ are direct sums of direct summands of objects of the form*

$$\widehat{\Delta}_{w^{\min}_{\beta},t}\widehat{\star}\widehat{\mathscr{T}}_{s_r,t}\widehat{\star}(\cdots)\widehat{\star}\widehat{\mathscr{T}}_{s_1,t}\widehat{\star}\Delta(e)_{\mathscr{L}^T_t}$$

for any family $\{s_i\}_i$ of simple reflections in W°_t.

Proof. Thanks to lemma 9.7.1, the functor $\widehat{\Delta}_{w^{\min}_{\beta},t}\widehat{\star}(-)$ induces an equivalence on tilting categories, thus it suffices to prove the corollary for $\beta = W^{\circ}_t$. Lemma 9.9.1 implies that $\langle \widehat{\mathscr{T}}_{s,t} \mid s \in S_t \rangle_{\widehat{\star},\oplus,\overset{\oplus}{\subseteq}}$ is a full subcategory of $\widehat{T}^{\circ}_{[t,t]}$ (one could alternatively use lemma 9.7.5 to see this fact). Indeed, consider an object $\widehat{\mathscr{T}} \in \langle \widehat{\mathscr{T}}_{s,t} \mid s \in S_t \rangle_{\widehat{\star},\oplus,\overset{\oplus}{\subseteq}}$ given as a successive convolution product of object of the form $\widehat{\mathscr{T}}_{s,t}$. The description of the standard filtrations of $\widehat{\mathscr{T}}_{s,t}$ given in subsection 9.8.1 together with lemma 9.9.1 imply that the only standard objects that can appear in standard filtration of $\widehat{\mathscr{T}}$ are indexed by element of W°_t. Now $\widehat{\mathscr{T}}'$ is a direct summand of $\widehat{\mathscr{T}}$, it is tilting too and the standard appearing in a standard filtration have to be chosen in the standard appearing in a standard filtration of $\widehat{\mathscr{T}}$. By definition of the neutral block, we get our assertion.

Now to prove the corollary, it suffices to show that any indecomposable tilting perverse sheaf in $\widehat{T}^{\circ}_{[t,t]}$ is a direct summand in an object of the form $\widehat{\mathscr{T}}_{s_r,t}\widehat{\star}(\cdots)\widehat{\star}\widehat{\mathscr{T}}_{s_1,t}$. Recall that these indecomposable objects are parametrized by elements in W°_t. For w in this group, choose a reduced expression $\underline{w} = (s_r, \dots, s_1)$ in W°_t (that is, with $s_i \in S_t$ for all i). Set

$$\widehat{\mathscr{T}}(\underline{w}) := \widehat{\mathscr{T}}_{s_r,t}\widehat{\star}(\cdots)\widehat{\star}\widehat{\mathscr{T}}_{s_1,t};$$

it readily follows from lemma 9.9.1 that the object $\widehat{\Delta}_{w,t}$ appears in a pro-standard filtration of $\widehat{\mathscr{T}}(\underline{w})$ with multiplicity one, and that w is the unique maximal element in the set $\{v \in W^{\circ}_t \mid (\widehat{\mathscr{T}}(\underline{w}) : \widehat{\Delta}_{v,t}) \neq 0\}$ (for the Bruhat order of W°_t, hence also for the Bruhat order of W, see (5.1.1)). These considerations imply that $\widehat{\mathscr{T}}_{w,t}$ appears as a direct summand in $\widehat{\mathscr{T}}(\underline{w})$. This settles the proof of the first assertion, and the second one immediately follows. □

Corollary 9.9.3. *For any $w \in W_t^\circ$, the objects $\widehat{\Delta}_{v,t}$ appearing in a pro-standard filtration of $\widehat{\mathscr{T}}_{w,t}$ are indexed by elements $v \in W_t^\circ$ satisfying $v \leq_t w$. A similar statement holds for $\mathscr{T}(w)_{\mathscr{L}_t^T}$.*

Proof. Let us fix a reduced expression $\underline{w} = (s_r, \dots, s_1)$ of w in W_t°. As in the proof of corollary 9.9.2, we see that the object $\widehat{\mathscr{T}}_{w,t}$ appears as a direct summand in $\widehat{\mathscr{T}}(\underline{w}) := \widehat{\mathscr{T}}_{s_r,t}\widehat{\star}(\cdots)\widehat{\star}\widehat{\mathscr{T}}_{s_1,t}$. It follows that the pro-standard appearing in a $\widehat{\Delta}$-filtration of $\widehat{\mathscr{T}}_{w,t}$ already appear in a standard filtration of $\widehat{\mathscr{T}}(\underline{w})$. Lemma 9.9.1 ensures that the latter objects are of the form $\widehat{\Delta}_{v,t}$ with $v \leq_t w$. This concludes the proof.

The fact that the lemma is true for $\mathscr{T}(w)_{\mathscr{L}_t^T}$ can be seen using the exact same kind of arguments, or as a direct consequence of lemma 7.8.9. □

Corollary 9.9.4. *Let $w \in W_t^\circ$ and $(s_r, \dots, s_1)$ any reduced expression of w in (W_t°, S_t). The object $\widehat{\mathscr{T}}_{w,t}$ is characterized as the only direct summand in $\widehat{\mathscr{T}}_{s_r,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s_1,t}$ that does not appear as a direct summand in any product of the form $\widehat{\mathscr{T}}_{s'_j,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s'_1,t}$ with $s'_i \in S_t$ and $j < r$.*

A similar statement holds for $\mathscr{T}(w)_{\mathscr{L}_t^T}$.

Proof. The arguments used in the proof of corollary 9.9.2 imply that $\widehat{\mathscr{T}}_{w,t}$ cannot appear as a direct summand in a convolution product $\widehat{\mathscr{T}}_{s'_j,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s'_1,t}$ with $j < r$, because $\widehat{\Delta}_{w,t}$ cannot appear in a standard filtration of this object. Since for any $v \in W_t^\circ$ and any reduced expression $(s'_j, \dots, s'_1)$ of v in W_t° the object $\widehat{\mathscr{T}}_{v,t}$ does occur as a direct summand in $\widehat{\mathscr{T}}_{s'_j,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s'_1,t}$, the unicity of the statement is clear.

The proof in the Lusztig–Yun case is similar. □

9.10 The functors $\mathbb{V}$

We introduce here our Soergel functors on tilting objects. Fix $t, t' \in T_{\Bbbk}^\vee$ in the same W-orbit and choose a block $\beta \in {}_{t'}\underline{W}_t$. Let $A_t^\beta := \mathrm{End}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T})$ and $\widehat{A}_t^\beta := \mathrm{End}(\widehat{\mathscr{T}}_{w_\beta^{\max},t})$. According to lemma 9.4.2, we have an isomorphism

$$A_t^\beta \cong \widehat{A}_t^\beta \otimes_{\widehat{R}_t} \Bbbk_t\,. \tag{9.10.1}$$

We define two functors:

$$\begin{aligned}\mathbb{V}_t^\beta : \mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]} &\longrightarrow \mathrm{Mod}(A_t^\beta)^{\mathrm{fg}},\\ \mathscr{F} &\mapsto \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}, \mathscr{F})\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathbb{V}}_t^\beta : \widehat{\mathrm{P}}^\beta_{[-,t]} &\longrightarrow \mathrm{Mod}(\widehat{A}_t^\beta)^{\mathrm{fg}},\\ \widehat{\mathscr{F}} &\mapsto \mathrm{Hom}_{\widehat{\mathrm{P}}^\beta_{[-,t]}}(\widehat{\mathscr{T}}_{w_\beta^{\max},t}, \widehat{\mathscr{F}}).\end{aligned}$$

Here, $\mathrm{Mod}(A_t^\beta)^{\mathrm{fg}}$ denotes the category of finitely generated right A_t^β-modules (and similarly for $\mathrm{Mod}(\widehat{A}_t^\beta)^{\mathrm{fg}}$). In the case $t = t'$ and $\beta = W_t^\circ$, we denote $\mathbb{V}_t^\circ$ and $\widehat{\mathbb{V}}_t^\circ$ the corresponding functors.

We aim to show that the various functors $\mathbb{V}$ are fully faithful. We can define functors Υ_t^β and $\widehat{\Upsilon}_t^\beta$ in the other direction. We detail the construction of Υ_t^β below.

Let $\mathrm{Free}(A_t^\beta)$ be the full subcategory of $\mathrm{Mod}^{\mathrm{fg}}(A_t^\beta)$ whose objects are finite-type free A_t^β-modules. Note that this category is equivalent to the category $\mathbb{Z}_{\geq 0}^R$ whose objects are the integers in $\mathbb{Z}_{\geq 0}$ and with morphisms given by

$$\mathrm{Hom}_{\mathbb{Z}_{\geq 0}^R}(m,n) := \mathrm{Mat}_{n\times m}(A_t^\beta),$$

the $n\times m$ matrices with coefficients in A_t^β. By definition of A_t^β, we have a natural additive functor $\mathrm{Free}(A_t^\beta) \to \mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}$ sending A_t^β to $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}$. This functor upgrades to

$$K^-\mathrm{Free}(A_t^\beta) \to K^-\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]} \to D^-\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]},$$

where $K^-(-)$ denotes the (bounded below) homotopy category. Now one can check that the natural functor $K^-\mathrm{Free}(A_t^\beta) \to D^-\,\mathrm{Mod}^{\mathrm{fg}}(A_t^\beta)$ is an equivalence For $M \in \mathrm{Mod}^{\mathrm{fg}}(A_t^\beta)$, we then let $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{A_t^\beta} M$ be the image of M under the composition

$$\begin{aligned}\mathrm{Mod}^{\mathrm{fg}}(A_t^\beta) \to D^-\,\mathrm{Mod}^{\mathrm{fg}}(A_t^\beta) \cong K^-\mathrm{Free}(A_t^\beta) \to K^-\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}\\ \to D^-\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]} \xrightarrow{{}^p\mathscr{H}^0(-)} \mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}.\end{aligned}$$

This defines a functor denoted $\Upsilon_t^\beta(M) : \mathrm{Mod}^{\mathrm{fg}}(A_t^\beta) \to \mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}$. In a completely analogous way, we define a functor $\widehat{\Upsilon}_t^\beta : \mathrm{Mod}^{\mathrm{fg}}(\widehat{A}_t^\beta) \to \widehat{P}^\beta_{[-,t]}$ by setting $\widehat{\Upsilon}_t^\beta(M') := \widehat{\mathscr{T}}_{w_\beta^{\max},t} \otimes_{\widehat{A}_t^\beta} M'$ for M' an $\widehat{A}_t^\beta$-module.

Note that for a finitely generated A_t^β-module M, since A_t^β is noetherian, we can find two finite dimensional $\Bbbk_t$-vector spaces V_1, V_2 and an exact sequence in $\mathrm{Mod}(A_t^\beta)^{\mathrm{fg}}$

$$A_t^\beta \otimes_{\Bbbk_t} V_1 \to A_t^\beta \otimes_{\Bbbk_t} V_2 \to M \to 0.$$

This yields a map

$$\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{\Bbbk_t} V_1 \longrightarrow \mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{\Bbbk_t} V_2;$$

and $\Upsilon_t^\beta(M)$ can be described as the cokernel of this map. A similar description holds for $\widehat{\Upsilon}_t^\beta$; moreover, one can check that this description does not depend on the choices of V_1 and V_2, see [KS2, §8.5] for more details.

Lemma 9.10.1. *1) The couples of functors $(\Upsilon_t^\beta, \mathbb{V}_t^\beta)$ and $(\widehat{\Upsilon}_t^\beta, \widehat{\mathbb{V}}_t^\beta)$ form adjoint pairs.*

2) The adjunction morphism $\mathrm{id} \to \mathbb{V}_t^\beta \circ \Upsilon_t^\beta$ *is an isomorphism.*
3) For any finitely generated right $\widehat{A}_t^\beta$*-module* M*, we have*

$$^p\mathscr{H}^0(\pi_\dagger^t(\widehat{\Upsilon}_t^\beta(M))) \cong \Upsilon_t^\beta(M \otimes_{\widehat{A}_t^\beta} \Bbbk_t).$$

Proof. We prove 1) in the case $(\Upsilon_t^\beta, \mathbb{V}_t^\beta)$, the other one is similar. We must show that for any $M \in \mathrm{Mod}(A_t^\beta)^{\mathrm{fg}}$ and $\mathscr{F} \in \mathfrak{P}(\mathscr{X} /\!\!/ T)^\beta_{[-,\mathscr{L}_t^T]}$, we have an isomorphism $\mathrm{Hom}_{A_t^\beta}(M, \mathbb{V}_t^\beta(\mathscr{F})) \cong \mathrm{Hom}_{\mathfrak{P}(\mathscr{X} /\!\!/ T)^\beta_{[-,\mathscr{L}_t^T]}}(\Upsilon_t^\beta(M), \mathscr{F})$. As above, we can identify M with the cokernel of some map $A_t^\beta \otimes_{\Bbbk_t} V_1 \to A_t^\beta \otimes_{\Bbbk_t} V_2$, so that we have an exact sequence

$$\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{\Bbbk_t} V_1 \longrightarrow \mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{\Bbbk_t} V_2 \longrightarrow \Upsilon_t^\beta(M) \to 0$$

for some finite dimensional $\Bbbk_t$-vector spaces V_1, V_2. Apply the Hom-functor $\mathrm{Hom}_{\mathfrak{P}(\mathscr{X} /\!\!/ T)^\beta_{[-,\mathscr{L}_t^T]}}(-, \mathscr{F})$ to obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{P}(\mathscr{X} /\!\!/ T)^\beta_{[-,\mathscr{L}_t^T]}}(\Upsilon_t^\beta(M), \mathscr{F}) \to \mathbb{V}_t^\beta(\mathscr{F}) \otimes_{\Bbbk_t} V_2^* \to \mathbb{V}_t^\beta(\mathscr{F}) \otimes_{\Bbbk_t} V_1^*. \tag{9.10.2}$$

(where V_i^* denotes the dual vector space). Thus to conclude, we must show that $\mathrm{Hom}_{A_t^\beta}(M, \mathbb{V}_t^\beta(\mathscr{F}))$ identifies with the kernel of the above map $\mathbb{V}_t^\beta(\mathscr{F}) \otimes_{\Bbbk_t} V_2 \to \mathbb{V}_t^\beta(\mathscr{F}) \otimes_{\Bbbk_t} V_1$. Start from the exact sequence

$$A_t^\beta \otimes_{\Bbbk_t} V_1 \to A_t^\beta \otimes_{\Bbbk_t} V_2 \to M \to 0$$

and apply the functor $\mathrm{Hom}_{A_t^\beta}(-, \mathbb{V}_t^\beta(\mathscr{F}))$. We obtain an exact sequence

$$\mathrm{Hom}_{A_t^\beta}(M, \mathbb{V}(\mathscr{F})) \hookrightarrow \mathrm{Hom}_{A_t^\beta}(A_t^\beta \otimes_{\Bbbk_t} V_2, \mathbb{V}(\mathscr{F})) \to \mathrm{Hom}_{A_t^\beta}(A_t^\beta \otimes_{\Bbbk_t} V_1, \mathbb{V}(\mathscr{F})). \tag{9.10.3}$$

Under the canonical identifications $\mathrm{Hom}_{A_t^\beta}(A_t^\beta \otimes_{\Bbbk_t} V_i, \mathbb{V}_t^\beta(\mathscr{F})) \cong \mathbb{V}_t^\beta \otimes_{\Bbbk_t} V_i^*$ for $i = 1, 2$, the second map in (9.10.3) identifies with the second map in (9.10.2) and thus we have an isomorphism

$$\mathrm{Hom}_{A_t^\beta}(M, \mathbb{V}_t^\beta(\mathscr{F})) \cong \mathrm{Hom}_{\mathfrak{P}(\mathscr{X} /\!\!/ T)^\beta_{[-,\mathscr{L}_t^T]}}(\Upsilon_t^\beta(M), \mathscr{F});$$

this concludes the proof.

Point 2) is clear from the description of Υ_t^β using cokernels and the fact that $\mathbb{V}_t^\beta$ is exact since $\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}$ is a projective object.

Finally, we prove 3). For M as in the statement, we find V_1, V_2 two $\Bbbk_t$-vector spaces yielding an exact sequence

$$\widehat{A}_t^\beta \otimes_{\Bbbk_t} V_1 \to \widehat{A}_t^\beta \otimes_{\Bbbk_t} V_2 \to M \to 0. \tag{9.10.4}$$

We deduce an exact sequence in $\widehat{P}^\beta_{[-,t]}$

$$\widehat{\mathscr{T}}_{w_\beta^{\max},t} \otimes_{\Bbbk_t} V_1 \to \widehat{\mathscr{T}}_{w_\beta^{\max},t} \otimes_{\Bbbk_t} V_2 \to \widehat{\Upsilon}_t^\beta \to 0.$$

Apply ${}^p\mathscr{H}^0\circ\pi_\dagger^t$ to this exact sequence; this functor is right-exact for the perverse t-structure (see corollary 7.8.5) and by definition, we have

$$ {}^p\mathscr{H}^0 \circ \pi_\dagger^t(\widehat{\mathscr{T}}_{w_\beta^{\max},t}) = \pi_\dagger^t(\widehat{\mathscr{T}}_{w_\beta^{\max},t}) = \mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}. $$

Thus we get an exact sequence in $\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}$

$$ \mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{\Bbbk_t} V_1 \longrightarrow \mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T} \otimes_{\Bbbk_t} V_2 \longrightarrow {}^p\mathscr{H}^0(\pi_\dagger^t(\widehat{\Upsilon}_t^\beta)) \to 0. $$

Now, as stated above, we have $A_t^\beta \cong \widehat{A}_t^\beta \otimes_{\widehat{R}_t} \Bbbk_t$, and using (9.10.4), we see that ${}^p\mathscr{H}^0(\pi_\dagger^t(\widehat{\Upsilon}_t^\beta))$ indeed identifies with $\Upsilon_t^\beta(M \otimes_{\widehat{A}_t^\beta} \Bbbk_t)$. □

Proposition 9.10.2. *Fix $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit and choose a block $\beta \in {}_{t'}W_t$. Then the functors $\mathbb{V}_t^\beta$ and $\widehat{\mathbb{V}}_t^\beta$ are fully faithful on tilting objects.*

Said otherwise, for any $\mathscr{T}, \mathscr{T}' \in \mathfrak{T}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}$, the functor $\mathbb{V}_t^\beta$ induces an isomorphism

$$ \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}}(\mathscr{T}, \mathscr{T}') \cong \mathrm{Hom}_{A_t^\beta}(\mathbb{V}_t^\beta(\mathscr{T}), \mathbb{V}_t^\beta(\mathscr{T}')). $$

Similarly, for any $\widehat{\mathscr{T}}, \widehat{\mathscr{T}}' \in \widehat{\mathrm{T}}^\beta_{[-,t]}$, the functor $\widehat{\mathbb{V}}_t^\beta$ induces an isomorphism

$$ \mathrm{Hom}_{\widehat{\mathrm{P}}^\beta_{[-,t]}}(\widehat{\mathscr{T}}, \widehat{\mathscr{T}}') \cong \mathrm{Hom}_{\widehat{A}_t^\beta}(\widehat{\mathbb{V}}_t^\beta(\widehat{\mathscr{T}}), \widehat{\mathbb{V}}_t^\beta(\widehat{\mathscr{T}}')). $$

Proof. The proof given here is similar to the one in [BY, §4.7]. We prove first the case of the functor $\mathbb{V}_t^\beta$. Consider thus two tilting objects $\mathscr{T}, \mathscr{T}' \in \mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}$. We have an adjunction map

$$ \Upsilon_t^\beta(\mathbb{V}_t^\beta(\mathscr{T})) \longrightarrow \mathscr{T}\,. \tag{9.10.5} $$

Since $\mathbb{V}_t^\beta \circ \Upsilon_t^\beta \cong \mathrm{id}$, one can see that the image of this map under $\mathbb{V}_t^\beta$ identifies with the identity of $\mathbb{V}_t^\beta(\mathscr{T})$. Thus the kernel and cokernel of (9.10.5) are killed by $\mathbb{V}_t^\beta$. By virtue of proposition 9.3.4, this means that this kernel and cokernel do not admit $\mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^T}$ as composition factor. This already tells us that the cokernel is zero: as $\mathscr{T}$ is a direct sum of indecomposable tilting in $\mathfrak{T}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}$, proposition 9.3.4 implies that there is no nonzero morphisms from $\mathscr{T}$ to an $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ with $w \neq w_\beta^{\min}$, and thus similarly for an object that is an extension of such simples.

Let $\mathscr{K}$ denote the kernel of (9.10.5). To conclude, it suffices to show that we have

$$ \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}}(\mathscr{K}, \mathscr{T}') \cong 0 $$

for any tilting object $\mathscr{T}'$. But this follows directly from proposition 8.6.1 again. Thus we have

$$ \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}}(\mathscr{T}, \mathscr{T}') \cong \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)^\beta_{[-,\mathscr{L}_t^T]}}(\Upsilon_t^\beta(\mathbb{V}_t^\beta(\mathscr{T})), \mathscr{T}') $$

and we conclude by adjunction.

We follow the same strategy for $\widehat{\mathbb{V}}_t^\beta$: consider two tilting objects $\widehat{\mathscr{T}}, \widehat{\mathscr{T}}'$. Once again, we have an adjunction morphism

$$\widehat{\Upsilon}_t^\beta(\widehat{\mathbb{V}}_t^\beta(\widehat{\mathscr{T}})) \longrightarrow \widehat{\mathscr{T}}. \tag{9.10.6}$$

Let $\widehat{\mathscr{K}}$ and $\widehat{\mathscr{C}}$ be respectively the kernel and cokernel of this map, so that we have an exact sequence

$$0 \to \widehat{\mathscr{K}} \to \widehat{\Upsilon}_t^\beta(\widehat{\mathbb{V}}_t^\beta(\widehat{\mathscr{T}})) \to \widehat{\mathscr{T}} \to \widehat{\mathscr{C}} \to 0 \tag{9.10.7}$$

in $\widehat{\mathrm{P}}_{[-,t]}$. This time we cannot say that the kernel and cokernel are killed by the functor $\widehat{\mathbb{V}}_t^\beta$ directly. Instead, we apply the functor ${}^p\mathscr{H}^0 \circ \pi_\dagger^t$ to the central map of (9.10.7). Using the point 3) of lemma 9.10.1 and point 2) of lemma 9.4.2, what we get is the adjunction map

$$\Upsilon_t^\beta(\mathbb{V}_t^\beta(\pi_\dagger^t(\widehat{\mathscr{T}}))) \longrightarrow \pi_\dagger^t(\widehat{\mathscr{T}}) \tag{9.10.8}$$

(since $\widehat{\mathscr{T}}$ is tilting, $\pi_\dagger^t(\widehat{\mathscr{T}})$ is already perverse, so no need for ${}^p\mathscr{H}^0$ on the right side). Thanks to the first part of this proof, we know that this map is surjective. Thus the (perverse) cokernel $\widehat{\mathscr{C}}$ is killed by ${}^p\mathscr{H}^0 \circ \pi_\dagger^t$ and is thus zero according to lemma 7.5.6. Once again, in order to conclude our proof, it suffices to show that there are no non-zero map from $\widehat{\mathscr{K}}$ to any tilting object in $\widehat{\mathrm{P}}_{[-,t]}^\beta$. Since any tilting $\mathscr{T}'$ admits a pro-standard filtration, it suffices to show that

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \widehat{\Delta}_{w,t}) = 0$$

for any $w \in \beta$. By definition of the morphism-spaces between pro-objects, we have

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \widehat{\Delta}_{w,t}) = \varprojlim_n \mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \Delta_{w,t}^n).$$

We can then reduce our calculation to $\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \Delta_{w,t}^n) = 0$ for any $n > 0$. Even more, recall that $\Delta_{w,t}^n$ is the (shifted) extension by zero of the local system $\mathscr{L}_{t,n}^w$ on $\mathscr{X}_w$, and the latter local system is a successive extension of $\mathscr{L}_t^w$. We can finally just show that

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \Delta_{w,t}^1) = \mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \Delta_{w,t}) = 0.$$

But now $\Delta_{w,t} = \mathrm{For}_t(\Delta(w)_{\mathscr{L}_t^T})$; using the adjunction $(\pi_\dagger^t, \mathrm{For}_t)$ proved in lemma 7.1.3, we get

$$\mathrm{Hom}_{\widehat{D}_{[-,t]}}(\widehat{\mathscr{K}}, \Delta_{w,t}) \cong \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\pi_\dagger^t(\widehat{\mathscr{K}}), \Delta(w)_{\mathscr{L}_t^T}).$$

Since $\pi_\dagger^t$ is right t-exact (and $\Delta(w)_{\mathscr{L}_t^T}$ is perverse), the later space is isomorphic to

$$\begin{aligned}\mathrm{Hom}_{{}^p\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}(\geq 0)}&({}^p\tau^{\geq 0}(\pi_\dagger^t(\widehat{\mathscr{K}})), \Delta(w)_{\mathscr{L}_t^T}) \\ &= \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}({}^p\mathscr{H}^0(\pi_\dagger^t(\widehat{\mathscr{K}})), \Delta(w)_{\mathscr{L}_t^T}).\end{aligned}$$

The exact sequence (9.10.7) can be written as a distinguished triangle

$$\widehat{\mathscr{K}} \to \widehat{\Upsilon}_t^{\beta}(\widehat{\mathbb{V}}_t^{\beta}(\widehat{\mathscr{T}})) \to \widehat{\mathscr{T}} \xrightarrow{+1} .$$

Applying ${}^{p}\mathscr{H}^{0} \circ \pi_{\dagger}^{t}$ to this triangle and using the long exact sequence for perverse cohomology, we get

$$\cdots \to {}^{p}\mathscr{H}^{-1}(\pi_{\dagger}^{t}(\widehat{\mathscr{T}})) \to {}^{p}\mathscr{H}^{0}(\pi_{\dagger}^{t}(\widehat{\mathscr{K}})) \to \Upsilon_t^{\beta}(\mathbb{V}_t^{\beta}(\pi_{\dagger}^{t}(\widehat{\mathscr{T}}))) \to \cdots .$$

The first term is zero, so that ${}^{p}\mathscr{H}^{0}(\pi_{\dagger}^{t}(\widehat{\mathscr{K}}))$ identifies with the kernel of (9.10.8). From the first part of this proof, we then know that this kernel does not admits $\mathrm{IC}(w_{\beta}^{\min})_{\mathscr{L}_t^T}$ as composition factor. We can finally conclude that

$$\mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}({}^{p}\mathscr{H}^{0}(\pi_{\dagger}^{t}(\widehat{\mathscr{K}})), \Delta(w)_{\mathscr{L}_t^T}) = 0,$$

which finishes the proof. □

Chapter 10

Constructions on modules and Endomorphismensatz

10.1 Variations on the Pittie–Steinberg theorem

10.1.1 General setting

Let H be a semisimple simply-connected algebraic group over an algebraically closed field $\mathbb{K}$. Fix a maximal torus T_H in H and let $\mathrm{X}^*(T_H)$ be the character lattice of T_H. We also denote by W_H the Weyl group of the pair (H, T_H). The algebra $\mathbb{Z}[\mathrm{X}^*(T_H)]$ admits a natural action of W_H by algebra automorphisms; we set $\mathbb{Z}[\mathrm{X}^*(T_H)]^{W_H}$ for the subalgebra of W_H-invariants elements (more generally, for algebras endowed with an action of W_H, a superscript W_H will denote W_H-invariants). In this setting we have the following result (see [St, Theorems 1.1 and 1.3]), known as the Pittie–Steinberg theorem:

Theorem 10.1.1. *The $\mathbb{Z}[\mathrm{X}^*(T_H)]^{W_H}$-module $\mathbb{Z}[\mathrm{X}^*(T_H)]$ is free of rank $|W_H|$.*

In [BeR], Bezrukavnikov and Riche consider a variation of theorem 10.1.1. They start with a complex semisimple adjoint algebraic group G and a fixed maximal torus T; the $\mathbb{K}$-dual of such a group is then a semisimple simply-connected $\mathbb{K}$-group. Denote by $\mathbb{K}[\mathrm{X}_*(T)]_1^\wedge$ the completion of $\mathbb{K}[\mathrm{X}_*(T)]$ with respect to the maximal augmentation ideal $\langle e^\lambda - 1 \mid \lambda \in \mathrm{X}_*(T)\rangle$. This completion admits a natural action of the Weyl group W_G of G. From theorem 10.1.1, the authors of *loc. cit.* then deduce the following consequence (see [BeR, Theorem 8.1]):

Theorem 10.1.2. *The $(\mathbb{K}[\mathrm{X}_*(T)]_1^\wedge)^{W_G}$-module $\mathbb{K}[\mathrm{X}_*(T)]_1^\wedge$ is free of rank $|W_G|$.*

We aim to show that theorem 10.1.2 is in fact true in a different setting, namely for any complex reductive group for which the characteristic ℓ of $\mathbb{K}$ is good, and instead of 1, any central element.

We start by giving a slightly different version of the (usual) Pittie–Steinberg theorem. The following arguments are almost verbatim those of Steinberg; we give a detailed exposition essentially because we consider only a particular case (namely $W' = \{1\}$ with the notation of [St, §2]). Let us first set some notation: as above, let H be a reductive algebraic group over an algebraically closed field $\mathbb{K}$ and fix a maximal torus $T \subseteq H$; we have an associated root datum $(\mathrm{X}^*(T), \Phi, \mathrm{X}_*(T), \Phi^\vee)$. Let W denote the Weyl group of (H, T); Φ_+ be a subset of positive roots and let $\Pi \subseteq \Phi_+$ be the subset of simple roots. Recall that $\mathrm{X}^*(T)$ identifies with the dual of the cocharacter lattice $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}_*(T), \mathbb{Z})$ via the usual duality pairing $\langle \cdot, \cdot \rangle : \mathrm{X}^*(T) \times \mathrm{X}_*(T) \to \mathbb{Z}$. We will consider the usual partial order on $\mathrm{X}^*(T)$: recall that we say that $\lambda \geq \mu$ if $\lambda - \mu$ is a sum of simple roots.

Assume that the quotient

$$\mathrm{X}_*(T) / \mathbb{Z}\,\Phi^\vee$$

is free as a $\mathbb{Z}$-module, i.e. that the derived subgroup of H is simply connected. (It follows from the definition of "simply connected", see [J, II, §1.6, (4)], and from the description of the cocharacter lattice of the derived subgroup, given e.g. in [J, II, §, 1.18] that the two conditions are indeed equivalent.) Then there exists a complement E to $\mathbb{Z}\,\Phi^\vee$ in $\mathrm{X}_*(T)$. For any simple root α, we define a "fundamental weight" $\varsigma_\alpha \in \mathrm{X}^*(T)$ by

$$\langle \varsigma_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}, \qquad (\varsigma_\alpha)_{|E} = 0$$

for $\beta^\vee$ any simple coroot; note that this character is dominant. For any $v \in W$, define an element $\lambda_v \in \mathrm{X}^*(T)$ as

$$\lambda_v := \sum_{\substack{\alpha \in \Pi \\ v^{-1}(\alpha) < 0}} \varsigma_\alpha \,. \tag{10.1.1}$$

Set $\varsigma := \sum_{\alpha \in \Pi} \varsigma_\alpha$ and $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Note that ρ is not a character in general; it is only defined in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{X}^*(T)$.

Theorem 10.1.3. *In the above setting, $\mathbb{Z}[\mathrm{X}^*(T)]$ is a free $\mathbb{Z}[\mathrm{X}^*(T)]^W$-module, of rank $|W|$. Moreover, there exists a basis $(c_w)_{w \in W}$ of $\mathbb{Z}[\mathrm{X}^*(T)]$ over $\mathbb{Z}[\mathrm{X}^*(T)]^W$ such that*

$$\det((v(c_w))_{v,w \in W}) = (e^\varsigma)^{|W|/2} \Big(\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}) \Big)^{|W|/2}.$$

Before the proof of the theorem, we will give some intermediary results concerning $\mathbb{Z}[\mathrm{X}^*(T)]$.

Lemma 10.1.4. *We use the above notation. For any $w \in W$, we have*

$$w(\varsigma - \rho) = \varsigma - \rho.$$

Proof. We know that $W = \langle s_\alpha \mid \alpha \in \Pi \rangle$. If we show that the statement of the lemma is true for any s_α with α simple, then the lemma will be proved. For any $\lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{X}^*(T)$, the action of s_α on λ is by definition

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Thus it is enough to show that $\langle \varsigma - \rho, \alpha^\vee \rangle = 0$ for any simple root α. But it is immediate by definition of the element ς that $\langle \varsigma, \alpha^\vee \rangle = 1$ and it is well known that $\langle \rho, \alpha^\vee \rangle = 1$. This concludes the proof. □

For an element $\sum_\lambda x_\lambda e^\lambda \in \mathbb{Z}[\mathrm{X}^*(T)]$, we say that the term $x_\lambda e^\lambda$ is maximal in this expression if λ is maximal in the set

$$\{\lambda \in \mathrm{X}^*(T) \mid x_\lambda \neq 0\}$$

(which we view as a poset with the restriction of $\leq$). We will also consider minimal terms; the definition is the exact analogue of the above one. Consider $x, y \in \mathbb{Z}[\mathrm{X}^*(T)]$. Write $x = \sum_\lambda x_\lambda e^\lambda$ and $y = \sum_\mu y_\mu e^\mu$ (where the sums are finite and where each x_λ or y_μ appearing is non-zero). Assume that there is a unique maximal term in the above expression of x corresponding to a character λ_0 and fix a character μ_0 such that $y_{\mu_0} e^{\mu_0}$ is a maximal term in the expression of y.

Lemma 10.1.5. *In the above setting, the term $x_{\lambda_0} y_{\mu_0} e^{\lambda_0 + \mu_0}$ is maximal in the expression of xy.*

Proof. We write

$$xy = \sum_\nu \left(\sum_{\lambda + \mu = \nu} x_\lambda y_\mu \right) e^\nu.$$

We show that there is no pair $(\lambda, \mu) \neq (\lambda_0, \mu_0)$ with λ appearing in the expression of x and μ in the expression of y such that

$$\lambda + \mu \geq \lambda_0 + \mu_0. \tag{10.1.2}$$

Consider (λ, μ) satisfying (10.1.2). As λ_0 is the unique maximal term in the expression of x, $\lambda_0 - \lambda$ is a sum of positive roots so we get that

$$\mu \geq \mu_0 + (\lambda_0 - \lambda) \geq \mu_0.$$

Since μ_0 is maximal, this is possible only if $\mu = \mu_0$, and then $\lambda = \lambda_0$. Thus (λ_0, μ_0) is the only pair of characters (λ, μ) such that $x_\lambda y_\mu$ contributes to the coefficient of $e^{\lambda_0 + \mu_0}$. This shows at once that this coefficient is non-zero and that the term $x_{\lambda_0} y_{\mu_0} e^{\lambda_0 + \mu_0}$ is maximal in the above expression of xy. □

Corollary 10.1.6. *Consider two non-zero elements $x, z \in \mathbb{Z}[\mathrm{X}^*(T)]$ such that x divides z in this algebra. Assume that both x and z have a unique maximal term and a unique minimal term, and assume moreover that the maximal and minimal terms of x and z coincide up to a scalar. Then $x = y_0 z$ with $y_0 \in \mathbb{Z} \smallsetminus \{0\}$.*

Proof. By assumption, there exists $y \in \mathbb{Z}[\mathrm{X}^*(T)]$ such that $xy = z$. Thanks to lemma 10.1.5, we see that any maximal element in an expression of y gives rise to a maximal element in z. But there is a unique maximal element in z, so there must be a unique maximal element in y. Moreover, we deduce immediately from the lemma that this maximal element of y can only be of the form $y_0 e^0$ for some $y_0 \in \mathbb{Z}\setminus\{0\}$. With the same kind of reasoning, but replacing everywhere "maximal" by "minimal", we get that $y = y_0 \in \mathbb{Z}\setminus\{0\}$ and hence conclude the proof. □

For any $v \in W$, set $c_v := v^{-1}(e^{\lambda_v}) \in \mathbb{Z}[\mathrm{X}^*(T)]$, where λ_v is as in (10.1.1).

Lemma 10.1.7. *Let $A := (uc_v)_{u,v\in W}$. For any family $\mathcal{F} = \{f_v\}_{v\in W}$ of elements in $\mathbb{Z}[\mathrm{X}^*(T)]$, we set $B_{\mathcal{F}} = (uf_v)_{u,v\in W}$. Let $D = \det(A)$ and $E_{\mathcal{F}} = \det(B_{\mathcal{F}})$. Then,*

1. *$D = (e^{\varsigma})^{\frac{|W|}{2}} \left(\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})\right)^{\frac{|W|}{2}} \neq 0$,*
2. *D divides $E_{\mathcal{F}}$ in $\mathbb{Z}[\mathrm{X}^*(T)]$ and the quotient belongs to $\mathbb{Z}[\mathrm{X}^*(T)]^W$.*

Before the proof, let us choose a total order $\preceq$ on W which refines the Bruhat order and for which the length function is increasing, that is to say $\ell(u) \leq \ell(u')$ when $u \preceq u'$. We order the rows and columns of A according to this order, saying that row u is above row u' if $u \preceq u'$ (and similarly for the columns). Note that this implies in particular that row u is above row u' as soon as $\ell(u) < \ell(u')$.

Remark 10.1.8. Since for any root α the element e^{α} is invertible, $e^{\lambda} - s_\alpha(e^{\lambda})$ is divisible by $(e^{\alpha}-1)$ and $(1-e^{-\alpha})$. Indeed, we have $s_\alpha(e^{\lambda}) = e^{\lambda}e^{-n\alpha}$ with $n = \langle\lambda, \alpha^\vee\rangle$. We now write

$$\begin{aligned} e^{\lambda} - s_\alpha(e^{\lambda}) &= e^{\lambda}\left(1-e^{-n\alpha}\right) \\ &= e^{\lambda}\left(e^{\alpha}-1\right)\left(1+\cdots+e^{(n-1)\alpha}\right)e^{-n\alpha}. \end{aligned}$$

Proof. We show that D admits a unique maximal term, which is $(e^{\varsigma})^{\frac{|W|}{2}}$ and a unique minimal term which is $\pm(e^{\varsigma})^{\frac{|W|}{2}}(e^{-2\rho})^{\frac{|W|}{2}}$. Consider the column $(uc_v)_u$ of A. For any $u \in W$, we have

$$\lambda_v \geq uv^{-1}\lambda_v \tag{10.1.3}$$

since λ_v is dominant (see [H1, 13.2, Lemma A]); in particular, any diagonal element is maximal in the column it belongs to.

We claim that if $\ell(u) \leq \ell(v)$, we have equality in (10.1.3) only for $u = v$. Assume that equality holds, i.e. that uv^{-1} lies in the stabilizer of λ_v; it is well known that this element is then in the subgroup of W generated by the subset $I = \{s_\alpha \mid \alpha \in \Pi,\ \langle\lambda_v, \alpha^\vee\rangle = 0\}$. By definition, this is the parabolic subgroup $W_I := \langle s_\alpha \mid \alpha \in \Pi,\ v^{-1}\alpha > 0\rangle$. [H3, 5.7, Proposition] tells us that $s_\alpha v \geq v$ for any $s_\alpha \in I$. Hence with [H3, 5.12], we know that v is minimal (for the Bruhat

order) in the coset $W_I v$ and that for any $w \in W_I$, we have $\ell(wv) = \ell(w) + \ell(v)$. We thus deduce that

$$\ell(u) = \ell(uv^{-1}v) = \ell(uv^{-1}) + \ell(v).$$

Under the assumption $\ell(u) \leq \ell(v)$, this is only possible for $\ell(u) = \ell(v)$, so $\ell(uv^{-1}) = 0$ and $u = v$.

Now we have (for ε the signature character)

$$\det(A) = \sum_{\sigma \in \mathfrak{S}(W)} \left(\varepsilon(\sigma) \prod_{u \in W} \sigma(u) c_u \right).$$

Consider a $\sigma \neq \mathrm{id}_W$. Then there exists an element $u \in W$ such that $\sigma(u) \prec u$, so the above discussion implies $\sigma(u)u^{-1}\lambda_u < \lambda_u$. Since for any $v \in W$, we have $\sigma(v)v^{-1}\lambda_v \leq \lambda_v$, one sees that $\sum_{u \in W} \sigma(u)u^{-1}\lambda_u < \sum_{u \in W} \lambda_u$; this exactly says that $\prod_{u \in W} uc_u = \prod_{u \in W} e^{\lambda_u}$ is the unique maximal term in $\det(A) = D$.

Now we need to determine how many times $e^{\varsigma\alpha}$ appears in this product. By definition of the elements λ_v, the term $e^{\varsigma\alpha}$ makes a contribution in e^{λ_v} for any $v \in W$ satisfying $v^{-1}\alpha < 0$. It is easy to see that the cardinality of the set $\{v \in W \mid v^{-1}\alpha < 0\}$ is $\frac{|W|}{2}$ for any simple root α. Thus, the maximal term of D is

$$\left(\prod_{\alpha \in \Pi} e^{\varsigma\alpha} \right)^{\frac{|W|}{2}} = (e^{\varsigma})^{\frac{|W|}{2}}.$$

Now note that each $w \in W$ permutes the rows of A and hence fixes D up to sign. We deduce that there is a unique lowest term, and that this term is $\pm w_0(\prod e^{\varsigma\alpha})^{\frac{|W|}{2}}$ where w_0 is the longest element of W. To determine this term more explicitly, we must understand the action of w_0 on $\sum_{\alpha \in \Pi} \varsigma\alpha$. But with lemma 10.1.4, we have

$$w_0(\varsigma) = \varsigma - 2\rho.$$

We deduce that the lowest term of D is

$$\pm(e^{\varsigma})^{\frac{|W|}{2}}(e^{-2\rho})^{\frac{|W|}{2}}.$$

Set

$$D_1 = (e^{\varsigma})^{\frac{|W|}{2}} \left(\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \right)^{\frac{|W|}{2}}.$$

We show that D_1 divides $E_{\mathcal{F}}$ and that $D = D_1$. Consider the $\frac{|W|}{2}$ rows of $B_{\mathcal{F}}$ indexed by elements $u \in W$ such that $u^{-1}\alpha \leq 0$. If we subtract, for such a u, row $s_\alpha u$ from row u, each entry is divisible by $(e^\alpha - 1)$ (see remark 10.1.8). We deduce that $(e^\alpha - 1)^{\frac{|W|}{2}}$ divides $E_{\mathcal{F}}$. Since $\mathbb{Z}[\mathrm{X}^*(T)]$ is a unique factorization domain, $\prod_{\alpha \in \Sigma^+}(e^\alpha - 1)^{\frac{|W|}{2}}$ divides $E_{\mathcal{F}}$ too. Since (e^{ς}) is an invertible element, $E_{\mathcal{F}}$ is divisible by D_1, and so is D. It is quite clear that D_1 has a unique

highest term and that this term coincides with the highest term of D, as well as a unique lowest term which coincides up to a sign, with the lowest terms of D. We deduce easily that $D = D_1$, thanks to corollary 10.1.6. Finally, any element of W permutes the rows of A and $B_{\mathcal{F}}$ in the same way, so that the quotient $\frac{E_{\mathcal{F}}}{D}$ is W-invariant. □

Proof of Theorem 10.1.3. Take $f \in \mathbb{Z}[\mathrm{X}^*(T)]$. If we can show that the system

$$(S) : \left(\sum_{v \in W} a_v u c_v \right)_{u \in W} = (uf)_{u \in W}$$

admits a unique solution $(a_v)_{v \in W}$ with $a_v \in \mathbb{Z}[\mathrm{X}^*(T)]^W$ for any v, then the theorem will be proved. We place ourselves in the field of fractions $F := \mathrm{Frac}(\mathbb{Z}[\mathrm{X}^*(T)])$. The matrix A is invertible in F since its determinant is non-zero (see lemma 10.1.7), hence, we know that in F the system S admits a unique solution $(a_v)_{v \in W}$.

For any $v \in W$, denote by A_v the matrix A where the column indexed by v is replaced by $(uf)_{u \in W}$. Using Cramer's rule, we also know that

$$a_v = \frac{\det(A_v)}{\det(A)}.$$

With lemma 10.1.7, we get that in fact a_v is in $\mathbb{Z}[\mathrm{X}^*(T)]^W$, and thus we obtain the theorem. □

10.1.2 Completed version

Let H be a reductive algebraic $\Bbbk$-group, fix a maximal torus T_H in H and let W_H denote the Weyl group of the pair (H, T_H). For any element $t \in T_H$, we set

$$\widehat{R}(H)_t := \varprojlim_n \Bbbk[\mathrm{X}^*(T_H)]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}^*(T_H)\rangle^n.$$

If t is stable under the action of W_H, e.g. if t is a central element, the ideal $\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}^*(T_H)\rangle$ is W_H-stable, so that we have a natural action of W_H on $\widehat{R}(H)_t$. In this case, let $\widehat{R}(H)_t^{W_H}$ be the subalgebra of W_H-invariants in this algebra.

Lemma 10.1.9. *Assume that $\mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$ is free as a $\mathbb{Z}$-module. Then, for any element t central in H, $\widehat{R}(H)_t$ is free of rank $|W_H|$ as a $\widehat{R}(H)_t^{W_H}$-module. There exists a basis $(c_v)_{w \in W_H}$ of $\widehat{R}(H)_t$ over $\widehat{R}(H)_t^{W_H}$ such that*

$$\det((v(c_w))_{v,w \in W_H}) = (e^\varsigma)^{|W_H|/2} \Big(\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}) \Big)^{|W_H|/2}.$$

Proof. The proof can be copied from [BeR, Theorem 8.1]; we do not reproduce it here. Let us however explain rapidly how the it goes: the first step is to show

that theorem 10.1.3 holds after extension of scalars to $\Bbbk$, which is easy. We obtain a basis $(c_v)_{v\in W_H}$ of $\Bbbk[\mathrm{X}^*(T_H)]$ over $\Bbbk[\mathrm{X}^*(T_H)]^{W_H}$ satisfying the equality in the statement. To deal with completions, the strategy is to come back to the definition, and view elements in $\widehat{R}(H)_t$ as sequences $(a_n)_n$ with a_n an element in $\Bbbk[\mathrm{X}^*(T_H)]/\langle e^\lambda - \lambda(t)\rangle^n$. For any n, we can express a_n as a linear combination of c_v's with coefficients in $\Bbbk[\mathrm{X}^*(T_H)]^{W_H}$. One obtains sequences $(b_n^v)_n$ in $\Bbbk[\mathrm{X}^*(T_H)]^{W_H}$ for any $v \in W_H$, and it remains to show that for each v, the sequence $(b_n^v)_n$ converges to an element in $\widehat{R}(H)_t^{W_H}$ in order to show that the elements $(c_v)_{v\in W_H}$ generate the $\widehat{R}(H)_t^{W_H}$-module $\widehat{R}(H)_t$. This in turn is done with the help of the matrix A of lemma 10.1.7 and the Artin–Rees lemma. □

Now we prove an analogue of lemma 10.1.9, but with no other assumption on H that it is reductive and some mild condition on the characteristic of $\Bbbk$ (that is to say, we want to drop the assumption that $\mathrm{X}_*(T_H)/\mathbb{Z}\,\Phi^\vee$ is free). The strategy is to construct a finite central isogeny from a reductive group H_{sc} satisfying the hypothesis of lemma 10.1.9 to H, and deduce the wished-for property from the one for H_{sc} and some standard tools of algebraic geometry.

Thus we consider again a general reductive group H over $\Bbbk$, we fix a maximal torus T_H. We keep the notation used above: we have the associated root datum $(\mathrm{X}^*(T_H), \Phi, \mathrm{X}_*(T_H), \Phi^\vee)$; we denote by $\langle\cdot,\cdot\rangle : \mathrm{X}^*(T_H) \times \mathrm{X}_*(T_H) \to \mathbb{Z}$ the usual perfect pairing, as well as its extension to $\mathbb{R}\otimes_{\mathbb{Z}} \mathrm{X}^*(T_H) \times \mathbb{R}\otimes_{\mathbb{Z}} \mathrm{X}_*(T_H)$. Let $\mathscr{D}(H)$ be the derived subgroup of H, and consider the maximal torus $T_{\mathscr{D}(H)}$ of $\mathscr{D}(H)$ contained in T_H. Throughout this section, we make the following assumption:

$$\text{The group } \mathrm{X}_*(T_{\mathscr{D}(H)})/\mathbb{Z}\cdot\Phi^\vee \text{ does not have } \ell\text{-torsion.}$$

Recall that ℓ denotes the characteristic of $\Bbbk$. Here, $\mathrm{X}_*(T_{\mathscr{D}(H)})$ is the cocharacter lattice of the derived subgroup of H; this is given by (see e.g. [J, §1.18])

$$\mathrm{X}_*(T_{\mathscr{D}(H)}) = \{x \in \mathrm{X}_*(T) \mid \mathbb{Z}\cdot x \cap \mathbb{Z}\cdot\Phi^\vee \neq 0\}. \tag{10.1.4}$$

Since $\mathscr{D}(H)$ is a semisimple group, the above quotient is finite and our statement amounts to saying that ℓ does not divide the order of this finite group.

We now proceed to construct effectively our group H_{sc}. Consider the quotient $\mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$: this is a finitely generated $\mathbb{Z}$-module, so it can be written as a direct sum $L \oplus M$ where L is a free (of finite rank) $\mathbb{Z}$-module and M is a finitely generated torsion (hence finite) $\mathbb{Z}$-module. The inverse image of L by the natural map $\mathrm{X}_*(T_H) \twoheadrightarrow \mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$ is a submodule L' of $\mathrm{X}_*(T_H)$. Clearly, we have $L' \cap \mathbb{Z}\cdot\Phi^\vee = \{0\}$. We set

$$\widetilde{\mathrm{X}_*(T_H)} := \mathbb{Z}\cdot\Phi^\vee \oplus L'.$$

This is a $\mathbb{Z}$-submodule of $\mathrm{X}_*(T_H)$; by construction, the quotient $\mathrm{X}_*(T_H)/\widetilde{\mathrm{X}_*(T_H)}$ is finite. Let $\widetilde{\mathrm{X}^*(T_H)} \subseteq \mathbb{R}\otimes_{\mathbb{Z}} \mathrm{X}^*(T_H)$ be its dual for the perfect paring $\langle\cdot,\cdot\rangle$.

Note that we have $\Phi^\vee \subseteq \widetilde{\mathrm{X}_*(T_H)}$ and $\Phi \subseteq \mathrm{X}^*(T_H) \subseteq \widetilde{\mathrm{X}^*(T_H)}$. Define H_{sc} to be the reductive group over $\Bbbk$ whose root datum is given by

$$(\widetilde{\mathrm{X}^*(T_H)}, \Phi, \widetilde{\mathrm{X}_*(T_H)}, \Phi^\vee).$$

By definition, the quotient $\widetilde{\mathrm{X}_*(T_H)}/\mathbb{Z}\cdot\Phi^\vee$ is free; in other words, the derived group $\mathscr{D}(H_{\mathrm{sc}})$ is simply connected.

Lemma 10.1.10. *The natural inclusion map* $\mathrm{X}_*(T_{\mathscr{D}(H)}) \hookrightarrow \mathrm{X}_*(T_H)$ *induces an isomorphism of* $\mathbb{Z}$*-module*

$$\mathrm{X}_*(T_{\mathscr{D}(H)})/\mathbb{Z}\cdot\Phi^\vee \xrightarrow{\sim} \mathrm{X}_*(T_H)/\widetilde{\mathrm{X}_*(T_H)}.$$

Proof. We prove first the injectivity. Consider $x, x' \in \mathrm{X}_*(T_{\mathscr{D}(H)})$; from (10.1.4), one sees that there exists a $n \in \mathbb{Z}$ such that $n(x-x')$ is in $\mathbb{Z}\cdot\Phi^\vee$. Thus, if $x-x'$ lies in $\widetilde{\mathrm{X}_*(T_H)} = L' \oplus \mathbb{Z}\cdot\Phi^\vee$, we get that $x - x'$ lies in $\mathbb{Z}\cdot\Phi^\vee$ and our map is injective.

Now for the surjectivity, choose $x \in \mathrm{X}_*(T_H)$. Recall that $\mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$ splits as a direct sum $L\oplus M$ with L a free $\mathbb{Z}$-module and M a torsion module. We can then write the image of x in the quotient $\mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$ as a sum $l+c$ with $l \in L$ and $c \in M$ (note that c identifies with the image of x in $\mathrm{X}_*(T_H)/\widetilde{\mathrm{X}_*(T_H)}$). For any element $l' \in L'$ whose image in $\mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$ is l, the cocharacter $x - l' \in \mathrm{X}_*(T_H)$ maps to c under the quotient map $\mathrm{X}_*(T_H) \twoheadrightarrow \mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi^\vee$; since c is of finite order, there exists $n \gg 0$ such that $n(x-l')$ belongs to $\mathbb{Z}\cdot\Phi^\vee$. In particular $x - l'$ belongs to $\mathrm{X}_*(T_{\mathscr{D}(H)})$. This concludes the proof of the lemma. □

Thanks to the assumption made at the beginning of this section, ℓ is prime to the order of the quotient $\mathrm{X}_*(T_H)/\widetilde{\mathrm{X}_*(T_H)}$.

Now the inclusion $\mathrm{X}^*(T_H) \hookrightarrow \widetilde{\mathrm{X}^*(T_H)}$ defines a morphism of root data (in the sense of [J, II, 1.13]). According to [J, II, 1.14, 1.15 and 1.17], this in turn defines a central isogeny

$$\nu_H : H_{\mathrm{sc}} \twoheadrightarrow H.$$

In fact, H_{sc} is exactly what is called a covering group in [J, §1.17]. Let $T_{H_{\mathrm{sc}}}$ be the maximal torus $\nu_H^{-1}(T_H)$, so that we have

$$\mathrm{X}^*(T_{H_{\mathrm{sc}}}) \cong \widetilde{\mathrm{X}^*(T_H)} \quad \text{and} \quad \mathrm{X}_*(T_{H_{\mathrm{sc}}}) \cong \widetilde{\mathrm{X}_*(T_H)}.$$

Note that we have a canonical identification of the Weyl groups of H_{sc} and H: $W_{H_{\mathrm{sc}}} \cong W_H$. We claim that the induced map

$$\nu_H : T_{H_{\mathrm{sc}}} \twoheadrightarrow T_H$$

is an étale map. To see this, we have to show that the induced morphism on tangent spaces at all points is an isomorphism; since we work with group

schemes, it suffices to consider the tangent space at the neutral element, that it to say, the Lie algebras $\mathfrak{t}_{H_{\mathrm{sc}}}$ and $\mathfrak{t}_H$. It is well known that we have

$$\mathfrak{t}_{H_{\mathrm{sc}}} \cong \Bbbk \otimes_{\mathbb{Z}} \mathrm{X}_*(T_{H_{\mathrm{sc}}}) \quad \text{and} \quad \mathfrak{t}_H \cong \Bbbk \otimes_{\mathbb{Z}} \mathrm{X}_*(T_H);$$

moreover the map $\mathfrak{t}_{H_{\mathrm{sc}}} \to \mathfrak{t}_H$ is induced by the natural morphism

$$\nu_H^{\#} : \mathrm{X}_*(T_{H_{\mathrm{sc}}}) \to \mathrm{X}_*(T_H); \quad \psi \mapsto \nu_H \circ \psi.$$

By construction, the map $\nu_H^{\#}$ is injective. Our assumption on the characteristic ℓ of $\Bbbk$ implies that it does not divide the order of the cokernel $\mathrm{X}_*(T_H)/\mathrm{X}_*(T_{H_{\mathrm{sc}}})$ (see lemma 10.1.10), so the map

$$\mathfrak{t}_{H_{\mathrm{sc}}} \cong \Bbbk \otimes_{\mathbb{Z}} \mathrm{X}_*(T_{H_{\mathrm{sc}}}) \xrightarrow{\Bbbk \otimes_{\mathbb{Z}} \nu_H^{\#}} \Bbbk \otimes_{\mathbb{Z}} \mathrm{X}_*(T_H) \cong \mathfrak{t}_H$$

is an isomorphism and the restriction

$$(\nu_H)_{|T_{H_{\mathrm{sc}}}} : T_{H_{\mathrm{sc}}} \to T_H$$

is an étale morphism.

We will need a standard result of commutative algebra during the proof of the next proposition:

Lemma 10.1.11. *Let A be a noetherian commutative ring and $\mathfrak{m}$ be a maximal ideal. Let $\widehat{A} := \varprojlim_n A/\mathfrak{m}^n$ be the completion of A with respect to $\mathfrak{m}$, $A_{\mathfrak{m}}$ be the localization of A with respect to $\mathfrak{m}$ and finally $\widehat{A_{\mathfrak{m}}} := \varprojlim_n A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^n$ be the completion of $A_{\mathfrak{m}}$ with respect to the maximal ideal $\mathfrak{m} \cdot A_{\mathfrak{m}}$. Then we have a canonical isomorphism of rings*

$$\widehat{A} \cong \widehat{A_{\mathfrak{m}}}.$$

Proof. We use the well known fact that localization is an exact functor, so that

$$A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^n \cong (A/\mathfrak{m}^n)_{\mathfrak{m}_n},$$

where $\mathfrak{m}_n$ is the image of $\mathfrak{m}$ in $A/\mathfrak{m}^n$. Now any element in $A/\mathfrak{m}^n$ which lies outside of $\mathfrak{m}_n$ is invertible. (This can be justified as follows: if the image of $x \in A$ in $A/\mathfrak{m}^n$ does not belong to $\mathfrak{m}_n$, then it is invertible in $A/\mathfrak{m}$, i.e. there exists y in A such that $xy-1 \in \mathfrak{m}$. Then $(xy-1)^n \in \mathfrak{m}^n$; using the commutativity of A, we see that the image of x in $A/\mathfrak{m}^n$ is invertible). Hence we get $(A/\mathfrak{m}^n)_{\mathfrak{m}_n} = A/\mathfrak{m}^n$, and it is not difficult to check that the following square is commutative for any $n \geq r$:

$$\begin{array}{ccc} A/\mathfrak{m}^n & \longrightarrow & A/\mathfrak{m}^r \\ \wr\uparrow & & \wr\uparrow \\ A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^n & \longrightarrow & A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^r. \end{array}$$

(The horizontal arrows being the natural transition functions, and the vertical ones are given by the identification justified above.) Finally, we have the following sequence of isomorphisms

$$\widehat{A_{\mathfrak{m}}} \cong \varprojlim_n A_{\mathfrak{m}}/(\mathfrak{m}\cdot A_{\mathfrak{m}})^n \cong \varprojlim A/\mathfrak{m}^n \cong \widehat{A},$$

which concludes the proof. □

Proposition 10.1.12. *Assume that $t \in T_H$ is central in H. Assume moreover that ℓ does not divide the order of the finite group $\mathrm{X}_*(T_{\mathscr{D}(H)})/\mathbb{Z}\cdot\Phi^\vee$. Then $\widehat{R}(H)_t$ is free of rank $|W_H|$ as a $\widehat{R}(H)_t^{W_H}$-module. There exists a basis $(c_v)_{w\in W_H}$ of $\widehat{R}(H)_t$ over $\widehat{R}(H)_t^{W_H}$ such that*

$$\det((v(c_w))_{v,w\in W_H}) = x(\prod_{\alpha\in\Phi_+}(1-e^{-\alpha}))^{|W|/2}$$

with $x \in \widehat{R}(H)_t$ is an invertible element.

Proof. As we noted above, there exists a reductive algebraic group H_{sc} such that $\mathrm{X}_*(T_{H_{\mathrm{sc}}})/\mathbb{Z}\cdot\Phi^\vee$ is free, and a finite central isogeny

$$\nu_H : H_{\mathrm{sc}} \twoheadrightarrow H$$

such that the restriction of this map to the maximal torus above T_H is an étale map. Now any element t' in the inverse image of t under ν_H is also central in H_{sc}, so lemma 10.1.9 tells us that the statement of the proposition is true for this group: $\widehat{R}(H_{\mathrm{sc}})_{t'}$ is free of rank $|W_{H_{\mathrm{sc}}}|$ as a module over $\widehat{R}(H_{\mathrm{sc}})_{t'}^{W_{H_{\mathrm{sc}}}}$.

We have a natural $W_{H_{\mathrm{sc}}}$-equivariant algebra morphism

$$\mathscr{O}(T_H)_t \to \mathscr{O}(T_{H_{\mathrm{sc}}})_{t'}.$$

Let $\Bbbk(t)$ be the residue field of the local ring $\mathscr{O}(T_H)_t$. Denote by $\widehat{\mathscr{O}}(T_H)_t$ the completion of the algebra $\mathscr{O}(T_H)_t$ with respect to the kernel of the natural map

$$\mathscr{O}(T_H)_t \twoheadrightarrow \Bbbk(t), \tag{10.1.5}$$

and similarly for $\widehat{\mathscr{O}}(T_{H_{\mathrm{sc}}})_{t'}$. Thanks to [Har, Chapter III, Exercice 10.4], the latter map induces an algebra isomorphism

$$\widehat{\mathscr{O}}(T_H)_t \xrightarrow{\sim} \widehat{\mathscr{O}}(T_{H_{\mathrm{sc}}})_{t'}. \tag{10.1.6}$$

(We used here the fact that $\Bbbk$ is algebraically closed.) It is a standard fact that for a torus, we have

$$\mathscr{O}(T_H) \cong \Bbbk[\mathrm{X}^*(T_H)].$$

Let $\mathfrak{m}_t$ be the maximal ideal $\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}^*(T_H)\rangle \subseteq \Bbbk[\mathrm{X}^*(T_H)]$. We have

$$\mathscr{O}(T_H)_t \cong \Bbbk[\mathrm{X}^*(T_H)]_{\mathfrak{m}_t}$$

and the map (10.1.5) identifies with the natural quotient

$$\Bbbk[\mathrm{X}^*(T_H)]_{\mathfrak{m}_t} \twoheadrightarrow \Bbbk[\mathrm{X}^*(T_H)]_{\mathfrak{m}_t}/\mathfrak{m}_t \cdot \Bbbk[\mathrm{X}^*(T_H)]_{\mathfrak{m}_t} \cong \Bbbk[\mathrm{X}^*(T_H)]/\mathfrak{m}_t.$$

Using lemma 10.1.11, we get an isomorphism

$$\Bbbk[\mathrm{X}^*(T_H)]^\wedge_{\mathfrak{m}_t} \cong \Bbbk[\mathrm{X}^*(T_H)]^{\wedge,\mathfrak{m}_t}$$

where the left hand side denotes the completion of the local ring $\Bbbk[\mathrm{X}^*(T_H)]_{\mathfrak{m}_t}$ with respect to its maximal ideal, and the right hand side is the completion of the ring $\Bbbk[\mathrm{X}^*(T_H)]$ with respect to its maximal ideal $\mathfrak{m}_t$. Thus we can write the isomorphism (10.1.6) as

$$\widehat{R}(H)_t \xrightarrow{\sim} \widehat{R}(H_{\mathrm{sc}})_{t'}, \tag{10.1.7}$$

and this map commutes with the action of $W_{H_{\mathrm{sc}}} = W_H$ on both sides. We readily deduce from the analogous result for H_{sc} that $\widehat{R}(H)_t$ is free of rank $|W_H|$ as a module over $\widehat{R}(H)_t^{W_H}$.

For the basis, note first that there exists a basis $(c'_w)_{w\in W_{H_{\mathrm{sc}}}}$ of $\widehat{R}(H_{\mathrm{sc}})_{t'}$ over $\widehat{R}(H_{\mathrm{sc}})_{t'}^{W_{H_{\mathrm{sc}}}}$ as in lemma 10.1.9. Consider the basis $(c_w)_{w\in W_H}$ of $\widehat{R}(H)_t$ over $\widehat{R}(H)_t^{W_H}$ defined to be the inverse image of $(c'_w)_{w\in W_{H_{\mathrm{sc}}}}$ under the isomorphism (10.1.7). Since this algebra isomorphism commutes with the actions of the Weyl groups, the determinant of the matrix $(v(c_w))_{w\in W_H}$ is the inverse image under (10.1.7) of the determinant of the matrix $(v(c'_w))_{w\in W_{H_{\mathrm{sc}}}}$. Now (10.1.7) is induced by the isogeny ν_H, which is itself constructed from a morphism of root data. In particular, $e^\alpha \in \widehat{R}(H)_t$ is mapped to $e^\alpha \in \widehat{R}(H_{\mathrm{sc}})_{t'}$. The element x is just the inverse image of the invertible element $(e^\varsigma)^{|W|/2} \in \widehat{R}(H_{\mathrm{sc}})_{t'}$ in $\widehat{R}(H)_t$. □

10.2 The endoscopic group

We now come back to the framework of subsection 5.1.1, so that G is a complex reductive connected algebraic group. **From now on, and until the end of this chapter, we make the following assumption:**

The characteristic ℓ of the field $\Bbbk$ is not a torsion prime for the dual group $G^\vee_\Bbbk$ (equivalently, for the derived subgroup $\mathscr{D}(G^\vee_\Bbbk)$).

Remark 10.2.1. This assumption implies the following: for any sub-root system Φ_1 integrally closed in Φ_H, the quotient $\mathrm{X}_*(T_H)/\mathbb{Z}\cdot\Phi_1$ is ℓ-torsion free (see [SS, I.4.3, I.4.4] for more details).

Let us fix an element $t \in T^\vee_\Bbbk$; recall the group W_t° defined in subsection 5.1.3. We have associated sets of roots Φ_t and coroots $\Phi_t^\vee$.

Definition 10.2.2. *Consider the reductive $\Bbbk$-group $H^\vee_{t,\Bbbk}$ defined as the identity component of the centralizer in $G^\vee_\Bbbk$ of t:*

$$H_t^\vee := C_{G^\vee_\Bbbk}(t)^\circ.$$

Define the endoscopic group H_t associated to t to be the (complex) Langlands dual to $H_t^\vee$.

According to [H4, Theorem 2.2], $H_t^\vee$ is generated by $T_{\Bbbk}^\vee$ and the root subgroups $U_{\alpha^\vee}$ with $\alpha \in \Phi_t$. The root datum of $H_t^\vee$ is then given by

$$(\mathrm{X}^*(T_{\Bbbk}^\vee), \Phi_t^\vee, \mathrm{X}_*(T_{\Bbbk}^\vee), \Phi_t).$$

The Weyl group of $H_t^\vee$ identifies canonically with W_t° (and so does the Weyl group of H_t). Finally, recall the ring $\widehat{R}_t$ defined in subsection 5.1.4.

Proposition 10.2.3. *$\widehat{R}_t$ is free of rank $|W_t^\circ|$ over $(\widehat{R}_t)^{W_t^\circ}$. There exists a basis $(c_w^t)_{w \in W_t^\circ}$ such that*

$$\det((v(c_w^t))_{v,w \in W_H}) = x\Big(\prod_{\alpha \in \Phi_{t,+}^\vee} (1 - e^{-\alpha^\vee})\Big)^{|W|/2},$$

with x an invertible element in $\widehat{R}_t$.

Proof. Using the identification $\Bbbk[\mathrm{X}_*(T)] \cong \Bbbk[\mathrm{X}^*(T_{\Bbbk}^\vee)]$, the statement of the proposition is exactly proposition 10.1.12 applied to the reductive group $H_t^\vee$ and the central element t. Note that we can indeed apply this theorem because remark 10.2.1 tells us that $\mathrm{X}_*(T_{\mathscr{D}(H_{t,\Bbbk}^\vee)})/\mathbb{Z} \cdot \Phi_t$ is ℓ-torsion free. □

Remark 10.2.4. It will be convenient later on to notice the following: we have an equality

$$\prod_{\alpha \in \Phi_{t,+}^\vee} (1 - e^{-\alpha^\vee}) = (-1)^{|\Phi_{t,+}^\vee|}\Big(\prod_{\alpha \in \Phi_{t,+}^\vee} (1 - e^{\alpha^\vee})\Big) e^{2\rho_t^\vee},$$

with $\rho_t^\vee = \frac{1}{2}\sum_{\alpha^\vee \in \Phi_{t,+}^\vee} \alpha^\vee$, and $2\rho_t^\vee$ is in $\mathrm{X}_*(T)$.

The preceding proposition immediately implies the following:

1. the $\widehat{R}_t$-module $\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t$ is free of rank $|W_t^\circ|$,
2. if $(\widehat{R}_t)_+^{W_t^\circ}$ denotes the kernel of the map $(\widehat{R}_t)^{W_t^\circ} \hookrightarrow \widehat{R}_t \twoheadrightarrow \Bbbk$ where the second map is induced by $e^\lambda \mapsto \lambda(t)$, then the $\Bbbk$-vector space $\widehat{R}_t/(\widehat{R}_t)_+^{W_t^\circ}$ has dimension $|W_t^\circ|$.

10.3 Modules

In this section, we follow very closely [BeR, §8], itself derived from the results of [KK].

Recall that S_t denotes a subset of simple reflections in W_t° (see subsection 5.1.3) and that this subset defines a Bruhat order $\leq_t$ and a length function ℓ_t on W_t°. Denote by $\widehat{Q}_t$ the fraction field of $\widehat{R}_t$. Let $Q_{W_t^\circ}$ be a $\widehat{Q}_t$-vector space

of dimension $|W_t^\circ|$ and let $(\delta_w^t)_{w\in W_t^\circ}$ be a basis. We define a multiplication on $Q_{W_t^\circ}$: for $a, b \in \widehat{Q}_t$ and $w, v \in W_t^\circ$, set

$$(a\delta_w^t)\cdot(b\delta_v^t) = aw(b)\delta_{wv}^t.$$

We also define an anti-involution ι of $Q_{W_t^\circ}$ by

$$\iota(a) = a \qquad \text{and} \qquad \iota(\delta_w^t) = \delta_{w^{-1}}^t.$$

For any $s \in S_t$ set

$$y_s^t := (\delta_e^t + \delta_s^t)\frac{1}{1-e^{-\alpha_s^\vee}} = \frac{1}{1-e^{-\alpha_s^\vee}}(\delta_e^t - e^{-\alpha_s^\vee}\delta_s^t).$$

These elements satisfy the braid relations of W_t°, so we can set

$$y_w^t := y_{s_1}^t \cdot (\cdots) \cdot y_{s_r}^t$$

where $(s_1, \dots, s_r)$ is any reduced expression of w in W_t° (see [KK, Proposition 2.4], and references therein). The family $(y_w^t)_{w\in W_t^\circ}$ is a $\widehat{Q}_t$-basis of $Q_{W_t^\circ}$, so

$$Y_{W_t^\circ} := \bigoplus_{w\in W_t^\circ} \widehat{R}_t \cdot y_w^t$$

is a free $\widehat{R}_t$-module. One can show that $Y_{W_t^\circ}$ is a subring of $Q_{W_t^\circ}$ (see [KK, Corollary 2.5]).

Proposition 10.3.1. *For any $w \in W_t^\circ$, we can write (in a unique way)*

$$y_w^t = \sum_{v\in W_t^\circ} b_{w,v}\delta_v^t.$$

Then we have $b_{w,v} = 0$ unless $v \le_t w$ and

$$b_{w,w} = \Big(\prod_{\alpha\in\Phi_{t,+}^\vee \cap w(-\Phi_{t,+}^\vee)} (1-e^{\alpha^\vee})\Big)^{-1}.$$

Proof. Let us choose a reduced expression $s_1\cdots s_r$ of w in W_t°. We can write $y_w^t := y_{s_1}^t \cdot (\cdots) \cdot y_{s_r}^t$. For brevity, we set, in this proof, $\alpha_{s_i}^\vee = \alpha_i^\vee$. We use the expression $y_{s_i}^t = \frac{1}{1-e^{-\alpha_i^\vee}}(\delta_e^t - e^{-\alpha_i^\vee}\delta_s^t)$.

We prove the first fact: it follows from the definition of the multiplication in $\widehat{Q}_t$ that the only elements v such that $b_{w,v}$ is nonzero are those that can be written as products

$$s_{i_1} \cdot (\cdots) \cdot s_{i_n}$$

with $1 \le i_1 < \cdots < i_n \le r$. Such a v obviously satisfies $v \le_t w$, so this concludes the proof of this assertion. Now for $b_{w,w}$: this coefficient is the one appearing before the product

$$\frac{-e^{-\alpha_1^\vee}}{1-e^{-\alpha_1^\vee}}\delta_{s_1} \cdot (\cdots) \cdot \frac{-e^{-\alpha_r^\vee}}{1-e^{-\alpha_r^\vee}}\delta_{s_r}.$$

Set $w_i := s_1 \cdots s_i$; the above product is then equal to

$$\frac{-e^{-\alpha_1^\vee}}{1-e^{-\alpha_1^\vee}} \cdot \frac{-e^{-w_1(\alpha_2^\vee)}}{1-e^{-w_1(\alpha_2^\vee)}} \cdot (\cdots) \cdot \frac{-e^{-w_{r-1}(\alpha_r^\vee)}}{1-e^{-w_{r-1}(\alpha_r^\vee)}} \delta_w^t.$$

Write $\frac{-e^{-w_i(\alpha_{i+1}^\vee)}}{1-e^{-w_i(\alpha_{i+1}^\vee)}} = \frac{1}{1-e^{w_i(\alpha_{i+1}^\vee)}}$; we now have to identify the set of coroots $\{\alpha_1^\vee, w_1(\alpha_2^\vee), \ldots, w_{r-1}(\alpha_r^\vee)\}$. But according to [H3, §5.6, Exercice 1], we precisely have

$$\Phi_{t,+}^\vee \cap w(-\Phi_{t,+}^\vee) = \{\alpha_1^\vee, w_1(\alpha_2^\vee), \ldots, w_{r-1}(\alpha_r^\vee)\}.$$

The lemma is now proved. □

Consider

$$\Omega_{W_t^\circ} := \mathrm{Hom}_{\widehat{Q}_t}(Q_{W_t^\circ}, \widehat{Q}_t)$$

where $Q_{W_t^\circ}$ is viewed as a $\widehat{Q}_t$-vector space via right multiplication. We put a structure of $\widehat{Q}_t$-vector space on $\Omega_{W_t^\circ}$ by $(a \cdot \psi)(b) = \psi(ba)$. It will be convenient to identify the $\widehat{Q}_t$-vector space $\Omega_{W_t^\circ}$ with the $\widehat{Q}_t$-vector space $\mathrm{Fun}(W_t^\circ, \widehat{Q}_t)$ of functions from W_t° to $\widehat{Q}_t$, via the map $\psi \mapsto (w \mapsto \psi(\delta_w^t))$. We have an action of $Q_{W_t^\circ}$ on $\Omega_{W_t^\circ}$ defined by

$$(y \cdot \psi)(x) = \psi(\iota(y) \cdot x)$$

for any $\psi \in \Omega_{W_t^\circ}$ and $y, x \in Q_{W_t^\circ}$. This formula in particular gives:

$$(y_s^t \cdot \psi)(\delta_w) = \frac{\psi(\delta_w^t) - e^{-w^{-1}\alpha_s^\vee}\psi(\delta_{sw}^t)}{1-e^{-w^{-1}\alpha_s^\vee}}.$$

Finally, set

$$\Psi_{W_t^\circ} := \{\psi \in \Omega_{W_t^\circ} \mid \forall y \in Y_{W_t^\circ},\ \psi(\iota(y)) \in \widehat{R}_t\}.$$

Remark that this is a $\widehat{R}_t$-submodule of $\Omega_{W_t^\circ}$ (for the action of $\widehat{R}_t$ induced by the one of $\widehat{Q}_t$). It is also stable by the action of $Y_{W_t^\circ} \subseteq Q_{W_t^\circ}$. Let $\bar{y}_w^t := \iota(y_{w^{-1}}^t)$, clearly $(\bar{y}_w^t)_{w \in W_t^\circ}$ is a basis of $\iota(Y_{W_t^\circ})$ as a right $\widehat{R}_t$-module. This implies in particular that $\Psi_{W_t^\circ}$ is free as a $\widehat{R}_t$-module, with a basis $(\psi_w^t)_{w \in W_t^\circ}$ uniquely determined by

$$\psi_w(\bar{y}_v^t) = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 10.3.2. *(1) For any $v, w \in W_t^\circ$, the element $\psi_v^t(\delta_w^t)$ belongs to $\widehat{R}_t$ and vanishes unless $v \leq_t w$.*

(2) For any $w \in W_t^\circ$, we have

$$\psi_w^t(\delta_w^t) = \prod_{\substack{\alpha^\vee \in \Phi_{t,+}^\vee \\ w(\alpha^\vee) \in -\Phi_{t,+}^\vee}} (1 - e^{\alpha^\vee}).$$

(3) For any $w \in W_t^\circ$ and $s \in S_t$, we have

$$y_s^t \cdot \psi_w = \begin{cases} \psi_w^t + \psi_{sw}^t & \text{if } sw <_t w \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove statements *(1)* and *(2)* by induction on $\ell_t(w)$. The case $\ell_t(w) = 0$, that is, $w = e$, is clear since $\delta_e^t = \bar{y}_e^t$.

Consider $w \in W_t^\circ$ and assume that the statements are true for all $u \in W_t^\circ$ such that $\ell_t(u) < \ell_t(w)$. Consider $v \in W_t^\circ$ such that $v \not\leq_t w$. By induction hypothesis, we have $\psi_v^t(\delta_u^t) = 0$ for any $u <_t w$: remark that $u <_t w$ implies $\ell_t(u) < \ell_t(w)$ and $v \not\leq_t u$, for otherwise we would have $v \leq_t w$.

Thanks to proposition 10.3.1, we can write

$$\delta_{w^{-1}}^t = \prod_{\substack{\alpha^\vee \in \Phi_{t,+}^\vee \\ w(\alpha^\vee) \in -\Phi_{t,+}^\vee}} (1 - e^{\alpha^\vee}) y_{w^{-1}}^t + \sum_{u <_t w^{-1}} q_u \delta_u^t$$

for some $q_u \in \widehat{Q}_t$. Apply ι to this equality to get an equality

$$\delta_w^t = \prod_{\substack{\alpha^\vee \in \Phi_{t,+}^\vee \\ w(\alpha^\vee) \in -\Phi_{t,+}^\vee}} (1 - e^{\alpha^\vee}) \bar{y}_w^t + \sum_{u <_t w} q'_u \delta_u^t$$

(for some $q'_u \in \widehat{Q}_t$). Apply ψ_v^t to this new equality to get $\psi_v^t(\delta_w^t) = 0$; then apply ψ_w^t to finally get *(2)*.

One can prove *(3)* exactly as in [KK, proposition 2.22, (d)]. □

Point *(1)* in proposition 10.3.2 implies in particular that under the identification $\Omega_{W_t^\circ} = \mathrm{Fun}(W_t^\circ, \widehat{Q}_t)$, the subset $\Psi_{W_t^\circ} \subseteq \Omega_{W_t^\circ}$ is contained in the subset $\mathrm{Fun}(W_t^\circ, \widehat{R}_t)$ of functions whose image lies in $\widehat{R}_t$.

The following lemma is proved in [BeR, Lemma 8.3]; the proof of *loc. cit.* can be copied verbatim.

Lemma 10.3.3. *Let $f \in \widehat{R}_t$ and $\alpha^\vee, \beta^\vee$ two distinct positive coroots in $\Phi_t^\vee$. If $(1 - e^{\alpha^\vee}) \cdot f$ is divisible by $(1 - e^{\beta^\vee})$ in $\widehat{R}_t$, then so is f.*

Theorem 10.3.4. *The map*

$$\tau_t : \widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \longrightarrow \mathrm{Fun}(W_t^\circ, \widehat{R}_t)$$

sending $a \otimes b$ to $w \mapsto a \cdot w^{-1}(b)$ is injective. Its image consists of the functions f such that

$$f(w) \equiv f(w s_{\alpha^\vee}) \bmod (1 - e^{\alpha^\vee})$$

for any $w \in W_t^\circ$ and any coroot $\alpha^\vee$ in $\Phi_t^\vee$.

We will need an intermediary result in the proof of the theorem:

Lemma 10.3.5. *Consider $f \in \mathrm{Fun}(W_t^\circ, \widehat{R}_t)$ such that $f(w) \equiv f(w s_{\alpha^\vee}) \bmod (1 - e^{\alpha^\vee})$ for any coroot $\alpha^\vee$ in $\Phi_t^\vee$. Then f is a linear combination of the elements $\{\psi_w^t\}_{w \in W_t^\circ}$ with coefficients in $\widehat{R}_t$.*

Proof. Choose a total order $\preceq$ on W_t° that extends the Bruhat order. We use an induction argument on the smallest $w \in W_t^\circ$ satisfying $f(w) \neq 0$. If $\alpha^\vee \in \Phi_{t,+}^\vee$ is such that $w(\alpha^\vee) \in -\Phi_{t,+}^\vee$, then we have $ws_\alpha \prec w$ (see [H3, §5.7 Proposition]) so that $f(ws_\alpha) = 0$. Now considering our choice of f, this implies that $(1-e^{\alpha^\vee})$ divides $f(w)$. Using lemma 10.3.3, we see that there exists an element $a \in \widehat{R}_t$ such that

$$f(w) = a\Big(\prod_{\substack{\alpha^\vee \in \Phi_{t,+}^\vee \\ w(\alpha^\vee) \in -\Phi_{t,+}^\vee}} (1 - e^{\alpha^\vee})\Big) = a\psi_w^t(\delta_w^t),$$

where the second equality follows from proposition 10.3.2 *(2)*. The function $f - a\psi_w^t$ still satisfies the condition of our theorem, and vanishes on w and on all the elements $v \prec w$: this follows from proposition 10.3.2 *(1)*. By induction hypothesis, we get that $f - a\psi_w^t$ is a $\widehat{R}_t$-linear combination of ψ_v^t with $v \in W_t^\circ$. The proof is now complete. □

Proof of theorem 10.3.4. We consider the basis $\{c_w^t\}_{w \in W_t^\circ}$ of $\widehat{R}_t$ (as a $(\widehat{R}_t)^{W_t^\circ}$-module) of proposition 10.2.3. The family $\{1 \otimes c_w^t\}_{w \in W_t^\circ}$ is then a basis of the $\widehat{R}_t$-module $\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t$. The function $\tau(1 \otimes c_w^t)$ is the function defined by $v \mapsto v^{-1}(c_w^t)$. Thanks to proposition 10.2.3, we have $\det((v^{-1}(c_w^t))_{v,w}) \neq 0$ so that the functions $\{\tau(1 \otimes c_w^t)\}_w$ are linearly independent over $\widehat{R}_t$; hence τ is injective.

Let us show that $\psi_{w_{t,\circ}}^t$ lies in the image of τ. This amounts to showing that there exists a family $\{p_w^t\}_{w \in W_t^\circ}$ in $\widehat{R}_t$ such that $\tau(\sum_w p_w^t \otimes c_w^t) = \psi_{w_{t,\circ}}$; or in other terms, such that

$$\sum_w p_w^t v(c_w^t) = \begin{cases} \prod_{\alpha^\vee \in \Phi_{t,+}^\vee} (1 - e^{\alpha^\vee}) & \text{if } v = w_{t,\circ}, \\ 0 & \text{otherwise.} \end{cases}$$

We can reformulate the problem: we have to solve the system

$$A \cdot p^t = q^t$$

where A is the matrix $(v(c_w^t))_{w,v \in W_t^\circ}$, q^t is the vector indexed by W_t° with $q_w^t = 0$ for $w \neq w_{t,\circ}$ and $q_{w_{t,\circ}}^t = \prod(1 - e^{\alpha^\vee})$. Since the matrix A is invertible in $\widehat{Q}_t$ (its determinant being nonzero, as stated above), there exist such a family $\{p_w^t\}$, unique, in $\widehat{Q}_t$. The inverse of A is given by $\frac{1}{\det(A)}\mathrm{C}(A)^{\mathrm{tr}}$ where $\mathrm{C}(A)^{\mathrm{tr}}$ is the transpose of the cofactor matrix of A. Thus, if we show that $\det(A)$ divides each entry of the vector $\mathrm{C}(A)^{\mathrm{tr}}q^t$ in $\widehat{R}_t$, we will get that p^t has its entries in $\widehat{R}_t$. Let $m_A(w,v)$ be the determinant of the matrix obtained from A by removing the row w and the column v. Using remark 10.2.4, it suffices to see that $\prod_{\alpha^\vee \in \Phi_{t,+}^\vee} (1 - e^{\alpha^\vee})^{|W|/2-1}$ divides $m_A(w,v)$ in $\widehat{R}_t$. Now, fix a positive coroot $\alpha^\vee$; let s be the associated reflection in W_t°. We can write $W \smallsetminus \{sv, v\}$ as the disjoint union of $|W|/2-1$ s-orbits $\{sw, w\}$ (for $w \in W$). For any $x, y \in W_t^\circ$, the element $sx(e_y^t) - x(e_y^t)$ is divisible in $\widehat{R}_t$ by $(1-e^{\alpha^\vee})$; subtracting the column

x to the column sx for any s-orbit in $W \smallsetminus \{sv, v\}$ we get that $(1-e^{\alpha^\vee})^{|W|/2-1}$ divides $m_A(w,v)$ for any positive coroot $\alpha^\vee$. Finally, using lemma 10.3.3, we deduce the wished-for divisibility.

We note the following: for any $a, b \in \widehat{R}_t$ and any $s \in S_t$, we have

$$y_s \cdot \tau_t(a \otimes b) = \tau_t\left(a \otimes \frac{b - e^{-\alpha_s^\vee} s(b)}{1 - e^{-\alpha_s^\vee}}\right).$$

Thus the image of τ_t is stable under the action of the y_s (remark that the quotient on the right is an element of $\widehat{R}_t$). Since $\psi^t_{w_{t,\circ}}$ belongs to this image, so do the various $y_s \cdot \psi^t_{w_{t,\circ}}$ for $s \in S_t$. Thanks to proposition 10.3.2 *(3)*, this implies that $\psi^t_w \in \mathrm{im}(\tau_t)$ for any $w \in W_t^\circ$. Thus we have $\Psi_{W_t^\circ} \subseteq \mathrm{im}(\tau_t)$.

We finally prove the asserted fact about the image. Using lemma 10.3.5, is f satisfies the condition of the theorem, then it belongs to the image of τ_t. Reciprocally, as $b - s_{\alpha^\vee}(b)$ (for $\alpha^\vee$ a coroot in $\Phi_t^\vee$) is divisible by $1 - e^{\alpha^\vee}$ in $\widehat{R}_t$, we get that any f in the image of τ_t satisfies the condition of the theorem; this concludes the proof. □

10.4 Completed Endomorphismensatz

Theorem 10.4.1. *Consider t in $T^\vee_{\Bbbk}$. The morphism of lemma 9.6.5 is an algebra isomorphism for $\widehat{\mathscr{T}}_{w_{t,\circ},t}$:*

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \xrightarrow{\sim} \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}).$$

Before discussing the proof of the theorem, we state a similar description for the endomorphisms of pro-tilting objects associated to maximal elements in all blocks, as well as endomorphisms of maximal tilting objects in the Lusztig–Yun category:

Corollary 10.4.2. *Consider $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit and $\beta \in {}_{t'}\underline{W}_t$. The morphism of lemma 9.6.5 is an algebra isomorphism for $\widehat{\mathscr{T}}_{w_\beta^{\max},t}$:*

$$\widehat{R}_{t'} \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \xrightarrow{\sim} \mathrm{End}(\widehat{\mathscr{T}}_{w_\beta^{\max},t}).$$

The proof of corollary 10.4.2 will directly follow from the proof of theorem 10.4.1, particularly from diagram (10.4.4) and lemma 10.4.5 below.

Corollary 10.4.3. *Consider $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit and $\beta \in {}_{t'}\underline{W}_t$. The left monodromy morphism $\varphi_{\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T},t}$ is an algebra isomorphism*

$$\widehat{R}_{t'} \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \otimes_{\widehat{R}_t} \Bbbk_t \xrightarrow{\sim} \mathrm{End}(\mathscr{T}(w_\beta^{\max})_{\mathscr{L}_t^T}).$$

Proof. This is a direct consequence of corollary 10.4.2 and point 2. in lemma 9.4.2. □

Let us now establish the proof of the theorem. According to lemma 9.3.3, we have an isomorphism

$$\mathrm{gr}(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \cong \bigoplus_{w\in W_t^\circ} \widehat{\Delta}_{w,t}\,.$$

In view of lemma 7.8.1 and lemma 9.6.3, we obtain

$$\mathrm{End}(\mathrm{gr}(\widehat{\mathscr{T}}_{w_{t,\circ},t})) \cong \bigoplus_{w\in W_t^\circ} \widehat{R}_t. \tag{10.4.1}$$

We will consider the following composition

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \xrightarrow{\sim} \widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \to \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \to \bigoplus_{w\in W_t^\circ} \widehat{R}_t. \tag{10.4.2}$$

Here the first map is given by $x\otimes y \mapsto y\otimes x$, the second one is the morphism of theorem 10.4.1, and the third map is induced by the functor gr and (10.4.1). Under the natural indentification $\bigoplus_{w\in W_t^\circ} \widehat{R}_t \cong \mathrm{Fun}(W_t^\circ, \widehat{R}_t)$, the map (10.4.2) is the morphism of theorem 10.3.4: this composition maps $a\otimes b$ to the vector $(\varphi_{l,\widehat{\Delta}_{w,t}}(b)\circ\varphi_{r,\widehat{\Delta}_{w,t}}(a))_{w\in W_t^\circ} \in \bigoplus_{w\in W_t^\circ} \mathrm{End}(\widehat{\Delta}_{w,t})$. Now, lemma 5.2.1 tells us that the endomorphism $\varphi_{l,\widehat{\Delta}_{w,t}}(b)\circ\varphi_{r,\widehat{\Delta}_{w,t}}(a)$ of $\widehat{\Delta}_{w,t}$ corresponds to the element $aw^{-1}(b)$ under the isomorphism of lemma 7.8.1 (induced by right monodromy). Thus we indeed find the morphism of theorem 10.3.4, which is injective. In particular the morphism of theorem 10.4.1 is also injective. Note that according to lemma 9.6.4, the functor gr is faithful, so that the third map is also an injection. Moreover, the above discussion implies that the image of this third morphism in (10.4.2) contains the subset of vectors $(a_w)_{w\in W_t^\circ}$ such that

$$a_{ws_{\alpha^\vee}} \equiv a_w \mod(1-e^{\alpha^\vee}) \tag{10.4.3}$$

for $\alpha^\vee$ any coroot in $\Phi_t^\vee$.

We will prove the converse: if $(a_w)_{w\in W_t^\circ}$ belongs to the image of the third map in (10.4.2), then (10.4.3) holds for any coroot $\alpha^\vee \in \Phi_t^\vee$.

We first need a few preliminary results. Assume that $s := s_{\alpha^\vee}$ is simple in W, that is to say, s is a reflection of W_t° associated to a coroot $\alpha^\vee$ simple in $\Phi^\vee$. Let $\bar{\jmath}_s : \mathscr{X}_e \sqcup \mathscr{X}_s \hookrightarrow \mathscr{X}$ be the inclusion map; this is a closed embedding.

Lemma 10.4.4. *We have $\bar{\jmath}_s^*(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \cong \widehat{\mathscr{T}}_{s,t}$; for $w\in W_t^\circ$, the morphism*

$$\mathrm{gr}_w(\widehat{\mathscr{T}}_{w_{t,\circ},t}) \to \mathrm{gr}_w(\widehat{\mathscr{T}}_{s,t})$$

induced by adjunction is an isomorphism if $w = e$ or s and is zero otherwise.

Proof. To prove the first point, we use lemma 7.5.1 and proposition 9.5.2: it suffices to show that $\bar{\jmath}_s^*(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T})) \cong \mathscr{T}(s)_{\mathscr{L}_t^T}$. Now $\bar{\jmath}_s^*(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T}))$ has a Δ-filtration, and so is perverse; moreover the standard objects in this filtration are

$\Delta(e)_{\mathscr{L}_t^T}$ and $\Delta(s)_{\mathscr{L}_t^T}$, each with multiplicity 1 (this follows from lemma 9.3.3). Thus $\bar{\jmath}_s^*(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T}))$ is an extension in $\mathrm{Ext}^1_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\Delta(e)_{\mathscr{L}_t^T}, \Delta(s)_{\mathscr{L}_t^T})$. Now, as explained in subsection 9.8.1, it follows from lemma A.0.6 that this Ext space is 1-dimensional, with only nontrivial object given by $\mathscr{T}(s)_{\mathscr{L}_t^T}$. The object $\bar{\jmath}_s^*(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T}))$ is then either isomorphic to $\mathscr{T}(s)_{\mathscr{L}_t^T}$ or to the direct sum $\Delta(e)_{\mathscr{L}_t^T} \oplus \Delta(s)_{\mathscr{L}_t^T}$. The latter is impossible however, since by adjunction, we have

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\bar{\jmath}_s^*(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T})), \mathrm{IC}(s)_{\mathscr{L}_t^T}) \\ = \mathrm{Hom}_{\mathfrak{P}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]}}(\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^T}, \mathrm{IC}(s)_{\mathscr{L}_t^T}) = 0\end{aligned}$$

according to proposition 9.3.4.

The fact about the graded functor is an easy consequence of the above reasoning. □

Consider two elements $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit. Fix a block $\beta \in {}_{t'}\underline{W}_t$. For any v, w in β we define four morphisms $\kappa^i_\beta(v,w)$ $(i = 1,2,3,4)$ making the following a commutative diagram, where the rows are given by (10.4.2):

$$\begin{array}{ccccccc}
\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t & \longrightarrow & \widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t & \xrightarrow{f_t} & \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}}) & \xrightarrow{g_t} & \bigoplus_{x\in W_t^\circ} \widehat{R}_t \\
\wr\downarrow \kappa^1_\beta(v,w) & & \wr\downarrow \kappa^2_\beta(v,w) & & \wr\downarrow \kappa^3_\beta(v,w) & & \wr\downarrow \kappa^4_\beta(v,w) \\
\widehat{R}_{t'} \otimes_{(\widehat{R}_{t'})^{W_{t'}^\circ}} \widehat{R}_{t'} & \longrightarrow & \widehat{R}_{t'} \otimes_{(\widehat{R}_{t'})^{W_{t'}^\circ}} \widehat{R}_{t'} & \xrightarrow{f_{t'}} & \mathrm{End}(\widehat{\mathscr{T}}_{w_{t',\circ}}) & \xrightarrow{g_{t'}} & \bigoplus_{y\in W_{t'}^\circ} \widehat{R}_{t'}.
\end{array} \tag{10.4.4}$$

The definitions are easy for $i = 1, 2, 4$:

$$\kappa^1_\beta(v,w)(a\otimes b) = v(a)\otimes w(b) \qquad \kappa^2_\beta(v,w)(a\otimes b) = w(a)\otimes v(b)$$

$$\kappa^4_\beta(v,w)((c_x)_{x\in W_t^\circ}) = (v(c_{w^{-1}uv}))_{u\in W_{t'}^\circ}.$$

Let us define $\kappa^3_\beta(v,w)$. According to proposition 9.7.6, we have an isomorphism

$$\widehat{\Delta}_{w,t} \widehat{\star} \widehat{\mathscr{T}}_{w_{t,\circ}} \widehat{\star} \widehat{\Delta}_{v^{-1},t'} \cong \widehat{\mathscr{T}}_{w_{t',\circ}}. \tag{10.4.5}$$

We then set

$$\kappa^3_\beta(v,w) : \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}}) \xrightarrow{\sim} \mathrm{End}(\widehat{\Delta}_{w,t} \widehat{\star} \widehat{\mathscr{T}}_{w_{t,\circ}} \widehat{\star} \widehat{\Delta}_{v^{-1},t'}) \xrightarrow{\sim} \mathrm{End}(\widehat{\mathscr{T}}_{w_{t',\circ}}).$$

Here the first arrow is induced by the functor $\widehat{\Delta}_{w,t} \widehat{\star}(-) \widehat{\star} \widehat{\Delta}_{v^{-1},t'}$ and the second one by a choice of an isomorphism as in (10.4.5). If $\beta = W_t^\circ$ then we set $\kappa^i_\beta = \kappa^i_{t,\circ}$.

Lemma 10.4.5. *The isomorphism $\kappa^3_\beta(v,w)$ does not depend on the choice made for the isomorphism as in* (10.4.5)*. We have*

$$\kappa^3_\beta(v,w) \circ f_t = f_{t'} \circ \kappa^2_\beta(v,w), \qquad \kappa^4_\beta(v,w) \circ g_t = g_{t'} \circ \kappa^3_\beta(v,w).$$

Proof. Let us fix a choice of $\kappa^3_\beta(v,w)$ (that is, choose and fix an isomorphism as in (10.4.5)). It is easy to see that we have

$$\kappa^4_\beta(v,w)\circ g_t\circ f_t = g_{t'}\circ f_{t'}\circ\kappa^2_\beta(v,w).$$

We show that $\kappa^3_\beta(v,w)\circ f_t = f_{t'}\circ\kappa^2_\beta(v,w)$. The map f_t (resp. $f_{t'}$) maps $a\otimes b\in \widehat{R}_t\otimes_{(\widehat{R}_t)^{W_t^\circ}}\widehat{R}_t$ (resp. $a'\otimes b'\in \widehat{R}_{t'}\otimes_{(\widehat{R}_t)^{W_{t'}^\circ}}\widehat{R}_{t'}$) to $\varphi_{l,\widehat{\mathscr{T}}_{w_{t,\circ}}}(a)\circ\varphi_{r,\widehat{\mathscr{T}}_{w_{t,\circ}}}(b)$ (resp. $\varphi_{l,\widehat{\mathscr{T}}_{w_{t',\circ}}}(a)\circ\varphi_{r,\widehat{\mathscr{T}}_{w_{t',\circ}}}(b)$). We then have

$$\begin{aligned}
\mathrm{id}_{\widehat{\Delta}_{w,t}}\,\widehat{\star}\,(\varphi_{l,\widehat{\mathscr{T}}_{w_{t,\circ}}}(a)\circ\varphi_{r,\widehat{\mathscr{T}}_{w_{t,\circ}}}(b))\widehat{\star}\,\mathrm{id}_{\widehat{\Delta}_{v^{-1},t'}}&\\
&= \varphi_{r,\widehat{\Delta}_{w,t}}(a)\,\widehat{\star}\,\mathrm{id}_{\widehat{\mathscr{T}}_{w_{t,\circ}}}\,\widehat{\star}\,\varphi_{l,\widehat{\Delta}_{v^{-1},t'}}(b)\\
&= \varphi_{l,\widehat{\Delta}_{w,t}}(w(a))\,\widehat{\star}\,\mathrm{id}_{\widehat{\mathscr{T}}_{w_{t,\circ}}}\,\widehat{\star}\,\varphi_{r,\widehat{\Delta}_{v^{-1},t'}}(v(b))\\
&= \varphi_{l,\widehat{\Delta}_{w,t}\,\widehat{\star}\,\widehat{\mathscr{T}}_{w_{t,\circ}}\,\widehat{\star}\,\widehat{\Delta}_{v^{-1},t'}}(w(a))\circ\varphi_{r,\widehat{\Delta}_{w,t}\,\widehat{\star}\,\widehat{\mathscr{T}}_{w_{t,\circ}}\,\widehat{\star}\,\widehat{\Delta}_{v^{-1},t'}}(v(b)).
\end{aligned}$$

Here the equalities are respectively justified by lemma 6.2.1 (1), lemma 5.2.1 and lemma 6.2.1 (3). Since monodromy commutes with any morphism (see lemma 1.5.1), conjugation of the above morphism with any choice of isomorphism

$$\widehat{\Delta}_{w,t}\,\widehat{\star}\,\widehat{\mathscr{T}}_{w_{t,\circ}}\,\widehat{\star}\,\widehat{\Delta}_{v^{-1},t'}\cong\widehat{\mathscr{T}}_{w_{t',\circ}}$$

yields $\varphi_{l,\widehat{\mathscr{T}}_{w_{t',\circ}}}(w(a))\circ\varphi_{r,\widehat{\mathscr{T}}_{w_{t',\circ}}}(v(b)) = f_{t'}(\kappa^2_\beta(v,w)(a\otimes b))$.

Now to prove that $\kappa^4_\beta(v,w)\circ g_t = g_{t'}\circ\kappa^3_\beta(v,w)$, we remark that all the spaces involved in the top row of (10.4.4) are free of rank $|W_t^\circ| = |W_{t'}^\circ|$ over $\widehat{R}_t$, and all the spaces on the bottom row are free of rank $|W_t^\circ| = |W_{t'}^\circ|$ over $\widehat{R}_{t'}$. Moreover, we can make $\widehat{R}_{t'}$ into a $\widehat{R}_t$-module, as in lemma 9.6.5, via the map

$$\widehat{R}_t\xrightarrow{w_\beta^{\min}(-)}\widehat{R}_{t'};$$

this is an algebra isomorphism and thus $\widehat{R}_{t'}$ is free of rank one as a $\widehat{R}_t$-module. We apply the functor $\widehat{Q}_t\otimes_{\widehat{R}_t}(-)$ to diagram (10.4.4). One obtains a diagram where all the spaces are $\widehat{Q}_t$-vector spaces of dimension $|W_t^\circ|$. Since all the morphisms on the top row of diagram (10.4.4) are injective and $\widehat{Q}_t$ is flat over $\widehat{R}_t$, we obtain injective morphisms between $\widehat{Q}_t$-vector spaces of the same dimension, thus isomorphisms. Similarly, the maps in the bottom row also become isomorphisms after tensoring with $\widehat{Q}_t$ over $\widehat{R}_t$. For any map ψ appearing in this diagram, we let $\psi^{\widehat{Q}_t} := \mathrm{id}_{\widehat{Q}_t}\otimes_{\widehat{R}_t}\psi$. Now thanks to the beginning of this proof, we have

$$\kappa^4_\beta(v,w)^{\widehat{Q}_t}\circ g_t^{\widehat{Q}_t}\circ f_t^{\widehat{Q}_t} = g_{t'}^{\widehat{Q}_t}\circ f_{t'}^{\widehat{Q}_t}\circ\kappa^2_\beta(v,w)^{\widehat{Q}_t}$$

and

$$\kappa^3_\beta(v,w)^{\widehat{Q}_t}\circ f_t^{\widehat{Q}_t} = f_{t'}^{\widehat{Q}_t}\circ\kappa^2_\beta(v,w)^{\widehat{Q}_t}.$$

This readily implies that

$$\kappa^4_\beta(v,w)^{\widehat{Q}_t} \circ g_t^{\widehat{Q}_t} = g_{t'}^{\widehat{Q}_t} \circ \kappa^3_\beta(v,w)^{\widehat{Q}_t}.$$

Using the natural injections

$$\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}}) \cong \widehat{R}_t \otimes_{\widehat{R}_t} \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}}) \hookrightarrow \widehat{Q}_t \otimes_{\widehat{R}_t} \mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}})$$

and

$$\bigoplus_{x\in W_t^\circ} \widehat{R}_t \cong \widehat{R}_t \otimes_{\widehat{R}_t} (\bigoplus_{x\in W_t^\circ} \widehat{R}_t) \hookrightarrow \widehat{Q}_t \otimes_{\widehat{R}_t} (\bigoplus_{x\in W_t^\circ} \widehat{R}_t)$$

(and similarly for t') we get $\kappa^4_\beta(v,w) \circ g_t = g_{t'} \circ \kappa^3_\beta(v,w)$. Finally, as $g_{t'}$ is injective (and as $\kappa^4_\beta(v,w)$ does not require any choice), we see that $\kappa^3_\beta(v,w)$ does not depend on the chosen isomorphism as in (10.4.5). □

Lemma 10.4.6. *Assume that $t,t',t'' \in T^\vee_{\Bbbk}$ are three elements in the same W-orbit and fix two blocks $\beta \in {}_{t'}\underline{W}_t$ and $\gamma \in {}_{t''}\underline{W}_{t'}$. Take $v,w \in \beta$ and $v',w' \in \gamma$. For $i = 1,2,3,4$, we have*

$$\kappa^i_\gamma(v',w') \circ \kappa^i_\beta(v,w) = \kappa^i_{\gamma\beta}(v'v, w'w).$$

In particular, the family $(\kappa^i_{t,\circ}(v,w))_{v,w\in W_t^\circ}$ defines an action of $W_t^\circ \times W_t^\circ$ on the corresponding space in (10.4.4).

Proof. The cases $i = 1,2,4$ are immediate. For $i = 3$, we use lemma 10.4.5 and the case $i = 4$: we have

$$\begin{aligned} g_{t''} \circ \kappa^3_\beta(v',w') \circ \kappa^3_\beta(v,w) &= \kappa^4_\beta(v',w') \circ \kappa^4_\beta(v,w) \circ g_t \\ &= \kappa^4_\beta(v'v, w'w) \circ g_t \\ &= g_{t''} \circ \kappa^3_\beta(v'v, w'w). \end{aligned}$$

Here g_t is the morphism appearing in diagram (10.4.4) and and $g_{t''}$ is the analogue for t''. Since $g_{t''}$ is an injective map, we get $\kappa^3_\beta(v',w') \circ \kappa^3_\beta(v,w) = \kappa^3_\beta(v'v, w'w)$. □

We can now give a proof of a partial version of the announced result, namely:

Lemma 10.4.7. *Consider any $t \in T^\vee_{\Bbbk}$. If $(a_w)_{w\in W_t^\circ}$ belongs to the image of the third morphism in* (10.4.2)*, then* (10.4.3) *holds for $w = e$ and any coroot $\alpha^\vee \in \Phi_t^\vee$ simple in $\Phi_t^\vee$.*

Proof. In this proof, we will use freely the notation of diagram (10.4.4). Once again, the first step is to consider the case $s \in W_t^\circ$ simple in W, associated to a simple coroot $\alpha^\vee$. The proof is exactly the one of [BeR, Proposition 9.3] (exposed in §9.2 in loc.cit.). Lemma 10.4.4 implies that the composition

$$\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}}) \to \bigoplus_{w\in W_t^\circ} \widehat{R}_t \xrightarrow{(a_w)_w \mapsto (a_e, a_s)} \widehat{R}_t \oplus \widehat{R}_t$$

factors as

$$\mathrm{End}(\widehat{\mathscr{T}}_{w_{t,\circ}}) \xrightarrow{j_s^*} \mathrm{End}(\widehat{\mathscr{T}}_{s,t}) \xrightarrow{\mathrm{gr}} \widehat{R}_t \oplus \widehat{R}_t. \tag{10.4.6}$$

Thanks to proposition 9.8.2, the map

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t \to \mathrm{End}(\widehat{\mathscr{T}}_{s,t})$$

induced by monodromy is surjective; moreover its composition with the last map in (10.4.6) is given by

$$a \otimes b \mapsto (ab,\, s(a)b).$$

Now, we have $ab \equiv s(a)b \mod (1 - e^{\alpha^\vee})$, thus we can conclude that

$$c_e \equiv c_s \mod (1 - e^{\alpha^\vee})$$

for any $(c_w)_{w \in W_t^\circ}$ in the image of the map (10.4.2).

Assume now that s is simple in W_t° but not in W. Thanks to lemma 5.1.5, we can write $s = w_\beta^{\min} \cdot r \cdot w_\beta^{\min,-1}$ with r a simple reflection in $W_{t'}^\circ$ (t' an element in the same W-orbit of t) and β a block in ${}_{t'}\underline{W}_t$. Let $\alpha^\vee$ be the coroot associated to s; the reflection r is then associated to the coroot $w_\beta^{\min,-1}(\alpha^\vee)$. We claim that the latter coroot is simple (and not the opposite of a simple coroot, as it could a priori be): the coroot $\alpha^\vee$ lies in $\Phi_{t,+}^\vee = \Phi_+^\vee \cap \Phi_t^\vee$ and $w_\beta^{\min,-1}$ is minimal in its block; according to [LY, Lemma 4.2], this element maps $\Phi_{t,+}^\vee$ into $\Phi_+^\vee$ and so maps $\alpha^\vee$ to a simple coroot. Note that conjugation with $w_\beta^{\min}$ gives a bijection $W_{t'}^\circ \to W_t^\circ$. We let $\bar{\beta}$ be the block of ${}_t\underline{W}_{t'}$ whose elements are given by the inverses of the elements in β. Its minimal element is then $w_\beta^{\min,-1}$.

The first part of the proof tells us that if a $|W_{t'}^\circ|$-tuple $(c'_w)_{w \in W_{t'}^\circ}$ belongs to the image of the morphism $g_{t'}$, then we have

$$c'_e \equiv c'_r \mod (1 - e^{w_\beta^{\min,-1}(\alpha^\vee)}).$$

Now any $(c_v)_{v \in W_t^\circ}$ in the image of g_t is of the form

$$\kappa_{\bar{\beta}}^4(w_\beta^{\min,-1}, w_\beta^{\min,-1})((c'_w)_{w \in W_{t'}^\circ})$$

for some $(c'_w)_{w \in W_{t'}^\circ}$. In particular, we have

$$c_e = w_\beta^{\min}(c'_e) \quad \text{and} \quad c_s = w_\beta^{\min}(c'_r).$$

Since $c'_e \equiv c'_r \mod (1 - e^{w_\beta^{\min,-1}(\alpha^\vee)})$, one deduces that

$$c_e \equiv c_s \mod (w_\beta^{\min}(1 - e^{w_\beta^{\min,-1}(\alpha^\vee)})),$$

i.e. that we have

$$c_e \equiv c_s \mod (1 - e^{\alpha^\vee}).$$

The lemma is proved. □

Proposition 10.4.8. *If $(c_w)_{w\in W_t^\circ}$ belongs to the image of the third map in* (10.4.2)*, then* (10.4.3) *holds for any $w \in W_t^\circ$ and any coroot $\alpha^\vee \in \Phi_t^\vee$.*

Proof. Once again, we follow [BeR, §9.3]: we use the morphisms $\kappa^i_{t,\circ}(v,w)$ for $v, w \in W_t^\circ$. According to lemma 10.4.6, these maps define actions of $W_t^\circ \times W_t^\circ$ on the various spaces in (10.4.2), and all the arrows involved in this diagram are equivariant with respect to these actions. In particular, the image of the third morphism (which is denoted by g_t in (10.4.4)) is stable under the action of $W_t^\circ \times W_t^\circ$, that is to say under the various $\kappa^4_{t,\circ}(v,w)$ with $v, w \in W_t^\circ$.

Now assume that $\alpha^\vee$ is a simple coroot in $\Phi_t^\vee$. For any $w \in W_t^\circ$ and any $(c_w)_{w\in W_t^\circ} \in \mathrm{im}(g_t)$, the $|W_t^\circ|$-tuple $\kappa^4_{t,\circ}(e, w^{-1})((c_w)_w)$ still belongs to this image. From lemma 10.4.7, we deduce that

$$\big(\kappa^4_{t,\circ}(e,w^{-1})((c_w)_w)\big)_e \equiv \big(\kappa^4_{t,\circ}(e,w^{-1})((c_w)_w)\big)_{s_{\alpha^\vee}} \mod (1-e^{\alpha^\vee});$$

this can be rewritten as

$$c_w \equiv c_{ws_{\alpha^\vee}} \mod (1-e^{\alpha^\vee})$$

and deals with the case $\alpha^\vee$ simple.

If $\alpha^\vee$ is not simple, choose $v \in W_t^\circ$ such that $\sigma^\vee := v(\alpha^\vee)$ is a simple coroot in $\Phi_t^\vee$. If we prove that

$$v(c_w) \equiv v(c_{ws_{\alpha^\vee}}) \mod (1-e^{\sigma^\vee}),$$

we will be done (it then suffices to apply v^{-1} to this equality). We have $ws_{\alpha^\vee} = wv^{-1}s_{\sigma^\vee}v$ and $\kappa^4_{t,\circ}(v,e)((c_w)_w)$ belongs to the image of g_t; from the simple coroot case treated above, we have

$$c_{wv^{-1}} \equiv c_{wv^{-1}s_{\alpha^\vee}} \mod (1-e^{\sigma^\vee})$$

and thus

$$\big(\kappa^4_{t,\circ}(e,w^{-1})((c_w)_w)\big)_{wv^{-1}} \equiv \big(\kappa^4_{t,\circ}(e,w^{-1})((c_w)_w)\big)_{wv^{-1}s_{\alpha^\vee}} \mod (1-e^{\sigma^\vee});$$

which is the same as

$$c_w \equiv c_{ws_{\alpha^\vee}} \mod (1-e^{\sigma^\vee}).$$

The proposition is proved. □

Proof of theorem 10.4.1. We know from the discussion below the statement of the theorem that the considered morphism is injective, as is the third arrow in (10.4.2). From proposition 10.4.8, we also know that the image of this third map coincides with the image of the whole diagram (10.4.2), which is the morphism of theorem 10.3.4. This implies that the composition of the first two maps in (10.4.2) is surjective, which in turn implies that the second map (i.e. the map of the theorem) is surjective. Being both surjective and injective, this map is an isomorphism and the theorem is proved. □

Assume that $s \in W$ is a simple reflection of W_t°. Copying [BeR, Lemma 11.4], we derive the following description of $\widehat{\mathbb{V}}_t^\circ(\widehat{\mathscr{T}}_{s,t})$:

Lemma 10.4.9. *We have an isomorphism*

$$\widehat{\mathbb{V}}_t^\circ(\widehat{\mathscr{T}}_{s,t}) \cong \widehat{R}_t \otimes_{(\widehat{R}_t)^s} \widehat{R}_t$$

where $(\widehat{R}_t)^s$ denotes the elements of $\widehat{R}_t$ invariant under the action of $s \in W_t^\circ$.

Proof. We only sketch the proof. If s is simple in W, using lemma 10.4.4 and adjunction, we get $\widehat{\mathbb{V}}_t^\circ(\widehat{\mathscr{T}}_{s,t}) \cong \mathrm{End}(\widehat{\mathscr{T}}_{s,t})$. We can then conclude using theorem 10.4.1 applied to the Levi subgroup L_s associated to the simple reflection s.

If s is simple in W_t° but not in W, thanks to lemma 5.1.5, we can write $s = w_\beta^{\min} r w_\beta^{\min,-1}$, with β a block in ${}_t\underline{W}_{t'}$ and $r \in W_{t'}^\circ$ a reflection simple in W. Then we use lemma 9.7.1 to see that

$$\widehat{\mathscr{T}}_{s,t} \cong \widehat{\Delta}_{w_\beta^{\min},t'} \widehat{\star} \widehat{\mathscr{T}}_{r,t'} \widehat{\star} \widehat{\Delta}_{w_\beta^{\min,-1},t}$$

for β some block in ${}_{t'}\underline{W}_{t'}$ and r a reflection simple in W. Thanks to proposition 9.7.6, we have a commutative diagram (see the proof of lemma 10.4.5)

$$\begin{array}{ccc} \widehat{\mathbb{V}}_{t'}^\circ(\widehat{\mathscr{T}}_{r,t'}) & \xleftarrow{\varphi_{\widehat{\mathscr{T}}_{r,t}}} & \widehat{R}_{t'} \otimes \widehat{R}_{t'} \\ \Big\downarrow \wr & & \wr \Big\downarrow \\ \widehat{\mathbb{V}}_t^\circ(\widehat{\mathscr{T}}_{s,t}) & \xleftarrow{\varphi_{\widehat{\mathscr{T}}_{s,t}}} & \widehat{R}_t \otimes \widehat{R}_t. \end{array}$$

(left vertical: $\widehat{\Delta}_{w_\beta^{\min},t'} \widehat{\star}(-) \widehat{\star} \widehat{\Delta}_{w_\beta^{\min,-1},t}$; right vertical: $w_\beta^{\min}(-) \otimes w_\beta^{\min}(-)$)

Moreover, the right vertical arrow clearly induces an isomorphism

$$\widehat{R}_{t'} \otimes_{(\widehat{R}_{t'})^r} \widehat{R}_{t'} \xrightarrow{\sim} \widehat{R}_t \otimes_{(\widehat{R}_t)^s} \widehat{R}_t.$$

The lemma follows. □

Chapter 11

Comparison results

11.1 Rough strategy

Our ultimate goal in this thesis is to understand (say, for now, the neutral block in the category of) monodromic tilting (pro-)perverse objects, with an arbitrary fixed monodromy. For a fixed t, we already constructed functors $\mathbb{V}_t^\circ$ and $\widehat{\mathbb{V}}_t^\circ$ in section 9.10, and showed that they are fully faithful on tilting objects (see proposition 9.10.2). We now have to understand the essential image of these two functors. Corollary 9.9.2 and lemma 10.4.9 imply that we will be able to determine this essential image if we know that our functor $\widehat{\mathbb{V}}_t^\circ$ admits a monoidal structure. This chapter is a first step in this direction. Below, we present the strategy that we will follow in order to obtain the wished-for monoidality. Our proof will be based on "Whittaker-type" constructions, a feature that is only available in the different setting of schemes over fields of positive characteristic, on which we have to consider the étale topology. Thus we have to relate in a way or another our current complex-analytic setting to an appropriate étale framework. We will use a strategy similar to the ones in [BBD, §6.1.2] and [BBD, §6.1.7, 6.1.8]: the former will allow us to switch between the complex-analytic world (i.e. what we have done so far, and what we are interested in) and the complex-étale setting; the latter provides tools to relate the complex-étale case to the setting of $\mathbb{F}$-schemes endowed with the étale topology, where $\mathbb{F}$ is algebraically closed of positive characteristic. Let us give some more details.

From $\mathbb{C}$-analytic to $\mathbb{C}$-étale

Results from [SGA4] and [BBD] provide a natural way to transfer information from $\mathbb{C}$-schemes endowed with the étale topology to $\mathbb{C}$-varieties endowed with their classical topology. The point is that we have a natural way of comparing the categories of sheaves for the étale topology and the analytic topology (that is to say, a morphism of topoi); and we can very formally transfer from the analytic setting to the étale one the constructions made previously. In fact, we will adapt the constructions made in earlier chapters, such as monodromic

and equivariant Lusztig–Yun categories over $\mathscr{X}$, but now considering complex schemes endowed with the étale topology. We will show that these constructions transfer to the correct objects in the complex-analytic world.

From $\mathbb{C}$-étale to $\mathbb{F}$-étale: overview

This step is slightly more involved but essentially follows from general results from [BBD, Chapitre 6]. The schemes we are interested in admit $\mathbb{Z}$-versions, meaning here that we there exist $\mathbb{Z}$-schemes of finite type whose base change to $\mathbb{C}$ yield the complex schemes we consider: for example, the basic affine space $\mathscr{X}$ (viewed as a scheme over $\mathbb{C}$) can be "defined over $\mathbb{Z}$ as a $\mathbb{Z}$-scheme $\mathscr{X}_{\mathbb{Z}}$ such that $\mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathscr{X}_{\mathbb{Z}} \cong \mathscr{X}$. In particular, this allows us to consider versions of our schemes over various fields. Roughly, the results from *loc. cit.* then ensures that we can relate some subcategories of the derived categories of étale sheaves on a scheme over $\mathbb{C}$ and its analogue over an algebraically closed field of positive characteristic. To do so, a transition step on schemes defined over the spectrum of a strictly henselian discrete valuation local ring is necessary. This will force us to adapt some of our constructions on schemes defined over rings.

Constructions of local systems

The schemes we will consider are obviously $\mathscr{X}$ (viewed as a $\mathbb{C}$-scheme endowed with its étale topology) and its locally closed subsets that are union of $\mathscr{X}_w$'s. For each stratum, we have to define a finite family of irreducible (étale) local systems. These will be given by the étale analogue of the local systems $\mathscr{L}_t^w$. In fact, we will construct such analogues in a quite general setting: the scheme $\mathscr{X}$ can be defined already over $\mathbb{Z}$, thus we can consider its base change to any ring. We consider an appropriate subring A of finite type over $\mathbb{Z}$ on which we will be able to construct "by hand" the local systems we wish for.

Monodromic étale categories

Once the above constructions are fulfilled, we obtain an equivalence between appropriate subcategories of the derived categories of étale sheaves, for one side defined over complex schemes, on the other side, defined over $\mathbb{F}$-schemes. In these two settings, we can define a monodromy action in a way similar to the constructions of chapter 1, which, fortunately, coincide, in an appropriate sense.

Equivariant considerations

Above, we explain how one can find a way to relate (some subcategories of the étale) constructible categories on the algebraic closure $\mathbb{F}$ of a finite field and $\mathbb{C}$. Obtaining such a comparison result for equivariant categories is less immediate, mainly because the general results from [BBD] (notably [BBD, Lemme 6.1.9]) require an intermediary step on schemes defined over a ring. The definition of étale equivariant categories and their Lusztig–Yun subcategories is more or less

straightforward for schemes over algebraically closed fields, it is far less clear (for the author at least) that one can make the same constructions over a ring.

However, we will only have to consider such a result for objects equivariant for the action of a split algebraic torus T. The advantage here is that constructions can then be made quite explicitly, and some "equivariant category" can be defined already over $\mathbb{Z}$. Once this is done, there is still some difficulty to overcome: the definition of the equivariant category is as an inductive limit of categories, and as such requires considering an *infinite* family of schemes. We thus need to show that "[BBD] style" arguments can be applied. Then, the equivalence between the complex T-equivariant category and the one defined over $\mathbb{F}$ is formal. Once this is obtained, we can consider the Lusztig–Yun subcategories on both sides and we show that these are also equivalent.

11.2 From $\mathbb{C}$ étale to $\mathbb{C}$ analytic

11.2.1 Exposition

So far, we have considered only the setting (which is our main interest) which involves complex algebraic groups and complex varieties endowed with their usual, "analytic", topology. We will now also consider an étale setting, involving $\mathbb{C}$-schemes endowed with the étale topology. In this section, we present a way to relate these two settings. The arguments there are general, and come from [SGA4, Exposés XI et XVI] and [BBD, §6.1.2]. Let X be a $\mathbb{C}$-scheme of finite type.

Very roughly, the main tool here is a morphism of topoi relating the category of sheaves on the étale site of the $\mathbb{C}$-scheme X and the category of sheaves on the complex variety $X(\mathbb{C})$ with its analytic topology. We recall here the content of [SGA4, Exposé XI, §4]. We refer to *loc. cit.* and references therein for more details.

By abuse of notation, we will use $X(\mathbb{C})$ to denote both the site of open subsets of $X(\mathbb{C})$ (that is, of open immersions to $X(\mathbb{C})$ in the analytic topology) and the underlying complex variety.

We can consider more generally the category whose objects are morphisms $f : U \to X(\mathbb{C})$ where U is a $\mathbb{C}$-variety and f is a local homeomorphism, meaning that for any point $x \in U$, there exists an open neighbourhood U_x of x in U that is mapped homeomorphically to an open neighbourhood of $f(x)$ in $X(\mathbb{C})$. The definition of morphisms in this category is obvious.

For $V \to X(\mathbb{C})$, say that a family of morphisms $\{U_\nu \to V\}$ is a covering if V is the union of the images of the U_ν. This defines a Grothendieck topology on this category and we let $X(\mathbb{C})_{\mathrm{cl}}$ denote the obtained site.

Since an open immersion is in particular a local homeomorphism, a sheaf for this new Grothendieck topology induces (by restriction to open immersions) a sheaf in the usual analytic topology. Moreover, if $U \to X(\mathbb{C})$ is a local homeomorphism, by definition, for any $y = f(x)$ in the image of U, we can find an open subset V_y of $X(\mathbb{C})$ and a section $V_y \to U$ such that the composition

$V_y \to U \to X(\mathbb{C})$ is an open immersion: we can "cover U by open subsets of $X(\mathbb{C})$". Then a sheaf for the analytic topology defines a sheaf on the site $X(\mathbb{C})_{\mathrm{cl}}$, and in fact, it follows from [SGA4, Exposé XI, §4] that this procedure yields an equivalence of categories between the categories of sheaves on these sites.

Now consider an étale morphism $f : Y \to X$ of $\mathbb{C}$-schemes. Then the morphism $f(\mathbb{C}) : Y(\mathbb{C}) \to X(\mathbb{C})$ induced on complex points is a local homeomorphism. Thus $Y \mapsto Y(\mathbb{C})$ defines a functor from the étale site X_{et} of X (that is, the category whose objects are étale morphisms of $\mathbb{C}$-schemes to X and with natural morphisms and the usual Grothendieck topology) to the site $X(\mathbb{C})_{\mathrm{cl}}$, meaning that we have a natural way to associate a covering in $X(\mathbb{C})_{\mathrm{cl}}$ to a covering in X_{et}. Thus, we have a functor ε_* that associates an étale sheaf on X_{et} to a sheaf on $X(\mathbb{C})_{\mathrm{cl}}$. We also have an adjoint functor ε^* from the étale site to the classical one (we keep the notation of [SGA4] and [BBD]). With the above identification of the sites $X(\mathbb{C})$ and $X(\mathbb{C})_{\mathrm{cl}}$, this latter functor is the one considered in [BBD, §6.1.2, (A)].

The following result is [BBD, §6.1.2, (A'), (B')]; see also [SGA4, Exposé XI, Théorème 4.4, (i)]. For R a finite ring, we let $D_c^{b,\mathrm{et}}(X, R)$ be the constructible derived category of étale sheaves of R-modules on X and $D_c^b(X(\mathbb{C}), R)$ be the usual constructible derived category of sheaves of R-modules on $X(\mathbb{C})$.

Theorem 11.2.1. *Let R be a finite ring. The functors ε^* and ε_* induce inverse equivalences between the categories of étale constructible sheaves of R-modules on X and constructible sheaves of R-modules on $X(\mathbb{C})$. Moreover, these equivalences upgrade to the derived categories: $D_c^{b,\mathrm{et}}(X, R) \cong D_c^b(X(\mathbb{C}), R)$.*

[BBD, §6.1.2, (C')] (see also [SGA4, Exposé XVI, Théorème 4.1]) implies that if $f : Y \to X$ is a morphism between $\mathbb{C}$-schemes of finite type, the equivalences of theorem 11.2.1 commutes with the various (derived) functors $f^!, f^*, f_!, f_*, \otimes$ and $\mathcal{H}om$.

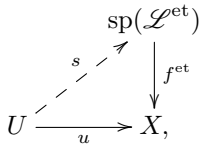
Combining this fact with theorem 11.2.1, one sees that the equivalences of the theorem preserve sheaves (or complexes of sheaves) constructible for a *fixed* stratification, as well as perverse sheaves.

11.2.2 Local systems

Let R be a finite ring, and consider a connected $\mathbb{C}$-scheme X, endowed with the étale topology. We consider the étale fundamental group $\pi_1^{\mathrm{et}}(X) := \pi_1^{\mathrm{et}}(X, \overline{x_0})$, where $\overline{x_0}$ is any geometric point of X. Let $\mathrm{Loc}^{\mathrm{et}}(X, R)$ be the category of étale local systems of R-modules on X. This category is equivalent to the category of finite R-modules endowed with a continuous action of $\pi_1^{\mathrm{et}}(X)$, i.e. with the category $\mathrm{Mod}^{\mathrm{f}}(R[\pi_1^{\mathrm{et}}(X)])$ of $R[\pi_1^{\mathrm{et}}(X)]$-modules, finite over R.

Consider an étale R-local system $\mathscr{L}^{\mathrm{et}}$ on X; there exists a $\mathbb{C}$-scheme $\mathrm{sp}(\mathscr{L}^{\mathrm{et}})$ and a finite étale morphism of $\mathbb{C}$-schemes $\mathrm{sp}(\mathscr{L}^{\mathrm{et}}) \xrightarrow{f^{\mathrm{et}}} X$ such that the sections of $\mathscr{L}^{\mathrm{et}}$ over any étale open subset $u : U \to X$ in X_{et} are given by commutative

diagrams



i.e. by sections of f^{et}.

Now by definition, if we let $\mathrm{Aut}(f^{\mathrm{et}})$ be the finite group of automorphisms of f^{et} in the category of étale coverings of X, we have $\mathrm{Aut}(f^{\mathrm{et}}) = \mathrm{Aut}(\mathscr{L}^{\mathrm{et}})$ and there is a natural group morphism $\pi_1^{\mathrm{et}}(X) \to \mathrm{Aut}(f^{\mathrm{et}})$ (this is the action of the étale fundamental group actually defining the local system $\mathscr{L}^{\mathrm{et}}$). Applying the functor ε^*, we obtain an analytic local system $\mathscr{L} := \varepsilon^* \mathscr{L}^{\mathrm{et}}$, corresponding to the local homeomorphism

$$\mathrm{sp}(\mathscr{L}^{\mathrm{et}})(\mathbb{C}) =: \mathrm{sp}(\mathscr{L}) \xrightarrow{f^{\mathrm{et}}(\mathbb{C})=:f} X(\mathbb{C}).$$

Let $\pi_1(X(\mathbb{C})) = \pi_1(X(\mathbb{C}), x_0)$ (where now x_0 denotes the complex point associated to $\overline{x_0}$) be the topological fundamental group of $X(\mathbb{C})$, and set $\widehat{\pi_1(X(\mathbb{C}))}$ for its profinite completion. It is well known that we have a natural identification $\widehat{\pi_1(X(\mathbb{C}))} \cong \pi_1^{\mathrm{et}}(X)$ (see e.g. [Mi, I, Remark 5.1, (c)]). The action of $\pi_1(X(\mathbb{C}))$ on $\mathrm{sp}(\mathscr{L}) \xrightarrow{f} X(\mathbb{C})$ is then given by the natural map

$$\pi_1(X(\mathbb{C})) \to \widehat{\pi_1(X(\mathbb{C}))} = \pi_1^{\mathrm{et}}(X) \to \mathrm{Aut}(f^{\mathrm{et}}) = \mathrm{Aut}(f).$$

We deduce the following result:

Lemma 11.2.2. *Let $\rho : \pi_1(X(\mathbb{C})) \to \widehat{\pi_1(X(\mathbb{C}))} = \pi_1^{\mathrm{et}}(X)$ be the natural map. We have a commutative diagram of functors*

$$\begin{array}{ccc} \mathrm{Loc}(X(\mathbb{C}), R) & \underset{\varepsilon^*}{\overset{\varepsilon_*}{\rightleftarrows}} & \mathrm{Loc}^{\mathrm{et}}(X, R) \\ \wr \uparrow\downarrow & & \uparrow\downarrow \wr \\ \mathrm{Mod}^{\mathrm{f}}(R[\pi_1(X(\mathbb{C}))]) & \underset{\rho^*}{\overset{\rho_*}{\rightleftarrows}} & \mathrm{Mod}^{\mathrm{f}}(R[\pi_1^{\mathrm{et}}(X_{\mathrm{et}})]) \end{array}$$

where ρ^* is the restriction of scalars functor along the map ρ and ρ_* is the extension of scalars functor.

11.3 From $\mathbb{F}$-étale to $\mathbb{C}$-étale

This is a brief exposition of the results from [BBD, §6.1.8, 6.1.9, 6.1.10]; we refer to *loc. cit.* for more details. We keep our $\mathbb{C}$-scheme X of finite type from the preceding subsection, and consider a finite field $\mathbb{K}$. We want to relate (some full subcategory of) the derived constructible category of étale sheaves of $\mathbb{K}$-vector

spaces on X with analogous objects defined over the algebraic closure $\mathbb{F}$ of a finite field.

The strategy is as follows: assume that X is endowed with an algebraic stratification $\mathcal{S}_X$ (in particular, the stratification is finite) with smooth connected strata. Consider given, for each stratum $T \in \mathcal{S}_X$, a finite family of irreducible $\mathbb{K}$-local systems $L_X(T)$. Assume that the triple $(X, \mathcal{S}_X, L_X)$ descends to a ring of finite type over $\mathbb{Z}$: there exists a ring $\mathbb{Z} \subseteq A \subseteq \mathbb{C}$, of finite type over $\mathbb{Z}$ and a scheme X_S of finite type over $S := \mathrm{Spec}(A)$, endowed with a stratification $\mathcal{S}_{X_S}$ with strata smooth over S with connected geometric fibers, and for each $T \in \mathcal{S}_{X_S}$, a finite family $L_{X_S}(T)$ of irreducible $\mathbb{K}$-local systems on T such that the base change of $(X_S, \mathcal{S}_{X_S}, L_{X_S})$ to $\mathbb{C}$ yields $(X, \mathcal{S}_X, L_X)$.

For any finite residue field $\mathbb{L}$ of A (i.e. quotient of A by some maximal ideal), we can find a strictly henselian discrete valuation ring $A \subseteq V \subseteq \mathbb{C}$ whose residue field is an algebraic closure $\mathbb{F}$ of $\mathbb{L}$ (see the discussion in [BBD, §6.1.8]). We obtain a diagram of schemes

$$X \to X_V \leftarrow X_{\mathbb{F}}$$

where $X_?$ denotes the base change $X_S \times_S \mathrm{Spec}(?)$.

Let $D^{b,\mathrm{et}}_{\mathcal{S}_X, L_X}(X, \mathbb{K})$ be the full subcategory of the derived category $D^{b,\mathrm{et}}(X, \mathbb{K})$ of étale $\mathbb{K}$-sheaves whose objects are those complexes $\mathscr{F}$ such that the cohomology sheaves of $\mathscr{F}_{|T}$ are finite successive extension of objects in $L_X(T)$ for any T in $\mathcal{S}_X$; similar (and obvious) notation apply on X_V and $X_{\mathbb{F}}$. Considering the pullbacks along the above morphisms, we obtain functors

$$D^{b,\mathrm{et}}_{\mathcal{S}_X, L_X}(X, \mathbb{K}) \leftarrow D^{b,\mathrm{et}}_{\mathcal{S}_{X_V}, L_{X_V}}(X_V, \mathbb{K}) \to D^{b,\mathrm{et}}_{\mathcal{S}_{X_{\mathbb{F}}}, L_{X_{\mathbb{F}}}}(X_{\mathbb{F}}, \mathbb{K}).$$

Choosing an appropriate V (see [BBD, Lemme 6.1.9]), one can show that these functors then define equivalences of categories.

11.4 Monodromic categories

11.4.1 Preliminaries on coefficients

The category of étale sheaves is well behaved only for certain coefficients rings. We fix here an appropriate field of coefficient for our purposes; recall that $\Bbbk$ is a fixed algebraic closure of a finite field of characteristic $\ell > 0$.

Consider a W-orbit $\mathfrak{o}$ in $T^\vee_{\Bbbk}$. As mentionned in remark 3.3.1, the elements of $\mathfrak{o}$ share a common finite order, which we denote $n(\mathfrak{o})$, and this order satisfies $\gcd(\ell, n(\mathfrak{o})) = 1$.

Now any element t in $\mathfrak{o}$ is defined over some finite subfield of $\Bbbk$ (this follows from the fact that $\Bbbk$ is the union of its finite subfields). We can then find a finite subfield $\mathbb{K} = \mathbb{K}(\mathfrak{o})$ of $\Bbbk$ such that any element in $\mathfrak{o}$ is defined over $\mathbb{K}$. This will be our field of coefficients for the duration of this chapter.

11.4.2 Preliminaries on schemes and groups

For a scheme $X_{\mathbb{Z}}$ over $\mathbb{Z}$, of finite type, we will set $X_{\mathfrak{R}} := \mathrm{Spec}(\mathfrak{R}) \times_{\mathrm{Spec}(\mathbb{Z})} X_{\mathbb{Z}}$ for the base change of $X_{\mathbb{Z}}$, where $\mathfrak{R}$ is any commutative ring. We will denote by $D^{b,\mathrm{et}}(X_{\mathfrak{R}}, \mathbb{K})$ the derived category of étale sheaves of $\mathbb{K}$-vector spaces on the scheme $X_{\mathfrak{R}}$.

Recall that we are interested in a complex connected reductive algebraic group G, and that we fixed a (positive) Borel subgroup B and a maximal torus $T \subseteq B$.

We can find a split connected reductive group $\mathbf{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ and choose a Borel subgroup $\mathbf{B}_{\mathbb{Z}}$ and a maximal torus $\mathbf{T}_{\mathbb{Z}} \subseteq \mathbf{B}_{\mathbb{Z}}$ of $\mathbf{G}_{\mathbb{Z}}$ such that extension of scalars to $\mathbb{C}$ yields the triplet $T \subseteq B \subseteq G$. Let $\mathbf{U}_{\mathbb{Z}}$ be the unipotent radical of $\mathbf{B}_{\mathbb{Z}}$ and set $\mathscr{X}_{\mathbb{Z}} := \mathbf{G}_{\mathbb{Z}}/\mathbf{U}_{\mathbb{Z}}$. For $\mathfrak{R}$ a commutative ring, we have $\mathscr{X}_{\mathfrak{R}} = \mathbf{G}_{\mathfrak{R}}/\mathbf{U}_{\mathfrak{R}}$. This scheme admits a stratification indexed by the Weyl group W, induced by the Bruhat stratification on $\mathbf{G}_{\mathfrak{R}}$; we denote the strata $\mathscr{X}_{w,\mathfrak{R}}$ for $w \in W$.

Finally, if $\mathrm{X}_*(\mathbf{T}_{\mathbb{Z}})$ is the cocharacter group of the $\mathbb{Z}$-torus $\mathbf{T}_{\mathbb{Z}}$, as soon as $\mathfrak{R}$ is an integral domain, we have an isomorphism $\mathrm{X}_*(\mathbf{T}_{\mathbb{Z}}) \cong \mathrm{X}_*(\mathbf{T}_{\mathfrak{R}})$ (see e.g. [J, I, §2.5, (1)]; see also [J, II, §1.1]). Thus we will simply denote this group as $\mathrm{X}_*(\mathbf{T})$ without mentioning the base ring.

11.4.3 Preliminaries on local systems

From now on, we let $A \subseteq \mathbb{C}$ be any fixed ring of finite type over $\mathbb{Z}$ containing a primitive $n(\mathfrak{o})$-th root of unity ξ and such that $n(\mathfrak{o}) \in \mathbb{Z}$ is invertible in A (for example, one could take $A = \mathbb{Z}[\frac{1}{n(\mathfrak{o})}, \xi]$). In what follows, we will have to consider two different settings: schemes over rings and schemes over algebraically closed fields; unless specified otherwise, $\mathfrak{R}$ will denote either any ring $A \subseteq \mathfrak{R} \subseteq \mathbb{C}$ or an algebraically closed field of characteristic exponent p satisfying $\gcd(p, n(\mathfrak{o})) = 1$. In any case, $\mathfrak{R}$ contains a primitive $n(\mathfrak{o})$-th root of unity ξ: if $\mathfrak{R}$ is algebraically closed, this is clear thanks to our assumption on p and $n(\mathfrak{o})$; if $\mathfrak{R}$ is a subring of $\mathbb{C}$ containing A, then it follows from the existence of such a ξ in A.

Our assumption implies that the application $e_{n(\mathfrak{o}),\mathfrak{R}}$ mapping an element to its $n(\mathfrak{o})$-th power is a finite étale map $\mathbf{T}_{\mathfrak{R}} \to \mathbf{T}_{\mathfrak{R}}$. Let us justify this fact in the case $\mathbf{T} \cong \mathbb{G}_{\mathrm{m}}$. The morphism of affine $\mathfrak{R}$-schemes $e_{n(\mathfrak{o}),\mathfrak{R}}$ corresponds to the algebra map

$$\mathfrak{R}[x, x^{-1}] \to \mathfrak{R}[x, x^{-1}],\ x \mapsto x^{n(\mathfrak{o})}.$$

One can then write $\mathfrak{R}[x, x^{-1}] = \mathfrak{R}[x^{n(\mathfrak{o})}, x^{-n(\mathfrak{o})}][u]/(u^{n(\mathfrak{o})} - x^{n(\mathfrak{o})})$, where u is an indeterminate. Moreover, the formal derivative of $u^{n(\mathfrak{o})} - x^{n(\mathfrak{o})}$ is $n(\mathfrak{o})u^{n(\mathfrak{o})-1}$, which is invertible in $\mathfrak{R}[x^{n(\mathfrak{o})}, x^{-n(\mathfrak{o})}][u]/(u^{n(\mathfrak{o})} - x^{n(\mathfrak{o})})$. Thus [Mi, I, Example 3.4] tells us that $\mathfrak{R}[x^{n(\mathfrak{o})}, x^{-n(\mathfrak{o})}][u]/(u^{n(\mathfrak{o})} - x^{n(\mathfrak{o})})$ is étale over $\mathfrak{R}[x^{n(\mathfrak{o})}, x^{-n(\mathfrak{o})}]$, which implies our claim. Choosing a trivialisation $\mathbf{T}_{\mathfrak{R}} \cong (\mathbb{G}_{\mathrm{m},\mathfrak{R}})^r$, the general case follows from the preceding discussion.

Now our assumption of the existence of a primitive $n(\mathfrak{o})$-th root of unity in $\mathfrak{R}$ implies that $\ker(e_{n(\mathfrak{o}),\mathfrak{R}})$ is a constant group scheme. Then the finite group $\ker(e_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R})$ identifies with the Galois group of the finite étale covering

$\mathbf{T}_{\mathfrak{R}} \xrightarrow{e_{n(\mathfrak{o}),\mathfrak{R}}} \mathbf{T}_{\mathfrak{R}}$, so we have a natural surjective map $\pi_1^{\mathrm{et}}(\mathbf{T}_{\mathfrak{R}}) \to \ker(e_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R})$. Moreover, if we let $e^{\mathbb{G}_{\mathrm{m}}}_{n(\mathfrak{o}),\mathfrak{R}}$ denote the $n(\mathfrak{o})$-th power map on $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$, $\ker(e^{\mathbb{G}_{\mathrm{m}}}_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R})$ is the finite abelian group of $n(\mathfrak{o})$-th roots of unity in $\mathfrak{R}^*$ (which is a cyclic group of order $n(\mathfrak{o})$ by hypothesis) and we have a natural isomorphism of finite abelian groups

$$\ker(e^{\mathbb{G}_{\mathrm{m}}}_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R}) \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathbf{T}) \xrightarrow{\sim} \ker(e_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R})$$

induced by $x \otimes \lambda \mapsto \lambda(x)$. For any $t \in \mathfrak{o}$, we can then construct an étale $\mathbb{K}$-local system $\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}$ on $\mathbf{T}_{\mathfrak{R}}$ mimicking the construction of section 3.3. Namely, we consider the $\mathbb{K}$-vector space

$$\mathbb{K}[\mathrm{X}_*(\mathbf{T})]/\langle e^{\lambda} - \lambda(t) \mid \lambda \in \mathrm{X}_*(\mathbf{T})\rangle.$$

Since t is of order $n(\mathfrak{o})$, there is an action of $\ker(e_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R}) \cong \ker(e^{\mathbb{G}_{\mathrm{m}}}_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R}) \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathbf{T})$ on this vector space induced by the natural action of $\mathrm{X}_*(\mathbf{T})$: any element in $\ker(e^{\mathbb{G}_{\mathrm{m}}}_{n(\mathfrak{o}),\mathfrak{R}})$ can be written as ξ^n for some (non-unique) integer $n > 0$. Then, $\xi^n \otimes \lambda$ will act as $\lambda^n(t) = \lambda(t^n)$. We deduce a natural continuous action of $\pi_1^{\mathrm{et}}(\mathbf{T}_{\mathfrak{R}})$. In view of [FK, Proposition A.I.7], this defines an étale local system on $\mathbf{T}_{\mathfrak{R}}$. Note that by construction, the kernel $\ker(e_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R})$ acts on $\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}$ via a character $\chi_{t,\mathfrak{R}}$, induced by $\lambda \mapsto \lambda(t)$ for $\lambda \in \mathrm{X}_*(\mathbf{T})$.

We will need later a little technical lemma:

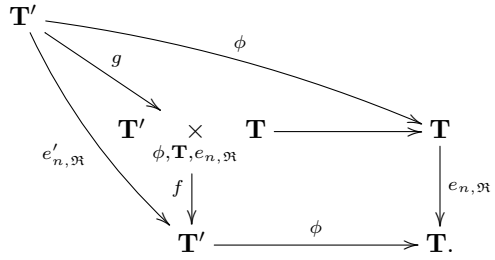
Lemma 11.4.1. *Consider a group morphism between split $\mathfrak{R}$-tori $\mathbf{T}' \xrightarrow{\phi} \mathbf{T}$. The map ϕ induces a morphism $\pi_1^{\mathrm{et}}(\mathbf{T}') \xrightarrow{\pi_1^{\mathrm{et}}(\phi)} \pi_1^{\mathrm{et}}(\mathbf{T})$ (resp. a morphism $\mathrm{X}_*(\mathbf{T}') \xrightarrow{\mathrm{X}_*(\phi)} \mathrm{X}_*(\mathbf{T})$). Then for any integer $n > 0$ such that n is invertible in $\mathfrak{R}$, the following diagram is commutative*

$$\begin{array}{ccc} \pi_1^{\mathrm{et}}(\mathbf{T}') & \xrightarrow{\pi_1^{\mathrm{et}}(\phi)} & \pi_1^{\mathrm{et}}(\mathbf{T}) \\ \downarrow & & \downarrow \\ \mathrm{X}_*(\mathbf{T}') \otimes_{\mathbb{Z}} \ker(e^{\mathbb{G}_{\mathrm{m}}}_{n,\mathfrak{R}})(\mathfrak{R}) & \xrightarrow{\mathrm{X}_*(\phi)\otimes_{\mathbb{Z}}\mathrm{id}} & \mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \ker(e^{\mathbb{G}_{\mathrm{m}}}_{n,\mathfrak{R}})(\mathfrak{R}). \end{array}$$

Proof. First, remark that our assumption on n implies that the power-n map $e_{n,\mathfrak{R}}$ on $\mathbf{T}$ (resp. $e'_{n,\mathfrak{R}}$ on $\mathbf{T}'$) is a finite étale covering, with Galois group $\mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \ker(e^{\mathbb{G}_{\mathrm{m}}}_{n,\mathfrak{R}})(\mathfrak{R})$ (resp. $\mathrm{X}_*(\mathbf{T}') \otimes_{\mathbb{Z}} \ker(e^{\mathbb{G}_{\mathrm{m}}}_{n,\mathfrak{R}})(\mathfrak{R})$).

By definition, the morphism at the fundamental groups level is constructed as follows: for an étale cover $L \to \mathbf{T}$ of $\mathbf{T}$, the base change of L along ϕ yields an étale cover L' of $\mathbf{T}'$, and we have then a natural map from the Galois group of L' above $\mathbf{T}'$ to the Galois group of L above $\mathbf{T}$. This construction then passes

to the limit and gives $\pi_1^{\mathrm{et}}(\phi)$. Now, we have a commutative diagram



Here, f is the base change of $e_{n,\mathfrak{R}}$ and g is the map obtained from the universal property of the fiber product. By definition, the map

$$\pi_1^{\mathrm{et}}(\mathbf{T}') \xrightarrow{\pi_1^{\mathrm{et}}(\phi)} \pi_1^{\mathrm{et}}(\mathbf{T}) \twoheadrightarrow \mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \ker(e_{n,\mathfrak{R}}^{\mathbb{G}_{\mathrm{m}}})(\mathfrak{R})$$

factors as

$$\pi_1^{\mathrm{et}}(\mathbf{T}') \twoheadrightarrow \mathrm{Gal}(f) \to \mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \ker(e_{n,\mathfrak{R}}^{\mathbb{G}_{\mathrm{m}}})(\mathfrak{R}).$$

But now the natural surjection $\pi_1^{\mathrm{et}}(\mathbf{T}') \twoheadrightarrow \mathrm{Gal}(f)$ itself factors as

$$\pi_1^{\mathrm{et}}(\mathbf{T}') \twoheadrightarrow \mathrm{Gal}(e'_{n,\mathfrak{R}}) \twoheadrightarrow \mathrm{Gal}(f);$$

the composition

$$\begin{aligned}\mathrm{X}_*(\mathbf{T}') \otimes_{\mathbb{Z}} \ker(e_{n,\mathfrak{R}}^{\mathbb{G}_{\mathrm{m}}})(\mathfrak{R}) \cong \mathrm{Gal}(e'_{n,\mathfrak{R}}) &\twoheadrightarrow \mathrm{Gal}(f) \\ &\to \mathrm{Gal}(e_{n,\mathfrak{R}}) \cong \mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \ker(e_{n,\mathfrak{R}}^{\mathbb{G}_{\mathrm{m}}})(\mathfrak{R})\end{aligned}$$

is clearly induced by the morphism $\mathrm{X}_*(\phi)$. This concludes the proof of the lemma. □

Remark 11.4.2. We keep the notation of lemma 11.4.1. This lemma has a nice consequence: assume that $\mathscr{L}$ is a local system on $\mathbf{T}$ whose associated $\pi_1^{\mathrm{et}}(\mathbf{T})$-representation factors through the natural quotient $\pi_1^{\mathrm{et}}(\mathbf{T}) \twoheadrightarrow \ker(e_{n,\mathfrak{R}})$. Then if $\phi : \mathbf{T}' \to \mathbf{T}$ is a group morphism between tori, the pullback $\phi^* \mathscr{L}$ is a local system on $\mathbf{T}'$ whose associated $\pi_1^{\mathrm{et}}(\mathbf{T}')$-representation factors through the natural quotient $\pi_1^{\mathrm{et}}(\mathbf{T}') \twoheadrightarrow \ker(e'_{n,\mathfrak{R}})$.

For us, this has two immediate implications: for any $t \in \mathfrak{o}$, the pullback of $\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}$ along $e_{n(\mathfrak{o}),\mathfrak{R}}$ is the constant $\mathbb{K}$-local system $\underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}}$; the pullback of $\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}$ along the multiplication map $m : \mathbf{T}_{\mathfrak{R}} \times \mathbf{T}_{\mathfrak{R}} \to \mathbf{T}_{\mathfrak{R}}$ is isomorphic to $\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}} \boxtimes \mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}$.

11.4.4 Constructible categories

Recall that we considered natural morphisms of $\mathbb{C}$-varieties $p_w : \mathscr{X}_w \to T$, see subsection 5.1.1; these morphisms are induced by isomorphisms $\mathscr{X}_w \cong U_w \times T$. According to [J, II, §§13.1, 13.2], the same identifications hold already over any

principal ideal domain. In practice, we fix such a morphism over $\mathbb{Z}$, and we derive morphisms of $\mathfrak{R}$-schemes $p_{w,\mathfrak{R}} : \mathscr{X}_{w,\mathfrak{R}} \to \mathbf{T}_{\mathfrak{R}}$ by base change. We set

$$L(w, \mathfrak{o}, \mathfrak{R}) := \{\mathscr{L}_t^{w,\mathrm{et}} := p_{w,\mathfrak{R}}^* \mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}} \mid t \in \mathfrak{o}\}.$$

This is a finite family of local systems on the stratum $\mathscr{X}_{w,\mathfrak{R}}$. Then, consider

$$L(W, \mathfrak{o}, \mathfrak{R}) := \bigcup_{w \in W} L(w, \mathfrak{o}, \mathfrak{R}) = \{\mathscr{L}_t^{w,\mathrm{et}} \mid w \in W, t \in \mathfrak{o}\}.$$

Finally, we set

$$D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$$

for the full subcategory of the derived category of étale sheaves of $\mathbb{K}$-vector spaces on $\mathscr{X}_{\mathfrak{R}}$ whose objects are those complexes $\mathscr{F}$ such that for any $i \in \mathbb{Z}$ and any $w \in W$, the sheaf $\mathscr{H}^i(\mathscr{F})_{|\mathscr{X}_{w,\mathfrak{R}}}$ is a successive finite extension of objects in $L(w, \mathfrak{o}, \mathfrak{R})$. Note that by definition, the category $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$ is a triangulated subcategory of $D^{b,\mathrm{et}}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$.

11.4.5 Standard and costandard objects

For any $w \in W$, we set $j_{w,\mathfrak{R}} : \mathscr{X}_{w,\mathfrak{R}} \hookrightarrow \mathscr{X}_{\mathfrak{R}}$ for the natural embedding. Then, we set

$$\Delta^{\mathfrak{R},\mathrm{et}}_{w,t} := (j_{w,\mathfrak{R}})_!(\mathscr{L}_t^{w,\mathrm{et}})[\dim(\mathscr{X}_{w,\mathfrak{R}})]$$

and

$$\nabla^{\mathfrak{R},\mathrm{et}}_{w,t} := (j_{w,\mathfrak{R}})_*(\mathscr{L}_t^{w,\mathrm{et}})[\dim(\mathscr{X}_{w,\mathfrak{R}})].$$

We will call the $\Delta^{\mathfrak{R},\mathrm{et}}_{w,t}$'s standard objects and the $\nabla^{\mathfrak{R},\mathrm{et}}_{w,t}$'s costandard objects. We have a version of lemma 5.2.2 in this new setting, with essentially the same proof:

$$\{\Delta^{\mathfrak{R},\mathrm{et}}_{w,t} \mid w \in W,\, t \in \mathfrak{o}\}$$

generates $D^{b,\mathrm{et}}_{L(W,\mathfrak{o})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$ as a triangulated category. We will justify later that for any $w \in W$ and $t \in \mathfrak{o}$, the object $\nabla^{\mathfrak{R},\mathrm{et}}_{w,t}$ belongs to $D^{b,\mathrm{et}}_{L(W,\mathfrak{o})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$, and that shifts of costandard objects also generate this category.

11.4.6 Monodromy

In this section, we assume that $\mathfrak{R}$ is an algebraically closed field; we let p be its characteristic exponent. Assume moreover that $\gcd(\ell, p) = 1$. We expose briefly here how one can define monodromy in the étale setting. We freely use the notation of section 1.2. What follows is written considering the right action of $\mathbf{T}_{\mathfrak{R}}$ on $\mathscr{X}_{\mathfrak{R}}$, i.e. for right monodromy, but the exact same arguments work as well for the left action.

We want to show that proposition 1.2.1 holds in $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$, that is to say, the following: for any $\mathscr{F} \in D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$ there exists an integer

$n > 0$ such that $a(n)^*\mathscr{F} \cong \mathrm{pr}_2^*\mathscr{F}$ on $\mathscr{X}_{\mathfrak{R}} \times \mathbf{T}_{\mathfrak{R}}$, moreover, the restriction of this isomorphism to $\{1\} \times \mathscr{X}_{\mathfrak{R}}$ is the identity of $\mathscr{F}$. The strategy is quite the same as in chapter 1. As in the proof of proposition 1.2.1, one reduces first to the case where $\mathscr{F}$ is a (possibly shifted) sheaf on $\mathscr{X}_{\mathfrak{R}}$. Analogues of lemma 1.1.6 and lemma 1.1.7 still hold in our current setting, although lemma 1.1.6 needs a slight adaptation (mainly because, what we need to consider now is the étale fundamental group; which is a slightly more complicated object than the topological one; for example, the étale fundamental group does not "commute with product" the way the topological one does).

Lemma 11.4.3. *The endomorphism of* $\mathbf{H}^1(\mathbf{T}_{\mathfrak{R}}, \mathbb{K}) = \mathrm{Ext}^1(\underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}}, \underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}})$ *induced by* e_ℓ^* *is given by multiplication by* ℓ*, hence is zero (where the* Ext*-space is taken in the category of étale sheaves on* $\mathbf{T}_{\mathfrak{R}}$*).*

Proof. First, we can reduce to the case $r = 1$, that is to say, $\mathbf{T}_{\mathfrak{R}} = \mathbb{G}_{\mathrm{m},\mathfrak{R}}$. This is a consequence of Künneth's formula; the arguments being essentially the same as those used in the end of the proof of lemma 1.2.2. Now, if $r = 1$, the space

$$\mathbf{H}^1(\mathbf{T}_{\mathfrak{R}}, \mathbb{K}) = \mathrm{Ext}^1(\underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}}, \underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}})$$

is isomorphic to $\mathbb{K}$.

Consider for a moment a non-split extension L in $\mathrm{Ext}^1_{\ker(e_\ell)}(\underline{\mathbb{K}}, \underline{\mathbb{K}})$ (the latter space being a space of extensions in the category of representations of the finite abelian group $\ker(e_\ell)$). This representation is then 2 dimensional, each element of $\ker(e_\ell)$ acts as a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, in particular, multiplication by ℓ is zero on this space. Now, via the natural surjective morphism $\pi_1^{\mathrm{et}}(\mathbf{T}_{\mathfrak{R}}) \to \ker(e_\ell)$, L defines a non-split representation of $\pi_1^{\mathrm{et}}(\mathbf{T}_{\mathfrak{R}})$, which in turn defines an étale local system $\mathscr{L}$ on $\mathbf{T}_{\mathfrak{R}}$. Moreover, lemma 11.4.1 implies that $e_\ell^*\mathscr{L}$ is trivial. The sheaf $\mathscr{L}$ then defines a nonzero element in the one-dimensional $\mathbb{K}$-vector space $\mathrm{Ext}^1(\underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}}, \underline{\mathbb{K}}_{\mathbf{T}_{\mathfrak{R}}})$, and as such, it generates this space. Since $e_\ell^*\mathscr{L}$ is trivial, the lemma follows. □

Lemma 11.4.4. *Let* $\mathscr{F}$ *be a sheaf on* $\mathscr{X}_{\mathfrak{R}}$ *whose restriction to each* $\mathscr{X}_{w,\mathfrak{R}}$ *is a finite extension of successive objects in* $L(W, \mathfrak{o}, \mathfrak{R})$*. Then there exists an integer* n *such that* $a(n)^*\mathscr{F}$ *is constant on any fiber of* pr_2^**. Moreover,* n *can be chosen of the form* $n = n(\mathfrak{o})\ell^s$ *for* $s \geq 0$*.*

Proof sketch. Since we only have a finite number of strata on $\mathscr{X}_{\mathfrak{R}}$, one easily sees that it suffices to prove the lemma with $\mathscr{X}_{\mathfrak{R}}$ replaced by $\mathscr{X}_{w,\mathfrak{R}}$ for any w. We can thus assume that $\mathscr{F}$ lives on some stratum $\mathscr{X}_{w,\mathfrak{R}}$, and that it is a successive finite extension of objects of $L(w, \mathfrak{o}, \mathfrak{R})$. Consider an exact sequence $\mathscr{L} \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{L}'$ of local systems on $\mathscr{X}_{w,\mathfrak{R}}$. If the result is known for both $\mathscr{L}$ and $\mathscr{L}'$, then there exists $n \geq 0$ sufficiently divisible such that the restriction of $a(n)^*\mathscr{F}$ to any fiber of pr_2 identifies with an extension between two copies of a constant sheaf on $\mathbf{T}_{\mathfrak{R}}$. Then lemma 11.4.3 readily implies that our statement is true for $\mathscr{F}$ too. We can thus further reduce to the case $\mathscr{F} = \mathscr{L}_t^{w,\mathrm{et}}$ for some $t \in \mathfrak{o}$. The definition of $\mathscr{L}_t^{w,\mathrm{et}}$ together with remark 11.4.2 easily imply that

$a(n)^* \mathscr{L}_t^{w,\mathrm{et}} \cong e_{n,\mathfrak{R}}^* \mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}} \boxtimes \mathscr{L}_t^{w,\mathrm{et}}$. For $n = n(\mathfrak{o})$ using remark 11.4.2 one more time, one sees that the latter object is isomorphic to $\mathrm{pr}_2^*(\mathscr{L}_t^{w,\mathrm{et}})$. The fact about n is clear from the above arguments. □

Lemma 11.4.3 also readily implies that an étale version of lemma 1.2.2 also holds. Then one can copy the proof of proposition 1.2.1 to see that this result is true in $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$.

Now, for any n such that $\gcd(n,p)=1$, let $\mathbb{G}_{\mathrm{m},\mathfrak{R}}[n]$ denotes (the $\mathfrak{R}$-points of) the kernel of the power n map on the multiplicative group. This is a finite abelian group; for $n > m$ such that m divides n, the map $e_{n/m}$ induces a natural morphism $\mathbb{G}_{\mathrm{m},\mathfrak{R}}[n] \to \mathbb{G}_{\mathrm{m},\mathfrak{R}}[m]$. We will consider the object $\varprojlim_{s\geq 0} \mathbb{G}_{\mathrm{m},\mathfrak{R}}[n(\mathfrak{o})\ell^s]$; let $\{x_s\}_{s\geq 0}$ be any fixed topological generator of this profinite group (we can construct one as follows: let ξ_0 be a primitive $n(\mathfrak{o})$-root of unity, then choose ξ_1 in $\mathbb{G}_{\mathrm{m},\mathfrak{R}}[n(\mathfrak{o})\ell]$ such that $\xi_1^\ell = \xi_0$ so ξ_1 is a primitive $n(\mathfrak{o})\ell$-th root of unity, and so on). We are ready to define monodromy for objects of $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$. For $\mathscr{F}$ there, consider $n = n(\mathfrak{o})\ell^s$ such that we have an isomorphism

$$\iota(n) : \mathrm{pr}_2^* \mathscr{F} \to a(n)^* \mathscr{F}$$

as in proposition 1.2.1. For any $\lambda \in \mathrm{X}_*(\mathbf{T})$, we then set

$$\varphi_{\mathscr{F}}(\lambda) := \iota(n)_{|\{\lambda(x_s)\}\times \mathscr{X}_{\mathfrak{R}}}.$$

The results of section 1.5 still hold for this definition, with essentially the same proofs.

Remark 11.4.5. Any choice of topological generator $\{x_s\}_{s\geq 0}$ as above will yield a monodromy action, a priori different. Later on, we will choose a topological generator in a slightly more specific way, but the definition and notation are somehow independent of this choice.

In analogy with the analytic case, we define

$$D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})_{[-,t]}$$

to be the full subcategory of $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$ whose objects are those $\mathscr{F}$ whose right monodromy action factors through a quotient

$$\mathbb{K}[\mathrm{X}_*(\mathbb{T})]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(\mathbf{T})\rangle^n$$

for some n; one can check that this identifies with the full subcategory of $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$ with objects those complexes $\mathscr{F}$ such that $\mathscr{H}^i(\mathscr{F})_{|\mathscr{X}_{w,\mathfrak{R}}}$ is a finite successive extension of $\mathscr{L}_t^{w,\mathrm{et}}$ for any $i \in \mathbb{Z}$ and $w \in W$. We could obviously define similarly categories $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})_{[-,\underline{t}]}$ or $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})_{[t,-]}$ as well; see subsection 5.2.1 for the definition in the analytic case.

11.4.7 Convolution

We address convolution in our étale category; here again, $\mathfrak{R}$ is either a subring of $\mathbb{C}$ containing our fixed ring A, or an algebraically closed field of characteristic exponent p, with $\gcd(p, n(\mathfrak{o})) = 1$. We set $q_{\mathfrak{R}}$ for the natural quotient morphism $\mathbf{G}_{\mathfrak{R}} \times \mathscr{X}_{\mathfrak{R}} \xrightarrow{q_{\mathfrak{R}}} \mathbf{G}_{\mathfrak{R}} \times^{\mathbf{U}_{\mathfrak{R}}} \mathscr{X}_{\mathfrak{R}}$, and $\varpi_{\mathfrak{R}}$ for the quotient morphism $\mathbf{G}_{\mathfrak{R}} \to \mathscr{X}_{\mathfrak{R}}$. Finally, let $m_{\mathfrak{R}}$ be the morphism $\mathbf{G}_{\mathfrak{R}} \times^{\mathbf{U}_{\mathfrak{R}}} \mathscr{X}_{\mathfrak{R}} \to \mathscr{X}_{\mathfrak{R}}$ induced by the action of $\mathbf{G}_{\mathfrak{R}}$ on $\mathscr{X}_{\mathfrak{R}}$.

For two complexes $\mathscr{F}, \mathscr{G} \in D^{b,\mathfrak{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$, we then define the ("$\mathbf{U}_{\mathfrak{R}}$-equivariant") convolution product to be

$$\mathscr{F} \star^{\mathbf{U}_{\mathfrak{R}}} \mathscr{G} := (m_{\mathfrak{R}})_!(q_{\mathfrak{R}})_*((\varpi_{\mathfrak{R}})^*(\mathscr{F}) \boxtimes \mathscr{G})[r].$$

Remark 11.4.6. It is clear that, in the analytic setting, the above description of convolution coincides with the one given in section 6.1: using the notation introduced there, the point is that we have an identification $q_*(\varpi^*(\mathscr{F}) \boxtimes \mathscr{G}) \cong \mathscr{F} \boxtimes^U \mathscr{G}$ (where q and ϖ are here defined between the corresponding complex algebraic varieties).

11.4.8 Equivalences

Let us come back for a moment to the complex analytic world of the first chapters. We let $D^b_{L(W,\mathfrak{o})}(\mathscr{X}, \mathbb{K})$ denote the full subcategory of $D^b_{(B)}(\mathscr{X}, \mathbb{K})$ whose objects are those complex $\mathscr{F}$ such that $\mathscr{H}^i(\mathscr{F})_{|\mathscr{X}_w}$ is a successive finite extension of shifted local systems of the form $\mathscr{L}^w_t$ with $t \in \mathfrak{o}$, for any $i \in \mathbb{Z}$ and $w \in W$.

Lemma 11.4.7. *The category $D^b_{(B)}(\mathscr{X}, \mathbb{K})_{[-,t]}$ is a full triangulated subcategory of $D^b_{L(W,\mathfrak{o})}(\mathscr{X}, \mathbb{K})$.*

Proof. This follows immediately from lemma 5.2.2. □

One then easily deduces that the category $D^b_{L(W,\mathfrak{o})}(\mathscr{X}, \mathbb{K})$ decomposes as the direct sum over $t \in \mathfrak{o}$ of the subcategories $D^b_{(B)}(\mathscr{X}, \mathbb{K})_{[-,t]}$.

We start by noticing that the local system $\mathscr{L}^T_t$ behaves as expected under the equivalence of theorem 11.2.1:

Corollary 11.4.8. *The analytic local system $\mathscr{L}^T_t$ on $T = \mathbf{T}(\mathbb{C})$ corresponds to the étale local system $\mathscr{L}^{\mathbf{T}_{\mathbb{C}},\mathrm{et}}_t$ on $\mathbf{T}_{\mathbb{C}}$ via the equivalences of theorem 11.2.1.*

Proof. This is a direct consequence of the construction of $\mathscr{L}^{\mathbf{T}_{\mathbb{C}},\mathrm{et}}_t$ in subsection 11.4.3 and lemma 11.2.2. □

As stated in remark 11.4.5, for any topological generator of the profinite group $\varprojlim_{s\geq 0} \mathbb{G}_{\mathrm{m},\mathbb{C}}[n(\mathfrak{o})\ell^s]$, we obtain a monodromy action on the category $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}}, \mathbb{K})$. We fix such a generator by $(x_s)_{s\geq 0} = (e^{\frac{2i\pi}{n(\mathfrak{o})\ell^s}})_{s\geq 0}$.

Proposition 11.4.9. *We have an equivalences of categories*

$$D^b_{L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K}) \xleftrightarrow{\sim} D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K}).$$

This equivalence swaps $\Delta_{w,t}$ and $\Delta^{\mathbb{C},\mathrm{et}}_{w,t}$ for any $w \in W$ and $t \in \mathfrak{o}$. Moreover, for $\mathscr{F} \in D^b_{L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K})$, if we let $\mathscr{F}^{\mathbb{C}}$ be the object of $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})$ corresponding to it via the above equivalence, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}(\mathscr{F}) & \xleftrightarrow{\quad\sim\quad} & \mathrm{End}(\mathscr{F}^{\mathbb{C}}) \\ & {\scriptstyle\varphi_{\mathscr{F}}} \nwarrow \quad \nearrow {\scriptstyle\varphi_{\mathscr{F}^{\mathbb{C}}}} & \\ & \mathbb{K}[\mathrm{X}_*(T)] = \mathbb{K}[\mathrm{X}_*(\mathbf{T})]. & \end{array}$$

Proof. Theorem 11.2.1 gives us an equivalence of constructible derived categories $D^b_c(\mathscr{X},\mathbb{K}) \xleftrightarrow{\sim} D^{b,\mathrm{et}}_c(\mathscr{X}_{\mathbb{C}},\mathbb{K})$. Combining the discussion that follows theorem 11.2.1 and corollary 11.4.8, we get that this equivalence restricts to an equivalence

$$D^b_{L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K}) \xleftrightarrow{\sim} D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K}).$$

The facts about standard objects follows from the commutation of this equivalence with various pushforward functors (see the discussion after theorem 11.2.1) and corollary 11.4.8. The fact about monodromy is clear by construction, and our choice of $(x_s)_{s\geq 0}$ made above. □

It follows readily from the previous proposition and lemma 11.4.7 that the subcategory with monodromy t is preserved via our equivalences:

Corollary 11.4.10. *The equivalence of proposition 11.4.9 restricts to an equivalence*

$$D^b_{(B)}(\mathscr{X},\mathbb{K})_{[-,t]} \xleftrightarrow{\sim} D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})_{[-,t]}. \tag{11.4.1}$$

We can now deduce information about costandard objects:

Corollary 11.4.11. *For any $t \in \mathfrak{o}$ and any $w \in W$, the object $\nabla^{\mathbb{C},\mathrm{et}}_{w,t}$ belongs to $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})_{[-,t]}$. Moreover, the family $\{\nabla^{\mathbb{C},\mathrm{et}}_{w,t}[n] \mid w \in W, n \in \mathbb{Z}\}$ generates the triangulated category $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})_{[-,t]}$.*

Proof. This follows directly from corollary 11.4.10 and the analogous facts in $D^b_{(B)}(\mathscr{X},\mathbb{K})_{[-,t]}$ (see lemma 5.2.2). □

According to [BBD, item (c) on page 70], corollary 11.4.11 allows us to consider a perverse t-structure on $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})_{[-,t]}$ (which is then defined in the usual way). We get the following immediate corollary:

Corollary 11.4.12. *For any $t \in \mathfrak{o}$ and any $w \in W$, the object $\Delta^{\mathbb{C},\mathrm{et}}_{w,t}$ and $\nabla^{\mathbb{C},\mathrm{et}}_{w,t}$ are perverse in $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})_{[-,t]}$. The ≤ 0 part of the perverse t-structure*

is generated under extension by the family $\{\Delta^{\mathbb{C},\mathrm{et}}_{w,t}[n] \mid w \in W, n \geq 0\}$*, and the* ≥ 0 *part of the perverse t-structure is generated under extension by the family* $\{\nabla^{\mathbb{C},\mathrm{et}}_{w,t}[n] \mid w \in W, n \leq 0\}$.

The equivalence of corollary 11.4.10 is perverse t-exact.

The next step is to compare the different étale settings. The following equivalence follows from arguments similar to those of [BBD, §6.1.8, 6.1.9], and briefly recalled in section 11.3. The arguments involve a step on schemes defined over a strictly henselian discrete valuation ring $\mathfrak{R} \subseteq \mathbb{C}$ whose residue field is $\mathbb{F}$, and are consequences of a comparison of cohomology of the strata over $\mathbb{C}$, $\mathbb{F}$ and $\mathfrak{R}$. We refer to *loc. cit* for more details. As in proposition 11.4.9, we wish to obtain a commutative diagram relating monodromy in the two different settings involved (i.e. over $\mathbb{C}$ and over $\mathbb{F}$), and as for proposition 11.4.9, the commutativity of our diagram (see below) will be ensured by a particular choice of topological generator. Let us explain a bit: as we said above, the monodromy actions on the categories $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})$ and $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$ depend on the choice of a topological generator of the profinite group $\varprojlim_{s\geq 0}\mathbb{G}_{\mathrm{m},\mathbb{C}}[n(\mathfrak{o})\ell^s]$, resp. $\varprojlim_{s\geq 0}\mathbb{G}_{\mathrm{m},\mathbb{F}}[n(\mathfrak{o})\ell^s]$. We fixed such a generator for the complex category, setting $(x_s)_{s\geq 0} = (e^{\frac{2i\pi}{n(\mathfrak{o})\ell^s}})_{s\geq 0}$. We claim that we can fix a topological generator of $\varprojlim_{s\geq 0}\mathbb{G}_{\mathrm{m},\mathbb{F}}[n(\mathfrak{o})\ell^s]$ such that diagram (11.4.2) below is commutative. This is a consequence of the following fact: the polynomials $x^{n(\mathfrak{o})\ell^s}-1$ for $s \geq 0$ split in a product of coprime monic polynomials over $\mathbb{F}$ since the latter field is algebraically closed of characteristic coprime to $n(\mathfrak{o})\ell^s$. Since $\mathbb{F}$ is the residue field of the henselian (discrete valuation) ring $\mathfrak{R}$, we can lift this splitting of polynomials in $\mathfrak{R}[x]$. This implies that $\mathfrak{R} \subseteq \mathbb{C}$ contains all the $n(\mathfrak{o})\ell^s$-th roots of unity, for $s \geq 0$, and we have natural identification $\mathbb{G}_{\mathrm{m},\mathfrak{R}}[n(\mathfrak{o})\ell^s] \cong \mathbb{G}_{\mathrm{m},\mathbb{F}}[n(\mathfrak{o})\ell^s]$. In particular, letting $x^{\mathbb{F}}_s$ be the image of $e^{\frac{2i\pi}{n(\mathfrak{o})\ell^s}}$ in $\mathbb{G}_{\mathrm{m},\mathbb{F}}[n(\mathfrak{o})\ell^s]$ for any $s \geq 0$, we obtain a topological generator $(x^{\mathbb{F}}_s)_{s\geq 0}$ of $\varprojlim_{s\geq 0}\mathbb{G}_{\mathrm{m},\mathbb{F}}[n(\mathfrak{o})\ell^s]$. This is the generator we use for defining the monodromy action on $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$, and such a choice clearly makes the diagram (11.4.2) commutative.

Proposition 11.4.13. *The categories* $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})$ *and* $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$ *are equivalent. Moreover, this equivalence commutes with convolution and swaps* $\Delta^{\mathbb{C},\mathrm{et}}_{w,t}$ *and* $\Delta^{\mathbb{F},\mathrm{et}}_{w,t}$ *as well as* $\nabla^{\mathbb{C},\mathrm{et}}_{w,t}$ *and* $\nabla^{\mathbb{F},\mathrm{et}}_{w,t}$ *for any* $w \in W$ *and* $t \in \mathfrak{o}$*. In particular, the objects* $\nabla^{\mathbb{F},\mathrm{et}}_{w,t}$ *belong to the category* $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$ *for any* $w \in W$ *and* $t \in \mathfrak{o}$*. For* $\mathscr{F}^{\mathbb{C}} \in D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})$*, if we let* $\mathscr{F}^{\mathbb{F}}$ *be the object of* $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$ *corresponding to it via the above equivalence, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{End}(\mathscr{F}^{\mathbb{C}}) & \xleftarrow{\quad\sim\quad} & \mathrm{End}(\mathscr{F}^{\mathbb{F}}) \\ & {\scriptstyle\varphi_{\mathscr{F}^{\mathbb{C}}}} \nwarrow \quad \nearrow {\scriptstyle\varphi_{\mathscr{F}^{\mathbb{F}}}} & \\ & \mathbb{K}[\mathrm{X}_*(\mathbf{T})]. & \end{array} \tag{11.4.2}$$

An immediate consequence of proposition 11.4.13 is that we can define a perverse t-structure on the category $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$, and that the equivalence of the proposition is t-exact for the perverse t-structure.

Putting proposition 11.4.9 and proposition 11.4.13 together, we finally get:

Proposition 11.4.14. *We have a perverse t-exact equivalence of categories*

$$D^b_{L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K}) \tilde{\leftrightarrow} D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K}).$$

This equivalence swaps $\Delta_{w,t}$ and $\Delta^{\mathbb{F},\mathrm{et}}_{w,t}$ as well as $\nabla_{w,t}$ and $\nabla^{\mathbb{F},\mathrm{et}}_{w,t}$ for any $w \in W$ and $t \in \mathfrak{o}$ and commutes with convolution. Moreover, it restricts to an equivalence $D^b_{(B)}(\mathscr{X},\mathbb{K})_{[-,t]} \xleftrightarrow{\sim} D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})_{[-,t]}$. For $\mathscr{F} \in D^b_{L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K})$, if we let $\mathscr{F}^{\mathbb{F}}$ be the object of $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})$ corresponding to it via the above equivalence, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}(\mathscr{F}) & \xleftrightarrow{\quad\sim\quad} & \mathrm{End}(\mathscr{F}^{\mathbb{F}}) \\ & {\scriptstyle\varphi_{\mathscr{F}}} \nwarrow \quad \nearrow {\scriptstyle\varphi_{\mathscr{F}^{\mathbb{F}}}} & \\ & \mathbb{K}[\mathrm{X}_*(T)] = \mathbb{K}[\mathrm{X}_*(\mathbf{T})]. & \end{array}$$

11.5 Lusztig–Yun equivariant categories

11.5.1 Tori-equivariant categories

In this section, we detail the construction of a torus-equivariant derived category (meaning, for the action of a split algebraic torus). Though our definition is the one originally introduced in [BL], we follow quite closely [RW, §3.4] for the details.

As above, we make all of our construction on $\mathfrak{R}$ for $A \subseteq \mathfrak{R} \subseteq \mathbb{C}$ or an algebraically closed field. Fix a $\mathfrak{R}$-split torus $\mathbf{H}_{\mathfrak{R}}$, and a $\mathfrak{R}$-scheme X of finite type endowed with an action of $\mathbf{H}_{\mathfrak{R}}$. Finally, let us fix once and for all a trivialisation $\mathbf{H}_{\mathfrak{R}} \cong (\mathbb{G}_{\mathrm{m},\mathfrak{R}})^r$, where r denotes the rank of $\mathbf{H}_{\mathfrak{R}}$.

For any $n \in \mathbb{Z}_{>0}$, set $V_n = \mathbb{A}^n_{\mathfrak{R}} \setminus \{0\}$. We have a canonical map $V_n \to \mathbb{P}^{n-1}_{\mathfrak{R}}$, which is known to be a Zariski locally trivial principal $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-bundle. Taking a product of r copies of V_n, we get a scheme V_n^r endowed with a natural action of $\mathbf{H}_{\mathfrak{R}}$ (we use our trivialisation of $\mathbf{H}_{\mathfrak{R}}$ here). Consider $P_n^X := V_n^r \times X$ with the diagonal action of $\mathbf{H}_{\mathfrak{R}}$. One can construct a scheme $\overline{P}_n^X$ and a natural morphism $q_n^X : P_n^X \to \overline{P}_n^X$ making P_n^X a principal $\mathbf{H}_{\mathfrak{R}}$-bundle: we have an open cover of $\mathbb{P}^{n-1}_{\mathfrak{R}}$ by affine $n-1$-spaces; over such an open, the map $V_n \to \mathbb{P}^{n-1}_{\mathfrak{R}}$ identifies with the projection $\mathbb{A}^{n-1}_{\mathfrak{R}} \times \mathbb{G}_{\mathrm{m},\mathfrak{R}} \to \mathbb{A}^{n-1}_{\mathfrak{R}}$. In fact, this is the well known construction of the projective n-space, by gluing copies of $\mathbb{A}^{n-1}_{\mathfrak{R}}$ along the usual transition functions, defined on nice open subsets $\mathbb{A}^{n-1}_{\mathfrak{R}} \xrightarrow{\sim} \mathbb{A}^{n-1}_{\mathfrak{R}}$. These identifications upgrade to transition functions $(\mathbb{A}^{n-1}_{\mathfrak{R}})^r \times (\mathbb{G}_{\mathrm{m},\mathfrak{R}})^r \times X \xrightarrow{\sim} (\mathbb{A}^{n-1}_{\mathfrak{R}})^r \times \mathbf{H}_{\mathfrak{R}} \times X$; the scheme $\overline{P}_n^X$ is then obtained by gluing, using these new data.

For a given map $Z \xrightarrow{f} X$ of $\mathfrak{R}$-schemes of finite type; for any X-scheme $Y \to X$ of finite type, let $f_Y : Z \times_X Y \to Y$ be the induced map. Consider $n \geq 0$. We say that f is n-acyclic if for any X-scheme Y étale of finite type over X, and any $\mathbb{K}$-sheaf $\mathscr{F}$ on Y, the morphism induced by adjunction

$$\mathscr{F} \to \tau^{\leq n}(f_Y)_* f_Y^* \mathscr{F}$$

is an isomorphism in $D^{b,\mathrm{et}}(Y, \mathbb{K})$ (here, $\tau^{\leq n}$ is the truncation functor for the natural t-structure on $D^{b,\mathrm{et}}(Y, \mathbb{K})$).

Lemma 11.5.1. *The projection $p_n^X : P_n^X := V_n^r \times X \to X$ is $(2n-2)$-acyclic for any $n \geq 2$.*

Proof. We treat first the case $r = 1$. We need to show that for any $\mathbb{K}$-sheaf $\mathscr{F}$ on X, the canonical map $\mathscr{F} \to \tau^{\leq 2n-2}(p_n^X)_*(p_n^X)^* \mathscr{F}$ is an isomorphism. We use the fact that the projection $p : \mathbb{A}^n \times X \to X$ is acyclic (see [Mi, VI, Corollary 4.20]). We have the following commutative diagram

$$\begin{array}{ccccc} \{0\} \times X & \xrightarrow{i} & \mathbb{A}^n \times X & \xleftarrow{j} & (\mathbb{A}^n \setminus \{0\}) \times X \\ & \searrow^{=} & \downarrow{\scriptstyle p} & \swarrow_{p_n^X} & \\ & & X & & \end{array}$$

where i and j denotes the complementary closed and open inclusion, respectively. We apply the distinguished triangle $i_* i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+1}$ to the sheaf $p^* \mathscr{F}$: since p is smooth of relative dimension n, we have $p^* \cong p^![-2n]$. Identifying $\{0\} \times X$ and X via $p \circ i$, we get $i_* \mathscr{F}[-2n] \to p^* \mathscr{F} \to j_*(p_n^X)^* \mathscr{F}$. Finally, apply p_* to this triangle to obtain

$$\mathscr{F}[-2n] \to p_* p^* \mathscr{F} \to (p_n^X)_*(p_n^X)^* \mathscr{F} \xrightarrow{+1} .$$

Applying the truncation functor $\tau^{\leq 2n-2}$ to this triangle, we get an isomorphism

$$\mathscr{F} \cong \tau^{\leq 2n-2} p_* p^* \mathscr{F} \xrightarrow{\sim} \tau^{\leq 2n-2}(p_n^X)_*(p_n^X)^* \mathscr{F},$$

which allows us to conclude in the case $r = 1$. The general case $r \geq 1$ follows from the case $r = 1$, remarking that we can decompose p_n^X as a composition

$$V_n^r \times X \to V_n^{r-1} \times X \to \cdots \to V_r^1 \times X \to X$$

and that a composition of m-acyclic maps is still m-acyclic for $m > 0$. □

For $m > n \geq 2$, the space $P_{n,m}^X := P_n^X \times_X P_m^X = V_n^r \times V_m^r \times X$ is a scheme endowed with a $\mathbf{H}_{\mathfrak{R}}$-action; the map $p_{n,m}^X : P_{n,m}^X \to X$ is again $2n-2$-acyclic and we can as above construct a quotient scheme $q_{n,m}^X : P_{n,m}^X \to \overline{P}_{n,m}^X$.

We now define our $\mathbf{H}_{\mathfrak{R}}$-equivariant derived category on X. For any $n > 0$, set

$$D^{\mathrm{et}}(X, n, \mathbf{H}_{\mathfrak{R}}, \mathbb{K})$$

for the category whose objects are triples $(\mathscr{F}, \overline{\mathscr{F}}, \beta)$ where

- $\mathscr{F}$ is an object in $D^{b,\mathrm{et}}(X,\mathbb{K})$,
- $\overline{\mathscr{F}}$ is an object in $D^{b,\mathrm{et}}(\overline{P}_n^X,\mathbb{K})$,
- β is an isomorphism $(q_n^X)^*\overline{\mathscr{F}} \cong (p_n^X)^*\mathscr{F}$.

A morphism $(\mathscr{F},\overline{\mathscr{F}},\beta) \to (\mathscr{G},\overline{\mathscr{G}},\beta')$ in $D^{\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ is given by a pair $(\varphi,\overline{\varphi})$ of morphisms, with $\varphi : \mathscr{F} \to \mathscr{G}$ in $D^{b,\mathrm{et}}(X,\mathbb{K})$ and $\overline{\varphi} : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ in $D^{b,\mathrm{et}}(\overline{P}_n^X,\mathbb{K})$, satisfying an obvious compatibility condition with respect to the isomorphisms β and β'.

For two integers $n,m \geq 2$, one can exactly as above define a category $D^{\mathrm{et}}(X,n\times m,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$, replacing n in the definition above by the pair (n,m).

For any interval $I \subseteq \mathbb{Z}$, set

$$D^{I,\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$$

for the full subcategory of $D^{\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ whose objects are those triples $(\mathscr{F},\overline{\mathscr{F}},\beta)$ such that $\mathscr{H}^i(\mathscr{F})$ vanishes unless $i \in I$. If n is such that $2n-2$ is bigger than the length of I then $D^{I,\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ does not depend on the choice of n: for two integer n,m such that both $(2n-2)$ and $(2m-2)$ are larger than the length of I, the exact same arguments as in [BL, §2.3.4] show that the categories $D^{I,\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ and $D^{I,\mathrm{et}}(X,m,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ are equivalent. In fact, the pullback functors $D^{I,\mathrm{et}}(X,?,\mathbf{H}_{\mathfrak{R}},\mathbb{K}) \to D^{I,\mathrm{et}}(X,n\times m,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ induced by the natural projections $P_{n,m}^X \to P_?^X$ for $? \in \{n,m\}$ are equivalences of categories.

Now for two bounded intervals $I \subseteq J$, we have a functor

$$D^{I,\mathrm{et}}(X,\mathbf{H}_{\mathfrak{R}},\mathbb{K}) \to D^{J,\mathrm{et}}(X,\mathbf{H}_{\mathfrak{R}},\mathbb{K}),$$

(identifying both as subcategories of $D^{\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ for $n \gg 0$). Thus we can finally set

$$D^{b,\mathrm{et}}_{\mathbf{H}_{\mathfrak{R}}}(X,\mathbb{K}) = \varinjlim_I D^{I,\mathrm{et}}(X,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$$

where the limit is taken over the bounded intervals $I \subseteq \mathbb{Z}$. Note that we have a natural forgetful functor $\mathrm{For}_{\widetilde{\mathbf{H}}_{\mathfrak{R}}} : D^{b,\mathrm{et}}_{\mathbf{H}_{\mathfrak{R}}}(X,\mathbb{K}) \to D^{b,\mathrm{et}}(X,\mathbb{K})$ defined in the following way: an object in the equivariant category is given by a triple $(\mathscr{F},\overline{\mathscr{F}},\beta) \in D^{I,\mathrm{et}}(X,n,\mathbf{H}_{\mathfrak{R}},\mathbb{K})$ for some $n \gg 0$; $\mathrm{For}_{\widetilde{\mathbf{H}}_{\mathfrak{R}}}$ is then defined by $(\mathscr{F},\overline{\mathscr{F}},\beta) \mapsto \mathscr{F}$. Its action on morphisms is obvious.

In the following, we will sometimes consider the equivariant category with $X = \mathscr{X}_{\mathfrak{R}}$ endowed with a twisted action: we will consider the $\mathbf{T}_{\mathfrak{R}}$-equivariance with respect to the natural action of $\mathbf{T}_{\mathfrak{R}}$ twisted by the morphism $e_{n(\mathfrak{o}),\mathfrak{R}}$. We will then denote our equivariant category with a $\widetilde{\mathbf{T}}_{\mathfrak{R}}$ (i.e. $D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$ will denote, for example, the $\mathbf{T}_{\mathfrak{R}}$-equivariant derived category as defined above, where $\mathbf{T}_{\mathfrak{R}}$ acts on $\mathscr{X}_{\mathfrak{R}}$ via $t\cdot x = e_{n(\mathfrak{o}),\mathfrak{R}}(t)x$).

11.5.2 Constructible $\mathbf{T}_{\mathfrak{R}}$-equivariant derived category

In fact, we are not interested in the full category $D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$, but rather in a constructible version of it. We use the notation of subsection 11.4.4 here: let

$$D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}},L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$$

be the full triangulated subcategory of $D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$ whose objects are those $(\mathscr{F},\overline{\mathscr{F}},\beta)$ such that $\mathscr{F}\in D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$.

11.5.3 Lusztig–Yun equivariant category

Let $\overline{\mathbf{T}}_{\mathfrak{R}}\xrightarrow{nu}\mathbf{T}_{\mathfrak{R}}$ be an étale morphism of algebraic $\mathfrak{R}$-tori with finite central kernel K, and assume that we have a character χ of K. In a way similar to the arguments of section 2.1, one can see that the finite group $K(\mathfrak{R})$ acts on any object $\mathscr{F}\in D^{b,\mathrm{et}}_{\overline{\mathbf{T}}_{\mathfrak{R}},L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$, and that this action is functorial, in the sense of lemma 2.1.1. We can thus consider the full subcategory whose objects are those $\mathscr{F}$ for which this action factors through the character χ.

In the particular case where ν is chosen to be $e_{n(\mathfrak{o},\mathfrak{R}}$ so that $K=\ker(e_{n(\mathfrak{o}),\mathfrak{R}})(\mathfrak{R})$, and χ is chosen to be $\chi_{t,\mathfrak{R}}$, we denote the so obtained category as

$$\mathfrak{D}^{\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}/\!\!/\mathbf{T}_{\mathfrak{R}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}]}.$$

Once again, we have a forgetful functor from this category to the constructible category $\mathrm{For}_t^{\mathfrak{R}}:=\mathrm{For}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}[r]$, where $\mathrm{For}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}$ is defined as above. We can define standard and costandard objects in this context too: for any $w\in W$, set

$$\Delta(w)^{\mathfrak{R},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}}:=(j_{w,\mathfrak{R}})_!(\mathscr{L}_t^{w,\mathrm{et}})[\dim(\mathscr{X}_{w,\mathfrak{R}})-r]$$

and

$$\nabla(w)^{\mathfrak{R},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}}:=(j_{w,\mathfrak{R}})_*(\mathscr{L}_t^{w,\mathrm{et}})[\dim(\mathscr{X}_{w,\mathfrak{R}})-r].$$

We then have $\mathrm{For}_t^{\mathfrak{R}}(\Delta(w)^{\mathfrak{R},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}})=\Delta^{\mathfrak{R},\mathrm{et}}_{w,t}$.

Remark 11.5.2. We will call the latter category an (étale) Lusztig–Yun category associated to the local system $\mathscr{L}_t^{\mathbf{T}_{\mathbb{R}},\mathrm{et}}$. However, so far, and contrary to the complex analytic case of chapter 2, we did not prove that this category is independent of the choice made for the isogeny $\widetilde{\mathbf{T}}_{\mathfrak{R}}\to\mathbf{T}_{\mathfrak{R}}$ and for the character (here $\chi_{t,\mathbb{F}}$). Thus it may be more appropriate to speak of $\chi_{t,\mathbb{F}}$-equivariant category; however, to mimic the complex analytic case, which is our primary interest, we stick to the former denomination. We will justify a "well-definiteness" of our category later, at the very end of subsection 11.5.4.

The arguments of section 2.2 show that $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathfrak{R}}/\!\!/\mathbf{T}_{\mathfrak{R}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}]}$ is a direct summand subcategory in the equivariant category $D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}},L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}},\mathbb{K})$. In particular, the Lusztig–Yun category is a full triangulated subcategory of the

full equivariant category. Exactly as in lemma 5.2.2, one can then show that the objects

$$\{\Delta(w)^{\mathfrak{R},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}}[n] \mid w \in W,\, n \in \mathbb{Z}\}$$

belong to $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathfrak{R}} /\!\!/ \mathbf{T}_{\mathfrak{R}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R}},\mathrm{et}}]}$, and in fact, generate this category (as a triangulated category).

11.5.4 Equivalences

We keep the notation introduced so far in section 11.5. We want to show that we have an equivalence of categories

$$\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]} \cong \mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}]}.$$

As a first step, we need once again to define the appropriate objects in the analytic world: let us define the category $D^b_{\widetilde{T},L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K})$ as the full subcategory of $D^b_{\widetilde{T}}(\mathscr{X},\mathbb{K})$ whose objects are those equivariant complexes $\mathscr{F}$ such that $\mathrm{For}_{\widetilde{T}}(\mathscr{F})$ lies in $D^b_{L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K})$.

Remark that $\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]}$ is a full subcategory of $D^b_{\widetilde{T},L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K})$. In fact, we prove first an analogue result considering the whole tori-equivariant categories. The strategy is the same as in the monodromic case; and proceeds in two steps: first we relate the complex-analytic setting to the complex étale case that is, the categories $D^b_{\widetilde{T},L(W,\mathfrak{o})}(\mathscr{X},\mathbb{K})$ and $D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathbb{C}},L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K})$ using essentially theorem 11.2.1; then we relate the étale categories over $\mathbb{C}$ and $\mathbb{F}$, using the general principles of section 11.3.

Analytic-étale

Below, we identify $\mathscr{X}$ with the $\mathbb{C}$-points of $\mathscr{X}_{\mathbb{C}}$, that is, we write $\mathscr{X}_{\mathbb{C}}(\mathbb{C})$ instead of $\mathscr{X}$ (this is just in order to make notation more consistent). The first step in given by direct applications of theorem 11.2.1.

Recall the $\mathbb{C}$-schemes $P_n^{\mathscr{X}_{\mathbb{C}}}$ and $P_{n,m}^{\mathscr{X}_{\mathbb{C}}}$ from subsection 11.5.1. For any $n, m > 2$, these are of finite type over $\mathbb{C}$; moreover, it is clear that the $\mathbb{C}$-points $P_n^{\mathscr{X}_{\mathbb{C}}}(\mathbb{C})$ are given by $P_n^{\mathscr{X}_{\mathbb{C}}(\mathbb{C})} = V_n^r(\mathbb{C}) \times \mathscr{X}_{\mathbb{C}}(\mathbb{C})$. In particular, $P_n^{\mathscr{X}_{\mathbb{C}}}(\mathbb{C}) \xrightarrow{p_n^{\mathscr{X}_{\mathbb{C}}}(\mathbb{C})} \mathscr{X}_{\mathbb{C}}$ is a smooth $T = \mathbf{T}_{\mathbb{C}}(\mathbb{C})$-resolution of $\mathscr{X}_{\mathbb{C}}(\mathbb{C})$, which is $2n-2$-acyclic (similar considerations apply for $P_{n,m}^{\mathscr{X}_{\mathbb{C}}}$). The point there is that the construction of subsection 11.5.1, performed replacing "schemes" with "$\mathbb{C}$-points" everywhere, yields the usual $\widetilde{T}$-equivariant category $D^b_{\widetilde{T}}(\mathscr{X}_{\mathbb{C}}(\mathbb{C}),\mathbb{K})$. Thus what we need to check is (quite roughly) that the functor ε^* of theorem 11.2.1 commutes with any step in the constructions of 11.5.1. This is the object of the following lemma.

Lemma 11.5.3. *We have an equivalence of triangulated categories*

$$D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathbb{C}},L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}},\mathbb{K}) \xrightarrow{\sim} D^b_{\widetilde{T},L(W,\mathfrak{o})}(\mathscr{X}_{\mathbb{C}}(\mathbb{C}),\mathbb{K}).$$

For any $t \in \mathfrak{o}$, this equivalence restricts to an equivalence of categories

$$\mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathbb{C}}/\!\!/\mathbf{T}_{\mathbb{C}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}_{\mathbb{C}}(\mathbb{C})/\!\!/T)_{[-,\mathscr{L}_t^{T}]}, \tag{11.5.1}$$

swapping $\Delta(w)_{\mathscr{L}_t^T}$ and $\Delta(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}$ for any $w \in W$.

In the following proof, we will use the notation of subsection 11.5.1, in the étale and in the analytic settings; though we did not formally introduce any notation for the latter, by similarity, it should be clear what objects and category we consider.

Proof. Let X be any of the schemes $\mathscr{X}_{\mathbb{C}}$, $P_n^{\mathscr{X}_{\mathbb{C}}}$ or $P_{n,m}^{\mathscr{X}_{\mathbb{C}}}$ for $n, m > 2$; then X is of finite type over $\mathbb{C}$, and theorem 11.2.1 gives us an equivalence of categories $\varepsilon_X^* : D_c^{b,\mathrm{et}}(X, \mathbb{K}) \xrightarrow{\sim} D_c^b(X(\mathbb{C}), \mathbb{K})$, where the "$c$" stands for "constructible". Moreover, these equivalences commute in a natural sense to the pullback functors $(q_n^{\mathscr{X}_{\mathbb{C}}})^*$, $(q_{n,m}^{\mathscr{X}_{\mathbb{C}}})^*$ and $(p_n^{\mathscr{X}_{\mathbb{C}}})^*$, $(p_{n,m}^{\mathscr{X}_{\mathbb{C}}})^*$: for example, we have an isomorphism of functors $(\varepsilon^*_{P_n^{\mathscr{X}_{\mathbb{C}}}})(p_n^{\mathscr{X}_{\mathbb{C}}})^* \cong (p_n^{\mathscr{X}_{\mathbb{C}}}(\mathbb{C}))^*(\varepsilon^*_{\mathscr{X}_{\mathbb{C}}})$ from $D_c^{b,\mathrm{et}}(\mathscr{X}_{\mathbb{C}}, \mathbb{K}) \to D_c^b(P_n^{\mathscr{X}_{\mathbb{C}}}(\mathbb{C}), \mathbb{K})$. We readily deduce, for all $n > 2$, an equivalence of categories:

$$D^{\mathrm{et}}(\mathscr{X}_{\mathbb{C}}, n, \widetilde{\mathbf{T}}_{\mathbb{C}}, \mathbb{K}) \xrightarrow{\sim} D(\mathscr{X}_{\mathbb{C}}(\mathbb{C}), n, \widetilde{T}, \mathbb{K}). \tag{11.5.2}$$

Since the functor $\varepsilon^*_{\mathscr{X}_{\mathbb{C}}}$ is exact, for $n \gg 0$ and any bounded interval $I \subseteq \mathbb{Z}$, the equivalence (11.5.2) restricts to an equivalence

$$D^{I,\mathrm{et}}(\mathscr{X}_{\mathbb{C}}, \widetilde{\mathbf{T}}_{\mathbb{C}}, \mathbb{K}) \xrightarrow{\sim} D^I(\mathscr{X}_{\mathbb{C}}(\mathbb{C}), \widetilde{T}, \mathbb{K}).$$

For another bounded interval $I \subseteq J \subseteq \mathbb{Z}$, the latter equivalence clearly commutes with the naturals functors $D^{I,\mathrm{et}}(\mathscr{X}_{\mathbb{C}}, \widetilde{\mathbf{T}}_{\mathbb{C}}, \mathbb{K}) \to D^{J,\mathrm{et}}(\mathscr{X}_{\mathbb{C}}, \widetilde{\mathbf{T}}_{\mathbb{C}}, \mathbb{K})$. By definition, we then obtain an equivalence of triangulated categories

$$D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathbb{C}},c}(\mathscr{X}_{\mathbb{C}}, \mathbb{K}) \xrightarrow{\sim} D^b_{\widetilde{T},c}(\mathscr{X}_{\mathbb{C}}(\mathbb{C}), \mathbb{K}).$$

The t-exactness is a formal consequence of the fact that $\varepsilon^*_{\mathscr{X}_{\mathbb{C}}}$ is perverse t-exact.

Finally, proposition 11.4.9 readily implies that this equivalence restricts to

$$D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathbb{C}},L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}}, \mathbb{K}) \xrightarrow{\sim} D^b_{\widetilde{T},L(W,\mathfrak{o})}(\mathscr{X}_{\mathbb{C}}(\mathbb{C}), \mathbb{K}). \tag{11.5.3}$$

Now for the restriction to the Lusztig–Yun subcategories, we remark that (11.5.3) clearly swaps the objects $\Delta(w)_{\mathscr{L}_t^T}$ and $\Delta(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}$, for any $w \in W$ and $t \in \mathfrak{o}$. Since these objects generate their respective Lusztig–Yun subcategories, (11.5.1) is a consequence of (11.5.3). □

Let us define a perverse t-structure on $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathbb{C}}/\!\!/\mathbf{T}_{\mathbb{C}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}]}$ as the transport along the equivalence (11.5.1) of the perverse t-structure on $\mathfrak{D}(\mathscr{X}_{\mathbb{C}}(\mathbb{C})/\!\!/T)_{[-,\mathscr{L}_t^T]}$. The standard objects $\Delta(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}$ are then perverse for any $t \in \mathfrak{o}$ and $w \in$

W. Arguments similar to those used in subsection 11.4.8 allow us to derive informations on costandard objects, in particular, that $\nabla(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}$ belongs to $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathbb{C}}/\!\!/\mathbf{T}_{\mathbb{C}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}]}$, and are perverse in this category. Once again, one can show that the latter objects generate the whole Lusztig–Yun category, and that moreover the family $\{\Delta(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}[n] \mid t \in \mathfrak{o}, w \in W, n \geq 0\}$ (resp. $\{\nabla(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}[n] \mid t \in \mathfrak{o}, w \in W, n \leq 0\}$) generates the ≤ 0 (resp. ≥ 0) part of the t-structure. The following is then clear:

Corollary 11.5.4. *The equivalence* (11.5.1) *swaps the objects* $\Delta(w)_{\mathscr{L}_t^T}$ *and* $\Delta(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}$ *and* $\nabla(w)_{\mathscr{L}_t^T}$ *and* $\nabla(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}}$ *for any* $w \in W$ *and* $t \in \mathfrak{o}$.

$\mathbb{C}$-étale–$\mathbb{F}$-étale

We want to compare the étale Lusztig–Yun category over $\mathbb{C}$ and over $\mathbb{F}$. As for proposition 11.4.13, one needs to consider a strictly Henselian discrete valuation ring $\mathfrak{R} \subseteq \mathbb{C}$ whose residue field is $\mathbb{F}$. We start by considering the $\widetilde{\mathbf{T}}$-equivariant categories. Consider $n > 0$. The resolutions P_n^X behave nicely with respect to base change: for $\mathfrak{R} \to \mathfrak{R}'$ a ring morphism, the base change to $\mathrm{Spec}(\mathfrak{R}')$ of $P_n^{\mathscr{X}_{\mathfrak{R}}} \to \mathscr{X}_{\mathfrak{R}}$ and $P_n^{\mathscr{X}_{\mathfrak{R}}} \to \overline{P}_n^{\mathscr{X}_{\mathfrak{R}}}$ yields $P_n^{\mathscr{X}_{\mathfrak{R}'}} \to \mathscr{X}_{\mathfrak{R}'}$ and $P_n^{\mathscr{X}_{\mathfrak{R}'}} \to \overline{P}_n^{\mathscr{X}_{\mathfrak{R}'}}$. Similarly, the base change to $\mathfrak{R}'$ of the action morphism $\mathscr{X}_{\mathfrak{R}} \times \widetilde{\mathbf{T}}_{\mathfrak{R}} \to \mathscr{X}_{\mathfrak{R}}$ yields the action morphism $\mathscr{X}_{\mathfrak{R}'} \times \widetilde{\mathbf{T}}_{\mathfrak{R}'} \to \mathscr{X}_{\mathfrak{R}'}$. We obtain equivalences

$$D^{b,\mathrm{et}}(P_n^{\mathscr{X}_{\mathbb{C}}}, \mathbb{K}) \xleftarrow{\sim} D^{b,\mathrm{et}}(P_n^{\mathscr{X}_{\mathfrak{R}}}, \mathbb{K}) \xrightarrow{\sim} D^{b,\mathrm{et}}(P_n^{\mathscr{X}_{\mathbb{F}}}, \mathbb{K}) \tag{11.5.4}$$

and

$$D^{b,\mathrm{et}}(\overline{P}_n^{\mathscr{X}_{\mathbb{C}}}, \mathbb{K}) \xleftarrow{\sim} D^{b,\mathrm{et}}(\overline{P}_n^{\mathscr{X}_{\mathfrak{R}}}, \mathbb{K}) \xrightarrow{\sim} D^{b,\mathrm{et}}(\overline{P}_n^{\mathscr{X}_{\mathbb{F}}}, \mathbb{K}). \tag{11.5.5}$$

Now, the equivalences (11.5.4) and (11.5.5) commute with the various pullback functors constructed from the projection and quotient morphisms $p_n^{\mathscr{X}}$ and $q_n^{\mathscr{X}}$. It follows from this fact that we have equivalences of categories

$$D^{\mathrm{et}}(\mathscr{X}_{\mathbb{C}}, n, \mathbf{T}_{\mathbb{C}}, \mathbb{K}) \xleftarrow{\sim} D^{\mathrm{et}}(\mathscr{X}_{\mathfrak{R}}, n, \mathbf{T}_{\mathfrak{R}}, \mathbb{K}) \xrightarrow{\sim} D^{\mathrm{et}}(\mathscr{X}_{\mathbb{F}}, n, \mathbf{T}_{\mathbb{F}}, \mathbb{K}),$$

and one can then deduce equivalences

$$D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathbb{C}}}(\mathscr{X}_{\mathbb{C}}, \mathbb{K}) \xleftarrow{\sim} D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K}) \xrightarrow{\sim} D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathbb{F}}}(\mathscr{X}_{\mathbb{F}}, \mathbb{K}). \tag{11.5.6}$$

To derive the Lusztig–Yun case from this one, we remark that the base change along a ring morphism $\mathfrak{R} \to \mathfrak{R}'$ of $e_{n(\mathfrak{o}),\mathfrak{R}}$, $\ker(e_{n(\mathfrak{o}),\mathfrak{R}})$ and $\chi_{t,\mathfrak{R}}$ yields the corresponding objects on $\mathfrak{R}'$.

From the definition of the action of $\ker(e_{a,\mathfrak{R}})$ on $D^{b,\mathrm{et}}_{\widetilde{\mathbf{T}}_{\mathfrak{R}}}(\mathscr{X}_{\mathfrak{R}}, \mathbb{K})$, we finally get that the equivalences (11.5.6) commute with the action of this kernel. We can then conclude that there are equivalences of categories

$$\begin{aligned}\mathfrak{D}^{\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{C})}(\mathscr{X}_{\mathbb{C}}/\!\!/\mathbf{T}_{\mathbb{C}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{C},\mathrm{et}}}]} &\xleftarrow{\sim} \mathfrak{D}^{\mathrm{et}}_{L(W,\mathfrak{o},\mathfrak{R})}(\mathscr{X}_{\mathfrak{R}}/\!\!/\mathbf{T}_{\mathfrak{R}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathfrak{R},\mathrm{et}}}]} \\ &\xrightarrow{\sim} \mathfrak{D}^{\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F},\mathrm{et}}}]}. \end{aligned} \tag{11.5.7}$$

Quite similarly as in section 11.4.8, and as in the end of the analytic-to-étale case treated above, one can show that the costandard objects $\nabla(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$ belong to the category $\mathfrak{D}^{\mathrm{et}}_{L(W,\mathtt{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}]}$, and that the equivalences (11.5.7) swap $\nabla(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C}},\mathrm{et}}}$ and $\nabla(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$ as well as $\Delta(w)^{\mathbb{C},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{C}},\mathrm{et}}}$ and $\Delta(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$. This implies in particular that these equivalences preserve the perverse t-structures on both sides.

Finally, as in the monodromic case, combining lemma 11.5.3 and the equivalences (11.5.6), we get the following:

Proposition 11.5.5. *We have a perverse t-exact equivalence of triangulated categories*

$$\mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} \cong \mathfrak{D}^{\mathrm{et}}_{L(W,\mathtt{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}]}.$$

This equivalence swaps the objects $\Delta(w)_{\mathscr{L}_t^T}$ and $\Delta(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$ as well as the objects $\nabla(w)_{\mathscr{L}_t^T}$ and $\nabla(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$ for any $w \in W$.

Remark 11.5.6. The equivalence of proposition 11.5.5 restricts to an equivalence between perverse subcategories, so we deduce that the heart $\mathfrak{P}^{\mathrm{et}}_{L(W,\mathtt{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}]}$ admits a highest weight structure with weight poset $(W,\leq)$, standard objects given by the Δ's and costandard objects given by the ∇'s. In particular, we can define tilting objects, and our equivalence clearly restricts to tilting subcategories. If we let $\mathscr{T}(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$ be the indecomposable tilting object associated to the element $w \in W$, then our equivalence maps $\mathscr{T}(w)_{\mathscr{L}_t^T}$ to this $\mathscr{T}(w)^{\mathbb{F},\mathrm{et}}_{\mathscr{L}^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}}$ for any $w \in W$.

Finally, let us note that the general results from [BBD, §6.1.8, 6.1.10] imply that the monodromic equivalence commutes with convolution; the same kind of results then also implies that that our equivalences intertwine the projection functors from the monodromic to the Lusztig–Yun categories, i.e. we have a commutative diagram of categories and functors

$$\begin{array}{ccc} D^b_{(B)}(\mathscr{X},\mathbb{K})_{[-,t]} & \xleftarrow{\ \sim\ } & D^{b,\mathrm{et}}_{L(W,\mathtt{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}},\mathbb{K})_{[-,t]} \\ \Big\downarrow{\scriptstyle \pi^t_\dagger} & & \Big\downarrow{\scriptstyle \pi^{t,\mathrm{et}}_\dagger} \\ \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} & \xleftarrow{\ \sim\ } & \mathfrak{D}^{\mathrm{et}}_{L(W,\mathtt{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}]}. \end{array} \tag{11.5.8}$$

(The top and bottom horizontal arrows are respectively the equivalences of proposition 11.4.14 and proposition 11.5.5.)

We conclude this chapter by an observation. Assume that you have a finite central isogeny $\overline{T} \xrightarrow{\nu} T$ from a complex torus $\overline{T}$ to T; let K be its kernel. Assume moreover that there exists a character $\chi : K \to \mathbb{K}^*$ of K such that $\mathscr{L}_t^T$ identifies with the χ-isotypic component $\nu_*\underline{\mathbb{K}}_{\overline{T}}[\chi]$. from these data, we can construct the (usual, complex analytic) Lusztig–Yun categoy, as explained in chapter 2.

Now, it follows from [SGA4, Exposé 11, Théorème 4.3] that the above data define a finite étale covering $\overline{\mathbf{T}}_{\mathbb{C}} \xrightarrow{\nu_{\mathbb{C}}} \mathbf{T}_{\mathbb{C}}$ between $\mathbb{C}$-schemes. The scheme $\overline{\mathbf{T}}_{\mathbb{C}}$ is a complex group scheme, in fact, a torus, and the kernel $K_{\mathbb{C}}$ of $\nu_{\mathbb{C}}$ identifies with K (remarking that the group K identifies with the automorphism group of ν in the category of finite étale covering of T, this follows from *loc. cit.*). In particular, χ defines a character $\chi_{\mathbb{C}}$ of $K_{\mathbb{C}}$. We can consider a the $\chi_{\mathbb{C}}$-equivariant category. Once again, our general results from [BBD] tell us that there exist analogous objects over $\mathfrak{R}$ and $\mathbb{F}$, and that all the resulting categories are equivalent. In particular, and with obvious notation, we obtain an equivalence

$$\mathfrak{D}(\mathscr{X} /\!\!/ T)_{[-,\mathscr{L}_t^T]} \xleftrightarrow{\sim} D^{b,\mathrm{et}}_{\overline{\mathbf{T}}_{\mathbb{F}},L(W,\mathfrak{o},\mathbb{F}),\chi_{\mathbb{F}}}(\mathscr{X}_{\mathbb{F}},\mathbb{K}).$$

Taking into account proposition 11.5.5, we get an equivalence of categories

$$\mathfrak{D}^{\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T}_{\mathbb{F}})_{[-,\mathscr{L}_t^{\mathbf{T}_{\mathbb{F}},\mathrm{et}}]} \cong D^{b,\mathrm{et}}_{\overline{\mathbf{T}}_{\mathbb{F}},L(W,\mathfrak{o},\mathbb{F}),\chi_{\mathbb{F}}}(\mathscr{X}_{\mathbb{F}},\mathbb{K}).$$

Thus, in case of tori-equivariant categories on $\mathscr{X}_{\mathbb{F}}$, the Lusztig–Yun category does not depend on the choices one can make for the isogeny ν and the character χ. In particular, later in chapter 12, we will have to perform the Lusztig–Yun constructions for some isogeny $\overline{\mathbf{T}}_{\mathbb{F}} \to \mathbf{T}_{\mathbb{F}}$ which is not a priori given by some n-th power map. The above discussion ensures that we will still obtain a category canonically equivalent to the one constructed in subsection 11.5.3.

Chapter 12

Étale setting

In this chapter, we keep the setting of chapter 11: schemes over algebraically closed fields endowed with the étale topology. **From now on and until the end of the chapter, unless stated explicitly otherwise, we assume that our varieties are defined over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$.** Let also $\mathbb{K}$ be the finite field of characteristic ℓ constructed in subsection 11.4.1. We then have $\gcd(\ell, p) = 1$.

12.1 Coefficients

The theory of étale sheaves is well-behaved only for finite rings of coefficients, however, we would like to consider étale categories with an algebraically closed field of coefficients. In order to do so, we mimic the construction of constructible ℓ-adic complexes.

Let X be any $\mathbb{F}$-scheme of finite type. By assumption, we have $\overline{\mathbb{K}} \cong \Bbbk$, our fixed algebraic closure of $\mathbb{K}$. To emphasize the fact that we consider algebraically closed coefficients (which is *non-standard* for étale sheaves) we keep the notation $\overline{\mathbb{K}}$ in this chapter. This field is the inductive limit of its finite subfields, for such a subfield $\mathbb{K}'$, we set $D_c^{b,\mathrm{et}}(X, \mathbb{K}')$ for the usual constructible derived category of sheaves of $\mathbb{K}'$-vector spaces on X. If $\mathbb{K}' \subseteq \mathbb{K}''$ is a field extension between two finite subfields of $\overline{\mathbb{K}}$, we have a natural t-exact functor of extension of scalars

$$\mathbb{K}'' \otimes_{\mathbb{K}'} (-) : D_c^{b,\mathrm{et}}(X, \mathbb{K}') \to D_c^{b,\mathrm{et}}(X, \mathbb{K}'').$$

The category $D_c^{b,\mathrm{et}}(X, \overline{\mathbb{K}})$ is then defined as the inductive limit of the categories $D_c^{b,\mathrm{et}}(X, \mathbb{K}')$, where the limit is taken over the finite subfields of $\overline{\mathbb{K}}$:

Definition 12.1.1. *Set $D_c^{b,\mathrm{et}}(X, \overline{\mathbb{K}})$ for the category defined as follows:*

1. *the objects are given by pairs $\mathscr{F} = (\mathbb{K}', \mathscr{F}_{\mathbb{K}'})$ where $\mathbb{K}'$ is a finite subfield of $\overline{\mathbb{K}}$ and $\mathscr{F}_{\mathbb{K}'}$ is an object in $D_c^{b,\mathrm{et}}(X, \mathbb{K}')$;*

2. *the morphisms are defined as follows: for* $\mathscr{F} = (\mathbb{K}', \mathscr{F}_{\mathbb{K}'})$ *and* $\mathscr{G} = (\mathbb{K}'', \mathscr{G}_{\mathbb{K}''})$ *two objects, set*

$$\mathrm{Hom}_{D_c^{b,\mathrm{et}}(X,\overline{\mathbb{K}})}(\mathscr{F},\mathscr{G}) := \varinjlim_{\mathbb{K}',\mathbb{K}''\subseteq\mathbb{L}} \mathrm{Hom}_{D_c^{b,\mathrm{et}}(X,\mathbb{L})}(\mathbb{L}\otimes_{\mathbb{K}'}\mathscr{F}_{\mathbb{K}'}, \mathbb{L}\otimes_{\mathbb{K}''}\mathscr{G}_{\mathbb{K}''})$$

where the limit is taken over the finite subfields $\mathbb{L}$ *of* $\overline{\mathbb{K}}$ *containing both* $\mathbb{K}'$ *and* $\mathbb{K}''$.

We will call this category the constructible derived category of $\overline{\mathbb{K}}$*-étale sheaves on* X.

Remark 12.1.2. 1. The Hom-spaces in $D_c^{b,\mathrm{et}}(X,\overline{\mathbb{K}})$ are naturally $\overline{\mathbb{K}}$ vector spaces. If $\mathscr{F} = (\mathbb{K}', \mathscr{F}_{\mathbb{K}'})$ and $\mathscr{G} = (\mathbb{K}'', \mathscr{G}_{\mathbb{K}''})$ are as in the definition, the inductive limit appearing in the Hom-space between $\mathscr{F}$ and $\mathscr{G}$ is superfluous: it suffices to find a finite subfield $\mathbb{L}$ of $\overline{\mathbb{K}}$ containing both $\mathbb{K}'$ and $\mathbb{K}''$ and we have

$$\mathrm{Hom}_{D_c^{b,\mathrm{et}}(X,\overline{\mathbb{K}})}(\mathscr{F},\mathscr{G}) = \left(\mathrm{Hom}_{D_c^{b,\mathrm{et}}(X,\mathbb{L})}(\mathbb{L}\otimes_{\mathbb{K}'}\mathscr{F}_{\mathbb{K}'}, \mathbb{L}\otimes_{\mathbb{K}''}\mathscr{G}_{\mathbb{K}''})\right)\otimes_{\mathbb{L}}\overline{\mathbb{K}}.$$

2. Note that the description of morphisms implies the following: $\mathscr{F}$ is identified with $(\mathbb{K}'', \mathbb{K}''\otimes_{\mathbb{K}'}\mathscr{F}_{\mathbb{K}'})$ for any finite subfield $\mathbb{K}''$ of $\overline{\mathbb{K}}$ such that $\mathbb{K}'\subseteq\mathbb{K}''$.

3. We will also consider equivariant categories below; we also have functors of extension of scalars on equivariant categories, and these commute in a natural sense with forgetful functors. We can then define, exactly as above, equivariant derived categories of étale sheaves, with algebraically closed coefficients.

4. In what follows, we will use the somehow abusive denomination of étale $\overline{\mathbb{K}}$-sheaf, étale $\overline{\mathbb{K}}$-local system, étale $\overline{\mathbb{K}}$-equivariant sheaf: this is to be understood as objects in the corresponding categories with algebraically closed coefficients as defined above.

The functors of extension of scalars also commute with all usual six operations $f^!$, f^*, $f_!$, f_*, $\otimes$ and $\mathcal{H}om$; one gets that these six operations will preserve (in an obvious sense) our $\overline{\mathbb{K}}$-étale categories.

We will denote objects in $D_c^{b,\mathrm{et}}(X,\overline{\mathbb{K}})$ as usual complexes (that is to say, with letters like $\mathscr{F},\mathscr{G}...$) and forget about the notion of pairs introduced in the above definition. Finally, if we have a stratification $\mathcal{S}$ on X, we can define the $\mathcal{S}$-constructible category $D_{\mathcal{S}}^{b,\mathrm{et}}(X,\overline{\mathbb{K}})$ as the full subcategory of $D_c^{b,\mathrm{et}}(X,\overline{\mathbb{K}})$ whose objects are given by the complexes $\mathscr{F}$ whose cohomology sheaves are constructible with respect to $\mathcal{S}$.

12.2 Notation

We will use similar notation to that of the previous chapter: we can find a split connected reductive group $\mathbf{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ and choose a Borel subgroup $\mathbf{B}_{\mathbb{Z}}$ and a maximal torus $\mathbf{T}_{\mathbb{Z}} \subseteq \mathbf{B}_{\mathbb{Z}}$ of $\mathbf{G}_{\mathbb{Z}}$ such that extension of scalars to $\mathbb{C}$ yields the triplet $T \subseteq B \subseteq G$. Let $\mathbf{U}_{\mathbb{Z}}$ be the unipotent radical of $\mathbf{B}_{\mathbb{Z}}$. We consider the base change to $\mathbb{F}$ of these groups, we obtain a connected reductive algebraic group $\mathbf{G}$ over $\mathbb{F}$, a Borel subgroup $\mathbf{B}$ and a maximal torus $\mathbf{T}$ contained in $\mathbf{B}$. We also have $\mathbf{U}$, the unipotent radical of $\mathbf{B}$; W denote the Weyl group of the pair $(\mathbf{G}, \mathbf{T})$. We will consider the root datum $(\mathrm{X}^*(\mathbf{T}), \Phi, \mathrm{X}_*(\mathbf{T}), \Phi^\vee)$ of the reductive group $\mathbf{G}$, and the dual $\overline{\mathbb{K}}$-torus $\mathbf{T}^\vee_{\overline{\mathbb{K}}}$, defined to be the $\overline{\mathbb{K}}$-torus with characters given by $\mathrm{X}_*(\mathbf{T})$ and cocharacters given by $\mathrm{X}^*(\mathbf{T})$. Note that, as in remark 3.3.1, any element in $\mathbf{T}^\vee_{\overline{\mathbb{K}}}$ is defined over some finite subfield of $\overline{\mathbb{K}}$. We will also consider the Borel subgroup $\mathbf{B}^-$, opposite to $\mathbf{B}$ with respect to $\mathbf{T}$.

We set $\mathscr{X}_{\mathbb{F}} := \mathbf{G}/\mathbf{U}$, this scheme admits a natural stratification

$$\mathscr{X}_{\mathbb{F}} = \bigsqcup_{w \in W} \mathscr{X}_{w,\mathbb{F}}$$

induced by the Bruhat decomposition of $\mathbf{G}$; for $w \in W$, we let $j_{w,\mathbb{F}} : \mathscr{X}_{w,\mathbb{F}} \hookrightarrow \mathscr{X}_{\mathbb{F}}$ be the inclusion map. Set $D^{b,\mathrm{et}}_{(\mathbf{B})}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ for the constructible (with respect to this stratification) derived category of étale $\overline{\mathbb{K}}$-sheaves on $\mathscr{X}_{\mathbb{F}}$.

We consider a W-orbit $\mathfrak{o}$ in $\mathbf{T}^\vee_{\overline{\mathbb{K}}}$; all the elements of $\mathfrak{o}$ are defined over some finite subfield $\mathbb{K}'$ of $\overline{\mathbb{K}}$ and share a common finite order $n(\mathfrak{o})$, coprime to ℓ. Then, exactly as in subsection 11.4.3, we can construct an object $\mathscr{L}_t^{\mathbf{T},\mathrm{et},\mathbb{K}'}$ over any finite subfield $\mathbb{K}' \subseteq \overline{\mathbb{K}}$, containing a primitive $n(\mathfrak{o})$-th root of unity. Moreover, we clearly have $\mathbb{K}'' \otimes_{\mathbb{K}'} \mathscr{L}_t^{\mathbf{T},\mathrm{et},\mathbb{K}'} \cong \mathscr{L}_t^{\mathbf{T},\mathrm{et},\mathbb{K}''}$ for any extension $\mathbb{K}' \subseteq \mathbb{K}''$ of finite subfields in $\overline{\mathbb{K}}$. Thus the collection of the $\{\mathscr{L}_t^{\mathbf{T},\mathrm{et},\mathbb{K}'}\}_{\mathbb{K}'}$ defines an object in the category of étale $\overline{\mathbb{K}}$-local systems on $\mathbf{T}$, which we denote $\mathscr{L}_t^{\mathbf{T},\mathrm{et}}$.

Starting from the étale local systems $\mathscr{L}_t^{\mathbf{T},\mathrm{et}}$ on $\mathbf{T}$ for any $t \in \mathfrak{o}$, we obtain $\overline{\mathbb{K}}$-local systems $\mathscr{L}_t^{w,\mathrm{et}}$ on $\mathscr{X}_{w,\mathbb{F}}$ for any $w \in W$, as in subsection 11.4.4.

12.3 Monodromy and Lusztig–Yun category

As in subsection 11.4.4, we consider $D^{b,\mathrm{et}}_{L(W,\mathfrak{o},\mathbb{F})}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ for the full subcategory of $D^{b,\mathrm{et}}_c(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ whose objects are those $\mathscr{F}$ for which the restriction of the cohomology objects to any $\mathscr{X}_{w,\mathbb{F}}$ are finite successive extensions of the local systems $\mathscr{L}_t^{w,\mathrm{et}}$ for $t \in \mathfrak{o}$. In this chapter, we abbreviate the notation for this category by $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$, where the $\mathcal{M}$ stands for "monodromic". Let us detail a bit the monodromy action on $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$. We showed in subsection 11.4.6 that it is possible to define a canonical monodromy action on $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \mathbb{K}')$, where $\mathbb{K}'$ is finite. For an object $\mathscr{F} = (\mathbb{K}', \mathscr{F}_{\mathbb{K}'}) \in D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$, we then have left and right monodromy morphisms

$$\mathbb{K}'[\mathrm{X}_*(\mathbf{T})] \longrightarrow \mathrm{End}_{D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \mathbb{K}')}(\mathscr{F}_{\mathbb{K}'}).$$

Extending the scalars to $\overline{\mathbb{K}}$, the above monodromy morphisms induce morphisms

$$\overline{\mathbb{K}}[\mathrm{X}_*(\mathbf{T})] \longrightarrow \mathrm{End}_{D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})}(\mathscr{F}).$$

This is the monodromy action we will consider on the category $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})$.

We let $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ for the full subcategory of $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})$ whose objects are those complexes $\mathscr{F}$ with generalized right monodromy given by t. This coincides with the full subcategory with objects the complexes $\mathscr{F}$ for which the restriction of the cohomology objects to any $\mathscr{X}_{w,\mathbb{F}}$ are finite successive extensions of the local system $\mathscr{L}^{w,\mathrm{et}}_t$ (with t fixed). Note that for an object in this category, the right monodromy map factors through the completion

$$\widehat{R}_t \longrightarrow \mathrm{End}_{D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})}(\mathscr{F});$$

recall that

$$\widehat{R}_t := \varprojlim_n \overline{\mathbb{K}}[\mathrm{X}_*(\mathbf{T})]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(\mathbf{T})\rangle^n$$

(we have $\Bbbk \cong \overline{\mathbb{K}}$ by definition).

The monodromic category splits as a direct sum of subcategories with monodromy $t \in \mathfrak{o}$:

$$D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}}) = \bigoplus_{t\in\mathfrak{o}} D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}.$$

One can replace $\mathscr{X}_{\mathbb{F}}$ by any locally closed subscheme $Z \subseteq \mathscr{X}_{\mathbb{F}}$ union of strata to obtain a monodromic category $D^{b,\mathrm{et}}_{\mathcal{M}}(Z,\overline{\mathbb{K}})_{[-,t]}$. We will also consider left monodromy, and hence categories of the form $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[t',t]}$ for $t,t' \in \mathfrak{o}$. This is defined as the full subcategory of $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ whose objects have generalized left monodromy t'.

The arguments of subsections 11.4.6 and 11.5.3 allow us to also define a Lusztig–Yun $\mathscr{L}^{\mathbf{T},\mathrm{et}}_t$-equivariant category for any $t \in \mathfrak{o}$. Recall that we can fix a finite subfield $\mathbb{K}'$ of $\overline{\mathbb{K}}$ and consider that $\mathscr{L}^{\mathbf{T},\mathrm{et}}_t$ is represented by the $\mathbb{K}'$-étale local system $\mathscr{L}^{\mathbf{T},\mathrm{et},\mathbb{K}'}_t$. Then, there exists a finite central isogeny $\widetilde{\mathbf{T}} \xrightarrow{\nu} \mathbf{T}$ (given by the power $n(\mathfrak{o})$ map) and a character χ_t of the kernel of this isogeny as in subsection 11.4.6; the local system $\mathscr{L}^{\mathbf{T},\mathrm{et},\mathbb{K}'}_t$ identifies with the χ_t-isotypic component of the pushforward $\nu_*\underline{\mathbb{K}'}_{\widetilde{\mathbf{T}}}$. Moreover, note that the above description is valid for any finite subfield $\mathbb{K}'$. Thus we in fact have

$$\mathscr{L}^{\mathbf{T},\mathrm{et}}_t \cong \nu_*\underline{\overline{\mathbb{K}}}_{\widetilde{\mathbf{T}}}[\chi_t],$$

where $\underline{\overline{\mathbb{K}}}_{\widetilde{T}}$ denote the $\overline{\mathbb{K}}$-local system on $\widetilde{\mathbf{T}}$ represented by the family $\{\underline{\mathbb{K}'}_{\mathbf{T}}\}_{\mathbb{K}'\subseteq\overline{\mathbb{K}}}$.

We will consider the category

$$\mathfrak{D}^{\mathrm{et}}_{(\mathbf{B})}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_t]} := D^{b,\mathrm{et}}_{(\mathbf{B}),\widetilde{\mathbf{T}},\chi_t}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}});$$

this is a full subcategory of the usual $\widetilde{\mathbf{T}}$-equivariant category $D^{b,\mathrm{et}}_{(\mathbf{B}),\widetilde{\mathbf{T}}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})$; once again we have a natural forgetful functor

$$\mathfrak{D}^{\mathrm{et}}_{(\mathbf{B})}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_t]} \to D^{b,\mathrm{et}}_{(\mathbf{B})}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}}).$$

We set $\Delta(e)^{\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} := (j_{e,\mathbb{F}})_! \mathscr{L}_t^{e,\mathrm{et}}[r]$, an object in the Lusztig–Yun étale category.

We will also consider the category $\mathfrak{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ defined to be the full subcategory of $\mathfrak{D}^{\mathrm{et}}_{(\mathbf{B})}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ whose objects are the complexes $\mathscr{F}$ whose image under the above forgetful functor is in $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$.

The various convolution products introduced in earlier chapters have analogues in this étale setting: we have a convolution $(-)\star^{\mathrm{LY}}(-)$ on the Lusztig–Yun category and a "$\mathbf{U}$-equivariant" convolution product $(-)\star^{\mathbf{U}}(-)$ as in chapter 6, note that the $\mathbf{U}$-equivariant convolution product defined in subsection 11.4.7 coincides with this $(-)\star^{\mathbf{U}}(-)$.

Remark 12.3.1. As explained in the very end of subsection 11.5.4, once we know that we can define a $\mathbf{T}$-equivariant Lusztig–Yun category as above, then this category actually does not depend on the isogeny chosen nor on the character of its kernel. In fact, later, in section 12.8, we will implicitly consider a Lusztig–Yun category constructed from an isogeny different from the $n(\mathfrak{o})$-th power map. This will then yield a category canonically equivalent to the one constructed in the current section; and thus we do not (and will not) need to specify what isogeny we consider in order to define the category $\mathfrak{D}^{\mathrm{et}}_{(\mathbf{B})}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$.

12.4 An analytic interlude

Later on, we will need to compare our current étale setting to our previous analytic framework. There is a problem, a priori: our étale monodromic category with algebraically closed coefficient is not defined exactly as our analytic monodromic category (on the analytic side, no inductive limit of categories is required). It turns out that this is not an issue.

Let us define an analytic monodromic category the way we did in the étale setting above: for $\mathbb{K}' \subseteq \overline{\mathbb{K}}$ a finite subfield, let $D^b_{\mathcal{M}}(\mathscr{X}, \mathbb{K}')_{[-,t]}$ be the full subcategory of $D^b_{(B)}(\mathscr{X}, \mathbb{K}')$ whose objects are those complexes $\mathscr{F}$ such that $\mathscr{H}^i(\mathscr{F})_{|\mathscr{X}_w}$ is a successive finite extension of $\mathscr{L}_t^w$, for any $i \in \mathbb{Z}$ and any $w \in W$.

Then, let $D^b_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}$ be the triangulated category whose objects are pairs $(\mathbb{K}', \mathscr{F}_{\mathbb{K}'})$, with $\mathbb{K}'$ a finite subfield of $\overline{\mathbb{K}}$ and $\mathscr{F}_{\mathbb{K}'}$ an object in $D^b_{\mathcal{M}}(\mathscr{X}, \mathbb{K}')_{[-,t]}$. The morphisms in this category are described as in point 2. of definition 12.1.1.

Completely similarly, we can define a Lusztig–Yun category following the same pattern, denoted $\mathfrak{D}_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,\mathscr{L}_t^T]}$, we have a projection functor

$$\pi_\dagger^{t,\mathcal{M}} : D^b_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]} \to \mathfrak{D}_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,\mathscr{L}_t^T]}.$$

Once again, we can define a completed category with these data, denoted $\widehat{D}_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}$.

Lemma 12.4.1. *We have an equivalence of categories*

$$D^b_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]} \xrightarrow{\sim} D^b_{(B)}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]};$$

this equivalence commutes with convolution.

Proof. For any finite subfield $\mathbb{K}' \subseteq \overline{\mathbb{K}}$, we have standard and costandard objects $\Delta^{\mathbb{K}'}_{w,t}$ and $\nabla^{\mathbb{K}'}_{w,t}$ (with $w \in W$), moreover, these objects generate the category $D^b_{\mathcal{M}}(\mathscr{X}, \mathbb{K}')_{[-,t]}$ as a triangulated category. For an extension of finite subfields $\mathbb{K}' \subseteq \mathbb{K}''$ in $\overline{\mathbb{K}}$, we clearly have

$$\mathbb{K}'' \otimes_{\mathbb{K}'} \Delta^{\mathbb{K}'}_{w,t} \cong \Delta^{\mathbb{K}''}_{w,t}$$

and similarly for costandard. We deduce that the category $D^b_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}$ is generated by the objects $\Delta^{\overline{\mathbb{K}}}_{w,t}$ and $\nabla^{\overline{\mathbb{K}}}_{w,t}$ for $w \in W$, where $\Delta^{\overline{\mathbb{K}}}_{w,t}$ is the object represented by any of the object $\Delta^{\mathbb{K}'}_{w,t}$ with $\mathbb{K}' \subseteq \overline{\mathbb{K}}$ finite. On the other hand, the objects $\Delta_{w,t}$ and $\nabla_{w,t}$ of subsection 5.2.1 generate the category $D^b_{(B)}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}$.

Now, we have a natural triangulated functor

$$D^b_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]} \to D^b_{(B)}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]},$$

defined by $\mathscr{F} = (\mathbb{K}', \mathscr{F}_{\mathbb{K}'}) \mapsto \overline{\mathbb{K}} \otimes_{\mathbb{K}'} \mathscr{F}$ on objects, and with obvious action on morphisms. This functor maps $\Delta^{\overline{\mathbb{K}}}_{w,t}$ to $\Delta_{w,t}$ and $\nabla^{\overline{\mathbb{K}}}_{w,t}$ to $\nabla_{w,t}$ for any $w \in W$. We will evaluate morphisms from Δ's to ∇'s in our two categories; in both cases, using adjunction, it suffices to consider the morphisms between $\Delta^{?}_{w,t}$ and $\nabla^{?}_{w,t}$ (? being either $\overline{\mathbb{K}}$ or $\emptyset$): in any other case, the morphism-space will be zero. We have the following sequence of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D^b_{\mathcal{M}}(\mathscr{X},\overline{\mathbb{K}})_{[-,t]}}(\Delta^{\overline{\mathbb{K}}}_{w,t}, \nabla^{\overline{\mathbb{K}}}_{w,t}) &= \mathrm{Hom}_{D^b_{\mathcal{M}}(\mathscr{X},\mathbb{K}')_{[-,t]}}(\Delta^{\mathbb{K}'}_{w,t}, \nabla^{\mathbb{K}'}_{w,t}) \otimes_{\mathbb{K}'} \overline{\mathbb{K}} \\
&\cong \mathrm{Hom}_{D^b_{\mathcal{M}}(\mathscr{X}_w,\mathbb{K}')_{[-,t]}}(\mathscr{L}^{w,\mathbb{K}'}_t, \mathscr{L}^{w,\mathbb{K}'}_t) \otimes_{\mathbb{K}'} \overline{\mathbb{K}} \\
&\cong \mathbf{H}^\bullet(T, \mathbb{K}') \otimes_{\mathbb{K}'} \overline{\mathbb{K}} \\
&\cong \mathbf{H}^\bullet(T, \overline{\mathbb{K}}) \\
&\cong \mathrm{Hom}_{D^b_{(B)}(\mathscr{X},\overline{\mathbb{K}})_{[-,t]}}(\Delta_{w,t}, \nabla_{w,t}).
\end{aligned}$$

Here, the second and last isomorphisms follow from the fact that $\mathscr{L}^T_t$ is invertible on T, and the isomorphism

$$\mathrm{Hom}(\mathscr{L}^w_t, \mathscr{L}^w_t) = \mathrm{Hom}(p^*_w \mathscr{L}^T_t, p^*_w \mathscr{L}^T_t) \cong \mathrm{Hom}(\mathscr{L}^T_t, \mathscr{L}^T_t),$$

(see subsection 5.2.1 for the notation). We deduce first that our functor is fully faithful, and then essentially surjective, that is to say, an equivalence of categories.

The commutation with convolution is immediate from the definition of our equivalence and the commutation of the extension of scalars functor $\overline{\mathbb{K}} \otimes (-)$ with all six usual operations. □

The same kind of arguments gives us an equivalence of categories

$$\mathfrak{D}_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}, \overline{\mathbb{K}})_{[-,\mathscr{L}^T_t]}, \tag{12.4.1}$$

(using obvious notation).

Moreover, the equivalences of lemma 12.4.1 and (12.4.1) intertwine the projection functors $\pi_\dagger^t$ on the usual and monodromic categories.

We finally deduce the following result:

Lemma 12.4.2. *We have a monoidal equivalence of categories*

$$(\widehat{D}_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}, \widehat{\star}) \xrightarrow{\sim} (\widehat{D}_{[-,t]}, \widehat{\star}),$$

swapping the pro-standard objects associated to w on both sides, and swapping the pro-costandard objects associated to w on both sides, for any $w \in W$.

This equivalence restricts to a a monoidal equivalence

$$(\widehat{T}_{\mathcal{M}}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}, \widehat{\star}) \xrightarrow{\sim} (\widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]}, \widehat{\star})$$

between the tilting subcategories, preserves block subcategories, and swaps $\widehat{\mathscr{T}}^{\mathrm{et}}_{w,t}$ and $\widehat{\mathscr{T}}^{\mathrm{et}}_{w,t}$ for any $w \in W$.

Proof. The lemma follows readily from the previous statements; the fact about the restriction to tilting objects is formal consequence of the fact that our equivalence maps pro-standard to pro-standard and pro-costandard to pro-costandard. Finally, to see that the equivalence maps $\widehat{\mathscr{T}}^{\mathrm{et}}_{w,t}$ to $\widehat{\mathscr{T}}^{\mathrm{et}}_{w,t}$, one notes that these objects are the unique (up to isomorphism) lifts of the objects $\mathscr{T}(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$ and $\mathscr{T}(w)_{\mathscr{L}_t^{T}}$ respectively. Then our claim follows once again from the fact that the equivalences of lemma 12.4.1 and (12.4.1) intertwine the projection functors $\pi_\dagger^t$ on the usual and monodromic categories. □

12.5 The étale projection functor and étale completed categories

12.5.1 Definitions

We detail below the construction of the pro-local system $\widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}}$, which will then allow us to define pro-standard and pro-costandard.

Recall the maps

$$e_n^{\mathbb{G}_{\mathrm{m}}} : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}},\ x \mapsto x^n$$

and similarly

$$e_n : \mathbf{T} \to \mathbf{T},\ x \mapsto x^n.$$

We set $\mathbb{G}_{\mathrm{m}}[n]$ (resp. $\mathbf{T}[n]$) for the kernel of $e_n^{\mathbb{G}_{\mathrm{m}}}$ (resp. e_n). For any integer n coprime to p we have a natural identification $\mathbb{G}_{\mathrm{m}}[n] \cong \mathbb{Z}/n\mathbb{Z}$; this group identifies naturally with the Galois group of the finite étale cover $e_n^{\mathbb{G}_{\mathrm{m}}}$. Finally, we have once again an isomorphism of finite abelian groups $\mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{G}_{\mathrm{m}}[n] \cong \mathbf{T}[n]$, the latter identifying with the Galois group of e_n.

As the elements of $\mathfrak{o}$ have finite orders, we can find a finite subfield $\mathbb{K}'$ of $\overline{\mathbb{K}}$ containing the set $\{\lambda(t) \mid \lambda \in \mathrm{X}_*(\mathbf{T})\}$. For any $\lambda \in \mathrm{X}_*(\mathbf{T})$ and any integer $s \geq 0$,

we have $(e^\lambda - \lambda(t))^{\ell^s} = e^{\ell^s \lambda} - \lambda(t^{\ell^s})$. This has the following consequence: for any $\lambda \in \mathrm{X}_*(\mathbf{T})$, the action of $\ell^s n(\mathfrak{o})\lambda$ is given by $\lambda(t^{n(\mathfrak{o})\ell^s}) = 1$. In particular, the natural action of $\mathrm{X}_*(\mathbf{T})$ on $\mathbb{K}'[\mathrm{X}_*(\mathbf{T})]/\langle e^\lambda - \lambda(t) \mid \lambda \in \mathrm{X}_*(\mathbf{T})\rangle^{\ell^s}$ factors through the quotient

$$\mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^s n(\mathfrak{o})\,\mathbb{Z}\,.$$

We deduce a continuous action of $\pi_1^{\mathrm{et}}(\mathbf{T})$ on $\mathbb{K}'[\mathrm{X}_*(\mathbf{T})]/\langle e^\lambda - \lambda(t)\rangle^{\ell^s}$ via the natural map

$$\pi_1^{\mathrm{et}}(\mathbf{T}) \twoheadrightarrow \mathrm{X}_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^s n(\mathfrak{o})\,\mathbb{Z}\,.$$

In this way, we obtain an étale local system $\mathscr{L}_{t,s}^{\mathbf{T},\mathrm{et},\mathbb{K}'}$ on $\mathbf{T}$, and if $\mathbb{K}' \subseteq \mathbb{K}''$ are two finite subfields of $\overline{\mathbb{K}}$, we have

$$\mathbb{K}'' \otimes_{\mathbb{K}'} \mathscr{L}_{t,s}^{\mathbf{T},\mathrm{et},\mathbb{K}'} \cong \mathscr{L}_{t,s}^{\mathbf{T},\mathrm{et},\mathbb{K}''}\,.$$

We let $\mathscr{L}_{t,s}^{\mathbf{T},\mathrm{et}}$ for the $\overline{\mathbb{K}}$-étale local system on $\mathbf{T}$ defined by the family of local systems $\{\mathscr{L}_{t,s}^{\mathbf{T},\mathrm{et},\mathbb{K}'}\}_{\mathbb{K}' \subseteq \overline{\mathbb{K}}}$. We then set

$$\widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et},} := \text{“}\varprojlim_{s \geq 0}\text{”}\, \mathscr{L}_{t,s}^{\mathbf{T},\mathrm{et}}\,.$$

This defines a $\overline{\mathbb{K}}$-pro-étale local system on $\mathbf{T}$. Note that by definition, we have $\mathscr{L}_t^{\mathbf{T},\mathrm{et}} := \mathscr{L}_{t,0}^{\mathbf{T},\mathrm{et}}$. Then, lemma 7.7.5 still holds in our current étale setting, with essentially the same proof.

First, let us remark that we can consider a *constructible* Lusztig–Yun equivariant category for any algebraic stratification of $\mathscr{X}_{\mathbb{F}}$ whose strata are $\mathbf{T}$-stable for the natural right action: for example, below, we will define a Lusztig–Yun equivariant Whittaker category, imposing a constructibility condition for the stratification given by orbits of the Borel subgroup $\mathbf{B}^-$ opposed to $\mathbf{B}$ (with respect to $\mathbf{T}$). For now, let $\mathcal{S}$ be such a stratification on $\mathscr{X}_{\mathbb{F}}$ and set $D_{\mathcal{S}}^{b,\mathrm{et}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ for the associated constructible derived category.

Let us now define our projection functor: for any $Z \subseteq \mathscr{X}_{\mathbb{F}}$, locally closed and union of strata, the mapping

$$\pi_\dagger^{t,\mathrm{et}}(\mathscr{F}) := \mathscr{F} \star^{\mathbf{U}} \Delta(e)^{\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$$

defines a functor $\pi_\dagger^{t,\mathrm{et}} : D_{\mathcal{S}}^{b,\mathrm{et}}(Z, \overline{\mathbb{K}}) \to \mathfrak{D}_{\mathcal{S}}^{\mathrm{et}}(Z /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$. (Note that this convolution product makes sense because $\mathscr{F}$, as an object on the quotient $\mathbf{G}/\mathbf{U}$ admits a natural right $\mathbf{U}$-equivariant structure.) We will essentially use this functor in case $\mathcal{S}$ is given either by the stratification by $\mathbf{B}$-orbits or $\mathbf{B}^-$-orbits; the notation will stay the same however. The relations between $\pi_\dagger^{t,\mathrm{et}}$ and these convolution functors are still valid here, i.e. lemma 7.1.5 holds in this new setting.

This functor allows us to use the constructions of chapter 7.2 in our étale setting. In particular for any $t \in \mathfrak{o}$, we can define a monodromic completed category $\widehat{D}_{\mathcal{M}}^{\mathrm{et}}(Z, \overline{\mathbb{K}})_{[?,t]}$, where ? is in the set $\{-\} \cup \mathfrak{o}$. The “$\mathbf{U}$-equivariant”

convolution product extends to a completed convolution functor $(-)\widehat{\star}(-)$ on the completed category as in section 7.3.

Finally, the same definition as in subsection 5.1.4 gives us a monodromy action on the completed category, derived from the one on the monodromic constructible category.

12.5.2 Pro-standard and costandard objects, perverse t-structure and tilting objects

In this subsection, we define the pro-standard and costandard objects in the étale setting, and show that we can define a perverse t-structure on the completed category, as in the analytic case.

For $w \in W$ and $t \in \mathfrak{o}$, we let

$$\widehat{\Delta}^{\mathrm{et}}_{w,t} := (j_{w,\mathbb{F}})_! p^*_{w,\mathbb{F}} \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}}[\dim(\mathscr{X}_{w,\mathbb{F}})]$$

and

$$\widehat{\nabla}^{\mathrm{et}}_{w,t} := (j_{w,\mathbb{F}})_* p^*_{w,\mathbb{F}} \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}}[\dim(\mathscr{X}_{w,\mathbb{F}})].$$

We will call these objects respectively pro-standard and pro-costandard objects. One can check that corollary 7.7.4 is still valid, i.e. that we have

$$\pi^{t,\mathrm{et}}_\dagger(\widehat{\Delta}^{\mathrm{et}}_{w,t}) \cong \Delta(w)^{\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \qquad \pi^{t,\mathrm{et}}_\dagger(\widehat{\nabla}^{\mathrm{et}}_{w,t}) \cong \nabla(w)^{\mathrm{et}}_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}.$$

Now, for the perverse t-structure, we will need an analogue of proposition 7.4.2; we consider the category $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathbf{T},\overline{\mathbb{K}})_{[t]} = \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{e,\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$.

Lemma 12.5.1. *There exists an equivalence of categories*

$$\Phi^{\mathrm{et}}_t : D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) \cong \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathbf{T},\overline{\mathbb{K}})_{[t]}$$

sending the object $\widehat{R}_t$ to the shifted pro-local system $\widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}}[r]$.

Proof sketch. We keep the notation of section 7.4. The main step is to obtain an analogue of lemma 7.4.1, i.e. an equivalence of categories

$$D^b \operatorname{Mod}^{\mathrm{nil}}(\widehat{R}_t) \to \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathbf{T},\overline{K})_{[t]},$$

where $\operatorname{Mod}^{\mathrm{nil}}(\widehat{R}_t)$ denotes the category whose objects are the finitely generated $\widehat{R}_t$-modules annihilated by a power of $\widehat{\mathfrak{m}}_t$. Now, this latter category is clearly equivalent to the category $\operatorname{Mod}^{\mathrm{nil},t}(\overline{\mathbb{K}}[\mathrm{X}_*(\mathbf{T})])$ of finitely generated $\overline{\mathbb{K}}[\mathrm{X}_*(\mathbf{T})]$-modules annihilated by a power of $\langle e^\lambda - \lambda(t)\rangle$. In turn this category identifies with the inductive limit of the categories $\operatorname{Mod}^{\mathrm{nil},t}(\mathbb{K}'[\mathrm{X}_*(\mathbf{T})])$ (with obvious notation) where the limit is taken over the finite subfields of $\overline{\mathbb{K}}$. Note that the category $\operatorname{Mod}^{\mathrm{nil},t}(\mathbb{K}'[\mathrm{X}_*(\mathbf{T})])$ is generated under extensions by the object $\mathbb{K}' = \mathbb{K}'[\mathrm{X}_*(\mathbf{T})]/\langle e^\lambda - \lambda(t)\rangle$; similarly, the category $\operatorname{Mod}^{\mathrm{nil}}(\widehat{R}_t)$ is generated under extensions by the object $\overline{\mathbb{K}} = \widehat{R}_t/\langle e^\lambda - \lambda(t)\rangle \cdot \widehat{R}_t$.

To any object in $\mathrm{Mod}^{\mathrm{nil},t}(\mathbb{K}'[\mathrm{X}_*(\mathbf{T})])$ we can associate a local system on $\mathbf{T}$ in a natural way, in particular, the object $\mathbb{K}'_t$ is mapped to $\mathscr{L}_t^{\mathbf{T},\mathrm{et},\mathbb{K}'}$. We readily deduce the existence of a functor

$$D^b\,\mathrm{Mod}^{\mathrm{nil}}(\widehat{R}_t) \to \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathbf{T},\overline{\mathbb{K}})_{[t]},$$

mapping $\overline{\mathbb{K}}_t$ to $\mathscr{L}_t^{\mathbf{T},\mathrm{et}}$. The fact that this functor is an equivalence can be checked as in [BeR, Lemma 4.1]. The equivalence stated in the lemma then follows formally, as for proposition 7.4.2, from the arguments of [BeR, §4.2, 4.3]. □

We can now define a perverse t-structure on the completed category exactly as in section 7.5 (the pro-standard and pro-costandard objects are then perverse), and a subcategory of tilting objects; lemma 9.4.2 holds again here. The construction of lifts of tilting objects in the Lusztig–Yun category to the completed category, as in section 9.5 (see particularly proposition 9.5.2), can still be performed. We let $\widehat{\mathscr{T}}^{\mathrm{et}}_{w,t}$ be the lift of $\mathscr{T}(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$, that is to say, we have

$$\pi_\dagger^{t,\mathrm{et}}(\widehat{\mathscr{T}}^{\mathrm{et}}_{w,t}) \cong \mathscr{T}(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$$

for any $w \in W$.

12.5.3 Equivalences

Proposition 11.4.14 and proposition 11.5.5 give us perverse t-exact equivalences of categories of the type (complex-analytic)$\widetilde{\leftrightarrow}$($\mathbb{F}$-étale) between the monodromic and Lusztig–Yun categories respectively. We derive from those equivalences at the completed level.

Coupling diagram (11.5.8) with the results of section 12.4, one easily obtains a commutative diagram

$$\begin{array}{ccc} D^b_{(B)}(\mathscr{X},\overline{\mathbb{K}})_{[-,t]} & \xleftarrow{\ \sim\ } & D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]} \\ \Big\downarrow{\scriptstyle \pi_\dagger^t} & & \Big\downarrow{\scriptstyle \pi_\dagger^{t,\mathrm{et}}} \\ \mathfrak{D}(\mathscr{X}/\!\!/T)_{[-,\mathscr{L}_t^T]} & \xleftarrow{\ \sim\ } & \mathfrak{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}. \end{array} \tag{12.5.1}$$

(Here, in contrast with (11.5.8), the coefficients are algebraically closed.)

Remark 12.5.2. In all the étale categories we considered so far (monodromic, Lusztig–Yun and completed), it is obvious that the notion of block subcategories still makes sense, exactly as in section 8.5. Moreover, it is clear that the equivalences given by the horizontal arrows of the above diagram(12.5.1) preserve these block decompositions, as they preserve standard objects.

From [BY, Proposition A.3.3], we get the following result:

Proposition 12.5.3. *We have an equivalence of triangulated categories*

$$\widehat{D}_{[-,t]} \xleftrightarrow{\sim} \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}.$$

This equivalence maps $\widehat{\Delta}_{w,t}$ to $\widehat{\Delta}^{\mathrm{et}}_{w,t}$ and $\widehat{\nabla}_{w,t}$ to $\widehat{\nabla}^{\mathrm{et}}_{w,t}$ for any $w \in W$, and preserve block subcategories on both sides. Moreover, it restricts to an equivalence between tilting subcategories

$$\widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})_{[-,t]} \xleftrightarrow{\sim} \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}.$$

For $\mathscr{F} \in \widehat{D}_{[-,t]}$, if we let $\mathscr{F}^{\mathbb{F}}$ be the object of $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$ corresponding to it via the above equivalence, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}(\mathscr{F}) & \xleftrightarrow{\quad\sim\quad} & \mathrm{End}(\mathscr{F}^{\mathbb{F}}) \\ & {\scriptstyle \varphi_{\mathscr{F}}} \nwarrow \quad \nearrow {\scriptstyle \varphi_{\mathscr{F}^{\mathbb{F}}}} & \\ & \overline{\mathbb{K}}[\mathrm{X}_*(T)] = \overline{\mathbb{K}}[\mathrm{X}_*(\mathbf{T})]. & \end{array}$$

Proof. The existence of the equivalence was justified above; the facts about pro-standard and pro-costandard objects follows from diagram (11.5.8) and the fact that the pro-standard (resp. pro-costandard) are lifts to the completed categories of the Lusztig–Yun standard and costandard: we can conclude using proposition 11.5.5 to get our claim. The fact about tilting objects can be justified similarly, using remark 11.5.6, whereas the statement about blocks is obvious from the definition of blocks and the fact that our equivalence preserve pro-standard objects. Finally, the commutative diagram is a consequence of the corresponding diagram in proposition 11.4.14. □

We will explicitly use only a particular case of the above proposition; we state it for reference convenience.

Corollary 12.5.4. *We have an equivalence of categories*

$$\widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})^{\circ}_{[-,t]} \xleftrightarrow{\sim} \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^{\circ}_{[-,t]}.$$

12.6 The Whittaker derived category

Let $\mathbf{U}^-$ be the unipotent radical of the Borel subgroup opposite to $\mathbf{B}$ (with respect to the maximal torus $\mathbf{T}$) in $\mathbf{G}$. For any simple root $\alpha \in S$, choose an isomorphism between $\mathbb{G}_a$ and the root subgroup of $\mathbf{G}$ associated to $-\alpha$, which we will denote $\mathbf{U}^-_\alpha$. In this way, one obtains an isomorphism

$$\prod_{\alpha \in S} \mathbf{U}^-_\alpha \cong (\mathbb{G}_a)^S.$$

Composing with the addition map $(\mathbb{G}_a)^S \to \mathbb{G}_a$, we get an additive character

$$\chi : \mathbf{U}^- \twoheadrightarrow \mathbf{U}^-/\mathscr{D}(\mathbf{U}^-) \cong \prod_{\alpha \in S} \mathbf{U}^-_\alpha \cong (\mathbb{G}_a)^S \to \mathbb{G}_a\,.$$

This morphism is in particular nontrivial on any $\mathbf{U}^-_\alpha$ for α simple.

The $\mathbf{U}^-$-orbits (for left multiplication) on $\mathbf{G}/\mathbf{B}$ are affine spaces parametrized by W; their inverse images under the natural projection $\mathscr{X}_{\mathbb{F}} \twoheadrightarrow \mathbf{G}/\mathbf{B}$ will be denoted $\{\mathscr{X}^-_{\mathbb{F},w}\}_{w\in W}$ (note that these locally closed smooth subschemes define a stratification on $\mathscr{X}_{\mathbb{F}}$, and the strata are stable under the right action of $\mathbf{T}$); we have $\mathscr{X}^-_{\mathbb{F},w} \cong \mathbb{A}^{\ell(w_\circ)-\ell(w)}_{\mathbb{F}} \times \mathbf{T}$. Let $j^-_w : \mathscr{X}^-_{\mathbb{F},w} \hookrightarrow \mathscr{X}_{\mathbb{F}}$ be the inclusion map; note that j^-_e is an open embedding. We let $D^{b,\mathrm{et}}_{(\mathbf{B}^-)}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ be the derived category of étale $\overline{\mathbb{K}}$-sheaves on $\mathscr{X}_{\mathbb{F}}$, constructible with respect to the stratification $\{\mathscr{X}^-_{\mathbb{F},w}\}_{w\in W}$.

There exists a primitive p-th root of unity in $\overline{\mathbb{K}}$, contained in some finite subfield $\mathbb{K}'$. This root determines an Artin–Schreier $\mathbb{K}'$-local system $\mathscr{L}_{\mathrm{AS}}$ on $\mathbb{G}_a$ (and hence a $\overline{\mathbb{K}}$-local system on $\mathbb{G}_a$); we denote its pullback to $\mathbf{U}^-$ via χ by $\mathscr{L}_\chi$.

We define the Whittaker derived category $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ to be the category of $(\mathbf{U}^-, \mathscr{L}_\chi)$-equivariant complexes (see [AR1, Appendix A] for a definition, construction and properties of such a category; note that this construction is formally different from the Lusztig–Yun construction). Let us recall here that an object in this category is given by a pair $(\mathscr{F}, \beta)$ with $\mathscr{F} \in D^{b,\mathrm{et}}_{(\mathbf{B}^-)}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ and with β an isomorphism

$$(a^{\mathbf{U}^-})^* \mathscr{F} \xrightarrow[\beta]{\sim} \mathscr{L}_\chi \boxtimes \mathscr{F}$$

on $\mathbf{U}^- \times \mathscr{X}_{\mathbb{F}}$, satisfying a usual cocycle condition (where $a^{\mathbf{U}^-} : \mathbf{U}^- \times \mathscr{X}_{\mathbb{F}} \to \mathscr{X}_{\mathbb{F}}$ is the action map). In practice, we will forget the isomorphism β and refer only to $\mathscr{F}$ as an object in $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$.

One can of course replace $\mathscr{X}_{\mathbb{F}}$ by any locally closed subscheme Z union of strata in $\{\mathscr{X}^-_{w,\mathbb{F}}\}_{w\in W}$ to obtain a triangulated category $D^{b,\mathrm{et}}_{\mathrm{Wh}}(Z, \overline{\mathbb{K}})$. The $*$ and !-restriction (resp. extension) functors relating $D^{b,\mathrm{et}}_{(\mathbf{B}^-)}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ and $D^{b,\mathrm{et}}_{(\mathbf{B}^-)}(Z, \overline{\mathbb{K}})$ then restrict to the Whittaker categories.

Lemma 12.6.1. *The category $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},w}, \mathbb{K})$ is zero unless $w = e$. In particular, for an object $\mathscr{F}$ in $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$, one has $(j^-_w)^* \mathscr{F} \cong 0 \cong (j^-_w)^! \mathscr{F}$ unless $w = e$.*

Proof. The key point is that for $w \neq e$, there exists a simple root $\alpha \in S$ such that $w^{-1}(-\alpha) \in \Phi^+$; this implies that $\mathbf{U}^-_\alpha$ lies in the stabilizer of $\dot{w}\mathbf{U}/\mathbf{U}$. Thus the (restriction of the) action map

$$a^{\mathbf{U}^-} : \mathbf{U}^- \times \{\dot{w}\mathbf{U}/\mathbf{U}\} \to \mathscr{X}^-_{\mathbb{F},w}$$

factors through the natural map

$$\mathbf{U}^- \times \{\dot{w}\mathbf{U}/\mathbf{U}\} \to \mathbf{U}^-/\mathbf{U}^-_\alpha \times \{\dot{w}\mathbf{U}/\mathbf{U}\}.$$

This implies in particular that any object $\mathscr{F}$ on this stratum satisfies

$$((a^{\mathbf{U}^-})^* \mathscr{F})_{|\mathbf{U}^-_\alpha \times \{\dot{w}\mathbf{U}/\mathbf{U}\}} \cong \overline{\mathbb{K}}_{\mathbf{U}^-_\alpha} \boxtimes \mathscr{F}_{\dot{w}\mathbf{U}/\mathbf{U}}\,.$$

Here, $\mathscr{F}_{\dot{w}\mathbf{U}/\mathbf{U}}$ is the stalk of the object $\mathscr{F}$ at the point $\dot{w}\mathbf{U}/\mathbf{U} \in \mathscr{X}_{\mathbb{F}}$.

Now by definition, we have

$$((a^{\mathbf{U}^-})^* \mathscr{F})_{|\mathbf{U}_\alpha^- \times \{\dot{w}\mathbf{U}/\mathbf{U}\}} \cong (\mathscr{L}_\chi)_{|\mathbf{U}_\alpha^-} \boxtimes \mathscr{F}_{\dot{w}\mathbf{U}/\mathbf{U}};$$

this implies that we have an isomorphism

$$(\mathscr{L}_\chi)_{|\mathbf{U}_\alpha^-} \boxtimes \mathscr{F}_{\dot{w}\mathbf{U}/\mathbf{U}} \cong \overline{\mathbb{K}}_{\mathbf{U}_\alpha^-} \boxtimes \mathscr{F}_{\dot{w}\mathbf{U}/\mathbf{U}}\,. \tag{12.6.1}$$

This is impossible if $\mathscr{F}$ is nonzero: indeed, in this case, there is a cohomology sheaf of $\mathscr{F}$ whose stalk is nonzero at $\dot{w}\mathbf{U}/\mathbf{U}$ (since these cohomology sheaves are local systems on $\mathscr{X}^-_{\mathbb{F},w}$ and $\mathscr{F} \neq 0$), in particular, the stalk of $\mathscr{F}$ at this point is nonzero. Thus, if we take the pushforward of (12.6.1) to $\{\dot{w}\mathbf{U}/\mathbf{U}\}$, the left hand side is zero (because $\mathscr{L}_\chi$ has no cohomology on $\mathbb{A}^1_{\mathbb{F}}$) whereas the right hand side is nonzero. Thus the only valid object in $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},w}, \overline{\mathbb{K}})$ is the zero object and our statement is proved. The other statement of the lemma follows easily. □

For $\mathscr{F}$ as in lemma 12.6.1, one deduces that the adjunction morphisms

$$(j_e^-)_!(j_e^-)^* \mathscr{F} \to \mathscr{F} \to (j_e^-)_*(j_e^-)^* \mathscr{F} \tag{12.6.2}$$

are isomorphisms. In particular, we have the following immediate corollary:

Corollary 12.6.2. *The canonical morphism $(j_e^-)_! \to (j_e^-)_*$ of functors from $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},e}, \overline{\mathbb{K}})$ to $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ is an isomorphism.*

Using the preceding results, we can define a monodromic Whittaker category: note that we have an isomorphism $\mathscr{X}^-_{\mathbb{F},e} \cong \mathbf{U}^- \times \mathbf{T}$, given by $(u,t) \mapsto ut\mathbf{U}/\mathbf{U}$. For $t \in \mathfrak{o}$, we set $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{e,\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$ for the full subcategory of $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{e,\mathbb{F}}, \overline{\mathbb{K}})$ whose objects are those complexes $\mathscr{F}$ such that for any $i \in \mathbb{Z}$, the cohomology sheaf $\mathscr{H}^i(\mathscr{F})$ is a successive finite extension of $\mathscr{L}_\chi \boxtimes \mathscr{L}_t^{\mathbf{T},\mathrm{et}}$. Lemma 12.6.1 tells us that the categories $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ and $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{e,\mathbb{F}}, \mathbb{K})$ are equivalent, thus it makes sense to denote $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$ for the full subcategory of $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ whose objects are of the form $(j_e^-)_! \mathscr{F}$ with $\mathscr{F} \in D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{e,\mathbb{F}}, \overline{\mathbb{K}})$. The notation is suggestive: using the same arguments as those in subsection 11.4.6 (see also section 12.3), one easily sees that one can define a monodromy action on $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$, and that any object in this category has generalised monodromy t. Also, it is easy to see that the monodromic categories acts on the Whittaker category by convolution on the right, i.e. that convolution induces a bifunctor, triangulated in both entries,

$$(-) \star^{\mathbf{U}} (-) : D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]} \times D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[t,?]} \to D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,?]}, \tag{12.6.3}$$

where $? \in \{-\} \cup \mathfrak{o}$.

We can similarly define the category of complexes on $\mathscr{X}_{\mathbb{F}}$ that are both $(\mathbf{U}^-, \mathscr{L}_\chi)$-equivariant and Lusztig–Yun equivariant for some local system $\mathscr{L}_t^{\mathbf{T},\mathrm{et}}$

on $\mathbf{T}$. More precisely, as explained at the end of section 12.3, we can consider a Lusztig–Yun $\mathbf{B}^-$-constructible category $\mathfrak{D}^{\mathrm{et}}_{(\mathbf{B}^-)}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$. We define $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ to be the category of pairs $(\mathscr{F}, \beta)$, with $\mathscr{F} \in \mathfrak{D}^{\mathrm{et}}_{(\mathbf{B}^-)}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ and β is an isomorphism

$$(a^{\mathbf{U}^-})^* \mathscr{F} \xrightarrow[\beta]{\sim} \mathscr{L}_\chi \boxtimes \mathscr{F},$$

once again satisfying the usual cocycle condition. Arguments similar as those used in [AR1, Appendix A] imply that the category $\mathfrak{D}^{\mathrm{et}}_{(\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ identifies with a full triangulated subcategory of $\mathfrak{D}^{\mathrm{et}}_{(\mathbf{B}^-)}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$. The above definitions still make sense if one replaces $\mathscr{X}_{\mathbb{F}}$ by any locally closed $\mathbf{U}^- \times \mathbf{T}$-stable subvariety.

Lemma 12.6.3. *Let Z be a locally closed union of strata in $\{\mathscr{X}^-_{w,\mathbb{F}}\}_{w\in W}$. The functor $\pi^{t,\mathrm{et}}_\dagger : D^{b,\mathrm{et}}_{(\mathbf{B}^-)}(Z, \overline{\mathbb{K}}) \to \mathfrak{D}^{\mathrm{et}}_{(\mathbf{B}^-)}(Z /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ induces a functor*

$$D^{b,\mathrm{et}}_{\mathrm{Wh}}(Z, \overline{\mathbb{K}})_{[-,t]} \to \mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(Z /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}.$$

The proof is similar to the one made in lemma 7.1.1; this allows us to consider a completed Whittaker category (defined in a similar way as in definition 7.2.1), which we denote $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(Z, \overline{\mathbb{K}})_{[-,t]}$. We deduce from (12.6.3) that the completed monodromic category acts on the completed Whittaker category: the completed convolution product induces a bifunctor, triangulated in both entries

$$\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]} \times \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[t,t']} \to \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t']}, \tag{12.6.4}$$

for $t, t' \in \mathfrak{o}$.

An analogue of lemma 12.6.1 holds for both the Lusztig–Yun and completed Whittaker category, with the same proof:

Lemma 12.6.4. *The category $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},w} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$, resp. $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},w}, \overline{\mathbb{K}})_{[-,t]}$ is zero unless $w = e$.*

For $\mathscr{F}$ in $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},w} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$, resp. $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$, one has $(j_w^-)^* \mathscr{F} \cong 0 \cong (j_w^-)^! \mathscr{F}$ unless $w = e$.

We introduce standard objects in our Whittaker categories; set:

$$\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} := (j_e^-)_!(\mathscr{L}_\chi \boxtimes \mathscr{L}_t^{\mathbf{T},\mathrm{et}})[\dim(\mathbf{U}^-)] = (j_e^-)_*(\mathscr{L}_\chi \boxtimes \mathscr{L}_t^{\mathbf{T},\mathrm{et}})[\dim(\mathbf{U}^-)]$$

and

$$\widehat{\Delta}_t^\chi := (j_e^-)_!(\mathscr{L}_\chi \boxtimes \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}})[\dim(\mathbf{U}^-) + r] = (j_e^-)_*(\mathscr{L}_\chi \boxtimes \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}})[\dim(\mathbf{U}^-) + r],$$

respectively in $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ and $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \mathbb{K})_{[-,t]}$. One can check that we have an identification

$$\pi^{t,\mathrm{et}}_\dagger(\widehat{\Delta}_t^\chi) = \widehat{\Delta}_t^\chi \star^{\mathbf{U}} \Delta(e)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \cong \Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}. \tag{12.6.5}$$

Lemma 12.6.5. *There exists an equivalence of categories* $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t) \cong \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$ *sending the object* $\widehat{R}_t$ *to* $\widehat{\Delta}^{\chi}_t$.

Proof. Let pr denote the projection $\mathscr{X}^-_{\mathbb{F},e} \cong \mathbf{U}^- \times \mathbf{T} \twoheadrightarrow \mathbf{T}$. The functor $F := (\mathscr{L}_\chi \boxtimes \overline{\mathbb{K}}_{\mathbf{T}}) \otimes \mathrm{pr}^*(-)$ is then an equivalence of categories from $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathbf{T}, \overline{\mathbb{K}})_{[t]}$ to $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},e}, \overline{\mathbb{K}})_{[-,t]}$. Now corollary 12.6.2 tells us that the functor

$$(j_e^-)_* \cong (j_e^-)_! : \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^-_{\mathbb{F},e}, \overline{\mathbb{K}})_{[-,t]} \to \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$$

is an equivalence of categories. Finally, the equivalence we wish for is given by the composition $(j_e^-)_* \circ F \circ \Phi^{\mathrm{et}}_t$, where Φ^{et}_t is as in lemma 12.5.1. □

We define the perverse t-structure on $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$ to be the transport of the natural t-structure on $D^b \operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t)$ under the above equivalence, making this same equivalence t-exact by definition.

Remark 12.6.6. Exactly as in lemma 7.8.4, one can show that the non-positive perverse subcategory in $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t]}$ is generated under extension by the objects $\widehat{\Delta}^{\chi}_t[n]$ with $n \geq 0$.

In complete analogy with the complex case, we define the t-structure on $D^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ as "the shift by r of the natural perverse t-structure on $D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$" (see definition 4.1.2 for a more precise statement). In particular, the object $\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$ is perverse.

Lemma 12.6.7. *We have a t-exact equivalence of categories*

$$\begin{array}{ccc} \mathfrak{D}^{\mathrm{et}}(\mathbf{T}/\!\!/\mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]} & \longrightarrow & \mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}}/\!\!/\mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]} \\ \mathscr{F} & \longmapsto & (j_e^-)_!(\mathscr{L}_\chi \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)]. \end{array}$$

Proof. The fact that our functor (let us call it Ψ) is fully faithful can be checked in the following way. Recall from section 12.3 that we constructed an isogeny $\widetilde{\mathbf{T}} \to \mathbf{T}$ and that we view our Lusztig–Yun equivariant categories as full subcategories of $\widetilde{\mathbf{T}}$-equivariant categories. Let $\mathfrak{D}^{b,\mathrm{et}}_{\mathbf{U}^-}(\mathbf{U}^- \times \mathbf{T}/\!\!/\mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ denote the full subcategory of the $\mathbf{U}^- \times \widetilde{\mathbf{T}}$-equivariant derived category of étale $\mathbb{K}$-sheaves on $\mathbf{U}^- \times \mathbf{T}$ (where the action of $\mathbf{U}^- \subseteq \mathbf{U}^- \times \widetilde{\mathbf{T}}$ is by multiplication on the left factor of $\mathbf{U}^- \times \mathbf{T}$, and the action of $\widetilde{\mathbf{T}} \subseteq \mathbf{U}^- \times \widetilde{\mathbf{T}}$ is induced by the natural action of $\widetilde{\mathbf{T}}$ on $\mathbf{T}$) whose objects are right $\mathbf{T}$-Lusztig–Yun equivariant complexes. Now, we can write

$$\Psi \cong (j_e^-)_! \circ (\mathrm{pr}_1^* \mathscr{L}_\chi \otimes (-)) \circ \mathrm{pr}_2^*$$

where pr_i denotes the projections from $\mathbf{U}^- \times \mathbf{T}$ onto the i-th factor. It is well known that $(j_e^-)_!$ is fully faithful; $\mathrm{pr}_1^* \mathscr{L}_\chi \otimes (-)$ is an equivalence of categories $\mathfrak{D}^{b,\mathrm{et}}_{\mathbf{U}^-}(\mathbf{U}^- \times \mathbf{T}/\!\!/\mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]} \cong \mathfrak{D}^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathbf{U}^- \times \mathbf{T}/\!\!/\mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$, and thanks to [AR1, Remark A.2], pr_2^* gives an equivalence of categories $\mathfrak{D}^{b,\mathrm{et}}(\mathbf{T}/\!\!/\mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]} \cong \mathfrak{D}^{b,\mathrm{et}}_{\mathbf{U}^-}(\mathbf{U}^- \times \mathbf{T}/\!\!/\mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ (in fact, in these assertions, the Lusztig–Yun equivariance does not play any role, as all of our functors clearly preserve this structure).

We show the essential surjectivity: according to lemma 12.6.1, for $\mathscr{G}$ in $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$, we have $\mathscr{G} \cong (j_e^-)_!(j_e^-)^*\mathscr{G}$. Thus we can assume that $\mathscr{G}$ is an object in $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathbf{U}^- \times \mathbf{T} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$, again with the action of $\mathbf{U}^-$ being by multiplication on itself. By definition, we have an isomorphism

$$(m \times \mathrm{id}_{\mathbf{T}})^* \mathscr{G} \cong \mathscr{L}_\chi \boxtimes \mathscr{G}\,.$$

(Here we let $m : \mathbf{U}^- \times \mathbf{U}^- \to \mathbf{U}^-$ be the multiplication map.) Restricting this isomorphism to $\mathbf{U}^- \times \mathbf{T}$ under the embedding $(u,t) \mapsto (u,e,t)$ then shows that $\mathscr{G}$ lies in the essential image of the functor $\mathscr{L}_\chi \boxtimes (-)$.

Finally, to prove the t-exactness, we show that any perverse sheaf in the category $\mathfrak{D}^{\mathrm{et}}(\mathbf{T} /\!\!/ \mathbf{T})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$ is mapped to a perverse object. But the perverse sheaves in the former category are direct sums of copies of $\mathscr{L}_t^{\mathbf{T},\mathrm{et}}$. The image of the latter local system is $\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$, which is perverse. This concludes the proof. □

Remark 12.6.8. As an immediate corollary of the above lemma, we get an equivalence

$$\mathfrak{P}^{\mathrm{et}}(\mathbf{T} /\!\!/ \mathbf{T})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \xrightarrow{\sim} \mathfrak{P}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}. \tag{12.6.6}$$

In particular, any object in $\mathfrak{P}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ is isomorphic to a direct sum of copies of $\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$.

Remark 12.6.9. Using the same kind of arguments as those used in section 7.4, on can show that we have a t-exact equivalence of categories $\mathfrak{D}^{\mathrm{et}}(\mathbf{T} /\!\!/ \mathbf{T})_{[\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]} \cong D^b\mathrm{Vect}^{\mathrm{fd}}_{\overline{\mathbb{K}}}$, for the perverse t-structure on the left hand side and the usual t-structure on the right hand side.

12.7 $\mathbf{U}^-$ and χ-averaging

We now introduce several classical averaging and forgetful functors. We let $D^{b,\mathrm{et}}_{\mathbf{U}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})$ denote the $\mathbf{U}$-equivariant derived category on $\mathscr{X}_{\mathbb{F}}$, where the action of $\mathbf{U}$ is by multiplication on the left. For $? \in \{*, !\}$, set

$$\begin{array}{rccc} \mathrm{av}^{\mathbf{U}}_? : & D^{b,\mathrm{et}}_c(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}}) & \longrightarrow & D^{b,\mathrm{et}}_{\mathbf{U}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}}) \\ & \mathscr{F} & \longmapsto & a^{\mathbf{U}}_?(\underline{\overline{\mathbb{K}}}_{\mathbf{U}} \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)]. \end{array}$$

(Here again, we denoted $\underline{\overline{\mathbb{K}}}_{\mathbf{U}}$ the $\overline{\mathbb{K}}$-local system on $\mathbf{U}$ represented by any of the local systems $\underline{\mathbb{K}'}_{\mathbf{U}}$, with $\mathbb{K}'$ a finite subfield of $\overline{\mathbb{K}}$.)

In the other direction, we have the forgetful functor

$$\mathrm{For}^{\mathbf{U}} : D^{b,\mathrm{et}}_{\mathbf{U}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}}) \to D^{b,\mathrm{et}}_c(\mathscr{X}_{\mathbb{F}}, \mathbb{K}),$$

which gives adjoint pairs $(\mathrm{av}^{\mathbf{U}}_!, \mathrm{For}^{\mathbf{U}}[\dim(\mathbf{U}^-)])$ and $(\mathrm{For}^{\mathbf{U}}[-\dim(\mathbf{U}^-)], \mathrm{av}^{\mathbf{U}}_*)$ (see for example [AR1, Lemma A.3]).

Remark 12.7.1. The functors $\mathrm{For}^{\mathbf{U}}$ and $\mathrm{av}_?^{\mathbf{U}}$ relate the constructible category to the $\mathbf{U}$-*equivariant* category. However, we will later consider them on monodromic categories, where no $\mathbf{U}$-equivariance is a priori required. But using e.g. [AR1, Remarks A.2, A.6], one sees that the $\mathbf{U}$-equivariant category identifies with a full subcategory of the constructible category, and one can show that our monodromic category $D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})$ in turn identifies with a full subcategory of this $\mathbf{U}$-equivariant subcategory.

We can also average along $(\mathbf{U}^-,\mathscr{L}_\chi)$: for $? \in \{*,!\}$, set

$$\begin{array}{rccc} \mathrm{av}_?^{\chi}: & D^{b,\mathrm{et}}_c(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}}) & \longrightarrow & D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}}) \\ & \mathscr{F} & \longmapsto & a_?^{\mathbf{U}^-}(\mathscr{L}_\chi \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)]. \end{array}$$

Here again, we have a forgetful functor in the other direction

$$\mathrm{For}^{\chi}: D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}}) \to D^{b,\mathrm{et}}_c(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}}),$$

giving adjoint pairs $(\mathrm{av}^{\chi}_!, \mathrm{For}^{\chi}[\dim(\mathbf{U}^-)])$ and $(\mathrm{For}^{\chi}[-\dim(\mathbf{U}^-)], \mathrm{av}^{\chi}_*)$. Note that all the above functors preserve both monodromy and Lusztig–Yun equivariance (whenever we can consider them); i.e. these functors restrict to monodromic categories: for $\mathscr{F} \in D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ one can check that we have $\mathrm{av}^{\chi}_?(\mathscr{F}) \in D^{b,\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$, and a similar statement for right monodromy replaced by right Lusztig–Yun-equivariance.

Using arguments similar to those used in the proof of lemma 7.5.1, (see also the original argument in [BY, Corollary A.3.4]), it is easy to see that we have isomorphisms of functors

$$\mathrm{av}^{\chi}_? \circ \mathrm{For}^{\mathbf{U}} \circ \pi^{t,\mathrm{et}}_\dagger \cong \pi^{t,\mathrm{et}}_\dagger \circ \mathrm{av}^{\chi}_? \circ \mathrm{For}^{\mathbf{U}} \quad \text{and} \quad \mathrm{av}^{\mathbf{U}}_? \circ \mathrm{For}^{\chi} \circ \pi^{t,\mathrm{et}}_\dagger \cong \pi^{t,\mathrm{et}}_\dagger \circ \mathrm{av}^{\mathbf{U}}_? \circ \mathrm{For}^{\chi}$$

for $? \in \{*,!\}$. Thus the various functors "$\mathrm{av} \circ \mathrm{For}$" extend to the various completed categories involved. We get new functors, that we denote with capital letters:

$$\begin{array}{rccc} \mathrm{Av}^{\mathbf{U}}_{!,[-,t]}, \mathrm{Av}^{\mathbf{U}}_{*,[-,t]}: & \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]} & \longrightarrow & \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}, \\ \mathrm{Av}^{\chi}_{!,[-,t]}, \mathrm{Av}^{\chi}_{*,[-,t]}: & \widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]} & \longrightarrow & \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}. \end{array}$$

Lemma 12.7.2. *Let $t,t' \in \mathfrak{o}$. There exist natural isomorphisms of functors $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[t,t']} \to \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t']}$*

$$\mathrm{Av}^{\chi}_{!,[-,t']}(-) \cong \widehat{\Delta}^{\chi}_t \widehat{\star}(-) \cong \mathrm{Av}^{\chi}_{*,[-,t']}(-).$$

Proof. The proof is borrowed from [BY, Lemma 4.4.3]. It suffices to prove that there are functorial isomorphisms $\mathrm{Av}^{\chi}_{!,[-,t']}(\mathscr{F}) \cong \widehat{\Delta}^{\chi}_t \widehat{\star} \mathscr{F} \cong \mathrm{Av}^{\chi}_{*,[-,t']}(\mathscr{F})$ for any $\mathscr{F} \in D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[t,t']}$. Recall that $\widehat{\Delta}^{\chi}_t$ can be defined either as the (shifted) ! or $*$-extension of the object $\mathscr{L}_\chi \boxtimes \widehat{\mathscr{L}}^{\mathbf{T},\mathrm{et}}_t$. We note first that the image in $\mathbf{G}$ of $\mathbf{U}^- \times \mathbf{T}$ under the multiplication map $(u,t) \mapsto ut$ maps bijectively to $\mathscr{X}^-_{\mathbb{F},e}$

under the quotient map $\mathbf{G} \to \mathscr{X}_{\mathbb{F}}$. Let $a^{\mathbf{T}}$ denote the action map of $\mathbf{T}$ by multiplication on the left of $\mathscr{X}_{\mathbb{F}}$; the following diagram is then commutative

$$\begin{array}{ccccc}
\mathbf{U}^- \times \mathbf{T} \times \mathscr{X}_{\mathbb{F}} & \hookrightarrow & \mathbf{G} \times^{\mathbf{U}} \mathscr{X}_{\mathbb{F}} & & \\
\downarrow{\scriptstyle \mathrm{id} \times a^{\mathbf{T}}} & & \downarrow & \searrow^{m_{\mathbf{U}}} & \\
\mathbf{U}^- \times \mathscr{X}_{\mathbb{F}} & \xhookrightarrow{j_{\mathbf{U}^-} \times \mathrm{id}} & \mathbf{G} \times^{\mathbf{B}} \mathscr{X}_{\mathbb{F}} & \xrightarrow[m_{\mathbf{B}}]{} & \mathscr{X}_{\mathbb{F}}.
\end{array}$$

The top horizontal arrow is given by $(u, t, gU) \mapsto [ut, gU]$. Finally, note that the composition of the two horizontal bottom arrows is just $a^{\mathbf{U}^-}$, the action map of $\mathbf{U}^-$ on $\mathscr{X}_{\mathbb{F}}$.

Now we have

$$\begin{aligned}
\widehat{\Delta}_t^{\chi} \widehat{\star} \mathscr{F} &= (m_{\mathbf{U}})_!((j_e^-)_!(\mathscr{L}_\chi \boxtimes \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}})[\dim(\mathbf{U}^-) + r] \boxtimes^{\mathbf{U}} \mathscr{F})[r] \\
&\cong (m_{\mathbf{B}})_!(j_{\mathbf{U}^-} \times \mathrm{id})_!(\mathrm{id} \times a^{\mathbf{T}})_!(\mathscr{L}_\chi \boxtimes \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}} \boxtimes \mathscr{F})[\dim(\mathbf{U}^-) + 2r] \\
&\cong a_!^{\mathbf{U}^-}(\mathscr{L}_\chi \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)] = \mathrm{Av}^{\chi}_{!,[-,t']}(\mathscr{F}).
\end{aligned}$$

We used lemma 7.7.5 for the last isomorphism. It is clear that the above identification is functorial in $\mathscr{F}$, so that the first isomorphism of the statement is proved; we proceed to get the second one. As above, we have

$$\begin{aligned}
\widehat{\Delta}_t^{\chi} \widehat{\star} \mathscr{F} &\cong (m_{\mathbf{B}})_!(j_{\mathbf{U}^-} \times \mathrm{id})_!(\mathrm{id} \times a^{\mathbf{T}})_!(\mathscr{L}_\chi \boxtimes \widehat{\mathscr{L}}_t^{\mathbf{T},\mathrm{et}} \boxtimes \mathscr{F})[\dim(\mathbf{U}^-) + 2r] \\
&\cong (m_{\mathbf{B}})_!(j_{\mathbf{U}^-} \times \mathrm{id})_!(\mathscr{L}_\chi \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)].
\end{aligned}$$

Now arguments similar to those used in the proof of lemma 12.6.1 show that the ! and $*$-extension of $\mathscr{L}_\chi$ along the map (induced by the action on the element $\dot{e} = 1$) $\mathbf{U}^- \to \mathbf{G}/\mathbf{B}$ coincide. Using moreover that the map $m_{\mathbf{B}}$ is proper, we get

$$\begin{aligned}
\widehat{\Delta}_t^{\chi} \widehat{\star} \mathscr{F} &\cong (m_{\mathbf{B}})_*(j_{\mathbf{U}^-} \times \mathrm{id})_*(\mathscr{L}_\chi \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)] \\
&\cong (a^{\mathbf{U}^-})_*(\mathscr{L}_\chi \boxtimes \mathscr{F})[\dim(\mathbf{U}^-)] \cong \mathrm{Av}^{\chi}_{*,[-,t']}(\mathscr{F}),
\end{aligned}$$

which concludes the proof. □

The previous lemma allows us to identify $\mathrm{Av}^{\chi}_{!,[-,t']} \cong \mathrm{Av}^{\chi}_{*,[-,t']}$, we denote this functor simply by $\mathrm{Av}^{\chi}_{[-,t']}$.

Remark 12.7.3. We emphasize a subtlety in the notation: the functor $\mathrm{Av}^{\chi}_{[-,t']}$ is defined on the whole category $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})_{[-,t']}$. The lemma says that when one restricts this functor to a subcategory with fixed *left* monodromy, one can view it as convolution with the Whittaker standard object associated to this *left* monodromy (i.e. the right monodromy of the standadr Whittaker object is given by the fixed left monodromy f our subcategory). Thus, it seems legitimate to denote $\mathrm{Av}^{\chi}_{[t,t']}$ for this restriction.

Exactly as in [BY, Corollary 4.4.4], we deduce from the preceding result the t-exactness of $\mathrm{Av}^{\chi}_{[-,t']}$:

Corollary 12.7.4. *The functor $\mathrm{Av}^{\chi}_{[-,t']}$ is t-exact for the perverse t-structures on $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t']}$ and $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t']}$.*

Proof. We show that $\mathrm{Av}^{\chi}_{!,[-,t']}$ is right t-exact and that $\mathrm{Av}^{\chi}_{*,[-,t']}$ is left t-exact; thanks to lemma 12.7.2, this will imply the claim. Moreover, according to lemma 7.5.4, it suffices to show the t-exactness at the non-completed level. The functor $\mathrm{Av}^{\chi}_{!,[-,t']}$ can be written as the composition

$$a^{\mathbf{U}^-}_{!} \circ (\mathrm{pr}_1^* \mathscr{L}_{\chi} \otimes (-)) \circ \mathrm{pr}_2^*[\dim(\mathbf{U}^-)],$$

where pr_i is the i-th projection map on $\mathbf{U}^- \times \mathscr{X}_{\mathbb{F}}$. This projection map is smooth with connected fibers, of relative dimension $\dim(\mathbf{U}^-)$, so that $\mathrm{pr}_2^*[\dim(\mathbf{U}^-)]$ is perverse t-exact; the functor $\mathrm{pr}_1^* \mathscr{L}_{\chi} \otimes (-)$ is clearly t-exact; finally, the map $a^{\mathbf{U}^-}$ is affine, hence $a^{\mathbf{U}^-}_{!}$ is right t-exact thanks to [BBD, Corollaire 4.1.2]. The same arguments show that $\mathrm{Av}^{\chi}_{*,[-,t']}$ is left t-exact, whence the result. □

Lemma 12.7.5. *We have adjoint pairs $(\mathrm{Av}^{\mathbf{U}}_{!,t'}, \mathrm{Av}^{\chi}_{[-,t']})$ and $(\mathrm{Av}^{\chi}_{[-,t']}, \mathrm{Av}^{\mathbf{U}}_{*,t'})$ of functors between the categories on $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t']}$ and $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t']}$.*

Proof. This results formally from the adjunction stated at the beginning of section 12.7, exactly as in [BY, Lemma 4.4.5]. □

Remark 12.7.6. Using the forgetful functor $\mathrm{For}_{t'} : \mathfrak{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}]} \to D^{b,\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t']}$, we can extend the functor $\mathrm{Av}^{\chi}_{[-,t']}$ to $\mathfrak{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}]}$ via $\mathrm{Av}^{\chi}_{[-,t']}(\mathscr{F}) := \mathrm{Av}^{\chi}_{[-,t']}(\mathrm{For}_{t'}(\mathscr{F}))$. In what follows, we will always omit $\mathrm{For}_{t'}$ to simplify notation.

12.8 $\mathbf{B}_s$ and $\mathbf{L}_s$-equivariance

We state here some results analogous to those of subsections 5.3.2 and 5.3.3, in our current étale setting.

We begin by a result about an extension property of local systems on reductive algebraic groups, taken from [LY, Lemma 2.3] (eventually, we will apply the following lemma to the local system $\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}$, but it is stated in a slightly greater generality). Let $\mathbf{H}$ be a reductive algebraic group over our algebraically closed field $\mathbb{F}$, and let $\mathbf{A} \subseteq \mathbf{H}$ be a maximal torus. We consider the root datum $(\mathrm{X}^*(\mathbf{A}), \Phi, \mathrm{X}_*(\mathbf{A}), \Phi^{\vee})$ for $\mathbf{H}$. Let $\mathscr{L}$ be a one dimensional étale $\overline{\mathbb{K}}$-local system on $\mathbf{A}$. Assume that we have a character $\psi : \ker(e_n) = \mathbf{A}[n] \to \overline{\mathbb{K}}^*$ such that $\mathscr{L} \cong ((e_n)_* \underline{\overline{\mathbb{K}}}_{\mathbf{A}})[\psi]$, for some $n > 0$ satisfying $\gcd(p, n) = 1$ (here again, $[\psi]$ denotes the ψ-isotypic component; note that ψ necessarily has its image contained in some finite subfield of $\overline{\mathbb{K}}$).

Lemma 12.8.1. *Consider the above setting. If $(\alpha^\vee)^*\mathscr{L}$ is isomorphic to $\underline{\overline{\mathbb{K}}}_{\mathbb{G}_\mathrm{m}}$ for any $\alpha^\vee \in \Phi^\vee$, then there exists a $\overline{\mathbb{K}}$-local system $\mathscr{L}_\mathbf{H}$ on $\mathbf{H}$ such that $(\mathscr{L}_\mathbf{H})_{|\mathbf{A}} \cong \mathscr{L}$. Moreover, there exists a finite central isogeny $\overline{\mathbf{H}} \xrightarrow{\nu} \mathbf{H}$ and a character $\chi_\mathbf{H}$ of $\ker(\nu)$ such that $\mathscr{L}_\mathbf{H} \cong \nu_*\underline{\overline{\mathbb{K}}}_{\overline{\mathbf{H}}}[\chi_\mathbf{H}]$.*

Remark 12.8.2. An application of the proper base change theorem yields an isomorphism $e_n^*(e_n)_*\underline{\overline{\mathbb{K}}}_\mathbf{A} \cong \underline{\overline{\mathbb{K}}}_\mathbf{A} \otimes \mathbf{H}^\bullet(\mathbf{A}[n], \overline{\mathbb{K}}) \cong \underline{\overline{\mathbb{K}}}_\mathbf{A}^{\oplus n}$. Thus we have $e_n^*\mathscr{L} \cong \underline{\overline{\mathbb{K}}}_\mathbf{A}$. This could also be seen as a direct consequence of the fact that the action of the fundamental group of $\mathbf{A}$ on $\mathrm{Aut}(\mathscr{L})$ is via $\mathbf{A}[n]$.

Proof. Choose a finite subfield $\mathbb{K}'$ of $\overline{\mathbb{K}}$ containing the image of ψ; enlarging $\mathbb{K}'$ if necessary, we can assume that $\mathscr{L}$ is represented by some one dimensional $\mathbb{K}'$-local system, still denoted $\mathscr{L}$. It then suffices to prove the lemma for $\mathbb{K}'$ as coefficients.

Note first that since $\gcd(p, n) = 1$, the map e_n is an étale covering, with Galois group $\ker(e_n)$ (and similarly for $e_n^{\mathbb{G}_\mathrm{m}}$). Using the isomorphism $\ker(e_n) \cong \ker(e_n^{\mathbb{G}_\mathrm{m}}) \otimes_\mathbb{Z} \mathrm{X}_*(\mathbf{A})$, the character χ induces a morphism

$$\overline{\chi} : \mathrm{X}_*(\mathbf{A}) \to \mathrm{Hom}(\ker(e_n^{\mathbb{G}_\mathrm{m}}), \mathbb{K}'^*).$$

The image of $\lambda \in \mathrm{X}_*(\mathbf{A})$ under $\overline{\chi}$ gives the character of $\ker(e_n^{\mathbb{G}_\mathrm{m}})$ obtained as the composition

$$\ker(e_n^{\mathbb{G}_\mathrm{m}}) \xrightarrow{\lambda_\star} \ker(e_n) \xrightarrow{\chi} \mathbb{K}'^*,$$

where $\lambda_\star$ is the map induced by λ. In turn, this character of $\ker(e_n^{\mathbb{G}_\mathrm{m}})$ corresponds to an étale local system on $\mathbb{G}_\mathrm{m}$ (since $\ker(e_n^{\mathbb{G}_\mathrm{m}})$ is the Galois group of a finite étale covering), given by $\lambda^*\mathscr{L}$. By assumption, we have $\mathbb{Z}\cdot\Phi^\vee \subseteq \Lambda := \ker(\overline{\chi})$.

Now there exists a reductive algebraic group $\overline{\mathbf{H}}$ over $\mathbb{F}$ with cocharacter lattice given by Λ and coroots $\Phi^\vee$, and the natural inclusion $\Lambda \hookrightarrow \mathrm{X}_*(T)$ defines a finite central isogeny $\nu : \overline{\mathbf{H}} \to \mathbf{H}$.

The map $\mathrm{X}_*(\mathbf{A}) \to \mathrm{X}_*(\mathbf{A})$ induced by e_n is given by multiplication by n, thus is injective. Let us consider $\lambda \in \mathrm{X}_*(\mathbf{A})$. The image $\overline{\chi}(e_n \circ \lambda)$ in $\mathrm{Hom}(\ker(e_n^{\mathbb{G}_\mathrm{m}}), \mathbb{K}'^*)$ corresponds to the local system $\lambda^* e_n^* \mathscr{L}$ on $\mathbb{G}_\mathrm{m}$. One sees, e.g. using remark 12.8.2, that $\lambda^* e_n^* \mathscr{L}$ is isomorphic to $\underline{\mathbb{K}'}_{\mathbb{G}_\mathrm{m}}$. Thus e_n maps (injectively) $\mathrm{X}_*(\mathbf{A})$ into Λ. Set $\overline{\mathbf{A}} := (\nu^{-1}(\mathbf{A}))^\circ$; we readily deduce that we have a factorisation of e_n as in the following commutative diagram, with β a central isogeny:

$$\begin{array}{ccccc} & & \overline{\mathbf{A}} & \hookrightarrow & \overline{H} \\ & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle \nu_{|\overline{\mathbf{A}}}} & & \downarrow{\scriptstyle \nu} \\ \mathbf{A} & \xrightarrow[e_n]{} & \mathbf{A} & \hookrightarrow & H. \end{array}$$

Thus we have $\ker(\beta) = \ker(\chi)$, and we obtain a character

$$\chi_\mathbf{H} : \ker(\nu) = \ker(\nu_{\overline{\mathbf{A}}}) \cong \ker(e_n)/\ker(\beta) \cong \ker(\nu)/\ker(\chi) \to \mathbb{K}'^*$$

where the last arrow is the map induced by χ. We then set $\mathscr{L}_{\mathbf{H}} := (\nu_* \underline{\mathbb{K}'}_{\overline{\mathbf{H}}})[\chi_{\mathbf{H}}]$. This is a rank one local system on $\mathbf{H}$, whose restriction to $\mathbf{A}$ is given by $((\nu_{|\overline{\mathbf{A}}})_* \underline{\mathbb{K}'}_{\overline{\mathbf{A}}})[\chi_H]$. By construction, this local system is associated to the character χ of $\ker(e_n)$. This implies $((\nu_{|\overline{\mathbf{A}}})_* \underline{\mathbb{K}'}_{\overline{\mathbf{A}}})[\chi_H] \cong \mathscr{L}$. □

We come back to our group $\mathbf{G}$. For any $t \in \mathfrak{o}$, let $\mathbf{P}_{\Phi_t}$ be the standard parabolic subgroup of $\mathbf{G}$ associated to the subset $\Phi_t \subseteq \Phi$, and denote by $\mathbf{L}_{\Phi_t}$ the Levi factor of $\mathbf{P}_{\Phi_t}$ containing $\mathbf{T}$. Note that the group $\mathbf{P}_{\Phi_t}$ contains the standard parabolic subgroups $\mathbf{P}_s$ for any reflection $s \in W_t^\circ$, and that their Levi factors $\mathbf{L}_s$ containing $\mathbf{T}$ are subgroups of $\mathbf{L}_{\Phi_t}$.

Corollary 12.8.3. *There exists a finite central isogeny*

$$\overline{\mathbf{L}}_{\Phi_t} \xrightarrow{\nu} \mathbf{L}_{\Phi_t}$$

and a character χ of $\ker(\nu)$ such that the local system $\mathscr{L}_{\Phi_t} := \nu_* \underline{\mathbb{K}}_{\overline{\mathbf{L}}_{\Phi_t}}[\chi]$ satisfies:

1. $\mathscr{L}_t^{\mathbf{T},\mathrm{et}} \cong (\mathscr{L}_{\Phi_t})_{|\mathbf{T}}$,
2. *for any $s \in W_t^\circ$ simple in W with associated root $\alpha_s \in \Phi_t$, the restriction of $\mathscr{L}_{\Phi_t}$ to any $\mathbf{U}_{\alpha_s}$ or $\mathbf{U}_{\alpha_s}^-$ is the constant sheaf $\underline{\mathbb{K}}_{\mathbf{U}_{\alpha_s}^\pm}$.*

Proof. The existence and description of $\mathscr{L}_{\Phi_t}$ is a direct consequence of lemma 12.8.1, only point 2. above needs a proof; we detail the case of $\mathbf{U}_{\alpha_s}$, the argument for $\mathbf{U}_{\alpha_s}^-$ is totally similar. The inverse image of $\mathbf{U}_{\alpha_s}$ by ν is $\mathbf{U}_{\alpha_s} \times \ker(\nu)$, and the restriction of ν to any of the copies of $\mathbf{U}_{\alpha_s}$ appearing in this fiber is an isomorphism. The restriction of $\mathscr{L}_{\Phi_t}$ to $\mathbf{U}_{\alpha_s}$ is then given by a one dimensional direct summand in $((\nu_{|\nu^{-1}(\mathbf{U}_{\alpha_s})})_* \underline{\mathbb{K}}_{\nu^{-1}(\mathbf{U}_{\alpha_s})} \cong \underline{\mathbb{K}}_{\mathbf{U}_{\alpha_s}}^{\oplus |\ker(\nu)|}$. The result follows. □

For any $s \in W_t^\circ$, simple in W, we set

$$\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}} := (\mathscr{L}_{\Phi_t})_{|\mathbf{L}_s}, \qquad \mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}} := (\mathscr{L}_{\Phi_t})_{|\mathbf{B}_s}.$$

Then the local systems $\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}}$ on $\mathbf{L}_s$ and $\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}$ on $\mathbf{B}_s$ both extend $\mathscr{L}_t^{\mathbf{T},\mathrm{et}}$. Moreover, the restriction of $\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}}$ to either $\mathbf{U}_{\alpha_s}$ or $\mathbf{U}_{\alpha_s}^-$ is the constant sheaf $\underline{\mathbb{K}}_{\mathbf{U}_{\alpha_s}^\pm}$. Finally, set

$$\overline{\mathbf{L}}_s := \mathbf{L}_s \times_{\mathbf{L}_{\Phi_t}} \overline{\mathbf{L}}_{\Phi_t}, \qquad \overline{\mathbf{B}}_s := \mathbf{B}_s \times_{\mathbf{L}_{\Phi_t}} \overline{\mathbf{L}}_{\Phi_t}.$$

The finite central isogeny ν of lemma 12.8.3 defines finite central isogenies

$$\overline{\mathbf{L}}_s \to \mathbf{L}_s, \qquad \overline{\mathbf{B}}_s \to \mathbf{B}_s;$$

we denote $K_{\mathbf{L}_s}$ and $K_{\mathbf{B}_s}$ their respective kernels. Similarly, the character χ of $\ker(\nu)$ (again, see lemma 12.8.3 for the notation) induces characters $\chi_{\mathbf{L}_s}$ and $\chi_{\mathbf{B}_s}$ of $K_{\mathbf{L}_s}$ and $K_{\mathbf{B}_s}$ respectively.

As above, let $\mathbf{P}_s$ be the standard parabolic subgroup of $\mathbf{G}$ associated to the simple reflection s, and set $\mathbf{U}^s$ for its unipotent radical. In analogy with the complex case, we set

$$\mathscr{X}^s_{\mathbb{F}} := \mathbf{G}/\mathbf{U}^s.$$

We define the right $\mathbf{L}_s$-equivariant (resp. $\mathbf{B}_s$-equivariant) Lusztig–Yun categories associates to the local system $\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}}$ (resp. $\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}$) as the $\chi_{\mathbf{L}_s}$-equivariant category (resp. $\chi_{\mathbf{B}_s}$-equivariant category):

$$\mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{L}_s)_{[-,\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}}]} := D^{b,\mathrm{et}}_{\overline{\mathbf{L}}_s,\chi_{\mathbf{L}_s}}(\mathscr{X}^s_{\mathbb{F}},\overline{\mathbb{K}})$$

resp.

$$\mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{B}_s)_{[-,\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}]} := D^{b,\mathrm{et}}_{\overline{\mathbf{B}}_s,\chi_{\mathbf{B}_s}}(\mathscr{X}^s_{\mathbb{F}},\overline{\mathbb{K}}),$$

where the action of $\overline{\mathbf{L}}_s$ (resp. $\overline{\mathbf{B}}_s$) is by on the right of $\mathscr{X}^s_{\mathbb{F}}$. We also have left versions of these categories, considering a left action.

As in subsections 6.3.2 and 6.3.3, we can define convolution products on these categories, denoted respectively $(-)\star^{\mathrm{LY}}_{\mathbf{L}_s}(-)$ and $(-)\star^{\mathrm{LY}}_{\mathbf{B}_s}(-)$.

The definitions and results of subsection 5.3.2 are still valid in the étale case, in particular, lemma 5.3.4 still holds, with identical proof, so the categories $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{B}_s)_{[-,\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}]}$ and $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$ are canonically equivalent; moreover, this equivalence commutes with the convolution products $\star^{\mathrm{LY}}$ and $\star^{\mathrm{LY}}_{\mathbf{B}_s}$, as in lemma 6.3.2.

Similarly, we again have forgetful and averaging functors between the categories $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{B}_s)_{[-,\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}]}$ and $\mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{L}_s)_{[-,\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}}]}$, as introduced in subsection 8.1.3; we will keep the notation used there: we have the right averaging and forgetful functors

$$\mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{B}_s)_{[-,\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}]} \underset{\mathrm{For}^s_t}{\overset{\mathrm{Av}^s_t}{\rightleftarrows}} \mathfrak{D}^{\mathrm{et}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{L}_s)_{[-,\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}}]};$$

as well as left versions:

$$\mathfrak{D}^{\mathrm{et}}(\mathbf{B}_s \setminus\!\!\setminus \mathscr{X}^s_{\mathbb{F}})_{[\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}},-]} \underset{\mathrm{For}^t_s}{\overset{\mathrm{Av}^t_s}{\rightleftarrows}} \mathfrak{D}^{\mathrm{et}}(\mathbf{B}_s \setminus\!\!\setminus \mathscr{X}^s_{\mathbb{F}})_{[\mathscr{L}_t^{\mathbf{L}_s,\mathrm{et}},-]}.$$

Corollary 12.8.3 implies that $\mathrm{IC}(s)_{\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}}$ lies in the image of For^t_s; lemma 8.3.6 holds in the étale setting. Thus, any simple complex of the form $\mathrm{IC}(sw)_{\mathscr{L}_{t'}^{\mathbf{B}_s,\mathrm{et}}}$ with $sw > w$ is in the image of the functor For^t_s.

Remark 12.8.4. The reader probably noticed that we did not make any mention of monodomic complexes (or monodromy) in this $\mathbf{B}_s$ or $\mathbf{L}_s$-equivariant setting. This is no problem since these categories are just used as a tool to deduce information about the $\mathbf{T}$-Lusztig–Yun-equivariant monodromic category, which is a full subcategory in the $\mathbf{T}$-Lusztig–Yun-equivariant category.

The scheme $\mathscr{X}^s_{\mathbb{F}}$ is endowed with a $\mathbf{U}^-$-action on the left, and we can define Whittaker (resp. Whittaker-monodromic, Whittaker and Lusztig–Yun equivariant) categories as above. For example, below, we will consider the category $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{B}_s)_{[-,\mathscr{L}^{\mathbf{B}_s,\mathrm{et}}_t]}$ of Lusztig–Yun right-$\mathbf{B}_s$-equivariant and left-Whittaker category. Finally, for $w \in W$, let $\mathscr{X}^{s,-}_{\mathbb{F},w}$ be the inverse image of $\mathscr{X}^-_{\mathbb{F},w}$ under the natural projection $\mathscr{X}^s_{\mathbb{F}} \to \mathscr{X}_{\mathbb{F}}$, and let $\Delta(\chi)_{\mathscr{L}^{\mathbf{B}_s,\mathrm{et}}_t}$ be the pullback of $\Delta(\chi)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_t}$ to $\mathscr{X}^s_{\mathbb{F}}$.

Lemma 12.8.5. *Consider $\mathscr{G} \in \mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{B}_s)_{[-,\mathscr{L}^{\mathbf{B}_s,\mathrm{et}}_t]}$. We have $\mathrm{Av}^s_t(\mathscr{G}) = 0$.*

Proof. The object $\mathscr{F} := \mathrm{Av}^s_t(\mathscr{G})$ lies in the category $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}^s_{\mathbb{F}} /\!\!/ \mathbf{L}_s)_{[-,\mathscr{L}^{\mathbf{L}_s,\mathrm{et}}_t]}$; any object there has its cohomology objects locally constant along the strata $\{\mathscr{X}^{s,-}_{\mathbb{F},w}\}_w$. We will show that the restrictions of $\mathscr{F}$ along any stratum $\mathscr{X}^{s,-}_{\mathbb{F},w}$ is zero, this will imply the lemma.

We have to separate three cases. First, assume that there exists a simple root β such that $w^{-1}(\beta) \in -\Phi^+$ and $w^{-1}(\beta) \neq \alpha_s$. Then $\mathbf{U}^-_\beta$ lies in the stabilizer of $\dot{w}\mathbf{U}^s/\mathbf{U}^s$ and we can conclude as in the proof of lemma 12.6.1 that $\mathscr{F}$ is zero along this stratum.

Now assume that there exists a simple root β which is mapped to $-\alpha_s$ by w^{-1}. The root subgroup $\mathbf{U}^-_\beta$ does not belong to the stabilizer of $\dot{w}\mathbf{U}^s/\mathbf{U}^s$, but we have a commutative diagram

$$\begin{array}{ccccc}
\mathbf{U}^-_\beta \times \{\dot{w}\mathbf{U}^s/\mathbf{U}^s\} & \hookrightarrow & & & \mathbf{U}^- \times \mathscr{X}^s_{\mathbb{F}} \\
\wr \downarrow & & & & \downarrow a^{\mathbf{U}^-} \\
\mathbf{U}_\alpha \times \{\dot{w}\mathbf{U}^s/\mathbf{U}^s\} & \hookrightarrow & \mathbf{L}_s \times \mathscr{X}^s_{\mathbb{F}} & \xrightarrow{a^{\mathbf{L}_s}} & \mathscr{X}^s_{\mathbb{F}}.
\end{array}$$

The vertical isomorphism on the left is induced by conjugation by $\dot{w}^{-1} : \mathbf{U}^-_\beta \cong \mathbf{U}_\alpha$. Using the definition of the Whittaker category and lemma 2.9.3, we thus deduce an isomorphism

$$\mathscr{L}^{\mathbf{L}_s,\mathrm{et}}_{w(t)|\mathbf{U}^-_\beta} \boxtimes \mathscr{F}_{\dot{w}\mathbf{U}^s/\mathbf{U}^s} \cong \mathscr{L}_{\chi|\mathbf{U}^-_\beta} \boxtimes \mathscr{F}_{\dot{w}\mathbf{U}^s/\mathbf{U}^s}.$$

Using once again arguments similar to those of lemma 12.6.1, one sees that this is impossible.

Finally, the same kind of arguments considering the action of $\mathbf{U}^-_\alpha$ allow us to conclude in the case where w maps all the negative roots to negative roots, that is, when $w = e$. □

Corollary 12.8.6. *Consider $t, t' \in \mathfrak{o}$; whoose $w \in W$ such that $t = w(t')$. Assume that w is not minimal in the block of ${}_t\underline{W}_{t'}$ it belongs to. Then*

$$\mathrm{Av}^\chi_{[t,t']}(\mathrm{IC}(w)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}}) = 0.$$

Proof. Thanks to lemma 12.7.2, it suffices to prove that $\widehat{\Delta}^\chi_t \widehat{\star} \mathrm{IC}(w)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}} = 0$. Using lemma 7.1.5, this reduces to showing that $\Delta(\chi)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_t} \star^{\mathrm{LY}} \mathrm{IC}(w)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}} = 0$.

The point is the following: if w is not minimal in its block, using lemma 5.1.2, we can write $w = w_\beta^{\min} sv$ with $\beta \in {}_t\underline{W}_{sv(t')}$, $s \in W^\circ_{v(t')}$ (so in fact $sv(t') = v(t')$) and v some element of W. We then have

$$\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \star^{\mathrm{LY}} \mathrm{IC}(w)_{\mathscr{L}_{t'}^{\mathbf{T},\mathrm{et}}} \cong (\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \star^{\mathrm{LY}} \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_{v(t')}^{\mathbf{T},\mathrm{et}}}) \star^{\mathrm{LY}} \mathrm{IC}(sv)_{\mathscr{L}_{t'}^{\mathbf{T},\mathrm{et}}}.$$

Now, using the (étale version of the) equivalence of lemma 5.3.4 between the $\mathbf{B}_s$-Lusztig–Yun-equivariant category over $\mathscr{X}_{\mathbb{F}}^s$ and $\mathbf{T}$-Lusztig–Yun-equivariant category over $\mathscr{X}_{\mathbb{F}}$, the simple perverse sheaf $\mathrm{IC}(sv)_{\mathscr{L}_{t'}^{\mathbf{T},\mathrm{et}}}$ identifies with the object $\mathrm{For}_s^{v(t')}(\mathrm{IC}(sv)_{\mathscr{L}_{t'}^{\mathbf{L}_s,\mathrm{et}}})$. According to lemma 8.1.9, we then have

$$\Delta(\chi)_{\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}} \star_{\mathbf{B}_s}^{\mathrm{LY}} \mathrm{IC}(w)_{\mathscr{L}_{t'}^{\mathbf{B}_s,\mathrm{et}}} \cong \mathrm{Av}^s_{v(t')}(\Delta(\chi)_{\mathscr{L}_t^{\mathbf{B}_s,\mathrm{et}}} \star_{\mathbf{B}_s}^{\mathrm{LY}} \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_{v(t')}^{\mathbf{B}_s,\mathrm{et}}}) \star_{\mathbf{L}_s}^{\mathrm{LY}} \mathrm{IC}(sv)_{\mathscr{L}_{t'}^{\mathbf{L}_s,\mathrm{et}}}.$$

This is zero thanks to lemma 12.8.5. □

12.9 An explicit construction of maximal tilting objects

In this section, we use standard arguments to give an explicit construction for maximal (pro)-tilting objects in the neutral block. This construction will eventually lead to a proof of the existence of a monoidal structure on $\widehat{\mathbb{V}}_t^\circ$.

Lemma 12.9.1. *Consider $t, t' \in \mathfrak{o}$ and an element w lying in some block $\beta \in {}_{t'}\underline{W}_t$. Then we have*

$$\mathrm{Av}^\chi_{[t',t]}(\Delta(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}) \cong \mathrm{Av}^\chi_{[t',t]}(\Delta(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}),$$

and similarly,

$$\mathrm{Av}^\chi_{[t',t]}(\nabla(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}) \cong \mathrm{Av}^\chi_t(\nabla(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}).$$

Proof. We sketch the proof only in the case of standard objects, the argument being the exact same one as [BY, Lemma 4.4.8].

We know from proposition 8.6.1 that we have an embedding $\mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \hookrightarrow \Delta(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$ whose cokernel has composition factors indexed by elements $v \in \beta$ such that $v > w_\beta^{\min}$. Recalling from lemma 8.5.2 that we have identifications

$$\Delta(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \cong \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}} \cong \nabla(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}},$$

lemma 12.8.6 thus allows us to conclude. □

Lemma 12.9.2. *Consider $t, t' \in \mathfrak{o}$ and β a block of ${}_{t'}\underline{W}_t$. We have*

$$\mathrm{Av}^\chi_{[t',t]}(\Delta(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}) \cong \Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}.$$

Proof. First we notice that the endomorphisms of $\Delta(\chi)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}}$ in the Whittaker category $\mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}]}$ are given by $\overline{\mathbb{K}}$: this is a direct consequence of lemma 12.6.7. Then, by (the étale analogue of) lemma 8.3.9, convolution on the right with $\Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}}$ will induce an equivalence

$$(-) \star^{\mathrm{LY}} \Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}} : \mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}]} \cong \mathfrak{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}]}.$$

(The Whittaker condition does not play any role here.) Using corollary 12.7.4 and remark 12.6.8, we see that the object

$$\mathrm{Av}^{\chi}_{[t',t]}(\Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}})$$

is a direct sum $(\Delta(\chi)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}})^{\oplus n}$, for $n \geq 0$. Now, with the convolution description of the functor $\mathrm{Av}^{\chi}_{[t',t]}$ (see lemma 12.7.2) and (12.6.5), we have

$$\begin{aligned}
\mathrm{Av}^{\chi}_{[t',t]}(\Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}}) &\cong \mathrm{Av}^{\chi}_{[t',t]}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}} \star^{\mathrm{LY}} \Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}}) \\
&\cong \mathrm{Av}^{\chi}_{[t',t']}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}}) \star^{\mathrm{LY}} \Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}} \\
&\cong \Delta(\chi)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}} \star^{\mathrm{LY}} \Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}}.
\end{aligned}$$

Consider $\mathrm{End}(\mathrm{Av}^{\chi}_{[t',t]}(\Delta(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}}))$. The above calculation shows us that this Hom-space is isomorphic to $\mathrm{End}(\Delta(\chi)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t'}}) \cong \overline{\mathbb{K}}$. This implies that $n = 1$ and concludes the proof. □

Proposition 12.9.3. *We have isomorphisms*

$$\mathrm{Av}^{\mathbf{U}}_{*,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\widehat{\Delta}_{e,t}) \cong \bigoplus_{t' \in \mathfrak{o}} \left(\bigoplus_{\beta \in {}_{t'}\underline{W}_{t}} \widehat{\mathscr{T}}^{\mathrm{et}}_{w^{\max}_{\beta},t} \right) \cong \mathrm{Av}^{\mathbf{U}}_{!,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\widehat{\Delta}_{e,t})$$

and

$$\begin{aligned}
\mathrm{Av}^{\mathbf{U}}_{*,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}}) &\cong \bigoplus_{t' \in \mathfrak{o}} \left(\bigoplus_{\beta \in {}_{t'}\underline{W}_{t}} \mathscr{T}(w^{\max}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}} \right) \\
&\cong \mathrm{Av}^{\mathbf{U}}_{!,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}}).
\end{aligned}$$

Proof. Using lemma 9.4.2 and proposition 9.5.2, one sees that it suffices to prove the isomorphisms in the non-completed case. The strategy is almost the same for the two isomorphisms: we prove that $\mathrm{Av}^{\mathbf{U}}_{!,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}})$ is the direct sum of the projective covers of the $\mathrm{IC}(w^{\min}_{\beta})_{\mathscr{L}^{\mathbf{T},\mathrm{et}}_{t}}$'s for β as in the above direct sums and that $\mathrm{Av}^{\mathbf{U}}_{*,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}})$ is the direct sum of their injective envelopes. In view of proposition 9.3.4 (and remark 9.3.5), this will complete the proof. We will thus only prove the first isomorphism. Set $\mathscr{A} := \mathrm{Av}^{\mathbf{U}}_{!,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\Delta(e)_{\mathscr{L}^{\mathbf{T},\mathrm{et}}})$.

Consider $\mathscr{F} \in \mathfrak{P}^{\mathrm{et}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$. We have

$$\mathrm{Hom}(\mathscr{A}, \mathscr{F}) \cong \mathrm{Hom}(\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}, \mathrm{Av}^{\chi}_{[-,t]}(\mathscr{F}))$$

by adjunction, where the first Hom-space is in the Lusztig–Yun perverse category, and the second one in the Whittaker category. Now, we showed (see corollary 12.7.4) that $\mathrm{Av}^{\chi}_{[-,t]}$ is t-exact for the perverse t-structures, so that the right hand side above is a Hom-space in the category $\mathfrak{P}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}} /\!\!/ \mathbf{T})_{[-,\mathscr{L}_t^{\mathbf{T},\mathrm{et}}]}$. Remark 12.6.9 implies that the latter is equivalent to the category $\mathrm{Vect}^{\mathrm{fd}}_{\overline{\mathbb{K}}}$, and that this equivalence (let us denote it ι) maps $\overline{\mathbb{K}}$ to $\Delta(\chi)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$. We then get

$$\mathrm{Hom}(\mathscr{A}, \mathscr{F}) \cong \mathrm{Hom}_{\overline{\mathbb{K}}}(\overline{\mathbb{K}}, \iota^{-1}(\mathrm{Av}^{\chi}_{[-,t]}(\mathscr{F}))) \cong \iota^{-1}(\mathrm{Av}^{\chi}_{[-,t]}(\mathscr{F})).$$

Thus $\mathrm{Hom}(\mathscr{A}, -)$ is an exact functor, in other words, $\mathscr{A}$ is a projective object.

Let us evaluate the morphisms from $\mathscr{A}$ to various IC complexes, i.e. evaluate $\mathrm{Hom}(\mathscr{A}, \mathrm{IC}(w)_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}})$ with $w \in W$. According to corollary 12.8.6 and the above isomorphism, this is zero as soon as w is not minimal in the block of ${}_{w(t)}\underline{W}_t$ it belongs to. Thus we need to consider $\mathrm{Hom}(\mathscr{A}, \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}})$ for any $\beta \in {}_{t'}\underline{W}_t$ and $t' \in \mathfrak{o}$. Using adjunction once again and lemma 12.9.2, we see that

$$\mathrm{Hom}(\mathscr{A}, \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}) \cong \overline{\mathbb{K}}.$$

(Recall that for elements of W that are minimal in their block, the standard objects coincide with the IC-complex, as well as with the costandard objects, see lemma 8.5.2.) From (8.5.4), we have a decomposition $\mathscr{A} = \bigoplus_\beta \mathscr{A}_\beta$ where β runs through the blocks of the various ${}_{t'}\underline{W}_t$, for $t' \in \mathfrak{o}$. Since $\mathrm{Hom}(\mathscr{A}, \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}})$ $= \mathrm{Hom}(\mathscr{A}_\beta, \mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}})$, we deduce from the above calculations that $\mathscr{A}_\beta$ is the projective cover of $\mathrm{IC}(w_\beta^{\min})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$. This concludes our proof. □

12.10 Monoidal structure for $\widehat{\mathbb{V}}_t^\circ$

12.10.1 Étale case

We stick to our étale setting. We will consider the (étale analogue of the) functor $\widehat{\mathbb{V}}_t^\circ$ introduced in section 9.10. Let $\widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^\circ_{[t,t]}$ be the full subcategory of tilting perverse sheaves in the completed category $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^\circ_{[t,t]}$. In order to define the étale version $\widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}$ of $\widehat{\mathbb{V}}_t^\circ$, we use proposition 12.9.3: we can choose the object $\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}$ as the projection onto the neutral block of $\mathrm{Av}^{\mathbf{U}}_{!,t} \circ \mathrm{Av}^{\chi}_{[t,t]}(\widehat{\Delta}_{e,t})$. We effectively do so and then define

$$\begin{array}{rccl} \widehat{\mathbb{V}}_t^{\circ,\mathrm{et}} : & \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^\circ_{[t,t]} & \to & \mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t), \\ & \widehat{\mathscr{T}} & \mapsto & \mathrm{Hom}_{\widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^\circ_{[t,t]}}(\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}, \widehat{\mathscr{T}}). \end{array}$$

If we choose $\mathscr{T}(w_{t,\circ})_{\mathscr{L}_t^{\mathbf{T},\mathrm{et}}}$ to be $\pi_\dagger^{t,\mathrm{et}}(\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t})$, then the results of section 9.10 still hold in this setting, with identical proofs.

Proposition 12.10.1. *The functor* $\widehat{\mathbb{V}}_t^{\circ,\mathrm{et}} : \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})^\circ_{[t,t]} \to \mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{\widehat{R}_t^{W_t^\circ}} \widehat{R}_t)$ *admits a canonical monoidal structure.*

Proof. The proof is almost exactly the same as [BeR, Proposition 11.5], and uses the Whittaker construction in a crucial way. Recall from lemma 12.6.5 that we have a t-exact equivalence $D^b\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t) \xrightarrow{\sim} \widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ (let us denote it Ω) sending the object $\widehat{R}_t$ to $\widehat{\Delta}_t^\chi$. Recall also the functor $\mathrm{Av}^\chi_{[t,t]}$ of section 12.7. Finally, recall that the category $\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[t,t]}$ acts on the right of $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ by convolution $\widehat{\star}$, as stated in (12.6.4).

Let $\widehat{F}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ denote the full subcategory of $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ whose objects are the images under Ω of the free finitely generated $\widehat{R}_t$-modules (equivalently, the direct sums of $\widehat{\Delta}_t^\chi$). For any $\widehat{\mathscr{T}} \in \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})^\circ_{[t,t]}$, the endofunctor $(-)\widehat{\star}\widehat{\mathscr{T}}$ of $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$ restricts to an endofunctor of $\widehat{F}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$. To see that, it suffices to check that $\widehat{\Delta}_t^\chi \widehat{\star} \widehat{\mathscr{T}} = \mathrm{Av}^\chi_{[t,t]}(\widehat{\mathscr{T}})$ is in $\widehat{F}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\mathbb{K})_{[-,t]}$. Now for $i \in \mathbb{Z}$ we have

$$\begin{aligned}\mathbf{H}^i(\Omega^{-1}(\mathrm{Av}^\chi_{[t,t]}(\widehat{\mathscr{T}}))) &= \mathrm{Hom}_{\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}}(\widehat{\Delta}_t^\chi, \mathrm{Av}^\chi_{[t,t]}(\widehat{\mathscr{T}})[i]) \\ &\cong \mathrm{Hom}_{\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}}(\mathrm{Av}^{\mathbf{U}}_!(\widehat{\Delta}_t^\chi), \widehat{\mathscr{T}}[i]) \\ &\cong \mathrm{Hom}_{\widehat{D}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})^\circ_{[t,t]}}(\widehat{\mathscr{T}}_{w_{t,\circ},t}, \widehat{\mathscr{T}}[i]).\end{aligned}$$

The last isomorphism follows from lemma 12.9.2, proposition 12.9.3 and the fact that we assumed that $\widehat{\mathscr{T}}$ is in the neutral block.

According to lemma 9.4.2, the latter Hom-space is free over $\widehat{R}_t$ if $i = 0$ and is zero otherwise. Thus $\Omega^{-1}(\mathrm{Av}^\chi_{[t,t]}(\widehat{\mathscr{T}}))$ itself is free over $\widehat{R}_t$ and $\mathrm{Av}^\chi_{[t,t]}(\widehat{\mathscr{T}}) = \widehat{\Delta}_t^\chi \widehat{\star} \widehat{\mathscr{T}}$ lies in $\widehat{F}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$.

Using remark 12.6.6, one deduces that the functor $(-)\widehat{\star}\widehat{\mathscr{T}}$ is right t-exact for the perverse t-structure on $\widehat{D}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[-,t]}$. Thus, we obtain a right exact (additive) functor

$$\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t) \xrightarrow[\sim]{\Omega} \widehat{\mathrm{P}}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[t,t]} \xrightarrow{{}^p\mathscr{H}^0((-)\widehat{\star}\widehat{\mathscr{T}})} \widehat{\mathrm{P}}^{\mathrm{et}}_{\mathrm{Wh}}(\mathscr{X}_{\mathbb{F}},\overline{\mathbb{K}})_{[t,t]} \xrightarrow[\sim]{\Omega^{-1}} \mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t). \tag{12.10.1}$$

It is well known that such an endofunctor on some category of modules is isomorphic to the functor of tensoring on the right with the image of the base ring, here $\widehat{R}_t$ (this is known as the Eilenberg–Watt's theorem). Thus, for any M in $\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t)$, we have

$$\Omega^{-1}({}^p\mathscr{H}^0(\Omega(M)\widehat{\star}\widehat{\mathscr{T}})) \cong M \otimes_{\widehat{R}_t} \Omega^{-1}(\widehat{\Delta}_t^\chi \widehat{\star} \widehat{\mathscr{T}}) \cong M \otimes_{\widehat{R}_t} \widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}(\widehat{\mathscr{T}}).$$

Now consider two tilting objects $\widehat{\mathscr{T}}, \widehat{\mathscr{T}}'$; usual features of (right) t-exact functors tells us that the functor (12.10.1) obtained from the tilting object $\widehat{\mathscr{T}}\widehat{\star}\widehat{\mathscr{T}}'$ coincide with the composition of the functors obtained first from $\widehat{\mathscr{T}}$ and then $\widehat{\mathscr{T}}'$. this implies that we have a canonical isomorphism

$$\widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}(\widehat{\mathscr{T}}\widehat{\star}\widehat{\mathscr{T}}') \cong \widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}(\widehat{\mathscr{T}}) \otimes_{\widehat{R}_t} \widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}(\widehat{\mathscr{T}}').$$

the functor $\widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}$ thus respects the monoidal products.

Moreover, for $\widehat{\mathscr{T}} = \widehat{\Delta}_{e,t}$, the functor (12.10.1) is just the identity; this readily implies that

$$\widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}(\widehat{\Delta}_{e,t}) = \widehat{R}_t.$$

The verification that the above isomorphism respect the associativity and unit constraint is easy, and left to the reader. □

12.10.2 Analytic case

We now want to obtain the monoidality of $\widehat{\mathbb{V}}_t^{\circ}$ on the non-étale (that is, analytic) setting.

Putting lemma 12.4.2 and corollary 12.5.4 together, we obtain a monoidal equivalence of categories

$$(\widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})^{\circ}_{[t,t]}, \widehat{\star}) \leftrightarrow (\widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^{\circ}_{[t,t]}, \widehat{\star}). \tag{12.10.2}$$

From proposition 12.9.3 (and its proof for notation), we have a canonical object $\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t} := \mathscr{A}_{W_t^{\circ}}$ in $\widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^{\circ}_{[t,t]}$. We let $\widehat{\mathscr{T}}_{w_{t,\circ},t}$ be the corresponding object in $\widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})^{\circ}_{[t,t]}$ under the equivalence (12.10.2). It follows from proposition 12.5.3 that the action of an element $x \in \widehat{R}_t$ on $\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}$ corresponds to the action of the same element on $\widehat{\mathscr{T}}_{w_{t,\circ},t}$, that is, the image of the automorphism $\varphi_{\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}}(r)$ under (12.10.2) is $\varphi_{\widehat{\mathscr{T}}_{w_{t,\circ},t}}(r)$. This readily implies that the structure of $\widehat{R}_t \otimes_{\widehat{R}_t^{W_t^{\circ}}} \widehat{R}_t$-module on $\mathrm{Hom}(\widehat{\mathscr{T}}^{\mathrm{et}}_{w_{t,\circ},t}, \widehat{\mathscr{T}}^{\mathrm{et}})$ and $\mathrm{Hom}(\widehat{\mathscr{T}}_{w_{t,\circ},t}, \widehat{\mathscr{T}})$ (where $\widehat{\mathscr{T}}^{\mathrm{et}} \in \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^{\circ}_{[t,t]}$ is any object and $\widehat{\mathscr{T}}$ is its image under (12.10.2)) coincide. We can restate this fact as the commutativity of the following diagram

$$\begin{array}{ccc} \widehat{T}_{(B)}(\mathscr{X}, \overline{\mathbb{K}})^{\circ}_{[t,t]} & \xrightarrow{\sim} & \widehat{T}^{\mathrm{et}}_{\mathcal{M}}(\mathscr{X}_{\mathbb{F}}, \overline{\mathbb{K}})^{\circ}_{[t,t]} \\ & {\scriptstyle \widehat{\mathbb{V}}_t^{\circ,\overline{\mathbb{K}}}} \searrow \quad \swarrow {\scriptstyle \widehat{\mathbb{V}}_t^{\circ,\mathrm{et}}} & \\ & \mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^{\circ}}} \widehat{R}_t). & \end{array}$$

We deduce from proposition 12.10.1 the following key result:

Proposition 12.10.2. *The functor*

$$\widehat{\mathbb{V}}_t^\circ : \widehat{T}_{(B)}(\mathscr{X}, \mathbb{K})^\circ_{[t,t]} \to \mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t)$$

has a monoidal structure for the convolution product $(-)\widehat{\star}(-)$ *on the left and the tensor product* $(-) \otimes_{\widehat{R}_t} (-)$ *on the right.*

Chapter 13

Soergel theory and main results

In this chapter, we make the following assumption:

The characteristic ℓ of the field $\Bbbk$ is not a torsion prime for the dual group $G^\vee_\Bbbk$ (equivalently, for the derived subgroup $\mathscr{D}(G^\vee_\Bbbk)$).

13.1 Main theorems

We now have all the ingredients needed in order to show our main theorem, which gives the description of the tilting subcategories in the neutral block. Let us fix an element $t \in T^\vee_\Bbbk$.

Theorem 13.1.1. *1. The functor $\widehat{\mathbb{V}}^\circ_t$ induces a monoidal equivalence of categories*

$$(\widehat{T}_{(B)}(\mathscr{X}, \Bbbk)^\circ_{[t,t]}, \widehat{\star}) \xrightarrow{\sim} (\mathbb{S}\operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^\circ_t}} \widehat{R}_t), \otimes_{\widehat{R}_t}),$$

where $\mathbb{S}\operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^\circ_t}} \widehat{R}_t)$ is the full subcategory of $\operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^\circ_t}} \widehat{R}_t)$ generated under tensor products, direct sums and direct summands by the objects $\widehat{R}_t$ and $\widehat{R}_t \otimes_{(\widehat{R}_t)^s} \widehat{R}_t$ for $s \in S_t$.

2. The functor $\mathbb{V}^\circ_t$ induces an equivalence of categories

$$\mathfrak{T}(\mathscr{X} /\!\!/ T)^\circ_{[t,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathbb{S}\operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)^{W^\circ_t}_+),$$

where $\mathbb{S}\operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)^{W^\circ_t}_+)$ is the full subcategory of $\operatorname{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^\circ_t}} \Bbbk_t)$ generated under direct sums, direct summands and application of $\widehat{R}_t \otimes_{(\widehat{R}_t)^s} (-)$ for $s \in S_t$ to the object $\Bbbk_t$.

3. The following diagram is commutative up to isomorphism

$$\begin{array}{ccc} \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} & \xrightarrow[\sim]{\widehat{\mathbb{V}}^{\circ}_t} & \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t) \\ \Big\downarrow{\scriptstyle \pi^t_\dagger} & & \Big\downarrow{\scriptstyle (-)\otimes_{\widehat{R}_t}\Bbbk_t} \\ \mathfrak{T}(\mathscr{X}\,/\!\!/T)^{\circ}_{[t,\mathscr{L}^T_t]} & \xrightarrow[\sim]{\mathbb{V}^{\circ}_t} & \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)^{W^{\circ}_t}_{+}), \end{array}$$

and the action of $\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]}$ by convolution on the left of the Lusztig–Yun category $\mathfrak{T}(\mathscr{X}\,/\!\!/T)^{\circ}_{[t,\mathscr{L}^T_t]}$ identifies with the action of $\mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t)$ on $\mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)^{W^{\circ}_t}_{+})$ by tensoring over $\widehat{R}_t$ on the left.

Proof. We prove point 1. First, thanks to theorem 10.4.1, the functor $\widehat{\mathbb{V}}^{\circ}_t$ indeed lands in $\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \widehat{R}_t)$. Now, according to proposition 9.10.2, this functor is fully faithful. Since it is monoidal, in order to prove our point, we have to determine its essential image. Corollary 9.9.2 implies that in order to determine the essential image of $\widehat{\mathbb{V}}^{\circ}_t$, it suffices to describe the images of the objects $\widehat{\mathscr{T}}_{s,t}$ with $s \in W^{\circ}_t$ (then, we just need to allow direct sums, tensor products over $\widehat{R}_t$ and direct summands). But lemma 10.4.9 gives us precisely $\widehat{\mathbb{V}}^{\circ}_t(\widehat{\mathscr{T}}_{s,t}) \cong \widehat{R}_t \otimes_{(\widehat{R}_t)^s} \widehat{R}_t$ for $s \in S_t$, whence our claim.

The proof of points 2. is similar: using corollary 10.4.3, the functor $\mathbb{V}^{\circ}_t$ takes its values in $\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W^{\circ}_t}} \Bbbk_t)$; proposition 9.10.2 asserts that it is fully faithful. To determine its essential image, we use corollary 9.9.2 once again: it suffices to determine the image of

$$\widehat{\mathscr{T}}_{s_1,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s_r,t}\widehat{\star}\,\Delta(e)_{\mathscr{L}^T_t} \cong \pi^t_\dagger(\widehat{\mathscr{T}}_{s_1,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s_r,t}))$$

for any sequence $(s_1,\dots,s_r)$ of elements of S_t. But it follows from lemma 9.4.2, point 2., that

$$\mathbb{V}^{\circ}_t(\pi^t_\dagger(\widehat{\mathscr{T}}_{s_1,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s_r,t})) \cong \widehat{\mathbb{V}}^{\circ}_t(\widehat{\mathscr{T}}_{s_1,t}\widehat{\star}\cdots\widehat{\star}\widehat{\mathscr{T}}_{s_r,t}) \otimes_{\widehat{R}_t} \Bbbk_t\,.$$

We deduce the wished for description of the essential image of $\mathbb{V}^{\circ}_t$ from the description of the essential image of $\widehat{\mathbb{V}}^{\circ}_t$ established in point 1.

Finally, 3. follows directly from points 1. and 2. using lemma 9.4.2. □

Now, fix three elements $t,t',t'' \in T^\vee_{\Bbbk}$ in the same W-orbit and two blocks $\beta \in {}_{t'}\underline{W}_t$ and $\gamma \in {}_{t''}\underline{W}_{t'}$. For a right $\widehat{R}_{t'}$-module M and a left $\widehat{R}_t$-module N, we set

$$M \otimes^{\beta}_{\widehat{R}_t} N$$

for the tensor product over $\widehat{R}_t$, where M is viewed as a $\widehat{R}_t$-module under the action $m \cdot x = m w^{\min}_\beta(x)$ (here, the right hand side denotes the natural action of the element $w^{\min}_\beta(x) \in \widehat{R}_{t'}$ on $m \in M$). We will apply the same convention for tensor product over any subring of $\widehat{R}_t$, and more generally to any block.

In a way completely analogous to the study of diagram (10.4.4) (and with proposition 9.7.6) one sees that the functor $\widehat{\Delta}_{w_\gamma^{\min},t'}\,\widehat{\star}(-)\widehat{\star}\,\widehat{\Delta}_{w_\beta^{\min},t}$ induces an isomorphism $\mathrm{End}(\widehat{\mathscr{T}}_{w_{\circ,t'},t'}) \xrightarrow{\sim} \mathrm{End}(\widehat{\mathscr{T}}_{w_{\gamma\beta}^{\max},t})$ independent of any choice, and that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{R}_{t'} \otimes_{(\widehat{R}_{t'})^{W_{t'}^\circ}} \widehat{R}_{t'} & \xrightarrow[\sim]{\varphi_{\widehat{\mathscr{T}}_{w_{\circ,t'},t'}}} & \mathrm{End}(\widehat{\mathscr{T}}_{w_{\circ,t'},t'}) \\ {\scriptstyle w_\gamma^{\min}(-)\otimes w_\beta^{\min,-1}(-)}\downarrow & & \downarrow{\scriptstyle \widehat{\Delta}_{w_\gamma^{\min},t'}\,\widehat{\star}(-)\widehat{\star}\,\widehat{\Delta}_{w_\beta^{\min},t}} \\ \widehat{R}_{t''} \otimes^{\gamma\beta}_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t & \xrightarrow[\sim]{\varphi_{\widehat{\mathscr{T}}_{w_{\gamma\beta}^{\max},t}}} & \mathrm{End}(\widehat{\mathscr{T}}_{w_{\gamma\beta}^{\max},t}). \end{array} \tag{13.1.1}$$

The above discussion and lemma 9.7.1 readily imply that the following diagram is commutative (up to isomorphism):

$$\begin{array}{ccc} \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^\circ_{[t',t']} & \xrightarrow[\sim]{\widehat{\mathbb{V}}^\circ_{t'}} & \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_{t'} \otimes_{(\widehat{R}_{t'})^{W_{t'}^\circ}} \widehat{R}_{t'}) \\ {\scriptstyle \widehat{\Delta}_{w_\gamma^{\min},t'}\,\widehat{\star}(-)\widehat{\star}\,\widehat{\Delta}_{w_\beta^{\min},t}}\downarrow & \wr & \downarrow{\scriptstyle \widehat{R}_{t''}\otimes^\gamma_{\widehat{R}_{t'}}(-)\otimes^\beta_{\widehat{R}_t}\widehat{R}_t} \\ \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\gamma\beta}_{[t'',t]} & \xrightarrow{\widehat{\mathbb{V}}^{\gamma\beta}_t} & \mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_{t''} \otimes^{\gamma\beta}_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t). \end{array} \tag{13.1.2}$$

We can now proceed to describe the monodromic categories for any block $\beta \in {}_{t'}\underline{W}_t$:

Theorem 13.1.2. *1. We have an equivalence of categories*

$$\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^\beta_{[t',t]} \xrightarrow[\sim]{\widehat{\mathbb{V}}^\beta_t} \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}_\beta(\widehat{R}_{t'} \otimes^\beta_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t)$$

where $\mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}_\beta(\widehat{R}_{t'} \otimes^\beta_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t)$ *is the full subactegory of* $\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_{t'} \otimes^\beta_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t)$ *whose objects are the* $\widehat{R}_{t'} \otimes^\beta_{\widehat{R}_t} N$ *with* $N \in \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t)$.

2. The functor $\mathbb{V}^\beta_t$ *induces an equivalence of categories*

$$\mathfrak{T}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]} \xrightarrow{\sim} \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}_\beta(\widehat{R}_{t'}/(\widehat{R}_t)^{W_t^\circ}_+),$$

where $\mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}_\beta(\widehat{R}_{t'}/(\widehat{R}_t)^{W_t^\circ}_+)$ *is the full subcategory of* $\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_{t'} \otimes^\beta_{(\widehat{R}_t)^{W_t^\circ}} \Bbbk_t)$ *whose objects are those* $\widehat{R}_{t'} \otimes^\beta_{\widehat{R}_t} N$ *with* $N \in \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}(\widehat{R}_t/(\widehat{R}_t)^{W_t^\circ}_+)$.

3. The following diagram is commutative up to isomorphism

$$\begin{array}{ccc} \widehat{T}_{(B)}(\mathscr{X},\Bbbk)^\beta_{[t',t]} & \xrightarrow[\sim]{\widehat{\mathbb{V}}^\beta_t} & \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}_\beta(\widehat{R}_{t'} \otimes^\beta_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t) \\ {\scriptstyle \pi^t_\dagger}\downarrow & & \downarrow{\scriptstyle (-)\otimes_{\widehat{R}_t}\Bbbk_t} \\ \mathfrak{T}(\mathscr{X}/\!\!/T)^\beta_{[t',\mathscr{L}^T_t]} & \xrightarrow[\sim]{\mathbb{V}^\beta_t} & \mathbb{S}\,\mathrm{Mod}^{\mathrm{fg}}_\beta(\widehat{R}_{t'}/(\widehat{R}_t)^{W_t^\circ}_+). \end{array}$$

Proof. The theorem is a direct consequence of theorem 13.1.1 and diagrams (13.1.1) and (13.1.2) □

Consider $\underline{w} = (s_1, \dots, s_r)$ any expression of simple reflections in W_t° (that is, a word in the alphabet S_t). Let

$$\mathcal{BS}(\underline{w}) := \widehat{R}_t \otimes_{(\widehat{R}_t)^{s_1}} \widehat{R}_t \otimes_{(\widehat{R}_t)^{s_2}} \cdots \otimes_{(\widehat{R}_t)^{s_r}} \Bbbk_t \,.$$

In the following theorem, we fix (arbitrarily) a reduced expression $\underline{w}$ in W_t° for any w in this group.

Recall that a projective generator P in an abelian category $\mathcal{A}$ admitting arbitrary coproducts is a projective object such that any object X in $\mathcal{A}$ is a quotient of a direct sum $P^{(I)}$, for some set I depending on X (see e.g. [Ba, Chapter II] for more information about projective generators). If one assumes that $\mathcal{A}$ is finite length (i.e. its objects have the finite length property), the set I as above can be chosen to be finite; it is then a standard fact (see [Ba, Exercise on p. 55]) that if P is a projective generator, the functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Mod}^{\mathrm{fg}}(\mathrm{End}_{\mathcal{A}}(P)^{\mathrm{op}})$ from $\mathcal{A}$ to the category of finitely generated left $\mathrm{End}_{\mathcal{A}}(P)^{\mathrm{op}}$-modules is an equivalence of categories.

Theorem 13.1.3. *Consider $t, t' \in T^\vee_{\Bbbk}$ in the same W-orbit and $\beta \in {}_{t'}\underline{W}_t$. Set*

$$A := \left(\mathrm{End}_{\widehat{R}_t} \left(\bigoplus_{w \in W_t^\circ} \mathcal{BS}(\underline{w}) \right) \right)^{\mathrm{op}} .$$

Then the category $\mathfrak{P}^\beta_{[t', \mathscr{L}_t^T]}$ is equivalent to the category $\mathrm{Mod}^{\mathrm{fg}}(A)$ of finitely generated left A-modules.

Proof. For any sequence $\underline{w} = (s_1, \dots, s_r)$ of elements of S_t, we set

$$\mathscr{T}(\underline{w}) := \widehat{\mathscr{T}}_{s_1,t} \widehat{\star} (\cdots) \widehat{\star} \widehat{\mathscr{T}}_{s_r,t} \widehat{\star} \Delta(e)_{\mathscr{L}_t^T} = \pi_t^t (\widehat{\mathscr{T}}(\underline{w})).$$

This object is tilting in the neutral block, so proposition 9.10.2 gives us an isomorphism

$$\mathrm{End}(\mathscr{T}(\underline{w})) \cong \mathrm{End}(\mathbb{V}_t^\circ(\mathscr{T}(\underline{w}))).$$

Now thanks to theorem 13.1.1, $\mathbb{V}_t^\circ(\mathscr{T}(\underline{w}))$ is given by $\mathcal{BS}(\underline{w})$. Hence, we have $\mathrm{End}(\bigoplus_{w \in W_t^\circ} \mathscr{T}(\underline{w})) \cong A^{\mathrm{op}}$.

Using proposition 9.2.2, we see that the object

$$\mathscr{P}_t^\beta := \widehat{\Delta}_{w_\beta^{\max}, t} \widehat{\star} \left(\bigoplus_{w \in W_t^\circ} \mathscr{T}(\underline{w}) \right)$$

is projective; in fact, it is a projective generator of $\mathfrak{P}^\beta_{[t', \mathscr{L}_t^T]}$: indeed, by construction, it admits a nonzero map to any simple object in $\mathfrak{P}^\beta_{[t', \mathscr{L}_t^T]}$. Moreover, its endomorphism ring is isomorphic to A^{op} since $\widehat{\Delta}_{w_\beta^{\max}, t} \widehat{\star}(-)$ is an equivalence according to lemma 8.4.3. The theorem then follows from the general fact about projective generators recalled before the statement of the theorem. □

13.2 Equivalence for the neutral block

In this section, we focus one more time on the neutral block: we show that the Lusztig–Yun equivariant derived category is equivalent to the derived bounded category of Bruhat constructible sheaves on the flag variety of the endoscopic group.

Recall the endoscopic group H_t defined in 10.2. This is a complex reductive group with maximal torus T and whose Weyl group is canonically identified with W_t°. We consider the Borel subgroup B_t of H_t corresponding to the positive roots Φ_t^+.

We obviously have the analogue of theorem 13.1.1 for H_t and monodromy given by $[1, \underline{\Bbbk}_T]$. We will denote the category constructed from H_t as the ones for G, but with a superscript $\mathrm{end}(t)$. Finally, let $\pi_\dagger^{1,\mathrm{end}(t)}$ be the analogue of $\pi_\dagger^t$ on the endoscopic side, that is to say, the functor $\widehat{D}_{[1,1]}^{\mathrm{end}(t)} \to \mathfrak{D}_{[1,\underline{\Bbbk}_T]}^{\mathrm{end}(t)}$ given by convolution on the right with $\Delta(e)_{\underline{\Bbbk}_T}^{\mathrm{end}(t)}$. In the next theorem, we consider the restriction of this functor to the subcategory of tilting objects.

Theorem 13.2.1. *1. There is a monoidal equivalence of categories*

$$\widehat{\mathfrak{e}} : (\widehat{T}_{[t,t]}^\circ, \widehat{\star}) \xrightarrow{\sim} (\widehat{T}_{[1,1]}^{\mathrm{end}(t)}, \widehat{\star})$$

mapping $\widehat{\mathscr{T}}_{w,t}$ *to* $\widehat{\mathscr{T}}_{w,1}^{\mathrm{end}(t)}$.

2. There is an equivalence of categories

$$\mathfrak{e} : \mathfrak{T}_{[t,\mathscr{L}_t^T]}^\circ \xrightarrow{\sim} \mathfrak{T}_{[1,\underline{\Bbbk}_T]}^{\mathrm{end}(t)}$$

mapping $\mathscr{T}(w)_{\mathscr{L}_t^T}$ *to* $\mathscr{T}(w)_{\underline{\Bbbk}_T}^{\mathrm{end}(t)}$ *for any* $w \in W_t^\circ$.

3. We have an isomorphism of functors $\mathfrak{e} \circ \pi_\dagger^t \cong \pi_\dagger^{1,\mathrm{end}(t)} \circ \widehat{\mathfrak{e}}$.

Before the proof, let us state a fact: consider $w \in W_t^\circ$ and any reduced expression $(s_1, \dots, s_r)$ of w in this group. Then $\widehat{\mathbb{V}}_t^\circ(\widehat{\mathscr{T}}_{w,t})$ can be characterized as the unique direct summand in $\widehat{\mathcal{BS}}(\underline{w}) = \widehat{R}_t \otimes_{(\widehat{R}_t)^{s_1}} \widehat{R}_t \otimes_{(\widehat{R}_t)^{s_2}} \cdots \otimes_{(\widehat{R}_t)^{s_r}} \widehat{R}_t$ that does not appear as a direct summand in any $\widehat{\mathcal{BS}}(\underline{v})$ for any $v \in W_t^\circ$ such that $\ell_t(v) < \ell_t(w)$ (with ℓ_t the length function of the Coxeter system (W_t°, S_t)). This is an immediate consequence of corollary 9.9.4. Of course, similar statements hold for $\mathscr{T}(w)_{\mathscr{L}_t^T}$, replacing $\widehat{\mathcal{BS}}(?)$ by $\mathcal{BS}(?)$ (for $? \in W_t^\circ$), as well as on the endoscopic side.

Proof. The existence of $\widehat{\mathfrak{e}}$ and $\mathfrak{e}$ are immediate consequences of theorem 13.1.1 for G and H_t and the following fact: there exists a W_t°-equivariant isomorphism $\widehat{R}_t \cong \widehat{R}_1$, which induces an additive and monoidal equivalence of categories between the category of $\widehat{R}_t \otimes_{(\widehat{R}_t)^{W_t^\circ}} \widehat{R}_t$-modules and $\widehat{R}_1 \otimes_{(\widehat{R}_1)^{W_t^\circ}} \widehat{R}_1$. In order to

see that, one considers the map given by multiplication by t on $T^\vee_{\Bbbk}$, sending x to xt. This map is an isomorphism (in particular, it is étale), is W_t°-equivariant and maps 1 to t. Both of these points are stabilized by W_t°, thus our morphism induces a W_t°-equivariant isomorphism on the completions of the local rings at 1 and t at their respective maximal ideals, whence our claim.

The proofs that $\widehat{\mathscr{T}}_{w,t}$ is mapped to $\widehat{\mathscr{T}}^{\mathrm{end}(t)}_{w,1}$ and $\mathscr{T}(w)_{\mathscr{L}^T_t}$ to $\mathscr{T}(w)^{\mathrm{end}(t)}_{\underline{\Bbbk}_T}$ are similar, thus we only consider the completed case. The fact that $\widehat{\mathfrak{e}}(\widehat{\mathscr{T}}_{w,t}) \cong \widehat{\mathscr{T}}^{\mathrm{end}(t)}_{w,1}$ follows from the following observation: the equivalence induced by the W_t°-equivariant isomorphism $\widehat{R}_t \cong \widehat{R}_1$ described above clearly swaps the modules

$$\widehat{R}_t \otimes_{(\widehat{R}_t)^{s_1}} \cdots \otimes_{(\widehat{R}_t)^{s_r}} \widehat{R}_t$$

and

$$\widehat{R}_1 \otimes_{(\widehat{R}_1)^{s_1}} \cdots \otimes_{(\widehat{R}_1)^{s_r}} \widehat{R}_1$$

for any $s_1, \dots, s_r$ in S_t. Now, using the characterisation of $\widehat{\mathbb{V}}^\circ_t(\widehat{\mathscr{T}}_{w,t})$ stated above the proof, our claim is clear.

Finally, note that since any tilting object is a direct sum of indecomposable ones and that $\pi^t_\dagger$ maps $\widehat{\mathscr{T}}_{w,t}$ to $\mathscr{T}(w)_{\mathscr{L}^T_t}$ (resp. $\pi^{1,\mathrm{end}(t)}_\dagger$ maps $\widehat{\mathscr{T}}^{\mathrm{end}(t)}_{w,1}$ to $\mathscr{T}(w)^{\mathrm{end}(t)}_{\underline{\Bbbk}_T}$), the above reasoning (together with theorem 13.1.1 for G and H_t) implies point 3. of our statement. □

We consider the flag variety of H_t: set $\mathscr{Y}^{\mathrm{end}(t)} := H_t/B_t$ and $\mathscr{Y}^{\mathrm{end}(t)}_w := B_t w B_t/B_t$ for $w \in W_t^\circ$. Recall our fixed Borel subgroup B_t of H_t; let U_t be its unipotent radical. Note that we have a canonical, t-exact, equivalence of triangulated categories $\mathfrak{D}^{\mathrm{end}(t)}_{[1,\underline{\Bbbk}_T]} = D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)}, \Bbbk)$. Indeed, it follows directly from the construction of the Lusztig–Yun equivariant category performed in sections 2.2 and 2.9 that $\mathfrak{D}^{\mathrm{end}(t)}_{[1,\underline{\Bbbk}_T]}$ identifies with the (right) T-equivariant category $D^b_{(B_t),T}(H_t/U_t, \Bbbk)$, which in turn is equivalent to $D^b_{(B_t)}(H_t/B_t, \Bbbk)$ thanks to the quotient equivalence of [BL, §2.6.2, Theorem]. This equivalence is perverse t-exact; for $w \in W_t^\circ$, we denote the image of the objects $\Delta(w)^{\mathrm{end}(t)}_{\underline{\Bbbk}_T}$, $\nabla(w)^{\mathrm{end}(t)}_{\underline{\Bbbk}_T}$, $\mathrm{IC}(w)^{\mathrm{end}(t)}_{\underline{\Bbbk}_T}$ and $\mathscr{T}(w)^{\mathrm{end}(t)}_{\underline{\Bbbk}_T}$ respectively by $\Delta^{\mathrm{end}(t)}_w$, $\nabla^{\mathrm{end}(t)}_w$, $\mathrm{IC}^{\mathrm{end}(t)}_w$ and $\mathscr{T}^{\mathrm{end}(t)}_w$.

Lemma 13.2.2. *There exists equivalences of triangulated categories*

$$K^b\widehat{T}_{(B)}(\mathscr{X}, \Bbbk)^\circ_{[t,t]} \xrightarrow{\sim} \widehat{D}^\circ_{[t,t]},$$

$$K^b\mathfrak{T}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{D}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}^T_t]}.$$

Proof sketch. We use the fact that both $D^b_{(B)}(\mathscr{X}, \Bbbk)^\circ_{[t,t]}$ and $\mathfrak{D}(\mathscr{X}/\!\!/T)^\circ_{[t,\mathscr{L}^T_t]}$ admit filtered versions. The argument is the same in both cases; we treat only the completed case. The exact same procedure as in [BY, Remark A.2.3]

allows one to construct a filtered version (in the sense of [Bei, Definition A.1, (c)]) of the category $\widehat{D}^{\circ}_{[t,t]}$. As explained in [AMRW, Proposition 2.2], we are then able to construct a functor $K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} \xrightarrow{\sim} \widehat{D}^{\circ}_{[t,t]}$ whose restriction to $\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} \subseteq K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]}$ is the natural inclusion functor.

Let us justify the fact that this functor is an equivalence: full faithfulness is a direct consequence of the description of Hom-spaces between tilting objects in the completed category given in lemma 9.4.2, 2.: for $\widehat{\mathscr{T}}, \widehat{\mathscr{T}}'$ to such objects, we have $\mathrm{Hom}_{\widehat{D}^{\circ}_{[t,t]}}(\widehat{\mathscr{T}}, \widehat{\mathscr{T}}'[k]) = 0$ for $k \neq 0$. Now for the essential surjectivity, one sees that the subcategory of tilting objects in $\widehat{D}^{\circ}_{[t,t]}$ generates this category. Indeed, we know that the family $\{\widehat{\Delta}_{w,t} \mid w \in W_t^{\circ}\}$ generates $\widehat{D}^{\circ}_{[t,t]}$ thanks to lemma 7.8.1, and any such standard object appear as a subobject in some pro-tilting. We are done. □

From lemma 9.7.5, one get that the convolution product $(-)\widehat{\star}(-)$ on $\widehat{D}^{\circ}_{[t,t]}$ restrict to a bifunctor $\widehat{T}^{\circ}_{[t,t]} \times \widehat{T}^{\circ}_{[t,t]} \to \widehat{T}^{\circ}_{[t,t]}$. We deduce a "convolution bifunctor", triangulated in both entries

$$(-)\widehat{\star}_{K^b}(-) : K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} \times K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} \to K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]};$$

one easily checks that the equivalence $K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} \xrightarrow{\sim} \widehat{D}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]}$ of lemma 13.2.2 intertwines $(-)\widehat{\star}_{K^b}(-)$ and $(-)\widehat{\star}(-)$.

Similarly, we obtain a bifunctor, triangulated in both entries

$$K^b\widehat{T}_{(B)}(\mathscr{X},\Bbbk)^{\circ}_{[t,t]} \times K^b\mathfrak{T}(\mathscr{X}/\!\!/T)^{\circ}_{[t,\mathscr{L}_t^T]} \to K^b\mathfrak{T}(\mathscr{X}/\!\!/T)^{\circ}_{[t,\mathscr{L}_t^T]},$$

that is compatible, in an obvious sense, with the equivalences of lemma 13.2.2.

Theorem 13.2.3. *1. There exists a monoidal equivalence of categories*

$$\widehat{\mathfrak{E}} : (\widehat{D}^{\circ}_{[t,t]}, \widehat{\star}) \xrightarrow{\sim} (\widehat{D}^{\mathrm{end}(t)}_{[1,1]}, \widehat{\star})$$

mapping $\widehat{\mathscr{T}}_{w,t}$ to $\widehat{\mathscr{T}}^{\mathrm{end}(t)}_{w,1}$, $\widehat{\Delta}_{w,t}$ to $\widehat{\Delta}^{\mathrm{end}(t)}_{w,1}$ and $\widehat{\nabla}_{w,t}$ to $\widehat{\nabla}^{\mathrm{end}(t)}_{w,1}$ for any $w \in W_t^{\circ}$.

2. There exists a t-exact (for the perverse t-structures) equivalences of categories

$$\mathfrak{E} : \mathfrak{D}^{\circ}_{[t,\mathscr{L}_t^T]} \xrightarrow{\sim} \mathfrak{D}^{\mathrm{end}(t)}_{[1,\underline{\Bbbk}_T]} = D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)}, \Bbbk)$$

mapping $\mathscr{T}(w)_{\mathscr{L}_t^T}$ to $\mathscr{T}^{\mathrm{end}(t)}_w$, $\Delta(w)_{\mathscr{L}_t^T}$ to $\Delta^{\mathrm{end}(t)}_w$, $\nabla(w)_{\mathscr{L}_t^T}$ to $\nabla^{\mathrm{end}(t)}_w$ and $\mathrm{IC}(w)_{\mathscr{L}_t^T}$ to $\mathrm{IC}^{\mathrm{end}(t)}_w$ for any $w \in W_t^{\circ}$.

3. *These equivalences satisfy* $\pi_\dagger^{1,\mathrm{end}(t)} \circ \widehat{\mathfrak{E}} \cong \mathfrak{E} \circ \pi_\dagger^t$, *i.e. we have a commutative (up to isomorphism of functors) diagram*

$$\begin{array}{ccc} \widehat{D}^\circ_{[t,t]} & \xrightarrow[\sim]{\widehat{\mathfrak{E}}} & \widehat{D}^{\mathrm{end}(t)}_{[1,1]} \\ \Big\downarrow{\scriptstyle \pi_\dagger^t} & & \Big\downarrow{\scriptstyle \pi_\dagger^{1,\mathrm{end}(t)}} \\ \mathfrak{D}^\circ_{[t,\mathscr{L}_t^T]} & \xrightarrow[\mathfrak{E}]{\sim} & \mathfrak{D}^{\mathrm{end}(t)}_{[1,\mathbb{k}_T]}. \end{array}$$

4. *The action of* $\widehat{D}^\circ_{[t,t]}$ *by convolution on the left on any on the category on the left column of the above diagram identifies with the action of* $\widehat{D}^{\mathrm{end}(t)}_{[1,1]}$ *by convolution on the left on the corresponding category on the right column.*

We will need an intermediary result before the proof of the theorem.

Lemma 13.2.4. *Assume that we have an object* $\mathscr{F} \in D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)}, \mathbb{k})$ *and a distinguished triangle*

$$\mathscr{F} \to \mathscr{T}_w^{\mathrm{end}(t)} \to C \xrightarrow{+1} \tag{13.2.1}$$

such that:

1. *the object* C *is supported on* $\mathscr{Y}^{\mathrm{end}(t)}_{<_t w} = \sqcup_{v <_t w} \mathscr{Y}^{\mathrm{end}(t)}_v$,
2. $\mathrm{Hom}(\mathscr{F}, \mathscr{G}[n]) = 0$ *for any object* $\mathscr{G}$ *supported on* $\mathscr{Y}^{\mathrm{end}(t)}_{<_t w}$ *and* $n \in \mathbb{Z}$.

Then we have $\mathscr{F} \cong \Delta_w^{\mathrm{end}(t)}$.

Similarly, assume that we have an object $\mathscr{G} \in D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)}, \mathbb{k})$ *and a distinguished triangle*

$$K \to \mathscr{T}_w^{\mathrm{end}(t)} \to \mathscr{G} \xrightarrow{+1} \tag{13.2.2}$$

such that:

1. *the object* K *is supported on* $\mathscr{Y}^{\mathrm{end}(t)}_{<_t w}$,
2. $\mathrm{Hom}(\mathscr{G}[n], \mathscr{G}) = 0$ *for any object* $\mathscr{G}$ *supported on* $\mathscr{Y}^{\mathrm{end}(t)}_{<_t w}$ *and* $n \in \mathbb{Z}$.

Then we have $\mathscr{G} \cong \nabla_w^{\mathrm{end}(t)}$.

Proof. We only prove the first statement, the proof of the second one being similar. Note that the hypotheses of the lemma hold for the object $\Delta_w^{\mathrm{end}(t)}$. Applying the functor $\mathrm{Hom}(\Delta_w^{\mathrm{end}(t)}, -)$ to the triangle (13.2.1), we get a map $a : \Delta_w^{\mathrm{end}(t)} \to \mathscr{F}$ such that the composition

$$\Delta_w^{\mathrm{end}(t)} \to \mathscr{F} \to \mathscr{T}_w^{\mathrm{end}(t)}$$

identifies with the canonical morphism $\Delta_w^{\mathrm{end}(t)} \to \mathscr{T}_w^{\mathrm{end}(t)}$. In particular, $j_w^*(a) : j_w^* \Delta_w^{\mathrm{end}(t)} \to j_w^* \mathscr{F}$ is an isomorphism; we deduce that the cone $C(a)$ of a is

supported on $\mathscr{X}^{\mathrm{end}(t)}_{<_t w}$. Using point *2.* of the statement, we see that the map $\mathscr{F} \to C(a)$ is zero, so that the triangle

$$C(a)[-1] \to \Delta^{\mathrm{end}(t)}_w \xrightarrow{a} \mathscr{F} \xrightarrow{+1}$$

splits. As $\Delta^{\mathrm{end}(t)}_w$ does not admit any nontrivial morphism to an object supported on $\mathscr{Y}^{\mathrm{end}(t)}_{<_t w}$, this implies $C(a) = 0$; in other words, a is an isomorphism. □

Proof of theorem 13.2.3. The strategy is the following: we prove first the existence of the equivalences stated in the theorem, and the fact that they preserve tilting objects. Then, the fact that $\widehat{\mathfrak{E}}$ preserves standard and costandard objects will follow formally from point 3. of the theorem.

Let us show the existence of the equivalences. Thanks to lemma 13.2.2, we have an equivalence

$$K^b\mathfrak{T}^\circ_{[t,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{D}^\circ_{[t,\mathscr{L}^T_t]}.$$

Similarly, we have

$$K^b\mathfrak{T}^{\mathrm{end}(t)}_{[1,\underline{\Bbbk}_T]} \xrightarrow{\sim} \mathfrak{D}^{\mathrm{end}(t)}_{[1,\underline{\Bbbk}_T]}.$$

Using point 2. of theorem 13.2.1, we get an equivalence $\mathfrak{D}^\circ_{[t,\mathscr{L}^T_t]} \xrightarrow{\sim} \mathfrak{D}^{\mathrm{end}(t)}_{[1,\underline{\Bbbk}_T]}$; this equivalence clearly preserve tilting objects.

The existence of $\widehat{\mathfrak{E}}$ follows from similar considerations, using now point 1. in theorem 13.2.1. The fact that $\pi^{1,\mathrm{end}(t)}_\dagger \circ \widehat{\mathfrak{E}} \cong \mathfrak{E} \circ \pi^t_\dagger$ is an consequences of the definition of $\mathfrak{E}$ and $\widehat{\mathfrak{E}}$ and points 3. of theorem 13.1.1 and theorem 13.2.1. Finally, the monoidality of $\widehat{\mathfrak{E}}$ and the compatibility asserted in point 4. are consequence of the facts established before the statement of the theorem.

We should now prove the facts about standard and costandard objects. We claim that it suffices to consider the non-completed case. First, this fact in the non-completed case will imply that $\mathfrak{E}$ sends $\mathrm{IC}(w)_{\mathscr{L}^T_t}$ to $\mathrm{IC}^{\mathrm{end}(t)}_w$ for any $w \in W^\circ_t$, by definition of IC-sheaves; it will aslo imply t-exactness of $\mathfrak{E}$. Finally, using point 3. and lemma 7.7.4, we see that $\mathfrak{E}(\Delta(w)_{\mathscr{L}^T_t}) \cong \Delta^{\mathrm{end}(t)}_w$ implies $\widehat{\mathfrak{E}}(\widehat{\Delta}_{w,t}) \cong \widehat{\Delta}^{\mathrm{end}(t)}_{w,1}$, and similarly for $\widehat{\nabla}$'s.

The proof is copied from [AR2, Lemma 5.2], and is by induction on w in the poset $(W^\circ_t, \leq_t)$. More precisely, we want to show that $\mathfrak{E}(\Delta(w)_{\mathscr{L}^T_t}) \cong \Delta^{\mathrm{end}(t)}_w$ for any $w \in W^\circ_t$, and we proceed by induction on the length of w in W°_t.

For $w = e$, we have $\Delta(e)_{\mathscr{L}^T_t} = \mathscr{T}(e)_{\mathscr{L}^T_t}$ and $\Delta^{\mathrm{end}(t)}_e = \mathscr{T}^{\mathrm{end}(t)}_e$; as our equivalence preserves tilting objects, these equalities settle this case.

Assume that the result is true for any element $v \in W^\circ_t$ such that $v <_t w$. Consider the full subcategory of $\mathfrak{D}(\mathscr{X} /\!\!/ T)^\circ_{[t,\mathscr{L}^T_t]}$ generated (as a triangulated category) by the objects $\{\Delta(v)_{\mathscr{L}^T_t} \mid v \in W^\circ_t,\ v <_t w\}$. Similarly, we have the subcategory of $D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)}, \Bbbk)$ generated as a triangulated category by the objects $\{\Delta^{\mathrm{end}(t)}_v \mid v \in W^\circ_t,\ v <_t w\}$. By induction hypothesis, we have

$$\mathfrak{E}(\langle \Delta(v)_{\mathscr{L}^T_t} \mid v \in W^\circ_t,\ v <_t w\rangle_{\mathrm{triang.}}) \subseteq \langle \Delta^{\mathrm{end}(t)}_v \mid v \in W^\circ_t,\ v <_t w\rangle_{\mathrm{triang.}}. \tag{13.2.3}$$

Now we have a distinguished triangle

$$\Delta(w)_{\mathscr{L}_t^T} \to \mathscr{T}(w)_{\mathscr{L}_t^T} \to C_w \xrightarrow{+1}$$

and thanks to corollary 9.9.3, C_w belongs to the subcategory $\langle \Delta(v)_{\mathscr{L}_t^T} \mid v \in W_t^\circ,\ v <_t w \rangle_{\text{triang.}}$. We apply $\mathfrak{E}$ to this triangle to get

$$\mathfrak{E}(\Delta(w)_{\mathscr{L}_t^T}) \to \mathscr{T}_w^{\mathrm{end}(t)} \to \mathfrak{E}(C_w) \xrightarrow{+1}.$$

The inclusion (13.2.3) tells us that $\mathfrak{E}(C_w)$ belongs to $\langle \Delta_v^{\mathrm{end}(t)} \mid v \in W_t^\circ,\ v <_t w \rangle_{\text{triang.}}$.

Consider $v \in W_t^\circ$ such that $v <_t w$ and $n \in \mathbb{Z}$; using the induction hypothesis, we find that

$$\begin{aligned} \mathrm{Hom}(\mathfrak{E}(\Delta(w)_{\mathscr{L}_t^T}), \Delta_v^{\mathrm{end}(t)}[n]) &\cong \mathrm{Hom}(\mathfrak{E}(\Delta(w)_{\mathscr{L}_t^T}), \mathfrak{E}(\Delta(v)_{\mathscr{L}_t^T})[n]) \\ &\cong \mathrm{Hom}(\Delta(w)_{\mathscr{L}_t^T}, \Delta(v)_{\mathscr{L}_t^T}[n]) \\ &\cong 0. \end{aligned}$$

Thus the object $\mathfrak{E}(\Delta(w)_{\mathscr{L}_t^T})$ satisfies the hypothesis of lemma 13.2.4, so this same lemma gives us

$$\mathfrak{E}(\Delta(w)_{\mathscr{L}_t^T}) \cong \Delta_w^{\mathrm{end}(t)}.$$

The statement about costandard objects can be proved similarly, using a triangle

$$K_w \to \mathscr{T}(w)_{\mathscr{L}_t^T} \to \nabla(w)_{\mathscr{L}_t^T} \xrightarrow{+1}.$$

This concludes the proof of the theorem. □

13.3 Description of maximal IC-sheaves

Fix $t, t' \in T_{\Bbbk}^\vee$ in the same W-orbit. As a corollary of the theorem, we obtain the following description of the object $\mathrm{IC}_{w_\beta^{\max},t}$, for any block $\beta \in_{t'} \underline{W}_t$. Recall from subsection 5.1.3 that ℓ_t denotes the length function of the Coxeter system (W_t°, S_t). Now, if $\beta \in_{t'} \underline{W}_t$ is a block, for any $w \in \beta$, there exists a unique $w' \in W_t^\circ$ such that $w = w_\beta^{\min} w'$. We then set, following [LY, §4.5],

$$\ell_\beta(w) := \ell_t(w').$$

For example, using lemma 5.1.2 we have $\ell_\beta(w_\beta^{\min}) = \ell_t(e) = 0$ and $\ell_\beta(w_\beta^{\max}) = \ell_t(w_{t,\circ})$.

Corollary 13.3.1. *If $w \notin \beta$, we have $j_w^* \mathrm{IC}_{w_\beta^{\max},t} \cong j_w^! \mathrm{IC}_{w_\beta^{\max},t} \cong 0$. For any $w \in \beta$, we have*

$$j_w^* \mathrm{IC}_{w_\beta^{\max},t} \cong \mathscr{L}_t^w[\dim(\mathscr{X}_w) + \ell_\beta(w_\beta^{\max}) - \ell_\beta(w)],$$

$$j_w^! \mathrm{IC}_{w_\beta^{\max},t} \cong \mathscr{L}_t^w[\dim(\mathscr{X}_w) - \ell_\beta(w_\beta^{\max}) + \ell_\beta(w)].$$

Proof. Using the equivalence from lemma 4.2.4, we can work in the Lusztig–Yun category, and replace $\mathrm{IC}_{w_\beta^{\max},t}$ by $\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T}$. The first assertion is clear since objects of $\mathfrak{D}^\beta_{[t',\mathscr{L}_t^T]}$ are necessarily supported on strata associated to elements of β.

Consider now $w \in \beta$. We show only the first isomorphism, the second follows by similar arguments (or Verdier duality). Working now with Lusztig–Yun equivariant objects, we should show

$$j_w^* \,\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T} \cong \mathscr{L}_t^w[\ell(w) + \ell_\beta(w_\beta^{\min}) - \ell_\beta(w)].$$

Let us fix $w \in \beta$; the strategy is to study $\mathrm{Hom}_{\mathfrak{D}^\beta_{[t',\mathscr{L}_t^T]}}(\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T}, \nabla(w)_{\mathscr{L}_t^T}[n])$ for $n \in \mathbb{Z}$. We have

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{D}^\beta_{[t',\mathscr{L}_t^T]}}(\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T}, \nabla(w)_{\mathscr{L}_t^T}[n]) \\
&\qquad \cong \mathrm{Hom}_{\mathfrak{D}(\mathscr{X}_w /\!\!/ T)_{[t',\mathscr{L}_t^T]}}(j_w^* \,\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T}, \mathscr{L}_t^w[\ell(w)+n]) \\
&\qquad\qquad \cong \mathrm{Hom}_{D^b\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}}}((\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T})_{\dot{w}U}, \Bbbk[\ell(w)+n]) \\
&\qquad\qquad\qquad \cong \mathbf{H}^{\ell(w)+n}((\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T})_{\dot{w}U}).
\end{aligned}$$

Here, $(\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T})_{\dot{w}U}$ denotes the stalk at $\dot{w}U \in \mathscr{X}_w$ of the object $\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T}$, and $\mathbf{H}^{\ell(w)+n}((\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T})_{\dot{w}U})$ its degree-$\ell(w)+n$ cohomology. The first isomorphism is induced by adjunction, and the second one comes from the equivalence $\mathfrak{D}(\mathscr{X}_w /\!\!/ T)_{[t',\mathscr{L}_t^T]} \cong D^b\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}}$, as in section 7.4.

Now, on the other hand, we can write $w = w_\beta^{\min} w'$ for some (unique) $w' \in W_t^\circ$. Using proposition 8.5.2, one obtains the following identifications:

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{D}^\beta_{[t',\mathscr{L}_t^T]}}(\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T}, \nabla(w)_{\mathscr{L}_t^T}[n]) \\
&\cong \mathrm{Hom}_{\mathfrak{D}^\beta_{[t',\mathscr{L}_t^T]}}(\nabla(w_\beta^{\min})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \mathrm{IC}(w_{t,\circ})_{\mathscr{L}_t^T}, \nabla(w_\beta^{\min})_{\mathscr{L}_t^T} \star^{\mathrm{LY}} \nabla(w')_{\mathscr{L}_t^T}[n]) \\
&\qquad\qquad \cong \mathrm{Hom}_{\mathfrak{D}^\circ_{[t,\mathscr{L}_t^T]}}(\mathrm{IC}(w_{t,\circ})_{\mathscr{L}_t^T}, \nabla(w')_{\mathscr{L}_t^T}[n]).
\end{aligned}$$

We can then use theorem 13.2.3 to reduce our problem to a study of objects on $\mathscr{Y}^{\mathrm{end}(t)}$: we have an isomorphism

$$\mathrm{Hom}_{\mathfrak{D}^\circ_{[t,\mathscr{L}_t^T]}}(\mathrm{IC}(w_{t,\circ})_{\mathscr{L}_t^T}, \nabla(w')_{\mathscr{L}_t^T}[n]) \cong \mathrm{Hom}_{D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)},\Bbbk)}(\mathrm{IC}^{\mathrm{end}(t)}_{w_{t,\circ}}, \nabla^{\mathrm{end}(t)}_{w'}[n]).$$

Since the variety $\mathscr{Y}^{\mathrm{end}(t)}$ is smooth of dimension $\ell_t(w_{t,\circ})$, we have $\mathrm{IC}^{\mathrm{end}(t)}_{w_{t,\circ}} = \underline{\Bbbk}_{\mathscr{Y}^{\mathrm{end}(t)}}[\ell_t(w_{t,\circ})]$. thus, using adjunction, the above Hom-space is given by

$$\begin{aligned}
&\mathrm{Hom}_{D^b_{(B_t)}(\mathscr{Y}^{\mathrm{end}(t)}_{w'},\Bbbk)}(\underline{\Bbbk}_{\mathscr{Y}^{\mathrm{end}(t)}_{w'}}[\ell_t(w_{t,\circ})], \underline{\Bbbk}_{\mathscr{Y}^{\mathrm{end}(t)}_{w'}}[\ell_t(w')+n]) \\
&\qquad\qquad \cong \begin{cases} \Bbbk & \text{if} \quad n = \ell_t(w_{t,\circ}) - \ell_t(w') \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This readily implies that

$$\mathbf{H}^{\ell(w)+n}((\mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T})_{\dot{w}U}) \cong \begin{cases} \Bbbk & \text{if} \quad n = \ell_t(w_{t,\circ}) - \ell_t(w') \\ 0 & \text{otherwise.} \end{cases}$$

Since by definition, we have $\ell_t(w_{t,\circ}) = \ell_\beta(w_\beta^{\max})$ and $\ell_t(w') = \ell_\beta(w)$, we deduce that $j_w^* \mathrm{IC}(w_\beta^{\max})_{\mathscr{L}_t^T} \cong \mathscr{L}_t^w[\ell(w) + \ell_\beta(w_\beta^{\max}) - \ell_\beta(w')]$ for any $w \in W_t^\circ$. This concludes the proof. □

Part III

Appendix

Appendix A

Highest weight categories

We recall here the definition of a highest weight category, and state usual properties of some objects of particular interest in such a category. Our references are [BGS, §3.1, 3.2] and [R, §7]. Let $\mathcal{A}$ be a finite-length $\mathbb{K}$-linear abelian category with $\mathbb{K}$ a field such that for any objects $M, N \in \mathcal{A}$, the $\mathbb{K}$-vector-space $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is finite-dimensional. Let $\mathcal{S}$ be the set of isomorphism classes of irreducible objects in $\mathcal{A}$ and assume that we have a partial order $\leq$ on $\mathcal{S}$. For any $s \in \mathcal{S}$, fix a representative L_s. Assume also that we have objects Δ_s, ∇_s together with morphisms

$$\Delta_s \to L_s \quad \text{and} \quad L_s \to \nabla_s \,.$$

For any subset $\mathcal{S}' \subseteq \mathcal{S}$, let $\mathcal{A}_{\mathcal{S}'}$ be the Serre subcategory generated by the L_s with $s \in \mathcal{S}'$. We say that $\mathcal{S}'$ is an ideal for the order $\leq$ if ($s \in \mathcal{S}'$ and $t \leq s$) implies ($t \in \mathcal{S}'$).

Definition A.0.1. *The data $(\mathcal{A}, (\mathcal{S}, \leq), L_s, \Delta_s \to L_s, L_s \to \nabla_s)$ defines a highest weight category with weight poset $(\mathcal{S}, \leq)$ if the following conditions hold:*

1. *for any $s \in \mathcal{S}$, the subset $\{t \leq s \mid t \in \mathcal{S}\}$ is finite,*
2. *for each $s \in \mathcal{S}$, we have $\mathrm{Hom}_{\mathcal{A}}(L_s, L_s) = \mathbb{K}$,*
3. *for any ideal $\mathcal{S}' \subseteq \mathcal{S}$ such that s is maximal in $\mathcal{S}'$, $\Delta_s \to L_s$ is a projective cover in $\mathcal{A}_{\mathcal{S}'}$ and $L_s \to \nabla_s$ is an injective hull in $\mathcal{A}_{\mathcal{S}'}$,*
4. *the kernel of $\Delta_s \to L_s$ and the cokernel of $L_s \to \nabla_s$ are in $\mathcal{A}_{<s}$,*
5. *for any $s, t \in \mathcal{S}$, we have $\mathrm{Ext}^2_{\mathcal{A}}(\Delta_s, \nabla_t) = 0$.*

The objects Δ_s for $s \in \mathcal{S}$ are called standard objects; the ∇_s's are called costandard objects.

We almost always forget part of the data above and just refer to $\mathcal{A}$ as a highest weight category.

The following result is a great part of what makes a highest-weight category particularly nice (this is [BGS, Theorem 3.2.1]).

Theorem A.0.2. *Assume that $(\mathcal{A}, (\mathcal{S}, \leq), L_s, \Delta_s \to L_s, L_s \to \nabla_s)$ is a highest-weight category with $\mathcal{S}$ finite. Then $\mathcal{A}$ has enough projective objects and any such object has a finite filtration with standard subquotients.*

Since we have enough projective objects in $\mathcal{A}$, we can consider the projective cover of some simple object L_s; denote it by P_s. Fix a standard filtration of P_s (that is, a finite filtration of P_s with standard objects as subquotients) and let

$$(P_s : \Delta_t)$$

be the multiplicity of Δ_t is this filtration. Then one has the reciprocity formula

$$(P_s : \Delta_t) = [\nabla_t : L_s]. \tag{A.0.1}$$

Indeed, both sides are equal to $\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(P_s, \nabla_t)$. In particular, the multiplicity of Δ_t in a standard filtration of P_s does not depend on the chosen filtration. The following proposition is a very useful result:

Proposition A.0.3. *For any $s, t \in \mathcal{S}$, we have*

$$\mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, \nabla_t) = \begin{cases} \mathbb{K} & \text{if } i = 0 \text{ and } s = t \\ \{0\} & \text{otherwise.} \end{cases}$$

There is another kind of objects (in any general highest weight category) that is of interest, namely the tilting objects.

Definition A.0.4. *Define an object in a highest weight category $(\mathcal{A}, (\mathcal{S}, \leq))$ to be tilting if it admits both a standard filtration and a costandard filtration.*

We set $\mathcal{T}$ for the full subcategory of tilting objects in $\mathcal{A}$. The following theorem is due to Ringel (see e.g. [R, §7.5, Theorem 7.14]).

Theorem A.0.5. *For any $s \in \mathcal{S}$, there exists up to isomorphism a unique indecomposable tilting object T_s such that*

$$[T_s, L_s] = 1 \quad \text{and} \quad [T_s : L_t] \neq 0 \Rightarrow t \leq s.$$

Moreover, there exists an embedding $\Delta_s \hookrightarrow T_s$ whose cokernel admits a standard filtration, and there exists a surjection $T_s \twoheadrightarrow \nabla_s$ whose kernel admits a costandard filtration. The subquotients appearing in these filtrations are indexed by indices $t < s$. Any indecomposable tilting object is isomorphic to one of the T_s's, and any tilting object is a direct sum of indecomposable tilting objects.

Lemma A.0.6. *Consider a highest weight category $(\mathcal{A}, (\mathcal{S}, \leq), \Delta_s \to L_s, L_s \to \nabla_s)$ and let $e \in \mathcal{S}$ be a minimal element. Assume that we have an element $s \in \mathcal{S}$ such that there exist two non-split short exact sequences*

$$L_e \hookrightarrow \Delta_s \twoheadrightarrow L_s \quad \text{and} \quad L_s \hookrightarrow \nabla_s \twoheadrightarrow L_e. \tag{A.0.2}$$

Then $\dim_{\mathbb{K}} \mathrm{Ext}^1_{\mathcal{A}}(\Delta_e, \Delta_s) = 1 = \dim_{\mathbb{K}} \mathrm{Ext}^1_{\mathcal{A}}(\nabla_s, \nabla_e)$.

Proof. We prove the first equality. Since e is minimal, we have $\Delta_e \cong L_e \cong \nabla_e$. We start by applying the functor $\mathrm{Hom}_{\mathcal{A}}(L_e, -)$ to the first exact sequence above to get a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(L_e, L_e) \longrightarrow \mathrm{Ext}^1_{A}(L_e, \Delta_s) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(L_e, L_s) \longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(L_e, L_e) \longrightarrow \cdots$$

the first Hom-space is zero, and the last one is as well. Thus we need to see that $\mathrm{Ext}^1_{\mathcal{A}}(L_e, L_s)$ is one-dimensional.

We apply the functor $\mathrm{Hom}_{\mathcal{A}}(L_e, -)$ to the second exact sequence in (A.0.2) to obtain

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(L_e, L_e) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(L_e, L_s) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(L_e, \nabla_s) \longrightarrow \cdots$$

The last term is zero (since $L_e \cong \Delta_e$, see proposition A.0.3), and the first one is $\Bbbk$. Thus the map $\mathrm{Hom}_{\mathcal{A}}(L_e, L_e) \to \mathrm{Ext}^1_{\mathcal{A}}(L_e, L_s)$ is surjective. Since $\mathrm{Ext}^1_{\mathcal{A}}(L_e, L_s)$ is nonzero, we deduce that this space is one dimensional. The proof for the second equality is similar. □

Lemma A.0.7. *We keep the assumption of lemma A.0.6. Then for any $r \in \mathcal{S}$, with $r \neq e$, we have*

$$\mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, \Delta_s) = 0.$$

Proof. We apply the functor $\mathrm{Hom}_{\mathcal{A}}(\Delta_r, -)$ to the first exact sequence in (A.0.2) to obtain a long exact sequence

$$\cdots \mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, L_e) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, \Delta_s) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, L_s) \cdots .$$

Since $L_e \cong \nabla_e$, the first space in the exact sequence is zero from proposition A.0.3. We then need to show the vanishing of the third space. To see that, we apply the same functor to the second exact sequence in (A.0.2) to obtain a long exact sequence

$$\cdots \mathrm{Hom}_{\mathcal{A}}(\Delta_r, L_e) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, L_s) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, \nabla_s) \cdots .$$

Here, as above, the first and third spaces are zero, and so is the middle one; this fact concludes the proof. □

Lemma A.0.8. *We keep the assumptions of lemma A.0.6. Then the only non trivial extension in $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_e, \Delta_s)$ is isomorphic to the tilting object T_s, as is the only non trivial extension in $\mathrm{Ext}^1_{\mathcal{A}}(\nabla_s, \nabla_e)$.*

Proof. Denote by T the unique (up to equivalence) non-split extension in the space $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_e, \Delta_s)$ (that is to say, choose a generator of this one-dimensional $\Bbbk$-vector space). We have to see that it is a tilting object i.e. that it admits a costandard filtration. To do so, we use [R, Proposition 7.10]: it suffices to see that $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, \mathsf{T}) = 0$ for any $r \in \mathcal{S}$. Assume first that $r \neq e$. Since $\Delta_e \cong \nabla_e$, we have $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, \Delta_e) = 0$ (in fact this is true for any r) and lemma A.0.7 tells

us that $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_r, \Delta_s) = 0$. Now, it suffices to recall that T is an extension of Δ_e by Δ_s to conclude.

We deal with the cases $r = e$. We apply the functor $\mathrm{Hom}(\Delta_r, -)$ to the short exact sequence $\Delta_s \hookrightarrow \mathsf{T} \twoheadrightarrow \Delta_e$ to obtain an exact sequence

$$\mathrm{Hom}(\Delta_r, \Delta_e) \xrightarrow{\partial} \mathrm{Ext}^1(\Delta_r, \Delta_s) \to \mathrm{Ext}^1(\Delta_r, \mathsf{T}) \to \mathrm{Ext}^1(\Delta_r, \Delta_e). \qquad (A.0.3)$$

If $r = e$, then the map ∂ is nonzero by definition, since T is a non-split extension, and both of the spaces involved are one-dimensional over $\Bbbk$, so this map is surjective. As the last Ext space of (A.0.3) is always zero, we have $\mathrm{Ext}^1(\Delta_r, \mathsf{T}) = 0$.

Thus T is a tilting object. Moreover, the largest $r \in \mathcal{S}$ such that $[\mathsf{T} : L_r] \neq 0$ is in fact s. According to [R, Remark 7.16], this implies that T admits a direct summand isomorphic to T_s. To conclude the proof, it thus suffices to see that T is indecomposable. Consider a direct summand T' of T which admits L_s as a composition factor. This direct summand is also tilting, in particular, it admits a standard filtration, and the subquotients are be Δ_e or Δ_s. Since each of these standard objects appears in T with multiplicity one, the only possibilities are $\mathsf{T}' \cong \Delta_e$ or $\mathsf{T}' \cong \Delta_s$. This cannot happen: Δ_s is not a tilting object and Δ_e does not admits L_s as a composition factor.

One can show in a similar manner that the unique (up to equivalence) non split extension in $\mathrm{Ext}^1(\nabla_s, \nabla_e)$ is isomorphic to T_s. □

Finally, let us state another useful properties of tilting objects. Let $K^b\mathcal{T}$ denote the bounded homotopy category of the additive category $\mathcal{T}$ of tilting objects in $\mathcal{A}$.

Proposition A.0.9. *The natural functor $K^b\mathcal{T} \to D^b\mathcal{A}$ is an equivalence of categories.*

See e.g. [R, Proposition 7.17] for a proof.

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