Cohomologie de Quillen des opérades enrichies

Manh Truong Hoang

ÉCOLE DOCTORALE GALILÉE

LABORATOIRE ANALYSE, GÉOMÉTRIE ET APPLICATIONS, UMR 7539

THÈSE

presentée et soutenue publiquement par

Manh Truong HOANG

le 25 octobre 2020

pour obtenir le grade de

Docteur de l'Université Sorbonne Paris Nord

Discipline: Mathématiques

COHOMOLOGIE DE QUILLEN DES OPÉRADES ENRICHIES

Directeur de thèse: **Yonatan HARPAZ**

Membres du Jury

Damien CALAQUE	PR, Université de Montpellier	Rapporteur
Yonatan HARPAZ	CR-HDR, Université Sorbonne Paris Nord	Directeur de thèse
Rune HAUGSENG	PR, NTNU	Rapporteur
Geoffroy HOREL	MCF, Université Sorbonne Paris Nord	Examinateur
Muriel LIVERNET	PR, Université Paris Diderot	Examinateur
Bruno VALLETTE	PR, Université Sorbonne Paris Nord	Président du jury

QUILLEN COHOMOLOGY OF ENRICHED OPERADS

by

Manh Truong HOANG

Submitted to Institut Galilée in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

Université Sorbonne Paris Nord

25 October 2020

Thesis Supervisor: **Yonatan HARPAZ**

Members of Jury

Damien CALAQUE	PR, Université de Montpellier	Rapporteur
Yonatan HARPAZ	CR-HDR, Université Sorbonne Paris Nord	Thesis supervisor
Rune HAUGSENG	PR, NTNU	Rapporteur
Geoffroy HOREL	MCF, Université Sorbonne Paris Nord	Examiner
Muriel LIVERNET	PR, Université Paris Diderot	Examiner
Bruno VALLETTE	PR, Université Sorbonne Paris Nord	Jury president

Abstract

A modern insight due to Quillen, which is further developed by Lurie, asserts that many cohomology theories of interest are particular cases of a single construction, which allows one to define cohomology groups in an abstract setting using only intrinsic properties of the category (or ∞-category) at hand. This universal cohomology theory is known as Quillen cohomology. In any setting, Quillen cohomology of a given object is classified by its cotangent complex.

The main purpose of this document is to study Quillen cohomology of enriched operads, when working in the model categorical framework. Our main result provides an explicit formula for computing Quillen cohomology of enriched operads, based on a procedure of taking certain infinitesimal models of their cotangent complexes. We are particularly interested in the Quillen cohomology of simplicial operads and dg operads. There is a natural construction of twisted arrow ∞-category of a simplicial operad, which extends the notion of twisted arrow ∞-category of an ∞-category introduced by Lurie. We assert that the cotangent complex of a simplicial operad can be represented as a spectrum valued functor on its twisted arrow ∞-category. Turning to the context of dg operads, the situation becomes simpler due to the stability of dg modules. We find that the cotangent complex of a dg operad $\mathcal{P}$ can be represented by a nice infinitesimal $\mathcal{P}$-bimodule, which is in fact closely related to the module of Kähler differentials of $\mathcal{P}$ via a cofiber sequence. Moreover, we prove the existence of an operadic version of the Dold-Kan correspondence, then due to this we find a connection between Quillen cohomology of a simplicial operad and Quillen cohomology of its associated dg operad. In the last section, we establish the relation between deformation theory and Quillen cohomology.

Acknowledgments

First of all, I would like to deeply thank my advisor, Yonatan Harpaz, for his great guidance and support throughout the past three years. I am greatly indebted to him for suggesting the problems and providing me with the tools that are needed to outline the comprehensive picture of the project.

I wish to express my attitude to Christian Ausoni for guiding me through the Master's program and for introducing me to my present advisor.

I would like to thank my thesis committee members for many valuable comments and suggestions.

I would like to thank Bruno Vallette for his great interest in my thesis, for his corrections, as well as suggesting directions for my future research.

I would like to thank David White and Joan Millès for helpful conversations on the topics concerning my work.

Finally, I would like to thank my family, my girlfriend and all my friends. Although words are never enough, thank you for your support and belief in me.

Contents

Chapter 1

Introduction

A widespread idea in the domain of homotopy theory is to study a given object of interest by associating to it various kinds of cohomology group. From generalized cohomology theories for spaces and various Ext groups in homological algebra, through group cohomology, sheaf cohomology, Hochschild cohomology and André-Quillen cohomology, such invariants vary from fairly useful to utterly indispensable. A modern insight due to Quillen [1], which is further developed by Lurie [3], asserts that all these cohomology theories are particular cases of a single universal construction, which allows one to define cohomology groups in an abstract setting using only intrinsic properties of the category (or ∞-category) at hand. This universal cohomology theory is known as **Quillen cohomology**.

In Quillen's approach, cohomology of an object of interest is classified by its derived abelianization. Suppose we are given a model category $\mathbf{M}$. Recall that an **abelian group object** in $\mathbf{M}$ is an object A equipped with two maps $* \longrightarrow A$ and $A \times A \longrightarrow A$ subject to the classical axioms of an abelian group. For each object $X \in \mathbf{M}$, the category of abelian group objects in $\mathbf{M}_{/X}$, denoted $\mathrm{Ab}(\mathbf{M}_{/X})$, possibly inherits a model structure transferred from that of $\mathbf{M}$. In this situation, the free-forgetful adjunction $\mathcal{F} : \mathbf{M}_{/X} \rightleftarrows \mathrm{Ab}(\mathbf{M}_{/X}) : \mathcal{U}$ forms a Quillen adjunction. The **cotangent complex** of X, denoted by L_X, is then defined to be $\mathrm{L}_X := \mathbb{L}\mathcal{F}(X)$ the **derived abelianization of** X. Moreover, given an object $M \in \mathrm{Ab}(\mathbf{M}_{/X})$, the n'th **Quillen cohomology group of** X **with coefficients in** M is formulated as

$$\mathrm{H}^n_Q(X, M) = \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathrm{Ab}(\mathbf{M}_{/X})}(\mathrm{L}_X, M[n])$$

where $M[n]$ refers to the n-suspension of M in $\mathrm{Ab}(\mathbf{M}_{/X})$.

The work of Quillen was first devoted to establishing a proper cohomology theory for rings and commutative algebra (cf. [2]), which is nowadays generalized into the operadic context as described in the following example.

Example 1.0.0.1. Let $\mathbf{k}$ be a commutative ring of characteristic 0 and let $\mathcal{P}$ be an operad enriched over dg $\mathbf{k}$-modules. For each $\mathcal{P}$-algebra A, it was known that the free-forgetful adjunction $(\mathrm{Alg}_{\mathcal{P}})_{/A} \rightleftarrows \mathrm{Ab}((\mathrm{Alg}_{\mathcal{P}})_{/A})$ is (homotopically) equivalent to the adjunction

$$\Omega^A(-) : (\mathrm{Alg}_{\mathcal{P}})_{/A} \rightleftarrows \mathrm{Mod}^{\mathcal{P}}_A : A \ltimes (-)$$

in which $\mathrm{Mod}^{\mathcal{P}}_A$ refers to the category of A**-modules over** $\mathcal{P}$, the left adjoint takes $B \in (\mathrm{Alg}_{\mathcal{P}})_{/A}$ to $\Omega^A(B)$ the **module of Kähler differentials** of B over A and the right adjoint takes $M \in \mathrm{Mod}^{\mathcal{P}}_A$ to

$A \ltimes M$ the **square-zero extension** of A by M. Therefore, after sending coefficients into $\mathrm{Mod}_A^{\mathcal{P}}$, the n'th Quillen cohomology group of A with coefficients in an object $M \in \mathrm{Mod}_A^{\mathcal{P}}$ is given by

$$\mathrm{H}_Q^n(A, M) = \pi_0 \, \mathrm{Map}^{\mathrm{h}}_{\mathrm{Mod}_A^{\mathcal{P}}}(\Omega^A(A^{\mathrm{cof}}), M[n])$$

where A^{cof} is a cofibrant resolution of A in $\mathrm{Alg}_{\mathcal{P}}$. (See, e.g., [11, 12, 7]).

The Quillen's approach has certain limitations, despite its success. Indeed, there is not a known traditional criterion assuring the existence of the transferred model structure on abelian group objects in a given model category and moreover, even when realized, this model category structure is not invariant under Quillen equivalences.

Improving the work of Quillen, Lurie [3] established the cotangent complex formalism, in the ∞-categorical framework, by extending the notion of abelianization to that of **stabilization**, which itself is inspired by the classical theory of spectra. Let $\mathcal{C}$ be a presentable ∞-category and let X be an object of $\mathcal{C}$. Consider the over ∞-category $\mathcal{C}_{/X}$. Conceptually, the **stabilization of** $\mathcal{C}_{/X}$ is the ∞-categorical limit of the tower

$$\cdots \xrightarrow{\Omega} (\mathcal{C}_{/X})_* \xrightarrow{\Omega} (\mathcal{C}_{/X})_* \xrightarrow{\Omega} (\mathcal{C}_{/X})_*$$

where Ω refers to the desuspension functor on $(\mathcal{C}_{/X})_*$ the pointed ∞-category associated to $\mathcal{C}_{/X}$. As in [3], we will refer to the stabilization of $\mathcal{C}_{/X}$ as the **tangent category to** $\mathcal{C}$ **at** X and denote it by $\mathcal{T}_X\mathcal{C}$. By construction, $\mathcal{T}_X\mathcal{C}$ is automatically a **stable ∞-category**. Moreover, the presentability of $\mathcal{C}$ implies that the canonical functor $\mathcal{T}_X\mathcal{C} \longrightarrow \mathcal{C}_{/X}$ admits a left adjoint, the **suspension spectrum functor**, written as $\Sigma^{\infty}_{+} : \mathcal{C}_{/X} \longrightarrow \mathcal{T}_X\mathcal{C}$. By this way, Lurie defined the **cotangent complex of** X to be $\mathrm{L}_X := \Sigma^{\infty}_{+}(X)$. By having that notion of cotangent complex, the n'th Quillen cohomology group of X with coefficients in a given object $M \in \mathcal{T}_X\mathcal{C}$ is now formulated as

$$\mathrm{H}_Q^n(X, M) := \pi_0 \, \mathrm{Map}_{\mathcal{T}_X\mathcal{C}}(\mathrm{L}_X, M[n]).$$

We refer the readers to [7] for a discussion on the naturality of the evolution from Quillen's approach to Lurie's, and also a comparison between them.

For necessary computations in abstract homotopy theory, model categories (or a bit more generally, **semi model categories** (cf., e.g., [52, 53, 54])) seem to be the most favorable environment, as far as we know. Fortunately, the Lurie's settings mentioned above were completely translated into the model categorical language, thanks to the recent works of Y. Harpaz, J. Nuiten and M. Prasma [6, 7]. Following the settings given in [6], tangent (model) categories come after a procedure of taking left Bousfield localizations of model categories of interest. Nevertheless, the obstacle is that left Bousfield localizing usually requires the left properness. The recent result of Batanin and White [16] allows one to take left Bousfield localizations, in the framework of semi model categories, without necessarily requiring the left properness. Inspired by this result, under our settings, tangent categories exist as semi model categories, which are basically convenient as well as (full) model categories.

The main purpose of this thesis is to formulate Quillen cohomology of operads enriched over a general symmetric monoidal model category, which we will refer to as the **base category**. Given a base category $\mathcal{S}$, we denote by $\mathrm{Op}_C(\mathcal{S})$ the category of **$\mathcal{S}$-enriched C-colored operads** with C being some fixed set of colors, yet the one we really care about is the category of **$\mathcal{S}$-enriched operads** (with non-fixed sets of colors), which will be denoted by $\mathrm{Op}(\mathcal{S})$. Under some suitable conditions, $\mathrm{Op}(\mathcal{S})$ admits the **canonical model structure**, according to Caviglia [17]. In particular, when $\mathcal{S}$ is the category of simplicial sets, Set_{Δ}, equipped with the standard (Kan-Quillen) model structure, the canonical model structure on $\mathrm{Op}(\mathrm{Set}_\Delta)$ agrees with the Cisinski-Moerdijk model structure, which was known to be a model for the theory of **∞-operads** (cf. [18]).

Given an $\mathcal{S}$-enriched C-colored operad $\mathcal{P}$, one can then consider $\mathcal{P}$ as either an object of $\mathrm{Op}_C(\mathcal{S})$ or an object of $\mathrm{Op}(\mathcal{S})$. As emphasized above, we are mostly concentrated in the latter case. Therefore, by **Quillen cohomology of** $\mathcal{P}$ we will mean the Quillen cohomology of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}(\mathcal{S})$. On other hand, by **reduced Quillen cohomology of** $\mathcal{P}$ we will mean the Quillen cohomology of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}_C(\mathcal{S})$. Some attention was given in the literature to the reduced Quillen cohomology of operads. For instance, in the context of dg operads, this was studied by Loday-Merkulov-Vallette [11, 12], (in which the resultant is described in terms of derivations, similarly as in Example 1.0.0.1). On the other hand, the problems of formulating Quillen cohomology of operads and investigating its applications, which are essentially more valuable, have not yet been considered so far.

The two base categories of most interest include the category of simplicial sets, Set_Δ, equipped with the standard (Kan-Quillen) model structure and the category of dg $\mathbf{k}$-modules, $\mathcal{C}(\mathbf{k})$, with $\mathbf{k}$ being a field of characteristic 0, equipped with the projective model structure. Operads which are enriched over Set_Δ (resp. $\mathcal{C}(\mathbf{k})$) will be called **simplicial operads** (resp. **dg operads**). These two contexts will also be our substantial concerns in the thesis.

It has been widely acknowledged that Quillen cohomology theory keeps a key role in the study of **deformation theory** and **obstruction theory**. Let us discuss on these in what below.

Naively, a deformation of an object of interest under "small perturbation" is an object of the same type which is "equivalent" to the original object. In our settings, a small perturbation is precisely an **artinian dg k-algebra** with $\mathbf{k}$ being a field of characteristic 0, i.e., a (connective) augmented commutative dg $\mathbf{k}$-algebra R of finite dimensional such that the augmentation map $R \to \mathbf{k}$ exhibits the 0'th homology of R as a local $\mathbf{k}$-algebra. For instance, in the context of algebraic objects (e.g., dg module, dg category, dg operad, etc.), a deformation of an object X under a small perturbation $R \to \mathbf{k}$ is by definition an object Y coming together with a weak equivalence $Y \otimes_R \mathbf{k} \xrightarrow{\simeq} X$. One can then organize all the deformations of X into a category such that every morphism is an equivalence of deformations. The ∞-groupoid associated to this category will be called the **space of deformations of X over R**, denoted by $\mathrm{Def}(X, R)$. In a somewhat more abstract setting given in § 5.4, we propose the notion of space of deformations for various types of object. We then show that Quillen cohomology of a given object indeed classifies the homotopy type of its space of deformations. Moreover, we show that the functor $R \mapsto \mathrm{Def}(X, R)$ forms a **formal moduli problem** in the sense of [5]. It implies that the deformations of X are "governed" by a single dg Lie algebra (cf. [5, 19]).

Given two topological spaces X and Y, understanding maps from X to Y (up to homotopy) is a classical problem in homotopy theory. Suppose that Y is simply connected. As the first step, one filters Y by its **Postnikov tower**:

$$\cdots \longrightarrow P_n(Y) \longrightarrow P_{n-1}(Y) \longrightarrow \cdots \longrightarrow P_1(Y) \longrightarrow P_0(Y).$$

The problem is therefore reduced to understanding maps $X \longrightarrow P_n(Y)$ which (up to homotopy) extend some given map $f : X \longrightarrow P_{n-1}(Y)$. It was known that the obstruction to a section $P_{n-1}(Y) \longrightarrow P_n(Y)$ is classified by a single cohomology class $k_{n-1} \in \mathrm{H}^{n+1}(P_{n-1}(Y); \pi_n Y)$. More generally, the obstruction to an extension $X \longrightarrow P_n(Y)$ of f is classified by the image of k_{n-1} under $f^* : \mathrm{H}^{n+1}(P_{n-1}(Y); \pi_n Y) \longrightarrow \mathrm{H}^{n+1}(X; \pi_n Y)$. The latter is called the **obstruction class of** f.

Note that the ordinary cohomology of spaces is nothing but a particular case of Quillen cohomology. The authors of [20] generalized the above procedures to study the obstruction theory of **simplicial**

categories with fixed set of objects, in which case obstruction classes are indeed contained in Quillen cohomology groups. The obstruction theory of **dg categories** was considered in [21], in which case Quillen cohomology groups play a central role, again. We hope that the present thesis may open the way to the study of obstruction theory of simplicial operads. We would like to leave this for future work.

We are now summarizing our main results with respect to some historical backgrounds. These are divided into two parts.

Part 1. In the first part (including Chapter 3 and Chapter 4), we study Quillen cohomology of enriched operads in general context and particularly, Quillen cohomology of simplicial operads. This work generalizes the study of Quillen cohomology of enriched categories (in particular, simplicial categories) carried out by Y. Harpaz, M. Prasma and J. Nuiten ([7]).

Suppose we are given a sufficiently nice base category $\mathcal{S}$ (cf. Conventions 3.1.0.2). As the starting point, we extend a result of [7], which we now recall. Let $\mathcal{C} \in \mathrm{Cat}(\mathcal{S})$ be a fibrant $\mathcal{S}$-enriched category. Denote by $\mathrm{Cat}_C(\mathcal{S})$ the category of $\mathcal{S}$-enriched categories with objects in $C \coloneqq \mathrm{Ob}(\mathcal{C})$ and by $\mathrm{BMod}(\mathcal{C})$ the category of **$\mathcal{C}$-bimodules**. There is a sequence of the obvious Quillen adjunctions

$$\mathrm{BMod}(\mathcal{C})_{\mathcal{C}/} \rightleftarrows \mathrm{Cat}_C(\mathcal{S})_{\mathcal{C}/} \rightleftarrows \mathrm{Cat}(\mathcal{S})_{\mathcal{C}/}.$$

Theorem 1.0.0.2. *(Y. Harpaz, M. Prasma and J. Nuiten [7]) The above sequence induces a sequence of Quillen equivalences connecting the associated tangent categories:*

$$\mathcal{T}_{\mathcal{C}}\,\mathrm{BMod}(\mathcal{C}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}_C(\mathcal{S}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathcal{S}) \tag{1.0.0.1}$$

Let us now fix $\mathcal{P}$ to be a fibrant and Σ-cofibrant C-colored operad in $\mathcal{S}$. We let $\mathrm{BMod}(\mathcal{P})$ and $\mathrm{IbMod}(\mathcal{P})$ respectively denote the categories of **$\mathcal{P}$-bimodules** and **infinitesimal $\mathcal{P}$-bimodules**. The induction-restriction functors form a sequence of Quillen adjunctions:

$$\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}/} \rightleftarrows \mathrm{BMod}(\mathcal{P})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}/}.$$

Theorem 1.0.0.3. *(3.2.4.1, 3.2.4.3) The above sequence induces a sequence of Quillen equivalences connecting the associated tangent categories:*

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}).$$

When $\mathcal{S}$ is in addition stable containing a strict zero object, all the terms above are Quillen equivalent to $\mathrm{IbMod}(\mathcal{P})$.

We then compute the derived image of the cotangent complex $\mathrm{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ of $\mathcal{P}$ under the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$. In the first step, our treatment is inspired by the following.

Theorem 1.0.0.4. *(Y. Harpaz, M. Prasma and J. Nuiten [7]) Let $\mathcal{C}$ be a fibrant $\mathcal{S}$-enriched category. Under the right Quillen equivalence $\mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{C}}\,\mathrm{BMod}(\mathcal{C})$, the cotangent complex $\mathrm{L}_{\mathcal{C}} \in \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathcal{S})$ is identified to $\mathrm{L}^b_{\mathcal{C}}[-1] \in \mathcal{T}_{\mathcal{C}}\,\mathrm{BMod}(\mathcal{C})$, i.e., the desuspension of $\mathrm{L}^b_{\mathcal{C}} \in \mathcal{T}_{\mathcal{C}}\,\mathrm{BMod}(\mathcal{C})$ the cotangent complex of $\mathcal{C}$ when regarded as a bimodule over itself.*

We prove that an analogue of this statement holds for the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ (cf. Proposition 3.4.0.5). However, the approach as in the loc.cit fails when extending to our problem. In particular, for our approach, the category $\mathcal{S}$ is technically required to satisfy the extra condition (S8) 3.3.0.2, which is inspired by the work of Dwyer and Hess [[24], Section 5]. After having proved that, it remains to describe the derived image of the cotangent complex of $\mathcal{P}$ (when regarded as a bimodule over itself) under the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$. The resultant is denoted by $\widetilde{\mathrm{L}}_{\mathcal{P}}$. When $\mathcal{S}$ is in addition stable, we compute further the derived image of $\widetilde{\mathrm{L}}_{\mathcal{P}}$ under the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \xrightarrow{\simeq} \mathrm{IbMod}(\mathcal{P})$, denoted $\overline{\mathrm{L}}_{\mathcal{P}}$. By having these "infinitesimal models" for $\mathrm{L}_{\mathcal{P}}$, we obtain the central result of the thesis stated as below.

Let S^n denote the **pointed n-sphere** in $\mathcal{S}$. Then, let S^n_C denote the C-collection which has $\mathrm{S}^n_C(c;c) = \mathrm{S}^n$ for every $c \in C$ and agrees with $\varnothing_{\mathcal{S}}$ on the other levels.

Theorem 1.0.0.5. *(3.4.0.17, 3.4.0.18) Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 3.3.0.2. The n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in a given fibrant object $M \in \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ is formulated as*

$$\mathrm{H}^n_Q(\mathcal{P}, M) \cong \pi_0\,\mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})}(\widetilde{\mathrm{L}}_{\mathcal{P}}, M[n+1])$$

in which $\widetilde{\mathrm{L}}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ is the prespectrum with $(\widetilde{\mathrm{L}}_{\mathcal{P}})_{n,n} = \mathcal{P} \circ \mathrm{S}^n_C$ for $n \geqslant 0$. If $\mathcal{S}$ is in addition stable containing a strict zero object 0 then the n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in a given fibrant object $M \in \mathrm{IbMod}(\mathcal{P})$ is formulated as

$$\mathrm{H}^n_Q(\mathcal{P}, M) \cong \pi_0\,\mathrm{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}, M[n+1])$$

where $\overline{\mathrm{L}}_{\mathcal{P}} \in \mathrm{IbMod}(\mathcal{P})$ is given by $\overline{\mathrm{L}}_{\mathcal{P}}(\overline{c}) = \mathcal{P}(\overline{c}) \otimes \mathrm{hocolim}_n\,\Omega^n[\,(\mathrm{S}^n)^{\otimes m} \times^{\mathrm{h}}_{1_{\mathcal{S}}} 0\,]$ for each C-sequence $\overline{c} := (c_1, \cdots, c_m; c)$.

Moreover, we find a connection between Quillen cohomology and reduced Quillen cohomology of $\mathcal{P}$, expressed as follows.

Theorem 1.0.0.6. *(3.5.0.2) Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 3.3.0.2. Given a fibrant object $M \in \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$, there is a long exact sequence of abelian groups of the form*

$$\cdots \longrightarrow \mathrm{H}^{n-1}_Q(\mathcal{P}, M) \longrightarrow \mathrm{H}^n_{Q,r}(\mathcal{P}, M) \longrightarrow \mathrm{H}^n_{Q,\mathrm{red}}(\mathcal{P}, M) \longrightarrow \mathrm{H}^n_Q(\mathcal{P}, M) \longrightarrow \mathrm{H}^{n+1}_{Q,r}(\mathcal{P}, M) \longrightarrow \cdots$$

where $\mathrm{H}^{\bullet}_{Q,r}(\mathcal{P}, -)$ refers to Quillen cohomology group of $\mathcal{P}$ when regarded as a right module over itself and $\mathrm{H}^{\bullet}_{Q,\mathrm{red}}(\mathcal{P}, -)$ refers to reduced Quillen cohomology group of $\mathcal{P}$.

Turning to the context of simplicial operads, our main result extends the following:

Theorem 1.0.0.7. *(Y. Harpaz, M. Prasma and J. Nuiten [7]) Let $\mathcal{C}$ be a fibrant simplicial category. There is an equivalence of ∞-categories $\mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathrm{Set}_{\Delta})_{\infty} \simeq \mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \textbf{Spectra})$ with **Spectra** being the ∞-category of spectra and $\mathrm{Tw}(\mathcal{C})$ being the **twisted arrow ∞-category** of $\mathcal{C}$. Furthermore, the cotangent complex $\mathrm{L}_{\mathcal{C}} \in \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathrm{Set}_{\Delta})$ is then identified to the constant functor $\mathrm{Tw}(\mathcal{C}) \longrightarrow \textbf{Spectra}$ on $\mathbb{S}[-1]$, i.e., the desuspension of the sphere spectrum. Consequently, the n'th Quillen cohomology group of $\mathcal{C}$ with coefficients in a given functor $\mathcal{F} : \mathrm{Tw}(\mathcal{C}) \longrightarrow \textbf{Spectra}$ is given by $\mathrm{H}^n_Q(\mathcal{C}, \mathcal{F}) = \pi_{-n-1}\lim \mathcal{F}$.*

The construction of twisted arrow ∞-categories (of ∞-categories) $\mathrm{Tw}(-) : \mathrm{Cat}_{\infty} \longrightarrow \mathrm{Cat}_{\infty}$ was originally introduced by Lurie [[3], §5.2]. We extend that to the construction of **twisted arrow ∞-categories of (fibrant) simplicial operads**. Let $\mathcal{P}$ be a fibrant simplicial operad. Conceptually, the twisted arrow ∞-category of $\mathcal{P}$, denoted by $\mathrm{Tw}(\mathcal{P})$, is defined to be the **covariant unstraightening** of the simplicial functor $\mathcal{P} : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathrm{Set}_{\Delta}$, which encodes the data of $\mathcal{P}$ as an infinitesimal bimodule over itself (see § 2.1.3). In particular, objects of $\mathrm{Tw}(\mathcal{P})$ are precisely the *operations* of $\mathcal{P}$ (i.e., the vertices of spaces of operations of $\mathcal{P}$). For examples, $\mathrm{Tw}(\mathcal{C}om)$ is equivalent to $\mathrm{Fin}_*^{\mathrm{op}}$ the (opposite) category of finite pointed sets, while $\mathrm{Tw}(\mathcal{A}ss)$ is equivalent to the simplex category Δ (cf. Proposition 4.2.0.15).

Theorem 1.0.0.8. *(4.3.0.1) Let $\mathcal{P}$ be a fibrant and Σ-cofibrant simplicial operad. Then there is an equivalence of ∞-categories*

$$\mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathrm{Set}_\Delta)_\infty \xrightarrow{\simeq} \operatorname{Fun}(\mathrm{Tw}(\mathcal{P}), \boldsymbol{Spectra}).$$

Moreover, under this equivalence, the cotangent complex $\mathrm{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathrm{Set}_\Delta)_\infty$ is identified to the desuspension of the functor $\mathcal{F}_{\mathcal{P}} : \mathrm{Tw}(\mathcal{P}) \longrightarrow \boldsymbol{Spectra}$ given on objects by sending each operation $\mu \in \mathcal{P}$ of arity m to $\mathcal{F}_{\mathcal{P}}(\mu) = \mathbb{S}^{\times m}$, i.e., the m-fold product of the sphere spectrum. Consequently, the n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in a given functor $\mathcal{F} : \mathrm{Tw}(\mathcal{P}) \longrightarrow \boldsymbol{Spectra}$ is formulated as

$$\mathrm{H}^n_Q(\mathcal{P};\mathcal{F}) = \pi_0 \operatorname{Map}_{\operatorname{Fun}(\mathrm{Tw}(\mathcal{P}),\boldsymbol{Spectra})}(\mathcal{F}_{\mathcal{P}}, \mathcal{F}[n+1]).$$

For example, we consider the **little ∞-cubes operad** E_∞. Denote by $\mathrm{Mod}_{\mathbf{k}}$ the category of vector spaces over a given field $\mathbf{k}$. A right **Γ-module** is by definition a functor $T : \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{\mathbf{k}}$. There is a significant invariant for right Γ-modules given by the **stable cohomotopy groups** (cf. [25]), which plays a key role in the **E_∞-obstruction theory** initiated by Robinson [26]. In Proposition 4.3.0.4, we show that for each integer n there is an isomorphism

$$\pi^n(T) \cong \mathrm{H}^{n-1}_Q(\mathrm{E}_\infty; \widetilde{T})$$

between the n'th stable cohomotopy group of T and the $(n-1)$'th Quillen cohomology group of E_∞ with coefficients in the induced functor $\widetilde{T} : \mathrm{Tw}(\mathrm{E}_\infty) = \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathbf{Spectra}$.

For more illustration, we consider the associative operad $\mathcal{A}ss$. Let $\mathcal{F} : \mathrm{Tw}(\mathcal{A}ss) = \Delta \longrightarrow \mathrm{Spectra}$ be a functor whose values are Ω-spectra. Then the Quillen cohomology groups of $\mathcal{A}ss$ with coefficients in $\mathcal{F}$ fit into a long exact sequence of the form

$$\cdots \longrightarrow \mathrm{H}^{-n-2}_Q(\mathcal{A}ss;\mathcal{F}) \longrightarrow \pi_n \operatorname{holim}\mathcal{F} \longrightarrow \pi_n\mathcal{F}([0]) \longrightarrow \mathrm{H}^{-n-1}_Q(\mathcal{A}ss;\mathcal{F}) \longrightarrow \pi_{n-1}\operatorname{holim}\mathcal{F} \longrightarrow \cdots,$$

(cf. Corollary 4.3.0.7).

A simplicial operad is said to be **unitally homotopy connected** if all its spaces of unary and 1-ary operations are weakly contractible. The following result in particular proves that Quillen cohomology of any little cubes operad with *constant coefficients* vanishes.

Corollary 1.0.0.9. *(4.3.0.8) Let $\mathcal{P}$ be a fibrant, Σ-cofibrant and unitally homotopy connected simplicial operad and $\mathcal{F}_0 : \mathrm{Tw}(\mathcal{P}) \longrightarrow \boldsymbol{Spectra}$ a constant functor. Then Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{F}_0$ vanishes.*

Part 2. In the second part (corresponding to Chapter 5), we study Quillen cohomology of dg operads and besides that, we outline the relation between deformation theory and Quillen cohomology. This is a joint work with Y. Harpaz.

Let us fix $\mathbf{k}$ to be a field of characteristic 0. By dg operads we mean the operads enriched over $\mathcal{C}(\mathbf{k})$ the category of dg $\mathbf{k}$-modules. In particular, dg operads which are concentrated in non-negative degrees will be said to be **connective**.

Let $\mathcal{P}$ be any C-colored dg operad. Since the category $\mathcal{C}(\mathbf{k})$ is stable, Theorem 1.0.0.3 tells us that there is a sequence of right Quillen equivalences $\mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{C}(\mathbf{k})) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\operatorname{Op}_C(\mathcal{C}(\mathbf{k})) \xrightarrow{\simeq} \mathrm{IbMod}(\mathcal{P})$. So the desired formula of Quillen cohomology of $\mathcal{P}$ can be expressed in terms of mapping spaces in $\mathrm{IbMod}(\mathcal{P})$, after having described the derived image of the cotangent complex of $\mathcal{P}$ in $\mathrm{IbMod}(\mathcal{P})$. Denote by $\mathcal{I}_C$ the initial C-colored operad.

Theorem 1.0.0.10. *(5.1.0.8) Let $M \in \mathrm{IbMod}(\mathcal{P})$ be an infinitesimal $\mathcal{P}$-bimodule, regarded as the module of coefficients. The n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in M is formulated as*

$$\mathrm{H}^n_Q(\mathcal{P}, M) \cong \pi_0 \, \mathrm{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}, M[n+1])$$

in which $\overline{\mathrm{L}}_{\mathcal{P}} = \mathcal{P} \circ_{(1)} \mathcal{I}_C$ with the infinitesimal $\mathcal{P}$-bimodule structure described as follows. As an infinitesimal left $\mathcal{P}$-module, it is free generated by $\mathcal{I}_C$. On the other hand, given an operation $\alpha \in \mathcal{P}(c_1, \cdots, c_n; c)$, the (infinitesimal) right action of an operation $\lambda \in \mathcal{P}(d_1, \cdots, d_m; c_j)$ on the element $(\alpha \, , \mathrm{id}_{c_i}) \in \mathcal{P} \circ_{(1)} \mathcal{I}_C$ is given by

$$(\alpha \, , \mathrm{id}_{c_i}) \circ^r_j \lambda := \begin{cases} (\alpha \circ_j \lambda \, , \mathrm{id}_{c_i}) & \text{if } j \neq i \\ \sum_{k=1}^m (\alpha \circ_j \lambda \, , \mathrm{id}_{d_k}) & \text{if } j = i. \end{cases}$$

Moreover, there is also a long exact sequence relating Quillen cohomology and reduced Quillen cohomology of $\mathcal{P}$, similarly as the one given in Theorem 1.0.0.6.

Now suppose further that $\mathcal{P}$ is a single-colored connective augmented dg operad. Let $\mathcal{Q}$ be another single-colored dg operad and $\alpha : \mathcal{P} \longrightarrow \mathcal{Q}$ a map of dg operads. We may take a cofibrant resolution for $\mathcal{P}$ of the form $\phi : \Omega(\mathcal{C}) \xrightarrow{\simeq} \mathcal{P}$ with $\Omega(\mathcal{C})$ being the cobar construction of $\mathcal{C}$ the dg cooperad characterized by having the same homotopy type as $\mathrm{B}(\mathcal{P})$ (i.e., the bar construction of $\mathcal{P}$). The composition $\alpha\phi : \Omega(\mathcal{C}) \longrightarrow \mathcal{Q}$ performs a Maurer-Cartan element of the **convolution dg Lie algebra** $\mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{Q})$, or of the reduced one $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \mathcal{Q})$ with $\overline{\mathcal{C}}$ being the coaugmented coideal of $\mathcal{C}$.

Recall that the **deformation complex of** α is defined to be $\mathrm{Hom}^{\alpha\phi}_\Sigma(\mathcal{C}, \mathcal{Q})$ the twisted dg Lie algebra of $\mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{Q})$ by $\alpha\phi$. On other hand, the **reduced deformation complex of** α is $\mathrm{Hom}^{\alpha\phi}_\Sigma(\overline{\mathcal{C}}, \mathcal{Q})$ the twisted dg Lie algebra of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \mathcal{Q})$ by $\alpha\phi$. (See [11] for more details). Consider $\mathcal{Q}$ as an infinitesimal $\mathcal{P}$-bimodule with the structure induced by α. According to the works of Loday-Merkulov-Vallette ([11, 12]), the reduced Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{Q} \in \mathrm{IbMod}(\mathcal{P})$ agrees with the homology of the reduced deformation complex of α. The following statement therefore fits into their works very naturally.

Theorem 1.0.0.11. *(5.2.0.7) The Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{Q} \in \mathrm{IbMod}(\mathcal{P})$ agrees with the homology of the deformation complex of α. More explicitly, for each $n \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{H}^{n-1}_Q(\mathcal{P}, \mathcal{Q}) \cong \mathrm{H}_{-n} \, Hom^{\alpha\phi}_\Sigma(\mathcal{C}, \mathcal{Q}).$$

The **Dold-Kan correspondence** asserts that there is an adjoint equivalence

$$\Gamma : \mathcal{C}_{\geqslant 0}(\mathbf{k}) \overset{\simeq}{\rightleftarrows} \mathrm{sMod}_{\mathbf{k}} : \mathrm{N}$$

between the categories of **connective dg k-modules** and **simplicial k-modules**, in which the functor N is the well known **normalized complex functor**. An important fact is that the functors Γ and N are no longer inverse equivalences of each other (or even adjunction) when descending to the categories of monoids. Despite this, the work of Schwede-Shipley [[58], Theorem 3.12] shows that the functor N is the right adjoint of a Quillen equivalence when considered as a functor between monoids. Improving their work, we shall prove the existence of an operadic version of the Dold-Kan correspondence. Note that while the normalized complex functor is lax symmetric monoidal, its inverse Γ does not admit any canonical lax symmetric monoidal structure. So one in general can not produce $\mathrm{sMod}_{\mathbf{k}}$-enriched operads just by applying Γ to $\mathcal{C}_{\geqslant 0}(\mathbf{k})$-enriched operads levelwise. The following statement is in particular not a trivial one.

Theorem 1.0.0.12. *(**Operadic Dold-Kan correspondence**, 5.3.2.4) The functor*

$$\mathrm{N} : \mathrm{Op}(\mathrm{sMod}_{\boldsymbol{k}}) \longrightarrow \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\boldsymbol{k})) \tag{1.0.0.2}$$

given by applying the normalized complex functor levelwise is a right Quillen equivalence.

For each simplicial operad $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_{\Delta})$, the image of $\mathcal{O}$ through the composite functor

$$\mathrm{Op}(\mathrm{Set}_{\Delta}) \xrightarrow{\mathbf{k}\{-\}} \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) \xrightarrow{\mathrm{N}} \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$$

is called the **dg version of** $\mathcal{O}$, and denoted by $\mathrm{d}\mathcal{O}$. We exploit the above assertion to find a connection between Quillen cohomology of a given simplicial operad and the corresponding of its dg version.

Let $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_{\Delta})$ be a fibrant and Σ-cofibrant simplicial operad and let $M \in \mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\})$ be given. The object M induces a coefficient functor $\mathcal{H}_M : \mathrm{Tw}(\mathcal{O}) \longrightarrow \mathbf{Spectra}$ in an obvious way.

Corollary 1.0.0.13. *(5.3.3.7) There is a canonical isomorphism*

$$\mathrm{H}^{\bullet}_{Q}(\mathcal{O}, \mathcal{H}_M) \xrightarrow{\simeq} \mathrm{H}^{\bullet}_{Q}(\mathrm{d}\mathcal{O}, \mathrm{N}(M))$$

*between Quillen cohomology of $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_{\Delta})$ with coefficients in $\mathcal{H}_M : \mathrm{Tw}(\mathcal{O}) \longrightarrow$ **Spectra** and Quillen cohomology of $\mathrm{d}\mathcal{O} \in \mathrm{Op}(\mathcal{C}(\boldsymbol{k}))$ with coefficients in $\mathrm{N}(M) \in \mathrm{IbMod}(\mathrm{d}\mathcal{O})$.*

This hence provides us with another approach for calculating Quillen cohomology of dg operads that come from simplicial operads. For example, using this result we observe that Quillen cohomology of the **dg little ∞-cubes operad** $\mathrm{E}_{\infty} \in \mathrm{Op}(\mathcal{C}(\mathbf{k}))$ with coefficients in itself vanishes (cf. Corollary 5.3.3.8). On other hand, consider the **dg associative operad** $\mathcal{A}ss \in \mathrm{Op}(\mathcal{C}(\mathbf{k}))$. Let A be an associative **k**-algebra and let End_A denote the **endomorphism operad** associated to A (regarded as an infinitesimal $\mathcal{A}ss$-bimodule via the canonical map $\mathcal{A}ss \longrightarrow \mathrm{End}_A$). We then observe that Quillen cohomology of $\mathcal{A}ss$ with coefficients in End_A can be described in terms of the **Hochschild cohomology groups** of A (cf. Corollary 5.3.3.9).

In the remainder, we establish the relation between deformation theory and Quillen cohomology. Here are some settings. Let $\mathbf{k}$ be a field of characteristic 0. We denote by $\mathrm{CAlg}^{\mathrm{art}}$ the category of artinian dg **k**-algebras and by ModCat the category whose objects are model categories and whose morphisms are Quillen adjunctions with the sources and targets being those of the left Quillen functors.

Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be a functor. As the first step, we propose the notion of **space of deformations** of a given object $X \in \mathcal{F}(\mathbf{k})$ over some $R \in \mathrm{CAlg}^{\mathrm{art}}$, denoted by $\mathrm{Def}(X, R)$ (cf. Definition 5.4.1.3). To make this notion well-defined, the functor $R \mapsto \mathrm{Def}(X, R)$ should form a **formal moduli problem** in the sense of [5]. To this end, we require that $\mathcal{F}$ forms a **formal moduli context** (cf. Definition 5.4.1.2 and Proposition 5.4.1.4). For each map $f : R \to S$ in $\mathrm{CAlg}^{\mathrm{art}}$, we will write $f_! : \mathcal{F}(R) \rightleftarrows \mathcal{F}(S) : f^*$ standing for the image of f through $\mathcal{F}$.

To state our main theorem, we will need the following extra construction. Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be a formal moduli context and $X \in \mathcal{F}(\mathbf{k})$ a fibrant object. Suppose that the tangent category $\mathcal{T}_X\mathcal{F}(\mathbf{k})$ exists. Given an Ω-spectrum $M : \mathbb{N} \times \mathbb{N} \longrightarrow \mathrm{CAlg}^{\mathrm{art}}$, we denote by $\mathrm{Def}(X, M) := \mathrm{Def}(X, \Omega^{\infty} M)$ and refer to it as the **space of first order deformations** of X in **direction** M. Moreover, we will denote by $M(X) \in \mathcal{T}_X\mathcal{F}(\mathbf{k})$ the spectrum object defined as

$$M(X)_{n,m} = u^*_{n,m}(u_{n,m})_! X \in \mathcal{F}(\mathbf{k})$$

where $u_{n,m} : \mathbf{k} \longrightarrow M_{n,m}$ is the unit of $M_{n,m} \in \mathrm{CAlg}^{\mathrm{art}}$. This object $M(X)$ forms an Ω-spectrum if either X is cofibrant or for every $(n,m) \in \mathbb{N} \times \mathbb{N}$ the induced functor $(u_{n,m})_! : \mathcal{F}(\mathbf{k}) \longrightarrow \mathcal{F}(M_{n,m})$ preserves weak equivalences (cf. Lemma 5.4.1.7 and Remark 5.4.1.9). Our main theorem is then stated as follows.

Theorem 1.0.0.14. *(5.4.1.8) The space* $\mathrm{Def}(X, M)$ *is weakly equivalent to the derived mapping space* $\mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})_{/X}}(X, \Omega^{\infty-1}(M(X)))$*. In particular, we have a canonical isomorphism*

$$\pi_0(\mathrm{Def}(X, M)) \cong \mathrm{H}^1_{\mathcal{Q}}(X, M(X)).$$

Moreover, the n'th homotopy group $\pi_n(\mathrm{Def}(X, M), *)$ *is isomorphic to* $\mathrm{H}^{1-n}_{\mathcal{Q}}(X, M(X))$*, where* $* \in \mathrm{Def}(X, M)$ *refers to the trivial deformation.*

The relation between deformation theory and Quillen cohomology of dg operads was considered in literature (cf. [11, 12, 62]). Theorem 1.0.0.14 allows us to recover their result in a particular case as follows. Let $A \in \mathcal{C}_{\geq 0}(\mathbf{k})$ be a finite dimensional connective dg $\mathbf{k}$-module and let $\mathcal{P}$ be a connective dg operad over $\mathbf{k}$. Then we have for each $n \geq 0$ an isomorphism

$$\pi_n(\mathrm{Def}(\mathcal{P}, \mathbf{k} \ltimes A), *) \cong \mathrm{H}^{1-n}_{\mathcal{Q}}(\mathcal{P}, \mathcal{P} \otimes A)$$

where $\mathcal{P} \otimes A$ is regarded as an infinitesimal $\mathcal{P}$-bimodule obtained by tensoring levelwise. In particular, when $A = \mathbf{k}$, we obtain that the homotopy type of $\mathrm{Def}(\mathcal{P}, \mathbf{k}[t]/(t^2))$ is classified by Quillen cohomology of $\mathcal{P}$ with coefficients in itself $\mathcal{P} \in \mathrm{IbMod}(\mathcal{P})$.

Organization of the thesis. In Chapter 2, we recall briefly some necessary facts relevant to enriched operads and various types of module over an operad. This chapter is also devoted to the most important concepts that we work with throughout the thesis including tangent category, cotangent complex and Quillen cohomology groups. In Chapter 3, we first concentrate on proving Theorem 1.0.0.3. We then set up an extra condition on the base category and by the way, provide several illustrations for this condition. The ultimate goal of this chapter is to prove Theorem 1.0.0.5. Chapter 4 is devoted to Quillen cohomology of simplicial operads. We shall discuss on the construction of twisted arrow ∞-categories of simplicial operads, after having described the unstraightening of simplicial (co)presheaves. We then explain how this construction classifies Quillen cohomology of simplicial operads. Our main purposes in Chapter 5 are to formulate Quillen cohomology of dg operads and to establish the relation between deformation theory and Quillen cohomology. Besides that, we consider Quillen cohomology of connective augmented dg operads. For other purposes, we prove the existence of an operadic version of the Dold-Kan correspondence, as well as give a connection between Quillen cohomology of a simplicial operad and Quillen cohomology of its dg version. In Appendix A, we recall some basic facts involving semi model categories and their localizations. Lastly, in Appendix B, we present the notion of homotopy Cartesian squares of model categories. This appendix is particularly devoted to the work of §5.4.

Chapter 2

Backgrounds and notations

This chapter is devoted to the basic concepts and notations we work with throughout the thesis. In the first section, we recall briefly fundamental notions relevant to enriched operads, various types of module over an operad and their homotopy theories. The second section is devoted to the needed concepts relevant to the Quillen cohomology theory.

2.1 Enriched operads and various types of operadic module

2.1.1 Enriched operads

Let $\mathcal{S}$ be a **symmetric monoidal category**. Given a set C, regarded as the set of **colors**, we denote by

$$\mathrm{Seq}(C) := \{(c_1, \cdots, c_n; c) \,|\, c_i, c \in C, n \geqslant 0\}$$

and refer to it as the collection of C-**sequences**.

For each $n \geq 0$, we let Σ_n denote the **symmetric group of degree** n.

Definition 2.1.1.1. A **symmetric** C-**collection** (also called a C-**symmetric sequence**) in $\mathcal{S}$ is a collection

$$M = \{M(c_1, \cdots, c_n; c)\}_{(c_1, \cdots, c_n; c) \in \mathrm{Seq}(C)}$$

of objects in $\mathcal{S}$ equipped with a **symmetric action** whose data consists of, for each $(c_1, \cdots, c_n; c) \in \mathrm{Seq}(C)$ and $\sigma \in \Sigma_n$, a map of the form $\sigma^* : M(c_1, \cdots, c_n; c) \longrightarrow M(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; c)$. These maps define a right action by Σ_n in the sense that $\sigma^* \tau^* = (\tau\sigma)^*$ and $\mu_n^* = \mathrm{Id}$ where $\mu_n \in \Sigma_n$ signifies the trivial permutation. With the obvious maps, symmetric C-collections in $\mathcal{S}$ form a category, denoted by $\mathrm{Coll}_C(\mathcal{S})$.

Remark 2.1.1.2. Let $P(C)$ denote the groupoid whose set of objects is given by

$$\mathrm{Ob}(P(C)) = \{\varnothing\} \sqcup (\bigsqcup_{n \geq 1} C^{\times n})$$

and whose morphisms just consist of the identity map $\mathrm{Id}_{\varnothing}$ and the morphisms of the form

$$\sigma^* : (c_1, \cdots, c_n) \longrightarrow (c_{\sigma(1)}, \cdots, c_{\sigma(n)})$$

with $\sigma \in \Sigma_n$, $n \geqslant 1$. Then the category $\mathrm{Coll}_C(\mathcal{S})$ is isomorphic to $\mathrm{Fun}(P(C) \times C, \mathcal{S})$ the category of $\mathcal{S}$-enriched functors from $P(C) \times C$ to $\mathcal{S}$.

The well known **composite product**

$$- \circ - : \mathrm{Coll}_C(\mathcal{S}) \times \mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathrm{Coll}_C(\mathcal{S})$$

endows $\mathrm{Coll}_C(\mathcal{S})$ with a monoidal structure. The monoidal unit will be denoted by $\mathcal{I}_C$, with $\mathcal{I}_C(c;c) = 1_{\mathcal{S}}$ for every $c \in C$ and agreeing with $\varnothing_{\mathcal{S}}$ on the other levels. (See, e.g., [8, 27, 29]).

Definition 2.1.1.3. A **symmetric C-colored operad** in $\mathcal{S}$ is a monoid in the monoidal category $(\mathrm{Coll}_C(\mathcal{S}), - \circ -, \mathcal{I}_C)$. We denote by $\mathrm{Op}_C(\mathcal{S})$ the category of symmetric C-colored operads.

Remark 2.1.1.4. Unwinding definition, a symmetric C-colored operad in $\mathcal{S}$ is a symmetric C-collection $\mathcal{P}$ equipped with

- a **composition** whose data consists of the maps of the form

$$\mathcal{P}(c_1, \cdots, c_n; c) \otimes \mathcal{P}(c_{1,1}, \cdots, c_{1,k_1}; c_1) \otimes \cdots \otimes \mathcal{P}(c_{n,1}, \cdots, c_{1,k_n}; c_n)$$
$$\longrightarrow \mathcal{P}(c_{1,1}, \cdots, c_{1,k_1}, \cdots, c_{n,1}, \cdots, c_{1,k_n}; c),$$

- and for each color $c \in C$, with a **unit operation** $\mathrm{id}_c : 1_{\mathcal{S}} \longrightarrow \mathcal{P}(c;c)$.

The composition maps are required to satisfy the essential axioms of Σ_*-equivariance, associativity and unitality. (Cf., e.g, [31, 29] for more details).

Given $\mathcal{P} \in \mathrm{Op}_C(\mathcal{S})$, each object $\mathcal{P}(c_1, \cdots, c_n; c)$ will be called a **space of n-ary operations** of $\mathcal{P}$. Recall that the collection of 1-ary operations of $\mathcal{P}$, denoted by $\mathcal{P}_1$, inherits an obvious $\mathcal{S}$-enriched category structure. We shall refer to $\mathcal{P}_1$ as the **underlying category** of $\mathcal{P}$.

The notion of a **nonsymmetric C-colored operad** (resp. **C-collection**) is the same as that of a symmetric C-colored operad (resp. C-collection) after forgetting the symmetric action. We denote by $\mathrm{nsColl}_C(\mathcal{S})$ (resp. $\mathrm{nsOp}_C(\mathcal{S})$) the category of nonsymmetric C-collections (resp. C-colored operads) in $\mathcal{S}$. The natural passage from nonsymmetric to symmetric context is performed by the **symmetrization functor** Sym. Namely, the functor $\mathrm{Sym} : \mathrm{nsColl}_C(\mathcal{S}) \longrightarrow \mathrm{Coll}_C(\mathcal{S})$ is given by sending each nonsymmetric C-collection M to $\mathrm{Sym}(M)$ with

$$\mathrm{Sym}(M)(c_1, \cdots, c_n; c) = \bigsqcup_{\sigma \in \Sigma_n} M(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; c).$$

The structure maps are induced by the multiplication of permutations in an evident way. By construction, the functor Sym forms a left adjoint to the associated forgetful functor.

Remark 2.1.1.5. Given two objects $M, N \in \mathrm{nsColl}_C(\mathcal{S})$, the composite product of unit maps $M \circ N \longrightarrow \mathrm{Sym}(M) \circ \mathrm{Sym}(N)$ induces a natural map $\mathrm{Sym}(M \circ N) \longrightarrow \mathrm{Sym}(M) \circ \mathrm{Sym}(N)$. It can then be verified that the latter is a natural isomorphism of symmetric C-collections with the inverse

$$\mathrm{Sym}(M) \circ \mathrm{Sym}(N) \xrightarrow{\cong} \mathrm{Sym}(M \circ N)$$

induced by the concatenation of linear orders

$$\Sigma_k \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \longrightarrow \Sigma_{n_1 + \cdots + n_k}.$$

On other hand, the functor Sym clearly preserves monoidal units. In conclusion, we get that Sym is strong monoidal.

By the above remark, the functor $\mathrm{Sym} : \mathrm{nsColl}_C(\mathcal{S}) \longrightarrow \mathrm{Coll}_C(\mathcal{S})$ descends to a functor between monoids

$$\mathrm{Sym} : \mathrm{nsOp}_C(\mathcal{S}) \longrightarrow \mathrm{Op}_C(\mathcal{S}),$$

which forms a left adjoint to the associated forgetful functor.

Conventions 2.1.1.6. Since we are almost concentrated in the symmetric context, throughout the forthcoming, unless otherwise specified, we shall omit the word "symmetric" when mentioning an object of $\mathrm{Op}_C(\mathcal{S})$ (or $\mathrm{Coll}_C(\mathcal{S})$).

One can integrate all the categories $\mathrm{Op}_C(\mathcal{S})$ for $C \in \mathrm{Sets}$ into a single category of $\mathcal{S}$-enriched operads, just like the way one establishes category of categories.

Definition 2.1.1.7. The **category of $\mathcal{S}$-enriched operads**, denoted by $\mathrm{Op}(\mathcal{S})$, is the (contravariant) **Grothendieck construction**

$$\mathrm{Op}(\mathcal{S}) := \int_{C \in \mathrm{Sets}} \mathrm{Op}_C(\mathcal{S})$$

in which for each map $\alpha : C \to D$ of sets, the corresponding functor $\alpha^* : \mathrm{Op}_D(\mathcal{S}) \longrightarrow \mathrm{Op}_C(\mathcal{S})$ is given by taking $\mathcal{Q} \in \mathrm{Op}_D(\mathcal{S})$ to $\alpha^*\mathcal{Q}$ with

$$\alpha^*\mathcal{Q}(c_1, \cdots, c_n; c) := \mathcal{Q}(\alpha(c_1), \cdots, \alpha(c_n); \alpha(c)).$$

The functor α^* will be called the **changing-colors functor** associated to α.

Unwinding definition, an object of $\mathrm{Op}(\mathcal{S})$ is a pair $(C, \mathcal{P})$ with $C \in \mathrm{Sets}$, $\mathcal{P} \in \mathrm{Op}_C(\mathcal{S})$ and moreover, a morphism $(C, \mathcal{P}) \longrightarrow (D, \mathcal{Q})$ consists of a map $\alpha : C \to D$ of sets and a map $f : \mathcal{P} \longrightarrow \alpha^*\mathcal{Q}$ of C-colored operads.

2.1.2 Various types of operadic module

Let $\mathcal{P}$ be a C-colored operad in $\mathcal{S}$. We let $\mathrm{LMod}(\mathcal{P})$ (resp. $\mathrm{RMod}(\mathcal{P})$) denote the categoy of **left (resp. right) $\mathcal{P}$-modules**. Besides that we let $\mathrm{BMod}(\mathcal{P})$ and $\mathrm{IbMod}(\mathcal{P})$ respectively denote the categories of **$\mathcal{P}$-bimodules** and **infinitesimal $\mathcal{P}$-bimodules**. Let us revisit these quickly.

Operadic left module (resp. right module, bimodule) is the usual notion of left module (resp. right module, bimodule) over an operad when one regards operads as monoids in the monoidal category of symmetric sequences. More explicitly,

Definition 2.1.2.1.
1. A **left $\mathcal{P}$-module** is a C-collection M equipped with a *left $\mathcal{P}$-action* map $\mathcal{P} \circ M \longrightarrow M$ whose data consists of the Σ_*-equivariant maps of the form

   $$\mathcal{P}(c_1, \cdots, c_n; c) \otimes M(d_{1,1}, \cdots, d_{1,k_1}; c_1) \otimes \cdots \otimes M(d_{n,1}, \cdots, d_{n,k_n}; c_n) \longrightarrow M(d_{1,1}, \cdots, d_{1,k_1}, \cdots, d_{n,1}, \cdots, d_{n,k_n}; c)$$

   satisfying the classical axioms of associativity and unitality for left modules.

2. Dually, a **right $\mathcal{P}$-module** is a C-collection M equipped with a *right $\mathcal{P}$-action* map $M \circ \mathcal{P} \longrightarrow M$ whose data consists of the Σ_*-equivariant maps of the form

   $$M(c_1, \cdots, c_n; c) \otimes \mathcal{P}(d_{1,1}, \cdots, d_{1,k_1}; c_1) \otimes \cdots \otimes \mathcal{P}(d_{n,1}, \cdots, d_{n,k_n}; c_n) \longrightarrow M(d_{1,1}, \cdots, d_{1,k_1}, \cdots, d_{n,1}, \cdots, d_{n,k_n}; c)$$

   satisfying the classical axioms of associativity and unitality for right modules.

3. A $\mathcal{P}$**-bimodule** is a C-collection M equipped with both a left and a right $\mathcal{P}$-module structure. These must satisfy the essential compatibility for bimodules.

Moreover, a $\mathcal{P}$-algebra is simply a left $\mathcal{P}$-module concentrated in level 0. More precisely,

Definition 2.1.2.2. A $\mathcal{P}$**-algebra** is an object $A \in \mathcal{S}^{\times C}$ which is equipped, for each $(c_1, \cdots, c_n; c)$, with a $\mathcal{P}$-action map

$$\mathcal{P}(c_1, \cdots, c_n; c) \otimes A(c_1) \otimes \cdots \otimes A(c_n) \longrightarrow A(c)$$

factoring through the tensor product over Σ_n. These maps must satisfy the essential axioms of associativity and unitality. We denote by $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$ the category of $\mathcal{P}$-algebras.

Remark 2.1.2.3. When $\mathcal{P}$ is concentrated in arity 1 then $\mathcal{P}$ is simply an $\mathcal{S}$-enriched category. In this situation, the category of $\mathcal{P}$-algebras is nothing but $\mathrm{Fun}(\mathcal{P}, \mathcal{S})$ the category of $\mathcal{S}$-valued enriched functors on $\mathcal{P}$.

Example 2.1.2.4. The collection of unary (= 0-ary) operations of $\mathcal{P}$, denoted by $\mathcal{P}_0$, inherits an obvious $\mathcal{P}$-algebra structure and moreover, $\mathcal{P}_0$ then becomes an initial object in $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$.

Example 2.1.2.5. It is noteworthy that for each set C, there exists a $\mathrm{Seq}(C)$-colored operad in $\mathcal{S}$, denoted by $\mathbf{O}_C$, whose algebras are precisely the C-colored operads, i.e., there is an isomorphism of categories $\mathrm{Alg}_{\mathbf{O}_C}(\mathcal{S}) \cong \mathrm{Op}_C(\mathcal{S})$. We will refer to $\mathbf{O}_C$ as the **operad of** C**-colored operads**. (Cf., e.g., [[35], §3]).

The structure of left modules (and hence, bimodules) over an operad is not abelian in general. The notion of *infinitesimal left modules (bimodules)*, as introduced by Merkulov-Vallette ([12]), essentially appears as the abelianization of the previous one. In terms of enriched colored operads, these are defined as follows.

Definition 2.1.2.6.

1. An **infinitesimal left** $\mathcal{P}$**-module** is a C-collection M equipped with the action maps of the form

$$\circ_i^l : \mathcal{P}(c_1, \cdots, c_n; c) \otimes M(d_1, \cdots, d_m; c_i) \longrightarrow M(c_1, \cdots, c_{i-1}, d_1, \cdots, d_m, c_{i+1}, \cdots, c_n; c)$$

which are Σ_*-equivariant and satisfy the classical axioms of associativity and unitality for left modules.

2. Dually, an **infinitesimal right** $\mathcal{P}$**-module** is a C-collection M equipped with the action maps of the form

$$\circ_i^r : M(c_1, \cdots, c_n; c) \otimes \mathcal{P}(d_1, \cdots, d_m; c_i) \longrightarrow M(c_1, \cdots, c_{i-1}, d_1, \cdots, d_m, c_{i+1}, \cdots, c_n; c)$$

which are Σ_*-equivariant and satisfy the classical axioms of associativity and unitality for right modules.

3. An **infinitesimal** $\mathcal{P}$**-bimodule** is a C-collection M equipped with both an infinitesimal left and an infinitesimal right $\mathcal{P}$-module structure which together satisfy the essential compatibility for bimodules.

Remark 2.1.2.7. The structure of an infinitesimal right $\mathcal{P}$-module is equivalent to that of a (non-infinitesimal) right $\mathcal{P}$-module.

The notion of an infinitesimal $\mathcal{P}$-bimodule can be reformulated using the diagramatical language, which has certain advantages over the above definition. To this end, one starts with the notion of **infinitesimal composite product**:

$$- \circ_{(1)} - : \mathrm{Coll}_C(\mathcal{S}) \times \mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathrm{Coll}_C(\mathcal{S}),$$

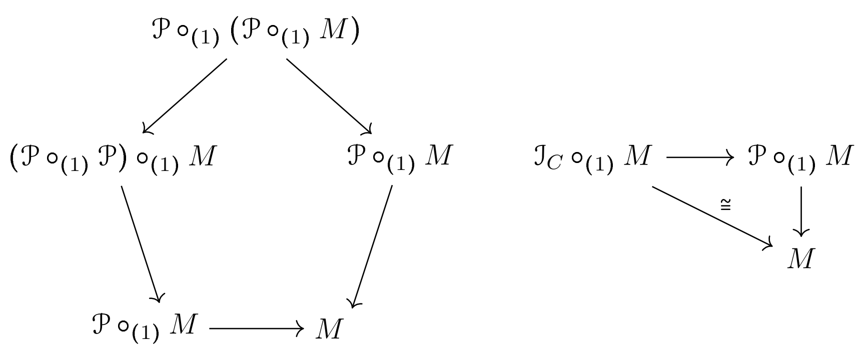
which can be thought of as the "right linearization" of the composite product $- \circ -$. Formally, given two C-collections M and N, $M \circ_{(1)} N$ is the sub C-collection of $M \circ (\mathcal{I}_C \sqcup N)$ linear in N. To be precise, looking at the explicit formula of $M \circ (\mathcal{I}_C \sqcup N)$, we have on each level that $(M \circ_{(1)} N)(c_1, \cdots, c_n; c)$ is the sub-object of $M \circ (\mathcal{I}_C \sqcup N)(c_1, \cdots, c_n; c)$ consisting of the *multi-tensor products* which contain one and only one factor in N. The readers can find out about this construction, in terms of single-colored dg operads, in [[11], Section 6.1].

Observe now that for each $M \in \mathrm{Coll}_C(\mathcal{S})$ there is a natural inclusion

$$\mathcal{P} \circ_{(1)} (\mathcal{P} \circ_{(1)} M) \longrightarrow (\mathcal{P} \circ_{(1)} \mathcal{P}) \circ_{(1)} M.$$

On other hand, the (partial) composition in $\mathcal{P}$ gives a map $\mu_{(1)} : \mathcal{P} \circ_{(1)} \mathcal{P} \longrightarrow \mathcal{P}$. The following is equivalent to Definition 2.1.2.6(i).

Definition 2.1.2.8. An **infinitesimal left $\mathcal{P}$-module** is a C-collection M equipped with an action map $\mathcal{P} \circ_{(1)} M \longrightarrow M$ satisfying the classical axioms of associativity and unitality for left modules, which are depicted as the commutativity of the following diagrams

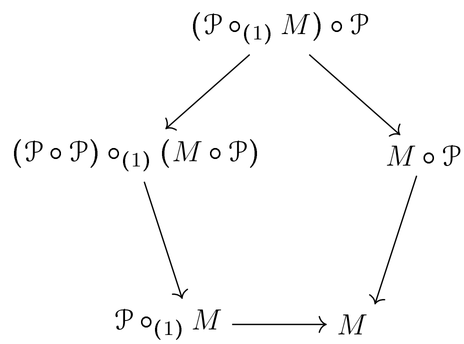


Now, notice that for each $M \in \mathrm{Coll}_C(\mathcal{S})$, there is a natural inclusion

$$(\mathcal{P} \circ_{(1)} M) \circ \mathcal{P} \longrightarrow (\mathcal{P} \circ \mathcal{P}) \circ_{(1)} (M \circ \mathcal{P}).$$

The following is equivalent to Definition 2.1.2.6(iii).

Definition 2.1.2.9. An infinitesimal $\mathcal{P}$-bimodule is a C-collection M endowed with an infinitesimal left $\mathcal{P}$-module structure, exhibited by a map $\mathcal{P} \circ_{(1)} M \longrightarrow M$ and with a right $\mathcal{P}$-module structure, exhibited by a map $M \circ \mathcal{P} \longrightarrow M$ which are subject to the essential compatibility, depicted as the commutativity of the following diagram



Remark 2.1.2.10. In the above diagram, the C-collection $(\mathcal{P} \circ_{(1)} M) \circ \mathcal{P}$ does not represent the free infinitesimal $\mathcal{P}$-bimodule generated by M. (This does not even carry any canonical infinitesimal $\mathcal{P}$-bimodule structure). To find out the exact one, we factor the free functor $\mathrm{Coll}_C(\mathcal{S}) \xrightarrow{\mathcal{F}^{ib}} \mathrm{IbMod}(\mathcal{P})$ as $\mathrm{Coll}_C(\mathcal{S}) \xrightarrow{\mathcal{F}_1} \mathrm{RMod}(\mathcal{P}) \xrightarrow{\mathcal{F}_2} \mathrm{IbMod}(\mathcal{P})$ where $\mathcal{F}_1$ ($\mathcal{F}_2$) refers to the left adjoint of the associated forgetful functor. Observe now that $\mathcal{F}_1 \cong (-) \circ \mathcal{P}$, while $\mathcal{F}_2 \cong \mathcal{P} \circ_{(1)} (-)$. In conclusion, the functor $\mathcal{F}^{ib}$ is given by $\mathcal{F}^{ib} = \mathcal{P} \circ_{(1)} (- \circ \mathcal{P})$. On other hand, the free infinitesimal left $\mathcal{P}$-module functor is simply $\mathcal{P} \circ_{(1)} (-)$.

Another important one is the notion of *modules over an operadic algebra*. Let A be a $\mathcal{P}$-algebra.

Definition 2.1.2.11. An **A-module over** $\mathcal{P}$ is an object $M \in \mathcal{S}^{\times C}$ equipped, for each sequence $(c_1, \cdots, c_n; c)$, with a mixed $(\mathcal{P}, A)$-action map of the form

$$\mathcal{P}(c_1, \cdots, c_n; c) \otimes \bigotimes_{i \in \{1, \cdots, n\} - \{k\}} A(c_i) \otimes M(c_k) \longrightarrow M(c)$$

factoring through the tensor product over Σ_n. These maps must satisfy the essential axioms of associativity and unitality. With the obvious maps, A-modules over $\mathcal{P}$ form a category, denoted by $\mathrm{Mod}_{\mathcal{P}}^A$.

To reformulate $\mathrm{Mod}_{\mathcal{P}}^A$ as a category of $\mathcal{S}$-valued enriched functors, one will need the construction of *enveloping operads*.

Denote by $\mathrm{Pairs}_C(\mathcal{S})$ the category whose objects are the pairs $(\mathcal{P}, A)$ with $\mathcal{P} \in \mathrm{Op}_C(\mathcal{S})$ and $A \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$, and whose morphisms are the pairs $(\varphi, f) : (\mathcal{P}, A) \longrightarrow (\mathcal{Q}, B)$ with $\varphi : \mathcal{P} \to \mathcal{Q}$ being a map in $\mathrm{Op}_C(\mathcal{S})$ and $f : A \to B$ a map of $\mathcal{P}$-algebras. There is a canonical functor $\delta : \mathrm{Op}_C(\mathcal{S}) \longrightarrow \mathrm{Pairs}_C(\mathcal{S})$ sending each C-colored operad $\mathcal{P}$ to the pair $(\mathcal{P}, \mathcal{P}_0)$ (see Example 2.1.2.4). According to [32], the functor δ admits a left adjoint $\mathrm{Env} : \mathrm{Pairs}_C(\mathcal{S}) \longrightarrow \mathrm{Op}_C(\mathcal{S})$ called the **enveloping functor**.

Definition 2.1.2.12. The **enveloping operad** associated to a pair $(\mathcal{P}, A) \in \mathrm{Pairs}_C(\mathcal{S})$ is defined to be $\mathrm{Env}(\mathcal{P}, A)$ the image of $(\mathcal{P}, A)$ through the enveloping functor.

Remark 2.1.2.13. Following Theorem 1.10 of loc.cit, there is a canonical isomorphism

$$\mathrm{Mod}_{\mathcal{P}}^A \cong \mathrm{Fun}(\mathrm{Env}(\mathcal{P}, A)_1, \mathcal{S}) \tag{2.1.2.1}$$

between the categories of A-modules over $\mathcal{P}$ and $\mathcal{S}$-valued enriched functors on $\mathrm{Env}(\mathcal{P}, A)_1$ the underlying category of $\mathrm{Env}(\mathcal{P}, A)$.

Remark 2.1.2.14. Another main interest in this construction is that there is a canonical isomorphism

$$\mathrm{Alg}_{\mathrm{Env}(\mathcal{P}, A)}(\mathcal{S}) \cong \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})_{A/}$$

between the categories of $\mathrm{Env}(\mathcal{P}, A)$-algebras and $\mathcal{P}$-algebras under A. On other hand, by construction there is a canonical map $j_A : \mathcal{P} \longrightarrow \mathrm{Env}(\mathcal{P}, A)$ of C-colored operads. This endows $\mathrm{Env}(\mathcal{P}, A)_0$ with a canonical $\mathcal{P}$-algebra structure and moreover, $\mathrm{Env}(\mathcal{P}, A)_0$ is isomorphic to A as $\mathcal{P}$-algebras. (See around [[32], Lemma 1.7]).

Finally, to sum up, we illustrate several links between mentioned operadic categories. Observe first that, as well as every type of monoid, there is a **restriction functor**

$$\mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \longrightarrow \mathrm{BMod}(\mathcal{P})_{\mathcal{P}/},$$

which admits a left adjoint usually called the **induction functor**.

On other hand, the (partial) composition in $\mathcal{P}$ endows $\mathcal{P}$ with the structure of an infinitesimal bimodule over itself. Let M be a $\mathcal{P}$-bimodule under $\mathcal{P}$. Then, M inherits a canonical infinitesimal $\mathcal{P}$-bimodule structure (under $\mathcal{P}$) induced by inserting the unit operations of $\mathcal{P}$ into M. This procedure determines a *restriction functor*

$$\mathrm{BMod}(\mathcal{P})_{\mathcal{P}/} \longrightarrow \mathrm{IbMod}(\mathcal{P})_{\mathcal{P}/},$$

which admits a left adjoint, the *induction functor* again.

Moreover, there is an adjunction

$$\mathcal{L}_{\mathcal{P}} : \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}/} : \mathcal{R}_{\mathcal{P}} \tag{2.1.2.2}$$

where the left adjoint is the obvious embedding functor and the right adjoint is given by the restriction of colors. Namely, let $\mathcal{P} \xrightarrow{f} \mathcal{Q}$ be an object of $\mathrm{Op}(\mathcal{S})_{\mathcal{P}/}$, then $\mathcal{R}_{\mathcal{P}}(\mathcal{Q})$ is given on each level as

$$\mathcal{R}_{\mathcal{P}}(\mathcal{Q})(c_1, \cdots, c_n; c) := \mathcal{Q}(f(c_1), \cdots, f(c_n); f(c)).$$

In conclusion, we obtain a sequence of adjunctions of the **induction-restriction** functors

$$\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}/} \rightleftarrows \mathrm{BMod}(\mathcal{P})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}/}. \tag{2.1.2.3}$$

2.1.3 Algebras encoding the categories IbMod($\mathcal{P}$), BMod($\mathcal{P}$), RMod($\mathcal{P}$) and LMod($\mathcal{P}$)

It is convenient that each of the categories $\mathrm{IbMod}(\mathcal{P})$, $\mathrm{BMod}(\mathcal{P})$, $\mathrm{LMod}(\mathcal{P})$ and $\mathrm{RMod}(\mathcal{P})$ can be encoded by an enriched operad (or category). In terms of single-colored operads, these constructions can be found in [46, 47].

Notations 2.1.3.1.

1. We let Fin denote the smallest skeleton of the category of finite sets whose objects consist of $\underline{0} := \varnothing$ and $\underline{m} := \{1, \cdots, m\}$ for $m \geqslant 1$.
2. We denote by Fin_* the category whose objects are finite pointed sets $\langle m \rangle := \{0, 1 \cdots, m\}$ (with 0 as the basepoint) for $m \geqslant 0$, and whose morphisms are basepoint-preserving maps. In other words, Fin_* is the smallest skeleton of the category of finite pointed sets.

Note that there is an obvious embedding functor $\mathrm{Fin} \longrightarrow \mathrm{Fin}_*$ taking each $\underline{m}$ to $\langle m \rangle$.

Construction 2.1.3.2. We now construct an $\mathcal{S}$-enriched category, $\mathbf{Ib}^{\mathcal{P}}$, which encodes infinitesimal $\mathcal{P}$-bimodules. The set of objects of $\mathbf{Ib}^{\mathcal{P}}$ is $\mathrm{Seq}(C)$, while its mapping spaces are defined as follows. For each map $\langle m \rangle \xrightarrow{f} \langle n \rangle$ in Fin_*, we denote by

$$\mathrm{Map}^f_{\mathbf{Ib}^{\mathcal{P}}}\left((c_1, \cdots, c_n; c), (d_1, \cdots, d_m; d) \right) := \mathcal{P}\left(c, \{d_j\}_{j \in f^{-1}(0)}; d\right) \otimes \bigotimes_{i=1,\cdots,n} \mathcal{P}\left(\{d_j\}_{j \in f^{-1}(i)}; c_i\right)$$

in which, for each $k \in \{0, \cdots, n\}$, the elements of $\{d_j\}_{j \in f^{-1}(k)}$ are put in the natural ascending order of j. Then, we define

$$\mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}\big(\,(c_1,\cdots,c_n;c)\,,(d_1,\cdots,d_m;d)\,\big) \coloneqq \coprod_{\langle m\rangle \xrightarrow{f} \langle n\rangle} \mathrm{Map}^{f}_{\mathbf{Ib}^{\mathcal{P}}}\big(\,(c_1,\cdots,c_n;c)\,,(d_1,\cdots,d_m;d)\,\big)\,.$$

Observe that

$$\mathrm{Map}^{\mathrm{Id}_{\langle n\rangle}}_{\mathbf{Ib}^{\mathcal{P}}}\big(\,(c_1,\cdots,c_n;c)\,,(c_1,\cdots,c_n;c)\,\big) = \mathcal{P}(c;c)\otimes\mathcal{P}(c_1;c_1)\otimes\cdots\otimes\mathcal{P}(c_n;c_n).$$

Due to this, we can define the unit morphisms of $\mathbf{Ib}^{\mathcal{P}}$ via the unit operations of $\mathcal{P}$. Moreover, the structure maps of $\mathbf{Ib}^{\mathcal{P}}$ are canonically defined via the composition in $\mathcal{P}$, along with the symmetric action on $\mathcal{P}$. (See also [46], §2).

Proposition 2.1.3.3. *There is a canonical isomorphism*

$$\mathrm{IbMod}(\mathcal{P}) \cong \mathrm{Fun}(\boldsymbol{Ib}^{\mathcal{P}}, \mathcal{S})$$

between the category of infinitesimal $\mathcal{P}$-bimodules and the category of $\mathcal{S}$-valued enriched functors on $\boldsymbol{Ib}^{\mathcal{P}}$.

Proof. (1) Let $M : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathcal{S}$ be an enriched functor given on objects by

$$M = \{M(c_1,\cdots,c_n;c)\}_{(c_1,\cdots,c_n;c)\in\mathrm{Seq}(C)}.$$

We establish its associated infinitesimal $\mathcal{P}$-bimodule, still denoted by M, as follows.

Each permutation $\alpha \in \Sigma_n$ determines a map $\langle n\rangle \xrightarrow{\alpha} \langle n\rangle$. Observe now that

$$\mathrm{Map}^{\alpha}_{\mathbf{Ib}^{\mathcal{P}}}\Big(\,(c_1,\cdots,c_n;c)\,,(c_{\alpha(1)},\cdots,c_{\alpha(n)};c)\,\Big) = \mathcal{P}(c;c)\otimes\mathcal{P}(c_1;c_1)\otimes\cdots\otimes\mathcal{P}(c_n;c_n).$$

In particular, the functor structure map of M

$$\mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}\Big(\,(c_1,\cdots,c_n;c)\,,(c_{\alpha(1)},\cdots,c_{\alpha(n)};c)\,\Big)\otimes M(c_1,\cdots,c_n;c) \longrightarrow M(c_{\alpha(1)},\cdots,c_{\alpha(n)};c)$$

has a component given by

$$\mathcal{P}(c;c)\otimes\mathcal{P}(c_1;c_1)\otimes\cdots\otimes\mathcal{P}(c_n;c_n)\otimes M(c_1,\cdots,c_n;c) \longrightarrow M(c_{\alpha(1)},\cdots,c_{\alpha(n)};c).$$

Now, the evaluation at the unit operations $\mathrm{id}_c, \mathrm{id}_{c_1}, \cdots, \mathrm{id}_{c_n}$ of $\mathcal{P}$ determines the symmetric action of typical form: $M(c_1,\cdots,c_n;c) \xrightarrow{\alpha^*} M(c_{\alpha(1)},\cdots,c_{\alpha(n)};c)$.

Next we define the infinitesimal right action of $\mathcal{P}$ on M. Observe that the structure map

$$\begin{gathered}\mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}\big(\,(c_1,\cdots,c_n;c)\,,(c_1,\cdots,c_{i-1},d_1,\cdots,d_m,c_{i+1},\cdots,c_n;c)\,\big)\otimes M(c_1,\cdots,c_n;c)\\ \longrightarrow M(c_1,\cdots,c_{i-1},d_1,\cdots,d_m,c_{i+1},\cdots,c_n;c)\end{gathered}$$

has a component given by

$$\begin{gathered}M(c_1,\cdots,c_n;c)\otimes\mathcal{P}(d_1,\cdots,d_m;c_i)\otimes\mathcal{P}(c;c)\otimes\mathcal{P}(c_1;c_1)\otimes\cdots\otimes\mathcal{P}(c_{i-1};c_{i-1})\otimes\mathcal{P}(c_{i+1};c_{i+1})\otimes\cdots\otimes\mathcal{P}(c_n;c_n)\\ \longrightarrow M(c_1,\cdots,c_{i-1},d_1,\cdots,d_m,c_{i+1},\cdots,c_n;c).\end{gathered}$$

This induces an infinitesimal right $\mathcal{P}$-action on M by evaluating at the unit operations of $\mathcal{P}$.

To define the infinitesimal left $\mathcal{P}$-action, observe that the structure map

$$\begin{gathered}\mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}\big(\,(d_1,\cdots,d_m;c_i)\,,(c_1,\cdots,c_{i-1},d_1,\cdots,d_m,c_{i+1},\cdots,c_n;c)\,\big)\otimes M(d_1,\cdots,d_m;c_i)\\ \longrightarrow M(c_1,\cdots,c_{i-1},d_1,\cdots,d_m,c_{i+1},\cdots,c_n;c)\end{gathered}$$

has a component given by

$$\mathcal{P}(c_i, c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n; c) \otimes \mathcal{P}(d_1; d_1) \otimes \cdots \otimes \mathcal{P}(d_m; d_m) \otimes M(d_1, \cdots, d_m; c_i)$$
$$\longrightarrow M(c_1, \cdots, c_{i-1}, d_1, \cdots, d_m, c_{i+1}, \cdots, c_n; c).$$

The latter induces an infinitesimal left action by evaluating at the unit operations of $\mathcal{P}$, again.

(2) Conversely, let M be an infinitesimal $\mathcal{P}$-bimodule. We want to see how M admits a canonical enriched functor structure $\mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathcal{S}$. We have to define the maps of the form

$$\mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}\left((c_1, \cdots, c_n; c), (d_1, \cdots, d_m; d) \right) \otimes M(c_1, \cdots, c_n; c) \longrightarrow M(d_1, \cdots, d_m; d).$$

This map must consist of, for each $\langle m \rangle \xrightarrow{f} \langle n \rangle$, a component map of the form

$$\mathcal{P}\left(c, \{d_j\}_{j \in f^{-1}(0)}; d\right) \otimes \bigotimes_{i=1,\cdots,n} \mathcal{P}\left(\{d_j\}_{j \in f^{-1}(i)}; c_i\right) \otimes M(c_1, \cdots, c_n; c) \longrightarrow M(d_1, \cdots, d_m; d).$$

The latter can be naturally defined using the (two sided) infinitesimal $\mathcal{P}$-action on M, along with the symmetric action on M (we will revisit this in Notation 4.2.0.7).

The explicit verifications are elephantine, but not complicated. □

Construction 2.1.3.4. The enriched category which encodes the category of right $\mathcal{P}$-modules will be denoted by $\mathbf{R}^{\mathcal{P}}$. Its set of objects is $\mathrm{Seq}(C)$, while its mapping objects are given by

$$\mathrm{Map}_{\mathbf{R}^{\mathcal{P}}}\left((c_1, \cdots, c_n; c), (d_1, \cdots, d_m; c) \right) := \coprod_{\underline{m} \xrightarrow{f} \underline{n}} \left[\bigotimes_{i=1,\cdots,n} \mathcal{P}\left(\{d_j\}_{j \in f^{-1}(i)}; c_i\right) \right]$$

where the coproduct ranges over the hom-set $\mathrm{Hom}_{\mathrm{Fin}}(\underline{m}, \underline{n})$. Observe that there is a map

$$\mathrm{Map}_{\mathbf{R}^{\mathcal{P}}}\left((c_1, \cdots, c_n; c), (d_1, \cdots, d_m; c) \right) \longrightarrow \mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}\left((c_1, \cdots, c_n; c), (d_1, \cdots, d_m; c) \right)$$

induced by the embedding $\mathrm{Fin} \longrightarrow \mathrm{Fin}_*$ and by inserting the unit operation id_c into the factor $\mathcal{P}(c; c)$ of the right hand side. The categorical structure of $\mathbf{R}^{\mathcal{P}}$ is then defined via the operad structure of $\mathcal{P}$, so that $\mathbf{R}^{\mathcal{P}}$ forms a subcategory of $\mathbf{Ib}^{\mathcal{P}}$.

It can then be verified the following, similarly as in the proof of Proposition 2.1.3.3.

Proposition 2.1.3.5. *There is a canonical isomorphism*

$$\mathrm{RMod}(\mathcal{P}) \cong \mathrm{Fun}(\boldsymbol{R}^{\mathcal{P}}, \mathcal{S})$$

between the category of right $\mathcal{P}$-modules and the category of $\mathcal{S}$-valued enriched functors on $\boldsymbol{R}^{\mathcal{P}}$.

We now wish to construct an operad encoding the category of $\mathcal{P}$-bimodules, yet it will be more convenient for us to start with the category $\mathcal{P}$-bimodules under $\mathcal{P}$, $\mathrm{BMod}(\mathcal{P})_{\mathcal{P}/}$.

Remark 2.1.3.6. As well as every type of monoid, for every $\mathcal{P}$-bimodule M there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{BMod}(\mathcal{P})}(\mathcal{P}, M) \cong \mathrm{Hom}^{c}_{\mathrm{Coll}_C(\mathcal{S})}(\mathcal{I}_C, M)$$

in which $\mathrm{Hom}^{c}_{\mathrm{Coll}_C(\mathcal{S})}(\mathcal{I}_C, M) \subseteq \mathrm{Hom}_{\mathrm{Coll}_C(\mathcal{S})}(\mathcal{I}_C, M)$ is the subset consisting of those $\varepsilon : \mathcal{I}_C \longrightarrow M$

making the following diagram commutative

$$\begin{array}{ccc} \mathcal{P} \circ \mathcal{I}_C = \mathcal{I}_C \circ \mathcal{P} & \xrightarrow{\varepsilon \circ \mathcal{P}} & M \circ \mathcal{P} \\ {\scriptstyle \mathcal{P}\circ\varepsilon}\downarrow & & \downarrow \\ \mathcal{P} \circ M & \longrightarrow & M \end{array}$$

Construction 2.1.3.7. The $\mathcal{S}$-enriched operad which encodes $\mathcal{P}$-bimodules under $\mathcal{P}$ will be denoted by $\mathbf{B}^{\mathcal{P}/}$. Its set of colors is again $\mathrm{Seq}(C)$. The unary operations of $\mathbf{B}^{\mathcal{P}/}$ agree with $\mathcal{P}$, i.e.,

$$\mathbf{B}^{\mathcal{P}/}(;(c_1,\cdots,c_n;c)) := \mathcal{P}(c_1,\cdots,c_n;c),$$

while its 1-ary operations coincide with those of $\mathbf{Ib}^{\mathcal{P}}$ (see Construction 2.1.3.2), i.e.,

$$\mathbf{B}^{\mathcal{P}/}\,(\,(c_1,\cdots,c_n;c)\,;(d_1,\cdots,d_m;d)\,) = \coprod_{\langle m\rangle \xrightarrow{f} \langle n\rangle} \left[\mathcal{P}\left(c,\{d_j\}_{j\in f^{-1}(0)};d\right) \otimes \bigotimes_{i=1,\cdots,n} \mathcal{P}\left(\{d_j\}_{j\in f^{-1}(i)};c_i\right)\right]$$

where f ranges over the set $\mathrm{Hom}_{\mathrm{Fin}_*}(\langle m\rangle,\langle n\rangle)$. Then we may extend the above formula to obtain the spaces of operations of higher arities. A typical space of n-ary operations of $\mathbf{B}^{\mathcal{P}/}$ is given by

$$\mathbf{B}^{\mathcal{P}/}\left(\left(c_1,\cdots,c_{r_1};c^{(1)}\right),\left(c_{r_1+1},\cdots,c_{r_1+r_2};c^{(2)}\right),\cdots,\left(c_{r_1+\cdots+r_{n-1}+1},\cdots,c_{r_1+\cdots+r_n};c^{(n)}\right);(d_1,\cdots,d_m;d)\,\right)$$
$$= \coprod_{\langle m\rangle \xrightarrow{f} \langle r_1+\cdots+r_n\rangle} \left[\mathcal{P}\left(c^{(1)},\cdots,c^{(n)},\{d_j\}_{j\in f^{-1}(0)};d\right) \otimes \bigotimes_{i=1,\cdots,r_1+\cdots+r_n} \mathcal{P}\left(\{d_j\}_{j\in f^{-1}(i)};c_i\right)\right].$$

The Σ_n-action is given by permuting the colors $c^{(1)},\cdots,c^{(n)}$ on the factor

$$\mathcal{P}\left(c^{(1)},\cdots,c^{(n)},\{d_j\}_{j\in f^{-1}(0)};d\right)$$

and simultaneously, permuting the terms $\{c_1,\cdots,c_{r_1}\},\cdots,\{c_{r_1+\cdots+r_{n-1}+1},\cdots,c_{r_1+\cdots+r_n}\}$ on the factor

$$\bigotimes_{i=1,\cdots,r_1+\cdots+r_n} \mathcal{P}\left(\{d_j\}_{j\in f^{-1}(i)};c_i\right).$$

The composition of $\mathbf{B}^{\mathcal{P}/}$ is canonically defined via the composition of $\mathcal{P}$, while its unit operations are exactly those of $\mathbf{Ib}^{\mathcal{P}}$.

Proposition 2.1.3.8. *There is a canonical isomorphism*

$$\mathrm{BMod}(\mathcal{P})_{\mathcal{P}/} \cong \mathrm{Alg}_{\boldsymbol{B}^{\mathcal{P}/}}(\mathcal{S})$$

between the category of $\mathcal{P}$-bimodules under $\mathcal{P}$ and the category of algebras over $\boldsymbol{B}^{\mathcal{P}/}$.

Proof. (1) Let M be a $\mathbf{B}^{\mathcal{P}/}$-algebra. Note first that since the underlying category of $\mathbf{B}^{\mathcal{P}/}$ agrees with $\mathbf{Ib}^{\mathcal{P}}$, M already inherits a canonical right $\mathcal{P}$-module structure (cf. Proposition 2.1.3.3).

Let us see how M comes equipped with a left $\mathcal{P}$-action. For simplicity, we only establish the action maps of the form

$$\mathcal{P}(c,d;e)\otimes M(c_1,\cdots,c_n;c)\otimes M(d_1,\cdots,d_m;d) \longrightarrow M(c_1,\cdots,c_n,d_1,\cdots,d_m;e) \tag{2.1.3.1}$$

To this end, observe first that the $\mathbf{B}^{\mathcal{P}/}$-algebra structure map of M of the form

$$\mathbf{B}^{\mathcal{P}/}\,(\,(c_1,\cdots,c_n;c)\,,(d_1,\cdots,d_m;d)\,;(c_1,\cdots,c_n,d_1,\cdots,d_m;e)\,)\,\otimes$$
$$\otimes\, M(c_1,\cdots,c_n;c)\otimes M(d_1,\cdots,d_m;d) \longrightarrow M(c_1,\cdots,c_n,d_1,\cdots,d_m;e)$$

has a component given by

$$\mathcal{P}(c,d;e)\otimes\mathcal{P}(c_1;c_1)\otimes\cdots\otimes\mathcal{P}(c_n;c_n)\otimes\mathcal{P}(d_1;d_1)\otimes\cdots\otimes\mathcal{P}(d_m;d_m)\otimes$$
$$\otimes M(c_1,\cdots,c_n;c)\otimes M(d_1,\cdots,d_m;d)\longrightarrow M(c_1,\cdots,c_n,d_1,\cdots,d_m;e).$$

The evaluation at the unit operations $\mathrm{id}_{c_1},\cdots,\mathrm{id}_{c_n},\mathrm{id}_{d_1},\cdots,\mathrm{id}_{d_m}$ of $\mathcal{P}$ to the latter gives us the action map (2.1.3.1) as desired.

Finally, the action of the unary operations of $\mathbf{B}^{\mathcal{P}/}$ on M gives us a canonical map $\mathcal{P}\to M$.

(2) Conversely, let M be a $\mathcal{P}$-bimodule under $\mathcal{P}$. We let the composite map $\varepsilon:\mathcal{I}_C\to\mathcal{P}\to M$ exhibit the images of the unit operations of $\mathcal{P}$ in M. In order to establish the $\mathbf{B}^{\mathcal{P}/}$-algebra structure on M, one will need to make use of the $\mathcal{P}$-bimodule structure of M, along with some suitable involvement of ε.

The explicit verifications are elephantine, but not complicated (with a help of Remark 2.1.3.6 at some points). □

Construction 2.1.3.9. The $\mathcal{S}$-enriched operad which encodes the category of $\mathcal{P}$-bimodules will be denoted by $\mathbf{B}^{\mathcal{P}}$. Its set of colors is again $\mathrm{Seq}(C)$. A typical space of n-ary operations is given by

$$\mathbf{B}^{\mathcal{P}}\left(\left(c_1,\cdots,c_{r_1};c^{(1)}\right),\left(c_{r_1+1},\cdots,c_{r_1+r_2};c^{(2)}\right),\cdots,\left(c_{r_1+\cdots+r_{n-1}+1},\cdots,c_{r_1+\cdots+r_n};c^{(n)}\right);(d_1,\cdots,d_m;d)\right)$$
$$:=\coprod_{\underline{m}\xrightarrow{f}\underline{r_1+\cdots+r_n}}\left[\mathcal{P}\left(c^{(1)},\cdots,c^{(n)};d\right)\otimes\bigotimes_{i=1,\cdots,r_1+\cdots+r_n}\mathcal{P}\left(\{d_j\}_{j\in f^{-1}(i)};c_i\right)\right]$$

where the coproduct ranges over the hom-set $\mathrm{Hom}_{\mathrm{Fin}}(\underline{m},\underline{r_1+\cdots+r_n})$. For each map $f:\underline{m}\longrightarrow\underline{r_1+\cdots+r_n}$, we will denote by $\mathbf{B}^{\mathcal{P}}_f(-)$ the component of $\mathbf{B}^{\mathcal{P}}(-)$ corresponding to f, taken from the above formula. As in Construction 2.1.3.7, the operad structure of $\mathbf{B}^{\mathcal{P}}$ is canonically defined via the structure of $\mathcal{P}$, so that $\mathbf{B}^{\mathcal{P}}$ is in fact a suboperad of $\mathbf{B}^{\mathcal{P}/}$. (See also [[47], §2.1.1]).

As in the proof of Proposition 2.1.3.8, it can be shown that:

Proposition 2.1.3.10. *There is a canonical isomorphism*

$$\mathrm{BMod}(\mathcal{P})\cong\mathrm{Alg}_{\boldsymbol{B}^{\mathcal{P}}}(\mathcal{S})$$

between the category of $\mathcal{P}$-bimodules and the category of algebras over $\boldsymbol{B}^{\mathcal{P}}$.

Finally, we construct an operad encoding the category of left $\mathcal{P}$-modules.

Construction 2.1.3.11. We denote by $\mathbf{L}^{\mathcal{P}}$ the $\mathcal{S}$-enriched operad whose set of colors is $\mathrm{Seq}(C)$ and whose spaces of operations are given as follows. For simplicity of equations, we just describe the spaces of 2-ary operations. These are concentrated in the following ones

$$\mathbf{L}^{\mathcal{P}}\left((c_1,\cdots,c_{r_1};c),(c_{r_1+1},\cdots,c_{r_1+r_2};c');\left(c_{\sigma(1)},\cdots,c_{\sigma(r_1+r_2)};d\right)\right):=\coprod_{\alpha}\mathcal{P}(c,c';d)$$

where $\sigma\in\Sigma_{r_1+r_2}$. The coproduct ranges over the subset of $\Sigma_{r_1+r_2}$ consisting of those α satisfying that for every $i\in\{1,\cdots,r_1+r_2\}$, the two colors $c_{\sigma(i)}$ and $c_{\alpha(i)}$ coincide. From the above formula, we denote by $\mathbf{L}^{\mathcal{P}}_\alpha(-)$ the component of $\mathbf{L}^{\mathcal{P}}(-)$ corresponding to α. Observe that there is a canonical map

$$\mathbf{L}^{\mathcal{P}}_\alpha\left((c_1,\cdots,c_{r_1};c),(c_{r_1+1},\cdots,c_{r_1+r_2};c');\left(c_{\sigma(1)},\cdots,c_{\sigma(r_1+r_2)};d\right)\right)=\mathcal{P}(c,c';d)\longrightarrow$$
$$\mathbf{B}^{\mathcal{P}}_\alpha\left((c_1,\cdots,c_{r_1};c),(c_{r_1+1},\cdots,c_{r_1+r_2};c');\left(c_{\sigma(1)},\cdots,c_{\sigma(r_1+r_2)};d\right)\right)=\mathcal{P}(c_1;c_1)\otimes\cdots\otimes\mathcal{P}(c_{r_1+r_2};c_{r_1+r_2})\otimes\mathcal{P}(c,c';d)$$

given by inserting the unit operations id_{c_i} into the factors $\mathcal{P}(c_i;c_i)$ for $i=1,\cdots,r_1+r_2$. In this way, the operad structure of $\mathbf{L}^{\mathcal{P}}$ is established via the structure of $\mathcal{P}$, so that $\mathbf{L}^{\mathcal{P}}$ forms a suboperad of $\mathbf{B}^{\mathcal{P}}$.

Proposition 2.1.3.12. *There is a canonical isomorphism*

$$\mathrm{LMod}(\mathcal{P}) \cong \mathrm{Alg}_{\boldsymbol{L}^{\mathcal{P}}}(\mathcal{S})$$

between the category of left $\mathcal{P}$-modules and the category of algebras over $\boldsymbol{L}^{\mathcal{P}}$.

Proof. The proof is similar to the ones above. Here we just note the following detail. Suppose we are given an $\mathbf{L}^{\mathcal{P}}$-algebra M and a permutation $\sigma \in \Sigma_n$. We wish to determine the symmetric action map $\sigma^* : M(c_1,\cdots,c_n;c) \longrightarrow M(c_{\sigma(1)},\cdots,c_{\sigma(n)};c)$. To this end, we make use of the structure map

$$\mathbf{L}^{\mathcal{P}}_{\sigma}((c_1,\cdots,c_n;c);(c_{\sigma(1)},\cdots,c_{\sigma(n)};c)) \otimes M(c_1,\cdots,c_n;c) = \mathcal{P}(c;c) \otimes M(c_1,\cdots,c_n;c) \longrightarrow M(c_{\sigma(1)},\cdots,c_{\sigma(n)};c)$$

where $\mathbf{L}^{\mathcal{P}}_{\sigma}(-)$ is the component of $\mathbf{L}^{\mathcal{P}}(-)$ corresponding to itself σ. The evaluation to the above map at the unit operation id_c gives us the expected action σ^*. □

2.1.4 Operadic transferred model structures

In this section, we assume further that $\mathcal{S}$ is a **symmetric monoidal model category** (cf. Hovey's [34]). Let $\mathcal{P}$ be a C-colored operad in $\mathcal{S}$ and let A be a $\mathcal{P}$-algebra. We collect here all the mentioned operadic categories, except the category of $\mathcal{S}$-enriched operads $\mathrm{Op}(\mathcal{S})$, including

$$\{\mathrm{Coll}_C(\mathcal{S}), \mathrm{Op}_C(\mathcal{S}), \mathrm{LMod}(\mathcal{P}), \mathrm{RMod}(\mathcal{P}), \mathrm{BMod}(\mathcal{P}), \mathrm{IbMod}(\mathcal{P}), \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}), \mathrm{Mod}^{A}_{\mathcal{P}}\} =: \mathbb{A}. \qquad (2.1.4.1)$$

Definition 2.1.4.1. Let $\mathcal{M}$ be any of the categories in $\mathbb{A}$. The (projective) **transferred model structure** on $\mathcal{M}$ is the one whose weak equivalences (resp. fibrations) are precisely the levelwise weak equivalences (resp. fibrations).

We wish to set up several suitable conditions on the base category $\mathcal{S}$ assuring the existence of the transferred model structure on every element of $\mathbb{A}$. As we have seen previously, each category $\mathcal{M} \in \mathbb{A}$ can be represented as the category of algebras over a certain operad (or category) (cf. Remarks 2.1.1.2, 2.1.2.13, Example 2.1.2.5, Propositions 2.1.3.3, 2.1.3.5, 2.1.3.10, 2.1.3.12). Consequently, one just needs to consider the transferred model structure on the category $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$. According to the literature, we know several criteria assuring the existence of that. Here are several settings.

Definition 2.1.4.2. A **symmetric monoidal fibrant replacement functor** on $\mathcal{S}$ is a symmetric monoidal functor $\mathbb{R} : \mathcal{S} \to \mathcal{S}$ together with a monoidal natural transformation $\varphi : \mathrm{Id} \to \mathbb{R}$ such that for each object $X \in \mathcal{S}$, the map $\varphi_X : X \longrightarrow \mathbb{R}(X)$ exhibits $\mathbb{R}(X)$ as a fibrant replacement of X.

Definition 2.1.4.3. A **functorial path data** on $\mathcal{S}$ is a symmetric monoidal functor $\mathbb{P} : \mathcal{S} \to \mathcal{S}$ together with monoidal natural transformations $s : \mathrm{Id} \to \mathbb{P}$ and $d_0, d_1 : \mathbb{P} \to \mathrm{Id}$ such that the composite map $X \xrightarrow{s_X} \mathbb{P}(X) \xrightarrow{(d_0,d_1)} X \times X$ exhibits $\mathbb{P}(X)$ as a path object for X.

The following is an analogue of [[36], Theorem 3.11].

Proposition 2.1.4.4. *Suppose that $\mathcal{S}$ is strongly cofibrantly generated (i.e., cofibrantly generated with domains of the generating cofibrations and trivial cofibrations being small). If $\mathcal{S}$ admits both a symmetric monoidal fibrant replacement functor $\mathbb{R}$ and a functorial path data $\mathbb{P}$ then the transferred model structure on $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$ exists for every operad $\mathcal{P}$.*

Proof. Consider the free-forgetful adjunction

$$\mathcal{F} : \mathcal{S}^{\times C} \rightleftarrows \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}) : \mathcal{U}.$$

By assumption, the product model structure on $\mathcal{S}^{\times C}$ is strongly cofibrantly generated. On other hand, it can be readily verified that the functor $\mathcal{U}$ creates filtered colimits. Thus, according to [[36], Lemma 3.3] (a version of the transfer principle for strongly cofibrantly generated model categories), we just need to show that there is a functorial $\mathcal{U}$-fibrant replacement in $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$ and that every $\mathcal{U}$-fibrant object $A \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$ admits a path object. (In our settings "$\mathcal{U}$-fibrant" means levelwise fibrant).

For the first condition, observe first that the symmetric monoidal functor $\mathbb{R}$ lifts to a functor $\mathbb{R} : \mathrm{Op}_C(\mathcal{S}) \longrightarrow \mathrm{Op}_C(\mathcal{S})$. Moreover, the monoidal natural transformation $\varphi : \mathrm{Id} \longrightarrow \mathbb{R}$ gives us a map of operads $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathbb{R}\mathcal{P}$. On other hand, $\mathbb{R}$ lifts to another functor $\mathbb{R} : \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}) \longrightarrow \mathrm{Alg}_{\mathbb{R}\mathcal{P}}(\mathcal{S})$. Due to the map $\varphi_{\mathcal{P}}$, for any $A \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$, the $\mathbb{R}\mathcal{P}$-algebra $\mathbb{R}A$ inherits a $\mathcal{P}$-algebra structure and moreover, the induced map $\varphi_A : A \longrightarrow \mathbb{R}A$ is then a map of $\mathcal{P}$-algebras. By construction, $\mathbb{R}A$ is indeed a $\mathcal{U}$-fibrant replacement of A.

It remains to verify the second condition. Let $A \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$ be a levelwise fibrant $\mathcal{P}$-algebra. By applying the symmetric monoidal functor $\mathbb{P}$ levelwise, we obtain a $\mathcal{P}$-algebra, $\mathbb{P}A$ (just as in the above paragraph). The monoidal natural transformations $s : \mathrm{Id} \longrightarrow \mathbb{P}$ and $d_0, d_1 : \mathbb{P} \longrightarrow \mathrm{Id}$ together determine a diagram in $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$:

$$A \xrightarrow{s_A} \mathbb{P}A \xrightarrow{(d_0,d_1)} A \times A,$$

which exhibits $\mathbb{P}A$ as a path object for A. □

There is another powerful criterion, thanks to the recent work of Pavlov-Scholbach. Here are several settings.

Recall from [[37], Definition 1.1] that a map $f : X \to Y$ in $\mathcal{S}$ is an **h-cofibration** if and only if for every commutative diagram of coCartesian squares in $\mathcal{S}$ of the form

$$\begin{array}{ccccc} X & \longrightarrow & A & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & A' & \xrightarrow[g']{} & B' \end{array}$$

the map g' is a weak equivalence whenever g is one. If f is in addition a weak equivalence then it is an **acyclic h-cofibration**.

Let $\overrightarrow{n} = (n_1, \cdots, n_k)$ be a finite sequence of natural numbers. For a family $s = (s_1, \cdots, s_k)$ of maps in $\mathcal{S}$, denote by

$$s^{\square \overrightarrow{n}} := \square_i \, s_i^{\square n_i},$$

where the subcript "$\square$" refers to the **pushout-product** operation of maps. The group $\Sigma_{\overrightarrow{n}} := \prod_i \Sigma_{n_i}$ acts on $s^{\square \overrightarrow{n}}$ in an evident way.

Definition 2.1.4.5. The symmetric monoidal model category $\mathcal{S}$ is said to be **symmetric h-monoidal** if for any finite family $s = (s_1, \cdots, s_k)$ of (resp. acyclic) cofibrations and for any object $X \in \mathcal{S}$ equipped with a right $\Sigma_{\overrightarrow{n}}$-action, the map $X \otimes_{\Sigma_{\overrightarrow{n}}} s^{\square \overrightarrow{n}}$ is a (resp. acyclic) h-cofibration.

Proposition 2.1.4.6. *(Pavlov-Scholbach, [[27], Theorem 5.10]) Suppose that $\mathcal{S}$ is a combinatorial symmetric monoidal model category such that weak equivalences are closed under transfinite compositions.*

If $\mathcal{S}$ is symmetric h-monoidal (the acyclic part is sufficient), then the transferred model structure on $\mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$ *exists for every operad* $\mathcal{P}$.

We end this section by listing some base categories of interest, which we will work with in the thesis, and discussing how they adapt to the criteria mentioned above.

Examples 2.1.4.7. 1. The base category of most interest is the Cartesian monoidal category of **simplicial sets**, $(\mathrm{Set}_{\Delta}, \times)$, equipped with the standard (Kan-Quillen) model structure. The model structure on Set_{Δ} is combinatorial (and hence strongly cofibrantly generated, in particular) and has weak equivalences being closed under filtered colimits. It admits a fibrant replacement functor given by $\mathrm{Ex}^{\infty} := \mathrm{Sing}|-|$ the composition of the realization and singular functors, and admits a functorial path data given by $(-)^{\Delta^1}$. Moreover, Set_{Δ} is as well symmetric h-monoidal, according to [[28], §7.1]. We hence get that Set_{Δ} satisfies the conditions of both two propositions 2.1.4.4 and 2.1.4.6.

2. The second one is the monoidal category of **simplicial** R**-modules**, $(\mathrm{sMod}_{\mathrm{R}}, \otimes)$, with R being a commutative ring, equipped with the standard model structure transferred from that of Set_{Δ}. As well as simplicial sets, simplicial R-modules satisfies the conditions of both two propositions 2.1.4.4 and 2.1.4.6. Indeed, note first that $\mathrm{sMod}_{\mathrm{R}}$ is combinatorial, has weak equivalences being closed under filtered colimits and moreover, it is a fibrant model category (i.e., all the objects are fibrant). A functorial path data for $\mathrm{sMod}_{\mathrm{R}}$ is given by $(-)^{\mathrm{R}\{\Delta^1\}}$. It is as well symmetric h-monoidal, according to [[28], §7.3].

3. Let $\mathbf{k}$ be a commutative ring of characteristic 0. Consider the monoidal category of **dg k-modules**, $(\mathcal{C}(\mathbf{k}), \otimes)$, equipped with the projective model structure. This is also a combinatorial fibrant model category and has weak equivalences being closed under filtered colimits. There is a functorial path data for $\mathcal{C}(\mathbf{k})$ given by $(-) \otimes \Omega^*(\Delta^1)$ where $\Omega^*(\Delta^1)$ is the *Sullivan's dg algebra of differentials* on the interval $\Delta^1 \in \mathrm{Set}_{\Delta}$ (see [[51], 5.3]). So we get that $\mathcal{C}(\mathbf{k})$ satisfies the conditions of Proposition 2.1.4.4. However, $\mathcal{C}(\mathbf{k})$ is in general not symmetric h-monoidal, unless $\mathbf{k}$ is a field of characteristic 0. (See the one below).

4. Suppose that $\mathbf{k}$ is a field of characteristic 0. We are also interested in the monoidal category of **connective dg k-modules**, $(\mathcal{C}_{\geqslant 0}(\mathbf{k}), \otimes)$, equipped with the projective model structure. This model category is combinatorial and has weak equivalences being closed under filtered colimits. Moreover, all its objects are both fibrant and cofibrant. However, as far as we know, $\mathcal{C}_{\geqslant 0}(\mathbf{k})$ does not adapt to the conditions of Proposition 2.1.4.4. The only thing missed is the existence of a functorial path data, (which we do not know about). Instead of that, $\mathcal{C}_{\geqslant 0}(\mathbf{k})$ satisfies the conditions of Proposition 2.1.4.6. Indeed, for any given finite group G, every $\mathbf{k}$-module equipped with a G-action is automatically projective as a $\mathbf{k}[G]$-module, (by the Maschke's theorem). This fact implies that $\mathcal{C}_{\geqslant 0}(\mathbf{k})$ is symmetric h-monoidal. (See [[28], §7.4] for more details).

5. More generally, let R be a commutative monoid in the monoidal category $(\mathcal{C}_{\geqslant 0}(\mathbf{k}), \otimes)$ with $\mathbf{k}$ being a field of characteristic 0. We are also interested in the monoidal category of R**-modules,** $(\mathrm{Mod}_R, - \otimes_R -)$, equipped with the projective model structure. This is a combinatorial fibrant model category and has weak equivalences being closed under filtered colimits. As well as $\mathcal{C}_{\geqslant 0}(\mathbf{k})$, the category Mod_R does not admit a functorial path data, but instead it satisfies the conditions of Proposition 2.1.4.6. To see the latter, one just needs to verify the symmetric h-monoidality. This is in fact transferred from the symmetric h-monoidality of $\mathcal{C}_{\geqslant 0}(\mathbf{k})$ (cf. [[28], Theorem 5.9]).

In conclusion, we see that all the base categories listed above are nice enough so that the transferred model structure on $\mathcal{P}$-algebras exists for every operad $\mathcal{P}$ and consequently, for every category $\mathcal{M} \in \mathbb{A}$ (2.1.4.1), the transferred model structure on $\mathcal{M}$ exists as well.

2.1.5 Dwyer-Kan and canonical model structures on enriched operads

Let $\mathcal{S}$ be a **monoidal model category** and let $\mathrm{Cat}(\mathcal{S})$ denote the category of (small) $\mathcal{S}$-enriched categories. For each $\mathcal{C} \in \mathrm{Cat}(\mathcal{S})$, the **homotopy category of** $\mathcal{C}$, denoted by $\mathrm{Ho}(\mathcal{C})$, is the ordinary category whose objects are the same as those of $\mathcal{C}$ and whose hom-set $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x,y)$, with $x, y \in \mathrm{Ob}(\mathcal{C})$, is defined to be

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x,y) := \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(1_{\mathcal{S}}, \mathrm{Map}_{\mathcal{C}}(x,y)).$$

By convention, a map $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}(\mathcal{S})$ is a *levelwise weak equivalence (fibration, trivial fibration, etc.)* if for every $x, y \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(f(x), f(y))$ is a weak equivalence (fibration, trivial fibration, etc.) in $\mathcal{S}$.

Definition 2.1.5.1. A map $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}(\mathcal{S})$ is called a **Dwyer-Kan equivalence** if it is a levelwise weak equivalence and such that the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ between homotopy categories is essentially surjective.

Definition 2.1.5.2. **Dwyer-Kan model structure on** $\mathrm{Cat}(\mathcal{S})$ is the one whose weak equivalences are the Dwyer-Kan equivalences and whose trivial fibrations are the levelwise trivial fibrations surjective on objects. (See, e.g., [38], [4]).

Definition 2.1.5.3. (Berger-Moerdijk [30]) **Canonical model structure on** $\mathrm{Cat}(\mathcal{S})$ is the one whose fibrant objects are the levelwise fibrant categories and whose trivial fibrations are the same as those of the Dwyer-Kan model structure.

By extending the two above, G. Caviglia [17] established both the Dwyer-Kan and canonical model structures on $\mathrm{Op}(\mathcal{S})$. Suppose further that $\mathcal{S}$ is a symmetric monoidal model category.

By convention, a map $f : \mathcal{P} \to \mathcal{Q}$ in $\mathrm{Op}(\mathcal{S})$ is called a *levelwise weak equivalence (fibration, trivial fibration, etc.)* if for every sequence $(c_1, \cdots, c_n; c)$ of colors in $\mathcal{P}$, the induced map

$$\mathcal{P}(c_1, \cdots, c_n; c) \longrightarrow \mathcal{Q}(f(c_1), \cdots, f(c_n); f(c))$$

is a weak equivalence (fibration, trivial fibration, etc.) in $\mathcal{S}$.

On other hand, the **homotopy category of** $\mathcal{P}$ is defined to be $\mathrm{Ho}(\mathcal{P}) := \mathrm{Ho}(\mathcal{P}_1)$ the homotopy category of its underlying category.

Definition 2.1.5.4. A map $f : \mathcal{P} \to \mathcal{Q}$ in $\mathrm{Op}(\mathcal{S})$ is called a **Dwyer-Kan equivalence** if it is a levelwise weak equivalence and such that the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(\mathcal{P}) \longrightarrow \mathrm{Ho}(\mathcal{Q})$ between homotopy categories is essentially surjective or alternatively, if f is a levelwise weak equivalence and has underlying map $f_1 : \mathcal{P}_1 \longrightarrow \mathcal{Q}_1$ being a Dwyer-Kan equivalence in $\mathrm{Cat}(\mathcal{S})$.

Definition 2.1.5.5. **Dwyer-Kan model structure on** $\mathrm{Op}(\mathcal{S})$ is the one whose weak equivalences are the Dwyer-Kan equivalences and whose trivial fibrations are the levelwise trivial fibrations surjective on colors.

Definition 2.1.5.6. Canonical model structure on $\mathrm{Op}(\mathcal{S})$ is the one whose fibrant objects are the levelwise fibrant operads and whose trivial fibrations are the same as those of the Dwyer-Kan model structure.

Remark 2.1.5.7. As originally introduced by G. Caviglia, a given map in $\mathrm{Op}(\mathcal{S})$ is a fibration (resp. weak equivalence) with respect to the canonical model structure if and only if it is a levelwise fibration (resp. weak equivalence) and such that its underlying map in $\mathrm{Cat}(\mathcal{S})$ is a fibration (resp. weak equivalence) with respect to that model structure. (See [[17], Definition 4.5] and [[30], §2.2]).

Following up his work, we give a set of conditions on the base category $\mathcal{S}$ assuring the existence of the canonical model structure.

Proposition 2.1.5.8. *(Caviglia, [17]) Let $\mathcal{S}$ be a combinatorial symmetric monoidal model category satisfying that:*

(S1) the class of weak equivalences is closed under filtered colimits,

(S2) either (a) $\mathcal{S}$ admits a symmetric monoidal fibrant replacement functor and a functorial path data, or (b) $\mathcal{S}$ is symmetric h-monoidal,

(S3) the monoidal unit is cofibrant, and

(S4) the model structure is right proper.

Then $\mathrm{Op}(\mathcal{S})$ admits the canonical model structure, which is as well right proper and combinatorial. Moreover, this model structure coincides then with the Dwyer-Kan model structure.

Proof. The combinatoriality of $\mathcal{S}$ implies that it is strongly cofibrantly generated and besides that, implies the existence of a set of **generating intervals** in the sense of [30] (cf. Lemma 1.12 of loc.cit). On other hand, by propositions 2.1.4.4 and 2.1.4.6, the condition (S2) ensures the existence of the transferred model structure on $\mathrm{Op}_C(\mathcal{S})$ for every set C. Then by [[17], Theorem 4.22 (1)], $\mathrm{Op}(\mathcal{S})$ admits the canonical model structure, which is combinatorial as well. The right properness follows by Proposition 5.3 of the loc.cit.

For the second claim, observe first that the condition (S1), together with the combinatoriality of $\mathcal{S}$, implies that $\mathcal{S}$ is **compactly generated** in the sense of [[30], Definition 1.2]. Combining this fact with (S2), we get that $\mathcal{S}$ is **adequate** in the sense of [[30], Definition 1.1]. The latter fact, along with the conditions (S3) and (S4), proves that the classes of Dwyer-Kan and canonical weak equivalences in $\mathrm{Cat}(\mathcal{S})$ coincide (cf. [30], propositions 2.20 and 2.24). Thus, by Remark 2.1.5.7 these two classes in $\mathrm{Op}(\mathcal{S})$ coincide as well. Combining the latter with the fact that the two model structures have the same trivial fibrations, we obtain the expected coincidence. □

Remark 2.1.5.9. Under the same assumptions as in Proposition 2.1.5.8, the canonical model structure on $\mathrm{Cat}(\mathcal{S})$ automatically exists and coincides with the Dwyer-Kan model structure.

Example 2.1.5.10. Some typical base categories satisfying the conditions of Proposition 2.1.5.8 include the ones of Examples 2.1.4.7

2.2 Tangent categories and Quillen cohomology

This section, based on the works of [6, 7], contains the most important concepts appearing throughout the thesis. Basically, *tangent category* comes after a procedure of taking stabilization of a model category. Note that under our settings, stabilizations exist only as semi model categories. Despite this, the needed results from those papers remain valid. We then get the notion of *cotangent complex*, which plays a central role in the Quillen cohomology theory.

Definition 2.2.0.1. A model category $\mathbf{M}$ is said to be **weakly pointed** if it contains a **weak zero object**, i.e., an object which is both homotopy initial and terminal.

Let $\mathbf{M}$ be a weakly pointed model category and let X be an $(\mathbb{N}\times\mathbb{N})$-diagram in $\mathbf{M}$. The diagonal squares of X are of the form

$$\begin{array}{ccc} X_{n,n} & \longrightarrow & X_{n,n+1} \\ \downarrow & & \downarrow \\ X_{n+1,n} & \longrightarrow & X_{n+1,n+1} \end{array}$$

Definition 2.2.0.2. An $(\mathbb{N}\times\mathbb{N})$-diagram in $\mathbf{M}$ is called

(1) a **prespectrum** if all its off-diagonal entries are weak zero objects in $\mathbf{M}$,

(2) an Ω**-spectrum** if it is a prespectrum and all its diagonal squares are homotopy Cartesian,

(3) a **suspension spectrum** if it is a prespectrum and all its diagonal squares are homotopy coCartesian.

The projective model category of $(\mathbb{N}\times\mathbb{N})$-diagrams in $\mathbf{M}$ will be denoted by $\mathbf{M}^{\mathbb{N}\times\mathbb{N}}_{\text{proj}}$.

Definition 2.2.0.3. ([6], Definition 2.1.2) Let $\mathbf{M}$ be a weakly pointed model category. A map $f: X \to Y$ in $\mathbf{M}^{\mathbb{N}\times\mathbb{N}}$ is said to be a **stable equivalence** if for every Ω-spectrum Z the induced map between derived mapping spaces

$$\mathrm{Map}^{\mathrm{h}}_{\mathbf{M}^{\mathbb{N}\times\mathbb{N}}_{\text{proj}}}(Y,Z) \longrightarrow \mathrm{Map}^{\mathrm{h}}_{\mathbf{M}^{\mathbb{N}\times\mathbb{N}}_{\text{proj}}}(X,Z)$$

is a homotopy equivalence. Note that a stable equivalence between Ω-spectra is always a levelwise equivalence.

Following [[6], Lemma 2.1.6], the Ω-spectra in $\mathbf{M}$ can be characterized as the **local objects** against a certain set of maps. Inspired by Definition 2.1.3 of the loc.cit, we give the following definition, which is valid due to Theorem A.0.0.14 (Batanin-White).

Definition 2.2.0.4. Let $\mathbf{M}$ be a weakly pointed combinatorial model category such that the domains of generating cofibrations are cofibrant. **Stabilization** of $\mathbf{M}$, denoted by $\mathrm{Sp}(\mathbf{M})$, is defined to be the left Bousfield localization of $\mathbf{M}^{\mathbb{N}\times\mathbb{N}}_{\text{proj}}$ with Ω-spectra as the local objects. Explicitly, $\mathrm{Sp}(\mathbf{M})$ is a **cofibrantly generated semi model category** (see Appendix A) whose

- weak equivalences are the stable equivalences, and whose
- (generating) cofibrations are the same as those of $\mathbf{M}^{\mathbb{N}\times\mathbb{N}}_{\text{proj}}$.

In particular, fibrant objects of $\mathrm{Sp}(\mathbf{M})$ are precisely the levelwise fibrant Ω-spectra.

Remark 2.2.0.5. When $\mathbf{M}$ is in addition left proper then the stabilization $\mathrm{Sp}(\mathbf{M})$ exists as a (full) model category. Nevertheless, we do not require the left properness throughout the thesis.

Definition 2.2.0.6. ([6]) A (semi) model category $\mathbf{M}$ is called **stable** if the following equivalent conditions hold:

1. The **underlying ∞-category** $\mathbf{M}_\infty$ of $\mathbf{M}$ (cf., [55, 42]) is stable in the sense of [3].
2. $\mathbf{M}$ is weakly pointed and such that a square in $\mathbf{M}$ is homotopy coCartesian if and only if it is homotopy Cartesian.
3. $\mathbf{M}$ is weakly pointed and such that the adjunction $\Sigma : \mathrm{Ho}(\mathbf{M}) \rightleftarrows \mathrm{Ho}(\mathbf{M}) : \Omega$ of suspension-desuspension functors is an adjoint equivalence.

Facts 2.2.0.7. (Y. Harpaz, J. Nuiten and M. Prasma [6]) Let $\mathbf{M}$ and $\mathbf{N}$ be two weakly pointed combinatorial model categories such that the domains of their generating cofibrations are cofibrant.

1. There is a Quillen adjunction $\Sigma^\infty : \mathbf{M} \rightleftarrows \mathrm{Sp}(\mathbf{M}) : \Omega^\infty$ where $\Omega^\infty(X) = X_{0,0}$ and $\Sigma^\infty(X)$ is the constant diagram with value X.
2. The induced functor $(\Omega^\infty)_\infty : \mathrm{Sp}(\mathbf{M})_\infty \longrightarrow \mathbf{M}_\infty$ exhibits $\mathrm{Sp}(\mathbf{M})_\infty$ as the stabilization of $\mathbf{M}_\infty$ in the sense of [3].
3. The stabilization $\mathrm{Sp}(\mathbf{M})$ is stable. Furthermore, if $\mathbf{M}$ is already stable then the adjunction $\Sigma^\infty : \mathbf{M} \rightleftarrows \mathrm{Sp}(\mathbf{M}) : \Omega^\infty$ is a Quillen equivalence.
4. A Quillen adjunction $\mathrm{F} : \mathbf{M} \rightleftarrows \mathbf{N} : \mathrm{G}$ lifts to a Quillen adjunction $\mathrm{Sp}(\mathrm{F}) : \mathrm{Sp}(\mathbf{M}) \rightleftarrows \mathrm{Sp}(\mathbf{N}) : \mathrm{Sp}(\mathrm{G})$ between stabilizations which is given by the adjunction $\mathrm{F}^{\mathbb{N}\times\mathbb{N}} \dashv \mathrm{G}^{\mathbb{N}\times\mathbb{N}}$ on underlying categories. Moreover, if $\mathrm{F} \dashv \mathrm{G}$ is a Quillen equivalence then $\mathrm{Sp}(\mathrm{F}) \dashv \mathrm{Sp}(\mathrm{G})$ is one. (We will sometimes write F^{Sp} instead $\mathrm{Sp}(\mathrm{F})$).

Let $\mathcal{I}$ be a diagram category. Then the left Quillen functor $\mathrm{F} : \mathbf{M} \longrightarrow \mathbf{N}$ gives rise to a left Quillen functor

$$[\mathcal{I}, \mathrm{Sp}(\mathrm{F})] : \mathrm{Fun}(\mathcal{I}, \mathrm{Sp}(\mathbf{M})) \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathrm{Sp}(\mathbf{N}))$$

between projective model categories given by postcomposition with $\mathrm{Sp}(\mathrm{F}) : \mathrm{Sp}(\mathbf{M}) \longrightarrow \mathrm{Sp}(\mathbf{N})$.

Observation 2.2.0.8. Suppose that F preserves weak equivalences. Then $\mathrm{Sp}(\mathrm{F})$ also preserves weak equivalences and therefore, so does the functor $[\mathcal{I}, \mathrm{Sp}(\mathrm{F})]$.

Proof. Since F preserves weak equivalences, so does the functor $\mathrm{F}^{\mathbb{N}\times\mathbb{N}} : \mathbf{M}^{\mathbb{N}\times\mathbb{N}}_{\mathrm{proj}} \longrightarrow \mathbf{N}^{\mathbb{N}\times\mathbb{N}}_{\mathrm{proj}}$. The proof is then straightforward using the definition of stable equivalences. □

Notations 2.2.0.9.

1. Let $\mathcal{C}$ be a category containing a terminal object $*$. We will denote by $\mathcal{C}_* := \mathcal{C}_{*/}$ the **pointed category** associated to $\mathcal{C}$.
2. Suppose that $\mathcal{C}$ contains an initial object $\varnothing$. We then denote by $\mathcal{C}^{\mathrm{aug}} := \mathcal{C}_{/\varnothing}$ the **augmented category** associated to $\mathcal{C}$.
3. Let $\mathcal{C}$ be a category containing an object X. We will denote by $\mathcal{C}_{X//X} := (\mathcal{C}_{/X})_*$ the pointed category associated to the over category $\mathcal{C}_{/X}$. More explicitly, objects of $\mathcal{C}_{X//X}$ are the diagrams $X \xrightarrow{f} A \xrightarrow{g} X$ in $\mathcal{C}$ such that $gf = \mathrm{Id}_X$. Alternatively, $\mathcal{C}_{X//X}$ is the same as $(\mathcal{C}_{X/})^{\mathrm{aug}}$ the augmented category associated to the under category $\mathcal{C}_{X/}$.

Note that a morphism $f: X \to Y$ in $\mathcal{C}$ gives rise to a canonical adjunction $f_!: \mathcal{C}_{X//X} \rightleftarrows \mathcal{C}_{Y//Y} : f^*$ in which $f_!(X \to A \to X) = A \coprod_X Y$ while $f^*(Y \to B \to Y) = B \times_Y X$.

It can be shown that if $\mathbf{M}$ is a combinatorial model category such that the domains of generating cofibrations are cofibrant then so is the transferred model structure on $\mathbf{M}_{A//A}$ (see Hirschhorn's [44]). This makes the following definition valid.

Definition 2.2.0.10. Let $\mathbf{M}$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let A be an object of $\mathbf{M}$. The **tangent category** to $\mathbf{M}$ at A, denoted by $\mathcal{T}_A\mathbf{M}$, is defined to be the stabilization of $\mathbf{M}_{A//A}$, i.e., $\mathcal{T}_A\mathbf{M} := \mathrm{Sp}(\mathbf{M}_{A//A})$.

There is a Quillen adjunction $\Sigma^\infty_+: \mathbf{M}_{/A} \rightleftarrows \mathcal{T}_A\mathbf{M} : \Omega^\infty_+$ given by the composition

$$\mathbf{M}_{/A} \underset{\text{forgetful}}{\overset{A\sqcup(-)}{\rightleftarrows}} \mathbf{M}_{A//A} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{T}_A\mathbf{M}.$$

Namely, for each $B \in \mathbf{M}_{/A}$, then $\Sigma^\infty_+(B) = \Sigma^\infty(A \longrightarrow A \sqcup B \longrightarrow A)$ the constant diagram with value $A \sqcup B$; and for each $X \in \mathcal{T}_A\mathbf{M}$, $\Omega^\infty_+(X) = [X_{0,0} \longrightarrow A]$.

Definition 2.2.0.11. Let $\mathbf{M}$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let A be an object of $\mathbf{M}$. The **cotangent complex** of A, denoted by L_A, is defined to be the derived suspension spectrum of A, i.e., $\mathrm{L}_A := \mathbb{L}\Sigma^\infty_+(A) \in \mathcal{T}_A\mathbf{M}$.

By Facts 2.2.0.7(iv), a given map $f: A \to B$ in $\mathbf{M}$ gives rise to a Quillen adjunction between tangent categories

$$f^{\mathrm{Sp}}_!: \mathcal{T}_A\mathbf{M} = \mathrm{Sp}(\mathbf{M}_{A//A}) \rightleftarrows \mathrm{Sp}(\mathbf{M}_{B//B}) = \mathcal{T}_B\mathbf{M} : f^*_{\mathrm{Sp}}.$$

Moreover, there is a commutative square of left Quillen functors

$$\begin{array}{ccc} \mathbf{M}_{/A} & \xrightarrow{f_!} & \mathbf{M}_{/B} \\ {\scriptstyle \Sigma^\infty_+}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty_+} \\ \mathcal{T}_A\mathbf{M} & \xrightarrow[f^{Sp}_!]{} & \mathcal{T}_B\mathbf{M} \end{array} \tag{2.2.0.1}$$

Definition 2.2.0.12. Let $\mathbf{M}$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let $f: A \to B$ be a map in $\mathbf{M}$. We will denote by

$$\mathrm{L}_{B/A} := \mathrm{hocofib}\,[\,\mathbb{L}\Sigma^\infty_+(f) \longrightarrow \mathrm{L}_B\,]$$

the homotopy cofiber of the map $\mathbb{L}\Sigma^\infty_+(f) \longrightarrow \mathrm{L}_B$ in $\mathcal{T}_B\mathbf{M}$ and refer to $\mathrm{L}_{B/A}$ as the **relative cotangent complex** of f.

Notice that the map $\mathbb{L}\Sigma^\infty_+(f) \longrightarrow \mathrm{L}_B$ can be identified to $f^{\mathrm{Sp}}_!(\mathrm{L}_A) \longrightarrow \mathrm{L}_B$, due to the commutativity of the square (2.2.0.1).

Remark 2.2.0.13. Suppose that A is cofibrant. Take a factorization $A \longrightarrow B^{\mathrm{cof}} \xrightarrow{\simeq} B$ in $\mathbf{M}$ of the map f into a cofibration followed by a weak equivalence. In particular, B^{cof} is a cofibrant resolution of B in $\mathbf{M}$. Consider the coCartesian square

$$\begin{array}{ccc} B \sqcup A & \longrightarrow & B \sqcup B^{\mathrm{cof}} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \coprod_A B^{\mathrm{cof}} \end{array}$$

When regarded as a coCartesian square in $\mathbf{M}_{B//B}$, this square is homotopy coCartesian because the map $A \longrightarrow B^{\mathrm{cof}}$ is a cofibration between cofibrant objects in $\mathbf{M}$ and in addition, B is a zero object in $\mathbf{M}_{B//B}$. Applying the functor $\Sigma^{\infty} : \mathbf{M}_{B//B} \longrightarrow \mathcal{T}_B\mathbf{M}$ to this square, we obtain a homotopy cofiber sequence in $\mathcal{T}_B\mathbf{M}$

$$\Sigma^{\infty}(B \sqcup A) \longrightarrow \Sigma^{\infty}(B \sqcup B^{\mathrm{cof}}) \longrightarrow \Sigma^{\infty}(B \coprod_A B^{\mathrm{cof}}).$$

In this sequence, the first term is a model for $\mathbb{L}\Sigma^{\infty}_{+}(f)$, while the second term is nothing but L_B. Hence, by definition $\Sigma^{\infty}(B \coprod_A B^{\mathrm{cof}})$ is a model for the relative cotangent complex $\mathrm{L}_{B/A}$.

Finally, the most important definition in this thesis is as follows:

Definition 2.2.0.14. ([7], Definition 2.2.1) Let $\mathbf{M}$ be a combinatorial model category such that the domains of generating cofibrations are cofibrant and let X be a fibrant object of $\mathbf{M}$. Suppose given a fibrant object $M \in \mathcal{T}_X\mathbf{M}$, regarded as the **$\Omega$-spectrum of coefficients**. For each $n \in \mathbb{Z}$, the **n'th Quillen cohomology group of** X with coefficients in M is defined to be

$$\mathrm{H}^n_Q(X, M) := \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathcal{T}_X\mathbf{M}}(\mathrm{L}_X, M[n])$$

where $M[n] := \Sigma^n M$, i.e, the n-suspension of M in $\mathcal{T}_X\mathbf{M}$.

Remark 2.2.0.15. By the Quillen adjunction $\Sigma^{\infty}_{+} : \mathbf{M}_{/X} \rightleftarrows \mathcal{T}_X\mathbf{M} : \Omega^{\infty}_{+}$, there is a canonical weak equivalence

$$\operatorname{Map}^{\mathrm{h}}_{\mathbf{M}_{/X}}(X, \Omega^{\infty}_{+} M[n]) \simeq \operatorname{Map}^{\mathrm{h}}_{\mathcal{T}_X\mathbf{M}}(\mathrm{L}_X, M[n]),$$

(cf. Proposition A.0.0.11). In particular, we have that

$$\mathrm{H}^n_Q(X, M) = \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathcal{T}_X\mathbf{M}}(\mathrm{L}_X, M[n]) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathbf{M}_{/X}}(X, \Omega^{\infty}_{+} M[n]).$$

Remark 2.2.0.16. Quillen cohomology is a homotopy invariant. Indeed, a weak equivalence $f : X \xrightarrow{\simeq} Y$ between fibrant objects induces a right Quillen equivalence $\mathbf{M}_{/Y} \xrightarrow{\simeq} \mathbf{M}_{/X}$. Therefore, for any fibrant object $M \in \mathcal{T}_Y\mathbf{M}$ we get a canonical weak equivalence

$$\operatorname{Map}^{\mathrm{h}}_{\mathbf{M}_{/Y}}(Y, \Omega^{\infty}_{+} M[n]) \xrightarrow{\simeq} \operatorname{Map}^{\mathrm{h}}_{\mathbf{M}_{/X}}(X, f^{*}\Omega^{\infty}_{+} M[n]) = \operatorname{Map}^{\mathrm{h}}_{\mathbf{M}_{/X}}(X, \Omega^{\infty}_{+}(f^{*}_{\mathrm{Sp}} M)[n]).$$

This gives us for each n an isomorphism $\mathrm{H}^n_Q(Y, M) \xrightarrow{\cong} \mathrm{H}^n_Q(X, f^{*}_{\mathrm{Sp}} M)$, by Remark 2.2.0.15.

Chapter 3

Quillen cohomology of enriched operads

This chapter contains the central results of the thesis. As the main goal, we give an explicit formula for computing Quillen cohomology of enriched operads. Besides that, we prove the existence of a long exact sequence relating Quillen cohomology and reduced Quillen cohomology of a given operad.

3.1 Conventions

We set up several suitable conditions on the base category that we work with throughout this chapter. We first recall from [6] the following definition, which itself is inspired by [[3], Definition 6.1.1.6].

Definition 3.1.0.1. A model category $\mathbf{M}$ is said to be **differentiable** if the derived colimit functor $\mathbb{L}\,\mathrm{colim} : \mathbf{M}^{\mathbb{N}} \longrightarrow \mathbf{M}$ preserves finite homotopy limits. Furthermore, a Quillen adjunction $\mathcal{L} : \mathbf{M} \rightleftarrows \mathbf{N} : \mathcal{R}$ is said to be **differentiable** if both $\mathbf{M}$ and $\mathbf{N}$ are differentiable and the right derived functor $\mathbb{R}\mathcal{R}$ preserves sequential homotopy colimits.

Conventions 3.1.0.2. In this chapter, we will work on the base category $\mathcal{S}$ which is assumed to be a combinatorial symmetric monoidal model category such that the domains of generating cofibrations are cofibrant. Moreover, $\mathcal{S}$ satisfies the conditions (S1)-(S4) of Proposition 2.1.5.8 and in addition, that

(S5) $\mathcal{S}$ is differentiable,

(S6) the unit $1_{\mathcal{S}}$ is **homotopy compact** in the sense that the functor $\pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathcal{S}}(1_{\mathcal{S}}, -)$ sends filtered homotopy colimits to colimits of sets, and

(S7) $\mathcal{S}$ satisfies the Lurie's **invertibility hypothesis** [[4], Definition A.3.2.12].

In particular, we will work on the canonical model structure on $\mathrm{Op}(\mathcal{S})$ $(\mathrm{Cat}(\mathcal{S}))$, which coincides with the Dwyer-Kan model structure by Proposition 2.1.5.8. On other hand, for any category

$$\mathcal{M} \in \{\mathrm{Coll}_C(\mathcal{S}), \mathrm{Op}_C(\mathcal{S}), \mathrm{LMod}(\mathcal{P}), \mathrm{RMod}(\mathcal{P}), \mathrm{BMod}(\mathcal{P}), \mathrm{IbMod}(\mathcal{P}), \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}), \mathrm{Mod}^A_{\mathcal{P}}\},$$

we will work with the (projective) transferred model structure on $\mathcal{M}$, which indeed exists as we discussed in §2.1.4.

Remark 3.1.0.3. Requiring the domains of generating cofibrations to be cofibrant is necessary for the existence of various types of operadic tangent category (cf. Section 2.2), which come after a procedure of taking left Bousfield localizations without left properness (see Definition 2.2.0.4).

Remark 3.1.0.4. The conditions (S5)-(S6) support to the differentiability of $\mathrm{Op}(\mathcal{S})$ and $\mathrm{Op}_C(\mathcal{S})$ (cf. Lemma 3.2.1.3), which we will need in proving Proposition 3.2.1.4.

Remark 3.1.0.5. The condition (S7) allows us to inherit [[7], Proposition 3.2.1] for the work of Section 3.4. Briefly, the invertibility hypothesis requires that, for any $\mathcal{C} \in \mathrm{Cat}(\mathcal{S})$ containing a morphism f, localizing $\mathcal{C}$ at f does not change the homotopy type of $\mathcal{C}$ whenever f is already an isomorphism in $\mathrm{Ho}(\mathcal{C})$. This condition is in fact pretty popular in practice. According to [39], if $\mathcal{S}$ is a combinatorial monoidal model category satisfying (S1) and such that every object is cofibrant then $\mathcal{S}$ satisfies the invertibility hypothesis. It also holds for dg modules over a commutative ring by [[40], Corollary 8.7], and for any simplicial monoidal model category, according to [[41], Theorem 0.9].

Example 3.1.0.6. Typical categories for Conventions 3.1.0.2 are again the ones of Examples 2.1.4.7.

3.2 Operadic tangent categories

Let $\mathcal{P}$ be a C-colored operad in $\mathcal{S}$. The sequence (2.1.2.3) induces a sequence of adjunctions connecting the associated augmented categories

$$\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathrm{BMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}.$$

Observe that each of the right adjoints in this sequence preserves fibrations and weak equivalences. So all the adjunctions in this sequence are Quillen adjunctions. We thus obtain a sequence of Quillen adjunctions connecting the associated tangent categories (cf. Facts 2.2.0.7(iv)) :

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})\,. \tag{3.2.0.1}$$

Recall that the operad $\mathcal{P} \in \mathrm{Op}_C(\mathcal{S})$ is said to be **Σ-cofibrant** if its underlying C-collection is cofibrant as an object of $\mathrm{Coll}_C(\mathcal{S})$. Our ultimate goal in this section is to prove that the adjunctions in the above sequence are all Quillen equivalences when $\mathcal{P}$ is fibrant and Σ-cofibrant. The work may require making use of the **Comparison theorem** [8], which we now recall.

Let $\mathbf{M}$ be a symmetric monoidal model category and let $\mathcal{O}$ be an $\mathbf{M}$-enriched operad. We denote by $\mathcal{O}_{\leqslant 1}$ the operad obtained from $\mathcal{O}$ by removing the operations of arity > 1. Recall that the collection of unary (= 0-ary) operations of $\mathcal{O}$, $\mathcal{O}_0$, inherits the obvious structure of an $\mathcal{O}$-algebra and then, becomes an initial object in the category $\mathrm{Alg}_{\mathcal{O}}(\mathbf{M})$.

Definition 3.2.0.1. The operad $\mathcal{O}$ is said to be **admissible** if the transferred model structure on $\mathrm{Alg}_{\mathcal{O}}(\mathbf{M})$ exists. Furthermore, $\mathcal{O}$ is called **stably (resp. semi) admissible** if it is admissible and the stabilization $\mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathbf{M}))$ exists as a (resp. semi) model category, where $\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathbf{M}) := \mathrm{Alg}_{\mathcal{O}}(\mathbf{M})_{\mathcal{O}_0//\mathcal{O}_0}$ the augmented category associated to the category $\mathrm{Alg}_{\mathcal{O}}(\mathbf{M})$ $(= \mathrm{Alg}_{\mathcal{O}}(\mathbf{M})_{\mathcal{O}_0/})$.

Note that there is a canonical isomorphism $\mathrm{Alg}_{\mathcal{O}_{\leqslant 1}}(\mathbf{M}) \cong \mathrm{Alg}_{\mathcal{O}_1}(\mathbf{M})_{\mathcal{O}_0/}$. The inclusion of operads $\varphi : \mathcal{O}_{\leqslant 1} \longrightarrow \mathcal{O}$ induces a Quillen adjunction

$$\varphi^{\mathrm{aug}}_! : \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}_{\leqslant 1}}(\mathbf{M}) = \mathrm{Alg}_{\mathcal{O}_1}(\mathbf{M})_{\mathcal{O}_0//\mathcal{O}_0} \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathbf{M})_{\mathcal{O}_0//\mathcal{O}_0} = \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathbf{M}) : \varphi^*_{\mathrm{aug}}.$$

Theorem 3.2.0.2. *(**Comparison theorem**) [Y. Harpaz, J. Nuiten and M. Prasma [8]] Let $\boldsymbol{M}$ be a differentiable, left proper and combinatorial symmetric monoidal model category and let $\mathcal{O}$ be a Σ-cofibrant stably admissible operad in $\boldsymbol{M}$. Assume either $\boldsymbol{M}$ is right proper or $\mathcal{O}_0$ is fibrant. Then the induced Quillen adjunction between stabilizations*

$$\varphi_!^{\mathrm{Sp}} : \mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}_{\leqslant 1}}(\boldsymbol{M})) \rightleftarrows \mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\boldsymbol{M})) : \varphi^*_{\mathrm{Sp}}$$

is a Quillen equivalence.

Remark 3.2.0.3. In fact, many model categories of interest are not left proper (where $\mathrm{Op}_C(\mathcal{S})$ and $\mathrm{Op}(\mathcal{S})$ are typical examples) and as a sequel, their stabilizations do not exist as (full) model categories (cf. Remark 2.2.0.5). In the loc.cit, the authors were aware of this fact, and made sure to include Corollary 4.1.4 saying that the restriction functor $\varphi^*_{\mathrm{aug}} : \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathbf{M}) \longrightarrow \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}_{\leqslant 1}}(\mathbf{M})$, under the same assumptions as in Proposition 3.2.0.2 except the left properness of $\mathbf{M}$, induces an equivalence of **relative categories** after taking stabilizations

$$\varphi^*_{\mathrm{Sp}'} : \mathrm{Sp}'(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathbf{M})) \xrightarrow{\simeq} \mathrm{Sp}'(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}_{\leqslant 1}}(\mathbf{M})).$$

In particular, when the stabilizations exist as semi model categories then $\varphi_!^{\mathrm{Sp}} \dashv \varphi^*_{\mathrm{Sp}}$ is indeed a Quillen equivalence. So keep in mind that the statement of the Comparison theorem remains valid when $\mathcal{P}$ is just stably semi admissible.

3.2.1 The first Quillen equivalence

The Quillen adjunction $\mathcal{L}_{\mathcal{P}} : \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}/} : \mathcal{R}_{\mathcal{P}}$ (2.1.2.2) lifts to a Quillen adjunction between the associated tangent categories

$$\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) : \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}.$$

Our goal in this subsection is to prove that $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ is a Quillen equivalence when provided that $\mathcal{P}$ is fibrant, yet let us start with the following simple observations.

Observations 3.2.1.1.

1. A given map between C-colored operads is a weak equivalence (resp. trivial fibration, cofibration) in $\mathrm{Op}_C(\mathcal{S})$ if and only if it is a weak equivalence (resp. trivial fibration, cofibration) in $\mathrm{Op}(\mathcal{S})$.

2. A given C-colored operad is cofibrant (resp. fibrant) as an object of $\mathrm{Op}_C(\mathcal{S})$ if and only if it is cofibrant (resp. fibrant) as an object of $\mathrm{Op}(\mathcal{S})$.

3. A cofibrant resolution $\mathcal{P}^{\mathrm{cof}} \xrightarrow{\simeq} \mathcal{P}$ of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}_C(\mathcal{S})$ is also a cofibrant resolution of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}(\mathcal{S})$.

Proof. Let $f : \mathcal{P} \to \mathcal{Q}$ be a map between C-colored operads.

1. If f is a weak equivalence in $\mathrm{Op}(\mathcal{S})$ then it is in particular a levelwise weak equivalence, and hence a weak equivalence in $\mathrm{Op}_C(\mathcal{S})$. Conversely, suppose that f is a weak equivalence in $\mathrm{Op}_C(\mathcal{S})$, i.e., a levelwise weak equivalence. Since f is the identity on colors, the induced map $\mathrm{Ho}(f)$ is automatically essentially surjective (cf. Definition 2.1.5.4). Thus, by definition f is indeed a Dwyer-Kan equivalence, i.e., a weak equivalence in $\mathrm{Op}(\mathcal{S})$.

   The claim about trivial fibrations immediately follows by definition.

We now assume that f is a cofibration in $\mathrm{Op}(\mathcal{S})$, and prove that f is one in $\mathrm{Op}_C(\mathcal{S})$. For any given trivial fibration $f' : \mathcal{P}' \to \mathcal{Q}'$ in $\mathrm{Op}_C(\mathcal{S})$, we have to show that f has the lifting property against f'. Since f' is also a trivial fibration in $\mathrm{Op}(\mathcal{S})$, the lifting problem is solved when considered in $\mathrm{Op}(\mathcal{S})$. But such a lift must be the identity on colors, so it is also a lift in $\mathrm{Op}_C(\mathcal{S})$. We just showed that f is a cofibration in $\mathrm{Op}_C(\mathcal{S})$. Conversely, if f is a cofibration in $\mathrm{Op}_C(\mathcal{S})$ then it is one in $\mathrm{Op}(\mathcal{S})$, just by the fact that the embedding $\mathcal{L}_{\mathcal{P}} : \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \longrightarrow \mathrm{Op}(\mathcal{S})_{\mathcal{P}/}$ is a left Quillen functor.

2. The claim about the fibrancy immediately follows by definition. Now, if $\mathcal{P}$ is cofibrant as an object of $\mathrm{Op}(\mathcal{S})$ then it is so as an object of $\mathrm{Op}_C(\mathcal{S})$, similarly as the first claim of the above paragraph. For the converse direction, by the last claim of (1), it suffices to show that the initial C-colored operad $\mathcal{I}_C$ is also cofibrant as an object of $\mathrm{Op}(\mathcal{S})$. Notice that a map in $\mathrm{Op}(\mathcal{S})$, from $\mathcal{I}_C$ to a given operad $\mathcal{O}$, is fully characterized by a map from C to the set of colors of $\mathcal{O}$. The claim hence follows by the fact that any trivial fibration in $\mathrm{Op}(\mathcal{S})$ has underlying map between colors being surjective.

3. This follows by the two above.

□

As a consequence, by the second part we will usually say a certain C-colored operad is (co)fibrant without (necessarily) indicating precisely it is (co)fibrant as an object of $\mathrm{Op}_C(\mathcal{S})$ or $\mathrm{Op}(\mathcal{S})$.

The main tool for proving the adjunction $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ is a Quillen equivalence will be [[6], Corollary 2.4.9]. To be able to use this tool, we have to show that the induced Quillen adjunction

$$\mathcal{L}^{\mathrm{aug}}_{\mathcal{P}} : \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}} : \mathcal{R}^{\mathrm{aug}}_{\mathcal{P}}$$

between the associated augmented categories is differentiable (cf. Definition 3.1.0.1).

Remark 3.2.1.2. By convention, the base category $\mathcal{S}$ is differentiable and has the class of weak equivalences being closed under sequential colimits. Thus, the (underived) colimit functor $\mathrm{colim} : \mathcal{S}^{\mathbb{N}} \longrightarrow \mathcal{S}$ already preserves homotopy Cartesian squares and homotopy terminal objects. An analogue does hold for the functor $\mathrm{colim} : \mathrm{Cat}(\mathcal{S})^{\mathbb{N}} \longrightarrow \mathrm{Cat}(\mathcal{S})$, due to Remark 3.1.0.4 and [[7], Lemma 3.1.10] (saying that weak equivalences in $\mathrm{Cat}(\mathcal{S})$ are closed under sequential colimits).

Lemma 3.2.1.3. *The Quillen adjunction $\mathcal{L}_{\mathcal{P}} : \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}/} : \mathcal{R}_{\mathcal{P}}$ is differentiable. Consequently, the induced Quillen adjunction $\mathcal{L}^{\mathrm{aug}}_{\mathcal{P}} : \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}} : \mathcal{R}^{\mathrm{aug}}_{\mathcal{P}}$ is differentiable as well.*

Proof. Observe first that sequential colimits of enriched operads are taken levelwise in the following sense. Consider a sequence of objects in $\mathrm{Op}(\mathcal{S})$

$$\mathcal{P}^{(0)} \longrightarrow \mathcal{P}^{(1)} \longrightarrow \mathcal{P}^{(2)} \longrightarrow \cdots.$$

We establish an operad $\mathcal{P}$ as follows. Take $\mathrm{Col}(\mathcal{P}) := \mathrm{colim}_n \mathrm{Col}(\mathcal{P}^{(n)})$ where $\mathrm{Col}(-)$ refers to set of colors. For each $\overline{c} = (c_1, \cdots, c_n; c)$ in $\mathrm{Seq}(\mathrm{Col}(\mathcal{P}))$, we pick n_0 large enough such that $\overline{c} \in \mathrm{Seq}(\mathrm{Col}(\mathcal{P}^{(n_0)}))$. Then we take $\mathcal{P}(\overline{c}) := \mathrm{colim}_{n \geqslant n_0} \mathcal{P}^{(n)}(\overline{c})$. The operad structures of the terms $\mathcal{P}^{(n)}$'s together determine an operad structure on $\mathcal{P}$. It can then be verified that $\mathcal{P} \cong \mathrm{colim}_n \mathcal{P}^{(n)}$. In particular, we get that $\mathcal{P}_1 \cong \mathrm{colim}_n \mathcal{P}^{(n)}_1$ (cf. the proof of [[7], Lemma 3.1.10]).

We now claim that weak equivalences in $\mathrm{Op}(\mathcal{S})$ are closed under sequential colimits. Indeed, recall that a map f in $\mathrm{Op}(\mathcal{S})$ is a weak equivalence if and only if it is a levelwise weak equivalence and such that the underlying map f_1 is a weak equivalence in $\mathrm{Cat}(\mathcal{S})$ (cf. Remark 2.1.5.7). Thus, the claim follows by the above paragraph and by the fact that weak equivalences in $\mathcal{S}$ (or $\mathrm{Cat}(\mathcal{S})$) are closed under sequential colimits.

Next, we claim that a given square in $\mathrm{Op}(\mathcal{S})$ is homotopy Cartesian if and only if the following two conditions hold

1. the induced squares of spaces of operations are homotopy Cartesian in $\mathcal{S}$, and

2. the induced square of underlying categories is homotopy Cartesian in $\mathrm{Cat}(\mathcal{S})$.

Notice that this statement is already correct when we forget the word "homotopy". So we get the claim, immediately by Remark 2.1.5.7. On other hand, it is not hard to show that an object of $\mathrm{Op}(\mathcal{S})$ is homotopy terminal if and only if all its spaces of operations being homotopy terminal in $\mathcal{S}$.

We now show that $\mathrm{Op}(\mathcal{S})$ is differentiable. By the second paragraph, it suffices to verify that the (underived) colimit functor $\mathrm{colim} : \mathrm{Op}(\mathcal{S})^{\mathbb{N}} \longrightarrow \mathrm{Op}(\mathcal{S})$ preserves homotopy Cartesian squares and homotopy terminal objects. This follows by combining the first and third paragraphs, along with Remark 3.2.1.2.

On other hand, the category $\mathrm{Op}_C(\mathcal{S})$ is also differentiable. Indeed, the situation is similar to the differentiability of $\mathrm{Op}(\mathcal{S})$. One will need to use the facts that sequential colimits of C-colored operads are taken levelwise (and hence weak equivalences in $\mathrm{Op}_C(\mathcal{S})$ are closed under sequential colimits), that a square in $\mathrm{Op}_C(\mathcal{S})$ is homotopy Cartesian if and only if the induced squares of spaces of operations are homotopy Cartesian in $\mathcal{S}$, that a C-colored operad is homotopy terminal if and only if all its spaces of operations being homotopy terminal in $\mathcal{S}$, and that the functor $\mathrm{colim} : \mathcal{S}^{\mathbb{N}} \longrightarrow \mathcal{S}$ preserves homotopy Cartesian squares and homotopy terminal objects (as discussed in Remark 3.2.1.2).

Now, the category $\mathrm{Op}(\mathcal{S})_{\mathcal{P}/}$ (resp. $\mathrm{Op}_C(\mathcal{S})_{\mathcal{P}/}$) is differentiable since $\mathrm{Op}(\mathcal{S})$ (resp. $\mathrm{Op}_C(\mathcal{S})$) is already so. Moreover, the restriction functor $\mathcal{R}_{\mathcal{P}}$ clearly preserves sequential homotopy colimits. Thus, the adjunction $\mathcal{L}_{\mathcal{P}} \dashv \mathcal{R}_{\mathcal{P}}$ is differentiable, and hence so is the $\mathcal{L}^{\mathrm{aug}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{aug}}_{\mathcal{P}}$. ∎

We are now in position to prove the main result of this subsection.

Proposition 3.2.1.4. *The adjunction* $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) : \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ *is a Quillen equivalence when provided that* $\mathcal{P}$ *is fibrant.*

Proof. Let $\mathcal{Q} \in \mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$ be a fibrant object, exhibited by a diagram $\mathcal{P} \to \mathcal{Q} \to \mathcal{P}$ in $\mathrm{Op}(\mathcal{S})$ such that the second map is a fibration. The same arguments as in the proof of [[7], Lemma 3.1.13] show that the map between the homotopy pullbacks $\mathcal{P} \times^{\mathrm{h}}_{\mathcal{L}_{\mathcal{P}}\mathcal{R}_{\mathcal{P}}(\mathcal{Q})} \mathcal{P} \longrightarrow \mathcal{P} \times^{\mathrm{h}}_{\mathcal{Q}} \mathcal{P}$ is a weak equivalence in $\mathrm{Op}(\mathcal{S})$. In particular, the induced map $\Omega\,\mathcal{L}^{\mathrm{aug}}_{\mathcal{P}}\mathcal{R}^{\mathrm{aug}}_{\mathcal{P}}(\mathcal{Q}) \longrightarrow \Omega\,\mathcal{Q}$ is a weak equivalence in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$. This fact, together with Lemma 3.2.1.3, allows us to apply [[6], Corollary 2.4.9] to deduce that the derived counit of the Quillen adjunction $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ is a stable equivalence for every fibrant Ω-spectrum.

It remains to show that the derived unit of $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ is a stable equivalence for any cofibrant object. In fact, we will show that this holds for the larger class of levelwise cofibrant objects. Since $\mathcal{R}^{\mathrm{aug}}_{\mathcal{P}} \circ \mathcal{L}^{\mathrm{aug}}_{\mathcal{P}}$ is isomorphic to the identity functor and since $\mathcal{R}^{\mathrm{aug}}_{\mathcal{P}}$ preserves weak equivalences, the derived unit of $\mathcal{L}^{\mathrm{aug}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{aug}}_{\mathcal{P}}$ is a weak equivalence. By the first part of [[6], Corollary 2.4.9], the derived unit of $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ is a stable equivalence for any levelwise cofibrant prespectrum. But every levelwise cofibrant object in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S})$ is stably equivalent to a levelwise cofibrant prespectrum (see [6], Remark 2.3.6), it therefore suffices to observe that $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}}$ preserves stable equivalences. But this follows immediately by Observation 2.2.0.8. ∎

3.2.2 The second Quillen equivalence

We wish to prove that $\mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{S})$ is Quillen equivalent to $\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P})$.

Proposition 3.2.2.1. *Suppose that $\mathcal{P}$ is a cofibrant C-colored operad. Then the adjunction*

$$\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\operatorname{Op}_C(\mathcal{S})$$

is a Quillen equivalence. Consequently, the adjunction $\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{S})$ is a Quillen equivalence when $\mathcal{P}$ is in addition fibrant.

Proof. We regard $\mathcal{P}$ as an algebra over $\mathbf{O}_C$ the operad of C-colored operads. Then we get a canonical isomorphism $\operatorname{Alg}_{\operatorname{Env}(\mathbf{O}_C,\mathcal{P})}(\mathcal{S}) \cong \operatorname{Op}_C(\mathcal{S})_{\mathcal{P}/}$ between the categories of algebras over the enveloping operad $\operatorname{Env}(\mathbf{O}_C,\mathcal{P})$ and C-colored operads under $\mathcal{P}$ (cf. §2.1.2). On other hand, the same arguments as in the proof of [[35], Proposition 3.5] show that the structure of an infinitesimal $\mathcal{P}$-bimodule is equivalent to that of a $\mathcal{P}$-module over $\mathbf{O}_C$. So we have a canonical isomorphism of categories $\operatorname{Alg}_{\operatorname{Env}(\mathbf{O}_C,\mathcal{P})_1}(\mathcal{S}) \cong \operatorname{IbMod}(\mathcal{P})$ (cf. Remark 2.1.2.13).

We are now applying the Comparison theorem 3.2.0.2 (along with noting Remark 3.2.0.3) to the operad $\operatorname{Env}(\mathbf{O}_C,\mathcal{P})$. The symmetric groups act freely on $\mathbf{O}_C$. In particular, $\mathbf{O}_C$ is Σ-cofibrant. Moreover, since $\mathcal{P}$ is cofibrant, it implies that $\operatorname{Env}(\mathbf{O}_C,\mathcal{P})$ is Σ-cofibrant as well (cf. [[27], Lemma 6.1]). This fact makes the Comparison theorem work in our data. The first paragraph shows that the functor

$$\operatorname{Alg}^{\operatorname{aug}}_{\operatorname{Env}(\mathbf{O}_C,\mathcal{P})_{\leqslant 1}}(\mathcal{S}) \longrightarrow \operatorname{Alg}^{\operatorname{aug}}_{\operatorname{Env}(\mathbf{O}_C,\mathcal{P})}(\mathcal{S})$$

turns out to coincide with the left Quillen functor $\operatorname{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \longrightarrow \operatorname{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$. So the adjunction $\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\operatorname{Op}_C(\mathcal{S})$ is indeed a Quillen equivalence. Finally, by combining the latter with Proposition 3.2.1.4, we deduce that the adjunction $\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{S})$ is a Quillen equivalence when provided that $\mathcal{P}$ is **bifibrant** (i.e., both fibrant and cofibrant). □

The cofibrancy of $\mathcal{P}$ as required in this proposition is very strict and should be refined in order that it can work in the larger class of Σ-cofibrant operads.

Proposition 3.2.2.2. *The second statement of Proposition 3.2.2.1 is already correct when $\mathcal{P}$ is fibrant and Σ-cofibrant.*

Proof. By Observations 3.2.1.1, we can take $f : \mathcal{Q} \xrightarrow{\simeq} \mathcal{P}$ to be a bifibrant resolution of $\mathcal{P}$ in $\operatorname{Op}(\mathcal{S})$ such that f is a map in $\operatorname{Op}_C(\mathcal{S})$. By Proposition 3.2.2.1, we have a Quillen equivalence

$$\mathcal{T}_{\mathcal{Q}}\operatorname{IbMod}(\mathcal{Q}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{Q}}\operatorname{Op}(\mathcal{S}).$$

Thus, by the naturality, it suffices to prove that the induced adjunctions $\operatorname{IbMod}(\mathcal{Q})_{\mathcal{Q}//\mathcal{Q}} \rightleftarrows \operatorname{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}$ and $\operatorname{Op}(\mathcal{S})_{\mathcal{Q}//\mathcal{Q}} \rightleftarrows \operatorname{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$ are a Quillen equivalence.

Let us start with the second one. We first show that the adjunction

$$f_! \colon \operatorname{Op}(\mathcal{S})_{\mathcal{Q}/} \rightleftarrows \operatorname{Op}(\mathcal{S})_{\mathcal{P}/} : f^*$$

is a Quillen equivalence. Since f^* creates weak equivalences, it suffices to verify that, for any cofibration $\mathcal{Q} \to \mathcal{R}$ in $\operatorname{Op}(\mathcal{S})$, the induced map $\mathcal{R} \longrightarrow \mathcal{R} \coprod_{\mathcal{Q}} \mathcal{P}$ is a weak equivalence. But this immediately follows by the **relative left properness** of $\operatorname{Op}(\mathcal{S})$ (cf. [17], Theorem 6.7). Now we want to prove that

$$f_!^{\operatorname{aug}} \colon \operatorname{Op}(\mathcal{S})_{\mathcal{Q}//\mathcal{Q}} \rightleftarrows \operatorname{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}} : f^*_{\operatorname{aug}}$$

is a Quillen equivalence. We will need the following observation, which can be readily verified using definition:

(*) Suppose we are given a Quillen equivalence $\mathrm{F} : \mathcal{M} \overset{\simeq}{\rightleftarrows} \mathcal{N} : \mathrm{G}$ between right proper model categories. Let $\alpha : \mathrm{F}(A) \xrightarrow{\simeq} B$ be a weak equivalence in $\mathcal{N}$ with $A \in \mathcal{M}$ being cofibrant and $B \in \mathcal{N}$ being fibrant. Then the induced adjunction $\mathrm{F}_\alpha : \mathcal{M}_{/A} \rightleftarrows \mathcal{N}_{/B} : \mathrm{G}_\alpha$ is a Quillen equivalence.

By applying this (*) to the data of $M = \mathrm{Op}(\mathcal{S})_{\mathcal{Q}/}$, $N = \mathrm{Op}(\mathcal{S})_{\mathcal{P}/}$, $A = \mathrm{Id}_{\mathcal{Q}}$ and $B = \mathrm{Id}_{\mathcal{P}}$, we deduce that the adjunction $f_!^{\mathrm{aug}} \dashv f^*_{\mathrm{aug}}$ is indeed a Quillen equivalence.

It remains to prove that the adjunction $\mathrm{IbMod}(\mathcal{Q})_{\mathcal{Q}//\mathcal{Q}} \rightleftarrows \mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}$ is a Quillen equivalence. Since both $\mathcal{Q}$ and $\mathcal{P}$ are levelwise cofibrant, the map $f : \mathcal{Q} \xrightarrow{\simeq} \mathcal{P}$ induces a weak equivalence $\mathbf{Ib}^{\mathcal{Q}} \xrightarrow{\simeq} \mathbf{Ib}^{\mathcal{P}}$ of $\mathcal{S}$-enriched categories (see Construction 2.1.3.2). So the induced adjunction

$$\mathrm{IbMod}(\mathcal{Q}) = \mathrm{Fun}(\mathbf{Ib}^{\mathcal{Q}}, \mathcal{S}) \rightleftarrows \mathrm{Fun}(\mathbf{Ib}^{\mathcal{P}}, \mathcal{S}) = \mathrm{IbMod}(\mathcal{P})$$

is a Quillen equivalence. The claim can then be verified in the same fashion as above. □

3.2.3 The third Quillen equivalence

Recall from §2.1.3 that the category $\mathrm{BMod}(\mathcal{P})_{\mathcal{P}/}$ can be represented as the category of algebras over $\mathbf{B}^{\mathcal{P}/}$, which has the underlying category $\mathbf{B}_1^{\mathcal{P}/}$ agreeing with $\mathbf{Ib}^{\mathcal{P}}$ (cf. Constructions 2.1.3.2 and 2.1.3.7). In light of this, we can demonstrate the following, which is the last piece for the proof Theorem 3.2.4.1.

Proposition 3.2.3.1. *The adjunction* $\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}/} \rightleftarrows \mathrm{BMod}(\mathcal{P})_{\mathcal{P}/}$ *induces a Quillen equivalence of the associated tangent categories* $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ *whenever* $\mathcal{P}$ *is* Σ*-cofibrant.*

Proof. When $\mathcal{P}$ is Σ-cofibrant, by construction $\mathbf{B}^{\mathcal{P}/}$ is also Σ-cofibrant. We are now applying the Comparison theorem 3.2.0.2, along with noting Remark 3.2.0.3, to the operad $\mathbf{B}^{\mathcal{P}/}$. The key point is that the adjunction $\mathrm{Alg}^{\mathrm{aug}}_{\mathbf{B}^{\mathcal{P}/}_{\leqslant 1}}(\mathcal{S}) \rightleftarrows \mathrm{Alg}^{\mathrm{aug}}_{\mathbf{B}^{\mathcal{P}/}}(\mathcal{S})$ which arises from the inclusion $(\mathbf{B}^{\mathcal{P}/})_{\leqslant 1} \longrightarrow \mathbf{B}^{\mathcal{P}/}$ is the same as the adjunction of induction-restriction functors

$$\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathrm{BMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}.$$

Thus, the adjunction $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ is indeed a Quillen equivalence. □

3.2.4 Main statements

The main result of this section is stated as follows.

Theorem 3.2.4.1. *The adjunctions in the sequence*

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \tag{3.2.4.1}$$

are all Quillen equivalences provided that $\mathcal{P}$ *is fibrant and* Σ*-cofibrant.*

Proof. Proposition 3.2.1.4 proves that the adjunction $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ is a Quillen equivalence when $\mathcal{P}$ is fibrant. On other hand, by Proposition 3.2.2.2 the adjunction $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ is a Quillen equivalence when $\mathcal{P}$ is fibrant and Σ-cofibrant, while $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ is one by Proposition 3.2.3.1. These facts, along with the 2-out-of-3 property, prove the theorem. □

Besides that we are interested in the case in which $\mathcal{S}$ is stable (cf. Definition 2.2.0.6). The following is an analogue of [[8], Lemma 2.2.3].

Lemma 3.2.4.2. *Suppose that $\mathcal{S}$ is in addition stable containing a strict zero object 0 and let $M \in \mathrm{IbMod}(\mathcal{P})$ be a levelwise cofibrant infinitesimal $\mathcal{P}$-bimodule. Then the adjunction*

$$(-) \sqcup M : \mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathrm{IbMod}(\mathcal{P})_{M//M} : \ker$$

is a Quillen equivalence, where the functor $\ker$ *is defined by sending $M \to P \to M$ to $P \times_M 0$, while its left adjoint takes $N \in \mathrm{IbMod}(\mathcal{P})$ to $M \xrightarrow{i_0} M \sqcup N \xrightarrow{\mathrm{Id}_M + 0} M$.*

Proof. As in the loc.cit, we take a cofibrant object $N \in \mathrm{IbMod}(\mathcal{P})$ and a fibrant object $(M \to P \to M) \in \mathrm{IbMod}(\mathcal{P})_{M//M}$. We then have to show that a given map $f : N \sqcup M \longrightarrow P$ in $\mathrm{IbMod}(\mathcal{P})_{M//M}$ is a weak equivalence if and only if its adjoint $f^{\mathrm{ad}} : N \longrightarrow P \times_M 0$ in $\mathrm{IbMod}(\mathcal{P})$ is one.

Consider the following diagram of infinitesimal $\mathcal{P}$-bimodules

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{f^{\mathrm{ad}}} & P \times_M 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & N \sqcup M & \xrightarrow[f]{} & P & \longrightarrow & M \end{array}$$

Since infinitesimal $\mathcal{P}$-bimodules can be represented as $\mathcal{S}$-valued enriched functors on $\mathbf{Ib}^{\mathcal{P}}$, their (co)limits are in particular taken levelwise. Hence, for each C-sequence $\overline{c} := (c_1, \cdots, c_n; c)$, we get an induced diagram in $\mathcal{S}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N(\overline{c}) & \xrightarrow{f^{\mathrm{ad}}(\overline{c})} & P(\overline{c}) \times_{M(\overline{c})} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M(\overline{c}) & \longrightarrow & N(\overline{c}) \sqcup M(\overline{c}) & \xrightarrow[f(\overline{c})]{} & P(\overline{c}) & \longrightarrow & M(\overline{c}) \end{array}$$

Observe now that since both $N(\overline{c})$ and $M(\overline{c})$ are cofibrant, the left square is homotopy coCartesian. On other hand, the right square is homotopy Cartesian by the facts that the map $P(\overline{c}) \longrightarrow M(\overline{c})$ is a fibration and that $\mathcal{S}$ is right proper. By the stability of $\mathcal{S}$ and by pasting laws, the middle square is homotopy (co)Cartesian as well. Thus, $f(\overline{c})$ is a weak equivalence if and only if $f^{\mathrm{ad}}(\overline{c})$ is one. □

Theorem 3.2.4.3. *Suppose that $\mathcal{S}$ is in addition stable containing a strict zero object 0 and that $\mathcal{P}$ is fibrant and Σ-cofibrant. The sequence (3.2.4.1) is then prolonged to a sequence of Quillen equivalences of the form*

$$\mathrm{IbMod}(\mathcal{P}) \underset{\ker}{\overset{(-)\sqcup\mathcal{P}}{\rightleftarrows}} \mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \overset{\simeq}{\rightleftarrows} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \tag{3.2.4.2}$$

Proof. The category $\mathrm{IbMod}(\mathcal{P})$ is stable since $\mathcal{S}$ is already so (cf. [[7], Remark 2.2.2] and Proposition 2.1.3.3), and hence the category $\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}$ is stable as well. Thus, by Facts 2.2.0.7 the adjunction

$$\Sigma^\infty : \mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) : \Omega^\infty$$

is a Quillen equivalence. The statement hence follows by Lemma 3.2.4.2 and Theorem 3.2.4.1. □

3.3 An extra condition

For the remainder of this chapter, we will need to set up an extra condition on the base category $\mathcal{S}$. Let $\mathcal{P}$ be a C-colored operad in $\mathcal{S}$.

Notation 3.3.0.1. We will denote by $\mathrm{BMod}(\mathcal{P})^* := \mathrm{BMod}(\mathcal{P})_{\mathcal{P}\circ\mathcal{P}/}$ the category of $\mathcal{P}$-bimodules under $\mathcal{P}\circ\mathcal{P}$ (which performs the free $\mathcal{P}$-bimodule generated by $\mathcal{I}_C$), and refer to it as the category of **pointed $\mathcal{P}$-bimodules**. Observe that the composition $\mu : \mathcal{P}\circ\mathcal{P} \longrightarrow \mathcal{P}$ exhibits $\mathcal{P}$ itself as a pointed $\mathcal{P}$-bimodule.

Let $f + g : \mathcal{P}\sqcup\mathcal{P} \longrightarrow \mathcal{Q}$ be a map in $\mathrm{Op}_C(\mathcal{S})$. Then $\mathcal{Q}$ inherits a $\mathcal{P}$-bimodule structure with the left (resp. right) $\mathcal{P}$-action induced by f (resp. g). In particular, there is a restriction functor

$$\mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\sqcup\mathcal{P}/} \longrightarrow \mathrm{BMod}(\mathcal{P})^*,$$

which admits a left adjoint denoted by

$$\mathrm{E} : \mathrm{BMod}(\mathcal{P})^* \longrightarrow \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\sqcup\mathcal{P}/}.$$

Observe then that E sends $\mathcal{P} \in \mathrm{BMod}(\mathcal{P})^*$ to itself $\mathcal{P} \in \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\sqcup\mathcal{P}/}$ equipped with the fold map $\mathrm{Id}_{\mathcal{P}} + \mathrm{Id}_{\mathcal{P}} : \mathcal{P}\sqcup\mathcal{P} \longrightarrow \mathcal{P}$. Dwyer and Hess ([[24], section 5]) proved that, in the context of **nonsymmetric simplicial operads**, the left derived functor of E sends $\mathcal{P}$ to itself $\mathcal{P}$. Inspired by their work, we set up an extra condition on the base category $\mathcal{S}$ as follows.

Condition 3.3.0.2. (S8) For any cofibrant object $\mathcal{P} \in \mathrm{Op}_C(\mathcal{S})$, the left derived functor

$$\mathbb{L}\,\mathrm{E} : \mathrm{BMod}(\mathcal{P})^* \longrightarrow \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\sqcup\mathcal{P}/}$$

of E sends $\mathcal{P}$ to itself $\mathcal{P}$.

We shall now extend the work of Dwyer and Hess to prove that the category of simplicial sets, Set_Δ, satisfies the condition (S8). Moreover, the category of simplicial R-modules, sMod_R (see Examples 2.1.4.7), also satisfies (S8).

Proposition 3.3.0.3. *The category Set_Δ satisfies the condition (S8) 3.3.0.2. Namely, for every simplicial C-colored operad $\mathcal{P}$ which is cofibrant, the left derived functor $\mathbb{L}\,\mathrm{E} : \mathrm{BMod}(\mathcal{P})^* \longrightarrow \mathrm{Op}_C(\mathrm{Set}_\Delta)_{\mathcal{P}\sqcup\mathcal{P}/}$ of E sends $\mathcal{P}$ to itself $\mathcal{P}$.*

The proof first requires constructing a nice cofibrant resolution of $\mathcal{P}$ as an object in $\mathrm{BMod}(\mathcal{P})^*$. For this, we will follow C. Rezk's [[48], § 3.7.2]. (However, note that the operadic model structures considered in the loc.cit are different from ours, so it should be used carefully). Let $\mathcal{M}$ be a simplicial model category. The **diagonal** (or **realization**) functor $\mathrm{diag} : \mathcal{M}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{M}$ is by definition the left adjoint to the functor $\mathcal{M} \longrightarrow \mathcal{M}^{\Delta^{\mathrm{op}}}$ taking each $X \in \mathcal{M}$ to the simplicial object $[n] \mapsto X^{\Delta^n}$. For each $Y_\bullet \in \mathcal{M}^{\Delta^{\mathrm{op}}}$, one defines the **latching object** $\mathrm{L}_n Y_\bullet$ as the coequalizer in $\mathcal{M}$ of the form

$$\bigsqcup_{0\leqslant i<j\leqslant n} Y_{n-1} \rightrightarrows \bigsqcup_{0\leqslant k\leqslant n} Y_n \longrightarrow \mathrm{L}_n Y_\bullet$$

in which one of the two maps sends the (i,j) summand to the j'th summand by s_i while the other sends the (i,j) summand to the i'th summand by s_{j-1}. By convention, one puts $\mathrm{L}_{-1} Y_\bullet := \varnothing$. By construction, there is a unique map $\mathrm{L}_n Y_\bullet \longrightarrow Y_{n+1}$ factoring the map $s_k : Y_n \longrightarrow Y_{n+1}$ for every $k = 0, \cdots, n$. One then establishes a filtration $\mathrm{diag}(Y_\bullet) = \mathrm{colim}_n\, \mathrm{diag}_n Y_\bullet$ of $\mathrm{diag}(Y_\bullet)$, inductively, built up by taking $\operatorname{diag}_0 Y_\bullet := Y_0$ and, for each $n \geqslant 1$, taking the pushout:

$$\begin{array}{ccc} d_n Y_\bullet & \longrightarrow & \Delta^n \otimes Y_n \\ \downarrow & & \downarrow \\ \operatorname{diag}_{n-1} Y_\bullet & \longrightarrow & \operatorname{diag}_n Y_\bullet \end{array} \tag{3.3.0.1}$$

in which $d_n Y_\bullet := \Delta^n \otimes \mathrm{L}_{n-1} Y_\bullet \bigsqcup_{\partial\Delta^n \otimes \mathrm{L}_{n-1} Y_\bullet} \partial\Delta^n \otimes Y_n$. As a consequence, if for every $n \geqslant 0$ the **latching map** $\mathrm{L}_{n-1} Y_\bullet \longrightarrow Y_n$ is a cofibration then $\operatorname{diag}(Y_\bullet)$ is cofibrant. More generally, we have the following observation.

Lemma 3.3.0.4. *Let $X_\bullet \longrightarrow Y_\bullet$ be a map of simplicial objects in $\mathcal{M}$. Suppose that for every $n \geqslant 0$ the (relative) latching map*

$$X_n \bigsqcup_{\mathrm{L}_{n-1} X_\bullet} \mathrm{L}_{n-1} Y_\bullet \longrightarrow Y_n \tag{3.3.0.2}$$

is a cofibration. Then the induced map $\operatorname{diag}(X_\bullet) \longrightarrow \operatorname{diag}(Y_\bullet)$ is a cofibration as well.

Proof. By the filtrations of $\operatorname{diag}(X_\bullet)$ and $\operatorname{diag}(Y_\bullet)$ mentioned above, the map $\operatorname{diag}(X_\bullet) \longrightarrow \operatorname{diag}(Y_\bullet)$ is a cofibration as soon as the map $\operatorname{diag}_n X_\bullet \longrightarrow \operatorname{diag}_n Y_\bullet$ is one for every $n \geqslant 0$. Note first that when $n = 0$ the map (3.3.0.2) coincides with the map $\operatorname{diag}_0 X_\bullet \longrightarrow \operatorname{diag}_0 Y_\bullet$. Let us assume by induction that the map $\operatorname{diag}_{n-1} X_\bullet \longrightarrow \operatorname{diag}_{n-1} Y_\bullet$ is a cofibration. Then, factor $\operatorname{diag}_n X_\bullet \longrightarrow \operatorname{diag}_n Y_\bullet$ as

$$\operatorname{diag}_n X_\bullet \longrightarrow \operatorname{diag}_n X_\bullet \bigsqcup_{\operatorname{diag}_{n-1} X_\bullet} \operatorname{diag}_{n-1} Y_\bullet \xrightarrow{\varphi} \operatorname{diag}_n Y_\bullet.$$

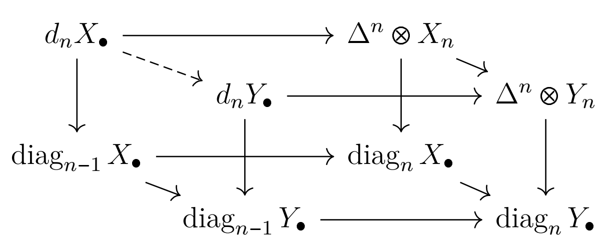
By the inductive assumption, the first map in this composition is a cofibration. Hence, it remains to show that φ is a cofibration. Let us denote by $L_{n-1}(X_\bullet, Y_\bullet) := X_n \bigsqcup_{\mathrm{L}_{n-1} X_\bullet} \mathrm{L}_{n-1} Y_\bullet$. We can then form a canonical map

$$\Delta^n \otimes L_{n-1}(X_\bullet, Y_\bullet) \bigsqcup_{\partial\Delta^n \otimes L_{n-1}(X_\bullet, Y_\bullet)} \partial\Delta^n \otimes Y_n \longrightarrow \Delta^n \otimes Y_n,$$

which is a cofibration by the pushout-product axiom. Unwinding computation, this map turns out to be isomorphic to the canonical map

$$d_n Y_\bullet \bigsqcup_{d_n X_\bullet} \Delta^n \otimes X_n \longrightarrow \Delta^n \otimes Y_n. \tag{3.3.0.3}$$

Now, consider the following commutative cube



whose the front and back squares are coCartesian. Applying the pasting law of pushouts iteratively, we find that φ turns out to be cobase change of the map (3.3.0.3), which is a cofibration as indicated there. We therefore get the conclusion. ☐

The category of simplicial C-collections admits a canonical simplicial model structure. It implies that the category $\mathrm{BMod}(\mathcal{P})$ admits a canonical simplicial model structure, by the transferred principle in simplicial version (see [[48], Propositions 3.1.5, 3.2.8]). One constructs the Hochschild resolution of $\mathcal{P}$ as follows.

Construction 3.3.0.5. Let $\mathrm{H}_\bullet \mathcal{P} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{BMod}(\mathcal{P})$ be the simplicial object of $\mathcal{P}$-bimodules with $\mathrm{H}_n \mathcal{P} := \mathcal{P}^{\circ(n+2)}$, the face map $d_i : \mathrm{H}_n \mathcal{P} \longrightarrow \mathrm{H}_{n-1} \mathcal{P}$ given by using the composition $\mu : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ to combine the factors $i+1$ and $i+2$ in $\mathrm{H}_n \mathcal{P}$ and with the degeneracy map s_i given by inserting the unit operations of $\mathcal{P}$ between the factors $i+1$ and $i+2$. The realization $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P}) \in \mathrm{BMod}(\mathcal{P})$ has n-simplices being those of $\mathrm{H}_n \mathcal{P}$. The map μ induces a canonical map of simplicial objects $\mathrm{H}_\bullet \mathcal{P} \longrightarrow \mathcal{P}$. The **augmentation map** $\psi : \mathrm{diag}(\mathrm{H}_\bullet \mathcal{P}) \longrightarrow \mathrm{diag}(\mathcal{P}) = \mathcal{P}$ is then a weak equivalence by [[48], Corollary 3.7.6], (this even comes with a contracting homotopy). The map ψ now exhibits $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ as the **Hochschild resolution of** $\mathcal{P} \in \mathrm{BMod}(\mathcal{P})$.

On other hand, since $\mathcal{P} \circ \mathcal{P} = \mathrm{H}_0 \mathcal{P}$, there is a unique map of simplicial objects $\mathcal{P} \circ \mathcal{P} \longrightarrow \mathrm{H}_\bullet \mathcal{P}$, which is the identity on degree 0. Now, the diagonal functor gives a map $\rho : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ of $\mathcal{P}$-bimodules, satisfying that the composition $\mathcal{P} \circ \mathcal{P} \xrightarrow{\rho} \mathrm{diag}(\mathrm{H}_\bullet \mathcal{P}) \xrightarrow[\simeq]{\psi} \mathcal{P}$ agrees with $\mu : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$.

Lemma 3.3.0.6. *Suppose that $\mathcal{P}$ is a Σ-cofibrant simplicial operad. The map ψ indeed exhibits* $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ *as a cofibrant resolution of $\mathcal{P}$ regarded as a bimodule over itself. Moreover, the map $\rho : \mathcal{P} \circ \mathcal{P} \longrightarrow$* $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ *is a cofibration of $\mathcal{P}$-bimodules. In particular,* $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ *is also a cofibration resolution of $\mathcal{P}$ when regarded as a pointed $\mathcal{P}$-bimodule.*

Proof. The first statement is an analogue of [[48], Corollary 3.7.6]. We now recall his arguments. It suffices to show that the latching map $\mathrm{L}_{n-1} \mathrm{H}_\bullet \mathcal{P} \longrightarrow \mathrm{H}_n \mathcal{P}$ is a cofibration for every $n \geqslant 0$. The case in which $n = 0$ is clear since $\mathrm{H}_0 \mathcal{P} = \mathcal{P} \circ \mathcal{P}$ is cofibrant as a $\mathcal{P}$-bimodule. When $n \geqslant 1$, the object $\mathrm{L}_{n-1} \mathrm{H}_\bullet \mathcal{P}$ can be written as $\mathrm{L}_{n-1} \mathrm{H}_\bullet \mathcal{P} = \mathcal{P} \circ K_{n-1} \circ \mathcal{P}$ where K_{n-1} is the object of [[48], Lemma 3.7.8]. Moreover, there is a canonical map $k_n : K_{n-1} \longrightarrow \mathcal{P}^{\circ n}$ such that the map $\mathrm{L}_{n-1} \mathrm{H}_\bullet \mathcal{P} \longrightarrow \mathrm{H}_n \mathcal{P}$ agrees with the free $\mathcal{P}$-bimodule map generated by k_n. It therefore suffices to show that k_n is a cofibration of symmetric sequences for every $n \geqslant 1$. This can be done using an inductive argument. Indeed, note that the map k_1 is the same as the unit map $\mathcal{I}_C \longrightarrow \mathcal{P}$, which is a cofibration by the assumption that $\mathcal{P}$ is Σ-cofibrant. Furthermore, by [[48], Lemma 3.7.8] the map k_{n+1} turns out to be the pushout-product of k_n with the unit map $\mathcal{I}_C \longrightarrow \mathcal{P}$. Finally, one will need the fact that, given two maps f and g of symmetric sequences in a sufficiently nice symmetric monoidal model category, the pushout-product of f with g is a cofibration as soon as both of them are one and, in addition, the domain of g is cofibrant (cf. [[52], Lemma 11.5.1]).

To prove that the map ρ is a cofibration, we make use of Lemma 3.3.0.4. Since $\mathcal{P} \circ \mathcal{P}$ is considered as a constant simplicial object, the latching map (3.3.0.2) is simply $\mathrm{L}_{n-1} \mathrm{H}_\bullet \mathcal{P} \longrightarrow \mathrm{H}_n \mathcal{P}$ when $n \geqslant 1$ and the identity map $\mathrm{Id}_{\mathcal{P} \circ \mathcal{P}}$ when $n = 0$. But the first map is a cofibration by the above paragraph. The proof is hence completed. □

Remark 3.3.0.7. Let $\mathcal{P}$ be any simplicial operad and let A be any $\mathcal{P}$-algebra. **Hochschild resolution of** A is the realization of the simplicial $\mathcal{P}$-algebra $\mathrm{H}^{\mathcal{P}}_\bullet A$ with $\mathrm{H}^{\mathcal{P}}_n A = \mathcal{P}^{\circ(n+1)} \circ A$. The augmentation map $\mathrm{diag}(\mathrm{H}^{\mathcal{P}}_\bullet A) \longrightarrow A$ is a weak equivalence by [[48], Corollary 3.7.4], and indeed exhibits $\mathrm{diag}(\mathrm{H}^{\mathcal{P}}_\bullet A)$ as a cofibrant resolution of A. To see this, we will show that the latching map $\mathrm{L}_{n-1} \mathrm{H}^{\mathcal{P}}_\bullet \mathcal{A} \longrightarrow \mathrm{H}^{\mathcal{P}}_n \mathcal{A}$ is a cofibration for every $n \geqslant 0$. The case $n = 0$ is clear since $\mathrm{H}^{\mathcal{P}}_0 \mathcal{A} = \mathcal{P} \circ A$ the free $\mathcal{P}$-algebra generated by A. Let us now pick up the objects K_n's and the maps k_n's mentioned in the above proof. When $n \geqslant 1$ the map $\mathrm{L}_{n-1} \mathrm{H}^{\mathcal{P}}_\bullet \mathcal{A} \longrightarrow \mathrm{H}^{\mathcal{P}}_n \mathcal{A}$ concides with the free $\mathcal{P}$-algebra map generated by $k_n \circ A : K_{n-1} \circ A \longrightarrow \mathcal{P}^{\circ n} \circ A$. It therefore suffices to show that the map $k_n \circ A$ is injective for every $n \geqslant 1$. The key point is that, in the monoidal category of simplicial symmetric sequences, pushout-product of any two injections is again an injection (cf. [[48], Proposition 3.4.5]). By an inductive argument we can show that the maps k_n are injective, and hence the maps $k_n \circ A$ are all injective as well.

Proof of Proposition 3.3.0.3. In the first step, we follow the arguments of [[24], section 5]. By applying the functor E to $\mathrm{H}_\bullet \mathcal{P}$ degreewise, one obtains a simplicial object, $\mathrm{E}\,\mathrm{H}_\bullet \mathcal{P}$, of operads under $\mathcal{P} \sqcup \mathcal{P}$. The realization $\mathrm{diag}(\mathrm{E}\,\mathrm{H}_\bullet \mathcal{P})$ is then an operad under $\mathcal{P} \sqcup \mathcal{P}$. Since $\mathrm{E}(\mathcal{P}) \cong \mathcal{P}$ in $\mathrm{Op}_C(\mathrm{Set}_\Delta)_{\mathcal{P}\sqcup\mathcal{P}/}$, there is a canonical map $\varphi_\mathcal{P} : \mathrm{diag}(\mathrm{E}\,\mathrm{H}_\bullet \mathcal{P}) \longrightarrow \mathcal{P}$ of operads under $\mathcal{P} \sqcup \mathcal{P}$. One observes that there is a canonical isomorphism $\mathrm{diag}(\mathrm{E}\,\mathrm{H}_\bullet \mathcal{P}) \cong \mathrm{E}(\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P}))$ of operads under $\mathcal{P} \sqcup \mathcal{P}$ and over $\mathcal{P}$ (cf. [[24], Proposition 5.3]). Since $\mathrm{E}(\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P}))$ is already a model for $\mathbb{L}\,\mathrm{E}(\mathcal{P})$ by Lemma 3.3.0.6, it just remains to show that the map $\varphi_\mathcal{P} : \mathrm{diag}(\mathrm{E}\,\mathrm{H}_\bullet \mathcal{P}) \longrightarrow \mathcal{P}$ is a weak equivalence of operads. By the diagonal principle, $\varphi_\mathcal{P}$ is a weak equivalence as soon as the map $\mathrm{E}\,\mathrm{H}_n \mathcal{P} \longrightarrow \mathcal{P}$ is one for every $n \geqslant 0$. Moreover, one finds that $\mathrm{E}\,\mathrm{H}_n \mathcal{P} \cong \mathcal{P} \sqcup \mathcal{F}_*(\mathcal{P}^{\circ n}) \sqcup \mathcal{P}$ where

$$\mathcal{F}_* : \mathrm{Coll}_C(\mathrm{Set}_\Delta)_{\mathcal{I}_C/} \longrightarrow \mathrm{Op}_C(\mathrm{Set}_\Delta)$$

refers to the free-operad functor on *pointed C-collections*. This tells us that if $\mathcal{Q} \xrightarrow{\simeq} \mathcal{P}$ is a weak equivalence between cofibrant operads then $\varphi_\mathcal{Q}$ is a weak equivalence if and only if $\varphi_\mathcal{P}$ is one. Applying the diagonal principle in the other direction, we get that $\varphi_\mathcal{P}$ is a weak equivalence as soon as the map $\varphi_{\mathcal{P}_{(n)}}$ is one for every $n \geqslant 0$ (where $\mathcal{P}_{(n)}$ is the operad of n-simplices of $\mathcal{P}$).

Now, consider $\mathcal{P}$ as an $\mathbf{O}_C$-algebra with $\mathbf{O}_C$ being the operad of simplicial C-colored operads. The above remark suggests that we can make use of $\mathcal{P}^c := \mathrm{diag}(\mathrm{H}^{\mathbf{O}_C}_\bullet \mathcal{P})$ as (another) cofibrant model for $\mathcal{P}$. By the first paragraph, it suffices to verify that the map $\varphi_{\mathcal{P}^c}$ is a weak equivalence. Again, the first paragraph tells us that it will suffice to show that the map $\varphi_{\mathcal{P}^c_{(n)}}$ is a weak equivalence for every n. More precisely, we have that

$$\mathcal{P}^c_{(n)} = ((\mathbf{O}_C)^{\circ(n+1)} \circ \mathcal{P})_{(n)} = (\mathbf{O}_C)^{\circ(n+1)} \circ \mathcal{P}_{(n)}.$$

(The second identification is because of the fact that $\mathbf{O}_C$ is a discrete operad). In particular, $\mathcal{P}^c_{(n)}$ is a discrete free $\mathbf{O}_C$-algebra. Note that a free $\mathbf{O}_C$-algebra is the same as the free operad generated by a *free symmetric sequence* (i.e., symmetrization of a nonsymmetric sequence).

By the second paragraph, we can assume without loss of generality that $\mathcal{P}$ is the free operad generated by a discrete free symmetric sequence or alternatively, $\mathcal{P}$ is the symmetrization of a discrete free nonsymmetric operad. Nevertheless, for the remainder we just need to assume that $\mathcal{P}$ is the symmetrization of a nonsymmetric operad $\mathcal{Q}$, i.e., $\mathcal{P} = \mathrm{Sym}(\mathcal{Q})$ (see §2.1.1). Back to the first paragraph, we therefore have to show that the map $\varphi_{\mathrm{Sym}(\mathcal{Q})} : \mathrm{diag}(\mathrm{E}\,\mathrm{H}_\bullet \mathrm{Sym}(\mathcal{Q})) \longrightarrow \mathrm{Sym}(\mathcal{Q})$ is a weak equivalence of operads. This is in fact equivalent to our original problem: proving that $\mathbb{L}\,\mathrm{E}\,\mathrm{Sym}(\mathcal{Q}) \cong \mathrm{Sym}(\mathcal{Q})$. Let us see how it goes.

Observe that the symmetrization functor lifts to the functors $\mathrm{Sym} : \mathrm{BMod}(\mathcal{Q})^* \longrightarrow \mathrm{BMod}(\mathrm{Sym}(\mathcal{Q}))^*$ and $\mathrm{Sym} : \mathrm{nsOp}_C(\mathrm{Set}_\Delta)_{\mathcal{Q}\sqcup\mathcal{Q}/} \longrightarrow \mathrm{Op}_C(\mathrm{Set}_\Delta)_{\mathrm{Sym}(\mathcal{Q})\sqcup\mathrm{Sym}(\mathcal{Q})/}$, which are left adjoints to the associated forgetful functors. Moreover, we have a commutative square of left Quillen functors

$$\begin{array}{ccc}
\mathrm{BMod}(\mathcal{Q})^* & \xrightarrow{\ \mathrm{E}\ } & \mathrm{nsOp}_C(\mathrm{Set}_\Delta)_{\mathcal{Q}\sqcup\mathcal{Q}/} \\
{\scriptstyle \mathrm{Sym}}\downarrow & & \downarrow{\scriptstyle \mathrm{Sym}} \\
\mathrm{BMod}(\mathrm{Sym}(\mathcal{Q}))^* & \xrightarrow[\ \mathrm{E}\ ]{} & \mathrm{Op}_C(\mathrm{Set}_\Delta)_{\mathrm{Sym}(\mathcal{Q})\sqcup\mathrm{Sym}(\mathcal{Q})/}
\end{array}$$

(for this, it suffices to verify the commutativity of the associated square of right adjoints). According to [[24], Proposition 5.4], we have that $\mathbb{L}\,\mathrm{E}\,\mathcal{Q} \cong \mathcal{Q}$. Thus, by the commutativity of the above square and by the fact that the symmetrization functor preserves weak equivalences, we obtain the expected identification $\mathbb{L}\,\mathrm{E}\,\mathrm{Sym}(\mathcal{Q}) \cong \mathrm{Sym}(\mathcal{Q})$. □

Remark 3.3.0.8. Hochschild resolutions work in the context of simplicial R-modules under a slightly different setting. Let $\mathcal{P} \in \mathrm{Op}_C(\mathrm{sMod}_R)$ be given such that the unit map $\mathcal{I}_C \longrightarrow \mathcal{P}$ is a cofibration of symmetric sequences. Then the composition $\mathcal{P} \circ \mathcal{P} \longrightarrow \mathrm{diag}(\mathrm{H}_\bullet \mathcal{P}) \xrightarrow{\simeq} \mathcal{P}$ exhibits $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ as a cofibrant resolution of $\mathcal{P} \in \mathrm{BMod}(\mathcal{P})^*$. On other hand, let A be a levelwise cofibrant $\mathcal{P}$-algebra. Again, the augmentation map $\mathrm{diag}(\mathrm{H}^{\mathcal{P}}_\bullet A) \xrightarrow{\simeq} A$ exhibits $\mathrm{diag}(\mathrm{H}^{\mathcal{P}}_\bullet A)$ as a cofibrant resolution of A. For the proof, one repeats the arguments given in the proof of Lemma 3.3.0.6 and Remark 3.3.0.7.

Proposition 3.3.0.9. *The category* sMod_R *satisfies the condition (S8) 3.3.0.2. Namely, for every cofibrant object* $\mathcal{P} \in \mathrm{Op}_C(\mathrm{sMod}_R)$*, the left derived functor* $\mathbb{L}\,\mathrm{E} : \mathrm{BMod}(\mathcal{P})^* \longrightarrow \mathrm{Op}_C(\mathrm{sMod}_R)_{\mathcal{P}\sqcup\mathcal{P}/}$ *of* E *sends* $\mathcal{P}$ *to itself* $\mathcal{P}$.

Proof. By the above remark, we can make use of $\mathrm{diag}(\mathrm{H}_\bullet \mathcal{P})$ as a cofibrant resolution of $\mathcal{P} \in \mathrm{BMod}(\mathcal{P})^*$ and $\mathrm{diag}(\mathrm{H}^{\mathbf{O}_C}_\bullet \mathcal{P})$ as (another) cofibrant model for $\mathcal{P} \in \mathrm{Op}_C(\mathrm{sMod}_R)$, (note that since the operad $\mathbf{O}_C$ is discrete Σ-cofibrant, its unit map is indeed a cofibration). We then pick up the first two paragraphs of the proof of Proposition 3.3.0.3. It hence suffices to prove that the map $\varphi_{\mathcal{P}} : \mathrm{diag}(\mathrm{E}\,\mathrm{H}_\bullet \mathcal{P}) \longrightarrow \mathcal{P}$ is a weak equivalence provided that $\mathcal{P}$ is the free operad generated by a discrete free symmetric sequence. Again, this is in fact equivalent to our original problem: proving that $\mathbb{L}\,\mathrm{E}\,\mathcal{P} \cong \mathcal{P}$. Let us see how it goes.

The free-forgetful adjunction $R\{-\} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{sMod}_R : \mathbb{U}$ lifts to a Quillen adjunction

$$R\{-\} : \mathrm{Op}_C(\mathrm{Set}_\Delta) \rightleftarrows \mathrm{Op}_C(\mathrm{sMod}_R) : \mathbb{U}$$

between operads. By the assumption on $\mathcal{P}$, there exists a simplicial operad $\mathcal{Q} \in \mathrm{Op}_C(\mathrm{Set}_\Delta)$ which is the free operad generated by a discrete free symmetric sequence such that $R\{\mathcal{Q}\} = \mathcal{P}$. On other hand, the functor $R\{-\}$ does lift to a left Quillen functor $R^b\{-\} : \mathrm{BMod}(\mathcal{Q})^* \longrightarrow \mathrm{BMod}(\mathcal{P})^*$, which fits into the following commutative square of left Quillen functors

$$\begin{array}{ccc}
\mathrm{BMod}(\mathcal{Q})^* & \xrightarrow{\mathrm{E}} & \mathrm{Op}_C(\mathrm{Set}_\Delta)_{\mathcal{Q}\sqcup\mathcal{Q}/} \\
{\scriptstyle R^b\{-\}}\downarrow & & \downarrow{\scriptstyle R\{-\}} \\
\mathrm{BMod}(\mathcal{P})^* & \xrightarrow[\mathrm{E}]{} & \mathrm{Op}_C(\mathrm{sMod}_R)_{\mathcal{P}\sqcup\mathcal{P}/}
\end{array}$$

Note that both the functors $R^b\{-\}$ and $R\{-\}$ preserve weak equivalences. Now, by the commutativity of the square we have that

$$\mathbb{L}R\{\mathbb{L}\,\mathrm{E}(\mathcal{Q})\} \cong \mathbb{L}\,\mathrm{E}(\mathbb{L}R^b\{\mathcal{Q}\}) = \mathbb{L}\,\mathrm{E}(\mathcal{P}).$$

On other hand, Proposition 3.3.0.3 proves that $\mathbb{L}\,\mathrm{E}(\mathcal{Q}) \cong \mathcal{Q}$, and hence we get $\mathbb{L}\,\mathrm{E}(\mathcal{P}) \cong \mathbb{L}R\{\mathcal{Q}\} \cong \mathcal{P}$ as expected. □

3.4 Cotangent complex and Quillen cohomology of enriched operads

Let $\mathcal{P}$ be an $\mathcal{S}$-enriched C-colored operad.

Notations 3.4.0.1. To avoid confusion, in the remainder of this chapter, we will use the symbol $\bigsqcup$ (resp. $\bigsqcup_b$, $\bigsqcup_{pb}$, $\bigsqcup_c$) standing for the coproduct operation in the category $\mathrm{Op}(\mathcal{S})$ (resp. $\mathrm{BMod}(\mathcal{P})$, $\mathrm{BMod}(\mathcal{P})^*$, $\mathrm{Op}_C(\mathcal{S})$).

Notations 3.4.0.2. We let $\mathrm{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ and $\mathrm{L}^b_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$, respectively, denote the **cotangent complexes of** $\mathcal{P}$ when regarded as an object of $\mathrm{Op}(\mathcal{S})$ and $\mathrm{BMod}(\mathcal{P})$. Besides that, we denote by $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S})$ the cotangent complex of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}_C(\mathcal{S})$ and refer to it as the **reduced cotangent complex of** $\mathcal{P}$.

Conventions 3.4.0.3. From now on, by **Quillen cohomology of** $\mathcal{P}$ we will mean the Quillen cohomology of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}(\mathcal{S})$, which is therefore classified by $\mathrm{L}_{\mathcal{P}}$. On other hand, by **reduced Quillen cohomology of** $\mathcal{P}$ we will mean the Quillen cohomology of $\mathcal{P}$ when regarded as an object of $\mathrm{Op}_C(\mathcal{S})$, which is classified by $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}$.

By Theorem 3.2.4.1, when $\mathcal{P}$ is fibrant and Σ-cofibrant, we have a sequence of Quillen equivalences connecting the tangent categories:

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \overset{\simeq}{\underset{\perp}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \overset{\simeq}{\underset{\perp}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \overset{\simeq}{\underset{\perp}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}).$$

Notations 3.4.0.4. We denote by $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \underset{\mathcal{U}^{ib}_{\mathcal{P}}}{\overset{\mathcal{F}^{ib}_{\mathcal{P}}}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ and $\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \underset{\mathcal{U}^{b}_{\mathcal{P}}}{\overset{\mathcal{F}^{b}_{\mathcal{P}}}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ two composed adjunctions taken from the above sequence.

In order to get the desired formula of Quillen cohomology of $\mathcal{P}$, we will compute the derived image of $\mathrm{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ under the composed right Quillen equivalence

$$\mathcal{U}^{ib}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}).$$

As the first step, we will show that the derived image of $\mathrm{L}_{\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ is weakly equivalent to $\mathrm{L}^b_{\mathcal{P}}$, up to a shift. Our work therefore extends [[7], Proposition 3.2.1], but in a different approach. For our approach, the base category $\mathcal{S}$ is technically required to satisfy the extra condition (S8) 3.3.0.2. After having done that first step, it just remains to compute the derived image of $\mathrm{L}^b_{\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$. Let us see how it goes.

As discussed above, we first wish to prove the following.

Proposition 3.4.0.5. *Suppose that* $\mathcal{S}$ *additionally satisfies the condition (S8) 3.3.0.2 and that* $\mathcal{P}$ *is fibrant and* Σ*-cofibrant. Then the left Quillen equivalence* $\mathcal{F}^b_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ *identifies* $\mathrm{L}^b_{\mathcal{P}}$ *to* $\mathrm{L}_{\mathcal{P}}[1]$ *(see Notations 3.4.0.2). Alternatively, the right Quillen equivalence* $\mathcal{U}^b_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ *identifies* $\mathrm{L}_{\mathcal{P}}$ *to* $\mathrm{L}^b_{\mathcal{P}}[-1]$.

The proof of Proposition 3.4.0.5 will require several technical lemmas.

Given two $\mathcal{S}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is by definition the $\mathcal{S}$-enriched category whose set of objects is $\mathrm{Ob}(\mathcal{C} \otimes \mathcal{D}) := \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and such that for every $c, c' \in \mathrm{Ob}(\mathcal{C})$ and $d, d' \in \mathrm{Ob}(\mathcal{D})$ we have

$$\mathrm{Map}_{\mathcal{C}\otimes\mathcal{D}}((c,d),(c',d')) := \mathrm{Map}_{\mathcal{C}}(c,c') \otimes \mathrm{Map}_{\mathcal{D}}(d,d').$$

Recall that the category of $\mathcal{C}$-bimodules is isomorphic to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathcal{S})$ the category of $\mathcal{S}$-valued enriched functors on $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}$. Under this identification, the functor $\mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathcal{S}, (x,y) \mapsto \mathrm{Map}_{\mathcal{C}}(x,y)$ is exactly $\mathcal{C}$ viewed as a bimodule over itself.

Lemma 3.4.0.6. *Let $\mathcal{C} \in \mathrm{Op}(\mathcal{S})$ be a fibrant and levelwise cofibrant operad concentrated in arity 1. Then there is a weak equivalence $\theta_{\mathcal{C}} : \mathcal{F}^b_{\mathcal{C}}(\mathrm{L}^b_{\mathcal{C}}) \xrightarrow{\simeq} \mathrm{L}_{\mathcal{C}}[1]$ in $\mathcal{T}_{\mathcal{C}}\,\mathrm{Op}(\mathcal{S})$.*

Proof. We also regard $\mathcal{C}$ as an $\mathcal{S}$-enriched category. The proof is then straightforward by observing that the category $\mathrm{Cat}(\mathcal{S})$ is already a "neighborhood" of $\mathcal{C}$ in $\mathrm{Op}(\mathcal{S})$. This idea is expressed as follows. There is a commutative square of left Quillen functors

$$\begin{array}{ccc} \mathcal{T}_{\mathrm{Map}_{\mathcal{C}}}\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathcal{S}) & \longrightarrow & \mathcal{T}_{\mathcal{C}}\,\mathrm{BMod}(\mathcal{C}) \\ \big\downarrow {\scriptstyle \mathcal{F}^b_{\mathrm{Map}_{\mathcal{C}}}} & & \big\downarrow {\scriptstyle \mathcal{F}^b_{\mathcal{C}}} \\ \mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathcal{S}) & \longrightarrow & \mathcal{T}_{\mathcal{C}}\,\mathrm{Op}(\mathcal{S}) \end{array}$$

The horizontal functors are the obvious embedding functors, which clearly preserve cotangent complexes, while the functor $\mathcal{F}^b_{\mathrm{Map}_{\mathcal{C}}}$ is the left Quillen equivalence appearing in [[7], Theorem 3.1.14]. According to Proposition 3.2.1 of loc.cit, there is a weak equivalence $\overline{\theta}_{\mathcal{C}} : \mathcal{F}^b_{\mathrm{Map}_{\mathcal{C}}}(\mathrm{L}_{\mathrm{Map}_{\mathcal{C}}}) \xrightarrow{\simeq} \mathrm{L}_{\mathcal{C}}[1]$ in $\mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathcal{S})$. Finally, the expected weak equivalence $\theta_{\mathcal{C}}$ is given by the image of $\overline{\theta}_{\mathcal{C}}$ under the embedding functor $\mathcal{T}_{\mathcal{C}}\,\mathrm{Cat}(\mathcal{S}) \longrightarrow \mathcal{T}_{\mathcal{C}}\,\mathrm{Op}(\mathcal{S})$. □

In what follows, we consider the case where $\mathcal{C} = \mathcal{I}_C$ the initial C-colored operad and describe the weak equivalence $\theta_{\mathcal{I}_C} : \mathcal{F}^b_{\mathcal{I}_C}(\mathrm{L}^b_{\mathcal{I}_C}) \xrightarrow{\simeq} \mathrm{L}_{\mathcal{I}_C}[1]$ of the above lemma.

Let us pick up several notations of [[7], §3.2]. We denote by $*$ the category which has a single object whose endomorphism object is $1_{\mathcal{S}}$. Moreover, let $[1]_{\mathcal{S}}$ denote the category with objects $0, 1$ and mapping spaces $\mathrm{Map}_{[1]_{\mathcal{S}}}(0,1) = 1_{\mathcal{S}}, \mathrm{Map}_{[1]_{\mathcal{S}}}(1,0) = \varnothing$ and $\mathrm{Map}_{[1]_{\mathcal{S}}}(0,0) = \mathrm{Map}_{[1]_{\mathcal{S}}}(1,1) = 1_{\mathcal{S}}$. Localizing $[1]_{\mathcal{S}}$ at the unique non-trivial morphism $0 \to 1$ gives us the category $[1]^{\sim}_{\mathcal{S}}$, which is the same as $[1]_{\mathcal{S}}$ except that $\mathrm{Map}_{[1]^{\sim}_{\mathcal{S}}}(1,0) = 1_{\mathcal{S}}$. By construction, the canonical map $[1]^{\sim}_{\mathcal{S}} \longrightarrow *$ is a weak equivalence.

Take a factorization $[1]_{\mathcal{S}} \longrightarrow \mathcal{E} \xrightarrow{\simeq} [1]^{\sim}_{\mathcal{S}}$ of the canonical map $[1]_{\mathcal{S}} \longrightarrow [1]^{\sim}_{\mathcal{S}}$ into a cofibration followed by a trivial fibration. We now obtain a sequence of maps $* \bigsqcup * \longrightarrow [1]_{\mathcal{S}} \longrightarrow \mathcal{E} \xrightarrow{\simeq} [1]^{\sim}_{\mathcal{S}} \xrightarrow{\simeq} *$ such that the first two maps are cofibrations, while the others are weak equivalences. Tensoring with $\mathcal{I}_C$ (viewed as an $\mathcal{S}$-enriched category) produces a sequence of maps in $\mathrm{Cat}(\mathcal{S})$

$$\mathcal{I}_C \bigsqcup \mathcal{I}_C \longrightarrow \mathcal{I}_C \otimes [1]_{\mathcal{S}} \longrightarrow \mathcal{I}_C \otimes \mathcal{E} \xrightarrow{\simeq} \mathcal{I}_C \otimes [1]^{\sim}_{\mathcal{S}} \xrightarrow{\simeq} \mathcal{I}_C.$$

The last two maps are again weak equivalences, while the others are again cofibrations because $\mathcal{I}_C$ is discrete.

By the above words, the pushout $\mathcal{I}_C \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}$ is a cofibrant model for $\mathcal{I}_C \overset{\mathrm{h}}{\bigsqcup}_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \in \mathrm{Op}(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C}$, and hence $\mathrm{L}_{\mathcal{I}_C}[1]$ is given by

$$\mathrm{L}_{\mathcal{I}_C}[1] \overset{\mathrm{def}}{=} \Sigma^{\infty}(\mathcal{I}_C \overset{\mathrm{h}}{\bigsqcup}_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C) \simeq \Sigma^{\infty}(\mathcal{I}_C \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}).$$

On the other hand, by definition $\mathrm{L}^b_{\mathcal{I}_C}$ is given by $\Sigma^{\infty}(\mathcal{I}_C \bigsqcup_b \mathcal{I}_C)$. Furthermore, note that the category $\mathrm{BMod}(\mathcal{I}_C)$ is isomorphic to $\mathrm{Coll}_C(\mathcal{S})$ the category of C-collections. So we find that $\mathcal{F}^b_{\mathcal{I}_C}(\mathrm{L}^b_{\mathcal{I}_C}) = \Sigma^{\infty}(\mathrm{Fr}(\mathcal{I}_C))$ where Fr is the free functor $\mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathrm{Op}_C(\mathcal{S})$ and $\mathrm{Fr}(\mathcal{I}_C)$ is regarded as an object in $\mathrm{Op}(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C}$. In fact, $\mathrm{Fr}(\mathcal{I}_C)$ is the same as the pushout $\mathcal{I}_C \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}}$. Finally, we find the expected map

$$\theta_{\mathcal{I}_C} : \mathcal{F}^b_{\mathcal{I}_C}(\mathrm{L}^b_{\mathcal{I}_C}) = \Sigma^{\infty}\Big(\mathcal{I}_C \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}}\Big) \xrightarrow{\simeq} \Sigma^{\infty}\Big(\mathcal{I}_C \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}\Big) = \mathrm{L}_{\mathcal{I}_C}[1] \qquad (3.4.0.1)$$

canonically induced by the map $[1]_{\mathcal{S}} \longrightarrow \mathcal{E}$.

Consider $\mu : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ as a map in $\mathrm{BMod}(\mathcal{P})$ and the unit map $\eta : \mathcal{I}_C \longrightarrow \mathcal{P}$ of $\mathcal{P}$ as a map in $\mathrm{Op}(\mathcal{S})$. Recall by notation that $\Sigma^\infty_+(\mu)$ is the image of μ under the left Quillen functor $\Sigma^\infty_+ : \mathrm{BMod}(\mathcal{P})_{/\mathcal{P}} \longrightarrow \mathcal{T}_\mathcal{P}\,\mathrm{BMod}(\mathcal{P})$. Note that $\Sigma^\infty_+(\mu)$ has already the right type since $\mathcal{P} \circ \mathcal{P} \in \mathrm{BMod}(\mathcal{P})$ is cofibrant. Also, $\Sigma^\infty_+(\eta)$ is the image of η under the left Quillen functor $\Sigma^\infty_+ : \mathrm{Op}(\mathcal{S})_{/\mathcal{P}} \longrightarrow \mathcal{T}_\mathcal{P}\,\mathrm{Op}(\mathcal{S})$, and has already the right type as well.

Lemma 3.4.0.7. *There is a weak equivalence $\mathcal{F}^b_\mathcal{P}(\Sigma^\infty_+(\mu)) \xrightarrow{\simeq} \Sigma^\infty_+(\eta)[1]$ in $\mathcal{T}_\mathcal{P}\,\mathrm{Op}(\mathcal{S})$.*

Proof. The map $\eta : \mathcal{I}_C \longrightarrow \mathcal{P}$ gives rise to a commutative square of left Quillen functors

$$\begin{array}{ccc} \mathcal{T}_{\mathcal{I}_C}\,\mathrm{BMod}(\mathcal{I}_C) & \xrightarrow{\mathcal{F}^b_{\mathcal{I}_C}} & \mathcal{T}_{\mathcal{I}_C}\,\mathrm{Op}(\mathcal{S}) \\ \Big\downarrow{\scriptstyle \eta^b_!} & & \Big\downarrow{\scriptstyle \eta^{op}_!} \\ \mathcal{T}_\mathcal{P}\,\mathrm{BMod}(\mathcal{P}) & \xrightarrow[\mathcal{F}^b_\mathcal{P}]{} & \mathcal{T}_\mathcal{P}\,\mathrm{Op}(\mathcal{S}) \end{array} \tag{3.4.0.2}$$

Let us start with the cotangent complex $\mathrm{L}^b_{\mathcal{I}_C} \in \mathcal{T}_{\mathcal{I}_C}\,\mathrm{BMod}(\mathcal{I}_C)$ of $\mathcal{I}_C \in \mathrm{BMod}(\mathcal{I}_C)$. Note that the functor $\mathrm{BMod}(\mathcal{I}_C) \longrightarrow \mathrm{BMod}(\mathcal{P})$ agrees with the free $\mathcal{P}$-bimodule functor $\mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathrm{BMod}(\mathcal{P})$, which in particular takes $\mathcal{I}_C$ to $\mathcal{P} \circ \mathcal{P}$. Due to this, we find that the functor $\eta^b_!$ sends $\mathrm{L}^b_{\mathcal{I}_C} = \Sigma^\infty(\mathcal{I}_C \coprod_b \mathcal{I}_C)$ to $\Sigma^\infty(\mathcal{P} \coprod_b (\mathcal{P} \circ \mathcal{P})) \in \mathcal{T}_\mathcal{P}\,\mathrm{BMod}(\mathcal{P})$, which is exactly $\Sigma^\infty_+(\mu)$. Thus, the commutativity of (3.4.0.2) shows that $\mathcal{F}^b_\mathcal{P}(\Sigma^\infty_+(\mu)) = \eta^{op}_! \mathcal{F}^b_{\mathcal{I}_C}(\mathrm{L}^b_{\mathcal{I}_C})$. On other hand, by the words after Definition 2.2.0.12 we have $\Sigma^\infty_+(\eta)[1] = \eta^{op}_!(\mathrm{L}_{\mathcal{I}_C}[1])$. Using $\Sigma^\infty(\mathcal{I}_C \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E})$ as a cofibrant model for $\mathrm{L}_{\mathcal{I}_C}[1]$ as discussed above, we find the desired weak equivalence given by

$$\mathcal{F}^b_\mathcal{P}(\Sigma^\infty_+(\mu)) = \eta^{op}_! \mathcal{F}^b_{\mathcal{I}_C}(\mathrm{L}^b_{\mathcal{I}_C}) \xrightarrow[\simeq]{\eta^{op}_!(\theta_{\mathcal{I}_C})} \eta^{op}_!(\mathrm{L}_{\mathcal{I}_C}[1]) = \Sigma^\infty_+(\eta)[1]$$

where $\theta_{\mathcal{I}_C}$ is the weak equivalence (3.4.0.1). □

Remark 3.4.0.8. It is necessary to give an explicit description of the map $\eta^{op}_!(\theta_{\mathcal{I}_C})$. Concretely, we have that

$$\mathcal{F}^b_\mathcal{P}(\Sigma^\infty_+(\mu)) = \eta^{op}_! \mathcal{F}^b_{\mathcal{I}_C}(\mathrm{L}^b_{\mathcal{I}_C}) = \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C} \mathrm{Fr}(\mathcal{I}_C)\right) = \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_\mathcal{S}\right).$$

On the other hand,

$$\Sigma^\infty_+(\eta)[1] = \eta^{op}_!(\mathrm{L}_{\mathcal{I}_C}[1]) = \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C} (\mathcal{I}_C \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E})\right) = \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}\right).$$

We now find the map $\eta^{op}_!(\theta_{\mathcal{I}_C})$ given by applying Σ^∞ to the map

$$\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_\mathcal{S} \longrightarrow \mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}$$

canonically induced by the map $[1]_\mathcal{S} \longrightarrow \mathcal{E}$.

Consider the Quillen adjunction $\mathcal{L}^{\mathrm{Sp}}_\mathcal{P} : \mathcal{T}_\mathcal{P}\,\mathrm{Op}_C(\mathcal{S}) \rightleftarrows \mathcal{T}_\mathcal{P}\,\mathrm{Op}(\mathcal{S}) : \mathcal{R}^{\mathrm{Sp}}_\mathcal{P}$.

Lemma 3.4.0.9. *The left Quillen functor $\mathcal{L}^{\mathrm{Sp}}_\mathcal{P}$ takes $\mathrm{L}^{\mathrm{red}}_\mathcal{P} \in \mathcal{T}_\mathcal{P}\,\mathrm{Op}_C(\mathcal{S})$ to $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$ the relative cotangent complex of the unit map $\eta : \mathcal{I}_C \longrightarrow \mathcal{P}$ (cf. § 2.2).*

Proof. Let $\mathcal{P}^c \xrightarrow{\simeq} \mathcal{P}$ be a cofibrant resolution of $\mathcal{P}$ in $\mathrm{Op}_C(\mathcal{S})$. So we get a factorization $\mathcal{I}_C \longrightarrow \mathcal{P}^c \xrightarrow{\simeq} \mathcal{P}$ in $\mathrm{Op}(\mathcal{S})$ of the map η into a cofibration followed by a weak equivalence. By Remark 2.2.0.13 we have that $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C} = \Sigma^\infty(\mathcal{P} \bigsqcup_{\mathcal{I}_C} \mathcal{P}^c)$. On other hand, by definition the reduced cotangent complex $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}} \mathrm{Op}_C(\mathcal{S})$ is given by $\Sigma^\infty(\mathcal{P} \bigsqcup_c \mathcal{P}^c)$ the suspension spectrum of $\mathcal{P} \bigsqcup_c \mathcal{P}^c \in \mathrm{Op}_C(\mathcal{S})$ considered as a C-colored operad over and under $\mathcal{P}$.

Observe now that since $\mathcal{I}_C$ is initial object in $\mathrm{Op}_C(\mathcal{S})$, the coproduct $\mathcal{P} \bigsqcup_c \mathcal{P}^c$ is nothing but $\mathcal{P} \bigsqcup_{\mathcal{I}_C} \mathcal{P}^c$ when regarded as an object in $\mathrm{Op}(\mathcal{S})$. We hence get the conclusion. □

Suppose that $\mathcal{P}$ is cofibrant. Then $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$ is simply given by $\Sigma^\infty(\mathcal{P} \bigsqcup_{\mathcal{I}_C} \mathcal{P})$. Nevertheless, we will need two more models for this. Let $\mathcal{R}$ and $\mathcal{R}'$ be in $\mathrm{Op}(\mathcal{S})$ with

$$\mathcal{R} = (\mathcal{P} \bigsqcup \mathcal{I}_C) \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}} \;\; , \;\; \mathcal{R}' = (\mathcal{P} \bigsqcup \mathcal{I}_C) \bigsqcup_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}.$$

We then form a diagram of coCartesian squares in $\mathrm{Op}(\mathcal{S})$ as follows

$$\begin{array}{ccccc}
\mathcal{I}_C & \longrightarrow & \mathcal{P} & & \\
{\scriptstyle i_0}\downarrow & & \downarrow & & \\
\mathcal{I}_C \bigsqcup \mathcal{I}_C & \longrightarrow & \mathcal{P} \bigsqcup \mathcal{I}_C & \longrightarrow & \mathcal{P} \bigsqcup \mathcal{P} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I}_C \otimes [1]_{\mathcal{S}} & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{R} \bigsqcup_{\mathcal{I}_C} \mathcal{P} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I}_C \otimes \mathcal{E} & \longrightarrow & \mathcal{R}' & \longrightarrow & \mathcal{R}' \bigsqcup_{\mathcal{I}_C} \mathcal{P} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\mathcal{I}_C & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P} \bigsqcup_{\mathcal{I}_C} \mathcal{P}
\end{array} \tag{3.4.0.3}$$

The middle column is nothing but the image of the first column through the left adjoint functor $\mathrm{Op}(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C} \longrightarrow \mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$ induced by the unit map $\eta : \mathcal{I}_C \longrightarrow \mathcal{P}$. We consider $\mathcal{R}$ and $\mathcal{R}'$ as objects in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$ via that way. Three of squares on the right hand side are considered as coCartesian squares in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$. Moreover, note that all the arrows in this diagram are cofibrations, except the three bottom vertical maps, which are all weak equivalences (the last two ones are homotopy cobase change of the weak equivalence $\mathcal{I}_C \otimes \mathcal{E} \xrightarrow{\simeq} \mathcal{I}_C$).

Lemma 3.4.0.10. *The map $\mathcal{R} \longrightarrow \mathcal{R}'$ induces a weak equivalence $\theta_{\mathcal{P}/\mathcal{I}_C} : \Sigma^\infty(\mathcal{R} \bigsqcup_{\mathcal{I}_C} \mathcal{P}) \xrightarrow{\simeq} \Sigma^\infty(\mathcal{R}' \bigsqcup_{\mathcal{I}_C} \mathcal{P})$ of spectrum objects in $\mathcal{T}_{\mathcal{P}} \mathrm{Op}(\mathcal{S})$. Moreover, the two are both weakly equivalent to the relative cotangent complex $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$.*

Proof. By the weak equivalence $\mathcal{R}' \bigsqcup_{\mathcal{I}_C} \mathcal{P} \xrightarrow{\simeq} \mathcal{P} \bigsqcup_{\mathcal{I}_C} \mathcal{P}$ of objects in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$ as indicated above, the suspension spectrum $\Sigma^\infty(\mathcal{R}' \bigsqcup_{\mathcal{I}_C} \mathcal{P})$ is indeed a model for $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$. It remains to show that $\theta_{\mathcal{P}/\mathcal{I}_C}$ is a weak equivalence. For this, we will follow the arguments given in the proof of [[7], Proposition 3.2.1]. It hence suffices to show that the map $\mathcal{R}\coprod_{\mathcal{I}_C}\mathcal{P} \longrightarrow \mathcal{R}'\coprod_{\mathcal{I}_C}\mathcal{P}$ is (-1)-cotruncated in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$. Since the latter is homotopy cobase change in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$ of the map $\mathcal{R} \longrightarrow \mathcal{R}'$, it now suffices to show that $\mathcal{R} \longrightarrow \mathcal{R}'$ is (-1)-cotruncated in $\mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$. Furthermore, since the map $\mathcal{R} \longrightarrow \mathcal{R}'$ agrees with the image of the map $\mathcal{I}_C \otimes [1]_{\mathcal{S}} \longrightarrow \mathcal{I}_C \otimes \mathcal{E}$ through the left Quillen functor $\mathrm{Op}(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C} \longrightarrow \mathrm{Op}(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$, the proof will be hence completed after showing that the latter map is a (-1)-cotruncated map (between cofibrant objects) in $\mathrm{Op}(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C}$. For this, it suffices to show that the fold map

$$\mathcal{I}_C \otimes \mathcal{E} \coprod_{\mathcal{I}_C \otimes [1]_{\mathcal{S}}} \mathcal{I}_C \otimes \mathcal{E} \longrightarrow \mathcal{I}_C \otimes \mathcal{E}$$

is a weak equivalence in $\mathrm{Op}(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C}$ or in $\mathrm{Cat}(\mathcal{S})$, alternatively. This follows by the fact that the fold map $\mathcal{E} \coprod_{[1]_{\mathcal{S}}} \mathcal{E} \longrightarrow \mathcal{E}$ is a weak equivalence in $\mathrm{Cat}(\mathcal{S})$, due to the convention that $\mathcal{S}$ satisfies the invertibility hypothesis (cf. § 3.1). □

We will denote by $\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ the relative cotangent complex of the map $\mu : \mathcal{P}\circ\mathcal{P} \longrightarrow \mathcal{P}$ regarded as a map in $\mathrm{BMod}(\mathcal{P})$. Take a factorization $\mathcal{P}\circ\mathcal{P} \longrightarrow \mathcal{P}^{\mathrm{cof}} \xrightarrow{\simeq} \mathcal{P}$ in $\mathrm{BMod}(\mathcal{P})$ of the map μ into a cofibration followed by a weak equivalence. By Remark 2.2.0.13 we have that $\Sigma^\infty(\mathcal{P}\coprod_{\mathcal{P}\circ\mathcal{P}}\mathcal{P}^{\mathrm{cof}})$ is a (cofibrant) model for $\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}}$.

Lemma 3.4.0.11. *Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 3.3.0.2 and that $\mathcal{P}$ is cofibrant. Then the left Quillen functor $\mathcal{F}^b_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ sends $\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ to $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}[1]$.*

Proof. Let us write $\mathcal{F}^b_{\mathcal{P}}$ as the composed left Quillen functor

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \xrightarrow{\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}).$$

As the first step, we will compute the image of $\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}}$ through the first functor. It will suffice to compute the image of $\mathcal{P}\coprod_{\mathcal{P}\circ\mathcal{P}}\mathcal{P}^{\mathrm{cof}}$ through the left Quillen functor $\mathrm{BMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \longrightarrow \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$, which is the same as the composition

$$\mathrm{BMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \cong \mathrm{BMod}(\mathcal{P})^*_{\mathcal{P}//\mathcal{P}} \xrightarrow{\widetilde{\mathrm{E}}} (\mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\coprod_c\mathcal{P}/})_{\mathcal{P}//\mathcal{P}} \cong \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$$

where $\widetilde{\mathrm{E}}$ is the one lifted by the functor $\mathrm{E} : \mathrm{BMod}(\mathcal{P})^* \longrightarrow \mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\coprod_c\mathcal{P}/}$ (see § 3.3). Note that the pushout $\mathcal{P}\coprod_{\mathcal{P}\circ\mathcal{P}}\mathcal{P}^{\mathrm{cof}}$, when regarded as an object in $\mathrm{BMod}(\mathcal{P})^*$, is exactly $\mathcal{P}\coprod_{pb}\mathcal{P}^{\mathrm{cof}}$ the coproduct of $\mathcal{P}$ with $\mathcal{P}^{\mathrm{cof}}$ as pointed $\mathcal{P}$-bimodules. Writing $\mathrm{E}(\mathcal{P}^{\mathrm{cof}}) := (\mathcal{P}\coprod_c\mathcal{P} \longrightarrow \mathcal{Q})$, we then have that the underlying C-colored operad of $\mathrm{E}(\mathcal{P}\coprod_{pb}\mathcal{P}^{\mathrm{cof}})$ is given by the pushout $\mathcal{Q}\coprod_{\mathcal{P}\coprod_c\mathcal{P}}\mathcal{P}$. The condition (S8) 3.3.0.2 implies that $\mathcal{Q}$ is a cofibrant resolution of $\mathcal{P}$ when considered as an object in $\mathrm{Op}_C(\mathcal{S})_{\mathcal{P}\coprod_c\mathcal{P}/}$ and hence, the latter pushout is a model for $\Sigma(\mathcal{P}\coprod_c\mathcal{P})$ the suspension of $\mathcal{P}\coprod_c\mathcal{P}$ considered as an object in $\mathrm{Op}_C(\mathcal{S})_{\mathcal{P}//\mathcal{P}}$. So we find that the derived image of $\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S})$ is given by $\Sigma^\infty(\Sigma(\mathcal{P}\coprod_c\mathcal{P}))$, which is nothing but $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}[1]$ the suspension of $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}$.

Finally, we deduce by using Lemma 3.4.0.9, which proves that the derived image of $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}[1]$ through the functor $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \longrightarrow \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ is exactly $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}[1]$. □

Proof of Proposition 3.4.0.5. We can take $f : \mathcal{Q} \xrightarrow{\simeq} \mathcal{P}$ to be a bifibrant resolution of $\mathcal{P}$ in $\mathrm{Op}(\mathcal{S})$ such that f is a map in $\mathrm{Op}_C(\mathcal{S})$ (cf. Observations 3.2.1.1). The map f gives rise to a commutative square of left Quillen equivalences

$$\begin{array}{ccc} \mathcal{T}_{\mathcal{Q}}\,\mathrm{BMod}(\mathcal{Q}) & \xrightarrow[\simeq]{\mathcal{F}^b_{\mathcal{Q}}} & \mathcal{T}_{\mathcal{Q}}\,\mathrm{Op}(\mathcal{S}) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P}) & \xrightarrow[\mathcal{F}^b_{\mathcal{P}}]{\simeq} & \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S}) \end{array} \tag{3.4.0.4}$$

(cf. Proposition 3.2.2.2). It is then not hard to see that the vertical functors preserve cotangent complexes. Therefore, if the statement holds for $\mathcal{Q}$ then it holds for $\mathcal{P}$ as well. So we can assume without loss of generality that $\mathcal{P}$ is bifibrant.

By the definition of relative cotangent complex, we have two cofiber sequences in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$:

$$\mathcal{F}^b_{\mathcal{P}}(\Sigma^\infty_+(\mu)) \longrightarrow \mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}}) \longrightarrow \mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}}) \;\;,\;\; \Sigma^\infty_+(\eta) \longrightarrow \mathrm{L}_{\mathcal{P}} \longrightarrow \mathrm{L}_{\mathcal{P}/\mathcal{I}_C}\,.$$

Let us consider the corresponding shifted cofiber sequences:

$$\mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}})[-1] \xrightarrow{\alpha} \mathcal{F}^b_{\mathcal{P}}(\Sigma^\infty_+(\mu)) \longrightarrow \mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}}) \;\;,\;\; \mathrm{L}_{\mathcal{P}/\mathcal{I}_C} \xrightarrow{\beta} \Sigma^\infty_+(\eta)[1] \longrightarrow \mathrm{L}_{\mathcal{P}}[1].$$

The map α is described as follows. From Remark 3.4.0.8 we have that

$$\mathcal{F}^b_{\mathcal{P}}(\Sigma^\infty_+(\mu)) = \Sigma^\infty(\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}}).$$

On other hand, by Lemma 3.4.0.11 we have an identification $\mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}}) \simeq \mathrm{L}_{\mathcal{P}/\mathcal{I}_C}[1]$ and hence, $\mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}})[-1]$ is identified to $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$. By Lemma 3.4.0.10 we have two models for $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$ given by $\Sigma^\infty(\mathcal{R} \coprod_{\mathcal{I}_C} \mathcal{P})$ and $\Sigma^\infty(\mathcal{R}' \coprod_{\mathcal{I}_C} \mathcal{P})$. We use the first one as a model for $\mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}})[-1]$, i.e.,

$$\mathcal{F}^b_{\mathcal{P}}(\mathrm{L}^b_{\mathcal{P}/\mathcal{P}\circ\mathcal{P}})[-1] = \Sigma^\infty(\mathcal{R}\coprod_{\mathcal{I}_C}\mathcal{P}) = \Sigma^\infty\left((\mathcal{P} \coprod \mathcal{P}) \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}}\right).$$

Under these identifications, α is given by the map of suspension spectra

$$\alpha : \Sigma^\infty\left((\mathcal{P} \coprod \mathcal{P}) \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}}\right) \longrightarrow \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes [1]_{\mathcal{S}}\right)$$

canonically induced by the fold map $\mathcal{P} \coprod \mathcal{P} \longrightarrow \mathcal{P}$.

Again by Remark 3.4.0.8, we have $\Sigma^\infty_+(\eta)[1] = \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}\right)$. Furthermore, when using $\Sigma^\infty(\mathcal{R}' \coprod_{\mathcal{I}_C} \mathcal{P}) = \Sigma^\infty\left((\mathcal{P} \sqcup \mathcal{P}) \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}\right)$ as a model for $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}$, then by construction β is given by the map of suspension spectra

$$\beta : \Sigma^\infty\left((\mathcal{P} \coprod \mathcal{P}) \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}\right) \longrightarrow \Sigma^\infty\left(\mathcal{P} \coprod_{\mathcal{I}_C \sqcup \mathcal{I}_C} \mathcal{I}_C \otimes \mathcal{E}\right)$$

canonically induced by the fold map $\mathcal{P} \coprod \mathcal{P} \longrightarrow \mathcal{P}$ again.

We now obtain a commutative square in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$

$$\begin{array}{ccc}
\Sigma^{\infty}\left((\mathcal{P}\sqcup\mathcal{P})\bigsqcup_{\mathcal{I}_C\sqcup\mathcal{I}_C}\mathcal{I}_C\otimes[1]_{\mathcal{S}}\right) & \xrightarrow{\alpha} & \Sigma^{\infty}\left(\mathcal{P}\bigsqcup_{\mathcal{I}_C\sqcup\mathcal{I}_C}\mathcal{I}_C\otimes[1]_{\mathcal{S}}\right)\\
\theta_{\mathcal{P}/\mathcal{I}_C}\downarrow\simeq & & \simeq\downarrow\eta_!^{op}(\theta_{\mathcal{I}_C})\\
\Sigma^{\infty}\left((\mathcal{P}\sqcup\mathcal{P})\bigsqcup_{\mathcal{I}_C\sqcup\mathcal{I}_C}\mathcal{I}_C\otimes\mathcal{E}\right) & \xrightarrow[\beta]{} & \Sigma^{\infty}\left(\mathcal{P}\bigsqcup_{\mathcal{I}_C\sqcup\mathcal{I}_C}\mathcal{I}_C\otimes\mathcal{E}\right)
\end{array}$$

in which the left and right vertical maps are the weak equivalences of lemmas 3.4.0.10 and 3.4.0.7, respectively. It hence induces a weak equivalence between homotopy cofibers of the maps α and β. We thus obtain a natural weak equivalence $\theta_{\mathcal{P}}:\mathrm{L}^b_{\mathcal{P}}\xrightarrow{\simeq}\mathrm{L}_{\mathcal{P}}[1]$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ as expected. □

In the next step, we will describe the cotangent complex $\mathrm{L}^b_{\mathcal{P}}\in\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$. We will need the following lemma.

Lemma 3.4.0.12. *The forgetful functor* $\mathrm{U}:\mathrm{BMod}(\mathcal{P})\longrightarrow\mathrm{LMod}(\mathcal{P})$ *is a left Quillen functor provided that* $\mathcal{P}$ *is* Σ*-cofibrant.*

Proof. The functor U is given by forgetting the right $\mathcal{P}$-action, which is linear. It implies that U preserves colimits. By the adjoint functor theorem and by the combinatoriality of $\mathcal{S}$, U is indeed a left adjoint.

Since U preserves weak equivalences, the proof will be completed after showing that it preserves cofibrations. To this end, we first prove that every cofibration in $\mathrm{RMod}(\mathcal{P})$ has underlying map in $\mathrm{Coll}_C(\mathcal{S})$ being a cofibration as well. Observe that the model structure on $\mathrm{RMod}(\mathcal{P})$ admits a set of generating cofibrations given by $\{i\circ\mathcal{P}:M\circ\mathcal{P}\longrightarrow N\circ\mathcal{P}\}_i$ where $i:M\to N$ ranges over the set of those of $\mathrm{Coll}_C(\mathcal{S})$. Since the forgetful functor $\mathrm{RMod}(\mathcal{P})\longrightarrow\mathrm{Coll}_C(\mathcal{S})$ is a left adjoint, it suffices to show that each map $i\circ\mathcal{P}:M\circ\mathcal{P}\longrightarrow N\circ\mathcal{P}$ is a cofibration in $\mathrm{Coll}_C(\mathcal{S})$. Let $\varnothing_C$ denote the initial C-collection, which agrees with $\varnothing_{\mathcal{S}}$ on every level. Then, factor the map $i\circ\mathcal{P}$ as

$$M\circ\mathcal{P}\longrightarrow N\circ\varnothing_C\bigsqcup_{M\circ\varnothing_C}M\circ\mathcal{P}\longrightarrow N\circ\mathcal{P}.$$

The map $i\circ\varnothing_C:M\circ\varnothing_C\longrightarrow N\circ\varnothing_C$ agrees with the underlying map of i between the collections of level 0, it is hence a cofibration. So the first map of the above composition is a cofibration. The second map is also a cofibration by [[52], Lemma 11.5.1], along with the Σ-cofibrancy of $\mathcal{P}$. Thus, $i\circ\mathcal{P}$ is indeed a cofibration.

It can be shown that the model structure on $\mathrm{BMod}(\mathcal{P})$ admits a set of generating cofibrations given by $\{\mathcal{P}\circ j:\mathcal{P}\circ K\longrightarrow\mathcal{P}\circ L\}_j$ where $j:K\to L$ ranges over the set of those of $\mathrm{RMod}(\mathcal{P})$. It therefore suffices to show that each map $\mathcal{P}\circ j:\mathcal{P}\circ K\longrightarrow\mathcal{P}\circ L$ is a cofibration in $\mathrm{LMod}(\mathcal{P})$. This is now clear since $\mathcal{P}\circ(-)$ is the free left $\mathcal{P}$-module functor $\mathrm{Coll}_C(\mathcal{S})\longrightarrow\mathrm{LMod}(\mathcal{P})$, which is a left Quillen functor, and since j is a cofibration in $\mathrm{Coll}_C(\mathcal{S})$ as indicated above. □

Suppose that $\mathcal{P}$ is fibrant and Σ-cofibrant. Properly, $\mathrm{L}^b_{\mathcal{P}}$ is given by $\Sigma^{\infty}(\mathcal{P}\bigsqcup_b\mathcal{P}^{\mathrm{cof}})$ with $\mathcal{P}^{\mathrm{cof}}$ being a cofibrant resolution of $\mathcal{P}$ in $\mathrm{BMod}(\mathcal{P})$. But the map $\mathcal{P}\bigsqcup_b\mathcal{P}^{\mathrm{cof}}\longrightarrow\mathcal{P}\bigsqcup_b\mathcal{P}$ is a weak equivalence of $\mathcal{P}$-bimodules due to Lemma 3.4.0.12, so we will exhibit $\Sigma^{\infty}(\mathcal{P}\bigsqcup_b\mathcal{P})$ as a model for $\mathrm{L}^b_{\mathcal{P}}$. According to [[6], Corollary 2.3.3], $\mathrm{L}^b_{\mathcal{P}}=\Sigma^{\infty}(\mathcal{P}\bigsqcup_b\mathcal{P})$ admits a **suspension spectrum replacement** simply given by fixing $\mathcal{P}\bigsqcup_b\mathcal{P}$ as its value at the bidegree $(0,0)$, and hence the value at the bidegree (n,n) is given by

$\Sigma^n(\mathcal{P}\bigsqcup_b\mathcal{P})$ the n-suspension of $\mathcal{P}\bigsqcup_b\mathcal{P}$ in $\mathrm{BMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}$. So $\mathrm{L}^b_{\mathcal{P}}$ is fully determined just by describing the $\mathcal{P}$-bimodule $\Sigma^n(\mathcal{P}\bigsqcup_b\mathcal{P})$ for every $n\geqslant 0$.

Notations 3.4.0.13. For each $n\geqslant 0$, we denote by $\mathrm{S}^n := \Sigma^n(1_{\mathcal{S}}\sqcup 1_{\mathcal{S}})\in\mathcal{S}$ with the suspension $\Sigma(-)$ computed in $\mathcal{S}_{1_{\mathcal{S}}//1_{\mathcal{S}}}$, and refer to S^n as the **pointed n-sphere** in $\mathcal{S}$. Furthermore, we will write S^n_C standing for the C-collection which has $\mathrm{S}^n_C(c;c)=\mathrm{S}^n$ for every $c\in C$ and agrees with $\varnothing_{\mathcal{S}}$ on the other levels.

Computations 3.4.0.14. By Lemma 3.4.0.12, the underlying left $\mathcal{P}$-module of $\Sigma^n(\mathcal{P}\bigsqcup_b\mathcal{P})$ is nothing but $\Sigma^n(\mathcal{P}\bigsqcup_l\mathcal{P})\in\mathrm{LMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}$, in which "$\bigsqcup_l$" refers to the coproduct operation in $\mathrm{LMod}(\mathcal{P})$. The good thing is that $\mathcal{P}$ is free (generated by $\mathcal{I}_C$) as a left module over itself. Thanks to this, we may compute $\Sigma^n(\mathcal{P}\bigsqcup_l\mathcal{P})$ as follows. First, note that $\mathcal{P}\bigsqcup_l\mathcal{P}\in\mathrm{LMod}(\mathcal{P})$ is isomorphic to $\mathcal{P}\circ\mathrm{S}^0_C$ the free left $\mathcal{P}$-module generated by S^0_C. We have further that

$$\Sigma(\mathcal{P}\bigsqcup_l\mathcal{P})\simeq\mathcal{P}\circ\left(\mathcal{I}_C\overset{\mathrm{h}}{\bigsqcup_{\mathrm{S}^0_C}}\mathcal{I}_C\right)\simeq\mathcal{P}\circ\mathrm{S}^1_C\,.$$

Inductively, we find that $\Sigma^n(\mathcal{P}\bigsqcup_l\mathcal{P})\simeq\mathcal{P}\circ\mathrm{S}^n_C$ the free left $\mathcal{P}$-module generated by S^n_C. In particular, for each C-sequence $\overline{c}:=(c_1,\cdots,c_m;c)$, we find that

$$\Sigma^n(\mathcal{P}\bigsqcup_l\mathcal{P})\,(\overline{c})\simeq\mathcal{P}\circ\mathrm{S}^n_C\,(\overline{c})=\mathcal{P}(\overline{c})\otimes(\mathrm{S}^n)^{\otimes m}.$$

Notations 3.4.0.15. We denote by $\widetilde{\mathrm{L}}_{\mathcal{P}}\in\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ the derived image of $\mathrm{L}^b_{\mathcal{P}}\in\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})$ under the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})\xrightarrow{\simeq}\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ (cf. Theorem 3.2.4.1). Furthermore, recall that when $\mathcal{S}$ is in addition stable containing a strict zero object 0, we have a sequence of right Quillen equivalences

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})\underset{\simeq}{\xrightarrow{\Omega^\infty}}\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}\underset{\simeq}{\xrightarrow{\ker}}\mathrm{IbMod}(\mathcal{P})$$

(cf. Theorem 3.2.4.3). In this situation, we will denote by $\overline{\mathrm{L}}_{\mathcal{P}}:=\mathbb{R}(\ker\circ\Omega^\infty)(\widetilde{\mathrm{L}}_{\mathcal{P}})$.

Computations 3.4.0.16. Let us compute $\widetilde{\mathrm{L}}_{\mathcal{P}}$ and $\overline{\mathrm{L}}_{\mathcal{P}}$.

(1) It is not difficult to show that the Quillen adjunction $\mathrm{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}\rightleftarrows\mathrm{BMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}}$ is differentiable (see Definition 3.1.0.1). The [[6], Corollary 2.4.8] hence shows that the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{BMod}(\mathcal{P})\xrightarrow{\simeq}\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ simply sends $\mathrm{L}^b_{\mathcal{P}}$ (which is now identified to its suspension spectrum replacement) to its underlying prespectrum of infinitesimal $\mathcal{P}$-bimodules.

(2) Suppose that $\mathcal{S}$ is in addition stable containing a strict zero object 0. Let us compute $\overline{\mathrm{L}}_{\mathcal{P}}=\mathbb{R}(\ker\circ\Omega^\infty)(\widetilde{\mathrm{L}}_{\mathcal{P}})$. By [[6], Remark 2.4.7], we have $\mathbb{R}\Omega^\infty(\widetilde{\mathrm{L}}_{\mathcal{P}})\simeq\mathrm{hocolim}_n\,\Omega^n(\widetilde{\mathrm{L}}_{\mathcal{P}})_{n,n}$. Now, by Computations 3.4.0.14 we find that

$$\mathbb{R}(\ker\circ\Omega^\infty)(\widetilde{\mathrm{L}}_{\mathcal{P}})\simeq(\mathrm{hocolim}_n\,\Omega^n(\mathcal{P}\circ\mathrm{S}^n_C))\overset{\mathrm{h}}{\times}_{\mathcal{P}}0\simeq\mathrm{hocolim}_n\,\Omega^n[\,(\mathcal{P}\circ\mathrm{S}^n_C)\overset{\mathrm{h}}{\times}_{\mathcal{P}}0)\,].$$

More explicitly, for each $\overline{c}=(c_1,\cdots,c_m;c)$ we find that

$$\mathbb{R}(\ker\circ\Omega^\infty)(\widetilde{\mathrm{L}}_{\mathcal{P}})\,(\overline{c})\simeq\mathrm{hocolim}_n\,\Omega^n[\,(\mathcal{P}(\overline{c})\otimes(\mathrm{S}^n)^{\otimes m})\overset{\mathrm{h}}{\times}_{\mathcal{P}(\overline{c})}0\,]\simeq\mathcal{P}(\overline{c})\otimes\mathrm{hocolim}_n\,\Omega^n[\,(\mathrm{S}^n)^{\otimes m}\overset{\mathrm{h}}{\times}_{1_{\mathcal{S}}}0\,]$$

in which the last desuspension $\Omega(-)$ is now computed in $\mathrm{IbMod}(\mathcal{P})$. (In the above formula, the second weak equivalence is because of the fact that homotopy pullbacks in $\mathcal{S}$ are also homotopy pushouts and that the functor $\mathcal{P}(\overline{c})\otimes(-)$ preserves colimits).

Corollary 3.4.0.17. *Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 3.3.0.2 and that $\mathcal{P}$ is fibrant and Σ-cofibrant. Then the right Quillen equivalence*

$$\mathcal{U}^{ib}_{\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P})$$

identifies the cotangent complex $\mathrm{L}_{\mathcal{P}}$ to $\widetilde{\mathrm{L}}_{\mathcal{P}}[-1] \in \mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P})$ in which $\widetilde{\mathrm{L}}_{\mathcal{P}}$ is the prespectrum with $(\widetilde{\mathrm{L}}_{\mathcal{P}})_{n,n} = \mathcal{P} \circ \mathrm{S}^n_C$ for $n \geq 0$. When $\mathcal{S}$ is in addition stable containing a strict zero object 0 then under the right Quillen equivalence

$$\mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{S}) \xrightarrow{\simeq} \operatorname{IbMod}(\mathcal{P}),$$

the cotangent complex $\mathrm{L}_{\mathcal{P}}$ is identified to $\overline{\mathrm{L}}_{\mathcal{P}}[-1]$ with $\overline{\mathrm{L}}_{\mathcal{P}} \in \operatorname{IbMod}(\mathcal{P})$ being given by

$$\overline{\mathrm{L}}_{\mathcal{P}}(\overline{c}) = \mathcal{P}(\overline{c}) \otimes \operatorname{hocolim}_n \Omega^n[\,(\mathrm{S}^n)^{\otimes m} \times^{\mathrm{h}}_{1_{\mathcal{S}}} 0\,]$$

for each C-sequence $\overline{c} := (c_1, \cdots, c_m; c)$.

Proof. By Proposition 3.4.0.5 (and by Notations 3.4.0.15), we get that $\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}}) \simeq \widetilde{\mathrm{L}}_{\mathcal{P}}[-1]$. By Computations 3.4.0.16(1), $\widetilde{\mathrm{L}}_{\mathcal{P}}$ agrees with $\mathrm{L}^b_{\mathcal{P}}$ on each level. The description of the latter is included in Computations 3.4.0.14.

In the case where $\mathcal{S}$ is in addition stable containing a strict zero object, the claim follows just by combining the above paragraph with Computations 3.4.0.16(2). □

By the definition of Quillen cohomology group 2.2.0.14, we give the following conclusion.

Theorem 3.4.0.18. *Suppose we are given a fibrant object $M \in \mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P})$. Under the same assumptions as in Proposition 3.4.0.17, the n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in M is formulated as*

$$\mathrm{H}^n_Q(\mathcal{P}, M) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P})}(\widetilde{\mathrm{L}}_{\mathcal{P}}[-1], M[n]) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P})}(\widetilde{\mathrm{L}}_{\mathcal{P}}, M[n+1]).$$

Furthermore, suppose that $\mathcal{S}$ is in addition stable containing a strict zero object 0. For a given fibrant object $M \in \operatorname{IbMod}(\mathcal{P})$, the n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in M is formulated as

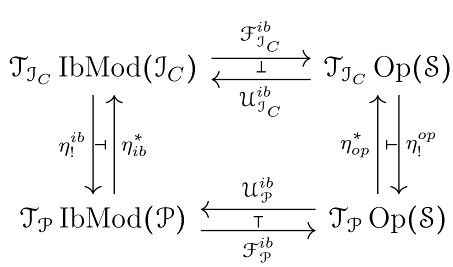
$$\mathrm{H}^n_Q(\mathcal{P}, M) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\operatorname{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}[-1], M[n]) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\operatorname{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}, M[n+1]).$$

3.5 Long exact sequence relating Quillen cohomology and reduced Quillen cohomology

The unit map $\eta : \mathcal{I}_C \longrightarrow \mathcal{P}$ gives rise to the Quillen adjunctions

$$\eta^{ib}_! : \mathcal{T}_{\mathcal{I}_C}\operatorname{IbMod}(\mathcal{I}_C) \rightleftarrows \mathcal{T}_{\mathcal{P}}\operatorname{IbMod}(\mathcal{P}) : \eta^*_{ib} \quad , \quad \eta^{op}_! : \mathcal{T}_{\mathcal{I}_C}\operatorname{Op}(\mathcal{S}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\operatorname{Op}(\mathcal{S}) : \eta^*_{op}.$$

Moreover, there is a commutative diagram of Quillen adjunctions of the form



The following is an analogue of [[7], Corollary 3.2.9].

Lemma 3.5.0.1. *Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 3.3.0.2 and that $\mathcal{P}$ is fibrant and Σ-cofibrant. There is a (homotopy) cofiber sequence in $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ of the form*

$$\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P},C}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}) \longrightarrow \mathbb{L}\eta^{ib}_{!}(\widetilde{\mathrm{L}}_{\mathcal{I}_C}) \longrightarrow \widetilde{\mathrm{L}}_{\mathcal{P}} \tag{3.5.0.1}$$

where $\mathcal{U}^{ib}_{\mathcal{P},C}$ is the right Quillen equivalence $\mathcal{U}^{ib}_{\mathcal{P},C} : \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{S}) \xrightarrow{\simeq} \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ appearing in Theorem 3.2.4.1.

Proof. Corollary 3.4.0.17 proves the existence of weak equivalences $\widetilde{\theta}_{\mathcal{I}_C} : \widetilde{\mathrm{L}}_{\mathcal{I}_C}[-1] \xrightarrow{\simeq} \mathbb{R}\mathcal{U}^{ib}_{\mathcal{I}_C}(\mathrm{L}_{\mathcal{I}_C})$ and $\widetilde{\theta}_{\mathcal{P}} : \widetilde{\mathrm{L}}_{\mathcal{P}}[-1] \xrightarrow{\simeq} \mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}})$ in $\mathcal{T}_{\mathcal{I}_C}\,\mathrm{IbMod}(\mathcal{I}_C)$ and $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$, respectively. Applying $\mathbb{L}\eta^{op}_{!}$ to $\widetilde{\theta}^{ad}_{\mathcal{I}_C} : \mathbb{L}\mathcal{F}^{ib}_{\mathcal{I}_C}(\widetilde{\mathrm{L}}_{\mathcal{I}_C}) \xrightarrow{\simeq} \mathrm{L}_{\mathcal{I}_C}[1]$ (i.e., the adjoint of $\widetilde{\theta}_{\mathcal{I}_C}$), and taking then the adjoint of the resultant, we obtain a weak equivalence in $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ of the form $\mathbb{L}\eta^{ib}_{!}(\widetilde{\mathrm{L}}_{\mathcal{I}_C}) \xrightarrow{\simeq} \mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}\mathbb{L}\eta^{op}_{!}(\mathrm{L}_{\mathcal{I}_C}[1])$.

On other hand, by the definition of relative cotangent complex, there is a cofiber sequence in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ of the form

$$\mathrm{L}_{\mathcal{P}/\mathcal{I}_C} \longrightarrow \mathbb{L}\eta^{op}_{!}(\mathrm{L}_{\mathcal{I}_C})[1] \longrightarrow \mathrm{L}_{\mathcal{P}}[1].$$

By applying $\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}$ to the latter and by the first paragraph, we get a cofiber sequence in $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$:

$$\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}) \longrightarrow \mathbb{L}\eta^{ib}_{!}(\widetilde{\mathrm{L}}_{\mathcal{I}_C}) \longrightarrow \widetilde{\mathrm{L}}_{\mathcal{P}}.$$

Now, note that the functor $\mathcal{U}^{ib}_{\mathcal{P}}$ is the same as the composition $\mathcal{U}^{ib}_{\mathcal{P},C} \circ \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$. Lemma 3.4.0.9 hence shows that there is a weak equivalence $\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P},C}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}) \xrightarrow{\simeq} \mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}/\mathcal{I}_C})$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$. So we get the desired cofiber sequence. □

We end this chapter with the following theorem.

Theorem 3.5.0.2. *Suppose that $\mathcal{S}$ additionally satisfies the condition (S8) 3.3.0.2 and that $\mathcal{P}$ is fibrant and Σ-cofibrant. Given a fibrant object $M \in \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$, there is a long exact sequence of abelian groups of the form*

$$\cdots \longrightarrow \mathrm{H}^{n-1}_{Q}(\mathcal{P},M) \longrightarrow \mathrm{H}^{n}_{Q,r}(\mathcal{P},M) \longrightarrow \mathrm{H}^{n}_{Q,\mathrm{red}}(\mathcal{P},M) \longrightarrow \mathrm{H}^{n}_{Q}(\mathcal{P},M) \longrightarrow \mathrm{H}^{n+1}_{Q,r}(\mathcal{P},M) \longrightarrow \cdots$$

where $\mathrm{H}^{\bullet}_{Q,r}(\mathcal{P},-)$ refers to Quillen cohomology group of $\mathcal{P}$ when regarded as a right module over itself, while $\mathrm{H}^{\bullet}_{Q}(\mathcal{P},-)$ refers to Quillen cohomology group of $\mathcal{P}$ and $\mathrm{H}^{\bullet}_{Q,\mathrm{red}}(\mathcal{P},-)$ refers to reduced Quillen cohomology group of $\mathcal{P}$ (cf. Conventions 3.4.0.3).

Proof. The cofiber sequence of Lemma 3.5.0.1 induces a fiber sequence of derived mapping spaces:

$$\mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})}(\widetilde{\mathrm{L}}_{\mathcal{P}},M) \longrightarrow \mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})}(\mathbb{L}\eta^{ib}_{!}(\widetilde{\mathrm{L}}_{\mathcal{I}_C}),M) \longrightarrow \mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})}(\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P},C}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}),M).$$

In this sequence, by notation $\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P},C}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}})$ classifies the reduced Quillen cohomology of $\mathcal{P}$, while $\widetilde{\mathrm{L}}_{\mathcal{P}}$ classifies the Quillen cohomology of $\mathcal{P}$, by Theorem 3.4.0.18. That fiber sequence will hence give rise to the desired long exact sequence after having that $\mathbb{L}\eta^{ib}_{!}(\widetilde{\mathrm{L}}_{\mathcal{I}_C})$ classifies the Quillen cohomology of $\mathcal{P}$ when regarded as a right module over itself. To this end, we first consider the Quillen adjunction $\mathcal{T}_{\mathcal{P}}\,\mathrm{RMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ which is induced by the free-forgetful adjunction $\mathrm{RMod}(\mathcal{P}) \rightleftarrows \mathrm{IbMod}(\mathcal{P})$. We denote by $\mathrm{L}^{r}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{RMod}(\mathcal{P})$ the cotangent complex of $\mathcal{P}$ when regarded as a right module over itself. It therefore suffices to prove that the derived image of $\mathrm{L}^{r}_{\mathcal{P}}$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ is weakly equivalent to $\mathbb{L}\eta^{ib}_{!}(\widetilde{\mathrm{L}}_{\mathcal{I}_C})$. For this last claim, observe first that $\eta^{ib}_{!}$ is the same as the functor $\mathcal{T}_{\mathcal{I}_C}\,\mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ induced by the free functor $\mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathrm{IbMod}(\mathcal{P})$. Moreover, under the identification $\mathcal{T}_{\mathcal{I}_C}\operatorname{IbMod}(\mathcal{I}_C) \simeq \mathcal{T}_{\mathcal{I}_C}\operatorname{Coll}_C(\mathcal{S})$, the object $\widetilde{\mathrm{L}}_{\mathcal{I}_C}$ is nothing but the cotangent complex of $\mathcal{I}_C$ when regarded as an object of $\operatorname{Coll}_C(\mathcal{S})$, which has the derived image through the left Quillen functor $\mathcal{T}_{\mathcal{I}_C}\operatorname{Coll}_C(\mathcal{S}) \longrightarrow \mathcal{T}_{\mathcal{P}}\operatorname{RMod}(\mathcal{P})$ being $\mathrm{L}^r_{\mathcal{P}}$ obviously. The proof is hence completed. □

Remark 3.5.0.3. When considering $\mathcal{P}$ as a right module over itself, it is free generated by $\mathcal{I}_C$. Thus, we have a canonical isomorphism

$$\mathrm{H}^n_{Q,r}(\mathcal{P}, M) \cong \mathrm{H}^n_{Q,col}(\mathcal{I}_C, \overline{M}),$$

where the right hand side is the n'th Quillen cohomology group of $\mathcal{I}_C$, when regarded as a C-collection, with coefficients in the derived image of M in $\mathcal{T}_{\mathcal{I}_C}\operatorname{Coll}_C(\mathcal{S})$. When $\mathcal{S}$ is further stable containing a strict zero object 0 (and hence, so is the category $\operatorname{Coll}_C(\mathcal{S})$), we have a sequence of right Quillen equivalences

$$\mathcal{T}_{\mathcal{I}_C}\operatorname{Coll}_C(\mathcal{S}) \xrightarrow[\simeq]{\Omega^\infty} \operatorname{Coll}_C(\mathcal{S})_{\mathcal{I}_C//\mathcal{I}_C} \xrightarrow[\simeq]{\ker} \operatorname{Coll}_C(\mathcal{S})$$

(cf. Theorem 3.2.4.3) and moreover, under this composed right Quillen equivalence, the cotangent complex of $\mathcal{I}_C$ is identified to itself $\mathcal{I}_C$ (cf. [[8], Corollary 2.2.4]). Therefore, in this situation we have that

$$\mathrm{H}^n_{Q,col}(\mathcal{I}_C, \overline{M}) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\operatorname{Coll}_C(\mathcal{S})}(\mathcal{I}_C, \mathbb{R}\ker\Omega^\infty(\Sigma^n\overline{M})) \cong \prod_{c\in C} \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathcal{S}}(1_{\mathcal{S}}, \Sigma^n(\overline{M}(c;c) \times_{1_{\mathcal{S}}} 0)).$$

Chapter 4

Quillen cohomology of simplicial operads

Simplicial operads are precisely operads enriched over the Cartesian monoidal category of simplicial sets, Set_Δ. This category comes equipped with the standard (Kan-Quillen) model structure and then, satisfies the conditions of Conventions 3.1.0.2 and also the extra condition (S8) 3.3.0.2 (cf. Example 2.1.4.7 and Proposition 3.3.0.3). We therefore inherit the results of §3.4 for the work of this chapter.

In the first section, we revisit the **straightening** and **unstraightening** constructions (in unmarked case), according to [[4], §2.2.1]. Given a simplicial (co)presheaf $\mathcal{F}$ over a simplicial category $\mathcal{C}$, the unstraightening of $\mathcal{F}$ is in particular a simplicial set over $\mathrm{N}\,\mathcal{C}$. As the first step, we will focus on describing the simplices of the unstraightening of $\mathcal{F}$ and furthermore, giving convenient models for its **spaces of (right) left morphisms**.

According to the work of Y. Harpaz, J. Nuiten and M. Prasma ([7]), the cotangent complex of an ∞-category (or a fibrant simplicial category) can be represented as a spectrum valued functor on its **twisted arrow ∞-category** (see Theorem 1.0.0.7). The construction of twisted arrow ∞-categories (of ∞-categories) $\mathrm{Tw}(-) : \mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}_\infty$ was originally introduced by Lurie [[3], § 5.2]. For a given fibrant simplicial category $\mathcal{C}$, the twisted arrow ∞-category of $\mathcal{C}$ is simply defined to be $\mathrm{Tw}(\mathcal{C}) := \mathrm{Tw}(\mathrm{N}\,\mathcal{C})$ with $\mathrm{N}\,\mathcal{C}$ being the **simplicial nerve** of $\mathcal{C}$. In the second section, we extend the latter to the construction of twisted arrow ∞-categories of (fibrant) simplicial operads. For a fibrant simplicial operad $\mathcal{P}$, the twisted arrow ∞-category $\mathrm{Tw}(\mathcal{P})$ is by definition the unstraightening of the simplicial copresheaf $\mathcal{P} : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathrm{Set}_\Delta$, which encodes the data of $\mathcal{P}$ as an infinitesimal bimodule over itself (see § 2.1.3). The results obtained from the first section provide us with a comprehensive view on the structure of $\mathrm{Tw}(\mathcal{P})$, on both the simplicial and ∞-categorical structures.

In the last section, for the main purpose we will show that the cotangent complex of a simplicial operad can be represented as a spectrum valued functor on its twisted arrow ∞-category.

4.1 Unstraightening of simplicial (co)presheaves

We denote by $\mathfrak{C}[-] : \mathrm{Set}_\Delta \longrightarrow \mathrm{Cat}(\mathrm{Set}_\Delta)$ the **rigidification functor**, which is left adjoint to the simplicial nerve functor N. Recall by definition that $\mathfrak{C}[\Delta^n]$ is the simplicial category whose set of objects is $[n] = \{0, \cdots, n\}$ and whose mapping spaces are defined as $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) \coloneqq \mathrm{N}\,\mathrm{P}_{i,j}$ the nerve of the poset

$$\mathrm{P}_{i,j} = \{A \,|\, \{i,j\} \subseteq A \subseteq [i,j]\},$$

with $[i,j] \coloneqq \{i, i+1, \cdots, j\}$. (In particular, $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = \varnothing$ when $i > j$). The composition in $\mathfrak{C}[\Delta^n]$ is induced by the maps $\mathrm{P}_{i,j} \times \mathrm{P}_{j,k} \longrightarrow \mathrm{P}_{i,k}$ taking (A,B) to $A \cup B$.

Depending if one wants to "unstraighten" simplicial presheaves or copresheaves, one would need to use **contravariant** or **covariant unstraightening** functor, respectively. Suppose we are given a simplicial set S and a simplicial category $\mathcal{C}$. Let $\phi : \mathfrak{C}[S] \longrightarrow \mathcal{C}$ be a simplicial functor. For each simplicial set X over S, one takes two simplicial categories

$$\mathcal{M}_X^{\triangleleft} \coloneqq \mathfrak{C}[X^{\triangleleft}] \coprod_{\mathfrak{C}[X]} \mathcal{C} \quad , \quad \mathcal{M}_X^{\triangleright} \coloneqq \mathfrak{C}[X^{\triangleright}] \coprod_{\mathfrak{C}[X]} \mathcal{C}$$

where $X^{\triangleleft}$ and $X^{\triangleright}$ are respectively the left and right cones of X. We will always use the letter "v" standing for the cone point, for both left and right cones.

The **covariant straightening functor** associated to ϕ, written as

$$\mathrm{St}_\phi^{\triangleleft} : (\mathrm{Set}_\Delta)_{/S} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta),$$

is defined by sending each simplicial set X over S to the functor $\mathrm{Map}_{\mathcal{M}_X^{\triangleleft}}(v, -) : \mathcal{C} \longrightarrow \mathrm{Set}_\Delta$. The functor $\mathrm{St}_\phi^{\triangleleft}$ admits a right adjoint, the **covariant unstraightening functor** associated to ϕ, denoted by $\mathrm{Un}_\phi^{\triangleleft}$. In fact, the adjunction $\mathrm{St}_\phi^{\triangleleft} \dashv \mathrm{Un}_\phi^{\triangleleft}$ forms a Quillen adjunction when one endows the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)$ with the projective model structure and the category $(\mathrm{Set}_\Delta)_{/S}$ with the **covariant model structure**. Moreover, this adjunction is a Quillen equivalence as long as ϕ is a weak equivalence of simplicial categories. (See [4], Chapter 3 for more details).

Dually, the **contravariant straightening functor** associated to ϕ, written as

$$\mathrm{St}_\phi^{\triangleright} : (\mathrm{Set}_\Delta)_{/S} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}_\Delta),$$

is defined by sending each simplicial set X over S to the functor $\mathrm{Map}_{\mathcal{M}_X^{\triangleright}}(-, v) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta$. The functor $\mathrm{St}_\phi^{\triangleright}$ admits a right adjoint, the **contravariant unstraightening functor** associated to ϕ, denoted by $\mathrm{Un}_\phi^{\triangleright}$. The adjunction $\mathrm{St}_\phi^{\triangleright} \dashv \mathrm{Un}_\phi^{\triangleright}$ forms a Quillen adjunction when one endows the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}_\Delta)$ with the projective model structure and the category $(\mathrm{Set}_\Delta)_{/S}$ with the **contravariant model structure** and moreover, becomes a Quillen equivalence if ϕ is a weak equivalence.

Notations 4.1.0.1.

1. We are almost concentrated in the case where $S = \mathrm{N}\,\mathcal{C}$ and $\phi = \varepsilon_{\mathcal{C}} : \mathfrak{C}[\mathrm{N}\,\mathcal{C}] \longrightarrow \mathcal{C}$ the counit map of $\mathfrak{C}[-] \dashv \mathrm{N}$. In this case, the corresponding covariant (resp. contravariant) straightening-unstraightening adjunction will be denoted by $\mathrm{St}_{\mathcal{C}}^{\triangleleft} \dashv \mathrm{Un}_{\mathcal{C}}^{\triangleleft}$ (resp. $\mathrm{St}_{\mathcal{C}}^{\triangleright} \dashv \mathrm{Un}_{\mathcal{C}}^{\triangleright}$).
2. When $\mathcal{C} = \mathfrak{C}[S]$ and ϕ is the identity functor, the corresponding covariant (resp. contravariant) straightening-unstraightening adjunction will be denoted by $\mathrm{St}_S^{\triangleleft} \dashv \mathrm{Un}_S^{\triangleleft}$ (resp. $\mathrm{St}_S^{\triangleright} \dashv \mathrm{Un}_S^{\triangleright}$).

Remark 4.1.0.2. Note that $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}$ is the same as the composed functor

$$\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta) \xrightarrow{\varepsilon_{\mathcal{C}}^*} \mathrm{Fun}(\mathfrak{C}[\mathrm{N}\,\mathcal{C}],\mathrm{Set}_\Delta) \xrightarrow{\mathrm{Un}_{\mathrm{N}\,\mathcal{C}}^{\triangleleft}} (\mathrm{Set}_\Delta)_{/\mathrm{N}\,\mathcal{C}},$$

(this follows from [4], 2.2.1.1(2)). The same thing holds for the contravariant case.

Let $\mathcal{F}:\mathcal{C}\longrightarrow\mathrm{Set}_\Delta$ be a simplicial functor. We wish to understand the structure of $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})$ as a simplicial set over $\mathrm{N}\,\mathcal{C}$. For this, we will follow [[48], §58], yet in the opposite convention. For each $n\in\mathbb{N}$, one establishes a simplicial functor

$$\mathfrak{D}_{\Delta^n}^{\triangleleft}:\mathfrak{C}[\Delta^n]\longrightarrow\mathrm{Set}_\Delta$$

given by sending each $i\in[n]$ to $\mathfrak{D}_{\Delta^n}^{\triangleleft}(i):=\mathrm{N}\,\mathrm{P}^{\triangleleft}(i)$ the nerve of the poset

$$\mathrm{P}^{\triangleleft}(i):=\{A\,|\,\{i\}\subseteq A\subseteq[0,i]\}$$

with $[0,i]:=\{0,1,\cdots,i\}$. The structure maps of simplicial functor are defined by applying the union operation of subsets in an obvious way. Moreover, for each map $\delta:\Delta^m\longrightarrow\Delta^n$, one defines a natural transformation

$$\mathfrak{D}_{\delta}^{\triangleleft}:\mathfrak{D}_{\Delta^m}^{\triangleleft}\longrightarrow\mathfrak{D}_{\Delta^n}^{\triangleleft}\circ\mathfrak{C}[\delta]$$

of the simplicial functors $\mathfrak{C}[\Delta^m]\longrightarrow\mathrm{Set}_\Delta$ given at each $i\in[m]$ by the map $\mathfrak{D}_{\delta}^{\triangleleft}(i):\mathfrak{D}_{\Delta^m}^{\triangleleft}(i)\longrightarrow\mathfrak{D}_{\Delta^n}^{\triangleleft}(\delta i)$ which is induced by the map of posets $\mathrm{P}^{\triangleleft}(i)\longrightarrow\mathrm{P}^{\triangleleft}(\delta i)$, $S\mapsto\delta(S)$.

Construction 4.1.0.3. The data of an n-simplex of $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})$ consists of

- an n-simplex $\varphi\in\mathrm{N}(\mathcal{C})$, i.e., a functor $\varphi:\mathfrak{C}[\Delta^n]\longrightarrow\mathcal{C}$, and
- a natural transformation $t:\mathfrak{D}_{\Delta^n}^{\triangleleft}\longrightarrow\mathcal{F}\circ\varphi$ between simplicial functors $\mathfrak{C}[\Delta^n]\longrightarrow\mathrm{Set}_\Delta$.

For each map $\delta:\Delta^m\longrightarrow\Delta^n$, the corresponding structure map $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})_n\longrightarrow\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})_m$ is given by sending each pair $(\varphi,t)\in\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})_n$ to the pair

$$\mathfrak{C}[\Delta^m]\xrightarrow{\mathfrak{C}[\delta]}\mathfrak{C}[\Delta^n]\xrightarrow{\varphi}\mathcal{C}\ ,\ \mathfrak{D}_{\Delta^m}^{\triangleleft}\xrightarrow{\mathfrak{D}_{\delta}^{\triangleleft}}\mathfrak{D}_{\Delta^n}^{\triangleleft}\circ\mathfrak{C}[\delta]\xrightarrow{t\circ\mathrm{Id}}\mathcal{F}\circ\varphi\circ\mathfrak{C}[\delta].$$

Explanations: First, argue that giving an n-simplex $z\in\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})$ is equivalent to giving a sequence of maps $\Delta^n\xrightarrow{z}\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})\longrightarrow\mathrm{N}\,\mathcal{C}$. Let us denote this composite map by $\varphi_z:\Delta^n\longrightarrow\mathrm{N}\,\mathcal{C}$, which is identified to a simplicial functor $\varphi_z:\mathfrak{C}[\Delta^n]\longrightarrow\mathcal{C}$. Observe now that for each such map φ_z there is a commutative square of right adjoint functors

$$\begin{array}{ccc}
\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta) & \xrightarrow{\mathrm{Un}_{\mathcal{C}}^{\triangleleft}} & (\mathrm{Set}_\Delta)_{/\mathrm{N}\,\mathcal{C}} \\
{\scriptstyle\varphi_z^*}\downarrow & & \downarrow{\scriptstyle\varphi_z^*} \\
\mathrm{Fun}(\mathfrak{C}[\Delta^n],\mathrm{Set}_\Delta) & \xrightarrow[\mathrm{Un}_{\Delta^n}^{\triangleleft}]{} & (\mathrm{Set}_\Delta)_{/\Delta^n}
\end{array}$$

(this follows by combining both two parts of [[4], Proposition 2.2.1.1], along with noting Remark 4.1.0.2). This proves the existence of a Cartesian square of the form

$$\begin{array}{ccc}
\mathrm{Un}_{\Delta^n}^{\triangleleft}(\mathcal{F}\circ\varphi_z) & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow \\
\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F}) & \longrightarrow & \mathrm{N}\,\mathcal{C}
\end{array}$$

So the data of z is equivalent to that of a map $\varphi_z : \Delta^n \longrightarrow \mathrm{N}\,\mathcal{C}$ and a map $\Delta^n \longrightarrow \mathrm{Un}^{\triangleleft}_{\Delta^n}(\mathcal{F}\circ\varphi_z)$ factoring the identity Id_{Δ^n}. By adjunction, the latter is identified to a natural transformation $\mathrm{St}^{\triangleleft}_{\Delta^n}(\mathrm{Id}_{\Delta^n}) \longrightarrow \mathcal{F}\circ\varphi_z$. But, by definition $\mathrm{St}^{\triangleleft}_{\Delta^n}(\mathrm{Id}_{\Delta^n}) = \mathrm{Map}_{\mathfrak{C}[(\Delta^n)^{\triangleleft}]}(v,-)$. Hence, it remains to establish for each n an isomorphism

$$\mathfrak{D}^{\triangleleft}_{\Delta^n} \xrightarrow{\simeq} \mathrm{Map}_{\mathfrak{C}[(\Delta^n)^{\triangleleft}]}(v,-)$$

between simplicial functors $\mathfrak{C}[\Delta^n] \longrightarrow \mathrm{Set}_{\Delta}$ compatible with simplicial maps $\Delta^m \longrightarrow \Delta^n$. For this, note that under the identification $(\Delta^n)^{\triangleleft} \cong \Delta^{n+1}$, the functor $\mathrm{Map}_{\mathfrak{C}[(\Delta^n)^{\triangleleft}]}(v,-)$ is isomorphic to the functor $[n] \ni i \mapsto \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(0,i+1) = \mathrm{N}\,\mathrm{P}_{0,i+1}$. For each $i \in [n]$, we define an isomorphism of posets

$$\mathrm{P}^{\triangleleft}(i) \xrightarrow{\simeq} \mathrm{P}_{0,i+1} \quad , \quad A \mapsto \{0\} \sqcup (A+1)$$

with $A+1 := \{a+1 \mid a \in A\}$. This indeed gives us the desired isomorphism $\mathfrak{D}^{\triangleleft}_{\Delta^n} \xrightarrow{\simeq} \mathrm{Map}_{\mathfrak{C}[(\Delta^n)^{\triangleleft}]}(v,-)$.

Remark 4.1.0.4. Unwinding definition, each vertex of $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$ is the choice of an object $x \in \mathrm{Ob}(\mathcal{C})$ and a vertex $\mu \in \mathcal{F}(x)$. Let x, y be two objects of $\mathcal{C}$. The data of an edge of $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$ from $\mu \in \mathcal{F}(x)$ to $\nu \in \mathcal{F}(y)$ consists of a vertex $\alpha \in \mathrm{Map}_{\mathcal{C}}(x,y)$ and an edge of $\mathcal{F}(y)$ of the form $t : \nu \longrightarrow \alpha_*(\mu)$ where $\alpha_* : \mathcal{F}(x) \longrightarrow \mathcal{F}(y)$ is the map induced by the simplicial functor structure of $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Set}_{\Delta}$.

Comments 4.1.0.5. Intuitively, there seems to be something illogical in the data of an edge of $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$ mentioned above. Indeed, if one thinks of the covariant unstraightening functor as something generalizing the classical (covariant) Grothendieck construction then it is natural to require that, in the data of such an edge of $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$, the edge t should be in the direction $\alpha_*(\mu) \longrightarrow \nu$ instead of $\nu \longrightarrow \alpha_*(\mu)$. In fact, this confusion arises from the convention, namely, when one chose the adjunction $\mathfrak{C}[-] \dashv \mathrm{N}$ to work with, for both covariant and contravariant cases. Let us try to modify this convention slightly. We define a new "rigidification functor" $\mathfrak{C}'[-]$ by taking, for each n, $\mathrm{Map}_{\mathfrak{C}'[\Delta^n]}(i,j) := \mathrm{N}\,\mathrm{P}^{\mathrm{op}}_{i,j}$, (instead of $\mathrm{N}\,\mathrm{P}_{i,j}$ as in the definition of $\mathfrak{C}[-]$). This leads to a new "simplicial functor" N' characterized as the right adjoint to $\mathfrak{C}'[-]$. Now, if instead of $\mathfrak{C}[-] \dashv \mathrm{N}$, one uses the adjunction $\mathfrak{C}'[-] \dashv \mathrm{N}'$ in establishing the covariant straightening, then the new covariant unstraightening will satisfy what one expects from the classical Grothendieck construction, i.e., one has the edge t indeed being in the direction $\alpha_*(\mu) \longrightarrow \nu$. We believe that when doing the covariant unstraightening, using the adjunction $\mathfrak{C}'[-] \dashv \mathrm{N}'$ is more natural than the $\mathfrak{C}[-] \dashv \mathrm{N}$. But if one does not expect a perfect generalization from the classical Grothendieck construction anymore, everything still works certainly.

Let $\mathcal{F}' : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}$ be a simplicial functor. Similarly as in the covariant case, one can get an explicit description of the contravariant unstraightening of $\mathcal{F}'$, $\mathrm{Un}^{\triangleright}_{\mathcal{C}}(\mathcal{F}')$. For each $n \in \mathbb{N}$, one defines a simplicial functor

$$\mathfrak{D}^{\triangleright}_{\Delta^n} : \mathfrak{C}[\Delta^n]^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}$$

taking each $i \in [n]$ to $\mathfrak{D}^{\triangleright}_{\Delta^n}(i) := \mathrm{N}\,\mathrm{P}^{\triangleright}_n(i)$ the nerve of the poset

$$\mathrm{P}^{\triangleright}_n(i) := \{A \mid \{i\} \subseteq A \subseteq [i,n]\}$$

with $[i,n] := \{i, i+1, \cdots, n\}$ For each map $\delta : \Delta^m \longrightarrow \Delta^n$, there is also a natural transformation $\mathfrak{D}^{\triangleright}_{\delta} : \mathfrak{D}^{\triangleright}_{\Delta^m} \longrightarrow \mathfrak{D}^{\triangleright}_{\Delta^n} \circ \mathfrak{C}[\delta]^{\mathrm{op}}$ of the simplicial functors $\mathfrak{C}[\Delta^m]^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}$. The structure maps are defined naturally as well as in the covariant case (see Construction 4.1.0.3).

Construction 4.1.0.6. The data of an n-simplex of $\mathrm{Un}^{\triangleright}_{\mathcal{C}}(\mathcal{F}')$ consists of

- an n-simplex $\varphi \in \mathrm{N}(\mathcal{C})$, i.e., a functor $\varphi : \mathfrak{C}[\Delta^n] \longrightarrow \mathcal{C}$, and

- a natural transformation $t : \mathfrak{D}^{\triangleright}_{\Delta^n} \longrightarrow \mathcal{F}' \circ \varphi^{\mathrm{op}}$ between simplicial functors $\mathfrak{C}[\Delta^n]^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta$.

The simplicial structure maps of $\mathrm{Un}^{\triangleright}_{\mathcal{C}}(\mathcal{F}')$ are induced by the natural transformations $\mathfrak{D}^{\triangleright}_{\delta}$ mentioned above. (See also [[48], §58]).

We now consider the case where $\mathcal{C} = \{*\}$ the terminal category. The corresponding covariant (resp. contravariant) straightening-unstraightening adjunction will be then denoted by $\mathrm{St}^{\triangleleft}_{*} \dashv \mathrm{Un}^{\triangleleft}_{*}$ (resp. $\mathrm{St}^{\triangleright}_{*} \dashv \mathrm{Un}^{\triangleright}_{*}$). By definition, these forms the self-adjunctions on Set_Δ

$$\mathrm{St}^{\triangleleft}_{*} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta : \mathrm{Un}^{\triangleleft}_{*} \quad , \quad \mathrm{St}^{\triangleright}_{*} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta : \mathrm{Un}^{\triangleright}_{*} .$$

Note that $\mathrm{St}^{\triangleright}_{*} \dashv \mathrm{Un}^{\triangleright}_{*}$ agrees with the adjunction $|-|_{Q^\bullet} \dashv \mathrm{Sing}_{Q^\bullet}$ of [[4], §2.2.2].

Let X be a simplicial set. We will need explicit descriptions of both $\mathrm{Un}^{\triangleleft}_{*}(X)$ and $\mathrm{Un}^{\triangleright}_{*}(X)$. Recall that the n-cube $(\Delta^1)^{\times\{1,\cdots,n\}}$ has total of n **end faces** (i.e., the faces containing the terminal vertex), including

$$T_i := (\Delta^1)^{\times\{1,\cdots,i-1\}} \times \{1\} \times (\Delta^1)^{\times\{i+1,\cdots,n\}} \subseteq (\Delta^1)^{\times\{1,\cdots,n\}}$$

for $i = 1, \cdots, n$. For each i, we denote by

$$T_i^{\circ} := \{1\} \times \cdots \times \{1\} \times (\Delta^1)^{\times\{i+1,\cdots,n\}} \subseteq T_i,$$

which is an $(n-i)$-subcube of T_i.

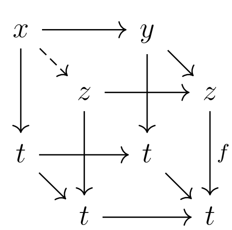
Definition 4.1.0.7. Let $\gamma : (\Delta^1)^{\times\{1,\cdots,n\}} \longrightarrow X$ be a simplicial map performing an n-cube in X. Then γ is called a **quasi n-simplex** of X if for every $i \in \{1, \cdots, n\}$, the end face of γ given by $\gamma\big|_{T_i}$ the restriction of γ to T_i is degenerate on $\gamma\big|_{T_i^{\circ}}$ the restriction of γ to T_i°.

Remark 4.1.0.8. Each n-simplex $\overline{\delta} : \Delta^n \longrightarrow X$ corresponds to a quasi n-simplex of X given by the composition

$$\delta : (\Delta^1)^{\times\{1,\cdots,n\}} \xrightarrow{\tau} \Delta^n \xrightarrow{\overline{\delta}} X$$

where the map τ is given by the canonical way of collapsing the n-cube to a nondegenerate n-subsimplex containing the initial vertex. Conversely, each quasi n-simplex of X agrees with an n-cube in X of such type δ on boundary. This phenomenon itself suggests us to the terminology "quasi simplex".

Example 4.1.0.9. A quasi 3-simplex of X is a 3-cube in X of the form



such that the front square is degenerate on the edge $z \xrightarrow{f} t$ and the bottom square is degenerate on the terminal vertex t.

The main interest in this notion is as follows.

Lemma 4.1.0.10. *1. There is a natural bijection between the n-simplices of $\mathrm{Un}^{\triangleright}_{*}(X)$ and the quasi n-simplices of X.*

2. Also, there is a natural bijection between the n-simplices of $\mathrm{Un}_*^{\lhd}(X)$ *and the quasi n-simplices of X.*

To get an explicit proof, we make use of $\mathfrak{D}_{\Delta^n}^{\rhd}(0) = \mathrm{N}\,\mathrm{P}_n^{\rhd}(0)$ as a specific model for the n-cube $(\Delta^1)^{\times\{1,\cdots,n\}}$ (see Construction 4.1.0.6). For each $i \in \{1,\cdots,n\}$, we take the subposets

$$\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \supseteq [0,i]\} \subseteq \{A \in \mathrm{P}_n^{\rhd}(0) \mid A \ni i\} \subseteq \mathrm{P}_n^{\rhd}(0).$$

Note that the inclusion $\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \supseteq [0,i]\} \subseteq \mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \ni i\}$ corresponds to the inclusion $T_i^\circ \subseteq T_i$ mentioned above. Moreover, there is an obvious identification of posets

$$\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \supseteq [0,i]\} \cong \mathrm{P}_n^{\rhd}(i).$$

Proof of Lemma 4.1.0.10. (1) By Construction 4.1.0.6, an n-simplex of $\mathrm{Un}_*^{\rhd}(X)$ is a natural transformation $t' : \mathfrak{D}_{\Delta^n}^{\rhd} \longrightarrow \{X\}$ between simplicial functors $\mathfrak{C}[\Delta^n]^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta$, with $\{X\}$ being the constant functor on X. Unwinding definition, the data of t' consists of the maps $t'(i) : \mathfrak{D}_{\Delta^n}^{\rhd}(i) = \mathrm{N}\,\mathrm{P}_n^{\rhd}(i) \longrightarrow X$ for $i \in [n]$, satisfying that:

(*) for every pair (i,j) such that $i > j$, the composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]^{\mathrm{op}}}(i,j) \times \mathfrak{D}_{\Delta^n}^{\rhd}(i) \xrightarrow{\mathfrak{D}_{\Delta^n}^{\rhd}} \mathfrak{D}_{\Delta^n}^{\rhd}(j) \xrightarrow{t'(j)} X \tag{4.1.0.1}$$

agrees with the composition $\mathrm{Map}_{\mathfrak{C}[\Delta^n]^{\mathrm{op}}}(i,j) \times \mathfrak{D}_{\Delta^n}^{\rhd}(i) \longrightarrow \mathfrak{D}_{\Delta^n}^{\rhd}(i) \xrightarrow{t'(i)} X$.

Note that the map (4.1.0.1) corresponds to the restriction of $t'(j)$ to $\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(j) \mid A \ni i\}$. When $j = i-1$, the condition (*) means that $t'(i)$ agrees with the restriction of $t'(i-1)$ to

$$\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(i-1) \mid A \ni i\} \cong \mathrm{N}\,\mathrm{P}_n^{\rhd}(i).$$

Inductively, for every $i \geqslant 1$, $t'(i)$ agrees with the restriction of $t'(0)$ to

$$\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \supseteq [0,i]\} \cong \mathrm{N}\,\mathrm{P}_n^{\rhd}(i).$$

In particular, the data of t' is fully displayed in that of $t'(0)$, which performs an n-cube in X. Now, fix $j = 0$, the condition (*) means that for every $i \in \{1,\cdots,n\}$, the end face of $t'(0)$ given by the restriction $t'(0)|_{\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \ni i\}}$ is degenerate on $t'(0)|_{\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \supseteq [0,i]\}}$. This is equivalent to saying that $t'(0)$ is a quasi n-simplex of X, by the words before the proof.

Conversely, let $t' : \mathfrak{D}_{\Delta^n}^{\rhd}(0) = \mathrm{N}\,\mathrm{P}_n^{\rhd}(0) \longrightarrow X$ be a quasi n-simplex of X. For each $i \in [n]$, we take $t'(i)$ to be the restriction of t' to $\mathrm{N}\{A \in \mathrm{P}_n^{\rhd}(0) \mid A \supseteq [0,i]\} \cong \mathrm{N}\,\mathrm{P}_n^{\rhd}(i)$. In particular, $t'(0)$ is nothing but t'. It can then be verified that $\{t'(i)\}_i$ forms a natural transformation $\mathfrak{D}_{\Delta^n}^{\rhd} \longrightarrow \{X\}$.

The two above paragraphs clearly provide the inverses of each other between the n-simplices of $\mathrm{Un}_*^{\rhd}(X)$ and the quasi n-simplices of X.

(2) By the first part, it suffices to prove the existence of a natural bijection between the n-simplices of $\mathrm{Un}_*^{\lhd}(X)$ and those of $\mathrm{Un}_*^{\rhd}(X)$. By Construction 4.1.0.3, an n-simplex of $\mathrm{Un}_*^{\lhd}(X)$ is a natural transformation $t : \mathfrak{D}_{\Delta^n}^{\lhd} \longrightarrow \{X\}$ between simplicial functors $\mathfrak{C}[\Delta^n] \longrightarrow \mathrm{Set}_\Delta$ and therefore, it consists of the maps $t(i) : \mathfrak{D}_{\Delta^n}^{\lhd}(i) = \mathrm{N}\,\mathrm{P}^{\lhd}(i) \longrightarrow X$ ($i \in [n]$) subject to the essential naturality with respect to every space of morphisms in $\mathfrak{C}[\Delta^n]$ (similarly as the condition (*) above).

For each $i \in [n]$, there is an isomorphism of posets $\mathrm{P}_n^{\rhd}(i) \xrightarrow{\simeq} \mathrm{P}^{\lhd}(n-i)$ taking $A \in \mathrm{P}_n^{\rhd}(i)$ to $n - A := \{n - a \mid a \in A\}$. We then take $t'(i)$ to be the composition $\mathrm{N}\,\mathrm{P}_n^{\rhd}(i) \cong \mathrm{N}\,\mathrm{P}^{\lhd}(n-i) \xrightarrow{t(n-i)} X$. It can then be verified that $t' = \{t'(i)\}_i$ forms a natural transformation $t' : \mathfrak{D}^{\triangleright}_{\Delta^n} \longrightarrow \{X\}$ and hence, performs an n-simplex of $\mathrm{Un}^{\triangleright}_*(X)$. Moreover, the assignment $t \mapsto t'$ indeed forms a natural bijection. □

Remark 4.1.0.11. Though the assignment $t \mapsto t'$ gives an isomorphism between $\mathrm{Un}^{\triangleleft}_*(X)$ and $\mathrm{Un}^{\triangleright}_*(X)$ on point-set level, it is not compatible with simplicial structures. Instead of that, this turns out to form a natural isomorphism $\mathrm{Un}^{\triangleleft}_*(X) \cong \mathrm{Un}^{\triangleright}_*(X)^{\mathrm{op}}$.

Remark 4.1.0.12. As mentioned in Remark 4.1.0.8, each n-simplex of X is associated to a quasi n-simplex. This determines a natural map $X \longrightarrow \mathrm{Un}^{\triangleright}_*(X)$, which is in fact a weak equivalence as long as X is a Kan complex (see [[4], 2.2.2.7, 2.2.2.8]). Combined with the above remark, we get a natural weak equivalence $X^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Un}^{\triangleleft}_*(X)$ when X is Kan. In other words, while the (derived) contravariant unstraightening $\mathrm{Un}^{\triangleright}_*$ is weakly equivalent to the identity functor, the covariant one $\mathrm{Un}^{\triangleleft}_*$ is weakly equivalent to the opposite functor $X \mapsto X^{\mathrm{op}}$.

Given a simplicial category $\mathcal{C}$ and a simplicial functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Set}_\Delta$, we turn back to consider the covariant unstraightening of $\mathcal{F}$. We will need the following in the last section.

Remark 4.1.0.13. As in [[4], Remark 2.2.2.11], for a given object $x \in \mathcal{C}$, there is a canonical isomorphism

$$\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F}) \times_{\mathrm{N}\,\mathcal{C}} \{x\} \cong \mathrm{Un}^{\triangleleft}_*(\mathcal{F}(x)).$$

This follows by the compatibility of the unstraightening functor with taking base change along the map $\{x\} \longrightarrow \mathrm{N}\,\mathcal{C}$ (see explanations after Construction 4.1.0.3). Combined with Remark 4.1.0.12, we get a weak equivalence

$$\mathcal{F}(x)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F}) \times_{\mathrm{N}\,\mathcal{C}} \{x\} \tag{4.1.0.2}$$

whenever $\mathcal{F}(x)$ is a Kan complex.

Let $x, y \in \mathrm{Ob}(\mathcal{C})$ be two objects of $\mathcal{C}$. Suppose we are given two vertices $\mu \in \mathcal{F}(x)$ and $\nu \in \mathcal{F}(y)$ regarded as vertices of $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$ (see Remark 4.1.0.4). We wish to give a convenient model for $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})}(\mu, \nu)$ the space of right morphisms from μ to ν in $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$. Recall by definition that an n-simplex of $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})}(\mu, \nu)$ is an $(n+1)$-simplex $T : \Delta^{n+1} \longrightarrow \mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$ of $\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})$ such that $d_{n+1}T$ is degenerate on μ and $T\big|_{\Delta^{\{n+1\}}} = \nu$. According to Construction 4.1.0.3, the data of T consists of:

- An $(n+1)$-simplex $H : \Delta^{n+1} \longrightarrow \mathrm{N}\,\mathcal{C}$ of $\mathrm{N}\,\mathcal{C}$ satisfying that $d_{n+1}H$ is degenerate on x and $H\big|_{\Delta^{\{n+1\}}} = y$. In other words, H is nothing but an n-simplex of $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x, y)$. We also regard H as a functor $\mathfrak{C}[\Delta^{n+1}] \longrightarrow \mathcal{C}$.
- In addition, a natural transformation $t : \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}} \longrightarrow \mathcal{F} \circ H$ between simplicial functors $\mathfrak{C}[\Delta^{n+1}] \longrightarrow \mathrm{Set}_\Delta$ satisfying that for every $i \in \{0, \cdots, n\}$ the map

$$t(i) : \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(i) \longrightarrow \mathcal{F} \circ H(i) = \mathcal{F}(x)$$

collapses its domain to the vertex μ and that the initial vertex of $t(n+1)$ agrees with ν. (Note that $t(n+1)$ performs an $(n+1)$-cube in $\mathcal{F}(y)$).

On other hand, we take ρ_μ to be the composition

$$\rho_\mu : \mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{\mathcal{F}} \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{F}(x), \mathcal{F}(y)) \xrightarrow{ev_\mu} \mathcal{F}(y)$$

with ev_μ being the evaluation at μ. Now, by [[4], 2.2.2.13] there is a canonical isomorphism $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x, y) \cong$

$\mathrm{Un}_*^{\triangleright}\,\mathrm{Map}_{\mathcal{C}}(x,y)$. So we can form a canonical map

$$\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x,y) \cong \mathrm{Un}_*^{\triangleright}\,\mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{\mathrm{Un}_*^{\triangleright}(\rho_\mu)} \mathrm{Un}_*^{\triangleright}\,\mathcal{F}(y)\ . \tag{4.1.0.3}$$

Let us analyze this map. Let H be an n-simplex of $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x,y)$ exhibited by a simplicial functor $H : \mathfrak{C}[\Delta^{n+1}] \longrightarrow \mathcal{C}$. The data of H consists of the maps of the form

$$H^i : \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \cong (\Delta^1)^{\times(n-i)} \longrightarrow \mathrm{Map}_{\mathcal{C}}(x,y)\ ,\ i = 0, \cdots, n.$$

Observe that each map H^i agrees with the restriction of H^0 to a certain $(n-i)$-cube. So the data of H is fully displayed in that of H^0. It can be shown that H^0 performs a quasi n-simplex of $\mathrm{Map}_{\mathcal{C}}(x,y)$ and hence, is an n-simplex of $\mathrm{Un}_*^{\triangleright}\,\mathrm{Map}_{\mathcal{C}}(x,y)$ by Lemma 4.1.0.10. Moreover, note that under the identification $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x,y) \cong \mathrm{Un}_*^{\triangleright}\,\mathrm{Map}_{\mathcal{C}}(x,y)$, H is simply identified to H^0. For each $i \in \{0, \cdots, n\}$, we denote by $H^i_*(\mu) \coloneqq H^i \circ \rho_\mu$. Finally, we get that the map (4.1.0.3) sends H to $H^0_*(\mu)$, (which performs a quasi n-simplex of $\mathcal{F}(y)$).

Lemma 4.1.0.14. *There is a canonical isomorphism*

$$\mathit{Hom}^{\mathrm{R}}_{\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})}(\mu,\nu) \cong (\mathrm{Un}_*^{\triangleright}\,\mathcal{F}(y))_{\nu/} \times_{\mathrm{Un}_*^{\triangleright}\,\mathcal{F}(y)} \mathit{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x,y) \eqqcolon \mathbb{P}_{\mu,\nu}.$$

Proof. With having chosen $H \in \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x,y)$, an n-simplex of $\mathbb{P}_{\mu,\nu}$ is a quasi $(n+1)$-simplex of $\mathcal{F}(y)$ which admits ν as its initial vertex and in addition, admits $H^0_*(\mu)$ as an end face. (To see this, we combine Lemma 4.1.0.10 with the above analyses).

On the other hand, with having chosen $H \in \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}\,\mathcal{C}}(x,y)$, an n-simplex of $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})}(\mu,\nu)$ (as analyzed earlier) is a natural transformation $t : \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}} \longrightarrow \mathcal{F} \circ H$ satisfying that for every $i \in \{0, \cdots, n\}$ the map $t(i) : \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(i) \longrightarrow \mathcal{F}(x)$ collapses its domain to the vertex μ and that the initial vertex of $t(n+1)$ agrees with ν. In particular, the data of t is fully enclosed in that of the map

$$t(n+1) : \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(n+1) = \mathrm{N}\,\mathrm{P}^{\triangleleft}(n+1) \longrightarrow \mathcal{F}(y)$$

subject to the following condition:

(**) for every $i \in \{0, \cdots, n\}$, the composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \times \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(i) \xrightarrow{\mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}} \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(n+1) \xrightarrow{t(n+1)} \mathcal{F}(y)$$

agrees with the composition $\mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \times \mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(i) \longrightarrow \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \xrightarrow{H^i_*(\mu)} \mathcal{F}(y)$.

Note that the first composition corresponds to the restriction of $t(n+1)$ to $\mathrm{N}\,\{A \in \mathrm{P}^{\triangleleft}(n+1) \,|\, A \ni i\}$, which is an end face of the $(n+1)$-cube $\mathfrak{D}^{\triangleleft}_{\Delta^{n+1}}(n+1) = \mathrm{N}\,\mathrm{P}^{\triangleleft}(n+1)$.

When $i = 0$, the condition (**) means that $H^0_*(\mu)$ agrees with the end face of $t(n+1)$ given by the restriction of itself to $\mathrm{N}\,\{A \in \mathrm{P}^{\triangleleft}(n+1) \,|\, A \ni 0\}$. In particular, for each $i \in \{0, \cdots, n\}$, $H^i_*(\mu)$ agrees with the restriction of $t(n+1)$ to $\mathrm{N}\,\{A \in \mathrm{P}^{\triangleleft}(n+1) \,|\, A \supseteq [0,i]\}$. So for every $i \in \{0, \cdots, n\}$, the condition (**) means that the end face of $t(n+1)$ given by the restriction $t(n+1)\big|_{\mathrm{N}\,\{A \in \mathrm{P}^{\triangleleft}(n+1) \,|\, A \ni i\}}$ is degenerate on $t(n+1)\big|_{\mathrm{N}\,\{A \in \mathrm{P}^{\triangleleft}(n+1) \,|\, A \supseteq [0,i]\}}$. This is equivalent to saying that $t(n+1)$ performs a quasi $(n+1)$-simplex of $\mathcal{F}(y)$. Therefore, by the first paragraph, the data of an n-simplex of $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Un}^{\triangleleft}_{\mathcal{C}}(\mathcal{F})}(\mu,\nu)$ is equivalent to that of an n-simplex of $\mathbb{P}_{\mu,\nu}$. The obtained identification is indeed compatible with simplicial structures. □

Remark 4.1.0.15. There is a canonical embedding $\mathrm{Un}_*^{\triangleright}(\mathcal{F}(y)_{\nu/}) \longrightarrow (\mathrm{Un}_*^{\triangleright}\mathcal{F}(y))_{\nu/}$ established as follows. By Lemma 4.1.0.10, an n-simplex of $\mathrm{Un}_*^{\triangleright}(\mathcal{F}(y)_{\nu/})$ is a quasi n-simplex of $\mathcal{F}(y)_{\nu/}$, which can be identified with a map

$$\gamma : \Delta^0 \star (\Delta^1)^{\times\{1,\cdots,n\}} \longrightarrow \mathcal{F}(y)$$

(where "$\star$" stands for the **join operation**) satisfying that for every $i \in \{1,\cdots,n\}$ the restriction $\gamma\big|_{\Delta^0 \star T_i}$ is degenerate on $\gamma\big|_{\Delta^0 \star T_i^\circ}$ and that $\gamma\big|_{\Delta^0}$ agrees with ν. We then associates to γ the composite map

$$\widetilde{\gamma} : (\Delta^1)^{\times\{1,\cdots,n+1\}} \longrightarrow \Delta^0 \star (\Delta^1)^{\times\{1,\cdots,n\}} \xrightarrow{\gamma} \mathcal{F}(y)$$

in which the first map refers to the canonical way of collapsing the $(n+1)$-cube to its subset given by joining the initial vertex with an end face. The map $\widetilde{\gamma}$ is indeed a quasi $(n+1)$-simplex of $\mathcal{F}(y)$ with ν as its initial vertex, and hence is an n-simplex of $(\mathrm{Un}_*^{\triangleright}\mathcal{F}(y))_{\nu/}$.

By the above remark and Remark 4.1.0.12, we can form a composite map

$$\mathcal{F}(y)_{\nu/} \times_{\mathcal{F}(y)} \mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Un}_*^{\triangleright}(\mathcal{F}(y)_{\nu/}) \times_{\mathrm{Un}_*^{\triangleright}\mathcal{F}(y)} \mathrm{Un}_*^{\triangleright}\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathbb{P}_{\mu,\nu} \tag{4.1.0.4}$$

Suppose now that $\mathcal{C}$ is fibrant and $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Set}_\Delta$ is levelwise fibrant. Then the map $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F}) \longrightarrow \mathrm{N}\,\mathcal{C}$ is a left fibration and $\mathrm{N}\,\mathcal{C}$ is an ∞-category. So $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})$ is an ∞-category as well. Our main interest in this section is the following:

Proposition 4.1.0.16. *Let $x, y \in \mathrm{Ob}(\mathcal{C})$ be two objects of $\mathcal{C}$. Given two vertices $\mu \in \mathcal{F}(x)$ and $\nu \in \mathcal{F}(y)$ regarded as vertices of $\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})$, there is a homotopy equivalence*

$$\{\nu\} \times^{\mathrm{h}}_{\mathcal{F}(y)} \mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Un}_{\mathcal{C}}^{\triangleleft}(\mathcal{F})}(\mu,\nu).$$

Proof. We make use of the pullback $\mathcal{F}(y)_{\nu/} \times_{\mathcal{F}(y)} \mathrm{Map}_{\mathcal{C}}(x,y)$ as a model for the left hand side. Since $\mathcal{F}(y)$ and $\mathrm{Map}_{\mathcal{C}}(x,y)$ are Kan complexes, that pullback is Kan as well. It implies that the first map of the composition (4.1.0.4) is a weak equivalence (see Remark 4.1.0.12). The second map of (4.1.0.4) is also a weak equivalence, since $\mathrm{Un}_*^{\triangleright}(\mathcal{F}(y)_{\nu/})$ and $(\mathrm{Un}_*^{\triangleright}\mathcal{F}(y))_{\nu/}$ are both contractible. So the map (4.1.0.4) is a weak equivalence. We deduce then by using Lemma 4.1.0.14. □

Remark 4.1.0.17. An analogue holds in the contravariant case. Let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta$ be a simplicial presheaf on $\mathcal{C}$. Let $x, y \in \mathrm{Ob}(\mathcal{C})$ be two objects of $\mathcal{C}$, and suppose we are given two vertices $\mu \in \mathcal{F}(x)$ and $\nu \in \mathcal{F}(y)$ regarded as vertices of $\mathrm{Un}_{\mathcal{C}}^{\triangleright}(\mathcal{F})$. We take ρ_ν to be the composite map

$$\rho_\nu : \mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{\mathcal{F}} \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{F}(y),\mathcal{F}(x)) \xrightarrow{ev_\nu} \mathcal{F}(x)$$

with ev_ν being the evaluation at ν. When $\mathcal{C}$ is fibrant and $\mathcal{F}$ is levelwise fibrant (so that $\mathrm{Un}_{\mathcal{C}}^{\triangleright}(\mathcal{F})$ is an ∞-category), there is a homotopy equivalence

$$(\mathcal{F}(x)_{\mu/} \times_{\mathcal{F}(y)} \mathrm{Map}_{\mathcal{C}}(x,y))^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Un}_{\mathcal{C}}^{\triangleright}(\mathcal{F})}(\mu,\nu).$$

4.2 Twisted arrow ∞-categories of simplicial operads

Let $\mathcal{E}$ be an ordinary category. The *(covariant) twisted arrow category* $\mathrm{Tw}(\mathcal{E})$ is by definition the category whose objects are the morphisms of $\mathcal{E}$ and such that maps from $f : x \to y$ to $f' : x' \to y'$ are given by commutative diagrams of the form

$$\begin{array}{ccc} x & \longleftarrow & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ y & \longrightarrow & y' \end{array} \tag{4.2.0.1}$$

This classical notion was generalized into the ∞-categorical framework due to Lurie [[3], §5.2.1]. For a given ∞-category $\mathcal{D}$, the **twisted arrow category** $\mathrm{Tw}(\mathcal{D})$ is the ∞-category whose n-simplices are the $(2n+1)$-simplices of $\mathcal{C}$. In particular, objects of $\mathrm{Tw}(\mathcal{D})$ are the morphisms of $\mathcal{D}$. A map from $f : x \longrightarrow y$ to $f' : x' \longrightarrow y'$ can be also depicted as a diagram of the type (4.2.0.1) commutative up to a chosen homotopy.

Furthermore, for a given fibrant simplicial category $\mathcal{C}$, the twisted arrow ∞-category of $\mathcal{C}$ is simply $\mathrm{Tw}(\mathcal{C}) := \mathrm{Tw}(\mathrm{N}\,\mathcal{C})$ the twisted arrow ∞-category of the nerve of $\mathcal{C}$. It turns out that $\mathrm{Tw}(\mathcal{C})$ can be represented as the unstraightening of a certain simplicial copresheaf. This phenomenon motivates us to establish *twisted arrow ∞-categories of (fibrant) simplicial operads*. Let us see how it arises.

We will need the following notations and conventions:

- For the remainder, we consider only the covariant unstraightening construction. So we will simply write $\mathrm{Un}_{\mathcal{C}}(-)$ standing for the covariant unstraightening of simplicial functors $\mathcal{C} \longrightarrow \mathrm{Set}_{\Delta}$.

- Let S be a simplicial set. We denote by $(\mathrm{Set}_{\Delta}^{\mathrm{cov}})_{/S}$ the **covariant model category of simplicial sets over** S whose cofibrations are monomorphisms and whose fibrant objects are left fibrations over S. (See [[4], §2.1.4] for more details).

- Let $\mathcal{C}$ be a simplicial category. Recall that the category of $\mathcal{C}$-bimodules can be represented as $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Set}_{\Delta})$ the category of simplicial functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Set}_{\Delta}$. Moreover, under this identification, the functor $\mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Set}_{\Delta}$, $(x, y) \mapsto \mathrm{Map}_{\mathcal{C}}(x, y)$ is nothing but $\mathcal{C}$, viewed as a bimodule over itself. We endow $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Set}_{\Delta})$ with the projective model structure.

Suppose now that $\mathcal{C}$ is a fibrant simplicial category. According to [[3], Proposition 5.2.1.11], the unstraightening functor

$$\mathrm{Un}_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Set}_{\Delta}) \xrightarrow{\simeq} (\mathrm{Set}_{\Delta}^{\mathrm{cov}})_{/\,\mathrm{N}\,\mathcal{C}^{\mathrm{op}} \times \mathrm{N}\,\mathcal{C}}, \tag{4.2.0.2}$$

which is a right Quillen equivalence, identifies $\mathrm{Map}_{\mathcal{C}}$ to $\mathrm{Tw}(\mathcal{C})$. In other words, the unstraightening of $\mathcal{C}$ (viewed as a bimodule over itself) is exactly a model for $\mathrm{Tw}(\mathcal{C})$.

We now fix $\mathcal{P}$ to be a fibrant simplicial C-colored operad. Recall from Section 3.2.2 that there is a canonical isomorphism $\mathrm{IbMod}(\mathcal{P}) \cong \mathrm{Fun}(\mathbf{Ib}^{\mathcal{P}}, \mathrm{Set}_{\Delta})$ between the categories of infinitesimal $\mathcal{P}$-bimodules and simplicial functors $\mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathrm{Set}_{\Delta}$. The unstraightening construction gives us a right Quillen equivalence

$$\mathrm{Un}_{\mathbf{Ib}^{\mathcal{P}}} : \mathrm{IbMod}(\mathcal{P}) = \mathrm{Fun}(\mathbf{Ib}^{\mathcal{P}}, \mathrm{Set}_{\Delta}) \xrightarrow{\simeq} (\mathrm{Set}_{\Delta}^{\mathrm{cov}})_{/\,\mathrm{N}(\mathbf{Ib}^{\mathcal{P}})}. \tag{4.2.0.3}$$

Since $\mathcal{P}$ is fibrant, it implies that $\mathbf{Ib}^{\mathcal{P}}$ is a fibrant simplicial category, and hence $\mathrm{N}(\mathbf{Ib}^{\mathcal{P}})$ is an ∞-category. Moreover, the functor

$$\mathcal{P} : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathrm{Set}_{\Delta}\ ,\ (c_1, \cdots, c_m; c) \mapsto \mathcal{P}(c_1, \cdots, c_m; c),$$

which encodes the data of $\mathcal{P}$ as an infinitesimal bimodule over itself, is levelwise fibrant. So we get a left fibration $\mathrm{Un}_{\mathbf{Ib}^{\mathcal{P}}}(\mathcal{P}) \longrightarrow \mathrm{N}(\mathbf{Ib}^{\mathcal{P}})$, and hence $\mathrm{Un}_{\mathbf{Ib}^{\mathcal{P}}}(\mathcal{P})$ is in particular an ∞-category.

Along the passage from simplicial categories to simplicial operads, the notion of categorical bimodules should correspond to that of operadic infinitesimal bimodules, which can be thought of as the linearization of the usual notion of operadic bimodules. With having this logic in mind, we propose the following concept.

Definition 4.2.0.1. The **twisted arrow ∞-category of** $\mathcal{P}$, denoted by $\mathrm{Tw}(\mathcal{P})$, is defined to be $\mathrm{Tw}(\mathcal{P}) := \mathrm{Un}_{\mathbf{Ib}^{\mathcal{P}}}(\mathcal{P})$ the unstraightening of the simplicial functor $\mathcal{P} : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathrm{Set}_{\Delta}$ or alternatively, unstraightening of $\mathcal{P}$ regarded as an infinitesimal bimodule over itself.

Examples 4.2.0.2. When $\mathcal{P}$ is discrete then $\mathrm{Tw}(\mathcal{P})$ is isomorphic to the nerve of an ordinary category. In this situation, we will identify $\mathrm{Tw}(\mathcal{P})$ to the corresponding ordinary category and refer to it as the **twisted arrow category of** $\mathcal{P}$. For example, it is not hard to show that the twisted arrow category of the **commutative operad** is equivalent to $\mathrm{Fin}_*^{\mathrm{op}}$. We will see that the twisted arrow category of the **associative operad** is equivalent to the simplex category Δ (cf. Proposition 4.2.0.15).

Proposition 4.2.0.3. *The construction* $\mathrm{Tw}(-)$ *determines a homotopy invariant from fibrant simplicial operads to* ∞*-categories.*

Proof. Let $f : \mathcal{P} \to \mathcal{Q}$ be a map between fibrant simplicial operads. By the compatibility of the unstraightening functor with taking base change along $f : \mathcal{P} \to \mathcal{Q}$, we obtain the induced map $\mathrm{Tw}(f) : \mathrm{Tw}(\mathcal{P}) \longrightarrow \mathrm{Tw}(\mathcal{Q})$ fitting into the following Cartesian square of ∞-categories

$$\begin{array}{ccc} \mathrm{Tw}(\mathcal{P}) & \longrightarrow & \mathrm{Tw}(\mathcal{Q}) \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathbf{Ib}^{\mathcal{P}}) & \longrightarrow & \mathrm{N}(\mathbf{Ib}^{\mathcal{Q}}) \end{array}$$

Note that this square is already homotopy Cartesian (with respect to the Joyal model structure), due to the fact that the right vertical map is a left fibration.

We are showing that the map $\mathrm{Tw}(f) : \mathrm{Tw}(\mathcal{P}) \longrightarrow \mathrm{Tw}(\mathcal{Q})$ is an equivalence when provided that $f : \mathcal{P} \to \mathcal{Q}$ is a weak equivalence. We first show that the induced map $\mathbf{Ib}^f : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathbf{Ib}^{\mathcal{Q}}$ is a weak equivalence of simplicial categories (i.e., a Dwyer-Kan equivalence). It is clear by construction that $\mathbf{Ib}^f$ is a levelwise weak equivalence. Hence it remains to show that $\mathbf{Ib}^f$ is essentially surjective. Suppose given an object $(d_1, \cdots, d_n; d_0)$ of $\mathbf{Ib}^{\mathcal{Q}}$. Since the underlying simplicial functor $f_1 : \mathcal{P}_1 \longrightarrow \mathcal{Q}_1$ of f is essentially surjective, for each $i \in \{0, \cdots, n\}$ there exists an object c_i of $\mathcal{P}$ together with an isomorphism $\theta_i : f(c_i) \xrightarrow{\simeq} d_i$ in the homotopy category of $\mathcal{Q}_1$. The morphisms θ_i's together form a morphism $\theta : (f(c_1), \cdots, f(c_n); f(c_0)) \longrightarrow (d_1, \cdots, d_n; d_0)$ in $\mathbf{Ib}^{\mathcal{Q}}$. It can then be verified by definition that θ is an isomorphism in the homotopy category of $\mathbf{Ib}^{\mathcal{Q}}$. We just proved that $\mathbf{Ib}^f$ is a weak equivalence (between fibrant simplicial categories). So the map $\mathrm{N}(\mathbf{Ib}^{\mathcal{P}}) \longrightarrow \mathrm{N}(\mathbf{Ib}^{\mathcal{Q}})$ is an equivalence of ∞-categories. This fact, together with the first paragraph, proves that the map $\mathrm{Tw}(f) : \mathrm{Tw}(\mathcal{P}) \longrightarrow \mathrm{Tw}(\mathcal{Q})$ is indeed an equivalence □

Again, let $\mathcal{P}$ be a fibrant simplicial C-colored operad. Due to the previous section, we can describe the simplicial structure of $\mathrm{Tw}(\mathcal{P})$.

Construction 4.2.0.4. The data of an n-simplex of $\mathrm{Tw}(\mathcal{P})$ consists of

- an n-simplex $\varphi \in \mathrm{N}(\mathbf{Ib}^{\mathcal{P}})$, i.e., a functor $\varphi : \mathfrak{C}[\Delta^n] \longrightarrow \mathbf{Ib}^{\mathcal{P}}$, and
- a natural transformation $t : \mathfrak{D}^{\triangleleft}_{\Delta^n} \longrightarrow \mathcal{P} \circ \varphi$ between simplicial functors $\mathfrak{C}[\Delta^n] \longrightarrow \mathrm{Set}_{\Delta}$.

For each map $\delta : \Delta^m \longrightarrow \Delta^n$, the simplicial structure map $\mathrm{Tw}(\mathcal{P})_n \longrightarrow \mathrm{Tw}(\mathcal{P})_m$ is induced by the natural transformation $\mathfrak{D}^{\triangleleft}_{\delta} : \mathfrak{D}^{\triangleleft}_{\Delta^m} \longrightarrow \mathfrak{D}^{\triangleleft}_{\Delta^n} \circ \mathfrak{C}[\delta]$, as in Construction 4.1.0.3.

To analyze the structure of $\mathrm{Tw}(\mathcal{P})$, we will need the following notations.

Notation 4.2.0.5. We always denote the permutations by listing their values. For instance, $\sigma = [i_1, \cdots, i_n]$ refers to the permutation $\sigma \in \Sigma_n$ with $\sigma(k) = i_k$.

Notations 4.2.0.6. Let $f : \langle n \rangle \longrightarrow \langle m \rangle$ be a map in Fin_*. For each $s \in \{0, 1, \cdots, m\}$, we let $[f^{-1}(s)]$ denote the increasing sequence of the elements of $f^{-1}(s)$, written as $[f^{-1}(s)] = [i^s_1 < \cdots < i^s_{k_s}]$. (Of course, this could be empty). Then f can be represented by the sequence obtained by concatenating those sequences for $s = 1, \cdots, m$, written as:

$$[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m)] = [i^1_1 < \cdots < i^1_{k_1} \mid i^2_1 < \cdots < i^2_{k_2} \mid \cdots \mid i^m_1 < \cdots < i^m_{k_m}],$$

or alternatively, by the extended sequence $[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m) \mid f^{-1}(0)^{\circ}]$ formed in the same manner, in which $[f^{-1}(0)^{\circ}] := [f^{-1}(0)] - \{0\}$. Moreover, we denote by σ_f the permutation $[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m) \mid f^{-1}(0)^{\circ}] \in \Sigma_n$.

Notation 4.2.0.7. Let $\overline{c} := (c_1, \cdots, c_m; c)$ and $\overline{d} := (d_1, \cdots, d_n; d)$ be two C-sequences and $f : \langle n \rangle \longrightarrow \langle m \rangle$ a map in Fin_*. Given a vertex

$$\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m) \in \mathrm{Map}^f_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c}, \overline{d})$$

with $\alpha_0 \in \mathcal{P}\big(c, \{d_j\}_{j \in f^{-1}(0)}; d\big)$ and $\alpha_i \in \mathcal{P}\big(\{d_j\}_{j \in f^{-1}(i)}; c_i\big)$ $(i = 1, \cdots, m)$ (see Construction 2.1.3.2), we denote by $\alpha^* : \mathcal{P}(\overline{c}) \longrightarrow \mathcal{P}(\overline{d})$ the image of α under the map $\mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c}, \overline{d}) \longrightarrow \mathrm{Map}_{\mathrm{Set}_{\Delta}}(\mathcal{P}(\overline{c}), \mathcal{P}(\overline{d}))$ which is part of the simplicial functor structure of $\mathcal{P} : \mathbf{Ib}^{\mathcal{P}} \longrightarrow \mathrm{Set}_{\Delta}$. By construction, for each simplex $\theta \in \mathcal{P}(\overline{c})$, we have that

$$\alpha^*(\theta) = (\, \alpha_0 \circ_1 \theta \circ (\alpha_1, \cdots, \alpha_m) \,)^{\sigma_f^{-1}} \in \mathcal{P}(\overline{d})$$

the action of $\sigma_f^{-1} \in \Sigma_n$ on $\alpha_0 \circ_1 \theta \circ (\alpha_1, \cdots, \alpha_m)$, where "$(\circ_1)\ \circ$" refers to the (partial) composition.

Unwinding definition we will see that $\mathrm{Tw}(\mathcal{P})$ indeed looks like something obtained by twisting "multiarrows" of $\mathcal{P}$.

• Objects of $\mathrm{Tw}(\mathcal{P})$ are precisely the **operations** of $\mathcal{P}$ (i.e., the vertices of the spaces of operations of $\mathcal{P}$).

• Let $\mu \in \mathcal{P}(\overline{c})$ and $\nu \in \mathcal{P}(\overline{d})$ be two operations of $\mathcal{P}$, the data of a morphism (edge) $\mu \to \nu$ in $\mathrm{Tw}(\mathcal{P})$ consists of

- a map $f : \langle n \rangle \longrightarrow \langle m \rangle$ in Fin_*,
- a tuple of operations $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m) \in \mathrm{Map}^f_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c}, \overline{d})$, and
- an edge $t : \nu \longrightarrow \alpha^*(\mu)$ in $\mathcal{P}(\overline{d})$, viewed as a homotopy from ν to $\alpha^*(\mu)$.

By convention, we will write $(f, \alpha, t) : \mu \longrightarrow \nu$ standing for such a typical morphism. It is convenient to depict this morphism as the diagram of the form

$$\begin{array}{ccc} (c_1, \cdots, c_m) & \xleftarrow{(\alpha_1 \cdots \alpha_m)} & (d_1, \cdots, d_n) \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \nu} \\ (c) & \xrightarrow[\alpha_0]{} & (d) \end{array} \tag{4.2.0.4}$$

which is "commutative up to a chosen homotopy".

• In general, a k-simplex of $\mathrm{Tw}(\mathcal{P})$ is the one, formally, depicted as the composition of k squares of the type (4.2.0.4) equipped with a collection of homotopies, homotopies between homotopies, and so forth.

We saw above the simplicial structure of $\mathrm{Tw}(\mathcal{P})$. So it remains to understand the ∞-categorical structure of $\mathrm{Tw}(\mathcal{P})$. As usual, this is best understood by reformulating equivalences in $\mathrm{Tw}(\mathcal{P})$ and describing its mapping spaces. We indeed hold these in hand, expressed by the two propositions below.

Again, let $\mu \in \mathcal{P}(\overline{c})$ and $\nu \in \mathcal{P}(\overline{d})$ be two operations of $\mathcal{P}$, regarded as objects of $\mathrm{Tw}(\mathcal{P})$.

Proposition 4.2.0.8. *A morphism $(f,\alpha,t) : \mu \longrightarrow \nu$ is an equivalence in $\mathrm{Tw}(\mathcal{P})$ if and only if the following conditions hold:*

1. *$f : \langle n \rangle \longrightarrow \langle m \rangle$ is bijective.*
2. *There exist $\beta \in \mathrm{Map}^{f^{-1}}_{\boldsymbol{Ib}^{\mathcal{P}}}(\overline{d},\overline{c})$, an edge $t' : \mu \longrightarrow \beta^*(\nu)$ and together with two edges*

$$[h : id_{\overline{c}} \longrightarrow \beta\alpha] \in \mathrm{Map}^{\mathrm{Id}_{\langle m\rangle}}_{\boldsymbol{Ib}^{\mathcal{P}}}(\overline{c},\overline{c}) \;\; , \;\; [h' : id_{\overline{d}} \longrightarrow \alpha\beta] \in \mathrm{Map}^{\mathrm{Id}_{\langle n\rangle}}_{\boldsymbol{Ib}^{\mathcal{P}}}(\overline{d},\overline{d})$$

which are all subject to the existence of two 2-simplices of the forms

$$\begin{array}{ccc} & \beta^*(\nu) & \\ {}^{t'}\nearrow & & \searrow^{\beta^*(t)} \\ \mu & \xrightarrow{h^*|_{\{\mu\}\times\Delta^1}} & \beta^*\alpha^*(\mu) \end{array} \qquad \begin{array}{ccc} & \alpha^*(\mu) & \\ {}^{t}\nearrow & & \searrow^{\alpha^*(t')} \\ \nu & \xrightarrow{(h')^*|_{\{\nu\}\times\Delta^1}} & \alpha^*\beta^*(\nu) \end{array} \tag{4.2.0.5}$$

belonging to $\mathcal{P}(\overline{c})$ and $\mathcal{P}(\overline{d})$, respectively. In this item, h^* (and similarly, $(h')^*$) is given by the composition

$$\mathcal{P}(\overline{c}) \times \Delta^1 \xrightarrow{\mathrm{Id}_{\mathcal{P}(\overline{c})} \times h} \mathcal{P}(\overline{c}) \times \mathrm{Map}^{\mathrm{Id}_{\langle m\rangle}}_{\boldsymbol{Ib}^{\mathcal{P}}}(\overline{c},\overline{c}) \xrightarrow{\mathcal{P}} \mathcal{P}(\overline{c}).$$

Proof. The proof is straightforward, once one knows that the data of a 2-simplex of $\mathrm{Tw}(\mathcal{P})$ of the form

$$\begin{array}{ccc} & \nu & \\ {}^{(f,\alpha,t)}\nearrow & & \searrow^{(g,\beta,t')} \\ \mu & \xrightarrow{s_0\mu} & \mu \;, \end{array}$$

with having (necessarily) required $fg = \mathrm{Id}_{\langle m\rangle}$, is equivalent to the choice of an edge $h : id_{\overline{c}} \longrightarrow \beta\alpha$ in $\mathrm{Map}^{\mathrm{Id}_{\langle m\rangle}}_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c},\overline{c})$ and in addition, a square in $\mathcal{P}(\overline{c})$ of the form

$$\begin{array}{ccc} \mu & \xrightarrow{s_0\mu} & \mu \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle h^*|_{\{\mu\}\times\Delta^1}} \\ \beta^*(\nu) & \xrightarrow[\beta^*(t)]{} & \alpha^*\beta^*(\nu) \;. \end{array}$$

Moreover, since $\mathcal{P}(\overline{c})$ is Kan, this square is equivalent to the choice of a 2-simplex of the form (4.2.0.5) (the first one). □

We take ρ_μ to be the composite map

$$\rho_\mu : \mathrm{Map}_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c},\overline{d}) \xrightarrow{\mathcal{P}} \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{P}(\overline{c}),\mathcal{P}(\overline{d})) \xrightarrow{ev_\mu} \mathcal{P}(\overline{d})$$

with ev_μ being the evaluation at μ.

Proposition 4.2.0.9. *There is a canonical homotopy equivalence*

$$\{\nu\} \times^{\mathrm{h}}_{\mathcal{P}(\overline{d})} \mathrm{Map}_{\boldsymbol{Ib}^{\mathcal{P}}}(\overline{c}, \overline{d}) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu).$$

Proof. This follows immediately by Proposition 4.1.0.16. □

Remark 4.2.0.10. Note that there is a canonical map of ∞-categories $\mathrm{Tw}(\mathcal{P}) \longrightarrow \mathrm{N}(\mathrm{Fin}_*^{\mathrm{op}})$ factoring through $\mathrm{N}(\mathbf{Ib}^{\mathcal{P}})$. Consider the induced map $\mathrm{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu) \longrightarrow \mathrm{Hom}_{\mathrm{Fin}_*^{\mathrm{op}}}(\langle m \rangle, \langle n \rangle)$. For each map $f : \langle n \rangle \longrightarrow \langle m \rangle$ in Fin_*, we denote by $\mathrm{Map}^f_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu)$ the component of $\mathrm{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu)$ over f, so that we can write

$$\mathrm{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu) = \coprod_{\langle n \rangle \xrightarrow{f} \langle m \rangle} \mathrm{Map}^f_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu).$$

By the above proposition, there is a homotopy equivalence

$$\{\nu\} \times^{\mathrm{h}}_{\mathcal{P}(\overline{d})} \mathrm{Map}^f_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c}, \overline{d}) \xrightarrow{\simeq} \mathrm{Map}^f_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu).$$

In what follows, we will survey some local properties of $\mathrm{Tw}(\mathcal{P})$ using the two propositions above.

Corollary 4.2.0.11. *Let $\mu \in \mathcal{P}(c_1, \cdots, c_m; c)$ and $\sigma \in \Sigma_m$ be given. Then there is a canonical equivalence $\mu \xrightarrow{\simeq} \mu^\sigma$ in* $\mathrm{Tw}(\mathcal{P})$.

Proof. We take a canonical edge $(f, \alpha, t) : \mu \longrightarrow \mu^\sigma$ as follows. The map $f : \langle m \rangle \longrightarrow \langle m \rangle$ agrees with σ on $\{1, \cdots, m\}$. Then, we take α to be the tuple of unit operations

$$\alpha := (\mathrm{id}_c, \mathrm{id}_{c_1} \cdots, \mathrm{id}_{c_m}) \in \mathrm{Map}^\sigma_{\mathbf{Ib}^{\mathcal{P}}}((c_1, \cdots, c_m; c), (c_{\sigma(1)}, \cdots, c_{\sigma(m)}; c)).$$

By construction, the induced map

$$\alpha^* : \mathcal{P}(c_1, \cdots, c_m; c) \longrightarrow \mathcal{P}(c_{\sigma(1)}, \cdots, c_{\sigma(m)}; c)$$

is nothing but the map defined by the action of the permutation σ on $\mathcal{P}$ (see the proof of Proposition 2.1.3.3). In particular, we get that $\alpha^*(\mu) = \mu^\sigma$. Finally, we take $t := s_0\mu^\sigma$. Using Proposition 4.2.0.8, we can readily verify that the obtained edge (f, α, t) is indeed an equivalence in $\mathrm{Tw}(\mathcal{P})$. □

By convention, a simplicial operad is said to be k-**connected** if all its spaces of operations are k-connected. For example, the **little n-cubes operad** E_n is $(n-2)$-connected (cf., e.g., [[9], Corollary 5.1.11]). As before, let $\mu \in \mathcal{P}(\overline{c})$ and $\nu \in \mathcal{P}(\overline{d})$ be two operations of $\mathcal{P}$, regarded as objects of $\mathrm{Tw}(\mathcal{P})$.

Corollary 4.2.0.12. *Suppose we are given an integer k and suppose further that $\mathcal{P}$ is k-connected. Then the induced map*

$$\mathrm{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu) \longrightarrow Hom_{\mathrm{Fin}_*^{\mathrm{op}}}(\langle m \rangle, \langle n \rangle) \tag{4.2.0.6}$$

is k-connected. In particular, whenever $k \geqslant 1$, the map $\mathrm{Tw}(\mathcal{P}) \longrightarrow \mathrm{N}(\mathrm{Fin}_*^{\mathrm{op}})$ induces an equivalence $\mathrm{Ho}(\mathrm{Tw}(\mathcal{P})) \xrightarrow{\simeq} \mathrm{Fin}_*^{\mathrm{op}}$ between (ordinary) homotopy categories.

Proof. For each map $f : \langle n \rangle \longrightarrow \langle m \rangle$ in Fin_*, by Remark 4.2.0.10 we have a fiber sequence

$$\mathrm{Map}^f_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu) \longrightarrow \mathrm{Map}^f_{\mathbf{Ib}^{\mathcal{P}}}(\overline{c}, \overline{d}) \longrightarrow \mathcal{P}(\overline{d}).$$

By assumption, both the second and third terms are k-connected. It implies that $\mathrm{Map}^f_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu)$ is $(k-1)$-connected, and hence the map (4.2.0.6) is indeed k-connected.

Consequently, when $k \geqslant 1$ we have that

$$\pi_0 \operatorname{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \nu) \cong \operatorname{Hom}_{\mathrm{Fin}_*^{\mathrm{op}}}(\langle m \rangle, \langle n \rangle).$$

In particular, this implies that any two operations of $\mathcal{P}$ that are of the same arity are equivalent as objects of $\mathrm{Tw}(\mathcal{P})$. The proof is therefore completed. $\square$

We find a somewhat large class of simplicial operads whose twisted arrow ∞-categories admit terminal objects.

Definition 4.2.0.13. A simplicial operad is said to be **homotopy unital** if all its spaces of unary operations are weakly contractible. Furthermore, a simplicial operad is said to be **unitally homotopy connected** if it is homotopy unital with weakly contractible spaces of 1-ary operations.

Typical examples for this definition include the little cubes operads E_n for $n = 0, \cdots, \infty$.

Corollary 4.2.0.14. *Let $\mathcal{P}$ be a fibrant and homotopy unital simplicial operad. Let d be a color of $\mathcal{P}$ and $\mu_d \in \mathcal{P}(d)$ an unary operation of $\mathcal{P}$. Then μ_d is a terminal object of $\mathrm{Tw}(\mathcal{P})$ if and only if for every color c, the space $\mathcal{P}(c; d)$ is contractible. Consequently, if $\mathcal{P}$ is fibrant and unitally homotopy connected then $\mathrm{Tw}(\mathcal{P})$ admits terminal objects being precisely the unary operations of $\mathcal{P}$.*

Proof. By definition, μ_d is a terminal object of $\mathrm{Tw}(\mathcal{P})$ precisely if for every operation $\mu \in \mathcal{P}$ the mapping space $\operatorname{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \mu_d)$ is contractible. Given any operation $\mu \in \mathcal{P}(c_1, \cdots, c_m; c)$, it suffices to show that there is a homotopy equivalence of spaces:

$$\mathcal{P}(c; d) \simeq \operatorname{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \mu_d).$$

By Proposition 4.2.0.9 there is a homotopy equivalence

$$\{\mu_d\} \times^h_{\mathcal{P}(d)} \operatorname{Map}_{\mathbf{Ib}^{\mathcal{P}}}((c_1, \cdots, c_m; c), (d)) \xrightarrow{\simeq} \operatorname{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \mu_d).$$

Since $\mathcal{P}(d)$ is contractible by assumption, the homotopy pullback is equivalent to $\operatorname{Map}_{\mathbf{Ib}^{\mathcal{P}}}((c_1, \cdots, c_m; c), (d))$. Furthermore, note that

$$\operatorname{Map}_{\mathbf{Ib}^{\mathcal{P}}}((c_1, \cdots, c_m; c), (d)) = \mathcal{P}(c; d) \times \mathcal{P}(c_1) \times \cdots \times \mathcal{P}(c_m),$$

which is therefore weakly equivalent to $\mathcal{P}(c; d)$, by assumption again. So we get indeed a homotopy equivalence $\mathcal{P}(c; d) \simeq \operatorname{Map}_{\mathrm{Tw}(\mathcal{P})}(\mu, \mu_d)$. $\square$

Finally, for more illustration, we prove the following:

Proposition 4.2.0.15. *There is a canonical equivalence $\varphi : \Delta \xrightarrow{\simeq} \mathrm{Tw}(\mathcal{A}ss)$ between the simplex category and the twisted arrow category of the associative operad $\mathcal{A}ss$.*

To this end, we first revisit some basic constructions.

Construction 4.2.0.16. There is a canonical functor $\iota : \Delta \longrightarrow \mathrm{Fin}_*^{\mathrm{op}}$ (which is essentially used to define underlying cosimplicial spaces of **gamma spaces**) defined by sending $[n] \in \Delta$ to $\langle n \rangle \in \mathrm{Fin}_*^{\mathrm{op}}$ and by giving the natural maps $\iota_{m,n} : \operatorname{Hom}_{\Delta}([m], [n]) \longrightarrow \operatorname{Hom}_{\mathrm{Fin}_*}(\langle n \rangle, \langle m \rangle)$ as follows. Given a map $f : [m] \longrightarrow [n]$, let us identify f with the increasing sequence $[j_1 < \cdots < j_k]$ of its values together with a tuple $(p_1, \cdots, p_k)$ with p_r being the cardinality of the fiber $f^{-1}(j_r)$. The map $\iota_{m,n}(f) : \langle n \rangle \longrightarrow \langle m \rangle$ is given by listing its nonempty fibers as follows:

$$\iota_{m,n}(f)^{-1}(p_1) = \{j_1 + 1, \cdots, j_2\} \ , \ \iota_{m,n}(f)^{-1}(p_1 + p_2) = \{j_2 + 1, \cdots, j_3\} \ , \cdots,$$

$$\iota_{m,n}(f)^{-1}(p_1 + p_2 + \cdots + p_{k-1}) = \{j_{k-1} + 1, \cdots, j_k\} \ , \ \iota_{m,n}(f)^{-1}(0) = \{0, 1, \cdots, n\} - \{j_1 + 1, \cdots, j_k\}.$$

Remark 4.2.0.17. It is straightforward to verify the following observations:

(1) The image $\mathrm{Im}(\iota_{m,n}) \subseteq \mathrm{Hom}_{\mathrm{Fin}_*}(\langle n\rangle, \langle m\rangle)$ of $\iota_{m,n}$ consists of precisely those maps g such that $[g^{-1}(1) \mid g^{-1}(2) \mid \cdots \mid g^{-1}(m)]$ (cf. Notations 4.2.0.6) is either empty or forms a sequence of consecutive natural numbers.

(2) Let $\mathrm{Hom}_{\Delta}^{\mathrm{const}}([m],[n]) \subseteq \mathrm{Hom}_{\Delta}([m],[n])$ denote the subset consisting of the constant maps and let $\mathrm{Hom}_{\Delta}^{\circ}([m],[n])$ denote the complement of the previous one in $\mathrm{Hom}_{\Delta}([m],[n])$. Likewise, we denote by $\mathrm{const}_0 \in \mathrm{Hom}_{\mathrm{Fin}_*}(\langle n\rangle, \langle m\rangle)$ the unique constant map (with value $0 \in \langle m\rangle$) and let $\mathrm{Im}^{\circ}(\iota_{m,n})$ be the complement $\mathrm{Im}(\iota_{m,n}) - \{\mathrm{const}_0\}$. Observe then that the restriction of $\iota_{m,n}$ to $\mathrm{Hom}_{\Delta}^{\circ}([m],[n])$ induces a natural bijection $\iota_{m,n}^{\circ} : \mathrm{Hom}_{\Delta}^{\circ}([m],[n]) \xrightarrow{\simeq} \mathrm{Im}^{\circ}(\iota_{m,n})$.

Recall that the associative operad $\mathcal{A}ss$ is the single-colored operad whose set of n-ary operations is given by $\mathcal{A}ss(n) = \Sigma_n$ for $n \geqslant 0$, equipped with the canonical right action of Σ_n on itself. The composition is defined by concatenating linear orders.

Proof of Proposition 4.2.0.15. By Corollary 4.2.0.11, $\mathrm{Tw}(\mathcal{A}ss)$ admits a skeleton whose objects are the trivial permutations $\mu_n := [1, \cdots, n] \in \Sigma_n$, $n \geqslant 0$. We define φ on objects by sending $[n] \in \Delta$ to μ_n. It remains to establish natural isomorphisms of the form

$$\varphi_{m,n} : \mathrm{Hom}_{\Delta}([m],[n]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Tw}(\mathcal{A}ss)}(\mu_m, \mu_n).$$

Let us analyze the right hand side. By definition, we write

$$\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A}ss)}(\mu_m, \mu_n) = \bigsqcup_{\langle n\rangle \xrightarrow{f} \langle m\rangle} \mathcal{A}_f$$

where $\mathcal{A}_f \subseteq \mathrm{Hom}^{f}_{\mathbf{Ib}^{\mathcal{A}ss}}((c_1, \cdots, c_m; c), (d_1, \cdots, d_n; d))$ denotes the subset consisting of those $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m)$ such that $\alpha^*(\mu_m) = \mu_n$ (see Notation 4.2.0.7). Unwinding definition, the latter is equivalent to the equation

$$\alpha_0 \circ_1 \mu_m \circ (\alpha_1, \cdots, \alpha_m) = [f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m) \mid f^{-1}(0)^{\circ}] \tag{4.2.0.7}$$

Observe that there is a unique choice of $(\alpha_1, \cdots, \alpha_m)$ such that this equation possibly admits solutions, being precisely $(\mu_{h(1)}, \cdots, \mu_{h(m)})$, i.e., the tuple of trivial permutations with $h(i)$ referring to the arity of α_i. Thus, by comparing the two sides of (4.2.0.7) after substituting $(\alpha_1, \cdots, \alpha_m) = (\mu_{h(1)}, \cdots, \mu_{h(m)})$ back to it, we realize that the set $\mathcal{A}_f$ is nonempty only if the sequence $[f^{-1}(1) \mid f^{-1}(2) \mid \cdots \mid f^{-1}(m)]$, whenever it is nonempty, forms a sequence of consecutive natural numbers. But this condition is equivalent to saying that $f \in \mathrm{Im}(\iota_{m,n})$ (see Remark 4.2.0.17), so we can rewrite

$$\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A}ss)}(\mu_m, \mu_n) = \bigsqcup_{f \in \mathrm{Im}(\iota_{m,n})} \mathcal{A}_f.$$

As in Remark 4.2.0.17, we write $\mathrm{Im}(\iota_{m,n}) = \{\mathrm{const}_0\} \sqcup \mathrm{Im}^{\circ}(\iota_{m,n})$. Unwinding computation, we have that

- when $f \in \mathrm{Im}^{\circ}(\iota_{m,n})$, there is also a unique choice of α_0 solving the equation (4.2.0.7), and
- when $f = \mathrm{const}_0$, there are $(n+1)$ choices of $\alpha_0 \in \Sigma_{n+1}$ solving (4.2.0.7) precisely given as

$$\alpha_0^i := [i, 1, \cdots, i-1, i+1, \cdots, n+1] \ , \ i = 1, \cdots, n+1.$$

So we can rewrite once more

$$\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A}ss)}(\mu_m, \mu_n) = \{\alpha_0^i\}_{i=1}^{n+1} \sqcup \mathrm{Im}^{\circ}(\iota_{m,n}).$$

Finally, we find the desired natural bijection $\varphi_{m,n}$ separated into two components as follows:

$$\varphi_{m,n} = \varphi^{\text{const}}_{m,n} \sqcup \iota^{\circ}_{m,n} : \text{Hom}^{\text{const}}_{\Delta}([m],[n]) \sqcup \text{Hom}^{\circ}_{\Delta}([m],[n]) \xrightarrow{\simeq} \{\alpha^i_0\}^{n+1}_{i=1} \sqcup \text{Im}^{\circ}(\iota_{m,n})$$

in which $\varphi^{\text{const}}_{m,n}$ sends each constant map $[m] \longrightarrow [n]$ with value i to α^{i+1}_0, while $\iota^{\circ}_{m,n} : \text{Hom}^{\circ}_{\Delta}([m],[n]) \xrightarrow{\simeq}$ $\text{Im}^{\circ}(\iota_{m,n})$ is the natural bijection mentioned in Remark 4.2.0.17. □

4.3 Main statements

Let $\mathcal{P}$ be a fibrant and Σ-cofibrant C-colored simplicial operad. We shall now prove that the cotangent complex $\text{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\text{Op}(\text{Set}_{\Delta})$ can be represented as a spectrum valued functor on $\text{Tw}(\mathcal{P})$. Our treatment is inspired by the work of [[7], §3.3].

As introduced in the previous section, $\text{Tw}(\mathcal{P})$ is defined to be the image of $\mathcal{P}$ through the unstraightening functor $\text{Un}_{\mathbf{Ib}^{\mathcal{P}}} : \text{IbMod}(\mathcal{P}) = \text{Fun}(\mathbf{Ib}^{\mathcal{P}}, \text{Set}_{\Delta}) \xrightarrow{\simeq} (\text{Set}^{\text{cov}}_{\Delta})_{/\,\text{N}(\mathbf{Ib}^{\mathcal{P}})}$. To avoid complication of notation, we will write $\mathcal{U}n$ instead $\text{Un}_{\mathbf{Ib}^{\mathcal{P}}}$. Recall by construction that there is a canonical left fibration $\text{Tw}(\mathcal{P}) = \mathcal{U}n(\mathcal{P}) \longrightarrow \text{N}(\mathbf{Ib}^{\mathcal{P}})$.

Observe first that $\mathcal{U}n$ induces a right Quillen equivalence (denoted by)

$$\mathcal{U}n_{\mathcal{P}//\mathcal{P}} : \text{IbMod}(\mathcal{P})_{\mathcal{P}//\mathcal{P}} \xrightarrow{\simeq} (\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})}$$

where $(\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})}$ refers to the **pointed model category** associated to $(\text{Set}^{\text{cov}}_{\Delta})_{/\,\text{Tw}(\mathcal{P})}$. This induces a right Quillen equivalence of stabilizations

$$\mathcal{U}n^{\text{Sp}}_{\mathcal{P}//\mathcal{P}} : \mathcal{T}_{\mathcal{P}}\,\text{IbMod}(\mathcal{P}) \xrightarrow{\simeq} \text{Sp}((\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})}).$$

Observe now that the straightening functor $\text{St}_{\text{Tw}(\mathcal{P})} : (\text{Set}^{\text{cov}}_{\Delta})_{/\,\text{Tw}(\mathcal{P})} \xrightarrow{\simeq} \text{Fun}(\mathfrak{C}[\text{Tw}(\mathcal{P})], \text{Set}_{\Delta})$ lifts to a left Quillen equivalence between the associated pointed model categories

$$(\text{St}_{\text{Tw}(\mathcal{P})})_* : (\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})} \xrightarrow{\simeq} \text{Fun}(\mathfrak{C}[\text{Tw}(\mathcal{P})], (\text{Set}_{\Delta})_*) \tag{4.3.0.1}$$

where $(\text{Set}_{\Delta})_*$ denotes the pointed model category associated to Set_{Δ}. The latter induces a left Quillen equivalence of stabilizations

$$\text{St}^{\text{Sp}}_{\text{Tw}(\mathcal{P})} : \text{Sp}((\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})}) \xrightarrow{\simeq} \text{Fun}(\mathfrak{C}[\text{Tw}(\mathcal{P})], \text{Spectra})$$

where Spectra refers to the stable model category of spectra.

We now obtain a sequence of right or left Quillen equivalences

$$\mathcal{T}_{\mathcal{P}}\,\text{Op}(\text{Set}_{\Delta}) \underset{\simeq}{\xrightarrow{\mathcal{U}^{ib}_{\mathcal{P}}}} \mathcal{T}_{\mathcal{P}}\,\text{IbMod}(\mathcal{P}) \underset{\simeq}{\xrightarrow{\mathcal{U}n^{\text{Sp}}_{\mathcal{P}//\mathcal{P}}}} \text{Sp}((\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})}) \underset{\simeq}{\xrightarrow{\text{St}^{\text{Sp}}_{\text{Tw}(\mathcal{P})}}} \text{Fun}(\mathfrak{C}[\text{Tw}(\mathcal{P})], \text{Spectra}) \tag{4.3.0.2}$$

Let us describe the derived image of $\text{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\text{Op}(\text{Set}_{\Delta})$ through this composition. We have seen that $\mathbb{R}\mathcal{U}^{ib}_{\mathcal{P}}(\text{L}_{\mathcal{P}})[1] \simeq \widetilde{\text{L}}_{\mathcal{P}}$ (cf. Corollary 3.4.0.17). Namely, $\widetilde{\text{L}}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\text{IbMod}(\mathcal{P})$ is the prespectrum with $(\widetilde{\text{L}}_{\mathcal{P}})_{k,k} = \mathcal{P} \circ \text{S}^k_C$ where S^k_C is the C-collection with $\text{S}^k_C(c;c) = \text{S}^k$ for every color c and agreeing with $\varnothing$ on the other levels. In our setting, we will refer to S^k as a Kan replacement of the (pointed) k-sphere so that $\mathcal{P} \circ \text{S}^k_C$ is fibrant. Furthermore, the right derived functor $\mathbb{R}\mathcal{U}n^{\text{Sp}}_{\mathcal{P}//\mathcal{P}}$ simply sends $\widetilde{\text{L}}_{\mathcal{P}}$ to the prespectrum

$$\mathcal{U}n(\mathcal{P} \circ \text{S}^{\bullet}_C) \in \text{Sp}((\text{Set}^{\text{cov}}_{\Delta})_{\text{Tw}(\mathcal{P})//\,\text{Tw}(\mathcal{P})})$$

with $\mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{\bullet}_{C})_{k,k} = \mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{k}_{C})$. Finally, let us denote by

$$\mathcal{F}_{\mathcal{P}} := \mathbb{L}\,\mathrm{St}^{\mathrm{Sp}}_{\mathrm{Tw}(\mathcal{P})}(\mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{\bullet}_{C})) \in \mathrm{Fun}(\mathfrak{C}[\mathrm{Tw}(\mathcal{P})], \mathrm{Spectra}) \tag{4.3.0.3}$$

the derived image of $\mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{\bullet}_{C})$ through $\mathrm{St}^{\mathrm{Sp}}_{\mathrm{Tw}(\mathcal{P})}$. So $\mathcal{F}_{\mathcal{P}}$ is the derived image of $\mathrm{L}_{\mathcal{P}}[1]$ under the composed functor (4.3.0.2).

To get a description of the functor $\mathcal{F}_{\mathcal{P}}$, observe first that there is an equivalence of ∞-categories

$$\mathrm{Fun}(\mathfrak{C}[\mathrm{Tw}(\mathcal{P})], \mathrm{Spectra})_{\infty} \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Tw}(\mathcal{P}), \mathbf{Spectra})$$

where **Spectra** is the ∞-category of spectra. So we will regard $\mathcal{F}_{\mathcal{P}}$ as an ∞-functor $\mathrm{Tw}(\mathcal{P}) \longrightarrow \mathbf{Spectra}$. Let $\mu \in \mathcal{P}(c_1, \cdots, c_m; c)$ be an operation of $\mathcal{P}$, regarded as an object of $\mathrm{Tw}(\mathcal{P})$. By construction, $\mathcal{F}_{\mathcal{P}}(\mu) \in$ **Spectra** is the prespectrum with $\mathcal{F}_{\mathcal{P}}(\mu)_{k,k}$ being given by the value at μ of the functor

$$\mathrm{St}_{\mathrm{Tw}(\mathcal{P})}(\mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{k}_{C})) : \mathfrak{C}[\mathrm{Tw}(\mathcal{P})] \longrightarrow \mathrm{Set}_{\Delta}\,.$$

By the fact that the map $\mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{k}_{C}) \longrightarrow \mathcal{U}n(\mathcal{P}) = \mathrm{Tw}(\mathcal{P})$ is a left fibration and by using [[4], 2.2.3.15] (with the opposite convention), we have that

$$\mathcal{F}_{\mathcal{P}}(\mu)^{\mathrm{op}}_{k,k} \simeq \mathcal{U}n(\mathcal{P}\circ \mathrm{S}^{k}_{C}) \times_{\mathcal{U}n(\mathcal{P})} \{\mu\}.$$

We have on each level that $(\mathcal{P}\circ \mathrm{S}^{k}_{C})(c_1, \cdots, c_m; c) = \mathcal{P}(c_1, \cdots, c_m; c) \times (\mathrm{S}^k)^{\times m}$. With a help of Remark 4.1.0.13, it can then be computed that $\mathcal{F}_{\mathcal{P}}(\mu)_{k,k} \simeq (\mathrm{S}^k)^{\times m}$ and hence, we find that $\mathcal{F}_{\mathcal{P}}(\mu) \simeq \mathbb{S}^{\times m}$ the m-fold product of the **sphere spectrum**.

Let $\nu \in \mathcal{P}(d_1, \cdots, d_n; d)$ be another operation of $\mathcal{P}$ and $\widetilde{f} : \mu \longrightarrow \nu$ a morphism in $\mathrm{Tw}(\mathcal{P})$ lying above a map $f : \langle n\rangle \longrightarrow \langle m\rangle$ in Fin_*. Then the structure map

$$\mathcal{F}_{\mathcal{P}}(\widetilde{f}) : \mathcal{F}_{\mathcal{P}}(\mu) = \mathbb{S}^{\times\{1,\cdots,m\}} \longrightarrow \mathbb{S}^{\times\{1,\cdots,n\}} = \mathcal{F}_{\mathcal{P}}(\nu) \tag{4.3.0.4}$$

is defined by, for each $i \in \{1, \cdots, m\}$, copying the i'th factor to the factors of position $j \in f^{-1}(i)$ when this fiber is nonempty or collapsing that factor to the zero spectrum otherwise.

We summarize the above steps in the following:

Theorem 4.3.0.1. *Let $\mathcal{P}$ be a fibrant and Σ-cofibrant simplicial operad. There is an equivalence of ∞-categories*

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathrm{Set}_{\Delta})_{\infty} \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Tw}(\mathcal{P}), \textbf{\textit{Spectra}}).$$

*Moreover, under this equivalence, the cotangent complex $\mathrm{L}_{\mathcal{P}}$ is identified to $\mathcal{F}_{\mathcal{P}}[-1]$ the desuspension of the functor $\mathcal{F}_{\mathcal{P}} : \mathrm{Tw}(\mathcal{P}) \longrightarrow$ **Spectra** (4.3.0.3), which is given on objects by sending each operation $\mu \in \mathcal{P}$ of arity m to $\mathcal{F}_{\mathcal{P}}(\mu) = \mathbb{S}^{\times m}$. Consequently, for a given functor $\mathcal{F} : \mathrm{Tw}(\mathcal{P}) \longrightarrow$ **Spectra**, the n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in $\mathcal{F}$ is computed by*

$$\mathrm{H}^n_Q(\mathcal{P}; \mathcal{F}) = \pi_0\,\mathrm{Map}_{\mathrm{Fun}(\mathrm{Tw}(\mathcal{P}), \textbf{\textit{Spectra}})}(\mathcal{F}_{\mathcal{P}}, \mathcal{F}[n+1]).$$

Example 4.3.0.2. By Proposition 4.2.0.3, the twisted arrow ∞-category of the little ∞-cubes operad E_{∞} is equivalent to that of $\mathcal{C}om$, and hence is equivalent to $\mathrm{Fin}^{\mathrm{op}}_*$. So the tangent category $\mathcal{T}_{\mathrm{E}_{\infty}}\,\mathrm{Op}(\mathrm{Set}_{\Delta})$ is (up to a zig-zag of Quillen equivalences) equivalent to $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}_*, \mathrm{Spectra})$ endowed with the projective model structure. The functor $\mathcal{F}_{\mathrm{E}_{\infty}} : \mathrm{Fin}^{\mathrm{op}}_* \longrightarrow \mathrm{Spectra}$ takes each object $\langle m\rangle$ to $\mathbb{S}^{\times m}$ and for a given map $f : \langle m\rangle \longrightarrow \langle n\rangle$ in Fin_*, the structure map $\mathcal{F}_{\mathrm{E}_{\infty}}(f) : \mathbb{S}^{\times n} \longrightarrow \mathbb{S}^{\times m}$ is the same as the map (4.3.0.4).

An immediate consequence of this example is that the stabilization of ∞-operads is equivalent to

$\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Spectra})$. For more illustration, we shall now relate Quillen cohomology of E_∞ to the one called **stable cohomotopy of right** Γ**-modules** (cf. [25, 26]).

Let $\mathbf{k}$ be a field. Recall that a **right** Γ**-module** is by definition a functor $\mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{\mathbf{k}}$. One particularly regards the right Γ-module $t : \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{\mathbf{k}}$ given by taking $\langle m \rangle$ to $t(\langle m \rangle) = [\langle m \rangle, \mathbf{k}]$ the $\mathbf{k}$-module of based maps from $\langle m \rangle$ to $\mathbf{k}$ (where $\mathbf{k}$ has basepoint $0_{\mathbf{k}}$).

Remark 4.3.0.3. The $\mathbf{k}$-module $t(\langle m \rangle)$ admits a canonical basis given by $\{\tau_i\}_{i=1}^m$ in which $\tau_i : \langle m \rangle \longrightarrow \mathbf{k}$ is defined by sending i to $1_{\mathbf{k}}$ and sending the others to $0_{\mathbf{k}}$. In particular, we have that

$$t(\langle m \rangle) = [\langle m \rangle, \mathbf{k}] \cong \mathbf{k}^{\oplus m}.$$

For a given map $f : \langle m \rangle \longrightarrow \langle n \rangle$ in Fin_*, under the above identification, the structure map

$$t(f) : t(\langle n \rangle) = \mathbf{k}^{\oplus n} \longrightarrow \mathbf{k}^{\oplus m} = t(\langle m \rangle)$$

is given by, for each $i \in \{1, \cdots, n\}$, copying the i'th factor to the factors of position $j \in f^{-1}(i)$ when this fiber is nonempty or collapsing that factor to the zero module otherwise (i.e., in the same fashion as the map $\mathcal{F}_{\mathrm{E}_\infty}(f)$ of Example 4.3.0.2).

We now fix $T : \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{\mathbf{k}}$ to be a right Γ-module. According to [[26], §3.4], the k**'th stable cohomotopy group** of T is formulated as

$$\pi^k T \cong \mathrm{Ext}_\Gamma^k(t, T) \cong \pi_0 \, \mathrm{Map}^{\mathrm{h}}_{\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathcal{C}(\mathbf{k}))}(t, T[k]).$$

Here we identified right Γ-modules with functors $\mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathcal{C}(\mathbf{k})$ from $\mathrm{Fin}_*^{\mathrm{op}}$ to dg $\mathbf{k}$-modules, via the embedding functor $\mathrm{Mod}_{\mathbf{k}} \longrightarrow \mathcal{C}(\mathbf{k})$. Besides that, $T[k] : \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathcal{C}(\mathbf{k})$ is the functor with $T[k](\langle m \rangle) = (T(\langle m \rangle))[k]$ the k-suspension of $T(\langle m \rangle)$ regarded as a chain complex concentrated in degree 0.

On other hand, the right Γ-module T corresponds to a functor $\overline{T} : \mathrm{Fin}_*^{\mathrm{op}} \longrightarrow \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}})$ given by the composition

$$\overline{T} : \mathrm{Fin}_*^{\mathrm{op}} \xrightarrow{T} \mathrm{Mod}_{\mathbf{k}} \longrightarrow \mathrm{sMod}_{\mathbf{k}} \xrightarrow{\overline{\Sigma^\infty}} \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}})$$

in which the second functor is the obvious embedding and $\overline{\Sigma^\infty}$ is the suspension spectrum replacement functor (cf. [6], § 2.3). Here we use $\overline{\Sigma^\infty}$ (instead of Σ^∞) to get that $\overline{T}$ is a diagram of suspension spectra, and hence a diagram of Ω-spectra. Furthermore, consider the adjunction

$$\mathrm{Sp}(\mathbb{F}_*) : \mathrm{Spectra} = \mathrm{Sp}((\mathrm{Set}_\Delta)_*) \rightleftarrows \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}}) : \mathrm{Sp}(\mathbb{U}_*)$$

induced by the free-forgetful adjunction $\mathbb{F}_* : (\mathrm{Set}_\Delta)_* \rightleftarrows (\mathrm{sMod}_{\mathbf{k}})_* = \mathrm{sMod}_{\mathbf{k}} : \mathbb{U}_*$ between pointed categories. We denote by $\widetilde{T}$ the composed functor

$$\widetilde{T} : \mathrm{Fin}_*^{\mathrm{op}} \xrightarrow{\overline{T}} \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}}) \xrightarrow{\mathrm{Sp}(\mathbb{U}_*)} \mathrm{Spectra}.$$

Proposition 4.3.0.4. *The $(k-1)$'th Quillen cohomology group of the little ∞-cubes operad E_∞ with coefficients in $\widetilde{T}$ is isomorphic to the k'th stable cohomotopy group of T:*

$$\mathrm{H}_Q^{k-1}(\mathrm{E}_\infty; \widetilde{T}) \cong \pi^k T.$$

Proof. By Theorem 4.3.0.1 and Example 4.3.0.2, we just need to prove the existence of a weak equivalence

$$\mathrm{Map}^{\mathrm{h}}_{\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Spectra})}(\mathcal{F}_{\mathrm{E}_\infty}, \widetilde{T}[k]) \simeq \mathrm{Map}^{\mathrm{h}}_{\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathcal{C}(\mathbf{k}))}(t, T[k]).$$

By the Quillen adjunction $\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Spectra}) \rightleftarrows \mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}}))$ induced by $\mathrm{Sp}(\mathbb{F}_*) \dashv \mathrm{Sp}(\mathbb{U}_*)$, we have that

$$\mathrm{Map}^{\mathrm{h}}_{\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Spectra})}(\mathcal{F}_{\mathrm{E}_\infty}, \widetilde{T}[k]) \simeq \mathrm{Map}^{\mathrm{h}}_{\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}}))}(\mathrm{Sp}(\mathbb{F}_*) \circ \mathcal{F}_{\mathrm{E}_\infty}, \overline{T}[k]).$$

Here we note that $\mathrm{Sp}(\mathbb{F}_*) \circ \mathcal{F}_{\mathrm{E}_\infty}$ has already the right type, by Observation 2.2.0.8. By construction, for each $\langle m \rangle \in \mathrm{Fin}_*$, $\mathrm{Sp}(\mathbb{F}_*) \circ \mathcal{F}_{\mathrm{E}_\infty}(\langle m \rangle) = \mathrm{Sp}(\mathbb{F}_*)(\mathbb{S}^{\times m}) \in \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}})$ is the prespectrum with

$$\mathrm{Sp}(\mathbb{F}_*)(\mathbb{S}^{\times m})_{n,n} = \mathrm{coker}\,(\mathbf{k} \longrightarrow \mathbf{k}\{\mathrm{S}^n\}^{\otimes m}) \in \mathrm{sMod}_{\mathbf{k}}\,. \tag{4.3.0.5}$$

On other hand, consider the sequence of right or left Quillen equivalences

$$\mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}}) \xrightarrow[\simeq]{\mathrm{Sp}(\mathrm{N})} \mathrm{Sp}(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \xrightarrow{\simeq} \mathrm{Sp}(\mathcal{C}(\mathbf{k})) \xrightarrow[\simeq]{\Omega^\infty} \mathcal{C}(\mathbf{k})$$

where the first arrow is the stabilization of the normalized complex functor N and the second arrow is the embedding functor. This induces a sequence of right or left Quillen equivalences

$$\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}})) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathcal{C}(\mathbf{k}))) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathcal{C}(\mathbf{k})).$$

Note that the derived image of $\overline{T}[k]$ in $\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathcal{C}(\mathbf{k}))$ is exactly $T[k]$. Combined with the first paragraph, it remains to show that the derived image of $\mathrm{Sp}(\mathbb{F}_*) \circ \mathcal{F}_{\mathrm{E}_\infty}$ in $\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathcal{C}(\mathbf{k}))$ is weakly equivalent to t.

We first compute the derived image of $\mathrm{Sp}(\mathbb{F}_*) \circ \mathcal{F}_{\mathrm{E}_\infty}$ in $\mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathcal{C}(\mathbf{k})))$. Observe that the actual image has already the right type, by Observation 2.2.0.8. So the expected derived image is given by the composition $\mathcal{G} := \mathrm{Sp}(\mathrm{N}) \circ \mathrm{Sp}(\mathbb{F}_*) \circ \mathcal{F}_{\mathrm{E}_\infty}$. Using the equation (4.3.0.5), $\mathcal{G}(\langle m \rangle) \in \mathrm{Sp}(\mathcal{C}(\mathbf{k}))$ is the prespectrum with

$$\mathcal{G}(\langle m \rangle)_{n,n} = \mathrm{N}\,(\mathrm{coker}\,(\mathbf{k} \longrightarrow \mathbf{k}\{\mathrm{S}^n\}^{\otimes m})) \simeq \mathrm{coker}\,(\mathbf{k} \longrightarrow (\mathbf{k} \oplus \mathbf{k}[n])^{\otimes m}) = \bigoplus_{i=1}^{m} \mathrm{C}^i_m\, \mathbf{k}[i\,n].$$

Here we made use of the fact that the normalized complex functor is (op)lax monoidal and that its (op)lax monoidal maps are weak equivalences (see § 5.3). Finally, we need to show that the derived image of $\mathcal{G}$ through the right Quillen equivalence $[\mathrm{Fin}_*^{\mathrm{op}}, \Omega^\infty] : \mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathrm{Sp}(\mathcal{C}(\mathbf{k}))) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Fin}_*^{\mathrm{op}}, \mathcal{C}(\mathbf{k}))$, denoted by $\overline{\mathcal{G}}$, is weakly equivalent to t. Namely, $\overline{\mathcal{G}}(\langle m \rangle)$ is given by $\mathbb{R}\Omega^\infty(\mathcal{G}(\langle m \rangle))$. According to [[6], Remark 2.4.7], the latter is given by

$$\mathbb{R}\Omega^\infty(\mathcal{G}(\langle m \rangle)) \simeq \mathrm{hocolim}_n\, \Omega^n \mathcal{G}(\langle m \rangle)_{n,n} \simeq \mathrm{hocolim}_n\, \Omega^n(\bigoplus_{i=1}^{m} \mathrm{C}^i_m\, \mathbf{k}[i\,n]) \simeq \bigoplus_{i=1}^{m} \mathrm{hocolim}_n\, \mathrm{C}^i_m\, \mathbf{k}[(i-1)n].$$

By the fact that the homology functor commutes with filtered colimits, $\mathrm{hocolim}_n\, \mathrm{C}^i_m\, \mathbf{k}[(i-1)n]$ vanishes whenever $i > 1$. So we get that $\overline{\mathcal{G}}(\langle m \rangle) = \mathbb{R}\Omega^\infty(\mathcal{G}(\langle m \rangle)) \simeq \mathbf{k}^{\oplus m}$, i.e., agrees with $t(\langle m \rangle)$. For each map $f : \langle m \rangle \longrightarrow \langle n \rangle$ in Fin_*, the map $\overline{\mathcal{G}}(f) : \mathbf{k}^{\oplus n} \longrightarrow \mathbf{k}^{\oplus m}$ indeed agrees with $t(f)$ (see Remark 4.3.0.3).

□

Example 4.3.0.5. Following Proposition 4.2.0.15, the twisted arrow category of the associative operad $\mathcal{A}ss$ is equivalent to Δ. So the tangent category $\mathcal{T}_{\mathcal{A}ss}\,\mathrm{Op}(\mathrm{Set}_\Delta)$ is (up to a zig-zag of Quillen equivalences) equivalent to $\mathrm{Fun}(\Delta, \mathrm{Spectra})$ endowed with the projective model structure. The functor $\mathcal{F}_{\mathcal{A}ss} : \Delta \longrightarrow \mathrm{Spectra}$ is given by the composition

$$\Delta \xrightarrow{\iota} \mathrm{Fin}_*^{\mathrm{op}} \xrightarrow{\mathcal{F}_{\mathrm{E}_\infty}} \mathrm{Spectra},$$

(see Example 4.3.0.2 and Construction 4.2.0.16 for notations).

Consider the functor $\eta_!(\mathbb{S}) : \Delta \longrightarrow \text{Spectra}$ which is the left Kan extension of $* \xrightarrow{\{\mathbb{S}\}} \text{Spectra}$ along the inclusion $\eta : * \xrightarrow{\{[0]\}} \Delta$. Concretely, we have $\eta_!(\mathbb{S})([n]) = \mathbb{S}^{\vee\{0,\cdots,n\}}$ the $(n+1)$-fold wedge sum of the sphere spectrum and for a given map $[n] \xrightarrow{f} [m]$, the structure map $\eta_!(\mathbb{S})(f) : \mathbb{S}^{\vee\{0,\cdots,n\}} \longrightarrow \mathbb{S}^{\vee\{0,\cdots,m\}}$ is given by taking the i'th summand to the $f(i)$'th summand. The following will be helpful in describing the Quillen cohomology groups of $\mathcal{A}ss$.

Lemma 4.3.0.6. *There is a homotopy cofiber sequence in* $\text{Fun}(\Delta, \text{Spectra})$ *of the form*

$$\mathcal{F}_{\mathcal{A}ss} \longrightarrow \eta_!(\mathbb{S}) \longrightarrow \underline{\mathbb{S}}$$

where $\underline{\mathbb{S}}$ *signifies the constant functor with value* $\mathbb{S}$*.*

Proof. We are finding another model for $\eta_!(\mathbb{S})$ which is more related to $\mathcal{F}_{\mathcal{A}ss}$. Let $sh : \Delta \longrightarrow \Delta$ denote the *shift functor* sending $[n]$ to $[n+1]$ and $[n] \xrightarrow{f} [m]$ to the map $sh(f) : [n+1] \longrightarrow [m+1]$ which agrees with f on $\{0,\cdots,n\}$ and takes $n+1$ to $m+1$. Let us denote by $\mathcal{F}^+_{\mathcal{A}ss} := \mathcal{F}_{\mathcal{A}ss} \circ sh$. By adjunction there exists a natural transformation $\theta : \eta_!(\mathbb{S}) \longrightarrow \mathcal{F}^+_{\mathcal{A}ss}$ which is the identity $\text{Id}_{\mathbb{S}}$ on degree 0. Unwinding definition the map $\theta_{[n]} : \mathbb{S}^{\vee\{0,\cdots,n\}} \longrightarrow \mathbb{S}^{\times\{1,\cdots,n+1\}}$ is given by taking the i'th summand $\mathbb{S}^{\{i\}}$ to $\mathbb{S}^{\times\{i+1,\cdots,n+1\}}$ for $i = 0,\cdots,n$. It is clear that $\theta_{[n]}$ is a stable homotopy equivalence and hence, θ is a weak equivalence in $\text{Fun}(\Delta, \text{Spectra})$.

Note that there is a natural transformation $\gamma : \text{Id}_\Delta \longrightarrow sh$ with $\gamma_{[n]}$ being given by the injection $\delta^{n+1} : [n] \longrightarrow [n+1]$. The γ now induces a natural transformation $\mathcal{F}_{\mathcal{A}ss} \longrightarrow \mathcal{F}^+_{\mathcal{A}ss}$ such that the map $\mathcal{F}_{\mathcal{A}ss}([n]) = \mathbb{S}^{\times\{1,\cdots,n\}} \longrightarrow \mathbb{S}^{\times\{1,\cdots,n+1\}} = \mathcal{F}^+_{\mathcal{A}ss}([n])$ is given by the obvious inclusion. So we obtain a homotopy cofiber sequence in $\text{Fun}(\Delta, \text{Spectra})$:

$$\mathcal{F}_{\mathcal{A}ss} \longrightarrow \mathcal{F}^+_{\mathcal{A}ss} \longrightarrow \underline{\mathbb{S}}.$$

The proof is therefore completed. ☐

Corollary 4.3.0.7. *Let* $\mathcal{F} : \Delta \longrightarrow \text{Spectra}$ *be a diagram of* Ω*-spectra. The Quillen cohomology groups of* $\mathcal{A}ss$ *with coefficients in* $\mathcal{F}$ *fit into a long exact sequence of the form*

$$\cdots \longrightarrow \text{H}^{-n-2}_Q(\mathcal{A}ss; \mathcal{F}) \longrightarrow \pi_n \operatorname{holim} \mathcal{F} \longrightarrow \pi_n \mathcal{F}([0]) \longrightarrow \text{H}^{-n-1}_Q(\mathcal{A}ss; \mathcal{F}) \longrightarrow \pi_{n-1} \operatorname{holim} \mathcal{F} \longrightarrow \cdots$$

Proof. By Lemma 4.3.0.6 we get a fiber sequence of derived mapping spaces

$$\text{Map}^{\text{h}}_{\text{Fun}(\Delta,\text{Spectra})}(\underline{\mathbb{S}}, \mathcal{F}) \longrightarrow \text{Map}^{\text{h}}_{\text{Fun}(\Delta,\text{Spectra})}(\eta_!(\mathbb{S}), \mathcal{F}) \longrightarrow \text{Map}^{\text{h}}_{\text{Fun}(\Delta,\text{Spectra})}(\mathcal{F}_{\mathcal{A}ss}, \mathcal{F}).$$

In this sequence, note that the first term is weakly equivalent to $\text{Map}^{\text{h}}_{\text{Spectra}}(\mathbb{S}, \operatorname{holim} \mathcal{F})$, while by adjunction the second one is weakly equivalent to $\text{Map}^{\text{h}}_{\text{Spectra}}(\mathbb{S}, \mathcal{F}([0]))$. Combined with Theorem 4.3.0.1, we get the expected long exact sequence induced by the above fiber sequence. ☐

We end this section by the following result, which in particular shows that Quillen cohomology of any little cubes operad with **constant coefficients** vanishes.

Corollary 4.3.0.8. *Suppose that* $\mathcal{P}$ *is fibrant,* Σ*-cofibrant and unitally homotopy connected (cf. Definition 4.2.0.13). Let* $\mathcal{F}_0 : \text{Tw}(\mathcal{P}) \longrightarrow$ ***Spectra*** *be a constant functor. Then Quillen cohomology of* $\mathcal{P}$ *with coefficients in* $\mathcal{F}_0$ *vanishes.*

Proof. By Theorem 4.3.0.1 and by the assumption that $\mathcal{F}_0$ is a constant functor, Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{F}_0$ is given by

$$\text{H}^\bullet_Q(\mathcal{P}; \mathcal{F}_0) = \pi_0 \text{Map}_{\mathbf{Spectra}}(\operatorname{colim} \mathcal{F}_{\mathcal{P}}, \mathcal{F}_0[\bullet + 1]).$$

By Lemma 4.2.0.14, $\mathrm{Tw}(\mathcal{P})$ admits terminal objects being precisely the unary operations of $\mathcal{P}$. It implies that $\operatorname{colim} \mathcal{F}_{\mathcal{P}}$ is weakly equivalent to $\mathcal{F}_{\mathcal{P}}(\mu_0)$ with $\mu_0 \in \mathcal{P}$ being an arbitrary unary operation. But $\mathcal{F}_{\mathcal{P}}(\mu_0)$ is just the zero spectrum, and hence $\mathrm{H}^{\bullet}_{Q}(\mathcal{P}; \mathcal{F}_0)$ vanishes as desired. □

Chapter 5

Quillen cohomology of dg operads

In this chapter, we will fix $\mathbf{k}$ to be a field of characteristic 0. By **dg operads** we shall mean the operads enriched over $\mathcal{C}(\mathbf{k})$ the monoidal category of dg $\mathbf{k}$-modules (cf. Examples 2.1.4.7). Our main goals in this chapter are to formulate Quillen cohomology of dg operads and to establish the relation between **deformation theory** and Quillen cohomology. Besides that, in the second section, we particularly regard Quillen cohomology of **connective augmented dg operads**. In the third section, we prove the existence of an operadic version of the Dold-Kan correspondence, then due to this we find a connection between Quillen cohomology of a simplicial operad and Quillen cohomology of its associated dg operad.

This chapter is part of joint work with Y. Harpaz.

5.1 Cotangent complex and Quillen cohomology of dg operads

Let $\mathcal{P} \in \mathrm{Op}_C(\mathcal{C}(\mathbf{k}))$ be a C-colored dg operad. Notice that $\mathcal{P}$ is automatically fibrant and Σ-cofibrant. By Theorem 3.2.4.3, there is a sequence of Quillen equivalences

$$\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{C}(\mathbf{k})) \underset{\mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}}{\overset{\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}}}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{C}(\mathbf{k})) \tag{5.1.0.1}$$

Notations 5.1.0.1. We will write $\mathrm{IbMod}(\mathcal{P}) \underset{\mathcal{U}_{\mathcal{P}}}{\overset{\mathcal{F}_{\mathcal{P}}}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{C}(\mathbf{k}))$ and $\mathrm{IbMod}(\mathcal{P}) \underset{\mathcal{U}^C_{\mathcal{P}}}{\overset{\mathcal{F}^C_{\mathcal{P}}}{\rightleftarrows}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{C}(\mathbf{k}))$ standing for two composed adjunctions taken from the above sequence.

As in Notations 3.4.0.2, we denote by $\mathrm{L}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{C}(\mathbf{k}))$ (resp. $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{C}(\mathbf{k}))$) the cotangent complex of $\mathcal{P}$ when considered as an object of $\mathrm{Op}(\mathcal{C}(\mathbf{k}))$ (resp. $\mathrm{Op}_C(\mathcal{C}(\mathbf{k}))$).

As in Conventions 3.4.0.3, by **Quillen cohomology of** $\mathcal{P}$ we shall mean the Quillen cohomology of $\mathcal{P}$ when considered as an object of $\mathrm{Op}(\mathcal{C}(\mathbf{k}))$, which is therefore classified by $\mathrm{L}_{\mathcal{P}}$. On the other hand, by **reduced Quillen cohomology of** $\mathcal{P}$ we shall mean the Quillen cohomology of $\mathcal{P}$ when considered as an object of $\mathrm{Op}_C(\mathcal{C}(\mathbf{k}))$, which is therefore classified by $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}$.

This section is devoted to the desired formula of Quillen cohomology of $\mathcal{P}$. To this end, we will give an explicit description of $\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}}) \in \mathrm{IbMod}(\mathcal{P})$ the derived image of $\mathrm{L}_{\mathcal{P}}$ under the right Quillen equivalence $\mathcal{U}_{\mathcal{P}}$. We would guess that the category $\mathcal{C}(\mathbf{k})$ also satisfies the condition (S8) 3.3.0.2, yet it could be complicated to verify this. We therefore do not inherit the results of §3.4. For our approach in this context, the key step is to find a cofiber sequence in $\mathrm{IbMod}(\mathcal{P})$ relating $\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}})$ and $\mathbb{R}\mathcal{U}^{C}_{\mathcal{P}}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}})$ and then, to observe that the latter was considered in literature under the name of **module of Kähler differentials**.

As the starting point, we revisit the construction of module of Kähler differentials. Let $\mathcal{O}$ be a colored dg operad and let A be an $\mathcal{O}$-algebra. For each $M \in \mathrm{Mod}^{A}_{\mathcal{P}}$, the (levelwise) coproduct $A \oplus M$ admits the canonical structure of an $\mathcal{O}$-algebra over A, usually called the **square zero extension of A by M** (cf., e.g., [[11], §12.3.3], [[7], §2.5]). This construction determines a right adjoint functor written as

$$A \ltimes (-) : \mathrm{Mod}^{A}_{\mathcal{P}} \longrightarrow (\mathrm{Alg}_{\mathcal{O}})_{/A}.$$

We will denote by $\Omega^{/A}$ the left adjoint of $A \ltimes (-)$ and refer to it as the **functor of Kähler differentials**.

Let $\mathcal{P}$ be a C-colored dg operad. Recall that the datum of $\mathcal{P}$ as a C-colored dg operad is equivalent to that as an algebra over $\mathbf{O}_C$, i.e., the **operad of C-colored (dg) operads** (cf. [35], Section 3). On other hand, the same arguments as in the proof of Proposition 3.5 of loc.cit show that the structure of a $\mathcal{P}$-module over $\mathbf{O}_C$ is the same as that of an infinitesimal $\mathcal{P}$-bimodule. Due to these translations, we obtain an adjunction

$$\Omega^{/\mathcal{P}} : \mathrm{Op}_C(\mathcal{C}(\mathbf{k}))_{/\mathcal{P}} \rightleftarrows \mathrm{IbMod}(\mathcal{P}) : \mathcal{P} \ltimes (-),$$

which is the operadic version of the adjunction $\Omega^{/A} \dashv A \ltimes (-)$ mentioned above. We will refer to $\Omega_{\mathcal{P}} := \Omega^{/\mathcal{P}}(\mathrm{Id}_{\mathcal{P}}) \in \mathrm{IbMod}(\mathcal{P})$ the **module of Kähler differentials of $\mathcal{P}$** (cf. [12], §8). In fact, this object is best understood in terms of **derivations**.

Definition 5.1.0.2. For a given infinitesimal $\mathcal{P}$-bimodule M, a **derivation** $d : \mathcal{P} \to M$ is a map of C-collections satisfying the classical derivative equations of the form

$$d(\alpha \circ_i \beta) = \alpha \circ^{l}_{i} d(\beta) + d(\alpha) \circ^{r}_{i} \beta \tag{5.1.0.2}$$

in which $\alpha \in \mathcal{P}(c_1, \cdots, c_n; c)$ and $\beta \in \mathcal{P}(d_1, \cdots, d_m; c_i)$ are taken arbitrarily, "$\circ_i$" refers to partial composition in $\mathcal{P}$ and "$\circ^{l}_{i}$" (resp. "$\circ^{r}_{i}$") refers to infinitesimal left (resp. right) $\mathcal{P}$-action on M (cf. Definition 2.1.2.6). The collection of such derivations will be denoted by $\mathrm{Der}(\mathcal{P}, M)$.

Proposition 5.1.0.3. *([12], Proposition 76) Up to isomorphisms, the module of Kähler differentials of $\mathcal{P}$ is an infinitesimal $\mathcal{P}$-bimodule equipped with a **universal derivation** $d_{\mathcal{P}} : \mathcal{P} \longrightarrow \Omega_{\mathcal{P}}$ and characterized by a universal property that for any $M \in \mathrm{IbMod}(\mathcal{P})$ and for any derivation $d : \mathcal{P} \to M$, there exists a unique map $\Omega_{\mathcal{P}} \longrightarrow M$ of infinitesimal $\mathcal{P}$-bimodules lifting the map d :*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{d_{\mathcal{P}}} & \Omega_{\mathcal{P}} \\ & \searrow^{d} & \downarrow \\ & & M . \end{array}$$

Proof. Fix M to be an infinitesimal $\mathcal{P}$-bimodule. Unwinding definition, the operad structure of $\mathcal{P} \ltimes M$ is set-theoretically given as follows. As a C-collection, it is given by the coproduct $\mathcal{P} \oplus M$. The unit operations are given by $(\mathrm{id}_c, 0) \in \mathcal{P}(c;c) \oplus M(c;c)$ for every color $c \in C$. Moreover, given $(\mu, m) \in (\mathcal{P} \oplus M)(c_1, \cdots, c_m; c)$ and $(\nu, n) \in (\mathcal{P} \oplus M)(d_1, \cdots, d_n; c_i)$, the partial composition is defined as

$$(\mu, m) \circ_i (\nu, n) = (\mu \circ_i \nu ; \mu \circ^{l}_{i} n + m \circ^{r}_{i} \nu).$$

The map $\mathcal{P} \ltimes M \longrightarrow \mathcal{P}$ of operads is given by the projection.

It can then be verified that the data of a map $\mathcal{P} \longrightarrow \mathcal{P} \ltimes M$ of operads over $\mathcal{P}$ is equivalent to the choice of a derivation $\mathcal{P} \longrightarrow M$ and moreover, this equivalence is natural in "M". Thus, by adjunction we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathrm{IbMod}(\mathcal{P})}(\Omega_{\mathcal{P}}, M) \cong \mathrm{Der}(\mathcal{P}, M),$$

which proves the proposition. □

Thanks to this, we are going to give an explicit description of $\Omega_{\mathcal{P}}$ on point-set level. Let $\mathcal{F}^{ib}(\mathcal{P})$ denote the free infinitesimal $\mathcal{P}$-bimodule generated by $\mathcal{P}$. Recall that $\mathcal{F}^{ib}(\mathcal{P}) \cong \mathcal{P} \circ_{(1)} (\mathcal{P} \circ \mathcal{P})$ as C-collections (cf. Remark 2.1.2.10). Each element of $\mathcal{F}^{ib}(\mathcal{P})$ is performed by a tuple of operations of $\mathcal{P}$ of the form $(\alpha\,;\beta\,;\beta_1,\cdots,\beta_n)$.

Furthermore, recall that the operad $\mathcal{P}$ is said to be **augmented** if it is equipped with a map $\varepsilon : \mathcal{P} \longrightarrow \mathcal{I}_C$ of C-colored dg operads. In this situation, the kernel $\overline{\mathcal{P}} := \ker(\mathcal{P} \xrightarrow{\varepsilon} \mathcal{I}_C)$ will be called the **augmentation ideal** of $\mathcal{P}$.

Corollary 5.1.0.4. *1. $\Omega_{\mathcal{P}}$ is isomorphic to the quotient of $\mathcal{F}^{ib}(\mathcal{P})$ under the relation*

$$(\,\alpha\,;\,\beta\circ_i\gamma\,;\,\beta_1,\cdots,\beta_{i-1},\gamma_1,\cdots,\gamma_n,\beta_{i+1},\cdots,\beta_m\,)$$
$$\approx (\,\alpha\circ_k\beta\circ(\beta_1,\cdots,\beta_m)\,;\,\gamma\,;\,\gamma_1,\cdots,\gamma_n\,) + (\,\alpha\,;\,\beta\,;\,\beta_1,\cdots,\beta_{i-1},\gamma\circ(\gamma_1,\cdots,\gamma_n),\beta_{i+1},\cdots,\beta_m\,)$$

in which β is grafted to the k'th leaf of α, while γ is grafted to the i'th leaf of β and the β_r's (resp. γ_t's) are grafted to β (resp. γ). (This can be viewed as the operadic version of the formula given in [[11], Lemma 12.3.20]).

2. *Under the isomorphism of (1), $\Omega_{\mathcal{P}}$ is spanned under **k**-linear combinations by the collection of elements of the form $(\alpha\,;\,\beta\,;\,\mathrm{id}_{d_1},\cdots,\mathrm{id}_{d_m})$ with $\alpha\in\mathcal{P}(c_1,\cdots,c_n;c)$ and $\beta\in\mathcal{P}(d_1,\cdots,d_m;c_i)$ such that β is not a unit operation.*

3. *Suppose further that $\mathcal{P}$ is augmented. Then $\Omega_{\mathcal{P}}$ is isomorphic to $\mathcal{P}\circ_{(1)}\overline{\mathcal{P}}$.*

Proof. (1) We first establish the universal derivation $d_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{F}^{ib}(\mathcal{P})/\approx$ as follows. Given an operation $\alpha\in\mathcal{P}(c_1,\cdots,c_n;c)$, we take $d_{\mathcal{P}}(\alpha) := (\mathrm{id}_c;\alpha;\mathrm{id}_{c_1},\cdots,\mathrm{id}_{c_n})$. Let $\beta\in\mathcal{P}(d_1,\cdots,d_m;c_i)$ be another operation. To show that $d_{\mathcal{P}}$ is a derivation, we have to verify the equation (5.1.0.2). By definition, we have that $\alpha\circ_i^l d_{\mathcal{P}}(\beta) = (\alpha\,;\,\beta\,;\,\mathrm{id}_{d_1},\cdots,\mathrm{id}_{d_m})$ and $d_{\mathcal{P}}(\alpha)\circ_i^r\beta = (\mathrm{id}_c\,;\,\alpha\,;\,\mathrm{id}_{c_1},\cdots,\mathrm{id}_{c_{i-1}},\beta,\mathrm{id}_{c_{i+1}},\cdots,\mathrm{id}_{c_n})$. It is now clear that

$$d_{\mathcal{P}}(\alpha\circ_i\beta) \approx \alpha\circ_i^l d_{\mathcal{P}}(\beta) + d_{\mathcal{P}}(\alpha)\circ_i^r\beta,$$

as expected. It remains to verify the universality of $d_{\mathcal{P}}$ as mentioned in Proposition 5.1.0.3. For any infinitesimal $\mathcal{P}$-bimodule M and a derivation $d : \mathcal{P} \to M$, the map d extends to a canonical map $\varphi_d : \mathcal{F}^{ib}(\mathcal{P}) \longrightarrow M$ of infinitesimal $\mathcal{P}$-bimodules, which passes through the relation "$\approx$" since d is a derivation, and hence φ_d descends to a map $\overline{\varphi_d} : \mathcal{F}^{ib}(\mathcal{P})/\approx\ \longrightarrow M$ of infinitesimal $\mathcal{P}$-bimodules. The latter is in fact the unique candidate satisfying the equation $\overline{\varphi_d}\circ d_{\mathcal{P}} = d$.

(2) For simplicity of equations, we shall now ignore the role of colors and besides that, we will not distinguish between partial composition and composition of operations in $\mathcal{P}$. Observe that an element in $\mathcal{F}^{ib}(\mathcal{P})$ of the form $(\alpha\,;\beta\,;\beta_1,\cdots,\beta_n)$ can be extracted as

$$(\alpha\,;\beta\,;\beta_1,\cdots,\beta_n) \approx (\alpha\,;\beta\circ(\beta_1,\cdots,\beta_n)\,;\mathrm{id},\cdots,\mathrm{id}) - \sum_{i=1}^{n}(\alpha\circ\beta\circ(\beta_1,\cdots,\beta_{i-1},\beta_{i+1},\cdots,\beta_n)\,;\,\beta_i\,;\,\mathrm{id},\cdots,\mathrm{id}).$$

On other hand, for any operation α, observe that

$$(\mathrm{id}\,;\,\alpha\,;\,\mathrm{id},\cdots,\mathrm{id}) = (\mathrm{id}\,;\,\alpha\circ\mathrm{id}\,;\,\mathrm{id},\cdots,\mathrm{id}) \approx (\alpha\,;\,\mathrm{id}\,;\,\mathrm{id}) + (\mathrm{id}\,;\,\alpha\,;\,\mathrm{id},\cdots,\mathrm{id}).$$

This means that $(\alpha\,;\,\mathrm{id}\,;\,\mathrm{id}) \approx 0$. So we get the conclusion.

(3) Let us first explain the infinitesimal $\mathcal{P}$-bimodule structure of $\mathcal{P}\circ_{(1)}\overline{\mathcal{P}}$. As an infinitesimal left $\mathcal{P}$-module, $\mathcal{P}\circ_{(1)}\overline{\mathcal{P}}$ is free generated by $\overline{\mathcal{P}}$ (cf. Remark 2.1.2.10). On other hand, let $(\alpha\,,\,\beta)$ perfom an element of $\mathcal{P}\circ_{(1)}\overline{\mathcal{P}}$ with $\alpha\in\mathcal{P}$ and $\beta\in\overline{\mathcal{P}}$ being grafted to some leaf of α. The infinitesimal right action of a given operation $\lambda\in\mathcal{P}$ on $(\alpha\,,\,\beta)$ is defined as follows:

$$(\alpha\,,\,\beta)\circ^r\lambda = \begin{cases}(\alpha\circ\lambda\,,\,\beta) & \text{if } \lambda \text{ is grafted to } \alpha\\ (\alpha\,,\,\beta\circ\lambda) - (\alpha\circ\beta\,,\,\lambda) & \text{if } \lambda \text{ is grafted to } \beta.\end{cases}$$

The universal derivation $d_{\mathcal{P}}:\mathcal{P}\longrightarrow\mathcal{P}\circ_{(1)}\overline{\mathcal{P}}$ is defined as

$$d_{\mathcal{P}}(\mu) = \begin{cases}0 & \text{if } \mu = \mathrm{id}\\ (\mathrm{id}\,,\mu) & \text{if } \mu \neq \mathrm{id}\,.\end{cases}$$

Suppose given any infinitesimal $\mathcal{P}$-bimodule M and a derivation $d:\mathcal{P}\to M$. The restriction of d to $\overline{\mathcal{P}}$ extends to a canonical map $\psi_d:\mathcal{P}\circ_{(1)}\overline{\mathcal{P}}\longrightarrow M$ of infinitesimal left $\mathcal{P}$-modules. Moreover, ψ_d is also a map of infinitesimal right $\mathcal{P}$-modules, by the assumption that d is a derivation. It is now clear that ψ_d is the unique map satisfying that $\psi_d\circ d_{\mathcal{P}} = d$ on $\overline{\mathcal{P}}$. But every derivation takes the unit operations to zero, and hence we get that $\psi_d\circ d_{\mathcal{P}} = d$ as desired. □

Observe that the unit map $\eta:\mathcal{I}_C\longrightarrow\mathcal{P}$ of $\mathcal{P}$ gives rise to a commutative square of Quillen adjunctions

$$\begin{array}{ccc}\mathrm{IbMod}(\mathcal{I}_C) & \underset{\mathcal{U}_{\mathcal{I}_C}}{\overset{\mathcal{F}_{\mathcal{I}_C}}{\rightleftarrows}} & \mathcal{T}_{\mathcal{I}_C}\,\mathrm{Op}(\mathcal{S})\\ \eta^{ib}_!\downarrow\dashv\uparrow\eta^*_{ib} & & \eta^*_{op}\uparrow\vdash\downarrow\eta^{op}_! \\ \mathrm{IbMod}(\mathcal{P}) & \underset{\mathcal{F}_{\mathcal{P}}}{\overset{\mathcal{U}_{\mathcal{P}}}{\leftrightarrows}} & \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})\end{array}$$

whose horizontal adjunctions are Quillen equivalences. Let us denote by $\overline{\mathrm{L}}_{\mathcal{P}} := \mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}})[1]\in\mathrm{IbMod}(\mathcal{P})$ the suspension of $\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}})$. The following is the key for understanding the cotangent complex $\mathrm{L}_{\mathcal{P}}$.

Lemma 5.1.0.5. *Let $\mathcal{P}$ be any C-colored dg operad. There is a homotopy cofiber sequence in* $\mathrm{IbMod}(\mathcal{P})$ *of the form*

$$\mathbb{R}\mathcal{U}^C_{\mathcal{P}}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}})\longrightarrow\mathcal{P}\circ_{(1)}\mathcal{P}\longrightarrow\overline{\mathrm{L}}_{\mathcal{P}},$$

where $\mathcal{P}\circ_{(1)}\mathcal{P}$ represents the free infinitesimal $\mathcal{P}$-bimodule generated by $\mathcal{I}_C$.

Proof. By Lemma 3.4.0.6 and by [[8], Corollary 2.2.4], there is a weak equivalence $\theta_{\mathcal{I}_C}:\mathcal{I}_C[-1]\xrightarrow{\simeq}\mathbb{R}\mathcal{U}_{\mathcal{I}_C}(\mathrm{L}_{\mathcal{I}_C})$ in $\mathrm{IbMod}(\mathcal{I}_C)$. Consider $\theta^{\mathrm{ad}}_{\mathcal{I}_C}:\mathbb{L}\mathcal{F}_{\mathcal{I}_C}(\mathcal{I}_C)\xrightarrow{\simeq}\mathrm{L}_{\mathcal{I}_C}[1]$ the adjoint map of $\theta_{\mathcal{I}_C}$, which is a weak equivalence in $\mathcal{T}_{\mathcal{I}_C}\,\mathrm{Op}(\mathcal{S})$. Applying $\mathbb{L}\eta^{op}_!$ to $\theta^{\mathrm{ad}}_{\mathcal{I}_C}$ and then, taking the adjoint of the resultant, we obtain a weak equivalence in $\mathrm{IbMod}(\mathcal{P})$ of the form $\mathbb{L}\eta^{ib}_!(\mathcal{I}_C)\xrightarrow{\simeq}\mathbb{R}\mathcal{U}_{\mathcal{P}}\mathbb{L}\eta^{op}_!(\mathrm{L}_{\mathcal{I}_C}[1])$.

On other hand, by the definition of relative cotangent complex we get a homotopy cofiber sequence in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{S})$ of the form $\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}\longrightarrow\mathbb{L}\eta^{op}_!(\mathrm{L}_{\mathcal{I}_C}[1])\longrightarrow\mathrm{L}_{\mathcal{P}}[1]$. Applying $\mathbb{R}\mathcal{U}_{\mathcal{P}}$ to the latter and by the above paragraph, we obtain a homotopy cofiber sequence in $\mathrm{IbMod}(\mathcal{P})$ of the form

$$\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}/\mathcal{I}_C}) \longrightarrow \mathbb{L}\eta_!^{ib}(\mathcal{I}_C) \longrightarrow \mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}}[1]) \simeq \overline{\mathrm{L}}_{\mathcal{P}}.$$

Note that $\eta_!^{ib}$ agrees with the free infinitesimal $\mathcal{P}$-bimodule functor $\mathrm{Coll}_C(\mathcal{S}) \longrightarrow \mathrm{IbMod}(\mathcal{P})$. We thus find that $\mathbb{L}\eta_!^{ib}(\mathcal{I}_C) = \mathcal{P} \circ_{(1)} \mathcal{P}$. It remains to show that $\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}/\mathcal{I}_C})$ is weakly equivalent to $\mathbb{R}\mathcal{U}^C_{\mathcal{P}}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}})$. Since the functor $\mathcal{U}_{\mathcal{P}}$ is the same as the composition

$$\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{C}(\mathbf{k})) \xrightarrow{\mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{C}(\mathbf{k})) \xrightarrow{\mathcal{U}^C_{\mathcal{P}}} \mathrm{IbMod}(\mathcal{P}),$$

it suffices to prove the existence of a weak equivalence $\mathrm{L}^{\mathrm{red}}_{\mathcal{P}} \xrightarrow{\simeq} \mathbb{R}\mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}/\mathcal{I}_C})$ in $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}_C(\mathcal{C}(\mathbf{k}))$. But this follows by Lemma 3.4.0.9, since $\mathcal{L}^{\mathrm{Sp}}_{\mathcal{P}} \dashv \mathcal{R}^{\mathrm{Sp}}_{\mathcal{P}}$ is a Quillen equivalence. □

By applying [[7], Corollary 2.5.11] to the data of $\mathcal{C}(\mathbf{k})$, $\mathbf{O}_C$ and $\mathcal{P}$, we obtain that the object $\mathbb{R}\mathcal{U}^C_{\mathcal{P}}(\mathrm{L}^{\mathrm{red}}_{\mathcal{P}}) \in \mathrm{IbMod}(\mathcal{P})$ is weakly equivalent to $\Omega_{\mathcal{P}}$ the module of Kähler differentials of $\mathcal{P}$. (Here we note that, by the assumption that $\mathbf{k}$ is a field of characteristic 0, it is not necessarily required that $\mathcal{P}$ is cofibrant). We may now describe the derived image of $\mathrm{L}_{\mathcal{P}}$ in $\mathrm{IbMod}(\mathcal{P})$.

Proposition 5.1.0.6. *Let $\mathcal{P}$ be any C-colored dg operad. The object $\overline{\mathrm{L}}_{\mathcal{P}} = \mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}})[1] \in \mathrm{IbMod}(\mathcal{P})$ is weakly equivalent to $\mathcal{P} \circ_{(1)} \mathcal{I}_C$ whose infinitesimal $\mathcal{P}$-bimodule structure is described as follows. As an infinitesimal left $\mathcal{P}$-module, it is free generated by $\mathcal{I}_C$ (cf. Remark 2.1.2.10). On other hand, given an operation $\alpha \in \mathcal{P}(c_1, \cdots, c_n; c)$, the (infinitesimal) right action of an operation $\lambda \in \mathcal{P}(d_1, \cdots, d_m; c_j)$ on the element $(\alpha\,, \mathrm{id}_{c_i}) \in \mathcal{P} \circ_{(1)} \mathcal{I}_C$ is given by*

$$(\alpha\,, \mathrm{id}_{c_i}) \circ^r_j \lambda := \begin{cases} (\alpha \circ_j \lambda\,, \mathrm{id}_{c_i}) & \text{if } j \neq i \\ \sum_{k=1}^m (\alpha \circ_j \lambda\,, \mathrm{id}_{d_k}) & \text{if } j = i. \end{cases}$$

Proof. Combining the above words with Lemma 5.1.0.5, we obtain a homotopy cofiber sequence in $\mathrm{IbMod}(\mathcal{P})$ of the form

$$\Omega_{\mathcal{P}} \xrightarrow{\varphi_d} \mathcal{P} \circ_{(1)} \mathcal{P} \longrightarrow \overline{\mathrm{L}}_{\mathcal{P}}.$$

Let $\alpha \in \mathcal{P}(c_1, \cdots, c_n; c)$ and $\beta \in \mathcal{P}(d_1, \cdots, d_m; c_i)$ be two operations of $\mathcal{P}$. Unwinding the definitions the map φ_d is the one induced by the canonical derivation $d : \mathcal{P} \longrightarrow \mathcal{P} \circ_{(1)} \mathcal{P}$ taking the operation β to

$$d(\beta) = (\beta\,, \mathrm{id}_{d_1}) + \cdots + (\beta\,, \mathrm{id}_{d_m}) - (\mathrm{id}_{c_i}\,, \beta).$$

(cf. Proposition 5.1.0.3). Therefore, φ_d sends each generating element $\varepsilon := (\alpha\,; \beta\,; \mathrm{id}_{d_1}, \cdots, \mathrm{id}_{d_m})$ (cf. Corollary 5.1.0.4(2)) to

$$\varphi_d(\varepsilon) = (\alpha \circ_i \beta\,, \mathrm{id}_{d_1}) + \cdots + (\alpha \circ_i \beta\,, \mathrm{id}_{d_m}) - (\alpha\,, \beta).$$

In particular, φ_d is injective and hence, its cokernel is already a model for its homotopy cofiber. (For this, note that the category $\mathrm{IbMod}(\mathcal{P})$ is abelian and stable). Thus, $\overline{\mathrm{L}}_{\mathcal{P}}$ can be modeled by the quotient of $\mathcal{P} \circ_{(1)} \mathcal{P}$ under the relation:

$$(\alpha\,, \beta) \sim (\alpha \circ_i \beta\,, \mathrm{id}_{d_1}) + \cdots + (\alpha \circ_i \beta\,, \mathrm{id}_{d_m})$$

for every $\alpha \in \mathcal{P}(c_1, \cdots, c_n; c)$ and $\beta \in \mathcal{P}(d_1, \cdots, d_m; c_i)$. Unwinding computation, this quotient is exactly $\mathcal{P} \circ_{(1)} \mathcal{I}_C$ with the infinitesimal $\mathcal{P}$-bimodule structure as described in the statement. □

Remark 5.1.0.7. The object $\overline{\mathrm{L}}_{\mathcal{P}} = \mathcal{P} \circ_{(1)} \mathcal{I}_C \in \mathrm{IbMod}(\mathcal{P})$ was considered in literature for different purposes. B. Fresse [52] considered this as a right $\mathcal{P}$-module satisfying the universal property that, for every $\mathcal{P}$-algebra A, there is a canonical isomorphism $\Omega^{/A}(A) \cong \overline{\mathrm{L}}_{\mathcal{P}} \circ_{\mathcal{P}} A$ of A-modules over $\mathcal{P}$, where $- \circ_{\mathcal{P}} -$ refers to the *relative composite product* over $\mathcal{P}$ (cf. section 10.3 of loc.cit). The object $\overline{\mathrm{L}}_{\mathcal{P}}$ was also considered by J. Millès [56] named *functorial module of Kähler differential forms* (see section 5.2 of loc.cit). We will revisit several results of this author in the next section.

Immediately by the definition of Quillen cohomology groups, Proposition 5.1.0.6 leads to the following conclusion:

Theorem 5.1.0.8. *Let $\mathcal{P}$ be a C-colored dg operad and M an infinitesimal $\mathcal{P}$-bimodule, regarded as the module of coefficients. The n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in M is formulated as*

$$\mathrm{H}^n_Q(\mathcal{P}, M) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}[-1], M[n]) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}, M[n+1]).$$

Remark 5.1.0.9. On other hand, the n'th reduced Quillen cohomology group of $\mathcal{P}$ with coefficients in M is given by

$$\mathrm{H}^n_{Q,\mathrm{red}}(\mathcal{P}, M) \cong \pi_0 \operatorname{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\Omega_{\mathcal{P}}, M[n]).$$

This assertion therefore fits into the work of [12, 11, 7].

Moreover, we find a long exact sequence relating Quillen cohomology and reduced Quillen cohomology of $\mathcal{P}$, just like the one given in Theorem 3.5.0.2.

Theorem 5.1.0.10. *Let $\mathcal{P}$ be a C-colored dg operad and let M be an infinitesimal $\mathcal{P}$-bimodule, regarded as the module of coefficients. There is a long exact sequence of abelian groups of the form*

$$\cdots \longrightarrow \mathrm{H}^{n-1}_Q(\mathcal{P}, M) \longrightarrow \mathrm{H}^n_{Q,r}(\mathcal{P}, M) \longrightarrow \mathrm{H}^n_{Q,\mathrm{red}}(\mathcal{P}, M) \longrightarrow \mathrm{H}^n_Q(\mathcal{P}, M) \longrightarrow \mathrm{H}^{n+1}_{Q,r}(\mathcal{P}, M) \longrightarrow \cdots \quad (5.1.0.3)$$

where $\mathrm{H}^\bullet_{Q,r}(\mathcal{P}, -)$ refers to Quillen cohomology group of $\mathcal{P}$ when regarded as a right module over itself.

Proof. The cofiber sequence $\Omega_{\mathcal{P}} \longrightarrow \mathcal{P} \circ_{(1)} \mathcal{P} \longrightarrow \overline{\mathrm{L}}_{\mathcal{P}}$ induces a fiber sequence of mapping spaces

$$\operatorname{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}, M) \longrightarrow \operatorname{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\mathcal{P} \circ_{(1)} \mathcal{P}, M) \longrightarrow \operatorname{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\Omega_{\mathcal{P}}, M).$$

As indicated above, $\Omega_{\mathcal{P}}$ classifies the reduced Quillen cohomology of $\mathcal{P}$, while $\overline{\mathrm{L}}_{\mathcal{P}}$ classifies the Quillen cohomology of $\mathcal{P}$. Thus, that fiber sequence will give rise to the desired long exact sequence after showing that $\mathcal{P} \circ_{(1)} \mathcal{P}$ classifies Quillen cohomology of $\mathcal{P}$ when regarded as a right module over itself. Note that the category of right $\mathcal{P}$-modules, $\mathrm{RMod}(\mathcal{P})$, is stable. It implies that the derived image of the cotangent complex of $\mathcal{P}$ (when regarded as a right module over itself) through the right Quillen equivalence $\mathcal{T}_{\mathcal{P}} \mathrm{RMod}(\mathcal{P}) \xrightarrow{\simeq} \mathrm{RMod}(\mathcal{P})$ is nothing but $\mathcal{P} \in \mathrm{RMod}(\mathcal{P})$ (cf. [[8], Corollary 2.2.4]), which has the derived image through the free functor $\mathrm{RMod}(\mathcal{P}) \longrightarrow \mathrm{IbMod}(\mathcal{P})$ being exactly $\mathcal{P} \circ_{(1)} \mathcal{P}$. The proof is therefore completed. □

5.2 Quillen cohomology of connective augmented dg operads

In this section, we study Quillen cohomology of **connective augmented dg operads**, i.e., the augmented operads which are in addition concentrated in non-negative degrees. Nevertheless, we will restrict our attention to the case where the set of colors C is a singleton. Then, each (dg) C-collection will be called a **Σ_*-module**, the initial operad will be denoted by $\mathcal{I}$. Moreover, we will denote by $\mathrm{Op}_*(\mathcal{C}(\mathbf{k}))$ the category of *(single-colored) dg operads*.

It was known that there exists a **conilpotent dg cooperad** $\mathcal{C}$ and a **quasi free resolution** $\Omega(\mathcal{C}) \xrightarrow{\simeq} \mathcal{P}$ of $\mathcal{P}$, where $\Omega(-)$ refers to the **cobar construction**. In fact, the cooperad $\mathcal{C}$ can be characterized by having the same homotopy type as $\mathrm{B}(\mathcal{P})$ the **bar construction** of $\mathcal{P}$. When $\mathcal{P}$ is further **Koszul**, it admits a smaller quasi free resolution $\Omega(\mathcal{P}^{\mathrm{i}}) \xrightarrow{\simeq} \mathcal{P}$ where $\mathcal{P}^{\mathrm{i}}$ denotes the **Koszul dual cooperad** of $\mathcal{P}$. Moreover, when $\mathcal{P}$ is in addition connective, those two become cofibrant resolutions of $\mathcal{P}$. For more details about these, we refer the readers to [11]. The main result of this section is Theorem 5.2.0.7.

Definition 5.2.0.1. ([11]) Let $\mathcal{C}$ be a dg cooperad and $\mathcal{P}$ a dg operad. The **convolution dg Lie algebra** associated to $\mathcal{C}$ and $\mathcal{P}$ is given by $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}) \coloneqq (\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}),[\,,],\partial)$ where

$$\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}) = \prod_{n\geq 0} \mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n),\mathcal{P}(n)),$$

∂ is the canonical differential of internal hom of chain complexes and moreover, the bracket $[\,,]$ is given by $[f,g] \coloneqq f\star g - (-1)^{|f||g|} g\star f$ with $\star$ being the **operadic convolution product** defined as

$$f\star g \coloneqq \mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C}\circ_{(1)}\mathcal{C} \xrightarrow{f\circ_{(1)} g} \mathcal{P}\circ_{(1)}\mathcal{P} \xrightarrow{\mu_{(1)}} \mathcal{P}$$

in which $\Delta_{(1)}$ (resp. $\mu_{(1)}$) refers to the *infinitesimal decomposition* (resp. *composition*) map of $\mathcal{C}$ (resp. $\mathcal{P}$) (cf. Chapter 6 of loc.cit).

Suppose further that $\mathcal{C}$ is **coaugmented**, i.e., it is equipped with a **coaugmentation map** $\eta : \mathcal{I} \longrightarrow \mathcal{C}$. We will refer to $\overline{\mathcal{C}} \coloneqq \mathrm{coker}\,(\mathcal{I} \xrightarrow{\eta} \mathcal{C})$ the **coaugmentation coideal** of $\mathcal{C}$. By definition, the **reduced convolution dg Lie algebra** associated to $\mathcal{C}$ and $\mathcal{P}$ is given by $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\mathcal{P}) \coloneqq (\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\mathcal{P}),[\,,],\partial)$ with

$$\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\mathcal{P}) = \prod_{n\geq 0} \mathrm{Hom}_{\Sigma_n}(\overline{\mathcal{C}}(n),\mathcal{P}(n)),$$

and with the differential ∂ and bracket $[\,,]$ defined similarly as those of $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})$.

Remark 5.2.0.2. According to the proof of [[11], Theorem 6.5.10], a map $\phi : \Omega(\mathcal{C}) \longrightarrow \mathcal{P}$ of operads can be identified to a **Maurer-Cartan element** of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\mathcal{P})$, i.e., a map $\phi : \overline{\mathcal{C}} \longrightarrow \mathcal{P}$ of degree -1 satisfying the **Maurer-Cartan equation** $\partial(\phi) + \phi\star\phi = 0$. This map extends trivially to a map $\phi : \mathcal{C} \longrightarrow \mathcal{P}$, which is also a Maurer-Cartan element of $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})$.

In what follows, we fix $\mathcal{P} \in \mathrm{Op}_*(\mathcal{C}(\mathbf{k}))$ to be a connective augmented dg operad. Let $\phi : \Omega(\mathcal{C}) \xrightarrow{\simeq} \mathcal{P}$ be a cofibrant resolution of $\mathcal{P}$, with $\mathcal{C}$ being a conilpotent dg cooperad. As before, we denote by $\overline{\mathcal{P}}$ (resp. $\overline{\mathcal{C}}$) the augmentation ideal (resp. coaugmentation coideal) of $\mathcal{P}$ (resp. $\mathcal{C}$). Moreover, we will use the same notation to write $\phi : \mathcal{C} \longrightarrow \mathcal{P}$ standing for the corresponding Maurer-Cartan element of $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})$.

Construction 5.2.0.3. Following [56], the **functorial cotangent complex** of $\mathcal{P}$ is the quasi free infinitesimal $\mathcal{P}$-bimodule $(\mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P}),\delta)$ with the differential given by $\delta = d_{\mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P})} - d^l + d^r$ where d^l is the composition

$$\mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P}) \xrightarrow{\mathrm{Id}_{\mathcal{P}}\circ_{(1)}(\Delta_{(1)}\circ\mathrm{Id}_{\mathcal{P}})} \mathcal{P}\circ_{(1)}((\mathcal{C}\circ_{(1)}\mathcal{C})\circ\mathcal{P}) \xrightarrow{\mathrm{Id}_{\mathcal{P}}\circ_{(1)}((\phi\circ_{(1)}\mathrm{Id}_{\mathcal{C}})\circ\mathrm{Id}_{\mathcal{P}})}$$
$$\mathcal{P}\circ_{(1)}((\mathcal{P}\circ_{(1)}\mathcal{C})\circ\mathcal{P}) \longrightarrow (\mathcal{P}\circ\mathcal{P}\circ\mathcal{P})\circ_{(1)}(\mathcal{C}\circ\mathcal{P}) \longrightarrow \mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P})$$

and d^r is almost the same as d^l except that in the second arrow one takes the factor $\mathrm{Id}_{\mathcal{C}}\circ_{(1)}\phi$ instead of $\phi\circ_{(1)}\mathrm{Id}_{\mathcal{C}}$ (cf. Section 5.1 of the loc.cit). Along with that, he also considered the quasi free infinitesimal

$\mathcal{P}$-bimodule $(\mathcal{P}\circ_{(1)}(s^{-1}\overline{\mathcal{C}}\circ\mathcal{P}),\overline{\delta})$ whose differential is given by $\overline{\delta} = d_{\mathcal{P}\circ_{(1)}(s^{-1}\overline{\mathcal{C}}\circ\mathcal{P})} - d^l + d^r$, defined similarly as above. This will be called the **reduced functorial cotangent complex** of $\mathcal{P}$.

We will denote by

$$\overline{\mathrm{L}}_{\mathcal{P}}^{\mathrm{res}} := (\mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P}),\delta) \ \ , \ \ \Omega_{\mathcal{P}}^{\mathrm{res}} := (\mathcal{P}\circ_{(1)}(s^{-1}\overline{\mathcal{C}}\circ\mathcal{P}),\overline{\delta}).$$

According to [[56], Lemma 4.2.2], the composite map

$$\mathcal{P}\circ_{(1)}(s^{-1}\overline{\mathcal{C}}\circ\mathcal{P}) \xrightarrow{\mathrm{Id}_{\mathcal{P}}\circ_{(1)}(\phi\circ\mathrm{Id}_{\mathcal{P}})} \mathcal{P}\circ_{(1)}(\mathcal{P}\circ\mathcal{P}) \xrightarrow{\mathrm{proj}} \Omega_{\mathcal{P}} = \mathcal{P}\circ_{(1)}\overline{\mathcal{P}}$$

exhibits $\Omega_{\mathcal{P}}^{\mathrm{res}}$ as a quasi free resolution of $\Omega_{\mathcal{P}}$ (see also Corollary 5.1.0.4 (3)). On other hand, the composite map

$$\mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P}) \xrightarrow{\mathrm{Id}_{\mathcal{P}}\circ_{(1)}(\theta\circ\mathrm{Id}_{\mathcal{P}})} \mathcal{P}\circ_{(1)}(\mathcal{I}\circ\mathcal{P}) = \mathcal{P}\circ_{(1)}\mathcal{P} \xrightarrow{\mathrm{proj}} \overline{\mathrm{L}}_{\mathcal{P}} = \mathcal{P}\circ_{(1)}\mathcal{I}$$

with $\theta : \mathcal{C}\to\mathcal{I}$ being the counit map of $\mathcal{C}$, exhibits $\overline{\mathrm{L}}_{\mathcal{P}}^{\mathrm{res}}$ as a quasi free resolution of $\overline{\mathrm{L}}_{\mathcal{P}}$ (see p. 29 of loc.cit). Moreover, since $\mathcal{P}$ is connective, these two are in fact cofibrant resolutions.

According to Theorem 5.1.0.8, we get the following:

Corollary 5.2.0.4. *Let $M \in \mathrm{IbMod}(\mathcal{P})$ be an infinitesimal $\mathcal{P}$-bimodule, regarded as the module of coefficients. The n'th Quillen cohomology group of $\mathcal{P}$ with coefficients in M is computed by*

$$\mathrm{H}^n_Q(\mathcal{P},M) \cong \mathit{Hom}_{\mathrm{Ho(IbMod}(\mathcal{P}))}(\overline{\mathrm{L}}_{\mathcal{P}}^{\mathrm{res}}, M[n+1]) \tag{5.2.0.1}$$

Remark 5.2.0.5. On other hand, by Remark 5.1.0.9, we obtain that the n'th reduced Quillen cohomology group of $\mathcal{P}$ with coefficients in M is computed by

$$\mathrm{H}^n_{Q,\mathrm{red}}(\mathcal{P},M) \cong \mathrm{Hom}_{\mathrm{Ho(IbMod}(\mathcal{P}))}(\Omega_{\mathcal{P}}^{\mathrm{res}}, M[n]) \tag{5.2.0.2}$$

Let $\mathcal{Q} \in \mathrm{Op}_*(\mathcal{C}(\mathbf{k}))$ be another dg operad and let $\alpha : \mathcal{P} \longrightarrow \mathcal{Q}$ be a map in $\mathrm{Op}_*(\mathcal{C}(\mathbf{k}))$. By Remark 5.2.0.2, the composition $\alpha\phi : \Omega(\mathcal{C}) \longrightarrow \mathcal{Q}$ performs a Maurer-Cartan element of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\mathcal{Q})$ (or $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{Q})$).

Definition 5.2.0.6. ([11]) (i) The **deformation complex of** α is defined to be $\mathrm{Hom}^{\alpha\phi}_\Sigma(\mathcal{C},\mathcal{Q})$ the twisted dg Lie algebra of the convolution dg Lie algebra $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{Q})$ by the Maurer-Cartan element $\alpha\phi$. Namely, $\mathrm{Hom}^{\alpha\phi}_\Sigma(\mathcal{C},\mathcal{Q})$ has the same bracket $[\,,]$ as of $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{Q})$ and has the differential $\partial_{\alpha\phi}$ defined as $\partial_{\alpha\phi}(f) := \partial(f) + [\alpha\phi, f]$.
(ii) The **reduced deformation complex of** α is defined to be $\mathrm{Hom}^{\alpha\phi}_\Sigma(\overline{\mathcal{C}},\mathcal{Q})$ the twisted dg Lie algebra of the reduced convolution dg Lie algebra $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\mathcal{Q})$ by the Maurer-Cartan element $\alpha\phi$.
(iii) When α is the identity map on $\mathcal{P}$, we will refer to the two as the **deformation complex** and **reduced deformation complex of** $\mathcal{P}$, respectively.

We also consider $\mathcal{Q}$ as an infinitesimal $\mathcal{P}$-bimodule with the structure induced by the map α.

Theorem 5.2.0.7. *The Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{Q} \in \mathrm{IbMod}(\mathcal{P})$ agrees with the homology of the deformation complex of α. More explicitly, for each $n \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{H}^{n-1}_Q(\mathcal{P},\mathcal{Q}) \cong \mathrm{H}_{-n}\,\mathit{Hom}^{\alpha\phi}_\Sigma(\mathcal{C},\mathcal{Q}) \tag{5.2.0.3}$$

Let $\Omega(\Delta^\bullet)$ denote the *Sullivan simplicial dg algebra of differentials* on the standard simplices $\Delta^\bullet$ (cf. [57]). According to [[62], Proposition 2.5], the simplicial object $\mathcal{Q}_\bullet := \mathcal{Q} \otimes \Omega(\Delta^\bullet)$ of dg operads forms a simplicial resolution of $\mathcal{Q}$. Observe that $\mathcal{Q}_\bullet$ admits the canonical structure of a simplicial object in $\mathrm{IbMod}(\mathcal{Q})$ induced by the degeneracy maps $\mathcal{Q} = \mathcal{Q}_0 \longrightarrow \mathcal{Q}_n$, $n \geqslant 0$.

Lemma 5.2.0.8. *$\mathcal{Q}_\bullet \in \mathrm{IbMod}(\mathcal{Q})^{\Delta^{\mathrm{op}}}$ is a simplicial resolution of $\mathcal{Q}$ when regarded as an infinitesimal bimodule over itself. Consequently, $\mathcal{Q}_\bullet$ is again a simplicial resolution of $\mathcal{Q}$ when regarded as an infinitesimal $\mathcal{P}$-bimodule.*

Proof. We will need the following fact:

(*) Suppose we are given a model category $\mathbf{M}$ and a fibrant object $X \in \mathbf{M}$. Let $X_\bullet$ be a simplicial object in $\mathbf{M}$ with $X_0 = X$. Then $X_\bullet$ is a simplicial resolution of X as an object of $\mathbf{M}$ if and only if it is a simplicial resolution of X as an object of $\mathbf{M}_{X/}$.

To prove (*), it suffices to observe that limits in the under category $\mathbf{M}_{X/}$ are taken as those in $\mathbf{M}$.

Now, since the restriction functor $\mathrm{Op}_*(\mathcal{C}(\mathbf{k}))_{\mathcal{Q}/} \longrightarrow \mathrm{IbMod}(\mathcal{Q})_{\mathcal{Q}/}$ is a right Quillen functor, it preserves simplicial resolutions of $\mathcal{Q}$. The first claim hence follows by the fact (*), immediately. Combined with the fact that the restriction functor $\alpha^* : \mathrm{IbMod}(\mathcal{Q}) \longrightarrow \mathrm{IbMod}(\mathcal{P})$ is a right Quillen functor, we deduce that $\mathcal{Q}_\bullet$ is indeed a simplicial resolution of $\mathcal{Q}$ when regarded as an infinitesimal $\mathcal{P}$-bimodule. □

By definition, the **functorial nerve** of a given dg Lie algebra $\mathfrak{g}$ is the simplicial set $\mathrm{MC}_\bullet(\mathfrak{g}) := \mathrm{MC}(\mathfrak{g} \otimes \Omega(\Delta^\bullet))$ whose n-simplices are the Maurer-Cartan elements of $\mathfrak{g} \otimes \Omega(\Delta^n)$ (see, e.g., [43]).

Proof of Theorem 5.2.0.7. By the isomorphism (5.2.0.1) we just need to prove that

$$\pi_0 \, \mathrm{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}^{\mathrm{res}}_{\mathcal{P}}, \mathcal{Q}[n]) \cong \mathrm{H}_{-n}(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q})).$$

By Lemma 5.2.0.8 we have that $\mathrm{Map}^{\mathrm{h}}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}^{\mathrm{res}}_{\mathcal{P}}, \mathcal{Q}[n]) \cong \mathrm{Hom}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}^{\mathrm{res}}_{\mathcal{P}}, (\mathcal{Q}_\bullet)[n])$. For each $m \in \mathbb{Z}$, consider the (abelian) dg Lie algebra $\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \mathbf{k}[m]$ whose bracket is trivial. According to [[63], Theorem 1.1], there is a canonical isomorphism

$$\pi_0 \, \mathrm{MC}_\bullet(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \mathbf{k}[n-1]) \cong \mathrm{H}_0(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \mathbf{k}[n]).$$

Note that the right hand side is nothing but $\mathrm{H}_{-n}(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}))$. By these facts, it will suffice to establish for each n a canonical isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}^{\mathrm{res}}_{\mathcal{P}}, (\mathcal{Q}_\bullet)[n]) \cong \mathrm{MC}_\bullet(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \mathbf{k}[n-1]).$$

Moreover, it suffices to treat only the case $n = 0$. We shall now establish a canonical isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}^{\mathrm{res}}_{\mathcal{P}}, \mathcal{Q}_\bullet) \cong \mathrm{MC}_\bullet(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \mathbf{k}[-1]) \overset{\mathrm{def}}{=} \mathrm{MC}(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \Omega(\Delta^\bullet) \otimes \mathbf{k}[-1]).$$

Following the proof of [[62], Theorem 2.12], there is a canonical isomorphism of simplicial dg Lie algebras:

$$\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \Omega(\Delta^\bullet) \cong \mathrm{Hom}^{s_\bullet\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}_\bullet)$$

where for each $m \geq 0$, $\mathrm{Hom}^{s_m\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}_m)$ refers to the twisted dg Lie algebra of $\mathrm{Hom}_{\Sigma}(\mathcal{C}, \mathcal{Q}_m)$ by the composition $\mathcal{C} \xrightarrow{\phi} \mathcal{P} \xrightarrow{\alpha} \mathcal{Q} \xrightarrow{s_m} \mathcal{Q}_m$ with s_m being the degeneracy map $\mathcal{Q} = \mathcal{Q}_0 \longrightarrow \mathcal{Q}_m$. So we obtain that

$$\mathrm{MC}(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}) \otimes \Omega(\Delta^\bullet) \otimes \mathbf{k}[-1]) \cong \mathrm{MC}(\mathrm{Hom}^{s_\bullet\alpha\phi}_{\Sigma}(\mathcal{C}, \mathcal{Q}_\bullet) \otimes \mathbf{k}[-1]).$$

The right hand side is precisely $Z_0 \operatorname{Hom}_{\Sigma}^{s_\bullet \alpha\phi}(\mathcal{C}, \mathcal{Q}_\bullet)$ the simplicial set of 0-cycles of the simplicial complex $\operatorname{Hom}_{\Sigma}^{s_\bullet \alpha\phi}(\mathcal{C}, \mathcal{Q}_\bullet)$. Thus, it remains to establish the isomorphisms of sets

$$\operatorname{Hom}_{\mathrm{IbMod}(\mathcal{P})}(\overline{\mathrm{L}}_{\mathcal{P}}^{\mathrm{res}}, \mathcal{Q}_m) \cong Z_0 \operatorname{Hom}_{\Sigma}^{s_m \alpha\phi}(\mathcal{C}, \mathcal{Q}_m)\,,\ m \geq 0$$

which are compatible with simplicial structures. Observe that a map $f : \overline{\mathrm{L}}_{\mathcal{P}}^{\mathrm{res}} \longrightarrow \mathcal{Q}_m$ in $\mathrm{IbMod}(\mathcal{P})$ is identified to a map $\mathcal{C} \longrightarrow \mathcal{Q}_m$ of degree 0 making the following square commutative

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ f\ } & \mathcal{Q}_m \\
{\scriptstyle d_{\mathcal{C}} - d^l + d^r}\downarrow & & \downarrow{\scriptstyle d_{\mathcal{Q}_m}} \\
\mathcal{P} \circ_{(1)} (\mathcal{C} \circ \mathcal{P}) & \xrightarrow[\ f\ ]{} & \mathcal{Q}_m
\end{array}$$

Unwinding computation, this commutativity is equivalent to the equation

$$\partial(f) + (s_m \alpha\phi) \star f - f \star (s_m \alpha\phi) = 0$$

meaning that $\partial_{s_m\alpha\phi}(f) = 0$, i.e., $f \in Z_0 \operatorname{Hom}_{\Sigma}^{s_m \alpha\phi}(\mathcal{C}, \mathcal{Q}_m)$. The obtained isomorphisms are clearly compatible with simplicial structures.

□

Remark 5.2.0.9. Using the same arguments as in the above proof (with the starting point being now the equation (5.2.0.2)), we obtain that the reduced Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{Q} \in \mathrm{IbMod}(\mathcal{P})$ agrees with the homology of the reduced deformation complex of α. More explicitly, for each $n \in \mathbb{Z}$, there is a canonical isomorphism

$$\mathrm{H}_{Q,\mathrm{red}}^{n-1}(\mathcal{P}, \mathcal{Q}) \cong \mathrm{H}_{-n} \operatorname{Hom}_{\Sigma}^{\alpha\phi}(\overline{\mathcal{C}}, \mathcal{Q}) \tag{5.2.0.4}$$

In fact, this identification is pretty well-know, as already shown in the works of Loday-Merkulov-Vallette ([11, 12]).

We end this section with some computations using the formulas (5.2.0.3) and (5.2.0.4). For simplicity, let us assume that $\mathcal{P} := \mathcal{P}(E, R)$ is a **binary quadratic operad** with E being concentrated in degree 0 (and in arity 2). Moreover, we suppose that $\mathcal{P}$ is **Koszul** so that $\mathcal{P}$ admits a nice cofibrant resolution $\phi : \Omega(\mathcal{P}^{\mathsf{i}}) \xrightarrow{\simeq} \mathcal{P}$ with $\mathcal{P}^{\mathsf{i}}$ being the **Koszul dual cooperad** of $\mathcal{P}$. The interested readers may refer to [[11], Chapter 7] for more details about these notions.

Note that $\mathcal{P}^{\mathsf{i}}(n+1)$ is concentrated in degree n for every $n \geqslant 0$. We now fix a map $\alpha : \mathcal{P} \longrightarrow \mathcal{Q}$ and assume that $\mathcal{Q}$ is a **connected operad** (i.e., $\mathcal{Q}(0) = 0$ and $\mathcal{Q}(1) = \mathbf{k}$) concentrated in degree 0. As a complex, $\operatorname{Hom}_{\Sigma}^{\alpha\phi}(\mathcal{P}^{\mathsf{i}}, \mathcal{Q})$ is of the form

$$\mathbf{k} = \operatorname{Hom}_{\mathbf{k}}(\mathcal{P}^{\mathsf{i}}(1), \mathcal{Q}(1)) \xrightarrow{d_1} \operatorname{Hom}_{\Sigma_2}(\mathcal{P}^{\mathsf{i}}(2), \mathcal{Q}(2)) \xrightarrow{d_2} \operatorname{Hom}_{\Sigma_3}(\mathcal{P}^{\mathsf{i}}(3), \mathcal{Q}(3)) \xrightarrow{d_3} \cdots \tag{5.2.0.5}$$

in which $\operatorname{Hom}_{\Sigma_n}(\mathcal{P}^{\mathsf{i}}(n), \mathcal{Q}(n))$ is of degree $-(n-1)$. The differential is given by

$$d(f) := [\alpha\phi, f] = (\alpha\phi) \star f - (-1)^{|f|} f \star (\alpha\phi).$$

On other hand, the complex $\operatorname{Hom}_{\Sigma}^{\alpha\phi}(\overline{\mathcal{P}^{\mathsf{i}}}, \mathcal{Q})$ is of the form

$$0 \longrightarrow \operatorname{Hom}_{\Sigma_2}(\mathcal{P}^{\mathsf{i}}(2), \mathcal{Q}(2)) \xrightarrow{d_2} \operatorname{Hom}_{\Sigma_3}(\mathcal{P}^{\mathsf{i}}(3), \mathcal{Q}(3)) \xrightarrow{d_3} \cdots \tag{5.2.0.6}$$

Remark 5.2.0.10. Let us survey the (reduced) Quillen cohomology of $\mathcal{P}$ in some small degrees. Assume that $\operatorname{Hom}_{\Sigma_2}(\mathcal{P}^{\mathsf{i}}(2), \mathcal{Q}(2)) = \mathbf{k}\{\alpha\phi\}$ the one dimensional $\mathbf{k}$-module spanned by $\alpha\phi$. Since $\alpha\phi$ is a twisting morphism, the map d_2 must be trivial. Then, by the equation (5.2.0.4) we get that

$$\mathrm{H}^0_{Q,\mathrm{red}}(\mathcal{P},\mathcal{Q}) \cong \mathrm{H}_{-1}(\mathrm{Hom}^{\alpha\phi}_{\Sigma}(\overline{\mathcal{P}^{\text{i}}},\mathcal{Q})) = \mathbf{k}.$$

Now, by cutting the long exact sequence (5.1.0.3), starting from $\mathrm{H}^{-1}_{Q,\mathrm{red}}(\mathcal{P},\mathcal{Q})$ to $\mathrm{H}^1_{Q,r}(\mathcal{P},\mathcal{Q})$, which are both trivial, we obtain an exact sequence of the form

$$0 \longrightarrow \mathrm{H}^{-1}_{Q}(\mathcal{P},\mathcal{Q}) \longrightarrow \mathrm{H}^0_{Q,r}(\mathcal{P},\mathcal{Q}) \xrightarrow{p} \mathrm{H}^0_{Q,\mathrm{red}}(\mathcal{P},\mathcal{Q}) \longrightarrow \mathrm{H}^0_{Q}(\mathcal{P},\mathcal{Q}) \longrightarrow 0 \qquad (5.2.0.7)$$

Note that $\mathrm{H}^0_{Q,r}(\mathcal{P},\mathcal{Q}) \cong \pi_0 \mathrm{Map}^{\mathrm{h}}_{\mathrm{RMod}(\mathcal{P})}(\mathcal{P},\mathcal{Q}) \cong \mathbf{k}$, since $\mathcal{Q}$ is connected. Moreover, the map p is the identity map $\mathrm{Id}_{\mathbf{k}}$. These facts, together with the exactness of (5.2.0.7), prove that $\mathrm{H}^{-1}_{Q}(\mathcal{P},\mathcal{Q})$ and $\mathrm{H}^0_{Q}(\mathcal{P},\mathcal{Q})$ are both trivial. In summary, we obtain that

(*) $\mathrm{H}^0_{Q,\mathrm{red}}(\mathcal{P},\mathcal{Q}) = \mathbf{k}$ and $\mathrm{H}^{-1}_{Q}(\mathcal{P},\mathcal{Q}) = \mathrm{H}^0_{Q}(\mathcal{P},\mathcal{Q}) = 0$ whenever $\mathrm{Hom}_{\Sigma_2}(\mathcal{P}^{\text{i}}(2),\mathcal{Q}(2)) = \mathbf{k}\{\alpha\phi\}$.

Example 5.2.0.11. Note that all our stated results remain valid for nonsymmetric operads. We consider the augmented version of the **nonsymmetric associative operad**, $\mathcal{A}s_{\geqslant 1}$. Let α be the identity map on $\mathcal{A}s_{\geqslant 1}$. By definition, $\mathcal{A}s_{\geqslant 1}(n) = \mathbf{k}\{\mu_n\}$ for $n \geqslant 1$. The partial composition in $\mathcal{A}s_{\geqslant 1}$ is simple: $\mu_m \circ_i \mu_n = \mu_{m+n-1}$ for any i. The Koszul dual cooperad $\mathcal{A}s^{\text{i}}_{\geqslant 1}$ is given on each arity by $\mathcal{A}s^{\text{i}}_{\geqslant 1}(n) = \mathbf{k}\{\mu^c_n\}$ for $n \geqslant 1$. By [[11], Lemma 9.1.7], the decomposition of $\mathcal{A}s^{\text{i}}_{\geqslant 1}$ is given by

$$\Delta(\mu^c_n) = \sum_{i_1+\cdots+i_k=n} (-1)^{\Sigma(i_j+1)(k-j)}(\mu^c_k;\mu^c_{i_1},\cdots,\mu^c_{i_k}).$$

The twisting map ϕ sends μ^c_2 to μ_2. Unwinding computation, we find that

$$[\,\mathrm{Hom}(\mathcal{A}s^{\text{i}}_{\geqslant 1}(n),\mathcal{A}s_{\geqslant 1}(n)) \xrightarrow{d_n} \mathrm{Hom}(\mathcal{A}s^{\text{i}}_{\geqslant 1}(n+1),\mathcal{A}s_{\geqslant 1}(n+1))\,] = \begin{cases} 0 & \text{if n is even} \\ \mathrm{Id}_{\mathbf{k}} & \text{if n is odd.} \end{cases}$$

The chain complex $\mathrm{Hom}^{\phi}(\mathcal{A}s^{\text{i}}_{\geqslant 1},\mathcal{A}s_{\geqslant 1})$ is therefore acyclic. So we deduce that $\mathrm{H}^{\bullet}_{Q}(\mathcal{A}s_{\geqslant 1},\mathcal{A}s_{\geqslant 1}) = 0$ and $\mathrm{H}^{\bullet}_{Q,\mathrm{red}}(\mathcal{A}s_{\geqslant 1},\mathcal{A}s_{\geqslant 1}) = \mathbf{k}$ concentrated in degree 0.

Example 5.2.0.12. We consider the **Lie operad**, $\mathcal{L}ie$ (cf. [[11], §13.2]), and compute the Quillen cohomology groups of $\mathcal{L}ie$ with coefficients in itself. Let $\mathcal{L}ie\{x_1,\cdots,x_n\}$ be the free Lie algebra with generators $x_1,\cdots,x_n$. By definition, as a $\mathbf{k}$-module, $\mathcal{L}ie(n) \subseteq \mathcal{L}ie\{x_1,\cdots,x_n\}$ is (non-freely) generated by the bracket monomials containing each x_i exactly once. The symmetric group Σ_n acts on $\mathcal{L}ie(n)$ by permutations. The Koszul dual cooperad $\mathcal{L}ie^{\text{i}}$ is given on each arity by $\mathcal{L}ie^{\text{i}}(n) = \mathbf{k}\{l^c_n\}$, while the action of $\sigma \in \Sigma_n$ on l^c_n is given by $(l^c_n)^{\sigma} := \mathrm{sign}(\sigma)l^c_n$. Observe that $\mathrm{Hom}_{\Sigma_n}(\mathcal{L}ie^{\text{i}}(n),\mathcal{L}ie(n)) = 0$ for every $n \geq 3$ because there is not any non-zero vector $\lambda \in \mathcal{L}ie(n)$ $(n \geq 3)$ satisfying that $\lambda^{\sigma} = \mathrm{sign}(\sigma)\lambda$ for every $\sigma \in \Sigma_n$. Moreover, $\mathrm{Hom}_{\Sigma_2}(\mathcal{L}ie^{\text{i}}(2),\mathcal{L}ie(2)) = \mathbf{k}\{\phi\}$ where ϕ sends l^c_2 to $[x_1,x_2]$. Thus, by Remark 5.2.0.10 we deduce that $\mathrm{H}^{\bullet}_{Q}(\mathcal{L}ie,\mathcal{L}ie) = 0$ and $\mathrm{H}^{\bullet}_{Q,\mathrm{red}}(\mathcal{L}ie,\mathcal{L}ie) = \mathbf{k}$ concentrated in degree 0.

Example 5.2.0.13. Let us compute the Quillen cohomology groups of the augmented version of the **commutative operad**, $\mathcal{C}om_{\geqslant 1}$ (cf. [[11], §13.1]), with coefficients in itself. By definition $\mathcal{C}om_{\geqslant 1}(n) = \mathbf{k}\{\mu_n\}$ equipped with the trivial Σ_n-action. The Koszul dual cooperad $\mathcal{C}om^{\text{i}}_{\geqslant 1}$ is (up to suspension) isomorphic to the **Lie cooperad** $\mathcal{L}ie^c$, which encodes the Lie coalgebras. As a Σ_n-module, $\mathcal{C}om^{\text{i}}_{\geqslant 1}(n) \cong \mathcal{L}ie(n) \otimes sign_{\Sigma_n}$. The situation is similar to Example 5.2.0.12. Unwinding verification, we have $\mathrm{Hom}_{\Sigma_2}(\mathcal{C}om^{\text{i}}_{\geqslant 1}(2),\mathcal{C}om_{\geqslant 1}(2)) = \mathbf{k}\{\phi\}$ and $\mathrm{Hom}_{\Sigma_n}(\mathcal{C}om^{\text{i}}_{\geqslant 1}(n),\mathcal{C}om_{\geqslant 1}(n)) = 0$ for every $n \geq 3$. So we deduce that $\mathrm{H}^{\bullet}_{Q}(\mathcal{C}om_{\geqslant 1},\mathcal{C}om_{\geqslant 1}) = 0$ and $\mathrm{H}^{\bullet}_{Q,\mathrm{red}}(\mathcal{C}om_{\geqslant 1},\mathcal{C}om_{\geqslant 1}) = \mathbf{k}$ concentrated in degree 0.

5.3 Operadic Dold-Kan correspondence

The **Dold-Kan correspondence** asserts that there is an adjoint equivalence

$$\Gamma : \mathcal{C}_{\geqslant 0}(\mathbf{k}) \underset{\perp}{\overset{\simeq}{\rightleftarrows}} \mathrm{sMod}_{\mathbf{k}} : \mathrm{N}$$

between the categories of connective dg $\mathbf{k}$-modules and simplicial $\mathbf{k}$-modules (cf. Examples 2.1.4.7). The functor N is the well-known **normalized complex functor**. An important fact is that the functors Γ and N are no longer inverse equivalences of each other (or even adjunction) when descending to the categories of monoids (cf. [58]). Despite this, the work of Schwede and Shipley (Theorem 3.12 of loc.cit) demonstrates that the functor N is the right adjoint of a Quillen equivalence when considered as a functor between monoids. By improving their work, Tabuada [23] proved that the functor $\mathrm{N} : \mathrm{Cat}(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Cat}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ between enriched categories, given by applying the normalized complex functor levelwise, is again the right adjoint of a certain Quillen equivalence. In this section, we will give such a statement in the operadic context. Namely, we shall prove the existence of a Quillen equivalence between the categories of enriched operads

$$\mathcal{L} : \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \underset{\perp}{\rightleftarrows} \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) : \mathrm{N}$$

in which the right adjoint is given by applying the normalized complex functor levelwise, while $\mathcal{L}$ is a bit more complicated mixed between Γ and the free-operad functors. For the proof, the key step is as follows. Let $\mathcal{P}$ be an $\mathrm{sMod}_{\mathbf{k}}$-enriched operad. We shall first prove that the functor

$$\mathrm{N}_{\mathcal{P}} : \mathrm{Alg}_{\mathcal{P}}(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Alg}_{\mathrm{N}\,\mathcal{P}}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$$

given by applying the normalized complex functor levelwise, is a right Quillen equivalence from $\mathcal{P}$-algebras to $\mathrm{N}\,\mathcal{P}$-algebras. For this, our treatment is inspired by the arguments given in the proof of [[58], Proposition 5.1].

On other hand, there is an obvious Quillen adjunction $\mathrm{Op}(\mathrm{Set}_{\Delta}) \underset{\perp}{\rightleftarrows} \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$ induced by the free-forgetful adjunction $\mathrm{Set}_{\Delta} \underset{\perp}{\rightleftarrows} \mathrm{sMod}_{\mathbf{k}}$. We therefore get a link between simplicial operads and (connective) dg operads. In the third subsection, we find a connection between Quillen cohomology of a simplicial operad and the corresponding of the associated dg operad via that link.

5.3.1 Dold-Kan correspondence of operadic modules

Let R be a unital ring and let A and B be the simplicial right and left R-modules, respectively. Recall that the **Alexander-Whitney** and **shuffle** maps form the homotopy inverses of each other:

$$\mathrm{AW} : \mathrm{N}(A \otimes_R B) \longrightarrow \mathrm{N}\,A \otimes_R \mathrm{N}\,B \ \ , \ \ \nabla : \mathrm{N}\,A \otimes_R \mathrm{N}\,B \longrightarrow \mathrm{N}(A \otimes_R B) \tag{5.3.1.1}$$

(cf., e.g., [[58], §2.3], [[59], §8.5] and [[60], Chapter 4, § 2.2]).

Remark 5.3.1.1. Let K and L be dg right and left R-modules, respectively. There is a natural map

$$\Gamma(K \otimes_R L) \longrightarrow \Gamma(K) \otimes_R \Gamma(L) \tag{5.3.1.2}$$

given by the adjoint of the composition

$$K \otimes_R L \longrightarrow \mathrm{N}\,\Gamma(K) \otimes_R \mathrm{N}\,\Gamma(L) \xrightarrow{\nabla} \mathrm{N}(\Gamma(K) \otimes_R \Gamma(L))$$

where the first map is the tensor product over R of the unit maps of the adjunction $\Gamma \dashv \mathrm{N}$. The map (5.3.1.2) is a weak equivalence as well.

Now, let A and B be two simplicial $\mathbf{k}$-modules. The maps $\nabla : \mathrm{N}\,A \otimes \mathrm{N}\,B \longrightarrow \mathrm{N}(A \otimes B)$ and $\mathrm{AW} : \mathrm{N}(A \otimes B) \longrightarrow \mathrm{N}\,A \otimes \mathrm{N}\,B$ respectively equip the functor $\mathrm{N} : \mathrm{sMod}_{\mathbf{k}} \longrightarrow \mathcal{C}_{\geqslant 0}(\mathbf{k})$ with a lax and an oplax monoidal structure. Note that the lax monoidal structure of N is symmetric while, however, its oplax monoidal structure is not (see, e.g., [[58], §2.3]). In particular, the canonical lax monoidal structure of $\Gamma : \mathcal{C}_{\geqslant 0}(\mathbf{k}) \longrightarrow \mathrm{sMod}_{\mathbf{k}}$ induced by AW is not symmetric.

The Dold-Kan correspondence lifts to an adjoint equivalence between C-collections:

$$\Gamma : \mathrm{Coll}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \underset{\perp}{\overset{\simeq}{\rightleftarrows}} \mathrm{Coll}_C(\mathrm{sMod}_{\mathbf{k}}) : \mathrm{N} \tag{5.3.1.3}$$

Since the lax monoidal structure of the functor $\mathrm{N} : \mathrm{sMod}_{\mathbf{k}} \longrightarrow \mathcal{C}_{\geqslant 0}(\mathbf{k})$ is symmetric, the lifted functor $\mathrm{N} : \mathrm{Coll}_C(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Coll}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ admits an extended lax monoidal structure:

$$\mathrm{N}(\mathcal{A}) \circ \mathrm{N}(\mathcal{B}) \longrightarrow \mathrm{N}(\mathcal{A} \circ \mathcal{B})$$

for $\mathcal{A}, \mathcal{B} \in \mathrm{Coll}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$. Due to this lax monoidal structure, the latter descends to a functor between monoids, i.e., C-colored operads:

$$\mathrm{N} : \mathrm{Op}_C(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \tag{5.3.1.4}$$

Remark 5.3.1.2. On other hand, the functor $\Gamma : \mathcal{C}_{\geqslant 0}(\mathbf{k}) \longrightarrow \mathrm{sMod}_{\mathbf{k}}$ does not admit any canonical symmetric lax monoidal structure. As a sequel, one in general can not produce $\mathrm{sMod}_{\mathbf{k}}$-enriched operads just by applying Γ to $\mathcal{C}_{\geqslant 0}(\mathbf{k})$-enriched operads levelwise.

To state our main results, we will need the following construction.

Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories. Furthermore, let $\mathcal{L} : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : \mathcal{R}$ be an adjunction between underlying categories such that the right adjoint $\mathcal{R}$ is lax monoidal. In particular, the left adjoint $\mathcal{L}$ inherits an induced oplax monoidal structure. Since $\mathcal{R}$ is lax monoidal, it descends to a functor between monoids. Now, fix A to be a monoid in $\mathcal{D}$. Again, the functor $\mathcal{R}$ descends to a functor $\mathcal{R} : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathcal{R}A}$ from left A-modules to left $\mathcal{R}A$-modules. Suppose that coequalizers of left A-modules exist. We take a functor $\mathcal{L}^A : \mathrm{Mod}_{\mathcal{R}A} \longrightarrow \mathrm{Mod}_A$ given by sending $M \in \mathrm{Mod}_{\mathcal{R}A}$ to the coequalizer

$$A \otimes \mathcal{L}(\mathcal{R}A \otimes M) \rightrightarrows A \otimes \mathcal{L}M \longrightarrow \mathcal{L}^A M\,. \tag{5.3.1.5}$$

One of the two maps is given by applying $A \otimes \mathcal{L}(-)$ to the structure map $\mathcal{R}A \otimes M \longrightarrow M$ of M. The other map is the unique map of left A-modules extending the composition

$$\mathcal{L}(\mathcal{R}A \otimes M) \longrightarrow \mathcal{L}\mathcal{R}A \otimes \mathcal{L}M \longrightarrow A \otimes \mathcal{L}M$$

where the first map is an oplax monoidal map of $\mathcal{L}$ and the second map is the tensor product of the counit map $\mathcal{L}\mathcal{R}A \longrightarrow A$ with $\mathrm{Id}_{\mathcal{L}M}$. As in [[58], Section 3], we have the following.

Lemma 5.3.1.3. *The obtained functors $\mathcal{L}^A$ and $\mathcal{R}$ form an adjunction*

$$\mathcal{L}^A : \mathrm{Mod}_{\mathcal{R}A} \underset{\perp}{\rightleftarrows} \mathrm{Mod}_A : \mathcal{R}.$$

Moreover, the module left adjoint $\mathcal{L}^A$ and the original left adjoint $\mathcal{L}$ are related via the identification

$$\mathcal{L}^A \circ (\mathcal{R}A \otimes -) \cong (A \otimes -) \circ \mathcal{L}\,. \tag{5.3.1.6}$$

Proof. Given any $M \in \mathrm{Mod}_{\mathcal{R}A}$ and $N \in \mathrm{Mod}_A$, we need to establish a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_A}(\mathcal{L}^A M, N) \cong \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{R}A}}(M, \mathcal{R}N).$$

By construction, giving a map $f : \mathcal{L}^A M \longrightarrow N$ in Mod_A is equivalent to giving a map $A \otimes \mathcal{L} M \longrightarrow N$ of left A-modules compatible with the coequalizer diagram (5.3.1.5) (*). The latter is identified with a map $\mathcal{L} M \longrightarrow N$ in $\mathcal{D}$, and hence corresponds to a map $f^{\mathrm{ad}} : M \longrightarrow \mathcal{R} N$ in $\mathcal{C}$, due to the adjunction $\mathcal{L} \dashv \mathcal{R}$. To make it work, we first need to show that f^{ad} is indeed a map of left $\mathcal{R} A$-modules. This follows by the condition (*), along with making use of the lax naturality of $\mathcal{R}$ suitably. Conversely, let $g : M \longrightarrow \mathcal{R} N$ be a map in $\mathrm{Mod}_{\mathcal{R} A}$. Consider the corresponding map $\mathcal{L} M \longrightarrow N$ in $\mathcal{D}$, which is identified with a map $A \otimes \mathcal{L} M \longrightarrow N$ of left A-modules. As before, it can be readily verified that the latter is compatible with the coequalizer diagram (5.3.1.5) and therefore, it induces a map $g^{\mathrm{ad}} : \mathcal{L}^A M \longrightarrow N$ in Mod_A. It is clear by construction that the obtained assignments $f \mapsto f^{\mathrm{ad}}$ and $g \mapsto g^{\mathrm{ad}}$ are natural and moreover, form the inverses of each other.

By construction, there is a commutative square of right adjoints

$$\begin{array}{ccc} \mathrm{Mod}_A & \xrightarrow{\mathcal{R}} & \mathrm{Mod}_{\mathcal{R} A} \\ {\scriptstyle \text{forgetful}} \downarrow & & \downarrow {\scriptstyle \text{forgetful}} \\ \mathcal{D} & \xrightarrow[\mathcal{R}]{} & \mathcal{C} , \end{array}$$

which proves the identification (5.3.1.6). □

We now fix $\mathcal{P} \in \mathrm{Op}_C(\mathrm{sMod}_{\mathbf{k}})$ to be a C-colored operad in $\mathrm{sMod}_{\mathbf{k}}$. Consider the functor $\mathrm{N}_{\mathcal{P}} : \mathrm{LMod}(\mathcal{P}) \longrightarrow \mathrm{LMod}(\mathrm{N}\,\mathcal{P})$ from left $\mathcal{P}$-modules to left $\mathrm{N}\,\mathcal{P}$-modules given by applying the normalized complex functor levelwise. Due to Lemma 5.3.1.3, $\mathrm{N}_{\mathcal{P}}$ admits a left adjoint $\mathcal{L}_{\mathcal{P}} : \mathrm{LMod}(\mathrm{N}\,\mathcal{P}) \longrightarrow \mathrm{LMod}(\mathcal{P})$. By construction, $\mathrm{N}_{\mathcal{P}}$ descends to a functor $\mathrm{N}_{\mathcal{P}} : \mathrm{Alg}_{\mathcal{P}}(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Alg}_{\mathrm{N}\,\mathcal{P}}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ between full subcategories of $\mathcal{P}$-algebras and $\mathrm{N}\,\mathcal{P}$-algebras. On other hand, note that the embedding functor $\mathrm{Alg}_{\mathcal{P}}(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{LMod}(\mathcal{P})$ is a left adjoint, it in particular preserves colimits. This implies that $\mathcal{L}_{\mathcal{P}}$ descends to a functor $\mathcal{L}_{\mathcal{P}} : \mathrm{Alg}_{\mathrm{N}\,\mathcal{P}}(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \longrightarrow \mathrm{Alg}_{\mathcal{P}}(\mathrm{sMod}_{\mathbf{k}})$ as well. We thus obtain an adjunction $\mathcal{L}_{\mathcal{P}} \dashv \mathrm{N}_{\mathcal{P}}$ between $\mathrm{N}\,\mathcal{P}$-algebras and $\mathcal{P}$-algebras.

Proposition 5.3.1.4. *The adjunction $\mathcal{L}_{\mathcal{P}} : \mathrm{Alg}_{\mathrm{N}\,\mathcal{P}}(\mathcal{C}_{\geqslant 0}(\boldsymbol{k})) \rightleftarrows \mathrm{Alg}_{\mathcal{P}}(\mathrm{sMod}_{\boldsymbol{k}}) : \mathrm{N}_{\mathcal{P}}$ is a Quillen equivalence.*

Let A be an $\mathrm{N}\,\mathcal{P}$-algebra. Recall from § 2.1.2 that there is a canonical map $j_{\mathcal{L}_{\mathcal{P}} A} : \mathcal{P} \longrightarrow \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)$ in $\mathrm{Op}_C(\mathrm{sMod}_{\mathbf{k}})$ with $\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)$ being the enveloping operad associated to the pair $(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)$. On other hand, note that the initial $\mathrm{N}\,\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)$-algebra is nothing but $\mathrm{N}\,\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)_0 = \mathrm{N}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}} A$. Now, due to the adjunction $\mathrm{Env} \dashv \delta$, the map $\mathrm{N}(j_{\mathcal{L}_{\mathcal{P}} A})$ and the unit map $A \longrightarrow \mathrm{N}_{\mathcal{P}} \mathcal{L}_{\mathcal{P}} A$ together induce a canonical map $\varphi_A : \mathrm{Env}(\mathrm{N}\,\mathcal{P}, A) \longrightarrow \mathrm{N}\,\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)$ in $\mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$. Considering φ_A as a map of C-collections and then, taking the adjoint of the latter, we obtain a map

$$\Psi_A : \Gamma\ \mathrm{Env}(\mathrm{N}\,\mathcal{P}, A) \longrightarrow \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)$$

in $\mathrm{Coll}_C(\mathrm{sMod}_{\mathbf{k}})$. For the proof of Proposition 5.3.1.4, we first need to prove that Ψ_A is a weak equivalence provided A is a cofibrant $\mathrm{N}\,\mathcal{P}$-algebra.

Our argument exploits the description of **free extensions of operads** (cf. [27, 33, 54]). Suppose we are given a pushout square of C-colored operads in some symmetric monoidal category of the form

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(N) \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}' \end{array} \qquad (5.3.1.7)$$

where $\mathcal{F}(f):\mathcal{F}(M)\longrightarrow\mathcal{F}(N)$ is the free map generated by the map $f:M\to N$ of C-collections. Fix a C-sequence $\overline{c}:=(c_1,\cdots,c_n;c)$. According to [[27], Proposition 5.2], the object $\mathcal{O}'(\overline{c})$ is described as the colimit of the sequence of maps

$$\mathcal{O}'^{(k)}(\overline{c})\longrightarrow\mathcal{O}'^{(k+1)}(\overline{c})$$

(with $\mathcal{O}'^{(0)}(\overline{c})=\mathcal{O}(\overline{c})$) which are pushouts of the maps of the form

$$H:\coprod_T f_T^{*}\otimes_{\mathrm{Aut}(T)}\Sigma_{\{c_1,\cdots,c_n\}}\longrightarrow\coprod_T f_T\otimes_{\mathrm{Aut}(T)}\Sigma_{\{c_1,\cdots,c_n\}}\tag{5.3.1.8}$$

The coproduct ranges over all isomorphism classes of $\mathbf{Tree}^{(k)}_{\overline{c}}$ the collection of **planar trees** with k **marked vertices** and with **valency** $\overline{c}$ (i.e., with **root edge** colored by c and **leaves** colored by $c_1,\cdots,c_n$). Moreover, $\Sigma_{\{c_1,\cdots,c_n\}}$ is certain product of symmetric groups depending on $\{c_1,\cdots,c_n\}$ (cf. the notation Σ_s in loc.cit). In order to determine the group $\mathrm{Aut}(T)$, one partitions the set $\{T_1,\cdots,T_r\}$ into subsets of pairwise isomorphic trees

$$\{T^1_1,\cdots,T^1_{r_1}\}\sqcup\cdots\sqcup\{T^q_1,\cdots,T^q_{r_q}\}.$$

Then, one writes

$$\mathrm{Aut}(T):=\prod_{i=1}^{q}\mathrm{Aut}(T^i)^{r_i}\ltimes\prod_{i=1}^{q}\Sigma_{r_i}.$$

Now, to understand the map H (5.3.1.8), it suffices to determine the map $\varepsilon_T:f_T^{*}\longrightarrow f_T$. For each $T\in\mathbf{Tree}^{(k)}_{\overline{c}}$, one takes the canonical **grafting operation** $T=r_T(T_1,\cdots,T_r)$ with r_T being the **root vertex** of T and T_i being the subtree of T grafted to the i'th leaf of r_T. The map ε_T is inductively defined as an iterated pushout-product of the form

$$\varepsilon_T:=\varepsilon_{r_T}\,\square\,\varepsilon_{T_1}\,\square\cdots\square\,\varepsilon_{T_r}\tag{5.3.1.9}$$

with ε_{r_T} given by

$$\varepsilon_{r_T}:=\begin{cases}f(val(r_T)) & \text{if } r_T \text{ is marked}\\ \eta_{\mathcal{O}}(val(r_T)) & \text{if } r_T \text{ is unmarked}\end{cases}$$

in which $val(r_T)$ refers to the valency of r_T and $\eta_{\mathcal{O}}:\mathcal{I}_C\longrightarrow\mathcal{O}$ is the unit of $\mathcal{O}$.

For the proof below, we just need to concentrate in analyzing the map ε_T. The readers may refer to [27] (around Proposition 5.2) for more details about the above constructions.

Lemma 5.3.1.5. *For any cofibrant* $\mathrm{N}\,\mathcal{P}$*-algebra* A*, the map* $\Psi_A:\Gamma\,\mathrm{Env}(\mathrm{N}\,\mathcal{P},A)\xrightarrow{\simeq}\mathrm{Env}(\mathcal{P},\mathcal{L}_{\mathcal{P}}A)$ *is a weak equivalence in* $\mathrm{Coll}_C(\mathrm{sMod}_{\boldsymbol{k}})$.

Proof. It will suffice to treat only the case where A is a *cellular* $\mathrm{N}\,\mathcal{P}$*-algebra*. Recall that a cellular $\mathrm{N}\,\mathcal{P}$-algebra is a sequential colimit, starting from the initial $\mathrm{N}\,\mathcal{P}$-algebra $\mathrm{N}\,\mathcal{P}_0$, of pushouts of *free cofibrations* (i.e., images of cofibrations in $\mathcal{C}_{\geqslant 0}(\mathbf{k})^{\times C}$ under the free $\mathrm{N}\,\mathcal{P}$-algebra functor). According to [[27], Proposition 4.4(ii)], along with noting the fact that the forgetful functor $\mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))\longrightarrow\mathrm{Coll}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ preserves filtered colimits, both the functors $\Gamma\,\mathrm{Env}(\mathrm{N}\,\mathcal{P},-)$ and $\mathrm{Env}(\mathcal{P},\mathcal{L}_{\mathcal{P}}(-))$ preserve the filtration of A. It allows us to argue by induction on the filtration of A, because weak equivalences in $\mathrm{Coll}_C(\mathrm{sMod}_{\mathbf{k}})$ are closed under filtered colimits. By the adjunction $\mathrm{Env}\dashv\delta$, the map $\Psi_{\mathrm{N}\,\mathcal{P}_0}$ agrees with the counit map $\Gamma\,\mathrm{N}\,\mathcal{P}\longrightarrow\mathcal{P}$, which is an isomorphism. Hence, it remains to show that for any pushout square in

$\mathrm{Alg}_{\mathrm{N}\,\mathcal{P}}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ of the form

$$\begin{array}{ccc} \mathrm{N}\,\mathcal{P}\circ X & \xrightarrow{\mathrm{N}\,\mathcal{P}\circ f} & \mathrm{N}\,\mathcal{P}\circ Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \tag{5.3.1.10}$$

with the top horizontal map being the free cofibration generated by a given map $f : X \longrightarrow Y$ in $\mathcal{C}_{\geqslant 0}(\mathbf{k})^{\times C}$, the map Ψ_B is a weak equivalence provided that the map Ψ_A is one.

By [[27], Proposition 4.4(iii)], there is a pushout square in $\mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ of the form

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \downarrow & & \downarrow \\ \mathrm{Env}(\mathrm{N}\,\mathcal{P}, A) & \longrightarrow & \mathrm{Env}(\mathrm{N}\,\mathcal{P}, B) \end{array} \tag{5.3.1.11}$$

with the top horizontal map being the free cofibration generated by f. As discussed above, for each $\overline{c} := (c_1, \cdots, c_n; c)$, $\mathrm{Env}(\mathrm{N}\,\mathcal{P}, B)(\overline{c})$ is the colimit of the sequence of maps

$$\mathrm{Env}(\mathrm{N}\,\mathcal{P}, B)^{(k)}(\overline{c}) \longrightarrow \mathrm{Env}(\mathrm{N}\,\mathcal{P}, B)^{(k+1)}(\overline{c})$$

which are pushouts of the maps of the form

$$H : \coprod_{T \in \mathbf{Tree}_{\overline{c}}^{(k)}/\cong} f_T^{*} \otimes_{\mathrm{Aut}(T)} \Sigma_{\{c_1,\cdots,c_n\}} \longrightarrow \coprod_{T \in \mathbf{Tree}_{\overline{c}}^{(k)}/\cong} f_T \otimes_{\mathrm{Aut}(T)} \Sigma_{\{c_1,\cdots,c_n\}} \tag{5.3.1.12}$$

Since f is a cofibration, the map $\varepsilon_T : f_T^{*} \longrightarrow f_T$ is one for every tree T. In fact, this is a consequence of [[32], Lemma 3.1], yet it is also can be proved in similar fashion as we treat the claim (*) below. Now, since ε_T is a cofibration (and since $\mathbf{k}$ is a field of characteristic 0), so is the map H. Therefore, the mentioned pushouts are in fact homotopy pushouts.

On other hand, by applying $\mathcal{L}_{\mathcal{P}}$ to the pushout square (5.3.1.10) and noting that the map $\mathcal{L}_{\mathcal{P}}(\mathrm{N}\,\mathcal{P}\circ f)$ is canonically isomorphic to the free cofibration $\mathcal{P}\circ\Gamma(f) : \mathcal{P}\circ\Gamma(X) \longrightarrow \mathcal{P}\circ\Gamma(Y)$ (cf. Lemma 5.3.1.3), we hence get that $\mathcal{L}_{\mathcal{P}}A \longrightarrow \mathcal{L}_{\mathcal{P}}B$ is pushout of $\mathcal{P}\circ\Gamma(f)$. As in the above paragraph, $\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}B)(\overline{c})$ is the colimit of the sequence of maps

$$\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}B)^{(k)}(\overline{c}) \longrightarrow \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}B)^{(k+1)}(\overline{c})$$

which are (homotopy) pushouts of the maps of the form

$$\widetilde{H} : \coprod_{T \in \mathbf{Tree}_{\overline{c}}^{(k)}/\cong} (\Gamma f)_T^{*} \otimes_{\mathrm{Aut}(T)} \Sigma_{\{c_1,\cdots,c_n\}} \longrightarrow \coprod_{T \in \mathbf{Tree}_{\overline{c}}^{(k)}/\cong} (\Gamma f)_T \otimes_{\mathrm{Aut}(T)} \Sigma_{\{c_1,\cdots,c_n\}} \tag{5.3.1.13}$$

For our purpose, we have to prove that the map $\Psi_B(\overline{c}) : \Gamma\,\mathrm{Env}(\mathrm{N}\,\mathcal{P}, B)(\overline{c}) \longrightarrow \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}B)(\overline{c})$ is a weak equivalence for every C-sequence $\overline{c}$. The map $\Psi_B(\overline{c})$ is compatible with the mentioned filtrations of $\mathrm{Env}(\mathrm{N}\,\mathcal{P}, B)(\overline{c})$ and $\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}B)(\overline{c})$. As indicated in the two above paragraphs, on the filtration of $\mathrm{Env}(\mathrm{N}\,\mathcal{P}, B)(\overline{c})$ (resp. $\mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}B)(\overline{c})$), the $(k+1)$-skeleton is given by (homotopy) pushout of the k-skeleton along a map of type H (resp. $\widetilde{H}$). For the 0-skeleton, by construction the map $\Psi_B^{(0)}(\overline{c})$ agrees with $\Psi_A(\overline{c})$, which was assumed to be a weak equivalence. By an inductive argument, it suffices to prove the following:

(*) For every $k \in \mathbb{N}$, $\overline{c} \in \mathrm{Seq}(C)$ and for every $T \in \mathbf{Tree}_{\overline{c}}^{(k)}$, the canonical maps $\Gamma f_T^{*} \longrightarrow (\Gamma f)_T^{*}$ and $\Gamma f_T \longrightarrow (\Gamma f)_T$ are weak equivalences.

By construction, f^*_T and f_T are respectively the source and target of the map ε_T (5.3.1.9) which is inductively defined as:

$$\varepsilon_T = \varepsilon_{r_T} \square \varepsilon_{T_1} \square \cdots \square \varepsilon_{T_r}$$

where ε_{r_T} is given by

$$\varepsilon_{r_T} := \begin{cases} f(val(r_T)) & \text{if } r_T \text{ is marked} \\ \eta_{\mathrm{Env}(\mathrm{N}\,\mathcal{P},A)}(val(r_T)) & \text{if } r_T \text{ is unmarked .} \end{cases}$$

Similarly, $(\Gamma f)^*_T$ and $(\Gamma f)_T$ are respectively the source and target of the map

$$\widetilde{\varepsilon}_T := \widetilde{\varepsilon}_{r_T} \square \widetilde{\varepsilon}_{T_1} \square \cdots \square \widetilde{\varepsilon}_{T_r}$$

where $\widetilde{\varepsilon}_{r_T}$ is given by

$$\widetilde{\varepsilon}_{r_T} := \begin{cases} \Gamma f(val(r_T)) & \text{if } r_T \text{ is marked} \\ \eta_{\mathrm{Env}(\mathcal{P},\mathrm{L}_{\mathcal{P}}\,A)}(val(r_T)) & \text{if } r_T \text{ is unmarked .} \end{cases}$$

We shall now prove the claim (*) in inductive style. By an **elementary tree**, we shall mean a planar tree with the root vertex as its unique vertex.

• In the first step, we assume that T is elementary. First, consider the case where r_T is unmarked. Then, the map $\Gamma f^*_T \longrightarrow (\Gamma f)^*_T$ agrees with $\mathrm{Id}_{\mathbf{k}}$ if $val(r_T) = (c;c)$ and agrees with Id_0 otherwise; while the map $\Gamma f_T \longrightarrow (\Gamma f)_T$ agrees with

$$\Psi_A(val(r_T)) : \Gamma\,\mathrm{Env}(\mathrm{N}\,\mathcal{P},A)(val(r_T)) \longrightarrow \mathrm{Env}(\mathcal{P},\mathrm{L}_{\mathcal{P}}\,A)(val(r_T)),$$

which was assumed to be a weak equivalence. In the other case where r_T is marked, the map $\Gamma f^*_T \longrightarrow (\Gamma f)^*_T$ is simply $\mathrm{Id}_{\Gamma X(val(r_T))}$ and the map $\Gamma f_T \longrightarrow (\Gamma f)_T$ is $\mathrm{Id}_{\Gamma Y(val(r_T))}$.

• In the next step, we suppose that T is not elementary. In particular, T has at least one **internal edge** e. Consider a decomposition $T = (T_1,T_2)$ of T obtained by cutting itself at e. We are showing that if both T_1 and T_2 satisfy the condition of (*) then so does T. Indeed, observe that $\varepsilon_T = \varepsilon_{T_1} \square \varepsilon_{T_2}$ and $\widetilde{\varepsilon}_T = \widetilde{\varepsilon}_{T_1} \square \widetilde{\varepsilon}_{T_2}$. So the map $\Gamma f_T \longrightarrow (\Gamma f)_T$ can be written as

$$\Gamma(f_{T_1} \otimes f_{T_2}) \longrightarrow (\Gamma f)_{T_1} \otimes (\Gamma f)_{T_2},$$

which is in fact the composition

$$\Gamma(f_{T_1} \otimes f_{T_2}) \longrightarrow \Gamma f_{T_1} \otimes \Gamma f_{T_2} \longrightarrow (\Gamma f)_{T_1} \otimes (\Gamma f)_{T_2}.$$

The first map is an oplax monoidal map of Γ, which is a weak equivalence. Thus, $\Gamma f_T \longrightarrow (\Gamma f)_T$ is a weak equivalence whenever the maps $\Gamma f_{T_i} \longrightarrow (\Gamma f)_{T_i}$ $(i = 1,2)$ are weak equivalences. On the other hand, the map $\Gamma f^*_T \longrightarrow (\Gamma f)^*_T$ is isomorphic to the map between pushouts:

$$\Gamma\,(f^*_{T_1} \otimes f_{T_2}) \coprod_{\Gamma\,(f^*_{T_1} \otimes f^*_{T_2})} \Gamma\,(f_{T_1} \otimes f^*_{T_2}) \quad\longrightarrow\quad (\Gamma f)^*_{T_1} \otimes (\Gamma f)_{T_2} \coprod_{(\Gamma f)^*_{T_1} \otimes (\Gamma f)^*_{T_2}} (\Gamma f)_{T_1} \otimes (\Gamma f)^*_{T_2}.$$

Note that these pushouts are in fact homotopy pushouts, since the maps ε_{T_i} and $\widetilde{\varepsilon}_{T_i}$ are cofibrations (see the second paragraph). Therefore, by the same reason as discussed above, $\Gamma f^*_T \longrightarrow (\Gamma f)^*_T$ is a weak equivalence whenever all the maps $\Gamma f_{T_i} \longrightarrow (\Gamma f)_{T_i}$ and $\Gamma f^*_{T_i} \longrightarrow (\Gamma f)^*_{T_i}$ are weak equivalences for $i = 1,2$.

We will say that the decomposition $T = (T_1,T_2)$ is elementary if both T_1 and T_2 are so.

• Now, let T be any tree. If T is elementary then we are done by the first step. Otherwise, we take a decomposition $T = (T_1,T_2)$ of T as in the second step. If this decomposition is elementary then we are done by the first and second steps. Otherwise, we decompose the subtree T_i at its internal edge (if it is not elementary). By the finiteness of trees, this process must be stable at elementary decompositions. Hence, the claim is verified by the first and second steps, again. □

Proof of Proposition 5.3.1.4. The functor $\mathrm{N}_{\mathcal{P}}$ creates fibrations and weak equivalences. In particular, the adjunction $\mathcal{L}_{\mathcal{P}} \dashv \mathrm{N}_{\mathcal{P}}$ indeed forms a Quillen adjunction. To prove that this is a Quillen equivalence, it suffices to show that for every cofibrant $\mathrm{N}\,\mathcal{P}$-algebra A the unit map $\eta_A : A \longrightarrow \mathrm{N}_{\mathcal{P}}\,\mathcal{L}_{\mathcal{P}}(A)$ is a weak equivalence. Considering η_A as a map in $\mathcal{C}_{\geqslant 0}(\mathbf{k})^{\times C}$, it suffices to verify that the adjoint of η_A,

$$\eta_A^* : \Gamma(A) \longrightarrow \mathcal{L}_{\mathcal{P}}(A),$$

is a weak equivalence in $\mathrm{sMod}_{\mathbf{k}}^{\times C}$. As in the above proof, we can assume that A is a cellular $\mathrm{N}\,\mathcal{P}$-algebra and then, argue by induction on the filtration of A. Observe first that the map $\eta^*_{\mathrm{N}\,\mathcal{P}_0}$ coincides with the counit map $\Gamma\,\mathrm{N}\,\mathcal{P}_0 \longrightarrow \mathcal{P}_0$, which is an isomorphism. Hence, it remains to show that for any pushout square in $\mathrm{Alg}_{\mathrm{N}\,\mathcal{P}}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ of the form

$$\begin{array}{ccc} \mathrm{N}\,\mathcal{P} \circ X & \xrightarrow{\mathrm{N}\,\mathcal{P}\circ f} & \mathrm{N}\,\mathcal{P} \circ Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \tag{5.3.1.14}$$

with the top horizontal map being a free cofibration generated by a map $f : X \to Y$ in $\mathcal{C}_{\geqslant 0}(\mathbf{k})^{\times C}$, the map η_B^* is a weak equivalence when provided that η_A^* is one.

For simplicity of equations, in the remainder, we assume the set of colors C is a singleton. According to [[27], Proposition 5.7], the underlying dg $\mathbf{k}$-module of B is sequential colimit of maps $B_{n-1} \longrightarrow B_n$ with $B_0 = A$ and such that each map $B_{n-1} \longrightarrow B_n$ is cobase change of the map $\varphi_n := \mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} f^{\square n}$ with $f^{\square n}$ being the n-fold pushout-product of f. Since f is a cofibration, so is $f^{\square n}$ and hence, so is φ_n. (For this, note that we are working on a field of characteristic 0). In particular, the map $B_{n-1} \longrightarrow B_n$ is in fact a homotopy pushout of φ_n. Moreover, by extracting the iterated pushout-product $f^{\square n}$, the map φ_n is of the form

$$\varphi_n : \mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} Q_n(X,Y) \longrightarrow \mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} Y^{\otimes n}$$

in which $Q_n(X,Y)$ is the domain of $f^{\square n}$. We may regard $Q_n(X,Y)$ as the colimit of the *punctured n-cube* (i.e., the n-cube with the terminal vertex removed) $W_n(X,Y)$ whose vertices are of the form $C_1 \otimes C_2 \otimes \cdots \otimes C_n$ such that each C_i is either X or Y, and whose edges are given by multi-tensor product of the maps of types $\{f, \mathrm{Id}_X, \mathrm{Id}_Y\}$. The object $Q_n(X,Y)$ is in fact homotopy colimit of $W_n(X,Y)$, again due to the pushout-product axiom.

On other hand, by applying the left adjoint $\mathcal{L}_{\mathcal{P}}$ to (5.3.1.14), we obtain a pushout square in $\mathrm{Alg}_{\mathcal{P}}(\mathrm{sMod}_{\mathbf{k}})$:

$$\begin{array}{ccc} \mathcal{L}_{\mathcal{P}}(\mathrm{N}\,\mathcal{P} \circ X) & \longrightarrow & \mathcal{L}_{\mathcal{P}}(\mathrm{N}\,\mathcal{P} \circ Y) \\ \downarrow & & \downarrow \\ \mathcal{L}_{\mathcal{P}}(A) & \longrightarrow & \mathcal{L}_{\mathcal{P}}(B) \end{array} \tag{5.3.1.15}$$

By Lemma 5.3.1.3, the top horizontal map is canonically isomorphic to the free cofibration $\mathcal{P} \circ \Gamma(f) : \mathcal{P} \circ \Gamma(X) \longrightarrow \mathcal{P} \circ \Gamma(Y)$. As in the above paragraph, the underlying simplicial $\mathbf{k}$-module of $\mathcal{L}_{\mathcal{P}}(B)$ is sequential colimit of maps $\mathcal{L}_{\mathcal{P}}(B)_{n-1} \longrightarrow \mathcal{L}_{\mathcal{P}}(B)_n$ with $\mathcal{L}_{\mathcal{P}}(B)_0 = \mathcal{L}_{\mathcal{P}}(A)$ and such that each map

$\mathcal{L}_{\mathcal{P}}(B)_{n-1} \longrightarrow \mathcal{L}_{\mathcal{P}}(B)_n$ is (homotopy) pushout of the map

$$\widetilde{\varphi}_n : \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(A))(n) \otimes_{\Sigma_n} Q_n(\Gamma(X), \Gamma(Y)) \longrightarrow \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(A))(n) \otimes_{\Sigma_n} (\Gamma Y)^{\otimes n}$$

in which $Q_n(\Gamma(X), \Gamma(Y)) \in \mathrm{sMod}_{\mathbf{k}}$ is homotopy colimit of the punctured n-cube $W_n(\Gamma(X), \Gamma(Y))$, defined similarly as above.

The map $\eta_B^* : \Gamma(B) \longrightarrow \mathcal{L}_{\mathcal{P}}(B)$ is compatible with the filtrations of B and of $\mathcal{L}_{\mathcal{P}}(B)$ constructed in the two above paragraphs. Thus, η_B^* is a weak equivalence as soon as for each n the component map $(\eta_B^*)_n : \Gamma(B_n) \longrightarrow \mathcal{L}_{\mathcal{P}}(B)_n$ is one. Again, the latter will be proved by an inductive argument. When $n = 0$, the map $(\eta_B^*)_0$ concides with η_A^*, which was assumed to be a weak equivalence. Assume by induction that the map $(\eta_B^*)_{n-1} : \Gamma(B_{n-1}) \longrightarrow \mathcal{L}_{\mathcal{P}}(B)_{n-1}$ is a weak equivalence. As indicated above, the object $\Gamma(B_n)$ is homotopy pushout of $\Gamma(B_{n-1})$ along $\Gamma(\varphi_n)$, while $\mathrm{L}_{\mathcal{P}}(B)_n$ is homotopy pushout of $\mathrm{L}_{\mathcal{P}}(B)_{n-1}$ along the map $\widetilde{\varphi}_n$. It therefore suffices to show that the following two canonical maps

$$\Gamma(\mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} Q_n(X, Y)) \xrightarrow{\theta_1} \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)(n) \otimes_{\Sigma_n} Q_n(\Gamma(X), \Gamma(Y)),$$

$$\Gamma(\mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} Y^{\otimes n}) \xrightarrow{\theta_2} \mathrm{Env}(\mathcal{P}, \mathcal{L}_{\mathcal{P}} A)(n) \otimes_{\Sigma_n} (\Gamma Y)^{\otimes n}$$

are a weak equivalence.

(1) Concretely, the map θ_1 is given by the composition

$$\Gamma\left(\mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} Q_n(X, Y)\right) \longrightarrow \Gamma\,\mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \otimes_{\Sigma_n} \Gamma Q_n(X, Y) \longrightarrow \mathrm{Env}(\mathcal{P}, \mathrm{L}_{\mathcal{P}}\, A)(n) \otimes_{\Sigma_n} Q_n(\Gamma(X), \Gamma(Y)).$$

The first one is the canonical map of type (5.3.1.2), which is a weak equivalence. The second one is the tensor product over Σ_n of the map $\Psi_A(n) : \Gamma\,\mathrm{Env}(\mathrm{N}\,\mathcal{P}, A)(n) \longrightarrow \mathrm{Env}(\mathcal{P}, \mathrm{L}_{\mathcal{P}}\, A)(n)$, which is a weak equivalence by Lemma 5.3.1.5, with the canonical map $h_{X,Y} : \Gamma Q_n(X, Y) \longrightarrow Q_n(\Gamma(X), \Gamma(Y))$. Therefore, the map θ_1 is a weak equivalence as soon as $h_{X,Y}$ is one. As analyzed above, $Q_n(-,-)$ is homotopy colimit of the punctured n-cube $W_n(-,-)$ and moreover, the restriction of $h_{X,Y}$ on each vertex of the punctured n-cube diagram is an (iterated) oplax monoidal map of Γ of the form

$$\Gamma(C_1 \otimes C_2 \otimes \cdots \otimes C_n) \longrightarrow \Gamma C_1 \otimes \Gamma C_2 \otimes \cdots \otimes \Gamma C_n$$

in which each C_i is either X or Y. The latter is certainly a weak equivalence and hence, $h_{X,Y}$ is indeed a weak equivalence.

(2) The map θ_2 is also a weak equivalence, by the same fashion as discussed above. □

Recall from §2.1.3 that for any operad $\mathcal{O}$ enriched over some symmetric monoidal category $\mathcal{S}$, there exists an enriched category, denoted by $\mathbf{Ib}^{\mathcal{O}}$, such that the structure of an infinitesimal $\mathcal{O}$-bimodule is equivalent to that of an $\mathcal{S}$-valued enriched functor on $\mathbf{Ib}^{\mathcal{O}}$.

Corollary 5.3.1.6. *Let $\mathcal{P} \in \mathrm{Op}_C(\mathrm{sMod}_{\boldsymbol{k}})$ be given. The functor*

$$\mathrm{N}_{\mathcal{P}}^{ib} : \mathrm{IbMod}(\mathcal{P}) \longrightarrow \mathrm{IbMod}(\mathrm{N}\,\mathcal{P})$$

given by applying the normalized complex functor levelwise is a right Quillen equivalence.

Proof. Consider $\mathbf{Ib}^{\mathcal{P}}$ as an operad concentrated in arity 1. By Proposition 5.3.1.4, the normalized complex functor induces a right Quillen equivalence

$$\mathrm{Fun}(\mathbf{Ib}^{\mathcal{P}}, \mathrm{sMod}_{\mathbf{k}}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{N}(\mathbf{Ib}^{\mathcal{P}}), \mathcal{C}_{\geqslant 0}(\mathbf{k}))$$

between the projective model categories of enriched functors. On other hand, the shuffle map ∇ induces a $\mathcal{C}_{\geqslant 0}(\mathbf{k})$-enriched functor $\nabla^* : \mathbf{Ib}^{\mathrm{N}\,\mathcal{P}} \longrightarrow \mathrm{N}(\mathbf{Ib}^{\mathcal{P}})$, which is the identity on objects and is a weak equivalence in $\mathrm{Cat}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$. We hence obtain another right Quillen equivalence

$$\mathrm{Fun}(\mathrm{N}(\mathbf{Ib}^{\mathcal{P}}), \mathcal{C}_{\geqslant 0}(\mathbf{k})) \xrightarrow{\simeq} \mathrm{Fun}(\mathbf{Ib}^{\mathrm{N}\,\mathcal{P}}, \mathcal{C}_{\geqslant 0}(\mathbf{k}))$$

given by precomposition with ∇^*. Finally, the proof is completed by observing that the functor $\mathrm{N}^{ib}_{\mathcal{P}}$ agrees with the composition of the two right Quillen equivalences mentioned above. □

Moreover, we have seen from §2.1.3 that each of the categories $\mathrm{LMod}(\mathcal{O})$, $\mathrm{RMod}(\mathcal{O})$ and $\mathrm{BMod}(\mathcal{O})$ can be also represented as a category of algebras over a certain operad. Using the same arguments as in the above proof, we get the following.

Corollary 5.3.1.7. *Let $\mathcal{P} \in \mathrm{Op}_C(\mathrm{sMod}_{\boldsymbol{k}})$ be given. The functors*

$$\mathrm{N}^l_{\mathcal{P}} : \mathrm{LMod}(\mathcal{P}) \longrightarrow \mathrm{LMod}(\mathrm{N}\,\mathcal{P}) \ , \ \mathrm{N}^r_{\mathcal{P}} : \mathrm{RMod}(\mathcal{P}) \longrightarrow \mathrm{RMod}(\mathrm{N}\,\mathcal{P}) \ , \ \mathrm{N}^b_{\mathcal{P}} : \mathrm{BMod}(\mathcal{P}) \longrightarrow \mathrm{BMod}(\mathrm{N}\,\mathcal{P})$$

given by applying the normalized complex functor levelwise, are all right Quillen equivalences.

5.3.2 Dold-Kan correspondence of enriched operads

To specify the regarded sets of colors, for each set C, we will write

$$\mathrm{N}_C : \mathrm{Op}_C(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$$

standing for the functor given by applying the normalized complex functor levelwise.

Applying Proposition 5.3.1.4 to the case where $\mathcal{P} = \mathbf{O}_C$ the operad of C-colored operads in $\mathrm{sMod}_{\mathbf{k}}$, we get a right Quillen equivalence $\mathrm{Op}_C(\mathrm{sMod}_{\mathbf{k}}) \xrightarrow{\simeq} \mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ given by applying the normalized complex functor levelwise. The latter certainly agrees with N_C. We will denote by $\mathcal{L}_C$ the left adjoint of N_C. In particular, we obtain the following.

Corollary 5.3.2.1. *The adjunction $\mathcal{L}_C : \mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\boldsymbol{k})) \rightleftarrows \mathrm{Op}_C(\mathrm{sMod}_{\boldsymbol{k}}) : \mathrm{N}_C$ is a Quillen equivalence.*

We wish to integrate the collection of adjunctions $\{\mathcal{L}_C \dashv \mathrm{N}_C\}_{C \in \mathrm{Sets}}$ into a single adjunction between the categories of enriched operads. To this end, we first need to realize how the Grothendieck construction behaves with adjunctions. Let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ be a contravariant functor on a given category $\mathcal{C}$ valued in the category of small categories Cat. For each map $\alpha : c \to d$ in $\mathcal{C}$, we let $\alpha^* : \mathcal{F}(d) \longrightarrow \mathcal{F}(c)$ denote the functor corresponding to α through $\mathcal{F}$. We denote by $\mathrm{Groth}(\mathcal{F})$ the (contravariant) Grothendieck construction of $\mathcal{F}$. Recall by definition that objects of $\mathrm{Groth}(\mathcal{F})$ are given by

$$\mathrm{Ob}(\mathrm{Groth}(\mathcal{F})) = \bigsqcup_{c \in \mathrm{Ob}(\mathcal{C})} \mathrm{Ob}(\mathcal{F}(c)) \, ,$$

and for $X \in \mathrm{Ob}(\mathcal{F}(c))$ and $Y \in \mathrm{Ob}(\mathcal{F}(d))$, the hom-set is given by

$$\mathrm{Hom}_{\mathrm{Groth}(\mathcal{F})}(X, Y) = \bigsqcup_{c \xrightarrow{\alpha} d} \mathrm{Hom}_{\mathcal{F}(c)}(X, \alpha^* Y)$$

where the coproduct ranges over $\mathrm{Hom}_{\mathcal{C}}(c, d)$.

Construction 5.3.2.2. Suppose we are given two functors $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ and suppose that for every object $c \in \mathcal{C}$, there is an adjunction

$$\mathcal{L}_c : \mathcal{F}(c) \rightleftarrows \mathcal{G}(c) : \mathcal{N}_c$$

such that the collection of right adjoints $\{\mathcal{N}_c\}_{c\in\mathcal{C}}$ forms a natural transformation from $\mathcal{G}$ to $\mathcal{F}$. We establish two **integrated functors** $\mathcal{L} : \mathrm{Groth}(\mathcal{F}) \rightleftarrows \mathrm{Groth}(\mathcal{G}) : \mathcal{N}$ as follows.

On objects, $\mathcal{N}$ is simply given by the coproduct of the maps $\mathcal{N}_c : \mathrm{Ob}(\mathcal{G}(c)) \longrightarrow \mathrm{Ob}(\mathcal{F}(c))$ and similarly, $\mathcal{L}$ is given by the coproduct of the maps $\mathcal{L}_c : \mathrm{Ob}(\mathcal{F}(c)) \longrightarrow \mathrm{Ob}(\mathcal{G}(c))$.

Fix $\alpha : c \to d$ to be a map in $\mathcal{C}$. Let $A \in \mathcal{G}(c)$ and $B \in \mathcal{G}(d)$ be given objects. For each map $f : A \longrightarrow \alpha^* B$ in $\mathcal{G}(c)$ regarded as a map from A to B in $\mathrm{Groth}(\mathcal{G})$, we take $\mathcal{N}(f)$ to be the map

$$\mathcal{N}_c(A) \xrightarrow{\mathcal{N}_c(f)} \mathcal{N}_c(\alpha^* B) \cong \alpha^* \mathcal{N}_d(B).$$

On the other hand, let $X \in \mathcal{F}(c)$ and $Y \in \mathcal{F}(d)$ be given objects. For each map $g : X \longrightarrow \alpha^* Y$ in $\mathcal{F}(c)$ regarded as a map from X to Y in $\mathrm{Groth}(\mathcal{F})$, we take $\mathcal{L}(g) : \mathcal{L}_c(X) \longrightarrow \alpha^* \mathcal{L}_d(Y)$ to be the adjoint of the composition

$$X \xrightarrow{g} \alpha^* Y \xrightarrow{\alpha^*(\eta_Y)} \alpha^* \mathcal{N}_d \mathcal{L}_d(Y) \cong \mathcal{N}_c \alpha^* \mathcal{L}_d(Y)$$

where η_Y signifies unit map of the adjunction $\mathcal{L}_d \dashv \mathcal{N}_d$.

The main interest in this construction is that:

Lemma 5.3.2.3. *The obtained integrated functors $\mathcal{L}$ and $\mathcal{N}$ form an adjunction between the Grothendieck constructions*

$$\mathcal{L} : \mathrm{Groth}(\mathcal{F}) \rightleftarrows \mathrm{Groth}(\mathcal{G}) : \mathcal{N} .$$

Proof. Let $c, d \in \mathcal{C}$ be two objects of $\mathcal{C}$. For $X \in \mathcal{F}(c)$ and $A \in \mathcal{G}(d)$, regarded as objects of $\mathrm{Groth}(\mathcal{F})$ and $\mathrm{Groth}(\mathcal{G})$ respectively, we have to establish a natural isomorphism

$$\varphi_{X,A} : \mathrm{Hom}_{\mathrm{Groth}(\mathcal{F})}(\mathcal{L}_c(X), A) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Groth}(\mathcal{G})}(X, \mathcal{N}_d(A)).$$

Note first that for each map $\alpha : c \to d$ in $\mathcal{C}$, due to the adjunction $\mathcal{L}_c \dashv \mathcal{N}_c$, we have a natural isomorphism

$$\varphi^{\alpha}_{X,A} : \mathrm{Hom}_{\mathcal{G}(c)}(\mathcal{L}_c(X), \alpha^* A) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}(c)}(X, \mathcal{N}_c(\alpha^* A)) \cong \mathrm{Hom}_{\mathcal{F}(c)}(X, \alpha^* \mathcal{N}_d(A)).$$

We now take $\varphi_{X,A}$ to be the coproduct of the maps $\varphi^{\alpha}_{X,A}$ with α ranging over $\mathrm{Hom}_{\mathcal{C}}(c, d)$. It remains to verify the naturality of the obtained map $\varphi_{X,A}$. The verification is straightforward using the naturality of the maps $\varphi^{\alpha}_{X,A}$, along with manipulating basic properties of adjunction suitably. ☐

Using this lemma, we can now integrate all the adjunctions $\mathcal{L}_C : \mathrm{Op}_C(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \rightleftarrows \mathrm{Op}_C(\mathrm{sMod}_{\mathbf{k}}) : \mathrm{N}_C$ for $C \in \mathrm{Sets}$ into a single adjunction between the categories of enriched operads, written as

$$\mathcal{L} : \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) : \mathrm{N} \ .$$

The statement we really care about in this section is the following:

Theorem 5.3.2.4. *(**Operadic Dold-Kan correspondence**) The adjunction*

$$\mathcal{L} : \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\boldsymbol{k})) \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\boldsymbol{k}}) : \mathrm{N}$$

is a Quillen equivalence.

The free-forgetful adjunction $\mathbf{k}\{-\} : \mathrm{Set}_{\Delta} \rightleftarrows \mathrm{sMod}_{\mathbf{k}} : \mathbb{U}$ lifts to an adjunction

$$\mathbf{k}\{-\} : \mathrm{Op}(\mathrm{Set}_{\Delta}) \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) : \mathbb{U} \qquad (5.3.2.1)$$

between simplicial operads and operads in simplicial $\mathbf{k}$-modules.

Lemma 5.3.2.5. *Let $f : \mathcal{P} \longrightarrow \mathcal{Q}$ be a map in* $\mathrm{Op}(\mathrm{sMod}_{\boldsymbol{k}})$*. Then f is a fibration (resp. weak equivalence) in* $\mathrm{Op}(\mathrm{sMod}_{\boldsymbol{k}})$ *if and only if* $\mathbb{U}(f)$ *is a fibration (resp. weak equivalence) in* $\mathrm{Op}(\mathrm{Set}_\Delta)$.

Proof. It is clear by definition that f is a levelwise weak equivalence if and only if $\mathbb{U}(f)$ is one. On other hand, by adjunction the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(\mathcal{P}) \longrightarrow \mathrm{Ho}(\mathcal{Q})$ agrees with the functor $\mathrm{Ho}(\mathbb{U}(f)) : \mathrm{Ho}(\mathbb{U}(\mathcal{P})) \longrightarrow \mathrm{Ho}(\mathbb{U}(\mathcal{Q}))$ (cf. Section 2.1.5). These facts together show that f is a weak equivalence if and only if $\mathbb{U}(f)$ is one.

By transferring the Dwyer-Kan model structure on $\mathrm{Op}(\mathrm{Set}_\Delta)$ along the adjunction $\mathbf{k}\{-\} \dashv \mathbb{U}$, we get a model structure on $\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$, which in particular has trivial fibrations being precisely the levelwise trivial fibrations surjective on colors. Combining this with the above paragraph, we see that the obtained model structure agrees with the Dwyer-Kan model structure on $\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$. Consequently, the map f is a fibration if and only if $\mathbb{U}(f)$ is one. □

Proof of Theorem 5.3.2.4. Let $f : \mathcal{P} \longrightarrow \mathcal{Q}$ be a map in $\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$. The above lemma shows that f is a fibration if and only if it is a levelwise fibration and such that the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(\mathcal{P}) \longrightarrow \mathrm{Ho}(\mathcal{Q})$ is an isofibration (see also [61]). An analogue holds for fibrations in $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$, according to [22] (along with noting Remark 2.1.5.7).

By the Dold-Kan correspondence, there is a natural equivalence $\mathrm{Ho}(\mathcal{O}) \simeq \mathrm{Ho}(\mathrm{N}(\mathcal{O}))$ for every $\mathcal{O} \in \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$. Thus, the functor $\mathrm{Ho}(f) : \mathrm{Ho}(\mathcal{P}) \longrightarrow \mathrm{Ho}(\mathcal{Q})$ is essentially surjective (resp. an isofibration) if and only if the functor

$$\mathrm{Ho}(\mathrm{N}(f)) : \mathrm{Ho}(\mathrm{N}\,\mathcal{P}) \longrightarrow \mathrm{Ho}(\mathrm{N}\,\mathcal{Q})$$

is essentially surjective (resp. an isofibration). Combining this with the fact that the functor N creates levelwise weak equivalences and fibrations, we deduce further that N creates weak equivalences and fibrations. In particular, $\mathcal{L} \dashv \mathrm{N}$ indeed forms a Quillen adjunction. Finally, to prove this is a Quillen equivalence, it suffices to prove that the unit map $\eta_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathrm{N}\,\mathcal{L}(\mathcal{P})$ is a weak equivalence for every cofibrant operad $\mathcal{P} \in \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$. This directly follows from Corollary 5.3.2.1, because $\eta_{\mathcal{P}}$ is simply given by the unit map of the adjunction between operads with fixed set of colors. □

5.3.3 From Quillen cohomology of simplicial operads to Quillen cohomology of dg operads

Theorem 5.3.2.4 provides us with a Quillen equivalence between operads in $\mathcal{C}_{\geqslant 0}(\mathbf{k})$ and $\mathrm{sMod}_{\mathbf{k}}$

$$\mathcal{L} : \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) : \mathrm{N}\,.$$

Let us fix $\mathcal{P}$ to be an operad in $\mathrm{sMod}_{\mathbf{k}}$ and consider the induced Quillen adjunction

$$\mathcal{L}_{\mathcal{P}//\mathcal{P}} : \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))_{\mathrm{N}\,\mathcal{P}//\mathrm{N}\,\mathcal{P}} \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})_{\mathcal{P}//\mathcal{P}} : \mathrm{N}_{\mathcal{P}//\mathcal{P}}\,.$$

Lemma 5.3.3.1. *The adjunction $\mathcal{L}_{\mathcal{P}//\mathcal{P}} \dashv \mathrm{N}_{\mathcal{P}//\mathcal{P}}$ is a Quillen equivalence. Consequently, the induced adjunction between tangent categories*

$$\mathrm{Sp}(\mathcal{L}_{\mathcal{P}//\mathcal{P}}) : \mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\boldsymbol{k})) \rightleftarrows \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathrm{sMod}_{\boldsymbol{k}}) : \mathrm{Sp}(\mathrm{N}_{\mathcal{P}//\mathcal{P}})$$

is a Quillen equivalence as well.

Proof. We first take a cofibrant resolution $f : (\mathrm{N}\,\mathcal{P})^{\mathrm{cof}} \xrightarrow{\simeq} \mathrm{N}\,\mathcal{P}$ of $\mathrm{N}\,\mathcal{P}$ in $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ such that f is a trivial fibration. Since $\mathbf{k}$ is a field of characteristic 0, the model structure on $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ is left proper and hence, is in fact proper, (and so is the model structure on $\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$). This implies that the induced adjunction $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))_{(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}}//(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}}} \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})_{\mathrm{N}\,\mathcal{P}//\mathrm{N}\,\mathcal{P}}$ is a Quillen equivalence. It now suffices to show that the composed adjunction

$$\mathcal{L}'_{\mathcal{P}//\mathcal{P}} : \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))_{(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}}//(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}}} \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})_{\mathcal{P}//\mathcal{P}} : \mathrm{N}'_{\mathcal{P}//\mathcal{P}}$$

is a Quillen equivalence. More precisely, given $\mathcal{Q} \in \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))_{(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}}//(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}}}$, then $\mathcal{L}'_{\mathcal{P}//\mathcal{P}}(\mathcal{Q})$ is given by the pushout of $\mathcal{L}(\mathcal{Q})$ along the composition $\mathcal{L}(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}} \longrightarrow \mathcal{L}\,\mathrm{N}\,\mathcal{P} \longrightarrow \mathcal{P}$, which is a weak equivalence because $\mathcal{L} \dashv \mathrm{N}$ is a Quillen equivalence. In the other hand, given $\mathcal{R} \in \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})_{\mathcal{P}//\mathcal{P}}$, then $\mathrm{N}'_{\mathcal{P}//\mathcal{P}}(\mathcal{R})$ is given by the pullback of $\mathrm{N}(\mathcal{R})$ along the map $f : (\mathrm{N}\,\mathcal{P})^{\mathrm{cof}} \xrightarrow{\simeq} \mathrm{N}\,\mathcal{P}$.

Since f is a trivial fibration and since N creates weak equivalences, the functor $\mathrm{N}'_{\mathcal{P}//\mathcal{P}}$ creates weak equivalences as well. It hence suffices to show that the unit map $\eta_{\mathcal{Q}} : \mathcal{Q} \longrightarrow \mathrm{N}'_{\mathcal{P}//\mathcal{P}}\,\mathcal{L}'_{\mathcal{P}//\mathcal{P}}(\mathcal{Q})$ is a weak equivalence provided that $\mathcal{Q}$ is cofibrant. When $\mathcal{Q}$ is cofibrant (i.e., the structure map $(\mathrm{N}\,\mathcal{P})^{\mathrm{cof}} \longrightarrow \mathcal{Q}$ is a cofibration), the structure map $\mathcal{L}(\mathcal{Q}) \longrightarrow \mathcal{L}'_{\mathcal{P}//\mathcal{P}}(\mathcal{Q})$ is a weak equivalence in $\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$ because $\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}})$ is left proper. But $\mathcal{Q}$ is also cofibrant as an object of $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$, so the adjoint map $\eta'_{\mathcal{Q}} : \mathcal{Q} \longrightarrow \mathrm{N}\,\mathcal{L}'_{\mathcal{P}//\mathcal{P}}(\mathcal{Q})$ is again a weak equivalence in $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$. On other hand, since f is a trivial fibration, the structure map $\theta_{\mathcal{Q}} : \mathrm{N}'_{\mathcal{P}//\mathcal{P}}\,\mathcal{L}'_{\mathcal{P}//\mathcal{P}}(\mathcal{Q}) \longrightarrow \mathrm{N}\,\mathcal{L}'_{\mathcal{P}//\mathcal{P}}(\mathcal{Q})$ is in particular a weak equivalence in $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$. The claim is now proved by the fact that $\eta'_{\mathcal{Q}} = \theta_{\mathcal{Q}} \circ \eta_{\mathcal{Q}}$. □

Corollary 5.3.1.6 provides us with a right Quillen equivalence $\mathrm{N}^{ib}_{\mathcal{P}} : \mathrm{IbMod}(\mathcal{P}) \xrightarrow{\simeq} \mathrm{IbMod}(\mathrm{N}\,\mathcal{P})$. This lifts to a right Quillen functor between tangent categories

$$\mathrm{Sp}(\mathrm{N}^{ib}_{\mathcal{P}}) : \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) \longrightarrow \mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{IbMod}(\mathrm{N}\,\mathcal{P}).$$

Moreover, we have a commutative square of right Quillen functors

$$\begin{array}{ccc} \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) & \xrightarrow{\mathrm{Sp}(\mathrm{N}_{\mathcal{P}//\mathcal{P}})} & \mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k})) \\ {\scriptstyle \mathcal{U}_{\mathcal{P}}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \mathcal{U}_{\mathrm{N}\,\mathcal{P}}} \\ \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P}) & \xrightarrow[\mathrm{Sp}(\mathrm{N}^{ib}_{\mathcal{P}})]{} & \mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{IbMod}(\mathrm{N}\,\mathcal{P}) \end{array} \tag{5.3.3.1}$$

in which the vertical functors are the right Quillen equivalences appearing in Theorem 3.2.4.1. In particular, $\mathrm{Sp}(\mathrm{N}^{ib}_{\mathcal{P}})$ is also a right Quillen equivalence by the above lemma.

Proposition 5.3.3.2. *Let $\mathcal{M} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ be a fibrant object. The normalized complex functor induces an isomorphism*

$$\mathrm{H}^{\bullet}_{Q}(\mathcal{P}, \mathcal{M}) \xrightarrow{\simeq} \mathrm{H}^{\bullet}_{Q}(\mathrm{N}\,\mathcal{P}, \mathrm{N}\,\mathcal{M})$$

between Quillen cohomology of $\mathcal{P} \in \mathrm{Op}(\mathrm{sMod}_{\boldsymbol{k}})$ with coefficients in $\mathcal{M} \in \mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$ and Quillen cohomology of $\mathrm{N}\,\mathcal{P} \in \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\boldsymbol{k}))$ with coefficients in $\mathrm{N}\,\mathcal{M} \in \mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{IbMod}(\mathrm{N}\,\mathcal{P})$, where $\mathrm{N}\,\mathcal{M} := \mathrm{Sp}(\mathrm{N}^{ib}_{\mathcal{P}})(\mathcal{M})$ given by applying the normalized complex functor to $\mathcal{M}$ degreewise and levelwise.

Proof. Recall that, after sending coefficients into $\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})$, the Quillen cohomology of $\mathcal{P}$ with coefficients in $\mathcal{M}$ is formulated as

$$\mathrm{H}^{\bullet}_{Q}(\mathcal{P}, \mathcal{M}) = \pi_0\,\mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})}(\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}}), \mathcal{M}[\bullet]).$$

Similarly, the Quillen cohomology of $\mathrm{N}\,\mathcal{P}$ with coefficients in $\mathrm{N}\,\mathcal{M}$ is given by

$$\mathrm{H}^{\bullet}_{Q}(\mathrm{N}\,\mathcal{P},\mathrm{N}\,\mathcal{M}) = \pi_0\,\mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{IbMod}(\mathrm{N}\,\mathcal{P})}(\mathbb{R}\mathcal{U}_{\mathrm{N}\,\mathcal{P}}(\mathrm{L}_{\mathrm{N}\,\mathcal{P}}),\mathrm{N}\,\mathcal{M}[\bullet]).$$

Since the functor $\mathrm{N} : \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$ is a right Quillen equivalence, the right derived functor $\mathbb{R}\,\mathrm{Sp}(\mathrm{N}_{\mathcal{P}//\mathcal{P}})$ sends $\mathrm{L}_{\mathcal{P}}$ to $\mathrm{L}_{\mathrm{N}\,\mathcal{P}}$, and hence the right derived functor $\mathbb{R}\,\mathrm{Sp}(\mathrm{N}^{ib}_{\mathcal{P}})$ sends $\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}})$ to $\mathbb{R}\mathcal{U}_{\mathrm{N}\,\mathcal{P}}(\mathrm{L}_{\mathrm{N}\,\mathcal{P}})$, due to the commutativity of (5.3.3.1). Moreover, since $\mathrm{Sp}(\mathrm{N}^{ib}_{\mathcal{P}})$ is a right Quillen equivalence, it induces a homotopy equivalence

$$\mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathcal{P}}\,\mathrm{IbMod}(\mathcal{P})}(\mathbb{R}\mathcal{U}_{\mathcal{P}}(\mathrm{L}_{\mathcal{P}}),\mathcal{M}[\bullet]) \xrightarrow{\simeq} \mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_{\mathrm{N}\,\mathcal{P}}\,\mathrm{IbMod}(\mathrm{N}\,\mathcal{P})}(\mathbb{R}\mathcal{U}_{\mathrm{N}\,\mathcal{P}}(\mathrm{L}_{\mathrm{N}\,\mathcal{P}}),\mathrm{N}\,\mathcal{M}[\bullet]).$$

Combining this with the first paragraph, we get the conclusion. □

Consider the Quillen adjunction $\mathbf{k}\{-\} : \mathrm{Op}(\mathrm{Set}_{\Delta}) \rightleftarrows \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) : \mathbb{U}$.

Definition 5.3.3.3. Let $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_{\Delta})$ be a simplicial operad. We will denote by $\mathrm{d}\mathcal{O} := \mathrm{N}\,\mathbf{k}\{\mathcal{O}\}$ the image of $\mathcal{O}$ under the composite functor

$$\mathrm{Op}(\mathrm{Set}_{\Delta}) \xrightarrow{\mathbf{k}\{-\}} \mathrm{Op}(\mathrm{sMod}_{\mathbf{k}}) \xrightarrow{\mathrm{N}} \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$$

and refer to it as the **differential graded (dg) version of** $\mathcal{O}$.

In fact, many dg operads of interest come from this way, typically, including the dg version of the little n-cubes operad E_n for $n \in \mathbb{N}$.

Besides that, we also regard the induced Quillen adjunction between operadic infinitesimal bimodules $\mathbf{k}\{-\}_{ib} : \mathrm{IbMod}(\mathcal{O}) \rightleftarrows \mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\}) : \mathbb{U}_{ib}$. This induces a Quillen adjunction between the associated tangent categories (still denoted by)

$$\mathbf{k}\{-\}_{ib} : \mathcal{T}_{\mathcal{O}}\,\mathrm{IbMod}(\mathcal{O}) \rightleftarrows \mathcal{T}_{\mathbf{k}\{\mathcal{O}\}}\,\mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\}) : \mathbb{U}_{ib}.$$

Notation 5.3.3.4. Let $\mathcal{O}$ be a fibrant and Σ-cofibrant simplicial operad. For each $\mathcal{M} \in \mathcal{T}_{\mathbf{k}\{\mathcal{O}\}}\,\mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\})$, we will denote by $\mathcal{H}_{\mathcal{M}} : \mathrm{Tw}(\mathcal{O}) \longrightarrow \mathbf{Spectra}$ the image of $\mathcal{M}$ under the composed ∞-functor

$$\mathcal{T}_{\mathbf{k}\{\mathcal{O}\}}\,\mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\})_{\infty} \xrightarrow{(\mathbb{U}_{ib})_{\infty}} \mathcal{T}_{\mathcal{O}}\,\mathrm{IbMod}(\mathcal{O})_{\infty} \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Tw}(\mathcal{O}),\mathbf{Spectra})$$

in which the second functor is the equivalence indicated in §4.3.

There is a connection between Quillen cohomology of a simplicial operad and Quillen cohomology of its dg version, expressed as follows.

Proposition 5.3.3.5. *Let $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_{\Delta})$ be a fibrant and Σ-cofibrant simplicial operad and let $\mathcal{M} \in \mathcal{T}_{\boldsymbol{k}\{\mathcal{O}\}}\,\mathrm{IbMod}(\boldsymbol{k}\{\mathcal{O}\})$ be a fibrant object. There is a canonical isomorphism*

$$\mathrm{H}^{\bullet}_{Q}(\mathcal{O},\mathcal{H}_{\mathcal{M}}) \xrightarrow{\simeq} \mathrm{H}^{\bullet}_{Q}(\mathrm{d}\mathcal{O},\mathrm{N}\,\mathcal{M})$$

*between Quillen cohomology of $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_{\Delta})$ with coefficients in $\mathcal{H}_{\mathcal{M}} : \mathrm{Tw}(\mathcal{O}) \longrightarrow$ **Spectra** (cf. Theorem 4.3.0.1) and Quillen cohomology of its dg version $\mathrm{d}\mathcal{O} \in \mathrm{Op}(\mathcal{C}_{\geqslant 0}(\boldsymbol{k}))$ with coefficients in $\mathrm{N}\,\mathcal{M} \in \mathcal{T}_{\mathrm{d}\mathcal{O}}\,\mathrm{IbMod}(\mathrm{d}\mathcal{O})$, where $\mathrm{N}\,\mathcal{M}$ is given by applying the normalized complex functor to $\mathcal{M}$ degreewise and levelwise.*

Proof. By notation and by the adjunction $\mathbf{k}\{-\}_{ib} \dashv \mathbb{U}_{ib}$, we find a canonical isomorphism

$$\mathrm{H}^{\bullet}_{Q}(\mathcal{O},\mathcal{H}_{\mathcal{M}}) \xrightarrow{\simeq} \mathrm{H}^{\bullet}_{Q}(\mathbf{k}\{\mathcal{O}\},\mathcal{M}).$$

On other hand, Proposition 5.3.3.2 proves the existence of a canonical isomorphism

$$\mathrm{H}^{\bullet}_{Q}(\mathbf{k}\{\mathcal{O}\},\mathcal{M}) \xrightarrow{\simeq} \mathrm{H}^{\bullet}_{Q}(\mathrm{d}\mathcal{O},\mathrm{N}\,\mathcal{M}).$$

So we get the conclusion. □

For more illustration to obtained results, we take any $M \in \mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\})$ and consider the case $\mathcal{M} = \Sigma^{\infty}(\mathbf{k}\{\mathcal{O}\} \sqcup M) \in \mathcal{T}_{\mathbf{k}\{\mathcal{O}\}}\,\mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\})$ with the structure maps given by

$$\mathbf{k}\{\mathcal{O}\} \xrightarrow{i_0} \mathbf{k}\{\mathcal{O}\} \sqcup M \xrightarrow{\mathrm{Id}_{\mathbf{k}\{\mathcal{O}\}}+0} \mathbf{k}\{\mathcal{O}\}.$$

In this situation, we simply denote by $\mathcal{H}_M := \mathcal{H}_{\mathcal{M}}$ the corresponding functor $\mathrm{Tw}(\mathcal{O}) \longrightarrow \mathbf{Spectra}$. We now describe this functor.

Computations 5.3.3.6. The object $\mathcal{M}$, regarded as a suspension spectrum (see around Computations 3.4.0.14), is given at each bidegree (n,n) as $\mathcal{M}_{n,n} = \mathbf{k}\{\mathcal{O}\} \sqcup M[n]$ the coproduct of $\mathbf{k}\{\mathcal{O}\}$ with the n-suspension of M in $\mathrm{IbMod}(\mathbf{k}\{\mathcal{O}\})$. First, argue that since $\mathcal{M}$ is also a fibrant Ω-spectrum, the image $\mathbb{U}_{ib}(\mathcal{M}) \in \mathcal{T}_{\mathcal{O}}\,\mathrm{IbMod}(\mathcal{O})$ has already the right type. By construction, $\mathbb{U}_{ib}(\mathcal{M})$ is the Ω-spectrum whose value at each bidegree (n,n) is given by the pullback in $\mathrm{IbMod}(\mathcal{O})$:

$$\begin{array}{ccc} \mathbb{U}_{ib}(\mathcal{M})_{n,n} & \longrightarrow & \mathbf{k}\{\mathcal{O}\} \sqcup M[n] \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{k}\{\mathcal{O}\} \end{array} \tag{5.3.3.2}$$

It can be shown that, for each operation $\mu \in \mathcal{O}(\overline{c})$ regarded as an object of $\mathrm{Tw}(\mathcal{O})$, the Ω-spectrum $\mathcal{H}_M(\mu) \in \mathbf{Spectra}$ is given at each bidegree (n,n) as the fiber in Set_{Δ}:

$$\mathcal{H}_M(\mu)_{n,n} = \mathbb{U}_{ib}(\mathcal{M})_{n,n}(\overline{c}) \times_{\mathcal{O}(\overline{c})} \{\mu\},$$

(this is very similar to the computations given in §4.3). By the Cartesian square (5.3.3.2), $\mathcal{H}_M(\mu)_{n,n}$ agrees with the fiber

$$(\mathbf{k}\{\mathcal{O}(\overline{c})\} \sqcup M(\overline{c})[n]) \times_{\mathbf{k}\{\mathcal{O}(\overline{c})\}} \{\mu\}$$

in Set_{Δ}. Note that this fiber is the same as the kernel of the projection $\mathbf{k}\{\mathcal{O}(\overline{c})\} \sqcup M(\overline{c})[n] \longrightarrow \mathbf{k}\{\mathcal{O}(\overline{c})\}$ regarded as a map of simplicial $\mathbf{k}$-modules, which is given by $M(\overline{c})[n]$. We thus find that

$$\mathcal{H}_M(\mu)_{n,n} = M(\overline{c})[n]$$

the n'th suspension of $M(\overline{c}) \in \mathrm{sMod}_{\mathbf{k}}$. So we deduce that $\mathcal{H}_M(\mu) = M(\overline{c})[\bullet] \in \mathbf{Spectra}$. We will simply write $\mathcal{H}_M(\mu) = M(\overline{c})$, because it comes from $M(\overline{c})$ via the composition $\mathrm{sMod}_{\mathbf{k}} \longrightarrow \mathrm{Sp}(\mathrm{sMod}_{\mathbf{k}}) \longrightarrow$ Spectra.

Let $\overline{d} \in Seq(C)$ be another C-sequence and let $\nu \in \mathcal{O}(\overline{d})$ be another operation of $\mathcal{O}$. Recall by construction that the data of an edge $\mu \to \nu$ in $\mathrm{Tw}(\mathcal{O})$ contains a morphism $\overline{c} \longrightarrow \overline{d}$ in $\mathbf{Ib}^{\mathcal{O}}$ and hence, it determines the structure map

$$\mathcal{H}_M(\mu) = M(\overline{c}) \longrightarrow M(\overline{d}) = \mathcal{H}_M(\nu),$$

because M carries an infinitesimal $\mathcal{O}$-bimodule structure, i.e., a functor $\mathbf{Ib}^{\mathcal{O}} \longrightarrow \mathrm{Set}_{\Delta}$.

Note that when $\mathcal{M} = \Sigma^{\infty}(\mathbf{k}\{\mathcal{O}\} \sqcup M)$ and considering $\mathrm{d}\mathcal{O}$ as an object in $\mathrm{Op}(\mathcal{C}(\mathbf{k}))$ (instead of $\mathrm{Op}(\mathcal{C}_{\geqslant 0}(\mathbf{k}))$), then under the right Quillen equivalence $\mathcal{T}_{\mathrm{d}\mathcal{O}}\,\mathrm{IbMod}(\mathrm{d}\mathcal{O}) \xrightarrow{\simeq} \mathrm{IbMod}(\mathrm{d}\mathcal{O})$, the object

$\mathrm{N}\,\mathcal{M} \in \mathcal{T}_{\mathrm{d}\mathcal{O}}\,\mathrm{IbMod}(\mathrm{d}\mathcal{O})$ corresponds to nothing but $\mathrm{N}(M) \in \mathrm{IbMod}(\mathrm{d}\mathcal{O})$. Thus, with having the description of $\mathcal{H}_M$ and Theorem 5.3.3.5, we obtain that:

Corollary 5.3.3.7. *There is an isomorphism*

$$\mathrm{H}^\bullet_Q(\mathcal{O}, \mathcal{H}_M) \xrightarrow{\simeq} \mathrm{H}^\bullet_Q(\mathrm{d}\mathcal{O}, \mathrm{N}(M))$$

between Quillen cohomology of $\mathcal{O} \in \mathrm{Op}(\mathrm{Set}_\Delta)$ *with coefficients in* $\mathcal{H}_M : \mathrm{Tw}(\mathcal{O}) \longrightarrow$ ***Spectra*** *and Quillen cohomology of* $\mathrm{d}\mathcal{O} \in \mathrm{Op}(\mathcal{C}(\boldsymbol{k}))$ *with coefficients in* $\mathrm{N}(M) \in \mathrm{IbMod}(\mathrm{d}\mathcal{O})$.

For example, we let $\mathcal{O} = \mathrm{E}_\infty$ the little ∞-cubes operad let $M = \mathbf{k}\{\mathrm{E}_\infty\}$ regarded as an infinitesimal bimodule over itself. In this case, $\mathcal{H}_M : \mathrm{Tw}(\mathrm{E}_\infty) \longrightarrow$ **Spectra** is weakly equivalent to the constant functor with value $\mathbf{k}$ and $\mathrm{N}(M) = \mathrm{d}\,\mathrm{E}_\infty$. Combining the above one with Corollary 4.3.0.8, we have the following.

Corollary 5.3.3.8. *Quillen cohomology of* $\mathrm{d}\,\mathrm{E}_\infty \in \mathrm{Op}(\mathcal{C}(\boldsymbol{k}))$ *with coefficients in itself vanishes.*

For another example, we consider the case $\mathcal{O} = \mathcal{A}ss$ the associative operad. Let $A \in \mathrm{sMod}_{\mathbf{k}}$ be a discrete monoid. Recall that the **endomorphism operad** End_A is given by setting $\mathrm{End}_A(n) = \mathrm{Map}_{\mathrm{sMod}_{\mathbf{k}}}(A^{\otimes n}, A)$ with the composition and symmetric action respectively induced by substitution and permutation of the tensor factors. There is a map of operads $\mathbf{k}\{\mathcal{A}ss\} \longrightarrow \mathrm{End}_A$ classifying the $\mathbf{k}\{\mathcal{A}ss\}$-algebra structure of A. We consider End_A as an object of $\mathrm{IbMod}(\mathbf{k}\{\mathcal{A}ss\})$ via that map. By Corollary 5.3.3.7 we have an isomorphism

$$\mathrm{H}^\bullet_Q(\mathcal{A}ss; \mathcal{H}_{\mathrm{End}_A}) \xrightarrow{\simeq} \mathrm{H}^\bullet_Q(\mathrm{d}\,\mathcal{A}ss; \mathrm{End}_A)$$

between Quillen cohomology of $\mathcal{A}ss \in \mathrm{Op}(\mathrm{Set}_\Delta)$ with coefficients in $\mathcal{H}_{\mathrm{End}_A}$ and Quillen cohomology of the dg associative operad $\mathrm{d}\,\mathcal{A}ss$ (considered as an object of $\mathrm{Op}(\mathcal{C}(\mathbf{k}))$) with coefficients in End_A. Here we note that since A is discrete, the normalized complex functor $\mathrm{sMod}_{\mathbf{k}} \longrightarrow \mathcal{C}(\mathbf{k})$ preserves the construction of the endomorphism operad. Following Computations 5.3.3.6, $\mathcal{H}_{\mathrm{End}_A}$ is given by $\mathcal{H}_{\mathrm{End}_A}([n]) = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{k}}}(A^{\otimes n}, A)$ and moreover, when considered as a cosimplicial $\mathbf{k}$-module, $\mathcal{H}_{\mathrm{End}_A}$ exhibits the **Hochschild cochain complex** of A. Using the long exact sequence of Corollary 4.3.0.7, we get an isomorphism $\mathrm{H}^n_Q(\mathrm{d}\,\mathcal{A}ss; \mathrm{End}_A) \cong \mathrm{H\,H}^{n+2}(A)$ for each $n \geq 0$. Around the degree -1, the long exact sequence takes the form

$$0 \longrightarrow Z(A) \longrightarrow A \longrightarrow \mathrm{H}^{-1}_Q(\mathrm{d}\,\mathcal{A}ss; \mathrm{End}_A) \longrightarrow \mathrm{H\,H}^1(A) \longrightarrow 0$$

where $Z(A) = \mathrm{H\,H}^0(A)$ is the center of A and the second map is the inclusion. Recall that $\mathrm{H\,H}^1(A)$ is given by $\mathrm{Der}(A, A)/\mathrm{Ider}(A, A)$ the module of derivations modulo the inner derivations. This implies that $\mathrm{H}^{-1}_Q(\mathrm{d}\,\mathcal{A}ss; \mathrm{End}_A) \cong \mathrm{Der}(A, A)$, because $\mathrm{Ider}(A, A)$ agrees with the quotient $A/Z(A)$. In summary, we obtain that:

Corollary 5.3.3.9. *Quillen cohomology of the dg associative operad with coefficients in* End_A *is given by*

$$\mathrm{H}^n_Q(\mathrm{d}\,\mathcal{A}ss; End_A) = \begin{cases} 0 & \text{if } n \leqslant -2\,, \\ \mathrm{Der}(A, A) & \text{if } n = -1\,, \\ \mathrm{H\,H}^{n+2}(A) & \text{otherwise}\,. \end{cases}$$

Consequently, $\mathrm{H}^\bullet_Q(\mathrm{d}\,\mathcal{A}ss; End_A)$ *is (up to a shift) isomorphic to Quillen cohomology of* A *as an associative algebra with coefficients in itself (cf. [7], Example 3.2.12).*

5.4 Deformation theory and Quillen cohomology

Our main purpose in this section is to establish the relation between **deformation theory** and Quillen cohomology.

We will denote by $\mathrm{CAlg}^{\mathrm{aug}}$ the category of commutative monoids in $\mathcal{C}_{\geq 0}(\mathbf{k})$ equipped with a map to $\mathbf{k}$. Alternatively, $\mathrm{CAlg}^{\mathrm{aug}} \cong \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{C}om}(\mathcal{C}_{\geq 0}(\mathbf{k}))$ the augmented category of algebras over $\mathcal{C}om \in \mathrm{Op}(\mathcal{C}_{\geq 0}(\mathbf{k}))$. We endow $\mathrm{CAlg}^{\mathrm{aug}}$ with the projective transferred model structure. Then the stabilization $\mathrm{Sp}(\mathrm{CAlg}^{\mathrm{aug}})$ exists (at least) as a semi model category (cf. Section 2.2).

For each $R \in \mathrm{CAlg}^{\mathrm{aug}}$, we denote by $\pi_n(R)$ the n'th homology group of the underlying complex of R. Recall by definition that an augmented commutative algebra $\varepsilon : R \longrightarrow \mathbf{k}$ is said to be **artinian** if the underlying complex of R is finite dimensional and such that the map $\pi_0(\varepsilon) : \pi_0(R) \longrightarrow \pi_0(\mathbf{k}) = \mathbf{k}$ exhibits $\pi_0(R)$ as a local $\mathbf{k}$-algebra (i.e. the kernel of $\pi_0(\varepsilon)$ is the unique maximal ideal of $\pi_0(R)$). We will denote by $\mathrm{CAlg}^{\mathrm{art}} \subseteq \mathrm{CAlg}^{\mathrm{aug}}$ the full subcategory spanned by artinian algebras. Furthermore, we abuse the notation to write $\mathrm{Sp}(\mathrm{CAlg}^{\mathrm{art}})$ standing for the full subcategory of $\mathrm{Sp}(\mathrm{CAlg}^{\mathrm{aug}})$ spanned by the spectrum objects $X_{\bullet,\bullet} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathrm{CAlg}^{\mathrm{aug}}$ whose degrees are all artinian.

On other hand, we let ModCat denote the category whose objects are model categories and whose morphisms are Quillen adjunctions with the sources and targets being those of the left Quillen functors. We are interested in functors $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$. By convention, for each map $f : R \to S$ in $\mathrm{CAlg}^{\mathrm{art}}$, we will write $f_! : \mathcal{F}(R) \rightleftarrows \mathcal{F}(S) : f^*$ standing for the image of f under the functor $\mathcal{F}$.

As the first step, we propose the notion of **space of deformations** of a given object $X \in \mathcal{F}(\mathbf{k})$ over some $R \in \mathrm{CAlg}^{\mathrm{art}}$, denoted by $\mathrm{Def}(X, R)$, with $\mathcal{F}$ being a functor $\mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$. To make this well-defined, the functor $\mathcal{F}$ is required to be a **formal moduli context** (cf. Definition 5.4.1.2) so that the functor $R \mapsto \mathrm{Def}(X, R)$ forms a **formal moduli problem** in the Lurie's sense [5] (cf. Proposition 5.4.1.4). For the main purpose, we will show that Quillen cohomology of X classifies homotopy type of its spaces of deformations. In the second subsection, we review some examples of interest.

5.4.1 Statements

We denote by $\mathcal{S}$ the ∞-category of spaces and by $\mathrm{CAlg}^{\mathrm{art}}_{\infty}$ the full ∞-subcategory of $\mathrm{CAlg}^{\mathrm{aug}}_{\infty}$ spanned by artinian $\mathbf{k}$-algebras.

Definition 5.4.1.1. ([5]) A **formal moduli problem** is a functor $F : \mathrm{CAlg}^{\mathrm{art}}_{\infty} \longrightarrow \mathcal{S}$ satisfying that $F(\mathbf{k})$ is contractible and that, if σ is a Cartesian square of the form

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

such that the induced maps $\pi_0(S) \longrightarrow \pi_0(U)$ and $\pi_0(S) \longrightarrow \pi_0(U)$ are surjective, then $F(\sigma)$ is Cartesian as well.

Definition 5.4.1.2. A **formal moduli context** is a functor $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ satisfying the following properties:

(1) $\mathcal{F}$ sends weak equivalences in $\mathrm{CAlg}^{\mathrm{art}}$ to Quillen equivalences and satisfies that for every morphism $f : R \longrightarrow S$ in $\mathrm{CAlg}^{\mathrm{art}}$ the right adjoint functor $f^* : \mathcal{F}(S) \longrightarrow \mathcal{F}(R)$ preserves weak equivalences.

(2) For every homotopy Cartesian square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

in $\mathrm{CAlg}^{\mathrm{art}}$ such that the maps $\pi_0(S) \longrightarrow \pi_0(U)$ and $\pi_0(T) \longrightarrow \pi_0(U)$ are surjective, the corresponding diagram

$$\begin{array}{ccc} \mathcal{F}(R) & \longrightarrow & \mathcal{F}(S) \\ \downarrow & & \downarrow \\ \mathcal{F}(T) & \longrightarrow & \mathcal{F}(U) \end{array}$$

of model categories is homotopy Cartesian (in the sense of Definition B.0.0.1).

Definition 5.4.1.3. Suppose given a formal moduli context $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$. Let $X \in \mathcal{F}(\mathbf{k})$ be a fibrant object and let R be an object of $\mathrm{CAlg}^{\mathrm{art}}$ equipped with the augmentation map $\varepsilon : R \longrightarrow \mathbf{k}$.

1. A **deformation of** X **over** R is defined to be a pair (Y, η) with $Y \in \mathcal{F}(R)$ being cofibrant and $\eta : \varepsilon_! Y \xrightarrow{\simeq} X$ being a weak equivalence.

2. If (Y, η) and (Y', η') are two deformations of X over R then an **equivalence** from (Y, η) to (Y', η') is a weak equivalence $Y \xrightarrow{\simeq} Y'$ compatible with the structure maps η and η'.

3. The **space of deformations of** X **over** R, denoted by $\mathrm{Def}(X, R)$, is defined to be the *Kan replacement* of the nerve of the category whose objects are deformations of X over R and whose morphisms are equivalences of deformations. We will consider $\mathrm{Def}(X, R)$ as a pointed space whose base point is the **trivial deformation** $(u_! X, \eta) \in \mathrm{Def}(X, R)$ where $u : \mathbf{k} \to R$ is the unit of R and $\eta : \varepsilon_! u_! X \xrightarrow{\simeq} X$ is the natural isomorphism.

For each ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^{\simeq}$ the **maximal** ∞**-subgroupoid** of $\mathcal{C}$. Explicitly, $\mathcal{C}^{\simeq}$ is the simplicial subset of $\mathcal{C}$ such that a simplex $\sigma \in \mathcal{C}$ is belong to $\mathcal{C}^{\simeq}$ if and only if every edge of σ is an equivalence. It was known that the assignment $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ determines an ∞-categorical right adjoint from ∞-categories to ∞-groupoids (cf. [4]).

The main interest in the notion of a formal moduli context is that it leads to a formal moduli problem:

Proposition 5.4.1.4. *Let* $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ *be a formal moduli context and let* $X \in \mathcal{F}(\boldsymbol{k})$ *be a fibrant object. Then the functor* $R \mapsto \mathrm{Def}(X, R)$ *is a formal moduli problem.*

Proof. We have to verify that $\mathrm{Def}(X, \mathbf{k})$ is weakly contractible and that $\mathcal{F}$ preserves the type of Cartesian squares appearing in Definition 5.4.1.1(2).

For the first claim, note first that $\mathrm{Def}(X, \mathbf{k})$ is (Kan replacement) of the nerve of the category whose objects are cofibrant replacements of X. According to [[45], Theorem 14.6.2], the latter is indeed weakly contractible.

For each $R \in \mathrm{CAlg}^{\mathrm{art}}$, the map $\varepsilon : R \to \mathbf{k}$ induces $\widetilde{\varepsilon_!} : \mathcal{F}(R)^{\simeq}_{\infty} \longrightarrow \mathcal{F}(\mathbf{k})^{\simeq}_{\infty}$. We may identify $\mathrm{Def}(X, R)$ to the homotopy pullback

$$\mathrm{Def}(X, R) \simeq \mathcal{F}(R)^{\simeq}_{\infty} \times^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})^{\simeq}_{\infty}} (\mathcal{F}(\mathbf{k})^{\simeq}_{\infty})_{/X},$$

(this follows from the fact that section model categories have the right type as discussed in Appendix B). In particular, $\mathrm{Def}(X,R)$ is in fact weakly equivalent to the homotopy fiber of the map $\widetilde{\varepsilon}_! : \mathcal{F}(R)^{\simeq}_{\infty} \longrightarrow \mathcal{F}(\mathbf{k})^{\simeq}_{\infty}$ over the object X.

Now, consider a homotopy Cartesian square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

in $\mathrm{CAlg}^{\mathrm{art}}$ such that the maps $\pi_0(S) \longrightarrow \pi_0(U)$ and $\pi_0(T) \longrightarrow \pi_0(U)$ are surjective. By assumption, there is a homotopy Cartesian square of model categories

$$\begin{array}{ccc} \mathcal{F}(R) & \longrightarrow & \mathcal{F}(S) \\ \downarrow & & \downarrow \\ \mathcal{F}(T) & \longrightarrow & \mathcal{F}(U) \end{array}$$

By Remark B.0.0.2, this induces a homotopy Cartesian square of the underlying ∞-categories and hence, we obtain a homotopy Cartesian square of ∞-groupoids

$$\begin{array}{ccc} \mathcal{F}(R)^{\simeq}_{\infty} & \longrightarrow & \mathcal{F}(S)^{\simeq}_{\infty} \\ \downarrow & & \downarrow \\ \mathcal{F}(T)^{\simeq}_{\infty} & \longrightarrow & \mathcal{F}(U)^{\simeq}_{\infty} \end{array} \tag{5.4.1.1}$$

By the above paragraph, we get a homotopy Cartesian square

$$\begin{array}{ccc} \mathrm{Def}(X,R) & \longrightarrow & \mathrm{Def}(X,S) \\ \downarrow & & \downarrow \\ \mathrm{Def}(X,T) & \longrightarrow & \mathrm{Def}(X,U) \end{array}$$

as desired. □

Definition 5.4.1.5. Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be a formal moduli context, let $X \in \mathcal{F}(\mathbf{k})$ be a fibrant object and $M \in \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{art}})$ an Ω-spectrum. Then $\Omega^{\infty} M \longrightarrow \mathbf{k}$ is an artinian $\mathbf{k}$-algebra. We will call the ∞-groupoid $\mathrm{Def}(X,M) \coloneqq \mathrm{Def}(X,\Omega^{\infty}M)$ the **space of first order deformations** of X in direction M.

To state our main theorem we will need the following construction:

Definition 5.4.1.6. Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be a formal moduli context and let $X \in \mathcal{F}(\mathbf{k})$ be an object such that the tangent category $\mathcal{T}_X\mathcal{F}(\mathbf{k})$ exists (at least) as a semi model category. Given a spectrum object $M : \mathbb{N} \times \mathbb{N} \longrightarrow \mathrm{CAlg}^{\mathrm{art}}$, we will denote by $M(X) \in \mathcal{T}_X\mathcal{F}(\mathbf{k})$ the spectrum object defined as

$$M(X)_{n,m} = u^*_{n,m}(u_{n,m})_! X \in \mathcal{F}(\mathbf{k}),$$

where $u_{n,m} : \mathbf{k} \longrightarrow M_{n,m}$ is the unit of $M_{n,m} \in \mathrm{CAlg}^{\mathrm{art}}$.

Lemma 5.4.1.7. *Suppose further that $X \in \mathcal{F}(\mathbf{k})$ is cofibrant and that $M \in \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{art}})$ is an Ω-spectrum. Then $M(X)$ is an Ω-spectrum in $\mathcal{T}_X\mathcal{F}(\mathbf{k})$.*

Proof. We first show that $M(X)$ is a prespectrum. For any given pair (n,m) with $n \neq m$, we have to prove that the structure map $X \longrightarrow u_{n,m}^*(u_{n,m})_! X$ is a weak equivalence. Since M is in particular a prespectrum, the unit map $u_{n,m} : \mathbf{k} \longrightarrow M_{n,m}$ is a weak equivalence. Furthermore, since $\mathcal{F}$ is a formal moduli context, the adjunction $(u_{n,m})_! : \mathcal{F}(\mathbf{k}) \rightleftarrows \mathcal{F}(M_{n,m}) : (u_{n,m})^*$ is a Quillen equivalence such that $(u_{n,m})^*$ preserves weak equivalences. This fact, together with the cofibrancy of X, proves that the map $X \longrightarrow u_{n,m}^*(u_{n,m})_! X$ is indeed a weak equivalence.

It remains to show that for every $n \geq 0$ the square

$$\begin{array}{ccc} u_{n,n}^*(u_{n,n})_! X & \longrightarrow & u_{n,n+1}^*(u_{n,n+1})_! X \\ \downarrow & & \downarrow \\ u_{n+1,n}^*(u_{n+1,n})_! X & \longrightarrow & u_{n+1,n+1}^*(u_{n+1,n+1})_! X \end{array} \tag{5.4.1.2}$$

is homotopy Cartesian in $\mathcal{F}(\mathbf{k})$. Given $i,j \in \{n, n+1\}$, we write $\varphi_{i,j} : M_{n,n} \longrightarrow M_{i,j}$ standing for the map induced by the unique map $(n,n) \longrightarrow (i,j)$ in $\mathbb{N} \times \mathbb{N}$. Then we note that $u_{i,j} = \varphi_{i,j} \circ u_{n,n}$ and therefore, $u_{i,j}^* = u_{n,n}^* \circ \varphi_{i,j}^*$ while $(u_{i,j})_! = (\varphi_{i,j})_! \circ (u_{n,n})_!$. Consequently, the square (5.4.1.2) is the same as the image through $u_{n,n}^*$ of the following square

$$\begin{array}{ccc} X' & \longrightarrow & \varphi_{n,n+1}^*(\varphi_{n,n+1})_! X' \\ \downarrow & & \downarrow \\ \varphi_{n+1,n}^*(\varphi_{n+1,n})_! X' & \longrightarrow & \varphi_{n+1,n+1}^*(\varphi_{n+1,n+1})_! X' \end{array} \tag{5.4.1.3}$$

where $X' := (u_{n,n})_! X$. So we just need to verify that the latter is a homotopy Cartesian square in $\mathcal{F}(M_{n,n})$. By assumption, for every $n \geqslant 0$ the square

$$\begin{array}{ccc} M_{n,n} & \longrightarrow & M_{n,n+1} \\ \downarrow & & \downarrow \\ M_{n+1,n} & \longrightarrow & M_{n+1,n+1} \end{array} \tag{5.4.1.4}$$

is homotopy Cartesian in $\mathrm{CAlg}^{\mathrm{art}}$ and such that $M_{n,n+1} \simeq M_{n+1,n} \simeq \mathbf{k}$. Observe that $\mathrm{Sp}(\mathrm{CAlg}^{\mathrm{aug}}) \stackrel{\mathrm{def}}{=} \mathrm{Sp}(\mathrm{Alg}_{\mathcal{C}om}^{\mathrm{aug}}(\mathcal{C}_{\geq 0}(\mathbf{k})))$ is equivalent to the stabilization $\mathrm{Sp}(\mathrm{Alg}_{\mathcal{C}om}^{\mathrm{aug}}(\mathcal{C}(\mathbf{k})))$ (these two are both equivalent to $\mathcal{C}(\mathbf{k})$), and hence we can assume that M comes from an Ω-spectrum in $\mathrm{Sp}(\mathrm{Alg}_{\mathcal{C}om}^{\mathrm{aug}}(\mathcal{C}(\mathbf{k})))$. In particular, the underlying square of chain complexes of (5.4.1.4) is homotopy coCartesian. Moreover, since $M_{n,n}$ is artinian, there exists a (connective) chain complex N such that the underlying complex of $M_{n,n}$ factors as $M_{n,n} \cong \mathbf{k} \oplus N$. We thus find that the underlying complex of $M_{n+1,n+1}$ is weakly equivalent to $\mathbf{k} \oplus N[1]$. This implies that the maps $M_{n,n+1} \longrightarrow M_{n+1,n+1}$ and $M_{n+1,n} \longrightarrow M_{n+1,n+1}$ induce the isomorphisms on π_0. Now, since $\mathcal{F}$ is a formal moduli context, we get a homotopy Cartesian square of model categories

$$\begin{array}{ccc} \mathcal{F}(M_{n,n}) & \longrightarrow & \mathcal{F}(M_{n,n+1}) \\ \downarrow & & \downarrow \\ \mathcal{F}(M_{n+1,n}) & \longrightarrow & \mathcal{F}(M_{n+1,n+1}) \end{array}$$

Finally, we deduce by Remark B.0.0.4 that the square (5.4.1.3) is indeed homotopy Cartesian. □

We are now in position to prove the main theorem of this section:

Theorem 5.4.1.8. *Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be a formal moduli context and let $X \in \mathcal{F}(\boldsymbol{k})$ be a bifibrant object such that the tangent category $\mathcal{T}_X\mathcal{F}(\boldsymbol{k})$ exists. Moreover, let $M \in \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{art}})$ be an Ω-spectrum. Then $\mathrm{Def}(X, M)$ is weakly equivalent to the derived mapping space $\mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\boldsymbol{k})_{/X}}(X, \Omega^{\infty-1}M(X))$. In particular, we have a canonical isomorphism*

$$\pi_0(\mathrm{Def}(X, M)) \cong \mathrm{H}^1_{\mathrm{Q}}(X, M(X)).$$

Moreover, the n'th homotopy group $\pi_n(\mathrm{Def}(X, M), *)$ is isomorphic to $\mathrm{H}^{1-n}_{\mathrm{Q}}(X, M(X))$, where $*$ refers to the basepoint $(u_!X, \eta)$ mentioned in Definition 5.4.1.3 (iii).

Proof. Note first that $M_{1,0} \simeq M_{0,1} \simeq \mathbf{k}$, $M_{0,0} \simeq \Omega^\infty M$ and $M_{1,1} \simeq \Omega^\infty M[1]$. We will simply denote by $u_M : \mathbf{k} \longrightarrow \Omega^\infty M[1]$ the unit of $\Omega^\infty M[1]$ and by $\varepsilon_M : \Omega^\infty M[1] \longrightarrow \mathbf{k}$ the augmentation map. As in the proof of Lemma 5.4.1.7 , there is a homotopy Cartesian square of model categories

$$\begin{array}{ccc} \mathcal{F}(\Omega^\infty M) & \longrightarrow & \mathcal{F}(\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathcal{F}(\mathbf{k}) & \longrightarrow & \mathcal{F}(\Omega^\infty M[1]) \end{array}$$

As in the proof of Proposition 5.4.1.4, the latter induces a homotopy Cartesian square of the associated ∞-groupoids

$$\begin{array}{ccc} \mathcal{F}(\Omega^\infty M)^{\simeq}_\infty & \longrightarrow & \mathcal{F}(\mathbf{k})^{\simeq}_\infty \\ \downarrow & & \downarrow{\scriptstyle (\widetilde{u}_M)_!} \\ \mathcal{F}(\mathbf{k})^{\simeq}_\infty & \xrightarrow{(\widetilde{u}_M)_!} & \mathcal{F}(\Omega^\infty M[1])^{\simeq}_\infty \end{array}$$

Since $\mathrm{Def}(X, M)$ is equivalent to the homotopy fiber of the left vertical map over $X \in \mathcal{F}(\mathbf{k})^{\simeq}_\infty$ (see the proof of Proposition 5.4.1.4), it is also equivalent to the homotopy fiber of the right vertical map over $(\widetilde{u}_M)_!X$. Consider a diagram of ∞-groupoids of the form

$$\begin{array}{ccc} * & \xrightarrow{X} & \mathcal{F}(\mathbf{k})^{\simeq}_\infty \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle (\widetilde{u}_M)_!} \\ Z & \longrightarrow & \mathcal{F}(\Omega^\infty M)^{\simeq}_\infty \\ \downarrow & & \downarrow{\scriptstyle (\widetilde{\varepsilon}_M)_!} \\ * & \xrightarrow{X} & \mathcal{F}(\mathbf{k})^{\simeq}_\infty \end{array}$$

where Z is taken such that the bottom square is homotopy Cartesian. In particular, the top square is homotopy Cartesian as well. Hence we may identify $\mathrm{Def}(X, M)$ to the loop space $\Omega_{\iota(*)}Z$. By this way, we find that $\mathrm{Def}(X, M)$ is weakly equivalent to the homotopy fiber of the map between loop spaces:

$$\Omega_{(u_M)_!X}\,\mathcal{F}(\Omega^\infty M)^{\simeq}_\infty \longrightarrow \Omega_X\,\mathcal{F}(\mathbf{k})^{\simeq}_\infty$$

over the constant loop at X, which is identified to the homotopy fiber of the map

$$\mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\Omega^\infty M[1])}((u_M)_!X, (u_M)_!X) \longrightarrow \mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})}(X, X)$$

over Id_X. By the Quillen adjunction $(u_M)_! \dashv u_M^*$, the latter is weakly equivalent to the map

$$\mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})}(X, u_M^*(u_M)_!X) \longrightarrow \mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})}(X, X) \tag{5.4.1.5}$$

Now, since X is fibrant, the model category $\mathcal{F}(\mathbf{k})_{/X}$ has the right type in the sense that, for every weak equivalence $Y \xrightarrow{\simeq} X$ with Y being also fibrant, the induced adjunction $\mathcal{F}(\mathbf{k})_{/Y} \rightleftarrows \mathcal{F}(\mathbf{k})_{/X}$ is a Quillen equivalence. This fact proves that the homotopy fiber of the map (5.4.1.5) over Id_X is weakly equivalent to the derived mapping space

$$\mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})_{/X}}(X, u_M^*(u_M)_! X) \simeq \mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})_{/X}}(X, \Omega^{\infty-1}M(X)).$$

We just showed that $\mathrm{Def}(X,M)$ is weakly equivalent to $\mathrm{Map}_{\mathcal{F}(\mathbf{k})_{/X}}(X, \Omega^{\infty-1}M(X))$ and moreover, this equivalence identifies $* = (u_! X, \eta)$ to the unit map $\mu : X \longrightarrow \Omega^{\infty-1}M(X) = u_M^*(u_M)_! X$. Now, the Quillen adjunction $\Sigma^\infty_+ : \mathcal{F}(\mathbf{k})_{/X} \rightleftarrows \mathcal{T}_X\mathcal{F}(\mathbf{k}) : \Omega^\infty_+$ proves the existence of a canonical weak equivalence

$$\mathrm{Map}^{\mathrm{h}}_{\mathcal{F}(\mathbf{k})_{/X}}(X, \Omega^{\infty-1}M(X)) \simeq \mathrm{Map}^{\mathrm{h}}_{\mathcal{T}_X\mathcal{F}(\mathbf{k})}(\mathrm{L}_X, M(X)[1]),$$

which identifies μ to the zero map $\mathrm{L}_X \xrightarrow{0} M(X)[1]$. Finally, by the definition of Quillen cohomology group, we deduce that

$$\mathrm{H}^{1-n}_{\mathcal{Q}}(X, M(X)) \cong \pi_n(\mathrm{Map}_{\mathcal{T}_X\mathcal{F}(\mathbf{k})}(L_X, M(X)[1]), 0) \cong \pi_n(\mathrm{Def}(X,M), *).$$

□

Remark 5.4.1.9. If $M : \mathbb{N}\times\mathbb{N} \longrightarrow \mathrm{CAlg}^{\mathrm{art}}$ is an Ω-spectrum such that for every $(n,m) \in \mathbb{N}\times\mathbb{N}$ the induced left Quillen functor $(u_{n,m})_! : \mathcal{F}(\mathbf{k}) \longrightarrow \mathcal{F}(M_{n,m})$ preserves weak equivalences, then following the proof of Lemma 5.4.1.7, $M(X)$ is automatically an Ω-spectrum, even when X is not cofibrant. In this situation, the statement of Theorem 5.4.1.8 remains valid without necessarily requiring X to be cofibrant. In fact, this is usually the case. For instance, all the functors $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ appearing in the next subsection satisfy the property that, for every $R \in \mathrm{CAlg}^{\mathrm{art}}$ with the unit $u : \mathbf{k} \to R$, the induced functor $u_! : \mathcal{F}(\mathbf{k}) \longrightarrow \mathcal{F}(R)$ preserves weak equivalences.

5.4.2 Examples

In this subsection, we describe some interesting examples of formal moduli contexts. Repeatedly, the work requires verifying the two conditions of Definition 5.4.1.2.

Proposition 5.4.2.1. *Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be the functor $\mathcal{F}(R) = \mathrm{Mod}_R := \mathrm{Mod}_R(\mathcal{C}_{\geq 0}(\boldsymbol{k}))$. Then $\mathcal{F}$ is a formal moduli context.*

Proof. Let $\widetilde{R}$ be the category with a single object and with R as its mapping object. Note that the category Mod_R can be identified with $\mathrm{Fun}(\widetilde{R}, \mathcal{C}_{\geq 0}(\mathbf{k}))$ the category of $\mathcal{C}_{\geq 0}(\mathbf{k})$-valued enriched functors on $\widetilde{R}$. Using this identification, we can readily verify the first condition of Definition 5.4.1.2 on $\mathcal{F}$.

We shall now verify the second condition of Definition 5.4.1.2 on $\mathcal{F}$. Let

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow p \\ T & \xrightarrow{q} & U \end{array} \tag{5.4.2.1}$$

be a homotopy Cartesian square in $\mathrm{CAlg}^{\mathrm{art}}$ such that the maps p and q are surjective on π_0. We have to show that the corresponding square of model categories

$$\begin{array}{ccc} \mathrm{Mod}_R & \longrightarrow & \mathrm{Mod}_S \\ \downarrow & & \downarrow \\ \mathrm{Mod}_T & \longrightarrow & \mathrm{Mod}_U \end{array} \tag{5.4.2.2}$$

is homotopy Cartesian in the sense of Definition B.0.0.1. Let $\mathcal{F}' : [1] \times [1] \longrightarrow \mathrm{ModCat}$ represent the square (5.4.2.2). Let $\mathcal{I} \subseteq [1] \times [1]$ denote the full subcategory spanned by the vertices $(0,1), (1,0)$ and $(1,1)$. We need to show that the Quillen adjunction $\mathcal{L}^{\mathrm{coc}} : \mathrm{Mod}_R \rightleftarrows \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) : \mathcal{R}^{\mathrm{coc}}$ is a Quillen equivalence. For this, it suffices to show that $\mathbb{R}\mathcal{R}^{\mathrm{coc}}$ detects weak equivalences (between cofibrant objects) and that for every cofibrant object $X \in \mathrm{Mod}_R$ the derived unit map $X \longrightarrow \mathbb{R}\mathcal{R}^{\mathrm{coc}}\mathcal{L}^{\mathrm{coc}}(X)$ is a weak equivalence.

Let us start with the second claim. By Remark B.0.0.4 we just need to verify that the square

$$\begin{array}{ccc} X & \longrightarrow & X \otimes_R S \\ \downarrow & & \downarrow \\ X \otimes_R T & \longrightarrow & X \otimes_R U \end{array} \tag{5.4.2.3}$$

is homotopy Cartesian in Mod_R. Consider the functor $X \otimes_R (-) : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$. Firstly, since X is cofibrant, this functor preserves weak equivalences. Indeed, we may identify $X \otimes_R (-)$ to the functor $X \circ_{\widetilde{R}} (-) : \mathrm{Alg}_{\widetilde{R}}(\mathcal{C}_{\geq 0}(\mathbf{k})) \longrightarrow \mathrm{Alg}_{\widetilde{R}}(\mathcal{C}_{\geq 0}(\mathbf{k}))$ with $\widetilde{R}$ regarded as an operad in $\mathcal{C}_{\geq 0}(\mathbf{k})$. The latter indeed preserves weak equivalences (cf., e.g., [[52], Theorem 15.1.A]). The functor $X \otimes_R (-)$ is a left Quillen functor preserving weak equivalences, it hence preserves homotopy coCartesian squares.

Consider (5.4.2.1) as a homotopy Cartesian square in Mod_R. We are showing that this square is also homotopy coCartesian in Mod_R. Since the embedding functor

$$\iota_R : \mathrm{Mod}_R \overset{\mathrm{def}}{=} \mathrm{Mod}_R(\mathcal{C}_{\geq 0}(\mathbf{k})) \longrightarrow \mathrm{Mod}_R(\mathcal{C}(\mathbf{k}))$$

detects homotopy coCartesian squares, it suffices to show that (5.4.2.1) is homotopy coCartesian when regarded as a square $\mathrm{Mod}_R(\mathcal{C}(\mathbf{k}))$. But this is equivalent to saying that (5.4.2.1) is homotopy Cartesian in $\mathrm{Mod}_R(\mathcal{C}(\mathbf{k}))$, since $\mathrm{Mod}_R(\mathcal{C}(\mathbf{k}))$ is stable. The latter follows from the fact that a map in $\mathrm{Mod}_R(\mathcal{C}_{\geq 0}(\mathbf{k}))$ is surjective if and only if it is surjective in each positive degrees and in addition, surjective on π_0. We just showed that (5.4.2.1) is homotopy coCartesian in Mod_R. Combined with the above paragraph, we get that (5.4.2.3) is homotopy coCartesian. Finally, since the functor ι_R also detects homotopy Cartesian squares, the square (5.4.2.3) is indeed homotopy Cartesian as expected.

Let us now show that $\mathbb{R}\mathcal{R}^{\mathrm{coc}} : \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) \longrightarrow \mathrm{Mod}_R$ detects weak equivalences between cofibrant objects. Note first that a map in $\mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}})$ is a weak equivalence if and only if its cofiber is a weak 0-section (i.e. the section whose coordinates are weak 0-objects). Indeed, the property of being a weak equivalence and the formation of homotopy cofibers are both jointly created by the projections $\mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) \longrightarrow \mathrm{Mod}_S$, $\mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) \longrightarrow \mathrm{Mod}_T$ and $\mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) \longrightarrow \mathrm{Mod}_U$, and the characterization of weak equivalences via cofibers holds in any module category in $\mathcal{C}_{\geq 0}(\mathbf{k})$. It will hence suffice to show that $\mathbb{R}\mathcal{R}^{\mathrm{coc}}$ detects weak 0-objects. By definition, the data of a cofibrant section $s \in \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}})$ consists of a triple $(X_S, X_T, X_U) \in \mathrm{Mod}_S \times \mathrm{Mod}_T \times \mathrm{Mod}_U$ of cofibrant objects together with the maps $\varphi : X_T \otimes_T U \longrightarrow X_U$ and $\psi : X_S \otimes_S U \longrightarrow X_U$ in Mod_U. When s is in addition fibrant, the maps φ and ψ are weak equivalences. In summary, given such a bifibrant section $s = (X_S, X_T, X_U)$ such that the homotopy pullback $X_T \times^{\mathrm{h}}_{X_U} X_S$ is an acyclic complex, we need to show that each of X_S, X_T and X_U is an acyclic complex as well.

Observe first that since $\pi_k(X_T \times^{\mathrm{h}}_{X_U} X_S) = 0$, the induced map $f_k : \pi_k(X_T) \oplus \pi_k(X_S) \longrightarrow \pi_k(X_U)$ is injective for every $k \geqslant 0$. Let us assume that X_T is acyclic. By the weak equivalence $\varphi : X_T \otimes_T U \longrightarrow X_U$, we get that X_U is acyclic, and hence X_S is acyclic as well, by the injectivity of the maps f_k's. By symmetricity, the same thing holds if X_S is acyclic. In conclusion, we just need to verify that either X_T or X_S is acyclic. Assume by contradiction that both X_T and X_S are not so. Let $k \geq 0$ be the smallest integer such that at least one of $\pi_k(X_T), \pi_k(X_S)$ is non-trivial. Then we have $\pi_k(X_T \otimes_T U) \cong \pi_k(X_T) \otimes_{\pi_0(T)} \pi_0(U)$. In particular, the map $\pi_k(X_T) \longrightarrow \pi_k(X_U)$, which is the same as the composition

$$\pi_k(X_T) \longrightarrow \pi_k(X_T) \otimes_{\pi_0(T)} \pi_0(U) \xrightarrow[\simeq]{\pi_k(\varphi)} \pi_k(X_U)$$

is surjective. By symmetricity, the map $\pi_k(X_S) \longrightarrow \pi_k(X_U)$ is as well surjective. Combining these observations with the injectivity of f_k, we obtain that $\pi_k(X_T) = \pi_k(X_S) = 0$. But this is a contradiction. □

Proposition 5.4.2.2. *Let $\mathcal{P}$ be an operad in $\mathcal{C}_{\geq 0}(\boldsymbol{k})$. Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be the functor $\mathcal{F}(R) = \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_R)$. Then $\mathcal{F}$ is a formal moduli context.*

Proof. Each map $R \to R'$ in $\mathrm{CAlg}^{\mathrm{art}}$ gives rise to a symmetric monoidal Quillen adjunction $\mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_{R'}$, which then induces a Quillen adjunction $\mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_R) \rightleftarrows \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_{R'})$. The right adjoint functor is simply the restriction functor, and hence preserves weak equivalences. Moreover, as in the above proof, each weak equivalence $R \xrightarrow{\simeq} R'$ gives rise to a symmetric monoidal Quillen equivalence $\mathrm{Mod}_R \overset{\simeq}{\rightleftarrows} \mathrm{Mod}_{R'}$, which hence induces a Quillen equivalence $\mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_R) \overset{\simeq}{\rightleftarrows} \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_{R'})$. We just verified the first condition of Definition 5.4.1.2 on $\mathcal{F}$.

Now, observe that for each map $f : R \to S$ in $\mathrm{CAlg}^{\mathrm{art}}$ the induced Quillen adjunction

$$f_! : \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_R) \rightleftarrows \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_S) : f^*$$

is given on the underlying modules by the same adjunction $f_! : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_S : f^*$. In other words, both the square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_S) & \xrightarrow{f^*} & \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_R) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_S & \xrightarrow{f^*} & \mathrm{Mod}_R \end{array}$$

and the square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_R) & \xrightarrow{f_!} & \mathrm{Alg}_{\mathcal{P}}(\mathrm{Mod}_S) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_R & \xrightarrow{f_!} & \mathrm{Mod}_S \end{array}$$

commute. These facts allow us to inherit the proof of Proposition 5.4.2.1 for verifying the second condition of Definition 5.4.1.2 on $\mathcal{F}$. □

Proposition 5.4.2.3. *Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be the functor $\mathcal{F}(R) = \mathrm{Cat}(\mathrm{Mod}_R)$. Then $\mathcal{F}$ is a formal moduli context.*

Let $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ be a functor between Mod_R-enriched categories. We will say that F is an isofibration if the induced functor $\mathrm{Ho}(\mathrm{F}) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ between homotopy categories is an isofibration in the classical sense. By the free-forgetful adjunction $\mathcal{C}_{\geq 0}(\mathbf{k}) \rightleftarrows \mathrm{Mod}_R$, we see that the homotopy category of a Mod_R-enriched category is the same as that of its underlying $\mathcal{C}_{\geq 0}(\mathbf{k})$-enriched category. Moreover, note that a morphism $f : x \to y$ in $\mathcal{C}$ performs an isomorphism in $\mathrm{Ho}(\mathcal{C})$ if and only if for every object $z \in \mathrm{Ob}(\mathcal{C})$ the induced map $f \circ (-) : \mathrm{Map}_{\mathcal{C}}(z, x) \longrightarrow \mathrm{Map}_{\mathcal{C}}(z, y)$ is a weak equivalence. The proof of Proposition 5.4.2.3 will require the following lemma:

Lemma 5.4.2.4. *Let $p : R \to S$ be a map in $\mathrm{CAlg}^{\mathrm{art}}$ such that $\pi_0(p) : \pi_0(R) \longrightarrow \pi_0(S)$ is surjective and let $\mathcal{C}$ be a levelwise cofibrant Mod_R-enriched category. Then the induced functor $\mathcal{C} \longrightarrow \mathcal{C} \otimes_R S$ is an isofibration of Mod_R-enriched categories (where $\mathcal{C} \otimes_R S = p^* p_! \mathcal{C}$ the category obtained by tensoring all mapping objects with S over R). More generally, given a sequence of maps $R \longrightarrow S \xrightarrow{p} U$ in $\mathrm{CAlg}^{\mathrm{art}}$ such that p induces a surjection on π_0, then the induced functor $\mathcal{C} \otimes_R S \longrightarrow \mathcal{C} \otimes_R U$ is an isofibration of Mod_R-enriched categories.*

Proof. Unwinding definition, the functor $\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C} \otimes_R S)$ is the identity on objects and given on hom-object by the map

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y) \otimes_{\pi_0 R} \pi_0 S,$$

which can be identified to the projection

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y)/\mathfrak{a}(\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y))$$

with $\mathfrak{a}$ being the kernel of the surjection $\pi_0(R) \longrightarrow \pi_0(S)$. So we just need to show that, for any map $f : x \to y$ in $\mathrm{Ho}(\mathcal{C})$, if the corresponding map $\overline{f} : x \to y$ is an isomorphism in $\mathrm{Ho}(\mathcal{C} \otimes_R S)$ then f itself is one. For this, it suffices to show that for every object $z \in \mathrm{Ob}(\mathcal{C})$ the induced map $f \circ (-) : \mathrm{Map}_{\mathcal{C}}(z, x) \longrightarrow \mathrm{Map}_{\mathcal{C}}(z, y)$ is a weak equivalence as soon as the map

$$\mathrm{Map}_{\mathcal{C}}(z, x) \otimes_R S \longrightarrow \mathrm{Map}_{\mathcal{C}}(z, y) \otimes_R S$$

is one. Note by assumption that the map $f \circ (-)$ goes between cofibrant R-modules, hence the functor $(-) \otimes_R S$ preserves the homotopy cofiber of $f \circ (-)$. So the problem is reduced to proving that for any $M \in \mathrm{Mod}_R$, if $M \otimes_R S$ is acyclic then so is M. Assume by contradiction that M is not acyclic, and let $k \geq 0$ be the smallest integer such that $\pi_k M$ is non-trivial. We then have that $\pi_k M/\mathfrak{a}(\pi_k M) = \pi_k M \otimes_{\pi_0 R} \pi_0 S = \pi_k(M \otimes_R S) = 0$. But the equation $\pi_k M/\mathfrak{a}(\pi_k M) = 0$ implies the contradiction that $\pi_k M = 0$, because every ideal of a local artinian algebra is nilpotent. The second statement of the lemma is very similar. □

Proof of Proposition 5.4.2.3. As in the previous proofs, it is straightforward to verify the first condition of Definition 5.4.1.2 on $\mathcal{F}$. For the second condition, we let

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow p \\ T & \xrightarrow{q} & U \end{array} \tag{5.4.2.4}$$

be a homotopy Cartesian square in $\mathrm{CAlg}^{\mathrm{art}}$ such that the maps $\pi_0(p) : \pi_0(S) \longrightarrow \pi_0(U)$ and $\pi_0(q) : \pi_0(T) \longrightarrow \pi_0(U)$ are surjective. We need to show that the induced square of model categories

$$\begin{array}{ccc} \mathrm{Cat}(\mathrm{Mod}_R) & \longrightarrow & \mathrm{Cat}(\mathrm{Mod}_S) \\ \downarrow & & \downarrow p_! \\ \mathrm{Cat}(\mathrm{Mod}_T) & \xrightarrow{q_!} & \mathrm{Cat}(\mathrm{Mod}_U) \end{array} \tag{5.4.2.5}$$

is homotopy Cartesian in the sense of Definition B.0.0.1. Let $\mathcal{F}' : [1] \times [1] \longrightarrow \mathrm{ModCat}$ represent the square (5.4.2.5) and let $\mathcal{I} \subseteq [1] \times [1]$ denote the full subcategory spanned by $(0,1), (1,0)$ and $(1,1)$. We need to show that the Quillen adjunction $\mathcal{L}^{\mathrm{coc}} : \mathrm{Cat}(\mathrm{Mod}_R) \rightleftarrows \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) : \mathcal{R}^{\mathrm{coc}}$ is a Quillen equivalence. As in the proof of Proposition 5.4.2.1, we shall prove that $\mathcal{R}^{\mathrm{coc}}$ detects weak equivalences between bifibrant objects and that for every cofibrant object $\mathcal{C} \in \mathrm{Cat}(\mathrm{Mod}_R)$ the derived unit map $\mathcal{C} \longrightarrow \mathbb{R}\mathcal{R}^{\mathrm{coc}}\mathcal{L}^{\mathrm{coc}}(\mathcal{C})$ is a weak equivalence.

Let us start with the second claim. By Remark B.0.0.4, we have to show that the induced square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C} \otimes_R S \\ \downarrow & & \downarrow \\ \mathcal{C} \otimes_R T & \longrightarrow & \mathcal{C} \otimes_R U \end{array} \tag{5.4.2.6}$$

is homotopy Cartesian in $\mathrm{Cat}(\mathrm{Mod}_R)$. According to [[7], Lemma 3.1.11], we just need to verify that for every pair $x, y \in \mathrm{Ob}(\mathcal{C})$ the induced square of mapping objects

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x,y) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x,y) \otimes_R S \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(x,y) \otimes_R T & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x,y) \otimes_R U \end{array}$$

is homotopy Cartesian, and that the maps $\mathcal{C} \otimes_R T \longrightarrow \mathcal{C} \otimes_R U$ and $\mathcal{C} \otimes_R S \longrightarrow \mathcal{C} \otimes_R U$ are an isofibration. The first condition was indicated in the proof of Proposition 5.4.2.1 while the second one follows by Lemma 5.4.2.4.

We now show that $\mathcal{R}^{\mathrm{coc}}$ detects weak equivalences between bifibrant objects. Let $f : s \to s'$ be a map between bifibrant objects in $\mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}})$ such that $\mathcal{R}^{\mathrm{coc}}(s) \longrightarrow \mathcal{R}^{\mathrm{coc}}(s')$ is a weak equivalence in $\mathrm{Cat}(\mathrm{Mod}_R)$. We have to show that f itself is a weak equivalence. By definition, s is in particular a coCartesian section, and hence the homotopy type of $s(1,1)$ is determined by either $s(0,1)$ or $s(1,0)$ via (derived) tensoring. An analogue holds for s'. Therefore, it will suffice to show that the maps $f_{0,1} : s(0,1) \longrightarrow s'(0,1)$ and $f_{1,0} : s(1,0) \longrightarrow s'(1,0)$ are weak equivalences (in $\mathrm{Cat}(\mathrm{Mod}_T)$ and $\mathrm{Cat}(\mathrm{Mod}_S)$ respectively). By symmetricity, we just need to verify that $f_{0,1}$ is a weak equivalence. Let us see how it goes.

We first show that $f_{0,1}$ is fully-faithful. Let x, y be two objects of $s(0,1) \in \mathrm{Cat}(\mathrm{Mod}_T)$ and let x', y' denote their images in $s(1,1) \in \mathrm{Cat}(\mathrm{Mod}_U)$. By assumption, the map $s(1,0) \longrightarrow p^* s(1,1)$ is a fibration (cf. Observation B.0.0.3) and in particular, an isofibration; while its adjoint $p_! s(1,0) \longrightarrow s(1,1)$ is a weak equivalence and in particular, is essentially surjective. These facts together prove that the map $s(1,0) \longrightarrow p^* s(1,1)$ is surjective on objects. Let $x'', y'' \in s(1,0)$ be objects which map to $x', y' \in s(1,1)$ respectively. Since $\mathcal{R}^{\mathrm{coc}}(s) \longrightarrow \mathcal{R}^{\mathrm{coc}}(s')$ is a weak equivalence, the map

$$\mathrm{Map}_{s(0,1)}(x,y) \times^{\mathrm{h}}_{\mathrm{Map}_{s(1,1)}(x',y')} \mathrm{Map}_{s(1,0)}(x'',y'') \xrightarrow{\simeq} \mathrm{Map}_{s'(0,1)}(x,y) \times^{\mathrm{h}}_{\mathrm{Map}_{s'(1,1)}(x',y')} \mathrm{Map}_{s'(1,0)}(x'',y'')$$

is a weak equivalence in Mod_R. Following the proof of Proposition 5.4.2.1, the map $\mathrm{Map}_{s(0,1)}(x,y) \longrightarrow \mathrm{Map}_{s'(0,1)}(x,y)$ is a weak equivalence. We just verified that $f_{0,1}$ is fully-faithful. It remains to prove that it is essentially surjective. Let z be an object of $s'(0,1) \in \mathrm{Cat}(\mathrm{Mod}_T)$ and let z' denote its image in $s'(1,1)$. As above, there exists an object $z'' \in s'(1,0)$ which maps to z'. We thus find an object $\omega := (z, z', z'')$ of $\mathcal{R}^{\mathrm{coc}}(s') = s'(0,1) \times^h_{s'(1,1)} s'(1,0)$ in $\mathrm{Cat}(\mathrm{Mod}_R)$. Since $\mathcal{R}^{\mathrm{coc}}(s) \longrightarrow \mathcal{R}^{\mathrm{coc}}(s')$ is essentially surjective, there exists an object $\overline{\omega} \in \mathcal{R}^{\mathrm{coc}}(s)$ whose image in $\mathcal{R}^{\mathrm{coc}}(s')$ is equivalent to ω.

Now, we have a commutative square

$$\begin{array}{ccc} \mathcal{R}^{\mathrm{coc}}(s) & \xrightarrow{\varphi} & s(1,0) \\ \downarrow & & \downarrow f_{0,1} \\ \mathcal{R}^{\mathrm{coc}}(s') & \xrightarrow[\varphi']{} & s'(1,0) \end{array} \tag{5.4.2.7}$$

in which the left vertical map takes $\overline{\omega} \in \mathcal{R}^{\mathrm{coc}}(s)$ to $\omega \in \mathcal{R}^{\mathrm{coc}}(s')$ (up to an equivalence) while the bottom horizontal map takes ω to z. The commutativity of this square tells us that z is equivalent to the image under $f_{0,1}$ of $\varphi(\overline{\omega}) \in s(1,0)$. We just showed that $f_{0,1} : s(0,1) \longrightarrow s'(0,1)$ is essentially surjective, as expected. □

Finally, the example that we really care about is the following:

Proposition 5.4.2.5. *Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}$ be the functor $\mathcal{F}(R) = \mathrm{Op}(\mathrm{Mod}_R)$. Then $\mathcal{F}$ is a formal moduli context.*

Proof. As before, it is straightforward to verify the first condition of Definition 5.4.1.2 on $\mathcal{F}$. Verifying the second condition on $\mathcal{F}$ will be mainly supported by the propositions 5.4.2.1 and 5.4.2.3. Let us pick up the settings from the proofs of those. In brief, we have to show that the square of model categories

$$\begin{array}{ccc} \mathrm{Op}(\mathrm{Mod}_R) & \longrightarrow & \mathrm{Op}(\mathrm{Mod}_S) \\ \downarrow & & \downarrow p_! \\ \mathrm{Op}(\mathrm{Mod}_T) & \xrightarrow[q_!]{} & \mathrm{Op}(\mathrm{Mod}_U) \end{array} \tag{5.4.2.8}$$

is homotopy Cartesian. Again, we have to show that the adjunction

$$\mathcal{L}^{\mathrm{coc}} : \mathrm{Op}(\mathrm{Mod}_R) \rightleftarrows \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{J}}) : \mathcal{R}^{\mathrm{coc}}$$

is a Quillen equivalence where $\mathcal{F}' : [1] \times [1] \longrightarrow \mathrm{ModCat}$ represents the square (5.4.2.8).

We first show that the derived unit map $\mathcal{O} \longrightarrow \mathbb{R}\mathcal{R}^{\mathrm{coc}}\mathcal{L}^{\mathrm{coc}}(\mathcal{O})$ is a weak equivalence for every cofibrant object $\mathcal{O} \in \mathrm{Op}(\mathrm{Mod}_R)$. By Remark B.0.0.4, this is equivalent to saying that the square

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O} \otimes_R S \\ \downarrow & & \downarrow \\ \mathcal{O} \otimes_R T & \longrightarrow & \mathcal{O} \otimes_R U \end{array} \tag{5.4.2.9}$$

is homotopy Cartesian in $\mathrm{Op}(\mathrm{Mod}_R)$. Following the proof of Lemma 3.2.1.3, we need to verify that the induced square of underlying categories

$$\begin{array}{ccc} \mathcal{O}_1 & \longrightarrow & \mathcal{O}_1 \otimes_R S \\ \downarrow & & \downarrow \\ \mathcal{O}_1 \otimes_R T & \longrightarrow & \mathcal{O}_1 \otimes_R U \end{array}$$

is homotopy Cartesian in $\mathrm{Cat}(\mathrm{Mod}_R)$ and that the induced squares of spaces of operations

$$\begin{array}{ccc} \mathcal{O}(c_1, \cdots, c_n; c) & \longrightarrow & \mathcal{O}(c_1, \cdots, c_n; c) \otimes_R S \\ \downarrow & & \downarrow \\ \mathcal{O}(c_1, \cdots, c_n; c) \otimes_R T & \longrightarrow & \mathcal{O}(c_1, \cdots, c_n; c) \otimes_R U \end{array}$$

are all homotopy Cartesian in Mod_R. The first square is homotopy Cartesian by the proof of Proposition 5.4.2.3 while the second one is so by the proof of Proposition 5.4.2.1.

It remains to check that $\mathcal{R}^{\mathrm{coc}}$ detects weak equivalences between bifibrant objects. Recall from Remark 2.1.5.7 that a map between operads in Mod_R is a weak equivalence if and only if the induced map between underlying categories is one in $\mathrm{Cat}(\mathrm{Mod}_R)$ and the induced maps between spaces of operations are a weak equivalence in Mod_R. So the expected property of $\mathcal{R}^{\mathrm{coc}}$ just follows by the fact that both the functors $\mathbb{R}\mathcal{R}^{\mathrm{coc}} : \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) \longrightarrow \mathrm{Cat}(\mathrm{Mod}_R)$ (of Proposition 5.4.2.3) and $\mathbb{R}\mathcal{R}^{\mathrm{coc}} : \mathrm{Sec}^{\mathrm{coc}}(\mathcal{F}'|_{\mathcal{I}}) \longrightarrow \mathrm{Mod}_R$ (of Proposition 5.4.2.1) detect weak equivalences. □

To end the chapter, we give an illustration concerning deformations of dg operads using the obtained results. In particular, we are interested in the functor $\mathcal{F} : \mathrm{CAlg}^{\mathrm{art}} \longrightarrow \mathrm{ModCat}\,,\ R \mapsto \mathrm{Op}(\mathrm{Mod}_R)$ of the above proposition.

Let $A \in \mathcal{C}_{\geq 0}(\mathbf{k})$ be a finite dimensional connective dg $\mathbf{k}$-module. For each $n \in \mathbb{N}$, the square-zero extension $\mathbf{k} \ltimes A[n]$ of $\mathbf{k}$ by $A[n]$ is an artinian algebra. Consider the Ω-spectrum $M \in \mathrm{Sp}(\mathrm{CAlg}^{\mathrm{art}})$ with $M_{n,n} = \mathbf{k} \ltimes A[n]$ and $M_{n,m} = \mathbf{k}$ when $n \neq m$. Let $\mathcal{P}$ be an operad in $\mathcal{C}_{\geq 0}(\mathbf{k})$. Then the object $M(\mathcal{P}) \in \mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{C}_{\geq 0}(\mathbf{k}))$ of Definition 5.4.1.6 is an Ω-spectrum (cf. Lemma 5.4.1.7 and Remark 5.4.1.9). Concretely, we have for each $n \in \mathbb{N}$ that $M(\mathcal{P})_{n,n} = \mathcal{P} \ltimes (\mathcal{P} \otimes A[n])$ the square-zero extension of $\mathcal{P}$ by $\mathcal{P} \otimes A[n]$, where $\mathcal{P} \otimes A[n]$ is regarded as an infinitesimal $\mathcal{P}$-bimodule with

$$(\mathcal{P} \otimes A[n])(c_1, \cdots, c_r; c) = \mathcal{P}(c_1, \cdots, c_r; c) \otimes A[n]$$

for every sequence $(c_1, \cdots, c_r; c)$ of colors of $\mathcal{P}$ (cf. §5.1). Note that if we regard $\mathcal{P}$ as an object of $\mathrm{Op}(\mathcal{C}(\mathbf{k}))$ (instead $\mathrm{Op}(\mathcal{C}_{\geq 0}(\mathbf{k}))$), then under the right Quillen equivalence $\mathcal{T}_{\mathcal{P}}\,\mathrm{Op}(\mathcal{C}(\mathbf{k})) \xrightarrow{\simeq} \mathrm{IbMod}(\mathcal{P})$ (cf. §5.1), $M(\mathcal{P})$ corresponds to nothing but $\mathcal{P} \otimes A$. Theorem 5.4.1.8 hence tells us that:

(*) $\pi_0\,\mathrm{Def}(\mathcal{P}, \mathbf{k} \ltimes A) \cong \mathrm{H}^1_{\mathcal{Q}}(\mathcal{P}, \mathcal{P} \otimes A)$ and $\pi_n(\mathrm{Def}(\mathcal{P}, \mathbf{k} \ltimes A), *) \cong \mathrm{H}^{1-n}_{\mathcal{Q}}(\mathcal{P}, \mathcal{P} \otimes A)$ where $\mathrm{H}^{\bullet}_{\mathcal{Q}}(\mathcal{P}, \mathcal{P} \otimes A)$ refers to the Quillen cohomology group of $\mathcal{P}$ (considered as an object in $\mathrm{Op}(\mathcal{C}(\mathbf{k}))$) with coefficients in $\mathcal{P} \otimes A \in \mathrm{IbMod}(\mathcal{P})$.

In particular, when $A = \mathbf{k}$ then $\mathrm{Def}(\mathcal{P}, \mathbf{k} \ltimes A)$ represents the space of deformations of $\mathcal{P}$ over the **algebra of dual numbers** $\mathbf{k}[t]/(t^2)$. We hence get that the homotopy type of $\mathrm{Def}(\mathcal{P}, \mathbf{k}[t]/(t^2))$ is classified by Quillen cohomology of $\mathcal{P}$ with coefficients in itself $\mathcal{P} \in \mathrm{IbMod}(\mathcal{P})$.

Appendix A

Semi model categories and their left Bousfield localizations

We first recall briefly some basic facts involving **semi model categories**. For more details, we refer the readers to [53, 54].

Definition A.0.0.1. ([54]) A *semi model category* is a bicomplete category $\mathbf{M}$ which is equipped with three subcategories of *weak equivalences* W, *fibrations* F, and *cofibrations* C satisfying the following axioms:

(SM 0) The initial object of $\mathbf{M}$ is cofibrant.

(SM 1) The 2-out-of-3 and retract axioms hold.

(SM 2) (i) Cofibrations have the left lifting property with respect to trivial fibrations; (ii) trivial cofibrations whose domain is cofibrant have the left lifting property with respect to fibrations.

(SM 3) (i) Every map in $\mathbf{M}$ can be functorially factored into a cofibration followed by a trivial fibration; (ii) every map whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.

(SM 4) The fibrations are closed under pullbacks and transfinite (reverse) compositions.

Facts A.0.0.2. The following observations hold in a semi model category:

1. The class of cofibrations (resp. trivial fibrations) is characterized by the left (resp. right) lifting property with respect to the class of trivial fibrations (resp. cofibrations).

2. A map with cofibrant domain is a trivial cofibration if and only if it has the left lifting property with respect to the class of fibrations.

3. The class of cofibrations is stable under pushouts and transfinite compositions.

4. The class of trivial fibrations is stable under pullbacks and transfinite (reverse) compositions.

Proof. The same arguments as in the proof of [[34], Lemma 1.1.10] prove (1) and (2). (3) and (4) then follow by (1). □

In a semi model category $\mathbf{M}$, (resp. *path*) *cylinder objects* exist for every (resp. cofibrant) object. *Homotopy relations* in $\mathbf{M}$ are then well behaved in the sense that: if X is cofibrant and Y is bifibrant then left and right homotopy coincide and are equivalence relations on $\mathrm{Hom}(X,Y)$. Consequently, on the full subcategory of bifibrant objects, denoted $\mathbf{M}_{cf}$, the homotopy relation is an equivalence relation and moreover, compatible with composition. One can then prove that there is a canonical isomorphism ${\mathbf{M}_{cf}}_{/\sim} \cong \mathrm{Ho}(\mathbf{M}_{cf})$ and that the inclusion $\mathbf{M}_{cf} \longrightarrow \mathbf{M}$ induces an equivalence $\mathrm{Ho}(\mathbf{M}_{cf}) \xrightarrow{\simeq} \mathrm{Ho}(\mathbf{M})$ between the associated homotopy categories. (These facts can be found in [54]).

Definition A.0.0.3. An adjunction $\mathrm{L} : \mathbf{M} \rightleftarrows \mathbf{N} : \mathrm{R}$ between semi model categories is called a *Quillen adjunction* if the right adjoint R preserves fibrations and trivial fibrations.

Lemma A.0.0.4. *(K. Brown's lemma) Let* $\mathrm{F} : \boldsymbol{M} \longrightarrow \mathcal{C}$ *be a functor where* $\boldsymbol{M}$ *is a semi model category and* $\mathcal{C}$ *is a category equipped with a class of weak equivalences satisfying the 2-out-of-3 property. If* F *sends trivial cofibrations between cofibrant objects to weak equivalences then* F *preserves weak equivalences between cofibrant objects. Dually, if* F *sends trivial fibrations between fibrant objects to weak equivalences then* F *preserves the weak equivalences whose domain is bifibrant and whose codomain is fibrant.*

Proof. The claims follow by the same arguments as in the proof of the (original) K. Brown's lemma (cf. [34]). Here we just note that, for the second claim, one would need a help of the axiom (SM 4). □

Facts A.0.0.5.

1. The *left Quillen functor* L preserves cofibrations, but preserves only trivial cofibrations whose domain is cofibrant. Moreover, L preserves weak equivalences between cofibrant objects.

2. The *right Quillen functor* R preserves weak equivalences between fibrant objects.

3. The Quillen adjunction $\mathrm{L} \dashv \mathrm{R}$ induces an adjunction $\mathbb{L} : \mathrm{Ho}(\mathbf{M}) \rightleftarrows \mathrm{Ho}(\mathbf{N}) : \mathbb{R}$ between the associated homotopy categories.

Proof.

1. The first two claims follow by Facts A.0.0.2 (1, 2). The other follows by the K. Brown's lemma A.0.0.4.

2. Let $f : X \xrightarrow{\simeq} Y$ be a weak equivalence between fibrant objects in $\mathbf{N}$. By the axiom SM 3(i) we can take a commutative square in $\mathbf{N}$:

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ {\scriptstyle \widetilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widetilde{Y} & \longrightarrow & Y \end{array}$$

such that the horizontal maps are trivial fibrations and the map $\widetilde{f}$ is a weak equivalence between bifibrant objects. By the "fibrant half" of the K. Brown's lemma A.0.0.4 the map $\mathrm{R}(\widetilde{f})$ is a weak equivalence. On other hand, by definition R sends the horizontal maps to trivial fibrations in $\mathbf{M}$. The claim hence follows by the 2-out-of-3 axiom.

3. For this, one makes use of the two above and follows up the arguments given in the proof of [[34], Lemma 1.3.10].

□

Lemma A.0.0.6. *Let* $\mathrm{L} : \boldsymbol{M} \rightleftarrows \boldsymbol{N} : \mathrm{R}$ *be a Quillen adjunction between semi model categories. The following conditions are equivalent.*

1. *Given any cofibrant object* $A \in \boldsymbol{M}$ *and any fibrant object* $X \in \boldsymbol{N}$*, a map* $f : \mathrm{L}(A) \longrightarrow X$ *is a weak equivalence if and only if its adjoint* $f^{\mathrm{ad}} : A \longrightarrow \mathrm{R}(X)$ *is one.*

2. *Given any cofibrant object* $A \in \boldsymbol{M}$ *and any bifibrant object* $X \in \boldsymbol{N}$*, a map* $f : \mathrm{L}(A) \longrightarrow X$ *is a weak equivalence if and only if its adjoint* $f^{\mathrm{ad}} : A \longrightarrow \mathrm{R}(X)$ *is one.*

3. *For every cofibrant object* $A \in \boldsymbol{M}$ *the composition*
$$A \xrightarrow{\eta_A} \mathrm{R}\,\mathrm{L}(A) \xrightarrow{\mathrm{R}(i)} \mathrm{R}(X)$$
is a weak equivalence, where η *is the adjunction unit and* $i : \mathrm{L}(A) \longrightarrow X$ *is the trivial cofibration coming from the factorization* SM 3(ii) *of the terminal map* $\mathrm{L}(A) \longrightarrow *$ *and in addition, for every fibrant object* $X \in \boldsymbol{N}$ *the composition*
$$\mathrm{L}(\mathrm{R}(X)^{\mathrm{cof}}) \longrightarrow \mathrm{L}\,\mathrm{R}(X) \xrightarrow{\varepsilon} X$$
is a weak equivalence, where ε *is the adjunction counit and* $\mathrm{R}(X)^{\mathrm{cof}}$ *is a cofibrant resolution of* $\mathrm{R}(X)$*.*

4. *The induced adjunction* $\mathbb{L} : \mathrm{Ho}(\boldsymbol{M}) \rightleftarrows \mathrm{Ho}(\boldsymbol{N}) : \mathbb{R}$ *is an adjoint equivalence.*

Proof. (1) $\Longleftrightarrow$ (2): the direction (1) $\Longrightarrow$ (2) is clear. For the converse direction, let $A \in \mathbf{M}$ and $X \in \mathbf{N}$ be cofibrant and fibrant respectively, and let $f : \mathrm{L}(A) \longrightarrow X$ be given. Let us factor f into a cofibration $g : \mathrm{L}(A) \longrightarrow Y$ followed by a trivial fibration $Y \to X$. In particular, f^{ad} agrees with the composite map $A \xrightarrow{g^{\mathrm{ad}}} \mathrm{R}(Y) \longrightarrow \mathrm{R}(X)$ with the second map being a trivial fibration. By the 2-out-of-3 axiom, f (resp. f^{ad}) is a weak equivalence if and only if g (resp. g^{ad}) is one. Since Y is bifibrant, the condition (2) implies that g is a weak equivalence if and only if g^{ad} is one.

(2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4): this is done by following up the arguments of [[34], Proposition 1.3.13], along with making use of Facts A.0.0.5. □

Definition A.0.0.7. The Quillen adjunction $\mathrm{L} : \mathbf{M} \rightleftarrows \mathbf{N} : \mathrm{R}$ is called a *Quillen equivalence* if the equivalent conditions of Lemma A.0.0.6 are satisfied.

We now discuss on *derived mapping spaces* in a semi model category. As well as every **relative category**, any semi model category $\mathbf{M}$ has an **underlying ∞-category**, denoted by $\mathbf{M}_\infty$. Let $\mathrm{L}^{\mathrm{W}}\mathbf{M} \in \mathrm{Cat}(\mathrm{Set}_\Delta)$ denote the **hammock localization** associated to the pair $(\mathbf{M}, \mathrm{W})$ (cf. [55]). One then defines
$$\mathbf{M}_\infty := \mathrm{N}(\mathrm{L}^{\mathrm{W}}\mathbf{M})^{fib} \in \mathrm{Cat}_\infty$$
the coherent nerve of any fibrant resolution of $\mathrm{L}^{\mathrm{W}}\mathbf{M}$. By this way, the derived mapping space $\mathrm{Map}^{\mathrm{h}}_{\mathbf{M}}(X, Y)$ is given by
$$\mathrm{Map}^{\mathrm{h}}_{\mathbf{M}}(X, Y) := \mathrm{Map}_{\mathbf{M}_\infty}(X, Y), \qquad \text{(A.0.0.1)}$$
which is weakly equivalent to the simplicial set $\mathrm{Map}_{\mathrm{L}^{\mathrm{W}}\mathbf{M}}(X, Y)$.

On other hand, as well as model categories, the derived mapping spaces in $\mathbf{M}$ can be modeled by *homotopy function complexes*. The key point is the following:

Theorem A.0.0.8. *(M. Spitzweck [54]) Let $\mathcal{R}$ be a Reedy category. The category of $\mathcal{R}$-diagrams in $\boldsymbol{M}$ admits a semi model structure whose classes of weak equivalences and (co)fibrations are the same as those of the classical Reedy model structure.*

Let X be an object of $\mathbf{M}$. We let $cc(X) \in \mathbf{M}^{\Delta}$ and $cs(X) \in \mathbf{M}^{\Delta^{\mathrm{op}}}$ denote the constant cosimplicial and simplicial objects on X respectively.

Definition A.0.0.9. A *cosimplicial resolution* of X is a cofibrant resolution $X^{\bullet} \xrightarrow{\simeq} cc(X)$ of $cc(X)$ with respect to the Reedy model structure on $\mathbf{M}^{\Delta}$. A *simplicial resolution* of X is a fibrant resolution $cs(X) \xrightarrow{\simeq} X_{\bullet}$ of $cs(X)$ with respect to the Reedy model structure on $\mathbf{M}^{\Delta^{\mathrm{op}}}$.

Functorial cosimplicial resolutions exist for every object of $\mathbf{M}$. On the other hand, functorial simplicial resolutions in general exist only for cofibrant objects. In what follows, we will always assume that simplicial resolution of a given object exists. (Otherwise, one takes simplicial resolution of its cofibrant resolution). We have an analogue of [[34], Proposition 5.4.7].

Proposition A.0.0.10. *Let $X \in \boldsymbol{M}$ be a cofibrant object and $Y \in \boldsymbol{M}$ a fibrant object. Then there are homotopy equivalences of Kan complexes*

$$Hom_{\boldsymbol{M}}(X^{\bullet}, Y) \underset{\simeq}{\xrightarrow{\varphi_{X,Y}}} \operatorname{diag} Hom_{\boldsymbol{M}}(X^{\bullet}, Y_{\bullet}) \underset{\simeq}{\xleftarrow{\varphi'_{X,Y}}} Hom_{\boldsymbol{M}}(X, Y_{\bullet}) \tag{A.0.0.2}$$

Proof. We repeat the Hovey's arguments given in [[34], Section 5.4]. Using the same arguments as in the proof Proposition 5.4.1 (loc.cit), we obtain that:

(1a) The functor $X^{\bullet} \otimes (-) : \mathrm{Set}_{\Delta} \longrightarrow \mathbf{M}$ preserves cofibrations and trivial cofibrations. For this claim, one would need to use Facts A.0.0.2(3) saying that cofibrations in $\mathbf{M}$ are closed under pushouts and transfinite compositions.

(2a) The functor $Y_{\bullet}^{(-)} : \mathrm{Set}_{\Delta} \longrightarrow \mathbf{M}^{\mathrm{op}}$ preserves cofibrations and trivial cofibrations as well. For this, one would need to use the axiom (SM 4) saying that fibrations in $\mathbf{M}$ are closed under pullbacks and transfinite (reverse) compositions.

Note that there are canonical adjunctions $X^{\bullet} \otimes (-) \dashv \mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, -)$ and $Y_{\bullet}^{(-)} \dashv \mathrm{Hom}_{\mathbf{M}}(-, Y_{\bullet})$. We then argue as follows.

(1b) The functor $\mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, -) : \mathbf{M} \longrightarrow \mathrm{Set}_{\Delta}$ preserves trivial fibrations by (1a) and by Facts A.0.0.2(1). It also preserves fibrations. Indeed, to see this one would need to make use of the axiom SM 2(ii), along with the facts that the left adjoint $X^{\bullet} \otimes (-)$ preserves trivial cofibrations and that $X^{\bullet} \otimes K$ is cofibrant for every $K \in \mathrm{Set}_{\Delta}$.

(2b) The functor $\mathrm{Hom}_{\mathbf{M}}(-, Y_{\bullet}) : \mathbf{M}^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}$ preserves trivial fibrations by (1b) and by Facts A.0.0.2(1). It preserves fibrations as well, by similar arguments as above.

Consequently, $\mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, Y)$ and $\mathrm{Hom}_{\mathbf{M}}(X, Y_{\bullet})$ are indeed Kan complexes. In order to see that $\operatorname{diag} \mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, Y_{\bullet})$ is as well a Kan complex, one follows up the arguments as in the proof of [[45], Theorem 16.5.18].

To prove the maps $\varphi_{X,Y}$ and $\varphi'_{X,Y}$ are weak equivalences, the arguments as in the proof of [[34], Proposition 5.4.7] can be repeated. In particular, one would need to use the following observations:

(1c) By (1b) and by Facts A.0.0.5(2), the functor $\mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, -)$ preserves weak equivalences between fibrant objects. So each bisimplicial structure map of $\mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, Y_{\bullet})$ of the form

$$\mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, Y_k) \longrightarrow \mathrm{Hom}_{\mathbf{M}}(X^{\bullet}, Y_n)$$

is a weak equivalence.

(2c) By (2b) the functor $\mathrm{Hom}_{\mathbf{M}}(-, Y_\bullet) : \mathbf{M} \longrightarrow \mathrm{Set}_\Delta^{\mathrm{op}}$ sends trivial cofibrations to weak equivalences, and hence it preserves weak equivalences between cofibrant objects, according to the K. Brown's lemma A.0.0.4. So each bisimplicial structure map of $\mathrm{Hom}_{\mathbf{M}}(X^\bullet, Y_\bullet)$ of the form

$$\mathrm{Hom}_{\mathbf{M}}(X^k, Y_\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{M}}(X^n, Y_\bullet)$$

is a weak equivalence as well. □

A consequence of Proposition A.0.0.10 is that all the function complexes appearing in (A.0.0.2) are homotopy invariant with respect to the cofibrant objects of $\mathbf{M}$ in the left variable and fibrant objects of $\mathbf{M}$ in the right variable. In conclusion, for any $X, Y \in \mathbf{M}$ the derived mapping space $\mathrm{Map}^{\mathrm{h}}_{\mathbf{M}}(X, Y)$ can be modeled by any of three homotopy function complexes above (after taking suitable resolutions of X and Y). As in [[55], Proposition 4.4], it can be proved that these models and the model (A.0.0.1) are equivalent.

Proposition A.0.0.11. *Let* $\mathrm{L} : \boldsymbol{M} \rightleftarrows \boldsymbol{N} : \mathrm{R}$ *be a Quillen adjunction between semi model categories. Given a cofibrant object* $X \in \boldsymbol{M}$ *and a fibrant object* $A \in \boldsymbol{N}$*, then there is a natural homotopy equivalence between derived mapping spaces*

$$\mathrm{Map}^{\mathrm{h}}_{\boldsymbol{N}}(\mathrm{L}\,X, A) \simeq \mathrm{Map}^{\mathrm{h}}_{\boldsymbol{M}}(X, \mathrm{R}\,A).$$

Proof. According to [[45], Proposition 16.2.1], $\mathrm{L}\,X^\bullet$ is a cosimplicial resolution of $\mathrm{L}\,X$. By adjunction we get a natural isomorphism

$$\mathrm{Hom}_{\mathbf{N}}(\mathrm{L}\,X^\bullet, A) \cong \mathrm{Hom}_{\mathbf{M}}(X^\bullet, \mathrm{R}\,A)$$

in which the left and right hand sides are respectively models for $\mathrm{Map}^{\mathrm{h}}_{\mathbf{N}}(\mathrm{L}\,X, A)$ and $\mathrm{Map}^{\mathrm{h}}_{\mathbf{M}}(X, \mathrm{R}\,A)$, as indicated above. □

In the remainder, we discuss on *left Bousfield localizations* of semi model categories, according to the recent work of M. Batanin and D. White [16]. The Hirschhorn's [45] and Barwick's [53] are good references for left Bousfield localizations of model categories.

Definition A.0.0.12. ([16]) A semi model category $\mathbf{M}$ is said to be *cofibrantly generated* if there exists a set of *generating cofibrations* I and a set of *generating trivial cofibrations* J such that the domains of I are small relative to I*-cell* and the domains of J are small relative to the maps in J*-cell* whose domain is cofibrant and in addition, the class of fibrations (resp. trivial fibrations) is characterized by the right lifting property with respect to the maps in J (resp. I).

Definition A.0.0.13. ([16]) A semi model category $\mathbf{M}$ is said to be *combinatorial* if it is cofibrantly generated and such that its underlying category is *locally presentable.*

Let $\mathbf{M}$ be a semi model category and let C be a set of maps of $\mathbf{M}$. An object $X \in \mathbf{M}$ is said to be C-**local** if for every map $f : Y \to Z$ in C, the induced map between derived mapping spaces

$$f^* : \mathrm{Map}^{\mathrm{h}}_{\mathbf{M}}(Z, X) \longrightarrow \mathrm{Map}^{\mathrm{h}}_{\mathbf{M}}(Y, X)$$

is a homotopy equivalence. A given map $f : Y \to Z$ in $\mathbf{M}$ is called a C-**local equivalence** if the map f^* is a homotopy equivalence for every C-local object X.

Theorem A.0.0.14. *(M. Batanin and D. White [16]) Suppose that $\boldsymbol{M}$ is a combinatorial semi model category such that the domains of generating cofibrations are cofibrant. Then the left Bousfield localization $\mathrm{L_C}\,\boldsymbol{M}$ exists as a cofibrantly generated semi model category whose*

- weak equivalences are the C*-local equivalences,*

- (generating) cofibrations are the same as those of $\boldsymbol{M}$, and whose

- fibrations are characterized by the right lifting property with respect to a certain set $\mathrm{J_C}$ of C*-local equivalences.*

Moreover, fibrant objects of $\mathrm{L_C}\,\boldsymbol{M}$ are precisely the fibrant C*-local objects of $\boldsymbol{M}$.*

Proposition A.0.0.15. *Suppose that $\boldsymbol{M}$ is a combinatorial semi model category such that the domains of generating cofibrations are cofibrant, and let* C *be a set of maps of $\boldsymbol{M}$. Let $\mathrm{L} : \boldsymbol{M} \rightleftarrows \boldsymbol{N} : \mathrm{R}$ be a Quillen adjunction between semi model categories such that the functor* R *takes fibrant objects of $\boldsymbol{N}$ to C-local objects. Then the adjunction $\mathrm{L} \dashv \mathrm{R}$ descends to a Quillen adjunction between $\mathrm{L_C}\,\boldsymbol{M}$ and $\boldsymbol{N}$.*

Proof. Assuming that R takes fibrant objects of **N** to C-local objects in **M** is equivalent to saying that the left adjoint L takes C-local equivalences to weak equivalences in **N**. Hence, the claim follows just by [[16], Theorem 4.3]. □

Appendix B

Homotopy Cartesian squares of model categories

This section is devoted to the work of §5.4. We denote by ModCat the category whose objects are model categories and whose morphisms are Quillen adjunctions with the sources and targets being those of the left Quillen functors. Let $\mathcal{I}$ be a small category and let $\mathcal{F} : \mathcal{I} \longrightarrow \mathrm{ModCat}$ be a diagram of model categories. By convention, for each map $\alpha : i \to j$ in $\mathcal{I}$, we will write $\alpha_! : \mathcal{F}(i) \rightleftarrows \mathcal{F}(j) : \alpha^*$ standing for the image of α through $\mathcal{F}$.

By a **section** of $\mathcal{F}$ we shall mean a section of the natural projection $\int_{\mathcal{I}} \mathcal{F} \longrightarrow \mathcal{I}$ from the Grothendieck construction of $\mathcal{F}$ to $\mathcal{I}$. More explicitly, a section of $\mathcal{F}$ consists of a choice of an object $s(i) \in \mathcal{F}(i)$ for each $i \in \mathcal{I}$ and a morphism $f_\alpha : \alpha_! s(i) \longrightarrow s(j)$ for each map $\alpha : i \to j$ in $\mathcal{I}$ which are subject to a natural compatibility constraint for every composable pair of morphisms in $\mathcal{I}$. We will denote by $\mathrm{Sec}(\mathcal{F})$ the category of sections of $\mathcal{F}$ and morphisms between them.

When $\mathcal{F}(i)$ is combinatorial for every i one can endow the category $\mathrm{Sec}(\mathcal{F})$ with either the projective or the injective model structure, according to [53] (see also [10] and [64]). Here we will be interested in the injective model structure, denoted by $\mathrm{Sec}(\mathcal{F})^{\mathrm{inj}}$, in which a map $s \to s'$ is a cofibration (resp. weak equivalence) if and only if $s(i) \longrightarrow s'(i)$ is a cofibration (resp. weak equivalence) for every i. Recall that a section $s : \mathcal{I} \longrightarrow \int_{\mathcal{I}} \mathcal{F}$ is said to be **coCartesian** if the composed map

$$f_\alpha : \alpha_!(s(i)^{\mathrm{cof}}) \longrightarrow \alpha_!(s(i)) \xrightarrow{f_\alpha} s(j)$$

is a weak equivalence for every $\alpha : i \to j$ in $\mathcal{I}$, where $s(i)^{\mathrm{cof}} \longrightarrow s(i)$ is a cofibrant replacement of $s(i)$ in $\mathcal{F}(i)$. Moreover, this model category $\mathrm{Sec}(\mathcal{F})^{\mathrm{inj}}$ is combinatorial as well. When all the model categories $\mathcal{F}(i)$'s are furthermore left proper then $\mathrm{Sec}(\mathcal{F})^{\mathrm{inj}}$ is left proper and one can then left Bousfield localize $\mathrm{Sec}(\mathcal{F})^{\mathrm{inj}}$ so that the new fibrant objects are the injective fibrant coCartesian sections. In this case, we will denote the localized model category by $\mathrm{Sec}(\mathcal{F})^{\mathrm{coc}}$. If the $\mathcal{F}(i)$'s are combinatorial but not left proper, we may still define the localization $\mathrm{Sec}(\mathcal{F})^{\mathrm{coc}}$ as a semi model category. In any case, $\mathrm{Sec}(\mathcal{F})^{\mathrm{coc}}$ has the right type in the sense that its underlying ∞-category $\mathrm{Sec}(\mathcal{F})^{\mathrm{coc}}_\infty$ is a model for the limit of the diagram $\mathcal{I} \ni i \mapsto \mathcal{F}(i)_\infty \in \mathrm{Cat}_\infty$, in which for each map $\alpha : i \to j$ the functor $\mathcal{F}(i)_\infty \longrightarrow \mathcal{F}(j)_\infty$ is given by $(\alpha_!)_\infty$ (cf. [64], Corollary 3.45).

Now let $[1]\times[1]$ be the square category and let $\mathcal{F} : [1]\times[1] \longrightarrow \mathrm{ModCat}$ be a diagram of combinatorial model categories. Let $\mathcal{I} \subseteq [1] \times [1]$ be the full subcategory spanned by the objects $(0,1), (1,0), (1,1)$. Since $(0,0)$ is the initial object of $\mathcal{I}$, we have for each $i \in \mathcal{I}$ a unique map $\alpha_i : (0,0) \longrightarrow i$. We then obtain a natural functor $\mathcal{L} : \mathcal{F}(0,0) \longrightarrow \mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})$ which sends an object $X \in \mathcal{F}(0,0)$ to the section $\mathcal{L}(X)(i) = (\alpha_i)_! X \in \mathcal{F}(i)$. The functor $\mathcal{L}$ admits a right adjoint $\mathcal{R} : \mathrm{Sec}(\mathcal{F}|_{\mathcal{I}}) \longrightarrow \mathcal{F}(0,0)$ which sends a section $s : \mathcal{I} \longrightarrow \int_{\mathcal{I}} \mathcal{F}$ to the pullback

$$\mathcal{R}(s) := \lim_{i\in\mathcal{I}} \alpha_i^* s(i) \in \mathcal{F}(0,0).$$

Since $\mathcal{L}$ clearly sends (trivial) cofibrations to injective (trivial) cofibrations, we see that the adjunction $\mathcal{L} : F(0,0) \rightleftarrows \mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})^{\mathrm{inj}} : \mathcal{R}$ is a Quillen adjunction.

Definition B.0.0.1. Let $\mathcal{F} : [1] \times [1] \longrightarrow \mathrm{ModCat}$ be a diagram of combinatorial model categories. We will say that $\mathcal{F}$ is **homotopy Cartesian** if the composed left Quillen functor

$$\mathcal{L}^{\mathrm{coc}} : \mathcal{F}(0,0) \xrightarrow{\mathcal{L}} \mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})^{\mathrm{inj}} \longrightarrow \mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})^{\mathrm{coc}}$$

is a left Quillen equivalence, in which the functor $\mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})^{\mathrm{inj}} \longrightarrow \mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})^{\mathrm{coc}}$ is the identity functor, which is a left Quillen functor by construction.

Remark B.0.0.2. Let $\mathcal{F} : [1]\times[1] \longrightarrow \mathrm{ModCat}$ be a diagram of combinatorial model categories depicted as the square of left Quillen functors:

$$\begin{array}{ccc} \mathcal{F}(0,0) & \longrightarrow & \mathcal{F}(0,1) \\ \downarrow & & \downarrow \\ \mathcal{F}(1,0) & \longrightarrow & \mathcal{F}(1,1) \end{array}$$

If $\mathcal{F}$ is homotopy Cartesian in the sense of Definition B.0.0.1 then the corresponding square of underlying ∞-categories

$$\begin{array}{ccc} \mathcal{F}(0,0)_\infty & \longrightarrow & \mathcal{F}(0,1)_\infty \\ \downarrow & & \downarrow \\ \mathcal{F}(1,0)_\infty & \longrightarrow & \mathcal{F}(1,1)_\infty \end{array}$$

is homotopy Cartesian as well. This follows by the fact that the model category $\mathrm{Sec}(\mathcal{F}|_{\mathcal{I}})^{\mathrm{coc}}$ has the right type, as mentioned above.

Observation B.0.0.3. Let $\mathcal{G} : \mathcal{I} \longrightarrow \mathrm{ModCat}$ be a diagram of combinatorial model categories with $\mathcal{I} \subseteq [1]\times[1]$ being the category mentioned above. Let $s \in \mathrm{Sec}(\mathcal{G})$ be a section of $\mathcal{G}$. If s is injective fibrant, i.e., fibrant as an object of $\mathrm{Sec}(\mathcal{G})^{\mathrm{inj}}$ then the structure maps $s(0,1) \longrightarrow \beta^* s(1,1)$ and $s(1,0) \longrightarrow \gamma^* s(1,1)$ are fibrations, in which β and γ are the arrows $\beta : (0,1) \longrightarrow (1,1)$ and $\gamma : (1,0) \longrightarrow (1,1)$.

Proof. By symmetricity we just need to show that the map $s(0,1) \longrightarrow \beta^* s(1,1)$ is a fibration in $\mathcal{G}(0,1)$. Suppose we are given a commutative square in $\mathcal{G}(0,1)$ of the form

$$\begin{array}{ccc} A & \longrightarrow & s(0,1) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \beta^* s(1,1) \end{array}$$

with $A \to B$ being a trivial cofibration. Take two sections s' and s'' with $s'(0,1) = A$, $s'(1,1) = \beta_! A$ and

$s'(1,0) = \varnothing$, while $s''(0,1) = B$, $s''(1,1) = \beta_! B$ and $s''(1,0) = \varnothing$. The map $A \to B$ hence builds up to a map of sections $s' \longrightarrow s''$. By construction the latter is an injective trivial cofibration. On other hand, there is a canonical map of sections $s' \longrightarrow s$ consisting of $s'(0,1) = A \longrightarrow s(0,1)$, $s'(1,1) = \beta_! A \longrightarrow \beta_! s(0,1) \longrightarrow s(1,1)$ and $s'(1,0) = \varnothing \longrightarrow s(1,0)$. Now, since s is injective fibrant there exists a map $s'' \longrightarrow s$ lifting the map $s' \longrightarrow s$. In particular, we get a map $B = s''(0,1) \longrightarrow s(0,1)$, which is indeed a lift of the above square. □

Remark B.0.0.4. Let $\mathcal{F} : [1] \times [1] \longrightarrow \text{ModCat}$ be a diagram of combinatorial model categories and assume in addition that for every morphism α in $[1]\times[1]$ the right adjoint α^* preserves weak equivalences. Let $\mathcal{R}^{\text{coc}} : \text{Sec}(\mathcal{F}|_{\mathcal{J}})^{\text{coc}} \longrightarrow \mathcal{F}(0,0)$ be the right adjoint of $\mathcal{L}^{\text{coc}}$. Since $\mathcal{L} : \mathcal{F}(0,0) \longrightarrow \text{Sec}(\mathcal{F}|_{\mathcal{J}})^{\text{inj}}$ sends cofibrant objects to coCartesian sections, it follows that for a given cofibrant object $X \in \mathcal{F}(0,0)$, the derived unit map $X \longrightarrow \mathbb{R}\mathcal{R}^{\text{coc}}\mathbb{L}\mathcal{L}^{\text{coc}}(X)$ is a weak equivalence if and only if the derived unit map $X \longrightarrow \mathbb{R}\mathcal{R}\mathbb{L}\mathcal{L}(X)$ is a weak equivalence. Now, for a section $s : \mathcal{J} \longrightarrow \int_{\mathcal{J}} \mathcal{F}$, the value of the right derived functor $\mathbb{R}\mathcal{R}(s)$ can be identified to the homtopy fiber product $\text{holim}_{i\in\mathcal{J}}\, \alpha_j^* s(i)$, as a consequence of Observation B.0.0.3. In conclusion, if $X \in \mathcal{F}(0,0)$ is cofibrant then the derived unit map $X \longrightarrow \mathbb{R}\mathcal{R}^{\text{coc}}\mathbb{L}\mathcal{L}^{\text{coc}}(X)$ is a weak equivalence if and only if the square

$$\begin{array}{ccc} X & \longrightarrow & \alpha^*_{(0,1)}(\alpha_{(0,1)})_! X \\ \downarrow & & \downarrow \\ \alpha^*_{(1,0)}(\alpha_{(1,0)})_! X & \longrightarrow & \alpha^*_{(1,1)}(\alpha_{(1,1)})_! X \end{array}$$

is homotopy Cartesian in $\mathcal{F}(0,0)$.

Bibliography

[1] D. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics. 43 (1967).

[2] D. Quillen, *On the (co-)homology of commutative rings*, Applications of Categorical Algebra, Proc. Sympos. Pure Math. 17, New York 1968, Amer. Math. Soc., Providence, R.I., p. 65–87 (1970).

[3] J. Lurie, *Higher algebra*, preprint, available at author's homepage http://www.math.harvard.edu/~lurie/ (2011).

[4] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies. 170, Princeton University Press (2009).

[5] J. Lurie, *Derived Algebraic Geometry X: Formal moduli problems*, available at http://www.math.harvard.edu/~lurie/ (2011).

[6] Y. Harpaz, J. Nuiten and M. Prasma, *The tangent bundle of a model category*, Theory and Applications of Categories. 34, p. 1039-1072 (2019).

[7] Y. Harpaz, J. Nuiten and M. Prasma, *The abstract cotangent complex and Quillen cohomology of enriched categories*, Journal of Topology. 11, p. 752-798 (2018).

[8] Y. Harpaz, J. Nuiten and M. Prasma, *Tangent categories of algebras over operads*, The Israel Journal of Mathematics. 234, p. 691-742 (2019).

[9] Y. Harpaz, *Little cube algebras and factorization homology*, Lecture notes, available at author's homepage https://www.math.univ-paris13.fr/~harpaz/.

[10] Y. Harpaz, *Lax limits of model categories*, Theory and Applications of Categories. 35, p. 959-978 (2020).

[11] J.-L. Loday and B. Vallette, *Algebraic Operads*, Springer, (2012).

[12] S. Merkulov and B. Vallette, *Deformation theory of representations of prop(erad)s. II*, J. Reine Angew. Math. 636, p. 1-52 (2009).

[13] B. Fresse, *Modules over operads and functors*, Springer, (2009).

[14] C. Barwick, *On left and right model categories and left and right Bousfield localizations*, Homology, Homotopy Appl. 12(2), p. 245-320 (2010).

[15] M. Spitzweck, *Operads, algebras and modules in general model categories*, preprint, arXiv:math/0101102, (2001).

[16] M. Batanin and D. White, *Left Bousfield localization without left properness*, preprint, arXiv:2001.03764, (2020).

[17] G. Caviglia, *A model structure for enriched coloured operads*, available at author's homepage https://www.math.ru.nl/~gcaviglia/ (2015).

[18] D.C. Cisinski and I. Moerdijk, *Dendroidal sets and simplicial operads*, Journal of Topology. 6(3), p. 705-756 (2013).

[19] J. Pridham, *Unifying derived deformation theories*, Advances in Mathematics. 224, no. 3, p. 772-826 (2010).

[20] W. G. Dwyer, D. M. Kan, and J. H. Smith, *An obstruction theory for simplicial categories*, Indagationes Mathematicae (Proceedings). 89(2), North-Holland (1986).

[21] G. Tabuada, *Postnikov towers, k-invariants and obstruction theory for dg categories*, Journal of Algebra. 321(12), p. 3850-3877 (2009).

[22] G. Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Acad. Sci. Paris Sér. I Math. 340 (1), p.15-19 (2005).

[23] G. Tabuada, *Differential graded versus simplicial categories*, Topology Appl. 157, no. 3, p. 563- 593 (2010).

[24] W. G. Dwyer and K. Hess, *Long knots and maps between operads*, Geometric and Topology. 16, p. 919-955 (2012).

[25] T. Pirashvili, *Dold-Kan type theorems for* Γ*-groups*, Mathematische Annalen. 318, p. 277-298 (2000).

[26] A. Robinson, E_∞*-obstruction theory*, Homology, Homotopy and Applications, (20.1), p.155-184 (2018).

[27] D. Pavlov and J. Scholbach, *Admissibility and rectification of colored symmetric operads*, Journal of Topology. 11 (3), p. 559-601 (2018).

[28] D. Pavlov and J. Scholbach, *Homotopy theory of symmetric powers*, Homology, Homotopy Appl. 20(1), p. 359-397 (2018).

[29] D. Yau, *Colored Operads*, AMS, Graduate Studies in Math. 170 (2016).

[30] C. Berger and I. Moerdijk, *On the homotopy theory of enriched categories*, Quarterly Journal of Mathematics. 64, p. 805-846, (2013).

[31] C. Berger and I. Moerdijk, *Resolution of coloured operads and rectification of homotopy algebras*, Contemporary Mathematics. 431, p. 31-58 (2007).

[32] C. Berger and I. Moerdijk, *On the derived category of an algebra over an operad*, Georgian Mathematical Journal. 16(1), p. 13-28 (2009).

[33] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. 78, p. 805-831 (2003).

[34] M. Hovey, *Model categories*, American Mathematical Soc. 63 (1999).

[35] J. J. Gutiérrez and R. M. Vogt, *A model structure for coloured operads in symmetric spectra*, Mathematische Zeitschrift. 270, p. 223-239, (2012).

[36] M.W. Johnson and D. Yau, *On homotopy invariance for algebras over colored PROPs*, J. Homotopy and Related Structures. 4, p. 275-315 (2009).

[37] M. Batanin and C. Berger, *Homotopy theory for algebras over polynomial monads*, preprint, arXiv:1305.0086v7 (2017).

[38] F. Muro, *Dwyer-Kan homotopy theory of enriched categories*, Journal of Topology. 8, p. 377-413 (2015).

[39] T. Lawson, *Localization of enriched categories and cubical sets*, Theory and Application of Categories. 32, (2017).

[40] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. 167 (3), p.615-667 (2007).

[41] B. I. Dundas, *Localization of V-categories*, Theory and Application of Categories. 8, p. 284-312 (2001).

[42] V. Hinich, *Dwyer-Kan localization revisited*, Homology Homotopy Appl. 18, p. 27-48 (2016).

[43] V. Hinich, *DG coalgebras as formal stacks*, J. Pure Appl. Algebra. 162, p. 209-250 (2001).

[44] P. S. Hirschhorn, *Overcategories and undercategories of model categories*, preprint, available at `http://www-math.mit.edu/psh/undercat.pdf` (2005).

[45] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, xvi+457 pp. (2003).

[46] G. Arone and V. Turchin, *On the rational homology of high-dimensional analogues of spaces of long knots*, Geometric and Topology. 18, p. 1261-1322 (2014).

[47] J. Ducoulombier, B. Fresse and V. Turchin, *Projective and Reedy model category structures for (infinitesimal) bimodules over and operad*, preprint, arXiv:1911.03890 (2019).

[48] C. Rezk, *Spaces of algebra structures and cohomology of operads*, PhD Thesis, Massachusetts Institute of Technology, (1996).

[49] C. Rezk, *Stuff about quasicategories*, Lecture notes, available at `https://faculty.math.illinois.edu/~rezk/quasicats.pdf` (2019).

[50] J. J. Gutiérrez and R. M. Vogt, *A model structure for coloured operads in symmetric spectra*, Math. Z. 270, p. 223-239, (2012).

[51] B. Fresse, *Props in model categories and homotopy invariance of structures*, Georgian Math. J. 17, p. 79-160 (2010).

[52] B. Fresse, *Modules over operads and functors*, Springer, (2009).

[53] C. Barwick, *On left and right model categories and left and right Bousfield localizations*, Homology, Homotopy Appl. 12(2), p. 245-320 (2010).

[54] M. Spitzweck, *Operads, algebras and modules in general model categories*, preprint arXiv:math/0101102, (2001).

[55] W.G. Dwyer and D.M. Kan, *Function complexes in homotopical algebra*, Topology 19(4), p. 427-440 (1980).

[56] J. Milles, *Andre-Quillen cohomology of algebras over an operad*, Advances in Mathematics. 226.6, p. 5120-5164 (2011).

[57] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47, p. 269-331 (1977).

[58] S. Schwede and B. Shipley, *Equivalences of monoidal model categories*, Algebraic and Geometric Topology. 3, p. 287-334 (2003).

[59] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics. 38, Cambridge University Press, Cambridge (1994).

[60] P. Goerss and R. Jardine, *Simplicial homotopy theory*, Springer. 174 (2009).

[61] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. 359, p.2043-2058 (2007).

[62] S. Yalin, *Moduli stacks of algebraic structures and deformation theory*, J. Noncommutative Geometric. 10, p. 579-661 (2016).

[63] A. Berglund, *Rational homotopy theory of mapping spaces via Lie theory for L_∞ algebras*, Homology, Homotopy Appl. 17(2), p.343-369 (2015).

[64] E. Balzin, *Reedy model structures in families*, preprint arXiv:1803.00681v2, (2019).