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How to do the nature and the structure of information affect the optimal pricing algorithm to guarantee market efficiency and minimize fundamental prices volatility?

David Batista Soares

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Normandie Université

THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHÉMATIQUES

Préparée au sein de l'Université de Caen Normandie

How to do the nature and the structure of information affect the optimal pricing algorithm to guarantee market efficiency and minimize fundamental prices volatility ?

**Présentée et soutenue par
David BATISTA SOARES**

**Thèse soutenue le 12/11/2020
devant le jury composé de**

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UNIVERSITÉ
CAEN
NORMANDIE



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Résumé en français de la thèse

Les récents changements majeurs dans la structure des marchés financiers, réactualisent l'étude de l'efficacité des marchés financiers ainsi que celle de la volatilité des prix. Le travail conceptuel de cette thèse se concentre sur la modélisation de la dynamique des prix sur les marchés financiers. En particulier, nous examinons les problèmes posés par les formes institutionnelles d'organisation de marché, notamment la fragmentation des ordres, ou encore le nombre d'agents économiques présents sur les marchés qui influencent *in fine* la nature et la structure de l'information.

Ces travaux ont été menés avec le prisme de la littérature économique, mais aussi avec l'aide de la théorie de l'information très utilisée en sciences informatiques et ce qu'on appelle aujourd'hui l'éconophysique. Nous discutons également les propositions de l'analyse économique visant à améliorer la qualité informationnelle qui sous-tend la formation des prix à partir des deux types de modèles qui ont été construits. Le point de départ de ce travail est l'étude de l'information diffusée sur les marchés financiers. À cet effet, nous distinguons l'information qui peut être traitée à l'aide d'un algorithme (appelé information effective) de l'information qui ne peut l'être. Nous considérons également que sur certains marchés, comme les marchés de matières premières agricoles standardisées (appelées par l'anglicisme *commodités*) par exemple, le prix physique, et donc l'information qu'il se doit en principe de contenir, n'est pas actualisée aussi fréquemment que le cours des actions ou les cours des devises. En effet, la plupart des cours des actions sont mis à jour presque en permanence, mais sur les marchés de *commodités* agricoles, le prix est mis à jour relativement lentement. Ensuite, nous pouvons considérer l'information pertinente sur ces marchés comme n'étant pas continue (les données météorologiques par exemple). Dans ce travail, nous étudions la nature et la structure de l'information afin de mieux comprendre la structure actuelle des marchés financiers, et leurs implications.

La typologie des actifs que nous utilisons dans ce travail repose sur la tangibilité ou non de celui-ci. Elle nous permet notamment de définir un marché financier comme un lieu où s'échangent des actifs intangibles (voir définition 1 et 2). Sur ces marchés financiers, les agents économiques passent des ordres d'achat ou de vente à un prix fixé. Ce sont ces ordres qui permettent l'émergence d'un prix de marché qui est donc défini par le prix résultant d'au moins une transaction, qui elle-même est générée par l'appariement d'un ordre d'achat et d'un ordre de vente à des prix compatibles (voir définition 3). Un algorithme étant défini par une séquence finie d'opérations ordonnées (voir définition 4), une transaction est exécutée par un algorithme qui apparie les différents ordres. Le prix de marché sur un marché financier est donc déterminé par un algorithme qui a pour point de départ les ordres des agents. Le prix de marché est un concept a priori universel. Cependant, il dépend des règles d'échanges ou de transactions spécifiques à chaque marché, et plus particulièrement sur la méthode utilisée pour déterminer le prix (donc l'algorithme utilisé). Ainsi, le prix de marché du même actif peut être déterminé différemment sur Euronext Paris et à la Bourse de Londres.

En pratique, sur la plupart des marchés organisés gérés par des entreprises de marchés, deux algorithmes de détermination des prix sont utilisés: la détermination grâce au carnet d'ordres à cours limités, dénommé **Limit Order Book (LOB)** en anglais et le *fixing*. Depuis le début de l'utilisation de l'algorithme du LOB, qui est bien plus récent, certains économistes célèbres ont mis en exergue son potentiel effet déstabilisateur.¹

¹A titre d'illustration, nous pouvons citer la réponse partielle au mini-crash du 13 octobre 1989 de Maurice Allais, prix Nobel d'économie en 1988:

"Le marché en transactions continue est une aberration d'un point de vue économique et génère une potentielle instabilité permanente favorisant la fraude et la manipulation de

Les deux principaux algorithmes de détermination du prix actuellement utilisés sur les marchés réglementés sont basés sur la centralisation des ordres d'achat et des ordres de vente dans un carnet d'ordres. Ce dernier regroupe, à un moment donné, les ordres d'achat et ordres de vente exécutables mais non exécutés (voir définition 5).

Le fixing, qui est devenu la méthode la moins utilisée, détermine le prix qui maximise les quantités échangées un nombre de fois fixe par jour. Par conséquent, les quantités échangées ont un impact direct sur le prix de marché dans cet algorithme.

Le LOB quant à lui, exécute une transaction à l'instant ou un ordre est compatible avec un ordre existant. Le prix qui a permis cette transaction est immédiatement rendu public et devient le prix de marché jusqu'à la prochaine transaction. Il est important de souligner que la fréquence d'évolution du prix de marché est donc a priori inconnu si le LOB opère sur un marché.

Nous nous concentrons maintenant sur la régulation des marchés financiers, établie par les autorités nationales et supranationales.² Ces institutions visent à promouvoir la transparence de l'information et "le bon fonctionnement des marchés financiers" et de rendre plus profonds les marchés financiers afin de garantir un haut niveau de liquidité (voir définition 6). La liquidité revêt donc deux dimensions: la capacité à échanger immédiatement, qui augmente avec la fréquence des transactions, mais aussi la capacité à échanger la quantité souhaitée sans subir de variation de prix, qui augmente avec la profondeur de marché. Plus un marché est profond, plus il sera *a priori* capable d'absorber des ordres de grosses quantités sans subir de variation de prix.

L'un des plus grands consensus en économie financière repose sur les caractéristiques d'un marché efficient (voir Fama [15], Fama et al. [16]). Un marché est dit efficient si la dynamique de son prix intègre instantanément toute l'information disponible. La valeur fondamentale d'un actif à un moment donné est égale au meilleur et immédiat traitement de toutes les informations disponibles à ce même moment (voir définition 7). C'est cette définition de la valeur fondamentale qui nous permet de poser la définition suivante: sur une période donnée, un marché est dit efficient si le prix d'un actif est au plus près de sa valeur fondamentale (voir définition 8). Cette définition nous fait prendre conscience que ces travaux ne peuvent être déconnectés de l'étude de l'information économique (structure, quantité, fréquence, etc.) et de son évolution. Dans son travail, Fama introduit trois niveaux d'efficience de marché différents:

- La forme d'efficience faible qui rend toute variation de prix de marché impossible à prévoir grâce aux prix passés;
- La forme d'efficience semi-forte qui implique que le prix de marché contient l'ensemble de l'information publique (incluant l'information passée);
- La forme d'efficience forte qui implique que le prix de marché contient l'ensemble de l'information publique et privée.

marché. [Les marchés américains pourraient être améliorés] en éliminant le mécanisme de transaction en continu et en le remplaçant partout par un unique et quotidien prix de marché pour chaque actif dans chaque marché." – M. Allais (1989)

²Dans l'Union européenne, l' [European Securities and Markets Authority \(ESMA\)](#) est en charge du contrôle du respect de ce règlement. Dans le cadre donné par l'[ESMA](#), l'[Autorité des Marchés Financiers \(AMF\)](#) "assure la protection de l'épargne, l'information des investisseurs et le bon fonctionnement des marchés" en France.

La structure des marchés financiers est le résultat de deux composantes hiérarchiques. Premièrement, les institutions définissent un cadre général (voir infra pour la description des institutions au niveau de l'Union européenne). Deuxièmement, la décision des entreprises de marché sur l'organisation des marchés, ainsi que pour les marchés gérés par un [Authorized Investment Services Provider \(AISP\)](#).

L'objectif des entreprises de marché est de générer un profit maximum, tout en respectant la réglementation; Ce profit est essentiellement proportionnel au nombre de transactions (n.o.t) effectuées puisqu'elles facturent une commission par transaction ou une commission pour accéder au marché. Par conséquent, elles ont aussi un intérêt à ce que les marchés financiers soient liquides. Dans un marché efficient, le prix de marché doit changer à chaque nouvelle information. La capacité d'échanger immédiatement l'actif doit être absolue si l'on ne connaît pas a priori la fréquence d'arrivée de l'information. Une forte liquidité permet à l'acheteur ou au vendeur qui intègre les nouvelles informations dans le prix de son ordre, de trouver une contrepartie sans délai. Cela équivaut à considérer qu'au moins un acheteur et un vendeur doivent réagir simultanément aux nouvelles informations et sans délai en passant des ordres compatibles. À ce titre, le [LOB](#) est donc l'algorithme de détermination du prix de marché qui maximise la liquidité et est privilégié par les entreprises qui organisent les marchés financiers.

La multiplicité des marchés pour un même actif conjugués à des algorithmes de détermination de prix parfois différents sont une des sources d'apparition des stratégies de fragmentation des ordres. Cette dernière se définit par un découpage d'un ordre dit parent en plusieurs sous-ordres dits enfants (voir définition 9).

La fragmentation des ordres (ou fragmentation lorsqu'il n'y a pas d'ambiguïté) peut être effectuée sur un même marché dans le temps, et en même temps sur différents marchés ou simultanément dans le temps sur différents marchés. Elle peut être volontaire ou contrainte. La réduction des coûts de transactions grâce aux avancées technologiques par exemple, permet aux agents économiques de - volontairement - fragmenter leurs ordres. Cependant, sur un marché opérant avec un [LOB](#) où la profondeur du marché est relativement insuffisante (c'est-à-dire si la taille des ordres limites n'est pas suffisante), si un agent économique propose d'acheter ou vendre une grande quantité de l'actif, le prix évoluera. On dit alors que l'ordre a un impact sur le prix. Cet impact diminue lorsque la profondeur de marché augmente. La fragmentation des ordres apparaît comme une contrainte pour les agents économiques qui souhaiteraient échanger des relativement grandes quantités de l'actif. En d'autres termes, moins le marché est profond, plus un agent économique voulant vendre ou acheter à un prix proche du prix de marché actuel doit fragmenter son ordre. Les conséquences d'une telle pratique sont une diminution des quantités échangées par transaction et une augmentation du nombre de transactions (on utilisera l'abréviation [Number of transactions \(n.o.t\)](#) dans la suite de ce travail).

Sur les marchés financiers modernes cependant, l'information disponible est en partie contenue dans le carnet d'ordres, continuellement mis à jour (voir Cont and Larrard [\[8\]](#)). En plus de l'information passée et présente l'information disponibles aux opérateurs sur les marchés financiers, le carnet d'ordres est une indication sur le souhait d'achat et de vente des opérateurs;

Cette vision du carnet d'ordres comme vecteur d'information s'inscrit également dans le cadre du travail de Stiglitz qui déclare que “ (...) les actions ou choix [des acteurs] véhiculent de l'information. Les acteurs du marché le savent et agissent en conséquence.” - Stiglitz [\[28\]](#), p.468. Certaines formes de marchés hybrides, que nous n'aborderons pas dans cette thèse ont vu le jour pour permettre aux agents économiques de camoufler cer-

taines informations aux autres acteurs du marché, tout en passant des ordres.

La régulation sur les marchés financiers doit donc avoir plusieurs sous-objectifs permettant d'atteindre l'efficience:

- La transparence de l'information pertinente pour la détermination du prix de marché. Sans cela, il est impossible pour les agents économiques de l'incorporer au prix de marché à travers leurs ordres;
- La liquidité quant à elle, doit être suffisamment élevée pour permettre à toute nouvelle information d'être incorporée immédiatement au prix de marché à travers une transaction sur un marché fonctionnant en **LOB**. La séquence chronologique des changements de prix doit idéalement être superposée sur la séquence chronologique d'arrivée d'information nouvelle.

Dans ce travail, nous accordons une attention particulière aux commodités agricoles qui sont le sous-jacent de contrats standardisés (nous utiliserons l'anglicisme *commodity* agricole pour désigner une matière premières standardisée qui est sous-jacent d'au moins un contrat standardisé). Historiquement, ce sont leur capacité à être stockées et la saisonnalité de leurs récoltes qui ont permis le développement d'actifs financiers autour de certaines d'entre-elles (voir définition 10).

Les marchés de dérivés de matières premières agricoles sont intrinsèquement différents des autres marchés dérivés. Le sous-jacent étant un actif tangible, cela implique des coûts de transactions différents par rapport aux devises notamment, qui sont des sous-jacents intangibles. Dans le cas d'un actif tangible, les coûts de transactions représentent le coût à trouver un courtier et le payer pour s'assurer que la transaction ait lieu. Dans le cas d'un actif intangible, les coûts de transactions représentent les coûts d'accès au marché et de placement de l'ordre. Sur ce que nous définissons comme les marchés de commodités agricoles (cf. définition 11), quatre principaux types de contrats sont échangés:

- Le contrat forward qui est un contrat où l'on s'engage à acheter ou vendre la commodité à une date future. Les caractéristiques de ce contrat (date, transport, qualité, quantité) sont établies par les deux contractants. Cet actif financier est négocié sur les marchés de gré à gré et leur négociation est souvent effectuée par le biais d'un courtier;
- Le contrat futures qui est un contrat où l'on s'engage à acheter ou vendre la commodité à une date future. Les caractéristiques de ce contrat (date, transport, qualité, quantité) sont standardisées. Chaque position ouverte sur ce type de contrat doit être suivie d'un dépôt de la part de l'agent économique, auprès de la chambre de compensation. Chaque jour, le rendement du contrat à terme est payé par l'opérateur perdant à la chambre de compensation et reçu par l'agent perdant auprès de la chambre de compensation. Ce mécanisme se nomme l'appel de marge. À une même date, plusieurs contrats à termes sont simultanément disponibles, pour plusieurs maturités. Bien entendu, les prix de ces contrats sont liés par la structure par terme;
- Le contrat swap qui est un contrat où les deux contractants s'entendent pour échanger le flux d'argent différentiel sur deux instruments financiers portant sur une commodité agricole ou plusieurs;

- L'option qui est un contrat donnant à l'acheteur un droit spécifique. Les options les plus connues sont l'option put et l'option call. Le put donne le droit de vendre à un certain prix tandis que le call donne le droit d'acheter à un certain prix.

Alors que les contrats forward se terminent souvent par une livraison physique de la commodité concernée, les contrats futures, eux, se terminent très majoritairement par le versement du flux financier de l'agent économique perdant à la chambre de compensation, et de la chambre de compensation vers l'agent économique gagnant (le contrat futures étant un contrat dont les flux financiers sont à somme nulle). En effet, Working [30] indiquait déjà en 1953 que "l'échange de contrats à termes peut se définir comme des échanges effectués sous certaines régulations et conventions, plus restrictives que celles appliquées à n'importe qu'elle autre classe de transactions sur les commodités, qui servent en premier lieu à faciliter la couverture du risque et la spéculation d'une facilité remarquable (...)" (p.315). Le bénéfice d'un contrat standardisé tel que le futures est donc sa liquidité, ou tout du moins, la fréquence à laquelle on peut l'échanger. Il est donc plus facile pour n'importe quel acteurs subissant le risque de prix sur une commodité de céder ce risque de prix. C'est l'essence même de la création des premiers contrats futures créés à Chicago. La fréquence dans les changements de prix du contrat futures est la raison pour laquelle le prix à terme est souvent considéré comme le prix de référence (on entend par là, le prix que les agents considère comme contenant le plus d'information).

Sur les marchés de commodités, la fragmentation des ordres peut également intervenir (autant sur les marchés physiques que les marchés dérivés). On peut également concevoir que si les coûts de transactions diminuent, certains opérateurs fragmenteront davantage leurs ordres. Néanmoins, la dimension intrinsèquement différente du marché physique et du marché futures impliquera certainement une fragmentation différente et donc une modification de la série de prix sur ces marchés, malgré une relation fondamentalement étroite entre ces deux prix. Ce raisonnement conduit donc à supposer une évolution de la dynamique de la base sur de tels marchés, qui représente pour une maturité donnée, l'écart entre le prix futures à cette même maturité et le prix spot (voir définition 12). La probabilité pour que la fragmentation conduise à une évolution proportionnelle des fréquences de transaction sur ces deux marchés étant assez faible, on peut imaginer qu'un changement de stratégie de trading provoquera une évolution de la fréquence d'échange relative (appelée [Relative Trades Frequency \(RTF\)](#) dans ce manuscrit) ainsi que des quantités échangées à chaque transactions sur ces marchés. Un des objectifs centraux de ce travail est d'étudier l'impact de tels changements.

Les marchés de commodités sont l'objet d'institutions spécifiques comme par exemple le Système d'Information sur les marchés agricoles ([Agricultural Markets Information System \(AMIS\)](#)) qui sont une réponse institutionnelle à la forte hausse des prix des commodités de 2007/2008. L'Union Européenne a également composé un Groupe d'Expert sur les Dérivés et Marchés Spot de Commodités Agricoles ([Expert Group on Agricultural Commodity Derivatives and Spot Markets \(EGDSM\)](#)). Leurs objectifs sont de permettre l'efficacité des marchés et de mettre à disposition des informations permettant de réduire la volatilité des prix, mais aussi celle de la base, un déterminant fondamental des décisions de productions des agriculteurs (cf. Moschini and Hennessy [24], Bégue Turon et al. [5]).

Une des principales particularités des marchés agricoles est que le prix de marché du sous-jacent des contrats futures n'est pas déterminé de la même façon que le prix de marché du contrat futures. Ce dernier est déterminé à l'aide d'un [LOB](#), en continu, ce

qui n'est bien sûr pas le cas de la commodité sur le marché spot. On comprend immédiatement que cela provoque une dynamique de la base, l'écart entre ces deux prix, particulière sur ce point. L'information pertinente sur les marchés de commodités agricole concerne les prévisions faites par les institutions sur les futures récoltes mais aussi les prévisions météorologiques. Non seulement la fréquence d'arrivée de ces informations est fixe, mais aussi connue à l'avance. Les caractéristiques de l'information pertinente sur les marchés de commodités peut donc amener à modifier la perception de l'efficience du marché sur ces marchés. Pour finir, le niveau des stocks joue également un rôle important sur la dynamique de la base. Ce dernier étant directement impacté par la fréquence des transactions sur le marché physique, une évolution de la fréquence des transactions sur le marché physique influe au moins par deux canaux la dynamique de la base. Se sont ces différences de fréquences de transaction, ainsi que leurs rétro-actions sur le marché de la commodité auxquelles nous nous intéressons principalement dans ce travail.

Dans la plupart des travaux économiques contemporains, le cadre théorique du prix d'équilibre est utilisé pour modéliser la formation du prix. Au sein de ce cadre théorique, le prix d'équilibre se détermine en égalisant l'offre et la demande agrégés (définition 13). Cette détermination du prix, unique dans le cadre originel Walrasien, donne lieu à des transactions simultanées lorsque le prix est déterminé. Ce cadre originel efface donc totalement la notion de dynamique. L'information passée, présente et future est connue par l'ensemble des agents et est parfaitement *traitée* avant que les échanges aient lieu. De nombreux autres cadres théoriques ont été développés mais ont conservé l'idée que le prix se détermine avant que les transactions n'aient lieu, ces derniers reposant toujours sur le concept de l'équilibre. Certaines études ont également porté sur la convergence du prix vers le prix d'équilibre un des aspects de la dynamique des prix (cf. Ezekiel [14], Guesnerie [17]). Ces travaux conservent néanmoins l'idée d'une fréquence des transactions fixe.

Les travaux de Fama et al. [16] et Fama [15], peuvent ainsi être perçus comme l'extension de ce raisonnement aux marchés financiers. La définition d'un marché efficient peut être interprétée comme le passage à la limite d'un tel raisonnement. Si une période correspond à la plus petite unité de temps possible, le prix doit refléter à chaque unité de temps, l'ensemble de l'information disponible affectant l'offre et la demande agrégés. Dans ce raisonnement, Fama relâche l'hypothèse de fréquence des transactions fixes puisque la dynamique du prix doit être déterminée de façon endogène par la fréquence d'arrivée de l'information nouvelle. Dans le même temps, aucune spécification n'est apporté sur les moyens de déterminer le prix de marché. L'efficience d'un marché est donc appréhendée par la qualité de sa production: le prix de marché.

Hayek [18] quant à lui, porte une attention toute particulière au procédé de formation du prix de marché. En effet, ce sont les agents économiques, par leurs interactions, qui permettent au prix de marché de véhiculer de la meilleure façon possible, l'information disponible affectant la demande agrégée et/ou l'offre agrégée, présente et future. Nous pouvons extraire de ce raisonnement que plus il est facile d'exprimer un vœu marchand et de connaître le vœu des autres agents économiques, plus le prix de marché sera efficace dans la découverte du prix. Cette idée est précisément celle que nous appelons dans ce travail la force computationnelle. L'organisation de la société avec l'aide des marchés permet l'interaction marchande entre les agents à travers leurs souhaits d'acheter ou de vendre. Un prix de marché émergera tel qu'il véhiculera mieux l'information en moyenne sur une durée spécifiée que n'importe lequel des agents prix séparément. Toutefois, dans son travail, Hayek ne spécifie pas la séquence finie des opérations menant à la détermi-

nation. Finalement, aucun algorithme de détermination du prix n'est ici donné.

Plus récemment, le champs de recherche de la microstructure des marchés dont Madhavan [23] a rétrospectivement daté le commencement au début des années 1980, a produit des résultats très intéressants quant à la formation des prix et le rôle de l'information. Parmi ces champs de recherche, la compréhension de la formation du prix a mené les économistes à de nombreuses études sur l'écart achat vente (c'est-à-dire le prix d'achat proposé le plus élevé et le prix de vente le plus bas), mais aussi l'impact des [Market Maker \(MM\)](#). La plupart de ces études considèrent néanmoins le prix de marché comme étant un prix d'équilibre (voir Biais et al. [2]).

La comparaison de la théorie avec la formation des prix de marché sur les marchés financiers

La vision théorique consensuelle de la dynamique des prix peut apparaître comme souffrant de contradictions avec la réalité sur deux principaux aspects. La première concerne l'égalisation entre l'offre agrégée et la demande agrégée, qui ne peut être vérifiée par définition si l'algorithme du [LOB](#) est utilisé. La détermination du prix par fixing en revanche, repose sur un algorithme de maximisation des quantités échangées sur l'ensemble des ordres compatibles. Ce dernier constitue donc la méthode de détermination du prix la plus proche de la perception théorique de la détermination du prix de marché.³ La détermination du prix par le [LOB](#) étant majoritaire sur l'ensemble des marchés financiers, ce dernier est davantage compatible avec les caractéristiques d'un marché efficient, la fréquence d'évolution du prix n'étant pas fixée. Le second aspect repose sur le concept même de dynamique. La théorie de l'équilibre général considère un prix de marché unique, ou une dynamique dont la fréquence est fixée. La théorie des marchés financiers appréhende une dynamique des prix contingente à l'arrivée d'information. Au contraire, la dynamique d'un marché dont le prix de marché est déterminé selon le [LOB](#) est conditionné à la dynamique des transactions.

D'un autre côté, le carnet d'ordre est un moyen de faciliter les interactions marchandes, qui plus est parce qu'il est accompagné d'une assurance d'exécution à la meilleure offre. Le raisonnement d'Hayek ne spécifiant pas l'égalisation de l'offre et de la demande agrégée unique ou périodique, et ne présupant pas de fréquence fixe de transaction dans son raisonnement, il semblerait que sa vision soit en adéquation avec les caractéristiques actuelles des marchés financiers, peu importe l'algorithme de détermination de prix utilisé. La dynamique d'un marché financier est donc par conséquent identifiée par la série de prix de marché qu'il propose. Cette série de prix peut donc être considérée comme la production, la résultante d'un marché financier organisé par un teneur de marché, considéré comme le producteur de l'algorithme de prix utilisé, et de l'organisation du marché ou s'expriment les souhaits marchands. Ce raisonnement a pour conséquence d'inverser la causalité de la dynamique du prix de marché. Alors que la théorie économique appréhende un processus permettant l'émergence d'un prix d'équilibre en tant que prix de marché qui permettra à toutes les transactions d'avoir lieu à ce prix donné, nous nous retrouvons dans ce travail, sur une causalité inverse. Les transactions ont lieu et génèrent une variation de prix de marché. En fait, les agents économiques qui passent des transactions ne peuvent finalement pas être considérées comme purement "preneurs de prix", puisqu'ils influencent voire déterminent complètement le prix de marché en procédant à

³En fait, c'est la détermination du prix de marché par fixing qui inspira Walras dans sa représentation du prix d'équilibre (cf. Dupuy [11]).

l'expression de leurs vœux marchands. Chaque opérateur a par conséquent une probabilité non-nulle de devenir “faiseur de prix”, lorsque le nombre d’opérateur sur un marché est fini.

Le rôle clé de l’information sur les marchés financiers

En économie financière, la littérature est principalement divisée autour de deux courants: La finance classique et la finance comportementale. Le premier, repose sur l’hypothèse des marchés efficient. Sur de tels marchés, il n’y a pas d’arbitrage possible. Si les agents sont parfaitement capable d’identifier toute opération d’arbitrage, et que les coûts de transactions sont nuls, toute nouvelle information sera suivie d’une transaction. Le prix de marché reflétera alors parfaitement et instantanément l’information disponible. Le second courant a pour principale question de décrire les biais psychologiques dont peuvent faire preuve les agents économiques et d’exprimer la conséquence que cela peut avoir sur les marchés financiers. Mais finalement, ces deux courants ont tous deux le même point commun qui est que chaque actif a une valeur fondamentale (définition 7). Cette dernière est empiriquement et conventionnellement considérée comme étant égale au flux de trésorerie dégagés par l’actif actualisé à l’instant t . Alors que ces deux courants sont souvent présentés comme étant incompatibles, le second chapitre de cette thèse est construit sur l’idée qu’un cadre plus général peut être construit en partant non pas du comportement des agents mais bien la nature et la structure même de l’information qu’ils utilisent. Des recherches avaient déjà été menées en ce sens par Lo [22]. Dans son cadre théorique nommé l’hypothèse des marchés adaptatifs, les agents évoluent et sont capables de constater leurs erreurs a posteriori. Ils corrigent ainsi leurs comportements en espérant atteindre l’efficience. L’idée clé dans ce travail est que les agents ont un comportement évolutif et qu’ils sont poussés à s’adapter pour “survivre” sur le marché. Ce travail intègre un tel processus évolutif en considérant la survie comme étant la capacité à exécuter un ordre, à travers les outils offerts par les sciences informatiques. La partition de l’information utilisée dans ce travail repose sur sa capacité à être traitée par le biais d’un algorithme (information dite effective) ou non (information dite non effective). Cette partition donne lieu à la définition 15. La finance classique considère que les agents sont parfaitement rationnel et de façon latente, que l’information est parfaitement effective. Au contraire, la finance comportementale considère que toute l’information est traitée par les agents eux-mêmes, et donc qu’elle est totalement non effective.

Les sources d’information se sont ces dernières années multipliées, posant le problème de la caractérisation des marchés financiers. Actuellement, la littérature économique distingue les marchés financiers efficient de ceux qui ne le sont pas. Toutefois, un marché non efficient est-il nécessairement un “mauvais” convoyeur d’information par rapport aux agents pris séparément ? Cette simple question demande toutefois de définir ce que l’on entend par “mieux” convoyeur l’information que chaque agent séparément. Dans le sillage des travaux d’Hayek, nous définissons cela sous le terme de force computationnelle du marché (définition 14, puis 22). Un marché dans une telle configuration impliquerait qu’un marché non efficient ne serait pas détecté par les agents ou alors qu’ils ne parviennent pas à l’exploiter. Dans le cas contraire, cela signifierait que l’on pourrait laisser le/les agents qui véhiculent davantage d’information définir le prix de marché. Le modèle développé dans le chapitre 2 démontre l’importance du souhait d’échanger des agents économiques (i.e. leurs ordres) sur la détermination de la valeur d’un actif (voir Schaden [26]). Ce modèle permet de prendre en compte les agents comme “preneur de prix” mais aussi comme “faiseur de prix” avec une certaine probabilité (voir Choustova

[6]). Alors que la littérature des modèles basés sur les agents constitue aujourd'hui, la référence pour prendre en compte les interactions entre agents, le modèle construit ici permet de considérer une interaction particulière. Non seulement chaque état du marché est déterminé par l'ensemble des agents mais les anticipations des agents seront en retour, impactées par l'état du marché, dans un cycle sans fin. En revanche, considérer que l'information arrive continuellement est une hypothèse restrictive au regard de la réalité sur certains marchés. Sur les marchés de commodités par exemple, une grande partie de l'information peut être considérée comme exogène d'une part mais aussi relativement peu fréquente par rapport à celle des marchés classiques les plus liquides. Ainsi, la double typologie de l'information utilisée dans ce travail est donnée dans le tableau suivant:

	Exogène	Endogène
Effective		
Non effective		

Les faits stylisés influençant l'information sur les marchés financiers

Un ensemble de faits ou pratiques influence l'arrivée de l'information (rythme, fréquence, nombre de sources...). À ce titre, la fragmentation des ordres est un challenge théorique puisque le seul fait d'en considérer tous ses aspects empêche tout raisonnement basé sur un prix d'équilibre. Une étude d'un tel phénomène peut être généralisée à l'étude d'une variation dans les fréquences de transactions, une des dimensions de la liquidité. Une telle question est encore plus intéressante lorsque l'on sait que la réponse des prix à une variation des quantités n'est pas linéaire comme l'on montré Kempf and Korn [20] puis Tóth et al. [29]. Puisque envoyer un ordre dans un carnet d'ordre véhicule de l'information, les prix réagissent également aux quantités échangées. La fragmentation des ordres est contingente à certaines caractéristiques de marché. Le travail empirique de Ranaldo [25] sur les marchés Suisses souligne notamment que plus le côté opposé du carnet d'ordre par rapport à la position de l'agent est étroit, plus les agents agressifs vont placer des ordres de grande taille. Il montre également que l'agressivité des agents économiques augmente lorsque la période est particulièrement volatile mais aussi lorsque l'écart entre la meilleure offre et la meilleure demande est élevé. L'étude empirique de Comerton-Forde and Tang [7] sur les marchés financiers Australien souligne également que l'agressivité des ordres a diminué avec l'anonymisation des ordres passés, en même temps que la profondeur des marchés s'est améliorée. Pourtant la taille moyenne des ordres a elle augmenté. Cela peut s'expliquer par le fait que les agents sont plus enclin à dévoiler leurs vraies préférences puisqu'elles sont rendues transparentes mais de façon anonyme. Il est donc complexe d'identifier qui passe une même série d'ordre.

Ainsi, l'impact de la taille des ordres a également été étudié. Le travail empirique majeur de Easley and O'hara [12] montre que la taille des ordres peut être expliquée par une sélection adverse. Les agents les mieux informés consentent plus facilement à passer des ordres de grande taille à un prix donné lorsqu'ils jugent que cela est pertinent (c'est-à-dire lorsque leurs informations induit une variation de prix de marché significative). Ce résultat est cohérent avec le travail de Biais et al. [1] qui montre que des changements de

la meilleure offre et de la meilleure demande ont lieu suite à des transactions de grande taille. Ainsi, les agents ont intégré le fait que les transactions de grande taille véhiculent beaucoup d'information et ils ajustent leurs positions en fonction de ces transactions. Une extension de cela réside dans le fait que la fréquence des transactions véhicule également de l'information (dans une organisation de marché où les fréquences de transactions sont libres). Une étude empirique de Easley et al. [13] montre notamment que les agents extraient de l'information des intervalles à fréquence de transactions élevées mais aussi des intervalles de temps où les transactions sont absentes. Ces deux dimensions font partie de la relation volatilité-volume définie comme la relation croissante entre la volatilité des prix et les volumes échangés. Finalement, l'étude de l'impact de la fragmentation des ordres fait émerger la question de l'existence d'une fréquence optimale des transactions. Jusqu'ici, cette question a été posée par Du and Zhu [10]. Néanmoins, ce travail théorique est microfondé par un cadre d'enchère double, qui suppose que l'ensemble des souhaits des agents sont centralisés par un commissaire priseur qui détermine ensuite un prix d'équilibre qui égalise l'offre et la demande. Toutefois, ce cadre théorique se réfère davantage à un fixing comme nous l'avons déjà expliqué, et l'information concernant la valeur de l'actif est exogène au marché. Les travaux développés ici ont pour but de comprendre l'impact des fréquences de transaction lorsque celles-ci évoluent en considérant en particulier, que la fréquence des transactions et la taille des ordres sont source d'information. Le même raisonnement s'applique aux marchés de matières de commodités et jusqu'ici, les spécificités des marchés agricoles n'ont pas été prises en compte dans l'étude de cette question.

L'atomicité des agents est une des conditions de la compétition pure et parfaite. Si le marché est bien atomistique, les agents n'ont aucun pouvoir de marché et par définition, aucun impact individuellement sur le prix. Ils sont alors dit "preneur de prix". Toutefois, l'atomicité peut être interprétée différemment lorsque l'on considère que les actions des agents véhiculent de l'information. Si un nombre fini d'agents est présent sur un marché financier, tel qu'il n'y a pas d'atomicité, on peut de façon équivalente considérer que chacun des agents pris séparément véhicule une information non redondante comparée à celle fournie par les autres agents. En revanche, une infinité d'agents sur un marché financier ou l'atomicité serait vérifiée impliquerait réciproquement que chaque agent véhicule une information redondante comparée à l'information véhiculée par les autres agents. En d'autres termes, la considération de l'absence d'atomicité sur les marchés financiers est également en relation avec l'étude de l'information véhiculée par les agents selon les deux dimensions que nous avons introduites. Le nombre d'agents impactera l'information effective et non effective qui sera véhiculée de façon endogène par rapport aux fréquences de transactions qui évolueront également.

Ainsi, la crise financière de 2008 a été suivie par un pic des prix des commodités en particulier énergétique et des principales commodités agricoles. L'hypothèse de Masters est que l'entrée massive des investisseurs institutionnels et par extension l'affluence d'une soudaine liquidité considérable ont fait monter le prix de ces commodités de façon artificielle. Un autre moyen de comprendre ce problème est qu'un choc soudain dans le nombre d'agents économiques a pu altérer l'efficacité des marchés selon leur impact sur la fréquence et la taille des transactions effectuées. Si l'on se réfère une fois de plus aux résultats de Du and Zhu [10], plus les agents sont nombreux sur le marché, plus la fréquence de transaction optimale sur le marché est élevée. Néanmoins, la fréquence de l'information est également impactée par une évolution de la structure du marché financier sous le paramètre du nombre d'agents. La fréquence d'arrivée de l'information

peut donc être considérée comme endogène sur les marchés ou l'arrivée d'information ne peut être jugée comme continue. Ainsi, les chapitres de cette thèse s'articulent autour de la problématique suivante:

Comment la nature et la structure de l'information affectent-elles l'algorithme de détermination du prix de marché optimal garantissant l'efficacité des marchés et une volatilité fondamentale minimale ?

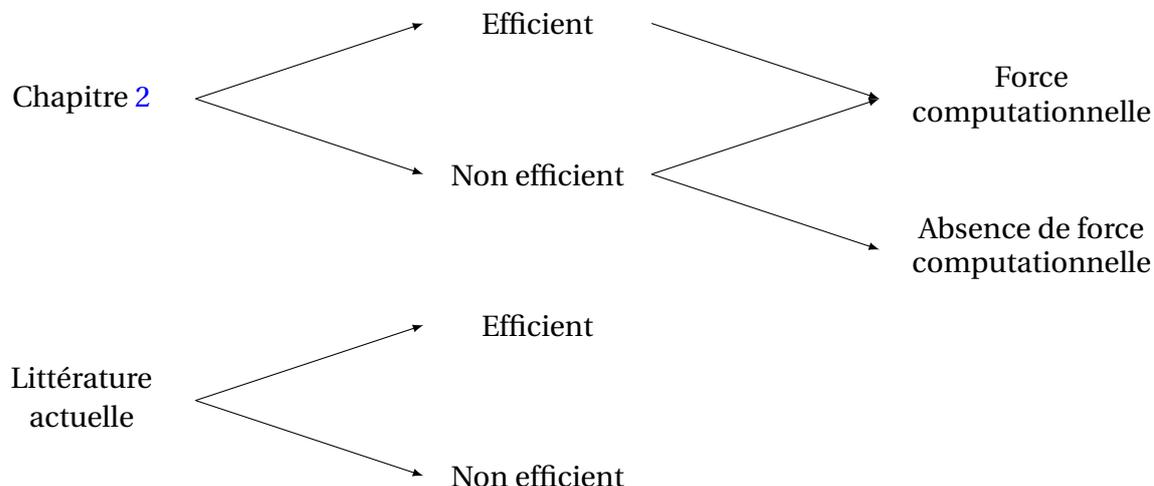
Chapitre 2: La nature et la structure de l'information évolue: Les marchés financiers sont-ils toujours efficaces ?

Le chapitre 2 de cette thèse présente une extension du modèle de Bretto et Priolon qui vise à étudier l'impact d'une information arrivant continuellement et de façon exogène, qui peut être effective ou non, sur un marché financier opérant grâce à un LOB. Cette approche bénéficie de certaines propriétés du formalisme de la physique quantique et surtout concernant les outils statistiques utilisés. Nous modélisons des agents indépendants qui influencent par les ordres qu'ils passent dans le carnet d'ordre, le prix de marché (au minima la distribution du prochain prix de marché) et chaque prix de marché en retour, influence les agents uns à uns dans un processus sans fin. Dans ce travail, les typologies de l'information utilisées sont données dans le tableau suivant:

	Exogène	Endogène
Effective	X	
Non effective	X	

Ce travail est compatible avec le cadre de pensée développé par Lo [22], dont l'idée est que les agents ont un comportement évolutif car ces derniers recherchent à atteindre l'efficacité. Le premier résultat de ce modèle basé sur l'information est qu'il permet d'unifier le courant de la finance classique et celui de la finance comportementale selon la quantité d'information effective et non effective. Ainsi, si la part de l'information effective augmente parmi l'information totale, le marché financier verra sa force computationnelle augmenter statistiquement (et tendra vers un marché efficace). Cela ne veut pas dire que les agents ont des anticipations identiques du prochain prix de marché mais le produit de leurs transactions (le prix de marché) sera aussi proche que possible de la valeur fondamentale de l'actif. Sous nos hypothèses, nous démontrons ainsi que si l'information non effective est négligeable devant l'information effective, ou que l'information est totalement effective, une fréquence des transactions très élevée permet au marché d'être semi-efficace (puisque nous considérons que l'information est parfaite entre les agents). Nous démontrons finalement qu'un marché semi-efficace repose sur des propriétés spécifiques de l'information lorsque la fréquence des transactions est élevée. Nous étendons les résultats originels du modèle en montrant que sous une croissance modérée de l'information non effective en comparaison de l'information effective, une bulle apparaît, et se termine par une crise (voir le chapitre pour les définitions rigoureuses de bulle et crise). Nous introduisons également le concept de force computationnelle dont la formulation la plus proche réside sans doute dans les travaux de Hayek [19]. Celui-ci nous permet de distinguer un marché sur lequel, la série de prix de marché issue de l'algorithme de détermination du prix mis en place, véhicule une information

plus complète que n'importe quel agent économique pris séparément. En d'autres termes, il nous permet de savoir si oui ou non, le marché remplit son rôle informationnel ou non. Cette force computationnelle est calculée en utilisant une fonction zeta spectrale. La figure suivante résume ce que change l'introduction du concept de force computationnelle quant à la caractérisation d'un marché financier:



À la lumière de ce concept et de ces conséquences théoriques, nous proposons également des indicateurs de la force computationnelle qui pourraient être implémentés par le régulateur sur plusieurs durées différentes à rebours (puisque la force computationnelle est calculée sur un intervalle de temps). La principale contribution de ce travail est de donner un argument supplémentaire à l'implémentation d'un algorithme de **LOB** sur les marchés où l'information arrive continuellement. Toutefois, le régulateur financier se doit de limiter les sources d'information non effective afin d'éviter l'émergence d'une bulle et sa transformation en crise.

Nous développons également les implications du concept de système de marché introduit initialement par Bretton and Priolon [4]. Nous définissons le système de marché adéquat à une situation de concurrence parfaite (avec un nombre d'agent fini ou infini). Nous menons ensuite des simulations afin de montrer que l'estimation d'un processus statistiques VARMA à partir des carnet d'ordres peut être utilisé pour anticiper le pouvoir de marché des agents économiques. Nos résultats montrent que le régulateur pourrait potentiellement anticiper le pouvoir de marché des opérateurs. Il faudrait alors distinguer si les opérateurs avec le pouvoir de marché le plus fort ont des caractéristiques en commun telles qu'être des traders à haute fréquence, échanger des volumes considérable,...

Les pistes de recherche possibles sont multiples. Des études empiriques pourraient être menée en utilisant le formalisme de ce modèle. On pourrait calculer la force computationnelle du marché de certains marchés financiers a priori et estimer la quantité relative d'information effective ou non traitée sur ladite période. On peut également utiliser des données de marché pour estimer le rythme de croissance d'une bulle dans le temps. Sur le champ théorique, la principale extension du modèle serait de formaliser l'asymétrie d'information.

Les deux chapitres qui suivent se concentrent sur les marchés de commodités. Nous considérons alors que le marché financier de commodité est composé avec des contrats à terme, c'est-à-dire les produits dérivés les plus liquides liés aux commodités, et côté en utilisant l'algorithme du **LOB**. Nous écarterons donc les swaps de commodités ou encore

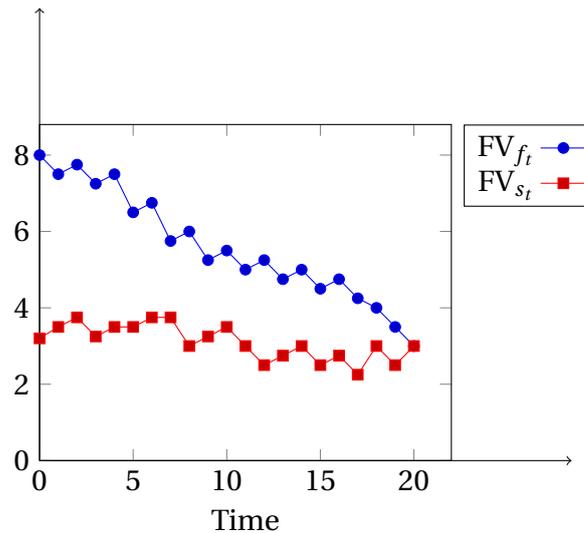
les options sur commodités dans un souci de simplicité.

Chapitre 3: Volatilité de la base et efficence des marchés de commodités: l'impact d'une différence des fréquences de transactions sur les marchés spot et à termes

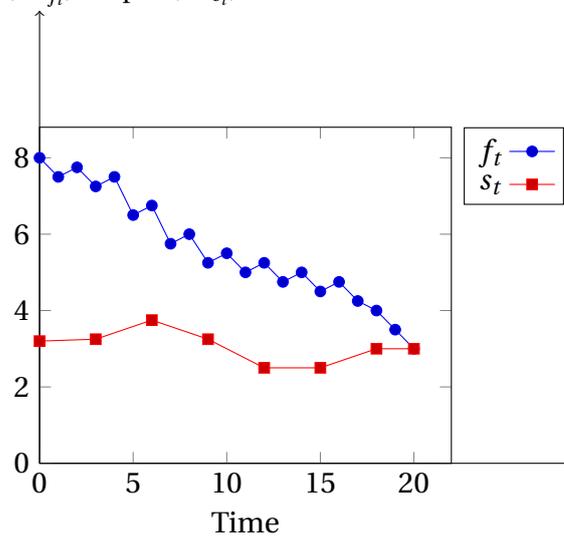
Le chapitre 3 présente un modèle permettant l'étude de l'impact des fréquence de transaction sur les marchés de commodités. En effet, ces marchés ont des structures particulières quant à l'information arrivante. En effet, la fréquence d'arrivée de l'information sur ces marchés est particulièrement lente et fixe. La production de la commodité est incertaine mais pas sa fréquence. Cette structure particulière de l'information est néanmoins couplée à des contrats à termes dont l'algorithme de détermination du prix est un **LOB**. Ainsi, la fréquence relativement lente d'arrivée de l'information réduit le coût d'une perte de liquidité comme Du and Zhu [10] l'ont souligné. Nous introduisons également dans notre modèle que les transactions vont générer à leur tour de l'information (très exactement, dans notre modèle, les transactions vont influencer la loi de probabilité d'arrivée de l'information). L'information est donc influencée par le degré de liquidité sur les deux marchés considérés: le marché physique spot et le marché financier sur lequel sont échangés les contrats à termes. La typologie de l'information utilisée dans ce travail est donnée dans le tableau suivant:

	Exogène	Endogène
Effective		X
Non effective		

Les coûts de transactions du sous-jacent d'un contrat à termes sur les commodités sont bien plus élevés que les coûts de transactions sur le contrat à termes en question. C'est une des raisons pour laquelle la fréquence des transactions sur le marché physique est significativement moins élevée que sur le marché à terme. Il est néanmoins légitime et naturel de penser que leurs valeurs fondamentales évoluent conjointement. Il est donc évident qu'une étude doit être menée sur la dissociation de l'évolution conjointe des valeurs fondamentales, qui se distingue de l'évolution à deux rythmes différents des séries de prix observées. Les schémas suivants illustrent ces deux dynamiques différentes.



(a) Dynamique des valeurs fondamentales des prix futures (FV_{f_t}) et spot (FV_{s_t})



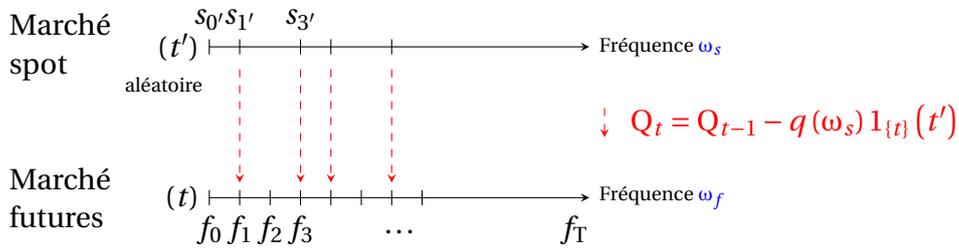
(b) Dynamique des prix de marchés réels futures (f_t) et spot (s_t)

Une des origines de cette différence de dynamique des prix réside dans la différence des algorithmes de détermination du prix de marché utilisés. Sur les marchés spot le prix de marché est en pratique déterminé sur plusieurs prix de transactions transmis par les courtiers (mais pas tous). Dans ce travail, à des fins de simplification, nous supposons que l'information sur le marché spot peut être centralisée par les agents et que le prix de chaque transaction influence le prix de marché. Malgré le fait que la modification du prix de marché sur chaque transaction sur le marché spot est une hypothèse relativement forte, supposer que les agents sont capables de centraliser l'information du marché spot ne semble pas l'être (cf. Simon [27]).

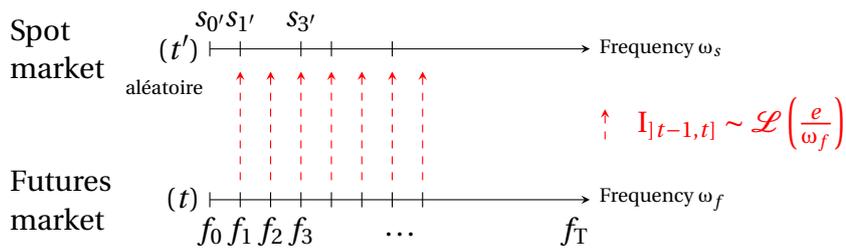
La base quant à elle, varie selon que le prix spot ou le prix à terme varient. Dans ce travail, nous étudions par extension l'impact d'une différence des fréquence de transactions sur la base. Nous nous intéressons particulièrement à l'augmentation de la fréquence des transactions, une dimension de la liquidité, résultant d'une fragmentation des ordres croissante ou encore d'une augmentation de la vitesse de transmission des ordres, qui peut avoir lieu sur les marchés de commodités. Nous étudions les conséquences de ces

changements sur l'efficacité informationnelle des marchés spot et à termes.

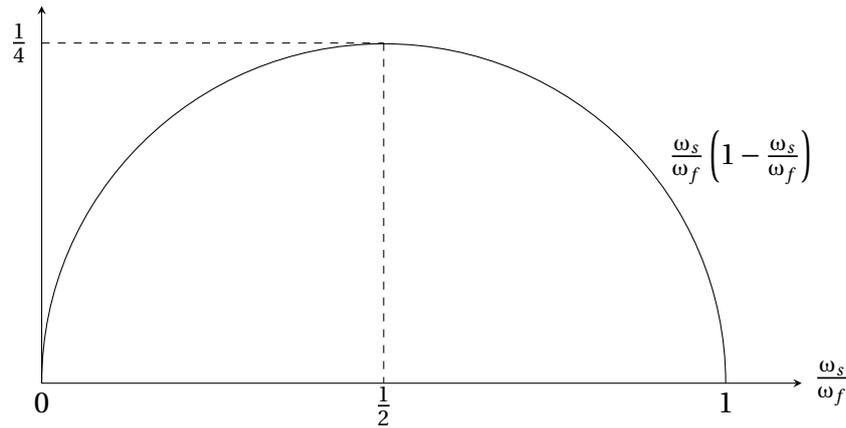
Nous avons construit pour cela, un modèle de pricing séquentiel simple dans l'esprit de celui de Black and Scholes [3] ou encore de Cox et al. [9], qui intègre la théorie du stockage. Ainsi, la base est fondamentalement déterminé par le convenience yield qui correspond à l'avantage lié à la détention d'une unité de commodité sur le temps qui sépare l'instant présent et la maturité du contrat à termes. Plus le stock disponible sera rare à l'instant t , et plus détenir une unité de stock aura une valeur élevée. Dans ce modèle, l'information qui a un impact sur le prix à termes est déterminée de façon endogène par les transactions qui ont lieu sur le marché physique. Cela est dû au fait qu'une transaction ayant lieu sur le marché spot à un prix s_t génère une diminution du stock disponible (noté Q_t) à l'instant t , à hauteur d'une quantité q si elle n'est pas spéculative. Cette diminution du stock influence par extension la base à travers le mécanisme de convenience yield, explicité précédemment. Cela est synthétisé par la figure qui suit, en considérant que les fréquences de transaction sur le marché spot et sur le marché à termes sont notées respectivement ω_s et ω_f :



De la même façon, la loi de probabilité de l'information arrivant entre deux transactions sur le marché à terme impactant le prix physique (et par absence d'arbitrage le prix à termes), notée $I_{|t-1,t|}$, est influencée de façon endogène par la fréquence des transactions ayant lieu sur le marché à termes (plus précisément par des transactions sur le marché à termes) dont la loi sera $\mathcal{L}\left(\frac{e}{\omega_f}\right)$. Cela est synthétisé par la figure qui suit:



Nous démontrons grâce à ce dernier que la volatilité fondamentale de la base peut augmenter suite à une augmentation de liquidité sur les deux marchés, et ce, même si cette augmentation est proportionnelle sur les deux marchés. La raison de cela est très simple. L'absence de synchronisation des transactions sur les deux marchés génère mécaniquement un risque (selon la définition du risque de Knight [21]) concernant l'anticipation de la dynamique du stock réel de la matière première. Cette idée est traduite par le graphique suivant qui traduit la volatilité de l'occurrence de deux transactions sur le marché spot et le marché à termes de la commodité à chaque transaction sur le marché à terme qui correspond à une loi de Bernoulli de paramètre $\left(\frac{\omega_s}{\omega_f}\right)$:



Or le niveau des stocks constitue une information primordiale afin d'évaluer la base. Cela se répercute donc sur le risque d'évolution de la base au niveau fondamental. Nous montrons également que la synchronisation des transactions sur les marchés de commodités et a fortiori de la fréquence de ces transactions peut permettre l'obtention de deux séries de prix non biaisées sur les marchés spot et à termes, une condition sine qua non pour l'efficacité des marchés. Sous ces conditions, la volatilité fondamentale sur le contrat à terme est également minimale. Ainsi, une augmentation non proportionnelle de la fréquence des transactions sur les marchés spot et à termes peut augmenter la volatilité de l'information sur le contrat à termes.

De ces résultats, nous extrapolons l'existence d'un arbitrage entre la liquidité sur les marchés à termes d'une part et l'efficacité informationnelle sur ces marchés ainsi que la volatilité fondamentale de la base d'autre part. En effet, la politique optimale qui permet la minimisation de la volatilité fondamentale de la base et la maximisation de l'efficacité informationnelle est la mise en place d'un algorithme de détermination de prix sur le marché à termes qui opérerait à la même fréquence que les transactions sur le marché spot associé. De façon équivalente, ce résultat souligne que l'algorithme de détermination du prix utilisé sur les marchés à termes, le **LOB**, n'est pas optimal au regard d'un objectif de maximisation de l'efficacité informationnelle et de la minimisation de la volatilité de la base d'autre part. Ce résultat n'est pas nécessairement en contradiction avec les travaux empiriques statuant que plus de liquidité, i.e. une fréquence de transaction plus élevée, permet une amélioration de l'efficacité informationnelle (cf. O'Hara and Ye [9]). En effet, selon la fréquence relative des transactions de départ, nous pouvons avoir un effet bénéfique de l'augmentation de la fréquence des transactions sur le marché à termes. Ce travail aboutit à la recommandation du passage d'un algorithme de détermination du prix des contrats à termes en **LOB**, à un algorithme de fixing, ou au moins un algorithme dont la détermination du prix ainsi que les transactions s'effectuent à une fréquence fixe, opérant à la même fréquence que les transactions sur le marché spot associé. Néanmoins, il nous permet aussi de mettre en exergue qu'une augmentation de la fréquence des transactions sur le marché à termes est possible conjointement à une augmentation de l'efficacité informationnelle des marchés et la minimisation de la volatilité de la base si l'élasticité fréquence des quantités échangées de la commodité sur le marché spot est suffisamment élevée en valeur absolue (c'est-à-dire que lorsque la fréquence des transactions sur le marché spot augmente, la diminution des quantités échangées doit permettre de réduire le risque lié à la dynamique du stock disponible de la commodité).

Les pistes de travaux liées à ce modèle sont nombreuses. Sur un plan empirique

tout d'abord, une calibration de ce modèle pourrait permettre d'évaluer l'arbitrage qui est actuellement effectué entre liquidité (sous l'aspect fréquence des transactions) d'une part et efficacité informationnelle ainsi que volatilité fondamentale d'autre part. Bien entendu, des tests de robustesses seraient à mener également. Sur la dimension théorique, plusieurs extensions pourraient être apportées. Par exemple, une formulation de la fonction des quantités échangées sur le marché spot pourrait être faite en fonction de la fréquence des transactions sur le marché à termes et non uniquement de la fréquence des transactions sur le marché physique. De la même façon, ce modèle qui repose sur une détermination des fréquences optimales de transactions sur des marchés de commodités pris un à un pourrait être formulé pour plusieurs commodités simultanément. L'étude de la fréquence optimale des transactions s'en trouverait certainement impactée. On pourrait également considérer que les transactions ont lieu de façon asynchrone malgré des fréquences identiques (ce qui n'est actuellement pas le cas) par l'utilisation de lois de Poisson. On pourrait ainsi tenter d'unifier ce modèle avec celui de Du and Zhu [10] par une microfondation similaire mais en considérant que l'information nouvelle n'est pas exogène et est fonction de la fréquence des transaction observée.

Chapitre 4: Volatilité de la base et efficacité sur les marchés de commodités: l'impact du nombre d'agents économiques

Le chapitre 4 correspond lui, à une des extensions possible du modèle construit et présenté dans le chapitre 3. On introduit à ce titre, un paramètre représentant le nombre d'agents sur le marché de la commodité et nous considérons l'impact de ce nombre d'agents sur les fréquences de transactions. Étant donné la différence de coûts de transactions sur le marché spot d'une commodité et sur le marché à termes, nous faisons l'hypothèse que les fréquences de transactions ne réagissent ni identiquement, ni proportionnellement face à l'entrée de nouveaux agents sur le marché de la commodité. En d'autres termes, les agents influenceront à la fois les deux fréquences de transactions mais aussi la fréquence relative des transactions. Enfin, nous considérons l'impact des agents sur les quantités échangées de façon indirecte, puisque la taille des transactions sur le marché spot est une fonction de la fréquence des transactions sur ce même marché. Par ce formalisme, nous supposons que le nombre d'agents influence la loi de probabilité de l'information nouvelle sur le marché spot mais aussi l'information nouvelle relative à la base, à travers leur impact sur la dynamique du stock disponible de la commodité. De la même façon que Stiglitz [28] statuait que les actions ou choix des individus véhiculent de l'information, nous supposons que l'entrée d'agents supplémentaires influence la volatilité de l'information arrivant sur le marché spot, pour une unité de temps donnée. Chaque agents sera finalement considéré comme une source d'information non redondante comparée à l'information apportée par les autres agents. Ici encore, une transaction générera en retour de l'information nouvelle. Enfin, nous étudions l'introduction d'un biais d'interprétation de l'information et de son impact sur le prix de marché de la commodité. Deux cas sont étudiés: la sous-estimation de l'impact sur le prix et la sur-estimation de l'impact sur le prix. Nous considérerons que ce biais sera corrigé par le nombre d'agents conformément à l'idée d'Hayek qui est que plus les agents sont nombreux, plus la capacité du marché à intégrer l'information par l'interaction des agents économiques est forte. La typologie de l'information utilisée peut donc être illustrée par le tableau suivant:

	Exogène	Endogène
Effective		X
Non effective		X

Tout comme dans le chapitre 3, nous considérons que chaque transaction sur le marché spot sera reportée et que le prix de marché sera modifié conformément à cette transaction. Nous nous intéressons particulièrement à l'impact de l'entrée de nouveaux agents que l'on pourrait par exemple assimiler à l'entrée des investisseurs institutionnels sur les principaux marchés de commodités en 2008. Nous étudions leur impact sur la volatilité fondamentale de la base, et celui sur l'efficacité informationnelle du marché.

Le premier résultat de ce formalisme est que la volatilité fondamentale de la base peut augmenter suite à l'entrée de nouveaux agents. L'intégration de l'ensemble des nouveaux paramètres nous empêche d'obtenir des expressions analytiques des valeurs optimales pour la plupart des sous-ensembles de paramètres. Cela nous permet d'exprimer que dans la plupart des cas, un arbitrage entre la liquidité (fréquence des transactions) et l'efficacité informationnelle ainsi que la minimisation de la volatilité fondamentale des deux prix est obligatoire. Cet arbitrage est fonction de l'aversion au risque du décideur politique. Nous effectuons une analyse de sensibilité quant au nombre optimal d'agents sur le marché en considérant que chaque paramètre peut évoluer. Ainsi, un nombre fini d'agents couplé à un algorithme de détermination du prix de marché qu'est le **LOB** sur le marché financier est dans la plupart des cas optimal. S'il y a plus d'agents que le nombre optimal en revanche, l'algorithme de détermination du prix optimal pourrait être un algorithme contraignant la fréquence des transactions sur le marché à termes selon les caractéristiques du marché considéré.

L'introduction d'un biais de sous-estimation ou de sur-estimation de l'impact de l'information sur les valeurs fondamentales par les agents qui se résorbe avec le nombre d'agents est un formalisme qui va venir influencer ces résultats. Un des premiers résultats de ce travail est de constater l'asymétrie du problème posé par l'évaluation incorrecte de l'impact de l'information. Lorsque les agents sur-estiment cet impact, le biais généré sur le marché spot par le retard de l'incorporation de l'information au prix peut finalement s'avérer compensé par une sur-estimation de l'impact de l'information, ce qui n'est pas le cas dans le cas où les agents sous-estiment cet impact. Ainsi, le marché spot peut se retrouver avec une dynamique de prix non biaisée qui est le résultat d'un retard d'incorporation de l'information dans le prix de marché parfaitement compensé par le fait que les agents sur-estiment l'impact de l'information (et donc exacerbent la variation du fondamental).

Encore une fois, la complexité du problème nous empêche d'avoir des solutions analytiques tractables, nous "obligeant" à mener des analyses de sensibilité du nombre optimal d'agents selon un set de paramètres. Le résultat global de cette recherche corrobore l'arbitrage entre la liquidité des transaction sur le marché à termes d'une part, et la volatilité fondamentale et l'efficacité informationnelle d'autre part. L'entrée de nouveaux agents sur le marché de la commodité augmente de facto la fréquence des transactions sur le marché à terme lorsque l'on utilise un algorithme de détermination du prix de marché qu'est le **LOB**. L'entrée de nouveaux agents favorise donc la liquidité sur le marché de la commodité. Toutefois, l'utilisation d'un algorithme qui détermine à une fréquence fixe un nouveau prix de marché, comme le **fixing**, pourrait être bénéfique pour l'efficacité informationnelle du marché ainsi que la stabilité fondamentale des prix selon les mêmes mécanismes à l'œuvre que dans le chapitre 3. Toutefois, ce chapitre 4 met en évidence une seconde boucle de rétroaction. Les agents augmentent la volatilité fondamentale du

prix spot sur un temps donné mais réduisent en contrepartie le biais du prix spot ainsi que celui de la base. Ces deux arbitrages différents sont pourtant tous deux liés sur un marché qui opère avec un algorithme de **LOB** ou les agents influencent à la fois les fréquences de transactions et l'information. L'implémentation d'un algorithme de détermination du prix de marché à fréquence fixe tel le **fixing**, permet néanmoins de dissocier ces deux arbitrages pour un nombre suffisamment élevé d'agents permettant d'être certain qu'une transaction ait lieu à chaque **fixing**.

De la même façon que pour le chapitre **3**, une calibration empirique du modèle pourrait être menée pour estimer dans un premier temps quel est l'arbitrage effectué entre la liquidité et l'efficacité informationnelle et la volatilité fondamentale. Les limites et possibles extensions de ce modèle quant à elles, sont identiques à celle du chapitre **3**.

Finalement, la principale limite transversale de ces modèles basés sur l'information réside dans le fait que l'information est parfaite entre tous les agents. Cette asymétrie d'information serait complexe à formuler dans le cadre théorique développé dans le chapitre **2**. Cela modifierait sans aucun doute l'ensemble des propriétés du modèle et apporterait un nouveau niveau de complexité. Sur le chapitre **3** (et le chapitre **4** par extension), de tels hypothèses pourraient être prises en compte par la microfondation de ce modèle et plus particulièrement celle de l'occurrence des transactions sur les deux marchés. Formaliser des coûts de transactions sur le marché physique supérieurs à ceux sur le marché à termes permettrait d'obtenir une fréquence de transactions moins élevée sur le marché physique.

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Chapter 1

General introduction

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Major changes in the structure of financial markets, have been reinvigorating the study of financial markets efficiency and of prices volatility. Our work focuses on price dynamics in financial markets. In particular, we examine the problems posed by the current institutional forms of market organization: fragmentation of orders, structure of financial information, nature of information, number of economic agents, etc. We analyze these issues with the prism of economic literature, but also with the help of computer sciences and what is called today econophysics. We also discuss proposals from contemporary economic analysis to improve the informational quality that underlies the formation of prices. The starting point of this work is the study of information that is released on financial market. To this purpose, we distinguish the information that can be processed with the help of an algorithm (also called effective information) from the information that cannot. We also consider that in some markets, like agricultural commodity markets for instance, the price, and so the information, of the underlying commodity is not updated as frequently as equity prices or currency prices. Indeed, most of equities prices are updated almost continuously, but in agricultural commodity markets, the underlying price is updated relatively slowly. Then, we can consider the information that is displayed on these markets as being not continuous. In this work, we study the nature and the structure of information in order to better understand the current financial market structure.

In this general introduction, we first proceed by depicting the current functioning of financial market with a focus on agricultural commodity markets, and defining the main terms of this thesis in section 1.1. Then, we expose the reasoning linking the chapters of this thesis, with the help of the current economic literature in section 1.2. We finish with a presentation of the three chapters of this thesis in section 1.3.

1.1 The context of this research

In this section we briefly present the contemporaneous functioning of financial markets and we define some concepts or notions that we use in this thesis. We first focus on general properties and characteristics of financial markets in subsection 1.1.1, and then we pay a particular attention to the specificity of agricultural commodity markets in subsection 1.1.2.

1.1.1 The actual functioning of financial markets

Definition 1. *An asset is an item of property. It has a value for its owner.*

Hence, an asset is a resource that the owner controls and from which s/he expects futures financial flows. Among several possible typologies, we choose to divide assets between tangible assets, such as a commodity, and intangible assets, such as financial assets.

Definition 2. *A financial market is a market in which intangible financial assets are traded.*

In these financial markets, economic agents place buy orders or sell orders.¹When a clearing house guarantees the completion of transactions the market is called an organized market or a listed market. In the absence of a clearing house the market is called an

¹Orders may be placed “at any price”. In the case of a buy order for a quantity x , this order will have for counterpart the sale orders which have no counterpart and whose prices are fixed, arranged in an increasing way, until the quantity x is obtained.

Over The Counter (OTC) market. In addition to these traditional organized or OTC markets, “hybrid” markets have been created over the last fifteen years and they are called **Multilateral Trading Facilities (MTF)s**.²

Definition 3. *A market price is a transaction price.*

A transaction is generated by the matching of at least one buy order with at least one sell order. In that case, the opposite orders are said to be compatible.

Definition 4. *An algorithm is a finite sequence of ordered operations.*

A transaction is generated by at least one compatible buy order and sell order. The market price on a financial market is therefore the result of an algorithm whose aim is the execution of compatible buy orders and sell orders.³ The *market price* is *a priori* a universal concept. However, it depends on the trading rules specific to each financial market, and more particularly on the method used to determine the price (therefore the algorithm used). Thus, the *market price* of the same asset can be determined differently on Euronext Paris and on the London Stock Exchange.

In practice, on most of the organized markets managed by **MU**s, two algorithms of pricing are used: the **LOB** pricing and the fixing pricing.⁴ But these two pricing algorithms have been processed with the help of computers relatively recently. Before, the pricing algorithm was implicit since transactions were not processed by the mean of computers but there was finite sequence of ordered operations systematically repeated (see Faye et al. [31] for specific details on the french futures market (Marché à Terme International France)). Despite that the fixing *algorithm* exists for a long time, the emergence of the **LOB** was more recent and can be approximately dated back to three decades ago on average between the different stock exchanges. Since the beginning of the **LOB** establishment, some famous economists argued that this pricing *algorithm* was a potential source of price instability. As an illustration we can quote the partial response to the October 13, 1989 mini-crash from Allais, the 1988 Economic Nobel Prize to the Los Angeles Times:

“The continuous trading market is an aberration from an economic viewpoint and generates a potentially permanent instability favoring fraud and manipulation of the market. [US markets could be improved] by eliminating the continuous market and replacing it everywhere with a single daily trading price for each stock in each market.” – M. Allais (1989)

The two current pricing algorithms used (the fixing and the **LOB** pricing algorithms), are based on the centralization of buy orders and sell orders in an *order book*.

Definition 5. *For a given asset, at a given moment, the order book gathers buy orders and sell orders that are executable. However, they have not found a counterpart yet.*

²In the french case, in accordance with Article L. 424-1 of the Code Monétaire et Financier, without being certified as a regulated market, **MTF**s can be managed by an **AISP** to provide the investment service or by a **Market Undertaking (MU)**. Source: <https://www.amf-france.org/Actors-and-Products/Financial-Market-and-Infrastructure/Other-Conflict-Services/Multilateral-Systems-of-Negotiation>

³On Euronext, the pricing is automatically ensured by the UTP computer exchange system. Source: <https://www.euronext.com/en/content/questions-with-the-quality-changing-the-changing-of-values>

⁴On Euronext, it can be a simple fixing (once a day at 03:00 P.M), or a double fixing (twice a day at 11:30 A.M and 04:30 P.M). Source: <https://www.euronext.com/en/content/questions-with-the-quality-changing-services>

The fixing, which has become the minor method used, determines the price that maximizes traded quantities. Therefore, quantities traded have a direct impact on the market price in that algorithm. We provide an example of this algorithm. Let us assume that the following *order book* is available for a given asset:

Ask		Bid	
Quantities	Price	Quantities	Price
400	At any price	400	At any price
200	156	250	150
250	155	400	151
500	154	500	152
850	153	600	153
1000	152	1250	154
3000	151	1700	155

Table 1.1 – Example of an order book.
Source : SBF, 1998, *La Bourse de Paris*

Hence, we aggregate ask orders quantities and bid orders quantities available for each level of price:

Ask		Bid	
Quantities	Price	Quantities	Price
400	At any price	400	At any price
600	156	650	150
850	155	1050	151
1350	154	1550	152
2200	153	2150	153
3200	152	3400	154
6200	151	5100	155

Table 1.2 – Example of a fixing algorithm first operation.
Source : SBF, 1998, *La Bourse de Paris*

Then, the following sequence of operations are processed:

- For a price of 151, there are 1050 quantities on the bid side and there are 6200 quantities on the ask side. There would be 5150 quantities remaining from trades at that price;
- For a price of 152, there are 1550 quantities on the bid side and there are 3200 quantities on the ask side. There would be 1650 quantities remaining from trades at that price;
- For a price of 153, there are 2150 quantities on the bid side and there are 2200 quantities on the ask side. There would be 50 quantities remaining from trades at that price;
- For a price of 154, there are 3400 quantities on the bid side and there are 1350 quantities on the ask side. There would be 2050 quantities remaining from trades at that price.

As soon as the remaining quantities are increasing, this operation ends and the price minimizing the remaining quantities (i.e. maximizing traded quantities) is known. Following our example, a price of 153 minimizes the remaining quantities and 2150 quantities will be traded at that price. The new *market price* resulting from the fixing algorithm is 153.

The **LOB** is an algorithm that instantly executes two compatible orders; as soon as a sell order is compatible with a buy order, a transaction is executed at the compatible price. The stock exchange immediately displays the price at which the last transaction has taken place and this price becomes the *market price* until the next transaction. The frequency of price evolution is *a priori* unknown.

We now focus on the modern structure of financial markets. It has an intrinsic impact on prices' dynamics. Regulation of financial markets is voted by national and supra-national institutions. In the European Union, the **ESMA** is in charge of monitoring the compliance with this regulation.⁵ Within the framework given by the **ESMA**, the **AMF** "ensures the protection of savings, investors' information and the proper functioning of financial markets" in France. These institutions aim to promote information transparency and "the good functioning of financial markets" and to make "capital markets deeper" in order to guarantee a high level of liquidity.⁶

Definition 6. *Liquidity can be defined as the capability to buy or sell a desired quantity of asset immediately, and with the smallest spread with its current market price.*

Then liquidity has two dimensions:

- (i) The capability to trade immediately which raises with trades frequency;
- (ii) The capability to trade the desired amount which raises with market depth. Market depth represents the capability of the market to absorb relatively large orders without impacting the *market price*.

One of the largest consensus in financial economics is based on the characteristics of an efficient market (cf. Fama [29], Fama et al. [30]). A market is said to be efficient if the dynamics of its price incorporates instantly all the available information.⁷

Definition 7. *The fundamental value of an asset at a given time, is equal to the best and immediate computation of all available information at the given time.*

Definition 8. *Over a period of time, a market is said to be efficient if the price of an asset is as close as possible to its fundamental value.*

One immediately understands that our research cannot be disconnected from the study of economic information (structure, quantity, frequency, etc.) and its evolution. In his work, Fama (see Fama [29], Fama et al. [30]) introduces three different levels of market efficiency:

⁵All missions of the **ESMA** are detailed at the following link: https://europa.eu/european-union/about-eu/agencies/esma_en

⁶Sources: <https://www.amf-france.org/L-amf/Missions-et-competences/Presentation>
https://ec.europa.eu/info/policies/banking-and-financial-services_fr

⁷An original quote is: "(...) under the assumption that security prices at any time "fully reflect" all available information. A market in which prices always "fully reflect" available information is called "efficient"." – Fama et al. [30], p.1.

- (i) The weak efficiency implies that the future market price cannot be expected according towards its past dynamics;
- (ii) The semi-strong efficiency implies that the market price contains all public information (including past prices);
- (iii) The strong efficiency implies that the market price contains all public and private information.

Hence, according to Fama, the typology of efficiency mainly relies on the availability of information.

The structure of financial markets is the result of two hierarchical components.

First, institutions define a general framework (see *infra* for the description of institutions at the level of the European Union). The objective of institutions is to establish a transparent framework in order to allow agents operating on the markets to dispose and to disclose correctly and without delay the information through their orders. In other words, the objective is to enable the efficiency of financial markets. However, there are some elements of the framework that can *a priori* limit market efficiency but have another purpose. The position limit regulation for instance, prevents any economic agent who has reached the limit for a given asset to buy a bigger quantity of the asset. This regulation alters the capability of agents to disclose information through orders as soon as they have reached the position limit. However, regulation aims at preventing any agents from having a *market power* and so, from having a significant price impact. Of course, there are other regulation representing a trade-off between the objective of “the good functioning of financial markets” and the objective of making “capital markets deeper”.

Second, the decision of the MUs on the organization of markets, as well as for markets handled by an AISP. We remind that MUs are private companies.⁸

The goal of MUs is to generate a profit, while respecting the regulation; This profit is essentially proportional to the *n.o.t* made since MUs charge a commission per transaction or a commission to access the market. Therefore they also have an interest in the maximum effective liquidity. To this end, some MM operates on some organized markets, insuring the role of systematic counterpart. It is equivalent to being a provider of liquidity when the timing of investors orders is asynchronous.

Formally, MUs develop market rules and pricing algorithms on these markets in respect to the prior regulation. Liquidity plays a key role here. In an efficient market, the market price must change as soon as a new information is released. The capability to trade immediately must be absolute if we do not know *a priori* the frequency of arrival of information. Liquidity allows the buyer or the seller that incorporates the new information in his/her order’s price, to find a counterpart without delay. This is equivalent to consider at least that a buyer and a seller must react to new information simultaneously and without delay by placing compatible orders. This reasoning pushes institutions to regulate financial markets in order to reach a maximal liquidity. The limit order book is then the pricing algorithm that seems to maximize the liquidity.

Hence, MUs favor the LOB quotation of assets, and this is justified by the fact that we do not know *a priori* the frequency of arrival of new information.⁹

⁸Some information on the property of Euronext Paris’ equity is available at the following link: <https://www.euronext.com/en/products/equities/NL0006294274-XP/company-information>

⁹Euronext authorizes any asset to be priced using a LOB as soon as a minimum of 2500 transactions are

The regulation of the [ESMA](#) aims to favor competition on regulated financial markets. The presence or increase in competition should reduce transaction costs and market access costs set by the [MUs](#) to investors.¹⁰ As a result, the [LOB](#) was preferred and several markets have been developed to compete with existing markets: [MTFs](#), and among them, some markets with specific rules that are commonly referred to as “dark pools”.¹¹ The main advantage of these [MTFs](#) is to avoid excessive fluctuations in market prices when massive trades occur. They are often restricted to institutional operators, who are the first to be exposed to such impacts because of the size of their portfolios (and *in fine* of their trades). Nevertheless they can sometimes be open to investors via brokers.

Such a multiplicity of markets in order to trade and quote a given asset leads investors to sometimes fragment their orders.

Definition 9. *The fragmentation of orders can be defined as the split of a buying (respectively selling order) – called parent order – into several smaller buying (respectively selling orders) – called children orders.*

Orders fragmentation (or fragmentation when there is no ambiguity) can be made on the same market across time, at the same time on different markets (if the same asset can be exchanged in different markets), or across time on different markets.¹² This fragmentation can be voluntary or constrained. The reduction of transaction costs due to advances in technology for instance, allow economic agents to – voluntary – fragment their orders. However, within a [LOB](#) pricing where market depth (a dimension of liquidity) is relatively insufficient (i.e. if limit orders size is not sufficient), an economic agent announcing a large size order at a limit price will upward prices if s/he passes a buying order or a downward if s/he passes a selling order. It is said that the order has a price impact. This price impact reduces with an increasing *market depth*. Then agents may be forced to fragment their orders to trade with the smallest spread with the *current price*, and enter multiple orders on one or more market using a [LOB](#) pricing algorithm. In other words, the lower the *market depth* is, the more an economic agent that wants to sell or buy at the smallest spread with the current *market price* has to fragment her/his order. The consequences of such a practice are a decrease in quantities exchanged per transaction and an increase in the [n.o.t.](#)

In modern financial markets however, available information is partly contained in the order book, continuously updated (see Cont and Larrard [18]). In addition to the past and present information available to operators in financial markets, the order book is an indication on the wish of buying and selling of operators;

“In contrast to looking at the volume traded, which shows what already happened, the order book provides an insight into the intention of the traders.” – Galanos [33], p.10

made on this asset annually at the request of the owners of this asset. Source: <https://www.euronext.com/en/content/questions-with-the-quality-changing-services>

¹⁰Source: http://www.parisschoolofeconomics.com/hautcoeur-pierre-cyrille/Revoir_MIF.pdf

¹¹The [AMF](#) defines the “dark pools” as “trading systems that operate without any pre-trade transparency”. Source: <https://www.amf-france.org/Presse-Space/Folders-and-presse-fiches/The-games-lies-for-emergence-of-dark-pools-and-crossing-networks>

¹²Orders fragmentation could also occur on physical markets for instance on spot agricultural commodity markets (see *infra*).

This vision of the order book as a vector of information is also in line with the work of Stiglitz which states that:

“(...) market actions or choices convey information. Market participants know this and respond accordingly.” – Stiglitz [71], p.468

Investors trading large volumes have integrated that the order book contains information. So they use dark pools to trade, to avoid conveying information. These specific market rules lead to a reduction in markets’ transparency since all or part of the information is hidden there.¹³ This emergence of new platforms has finally “fragmented” liquidity.¹⁴ Tóth et al. [75] show that in the vicinity of the market price, the financial markets are *illiquid* in the sense that a slight evolution in quantities provokes a more than linearly proportional price variation. To exchange a substantial volume of assets with the smallest spread with the market price, operators are therefore forced to fragment their orders in different markets, which leads to a rise in transaction costs.¹⁵ This result goes against the results expected by the competition policy of the ESMA.

To sum up, fragmented markets also fragment information. They provide market *prices* and *order books*. Nevertheless, quantities that appear in the order books do not convey all information. Operators who want to pass large size orders – or block orders – sometimes prefer to fragment their orders (thus fragmenting the information), or to pass through a dark pool (thus hiding information).

A multitude of economic agents operates on financial markets. We could give several typologies but economic theory focuses mainly on a distinction based on:

- (i) The operators decision variables: Fundamentalists who look at their available information and try to estimate the fundamental value or chartists who try to guess the movements of an asset price independently of the fundamental value;
- (ii) The operators objectives when they pass orders: Arbitrageurs who identify arbitrage opportunity, hedgers who act on financial market to hedge their output from a price risk, processors who insure themselves for having the raw material for their production, speculators who just try to make profit from price fluctuations, or spread fluctuations...

Agents can also be segmented according to the speed of their orders, which is particularly important for market price dynamics. Indeed, an overwhelming majority of orders now rely on information processing algorithms.¹⁶ What makes the operators heterogeneous is not only their access or not to an information processing technology but the

¹³Moreover, an asset can be quoted on multiple markets for the most traded assets (this has been favored by the emergence of MTFs as well). Then a given asset can have multiple price dynamics. Theoretically, however, these dynamics should be intricate, that is to say, should evolve exactly in the same way (frequency and variation), regardless of markets’ efficiency. Otherwise it is possible to buy the asset on the market where it is the cheapest and resell it on the market where it is the most expensive if the two markets use a LOB pricing. This reasoning was already applicable between MUs but it is now accentuated. A same regulated stock market can price the same asset according to different market rules whereas the price on these two markets should be identical. We will not consider multiple quotations for assets in our work.

¹⁴Source: http://www.parisschoolofeconomics.com/heart-heart-cyrille/Review_MIE.pdf

¹⁵Source: AMAFI - 10-01 - MIF REVISION - FLEURIOT MISSION - AMAFI CONTRIBUTION (p.15) <http://www.amafi.fr/topics/en/documentation/mif-1?page=1>

¹⁶In 2009, the proportion of orders placed through algorithmic trading represented between 50% and 75% of United States (see Leshik and Cralle [49], p.4). In 2012, 85% of the market volume was traded using algorithmic (see Glantz and Kissell [35])

technology that will make it possible to place orders faster than other operators. High Frequency Trading (HFT) is practiced by a handful of operators with sufficiently high means, and makes it possible to place executable orders faster than other operators in order to take advantage of the execution time of their orders – or simply react faster to new information. Of course, this segmentation does not exclude the idea that operators may be intrinsically different (arbitrageurs, portfolio managers, trend-followers etc.), but nevertheless highlights a new form of competition between operators studied in recent years (see Foucault et al. [32], Hoffmann [44], Roşu [65]). Among them, the technique of spoofing, well documented since it was the subject of trials in the United States, particularly highlights the risks of this new competition. Spoofing is a market manipulation technique that consists of offering an asset for sale (or buy), while the actual position is respectively buying (or selling), with the intention of canceling the order just before it is executed, to obtain a favorable movement of prices respectively downward (or upward). It is therefore potentially practiced on markets where the order book is available, and where a **LOB** pricing is implemented. The immediate consequence of such a practice is to blur or even distort information on these markets.

Hence, the heterogeneity of traders on the markets can also be distinguished by competition regarding their orders' lead time in fragmented markets using a **LOB** pricing.

Then, financial market regulation has to take several intermediate objectives into account in order to achieve market efficiency:

- (i) The transparency of information relevant to pricing: Otherwise it is impossible for traders to incorporate it into the *market price* through their orders;
- (ii) Liquidity: Any new information must be incorporated into prices by a transaction at a new price into a **LOB** pricing. The chronological sequence of price changes must ideally be superimposed on the chronological sequence of new information. Therefore, the inherent objective is to identify the frequency of incoming information. One can imagine, that the latest is different in the context of a stock belonging to an index and in the context of a commodity futures contract;

1.1.2 On the specificity of agricultural commodity markets

In this work, we give a particular attention to agricultural commodity which are the underlying of standardized contracts. Then, we exclude agricultural commodities, which are not the underlying of standardized contracts, from our analysis. Historically, two main characteristics of some agricultural commodities allowed the development of financial assets: their capability to be stored, and their harvest's seasonality. Hence, we give the following definition of an agricultural commodity.

Definition 10. *An agricultural commodity is a standardized tangible asset which is storable and whose harvest is seasonal.*

Agricultural commodities derivatives markets are intrinsically different of other derivatives markets. Since the underlying asset is a tangible asset, that implies structurally different transaction costs when compared to currencies underlying which are an intangible asset for instance. In the case of a tangible asset, transaction costs represent the cost to find a broker and to pay her/him to insure that the transaction take place. In the case of an intangible asset, transaction costs represent the costs to access the market and to place the order.

Definition 11. *An agricultural commodity market is a market where an agricultural commodity can be exchanged within two types of markets:*

- (i) *A spot market, where sellers and buyers trade the agricultural commodity itself;*¹⁷
- (ii) *A financial market, where financial contracts whose underlying is the agricultural commodity are traded.*

The principle types of contracts exchanged on the agricultural commodity financial market are the forward contracts, the futures contracts, the swap contracts and the option contracts.

A forward contract is a commitment to sell or buy the commodity at a later date, taken by respectively a seller and a buyer. Characteristics of the contract (date, transport, quality, quantity) are decided by both counterparts. This financial asset is negotiated in an **OTC** market and there often is a broker to facilitate transactions.

A futures contract is a commitment to sell or buy the commodity at a later date, taken by respectively a seller and a buyer. It is negotiated on an organized market where a clearing house operates and the characteristics of the contract (date, transport, quality, quantity) are standardized. Each open futures contract implies the economic agent to make a deposit to the clearing house. Each daily return is paid by the losing operator to the clearing house and received by the winning operator from the clearing house. It is called the margin call mechanism. At a given date, several futures contracts are open at several maturities. Of course, futures prices are linked by the price term structure also called the “Samuelson effect” (cf. for instance Lautier and Simon [47]).

A swap contract is a contract through which two parties exchange the cash flows or liabilities from two different financial instruments. Commodity swaps involve the exchange of a floating commodity price for a set price over an agreed-upon period.

An option contract on an agricultural commodity is a contract giving to the buyer a specific right at a given date or period relative to the agricultural commodity market. The best known option contracts are the calls and the puts. The owner of a call has the right – but is not obliged – to buy the underlying asset at a specific price at a given date or period. The owner of a put has the right – but is not obliged – to sell the underlying asset at a specific price at a given date or period.

In practice, forward contracts often end with an effective physical delivery of the agricultural commodity whereas futures contracts often do not end with physical delivery. In this case, they are said to be cash settled. As early pointed out by Working:

“Futures trading in commodities may be defined as trading conducted under special regulations and conventions, more restrictive than those applied to any other class of commodity transactions, which serve primarily to facilitate hedging and speculation by promoting exceptional convenience and economy of the transactions.” – Working [78](p.315)

The benefit of a standardized contract such as the futures is its liquidity, or at least the frequency of trades. At the opposite, a forward contract negotiated for the specific needs of both counterparts is *stricto sensu*, not liquid. Hence, it is easier for farmers or transformers to hedge their production (or respectively their input) to face price risk by finding a counterpart who accepts to take this risk. This is the very starting point of

¹⁷Some authors consider – agricultural commodity – spot markets as a financial market. We do not.

creation of futures contracts, created on the Chicago Board of Trade (which is now integrated to the [Chicago Mercantile Exchange \(CME\)](#)). The relatively high frequency of trades in futures contracts is the reason why futures prices is often considered as the reference price. Indeed, this high frequency of price evolution is by extension assumed to provide more frequently information to the valuation of the agricultural commodity. As stated by Bégué Turon et al. [12], the use of futures market by operators is increasing and is either direct (by operating on the futures market) or indirect (by referring to futures prices on [OTC](#) contracts) in order to manage the price risk in the French grain market.

Within an agricultural commodity futures market, orders fragmentation could also occur, leading to a rise of trades frequency. One can also assume that the reduction in transaction costs on agricultural commodity spot markets could allow orders fragmentation on those markets. Thus, orders fragmentation could occur on these two markets and lead to an evolution in prices' dynamics despite the two markets are fundamentally intricate. However, there are feedbacks between these two markets. As a consequence, orders fragmentation would influence the basis.

Definition 12. *The basis at a given maturity in an agricultural commodity market is defined as the difference between the futures price at the same maturity and the spot price.*

The basis' volatility is a major stake for farmers and processors, and a major component of their production and selling decisions (cf. Moschini and Hennessy [57], Bégué Turon et al. [12]). Although fragmentation of orders can be used in both spot and futures markets, transaction costs in these two markets are not the same since they are intrinsically different (cf. Working [78]). Ultimately, fragmentation of orders would be neither equal nor proportional in these two markets, impacting both relative trade frequency ([RTF](#)) and quantities exchanged per transaction in these markets. One of the objectives of this thesis is to study the impacts of orders fragmentation.

Agricultural commodity markets are the topic of specific institutions with specific goals. First the [AMIS](#), which is a response of the G-20 Ministers of Agriculture to the global food price hikes in 2007/2008.¹⁸ Its objectives are the following:

- Improve agricultural market information, analysis and short-term supply and demand forecasts at both national and international levels;
- Collect and analyze policy information affecting global commodity markets, and promote international policy dialogue and coordination;
- Report on critical conditions of international food markets, including structural weaknesses, and strengthen global early warning capacity on these movements;
- Build data collection capacity in participating countries by promoting best practices and improved methodologies, providing training to national stakeholders and facilitating the exchange of lessons learned among participating countries.

The [AMIS](#) achieves its goals by providing a “platform to coordinate policy action in times of market uncertainty”.¹⁹

¹⁸The website of the [AMIS](#): <http://www.amis-outlook.org/home/en/>

¹⁹Source : <http://www.amis-outlook.org/amis-about/en/>

The [European Union \(EU\)](#) also set up a consultative [EGDSM](#) in 2012.²⁰ Its role is to provide advice to the European Commission about:

- The functioning of the agricultural commodity derivatives and spot markets;
- The implementation of existing [EU](#) legislation and policies;
- The preparation of legislative proposals and policy initiatives in this field.

This consultative group objectives are the same as those of the [AMIS](#): the agricultural markets' efficiency and the minimization of the volatility of agricultural market's prices. However, this institution is not mainly focused on the information on agricultural commodity markets but on "all" their regulation policies and legislation. During the last meeting of these consultative Expert Group, Cordier insisted on the need of an updated public market information on agricultural commodity quantity and quality in order to reduce the basis' volatility.²¹

A particularity of agricultural commodity markets is the underlying asset market price dynamics. The spot price is discontinuous (not quoted with a [LOB](#) mechanism) whereas the futures price is quoted with a [LOB](#) mechanism (continuously by definition). One understanding that it gives to the basis, a peculiar dynamics. Then, the relevant information concerns the previsions made (for instance the one on the monthly report of the [USDA](#)) and meteorological data. Hence, not only the frequency of this kind of information is fixed, but also known in advance. One can imagine that such information characteristics can deeply modify the perception of the market efficiency on those markets. In the end, the level of storage also plays a key role in the basis dynamics but storage directly depends on the trades realized on the spot market. This difference in trades frequency on the spot market and on the futures market, as well as their retro-actions on the agricultural commodity market is one of our main topics.

1.2 Thesis' motivations: From academic contributions to the scientific research question of this thesis

In this section, we highlight important questions emerging from the current state of the art in financial economics. The first one concerns the modeling of the effective price formation. We provide in subsection [1.2.1](#) a short literature survey in order to highlight the evolution of the understanding of the formation of prices and compare it to actual pricing algorithms. The second one relies on the key role of information in financial economics (in particular [Hayek \[40\]](#) and [Fama et al. \[30\]](#)). We provide in subsection [1.2.2](#) a selective literature survey of some major works in financial economics and understand them in a information based reasoning to justify our approach. In the end, we highlight in subsection [1.2.3](#), how some stylized facts in both financial markets and agricultural commodity markets, modify the released information on these same markets.

²⁰Website : https://ec.europa.eu/agriculture/cereals/commodity-expert-group_en

²¹Presentation is available: https://ec.europa.eu/assets/agri/market-sectors/cereals/commodity-expert-group/2018-04-18/benefits_otc.pdf

1.2.1 The market price formation: A theoretical representation of pricing algorithms

Most of the time economic theory uses an *equilibrium* framework but, paradoxically, it gives little importance to the formation of market prices, if we except the literature dedicated to market microstructure. The way in which *actual prices* get closer to the equilibrium price is relatively unexplored.

The theoretical formation of the market price: The Equilibrium price

Economic theory focuses almost exclusively on the existence and determination of the *equilibrium price* which is defined as follows:

Definition 13. *The equilibrium price is the price that equalizes the aggregate supply and the aggregate demand.*²²

In the framework of the “Walrasian” general equilibrium theory, this equalization is *unique* (see Arrow and Debreu [1]) and is realized instantly as soon as the transactions are opened (in fact the auctioneer realizes or authorizes the transactions when and only when equilibrium prices are discovered); this model therefore rules out the very concept of market price *dynamics*. Past, present, and future information is known to all agents and is perfectly processed *before* the trades take place. The hypothesis of the uniqueness of the moment of transactions is sometimes released; In this respect, Hicks [42] work is emblematic. However, the assumption of equalization of aggregate supply and demand is maintained. The dynamics of the price has a fixed frequency such that at each period, the aggregate supply is equal to the aggregate demand and all information is incorporated into the price. Recall that at the beginning of a period (a week in the Hicks’ vocabulary), characteristics of the market are known: supply functions, demand functions, production functions, utility functions. A balance is formed for the week and an *equilibrium price* is determined; between the end of a period and the opening of the next one, new market conditions may appear so a new price would be determined. This model, known as the Hicksian traverse, combines the idea of market equilibrium with that of an evolutionary process. But this model keeps the idea that trades take place *after* the equilibrium price has been determined by processing all available information at the beginning of the period. Works based on the same assumptions concern the study of the convergence of a price towards its equilibrium price, an aspect of price dynamics (see Ezekiel [28], Guesnerie [37]). However, the dynamic aspect in these works still assumes a fixed frequency.

Fama et al. [30] and Fama [29] can be considered as an extension of these reasoning on financial markets. The definition of market efficiency can be understood as the passage to the limit of such reasoning. If we consider that a period is equal to the smallest unit of time, the price must reflect at each unit of time (= at each moment), all the available information affecting the aggregated supply and / or aggregated demand (present and future). In his thinking, E. Fama relaxes the assumption of fixed frequency since the price dynamics is endogenously determined by the frequency of new information. More precisely, a transaction is justified if and only if new information is released. It is therefore the - exogenous - timing of arrival of information that explains the frequency of transactions. On the other hand, no specification is made on the means for determining the *market price*. The efficiency of a market is therefore considered here only by its final outcome, the *market price*.

²²Aggregate supply is the sum of sales intentions for a given price; Aggregate demand is the sum of the buy intentions for a given price.

Hayek [39] pays particular attention to the process of the market price formation. Indeed, it is the market interactions between economic agents that allow the market price to incorporate at best available information affecting aggregate supply and / or aggregate demand (present and future). We can extract from this reasoning that the easier it is to express a merchant wish and to have the wishes of other *economic agents* available, the more effective the market will be in finding the “good” *market price*. This idea is precisely what we refer to as the *computational strength*. The organization of the society with the help of markets allows interactions between agents through wishes to buy or to sell. A price will emerge such that it will better convey information on average than any agent alone. However, in his work, Hayek does not precisely explain the finite sequence of ordered operations leading to the price determination. Hence, no pricing algorithm is given here.

More recently, the market microstructure field of research, that Madhavan [55] retrospectively date to the early 1980, has produced some significant results about the formation of prices and the role of information. Within these field of research, the understanding of the price formation leads economists to numerous study on the bid ask spread, the difference between the best bid (i.e. the lowest price on the supply side), and the best ask (i.e. the highest price on the demand side), and of course, the impact of MM. Most of those studies are however, considering the price as being an *equilibrium price* (cf. Biais et al. [8]).

The comparison with the modern formation of the market price on financial markets

The consensual theoretical view of price dynamics can suffer from a contradiction with the reality of price dynamics in financial markets in two main respects. The first concerns the equalization of aggregate supply and aggregate demand, that is not verified in the context of a LOB pricing. The fixing quotation is made on an aggregated order book, and the fixing algorithm deals with compatible orders only in order to determine the new *market price*. Thus, the “fixing” pricing is closer to the theoretical view of the meeting between aggregated supply and aggregated demand (determination of a repeated *equilibrium price*) than the LOB pricing.²³ Having a large majority on Euronext – but also on all organized market managed by MUs – the latter is more compatible with characteristics of the market efficiency because the frequency of the price dynamics is not fixed. The second aspect lies in the very concept of dynamics. The general equilibrium theory considers a single market price or even a market price dynamics whose frequency is fixed. The theory of the financial markets apprehends a price dynamics contingent to the arrival of information. On the contrary, the dynamics of a LOB pricing is conditioned to the dynamics of transactions.

On the other hand, an order book is a mean to facilitate market interactions, especially since it is accompanied by an execution insurance at the best offer. Moreover, Hayek’s reasoning does not postulate any equalization between single or periodic aggregated supply and aggregated demand, and does not assume any fixed frequency in his reasoning, so it seems in line with the current characteristics of the financial markets regardless of the pricing method.

The dynamics of a financial market is therefore reflected in a series of *market prices*. From an analytical point of view, it can be considered as an output of an organized financial market (by a MU) considered as the producer of a pricing algorithm as well as a

²³Actually, the fixing pricing algorithm inspired the work of Walras on the theoretical representation of the auctioneer (cf. Dupuy [23]).

market organization (which aims to maximize its profit). This reasoning has the major consequence of reversing the causality of the market price dynamics. While economic theory apprehends a grope process allowing the emergence of an *equilibrium price* as a market price that will allow transactions to take place, we find ourselves in a reverse causality; Transactions take place and generate a variation of the *market price*. We note in fact that the economic agents carrying out a transaction cannot be considered as “price takers”, since they determine the market price by carrying out a new transaction. Each operator therefore has a non-zero probability of becoming “price maker” when the number of operators is finite.

1.2.2 On the key role of information in financial economics

In financial economics, the literature is mainly divided into two theoretical streams. The first one is the classical finance which is based on the [Efficient Market Hypothesis \(EMH\)](#). In such a market, there is “no free lunch” or identically no possible arbitrage. Poncet et al. [63] define an arbitrage as an operation that guarantees a positive profit without risk of loss. If agents are perfectly able to distinguish any arbitrage opportunity and there are no transaction costs, then markets are efficient. As soon as new information is released, agents exploit the possible arbitrage (since a new information implies that an asset price becomes over-evaluated or under-evaluated) instantaneously, until the price reflects its no arbitrage value. This set of hypothesis is central to classical finance and allowed some major results to emerge. Among them, the derivatives asset pricing theory initiated by Black and Scholes [9], Merton [56] and by Cox et al. [19].²⁴ Classical finance developed some useful tools in treating the question of the incorporation of private information into prices such as the Bayesian theory. One can read the famous work of Vives [76] in this research field. Results of such research and tools are that private information of economic agents is perfectly integrated by a market price; But one must be aware that the market price as intended here corresponds to an *equilibrium price* (see below).

The second stream of research is the behavioral finance whose main research question is to describe psychological biases of agents in order to explain some characteristics of financial markets. This research field has two blocks as it is underlined by Barberis and Thaler:

“Limits to arbitrage, which argues that it can be difficult for rational traders to undo the dislocations caused by less rational traders; and psychology, which catalogues the kinds of derivations from full rationality we might expect to see.” – Barberis [2], p.1054

Behavioral finance has revealed itself to be a powerful tool in explaining positively how agents’ behavior conducted prices to diverge from their fundamental. One of the most powerful extension of such work is that irrational traders are not necessarily evicted from the market (since there are limits to arbitrage) leading to qualify as “persistent” market price imperfection. This element of response is well explained by Mullainathan and Thaler:

²⁴We recall that the Cox Ross Rubinstein discrete time derivatives pricing is converging to the Black and Scholes one when considering the time between two time-steps converging to zero. Proof has been given by Cox et al. [19].

“Behavioral Economics (...) investigates what happens in markets in which some of the agents display human limitations and complications. We begin with a preliminary question about relevance. Does some combination of market forces, learning and evolution render these human qualities irrelevant? No. Because of limits of arbitrage less than perfect agents survive and influence market outcomes.” – Mullainathan and Thaler [58], p.2

Finally, these two streams of research have the same starting point which is that assets have a *fundamental value* (see definition 7). The latest is empirically and conventionally considered to be equal to the discounted sum of future cash-flows. But in a surprising way, these two complementary literature do not take the best from each other and often create a gap between classical and behavioral finance as if their results could not be expressed in a unique framework (i.e. as if they fundamentally disagree the other point of view).

From a theoretical point of view, financial economics is in most of the cases, micro-funded by the behavior or utility functions of agents. The first chapter of this PhD thesis is constructed with the idea that a more general framework can be expressed by micro-funding it using not only the behavior or utility functions of agents but also the nature and the structure of information itself. Of course, within this framework, both classical finance and behavioral finance could hold but not at a given time. A conceptual framework unifying both streams of research named Adaptive Markets Hypothesis was developed by Lo [51]. In such a model, agents evolve and are able to discern *a posteriori* their errors. As a result, they correct their behavior with the objective of reaching the efficient one. The key idea of this model is that agents have an evolutionary behavior and that they are pushed to adapt themselves by the wish to survive the market. We integrate such an evolving process into our model by considering that survival is the capability to *execute* an order. As a consequence, agents try to guess the next transaction price (i.e. the next market price). The need of a framework able to assess both the objective dimension of prices and the subjective dimension of prices was early stated by Walter [77] who argue that the use of Pareto laws can be a useful tool to this end. We propose in this work another tool, based on computer sciences, to achieve this goal.

Then, information is a channel which can allow a conciliation between both streams of research and to rethink the debate. This is the starting point of this PhD thesis: We aim to identify what are the conditions for agents being able to integrate efficiently information to prices, and we start from the nature of information itself. Later in chapter 2, we divide information into two subsets: The effective information which can be rationally processed by an algorithm, and the non effective information which can not be processed by an algorithm (see definition 15 for a rigorous definition). For instance, a data-frame on the harvest previsions can be processed by an algorithm. On the other hand, a central bank governor statement must be at least partly processed by agents mind. On the first hand, the classical finance usually considers perfectly rational agents combined to –implicitly fully effective – information flows (also referred to as signals) leading to efficient financial markets (for instance Vives [76]). On the other hand, the behavioral finance focuses on the process of information by agents and often points at the limit of the market efficiency through agents limited rationality or even irrationality (cf. Shiller [68]). This irrationality takes many forms such as exuberance (cf. Shiller [69]), social moods (cf. Nofsinger [59], and Olson [60] for a review), ... We assimilate the possibility of limited rationality or even irrationality to the existence of non effective information, that requires a certain process by agents’ mind only. Hence, a market can be non efficient according to agents

rationality.²⁵

Furthermore, information has evolved across time as explained in the previous section 1.1. Information sources are increasing more and more and transactions occur permanently regardless of whether or not aggregate supply equals aggregate demand (O'Hara [62], Schütz et al. [67], Tóth et al. [75]). Considering these facts, we are confronted to a problem relative to the characterization of a financial market. Actual financial economics is able to distinguish between an efficient market or not. However, does a non efficient market display a price closest to the fundamental value than any agent alone? This question implies the definition of displaying “better” information than any agent separately. Considering that series of prices result from agents anticipations when they process information and then pass sell or buy orders, Hayek studied the formation of prices, and addressed the capability of markets to integrate information better than any agent considered individually (see Hayek [40]). In the wake of Hayek, we define the *computational strength* of a financial market.

Definition 14. *A financial market has a computational strength during a period of time if and only if it better integrates information in the market price than any agent separately during the period of time.*

One can immediately see that if agents are perfectly rational, a market having a computational strength is equivalent to having an efficient market, even if there is private information. This very simple definition has a strong meaning: On a first hand, if a financial market is not efficient, it is not enough to argue that it is useless. On the other hand, a thought experiment suggests that if there is an agent able to better forecast the fundamental value compared to the market on a given period, then it would be better to let him/her evaluate the asset such as other agents would have to make their trade at the price he/she gives (it could be an institutionally recognized analyst as well as a “simple” trader). The framework developed in chapter 2 emphasizes the importance of the wish to trade for determining the value of an asset (cf. Schaden [66]). This model also allows us to consider agents not just as price takers, but also as price makers with a certain probability (cf. Choustova [15]). As a consequence, agent's atomicity cannot hold except if there is an infinite *n.o.a.* The latest refinement is again the product of new tools whose starting point is our perception of information and rationality.

Agent-based models are the most common tool to address the heterogeneity of agents and to study how their interactions influence or determine prices on financial markets (see for instance LeBaron [48] for a review of agents-based modelling in economics, Chakraborti et al. [13] for a review of agent-based modelling in econophysics, Tesfatsion and Judd [72] for a review of agent-based modelling in computational economics). The model built allows us to consider a special case of heterogeneous agents' interactions: Not only each state of the market is determined by the set of agents' expectations but expectations of agents are – in turn – influenced by the state of the market, in a never ending movement. Such a theoretical frame where we consider the human imperfection (which is not a synonymous for irrational) in parallel to information which must be at least partly treated by agents minds is appropriate to the study of the *computational strength* of the market.

New information arrives with increasing speed with technological advances, but also the reduction of transaction costs. The number of sources of information also increases

²⁵An almost similar segmentation of information can be find in the famous work of Hayek [39] under the appellation of *objective* information and *subjective* information. He refers to objective information all the facts and to subjective information all that is relative to agents minds.

such that we can face exogenous sources of information and endogenous sources of information. We also recall that information arrives at different frequencies in different markets. For agricultural commodity markets for instance, new information on the supply is not as frequent as for the most liquid equity markets (cf. subsection 1.1.2). Hence, we distinguish between financial markets whose information arrives essentially exogenously to the transactions in the considered market, and financial markets whose information arrives essentially endogenously to the transactions in the considered market.

The typology of information used in this thesis can be represented with the help of the following table:

	Exogenous	Endogenous
Effective		
Not effective		

Table 1.3 – Typology of information

1.2.3 Some stylized facts influencing the information on financial markets

On financial markets, some facts or practices influence the release of information. Such facts influence then, the fundamental values of the concerned assets as that value is contingent to the information. We propose in this subsection a list – that we do not pretend to be exhaustive – of such phenomenons. We present the way in which they are usually studied in the literature but also, the principal results emerging from these studies.

The fragmentation of orders

The fragmentation of orders (cf. definition 9), that can be voluntary or forced, is a challenge for theoretical works as the simple fact of considering this practice prevents from being in an equilibrium framework and to consider the price as being an *equilibrium price* (cf. definition 13). The study of such a phenomenon can be generalized to the study of an evolution in the trades frequencies, one dimension of the market *liquidity*. This question is even more relevant when considering that the response of prices to quantities is not linear, as early pointed out by Kempf and Korn [45] in their study of the intraday data on German index futures and more recently by Tóth et al. [75] in their study of the futures market.

Because sending an order to an order book provides information, prices react to it. Hence, a stream of academic works started to investigate the problem of the optimal placement of child limit orders (see for instance Bayraktar and Ludkovski [6], Hendricks and Wilcox [41], Yam and Zhou [80]).

The fragmentation of orders, and equivalently the size of placed orders, are contingent to some market characteristics. The empirical work of Rinaldo [64] on the Swiss stock exchange underlies that the thinner the opposite side book is (opposite means the buy side when a trader wants to sell and the sell side when the trader wants to buy), the more patient traders place aggressive orders (i.e. large size orders). He also shows that orders aggressiveness increases with a wider spread, and within temporary volatile periods. The empirical study of Comerton-Forde and Tang [16] on the Australian stock exchange

highlights that order aggressiveness declined with the introduction of anonymous trading. However, they show that the order book depth (i.e. the *market depth*) has also increased, as well as the average limit order size. This is consistent with the fact that with anonymity, agents convey less information and then are more inclined to display their *real* wish to trade.

The impact of the size of trades was then investigated. A major empirical work of Easley and O'hara [24] shows that trade size can also be explained by an adverse selection. Informed traders are more inclined to trade large amounts at any given price when relevant (i.e. when their private information induces a significant price variation). This result is consistent with the empirical study of Biais et al. [7], who shows that shifts in both bid and ask quotes occur after large size transactions. Furthermore, Lin et al. [50] find that the adverse selection component increases uniformly with trade size. Hence, traders react accordingly to the displayed information. They integrated that large size orders often come from informed traders, and then, adjust their orders accordingly. An extension of this relies on the frequencies of the trades that also convey information, as stated by Easley et al. [25]. An empirical study of Easley et al. [26] also shows that information is extracted from trading time intervals but also no-trading time intervals, in other words, trades frequencies. These two dimensions of trading are parts of the volatility-volume relation defined as a positive relation between prices volatility and trading volumes. The empirical study of Chan and Fong [14] on the [New York Stock Exchange \(NYSE\)](#) and the Nasdaq underlies a higher significance of the size of trades, beyond that of the number of trades, in the volatility–volume relation.

Finally, the study of the impacts of orders fragmentation raises the question of the existence of an optimal frequency of trades. So far, to the best of our knowledge, this question has only been treated by Du and Zhu [22]. This theoretical work is however microfounded with a double auction framework assuming that all wishes of economic agents are centralized by an auctioneer who then determines the *equilibrium price* clearing the market. However, as previously explained in subsection 1.2.1, this theoretical tool is more adequate to a fixing pricing. Furthermore, in this work, information and signals about the asset value are assumed to be exogenous to the market. Then, positive works on the effect of an evolution in trades frequencies and normative works on the optimal trades frequencies are necessary to improve the understanding of current financial markets where trades frequencies evolve, particularly when considering that trades frequencies and size also provide information themselves. Indeed, a crucial point is that trades frequencies and sizes are a source of information for traders. Then, they also impact the fundamental value of the underlying asset. The same reasoning applies on agricultural commodity markets. As stated by Working [78], transaction costs are structurally different between the futures markets and the spot market of a given commodity. As a consequence, if liquidity evolve on the two markets (due to orders fragmentation or not), it may not evolve proportionally. To the best of our knowledge, no academic study deals with the latest question.

The number of agents

Atomicity is one condition of pure competition. If atomistic, agents have no *market power* and by definition, no impact on prices individually. They are said to be *price takers*. However, another interpretation of atomicity emerges when considering that agents actions convey information. If a finite number of agents ([n.o.a](#)) is present on a (financial) market

such that there is no atomicity, we can equivalently consider that each agent separately convey a no redundant information when compared to the other agents. But if there is an infinite *n.o.a* on the market such that there is atomicity, each agent separately conveys no new information. Indeed, all information conveyed is redundant when compared to the information conveyed by all the remaining agents. In other words, the consideration of the absence of atomicity in markets also deals with the study of the information conveyed by agents according to both dimensions that we introduced. The *n.o.a* influences the presence of *effective* and *non effective* information that can be displayed endogenously at a varying frequency. For instance, the more there are agents, the more information is released with a high frequency (if agents display information asynchronously). Likewise, the evolution of the *n.o.a* can naturally modify the relative quantity of *effective* and *non effective* information.

This reasoning leads to numerous theoretical questions that are connected to some empirical facts. For instance, the financial crisis of 2008 was followed by a spike of commodity prices, in particular on energy commodity markets and main agricultural commodity markets. The Masters' hypothesis, referring to claims by investment manager Michael Masters, argues that significant flows of cash coming from institutional investors into commodity index funds is a possible explanation.²⁶ Another way of understanding the problem pointed out by Masters is that the entrance of a sudden *n.o.a* (i.e. a shock in the *n.o.a*), has altered these markets' efficiency.

A wide literature investigate both theoretically and empirically the impacts of intrinsically different economic agents but also the impacts of an evolution of the composition of the market on financial markets and on commodity markets. On financial markets, several kinds of typology of agents are used (as previously mentioned in subsection 1.1.1).

Hence, numerous academic works study the impacts of a change in the structure of agents in a financial market, assuming that the *n.o.a* is fixed. Since the claim of M. Friedman that only irrational speculators are price destabilizing, a wide range of theoretical works investigated the possibility of rational speculators to destabilize prices. The work of De Long et al. [21] was one of the first (of a large series) to strongly invalidate this claim. It lead to the distinction of several type of speculators, whose impacts were investigated. Then, economists early studied how the market efficiency evolves according to the proportion of chartists (cf. Lux and Marchesi [53]), or noise traders (cf. Lux and Marchesi [54]), compared to fundamentalists traders on the financial market. These studies argued that the composition of the market matters, as the more chartists (respectively noise traders) there are, the more episodes of high volatility are frequent.

The contemporaneous increase in the weight of the so called institutional investors implied economists to introduce a new category of economic agents: institutional investors. They can be defined as traders who care about some prices indices and may have a large amount of liquidity.²⁷ If only the dimension of the large size of liquidity is selected, we face a distinction between large traders and small traders, like in Glebkin [36]. In his work, large traders do not act as *price takers* such that when increasing the market depth of the market, they diminish the informational efficiency of the market. When considering that is their utility function they integrate the value of some prices indices, Basak and Pavlova [4] showed that their presence induce an increase in indexed prices. Further-

²⁶The testimony of Michael W. Masters before the Committee on Homeland Security and Governmental Affairs of the United States Senate is available at the following link: <https://www.hsgac.senate.gov/imo/media/doc/052008Masters.pdf>

²⁷The OECD published in 1998 a consequent empirical study on institutional investors (see. Blommestein et al. [10]).

more, the more institutional investors represent a high proportion of the agents, the more indexed prices increase.

Behavioral finance plays a major role in these studies. De Bondt and Thaler [20] early showed the presence of overreaction traders in their study of NYSE monthly returns between 1926 and 1982. The major contribution of this work leads numerous theoretical and empirical studies on the importance of such type of traders behavior. Among them García et al. [34] recently exposed that an increase in overconfidence makes rational traders reduce their investment in information acquisition, the market price being less informative and not efficient *in fine*. Introducing a feedback loop between the information of traders (composed with past prices, and some exogenous information) and the current price through the price series, Wyart and Bouchaud [79] explain in this framework the occurrence of traders overreaction and show that it also leads to an excess volatility of the price.

Another explanation of the behavioral finance is the possibility of having mimetic traders, i.e. traders who consider past and present actions of other traders as the only source of information. Orléan [61] early showed how mimetic contagion happens when there is uncertainty. The latest pushes traders to copy the others as they are a certain source of information. Topol [74] and Lux [52] also investigated the impacts of mimetic contagion on prices. They respectively model the mimicry through the distance between the actual returns to the average one, and the wish to "capture information not available for himself and that he expects to be known by other investors and included in their prices" (Topol [74], p789). One of the most powerful tools to explain stylized facts on financial markets such as excess volatility puzzle, volatility clustering addressed by the behavioral finance is the Agent-Based Model as underlined in the work of Cont [17]. To this extent, Barberis et al. [3] developed the X-CAPM model, able to consider not only heterogeneity between different types of traders but also heterogeneity within each type of traders.

Last but not least, the heterogeneity between traders orders speed and their impact on market efficiency have become an increasing field of research with the ongoing of high frequency trading. As it is well explained in the work of Hoffmann [43], having the capability to place orders faster than other traders is a source of *market power*.

A particular attention was also given to commodity markets regarding the structure of agents operating, starting with the impact of speculators. The famous work of Hart and Kreps [38] early underscored the rational destabilization emerging from speculators whose actions are the storage of commodity. Ekeland et al. [27], with the model they proposed unifying the hedging pressure theory and the storage theory, highlight what are the conditions for speculators to increase the benefits of hedging strategies. Finally, the major commodity spike in 2007 and 2008 leads to some research on the impact of institutional investors into prices. In the multi-commodity model that they build, Basak and Pavlova [5] highlight the mechanism through which, the presence of institutional investors different from the others by having a utility depending on wealth but also price index, lead to an increase in all futures commodity prices – but even more for the indexed one – affecting in turn all the associated spot prices for storable commodities. On these works, the impacts on the fundamental commodity values are taken into account mainly through the storage level which can be assimilated to the *effective* dimension of information. By referring to the work of Du and Zhu [22], we recall that the more agents there are on a market the higher optimal trade frequency on the (futures commodity) markets is. However, we argue that frequency of displayed information is also impacted by an evolution of the structure of the market in terms of agents. As a consequence, the *frequency* dimension of

information released should be taken into account on markets whose information is not necessarily continuous that is, on agricultural commodity markets for instance.

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Following the previous reflection, this thesis aims to answer the following question:

How do the nature and the structure of information affect the optimal pricing algorithm to guarantee market efficiency and minimize fundamental prices volatility ?

1.3 Thesis contributions to scientific literature

This work is an information based work, and the scientific research question mentioned in section 1.2 can be decomposed in three stages. In chapter 2 we investigate the impacts of a continuous and exogenous information that can be effective or not, within a market where a **LOB** operates. A "big picture" of this chapter is given in subsection 1.3.1. In the remaining of this manuscript, we will focus on agricultural commodity markets. The following subsection 1.3.2 presents the chapter 3. That chapter deals with the specificity of agricultural commodity markets that are translated into information properties. We investigate the impact of a variation in trades frequency, one dimension of the liquidity. The last subsection 1.3.3 investigates the impact of an entrance of agents on the agricultural commodity markets. Again, we model agents by their impacts on information properties.

Compared to our presentation of contemporaneous financial and agricultural commodity markets we highlight that we do not deal in this thesis with the multiplicity of trading places – also called trading *venues* – as well as with dark pools and asymmetric information.

1.3.1 Investigating the impact of a continuous exogenous information (effective or not) with a limit order book pricing algorithm (chapter 2)

Markets where exogenous information arrives continuously must produce a price dynamics whose frequency of variation is permanent. Thus, a LOB pricing seems adapted. It is in this framework that we situate the first contribution of this thesis, starting from the theoretical work of Bretto and Priolon [11]. This approach takes advantage of some properties of the formalism of quantum physics and specifically some of its probabilistic properties. Then, each independent agent influences the price when s/he sends an order that is aggregated in a central order book and each price influences in turn all independent agents in a never ending process. In this work, information's nature and structure is given by the following table 1.4.

	Exogenous	Endogenous
Effective	X	
Not effective	X	

Table 1.4 – Typology of information used in chapter 2 (X represents the characteristics used)

Under our hypotheses, we demonstrate that if information is mainly or even fully treatable using algorithms, a high frequency of trades allows the market to be semi-strong efficient (information is publicly and freely available in the model). The capability of the market to be semi-strong efficient fully depends on the nature and structure of information. We extend the outputs of the initial model by showing that under a moderated growth of information that is non negligible compared to effective information, a bubble appears, and ends up with a financial crisis (see chapter 2 for rigorous definitions of *bubble* and *financial crisis*). We also introduce the concept of *computational strength* whose closest formulation comes from Hayek [40], to the best of our knowledge, which allows to distinguish between a computationally strong market and a no computationally strong market; In a no computationally strong market there is always at least one agent who is better at integrating relevant information into the price (its *order price*) than the market. Figure 1.1 summarizes what the introduction of the concept of *computational strength* changes to the description of a financial market.

In the light of this concept, within a period of bubble, the financial market can keep its computational strength until the bubble turns into a crisis. Above this threshold, within a financial crisis, it is impossible for the market to keep its computational strength. As a consequence, above this threshold, a financial market is neither efficient nor computationally strong.

Aside this new concept and its theoretical consequences we also propose indicators of the computational strength of a financial market that can be implemented by the financial market regulator at several given real-time backward (since the computational strength is computed on a given period).

We also develop the implications of the concept of *market system* introduced in Bretto and Priolon [11]. We define the *market system* adequate to a *perfect competition* – with a finite or infinite *n.o.a.* We proceed with simulations to show to what extent a VARMA process estimation can be used to anticipate agents *market power*.

The main contribution of this work is that it gives another argument in favor of the implementation of a LOB pricing for assets whose information arrives continuously. How-

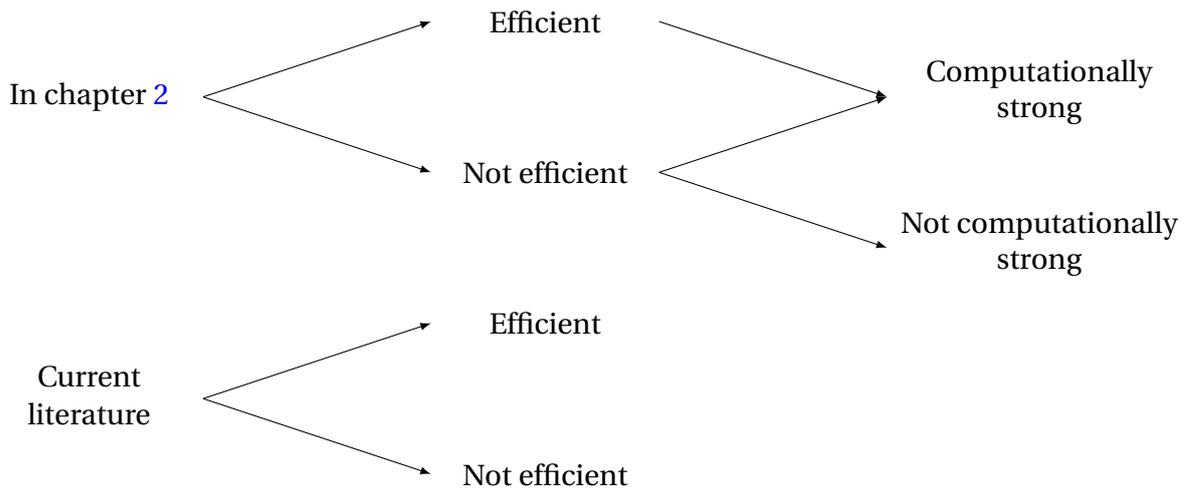


Figure 1.1 – Impact of the introduction of the computational strength concept on the characterization of a – financial – market.

ever, financial regulator must limit the sources of *non effective* information to avoid the emergence of a bubble and its transformation into a financial crisis.

The next two chapters focus on agricultural commodity markets (cf. definition 11); We consider that the financial commodity market is composed with futures contracts so we do not study commodity swaps and option contracts.

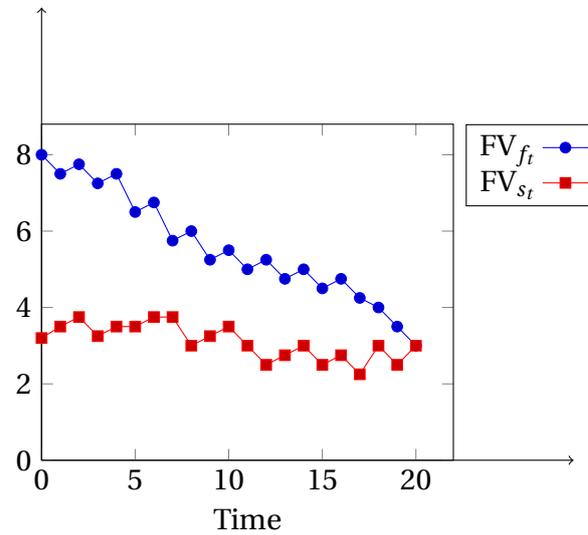
1.3.2 Investigating the impact of trades frequencies on an agricultural commodity market (chapter 3)

Agricultural commodity markets have a peculiar structure of information. Information arrives with a relatively low and fixed frequency. The production is uncertain but not its frequency. However, futures contract are priced with a **LOB** algorithm. However, the relatively low frequency of new information diminishes the cost of losing liquidity as stated by Du and Zhu [22]. Furthermore, we introduce the fact that a transaction generates, in return, new information. Hence, information's nature and structure are influenced by the degree of liquidity on both spot and futures markets. The typology used in this work is given by the following table 1.5.

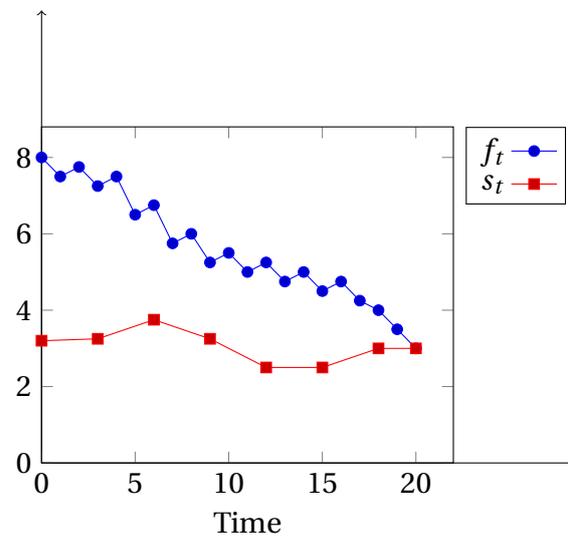
	Exogenous	Endogenous
Effective		X
Not effective		

Table 1.5 – Typology of information used in chapter 3

Transaction costs of the underlying of a futures agricultural commodity contract (namely the agricultural commodity itself) are higher than the transaction costs of the futures contract, such that trades frequency on the spot market is significantly lower. However, one can imagine that fundamental values have a synchronous dynamics. Figure 1.2 illustrates the latest.



(a) Futures (FV_{f_t}) and Spot (FV_{s_t}) fundamental values dynamics



(b) Futures (f_t) and Spot (s_t) real prices dynamics

Figure 1.2 – Representation of both the spot price and the futures contract price fundamental value dynamics and real prices dynamics.

Furthermore, the pricing algorithm used on the spot market is very different, as it is computed as an average of several (but not necessary all) transactions prices. In our work, we will assume that information of the spot market can be centralized by agents and that each transaction modifies the market price. Despite the fact that the modification of the spot market price at each transaction is a strong assumption, assuming that agents are able to centralize the information relative to the spot market is adequate (cf. Simon [70]). The basis (cf. definition 12) varies both according to futures price and to spot price. In this work, we study the impact of such differences on the basis. We particularly focus on the increase in liquidity resulting from fragmentation of orders, faster order transmission, increase in the number of traders, that occurs on agricultural commodity markets; The degree of liquidity results from trading behaviors as well as pricing methods and transaction costs. We study the impacts of these strategies on the fundamental basis volatility. In the wake of this study, we examine the impact of the increase in trade frequencies on the efficiency of spot and futures markets.

We build an original and simple sequential pricing model in the spirit of Black and Scholes [9], Cox et al. [19]. We demonstrate that the fundamental basis volatility can increase even if the increase in liquidity is proportional on both markets. The idea is as simple as it can be: Asynchronous trades on both spot and futures markets generate a risk (according to the definition of Knight [46]). We prove and defend that synchronization of transactions on agricultural commodity markets and *a fortiori* of the frequency of these transactions can allow unbiased prices dynamics on spot and futures markets, a *sine qua non* condition for having an efficient agricultural commodity market. Under these conditions, the fundamental volatility of the futures contract is also minimal. Then, a non-proportional increase in liquidity on spot market and futures market can increase the volatility of information on the futures contract.

From these results, we extrapolate the existence of a trade-off between futures market liquidity and either the fundamental basis volatility or the market efficiency. Indeed, the optimal policy in order to minimize the fundamental basis volatility and maximize the spot and futures market efficiency is to implement a pricing algorithm on the futures market determining the *market price* at the same frequency than transactions on the spot market.

1.3.3 Investigating the influence of the number of agents on an agricultural commodity market (chapter 4)

In this chapter, we extend the model built in chapter 3. We introduce a parameter representing the *n.o.a* and consider their impact on trades frequencies. As transaction costs in these two markets are different (cf. Working [78]), the *n.o.a* can influence trades frequencies in the two markets neither identically nor proportionally. In other words, agents also influence the ratio between these trades frequency (namely the relative trades frequencies). We also consider their impact on traded quantities which we assume to be indirect (since trades size depends on adverse selection and information of each trader as proved by Easley and O'hara [24]). We assume that the number of agents influence the spot market information as "(...)market actions or choices convey information" – Stiglitz [71], p.468.²⁸ Each "new" agent is assumed to convey a not fully redundant information when compared to the other agents. Again, a transaction generates in turn new information. Last but not least, we consider in a first stage that agents correctly interpret avail-

²⁸We could also link this idea with the rational destabilization mechanism in the work of De Long et al. [21] that depends on the proportion of speculators.

able information as well as they form rational expectations on the incoming information. However, we later relax this assumption with the possibility of having underestimating or overestimating biases. Hence, information's nature and structure is influenced by the [n.o.a](#). The typology used in this work is given by the following table [1.6](#).

	Exogenous	Endogenous
Effective		X
Not effective		X

Table 1.6 – Typology of information used in chapter 4

In our work, we also consider that each transaction on the spot price is reported and that the market price is modified accordingly. The basis (cf. definition [12](#)) varies both according to futures price and to spot price. In this work, we study the impact of such differences on the basis. We particularly focus on the increase in the [n.o.a](#), that we can assimilate for instance to the entrance of institutional investors on the main agricultural commodity markets in 2008. We study their impacts on the fundamental basis volatility. In the wake of this study, we examine the impact of the increase in the [n.o.a](#) on the efficiency of spot and futures markets. Our work is close to that of Glebkin [\[36\]](#) who showed to what extent more large traders and then more market depth – a dimension of the liquidity – can reduce the profit of every agent (small ones, and large ones).²⁹ In the end, we also consider that agents can form biased expectations on the spot market information (see Thaler [\[73\]](#) to understand the impacts of all different types of biased traders). Eventually, we assume that this bias is corrected by the [n.o.a](#) acting on the market, i.e. the size of the market. We defend this assumption by referring to Hayek [\[40\]](#). The more there are agents, the more the capability of the market to integrate information by agents economic interactions increases.

We extend the simple sequential pricing model developed in chapter [3](#) and consider in a first stage that agents form rational expectations and correctly interpret available information. We demonstrate that the fundamental basis volatility can increase with the [n.o.a](#). The introduction of new parameters prevents us from having analytical solutions for most of the subset of parameters values. In other words, there is in most of the cases, a mandatory trade-off between liquidity and the two objectives, according to the risk aversion of the policy maker. We proceed with sensitivity tests to study the sensitivity of the optimal [n.o.a](#) to the value of our parameters. In other words, we prove and defend that a finite [n.o.a](#) coupled to the actual [LOB](#) pricing algorithm in financial markets is in most of the cases optimal. If there are more agents than the optimal number however, the optimal pricing algorithm may be a pricing algorithm reducing trades frequency on the futures market. In other words, an entrance of agents on an agricultural commodity market can increase the volatility of information on the futures contract and reduce the agricultural market efficiency. When considering biased expectations and interpretation of information, we find that underestimating bias and overestimating bias have asymmetric effects and are not equivalent.³⁰ Within an overestimating bias case, the difference in trade frequencies can perfectly compensate the overestimation bias such that the spot price can be unbiased. It is not the case within an underestimating bias case. The complexity of the problem prevents us from having tractable analytical solutions. We proceed with sensitivity tests

²⁹In his model, large traders receive an identical and exogenous private information. Small traders receive a dispersed exogenous private information.

³⁰The overestimating bias could have been named *exuberance* bias.

to study the sensitivity of the optimal [n.o.a](#) to the value of our parameters. We study how this *a priori* incentive – since the bias is a decreasing function of the [n.o.a](#) – to let agents enter the market influences our results by comparing these sensitivity tests to the ones with correct interpretations and rational expectations on the spot market information.

From these results, we confirm the existence of a trade-off between futures market liquidity (which is an increasing function of the [n.o.a](#) on the agricultural commodity futures market), and either the fundamental basis volatility or the agricultural commodity market efficiency. This result is consistent with the one of [Glebkin \[36\]](#) who underscores the trade-off between market liquidity (more exactly the market depth) and market efficiency (namely informational efficiency in his work).

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Chapter 2

The nature and the structure of information evolve: Do financial markets remain efficient ?

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2.1 Introduction

Our work is motivated by the following question: Do prices on modern financial markets fully and instantaneously reflect the huge, growing and accelerating flows of available information? If prices fully and instantaneously reflect information, then the market is described as efficient. The market efficiency research programme proposed by Fama [10], Fama et al. [11] fifty years ago has been reinvigorated because of the changes in the nature and the "size" of information processed by economic agents. Information grows more and more and transactions occur permanently regardless of whether or not aggregate supply equals aggregate demand (cf. O'Hara [24], Schutz et al. [28], Tóth et al. [34]). That evolution is a challenge for financial economists who:

"want to think of asset prices as being determined by the interaction of rational agents – that is, as being determined as an economic equilibrium [...]" – cf. LeRoy [23] (p.1584).

Agent-based models are the most common tool to address the heterogeneity of agents and to study how their interactions influence or determine prices on financial markets (see for instance Oldham [25]). In the model that we present below not only each state of the market is determined by the set of agents' expectations but expectations of agents are – in turn – influenced by the state of the market, in a never ending movement.

For modern financial markets, the available information are mainly contained in the order books that are permanently updated (cf. Cont and Larrard [6]). In addition, and aside from past and present information, the series of order books provide a kind of insight into agents' expectations;

"In contrast to looking at the volume traded, which shows what already happened, the order book provides an insight into the intention of the traders" – cf. Galanos [13] (p.10).

The order book contains agents decisions and we model it as a vector of information. This idea is consistent with Stiglitz who argues that:

"(...) market actions or choices convey information. Market participants know this and respond accordingly." – cf. Stiglitz [31] (p. 468)

Our conceptual framework spans three important fields of economics i) the formation of prices ii) the market efficiency research programme iii) the economics of information.

Research about the formation of prices has produced some very significant results (see Biais et al. [3] for a review). In our model, prices result from the resolution of an order book which is regularly updated (cf. Cont et al. [7]). Series of prices result from agents expectations when they process information and then pass sell or buy orders. When Hayek studied the formation of prices, he addressed the capability of markets to integrate information better than any agent considered individually (see Hayek [15]). In the wake of Hayek, we call *computational strength* of the market the capability of the market to process information better than any agent considered individually.

The computational strength of the market is indirectly studied by the market efficiency research programme. Recall that a market is said to be efficient if all information is instantaneously integrated into the price. Hence, if a market is efficient, it has necessarily a computational strength. In this case, an agent can only "do as good as the market" but

cannot "do better than the market" at integrating information into the price.¹ This distinction leads to the following question: even if a market is not efficient, is it still better at (correctly) integrating information than any agent alone? The latest does not imply that if there is an agent doing "better than the market", s/he is making arbitrage operations. S/he can be on the selling side with a price superior to the market price (her/his order is not executed) and conversely. An illustration of the latest is the existence of rational bubbles (cf. Diba and Grossman [9], Jang and Kang [19]).

Our work also deals with economics of information. More precisely, the research question of this paper can be stated with the words of a synthetic work of Stiglitz (cf. Stiglitz [31]) on the role of information:

"At the beginning of time, the full equilibrium was solved, and everything from then on was an unfolding over time of what had been planned in each of the contingencies. In the real world, the critical question was : How and how well, do markets handle fundamental problems of information ?" – cf. Stiglitz [31] (p.467)

Our research is partly based on a previous work (see Bretto and Priolon [4]); The use of quantum tools allows to take the complexity of information into account (cf. Cotfas [8]). The use of statistical tools to consider information complexity was early suggested by Walter [36]. In our model, information is public and freely available at any time but we introduce a crucial distinction: one part of it can be processed with the help of an algorithm and it is called *effective information* but the other part cannot be processed with the help of an algorithm and it is called *non effective information*. That latter part has to be treated by agents' minds directly.

Formally, we model the incorporation of information in a sequential order book after information has been processed by agents, and we give a mathematical status to that information. Order book prices are considered as eigenvalues of a diagonal matrix that influences the market. Therefore, order prices become mathematical objects that have an influence on the market dynamics. In other words, an order book is not only a table of prices and quantities, it also records the market dynamics, which, in turn, has an effect on price' dynamics. This framework emphasizes the importance of the wish to trade for determining the value of an asset (cf. Schaden [27]). This model considers agents not just as price takers, but also as price makers with a certain probability (cf. Choustova [5]). From this consideration emerges what we refer to as the *market system*. The latest contains information on the state of the market at each measurement of the market (measure of traded quantities, measure of the new price and so on) and we can infer the statistical *market power* of each trader in a way described below. For instance, the estimation of the *market system* in a period of time and its prediction could help to study if the *market power* of high frequency traders is significant compared to the one of slow traders according to the work of Hoffmann [17]. That model equally gives an original basis for formal discussions about bubbles and crisis compatible with the existing literature. Finally, the main theoretical question is: What are the conditions on the nature and on the relative quantity of information in order to allow a financial market to be semi-strong efficient or to be at least better at integrating correctly information into the price than any agent considered individually?

¹It means that we compare the financial market price considered to be the outcome of the market, and the price entered by any agent individually into the order book of the same financial market. The first one is the process of the financial market as a whole and the second is the process of one agent considered individually.

Section 2.2 presents all the hypothesis of the model and mathematical tools used while section 2.3 underlines some of its intermediary outcomes. In section 2.4 we study the impact of the nature and the structure of information on the characteristics of the financial market. We demonstrate that information can generate a bubble or even a financial crisis. We also study within each case, what is the computational strength of the market. Section 2.5 provides some possible applications of this theoretical model for trader on the first hand, and for policy maker in the other hand. We conclude in section 2.6.

2.2 The model

We consider an order driven financial market on which one single asset is traded over a period that we denote $[0; T] \subset \mathbb{N}$. We then denote t every instant of time when one trade occurs with $t \in [0; T]$. The market evolves continuously under the influence of what we call the *market system* (see subsection 2.3.2) but prices p_t , quantities q_t etc. are discrete values observed at time t . Without losing generality, we normalize any time interval $[t_1; t_2]$ by $[1; t]$.

In subsection 2.2.1 we define in a natural way the information nature and in a rigorous way how we model it. Then, subsection 2.2.2 provide all the assumption we need to model agents decisions. In subsection 2.2.3, we provide a definition and some useful insight of what we call the *market system* to understand how we apprehend the market dynamics. Once the information based market is settled, we respectively define in a proper way the fundamental value of the asset in subsection 2.2.4, and what we call a semi-strong efficient market in subsection 2.2.5. By extension the latest subsection provides a formal definition of a financial market in crisis or in a bubble. Subsection 2.2.6 provide a conceptual and formal definition of a market endue with a computational strength. At the end, we provide some pure mathematical definitions that we need to extract results of such a model in subsection 2.2.7.

2.2.1 The information's nature

We denote by $I([t_i; t_{i+1}[)$ the *information* that is released during the period $[t_i; t_{i+1}[\subset [0; T] \subset \mathbb{N}$. We suppose that $I(\emptyset) = \emptyset$, and also that all information is publicly and freely available; However agents don't know perfectly the future.

An *information parameter* of the market denoted by pa is represented by a function capable of changing the state of the market through what we call the market system (see below). Let the following function define and model those parameters:

$$\begin{array}{rcl} pa: & I \subseteq [0; T] \subset \mathbb{N} & \rightarrow V \\ & t & \mapsto pa(t) \end{array}$$

The set V is a linear space (with a finite dimension), and I is the union of disjoint segments.

We now distinguish a typology of the nature of information itself.²

Definition 15. *Information parameters are characteristics of the market. They can be:*

²A similar typology can be found in the work of F. Hayek, who uses the terms objective and subjective information.

- (i) Effective information parameters *if an algorithm can process it and produce a decimal expansion with it (a price for instance)*,³
- (ii) Non effective information parameters *if not*.

Financial economics are mainly divided into two streams of research and our dichotomous typology of the nature of information can be understood by these two streams of research. On the first hand, the classical finance theoretical framework usually considers perfectly rational agents combined to –implicit– fully effective information flows leading to efficient financial markets (for instance Vives [35]). On the other hand, the behavioral finance theoretical framework focuses on the process of information by agents and often points at the limit of the market efficiency through agents limited rationality or even irrationality (cf. Shiller [30]).

Both types of *information parameters* denoted by ef and nef are represented by functions capable of changing the market system. Let the following functions define and model those parameters:

$$\begin{aligned} ef : I \subseteq [0; T] &\rightarrow V \\ t &\mapsto ef(t) \end{aligned}$$

The function ef is an effective information parameter function belonging to the piecewise class C^1 .

$$\begin{aligned} nef : I \subseteq [0; T] &\rightarrow V \\ t &\mapsto nef(t) \end{aligned}$$

The function nef is a non effective information parameter function belonging to the piecewise continuous functions class.

These two types of parameters have an impact on agents' expectations (see Lang [21, 22], Rudin [26], Weihrauch [37]). This typology distinguishes information that can be treated by an algorithm (for instance entering the volatility measure into the Black-Scholes-Merton formula) from the information that can only be processed by the economic agent's mind (for instance the speech of a central bank governor). The structure of information has a crucial role on the capability of economic agents to process it and *in fine* on the capability of the market to be semi-strong efficient.

We denote respectively the linear space on \mathbb{R} formed by the set of information parameters of the market, the linear space on \mathbb{R} formed by the set of effective information parameters of the market and the linear space on \mathbb{R} formed by the set of non effective information parameters of the market equipped with zero function by $\mathcal{PA}(I; V)$, $\mathcal{EF}(I; V)$ and $\mathcal{NEF}(I; V)$.

2.2.2 The individual decisions

Let $i \in \{1, 2, \dots, n\}$ denote the i^{th} agent acting on the market. Hence, n is the number of agents on the market and n is also the rank of the order book. Agents stand for an orthonormal basis of the Hilbert space (denoted by $e_i, \forall i \in \{1, \dots, n\}$) that constitutes the

³We recall that a real number is called *effective*, if there exists at least an algorithm which can produce its decimal expansion.

market, denoted by \mathcal{H} .⁴ This point is very important in the model: A market is not an external object with an auctioneer processing information centrally according to the wish to trade of agents. A market exists because agents are acting, they enter orders into the order book. This view is consistent with the reasoning of agent based model (cf. Tesfatsion [32]) but also with the point of view of F.A. Hayek who defines the market as:

" (...) A system of the utilization of knowledge which nobody can possess as a whole, which (...) leads people to aim at the needs of people whom they do not know, make use of facilities about which they have no direct information; all this condensed in abstract signals – cf. Hayek [16] (p. 80).

These agents are heterogeneous, independent, rational and equally informed. They may be for instance portfolio managers who must optimize the risk-adjusted performance of their portfolio (cf. Sharpe [29]); these agents have heterogeneous constraints so they do not necessarily take homogeneous decisions even if they have the same analysis of the state of the market. The objective function of any agent is simply defined: any successful adjustment of the portfolio increases the gains of the portfolio manager. In other words any manager must try to guess at what price the next trade will occur if s/he wants her/his order to be executed.

At any given time $t \in [0; T]$, each agent must make one decision which is entered into the order book of the market and is treated at instant t : i can enter a sell order, a buy order or choose to send no order. If the order fits with the instantaneous state of the market, then it is executed. If the order does not fit, it is canceled. If i wants to repeat an order which has not been executed, s/he must enter it again. Each agent has full and free access to the set of information $I([0; t])$ at time t . We call $M_{i,t}$ the model which allows i to decide which order s/he will enter for execution at time t . This model is defined by a function (see below). Agents may be heterogeneous; they do not need to use the same model to process the economic information. The agents' models are time evolving because agents are able to adapt and improve regularly the model they use. Hence, the couple $(i; M_{i,t})$ may be seen as an oracle machine. We write $E_i(\tilde{p}_t) = M_{i,t}(I([0; t]))$, where E_i represents the expectation of agent i once s/he has processed information. The $M_{i,t}$ is a combination of two components at time t :

- (i) The effective dimension of the model denoted by $M_{i,t}^{ef}$. Agent i uses it to analyze the effective subset of available information at time t , such that $M_{i,t}^{ef}(I([0; t])) = \mu_{i,t}$;
- (ii) The non effective dimension of the model denoted by $M_{i,t}^{nef}$. Agent i uses it to analyze the non effective subset of available information. This means that the agent has to rely on a qualitative judgment, such that $M_{i,t}^{nef}(I([0; t])) = \kappa_{i,t}$.

Then, $M_{i,t}(I([0; t])) = E_i(\tilde{p}_t) = \mu_{i,t} + \kappa_{i,t}$. The well known dichotomy between classical finance and behavioral finance can be viewed with the help of that typology of information. Implicitly, classical finance considers almost only effective information under scrutiny, and agents process it uniformly; in behavioral finance information is not totally effective and it is not perfectly processed by agents. As we theorize the two kinds of information in the same analytical framework we are able to jointly consider the two research programmes.

⁴Within an economic reality, agents need to make calculations but we do not know in advance what kind of calculations they have to do. They may be analytical, geometrical, topological, and so on. For this purpose, we situate the economy and by extension the financial market in a Hilbert space where all types of calculations can be done.

At time t , n orders are entered looking for being executed. If at least one order is compatible with the state of the market, a quantity q_t is traded at the price p_t . Prices and quantities are quantifiable properties of the market (see below). mh

Definition 16. A quantifiable property (or quantifiable for short) of a market is a characteristic of this market that can be evaluated, measured, read, ... at time t . For instance prices, quantities, discount rates, dividends, ... are quantifiable properties. We denote them by \mathcal{Q}_t (\mathcal{Q} when there is no ambiguity).

Remark 1. As stock prices are positive real number, eigenvalues of the quantifiable price also are. Hence the quantifiable matrix is positive-semidefinite.

To each order book we associate a diagonalized matrix assumed to be a quantifiable \mathcal{Q}_t that acts on \mathcal{H} ; the only possible outcome of the measurement of the quantifiable is one of its eigenvalues. In other words, the next market price is necessary one of the price that agents entered in the order book.

2.2.3 The market system

The concept of *market system*, denoted ms_t (ms when there is no ambiguity), is at the heart of the model. We call market system a mathematical function which explains the market dynamics over time. We consider that over $[0;T]$, p_t and q_t form discrete sequences, whose properties are determined by ms . Then, a market is completely defined at time t by a unitary vector ms_t belonging to the Hilbert space denoted by \mathcal{H} .

Definition 17. Let \mathcal{Q} be a quantifiable property of a market system whose eigenvectors are $(e_i)_{i \in \{1,2,\dots,n\}}$, an orthonormal basis of \mathcal{H} . Hence, the market system is defined as follows:

$$ms_t = \sum_i \beta_{i,t} e_i, \text{ with } \beta_{i,t} \in]0, 1[\subset \mathbb{R}^+ \forall (i, t), \text{ and } \sum_i \beta_{i,t} = 1 \quad (2.1)$$

The probability that the result of the measurement of the quantifiable \mathcal{Q}_t at time t be $E_j(\tilde{p}_t)$ is then $\sum_{i=1}^n \mathbf{1}_{\{E_i(\tilde{p}_t)=E_j(\tilde{p}_t)\}} \beta_i$, where $\sum_{i=1}^n \mathbf{1}_{\{E_i(\tilde{p}_t)=E_j(\tilde{p}_t)\}}$ gives the number of eigenvectors linearly independent (i.e. the number of independent agents) associated with the eigenvalue $E_j(\tilde{p}_t)$ (i.e. the number of agents who entered on the order book an order at the same price $E_j(\tilde{p}_t)$). In other words:

$$\Pr(\text{Measurement of } \mathcal{Q}_t = E_j(\tilde{p}_t)) = \sum_{i=1}^n \mathbf{1}_{\{E_i(\tilde{p}_t)=E_j(\tilde{p}_t)\}} \beta_i$$

Eigenvalues are the predictions of agents, the price they enter in the order book; Therefore, they have a probability strictly positive to be the measure of the quantifiable. These probabilities give the coefficients of the vector ms . This shows that:

- (i) The state of the market influences agents' decisions because agents process the information they have on the state of the market;
- (ii) All agents' decisions modify the state of the market.

We extend this definition to define the *effective market system* denoted by $ms_{t,ef}$ (ms_{ef} when there is no ambiguity), and the *non effective market system* denoted by $ms_{t,nef}$ (ms_{nef} when there is no ambiguity), such that:

$$\begin{aligned} ms_{t,ef} &= \sum_i \beta'_{i,t} e_i & \text{with } \beta_{i,t} \in]0, 1[\subset \mathbb{R}^+ \forall i, t & \text{ and } \sum_i \beta_{i,t} = 1 \\ ms_{t,nef} &= \sum_i \beta''_{i,t} e_i & \text{with } \beta_{i,t} \in]0, 1[\subset \mathbb{R}^+ \forall i, t & \text{ and } \sum_i \beta_{i,t} = 1 \end{aligned}$$

These different *market systems* have different coordinates $\beta_{i,t}$ at time t (i.e. $ms_t \neq ms_{t,ef} \neq ms_{t,nef}$). Despite the *market system*, ms , gathers theoretically all available information, the *effective market system*, as well as the *non effective market system* only gather respectively the *effective* part of information and the *non effective* part of information. There is obviously a relation between those market system and their coordinates but we cannot extrapolate it.

Remark 2. *The market system is linked to the notion of market power and by extension the degree of competition. If at a given period, the market system is such that $\beta_i = 1$ and $\beta_j = 0 \forall j \neq i, j \in \{1, n\}$, the market power of agent i is maximum, and others agent have no market power since only agent i is able to influence the market price. We denote with the exponent c characteristics of the perfect competition. A perfect competition situation is then characterized by the following property:*

- (i) *For a finite number of agents n , $\beta_i^c = \beta^c = \frac{1}{n} \forall i \in \{1, n\}$;*
- (ii) *For an infinite number of agents ($n \rightarrow +\infty$), $\beta_i^c = \beta^c = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0 \forall i \in \{1, n\}$.*

This tool is consistent with the following reasoning:

"In [the] model, [Fast Trader]s are able to avoid being adversely selected. However, because agents trade directly with each other in a dynamic setting, this effectively increases their market power and allows them to extract rents from slower market participants." – cf. Hoffmann [17] (p.157)

Despite we do not integrate an adverse selection mechanism, the *market system* allows to determine whose agents have statistically the more influence on prices and by definition, whose agents have statistically the more *market power*.

2.2.4 The fundamental value

In order to define precisely the concept of market efficiency (see below subsection 2.2.5), we first need to define or assume what the price of the asset should be, its so called *fundamental value*.

Definition 18. *The fundamental value of the asset at time t , or equivalently its best possible evaluation at time t , is equal to the processing of effective information parameters by the economic agent (= the oracle machine) whose order is executed (= who gives the measure of the quantifiable price) at time t .*

Then, if the measure at time t of the quantifiable price is given by agent i , denoted by $E_i(\tilde{p}_t)$, the fundamental value at time t of the quantifiable price is given by the processing of effective information by agent i 's algorithm, denoted by $\mu_{i,t}$.

This definition is fully compatible with that of Fama [10], but is also a response to the main critics that can be addressed to the standard definition of the fundamental value as being the discounted value of future cash-flows (see Walter [36] for a review). We assume that all information represent everything that may happen, i.e. the set of all possible outcomes. The effective information can be processed by an algorithm and the non effective information represents everything else and must be treated by human mind. However, no agent can perfectly evaluate the probability of occurrence of any event that may happen. Under that definition, the fundamental value of an asset results from the processing of the effective information.

2.2.5 The market semi-strong efficiency, the financial market crisis, and the bubble

We can now give a formal definition of a *semi-strong efficient market*.⁵

Definition 19. A semi-strong efficient market between 1 and t is such that if the price at time k is given by any agent i_k (i.e. $p_k = E_{i_k}(\tilde{p}_k)$), we have:

$$|E_{i_k}(\tilde{p}_k) - \mu_{i_k,k}| \leq \epsilon_{i_k}, \forall k \in [1; t], \text{ with } \epsilon_{i_k} \text{ "small"}^6 \quad (2.2)$$

Then, we call semi-strong efficient a market in which, prices and other quantifiables belong to a neighborhood of their fundamental value over the period that is considered.

Latest definition allows the study of the conditions on the nature and relative quantity of information which ensure that a market is semi-strong efficient.

We also define financial crisis as being a market whose characteristics are the opposite of a semi-strong efficient market. We consider a *financial crisis* as a period where the financial market (all quantifiable whose eigenvalues are positive) is as far as possible of a semi-strong efficient market (i.e. as far as possible of a perfect market). The latest assumption can also be understood as a financial market where there is a radical uncertainty about the valuation of the asset.

Definition 20. A financial crisis is a period of time in which prices and other quantifiables (whose all eigenvalues are positive) belong to a neighborhood of their non effective value. If the price is given by the agent i at time t (i.e. $p_t = E_i(\tilde{p}_t)$), then, a market is in crisis at time t if we have:

$$|E_i(\tilde{p}_t) - \kappa_{i,t}| \leq \epsilon_i, \text{ with } \epsilon_i \text{ "small"} \quad (2.3)$$

From this definition we have the real market system which is ms_{pa} , the semi-strong efficient market system which is ms_{ef} , and the crisis market system which is ms_{nef} , such as defined in subsection 2.2.3.

Our model makes it possible to give a formal definition of a bubble that is fully compatible with classical definitions given in Abreu and Brunnermeier [1], Allen and Gale [2], Hugonnier [18], Tirole [33]. A *bubble* is commonly defined as an excessive spread between the market price and the fundamental value of the asset traded in this market. We translate this definition by considering that if the financial market is neither semi-strong efficient or in a crisis, it is in a bubble.⁷

Definition 21. A bubble is a period of time $B \subseteq I$ such that $nef_i(k) \neq o(ef_i(k)), \forall k \in B$ for some $i \in \{1, 2, 3, \dots, n\}$. We assume that the non effective parameters follow a moderate growth with time such that:

$$nef_i(k) = \mathcal{O}\left((ef_i(k))^k\right) \quad (2.4)$$

⁵It is called semi-strong efficient because there is no private information (see Fama et al. [11]).

⁶In this case, the neighborhood is defined by a classical topology on \mathbb{R} .

⁷Of course we can consider that a financial market in crisis is by extension within a period of bubble.

2.2.6 The computational strength of the market

The concept of *computational strength* takes its origin in Hayek [15]. We now give it a formal definition but we first need an intermediary function. Let $(x_1, \dots, x_t), (\mu_1, \dots, \mu_t) \in \mathbb{R}^+{}^t$. We compute the sum of the inverse of distances between a source x whose fundamental value is μ and we add an ϵ to that distance (ϵ is "small"), powered by a factor s , $s \in \mathbb{S} \subset \mathbb{R}^+ \setminus 0$. This function, denoted by ζ_x , is a spectral zeta function, and it is strictly decreasing with the distance between the source and its fundamental, such that:

$$\begin{aligned} \zeta_x : \mathbb{S} \times [1; T] &\rightarrow \mathbb{R} \\ (s, t) &\mapsto \zeta_x(s, t) = \sum_{j=1}^t \frac{1}{(\epsilon + |x_j - \mu_j|)^s} \end{aligned} \quad (2.5)$$

Among other possible refinements, the use of the spectral zeta links the sum of distances' inverse to the determinant of the distance matrix between the source x and its fundamental μ .⁸ By using this general function, we then give a basis to further definitions of the computational strength.

Definition 22. *The computational strength of the market between 1 and t , denoted by C , is measured by the following function :*

$$\begin{aligned} C : \mathbb{S} \times [1; T] &\rightarrow \mathbb{R} \\ (s, t) &\mapsto C(s, t) = \begin{cases} \zeta_p(s, t) - \max_i \zeta_i(s, t) & \text{if } \zeta_p(s, t) \neq \max_i \zeta_i(s, t) \\ 0 & \text{if } \zeta_p(s, t) = \max_i \zeta_i(s, t) \end{cases} \end{aligned} \quad (2.6)$$

where p denote the price vector among time (i.e. (p_1, p_2, \dots, p_t)) and i the measurement of quantifiable price by agent i vector among time (i.e. $(E_i(\tilde{p}_1), E_i(\tilde{p}_2), \dots, E_i(\tilde{p}_t))$).

The computational strength studies the information integrated by the price relatively to its fundamental value and compare it to the agent that instantaneously integrates information at best relatively to the fundamental value. When an agent is able to anticipate the fundamental price with more accuracy than the market on the period on average, then the market has no computational strength (i.e. there is a computational strength when $C(s, t) \geq 0$). Again an agent can expect with more accuracy the fundamental value of a price without making profit. A counter example is the existence of rational bubbles. An agent correctly expecting the fundamental price is going to loose money against every remaining agents. This distinction is a pragmatic one. When agents are perfectly rational and information is totally effective, a market logically becomes a useful tool and even becomes efficient.⁹ However, if agents are not perfectly rational and information is not fully effective, can the financial market be efficient ? And if not efficient, is it equivalent to say that the market is useless ? The computational strength concept aims to highlight the spread between our theoretical understanding of what is a perfect market and what is a useless market. One can imagine that in some cases, a market can be useful is displaying information better than any agent separately without being perfect when agents are not perfectly rational and information is not fully effective.

⁸Let H be a matrix whose eigenvalues are $\lambda_i, i \in [1; n]$. We associate the spectral zeta function to these eigenvalues $(\zeta(s) = \sum_{i=1}^n \frac{1}{\lambda_i^s})$. Then we have $\det(H) = \exp(-\zeta'(0))$.

⁹In fact the following logic is a tautology. If each agent is perfectly rational and information is totally effective and freely available, the market is useless. All agents have access to all information. They can then, compute the fundamental value alone without interacting with each other which is the aim of a market.

2.2.7 The mathematical formalism

In the sequel of the chapter we use mathematical tools that are not commonly used in financial economics. These tools are not introduced for superfluous refinement but because we need to express some properties that are essential in our view:

- The state of the market at time t influences the decisions of agents;
- The decisions of agents modify the state of the market;
- We need a measure to make comparisons between different states of the market;
- We give a mathematical status to the notion of information.

In the sequel, map is equivalent to function.

This subsection is exclusively dedicated to a set of mathematical preliminaries and definitions which have no direct economic meaning but are necessary to develop the model and to study its properties.

Our main task is to give a mathematical definition of what *almost everywhere* precisely means: we define a measure that makes it possible to compare situations in which the market is driven by *effective* parameters to situations where *non effective* parameters become determinant. These two types of parameters have an impact on agents' expectations. For more precision see Lang [21, 22], Rudin [26], Weihrauch [37].

Measure space

In the model, we need to use a measure space in order to compare different sets or subsets. The object of this subsection is to give a definition of this type of space.

Let \mathcal{S} be a non-empty subset of the power set of X closed under the union of pairs of sets and under complementary set with X belonging to \mathcal{S} . The couple $(X; \mathcal{S})$ is called a *field of sets* or an *algebra of sets*. In other words $(X; \mathcal{S})$ is an algebra of sets if :

- 1) $X \in \mathcal{S}$;
- 2) If $A \in \mathcal{S}$ then $\bar{A} \in \mathcal{S}$, where $\bar{A} = X \setminus A$;
- 3) If $A, B \in \mathcal{S}$ then $A \cup B \in \mathcal{S}$.

Let X be a set and $\mathcal{F} \subseteq 2^X$; \mathcal{F} is a σ -algebra if it verifies:

- 1) $X \in \mathcal{F}$;
- 2) If $A \in \mathcal{F}$ then $\bar{A} \in \mathcal{F}$;
- 3) If $(A_i)_{i \in I}$ is a countable family of elements of \mathcal{F} then $\bigcup_i A_i \in \mathcal{F}$.

The set $(X; \mathcal{F})$ is called *measurable space*. It is rather easy to prove the following:

Lemma 1. *Let X be a set and let $(\mathcal{S})_i$ be a countable family of algebras of sets (resp. a countable family of σ -algebras) on X , then $\bigcap_i \mathcal{S}_i$ is an algebra of sets (resp. is a σ -algebra).*

Let X be a set and $\mathcal{F} \subseteq 2^X$; the *algebra of sets generated* (resp. the σ -*algebra generated*) by \mathcal{F} is the intersection of all algebras of sets (resp. of all σ -algebras) containing \mathcal{F} .

Let $\mathcal{I} = \{I = [a, b[: [a, b[\subset \mathbb{R}\}$. We will denote by $B(\mathbb{R})$ the σ -algebra generated by \mathcal{I} .

Let \mathcal{F} be an algebra of sets (a σ -algebra) over a set X . A function μ from \mathcal{F} to $[0; +\infty[$ is called a *measure* if it satisfies the following properties:

- a) Null empty set: $\mu(\emptyset) = 0$;
- b) Countable additivity (or σ -additivity): for all countable collections $\{A_i\}_{i \in I}$ of pairwise disjoint sets in \mathcal{F} , we have $\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i)$.

A measurable space $(X; \mathcal{F})$ with a measure μ is called a *measure space*; we denote it by: $(X; \mathcal{F}; \mu)$.

Let $(X; \mathcal{F}; \mu)$ be a measure space; a subset Y of X is *negligible* if there is an element $A \in \mathcal{F}$ such that $Y \subseteq A$ and $\mu(A) = 0$.

For $A \in \mathcal{F}$, if $\mu(A) < +\infty$ we say that A has a *finite measure*.

Let $(X; \mathcal{F}; \mu)$ be a measure space, then the measure μ is called *finite* if $\mu(X)$ is a finite real number; the measure μ is called σ -*finite* if X can be decomposed into a countable union of measurable sets of finite measure.

Properties true almost everywhere

Let $(X; \mathcal{F}; \mu)$ be a measure space. Let $P(x)$, $x \in X$ be a property on $(X; \mathcal{F}; \mu)$; This property is true *almost everywhere*, (denoted by $P_{a.e.}(x)$) if the set $Y = \{x \in X\}$ where this property is not true is negligible *i.e.* there is $A \in \mathcal{F}$ such that $Y \subseteq A$ and $\mu(A) = 0$.

Let

$$f, g: I \subseteq [0; 1] \longrightarrow V$$

be two functions and V be a topological space.

The function f is *negligible* compared to the function g at $a \in I$ if there is a neighborhood W of a and a function

$$\begin{aligned} \epsilon: W &\longrightarrow V \\ t &\longmapsto \epsilon(t) \end{aligned}$$

such that for all $t \in W$,

$f(t) = \epsilon(t).g(t)$, $\lim_{t \rightarrow a} \epsilon(t) = 0$ and $\lim_{t \rightarrow a} g(t) \neq 0$. We will denote this by $f = o(g)$.

We say that the function f is *negligible compared to the function g almost everywhere* on $I \subseteq [0; 1]$ if there is a sequence (a_k) , $a_k \in I$ and a sequence of neighborhoods (W_{a_k}) with $I \subseteq \bigcup_{a_k} W_{a_k}$ where f is negligible for all $a_k \in I$ except on a set $Y \subseteq I$ where there exists a measure μ such that Y is negligible for μ .

We denote this by $f = o_{a.e.}(g)$.

A function f is *equivalent* to a function g , denoted by $f \sim g$ at $a \in I$ if there is a neighborhood W of a and a function ϵ defined as above such that for all $t \in W$, $f(t) = (1 + \epsilon(t)).g(t)$, $\lim_{t \rightarrow a} \epsilon(t) = 0$. This relation is an equivalent relation. It is easy to see that $(f \sim g) \iff (f - g = o(g)) \iff (g - f = o(f))$. In the same way as above we say that a

function f is *equivalent to a function g almost everywhere* on $I \subseteq [0; 1]$; we will denote this relation as follows: $f \sim_{a.e.} g$.

Naturally we can extend these definitions to vector functions; it is enough to check the conditions for all coordinates $(f_1(t), f_2(t), \dots, f_k(t))$, where k is the dimension of the vector space.

More generally let H be a space of vectors with 2 bases (e_1, e_2, \dots, e_n) and (a_1, a_2, \dots, a_n) ; two vectors depending of a variable t , (for example the time) are equivalent at $t : v(t) \sim w(t) \iff \sum_{j=1}^n \beta_j e_j \sim \sum_{j=1}^n \alpha_j a_j$ if and only if $\beta_j \sim \alpha_j$ and $e_j \sim a_j$, (we drop the variable t in order to alleviate the notation).

Piecewise effective functions

An *algorithm* is a finite sequence of finite instructions.

Recall that a real number is called *effective*, if there exists an algorithm which can produce its decimal expansion. By extension a complex number is *effective* if there exists an algorithm which can produce the decimal expansion of both the real part and the imaginary part. These definitions can also be extended to a vector function.

Let $f : X \rightarrow Y$ be a function such that X is a set of effective elements (real, complex, vectors, ...). This function f is *effective* if $\forall z \in X, f(z)$ is *effective*, i.e. the ordered pair $(z; f(z))$ is an ordered pair of effective numbers for all z in the domain of f . We will also say that f is *computable*. Note that we can combine this definition with the definition of almost everywhere to get a definition of an *effective function almost everywhere*.

Addition, multiplication and composition of effective functions are effective functions. Moreover, addition, multiplication, ... of effective reals, complexes, vectors are effective reals, complexes, vectors. So the set of real numbers, (complex numbers) forms a field denoted by $\mathbb{R}_{ef}, (\mathbb{C}_{ef})$.

The *zero function* will be denoted by \mathbb{O} : for all x in the domain of $\mathbb{O}, \mathbb{O}(x) = 0$.

A function f is *piecewise effective* on $[a, b]$ if there exists a subdivision (a_i) of $[a, b]$ such that the restriction of f to each $]a_i, a_{i+1}[$ can be prolonged by an effective function on the corresponding closed set $[a_i, a_{i+1}]$.

2.3 The intermediary outcomes of the model

In this section, some interesting intermediary outcomes of the model are discussed. In subsection 2.3.1, we present an interesting result concerning the structure of information, induced by our assumptions. In subsection 2.3.2, we use our framework to equivalently define the market system as in subsection 2.2.3. Finally, we discuss in subsection 2.3.3 the implications of our formalism concerning the interactions between quantifiables and the market system.

2.3.1 The information's structure

The linear spaces formed by the set of information parameters, the set of effective information parameters, and the non effective set of information parameters are such that:

$$\mathcal{P}\mathcal{A}(I; V) = \mathcal{E}\mathcal{F}(I; V) \oplus \mathcal{N}\mathcal{E}\mathcal{F}(I; V)$$

Proof. This proof is fully extracted from Bretto and Priolon [4].

The sum and the product by an effective number of effective information parameters (effective functions) is an effective information parameter;

The sum and the product by an effective number of non effective information parameters (non effective functions) is a non effective information parameter;

The sum and the product by an effective number of functions in piecewise class \mathcal{C}^1 belong to piecewise class \mathcal{C}^1 .

So, these sets form linear spaces. It is the same for piecewise continuous functions.

Moreover, we have: $\mathcal{EF}(I;V); \mathcal{NEF}(I;V) \leq \mathcal{PA}(I;V)$.

An effective information parameter is not a non effective information parameter and a non effective information parameter is not an effective information parameter. Hence, $\mathcal{EF}(I;V) \cap \mathcal{NEF}(I;V) = \{\emptyset\}$. \square

Any information parameter can be expressed as the unique sum of two components. These two components are independent. For all parameters of the market system:

$$pa_i = ef_i + nef_i$$

Where $pa \in \mathcal{PA}(I;V)$, $ef \in \mathcal{EF}(I;V)$, and $nef \in \mathcal{NEF}(I;V)$. Let \mathcal{Q}_t be a quantifiable property at time $t \in I$. To this quantifiable, we can now associate 3 linear operators: $\mathcal{Q}_{t,pa}$, $\mathcal{Q}_{t,ef}$ and $\mathcal{Q}_{t,nef}$ (\mathcal{Q}_{pa} , \mathcal{Q}_{ef} and \mathcal{Q}_{nef} when there is no ambiguity).

2.3.2 The market system

Let $(pa_i)_{i \in \{1,2,\dots,n\}}$ be a family of parameters and $(ef_i)_{i \in \{1,2,\dots,n\}}$ (resp. $(nef_i)_{i \in \{1,2,\dots,n\}}$) be a family of effective parameters (resp. non effective parameters) such that $(pa_i)_{i \in \{1,2,\dots,n\}} = (ef_i)_{i \in \{1,2,\dots,n\}} + (nef_i)_{i \in \{1,2,\dots,n\}}$. We can redefine a *market system* as a function f in the following way:

$$\begin{aligned} pa: [0;T] &\rightarrow V^n \\ t &\mapsto pa = (pa_1(t), \dots, pa_n(t)) \end{aligned}$$

And:

$$\begin{aligned} f: V^n &\rightarrow \mathcal{H} \\ (pa_1(t), \dots, pa_n(t)) &\mapsto f(pa_1(t), \dots, pa_n(t)) = f(pa, t) \end{aligned}$$

The value of $f(ef_1(t), \dots, ef_n(t)) = f(ef, t)$ is the *effective market system* and the value of $f(nef_1(t), \dots, nef_n(t)) = f(nef, t)$ is the *non effective market system*. To each time t fixed, when there is no ambiguity we can write $ms = f(pa)$ (idem for $ms_{ef} = f(ef)$ and $ms_{nef} = f(nef)$). Hence we can assimilate a *market system* either to a vector or a function.

Remark 3. Since these n vectors are unitary, a market system is described on a sphere of dimension n called market sphere either by a vector (see definition in subsection 2.2.3) or a point (see present subsection).

Let \mathcal{Q}_t be a quantifiable price whose reading gives the eigenvalue $E_j(\tilde{p}_t)$ at time t . Then the state of the market *instantaneously* becomes:

$$f(pa, t) = \frac{1}{\sum_{i=1}^n 1_{\{E_i(\tilde{p}_t) = E_j(\tilde{p}_t)\}} \beta_i} \sum_{i=1}^n 1_{\{E_i(\tilde{p}_t) = E_j(\tilde{p}_t)\}} \beta_i e_i \quad (2.7)$$

If the reading of the quantifiable price \mathcal{Q}_t is the eigenvalue $E_j(\tilde{p}_t)$ at time t , then the market system is in the vector space spanned by the eigenvectors associated to the eigenvalue $E_j(\tilde{p}_t)$; That is, in the vector space spanned by agents that gave the same expectation of the next measure of the quantifiable price.

An illustration of this result can be the following: Suppose that a bookmaker knows that a great champion runs in a race, her/his prognostication is that s/he will win the race with a probability of 90%. A few minutes before departure the bookmaker learns that this great champion just suffered a slight sprain, then her/his anticipation that this champion wins the race passes instantaneously from 90% to nearly 0%.

Remark 4. *This intermediary outcome applies to all market systems ms , ms_{ef} and ms_{nef} (cf. subsection 2.2.3).*

2.3.3 The interaction between quantifiables and the market

Let \mathcal{P} be the quantifiable price and let \mathcal{Q} be the quantifiable quantity. What happens the quantifiable price is measured before the quantifiable quantity? What happens if it is the opposite?

Both \mathcal{P} and \mathcal{Q} are positive-semidefinite diagonal matrix. These two matrix are commutative, i.e. $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P}$ if and only if these matrix are diagonalized in the same basis of eigenvectors, i.e. if there is $\{e_i, i \in \{1, 2, \dots, n\}\}$ a basis of eigenvectors such that $\mathcal{P}.e_i = \lambda_i.e_i$ where λ_i is an eigenvalue of \mathcal{P} and $\mathcal{Q}.e_i = \mu_i.e_i$, where μ_i is an eigenvalue of \mathcal{Q} . It means that the eigenspaces of \mathcal{P} are invariant under the action of \mathcal{Q} and conversely. Indeed $\mathcal{P}\mathcal{Q}.e_i = \mathcal{Q}\mathcal{P}.e_i = \mathcal{Q}\lambda_i.e_i = \lambda_i.\mathcal{Q}.e_i$. Hence $\mathcal{Q}.e_i$ is an eigenvector of \mathcal{P} .

Another interpretation is that at time t the action of $\mathcal{Q}\mathcal{P}$ on the Hilbert space \mathcal{H} is exactly the same as the action of $\mathcal{P}\mathcal{Q}$ on the Hilbert space \mathcal{H} , i.e. at time t , $\mathcal{Q}\mathcal{P}.ms = \mathcal{P}\mathcal{Q}.ms$.

Consider for instance a commodity market where two categories of agents operate: some dealers whose job is to purchase and sell the "real commodity" and some "pure" speculators who trade on the futures market (see Johnson [20]). Dealers are mainly interested in the quantifiable quantity while speculators are principally interested in the quantifiable price. Consequently from our model the set of dealers represents a basis of eigenvectors $\{d_i, i \in \{1, 2, \dots, n\}\}$ and the set of speculators represents another (distinct) basis of eigenvectors $\{s_i, i \in \{1, 2, \dots, n\}\}$. Consequently matrix \mathcal{P} and \mathcal{Q} are not commutative and the action at time t of $\mathcal{P}\mathcal{Q}$ on ms is not the same as the action at time t of $\mathcal{Q}\mathcal{P}$ on ms . If agents are informed sequentially about the quantities on the first hand, and prices on the other hand, the states of the market will be different than if information were given in the opposite order.

2.4 The impact of information's nature and structure on the financial market

In this section, we present the major outcomes of the model which can give some insight to actual financial markets functioning. We provide in subsection 2.4.1 the condition on the nature and structure of information insuring a financial market where information arrive continuously and transactions are frequent to be qualified as semi-strong efficient as well as the study of the computational strength of such a financial market. Subsection 2.4.2 present what are the characteristics of information that explain a state of a financial

market crisis as well as the study of the computational strength of such a financial market. At the end, subsection 2.4.3 provides what the characteristics of information are within a bubble as well as the study of the computational strength of such a financial market. As the latest is time evolving, we provide some simulations which illustrate this dynamics.

2.4.1 The semi-strong efficient financial market: Condition on information and computational strength

The semi-strong efficient financial market: Condition on information

Theorem 1. Let \mathcal{Q}_t be a quantifiable property at time $k \in [1; t]$, and let \mathcal{Q}_{pa} , \mathcal{Q}_{ef} be the diagonal matrix associated to \mathcal{Q}_t . If at time k , $k \in [1; t]$, the eigenvalues of \mathcal{Q}_{pa} are $E_1(\tilde{p}_k) > E_2(\tilde{p}_k) \geq E_3(\tilde{p}_k) \geq \dots \geq E_n(\tilde{p}_k)$ and the eigenvalues of \mathcal{Q}_{ef} are $\mu_{1,k} > \mu_{2,k} \geq \mu_{3,k} \geq \dots \geq \mu_{n,k}$ then a "small" $\epsilon_{i,k}$ such that $|E_i(\tilde{p}_k) - \mu_{i,k}| \leq \epsilon_{i,k} \forall i \in \{1, 2, 3, \dots, n\}$ exists if and only if:

$$nef_i(k) = o(ef_i(k)), \forall i \in \{1, 2, 3, \dots, n\} \quad (2.8)$$

Proof. This proof is constructed using the similar argumentation than Bretto and Priolon [4].

Let $k \in [1; t]$ such that $nef_i(k) = o(ef_i(k)), \forall i \in \{1, 2, \dots, n\}$ and $f_{pa}(k) = f(pa_1(k), \dots, pa_n(k))$. We have:

$$\begin{aligned} \frac{f(pa, k)}{f(ef, k)} &= \frac{f(pa_1(k), \dots, pa_n(k))}{f(ef_1(k), \dots, ef_n(k))} \\ \Leftrightarrow \frac{f(pa, k)}{f(ef, k)} &= \frac{f(ef_1(k) + nef_1(k), \dots, ef_n(k) + nef_n(k))}{f(ef_1(k), \dots, ef_n(k))} \\ \Leftrightarrow \frac{f(pa, k)}{f(ef, k)} &= \frac{f(ef_1(k) + o(ef_1(k)), \dots, ef_n(k) + o(ef_n(k)))}{f(ef_1(k), \dots, ef_n(k))} \end{aligned}$$

According to subsection 2.2.7, $nef_i(k) = o(ef_i(k)) \Leftrightarrow nef_i(k) = \epsilon_i(k)ef_i(k)$. We get:

$$\begin{aligned} \frac{f(pa, k)}{f(ef, k)} &= \frac{f(ef_1(k) + \epsilon_1(k)ef_1(k), \dots, ef_n(k) + \epsilon_n(k)ef_n(k))}{f(ef_1(k), \dots, ef_n(k))} \\ \Leftrightarrow \frac{f(pa, k)}{f(ef, k)} &\simeq \frac{f(ef_1(a), \dots, ef_n(a))}{f(ef_1(a), \dots, ef_n(a))} = 1 \\ \Leftrightarrow f(pa, k) &\sim f(ef, k) \end{aligned}$$

Let $(e_i)_{i \in \{1, 2, \dots, n\}}$ and $(a_i)_{i \in \{1, 2, \dots, n\}}$ be the basis of orthonormal eigenvectors associated to $(E_i(\tilde{p}_k))_{i \in \{1, 2, \dots, n\}}$ and $(\mu_{i,k})_{i \in \{1, 2, \dots, n\}}$ respectively. For $k \in [1; t]$ we have: $f(pa, k) \sim f(ef, k) \Leftrightarrow \sum_{j=1}^n \beta_j e_j \sim \sum_{j=1}^n \alpha_j a_j \Leftrightarrow ms \sim ms_{ef} \sim \sum_{j=1}^n \beta_j e_j$.

If $\mathcal{Q}_{pa} = \mathcal{Q}_{ef}$, it is over.

Assume that $\mathcal{Q}_{pa} \neq \mathcal{Q}_{ef}$.

$$\begin{aligned} ms^k &= (\mathcal{Q}_{pa})^k ms \\ &= \sum_{i=1}^n (E_i(\tilde{p}))^k \beta_i e_i \\ &= (E_1(\tilde{p}))^k \left[\beta_1 e_1 + \sum_{i=2}^n \beta_i \left[\frac{E_i(\tilde{p})}{E_1(\tilde{p})} \right]^k e_i \right] \\ ms^k &= (E_1(\tilde{p}))^k \left[\beta_1 e_1 + \mathcal{O} \left(\left[\frac{E_i(\tilde{p})}{E_1(\tilde{p})} \right]^k \right) \right] \\ ms_{ef}^k &= (\mathcal{Q}_{ef})^k ms_{ef} \\ ms_{ef}^k &= (\mu_1)^k \left[\beta_1 e_1 + \mathcal{O} \left(\left[\frac{\mu_i}{\mu_1} \right]^k \right) \right] \end{aligned}$$

Let $\widetilde{ms}^k = \frac{(\mathcal{Q}_{pa})^k ms}{\|(\mathcal{Q}_{pa})^k ms\|}$ and $\widetilde{ms}_{ef}^k = \frac{(\mathcal{Q}_{ef})^k ms_{ef}}{\|(\mathcal{Q}_{ef})^k ms_{ef}\|}$. We have:

$$\widetilde{ms}^k = \frac{(E_1(\tilde{p}))^k \left[\beta_1 e_1 + \mathcal{O} \left(\left[\frac{E_i(\tilde{p})}{E_1(\tilde{p})} \right]^k \right) \right]}{\left\| (E_1(\tilde{p}))^k \left[\beta_1 e_1 + \mathcal{O} \left(\left[\frac{E_i(\tilde{p})}{E_1(\tilde{p})} \right]^k \right) \right] \right\|} \simeq \frac{\beta_1 e_1}{\beta_1}$$

When $k \rightarrow +\infty$.

We also have, when $k \rightarrow +\infty$:

$$\widetilde{ms}_{ef}^k \simeq \frac{\beta_1 e_1}{\beta_1}$$

Consequently when $k \rightarrow +\infty$ we have:

$$\widetilde{ms}^k \simeq \widetilde{ms}_{ef}^k$$

Hence:

$$(\mathcal{Q}_{pa})^k ms \simeq \frac{\|(\mathcal{Q}_{pa})^k ms\| \cdot (\mathcal{Q}_{ef})^k ms_{ef}}{\|(\mathcal{Q}_{ef})^k ms_{ef}\|}$$

When $k \rightarrow +\infty$, the latest expression is equivalent to $(E_1(\tilde{p}))^k \beta_1 e_1 \simeq t \times (\mu_1)^k \beta_1 e_1$.
So $(E_1(\tilde{p}))^k \simeq k \times (\mu_1)^k$.

Since $E_1(\tilde{p}), \mu_1, k > 1$ and $k^{\frac{1}{k}} \rightarrow 1$ when $k \rightarrow +\infty$ we have:

$$\boxed{E_1(\tilde{p}) \simeq \mu_1}$$

Let $\delta(\mathcal{Q}) = \mathcal{Q}_{pa} - \mathcal{Q}_{ef}$; These two matrix are diagonal and $\mathcal{Q}_{pa} \neq \mathcal{Q}_{ef}$ then $\delta(\mathcal{Q})$ is diagonal and its eigenvalues are real. From inequality of WEYL, Franklin [12], Golub and Loan [14] we have:

$$\mu_i(\mathcal{Q}_{ef}) + \lambda_{\min}(\delta(\mathcal{Q})) \leq \lambda_i(\mathcal{Q}_{pa}) \leq \mu_i(\mathcal{Q}_{ef}) + \lambda_{\max}(\delta(\mathcal{Q}))$$

We also have: $E_1(\tilde{p}) \simeq \mu_1$ and $\lambda_n \simeq \mu_n$, so $\lambda_{\min}(\delta(\mathcal{Q})) \simeq 0$.
From this we can conclude that:

$$\boxed{E_i(\tilde{p}) \simeq \mu_i, \forall i \in \{1, 2, 3, \dots, n\}}$$

There is ϵ_i, ϵ_i "small", such that : $|E_i(\tilde{p}) - \mu_i| \leq \epsilon_i$ for all $i \in \{1, 2, 3, \dots, n\}$.

It is immediate that this result does not hold when $nef_i(t) \neq o(ef_i(t))$, which proves the equivalence between having a semi-strong efficient market and verifying the condition: $nef_i(k) = o(ef_i(k)), \forall i \in \{1, 2, 3, \dots, n\}$. \square

To achieve semi-strong efficiency, the market system must be mainly or even fully determined by the effective information parameters; The non effective information must be negligible when compared to effective information. More generally any quantifiable (whose all eigenvalues are positive) tends to its optimal evaluation so theorem 1 gives the conditions on the nature and structure of information to allow the quantifiable price but also others quantifiables (whose all eigenvalues are positive) to be semi-strong efficient. In other words, agents can rely on qualitative judgment on the value of an asset if the non effective information is relatively negligible before the effective information treated by an algorithm.

The semi-strong efficient financial market: Computational strength

Corollary 1. *If a market is semi-strong efficient, it has a computational strength.*

Proof. From theorem 1, $nef_i(k) = o(ef_i(k)) \Leftrightarrow |E_i(\tilde{p}_k) - \mu_{i,k}| \leq \epsilon_{i,k}$, with $\epsilon_{i,k}$ "small" for all $i \in \{1, 2, 3, \dots, n\}$. For $k \in [1; t]$, price is given by one of the n agents denoted by " m_k " (i.e. $p_k = E_{m_k}(\tilde{p}_k)$).

If $|E_{m_k}(\tilde{p}_k) - \mu_{m_k,k}| = 0 \forall k \in [1; t]$, we get $\zeta_p(s, t) = +\infty$. Hence, $C(s, t) = \zeta_p(s, t) - \max_i \zeta_i(s, t) \geq 0$. It is over.

Assume that it exists at least one period k when $|E_{m_k}(\tilde{p}_k) - \mu_{m_k,k}| = |p_k - \mu_{m_k,k}| \neq 0$.

$$\begin{aligned}
 0 & \leq |p_k - \mu_{m_k,k}| \leq \epsilon_{m_k,j} \\
 \Leftrightarrow \epsilon & \leq \epsilon + |p_k - \mu_{m_k,k}| \leq \epsilon + \epsilon_{m_k,j} \\
 \Leftrightarrow \epsilon^s & \leq (\epsilon + |p_k - \mu_{m_k,k}|)^s \leq (\epsilon + \epsilon_{m_k,j})^s \\
 \Leftrightarrow \frac{1}{(\epsilon + \epsilon_{m_k,j})^s} & \leq \frac{1}{(\epsilon + |p_k - \mu_{m_k,k}|)^s} \leq \frac{1}{\epsilon^s} \\
 \Leftrightarrow \sum_{k=1}^t \frac{1}{(\epsilon + \epsilon_{m_k,j})^s} & \leq \sum_{k=1}^t \frac{1}{(\epsilon + |p_k - \mu_{m_k,k}|)^s} \leq \frac{1}{\epsilon^s} < +\infty
 \end{aligned}$$

Let $\bar{\epsilon}$ be the maximal value of all $\epsilon + \epsilon_{m_k,k}$ such that $\epsilon + \epsilon_{m_k,k} \leq \bar{\epsilon}, \forall \{k, m_k\} \in [1; t] \times [1; n]$, and so $\bar{\epsilon}$ is "small". We get:

$$\begin{aligned}
 \Leftrightarrow \sum_{k=1}^t \frac{1}{\bar{\epsilon}^s} & \leq \zeta_p(s, t) < +\infty \\
 \Leftrightarrow t \times \frac{1}{\bar{\epsilon}^s} & \leq \zeta_p(s, t) < +\infty
 \end{aligned}$$

We also have $\frac{t}{\bar{\epsilon}^s} \rightarrow +\infty$, then:

$$\zeta_p(s, t) \rightarrow +\infty$$

Therefore, the probability of an agent i to "do better than the market", is :

$$\Pr(C(s, t) < 0) = \Pr\left(\zeta_p(s, t) < \max_i \zeta_i(s, t) \leq +\infty\right)$$

But we have :

$$\Pr\left(\zeta_p(s, t) < \max_i \zeta_i(s, t) \leq +\infty\right) \leq \Pr\left(\frac{t}{\bar{\epsilon}^s} < \max_i \sum_{j=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k,j}|)^s} \leq +\infty\right)$$

And,

$$\begin{aligned} \Pr\left(\frac{t}{\bar{\epsilon}^s} < \max_i \sum_{j=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k,j}|)^s} \leq +\infty\right) &\simeq \Pr\left(\max_i \sum_{j=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k,j}|)^s} \rightarrow +\infty\right) \\ \Leftrightarrow \Pr\left(\frac{t}{\bar{\epsilon}^s} < \max_i \sum_{j=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k,j}|)^s} \leq +\infty\right) &\simeq \Pr\left(\bigcap_{j=1}^t (E_i(\tilde{p}_j) \simeq \mu_{m_k,j})\right) \\ \Leftrightarrow \Pr\left(\bigcap_{j=1}^t (E_i(\tilde{p}_j) \simeq \mu_{m_k,j})\right) &= \bigcap_{j=1}^t \Pr(E_i(\tilde{p}_j) \simeq \mu_{m_k,j}) \\ \Leftrightarrow \Pr\left(\bigcap_{j=1}^t (E_i(\tilde{p}_j) \simeq \mu_{m_k,j})\right) &\simeq \prod_{j=1}^t \left(\sum_{i=1}^n 1_{\{E_i(\tilde{p}_t) = \mu_{m_k,j}\}} \beta_i\right) \end{aligned}$$

Hence,

$$\Pr\left(\frac{t}{\bar{\epsilon}^s} < \max_i \sum_{j=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k,j}|)^s} \leq +\infty\right) \simeq \prod_{j=1}^t \left(\sum_{i=1}^n 1_{\{E_i(\tilde{p}_t) = \mu_{m_k,j}\}} \beta_i\right)$$

Since $0 < \sum_{i=1}^n 1_{\{E_i(\tilde{p}_t) = \mu_{m_k,j}\}} \beta_i < 1$, then $\lim_{t \rightarrow +\infty} \prod_{j=1}^t \left(\sum_{i=1}^n 1_{\{E_i(\tilde{p}_t) = \mu_{m_k,j}\}} \beta_i\right) = 0$.

Hence, we have:

$$\Pr(C(s, t) \geq 0) \rightarrow 1 \quad (2.9)$$

Then, corollary 1 is proved. \square

Consequently, one isolated agent can at best do "as good as the market" but s/he cannot do better; This result is consistent with classical finance results. If a market is semi-strong efficient, no agent can "do better than the market" since s/he has no private information.

2.4.2 The financial market crisis: Condition on information and computational strength

The financial market crisis: Condition on information

Theorem 2. Let \mathcal{Q}_t be a quantifiable property at time $k \in [1; t]$, let \mathcal{Q}_{pa} , \mathcal{Q}_{nef} be the diagonal matrix associated to \mathcal{Q} . If at time k , $k \in [1; t]$, eigenvalues of \mathcal{Q}_{pa} are $E_1(\tilde{p}_k) > E_2(\tilde{p}_k) \geq E_3(\tilde{p}_k) \geq \dots \geq E_n(\tilde{p}_k)$ and eigenvalues of \mathcal{Q}_{nef} are $\theta_{1,k} > \theta_{2,k} \geq \theta_{3,k} \geq \dots \geq \theta_{n,k}$ then a "small" $\epsilon_{i,k}$ such that $|E_i(\tilde{p}_k) - \theta_{i,k}| \leq \epsilon_{i,k}, \forall i \in \{1, 2, 3, \dots, n\}$ exists if and only if:

$$ef_i(k) = o(nef_i(k)), \forall i \in \{1, 2, 3, \dots, n\} \quad (2.10)$$

Proof. We first show that in a period of crisis, we face $f(pa, t) \sim f(nef, t)$. The rest of the proof is similar to Theorem 1. \square

During a semi-strong efficient functioning of the market, the real market system tends towards the effective market system and during a financial crisis, the real market system tends towards the non effective market system.

The financial market crisis: Computational strength

Corollary 2. *If a market is in crisis, it has no computational strength.*

Proof. We start with a market without computational strength, i.e. $C(s, t) < 0$. We have:

$$\begin{aligned} \zeta_p &< \max_i \zeta_i \\ \Leftrightarrow \sum_{k=1}^t \frac{1}{(\epsilon + |p_k - \mu_{m_k, k}|)^s} &< \max_i \sum_{k=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_k) - \mu_{m_k, k}|)^s} \\ \Leftrightarrow \sum_{k=1}^t (\epsilon + |p_k - \mu_{m_k, k}|)^s &> \min_i \sum_{k=1}^t (\epsilon + |E_i(\tilde{p}_k) - \mu_{m_k, k}|)^s \\ \Leftrightarrow \sum_{k=1}^t (\epsilon + |\kappa_{m_k, k}|)^s &> \min_i \sum_{k=1}^t (\epsilon + |E_i(\tilde{p}_k) - \mu_{m_k, k}|)^s \end{aligned}$$

Using properties of the absolute value we get:

$$\begin{aligned} |E_i(\tilde{p}_k) - \mu_{m_k, k}| &= |E_i(\tilde{p}_k) - \kappa_{m_k, k} + \kappa_{m_k, k} - \mu_{m_k, k}| \\ \Leftrightarrow |E_i(\tilde{p}_k) - \mu_{m_k, k}| &= |E_i(\tilde{p}_k) - \kappa_{m_k, k} - (\mu_{m_k, k} - \kappa_{m_k, k})| \\ \Leftrightarrow |E_i(\tilde{p}_k) - \mu_{m_k, k}| &\geq ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| \end{aligned}$$

Finally, we get:

$$C(s, t) < 0 \Leftrightarrow \sum_{k=1}^t (\epsilon + |\kappa_{m_k, k}|)^s > \min_i \sum_{k=1}^t (\epsilon + ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}||)^s$$

Recall that within a financial crisis $k \in [1, t]$, we have $|E_i(\tilde{p}_k) - \kappa_{i, k}| \leq \epsilon_{i, k} \Leftrightarrow |\mu_{i, k}| \leq \epsilon_{i, k}$, with $\epsilon_{i, k}$ "small" for all $i \in \{1, 2, 3, \dots, n\}$. We get:

$$\begin{aligned} |E_i(\tilde{p}_k) - \kappa_{m_k, k}| &= |\mu_{i, k} + \kappa_{i, k} - \kappa_{m_k, k}| \\ \Leftrightarrow |E_i(\tilde{p}_k) - \kappa_{m_k, k}| &= |\mu_{i, k} + (\kappa_{i, k} - \kappa_{m_k, k})| \\ \Leftrightarrow |E_i(\tilde{p}_k) - \kappa_{m_k, k}| &\leq |\mu_{i, k}| + |\kappa_{i, k} - \kappa_{m_k, k}| \\ \Leftrightarrow |E_i(\tilde{p}_k) - \kappa_{m_k, k}| &\leq \epsilon_{i, k} + |\kappa_{i, k} - \kappa_{m_k, k}| \end{aligned}$$

Then, we have:

$$\begin{aligned} ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq |\epsilon_{i, k} + |\kappa_{i, k} - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| \\ \Leftrightarrow ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq |\epsilon_{i, k}| + ||\kappa_{i, k} - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| \\ \Leftrightarrow ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq \epsilon_{i, k} + |\kappa_{i, k} - \kappa_{m_k, k} - (\mu_{m_k, k} - \kappa_{m_k, k})| \\ \Leftrightarrow ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq \epsilon_{i, k} + |\kappa_{i, k} - \mu_{m_k, k}| \\ \Leftrightarrow ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq \epsilon_{i, k} + |\kappa_{i, k} - \mu_{m_k, k}| \\ \Leftrightarrow ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq \epsilon_{i, k} + |\kappa_{i, k}| + |-\mu_{m_k, k}| \\ \Leftrightarrow ||E_i(\tilde{p}_k) - \kappa_{m_k, k}| - |\mu_{m_k, k} - \kappa_{m_k, k}|| &\leq \epsilon_{i, k} + |\kappa_{i, k}| + \epsilon_{m_k, k} \end{aligned}$$

Hence we can write that within a financial crisis, we have:

$$\min_i \sum_{k=1}^t (\epsilon + |E_i(\tilde{p}_k) - \kappa_{m_k,k}| - |\mu_{m_k,k} - \kappa_{m_k,k}|)^s \leq \min_i \sum_{k=1}^t (\epsilon + |\epsilon_{i,k}| + |\kappa_{i,k}| + \epsilon_{m_k,k})^s$$

We now compute the probability of having no computational strength within a financial crisis and demonstrate that it converges to 1. We know that:

$$\Pr \left(\sum_{k=1}^t (\epsilon + |\kappa_{m_k,k}|)^s > \min_i \sum_{k=1}^t (\epsilon + \epsilon_{i,k} + |\kappa_{i,k}| + \epsilon_{m_k,k})^s \right) \leq \Pr(C(s, t) < 0) \leq 1$$

Except if agents are homogeneous (i.e. $\kappa_{i,k} = \kappa_k$ for all $i \in \{1, \dots, n\}$ and $k \in [1, t]$), we face $\lim_{t \rightarrow +\infty} \Pr \left(\sum_{k=1}^t (\epsilon + |\kappa_{m_k,k}|)^s > \min_i \sum_{k=1}^t (\epsilon + \epsilon_{i,k} + |\kappa_{i,k}|)^s \right) = 1$. Finally, we get $\lim_{t \rightarrow +\infty} \Pr \left(\sum_{k=1}^t (\epsilon + |\kappa_{m_k,k}|)^s > \min_i \sum_{k=1}^t (\epsilon + \epsilon_{i,k} + |\kappa_{i,k}| + \epsilon_{m_k,k})^s \right) = 1$.

We can finally write that within a financial crisis:

$$\lim_{t \rightarrow +\infty} \Pr(C(s, t) < 0) = 1$$

Corollary is proved. □

Consequently, during a crisis, there is always at least one agent considered individually that can "do better than the market"; In other words, if a market is in crisis, agents interactions provoke a price dynamic as far as possible from the perfect price dynamic. Consequently it always exist at least one agent able to better forecast the fundamental value without interacting with the financial market.

2.4.3 The bubble: Condition on information and computational strength

The bubble: Condition on information

According to the definition of a bubble, theorem 1 does not hold for $k > 1$. Therefore we can say that quantifiables (whose eigenvalues are positive) do not reflect their fundamental values within a period of bubble. Hence, if non effective information becomes non negligible when compared to effective information, the market is no more semi-strong efficient.

Proposition 1. *A bubble ends with a financial crisis.*

Proof. At time $k \in B$, we have $nef_i(k) = \mathcal{O} \left((ef_i(k))^k \right)$ for some $i \in \{1, 2, \dots, n\}$ and $f(pa, k) = f(pa_1(k), \dots, pa_n(k))$. We have:

$$f(pa, k) = f(pa_1(k), \dots, pa_n(k))$$

$$\Leftrightarrow f(pa, k) = f(ef_1(k) + nef_1(k), \dots, ef_n(k) + nef_n(k))$$

$$\Leftrightarrow f(pa, k) = f \left(ef_1(k) + \mathcal{O} \left((ef_1(k))^k \right), \dots, ef_n(k) + \mathcal{O} \left((ef_n(k))^k \right) \right)$$

Hence, $f(pa, k) \simeq f \left(\mathcal{O} \left((ef_1(k))^k \right), \dots, \mathcal{O} \left((ef_n(k))^k \right) \right) = f(nef, k)$.

Consequently, it exists a number of periods $t_b \in B$, such that $\forall t \geq t_b, f(pa, t) \sim f(nef, t)$. Hence, t_b denotes the period when the bubble turns into a financial crisis. □

Our interest is to study the characteristics of the financial market before a bubble turns up into a crisis. It is immediate that the market becomes non efficient (cf. Theorem 1). However, is the financial market devoid of computational strength ? In our theoretical framework, the concept of computational strength of the market allows to study the statistical capability of an agent to individually "do better than the market".

The bubble: Computational strength

Theorem 3. *In a period of bubble $B \subseteq I$, and $\forall [1; t] \subset [1; t_b]$, the market can keep a computational strength on the period $[1; t]$.*

Proof. For $[1; t] \subset [1; t_b]$, price is given by one of the n agents denoted by " m_k " (i.e. $p_k = E_{m_k}(\tilde{p}_k) \forall k \in [1; t]$).

If $|E_{m_k}(\tilde{p}_k) - \mu_{i,k}| = |p_k - \mu_{i,k}| = 0 \forall k \in [1; t]$, we get $\zeta_p(s, t) = +\infty$. Hence, $C(s, t) = \zeta_p(s, t) - \max_i \zeta_i(s, t) \geq 0$. It is over.

Assume that there exists at least one period $k, k \in [1; t]$ when $|E_{m_k}(\tilde{p}_k) - \mu_{m_k,k}| = |p_k - \mu_{m_k,k}| \neq 0$. An agent i can "do better than the market" during the period, and the probability of that event is:

$$\Pr(C(s, t) < 0) = \Pr\left(\zeta_p(s, t) < \max_i \zeta_i(s, t)\right)$$

Then, $\forall k \in [1; t] \subset [1; t_b]$, the probability of the price to be the highest distance with the fundamental value is neither equal to 0 nor to 1 except if the market is perfectly determined (i.e. except if all agents model are the same at each period). Then:

$$0 < \Pr\left(|p_k - \mu_{m_k,k}| \geq \max_{E_i \neq E_{m_k}} |E_i(\tilde{p}_k) - \mu_{m_k,k}|\right) < 1$$

Then, we have:

$$\begin{aligned} 0 < \Pr\left((\epsilon + |p_k - \mu_{m_k,k}|)^s \leq \max_{E_i \neq E_m} (\epsilon + |E_i(\tilde{p}_k) - \mu_{m_k,k}|)^s\right) < 1 \\ \Rightarrow 0 < \Pr\left(\frac{1}{(\epsilon + |p_k - \mu_{m_k,k}|)^s} \geq \min_{E_i \neq E_m} \frac{1}{(\epsilon + |E_i(\tilde{p}_k) - \mu_{m_k,k}|)^s}\right) < 1 \end{aligned}$$

The previous inequality is valid $\forall k \in [1; t]$, then it is valid for the sum of inequalities. With $\sum_{k=1}^t \frac{1}{(\epsilon + |p_j - \mu_{m_k,j}|)^s} = \zeta_p(s, t)$, we get:

$$0 < \Pr\left(\zeta_p(s, t) \geq \sum_{j=1}^t \min_{E_i \neq E_m} \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k,j}|)^s}\right) < 1$$

$$\begin{aligned} &\Rightarrow 0 < \Pr \left(\zeta_p(s, t) \geq \min_{E_i \neq E_m} \sum_{j=1}^t \frac{1}{(\epsilon + |E_i(\tilde{p}_j) - \mu_{m_k, j}|)^s} \right) < 1 \\ &\Rightarrow 0 < \Pr \left(\zeta_p(s, t) < \max_{E_i \neq E_m} \zeta_i(s, t) \right) \leq 1 \\ &\Rightarrow 0 < \Pr(C(s, t) < 0) \leq 1 \end{aligned}$$

But we also have $\forall k \in [1; t] \subset [1; t_b]$,

$$0 < \Pr \left(|p_k - \mu_{m_k, k}| \leq \min_{E_i \neq E_{m_k}} |E_i(\tilde{p}_k) - \mu_{m_k, k}| \right) < 1$$

that leads to

$$0 < \Pr(C(s, t) \geq 0) \leq 1$$

But we have:

$$\Pr(C(s, t) \geq 0) + \Pr(C(s, t) < 0) = 1$$

As both probabilities are strictly positive, we have $0 < \Pr(C(s, t) \geq 0) < 1$ and $0 < \Pr(C(s, t) < 0) < 1$. \square

From theorem 3, we get that the probability for an agent to "do better than the market" is neither equal to 0, nor to 1. Hence, we are able to distinguish the efficiency of a market from its capability to "do better" than any isolated agent; semi-strong efficiency and computational strength are not equivalent. Although the market is not semi-strong efficient during a bubble, this does not necessarily means that an agent can be better at discovering the fundamental price on the period. Hence, even during a bubble, the market can be a better tool to correctly price an asset than any agent considered individually. Indeed, it can have a computational strength. However, in this case, there is no proof that another way to price the asset would not be better.

We want to illustrate how evolve the computational strength of the financial market within a period of bubble. Among numerous simulations possibilities, we have chosen to represent how the computational strength of the market evolves according to the time horizon of the bubble and so does, its size (see definition 21). Recall that a bubble ends with a financial crisis, and a market without computational strength (from proposition 1), we want to represent how the computational strength decreases with the duration of the bubble. For these simulations, we need a few more hypothesis.

Let $s = 1$. Information and interpretation parameters, respectively denoted by $I_{ef} | t-1; t]$, $I_{nef} | t-1; t]$, $\gamma_{i,t}^{ef}$ and $\gamma_{i,t}^{nef}$, follow Gaussian laws.¹⁰ At time t , expectation of agent i is given by $E_i(\tilde{p}_t) = p_{t-1} + \gamma_{i,t}^{ef} I_{ef} | t-1; t] + \gamma_{i,t}^{nef} I_{nef} | t-1; t]$. The n agents' expectations have a probability of becoming the market price p_{t+1} according to distances between all expectations and considering a Gaussian law centered in the current market price p_t

¹⁰We have: $I_{ef} | t-1; t] \sim \mathcal{N}(E(I_{ef} | t-1; t]), \sigma_{ef})$, $I_{nef} | t-1; t] \sim \mathcal{N}(E(I_{nef} | t-1; t]), \sigma_{nef})$, $\gamma_{i,t}^{ef} \sim \mathcal{N}(E(\gamma_{i,t}^{ef}), \sigma_{\gamma_{ef}})$ and $\gamma_{i,t}^{nef} \sim \mathcal{N}(E(\gamma_{i,t}^{nef}), \sigma_{\gamma_{nef}})$. $\mathcal{N}(a, b)$ the Gaussian law with mean a and standard variance b .

and with a standard deviation σ_{β_i} assumed to be the β_i 's exogenous standard deviation.¹¹ The fundamental value at time t , denoted by $f v(t)$, is unique for all agents and equal to $f v(t) = p_0 + \sum_{k=1}^t I_{ef}]k - 1; k]$. The set of parameters values used is given in Table A.1. A hundred simulations were run for each bubble duration in Figure A.1a. In Figure A.1b, a thousand simulations were run for each bubble duration. As expected, the longer the bubble is, the less computational strength tends to exist.¹²

2.5 Applications

This model can be used in different ways, either by traders or by the financial market regulator. Subsection 2.5.1 reviews possible applications of the model for the traders and subsection 2.5.2 discusses the possible use of the model by the financial market regulator.

2.5.1 Applications of the model for the traders

A trader's objective can be to predict what the next order book will "look like". In our model, predict the next market system is equivalent. Assuming a statistical distribution to get the agents' β_i among time, a trader can characterize the coordinates of the *market system* in the market sphere of dimension n . It can give her/him, who is the most likely to give the next market price that is, who is going to make a transaction. However, it supposes to be able to distinguish the actors on the market. Another possible way of predicting the next order book can be to parameterize a VARMA model, in order to extract information from the order book.

We provide an example of such an application.¹³ Information and interpretation parameters, respectively denoted by $I]t - 1; t]$ and $\gamma_{i,t}$, follow Gaussian laws.¹⁴ Let $\mathcal{U}(a, b)$ denote the uniform law with parameters a and b .¹⁵ We define the model of agent i such that at time t expectation of agent i is given by $E_i(\tilde{p}_t) = p_{t-1} + \mathcal{U}(\gamma_{i,t} I]t - 1; t] - \sqrt{3}\sigma_\gamma, \gamma_{i,t} I]t - 1; t]) + \sqrt{3}\sigma_\gamma$. The n agents' expectations have a probability of becoming the market price p_{t+1} according to the distances between all expectations and considering a Gaussian law centered in the current market price p_t and with a standard deviation σ_{β_i} assumed to be the β_i 's exogenous standard deviation.¹⁶ The fundamental value at time t , denoted by $f v(t)$, is unique for all agents and equal to $f v(t) = p_0 + \sum_{k=1}^t I]k - 1; k]$. The set of parameters values used is given in Table A.2.

We are interested in the capability of the model to predict the next agents β_i . We provide all estimated coefficients values, variances, standard errors, and last but not least, AIC and BIC values in appendix A.2.2 for a randomly chosen simulation (among a hundred), for relevant couple (p, q) . Appendix A.2.3, summarizes AIC and BIC values for all tested couples (p, q) that we choose with the help of the p-values of table of Extended

¹¹The construction of beta i's agents depend on all prices given by agents. Distances between all different eigenvalues are used to construct classes and probability to be in each class in given according to $\mathcal{N}(p_t, \sigma_{\beta_i})$. Then, if x agents are on the same class, their β_i at time t is equal the probability to be on the class divided by x .

¹²We can provide the algorithm, david.batista-soares@unicaen.fr.

¹³We can provide the algorithm, david.batista-soares@unicaen.fr.

¹⁴We have: $I]t - 1; t] \sim \mathcal{N}(E(I]t - 1; t]), \sigma_I)$ and $\gamma_{i,t} \sim \mathcal{N}(E(\gamma_{i,t}), \sigma_\gamma)$.

¹⁵Recall that the standard variance of a continuous uniform law $\mathcal{U}(a, b)$ is equal to $\frac{(b-a)^2}{12}$.

¹⁶The construction of beta i's agents depend on all prices given by agents. Distances between all different eigenvalues are used to construct classes and probability to be in each class in given according to $\mathcal{N}(p_t, \sigma_{\beta_i})$. Then, if x agents are on the same class, their β_i at time t is equal the probability to be on the class divided by x .

Cross-Correlation Matrices, and that do not return NA values. Based on these two criteria, our simulation of the β_i time series, the market system most likely fits a VARMA(1, 0). Despite our *a priori* independent periods and evolving agents behavior, the next market system evolution can be predicted.¹⁷

2.5.2 Applications of the model for the financial market regulator

Considering the financial market regulator, we can study the *computational strength* of a market *a posteriori* if a MU provides all order books on a time period as well as the identity of economic agents operating on the market (or at least distinguish actors who pass orders). By computing the fundamental value by the discounted dividends (effective information), we can study if at least an agent was better than the market in discovering the fundamental value. This work could be compared to the one of Shiller [30].

Then, we also can schedule an experimentation when information is given in an *effective* format and a *non effective* format to calibrate the model. This could give us some insights on the computational strength of financial markets by getting a statistical estimation of the computational strength of the market evolution with time. Monte Carlo simulations method can be used.

Finally, if a policy maker wants to study the *market power* of some agents, an estimation of the market system using a VARMA model can also be used to parameterize the market system dynamics aside with some Monte Carlo simulations for the $\beta_{i,t}$ distribution. If an agent's β_i previsions are significantly higher than the one in perfect competition (cf. remark 2, $\beta_i^c = \beta^c = \frac{1}{n}$), it means that this agent has a higher probability of driving the *market price*. These tests can be made regarding the volumes of shares of a given asset (or open positions for derivatives). If so, we should be able to estimate a threshold of quantity above which an agent become significantly market driver if agents β_i are indeed correlated to volumes or open positions. It also can be done to estimate the *market power* of fast traders compared to the one of slow traders following the statement of Hoffmann [17]. The policy maker can regulate according to it by limiting the volumes or open positions to the threshold obtained.

2.6 Conclusion

This work underlines the major influence of information's nature and structure on the capability of a market to be a useful tool (having a computational strength) as well as to be the best tool (being semi-strong efficient). We summarize the relation between the two characteristics in table 2.1.

There are several possible further steps. One can compute the computational strength of a financial market *a posteriori* and estimate the relative quantity of effective information and non effective information processed during the period. One can also use market data to test how a bubble evolves over time in order to parameterize its (moderate) growth.

We have established a condition on the nature and relative quantity of information given the assumption of fast transactions and freely available information that make it possible for financial markets to be semi-strong efficient. We also extended the implications of the concept of *market system* introduced in Bretto and Priolon [4]. We argued that it could help both traders and financial regulators respectively by allowing them to predict who is statistically the trader that will give the next market price and by estimating

¹⁷We can provide the full algorithm, david.batista-soares@unicaen.fr

	$\forall t \in [1; t]$ $ p_t - \mu_{i,t} \leq \epsilon_i$	$\exists t \in [1; t]$ $ p_t - \mu_{i,t} > \epsilon_i$
$C(s, t) \geq 0$	\Leftarrow $nef = o(ef)$? $\left\{ \begin{array}{l} nef \neq o(ef) \\ ef \neq o(nef) \end{array} \right.$
$C(s, t) < 0$	\emptyset	\Rightarrow $ef = o(nef)$

Table 2.1 – Relation between the semi-strong efficiency and the computational strength of a financial market

if traders *market power* is equally distributed. These elements might help improve financial markets regulation, through the regulation of information flows on markets where information is continuously updated. Such a model also corroborates the presence of market makers allowing transaction's frequency to raise within markets where information is continuously updated. Regulators also can create some computational strength index with different time horizon in order to study how is evolving the capability of the financial market to price the asset. The policy maker has to regulate in order to achieve the negligibility of non effective information in front of effective information, that is, obligate all information providers to a certain type and format for their communications.

From scientist perspective, the main extension of this model would be to consider imperfect information. As we already proved it in subsection 2.3.3, the lecture of quantifiable do not commute with time. Hence, having information on quantities and then on prices is not equivalent to the opposite. Asymmetric information could also generate some unexpected effect according to who is giving the market price first. Within this model we cannot introduce a cost to get the information but we could consider that all agents have a part of the information as for the bayesian equilibrium models (cf. Vives [35]). We also can extend the model to financial markets whose information do not arrive continuously.

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Chapter 3

Basis volatility and efficiency on agricultural commodity markets: The impact of a difference in trade frequencies between spot and futures markets

This chapter was presented during three academic conferences:

- (i) On an early stage at the “Journées de Recherche en Sciences Sociales”, organized by the Société Française d’Économie Rurale in 2016;*
 - (ii) The symposium “Coping with risks in agriculture: what challenges and prospects?”, organized by UnilaSalle Beauvais, in partnership with Groupama Paris Val de Loire in 2018;*
 - (iii) The annual meeting of the Commodity and Energy Market Association, organized by the Tepper School of Business in 2019.*
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3.1 Introduction

The spot and the futures markets of any agricultural commodity are connected by definition and there are complex feedbacks between the series of prices on the two markets. The evolution of the basis is a source of risk, and its volatility is a major stake for farmers and processors: it represents an important component of their production and selling decisions (cf. Moschini and Hennessy [19]). However, the price on the futures market evolves faster than the price on the spot market (cf. O'Hara [21], Working [27]) and thus, the basis has a peculiar dynamics. A rich economic literature has empirically proved that the basis is significantly determined by the convenience yield (cf. Fama and French [9], Gorton et al. [12], Wei and Zhu [25]), and so by the level of the available stocks but – to the best of our knowledge – the dynamics of the basis has not been studied in terms of the difference in trade rhythms. In this view, the evolution in the market liquidity through splitting orders strategies, available quantities of the commodity on the spot market, etc. plays an important role in the evolution of trading dynamics.

A splitting order strategy – or also order fragmentation in this paper – is defined as the division of one specific order – called a parent order – into several suborders – called children orders – and it occurs on both markets. Order fragmentation entails a decrease in quantities traded per transaction and an increase in the *n.o.t* (cf. O'Hara and Ye [20], O'Hara [21], Wang [24], Yam and Zhou [28]). Consequently, on a given period of time, trade frequency increases with order fragmentation. Although splitting order strategy is both used on spot and futures markets, transaction costs are much lower on the derivative market (cf. Working [27]) thereby order fragmentation is neither equal nor proportional on these two markets thus the ratio between the two frequencies is also impacted. Several empirical studies have analyzed the impact of order fragmentation on market liquidity and price volatility (cf. Bennett and Wei [2], Boehmer and Boehmer [5], Kwan et al. [15]), but only for equity markets. However, agricultural commodity markets differ in many ways from equity markets (cf. Back and Prokopcuk [1] for a review). Our main original contribution is that we show how an increase in the trade frequencies, resulting from splitting orders strategy for instance either on the spot market, or on the futures market, or proportionally on both markets modifies the fundamental prices (see below subsection 3.3.5 for a definition) and their fundamental volatility (see below subsection 3.3.6 for a definition). The dissociation between the information on the spot market coming slower than trades on the futures market leads agents to face a probabilistic risk (as intended by Knight [14]) about the level of stock at the next transaction on the futures market.

Another original aspect of our work is the mathematical link that we establish between information and prices. That link has been famously and originally formulated by Fama et al. [11] and Fama [8]:

"I take the market efficiency hypothesis to be the simple statement that security prices fully reflect all available information." – cf. Fama [8] (p.1575)

Once we have formally written the link between prices and information, we study the impact of modern trading on market efficiency. In fact the market efficiency hypothesis is a double one : first the market prices are supposed to be fundamentally determined by information and, second, the process that turns information into prices is supposed to be as efficient as it should be. The market efficiency research program belongs mainly to empirical economics. In this work, we use the central part of Fama's concept which considers that market prices result from the processing of financial information. We can sum up that idea in a very general manner: $p_t = g(I_t)$ where p_t is the price at period t , I_t

is the set of available information and g is an economic function that turns information into prices (see below subsection 3.3.5). The main difference with Fama's tradition is that we use the concept of market efficiency for a research that is, in a first stage, a theoretical framework; that framework is dedicated to the study of the characteristics of two series of market prices that are perfectly connected at the level of the underlying fundamental values but which are not in fact perfectly correlated. Spot and futures prices normally evolve in a like manner but they are not thoroughly linked at least because their dynamics are not perfectly synchronized.

Our study aims to examine the extent to which the difference in trade frequencies between the two markets and the evolution of that difference influence market efficiency as well as to examine the fundamental volatility of the basis. Thus, that study deals with the *market structure and design issues* on market microstructure (cf. Madhavan [16]). Our study of the trade frequencies depends directly on the mechanism that underlies the formation of price. Within a limit order book, trade frequencies are not constrained; on the opposite, in a periodic trading, like a fixing pricing, trade frequencies are by definition fixed. Finally, we consider whether there is an optimal level of trade frequencies that satisfies these two objectives. Hence, if optimal trade frequencies appear to be lower than actual empirical ones, we must examine the actual continuous trading structure implemented through a limit order book pricing. To some extent, our study is close to the work of Du and Zhu [7] who theoretically study the optimal trade frequency on futures markets assuming an exogenous arrival of information from an allocative efficiency point of view. However, our results differ because we introduce the spot market structure which influences the release of information flows. Within that slightly alternative framework, we first demonstrate that a proportional increase in liquidity on both spot and futures markets does not necessarily lead to a decrease in the fundamental basis volatility despite it increases the spot market efficiency (and by extension improves the basis efficiency).

We finally argue that synchronization of trade frequencies allows unbiased prices and a minimal fundamental basis volatility. From this result we extrapolate the existence of a dilemma between market liquidity and the two objectives. That result suggests that the implementation of a limit order book pricing in order to increase the liquidity on commodity futures market generates a higher fundamental basis volatility.

This work is organized as follows. Section 3.2 presents all parameters and variables whereas section 3.3 presents the definitions and hypothesis of the model. Section 3.4 derives the influence of trade frequencies and section 3.5 discusses in detail the impacts of an increase in liquidity on whether the spot market *ceteris paribus*, the futures market *ceteris paribus*, or on both markets proportionally *ceteris paribus*. In section 3.6, we study the existence of optimal trade frequencies. Section 3.7 concludes and discusses the limitations of the model.

3.2 Parameters and variables

Despite the mathematical simplicity of each element of the model, we need numerous parameters and variables. In this section, we introduce all parameters (subsection 3.2.1) and variables (subsection 3.2.2).

3.2.1 The parameters of the model

The following set of parameters is used for the futures and the spot agricultural commodity markets:

Futures market	t_r	ω_f	e	a	ϵ	Q_{op}	T_r
Spot market	t_r	ω_s	e		ϵ		

Table 3.1 – Set of parameters used for the futures and spot markets

- t_r Unit of the clock time line (CTL) considered, such as a second.
- ω_f Historical trade frequency on the futures market per unit of clock time: $\left(\frac{\text{Number of transactions on the futures market}}{t_r} \right)$
- ω_s Historical trade frequency on the spot market per unit of clock time: $\left(\frac{\text{Number of transactions on the spot market}}{t_r} \right)$
- e Trend of the spot market information delivered in monetary value per unit of clock time.
- a Advantage of possessing a unit of the storable agricultural commodity in monetary value, per unit of clock time.
- ϵ Absolute value of the elasticity of quantities traded per exchange on the spot market to the spot market trade frequency.
- Q_{op} Optimal stock level of the agricultural commodity for agents.
- T_r Maturity of the futures contract, expressed in the CTL.

3.2.2 The variables of the model

The following set of variables is used for the futures and the spot agricultural commodity markets:

Futures market	t		T	$I_{]t_1, t_2]}$
Spot market	t	t'		$I_{]t_1, t_2]}$

Futures market		Q_t	CY_t	\widetilde{f}_{t+1}	f_t	FV_{f_t}
Spot market	q	Q_t		\widetilde{s}_{t+1}	s_t	FV_{s_t}

Table 3.2 – Set of variables used for the futures and spot markets

- t Unit of the Transactions Time Line (TTL), which represents a transaction on the futures market: $t = t_r \times \omega_f$, and $t \in \mathbb{N}$.
- t' Latest period when there was a synchronized transaction in the spot and futures markets until t , expressed in the TTL.
- T Maturity expressed in the TTL: $T = T_r \times \omega_f$, and $T \in \mathbb{N}$.
- $I_{]t_1, t_2]}$ Spot market information delivered in monetary value between t_1 and t_2 , t_1 excluded.
- q Quantities traded on the spot market at each transaction on the spot market.
- Q_t Available stock of the commodity at time t .
- CY_t Convenience yield (advantage in detaining one unit of stock to face risk or uncertainty) in monetary value at time t until maturity.
- $\widetilde{s}_{t+1}, \widetilde{f}_{t+1}, \widetilde{Q}_t$ Agents' expectations of the spot price, futures price, and commodity available stock, respectively, at time t .
- s_t, f_t Observed spot price and futures price, respectively, at time t .
- FV_{s_t}, FV_{f_t} Fundamental values of the spot price and futures price, respectively, at time t .

3.3 Definitions and hypotheses of the model

This section introduces all the definitions and hypotheses of the model. Despite their simplicity, their combination deserves a meticulous presentation. We also justify all assumptions that we make. A particular attention is given to the time lines and the trade frequencies as they are the very essence of the model (subsection 3.3.1). Subsection 3.3.2 specifies the characteristics of the spot market information. In subsection 3.3.3, we define how quantities traded are determined and we justify their influence on the agricultural commodity market through the convenience yield. Subsection 3.3.4 defines the way agents form their expectations of prices dynamics and subsection 3.3.5 defines the effective prices and fundamental values dynamics. Finally, we define an efficient market criteria and what we call the fundamental volatility and we justify these formulations in subsection 3.3.6.

3.3.1 Time lines and trade frequencies

We define \mathbb{F}_s and \mathbb{F}_f as the sets of all possible frequencies on the spot and futures markets. We have $0 \notin (\mathbb{F}_s \cup \mathbb{F}_f)$, because the two markets exist. Therefore, $(\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f$. Furthermore, we assume that $\omega_s \leq \omega_f$. Futures are more frequently traded than the commodity itself since transaction costs are lower (cf. Working [27]). This explains why we choose the transaction on the futures as the time step (the smallest one). We assume that transactions on the futures markets are equally spaced out, according to the futures market trade frequency (FTF). For the sake of simplicity, the clock time between two transactions on the futures market is constant. The effective FTF during the period is known, and the effective spot market trade frequency (STF), is unknown. We assume that the market existence duration is sufficient to get a significant and robust value of the STF within a specific period of time. Thus, we assume that the historical STF converges in probability to the current STF. Hence, the historical STF, ω_s , gives the probability of having a transaction on the spot market per unit of clock time.

At each transaction on the futures market, there is an independent probability $\frac{\omega_s}{\omega_f}$ of having a transaction on the spot market. This probability is equal to the historical RTF.¹ For the sake of simplicity, we do not integrate a "catch up effect", that is, no one passes an order on the spot market because there is no transaction since "a long time" or because the actual delayed spot price is "far" from its actual value (i.e. since the last transaction on the spot market, the released spot market information is very important). The introduction of such a boundary could have some effect depending on the criteria (detection of a possible arbitrage, allocative efficiency operation according to the released information etc.) that we do not develop in this work. It is a possible extension of the model.

Previous information is summarized by figure 3.1.

Since transactions on the spot market occur with a certain probability, t' is a (Discrete

¹Let \ll defines a relation wherein t' represents the latest period when there was a synchronized transaction on the spot and futures markets until t so that the following is true: $t' \ll t$ if $t' \leq t$, and there is no element $y' \neq t'$ nor $y' \neq t$ such that $t' \leq y' \leq t$. Thus, $(t-1)'$ is the latest period when there was a synchronized transaction on the spot market and futures market until $t-1$, and $(t-1)' \ll t-1$. However, we do not have $(t-1)' \ll t'-1$. Indeed, t' and $(t-1)'$ can both be equal to 0 such that $0 \leq -1$, which is impossible.

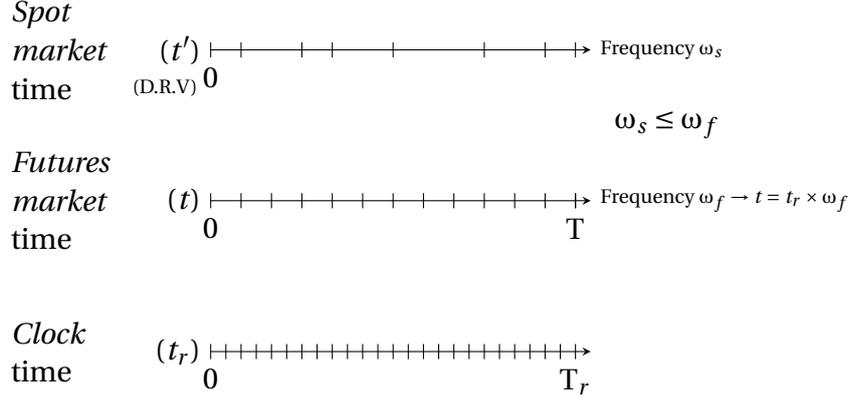


Figure 3.1 – Model time lines

Random Variable (D.R.V) which can take values in $[0; t] \subset \mathbb{N}$. Hence, we have:

$$\begin{cases} \Pr(t' = k) = \frac{\omega_s}{\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^{t-k} \quad \forall k \in [1; t] \\ \Pr(t' = 0) = \left(1 - \frac{\omega_s}{\omega_f}\right)^t \end{cases}$$

We can then express its mean, denoted by $E(\cdot)$, such that:

$$E(t') = \sum_{k=0}^t k \Pr(t' = k) = -\frac{\omega_s}{\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^{t+1} \sum_{k=0}^t -k \left(1 - \frac{\omega_s}{\omega_f}\right)^{-k-1}$$

Let $z = \left(1 - \frac{\omega_s}{\omega_f}\right)$, $0 \leq z < 1$. For a fixed t in the transaction time line (TTL), we have:

$$\sum_{k=0}^t -k z^{-k-1} = \sum_{k=0}^t \frac{d[(z^{-1})^k]}{dz} = \frac{(t+1)z^{-t} - tz^{-(t+1)} - 1}{(z-1)^2}$$

Then, we have:

$$\begin{aligned} E(t') &= -\frac{\omega_s}{\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^{t+1} \frac{(t+1) \left(1 - \frac{\omega_s}{\omega_f}\right)^{-t} - t \left(1 - \frac{\omega_s}{\omega_f}\right)^{-(t+1)} - 1}{\left(\frac{\omega_s}{\omega_f}\right)^2} \\ \Leftrightarrow E(t') &= t - \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t}{1 - \left(1 - \frac{\omega_s}{\omega_f}\right)} \end{aligned}$$

This can be rewritten as follows:

$$E(t') = t - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k \quad (3.1)$$

Remark 5. The *D.R.V* t' can be decomposed into two parts. First, t is the maximal value of t' (since $t' \leq t$). The second part is stochastic and can be assimilated to a truncated geometric law with t experiences – maximum – starting from t in backwardation and ending to 0.² Hence, its mean is equal to its maximal value minus the average delay for having a synchronized transaction looking backward in *n.o.t* on the futures market.

²This framework could have been modeled using a continuous clock time. If so, we would have used a Poisson process with the same property of "lack of memory" of our truncated geometrical component. It would also allow to integrate a catch up effect since the probability at each period won't be independent from the last one.

3.3.2 Spot market information

Let us define what we refer to as the spot market information.

We assume that information on the spot market is intrinsically released at each unit of the real time t_r . However, we naturally consider that information about the spot price is released at each transaction on the futures market. Then, we assume that the monetary impact – or the price impact – of incoming spot market information delivered at time t , denoted $I_{]t-1,t]}$, follows any probabilistic law \mathcal{L} whose mean is $E(I_{]t-1,t]}) = \frac{e}{\omega_f} (t - (t-1)) = \frac{e}{\omega_f}$. This mean is a strictly decreasing function of ω_f . As spot market information is released when a transaction occurs on the futures market, we can naturally assume that if transactions are more frequent on the futures market, spot market information between two transactions on the futures market has, on average, a lower price impact. This information can come from meteorological data, harvest prevision reports, etc. For the sake of simplicity, we consider that there is no minimal trade frequency on the futures market since we are reasoning at the scale of second or minutes despite the frequency of arrival of this type of information is daily (meteorological data for instance) or even monthly (USDA reports for instance). This decreasing mean is not in contradiction with the “Samuelson effect” since on the clock time line (CTL), the mean of information released is identical. Furthermore, we do not specify the impact of the FTF on the volatility of the spot market information released. If we compare two markets where the average price impact of the information is the same on the CTL, if a market has a higher trade frequency, it also has the lower average price impact of information between two transactions. This is a first feedback from the futures market to the spot market at a fundamental level. Hence, contrary to Du and Zhu [7], our spot fundamental value is endogenously determined by the FTF.

Previous information is summarized by figure 3.2.

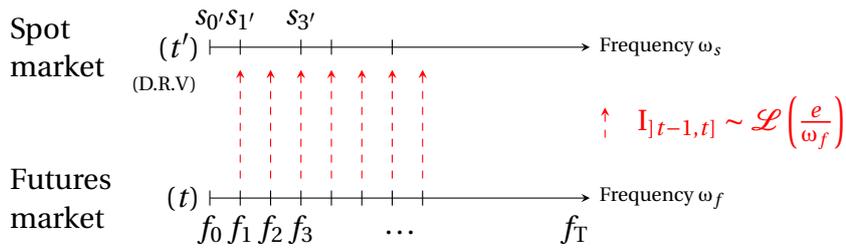


Figure 3.2 – Influence of ω_f on the spot market information, and so, on s_t

According to our definitions, $I_{]t',t]}$ represents the spot market information delivered in monetary value since the last transaction on the spot market at time t , denoted by t' .

3.3.3 Traded quantities and influence of stock dynamics

We now define q , which indicates the quantities traded per transaction on the spot market as a function of the STF, such that:

$$\begin{aligned} q: \mathbb{F}_s &\rightarrow \mathbb{R}^{+,*} \\ \omega_s &\mapsto q(\omega_s) = \omega_s^{-\epsilon}, \epsilon \in \mathbb{Q}^+ \end{aligned} \quad (3.2)$$

From the above function, quantities traded per transaction on the spot market are decreasing when the spot market trade frequency increases. Note that parameter ϵ corresponds to the absolute value of the elasticity of quantities traded per transactions on the spot market to the STF.³ If $\epsilon = 1$, an increase in the STF of one percent leads to a decrease in quantities traded at each transaction on the spot market of one percent. If $\epsilon > \frac{1}{2}$, an increase in the STF of one percent leads to a decrease in quantities traded at each transaction on the spot market of more than $\frac{1}{2}$ percent. If the value of this elasticity were higher than 0 (i.e. $\epsilon < 0$), it would mean that an increase in the STF would lead to an increase in quantities traded at each transaction on the spot market.

The available stock of the commodity at time t is defined using a D.R.V:

$$Q_0 > 0 \text{ and } Q_t := \begin{cases} Q_{t-1} & \text{if } t' \neq t \\ Q_{t-1} - q(\omega_s) & \text{if } t' = t \end{cases} = Q_{t-1} - q(\omega_s) 1_{\{t\}}(t') \quad \forall t \in [1, T] \quad (3.3)$$

We assume that Q_0 is such that for any ω_s in \mathbb{F}_s and any t in $[0; T - 1]$, $Q_t - q(\omega_s) \geq q(\omega_s)$. It is a constraint of non-negativity of the available commodity stock. We naturally assume that the available commodity is only purchased on the spot market during the period considered (at maturity, commodity is purchased and consumed according to the open positions on the futures market which are not cash-settled).⁴ This simple aspect of the model deals with another specificity of agricultural commodity markets when compared to other underlying assets. Indeed, an equity share, or currencies cannot be consumed such that agents choices do not influence their available stock. We assume that production or harvest of the commodity is less frequent than its consumption and then, has a symmetric effect to the stock consumption on expectations and fundamental values. For the sake of simplicity, we do not model them. Hence, the model applies to all storable commodity whose production frequency is lesser than the frequency of consumption. Consequently, the available stock evolves when and only when there is a transaction on the spot market. It is important to underline that this assumption requires us to neglect the cases of pure speculation when the commodity is purchased. For the sake of simplicity, there is no speculation on the spot market. Once again, we could integrate a consumption/speculation Bernoulli D.R.V.⁵

We assume that storage cost per unit of clock time is linear with time: Consequently, they are perfectly expected by agents under our assumption of rational expectations. For the sake of simplicity, we do not model them.

Despite positive stocks, the futures' price can be lower than the spot price because of the necessity for processors to maintain their stocks to face risk or uncertainty (cf. Kaldor [13]). This advantage (or disadvantage) to detain a unit of the agricultural commodity at

³This elasticity is given by: $\frac{\frac{dq(\omega_s)}{q(\omega_s)}}{\frac{d\omega_s}{\omega_s}} = \frac{dq(\omega_s)}{d\omega_s} \times \frac{\omega_s}{q(\omega_s)} = (-\epsilon \times \omega_s^{(-\epsilon-1)}) \times \frac{\omega_s}{\omega_s^{-\epsilon}} = -\epsilon$. This function is a simple Cobb-Douglas one.

⁴The fact that the agricultural commodity stock decreases each time there is a transaction on the spot market does not imply that the commodity is only purchased. However, it means that someone is selling the agricultural commodity and that the buying agent intends to consume it or to use it as an intermediary good, that is, he is not buying it to speculate.

⁵We can include a D.R.V denoted by C such that the probability of having a consumption of the commodity is given by:

$$\begin{cases} \Pr(C = 1) & = \gamma \\ \Pr(C = 0) & = 1 - \gamma \end{cases}$$

The following would be more accurate but does not alter the main outcomes on the impact of an increase in liquidity. However, the integration of such a variable modifies the optimal trade frequencies. We explain how, in the dedicated section 3.6.

transaction t in monetary value until maturity, also called convenience yield and denoted by CY_t , is given by the following equation:

$$CY_t := \frac{a}{\omega_f} (T - t) (Q_{op} - Q_t) \quad (3.4)$$

We assume that Q_{op} is exogenous to the model and is equal for all agents. When the stock is above this level, there is no advantage to detain more units, and the convenience yield is negative. When the stock is under this level, there is an advantage to detain the units of stock. This advantage also depends on the **n.o.t** on the futures market remaining until maturity weighted by the advantage per unit of clock time between a transaction on the futures market ($\frac{a}{\omega_f} (T - t) = a(T_r - t_r)$). Once again, this simple modeling has serious consequences since it considers a second feedback from the spot market to the futures market at the fundamental level. Contrary to Du and Zhu [7], our futures fundamental value is also endogenously determined by the **STF**.

Previous information is summarized by figure 3.3.

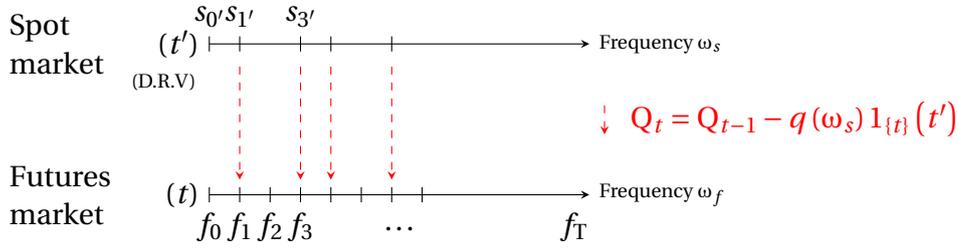


Figure 3.3 – Influence of ω_s on the Convenience Yield, and so, on f_t

3.3.4 Agents' expectations

We assume that all information is freely available and that, at time t , all agents know all the prices and all the market characteristics. Therefore, $\Phi_t = \{s_t, f_t, I_{[t',t]}, Q_t, CY_t\}$ is the common knowledge at time t . Furthermore, we assume that agents form rational expectations in the sense that, on average, their expectations reflect fundamental prices. They compute the average expected price evolution to do so.

The expectation of the spot price is computed by:

$$E(\widetilde{s}_{t+1} | \Phi_t) = s_{t'} + I_{[t',t]} + \frac{e}{\omega_f} \quad (3.5)$$

Recall that t' represents the latest synchronized transaction on the spot market and futures market until t . Agents add to the current spot price (at period t , $s_t = s_{t'}$), all spot market information that should have been included in the spot price $I_{[t',t]}$, and the expected incoming spot market information for its average value $E(I_{[t,t+1]}) = \frac{e}{\omega_f}$.

Expectations of the stock dynamics are given by:

$$E(\widetilde{Q}_{t+1} | \Phi_t) = Q_t - q(\omega_s) \frac{\omega_s}{\omega_f} \quad (3.6)$$

The expected stock evolution is given by quantities traded at each transaction on the spot market (that we suppose to be consumed) weighted by the probability of having a transaction on the spot market at the next transaction on the futures market. The convenience yield is expected such that:

$$E(\widetilde{CY}_{t+1}|\Phi_t) = \frac{a}{\omega_f} (T - (t + 1)) (Q_{op} - E(\widetilde{Q}_{t+1}|\Phi_t)) \quad (3.7)$$

To prevent an arbitrage operation (defined as an operation that guarantees a positive profit without risk of loss; cf. Poncet et al. [22]), the basis must include the current level of stock since the lower stocks are, the more agents have an interest in detaining it depending on the level of their optimal stock. Between each transaction on the futures market, the advantage in detaining stocks decreases by $\frac{a}{\omega_f}$ for the same level of stock than at t . However, there is a probability of $\frac{\omega_s}{\omega_f}$ of having a stock movement, and the average stock evolution is equal to q . Thus, expected stock evolution affects the expected advantage in detaining stocks until maturity.

Finally, expectations of the futures price for the next period are:

$$E(\widetilde{f}_{t+1}|\Phi_t) := E(\widetilde{s}_{t+1} - \widetilde{CY}_{t+1}|\Phi_t) \quad (3.8)$$

Equation (3.8) translates the fact that agents know what the no arbitrage condition is.

3.3.5 Prices dynamics and fundamental values

Prices evolve according to agents' expectations if and only if a transaction occurs. Agents expect the next prices but do not necessarily pass an order at these prices. In other words, expectations are fulfilled when agents act according to them. Thus:

$$\begin{aligned} s_{t+1} &:= \begin{cases} s_{t'} & \text{if } (t+1)' \neq t+1 \\ E(\widetilde{s}_{t+1}|\Phi_t) & \text{if } (t+1)' = t+1 \end{cases} = s_{t'} + 1_{\{t+1\}} ((t+1)') \left[I_{|t',t|} + \frac{e}{\omega_f} \right] \\ f_{t+1} &:= E(\widetilde{f}_{t+1}|\Phi_t) \end{aligned} \quad (3.9)$$

The fundamental value of a price corresponds to the instantaneous integration of all available information on the price (cf. Fama and Miller [10]). Assuming that $FV_{s_0} = s_0$ and $FV_{f_0} = f_0$, the fundamental values are naturally defined by:

$$\begin{aligned} FV_{s_{t+1}} &:= s_0 + I_{|0,t+1|} \\ FV_{f_{t+1}} &:= FV_{s_{t+1}} - CY_{t+1} \end{aligned} \quad (3.10)$$

The futures' fundamental value is classically computed using a classical view, by its no arbitrage value (cf. Black [3], Black and Scholes [4], Cox et al. [6], Merton [18]). However, the main difference with these famous models is that our fundamental values are endogenous to the model because they rely on feedbacks from one market to the other.

3.3.6 Market efficiency and fundamental volatility

We study prices bias and we refer to it as market efficiency, denoted by B . We are aware that an unbiased price does not ensure that at each time l , $l \in \{t, t_r\}$, the price fully reflects the information on the market, but unbiased price is a *sine qua non* condition to an efficient market. We give the following function for the market efficiency:

$$\begin{aligned} B: \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ p_l &\mapsto B(p_l) = |E(p_l - FV_{p_l})|, \quad p_t \in \{s_l, f_l\} \text{ where } |\cdot| \text{ denote the absolute value} \end{aligned} \quad (3.11)$$

Then, we consider the fundamental volatility criterion, denoted by V . It is computed using a standard variance. We are aware that the variance is not the only way of computing the volatility of a variable. We could have used a kurtosis measure, a value at risk, the spread between the maximum price and the minimum price on the period etc. However, it is one possible tool allowing to explain what is at stake. Then, we have at time l , $l \in \{t, t_r\}$:

$$\begin{aligned} V: \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ FV_{p_l} &\mapsto V(FV_{p_l}) = Var(FV_{p_l}), p_l \in \{s_l, f_l\} \end{aligned} \quad (3.12)$$

3.4 The impact of trade frequencies

In this section, we study the impact of trade frequencies on market efficiency and fundamental basis volatility. Market efficiency regarding splitting orders strategies and by extension their impact on liquidity has already been studied empirically in previous literature (cf. Bennett and Wei [2], Kwan et al. [15], O'Hara and Ye [20], O'Hara [21]). We provide a theoretical explanation of its impacts. Finally, we show within this framework that trade frequencies also impacts fundamental basis volatility.

3.4.1 Impact of trade frequencies on the market efficiency

Theorem 4. *Let SM and FM respectively be a spot market and a futures market under our hypothesis. According to the time line considered, the price on SM is biased such that:*

$$B(s_t) = \frac{|-e|}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k \quad (3.13)$$

$$B(s_{t_r}) = \frac{|-e|}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k \quad (3.14)$$

This bias is null if one of the two following subsets of conditions is verified:

- (i) For $e = 0$, $B(s_t) = B(s_{t_r}) = 0$;
- (ii) For $\omega_s = \omega_f$, $B(s_t) = B(s_{t_r}) = 0$.

Proof. Using equation (3.9), we have:

$$\begin{aligned} s_{(1)'} &= s_0 + 1_{\{1\}}(1') \left[I_{|0',0|} + \frac{e}{\omega_f} \right] \\ \Rightarrow s_{(2)'} &= s_1 + 1_{\{2\}}(2') \left[I_{|1',1|} + \frac{e}{\omega_f} \right] \\ \Leftrightarrow s_{(2)'} &= s_0 + 1_{\{1\}}(1') \left[I_{|0',0|} + \frac{e}{\omega_f} \right] + 1_{\{2\}}(2') \left[I_{|1',1|} + \frac{e}{\omega_f} \right] \end{aligned}$$

Hence, we express $s_{(t-1)'}$ according to s_0 as follows:

$$s_{(t-1)'} = s_0 + \sum_{k=1}^{(t-1)'} 1_{\{k\}}(k') \left[I_{|(k-1)',k-1|} + \frac{e}{\omega_f} \right] \quad (3.15)$$

We express the dynamics of s_t according to s_0 using equations (3.9) and (3.15):

$$s_t = s_0 + \sum_{k=1}^t 1_{\{k\}}(k') \left[I_{|(k-1)',k-1|} + \frac{e}{\omega_f} \right]$$

Then, its average value is:

$$\begin{aligned}
 E(s_t) &= E\left(s_0 + \sum_{k=1}^t 1_{\{k\}}(k') \left[I_{(k-1)',k-1} + \frac{e}{\omega_f} \right]\right) \\
 \Leftrightarrow E(s_t) &= s_0 + \sum_{k=1}^t E\left(1_{\{k\}}(k') \left[I_{(k-1)',k-1} + \frac{e}{\omega_f} \right]\right) \\
 \Leftrightarrow E(s_t) &= s_0 + \sum_{k=1}^t E(1_{\{k\}}(k')) E\left(I_{(k-1)',k-1} + \frac{e}{\omega_f}\right) \\
 \Leftrightarrow E(s_t) &= s_0 + \frac{\omega_s}{\omega_f} \frac{e}{\omega_f} \sum_{k=1}^t (k-1 - E((k-1)') + 1) \\
 \Leftrightarrow E(s_t) &= s_0 + \frac{\omega_s}{\omega_f} \frac{e}{\omega_f} \sum_{k=1}^t (k - E((k-1)'))
 \end{aligned}$$

We compute the average spot fundamental value at time t :

$$E(FV_{s_t}) = E(s_0 + I_{|0,t}) = s_0 + \frac{e}{\omega_f} t$$

We have the following expression of the bias:

$$\begin{aligned}
 B(s_t) &= \left| s_0 + \frac{\omega_s}{\omega_f} \frac{e}{\omega_f} \left[\sum_{k=1}^t (k - E((k-1)')) \right] - s_0 - \frac{e}{\omega_f} t \right| \\
 \Leftrightarrow B(s_t) &= \left| \frac{e}{\omega_f} \frac{\omega_s}{\omega_f} \sum_{k=1}^t \left(k - \frac{\omega_f}{\omega_s} - E((k-1)') \right) \right|
 \end{aligned} \tag{3.16}$$

Replacing $E((k-1)')$ by its value given in equation (3.1) in (3.16), we obtain the following expression of the spot price bias:

$$\begin{aligned}
 B(s_t) &= \left| \frac{e}{\omega_f} \frac{\omega_s}{\omega_f} \sum_{k=1}^t \left(k - \frac{\omega_f}{\omega_s} - \left[k-1 - \sum_{j=1}^{k-1} \left(1 - \frac{\omega_s}{\omega_f} \right)^j \right] \right) \right| \\
 \Leftrightarrow B(s_t) &= \left| \frac{e}{\omega_f} \frac{\omega_s}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_f}{\omega_s} + \sum_{j=1}^{k-1} \left(1 - \frac{\omega_s}{\omega_f} \right)^j \right) \right| \\
 \Leftrightarrow B(s_t) &= \left| \frac{-e}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} - \frac{\omega_s}{\omega_f} \sum_{j=1}^{k-1} \left(1 - \frac{\omega_s}{\omega_f} \right)^j \right) \right| \\
 \Leftrightarrow B(s_t) &= \left| \frac{-e}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} - \frac{\omega_s}{\omega_f} \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{1 - \left(1 - \frac{\omega_s}{\omega_f} \right)^{k-1}}{1 - \left(1 - \frac{\omega_s}{\omega_f} \right)} \right) \right| \\
 \Leftrightarrow B(s_t) &= \left| \frac{-e}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} - \left(1 - \frac{\omega_s}{\omega_f} \right) \left[1 - \left(1 - \frac{\omega_s}{\omega_f} \right)^{k-1} \right] \right) \right| \\
 \Leftrightarrow B(s_t) &= \frac{|-e|}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} \right)^k
 \end{aligned}$$

For $e \neq 0$ and $\omega_s \neq \omega_f$, $B(s_t) > 0$. Hence, the spot price is biased, and SM cannot be an efficient market.

To get $B(s_{t_r})$, we explain t in the clock time line (i.e. $t = t_r \times \omega_f$). The expression of the bias is null for the same subsets of conditions than $B(s_t)$. \square

If $e \neq 0$ and $\omega_s \neq \omega_f$, trade frequencies have an impact on spot market efficiency. They generate a bias on the spot price dynamics that results in an adjustment delay explained in a **n.o.t**. However, spot market information is released between transactions on the futures market. The adjustment delay in monetary value is then, strictly positive.

Remark 6. For $e = 0$, $B(s_t) = B(s_{t_r}) = 0$.

When incoming spot market information has no impact on average on the spot price, an adjustment delay in the **n.o.t** does not generate a bias on the market. As the released spot market information has on average no impact on monetary value, the adjustment delay in monetary value corresponding to the bias is null independently of the time line considered.

For $\omega_s = \omega_f$, $t' = t \forall t \in [1, T]$, we have $E(t') = t$; hence, $B(s_t) = B(s_{t_r}) = 0$.

When transactions are synchronized between spot and futures markets, there is no adjustment delay in **n.o.t** on the futures market. Thus, the rational expectations assumption ensures an unbiased spot price independently of the time line considered.

In reality, the futures market trade frequency (**FTF**) is higher than the spot market trade frequency (**STF**) (cf. Working [26]). Furthermore, commodity prices have a seasonal tendency such that $e \neq 0$ (cf. Back and Prokopczuk [1]). Then, the adjustment delay of the spot market can be important.

Corollary 3. Let FM be a futures market under our hypothesis. FM is an unbiased market such that:

$$B(f_t) = B(f_{t_r}) = 0 \quad (3.17)$$

Thus, FM can be an efficient market.

Spot market information arrives between two transactions on the futures market (impact of the futures market on the spot market). Under the assumption of rational expectations, agents perfectly expect, on average, fundamental prices. As the futures price always evolves according to expectations, futures price is unbiased independently of the time line considered.

Remark 7. The futures market efficiency is independent from trade frequencies. However, the **STF** has an indirect impact on stock dynamics and its expectations (cf. equations (3.3) and (3.6)). One can infer that this indirect impact is, on average, equal on both futures' price and fundamental value dynamics.

3.4.2 Impact of trade frequencies on the fundamental basis volatility

Theorem 5. Let SM and FM respectively be a spot market and a futures market under our hypothesis. The available stock and the fundamental value of the basis have the following properties:

- (i) The volatility of the available stock expressed respectively in the transactions time line (**TTL**), denoted by $\text{Var}(Q_t)$, and in the clock time line (**CTL**), denoted by $\text{Var}(Q_{t_r})$, are impacted by both **STF** and **FTF**.

$$\text{Var}(Q_t) = tq(\omega_s)^2 \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} \quad (3.18)$$

$$\text{Var}(Q_{t_r}) = t_r q(\omega_s)^2 \left(1 - \frac{\omega_s}{\omega_f}\right) \omega_s \quad (3.19)$$

This volatility is null if the following subset of conditions is verified:

- For $\omega_s = \omega_f$, $\text{Var}(Q_t) = \text{Var}(Q_{t_r}) = 0$.

(ii) The fundamental basis volatility in the *TTL*, denoted by $\text{Var}(FV_{f_t} - FV_{s_t})$, and in the *CTL*, denoted by $\text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}})$, are impacted by both *STF* and *FTF*.

$$\text{Var}(FV_{f_t} - FV_{s_t}) = \left(\frac{a}{\omega_f}\right)^2 ((T_r \omega_f) - t)^2 q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} \quad (3.20)$$

$$\text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = a^2 (T_r - t_r)^2 q(\omega_s)^2 t_r \left(1 - \frac{\omega_s}{\omega_f}\right) \omega_s \quad (3.21)$$

This volatility is null if one of the three following subsets of conditions is verified:

- For $\omega_s = \omega_f$, $\text{Var}(FV_{f_t} - FV_{s_t}) = \text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$;
- For $a = 0$, $\text{Var}(FV_{f_t} - FV_{s_t}) = \text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$;
- For $t = T$, or equivalently $t_r = T_r$, $\text{Var}(FV_{f_t} - FV_{s_t}) = \text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$.

Proof. First, we compute $\text{Var}(Q_t)$ and show that it depends on parameters ω_s and ω_f .

$$\begin{aligned} \text{Var}(Q_t) &= \text{Var}\left(Q_0 - q(\omega_s) \sum_{k=1}^t 1_{\{t\}}(t')\right) \\ \Leftrightarrow \text{Var}(Q_t) &= q(\omega_s)^2 \text{Var}\left(\sum_{k=1}^t 1_{\{t\}}(t')\right) = q(\omega_s)^2 \sum_{k=1}^t \text{Var}\left(1_{\{t\}}(t')\right) \\ \Leftrightarrow \text{Var}(Q_t) &= q(\omega_s)^2 \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} = q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} \end{aligned}$$

To get $\text{Var}(Q_{t_r})$, we explain t in the clock time line. We respectively obtain expressions (3.18) and (3.19).

For $\omega_s \neq \omega_f$, $\text{V}(Q_t) > 0$, and $\text{V}(Q_{t_r}) > 0$.

Thus, expressions of $\text{Var}(Q_t)$ and $\text{Var}(Q_{t_r})$ depend on parameters ω_s and ω_f , which proves the first assertion.

Then, we compute the fundamental variance of the spread between the futures price and the spot price. We show that it also depends on parameters ω_s and ω_f .

$$\begin{aligned} \text{V}(FV_{f_t} - FV_{s_t}) &= \text{Var}(CY_t) = \text{Var}\left(\frac{a}{\omega_f} (T - t) (Q_{op} - Q_t)\right) \\ \Leftrightarrow \text{V}(FV_{f_t} - FV_{s_t}) &= \left(\frac{a}{\omega_f}\right)^2 ((T_r \omega_f) - t)^2 \text{Var}(Q_{op} - Q_t) = \left(\frac{a}{\omega_f}\right)^2 ((T_r \omega_f) - t)^2 \text{Var}(Q_t) \end{aligned}$$

Knowing the value of $\text{Var}(Q_t)$ from the first assertion's proof above, we obtain the expression of equation (3.20).

To get $\text{V}(FV_{f_{t_r}} - FV_{s_{t_r}})$, we explain t in the clock time line. Knowing the value of $\text{Var}(Q_{t_r})$ from the first assertion's proof above, we obtain the expression of equation (3.21).

For $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, $\text{V}(FV_{f_t} - FV_{s_t}) > 0$.
For $a \neq 0$, $t_r < T_r$, and $\omega_s \neq \omega_f$, $\text{V}(FV_{f_{t_r}} - FV_{s_{t_r}}) > 0$.

Expressions of $\text{V}(FV_{f_t} - FV_{s_t})$ and $\text{V}(FV_{f_{t_r}} - FV_{s_{t_r}})$ depend on parameters ω_s and ω_f , which proves the second assertion. \square

If $a \neq 0$, $\omega_s \neq \omega_f$, and $t < T$ (or equivalently $t_r < T_r$), trade frequencies have an impact on the fundamental basis volatility, expressed either in the [TTL](#) or in the [CTL](#). They generate a probabilistic risk as defined by Knight [14] on the dynamics of the commodity stock. Furthermore, the [STF](#) has an impact on quantities traded on the spot market according to the absolute value of the elasticity ϵ .

A representation of the sensitivity of the fundamental basis volatility to trade frequencies according to the elasticity of quantities traded at each transaction on the spot market to the [STF](#) is given in figures 3.7 and 3.9 for the [TTL](#).

A representation of the sensitivity of the fundamental basis volatility to trade frequencies according to the elasticity of quantities traded at each transaction on the spot market to the [STF](#) is given in figure 3.13 for the [CTL](#).

Remark 8. For $a = 0$, $\text{Var}(FV_{f_t} - FV_{s_t}) = \text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$. If there is no advantage in detaining stocks until maturity both fundamental basis volatility and fundamental basis are null.⁶ The latest mechanism is valid independently of the time line considered.

For $\omega_s = \omega_f$, $\text{Var}(FV_{f_t} - FV_{s_t}) = \text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$. There are only synchronized transactions in the spot and futures agricultural commodity markets. Consequently, there is neither risk nor uncertainty; There is a transaction of q unit(s) of the commodity at each transaction on the spot market and de facto at each transaction on the futures market perfectly expected. The latest mechanism is valid independently of the time line considered.

For $t = T$, or equivalently $t_r = T_r$, $\text{Var}(FV_{f_t} - FV_{s_t}) = \text{Var}(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$. As a consequence of the no arbitrage condition, at maturity, the basis must be null as well as its volatility. The latest mechanism is valid independently of the time line considered.

In reality, the [FTF](#) is higher than the [STF](#). Transaction costs on the spot market are rather high, and they prevent the [STF](#) from increasing. Furthermore, the advantage of detaining stocks of the storable agricultural commodity is a major and significant component of the basis, such that $a \neq 0$.⁷ Then, basis fundamental volatility increases because of these market characteristics.

3.5 The impact of an increase in the liquidity

Theorems 4 and 5 prove that under our assumptions, trade frequencies influence the fundamental basis volatility and the spot price bias. In this section we study how this impact evolves when these frequencies increase and then when liquidity increases. Aside of the economic mechanisms, we provide all of the associated analytical results as well as some illustrations. We study the impact of the increase in liquidity on the spot market and on the futures market separately as well as the impact of a proportional increase in liquidity on the two markets on first, the spot price bias (subsection 3.5.1) and second, on the fundamental basis volatility (subsection 3.5.2).

⁶This result holds as we do not integrate storage costs. Otherwise, the basis would equal these storage costs.

⁷Fama and French [9], Gorton et al. [12], Wei and Zhu [25] made an empirical study respectively on 21 commodities, natural gas market and 31 commodities; Back and Prokopczuk [1] made a review.

3.5.1 Impact of an increase in the liquidity on the spot price bias

Theorem 6. *Let SM and FM be respectively a spot market and a futures market under our assumptions. For $e \neq 0$, $a \neq 0$, $\omega_s \neq \omega_f$, and according to the time line considered, the spot price bias evolves such that:*

- (i) *If the spot market trade frequency (STF) increases, the spot price bias strictly decreases, independently of the time line considered.*
- (ii) *Within the transaction time line (TTL) (t), if the futures market trade frequency (FTF) increases, we have:*

- *For $\frac{\omega_s}{\omega_f} \in \left] 0; 1 - \left(\frac{1}{t+1}\right)^{\frac{1}{t}} \right[\subset \left] 0; \frac{1}{2} \right[$, the spot price bias strictly decreases;*
- *For $\frac{\omega_s}{\omega_f} \in \left] 1 - \left(\frac{1}{t+1}\right)^{\frac{1}{t}}; 1 \right[\supset \left] \frac{1}{2}; 1 \right[$, the spot price bias strictly increases.*

Within the clock time line (CTL) (t_r), if the FTF increases the spot price strictly decreases if and only if the following condition is verified:

$$\frac{\omega_f + 1}{\omega_f} > \frac{\sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k}{\sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k} \quad (3.22)$$

- (iii) *Within the TTL, if both STF and FTF increase such that the relative trade frequency (RTF), equal to $\frac{\omega_s}{\omega_f}$, is unchanged, the spot price bias strictly decreases.*

Within the CTL, if both STF and FTF increase such that the RTF, equal to $\frac{\omega_s}{\omega_f}$, is unchanged, the spot price bias strictly decreases if and only if the following condition is verified:

$$\frac{\omega_f + 1}{\omega_f} > \frac{\sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}{\sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k} \quad (3.23)$$

Proof. To prove the first assertion, we calculate the derivation of the spot price bias expressed in the TTL according to ω_s , and we show that it is negative. Using equation (3.13) we have:

$$\begin{aligned} \frac{\partial B(s_t)}{\partial \omega_s} &= \frac{\partial \left[\frac{|-e|}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k \right]}{\partial \omega_s} \\ \Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_s} &= \frac{|-e|}{\omega_f} \left[\sum_{k=1}^t k \left(\frac{-1}{\omega_f}\right) \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} \right] < 0 \end{aligned}$$

The calculus is the same for $\frac{\partial B(s_{t_r})}{\partial \omega_s}$, starting from equation (3.14). Hence, first assertion is proved.

To prove the second assertion, we first calculate the derivation of the spot price bias expressed in the TTL according to ω_f , and we study its sign. Using equation (3.13) we have:

$$\begin{aligned}
\frac{\partial B(s_t)}{\partial \omega_f} &= \frac{\partial \left[\frac{|-e|}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k \right]}{\partial \omega_f} \\
\Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_f} &= |-e| \frac{\partial \left[\frac{1}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k \right]}{\partial \omega_f} \\
\Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_f} &= |-e| \left[-\frac{1}{\omega_f^2} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k + \frac{1}{\omega_f} \sum_{k=1}^t k \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} \frac{\omega_s}{\omega_f^2} \right] \\
\Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_f} &= \frac{|-e|}{\omega_f^2} \left[-\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k + \sum_{k=1}^t k \left(\frac{\omega_s}{\omega_f}\right) \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} \right] \\
\Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_f} &= \frac{|-e|}{\omega_f^2} \left[-\left(1 - \frac{\omega_s}{\omega_f}\right) \frac{1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t}{1 - \left(1 - \frac{\omega_s}{\omega_f}\right)} + \right. \\
&\quad \left. \frac{\omega_s}{\omega_f} \left[\frac{\left(1 - (t+1) \left(1 - \frac{\omega_s}{\omega_f}\right)^t\right) \left(1 - \left(1 - \frac{\omega_s}{\omega_f}\right)\right) + \left(\left(1 - \frac{\omega_s}{\omega_f}\right) - \left(1 - \frac{\omega_s}{\omega_f}\right)^{t+1}\right)}{\left(1 - \left(1 - \frac{\omega_s}{\omega_f}\right)\right)^2} \right] \right] \tag{3.24} \\
\Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_f} &= \frac{|-e|}{\omega_f \omega_s} \left[-\left(\left(1 - \frac{\omega_s}{\omega_f}\right) - \left(1 - \frac{\omega_s}{\omega_f}\right)^{t+1}\right) \right. \\
&\quad \left. + \left(1 - (t+1) \left(1 - \frac{\omega_s}{\omega_f}\right)^t\right) \left(\frac{\omega_s}{\omega_f}\right) + \left(\left(1 - \frac{\omega_s}{\omega_f}\right) - \left(1 - \frac{\omega_s}{\omega_f}\right)^{t+1}\right) \right] \\
\Leftrightarrow \frac{\partial B(s_t)}{\partial \omega_f} &= \frac{|-e|}{\omega_f^2} \left[1 - (t+1) \left(1 - \frac{\omega_s}{\omega_f}\right)^t \right]
\end{aligned}$$

Equation (3.24) is strictly negative if and only if:

$$\begin{aligned}
1 - (t+1) \left(1 - \frac{\omega_s}{\omega_f}\right)^t &< 0 \\
\Leftrightarrow 1 &< (t+1) \left(1 - \frac{\omega_s}{\omega_f}\right)^t \\
\Leftrightarrow \frac{\omega_s}{\omega_f} &< 1 - \left(\frac{1}{t+1}\right)^{\frac{1}{t}}
\end{aligned}$$

Let h be a function such that:

$$\begin{aligned}
h: [1; T_r \times \omega_f] \subset \mathbb{N} &\rightarrow \mathbb{R} \\
t &\mapsto h(t) = 1 - \left(\frac{1}{t+1}\right)^{\frac{1}{t}}
\end{aligned} \tag{3.25}$$

It is easy to prove that h is a C1 class function. Then,

$$\frac{dh(t)}{dt} = -\left(-\frac{1}{t^2}\right) \left(\frac{1}{t+1}\right)^{\frac{1}{t}-1} \frac{-1 \times 1}{(t+1)^2} = -\frac{1}{t^2(t+1)^2} \left(\frac{1}{t+1}\right)^{\frac{1-t}{t}} < 0.$$

The maximal value of h is then $h(1) = \frac{1}{2}$. The minimal value of h is then $\lim_{t \rightarrow T_r \times \omega_f \rightarrow \infty} h(t) = 1 - 0^0 = 0$.

Focusing on the price bias expressed in the CTL, the method used can not be the same for $B(s_{t_r})$, starting from equation (3.14). Indeed, ω_f can not have an infinitesimal evolution guaranteeing that $t_r \omega_f \in \mathbb{N}$, and then, guaranteeing that the sum exists. That is, we use the simple difference to study the evolution of the bias. We denote by $B(s_t)|_x$, $l \in \{t, t_r\}$, the value of the spot price bias with a FTF equal to x . It gives:

$$B(s_{t_r})|_{\omega_f+1} - B(s_{t_r})|_{\omega_f} = \frac{|-e|}{\omega_f+1} \sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k - \frac{|-e|}{\omega_f} \sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k$$

Putting to the same denominator and factorizing by $\frac{|-e|}{(\omega_f+1)\omega_f} > 0$, it gives:

$$B(s_{t_r})|_{\omega_f+1} - B(s_{t_r})|_{\omega_f} = \omega_f \sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k - (\omega_f+1) \sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k$$

This difference is strictly negative if and only if:

$$\begin{aligned} \omega_f \sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k &< (\omega_f+1) \sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k \\ \Leftrightarrow \frac{\sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k}{\sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k} &< \frac{\omega_f+1}{\omega_f} \end{aligned}$$

This proves the second assertion.

To prove the last assertion, we first focus on the **TTL** and we compute the total derivation of spot price bias and then study its sign. A proportional increase of the frequencies implies that the **RTF** is unchanged such that $\frac{\omega_s+d\omega_s}{\omega_f+d\omega_f} = \frac{\omega_s}{\omega_f} \Leftrightarrow \omega_f(\omega_s+d\omega_s) = \omega_s(\omega_f+d\omega_f) \Leftrightarrow d\omega_f = \frac{\omega_f d\omega_s}{\omega_s}$. This gives:

$$\begin{aligned} d(B(s_t)) &= \frac{\partial(B(s_t))}{\partial\omega_s} d\omega_s + \frac{\partial(B(s_t))}{\partial\omega_f} d\omega_f \\ \Leftrightarrow d(B(s_t)) &= \frac{|-e|}{\omega_f} \left[\sum_{k=1}^t k \left(\frac{-1}{\omega_f}\right) \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} \right] d\omega_s \\ &\quad + \frac{|-e|}{\omega_f^2} \left[-\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k + \sum_{k=1}^t k \left(\frac{\omega_s}{\omega_f}\right) \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} \right] \frac{\omega_f}{\omega_s} d\omega_s \\ \Leftrightarrow d(B(s_t)) &= \frac{|-e|}{\omega_f^2} d\omega_s \left[-\sum_{k=1}^t k \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} - \frac{\omega_f}{\omega_s} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k + \sum_{k=1}^t k \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} \right] \\ \Leftrightarrow d(B(s_t)) &= -\frac{|-e|}{\omega_s\omega_f} d\omega_s \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k < 0 \end{aligned}$$

We now focus on the **CTL**. Once again, we face the problem of the existence of the sum and we can not use the derivation. A variation of ω_f of one unit having a **RTF** unchanged implies that $\frac{\omega_s+\Delta\omega_s}{\omega_f+1} = \frac{\omega_s}{\omega_f} \Leftrightarrow \Delta\omega_s = \frac{\omega_s}{\omega_f}$. We denote by $B(s_l)|_{x,y}$, $l \in \{t, t_r\}$, the value of the spot price bias with a **FTF** equal to x and a **STF** equal to y . Using the simple difference, we get:

$$B(s_{t_r})|_{\{\omega_f+1, \omega_s+\frac{\omega_s}{\omega_f}\}} - B(s_{t_r})|_{\{\omega_f, \omega_s\}} = \frac{|-e|}{\omega_f+1} \sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k - \frac{|-e|}{\omega_f} \sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k$$

Putting to the same denominator and factorizing by $\frac{|-e|}{(\omega_f+1)\omega_f} > 0$, it gives:

$$B(s_{t_r})|_{\{\omega_f+1, \omega_s+\frac{\omega_s}{\omega_f}\}} - B(s_{t_r})|_{\{\omega_f, \omega_s\}} = \omega_f \sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f+1}\right)^k - (\omega_f+1) \sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k$$

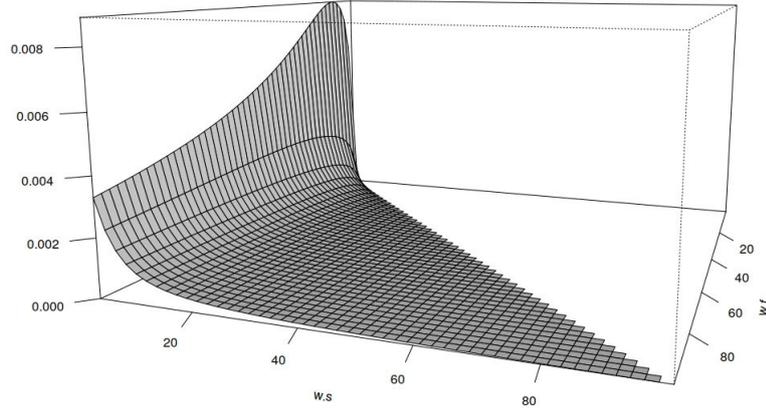


Figure 3.4 – Representation of the spot price bias $B(s_t)$. Parameters used are $e = 0.01$ and $t = 40$.

This difference is strictly negative if and only if:

$$\begin{aligned} \omega_f \sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f}\right)^k &< (\omega_f + 1) \sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k \\ \Leftrightarrow \frac{\sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}{\sum_{k=1}^{t_r\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k} &< \frac{\omega_f+1}{\omega_f} \end{aligned}$$

This proves the last assertion. □

Interpretation of the results for the transactions time line

An increase in the STF (i.e., in the liquidity on the spot market) decreases the average adjustment delay in **n.o.t** on the futures market. As the average spot market information impact between two transactions on the futures market is unchanged, the spot price bias decreases in monetary value (cf. figure 3.4; the bias is strictly decreasing when ω_s increases). This explains the first assertion of theorem 6.

An increase in the FTF (i.e., in the liquidity on the futures market) increases the average adjustment delay in **n.o.t** on the futures market. However, the average spot market information impact between two transactions on the futures market decreases. The first effect overcomes the second one when the **RTF** is initially sufficiently low. The impact of the **FTF** on the **RTF** and thus on the probability of having a transaction on the spot market at each transaction on the futures market is not linear. Figure 3.4 represents the upper level of the plan, wherein for a given (little) **STF**, there is an **FTF** above that the bias decreases. This explains the second assertion of theorem 6.

Remark 9. Note that the above condition depends on the **n.o.t** on the futures market considered, denoted by t . To have an increase in the liquidity on the futures market that reduces the spot price bias, the more we consider a high t , the lower the **RTF** must be (i.e., $\omega_f \gg \omega_s$).

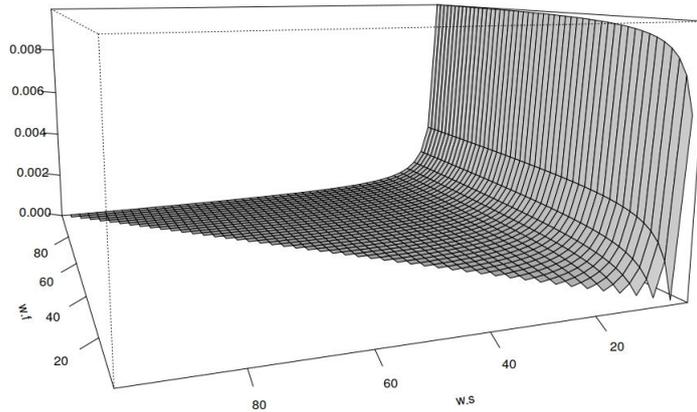


Figure 3.5 – Representation of the spot price bias $B(s_{t_r})$. Parameters used are $e = 0.01$ and $t_r = 40$.

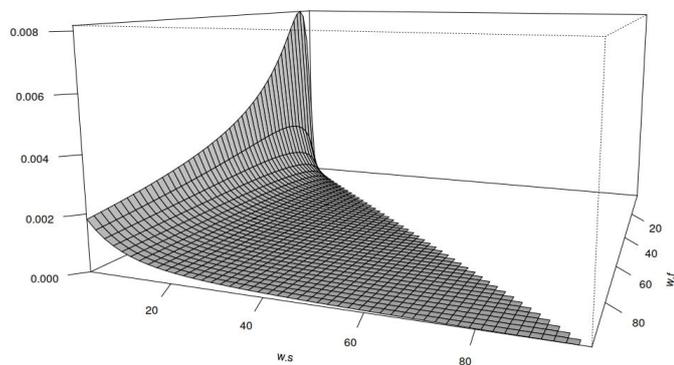
With a high initial *FTF*, an increase in the *FTF* has a small effect on the *RTF*. Figure 3.6 illustrates the effect of t . Looking at the upper part of the plan (when $\omega_f \gg \omega_s$), the higher t is, the less the spot price bias decreases.

A proportional increase in both *STF* and *FTF* such that the *RTF* is unchanged does not modify the probability of synchronized transaction's occurrence. However, between two transactions on the futures market, the average monetary impact of the spot market information on the futures market strictly decreases. Thus, the spot price bias strictly decreases. This result explains the third assertion of theorem 6, and figure 3.4 illustrates it. When following a parallel of the first bisector, the *RTF* is unchanged and the spot price bias decreases.

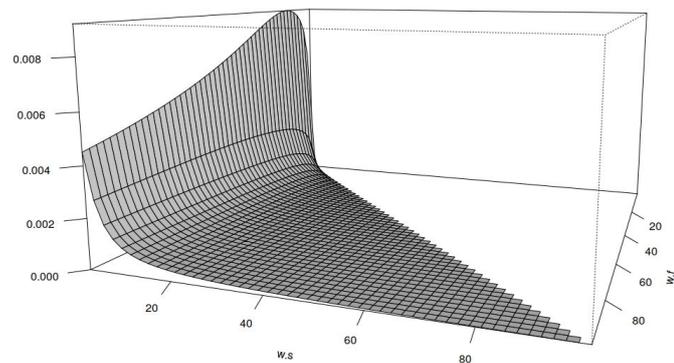
Interpretation of the results for the clock time line

An increase in the *STF* (i.e., in the liquidity on the spot market) decreases the average adjustment delay in *n.o.t* on the futures market. As the average spot market informational impact between two transactions on the futures market is unchanged, the spot price bias decreases in monetary value. This result explains the first assertion of theorem 6.

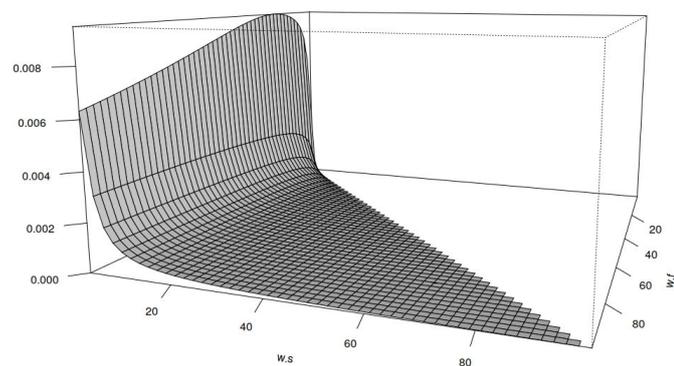
An increase in the *FTF* (i.e., in the liquidity on the futures market) increases the average adjustment delay in *n.o.t* on the futures market for a given *n.o.t* ($t_r \omega_f$). This effect is equal to $\frac{\sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f + 1}\right)^k}{\sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}$. Furthermore, as we consider a fixed clock time, there are more transactions on the futures market (exactly there are $t_r = t_r(\omega_f + 1) - t_r \omega_f$ more transactions). These two effects increase the spot price bias. They are jointly represented by the multiplier of the average delay in *n.o.t* $\frac{\sum_{k=1}^{t_r(\omega_f + 1)} \left(1 - \frac{\omega_s}{\omega_f + 1}\right)^k}{\sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}$. However, the average monetary impact of the spot market information between two transactions on the futures market decreases $\left(\frac{1-e|}{\omega_f + 1} < \frac{1-e|}{\omega_f}\right)$. This effect diminishes the spot price bias according to the multiplier of the *FTF*, $\frac{\omega_f + 1}{\omega_f}$. The latest effect has to overcome the first one to ensure that an increase in the liquidity on the futures market decreases the spot price



(a) $t = 20$



(b) $t = 60$



(c) $t = 100$

Figure 3.6 – Representation of the spot price bias $B(s_t)$ according to the value of t . Parameter used is $e = 0,01$.

bias. Hence, the second assertion of theorem 6 and its condition are explained.

A proportional increase in both STF and FTF such that the RTF is unchanged does not modify the average adjustment delay in n.o.t on the futures market at a given n.o.t ($t_r \omega_f$). This effect is equal to $\frac{\sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}{\sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k} = 1$. However, as we consider a fixed clock time, there are more transactions on the futures market (exactly there are $t_r = t_r(\omega_f + 1) - t_r \omega_f$ more transactions). This effect increases the spot price bias. It is represented by the multiplier of the average delay in n.o.t $\frac{\sum_{k=1}^{t_r(\omega_f+1)} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}{\sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k}$. Then, between two transactions on the futures market, the average monetary impact of the spot market information on the futures market strictly decreases (again according to the multiplier of the FTF $\frac{\omega_f+1}{\omega_f}$). The latest effect has to overcome the first one to ensure that a proportional increase in the liquidity on both spot and futures markets decreases the spot price bias. Hence, the third assertion of theorem 6, and its condition are explained.

3.5.2 Impact of an increase in the liquidity on the fundamental basis volatility

Theorem 7. *Let SM and FM be respectively a spot market and a futures market under our assumptions. For $e \neq 0$, $a \neq 0$, $\omega_s \neq \omega_f$, and according to the time line considered, the fundamental basis volatility evolves such that:*

(i) *If the STF increases, independently of the time line considered, we have:*

- *For $\frac{\omega_s}{\omega_f} \in]0, \frac{1}{2}[$, the fundamental basis volatility strictly decreases if the following condition is verified:*

$$-e < -\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} \quad (3.26)$$

- *For $\frac{\omega_s}{\omega_f} \in]\frac{1}{2}, 1[$, the fundamental basis volatility strictly decreases.*

(ii) *Within the TTL, if the FTF increases, we have:*

- *For $\frac{\omega_s}{\omega_f} \in]0, \frac{1}{2}[$, the fundamental basis volatility strictly decreases if the following condition is verified:*

$$-\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} < -\frac{t}{T_r \omega_f - t} \quad (3.27)$$

- *For $\frac{\omega_s}{\omega_f} \in]\frac{1}{2}, 1[$, the fundamental basis volatility increases.*

Within the CTL, if the FTF increases, the fundamental basis volatility strictly increases.

(iii) *Within the TTL, if both STF and FTF increase such that the RTF, equal to $\frac{\omega_s}{\omega_f}$, is unchanged, the fundamental basis volatility strictly decreases if the following condition is verified:*

$$-e < -\frac{t}{T_r \omega_f - t} \quad (3.28)$$

Within the CTL, if both STF and FTF increase such that the RTF, equal to $\frac{\omega_s}{\omega_f}$, is unchanged, the fundamental basis volatility strictly decreases if the following condition is verified:

$$-\epsilon < -\frac{1}{2} \quad (3.29)$$

Proof. To prove the first assertion, we calculate the derivation of the fundamental basis volatility according to ω_s , and we study its sign. Using equation (3.20) we have:

$$\begin{aligned} \frac{\partial V(FV_{f_t} - FV_{s_t})}{\partial \omega_s} &= \frac{\partial \left[\left(\frac{a}{\omega_f} \right)^2 ((T_r \omega_f) - t)^2 q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{\omega_s}{\omega_f} \right]}{\partial \omega_s} \\ \Leftrightarrow \frac{\partial V(FV_{f_t} - FV_{s_t})}{\partial \omega_s} &= \left(\frac{a}{\omega_f} \right)^2 ((T_r \omega_f) - t)^2 t \frac{1}{\omega_f} \frac{\partial \left[q(\omega_s)^2 \omega_s - \frac{q(\omega_s)^2 \omega_s^2}{\omega_f} \right]}{\partial \omega_s} \\ \Leftrightarrow \frac{\partial V(FV_{f_t} - FV_{s_t})}{\partial \omega_s} &= \left(\frac{a}{\omega_f} \right)^2 ((T_r \omega_f) - t)^2 t \frac{1}{\omega_f} \left[q(\omega_s)^2 + 2\omega_s q(\omega_s) \frac{dq(\omega_s)}{d\omega_s} \right. \\ &\quad \left. - \frac{1}{\omega_f} \left(2\omega_s q(\omega_s)^2 + 2\omega_s^2 q(\omega_s) \frac{dq(\omega_s)}{d\omega_s} \right) \right] \quad (3.30) \\ \Leftrightarrow \frac{\partial V(FV_{f_t} - FV_{s_t})}{\partial \omega_s} &= \left(\frac{a}{\omega_f} \right)^2 ((T_r \omega_f) - t)^2 t_r q(\omega_s) \times \\ &\quad \left[q(\omega_s) \left[1 - 2\frac{\omega_s}{\omega_f} \right] + 2\omega_s \frac{dq(\omega_s)}{d\omega_s} \left[1 - \frac{\omega_s}{\omega_f} \right] \right] \end{aligned}$$

Equation (3.30) is strictly negative if and only if:

$$\begin{aligned} q(\omega_s) \left[1 - 2\frac{\omega_s}{\omega_f} \right] + 2\omega_s \frac{dq(\omega_s)}{d\omega_s} \left[1 - \frac{\omega_s}{\omega_f} \right] &< 0 \\ \Leftrightarrow q(\omega_s) \left[\frac{\omega_f - 2\omega_s}{\omega_f} \right] &< -2\omega_s \frac{dq(\omega_s)}{d\omega_s} \left[\frac{\omega_f - \omega_s}{\omega_f} \right] \\ \Leftrightarrow \left[\frac{\omega_f - 2\omega_s}{\omega_f} \right] \left[\frac{\omega_f}{2(\omega_f - \omega_s)} \right] &< -\frac{\omega_s}{q(\omega_s)} \frac{dq(\omega_s)}{d\omega_s} \\ \Leftrightarrow -\epsilon &< -\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} \quad (3.31) \end{aligned}$$

This condition is always verified when $\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} < 0 \Leftrightarrow \frac{\omega_s}{\omega_f} > \frac{1}{2}$. Otherwise, the relation expressed in equation (3.31) must be verified. This proves the first assertion.

To prove the second assertion, we derive the fundamental basis volatility regarding ω_f

and study its sign.

$$\begin{aligned}
 \frac{\partial V(\widehat{FV}_{f_t} - \widehat{FV}_{s_t})}{\partial \omega_f} &= \frac{\partial \left[\left(\frac{a}{\omega_f} \right)^2 ((T_r \omega_f) - t)^2 q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{\omega_s}{\omega_f} \right]}{\partial \omega_f} \\
 \Leftrightarrow \frac{\partial V(\widehat{FV}_{f_t} - \widehat{FV}_{s_t})}{\partial \omega_f} &= a^2 q(\omega_s)^2 t \omega_s \frac{\partial \left[\frac{1}{\omega_f^2} ((T_r \omega_f) - t)^2 \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{1}{\omega_f} \right]}{\partial \omega_f} \\
 \Leftrightarrow \frac{\partial V(\widehat{FV}_{f_t} - \widehat{FV}_{s_t})}{\partial \omega_f} &= a^2 q(\omega_s)^2 t \omega_s \frac{\partial \left[((T_r \omega_f) - t)^2 \left(\frac{\omega_f - \omega_s}{\omega_f^4} \right) \right]}{\partial \omega_f} \tag{3.32} \\
 \Leftrightarrow \frac{\partial V(\widehat{FV}_{f_t} - \widehat{FV}_{s_t})}{\partial \omega_f} &= a^2 q(\omega_s)^2 t \omega_s \left[2T_r ((T_r \omega_f) - t) \left(\frac{\omega_f - \omega_s}{\omega_f^4} \right) \right. \\
 &\quad \left. + ((T_r \omega_f) - t)^2 \left(\frac{4\omega_s - 3\omega_f}{\omega_f^5} \right) \right] \\
 \Leftrightarrow \frac{\partial V(\widehat{FV}_{f_t} - \widehat{FV}_{s_t})}{\partial \omega_f} &= a^2 q(\omega_s)^2 t \omega_s ((T_r \omega_f) - t) \left[2T_r \left(\frac{\omega_f - \omega_s}{\omega_f^4} \right) \right. \\
 &\quad \left. + ((T_r \omega_f) - t) \left(\frac{4\omega_s - 3\omega_f}{\omega_f^5} \right) \right]
 \end{aligned}$$

This term is strictly negative if and only if:

$$\begin{aligned}
 2T_r \left(\frac{\omega_f - \omega_s}{\omega_f^4} \right) + ((T_r \omega_f) - t) \left(\frac{4\omega_s - 3\omega_f}{\omega_f^5} \right) &< 0 \\
 \Leftrightarrow 2T_r \left(\frac{\omega_f - \omega_s}{\omega_f^4} \right) &< -((T_r \omega_f) - t) \left(\frac{4\omega_s - 3\omega_f}{\omega_f^5} \right) \\
 \Leftrightarrow \frac{T_r \omega_f}{(T_r \omega_f) - t} &< - \left(\frac{4\omega_s - 3\omega_f}{\omega_f^4} \right) \left(\frac{\omega_f^4}{2(\omega_f - \omega_s)} \right) \tag{3.33} \\
 \Leftrightarrow \frac{T_r \omega_f}{T_r \omega_f - t} &< 1 + \frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} \\
 \Leftrightarrow -\frac{t}{T_r \omega_f - t} &> -\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)}
 \end{aligned}$$

For $t \in [0; T_r \times \omega_f - 1] \subset \mathbb{N}$, $\frac{t}{T_r \omega_f - t} \geq 0$. Then, this condition is never verified for:

$$\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} < 0 \Leftrightarrow \frac{\omega_s}{\omega_f} > \frac{1}{2} \tag{3.34}$$

For $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, $\frac{\partial V(\widehat{FV}_{f_t} - \widehat{FV}_{s_t})}{\partial \omega_f} > 0$ when $\frac{\omega_s}{\omega_f} > \frac{1}{2}$. Otherwise, the relation expressed in equation (3.33) must be verified. This proves the second assertion.

To prove the third assertion, we compute the total derivation of the fundamental basis volatility considering a proportional increase of the two frequencies.

Using equations (3.30), (3.32), and the variation of ω_f explained with the variation of

ω_s computed above, we get:

$$\begin{aligned}
 dV(FV_{f_t} - FV_{s_t}) &= \frac{\partial \text{Var}(FV_{f_t} - FV_{s_t})}{\partial \omega_s} d\omega_s + \frac{\partial \text{Var}(FV_{f_t} - FV_{s_t})}{\partial \omega_f} d\omega_f \\
 \Leftrightarrow dV(FV_{f_t} - FV_{s_t}) &= \left(\frac{a}{\omega_f}\right)^2 (T-t)^2 t \frac{1}{\omega_f} q(\omega_s) \times \\
 &\quad \left[q(\omega_s) \left[1 - 2\frac{\omega_s}{\omega_f}\right] + 2\omega_s \frac{dq(\omega_s)}{d\omega_s} \left[1 - \frac{\omega_s}{\omega_f}\right] \right] d\omega_s \\
 &\quad + a^2 q(\omega_s)^2 t \omega_s (T-t) \times \\
 &\quad \left[2T_r \left(\frac{\omega_f - \omega_s}{\omega_f^4}\right) + (T-t) \left(\frac{4\omega_s - 3\omega_f}{\omega_f^5}\right) \right] \frac{\omega_f}{\omega_s} d\omega_s \\
 \Leftrightarrow dV(FV_{f_t} - FV_{s_t}) &= a^2 (T_r - t_r)^2 t \frac{1}{\omega_f} q(\omega_s)^2 \left[1 - 2\frac{\omega_s}{\omega_f} + 2\frac{\omega_s}{q(\omega_s)} \frac{dq(\omega_s)}{d\omega_s} \left[1 - \frac{\omega_s}{\omega_f}\right] \right] d\omega_s \\
 &\quad + a^2 q(\omega_s)^2 t \omega_f (T_r - t_r)^2 \frac{\omega_f}{\omega_f^3} \\
 &\quad \left[2\frac{T_r}{(T_r - t_r)} \left(\frac{\omega_f - \omega_s}{\omega_f}\right) + \frac{(T-t)}{(T_r - t_r)} \left(\frac{4\omega_s - 3\omega_f}{\omega_f^2}\right) \right] d\omega_s \\
 \Leftrightarrow dV(FV_{f_t} - FV_{s_t}) &= a^2 (T_r - t_r)^2 \frac{t}{\omega_f} q(\omega_s)^2 d\omega_s \left[1 - 2\frac{\omega_s}{\omega_f} + 2\frac{\omega_s}{q(\omega_s)} \frac{dq(\omega_s)}{d\omega_s} \left(1 - \frac{\omega_s}{\omega_f}\right) \right. \\
 &\quad \left. + 2\frac{T_r}{(T_r - t_r)} \left(1 - \frac{\omega_s}{\omega_f}\right) + 4\frac{\omega_s}{\omega_f} - 3 \right] \\
 \Leftrightarrow dV(FV_{f_t} - FV_{s_t}) &= a^2 (T_r - t_r)^2 t_r q(\omega_s)^2 2 \left(1 - \frac{\omega_s}{\omega_f}\right) d\omega_s \left[\frac{\omega_s}{q(\omega_s)} \frac{dq(\omega_s)}{d\omega_s} + \frac{T_r}{(T_r - t_r)} - 1 \right]
 \end{aligned} \tag{3.35}$$

Equation (3.35) is strictly negative if and only if:

$$\begin{aligned}
 -1 + \frac{\omega_s}{q(\omega_s)} \frac{dq(\omega_s)}{d\omega_s} + \frac{T_r}{(T_r - t_r)} &< 0 \\
 \Leftrightarrow -\epsilon &< 1 - \frac{T_r}{(T_r - t_r)} \\
 \Leftrightarrow -\epsilon &< -\frac{t}{(T_r \omega_f - t)}
 \end{aligned}$$

This proves the third assertion. \square

Interpretation of the results for the transactions time line

An increase in the STF (i.e., in the liquidity on the spot market) reduces the volatility of the fundamental basis by two mechanisms if the RTF is superior to $\frac{1}{2}$ (cf. figure 3.7, the fundamental basis volatility curve decreases with an increasing STF when $\frac{\omega_s}{\omega_f} > \frac{1}{2}$). First, the volatility of the occurrence of synchronized transactions decreases (cf. figure 3.8). Second, quantities traded per transaction on the spot market are lower, and so smooth the stock dynamics. If the RTF is lower than $\frac{1}{2}$, the first mechanism does not hold but the second still holds. The volatility of the occurrence of synchronized transactions increases (cf. figure 3.8). The "smoothing per trade traded quantities" effect must overcome the "increasing occurrence of synchronized transaction volatility" effect. In other words, when initial market characteristics imply that $\frac{\omega_s}{\omega_f} < \frac{1}{2}$, more fragmentation on the spot market increases the volatility of the futures contract through the increase of the convenience yield volatility. However, more fragmentation on the spot market decreases traded quantities on the spot market. According to the elasticity parameter ϵ , and the original trade frequencies, the latest effect can overcome the first one but it is not always the case.

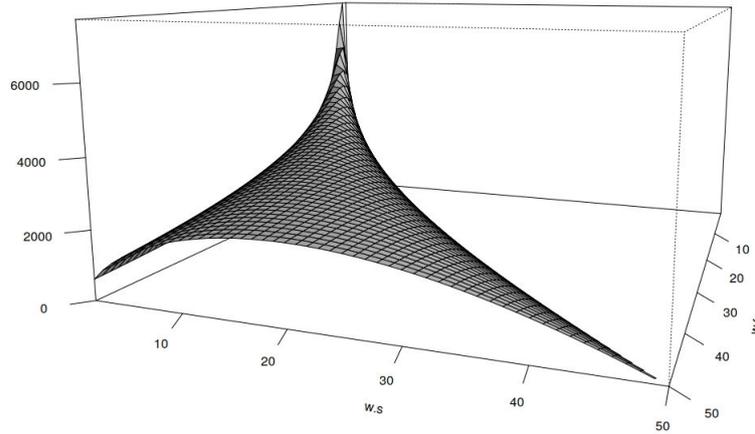


Figure 3.7 – Representation of the fundamental basis volatility $\text{Var}(FV_{f_t} - FV_{s_t})$. Parameters used are $a = 0.01$, $t = 50$, $T_r = 2500$, and $\epsilon = 0.2$.

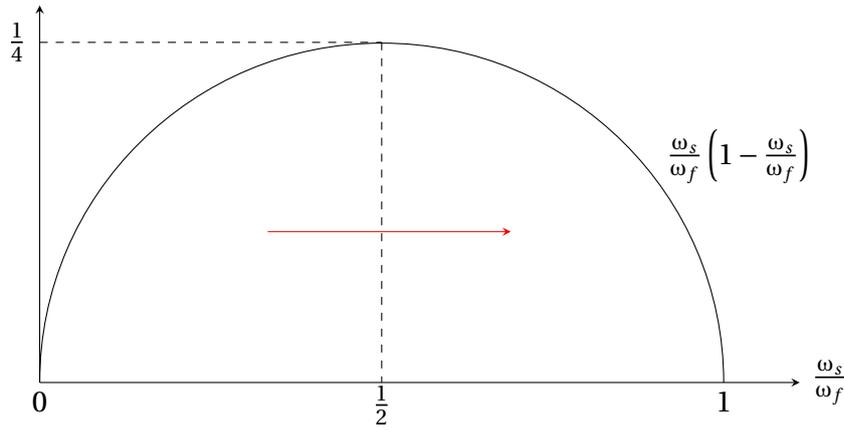


Figure 3.8 – Volatility of the occurrence of synchronized transactions at each transaction on the futures market. Red arrow gives the evolution of the volatility with an increasing STF.

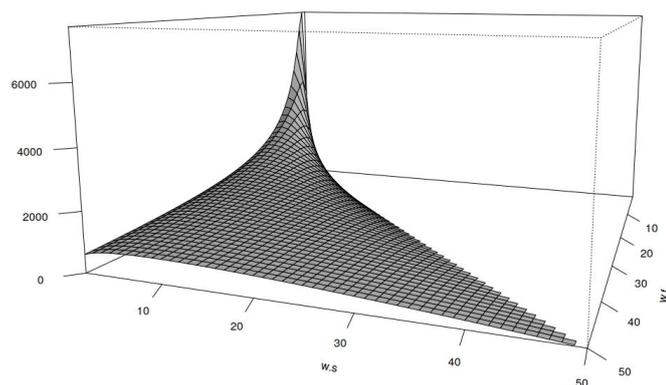
To smooth the stock dynamics enough, the elasticity must be under a threshold (sufficiently high in absolute value) such that quantities traded per trade sufficiently decrease. This condition is given by equation (3.26) and is illustrated in figures 3.7 and 3.9. For $\epsilon = 0,8$ (cf. figure 3.9b), the elasticity is high in absolute value, and the volatility is decreasing with an increasing ω_s . On the contrary, for $\epsilon = 0,2$ (cf. figure 3.7), the volatility is increasing when the STF is originally low. These results explain the first assertion of theorem 7. A study of the condition is presented in remark 10, and highlights the following remark.

Remark 10. Let f be a function such that:

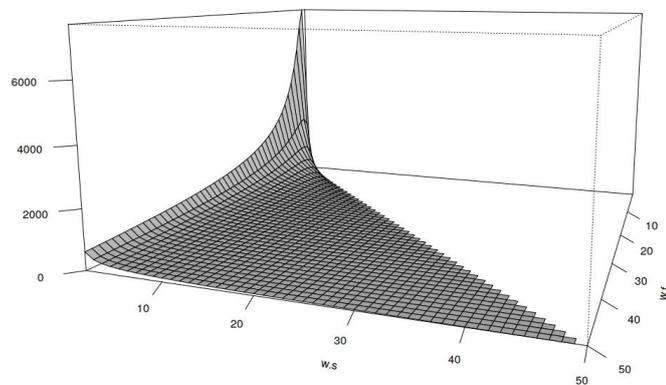
$$\begin{aligned} f : \mathbb{F}_s \setminus \{\omega_f\} \times \mathbb{F}_f &\rightarrow \mathbb{R} \\ (\omega_s, \omega_f) &\mapsto f(\omega_s, \omega_f) = -\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} \end{aligned}$$

It is easy to prove that f is a $C1$ class function.

For $\frac{\omega_s}{\omega_f} \in]0, \frac{1}{2}[$, $\frac{\partial f(\omega_s, \omega_f)}{\partial \omega_s} = \frac{-2(2(\omega_f - \omega_s)) - (\omega_f - 2\omega_s)(-2)}{(2(\omega_f - \omega_s))^2} = \frac{\omega_f}{2(\omega_f - \omega_s)^2} > 0$. The higher ω_s is, the



(a) $\epsilon = 0,4$



(b) $\epsilon = 0,8$

Figure 3.9 – Representation of the fundamental basis volatility $Var(FV_{f_t} - FV_{s_t})$ according to the value of ϵ . Parameters used are $a = 0.01$, $t = 50$ and $T_r = 2500$.

higher ϵ can be to verify equation (3.31).

For $\frac{\omega_s}{\omega_f} \in]0, \frac{1}{2}[$, $\frac{\partial f(\omega_s, \omega_f)}{\partial \omega_f} = -\frac{12(\omega_f - \omega_s) - (\omega_f - 2\omega_s)2}{4(\omega_f - \omega_s)^2} = -\frac{\omega_s}{2(\omega_f - \omega_s)^2} < 0$. The higher ω_f is, the lower ϵ has to be to verify equation (3.31).

Remark 11. This condition depends on the initial trade frequencies. The higher the STF initially is, the lower quantities traded per transaction on the spot market are. Thus, the higher the elasticity can be (the lower in absolute value) such that quantities traded will decrease enough, and overcome the increasing volatility in the stock dynamics. Near the first bisector, the fundamental basis volatility decreases with an increasing STF for all values of ϵ (cf. figures 3.7 and 3.9).

Based on the same reasoning, the higher the FTF is, the more restrictive the condition given by equation (3.26) is. The increasing part of the curve for a given STF is more important when the FTF increases. To overcome this increasing effect, the elasticity must be sufficiently high in absolute value to ensure a sufficient decrease in quantities traded per trade on the spot market. The analytic condition is given in equation (3.27). In figures 3.7 and 3.9, the plan twists according to the value of ϵ , representing this threshold. We observe that the higher ϵ is, the less the plan is twisted starting from low values of FTF.

An increase in the FTF (i.e., in the liquidity on the futures market) strictly increases the volatility of the fundamental basis by two mechanisms if the RTF is superior to $\frac{1}{2}$ (cf. figure 3.7; the fundamental basis volatility plan increases with an increasing ω_f when $\frac{\omega_s}{\omega_f} > \frac{1}{2}$). The volatility of the occurrence of synchronized transactions increases (cf. figure 3.10). Furthermore, a transaction on the spot market long before the maturity has more impact on the convenience yield than if it happens at maturity, as a consequence of the no arbitrage value and the time remaining until maturity (cf. figure 3.11). Here, we have the horizon t , expressed by a n.o.t on the futures market, to study the fundamental basis volatility. An increase in the FTF reduces the clock time at which the first t transactions are made. Each transaction takes place earlier, and there is more clock time remaining until maturity. An increase in the FTF gives more weight to the first t possible transactions on the spot market. It consequently increases the fundamental basis volatility. However, when the RTF is lower than $\frac{1}{2}$, an increasing FTF decreases the volatility of synchronized transaction's occurrence. This effect overcomes the "clock time interval" effect if the condition given by equation (3.27) is verified. Figure 3.12 illustrates the latest condition. These results explain the second assertion of theorem 7.

A proportional increase in the STF and the FTF such that $\frac{\omega_s}{\omega_f}$ is unchanged does not modify the occurrence of synchronized transaction's volatility. However, for a given clock time period, there are more transactions in the futures' market. When considering a given n.o.t on the futures market (or equivalently a given transaction time period) t , it reduces the clock time considered.

Therefore, each potential evolution of the stock has more impact on the convenience yield, since the clock time interval between the first t transactions and maturity is larger. The increase of the STF reduces quantities traded per transaction on the spot market reducing the stock dynamics volatility at an unchanged time horizon. To ensure that this effect overcomes the higher importance of stock evolution until maturity, the elasticity of quantities traded per transaction on the spot market to the STF must be under the threshold given by equation (3.28) (i.e., sufficiently high in absolute value). This result explains the third assertion of theorem 7. A study of this condition is presented in remark 12.

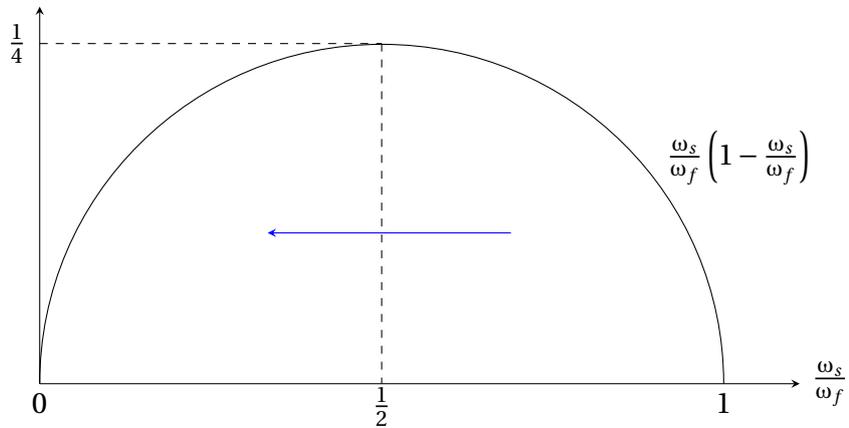
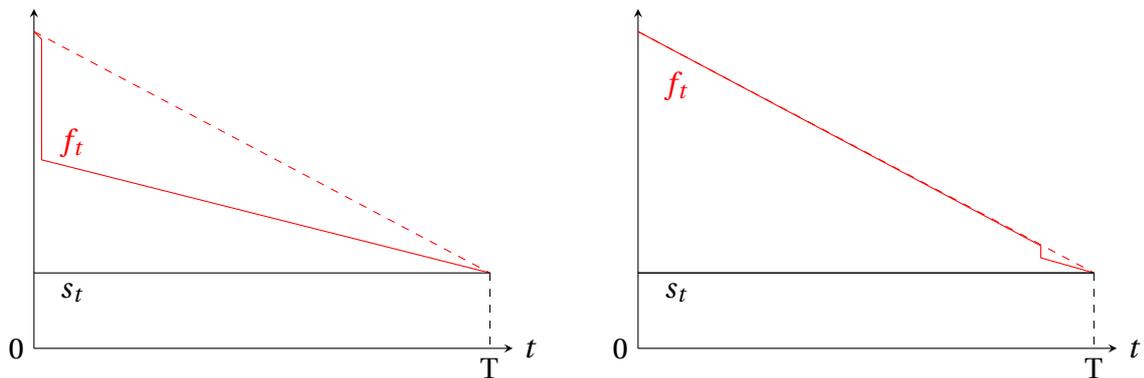
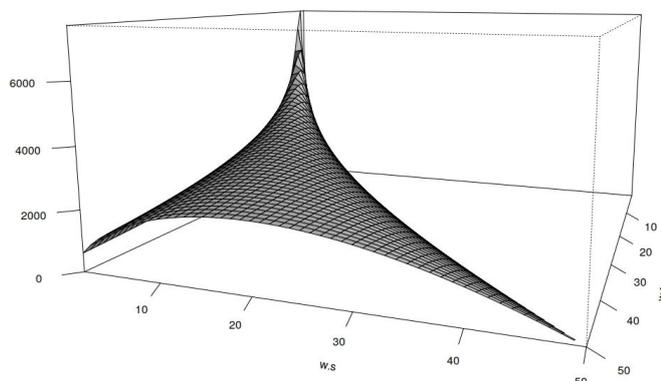


Figure 3.10 – Volatility of the occurrence of synchronized transactions at each transaction on the futures market. Blue arrow gives the evolution of the volatility with an increasing FTF.

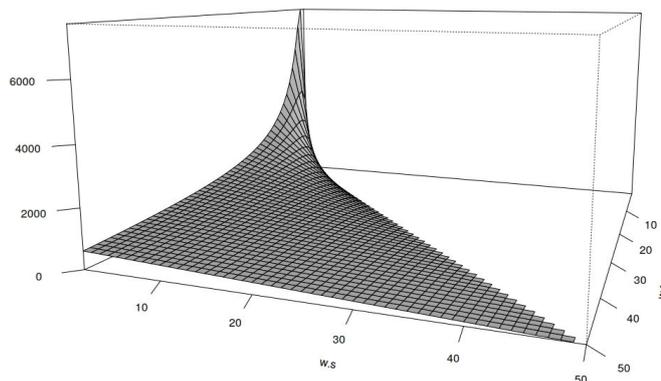


(a) Transaction away from the maturity ($t \rightarrow 0$) (b) Transaction close to the maturity ($t \rightarrow T$)

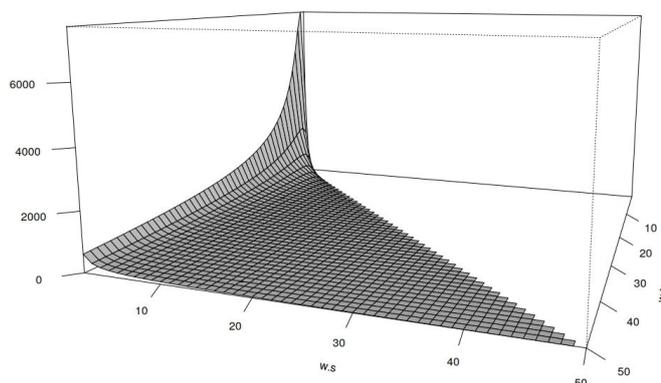
Figure 3.11 – Effect on the fundamental basis of a transaction of $\frac{1}{2}Q_t$. For simplicity's sake, we exclude here the spot market information ($s_k = s_0, \forall k \in [0; T] \Leftrightarrow e = 0$). The red dashed line represents the evolution of the futures price without transactions on the spot market



(a) $t = 25$



(b) $t = 100$



(c) $t = 200$

Figure 3.12 – Representation of the fundamental basis volatility $\text{Var}(FV_{f_t} - FV_{s_t})$ according to the value of t . Parameters used are $a = 0.01$, $\epsilon = 0.2$ and $T_T = 250$.

Remark 12. Let g be a function such that:

$$\begin{aligned} g : \mathbb{F}_f \times [1; (T_r \times \omega_f) - 1] &\rightarrow \mathbb{R} \\ (\omega_f, t) &\mapsto g(\omega_f, t) = -\frac{t}{(T_r \omega_f - t)} \end{aligned} \quad (3.36)$$

It is easy to prove that g is a C1 class function.

For $\frac{\omega_s}{\omega_f} \in]0, \frac{1}{2}[$, $\frac{\partial g(\omega_f, t)}{\partial \omega_f} = \frac{-t(-T_r)}{(T_r \omega_f - t)^2} > 0$. Referring to remark 10, the left term of the condition given by equation (3.27) is decreasing when ω_f increases. The higher ω_f is, the less restrictive the condition given by equation (3.27) is.

For $\frac{\omega_s}{\omega_f} \in]0, \frac{1}{2}[$, $\frac{\partial g(\omega_f, t)}{\partial t} = -\frac{1(T_r \omega_f - t) - t(-1)}{(T_r \omega_f - t)^2} = -\frac{T_r \omega_f}{(T_r \omega_f - t)^2} < 0$. The higher the *n.o.t* on the futures market considered t is, the more restrictive the condition of equation (3.27) is.

Remark 13. Equation (3.28) underlies the importance of the choice in the *n.o.t* considered to study the effect of a proportional increase in the *STF* and the *FTF*.

This threshold firstly depends on the period t . The nearer to maturity time t is, the more restrictive the condition given by equation (3.28) is. The "clock time interval" effect is higher if the initial clock time interval considered is important. Consequently, the longest the initial horizon is, the higher the quantity effect has to be.

Second, this threshold depends on the *FTF*. The "clock time interval" effect relies on the initial level of the *FTF*. Hence, a study of the condition shows that the higher the *FTF* is, the less restrictive the condition given by equation (3.28) is. If the t transactions considered occur with a high *FTF*, only a small clock time interval is originally considered. Consequently, an increase in the *FTF* reduces the clock time interval considered but not significantly, such that the quantity effect does not have to be so high.

Remark 14. The condition given by equation (3.28) is independent from the *RTF*. Since increases in the *STF* and in the *FTF* are proportional such that the *RTF* is unchanged, there is no effect on the volatility of occurrence of synchronized transactions. Then, the only condition to ensure a reducing fundamental basis volatility relies on the elasticity of quantities traded at each transaction on the spot market to the *STF* and the initial levels of both *STF* and *FTF*.

The latest result has strong implications. In subsection 3.5.1, we demonstrated that a proportional increase in the liquidity reduces spot market bias. One could think that a proportional increase in the liquidity should reduce fundamental basis volatility. However, this intuition is unverified if the decrease in quantities traded per transaction on the spot market is insufficient. This strong result shows the extent to which the spot market and its structure influences the futures market and can reduce the benefit of a proportional increase in liquidity on both markets. In other words, thinking that a proportional fragmentation in both spot and futures markets will lead to a decreasing volatility is *partial*. It is true that it decreases quantities traded per transaction and hence smooths the stock dynamics. But according to initial levels of trade frequencies, this effect can be more than compensated by an increase in the risk generated by the possibility of having more transactions long time before the maturity.

In our model, the elasticity of quantities traded per transaction on the spot market to the **STF** is constant. Otherwise, we could have a condition (given by equation (3.28)) depending on the current level of the **STF**. Based on the same reasoning, one could assume an elasticity depending on the **FTF**. Spot and futures markets are linked, and we can imagine that when there are more trades on the futures market, producers and buyers of the commodity adjust their orders and thus quantities traded. A further extension of this work could be to study the evolution of these results according to the form of the q function.

Interpretation of the results for the clock time line

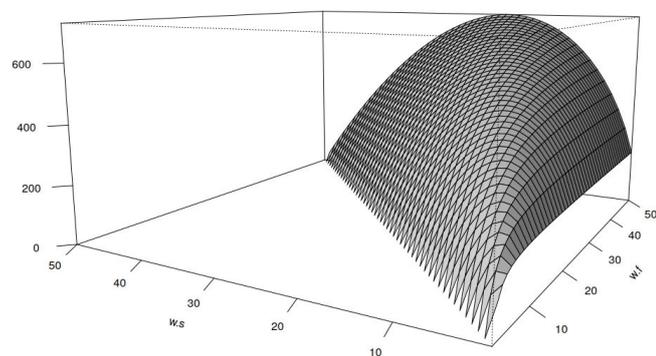
An increase in the STF (i.e., in the liquidity on the spot market) at a fixed clock time considered t_r , reduces the volatility of the fundamental basis by the two same mechanisms than for a fixed **n.o.t** on the futures market t , if the **RTF**, equal to $\frac{\omega_s}{\omega_f}$, is superior to $\frac{1}{2}$. First, the volatility of the occurrence of synchronized transactions decreases. Second, quantities traded per transaction on the spot market are lower, smoothing the stock dynamics.

If the **RTF** is lower than $\frac{1}{2}$, the first mechanism does not hold but the second still holds. The volatility of the occurrence of synchronized transactions increases. The "smoothing per trade traded quantities" effect must overcome the "increasing occurrence of transaction volatility" effect. To smooth the stock dynamics enough, the elasticity must be under a threshold (sufficiently high in absolute value) such that quantities traded per trade sufficiently decrease. These results explain the first assertion of theorem 7. Remark 10 also applies.

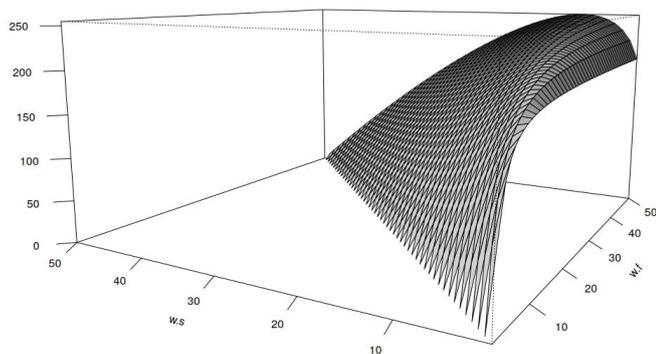
An increase in the FTF (i.e., in the liquidity on the futures market) strictly increases the volatility of the fundamental basis by two mechanisms if the **RTF** is superior to $\frac{1}{2}$. The volatility of the occurrence of synchronized transactions increases; and there are more transactions on the futures market for a given clock time. These two effects increase the fundamental basis volatility.

When the **RTF** is lower than $\frac{1}{2}$, the first mechanism does not hold and conversely, an increasing **FTF** reduces the volatility of occurrence of synchronized transactions. However, this effect is always more than compensated by the second, such that the fundamental basis volatility increases. These result explain the second assertion of theorem 7.

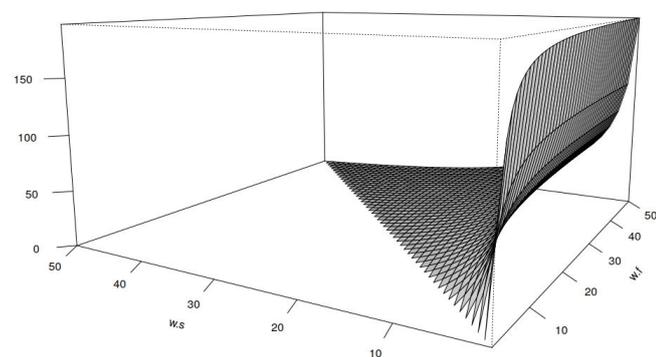
A proportional increase in the STF and the FTF such that the **RTF** is unchanged does not modify the volatility of occurrence of synchronized transaction. However, for a given *clock time*, there are more transactions in the futures' market. This increases the fundamental volatility of the basis. But the increase of the **STF** reduces quantities traded and consumed per transaction on the spot market such that it reduces the stock dynamics volatility, $Var(Q_{t_r})$. To ensure that this effect overcomes the increasing **n.o.t** on the futures market until t_r , the elasticity of quantities traded per transaction on the spot market to the **STF** must be under the threshold given by equation (3.29) (i.e., sufficiently high in absolute value). This result explains the third assertion of theorem 7.



(a) $\epsilon = 0.2$



(b) $\epsilon = 0.4$



(c) $\epsilon = 0.8$

Figure 3.13 – Representation of the fundamental basis volatility $\text{Var}(FV_{f_{tr}} - FV_{s_{tr}})$ according to the value of ϵ . Parameters used are $a = 0.01$, $T_r = 250$, and $t_r = 50$.

3.6 On the existence of an optimal spot market trades frequency and futures market trades frequency

The objectives of a policymaker on commodity markets are to minimize the fundamental basis volatility and increase market efficiency on both spot and futures markets. As the futures price is unbiased (cf. corollary 3), the policymaker focuses on spot market efficiency. We consider here that trade frequencies are the policymaker's tools, and thus the policymaker can implement a fixing pricing and by extension, transactions frequency on the futures market. The question of optimal trade frequency on futures market was treated by Du and Zhu [7], aiming at allocative efficiency. However, they do not consider the specificity of an agricultural commodity market in terms of trade frequency (i.e. on the specificity of the associated spot market). Considering the feedbacks between these two markets at the fundamental level, we find different levels of optimal trade frequency considering our objectives. This section has major issues. At some point, it questions the commodity futures pricing by limit order book; Optimal trade frequencies are lower than the actual futures market trade frequency (FTF) where a limit order book pricing operates.

Definition 23. We define a utility function $U_l, \forall l \in \{t, t_r\}$ in the following way:

$$U_l: \mathbb{F}_s \times \mathbb{F}_f \xrightarrow{U_1} \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{U_2} \mathbb{R}^- \quad (3.37)$$

$$(\omega_s, \omega_f) \longmapsto (B(s_l), V(FV_{f_l} - FV_{s_l})) \longmapsto U_l(B(s_l), V(FV_{f_l} - FV_{s_l}))$$

Hence, we have $U_l(\omega_s, \omega_f) = U_2 \circ U_1(\omega_s, \omega_f)$, verifying the two following properties:

- (i) $U_2(0, 0) = 0$;
- (ii) $\frac{\partial U_2(B(s_l), V(FV_{f_l} - FV_{s_l}))}{\partial B(s_l)} < 0$, and $\frac{\partial U_2(B(s_l), V(FV_{f_l} - FV_{s_l}))}{\partial V(FV_{f_l} - FV_{s_l})} < 0$.

We assume that a policymaker uses the utility function U_l as defined above to maximize the agricultural commodity market efficiency and minimize its fundamental basis volatility. Properties (i) and (ii) correspond respectively to the following natural insights:

- (i) U_2 reaches its maximal value on \mathbb{R}^- if and only if the spot price bias and the fundamental basis volatility are both null;⁸
- (ii) If either the spot price bias or the fundamental basis volatility increases, the policymaker's utility decreases.

Focus on the transactions time line (TTL)

The policymaker consequently has the following maximization program if $l = t$:

$$\begin{cases} \max_{\omega_s, \omega_f} U_t(\omega_s, \omega_f) = U_t \left(\frac{|-e|}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^k, \left(\frac{a}{\omega_f}\right)^2 (T_r \omega_f - t)^2 q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} \right) \\ \text{s.t. } \omega_f \geq \omega_s \end{cases} \quad (3.38)$$

⁸This utility function does not verify the axiom of non satiety. Indeed, if the market efficiency is maximized and the fundamental basis volatility is minimized, the utility cannot be improved. We could have used a more specified function as a mean-reverting one following Markowitz [17], Von Neumann et al. [23] such that if we denote Θ the risk aversion parameter, we have $U_l(B(s_l), V(FV_{f_l} - FV_{s_l})) = -B(s_l) - \Theta V(FV_{f_l} - FV_{s_l})$. Our results are general and also apply with such a function.

Theorem 8. For $t < T$, and $\mathbb{F}_s \cap \mathbb{F}_f \neq \emptyset$, a policymaker whose utility function is U_t has an optimal solution and an optimal solution by limit independently of market characteristics (e , a , and ϵ):

- (i) A policymaker does not have to fix trade frequencies of both spot and futures markets but has to synchronize them. This implies a relative trade frequency (RTF) equal to 1 (the subset of optimal solutions is $O_t = \left\{ (\omega_s, \omega_f) \in (\mathbb{F}_s \cap \mathbb{F}_f)^2 : \frac{\omega_s}{\omega_f} = 1 \right\}$);
- (ii) For any value of $\omega_s \in \mathbb{F}_s$ except $\omega_s \rightarrow +\infty$, a policymaker can allow an infinite liquidity in the futures contract. This implies an RTF converging to 0 (the subset of optimal solutions by limit is $O_{t,bl} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0 \text{ with } \omega_f \mapsto +\infty \right\}$).

Proof. The maximal value of the utility function is 0. According to properties (i) and (ii) of U_t , the maximal value of U_t is reached if and only if $B(s_t) = V(FV_{f_t} - FV_{s_t}) = 0$ (or $B(s_t) \rightarrow 0$ and $V(FV_{f_t} - FV_{s_t}) \rightarrow 0$). From theorems 4 and 5 as well as remarks 6 and 8, the only frequencies allowing $B(s_t) = V(FV_{f_t} - FV_{s_t}) = 0$ independently of the values of e , a , and ϵ are the subset $O_t = \left\{ (\omega_s, \omega_f) \in (\mathbb{F}_s \cap \mathbb{F}_f)^2 : \frac{\omega_s}{\omega_f} = 1 \right\}$. This proves the first assertion.

For $e \neq 0$, $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, we have $B(s_t) > 0$ and $V(FV_{f_t} - FV_{s_t}) > 0$. We look for optimal solutions by limit for the two arguments. For the second argument, it gives:

$$\begin{aligned}
 \left(\frac{a}{\omega_f}\right)^2 (T_r \omega_f - t)^2 q(\omega_s)^2 t \frac{\omega_s}{\omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right) &= 0 \\
 \Leftrightarrow q(\omega_s)^2 \frac{\omega_s}{\omega_f^3} (T_r \omega_f - t)^2 &= 0 \\
 \Leftrightarrow (\omega_s^{-\epsilon})^2 \frac{\omega_s}{\omega_f^3} (T_r \omega_f - t)^2 &= 0 \\
 \Leftrightarrow \frac{\omega_s^{\frac{1}{2}-\epsilon}}{\omega_f^{\frac{3}{2}}} (T_r \omega_f - t) &= 0
 \end{aligned} \tag{3.39}$$

As $\lim_{\substack{\omega_s \rightarrow 0 \\ \omega_f \rightarrow +\infty}} \frac{\omega_s^{\frac{1}{2}-\epsilon}}{\omega_f^{\frac{3}{2}}} (T_r \omega_f - t) = 0$, a subset of optimal solutions by limit exists for the

second argument of U_t . This subset is $O_{t,bl} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0, \omega_f \mapsto +\infty \right\}$, and it is independent of the values of e , a , and ϵ .

Focusing on the first argument gives the following:

$$\begin{aligned}
 \frac{|-e|}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} &= 0 \\
 \Leftrightarrow \frac{1}{\omega_f} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f}\right)^{k-1} &= 0 \\
 \Leftrightarrow \frac{1}{\omega_f} \frac{\left(1 - \frac{\omega_s}{\omega_f}\right) - \left(1 - \frac{\omega_s}{\omega_f}\right)^{t+1}}{1 - \left(1 - \frac{\omega_s}{\omega_f}\right)} &= 0 \\
 \Leftrightarrow \frac{1}{\omega_s} \left(1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t\right) &= 0
 \end{aligned} \tag{3.40}$$

The only subset of optimal solutions by limit for the second argument of U_t independent from the values of e , a , and ϵ is $O_{t,bl}$. We study the limit of the left term of equation (3.40) within $O_{t,bl}$. This gives the following:

$$\begin{aligned} \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0 \\ \omega_f \rightarrow +\infty}} \frac{1}{\omega_s} &= \frac{1}{\omega_s} \\ \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0 \\ \omega_f \rightarrow +\infty}} \frac{\omega_s}{\omega_f} &= 0 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0 \\ \omega_f \rightarrow +\infty}} 1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t = 0 \\ \text{By product} \quad \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0 \\ \omega_f \rightarrow +\infty}} \frac{1}{\omega_s} \left(1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t\right) &= 0 \end{aligned}$$

Hence, $O_{t,bl}$ is also a subset of optimal solutions by limit for the first argument of U_t . It is consequently the subset of all optimal solutions by limit independent from market characteristics (values of e , a , and ϵ). This proves the second assertion of the theorem. \square

This result implies that independently of the absolute value of both spot market trade frequency (STF) and FTF, the synchronization of trade frequencies ensures a minimal fundamental basis volatility and a minimal spot price bias, at a given $n.o.t$ on the futures market t .

Remark 15. *If $\omega_s = \omega_f$, there is neither risk nor uncertainty on the stock dynamics and no unexpected elements on the basis (cf. equation (3.6); $E(\widetilde{s_{t+1}}|\Phi_t) = s_{t+1} \forall t$). With synchronized transactions and rational expectations, the basis is always equal to its fundamental value.*

The links between a spot market and a futures market of a commodity responding to our framework prevent the spot market trade frequency (STF) and the FTF from being disconnected in order to achieve agricultural commodity market efficiency and a minimal fundamental basis volatility. More liquidity on the futures market (i.e., an increase in the FTF) *ceteris paribus* can lead to an increasing spot price bias and an increasing fundamental basis volatility (cf. section 3.5), except if the liquidity is relatively infinite (i.e. except if $(\omega_s, \omega_f) \in O_{t,bl}$). There are no optimal values for the STF and the FTF, but there are co-dependent conditions.

Theorem 9. *For $e \neq 0$, $a \neq 0$, and $t < T$, a policymaker whose utility function is U_t has other optimal solutions by limit relying on the following market characteristics:*

- (i) *For $\epsilon \leq \frac{1}{2}$, there is no conditional optimal solution different from unconditional optimal solutions to the policymaker's utility maximization program given in theorem 8;*
- (ii) *For $\epsilon > \frac{1}{2}$, the policymaker's utility is maximized by limit for all values in the following subset: $O_{t,bl}^{\epsilon > \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 1, \omega_s \mapsto +\infty, \omega_f \mapsto +\infty \right\}$.*

Proof. We use the proof of theorem 8 to identify the subsets of optimal and optimal by limit trade frequencies.

For $e \neq 0$, $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, we have $B(s_t) > 0$ and $V(FV_{f_t} - FV_{s_t}) > 0$.

Equation (3.39) gives the conditions for the second argument to be equal to 0.

If $\frac{1}{2} - \epsilon > 0 \Leftrightarrow -\epsilon > -\frac{1}{2}$, no ω_s exists that satisfies equation (3.39). However, the subset $O_{t,bl}^{\epsilon < \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0^+ \text{ with } \omega_s \mapsto 0^+ \right\}$ gives optimal solutions by limit for the second argument of the utility function U_t . This subset is such that:

$$O_{t,bl}^{\epsilon < \frac{1}{2}} \subset O_{t,bl}$$

Hence, there is no other optimal solution by limit for the second argument in the case of $-\epsilon > -\frac{1}{2}$.

If $\frac{1}{2} - \epsilon = 0 \Leftrightarrow -\epsilon = -\frac{1}{2}$, there is no ω_s satisfying equation (3.39). However, the subset $O_{t,bl}^{\epsilon = \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0, \omega_f \mapsto +\infty \right\}$ gives optimal solutions by limit for the second argument of the utility function U_t . However, this subset is such that:

$$O_{t,bl}^{\epsilon = \frac{1}{2}} = O_{t,bl}$$

Hence, there is no other subset of optimal solution by limit in the case of $-\epsilon = -\frac{1}{2}$. This proves the first assertion.

If $\frac{1}{2} - \epsilon < 0 \Leftrightarrow -\epsilon < -\frac{1}{2}$, there exists no ω_s satisfying equation (3.39). However, the subset $O_{t,bl}^{\epsilon > \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 1, \omega_s \mapsto +\infty, \omega_f \mapsto +\infty \right\}$ gives optimal solutions by limit for the second argument of U_t .

Equation (3.40) gives the condition for the first argument of U_t .

We study the limit of the left term of equation (3.40) within the subset $O_{t,bl}^{\epsilon > \frac{1}{2}}$. We get:

$$\begin{aligned} \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_f \rightarrow +\infty}} \frac{1}{\omega_s} &= 0 \\ \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_f \rightarrow +\infty}} \frac{\omega_s}{\omega_f} = 1 &\Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_f \rightarrow +\infty}} 1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t = 1 \\ \text{By product} \quad \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_f \rightarrow +\infty}} \frac{1}{\omega_s} \left(1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t\right) &= 0 \end{aligned}$$

Hence, within the subset $O_{t,bl}^{\epsilon > \frac{1}{2}}$ equation (3.40) is verified by limit. Hence, $O_{t,bl}^{\epsilon > \frac{1}{2}}$ is a subset of optimal solutions by limit which maximizes U_t . This subset is such that:

$$O_{t,bl}^{\epsilon > \frac{1}{2}} \cap O_{t,bl} = \emptyset$$

This proves the second assertion. □

There is also a set of optimal values by limit for the policymaker that are conditional to market characteristics (value of ϵ). The analytical proof of this result is presented in theorem 9. If $-\epsilon \geq -\frac{1}{2}$, there is no conditional optimal solution different from the unconditional optimal solutions of the policymaker's utility maximization program. However, if $-\epsilon < -\frac{1}{2}$, the optimal solution by limit for the policymaker can be to implement the centralization of orders by brokers and reduce transaction costs as much as possible on the spot market such that $\frac{\omega_s}{\omega_f} \rightarrow 1$, $\omega_s \rightarrow +\infty$ and $\omega_f \rightarrow +\infty$ independently of the FTF (the RTF is converging to 1). However, despite the possibility of having a high elasticity ϵ , implementing a market structure such that $\omega_s \rightarrow +\infty$ is far-fetched since there are structural transaction costs. We can qualify this conditional subset of optimal solutions as a limit case.

Remark 16. *When considering the possibility of speculation on the spot market, the synchronization of trade frequencies directly resulting from theorems 4 and 5 is not straightforward. There is still an ambiguity of having an available stock reduction at each transaction on the spot market. In this case, both $B(s_t)$ and $V(FV_{f_t} - FV_{s_t})$ cannot be simultaneously minimized ($B(s_t)$ is unchanged but $V(FV_{f_t} - FV_{s_t})$ is modified). The arbitrage between market efficiency (with an increasing liquidity due to fragmentation on the spot market) and fundamental volatility appears. In this case, we have to specify the policy maker utility function. One can chose a mean-reverting utility function such as:*

$$\begin{aligned} U_t: \mathbb{F}_s \times \mathbb{F}_f &\xrightarrow{U_1} \mathbb{R}^+ \times \mathbb{R}^+ && \xrightarrow{U_2} \mathbb{R}^- \\ (\omega_s, \omega_f) &\longmapsto (B(s_t), V(FV_{f_t} - FV_{s_t})) && \longmapsto U_t(\cdot, \cdot) = -B(s_t) - \frac{\alpha}{2}V(FV_{f_t} - FV_{s_t}) \end{aligned} \quad (3.41)$$

Where $\frac{\alpha}{2}$ represents the risk aversion of the policy maker. Notice that this mean reverting utility function verifies the properties of definition 23. Within such a case, no optimal trade frequencies exist independently of the market characteristics and we will have to calibrate the model according to the market characteristics as well as the risk aversion of the policy maker before estimating the optimal trade frequencies.

Focus on the clock time line (CTL)

The policymaker consequently has the following maximization program if $l = t_r$:

$$\begin{cases} \max_{\omega_s, \omega_f} U_{t_r}(\omega_s, \omega_f) = U_{t_r} \left(\frac{1-\epsilon l}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k, a^2 (T_r - t_r)^2 q(\omega_s)^2 t_r \left(1 - \frac{\omega_s}{\omega_f}\right) \omega_s \right) \\ \text{s.t. } \omega_f \geq \omega_s, t_r \omega_f \in \mathbb{N}^* \end{cases} \quad (3.42)$$

Theorem 10. *For $t_r < T_r$, $\mathbb{F}_s \cap \mathbb{F}_f \neq \emptyset$, and independently of the market characteristics (ϵ , a , and ϵ), a policymaker whose utility function is U_{t_r} does not have to fix the trade frequencies of both spot and futures markets but has to synchronize them. This implies an RTF equal to 1 (the set of optimal solutions is $O_{t_r} = \left\{ (\omega_s, \omega_f) \in (\mathbb{F}_s \cap \mathbb{F}_f)^2 : \frac{\omega_s}{\omega_f} = 1 \right\}$).*

Proof. The maximal value of the utility function is 0. According to properties (i) and (ii) of U_{t_r} , the maximal value of U_{t_r} is reached if and only if $B(s_{t_r}) = V(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$ (or $B(s_{t_r}) \rightarrow 0$ and $V(FV_{f_{t_r}} - FV_{s_{t_r}}) \rightarrow 0$). From theorems 4 and 5 as well as remarks 6 and 8, the only frequencies allowing $B(s_{t_r}) = V(FV_{f_{t_r}} - FV_{s_{t_r}}) = 0$ are the set $O_{t_r} = \left\{ (\omega_s, \omega_f) \in (\mathbb{F}_s \cap \mathbb{F}_f)^2 : \omega_s = \omega_f \right\}$. This proves the first assertion.

For $e \neq 0$, $a \neq 0$, $t_r < T_r$, and $\omega_s \neq \omega_f$, we have $B(s_{t_r}) > 0$ and $V(FV_{f_{t_r}} - FV_{s_{t_r}}) > 0$. We look for optimal solutions by limit for the two arguments. For the second argument, it gives:

$$\begin{aligned}
 a^2 (T_r - t_r)^2 q (\omega_s)^2 t_r \left(1 - \frac{\omega_s}{\omega_f}\right) \omega_s &= 0 \\
 \Leftrightarrow q (\omega_s)^2 \left(1 - \frac{\omega_s}{\omega_f}\right) \omega_s &= 0 \\
 \Leftrightarrow (\omega_s^{-\epsilon})^2 \omega_s \left(1 - \frac{\omega_s}{\omega_f}\right) &= 0 \\
 \Leftrightarrow \omega_s^{(1-2\epsilon)} \left(1 - \frac{\omega_s}{\omega_f}\right) &= 0
 \end{aligned} \tag{3.43}$$

The only subset ensuring the second term to be equal to 0 is the subset O_{t_r} . The first term value depends on the market characteristic ϵ . Hence, the only optimal solution independent from the market characteristics is the subset O_{t_r} .

The theorem is demonstrated. \square

This result implies that independently of the absolute value of both **STF** and **FTF**, the synchronization of trade frequencies ensures a minimal fundamental basis volatility and a minimal spot price bias at a given clock time t_r . Remark 15 also applies.

This result also corroborates that links between a spot market and a futures market of a commodity responding to our framework prevent the **STF** and the **FTF** from being disconnected in order to achieve market efficiency and a minimal fundamental basis volatility. More liquidity on the futures market (i.e., an increase in **FTF**) *ceteris paribus* can lead to an increasing spot price bias and always increases the fundamental basis volatility (cf. section 3.5). There are no optimal values for the **STF** and the **FTF**, but there are co-dependent conditions.

Theorem 11. For $e \neq 0$, $a \neq 0$, and $t_r < T_r$, a policymaker whose utility function is U_{t_r} has other optimal solutions by limit relying on the following market characteristics:

- (i) For $\epsilon \leq \frac{1}{2}$, there is no conditional optimal solution different from the unconditional optimal solutions to the policymaker's utility maximization program given in theorem 10;
- (ii) For $\epsilon > \frac{1}{2}$, the policymaker's utility is maximized by limit for all values in the following subset: $O_{t_r, bl}^{\epsilon > \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 1, \omega_s \mapsto +\infty, \omega_f \mapsto +\infty \right\}$.

Proof. We use the proof of theorem 10 to identify the other optimal and optimal by limit trade frequencies.

For $e \neq 0$, $a \neq 0$, $t_r < T_r$, and $\omega_s \neq \omega_f$, we have $B(s_{t_r}) > 0$ and $V(FV_{f_{t_r}} - FV_{s_{t_r}}) > 0$.

Equation (3.43) gives the conditions for the second argument of U_{t_r} to be equal to 0. We recall:

$$\omega_s^{(1-2\epsilon)} \left(1 - \frac{\omega_s}{\omega_f}\right) = 0$$

The first term is the one that we are interested in. If $1 - 2\epsilon > 0 \Leftrightarrow -\epsilon > -\frac{1}{2}$, an optimal solution by limit exists and is given by the following subset

$$O_{t_r, bl}^{\epsilon < \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0^+, \omega_s \mapsto 0^+ \right\}.$$

We study the limit of the first argument of U_{t_r} within $O_{t_r, bl}^{\epsilon < \frac{1}{2}}$. We get:

$$\lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0^+ \\ \omega_s \rightarrow 0^+}} \frac{|-e|}{\omega_f} = \frac{|-e|}{\omega_f}$$

$$\lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0^+ \\ \omega_s \rightarrow 0^+}} \left(1 - \frac{\omega_s}{\omega_f}\right) = 1 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0^+ \\ \omega_s \rightarrow 0^+}} \left(1 - \frac{\omega_s}{\omega_f}\right)^k = 1 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0^+ \\ \omega_s \rightarrow 0^+}} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k = t_r \omega_f$$

$$\text{By product} \quad \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 0^+ \\ \omega_s \rightarrow 0^+}} \frac{|-e|}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k = |-e| t_r \neq 0$$

Hence, there is no other optimal solution by limit for the first argument of U_{t_r} under the market characteristic $-\epsilon > -\frac{1}{2}$.

If $1 - 2\epsilon = 0 \Leftrightarrow -\epsilon = -\frac{1}{2}$, there is no other optimal solution by limit satisfying equation (3.43) and thus, for U_{t_r} . This proves the first assertion.

If $1 - 2\epsilon < 0 \Leftrightarrow -\epsilon < -\frac{1}{2}$, the subset $O_{t_r, bl}^{\epsilon > \frac{1}{2}} = \left\{ (\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 1, \omega_s \mapsto +\infty, \omega_f \mapsto +\infty \right\}$ gives optimal solutions by limit for the second argument of U_{t_r} (by satisfying equation (3.43) by limit).

We study the limit of the first argument of U_{t_r} within $O_{t_r, bl}^{\epsilon > \frac{1}{2}}$. We get:

$$\lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_s \rightarrow +\infty \\ \omega_f \rightarrow +\infty}} \frac{|-e|}{\omega_f} = 0$$

$$\lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_s \rightarrow +\infty \\ \omega_f \rightarrow +\infty}} \left(1 - \frac{\omega_s}{\omega_f}\right) = 0 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_s \rightarrow +\infty \\ \omega_f \rightarrow +\infty}} \left(1 - \frac{\omega_s}{\omega_f}\right)^k = 0 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_s \rightarrow +\infty \\ \omega_f \rightarrow +\infty}} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k = 0$$

$$\text{By product} \quad \lim_{\substack{\frac{\omega_s}{\omega_f} \rightarrow 1 \\ \omega_s \rightarrow +\infty \\ \omega_f \rightarrow +\infty}} \frac{|-e|}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k = 0$$

$O_{t_r, bl}^{\epsilon > \frac{1}{2}}$ is a subset of optimal solutions by limit for the first argument of U_{t_r} and thus, for U_{t_r} . This set is such that:

$$O_{t_r, bl}^{\epsilon > \frac{1}{2}} \cap O_{t_r} = \emptyset$$

This proves the second assertion. □

There is also a subset of optimal values by limit for the policymaker that are conditional to market characteristics (value of ϵ). The analytical proof of this result is presented in theorem 11. If $\epsilon \leq \frac{1}{2}$, there is no conditional optimal solution to the policymaker's utility maximization program. However, if $\epsilon > \frac{1}{2}$, the optimal solution by limit for the policymaker can be to implement the centralization of orders by brokers and reduce transaction costs as much as possible on the spot market such that $\frac{\omega_s}{\omega_f} \rightarrow 1$, $\omega_s \rightarrow +\infty$, and $\omega_f \rightarrow +\infty$ (the RTF is converging to 1). Again, despite the possibility of having a high elasticity ϵ , implementing a market structure such that $\omega_s \rightarrow +\infty$ is far-fetched since there are structural transaction costs. We can also qualify this conditional subset of optimal solutions as a limit case.

Finally, remark 16 also applies, such that considering speculation on the spot market prevents us from having analytical values on optimal trade frequencies. A calibration of the model must be done in the following case, and simulations can be made if some parameters are assumed to be stochastic such as quantities traded per transaction on the spot market.

Remark 17. *Note that:*

- (i) $O_t = O_{t_r}$; *The subset of optimal STF and FTF independent of market characteristics e , a and ϵ is independent from the time line considered;*
- (ii) $O_{t,bl}^{\epsilon > \frac{1}{2}} = O_{t_r,bl}^{\epsilon > \frac{1}{2}}$; *The subset of optimal STF and FTF by limit depending on the market characteristics e , a and ϵ is independent from the time line considered.*

The only subset of optimal or optimal by limit trade frequencies conditional to the time line is $O_{t,bl}$. The "clock time interval" effect only appears when considering a fixed n.o.t (i.e. considering the TTL) which is compensated when we consider a fixed clock time (i.e. considering the CTL). The other subsets of optimal or optimal by limit trade frequency are independent from the time line considered.

3.7 Conclusion

We have built a simple dynamics pricing model applied to an agricultural commodity market (spot and futures); more generally, that model is an analytical tool that can be applied to any storable commodity for which the frequency of production is lesser than the frequency of the consumption. As in Du and Zhu [7] we have studied the existence and the level of optimal trading frequencies on an agricultural commodity market, and the impact of the degree of liquidity on the market efficiency and on the fundamental basis volatility. In the model, information that determines the price of the futures is endogenously determined by trades occurring on the spot market and, conversely, the information that determines the price of the spot market is endogenously determined by trades occurring on the futures market. This framework is borrowed to the storage theory and models the links between the two markets. By feedback, the futures market trade frequency (FTF) influences the probability law of incoming spot market information that impacts the spot market. By extension the futures market will also be impacted by the no arbitrage condition. Hence, we consider the specificity of the underlying of a futures contract on an agricultural commodity.

This work has major implications. It shows that commodity futures pricing by limit order book is not an optimal policy regarding the objectives of maximizing market efficiency and of minimizing the fundamental prices' volatility. Previous empirical studies argued

that more liquidity allows for better market efficiency; their findings do not necessarily contradict our results, depending on the initial relative trade frequency (RTF) (cf. O'Hara and Ye [20], Yam and Zhou [28]). Our results recommend the implementation of a fixing pricing operating at the same frequency as trades on the underlying spot market, independently of the time line considered.⁹ From this result, we extrapolate a trade-off between liquidity on the futures market and achievement of the two objectives; we showed in section 3.5 that a proportional increase in both frequencies could lead to an increasing fundamental basis volatility according to the value of ϵ . A way to improve both liquidity and market efficiency of the spot market could be a regulation of the spot market structure to facilitate transactions on this market, allowing an increase in the absolute value of ϵ . To conclude, the simultaneous determination of spot and futures prices is optimal and independent from the market characteristics.

We are aware that this model presents some limits. Some extensions of this work can be made depending on the form of the q function as well as in the implementation of several commodity markets (or other markets in general) where information is released at different frequencies. Finally, we could also integrate the possibility of speculation on the spot market, such that trades on the spot market would not necessarily imply a commodity consumption.

⁹Otherwise, the liquidity on the futures market must be relatively infinite to ensure the maximization of the policy maker's utility when considering the transaction time line (TTL).

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Chapter 4

Basis volatility and efficiency on agricultural commodity markets: The impact of the number of agents

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4.1 Introduction

The spot and the futures markets of any agricultural commodity are connected by definition and there are complex feedbacks between the series of prices on the two markets. The evolution of the basis is a source of risk, and its volatility is a major stake for farmers and processors: it represents an important component of their production and selling decisions (cf. Moschini and Hennessy [15]). However, the price on the futures market evolves faster than the price on the spot market (cf. O'Hara [17], Working [23]) and thus, the basis has a peculiar dynamics. A rich economic literature has empirically proved that the basis is significantly determined by the convenience yield (cf. Fama and French [3], Gorton et al. [6], Wei and Zhu [22]). In a static reasoning and within the assumption of the simultaneity of determination of both spot and futures price, the work of Stein [18], demonstrates the existing link between spot and futures price through the level of the available stock. We already showed in the previous chapter 3 how the dynamics of the basis depends on the difference in trade rhythms. The higher available stock is, the lower is the basis. The idea is as simple as it can be: asynchronous trades on both spot and futures markets generate a risk (according to the definition of Knight [10]). This risk can be compensated or not according to the objectives of a policy maker. In this view, the fragmentation of orders which is responsible for the increase in trades frequencies, plays an important role in the evolution of trading dynamics and by extension in the evolution of the ratio between these two frequencies. This ratio also relies on the number of agents (*n.o.a*) on a market, except if an increase in the *n.o.a* is followed by a proportional increase in these trade frequencies.

Fragmentation of orders, defined as the division of one specific order – called a parent order – into several suborders – called children orders –, occurs on both markets. It entails a decrease in quantities exchanged per transaction and an increase in the *n.o.t* on a market (cf. O'Hara and Ye [16], O'Hara [17], Yam and Zhou [24]). Consequently, in order to buy the same quantity, agents pass more orders and trade frequency increases. An entrance of more agents on the market leads then, to an increase in these trade frequencies and do not directly modify quantities traded at each transactions. Although such fragmentation of orders is used on both spot and financial markets, transaction costs in these two markets are not the same (cf. Working [23]). Order fragmentation can be neither identical nor proportional in the two markets. It can impact both relative trades frequency (*RTF*) and quantities traded per transaction in these markets. An entrance of agents (traders, hedgers, speculators, transformers) on a commodity market modifies both trade's frequencies but not necessarily proportionally such that it also modifies the *RTF*. Glebkin [5] showed to what extent more large traders and then more liquidity, can reduce the profit of every agent (small ones, and large ones). In an empirical study, Irwin and Sanders [8] argued that the presence of index funds, which entered commodity markets massively in 2006, did not fuel the 2008 commodity price bubble. However, the presence of more agents on the market also impact fundamentally the commodity prices (spot and futures). As stated by J. Stiglitz:

"(...) market actions or choices convey information. Market participants know this and respond accordingly" – Stiglitz [19] (p. 468)

The latest assumption has strong implications. In the model that we build, we introduce an effect of the *n.o.a* on the volatility of the spot market information itself – and then on its price impact – and by extension on both the fundamental spot price and futures price volatility (see below). To the best of our knowledge, there is no study that simultaneously

links the [n.o.a](#) to the fundamentals volatility of both spot and futures prices, and the market efficiency.

In the end, we also consider that agents can form biased expectations on the spot market information. Early works such as the one of Thaler [20] emphasize the role of biased traders. Eventually, we assume that this bias is corrected by the [n.o.a](#) acting on the market, i.e. the size of the market. We defend this assumption by referring to the reasoning of Hayek [7]. The more agents they are, the more the capability of the market to integrate information by agents economic interactions increases.

This study aims to examine to what extent the [n.o.a](#) acting on a commodity market impacts the spot market efficiency and the futures market efficiency as well as to examine its impact on the fundamental volatility of the spot price and on the fundamental volatility of the futures price. Thus, that study deals with the *market structure and design issues* on market microstructure (cf. Madhavan [11]). Our study of the trade frequencies depends directly on the mechanism that underlies the formation of price. Within a limit order book, trade frequencies are not constrained; on the opposite, in a periodic trading, trade frequencies are by definition fixed. Finally, we consider whether there is an optimal [n.o.a](#) that satisfies the policy maker's objectives regarding these elements. If not, free entrance on the commodity market (including futures) should be implemented. Otherwise, the policy maker has to compare the current [n.o.a](#) with the optimal one to decide. We finally introduce the possibility of having biased expectations from agents that are corrected with an increasing [n.o.a](#). We study how this incentive to let agents enter the market influences our results.

To some extent, our study is close to the work of Glebkin [5] who theoretically examines the impact of large traders assuming that agents receive exogenous private information, on the profit of all traders. Within that slightly alternative framework, we first demonstrate that an infinite [n.o.a](#) does not necessarily ensure an unbiased market even under the assumption of rational expectations. We also show that more agents can fundamentally increase the fundamental basis volatility. When considering biased expectations, we find that underestimating bias and overestimating bias have asymmetric effects and are absolutely not equivalent. Within an overestimating bias case, the difference in trade frequencies can perfectly compensate the overestimation bias such that the spot price can be unbiased. It is not the case within an underestimating bias case. The complexity of the problem prevents us from having tractable analytical solutions. As all mechanisms at stake are explained in the following, we proceed with sensitivity tests to study the sensitivity of the optimal [n.o.a](#) to the value of our parameters. We also compare this sensitivity within the absence or not of the expectation bias.

This work is organized as follows. Section 4.2 presents all parameters and variables whereas section 4.3 presents the definitions and hypothesis of the model. Section 4.4 derives the influence of the [n.o.a](#) and section 4.5 discusses in detail the impacts of an entrance of agents on the commodity market. In section 4.6, we study the existence of an optimal [n.o.a](#). In section 4.7, we introduce the possibility of having biased expectations and we study how the latest modifies our previous results. Section 4.8 concludes and discusses the limitations of the model.

4.2 The Parameters and variables of the model

Despite the simplicity of the model, we need numerous parameters and variables. In this section, we introduce all parameters (subsection 4.2.1) and variables (subsection 4.2.2).

4.2.1 The parameters of the model

The following set of parameters is used for the futures and the spot agricultural commodity markets:

Futures market	t_r	ω_f	e	ξ				
Spot market	t_r	ω_s	e	ξ				
Futures market	a	ϵ	Q_{op}	T_r	n		σ_I	Θ
Spot market		ϵ			n	α	σ_I	Θ

Table 4.1 – Set of parameters used for the futures and spot markets

- t_r Unit of the clock time line (CTL) considered, such as a second.
- ω_f Minimal trade frequency on the futures market per unit of clock time (i.e. with the minimal number of agents (n.o.a) trading normalized to 1): $\left(\frac{\text{Number of transaction on the futures market}}{t_r} \right)$
- ω_s Minimal trade frequency on the spot market per unit of clock time (i.e. with the minimal n.o.a trading): $\left(\frac{\text{Number of transaction on the spot market}}{t_r} \right)$
- e Trend of the spot market information delivered in monetary value per unit of clock time.
- ξ Bias of agents in the process and the expectation of the spot market information.
- a Advantages of possessing a unit of the storable agricultural commodity in monetary value, per unit of clock time.
- ϵ Absolute value of the elasticity of quantities traded per exchange on the spot market to the spot market trade frequency.
- Q_{op} Optimal stock level of the agricultural commodity for agents.
- T_r Maturity of the futures contract, expressed in the CTL.
- n n.o.a on the market, $n \in \mathbb{N}^*$.
- α Elasticity of the spot market trade frequency to the n.o.a on the market.
- σ_I Maximal intrinsic volatility of the spot market information, per unit of clock time.
- Θ Risk aversion of the policy maker to the commodity fundamental value volatility.

4.2.2 The variables of the model

The following set of variables is used for the futures and the spot agricultural commodity markets:

Futures market	t		T	$I_{]t_1, t_2]}$			
Spot market	t	t'		$I_{]t_1, t_2]}$			
Futures market	γ_n		Q_t	CY_t	\widetilde{f}_{t+1}	f_t	FV_{f_t}
Spot market	γ_n	q	Q_t		\widetilde{s}_{t+1}	s_t	FV_{s_t}

Table 4.2 – Set of variables used for the futures and spot markets

t	Unit of the transactions time line (TTL), which represents a transaction on the futures market.
t'	Latest synchronized transaction on the spot and futures markets until t , expressed in the TTL.
T	Maturity expressed in the TTL.
$I_{]t_1, t_2]}$	Spot market information delivered in monetary value between t_1 and t_2 , t_1 excluded.
Υ_n	Capability of agents to process and expect the spot market information.
q	Quantities traded on the spot market at each transaction on the spot market.
Q_t	Available stock of the commodity at time t .
CY_t	Convenience yield (advantage in detaining one unit of stock to face risk or uncertainty) in monetary value at time t until maturity.
$\widetilde{s}_{t+1}, \widetilde{f}_{t+1}, \widetilde{Q}_t$	Agents' expectations of the spot price, the futures price, and the commodity available stock, respectively, at time t .
s_t, f_t	Observed spot price and futures price, respectively, at time t .
FV_{s_t}, FV_{f_t}	Fundamental values of the spot price and futures price, respectively, at time t .

4.3 The definitions and hypotheses of the model

This section introduces all the definitions and hypotheses of the model. We also justify all assumptions made. A particular attention is given to the time lines and the trade frequencies as they are the very essence of the model (subsection 4.3.1). Subsection 4.3.2 specifies the characteristics of the spot market information. In subsection 4.3.3, we define how quantities traded are determined and we justify their influence on the agricultural commodity market through the convenience yield. Subsection 4.3.4 defines the way agents form their expectations of prices dynamics and subsection 4.3.5 defines the effective prices and fundamental values dynamics. Finally, we define an efficient market criteria but also what we call the fundamental volatility and justify these formulations in subsection 4.3.6.

4.3.1 Time lines and trade frequencies

We define $\mathbb{F}_s \subset \mathbb{Q}^+ \setminus \{0\}$ and $\mathbb{F}_f \subset \mathbb{Q}^+ \setminus \{0\}$ as the sets of all possible frequencies on the spot market and on the futures market. We have $\{0\} \notin (\mathbb{F}_s \cup \mathbb{F}_f)$ which means that the two markets exist. Therefore, $(\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f$. Furthermore, we assume that $\omega_s \leq \omega_f$. Futures are more frequently traded than the commodity itself since transaction costs are structurally lower (cf. Working [23]). Both ω_s and ω_f depend on the technological advances allowing a reduction of these structural transaction costs. This explains why we choose the transaction on the futures as the time step (the smallest one). Then, we formally define the time line t and its maturity T according to the clock time line, denoted by t_r , and the frequency of trades on the futures market such that:

$$\begin{aligned} t &:= t_r \times \omega_f \times n \quad , \quad \text{such that } t \in \mathbb{N}^* \\ T &:= T_r \times \omega_f \times n \quad , \quad \text{such that } T \in \mathbb{N}^* \end{aligned} \quad (4.1)$$

We investigate the impact of the n.o.a trading on a spot and futures agricultural commodity market. We suppose the set \mathbb{N}^* to be the set of all possible number of agent, denoted

by n according to its unit. Then $n \in \mathbb{N}^*$. Again, if $n = 0$, there is no market. We normalize the **n.o.a** such that $n = 1$ represents the minimal **n.o.a** allowing for the market existence. We assume that both historical frequency of trades on the spot market, and on the futures market, are increasing functions of the **n.o.a**. However, transaction costs are higher on the spot market than on the futures market. Hence, we normalize to 1 the elasticity of the futures market trades frequency (**FTF**) to the **n.o.a** on the market. It leads us to assume that the increase in the spot market trades frequency (**STF**) is slower than the increase in the **FTF** using the parameter $\alpha \in]0, 1]$, to denote the elasticity of the **STF** to the **n.o.a** on the market. Hence, the **STF** is equal to $\omega_s \times n^\alpha$, $\alpha \in]0, 1]$, and the **FTF** is equal to $\omega_f \times n$.¹ We assume that transactions on the futures markets are equally spaced out, according to the **FTF**. The time between two transactions on the futures market is constant. The effective **FTF** on the period is known, and the effective **STF**, is unknown. The historical **STF** only gives the probability of having a transaction on the spot market per unit of time. At each transaction on the futures market, there is an independent probability $\frac{\omega_s \times n^\alpha}{\omega_f \times n} = \frac{\omega_s}{\omega_f} \times n^{\alpha-1}$ of having a transaction on the spot market. This probability also represents the historical relative trade frequency (**RTF**).² For the sake of simplicity, we do not integrate "catch up effect", that is, no one passes an order on the spot market because there is no transaction since "a long time" or because the actual delayed spot price is "far" from its actual value (i.e. since the last transaction on the spot market, the released spot market information is very important). The introduction of such a boundary could have some effect depending on the criteria (detection of a possible arbitrage, allocative efficiency operation according to the released information etc.) that we do not develop in this work. It is a possible extension of the model.

Remark 18. If $\alpha = 1$, $\frac{\omega_s \times n^\alpha}{\omega_f \times n} = \frac{\omega_s}{\omega_f}$. Both **STF** and **FTF** evolve proportionally and so the **RTF** is independent from the **n.o.a**.

Since transactions on the spot market occur with a certain probability, t' is a discrete random variable (**D.R.V**) which can take values in $[0; t] \subset \mathbb{N}$. Hence, we have:

$$\begin{cases} \Pr(t' = k) &= \frac{\omega_s}{\omega_f n^{1-\alpha}} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^{t-k} \quad \forall k \in [1; t] \\ \Pr(t' = 0) &= \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^t \end{cases}$$

Proceeding identically as in the previous chapter 3, we have:

$$E(t') = t - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \quad (4.2)$$

Remark 19. The **D.R.V** t' can be decomposed into two parts. First, t is the maximal value of t' (since $t' \leq t$). The second part is stochastic and can be assimilated to a truncated geometric law with t experiences – maximum – starting from t in backwardation and ending to 0. Hence, $E(t')$ is equal to its maximal value minus the average delay for having a synchronized transaction looking backward in number of transactions (**n.o.t**) on the futures market.

¹We have $\frac{\partial(\omega_s \times n^\alpha)}{\partial n} / \frac{(\omega_s \times n^\alpha)}{n} = \alpha$ and $\frac{\partial(\omega_f \times n)}{\partial n} / \frac{(\omega_f \times n)}{n} = 1$

²Let \leq define a relation where t' represents the latest period where there was a synchronized transaction on the spot market and the futures market until t such that:

$t' \leq t$ if $t' \leq t$ and there is no element $y' \neq t'$ nor $y' \neq t$ such that $t' \leq y' \leq t$. Thus, $(t-1)'$ is the latest period where there was a synchronized transaction on the spot market and the futures market until $t-1$, and $(t-1)' \leq t-1$. However, we do not have $(t-1)' \leq t'-1$. Indeed, t' and $(t-1)'$ can be both equal to 0 such that $0 \leq -1$ which is impossible.

4.3.2 The spot market information

We assume that information on the spot market is intrinsically released at each unit of the real time t_r . However, we naturally consider that information about the spot price is released at each transaction on the futures market. Then, we assume that the monetary impact – or the price impact – of incoming spot market information delivered at time t , denoted $I_{|t-1,t|}$, follows any probabilistic law \mathcal{L}_1 whose mean is $E(I_{|t-1,t|}) := E(I_{|0,t|}) - E(I_{|0,t-1|}) = \frac{e}{\omega_f \times n} t - \frac{e}{\omega_f \times n} (t-1) = \frac{e}{\omega_f n}$. This mean is a strictly decreasing function of the FTF. As spot market information arrives between each transaction on the futures market, we can naturally assume that if transactions are more frequent on the futures market, the spot market information released between two transactions on the futures market has, on average, a lower price impact. This information can come from meteorological data, harvest prevision reports, etc. For the sake of simplicity, we consider that there is no minimal trade frequency on the futures market since we are reasoning at the scale of second or minutes despite the frequency of arrival of this type of information is daily (meteorological data for instance) or even monthly (USDA reports for instance). The entrance of more agents on the market increases the frequency of the trades on the futures markets and then diminishes the average price impact of the spot market information released between two transactions such that:

$$\lim_{n \rightarrow +\infty} E(I_{|t-1,t|}) = \lim_{n \rightarrow +\infty} \frac{e}{\omega_f n} = 0$$

The standard error of this law is $\sqrt{\text{Var}(I_{|t-1,t|})} = \frac{\sigma_I}{\sqrt{\omega_f \times n}} \times \frac{\sqrt{n}}{\sqrt{1+n}} = \frac{\sigma_I}{\sqrt{\omega_f}} \times \frac{1}{\sqrt{1+n}}$, $\sigma_I \in \mathbb{R}^+ \setminus \{0\}$. Indeed, we assume that the higher the n.o.a on the spot and futures market is, the higher the volatility of the information released is. The idea is simple. The states of the market at a given later date, are more volatile when there are 10 000 agents acting on the market than when there are 2 agents. Then, the presence of more agents on the market also impact fundamentally the commodity prices (spot and futures) by conveying a more volatile information. It is consistent with the idea that market actions or choices convey information borrowed to the economics of information (cf. Stiglitz [19]). In other words, new agents do not convey a fully redundant information but at least partly a new one. It is equivalent to consider agents to be heterogeneous and thus, conveying more information. However, we can consider that this effect admit a limit such that the marginal information conveyed by the last agent is decreasing with the n.o.a.³ There is also a frequency impact smoothing the released information. This effect compensates the first one such that when the n.o.a is infinite we face:

$$\lim_{n \rightarrow +\infty} \sqrt{\text{Var}(I_{|t-1,t|})} = \lim_{n \rightarrow +\infty} \frac{\sigma_I}{\sqrt{\omega_f}} \times \frac{1}{\sqrt{1+n}} = \frac{\sigma_I}{\sqrt{\omega_f}} \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{1+n}} = 0$$

Then, the monetary impact of incoming spot market information delivered at time t is characterized by:

$$I_{|t-1,t|} \sim \mathcal{L}_1\left(\frac{e}{\omega_f \times n}, \frac{\sigma_I}{\sqrt{\omega_f}} \times \frac{1}{\sqrt{1+n}}\right) \quad (4.3)$$

So, $I_{|t',t|}$ represents the spot market information delivered in monetary value since the last transaction on the spot market (t'). If there is an infinite n.o.a, we have $I_{|t-1,t|} \sim \mathcal{L}_1(0,0) \Leftrightarrow I_{|t-1,t|} \approx 0$. This result holds because the influence of agents on the FTF is infinite and overcome their limited influence on spot market information volatility. Notice

³This assumption can be considered as a Malthusian assumption if one is risk averse, but not if one is risk neutral or risk lover.

that considering the clock time line (CTL), agents have no impact on the average spot market information delivered between two units of clock time such that:

$$E(I_{|t_r-1, t_r|}) := e \quad (4.4)$$

However, the influence of the n.o.a on the volatility of spot market information when considering the CTL is:⁴

$$\sqrt{\text{Var}(I_{|t_r-1, t_r|})} = \sigma_1 \times \sqrt{\frac{n}{1+n}} \quad (4.5)$$

Notice that $\lim_{n \rightarrow +\infty} \text{Var}(I_{|t_r-1, t_r|}) = \sigma_1^2$. Then, σ_1^2 represents the maximal fundamental volatility of the spot market within a unit of clock time. Indeed, the infinite effect of agents on the FTF is perfectly compensated by the fact that an infinite n.o.a leads to an infinite n.o.t within a unit of the CTL. Considering the CTL then, agents only affect the intrinsic volatility of the spot market information.

4.3.3 Traded quantities and influence of stock dynamics

We now define q , which indicates the quantities traded per transaction on the spot market as a function such that:

$$\begin{aligned} q: \mathbb{F}_s \times \mathbb{N}^* &\rightarrow \mathbb{R}^{+,*} \\ (\omega_s, n) &\mapsto q(\omega_s, n) = (\omega_s \times n^\alpha)^{-\epsilon}, \epsilon \in \mathbb{Q}^+ \end{aligned} \quad (4.6)$$

From the above function, quantities traded per transaction on the spot market are decreasing when the STF increases. Note that parameter ϵ corresponds to the absolute value of the elasticity of the quantities traded per transactions on the spot market to the STF. Focusing on the -indirect- influence of the n.o.a, we derive from equation (4.6) that the elasticity of quantities traded to the n.o.a on the market is also negative and equal to $\frac{\partial q(\omega_s, n)}{\partial n} / \frac{q(\omega_s, n)}{n} = -\frac{\epsilon \alpha}{\omega_s}$. This elasticity is constant. The higher ω_s is originally, the lower this elasticity is in absolute value. In other words, if quantities traded are originally low (because the STF is high), then the decreasing effect of more agents on quantities is low. Proceeding identically, the higher α and ϵ are, the higher the impact of an increasing n.o.a on the market is in absolute value.

The available stock of the commodity at time t is defined using a discrete random variable (D.R.V.). The effect of an entrance of more agents is not characterized by a direct evolution of quantities traded per transactions but only more frequent transactions.

$$Q_0 > 0 \text{ and } Q_t := \begin{cases} Q_{t-1} & \text{if } t' \neq t \\ Q_{t-1} - q(\omega_s, n) & \text{if } t' = t \end{cases} = Q_{t-1} - q(\omega_s, n) 1_{\{t\}}(t') \quad \forall t \in [1, T] \quad (4.7)$$

We assume that Q_0 is such that for any $(\omega_s, n) \in \mathbb{F}_s \times \mathbb{N}^*$ and any $t \in [0; T-1]$, $Q_t - q(\omega_s, n) \geq q(\omega_s, n)$. It is a constraint of non-negativity of the available commodity stock. We naturally assume that the available commodity is only purchased on the spot market during

⁴The calculus is the following:

$$\begin{aligned} \text{Var}(I_{|t_r-1, t_r|}) &= \text{Var}\left(I_{\left\lfloor \frac{t}{\omega_f n} - \omega_f n, \frac{t}{\omega_f n} \right\rfloor}\right) = \text{Var}\left(\sum_{k=\frac{t}{\omega_f n} - \omega_f n + 1}^{\frac{t}{\omega_f n}} I_{|k-1, k|}\right) \\ &= \sum_{k=\frac{t}{\omega_f n} - \omega_f n + 1}^{\frac{t}{\omega_f n}} \text{Var}(I_{|k-1, k|}) = \sum_{k=\frac{t}{\omega_f n} - \omega_f n + 1}^{\frac{t}{\omega_f n}} \frac{\sigma_1^2}{\omega_f} \times \frac{1}{1+n} \\ \text{Var}(I_{|t_r-1, t_r|}) &= \left[\frac{t}{\omega_f n} - \left(\frac{t}{\omega_f n} - \omega_f n + 1 \right) + 1 \right] \frac{\sigma_1^2}{\omega_f} \times \frac{1}{1+n} \end{aligned}$$

the period considered (at maturity, commodity is purchased and consumed according to the open positions on the futures market which are not cash-settled). We assume that production or harvest of the commodity is less frequent than the consumption.⁵ It has a symmetric – because the sign is opposite – effect to the stock consumption on expectations and fundamental value. For the sake of simplicity, we do not model them. Hence, the model applies to all storable agricultural commodity markets. Consequently, the available stock evolves when and only when there is a transaction on the spot market. It is important to underline that this assumption requires us to not consider that commodity is purchased only to be sold later. For the sake of simplicity, there is no speculation on the spot market. Once again, we could integrate a speculation Bernoulli D.R.V.⁶

We assume that storage costs per unit of clock time are linear with time. Consequently, they are perfectly expected by agents under our assumption of rational expectations. For the sake of simplicity, we do not integrate them in the model.

Despite positive stocks, the futures' price can be lower than the spot price because of the necessity for processors to maintain their stocks to face uncertainty (cf. Kaldor [9]). This advantage (or disadvantage) to detain a unit in stock at transaction t in monetary value until maturity, also called convenience yield and denoted by CY_t , is given by the following equation:

$$CY_t := \frac{a}{\omega_f n} (T - t) (Q_{op} - Q_t) \quad (4.8)$$

We assume that Q_{op} is exogenous to the model. This is the same for all agents. When the stock is above this threshold, there is no advantage to detain more units, and the convenience yield is negative. When the stock is under this level, there is an advantage to detain units of stock.

4.3.4 Agents' expectations

We assume that all information is freely available and that, at time t , all agents know all prices and market characteristics. Therefore, $\Phi_t = \{s_t, f_t, Q_t, I_{|t',t|}, CY_t\}$ is common knowledge at time t . Furthermore, we assume that agents form unbiased expectations in the sense that they are able to perfectly process the incoming information on average. This assumption is later relaxed on the paper. Then, we create a variable γ_n that denotes the capability to correctly interpret the monetary impact of the information. It follows an unknown probabilistic law \mathcal{L}_γ with $E(\gamma_n) = 1$. A perfect capability to interpret information implies $\gamma_n = 1$.

Agents expect the next spot price at time t , denoted by \widetilde{s}_{t+1} according to their information Φ_t such that:

$$E(\widetilde{s}_{t+1} | \Phi_t) = s_{t'} + \gamma_n I_{|t',t|} + \gamma_n \frac{e}{\omega_f n} \quad (4.9)$$

Recall that t' represents the latest period when there was a synchronized transaction on the spot market and futures market until t . Agents add to the current spot price (at pe-

⁵For most of agricultural commodities, there are a few harvests per year.

⁶We can include a D.R.V denoted by C such that the probability of having a consumption of the commodity is given by:

$$\begin{cases} Pr(C = 1) & = \gamma \\ Pr(C = 0) & = 1 - \gamma \end{cases}$$

The following would be more accurate but does not alter the main outcomes on the impact of the n.o.a on the market. However, the integration of such a variable modifies the optimal n.o.a. We explain how, in the dedicated section 4.6.

riod t) all spot market information that should have been included in the spot price denoted by $I_{[t',t]}$, and the expected incoming spot market information for its average value is $E(I_{[t,t+1]}) = \frac{e}{\omega_f n}$ according to their capability to process it given by the value of γ_n .

Expectations of the stock dynamics are given by:

$$E(\widetilde{Q}_{t+1}|\Phi_t) = Q_t - q(\omega_s, n) \frac{\omega_s}{\omega_f} n^{\alpha-1} \quad (4.10)$$

The expected stock evolution is given by the quantities exchanged at each transaction on the spot market (that we suppose to be consumed) weighted by the probability of having a transaction on the spot market at the next transaction on the futures market. Agents expect the next convenience yield at time t , denoted by \widetilde{CY}_{t+1} according to their information Φ_t such that:

$$E(\widetilde{CY}_{t+1}|\Phi_t) = \frac{a}{\omega_f n} (T - (t + 1)) (Q_{op} - E(\widetilde{Q}_{t+1}|\Phi_t)) \quad (4.11)$$

To prevent an arbitrage operation, the basis must include the current level of stock since the lower the stocks are, the more agents have an interest in detaining it depending on the level of their optimal stock. Between each transaction on the futures market, the advantage in detaining stocks decreases by $\frac{a}{\omega_f n}$ for the same level of stock than at t . However, there is a probability $\frac{\omega_s}{\omega_f} n^{\alpha-1}$ of having a stock movement, and the average stock evolution is not null. Thus, expected stock movement affects the expected advantage in detaining stocks until maturity.

Finally, expectations of the futures price for the next period are:

$$E(\widetilde{f}_{t+1}|\Phi_t) := E(\widetilde{s}_{t+1} - \widetilde{CY}_{t+1}|\Phi_t) \quad (4.12)$$

Recall that we do neither integrate – linear – storage costs nor harvest, this equation translates the fact that agents know what the no arbitrage conditions are.

4.3.5 Price dynamics and fundamental values

Prices evolve according to agents' expectations if and only if a transaction occurs. Agents expect the next prices but do not necessarily pass an order at this price, i.e. their expectations are not necessarily followed by an action. Thus,

$$s_{t+1} := \left\{ \begin{array}{ll} s_{t'} & \text{if } (t+1)' \neq t+1 \\ E(\widetilde{s}_{t+1}|\Phi_t) & \text{if } (t+1)' = t+1 \end{array} \right\} = s_{t'} + 1_{\{t+1\}} ((t+1)') \gamma_n \left[I_{[t',t]} + \frac{e}{\omega_f n} \right] \quad (4.13)$$

$$f_{t+1} := E(\widetilde{f}_{t+1}|\Phi_t) \quad (4.14)$$

The fundamental value of a price corresponds to the instantaneous integration of all available information on the price (cf. Fama and Miller [4]). Assuming that $FV_{s_0} = s_0$ and $FV_{f_0} = f_0$, the fundamental values are naturally defined by:

$$FV_{s_{t+1}} := s_0 + I_{[0,t+1]} \quad (4.15)$$

$$FV_{f_{t+1}} := FV_{s_{t+1}} - CY_{t+1} \quad (4.16)$$

The fundamental value of the futures is classically computed by its no arbitrage value (cf. Black and Scholes [1], Cox et al. [2], Merton [14]). However, the main difference with

these famous models is that our fundamental values are endogenous to the model because they rely on permanent feedbacks from one market to the other. Notice that the fundamental value of the spot market and the fundamental value of the basis do not covary (i.e. $\text{Cov}(FV_{s_t}, CY_t) = 0$). This independence relies on the fact that we assume e , the trend of the monetary impact of information on the spot market to be exogenous and so, uncorrelated to quantities exchanged on the market.⁷ However, spot and futures fundamental values are linked by the FTF. Another way to interpret the absence of correlation is to consider that information on quantities exchanged on the spot market does not modify the trend of the spot market information price impact. A way to make fundamental values to covary is to endogenize the value of parameter e .

Remark 20. *The fundamental value of the basis is equal to the opposite value of the convenience yield. Then, $FV_{f_{t+1}} - FV_{s_{t+1}} = -CY_{t+1}$. This result holds because of the absence of storage costs.*

4.3.6 Market efficiency and fundamental volatility

We study price bias and we refer to it as market efficiency, denoted by B . We are aware that an unbiased price does not ensure that at each time $l \in \{t, t_r\}$, the price fully reflects the information on the market, but unbiased price is a sine qua non condition.⁸ We give the following function for the market efficiency:

$$\begin{aligned} B: \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ p_l &\mapsto B(p_l) = |E(p_l - FV_{p_l})|, p_l \in \{s_l, f_l\} \text{ where } |\cdot| \text{ denote the absolute value} \end{aligned} \quad (4.17)$$

Then, we consider the fundamental volatility criteria, denoted by \mathcal{V} . It is computed using a standard variance. The variance is not the only way of considering the volatility of a variable. We could have used a kurtosis measure, a value at risk, the spread between the maximum price and the minimum price on the period etc. However, it is one possible tool allowing to explain for what is at stake. Then, we have at time $l \in \{t, t_r\}$:

$$\begin{aligned} \mathcal{V}: \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ FV_{p_l} &\mapsto \mathcal{V}(FV_{p_l}) = \text{Var}(FV_{p_l}), p_l \in \{s_l, f_l\} \end{aligned} \quad (4.18)$$

4.4 The impact of the number of agents

In this section, we study the impact of the number of agents (n.o.a) on market efficiency and the fundamental basis volatility. We provide here a new theoretical insight considering the effect of agents on the trade frequencies, on the market efficiency in subsection 4.4.1. Then, we study in subsection 4.4.2, their effects on the spot and the futures fundamental values volatility.

⁷There is no harvest in the model. Then we also can consider the trend of information to be equal to the average of quantities traded per unit of clock time on the spot market ($q(\omega_s, n) \times \omega_s$).

⁸This result holds as we do not integrate any risk premium. Otherwise, we could have an efficient price biased with a risk primum.

4.4.1 Impact of the number of agents on spot market efficiency and futures market efficiency

In this subsection we study separately the effect of agents on the spot market efficiency, and on the futures market efficiency.

Impact of the number of agents on spot market efficiency

Theorem 12. *Let a spot market under our hypothesis, be called SM. According to the time line considered, the price on SM is biased such that:*

$$B(s_t) = \frac{|-e|}{\omega_f n} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \quad (4.19)$$

$$B(s_{t_r}) = \frac{|-e|}{\omega_f n} \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \quad (4.20)$$

This bias is nul if one of the two following subsets of conditions is verified:

- (i) For $e = 0$, $B(s_t) = B(s_{t_r}) = 0$;
- (ii) For $\omega_s = \omega_f$ and $\alpha = 1$, $B(s_t) = B(s_{t_r}) = 0$.

Proof. As for the previous chapter, we first express the spot price dynamics according to its initial value s_0 . It gives:

$$s_{(t-1)'} = s_0 + \Upsilon_n \times \sum_{k=1}^{(t-1)'} 1_{\{k\}}(k') \left[I_{|(k-1)', k-1|} + \frac{e}{\omega_f n} \right] \quad (4.21)$$

We express the dynamics of s_t according to s_0 using equations (4.13) and (4.21):

$$s_t = s_0 + \Upsilon_n \times \sum_{k=1}^t 1_{\{k\}}(k') \left[I_{|(k-1)', k-1|} + \frac{e}{\omega_f n} \right]$$

Then, its average value is:

$$E(s_t) = s_0 + \frac{\omega_s}{\omega_f} n^{\alpha-1} \frac{e}{\omega_f n} \sum_{k=1}^t (k - E((k-1)'))$$

We compute the average spot fundamental value at time t :

$$E(FV_{s_t}) = E(s_0 + I_{|0,t|}) = s_0 + \frac{e}{\omega_f n} t$$

Using the expression of $E((k-1)')$ given in equation (4.2), we get:

$$E(s_t - FV_{s_t}) = \frac{-e}{\omega_f n} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k$$

Then, we compute the absolute value to express the spot price bias. It allows us to get equations (4.19) and (4.20), according to the time line considered. Hence, both equations (4.19) and (4.20) are equal to zero if and only if one of the two terms is equal to zero:

- (i) It is immediate that $e = 0$ is the only condition ensuring that the first term is equal to zero for both $B(s_t)$ and $B(s_{t_r})$;

- (ii) It is also immediate that $\omega_s = \omega_f$ and $\alpha = 1$ is the only condition ensuring that the second term is equal to zero for both $B(s_t)$ and $B(s_{t_r})$.

Theorem is proved. \square

The **n.o.a** acting on the spot market influences the spot market efficiency when $e \neq 0$. The **n.o.a** can generate a bias on the spot price dynamics that results in an adjustment delay in a **n.o.t** if $\omega_s \neq \omega_f$ and $\alpha \neq 1$. Spot market information is released between two transactions on the futures market and can not be included on the spot price by agents since spot market trades frequency (**STF**) is lower on probability. The condition $\alpha \neq 1$ prevents the **n.o.a** from having a proportional impact on both **STF** and futures market trades frequency (**FTF**) and so, generates a variation of the relative trades frequency (**RTF**) when the **n.o.a** evolves.

Remark 21. For $e = 0$, $B(s_t) = B(s_{t_r}) = 0$. When incoming information has no impact on average on the spot price (i.e. when spot market information has no trend), an adjustment delay in **n.o.t** does not generate a bias on the market. In other words, as the released spot market information has on average no impact in monetary value, the adjustment delay in monetary value corresponding to the bias is null.

Remark 22. We have $\lim_{\omega_f \rightarrow +\infty} B(s_t) = 0$. If intrinsic trade frequency on the futures market is infinite, the first t transactions take place in a negligible clock time. Then, information released is also negligible and the spot price is consequently unbiased.

Focusing on the spot price bias expressed in the clock time line (**CTL**), we have:

$$\lim_{\omega_f \rightarrow +\infty} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) = 1 \Rightarrow \lim_{\omega_f \rightarrow +\infty} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^j = 1$$

By sum and product, we respectively get:

$$\lim_{\omega_f \rightarrow +\infty} \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k = \lim_{\omega_f \rightarrow +\infty} t_r \omega_f n$$

Hence:

$$\lim_{\omega_f \rightarrow +\infty} B(s_{t_r}) = |-e| t_r > 0 \forall e \neq 0 \quad (4.22)$$

Remark 22 highlights to what extent technological advances that diminish the intrinsic transaction costs of the market (and allowing ω_f to raise) influence the spot market efficiency according to the time line. If a policy maker considers the transactions time line (**TTL**), an intrinsic infinite **FTF** allows for an unbiased market since the clock time line effect is infinite (the first t transactions take place in a negligible clock time). However, if s/he considers the **CTL**, it leads to a biased spot price despite the fact that spot market information released between two transactions on the futures price is equal to 0 – on average and variance. There is an infinite **n.o.t** on the futures market until t_r , and a relatively limited **n.o.t** on the spot market. Hence, the spot price is biased.

Remark 23. We have $\lim_{n \rightarrow +\infty} B(s_t) = 0$. If there is an infinite **n.o.a** on the market, the first t transactions take place in a negligible clock time ($\lim_{n \rightarrow +\infty} \omega_f n = 0$). Despite the **RTF** tends to 0, information released is negligible on average and the spot price is consequently unbiased.

Focusing on the spot price bias expressed in the **CTL** in the case $\alpha \in]0, 1[$, we have the same result than on remark 22:

$$\lim_{n \rightarrow +\infty} B(s_{t_r}) |_{\alpha \in]0, 1[} = |-e| t_r > 0 \forall e \neq 0 \quad (4.23)$$

However, in the case where $\alpha = 1$, we have:

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{\omega_s}{\omega_f} \right) = 1 - \frac{\omega_s}{\omega_f}$$

By sum and product, we respectively get:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right)^k &= \lim_{n \rightarrow +\infty} \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{\omega_f}{\omega_s} \left(1 - \left(1 - \frac{\omega_s}{\omega_f} \right)^{t_r \omega_f n} \right) = \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{\omega_f}{\omega_s} \\ \Rightarrow \lim_{n \rightarrow +\infty} \frac{|-e|}{\omega_f n} \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right)^k &= \left(1 - \frac{\omega_s}{\omega_f} \right) \frac{|-e|}{\omega_s} \lim_{n \rightarrow +\infty} \frac{1}{n} \end{aligned}$$

Hence:

$$\lim_{n \rightarrow +\infty} B(s_{t_r}) |_{\alpha=1} = 0 \quad (4.24)$$

Remark 23 highlights to what extent an entrance of an infinite **n.o.a** on the market (allowing for both **STF** and **FTF** to raise) influences the spot market efficiency according to the time line. If a policy maker considers the **TTL**, an infinite **n.o.a** allows for an unbiased spot market since the clock time line effect is infinite. However, if we consider the spot price expressed in the **CTL** and an elasticity of the **STF** to the **n.o.a** $\alpha < 1$, it leads to a biased spot price despite the fact that spot market information released between two transactions on the futures price is on average and variance equal to 0. There are an infinite **n.o.t** until t_r , and an infinite **n.o.t** on the futures market between two transactions on the spot market. Hence, the spot price is biased. But if the latest elasticity is equal to 1 ($\alpha = 1$), **STF** and **FTF** proportionally increases such that the **RTF** is unchanged. The average **n.o.t** on the futures market is unchanged but the clock time between two transactions on the spot market becomes negligible. Then, the released spot market information becomes negligible too on average, allowing for the spot price to be unbiased. This result underlies the major importance of the elasticity of the **STF** to the **n.o.a** α compared to the elasticity of the **FTF** to the **n.o.a** (normalized to 1 in the model).

Impact of the number of agents on futures market efficiency

Theorem 13. *Let a futures market under our hypothesis, be called FM. The price on FM is unbiased such that:*

$$B(f_t) = B(f_{t_r}) = 0 \quad (4.25)$$

Proof. We compute the average of the spread between the futures price and its fundamental value. Using equations (4.12) and (4.14), we have:

$$\begin{aligned} E(f_t - FV_{f_t}) &= E[E(\tilde{s}_t - \widetilde{CY}_t | \Phi_{t-1}) - (FV_{s_t} - CY_t)] \\ \Leftrightarrow E(f_t - FV_{f_t}) &= E[E(\tilde{s}_t | \Phi_{t-1}) - FV_{s_t}] - E[E(\widetilde{CY}_t | \Phi_{t-1}) - CY_t] \end{aligned}$$

Using equations (4.8), (4.10) and (4.12), we have:

$$\begin{aligned} E[E(\widetilde{CY}_t | \Phi_{t-1}) - CY_t] &= E\left[\frac{a}{\omega_f n} (T-t) (Q_{op} - E(\widetilde{Q}_t | \Phi_{t-1})) - \frac{a}{\omega_f n} (T-t) (Q_{op} - Q_t) \right] \\ \Leftrightarrow E[E(\widetilde{CY}_t | \Phi_{t-1}) - CY_t] &= \frac{a}{\omega_f n} (T-t) E[Q_t - E(\widetilde{Q}_t | \Phi_{t-1})] = 0 \end{aligned}$$

To compute the term $E[E(\tilde{s}_t | \Phi_{t-1}) - FV_{s_t}]$, we proceed with a recurrence and recall that $s_0 = FV_{s_0}$. The property $P(t)$ is: $E[E(\tilde{s}_t | \Phi_{t-1}) - FV_{s_t}] = 0 \forall t \in [1, T]$.

Initialization:

$$\begin{aligned} E[E(\tilde{s}_1 | \Phi_0) - FV_{s_1}] &= E\left[s_0' + \gamma_n I_{|0',0|} + \gamma_n \frac{e}{\omega_f n} - (s_0 + I_{|0,1|})\right] \\ \Leftrightarrow E[E(\tilde{s}_1 | \Phi_0) - FV_{s_1}] &= E\left[\gamma_n \frac{e}{\omega_f n} - I_{|0,1|}\right] = 0 \end{aligned}$$

P(1) is true. We assume that P(t) is true (i.e. $E\left[s_{(t-1)'} + \gamma_n I_{|(t-1)',t-1|} + \gamma_n \frac{e}{\omega_f n} - (s_0 + I_{|0,t|})\right] = 0$), we show that P($t+1$) is also true.

$$\begin{aligned} E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= E\left[s_{t'} + \gamma_n \left(I_{|t',t|} + \frac{e}{\omega_f n}\right) - (s_0 + I_{|0,t+1|})\right] \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= E\left[s_{(t-1)'} + \gamma_n 1_{\{t\}}(t') \left(I_{|(t-1)',t-1|} + \frac{e}{\omega_f n}\right) + \gamma_n \left(I_{|t',t|} + \frac{e}{\omega_f n}\right) - (s_0 + I_{|0,t|} + I_{|t,t+1|})\right] \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= E\left[s_{(t-1)'} + \gamma_n \left(I_{|(t-1)',t-1|} + \frac{e}{\omega_f n}\right) - (s_0 + I_{|0,t|})\right] \\ &\quad + E\left[\gamma_n (1_{\{t\}}(t') - 1) \left(I_{|(t-1)',t-1|} + \frac{e}{\omega_f n}\right) + \gamma_n \left(I_{|t',t|} + \frac{e}{\omega_f n}\right) - I_{|t,t+1|}\right] \end{aligned}$$

As we assumed that P(t) is true, we get:

$$\begin{aligned} E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= E\left[\gamma_n (1_{\{t\}}(t') - 1) \left(I_{|(t-1)',t-1|} + \frac{e}{\omega_f n}\right) + \gamma_n \left(I_{|t',t|} + \frac{e}{\omega_f n}\right) - I_{|t,t+1|}\right] \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= E\left[(1_{\{t\}}(t') - 1) \left(I_{|(t-1)',t-1|} + \frac{e}{\omega_f n}\right) + \left(I_{|t',t|} + \frac{e}{\omega_f n}\right) - I_{|t,t+1|}\right] \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} E\left[(1_{\{t\}}(t') - 1) (t - 1 - (t-1)' + 1) + (t - t' + 1) - (t + 1 - t)\right] \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} E\left[(1_{\{t\}}(t') - 1) (t - (t-1)') + (t - t')\right] \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} (E[(1_{\{t\}}(t') - 1) (t - (t-1)')] + E[t - t']) \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} (E[1_{\{t\}}(t') - 1] E[t - (t-1)'] + t - E[t']) \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} \left(\left(\frac{\omega_s}{\omega_f n^{1-\alpha}} - 1\right) (t - E[(t-1)']) + t - E[t']\right) \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} \left(t \frac{\omega_s}{\omega_f n^{1-\alpha}} - E[(t-1)'] \left(\frac{\omega_s}{\omega_f n^{1-\alpha}} - 1\right) - E[t']\right) \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} \left(t \frac{\omega_s}{\omega_f n^{1-\alpha}} + \left(t - 1 - \sum_{k=1}^{t-1} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k\right) \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) - E[t']\right) \\ \Leftrightarrow E[E(\tilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}}] &= \frac{e}{\omega_f n} \left(t - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - E[t']\right) = 0 \end{aligned}$$

If P(t) is true, P($t+1$) is also true. As, P(1) is true, P(t) is true for all $t \in [1, T]$.

Then, $E(f_t - FV_{f_t}) = 0 + 0 = 0$. Computing the absolute value and expressing it in both time lines leads to equation (4.25). Theorem is proved. \square

By assumption, agents form rational expectations on average (i.e. $E(\gamma_n) = 1$). Furthermore, spot market information arrives at the same rhythm than the transactions on the futures market. Hence, futures price is unbiased independently of the time line considered.

Corollary 4 (Corollary of theorems 12 and 13). *The basis on FM and SM is biased such that:*

$$B(f_t - s_t) = B(s_t) \quad (4.26)$$

$$B(f_{t_r} - s_{t_r}) = B(s_{t_r}) \quad (4.27)$$

Proof. Immediate from theorems 12 and 13. \square

We extract from corollary 4 that if agents reduce the spot price bias, they also reduce the basis bias.

4.4.2 Impact of the number of agents on spot and futures fundamental value volatility

In section 4.3.5, we expressed how the implication of an exogenous trend of spot market information (e) on the fundamental values dynamics leads to a spot fundamental value independent from the convenience yield. Then, the study of the volatility of the futures fundamental value is equivalent to study separately the spot fundamental value volatility on the one hand (cf. subsection 4.4.2), and the convenience yield volatility on the other hand (cf. subsection 4.4.2).

Impact of the number of agents on spot fundamental value volatility

Theorem 14. *On SM, the spot fundamental value volatility is such that:*

$$\mathcal{V}(FV_{s_t}) = t \times \frac{\sigma_I^2}{\omega_f} \times \frac{1}{1+n} > 0 \quad (4.28)$$

$$\mathcal{V}(FV_{s_{t_r}}) = t_r \times \sigma_I^2 \times \frac{n}{1+n} > 0 \quad (4.29)$$

Proof. Computing the variance of the spot price fundamental value at transaction t gives:

$$\begin{aligned} \mathcal{V}(FV_{s_t}) &= \text{Var}(s_0 + I_{|0,t|}) = \text{Var}\left(\sum_{k=1}^t I_{|k-1,k|}\right) = \sum_{k=1}^t \text{Var}(I_{|k-1,k|}) \\ \Leftrightarrow \mathcal{V}(FV_{s_t}) &= \sum_{k=1}^t \frac{\sigma_I^2}{\omega_f} \times \frac{1}{1+n} = t \times \frac{\sigma_I^2}{\omega_f} \times \frac{1}{1+n} \end{aligned}$$

Explaining this volatility in the CTL gives equation (4.29). It proves theorem 14. \square

Theorem 14 underlies to what extent the n.o.a modifies the spot fundamental value volatility according to the time line. If we are in the TTL (t), we find that agents reduce this volatility. The mechanism is that the FTF increases and there is a clock time effect, i.e. a compression of the clock time considered. It leads to an increase in the volatility of spot market information overcompensated by the compression of the clock time considered, constraining the spot volatility. This compensation does not hold when considering a fixed clock time such that agents increase the spot price fundamental volatility.

Impact of the number of agents on futures fundamental value volatility

Theorem 15. *On SM and FM, the available stock and the fundamental value volatility of the basis have the following properties:*

- (i) *The volatility of the available stock expressed respectively in the TTL, denoted by $\text{Var}(Q_t)$, and in the CTL, denoted by $\text{Var}(Q_{t_r})$, are impacted by the n.o.a on the market, and we have:*

$$\text{Var}(Q_t) = q(\omega_s, n)^2 t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \frac{\omega_s}{\omega_f n^{1-\alpha}} \quad (4.30)$$

$$\text{Var}(Q_{t_r}) = q(\omega_s, n)^2 t_r \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \omega_s n^\alpha \quad (4.31)$$

These volatility are null if the following condition is verified:

- For $\omega_s = \omega_f$ and $\alpha = 1$, $\text{Var}(Q_t) = \text{Var}(Q_{t_r}) = 0$.
- (ii) *The volatility of the fundamental value of the basis expressed respectively in the TTL, denoted by $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t})$, and in the CTL, denoted by $\mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}})$, are impacted by the n.o.a on the market and we have:*

$$\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \left(\frac{a}{\omega_f n}\right)^2 ((T_r \omega_f n) - t)^2 \text{Var}(Q_t) \quad (4.32)$$

$$\mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = a^2 (T_r - t_r)^2 \text{Var}(Q_{t_r}) \quad (4.33)$$

These volatility are null if one of the following condition is verified:

- For $a = 0$, $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = 0$;
- For $t = T_r \omega_f n = T \Leftrightarrow t_r = T_r$, $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = 0$;
- For $\omega_s = \omega_f$ and $\alpha = 1$, $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = 0$.

Proof. First, we compute $\text{Var}(Q_t)$ and show that it depends on n .

$$\text{Var}(Q_t) = \text{Var}\left(Q_0 - q(\omega_s, n) \sum_{k=1}^t 1_{\{t\}}(t')\right) = q(\omega_s, n)^2 t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \frac{\omega_s}{\omega_f n^{1-\alpha}}$$

Thus, $\text{Var}(Q_t)$ depends on n . By a time line transposition we get the expression of $\text{Var}(Q_{t_r})$. It also depends on n .

Hence, both equations (4.30) and (4.31) are equal to zero if and only if one of the terms is equal to 0. It is immediate that $\omega_s = \omega_f$ and $\alpha = 1$ is the only condition ensuring $\text{Var}(Q_t) = \text{Var}(Q_{t_r}) = 0$. It proves the first assertion.

Then, we compute the fundamental variance of the spread between the futures price and spot price. From remark 20, we get: $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(-CY_t) = \mathcal{V}(CY_t)$. We show that it depends on n .

$$\mathcal{V}(CY_t) = \text{Var}\left[\frac{a}{\omega_f n} (T - t) (Q_{op} - Q_t)\right] = \left(\frac{a}{\omega_f n}\right)^2 ((T_r \omega_f n) - t)^2 \text{Var}(Q_t)$$

Thus, $\mathcal{V}(CY_t)$ depends on n . By a time line transposition we get the expression of $\mathcal{V}(CY_{t_r})$. It also depends on n .

Hence, both equations (4.32) and (4.33) are equal to zero if and only if one of the terms is equal to 0:

- (i) It is immediate that $a = 0$ is the only condition ensuring the first term to be equal to 0 such that $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = 0$;
- (ii) It is immediate that $t = T_r \omega_f n = T \Leftrightarrow t_r = T_r$ is the only condition ensuring the second term to be equal to 0 such that $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = 0$;
- (iii) It is immediate that $\omega_s = \omega_f$ and $\alpha = 1$ is the only condition ensuring the last term to be equal to 0 such that $\mathcal{V}(\text{FV}_{f_t} - \text{FV}_{s_t}) = \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}}) = 0$.

Last assertion is proved. □

Theorem 15 demonstrates that the **n.o.a** on the commodity market impacts the fundamental basis volatility – here equal to the convenience yield. Recall that theorem 14 demonstrates that agents also influence the fundamental spot price volatility. In the end, we demonstrate that the **n.o.a** influences both spot and futures price volatility. In the previous chapter 3, we argued that the **RTF** influences the fundamental basis volatility. Here, we extend this result; Agents influence both **RTF** and the spot market information itself. These results can explain to what extent and in which cases the futures fundamental volatility is higher than the fundamental spot volatility. It provides an additional explanation to empirical tests showing the latest phenomena (cf. Matia et al. [13]).

4.5 The impact of the entrance of agents on the market

From theorems 12, 14 and 15, we proved that under the assumptions made, the number of agents (**n.o.a**) influences fundamentals volatility and spot market efficiency. In this section, we study how this impact evolves when the **n.o.a** increases. As the **n.o.a** is assumed to be in the set of natural numbers (i.e. $n \in \mathbb{N}^*$), our main mathematical tool is the simple difference. In subsection 4.5.1 we investigate the effect of an entrance of agents on spot and futures market efficiency. In subsection 4.5.2 we investigate the effect of an entrance of agents on spot and futures fundamental values volatility.

4.5.1 Impact of the entrance of agents on spot and futures market efficiency

In this subsection we study separately the effect of an entrance of agents on the spot market efficiency, and on the futures market efficiency.

Impact of the entrance of agents on spot market efficiency

Theorem 16. *An entrance of one – unit of – agent on the market has the following impact on the spot price bias of SM:*

- (i) *Within the transactions time line (**TTL**) (t), the spot price bias strictly decreases if and only if the following condition is verified:*

$$\frac{n+1}{n} > \frac{\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k}{\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k} \quad (4.34)$$

(ii) Within the clock time line (CTL) (t_r), the spot price bias strictly decreases if and only if the following condition is verified:

$$\frac{n+1}{n} > \frac{\sum_{k=1}^{t_r \omega_f (n+1)} \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k}{\sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k} \quad (4.35)$$

Proof. We calculate the simple difference of the spot price bias according to n , and we study its sign. Using equation (4.19) and factorizing by $\frac{1-\epsilon_l}{\omega_f} > 0$ we have:

$$B(s_t) |_{n+1} - B(s_t) |_n = \frac{1}{n+1} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - \frac{1}{n} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k$$

Putting to the same denominator and factorizing by $\frac{1}{n(n+1)} > 0$, we get:

$$B(s_t) |_{n+1} - B(s_t) |_n = n \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - (n+1) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k$$

This difference is strictly negative if and only if:

$$n \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k < (n+1) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k$$

It proves the first assertion.

Proceeding identically, we can prove the second assertion. Theorem is proved. \square

An increase in the **n.o.a** on the market not necessarily decreases the spot price bias.

Within the **TTL**, we found that the multiplier of the **n.o.a** ($\frac{n+1}{n}$), which equals the multiplier of the futures market trades frequency (**FTF**), must be higher than the increase of the average delay in **n.o.t** between two transactions on the spot market relatively to the **FTF**.

Within the **CTL**, this constraint is slightly modified. Indeed, an increase in the **n.o.a** increases the **FTF**. But as the clock time is fixed, there are more transactions on the futures market. This phenomena is translated on the terminal term of the sums ratio ($t_r \omega_f n$ and $t_r \omega_f (n+1)$) of the average delay in **n.o.t** between two transactions on the spot market relatively to the **FTF**. The left term still represents the increase in the **n.o.t** on the futures market per unit of clock time through the multiplier of the **FTF**.

Then, we demonstrate the major role of the "current state" of the market in determining the optimal **n.o.a** on the market according to the spot price bias objective (i.e. allowing for free entrance or not of more agents on the market).

Remark 24. If $\alpha = 1$, both inequalities (4.34) and (4.35) are verified. If the spot market trades frequency (**STF**) is proportionally modified by the entrance of agents compared to the **FTF**, the free entrance of agents is an optimal policy. Linking this remark with remark 23, not only the free entrance is an optimal policy but an infinite **n.o.a** allows for the spot price to be unbiased independently of the time line considered.

Impact of the entrance of agents on futures market efficiency

Corollary 5 (Corollary of theorem 13). *An entrance of one – unit of – agent on the market has no impact on the futures price bias of FM. Hence, we have:*

$$B(f_t) |_{n+1} - B(f_t) |_n = B(f_{t_r}) |_{n+1} - B(f_{t_r}) |_n = 0 \quad (4.36)$$

Proof. Immediate from theorem 13. □

In subsection 4.5.1, we demonstrated that an entrance of – a unit of – agents on a commodity market under our assumptions does not necessary allow for the spot price bias to decrease. In fact, if the increases in the FTF implied by this – unit of – agent does not overcome the average adjustment delay in n.o.t on the futures market multiplier, it leads to an increasing spot price bias. However, there is no trade-off between the futures price efficiency and the spot price efficiency since agents do not modify the futures price bias.

4.5.2 Impact of the entrance of agents on spot and futures fundamental volatility

In this subsection we study separately the effect of an entrance of agents on the spot fundamental value volatility in subsection 4.5.2, and on the futures fundamental value volatility in subsection 4.5.2.

Impact of the entrance of agents on spot fundamental value volatility

Theorem 17. *An entrance of one – unit of – agent on the market has the following impact on the fundamental spot price volatility of SM:*

(i) *Within the TTL (t), the fundamental spot price volatility strictly decreases such that:*

$$\mathcal{V}(FV_{s_t}) |_{n+1} - \mathcal{V}(FV_{s_t}) |_n = -t \times \frac{\sigma_I^2}{\omega_f} \times \frac{1}{(n+2)(n+1)} < 0 \quad (4.37)$$

(ii) *Within the CTL (t_r), the fundamental spot price volatility strictly increases such that:*

$$\mathcal{V}(FV_{s_{t_r}}) |_{n+1} - \mathcal{V}(FV_{s_{t_r}}) |_n = t_r \times \sigma_I^2 \times \frac{1}{(n+2)(n+1)} > 0 \quad (4.38)$$

Proof. We calculate the difference of the fundamental spot price volatility according to n , and we study its sign. Using equation (4.28) we have:

$$\mathcal{V}(FV_{s_t}) |_{n+1} - \mathcal{V}(FV_{s_t}) |_n = t \times \frac{\sigma_I^2}{\omega_f(1+n+1)} - t \times \frac{\sigma_I^2}{\omega_f(1+n)} = -t \times \frac{\sigma_I^2}{\omega_f} \times \frac{1}{(n+2)(n+1)}$$

As $\sigma_I > 0$, $\mathcal{V}(FV_{s_t}) |_{n+1} - \mathcal{V}(FV_{s_t}) |_n < 0$. Hence, first assertion is proved.

Proceeding identically, we can prove the second assertion. Theorem is proved. □

Theorem 17 underlies again the major importance of the time line considered. Within the TTL (t), the spot price volatility decreases with an entrance of more agents despite the increase of the spot market information volatility by agents, because there is a clock time line effect. The clock time between two transactions reduces such that the fundamental volatility of the spot price decreases. However, if the clock time is fixed, we only have an increase in the spot market information volatility and on the fundamental spot price volatility. Then, we extract from theorems 16 and 17 that a trade-off between decreasing the spot price bias and decreasing the fundamental spot price volatility may exist.

Impact of the entrance of agents on futures fundamental value volatility

Theorem 18. *An entrance of one – unit of – agent has the following impact on the fundamental basis volatility of SM and FM:*

- (i) *Within the TTL (t), the fundamental basis volatility strictly decreases if and only if the following condition is verified:*

$$\left(\frac{n+1}{n}\right)^{-2(1+\alpha\epsilon)} \left(\frac{T_r \omega_f(n+1) - t}{T_r \omega_f n - t}\right)^2 \frac{\left(1 - \frac{\omega_s}{\omega_f(n+1)^{1-\alpha}}\right) \frac{1}{(n+1)^{1-\alpha}}}{\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \frac{1}{n^{1-\alpha}}} < 1 \quad (4.39)$$

- (ii) *Within the CTL (t_r), the fundamental basis volatility strictly decreases if and only if the following condition is verified:*

$$\left(\frac{n+1}{n}\right)^{1-2\alpha\epsilon} \frac{\left(1 - \frac{\omega_s}{\omega_f(n+1)^{1-\alpha}}\right) \frac{1}{(n+1)^{1-\alpha}}}{\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \frac{1}{n^{1-\alpha}}} < 1 \quad (4.40)$$

Proof. We calculate the difference of the convenience yield volatility according to n within the TTL, and we study its sign. Using equations (4.30) and (4.32) and factorizing by $a^2 t \frac{\omega_s}{\omega_f^3} > 0$ we have:

$$\begin{aligned} \mathcal{V}(\text{CY}_t) |_{n+1} - \mathcal{V}(\text{CY}_t) |_n &= \frac{(n+1)^{\alpha-1}}{(n+1)^2} (T_r \omega_f(n+1) - t)^2 q(\omega_s, (n+1))^2 \left(1 - \frac{\omega_s}{\omega_f(n+1)^{1-\alpha}}\right) \\ &\quad - \frac{n^{\alpha-1}}{n^2} (T_r \omega_f n - t)^2 q(\omega_s, n)^2 \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \\ \Leftrightarrow \mathcal{V}(\text{CY}_t) |_{n+1} - \mathcal{V}(\text{CY}_t) |_n &= (n+1)^{\alpha(1-2\epsilon)-3} (T_r \omega_f(n+1) - t)^2 \omega_s^{-2\epsilon} \left(1 - \frac{\omega_s}{\omega_f(n+1)^{1-\alpha}}\right) \\ &\quad - n^{\alpha(1-2\epsilon)-3} (T_r \omega_f n - t)^2 \omega_s^{-2\epsilon} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \end{aligned}$$

Factorizing by $\omega_s^{-2\epsilon} > 0$, this difference is strictly negative if and only if:

$$\left(\frac{n+1}{n}\right)^{\alpha(1-2\epsilon)-3} (T_r \omega_f(n+1) - t)^2 \left(1 - \frac{\omega_s}{\omega_f(n+1)^{1-\alpha}}\right) < (T_r \omega_f n - t)^2 \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)$$

It proves the first assertion.

Proceeding identically, we can prove the second assertion. Theorem is proved. \square

Theorem 18 is a key result of this work. According to the elasticity of quantities traded to the STF (ϵ) and the elasticity of the STF to the n.o.a (α), i.e. the spot market characteristics, an entrance of more agents can lead to an increase in fundamental basis volatility. This condition also relies on the current level of agents n .

Within the TTL (t), there are four effects at stake:

- (i) A quantity effect. The entrance of agents modifies the quantities traded at each transaction on the spot market according to their effect on the STF. It is represented by the square of the multiplier of quantities traded at each transactions on the spot market (since $V(ax) = a^2 V(x)$), which is equal to $\left(\frac{n+1}{n}\right)^{-2\alpha\epsilon}$;

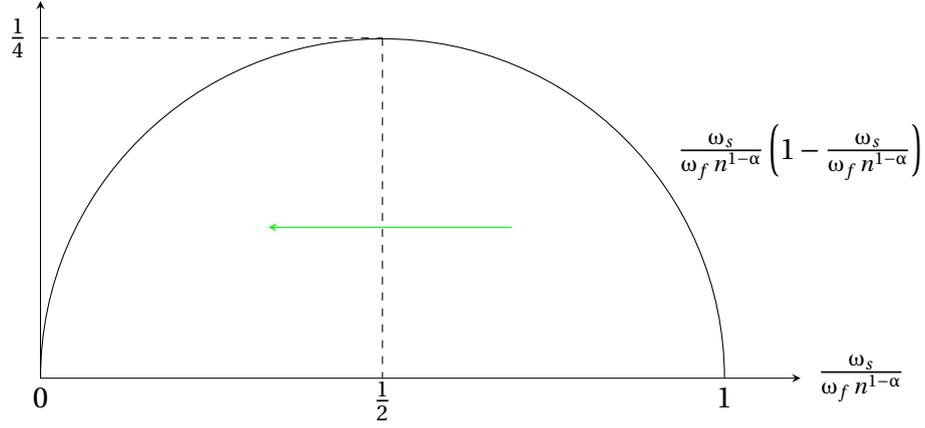


Figure 4.1 – Volatility of the occurrence of synchronized transactions at each transaction on the futures market. Green arrow gives the evolution of the volatility with an increasing **n.o.a**, $\alpha \in]0, 1[$.

- (ii) A clock time line effect. The first t transactions occur in less clock time according to the modification of the **FTF**. It is represented by the square of the multiplier of the **FTF**, which is equal to $\left(\frac{n+1}{n}\right)^{-2}$;
- (iii) A clock time line effect until maturity. The entrance of agents provokes more trades on the futures market. The second effect compresses the clock time considered until the t^{th} transaction on the futures market. Then, the clock time until maturity is more important. It is represented by the square of the multiplier of the **n.o.t** remaining until the maturity, equal to $\left(\frac{T_r \omega_f (n+1) - t}{T_r \omega_f n - t}\right)^2$;
- (iv) A risk of occurrence effect. The entrance of agents modifies the relative trades frequency (**RTF**) (if $\alpha \neq 1$). In other words, it represents the fundamental risk induced by the evolution of the **RTF**. It is represented by the multiplier of the volatility of the occurrence of transaction on the spot market at each transaction on the futures market induced by an increase in the **n.o.a**, equal to $\left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right) \frac{1}{(n+1)^{1-\alpha}} / \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \frac{1}{n^{1-\alpha}}$.

The first two effects decrease the fundamental basis volatility despite the third one increases it. However, the sign of the fourth is ambiguous and depends on current values of both parameters ω_s and ω_f , and the initial **n.o.a** n . The product of these four multipliers must give a multiplier strictly inferior to 1 in order to strictly decrease the fundamental basis volatility with an entrance of one more – unit of – agent.

Within the **CTL** (t_r), there are three effects at stake:

- (i) A quantity effect. The entrance of agents modifies the quantities traded at each transaction on the spot market according to their effect on the **STF**. It is represented by the square of the multiplier of quantities traded at each transactions on the spot market (since $V(ax) = a^2V(x)$), which is equal to $\left(\frac{n+1}{n}\right)^{-2\alpha\epsilon}$;
- (ii) A **FTF** effect. The entrance of agents increases the **n.o.t** on the futures market for a given clock time. This effect takes into account the increase in the **n.o.t** until t_r but also between t_r and the maturity T_r . It is represented by the multiplier of the **n.o.t** on the futures market during each unit of clock time t_r , equal to $\left(\frac{n+1}{n}\right)$;

- (iii) A risk of occurrence effect. The entrance of agents modifies the **RTF** (if $\alpha \neq 1$). In other words, it represents the fundamental risk induced by the evolution of the **RTF**. The evolution of the occurrence of synchronized transactions volatility with an increase in the **n.o.a** is represented in figure 4.1. It is represented by the multiplier of the volatility of the occurrence of transaction on the spot market at each transaction on the futures market induced by an increase in the **n.o.a**, equal to $\left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right) \frac{1}{(n+1)^{1-\alpha}} / \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \frac{1}{n^{1-\alpha}}$.

The first effect decreases the fundamental basis volatility despite the second one increases it. However, the sign of the third effect is ambiguous and depends on current values of both parameters ω_s and ω_f , and the initial **n.o.a** n . The product of these three multipliers must give a multiplier strictly inferior to 1 in order to strictly decrease the fundamental basis volatility with an entrance of one more – unit of – agent.

Remark 25. From theorem 15, the fundamental spot price and the convenience yield (i.e. the fundamental value of the basis) do not covary. The fundamental spot price volatility and the fundamental futures price volatility can evolve in opposite signs. From theorems 17 and 18, an increasing **n.o.a** on the market can have an opposite effect on the fundamental spot price volatility and the fundamental basis volatility, such that an increasing **n.o.a** on the market has an ambiguous effect on the fundamental futures price volatility.

4.6 On the existence of an optimal number of agents

The objectives of a policymaker on commodity markets are to minimize the fundamental values volatility and to increase market efficiency on both spot and futures markets. As the futures price is unbiased (cf. corollary 4), the policymaker focuses on spot market efficiency. We assume that the risk aversion of the policy maker is the same for both fundamental spot price volatility and basis volatility, such that it is equivalent to consider her/his risk aversion to the futures fundamental volatility (i.e. $\mathcal{V}(\text{FV}_{f_t}) = \mathcal{V}(\text{FV}_{s_t}) + \mathcal{V}(\text{CY}_t)$, $t \in \{t, t_r\}$). We consider here that the number of agents (**n.o.a**) on the market is the policymaker's tool, and thus the policymaker can implement a limit in the **n.o.a** or not (free entrance).

Considering the feedbacks between these two markets at the fundamental level, we investigate the existence of an optimal **n.o.a** and if any, if it is in adequacy with the current state of the market. As we explained all effects at stake within the transactions time line (**TTL**), we focus on the clock time line (**CTL**), as it is the time line considered by the policy maker. We find different levels of optimal **n.o.a** considering the objectives of the policy maker, and particularly her/his risk aversion.

Definition 24. We define a mean-variance utility function U_{t_r} (following Markowitz [12], Von Neumann et al. [21]) such that:

$$U_{t_r} : \mathbb{N}^* \xrightarrow{U_1} \mathbb{R}^+ \times \mathbb{R}^+ \quad \xrightarrow{U_2} \mathbb{R}^- \quad (4.41)$$

$$n \longmapsto \left(B(s_{t_r}), \mathcal{V}(\text{FV}_{f_{t_r}}) \right) \longmapsto U_2(\cdot, \cdot) = -B(s_{t_r}) - \Theta[\mathcal{V}(\text{FV}_{f_{t_r}})]$$

Hence, we have $U_{t_r}(n) = U_2 \circ U_1(n)$, verifying the two following properties:

- (i) $U_2(0, 0) = 0$;
- (ii) $\frac{\partial U_2(B(s_{t_r}), \mathcal{V}(\text{FV}_{f_{t_r}}))}{\partial B(s_{t_r})} < 0$, and $\frac{\partial U_2(B(s_{t_r}), \mathcal{V}(\text{FV}_{f_{t_r}}))}{\partial \mathcal{V}(\text{FV}_{f_{t_r}} - \text{FV}_{s_{t_r}})} < 0$.

We assume that a policymaker uses the utility function U_{t_r} as defined above to maximize the agricultural commodity market efficiency and to minimize its fundamental basis volatility. Properties (i) and (ii) correspond respectively to the following natural insights:

- (i) U_2 reaches its maximal value on \mathbb{R}^- if and only if the spot price bias and the fundamental futures volatility are both null;⁹
- (ii) If either the spot price bias or the fundamental futures volatility increases, the policymaker's utility decreases.

In this section, we provide some analytical results in subsection 4.6.1. However, in most of the cases, the optimal n.o.a depends on the set of all parameters of the model. Consequently, we provide some sensitivity analysis according to main parameters when necessary in subsection 4.6.2. All figures are also provided.

4.6.1 Analytical results

Theorem 19. *If a policymaker whose utility function is U_{t_r} has no risk aversion (i.e. if $\Theta = 0$), her/his utility function reaches its maximal value if one of the three following conditions given by the subset $\mathcal{O}_{\Theta=0}^{t_r}$ is verified:*

$$\mathcal{O}_{\Theta=0}^{t_r} = \left\{ (n, \Theta, \alpha, \omega_s, \omega_f, e) \in \mathbb{N}^+ \times \{0\} \times]0, 1] \times \mathbb{F}_s \times \mathbb{F}_f \times \mathbb{R} \mid (e = 0, n \in \mathbb{N}^*) \cup (\omega_s = \omega_f (\mathbb{F}_s \cap \mathbb{F}_f \neq \emptyset), \alpha = 1, n \in \mathbb{N}^*) \cup (\alpha = 1, n \rightarrow +\infty) \right\} \quad (4.42)$$

Proof. Within the CTL (t_r), let $\Theta = 0$. From theorem 12, we get

$$B(s_{t_r}) = 0 \Rightarrow (e = 0) \cup (\omega_s = \omega_f (\mathbb{F}_s \cap \mathbb{F}_f \neq \emptyset), \alpha = 1). \text{ From remark 23, we also get } \lim_{n \rightarrow +\infty} B(s_{t_r})|_{\alpha=1} = 0.$$

Let $\mathcal{O}_{\Theta=0}^{t_r}$ be the subset allowing $\lim_{n \rightarrow +\infty} U_{t_r}(n) = 0, \Theta = 0$. Hence, $\mathcal{O}_{\Theta=0}^{t_r}$ is as given in the second assertion.

Let $\Theta \neq 0$. From theorem 14 $\mathcal{V}(FV_{s_{t_r}}) > 0$, the policy maker's utility function can not take its maximal value. Then, the optimal n.o.a on the market fully depends on the parameters values. Second assertion is proved. \square

Remark 26. *When considering the possibility of speculation on the spot market, the optimal n.o.a on the market may change. There is still an ambiguity of having an available stock reduction at each transaction on the spot market, such that except if $a = 0$. In this case, all three arguments of U_{t_r} cannot be simultaneously equal to 0. The arbitrage between market efficiency and fundamental volatility becomes mandatory.*

We only get analytical results when considering no risk aversion. In the next subsection 4.6.2 we provide sensitivity analysis to the parameters of the model when there is risk aversion.

4.6.2 Sensitivity analysis

In this subsection, we provide some sensitivity analysis of the optimal n.o.a to different changes in the parameters values. Indeed, we cannot get tractable analytical solution from this model superimposing two kinds of complexity. The first one relies on the complexity of the expression of all terms in the utility function. The second one relies on the

⁹This utility function does not verify the axiom of non satiety. If the market efficiency is maximized and the fundamental futures volatility is minimized, the utility cannot be improved.

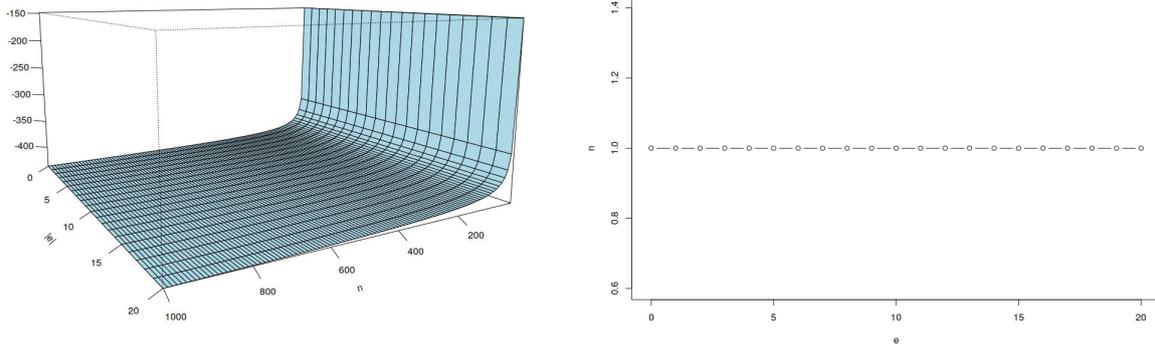


Figure 4.2 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter e ($e = \{0, 1, \dots, 20\}$).

introduction of the preference of the policy maker. Within the CTL, we consider a set of parameter to be in $\mathcal{O}_{\Theta=0}^{\bar{t}_r}$. The set of parameters default value used is given by the following table 4.3.

Parameter	e	a	ω_f	ω_s	t_r	σ_I	Θ	T_r	ϵ	α
Value	5	0,25	15	10	41	0,05	0,05	100	0,5	0,5

Table 4.3 – Set of parameters default values used for the sensitivity tests in $\mathcal{O}_{\Theta=0}^{\bar{t}_r}$, within the CTL

In order to have clear charts, we consider that $n^* \in \{1, 11, 21, \dots, 1001\}$.

Sensitivity analysis to e : We can observe on figure 4.2 that utility is a strictly decreasing function of n for all tested values of e . Despite the potential reduction of the spot price bias (cf. theorem 16) and the fundamental basis volatility (cf. theorem 18), the increase in the fundamental spot price volatility (cf. theorem 17) with an increasing n.o.a, compensate these effects. Then, the optimal n.o.a is always minimal ($n^* = 1$). In this case, we have:

$$\max_n U_{t_r}(n) = U_{t_r}(1) = -\frac{|-e|}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k - \Theta t_r \left[\frac{\sigma_I^2}{\omega_f} \frac{1}{2} + a^2 (T_r - t_r)^2 \omega_s^{1-2\epsilon} \left(1 - \frac{\omega_s}{\omega_f}\right) \right]$$

Sensitivity analysis to a : We can observe on figure 4.3 that utility is a strictly decreasing function of n for all tested values of $a \neq 0$. If there is no convenience yield (i.e. $a = 0$), the utility function fully depends on the spot price bias and on the fundamental spot price volatility. From remark 23 and first assertion of theorem 17, an infinite n.o.a leads to a biased spot price (but not necessarily maximal) and a maximal fundamental spot price volatility by limit. Hence, according to the risk aversion of the policy maker Θ , the increase in the fundamental spot price volatility is less important than the potential effect on the spot price bias (which is ambiguous from theorem 16). Within our set of parameters' default values, an entrance of an infinite n.o.a leads to a decrease in the spot price bias. In other words, the policy maker utility is maximized with an infinite n.o.a. Then, the optimal n.o.a is always minimal ($n^* = 1$), except if $a = 0 \Rightarrow n^* \rightarrow +\infty$.

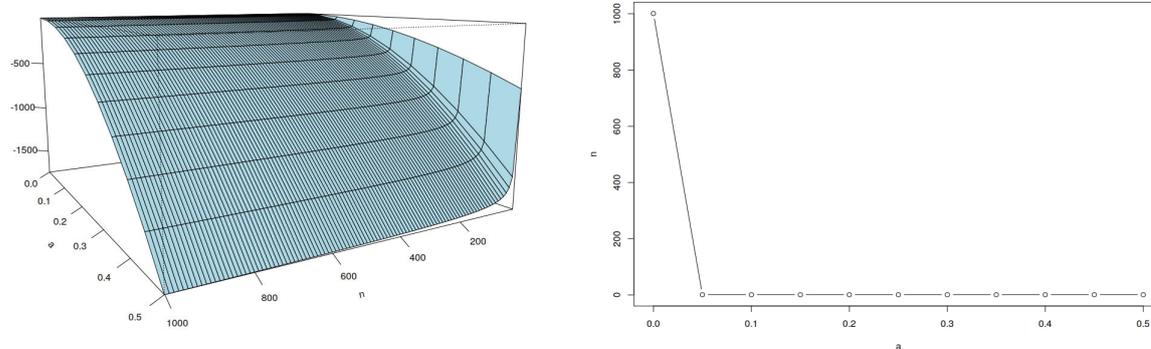


Figure 4.3 – Representation of both utility function $U_{tr}(n)$ (on left side), and n^* (on right side) according to the variation of parameter a ($a = \{0, 0.05, 0.1, \dots, 0.5\}$).

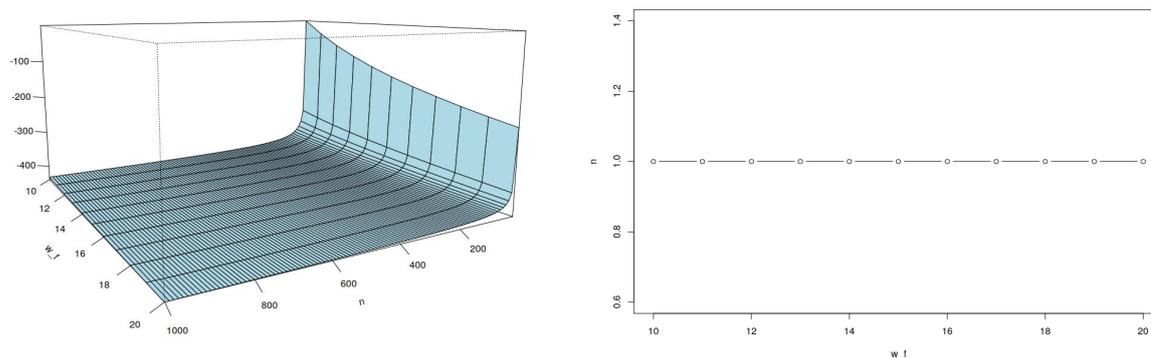
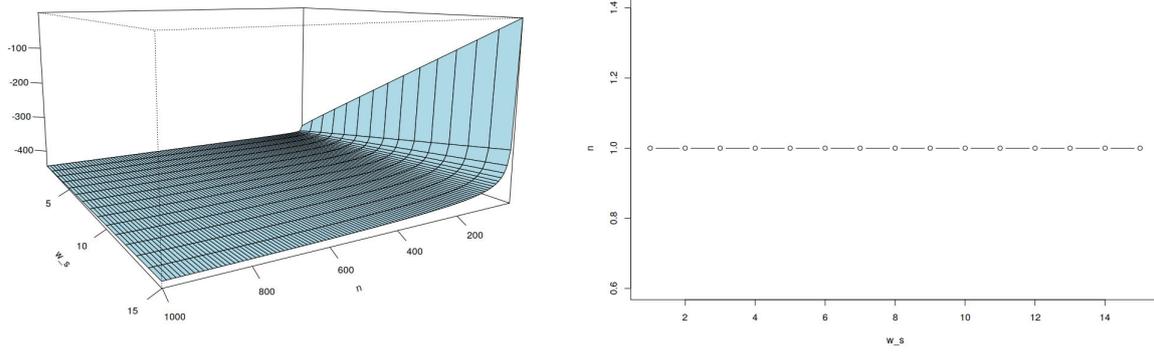


Figure 4.4 – Representation of both utility function $U_{tr}(n)$ (on left side), and n^* (on right side) according to the variation of parameter ω_f ($\omega_f = \{10, 11, \dots, 20\}$).



(a) $\omega_s = \{1, 2, \dots, 15\}$

Figure 4.5 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter ω_s ($\omega_s = \{1, 2, \dots, 15\}$).

Sensitivity analysis to ω_f : We can observe on figure 4.4 that utility is a strictly decreasing function of n for all tested values of ω_f . As $\alpha \neq 1$, the spot price is biased and an increase in the n.o.a increases the spot price bias in n.o.t. An increase in the n.o.a can secondly lead to an increase in the volatility of occurrence of synchronized transactions. Lastly, the increase in the n.o.a always increases the fundamental spot price volatility (cf. theorem 17).

Within our set of parameters' default values, an entrance of any n.o.a leads to a decrease in the utility. In other words, the policy maker utility is maximized with a minimal n.o.a. Then, the optimal n.o.a is minimal ($n^* = 1$) and not sensitive to the value of ω_f within our set of parameters' values. In this case, we get:

$$\max_n U_{t_r}(n) = U_{t_r}(1) = -\frac{|-e|}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k - \Theta t_r \left[\frac{\sigma_1^2}{\omega_f} \frac{1}{2} + a^2 (T_r - t_r)^2 \omega_s^{1-2\epsilon} \left(1 - \frac{\omega_s}{\omega_f}\right) \right]$$

Sensitivity analysis to ω_s : We can observe on figure 4.5 that utility is a strictly decreasing function of n for all tested values of ω_s . As $\alpha \neq 1$, the spot price is biased and an increase in the n.o.a increases the spot price bias in n.o.t. An increase in the n.o.a can secondly lead to an increase in the volatility of occurrence of synchronized transactions. Lastly, the increase in the n.o.a always increases the fundamental spot price volatility (cf. theorem 17).

Within our set of parameters' default values, an entrance of any n.o.a leads to a decrease in the utility. In other words, the policy maker utility is maximized with a minimal n.o.a. Then, the optimal n.o.a is minimal ($n^* = 1$) and not sensitive to the value of ω_s within our set of parameters' values. In this case, we get:

$$\max_n U_{t_r}(n) = U_{t_r}(1) = -\frac{|-e|}{\omega_f} \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k - \Theta t_r \left[\frac{\sigma_1^2}{\omega_f} \frac{1}{2} + a^2 (T_r - t_r)^2 \omega_s^{1-2\epsilon} \left(1 - \frac{\omega_s}{\omega_f}\right) \right]$$

Sensitivity analysis to t_r : We can observe on figure 4.6 that utility is a strictly decreasing function of n for all tested values of t_r . Then, n^* is not sensitive to the clock time targeted by the policy maker t_r . The optimal n.o.a is always minimal ($n^* = 1$).

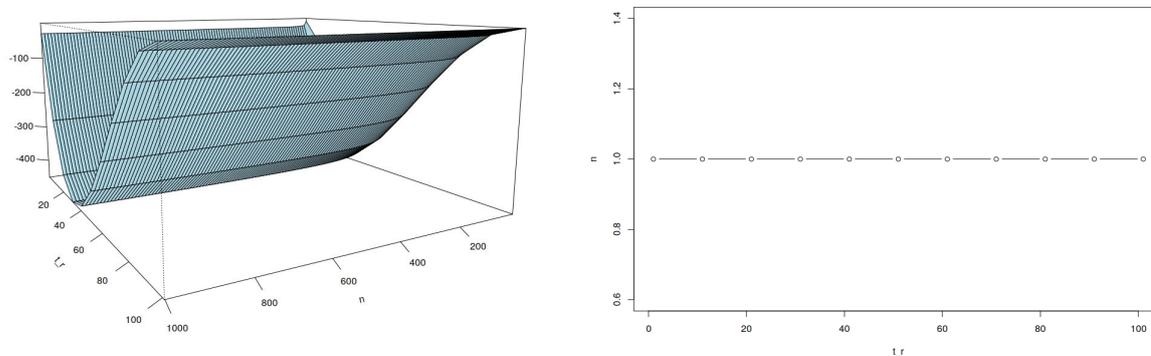


Figure 4.6 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter t ($t_r = \{1, 11, 21, \dots, 101\}$).

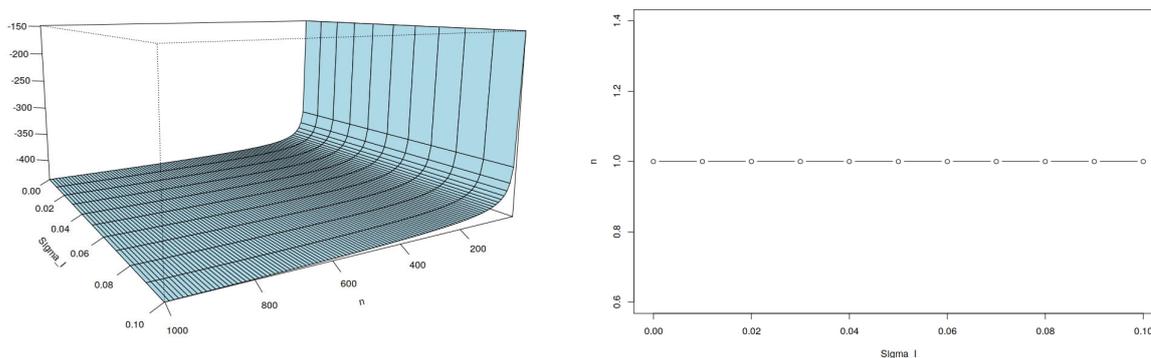


Figure 4.7 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter σ_I ($\sigma_I = \{0, 0.01, 0.02, \dots, 0.1\}$).

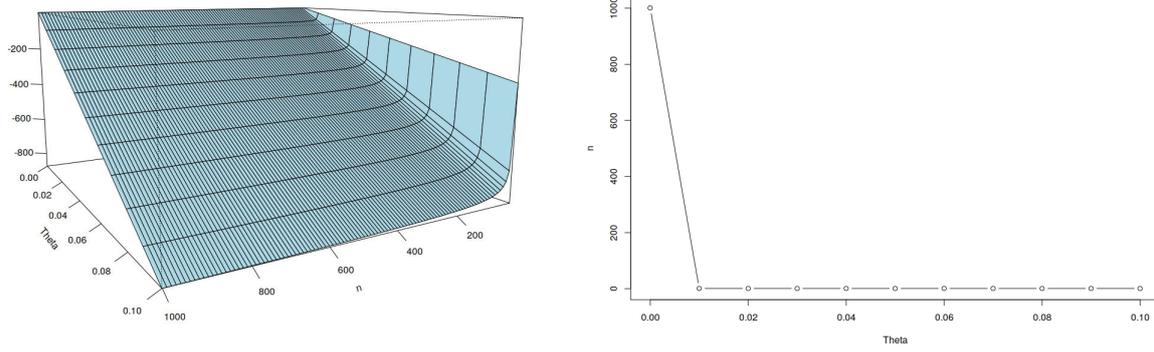


Figure 4.8 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter Θ ($\Theta = \{0, 0.01, 0.02, \dots, 0.1\}$).

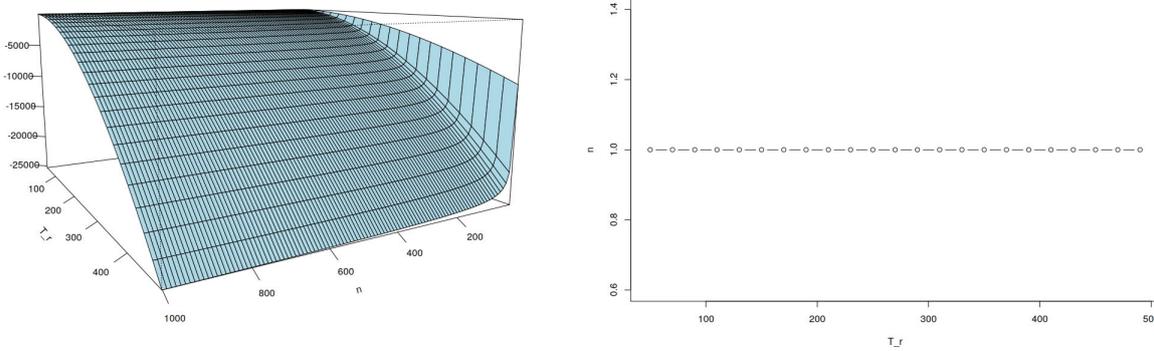


Figure 4.9 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter T_r ($T_r = \{50, 70, \dots, 500\}$).

Sensitivity analysis to σ_1 : We can observe on figure 4.7 that utility is a strictly decreasing function of n for all tested values of σ_1 , such that n^* is not sensitive to the maximal intrinsic volatility of the spot market information (per unit of real time), σ_1 , under our set of parameters' default values. Then, the optimal n.o.a is always minimal ($n^* = 1$).

Sensitivity analysis to Θ : We can observe on figure 4.8 that utility is a strictly decreasing function of n for all tested values of $\Theta \neq 0$. If there is no risk aversion (i.e. $\Theta = 0$), the utility function fully depends on the spot price bias. From remark 23, an infinite n.o.a allows for a biased spot price by limit. Under our set of parameters' default values, the spot price bias is minimized with a maximal n.o.a, such that the optimal n.o.a is always minimal ($n^* = 1$), except if $\Theta = 0 \Rightarrow n^* \rightarrow +\infty$.

Sensitivity analysis to T_r : We can observe on figure 4.9 that utility is a strictly decreasing function of n for all tested values of T_r (cf. theorem 15, equation (4.33) allows to write $\frac{\partial \mathcal{V}(FV_{f_{T_r}} - FV_{s_{T_r}})}{\partial T_r} > 0$). As for previous sensitivity analysis, the optimal n.o.a is minimal ($n^* = 1$) within our subset of parameters default values.

Sensitivity analysis to ϵ : We can observe on figure 4.10 that utility is not a monotonous function of n for all tested values of ϵ . For low values of ϵ , an increase in the n.o.a insufficiently smooths quantities traded at each transaction on the spot market. Then, it does

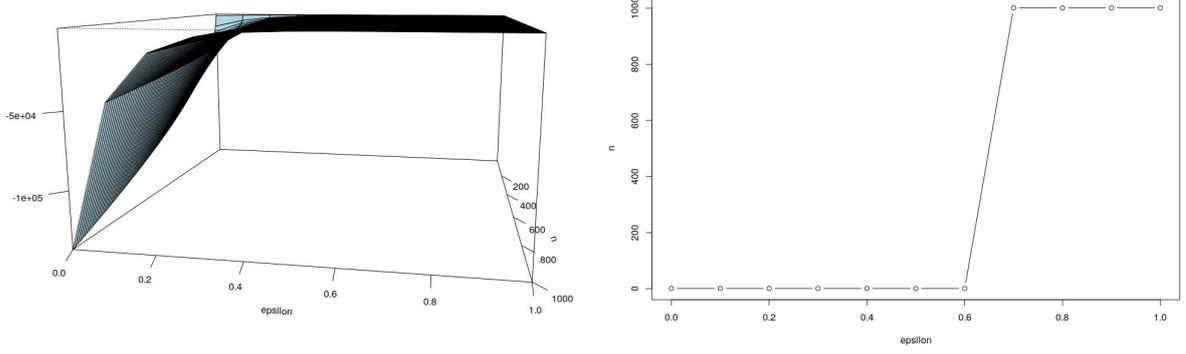


Figure 4.10 – Representation of both utility function $U_{tr}(n)$ (on left side), and n^* (on right side) according to the variation of parameter ϵ ($\epsilon = \{0, 0.1, 0.2, \dots, 1\}$).

not reduce sufficiently the fundamental basis volatility such that the optimal **n.o.a** is minimal ($n^* = 1$). However, as soon as ϵ is above a threshold, a maximal **n.o.a** allows for the fundamental basis volatility to tend to 0. All terms in equation (4.40) are strictly positive, such we can apply a strictly increasing function to this inequality. Isolating parameter ϵ we get the following threshold denoted by $\bar{\epsilon}$:

$$\bar{\epsilon} > \frac{1}{2} + \frac{1}{2\alpha} \left[\frac{\ln\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) - \ln\left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)}{\ln(n+1) - \ln(n)} \right] \quad (4.43)$$

This effect overcomes the increasing fundamental spot price volatility and the potential increase in the spot price bias, such that the optimal **n.o.a** strongly depends on value of parameter ϵ within the set of our parameters' default values.

Remark 27. Let g be a function such that:

$$g: \mathbb{F}_s \times \mathbb{F}_f \times]0, 1[\times \mathbb{N}^* \rightarrow \mathbb{R}$$

$$(\omega_s, \omega_f, \alpha, n) \mapsto g(\omega_s, \omega_f, \alpha, n) = \frac{1}{2} + \frac{1}{2\alpha} \left[\frac{\ln\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) - \ln\left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)}{\ln(n+1) - \ln(n)} \right]$$

It is easy to show that g is a $C1$ function. Then, we have:

$$\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial \omega_s} = \frac{1}{2\alpha(\ln(n+1) - \ln(n))} \left[\frac{(1 - \omega_s)}{\omega_f \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)} \frac{n^{1-\alpha} - (n+1)^{1-\alpha}}{(n(n+1))^{1-\alpha}} \right]$$

We have $\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial \omega_s} > 0 \forall \alpha \in]0, 1[$, and $\frac{\partial g(\omega_s, \omega_f, 1, n)}{\partial \omega_s} = 0$.

$$\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial \omega_f} = \frac{\omega_s}{2\alpha\omega_f(\ln(n+1) - \ln(n))} \times$$

$$\left[\frac{\left[\frac{1}{n^{1-\alpha}} - \frac{1}{(n+1)^{1-\alpha}} \right] + \frac{1}{\omega_f n^{2(1-\alpha)} (n+1)^{2(1-\alpha)}} \left[(n+1)^{1-\alpha} \left((n+1)^{1-\alpha} - \frac{\omega_s}{\omega_f} \right) - n^{1-\alpha} \left(n^{1-\alpha} - \frac{\omega_s}{\omega_f} \right) \right]}{\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)} \right]$$

We have $\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial \omega_f} > 0 \forall \alpha \in]0, 1[$, and $\frac{\partial g(\omega_s, \omega_f, 1, n)}{\partial \omega_f} = 0$.

$$\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial \alpha} = \frac{-1}{2\alpha^2} \left[\frac{\ln\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) - \ln\left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)}{\ln(n+1) - \ln(n)} \right] + \frac{1}{2\alpha} \left[\frac{-\frac{\omega_s \ln(n+1)}{\omega_f (n+1)^{1-\alpha} - \omega_s}}{\ln(n+1) - \ln(n)} \right]$$

We have $\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial \alpha} < 0$.

$$\begin{aligned} \frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial n} &= \frac{1}{2\alpha(\ln(n+1) - \ln(n))^2} \left[\left[\frac{\frac{1}{n^{2-\alpha}(n+1)^{2-\alpha}} \left[(n+1)^{2-\alpha} - n^{2-\alpha} - \frac{\omega_s}{\omega_f} \right]}{\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)} \right] (\ln(n+1) - \ln(n)) \right. \\ &\quad \left. - \left[\ln\left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right) - \ln\left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right) \right] \left(\frac{-1}{n(n+1)} \right) \right] \end{aligned}$$

We have $\frac{\partial g(\omega_s, \omega_f, \alpha, n)}{\partial n} > 0$.

Then, the minimal value of g is $g(\omega_s, \omega_f, 1, 1) = \frac{1}{2}$, and $g(\omega_s, \omega_f, \alpha, n) \geq \frac{1}{2}$.

From remark 27, the threshold $\bar{\epsilon}$ above which the fundamental basis volatility decreases is such that:

- (i) The higher the minimal trade frequency on the spot market ω_s is, the higher the elasticity of quantities traded to the spot market trades frequency (STF), ϵ , has to be to allow an entrance of – a unit of – agents to reduce the fundamental basis volatility;
- (ii) The higher the minimal trade frequency on the futures market ω_f is, the higher the elasticity of quantities traded to the STF, ϵ , has to be to allow an entrance of – a unit of – agents to reduce the fundamental basis volatility;
- (iii) The higher the elasticity of the STF to the n.o.a α is, the lower the elasticity of quantities traded to the STF, ϵ , has to be to allow an entrance of – a unit of – agents to reduce the fundamental basis volatility;
- (iv) The higher the n.o.a n is, the higher the elasticity of quantities traded to the STF, ϵ , has to be to allow an entrance of – a unit of – agents to reduce the fundamental basis volatility;

Finally, we find that the threshold $\bar{\epsilon}$ reaches its lower values when the relative trades frequency (RTF) is not sensitive to the n.o.a, and when the number of agent is minimal. Notice that it corresponds to the framework developed in previous chapter 3, such that by assuming that the RTF is not sensitive to the n.o.a, we underestimates the elasticity of quantities traded on the spot market to the STF necessary to allow a decrease in the fundamental basis volatility.

Sensitivity analysis to α : We can observe on figure 4.11 that utility is a strictly decreasing function of n for all tested values of $\alpha \neq 1$. If the RTF is not sensitive to the n.o.a (i.e. $\alpha = 1$), the spot price bias is constant in n.o.t and the volatility of occurrence of synchronized transaction is unchanged. From remark 23 and first assertion of theorem 17, an infinite n.o.a leads to an unbiased spot price, a maximal fundamental spot price volatility by limit, and a minimal basis volatility by limit (quantities traded per transaction on the spot market tend to 0 by limit). Hence, according to the risk aversion of the policy maker

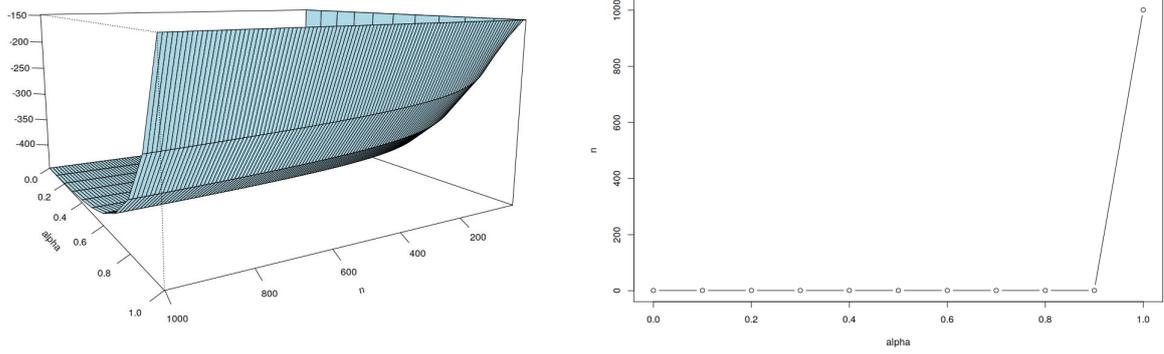


Figure 4.11 – Representation of both utility function $U_{tr}(n)$ (on left side), and n^* (on right side) according to the variation of parameter α ($\alpha = \{0, 0.1, 0.2, \dots, 1\}$).

⊕, the increase in the fundamental spot price volatility is compensated by the decrease in the spot price bias and in the fundamental basis volatility, within our set of parameters' default values. The optimal **n.o.a** is always minimal ($n^* = 1$), except if $\alpha = 1 \Rightarrow n^* \rightarrow +\infty$.

We introduced in this section a parameter of risk aversion for the policy maker such that a possible trade-off appears between the market efficiency objective, and the minimization of fundamental values volatility. We only got analytical solutions when the policy maker has no risk aversion. However, we conduct some sensitivity tests to study the degree of this trade-off through the optimal **n.o.a** on the market that the policy maker has to target. The optimal **n.o.a** is particularly sensitive to the risk aversion of the policy maker. As the entrance of agents can increase both spot and basis fundamental values volatility (and then even more the futures fundamental value volatility which is the sum of the two), the trade-off between the market efficiency and the fundamental values volatility influences the optimal **n.o.a** and decreases the optimal **n.o.a** on the market. We also point at the importance of the elasticity of the quantities traded on the spot market to the **STF** in determining the optimal **n.o.a**. Above a threshold, we find that the optimal **n.o.a** pass from the minimal one ($n^* = 1$), to the maximal one ($n^* = 1001$ within our sensitivity tests). Lastly, if the **RTF** is not sensitive to the **n.o.a** (i.e. if $\alpha = 1$), the optimal **n.o.a** is also the maximal one. If $\alpha \in]0, 1[$, the **RTF** converges to 0 and the market efficiency decreases with an increasing **n.o.a**, such that $n^* = 1$.

4.7 Introducing biased expectations

In this section, we consider that agents are neither able to perfectly process the spot market information, nor to expect it. We introduce a parameter ξ measuring the interpretation's bias and expectation's bias such that:

$$E(\gamma_n) := 1 + \frac{\xi}{n}, \xi \in]-1, 1[\quad (4.44)$$

The more agents there are on the market, the more the bias of interpretation $\frac{\xi}{n}$ tends to 0 (i.e. $\lim_{n \rightarrow +\infty} \frac{\xi}{n} = 0$), and so increases the capability of agents and by extension the market to process spot market information and discover its fundamental price impact.

Under this assumption, theorems 14 and 15 hold, as well as theorems 17 and 18. We present in subsection 4.7.1 the impact of biased expectations on the spot market effi-

ciency, the futures market efficiency, and the basis efficiency. We study in subsection 4.7.2, the impact of an entrance of – a unit of – agents and how these impacts differ when considering an expectation bias. The complexity of the problem prevents us from having analytical results. Finally, we present some sensitivity tests to the parameters of the model in subsection 4.7.3.

4.7.1 The impact of the number of agents

Theorem 20. *Let SM and FM respectively a spot market and a futures market under our hypothesis, be now submitted to agents biased expectations.*

(i) *The price on SM is biased such that:*

$$B(s_t) = \frac{|-e|}{\omega_f n} \left| \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right| \quad (4.45)$$

$$B(s_{t_r}) = \frac{|-e|}{\omega_f n} \left| \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t_r \omega_f \xi \right| \quad (4.46)$$

This bias is null if one of the three following subsets of conditions is verified:

(a) *For $e = 0$, $B(s_t) = B(s_{t_r}) = 0$;*

(b) *For $\xi = 0$, $\omega_s = \omega_f$ and $\alpha = 1$, $B(s_t) = B(s_{t_r}) = 0$;*

(c) *For $n \in \mathbb{O}_B^{s_t} = \left\{ (n, \xi) \in \mathbb{N}^* \times]0, 1[, n = \left\lfloor \xi \left(\frac{t}{\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k} - 1 \right) \right\rfloor \right\}$, $B(s_t) = 0$, where $\lfloor \cdot \rfloor$ denote the integer part.*

For $n \in \mathbb{O}_B^{s_{t_r}} = \left\{ (n, \xi) \in \mathbb{N}^ \times]0, 1[, n = \left\lfloor \xi \left(\frac{t_r \omega_f n}{\sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k} - 1 \right) \right\rfloor \right\}$, $B(s_{t_r}) = 0$.*

(ii) *The price on FM is biased such that:*

$$B(f_t) = t \frac{|e\xi|}{\omega_f n^2} \quad (4.47)$$

$$B(f_{t_r}) = t_r \frac{|e\xi|}{n} \quad (4.48)$$

This bias is null if one of the two following subsets of conditions is verified:

(a) *For $e = 0$, $B(f_t) = B(f_{t_r}) = 0$;*

(b) *For $\xi = 0$, $B(f_t) = B(f_{t_r}) = 0$.*

(iii) *The basis is biased such that:*

$$B(f_t - s_t) = \frac{|e|}{\omega_f n} \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \quad (4.49)$$

$$B(f_{t_r} - s_{t_r}) = \frac{|e|}{\omega_f n} \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \quad (4.50)$$

This bias is null if one of the two following subsets of conditions is verified:

- (a) For $e = 0$, $B(f_t - s_t) = B(f_{t_r} - s_{t_r}) = 0$;
 (b) For $\xi = 0$, $\omega_s = \omega_f$ and $\alpha = 1$, $B(f_t - s_t) = B(f_{t_r} - s_{t_r}) = 0$.

Proof. Proceeding identically as for theorem 12's proof, we have:

$$\begin{aligned} E(s_t) &= E\left[s_0 + \gamma_n \times \sum_{k=1}^t \mathbf{1}_{\{k\}}(k') \left[I_{|(k-1)', k-1|} + \frac{e}{\omega_f n} \right]\right] \\ \Leftrightarrow E(s_t) &= s_0 + \left(1 + \frac{\xi}{n}\right) \frac{\omega_s}{\omega_f} n^{\alpha-1} \frac{e}{\omega_f n} \sum_{k=1}^t (k - E((k-1)')) \end{aligned}$$

Using the expression of $E((k-1)')$ given in equation (4.2), we get:

$$E(s_t - \text{FV}_{s_t}) = \frac{-e}{\omega_f n} \left[\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right]$$

Then, we compute the absolute value to express the spot price bias. It allows us to get equations (4.45) and (4.46), according to the time line considered. Hence, both equations (4.45) and (4.46) are equal to zero if and only if one out of the two terms is equal to zero:

- (i) It is immediate that $e = 0$ is the only condition ensuring that the first term is equal to zero for both $B(s_t)$ and $B(s_{t_r})$;
- (ii) It is also immediate that $\xi = 0$, $\omega_s = \omega_f$ and $\alpha = 1$ is a condition ensuring that the second term is equal to zero for both $B(s_t)$ and $B(s_{t_r})$;
- (iii) Finally, equalizing the second term to zero for both $B(s_t)$ and $B(s_{t_r})$ gives us a condition on n that can be verified if and only if $\xi > 0$. It respectively leads to the subsets $\mathbb{O}_B^{s_t}$ and $\mathbb{O}_B^{s_{t_r}}$ for both $B(s_t)$ and $B(s_{t_r})$ according to the time line considered.

First assertion is proved.

Proceeding identically as for theorem 13's proof, we have $E(f_t - \text{FV}_{f_t}) = E[E(\tilde{s}_t | \Phi_{t-1}) - \text{FV}_{s_t}] - E[E(\widetilde{\text{CY}}_t | \Phi_{t-1}) - \text{CY}_t]$, and $E[E(\widetilde{\text{CY}}_t | \Phi_{t-1}) - \text{CY}_t] = 0$. To compute the term $E[E(\tilde{s}_t | \Phi_{t-1}) - \text{FV}_{s_t}]$, we proceed with a recurrence and recall that $s_0 = \text{FV}_{s_0}$. The property $P(t)$ is: $E[E(\tilde{s}_t | \Phi_{t-1}) - \text{FV}_{s_t}] = t \times \frac{e\xi}{\omega_f n} \forall t \in [1, T]$.

Initialization:

$$\begin{aligned} E[E(\tilde{s}_1 | \Phi_0) - \text{FV}_{s_1}] &= E\left[s_{0'} + \gamma_n I_{|0', 0|} + \gamma_n \frac{e}{\omega_f n} - (s_0 + I_{|0, 1|})\right] \\ \Leftrightarrow E[E(\tilde{s}_1 | \Phi_0) - \text{FV}_{s_1}] &= E\left[\gamma_n \frac{e}{\omega_f n} - I_{|0, 1|}\right] = \left(1 + \frac{\xi}{n}\right) \frac{e}{\omega_f n} - \frac{e}{\omega_f n} = 1 \times \frac{e\xi}{\omega_f n} \end{aligned}$$

$P(1)$ is true. We assume that $P(t)$ is true (i.e. $E\left[s_{(t-1)'} + \gamma_n I_{|(t-1)', t-1|} + \gamma_n \frac{e}{\omega_f n} - (s_0 + I_{|0, t|})\right] = t \times \frac{e\xi}{\omega_f n}$), we show that $P(t+1)$ is also true.

$$\begin{aligned} E[E(\widetilde{s}_{t+1} | \Phi_t) - \text{FV}_{s_{t+1}}] &= E\left[s_{(t-1)'} + \gamma_n \left(I_{|(t-1)', t-1|} + \frac{e}{\omega_f n}\right) - (s_0 + I_{|0, t|})\right] \\ &\quad + E\left[\gamma_n (\mathbf{1}_{\{t\}}(t') - 1) \left(I_{|(t-1)', t-1|} + \frac{e}{\omega_f n}\right) + \gamma_n \left(I_{|t', t|} + \frac{e}{\omega_f n}\right) - I_{|t, t+1|}\right] \end{aligned}$$

As we assumed that $P(t)$ is true, we get:

$$\begin{aligned} E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] &= E \left[\Upsilon_n \left(1_{\{t\}}(t') - 1 \right) \left(I_{|(t-1)', t-1]} + \frac{e}{\omega_f n} \right) + \Upsilon_n \left(I_{|t', t]} + \frac{e}{\omega_f n} \right) - I_{|t, t+1]} \right. \\ &\quad \left. + t \times \frac{e\xi}{\omega_f n} \right] \\ \Leftrightarrow E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] &= \left(1 + \frac{\xi}{n} \right) E \left[\left(1_{\{t\}}(t') - 1 \right) \left(I_{|(t-1)', t-1]} + \frac{e}{\omega_f n} \right) + I_{|t', t]} + \frac{e}{\omega_f n} \right] \\ &\quad - E \left[I_{|t, t+1]} \right] + t \times \frac{e\xi}{\omega_f n} \\ \Leftrightarrow E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] &= \frac{e}{\omega_f n} \left(1 + \frac{\xi}{n} \right) E \left[\left(1_{\{t\}}(t') - 1 \right) (t-1 - (t-1)' + 1) + t - t' + 1 \right] \\ &\quad - \frac{e}{\omega_f n} (t+1 - t) + t \times \frac{e\xi}{\omega_f n} \\ \Leftrightarrow E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] &= \frac{e}{\omega_f n} \left(1 + \frac{\xi}{n} \right) E \left[\left(1_{\{t\}}(t') - 1 \right) (t - (t-1)') + t - t' + 1 \right] \\ &\quad - \frac{e}{\omega_f n} + t \times \frac{e\xi}{\omega_f n} \\ \Leftrightarrow E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] &= \frac{e}{\omega_f n} \left(1 + \frac{\xi}{n} \right) \left[\left(\frac{\omega_s}{\omega_f n^{1-\alpha}} - 1 \right) (t - E[(t-1)']) + t - E[t'] + 1 \right] \\ &\quad - \frac{e}{\omega_f n} + t \times \frac{e\xi}{\omega_f n} \\ \Leftrightarrow E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] &= \frac{e}{\omega_f n} \left(1 + \frac{\xi}{n} \right) \left[t \frac{\omega_s}{\omega_f n^{1-\alpha}} + \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right) E[(t-1)'] - E[t'] + 1 \right] \\ &\quad - \frac{e}{\omega_f n} + t \times \frac{e\xi}{\omega_f n} \end{aligned}$$

Let $x = \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right) E[(t-1)'] - E[t']$, we have:

$$\begin{aligned} x &= \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right) \left[t - 1 - \sum_{k=1}^{t-1} \left(1 - \frac{\omega_s}{\omega_f} n^{\alpha-1} \right)^k \right] - t - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} n^{\alpha-1} \right)^k \\ \Leftrightarrow x &= -t \frac{\omega_s}{\omega_f n^{1-\alpha}} - \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right) - \sum_{k=1}^{t-1} \left(1 - \frac{\omega_s}{\omega_f} n^{\alpha-1} \right)^{k+1} - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} n^{\alpha-1} \right)^k \\ \Leftrightarrow x &= -t \frac{\omega_s}{\omega_f n^{1-\alpha}} - \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}} \right) - \sum_{k=2}^t \left(1 - \frac{\omega_s}{\omega_f} n^{\alpha-1} \right)^k - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f} n^{\alpha-1} \right)^k \\ \Leftrightarrow x &= -t \frac{\omega_s}{\omega_f n^{1-\alpha}} \end{aligned}$$

Then:

$$E \left[E(\widetilde{s}_{t+1} | \Phi_t) - FV_{s_{t+1}} \right] = (t+1) \times \frac{e\xi}{\omega_f n}$$

If $P(t)$ is true, $P(t+1)$ is also true. As, $P(1)$ is true, $P(t)$ is true for all $t \in [1, T]$.

Then, $E(f_t - FV_{f_t}) = 0 + t \times \frac{e\xi}{\omega_f n} = t \times \frac{e\xi}{\omega_f n}$. Computing the absolute value and expressing it in both time lines leads to equations (4.47) and (4.48). Hence, both equations (4.47) and (4.48) are equal to zero if and only if one of the following conditions is verified:

- (i) It is immediate that $e = 0$ is a condition ensuring that $B(f_t) = B(f_{t_r}) = 0$;
- (ii) It is also immediate that $\xi = 0$, is the only other condition ensuring that $B(f_t) = B(f_{t_r}) = 0$.

Second assertion is proved.

Recall that $E[f_t - s_t - (FV_{f_t} - FV_{s_t})] = E[f_t - FV_{f_t}] - E[s_t - FV_{s_t}]$. Using first and second assertions, we get:

$$\begin{aligned} E[f_t - FV_{f_t}] - E[s_t - FV_{s_t}] &= t \frac{e\xi}{\omega_f n^2} - \frac{-e}{\omega_f n} \left[\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right] \\ \Leftrightarrow E[f_t - FV_{f_t}] - E[s_t - FV_{s_t}] &= \frac{e}{\omega_f n} \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \end{aligned}$$

Then, we compute the absolute value to express the basis bias. We get equations (4.49) and (4.50), according to the time line. Hence, both equations (4.49) and (4.50) are equal to zero if and only if one out of the two terms are equal to zero:

- (i) It is immediate that $e = 0$ is the only condition ensuring that the first term is equal to zero for both $B(f_t - s_t)$ and $B(f_{t_r} - s_{t_r})$;
- (ii) It is also immediate that $\xi = 0$, $\omega_s = \omega_f$ and $\alpha = 1$ is the only condition ensuring that the second term is equal to zero for both $B(f_t - s_t)$ and $B(f_{t_r} - s_{t_r})$.

Last assertion is proved □

When $e \neq 0$, the number of agents (**n.o.a**) can generate a bias on the spot price dynamics that results in two possible effects. An adjustment delay if $\omega_s \neq \omega_f$ and $\alpha \neq 1$. Spot market information is released between two transactions on the futures market and can not be included in the spot price by agents since the spot market trades frequency (**STF**) is lower on probability. The condition $\alpha \neq 1$ prevents the **n.o.a** from having a proportional impact on both **STF** and futures market trades frequency (**FTF**) and so, generates a variation of the relative trades frequency (**RTF**). The second effect is linked to biased expectations on information by agents if $\xi \neq 0$.

Remark 28. For $\xi = 0$, $\omega_s = \omega_f$ and $\alpha = 1$, $B(s_t) = B(s_{t_r}) = 0$. This condition can be decomposed into two subsets of conditions. $\xi = 0$ can be understood as an interpretation subset of conditions. If agents have biased interpretations of the information, it obviously can lead to a biased spot price. $\omega_s = \omega_f$ and $\alpha = 1$ can be understood as a frequency subset of conditions. If the **RTF** is different from 1, it leads to a biased spot price. If these two conditions are simultaneously verified, it ensures an unbiased spot price.

Remark 29. When agents underestimate the impact of spot market information on the price (i.e. $-1 < \xi < 0$), there is no **n.o.a** allowing either $B(s_t) = 0$ or $B(s_{t_r}) = 0$. However, when agents overestimate the impact of spot market information on the price (i.e. $0 < \xi < 1$), it exists a subset of **n.o.a** respectively $\mathbb{O}_B^{s_t}$ and $\mathbb{O}_B^{s_{t_r}}$ allowing $B(s_t) = 0$ and $B(s_{t_r}) = 0$. Furthermore, this optimal **n.o.a** is an increasing function of agents' overestimation.¹⁰

The latest remark has strong implications. Overestimating the impact of the spot market information can be compensated by an average delay between two transactions on the spot relatively to transactions on the futures market. In other words, this delay allows for a variation of the spot fundamental value such that the spread between the spot price and the fundamental price on average is originally positive but becomes negative such that the average spread is null.

$$^{10}\text{We have } \left. \frac{\partial \xi \left(\frac{t}{\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k} - 1 \right)}{\partial \xi} \right|_{0 < \xi < 1} > 0 \text{ and } \left. \frac{\partial \xi \left(\frac{t_r \omega_f n}{\sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k} - 1 \right)}{\partial \xi} \right|_{0 < \xi < 1} > 0$$

4.7.2 The impact of an entrance of agents on the market

Theorem 21. SM and FM have the following properties:

- (i) Within the transactions time line (TTL) (t), an entrance of more agents on the markets strictly decreases the spot price bias if and only if:

$$\frac{n+1}{n} > \frac{\left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t \frac{\xi}{n+1}}{\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n}}, n \notin \mathbb{O}_B^{st} \quad (4.51)$$

- (ii) Within the CTL (t_r), an entrance of more agents on the markets strictly decreases the spot price bias if and only if:

$$\frac{n+1}{n} > \frac{\left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t_r \omega_f \xi}{\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^{t_r \omega_f n} \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t_r \omega_f \xi}}, n \notin \mathbb{O}_B^{st_r} \quad (4.52)$$

- (iii) Within the TTL (t), an entrance of more agents on the markets strictly decreases the futures price bias;
- (iv) Within the CTL (t_r), an entrance of more agents on the markets strictly decreases the futures price bias.

Proof. We calculate the simple difference of the spot price bias according to n , and we study its sign. Using equation (4.45), and factorizing by $\frac{|-e|}{\omega_f} > 0$ we have:

$$\begin{aligned} B(s_t) |_{n+1} - B(s_t) |_n &= \frac{1}{n+1} \left| \left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t \frac{\xi}{n+1} \right| \\ &\quad - \frac{1}{n} \left| \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right| \end{aligned}$$

Putting to the same denominator and factorizing by $\frac{1}{n(n+1)} > 0$, we get:

$$\begin{aligned} B(s_t) |_{n+1} - B(s_t) |_n &= n \left| \left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t \frac{\xi}{n+1} \right| \\ &\quad - (n+1) \left| \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right| \end{aligned}$$

If $\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} = 0 \Rightarrow n \in \mathbb{O}_B^{st}$, more agents cannot strictly diminish the spot price bias.

This difference is strictly negative if and only if $n \notin \mathbb{O}_B^{st}$ and:

$$\begin{aligned} \left| \left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t \frac{\xi}{n+1} \right| &< \frac{n+1}{n} \left| \left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right| \\ \Leftrightarrow \left| \frac{\left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t \frac{\xi}{n+1}}{\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n}} \right| &< \frac{n+1}{n} \end{aligned}$$

Let f be a function such that:

$$f:]-1, 1[\rightarrow \mathbb{R}$$

$$\xi \mapsto f(\xi) = \frac{\left(1 + \frac{\xi}{n+1}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - t \frac{\xi}{n+1}}{\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n}}$$

It is easy to show that f is a C1 function. Then, we have:

$$\frac{\partial f(\xi)}{\partial \xi} = \frac{\frac{n}{n+1} \left[\sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \right]}{\left[\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right]^2} + \frac{\frac{n}{n+1} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k \left[t - \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k \right]}{\left[\left(1 + \frac{\xi}{n}\right) \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k - t \frac{\xi}{n} \right]^2}$$

It is easy to show that $\frac{\partial f(\xi)}{\partial \xi} > 0 \forall \xi \in]-1, 1[$. Hence, the minimal value of f is given by $\lim_{\substack{\xi \rightarrow -1 \\ x > -1}} f(\xi)$ and we have:

$$\lim_{\substack{\xi \rightarrow -1 \\ x > -1}} f(\xi) = \frac{\frac{n}{n+1} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f (n+1)^{1-\alpha}}\right)^k + \frac{t}{n+1}}{\frac{n-1}{n} \sum_{k=1}^t \left(1 - \frac{\omega_s}{\omega_f n^{1-\alpha}}\right)^k + \frac{t}{n}} > 0$$

Then, the previous inequality can be rewritten to get the condition given by equation (4.51). It proves the first assertion.

We apply the same methodology to prove the second assertion and get condition (4.52).

To prove the third assertion, we compute the simple difference of the futures price bias according to n , and we study its sign. Using equation (4.47), and factorizing by $t \frac{|e\xi|}{\omega_f} > 0$ we get:

$$B(f_t) |_{n+1} - B(f_t) |_n = \frac{1}{(n+1)^2} - \frac{1}{n^2} = \frac{n^2 - (n+1)^2}{(n(n+1))^2} < 0$$

Third assertion is proved.

We apply the same methodology to prove the last assertion. □

First and second assertions of theorem 21, highlight that an entrance of agents can (but not necessarily) reduce the spot price bias either within the TTL or within the CTL.

Remark 30. *One could think that underestimating or overestimating the fundamental price impact of spot market information – in the same proportion – would have the same effect on the spot price bias. However, it is not equivalent.*

Within the *TTL*, the more agents underestimate the fundamental price impact of spot market information, the more an entrance of agents can reduce the spot price bias (i.e. the less restrictive equation (4.51) becomes, since $\frac{\partial f(\xi)}{\partial \xi} > 0 \forall \xi \in]-1, 1[$). On the contrary, the more agents overestimate the fundamental price impact of spot market information, the less an entrance of agents can reduce the spot price bias (i.e. the more restrictive equation (4.51) becomes). Hence, if agents underestimate on average the price impact of spot market information, the condition is less restrictive than when agents correctly interpret on average the price impact of spot market information. It is the opposite for overestimating agents.

The same reasoning applies within the *CTL*.

This is a major result underlying to what extent the expectations bias influences the impact of the entrance of more agents and then, later, the optimal policy for the policy maker regarding its objectives for agricultural commodity markets.

Conditions (4.51) and (4.52) can be interpreted in the same way than (4.34) and (4.35). The left term represents the multiplier of the *FTF*. However, the right term is different within both time lines (except if $\xi = 0$) and represents the multiplier of the average delay between two transactions on the spot market according to the frequency on the futures market added to the spot price bias generated by biased agents expectations. The spot price bias reduction induced by the *CTL* effect (multiplier of the *FTF*) must be higher than the increase in this bias induced by the increasing average delay corrected by the reduction of the agents expectations bias.

Third and fourth assertions of theorem 21 highlight that independently of market parameters and the time line, an entrance of agents on the market allows for the futures price bias to decrease. This result comes with the fact that futures price bias is due to biased expectations only. By assumption (equation (4.44)), this bias decreases with the *n.o.a.*

Finally, remark 26 also applies, such that considering speculation on the spot market prevents us from having an analytical optimal *n.o.a.* A calibration of the model also must be done in the following case, and simulations can be made if some parameters are assumed to be stochastic such as the spot market information or quantities traded per transaction on the spot market.

4.7.3 On the existence of an optimal number of agents under expectation bias hypothesis

The objectives of a policymaker on commodity markets are again to minimize the fundamental values volatility and increase market efficiency on both spot and futures markets. As the futures price is biased (cf. theorem 20), the policymaker focuses on both spot market efficiency and basis efficiency. It is not equivalent to focus on the futures market efficiency alone.¹¹ We assume that the risk aversion of the policy maker is the same for both fundamental spot price and basis volatility, such that it is equivalent to consider the futures fundamental volatility (i.e. $\mathcal{V}(FV_{f_l}) = \mathcal{V}(FV_{s_l}) + \mathcal{V}(CY_l)$, $l \in \{t, t_r\}$). We consider here that the *n.o.a.* on the market is the policymaker's tool, and thus the policymaker can implement a limit in the *n.o.a.* or not (free entrance). As in previous section 4.6, we focus on the clock time line (*CTL*), as it is the time line considered by the policy maker.

¹¹From properties of the absolute value, we have $B(f_t) \leq B(s_t) + B(f_t - s_t)$.

Definition 25. We define a mean-variance utility function U'_{t_r} (following Markowitz [12], Von Neumann et al. [21]) such that:

$$U'_{t_r} : \mathbb{N}^* \xrightarrow{U'_1} \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{U'_2} \mathbb{R}^-$$

$$n \longmapsto (B(s_{t_r}), B(f_{t_r} - s_{t_r}), \mathcal{V}(FV_{f_{t_r}})) \longmapsto U'_2(.,.,.) = - [B(s_{t_r}) + B(f_{t_r} - s_{t_r})] - \Theta [\mathcal{V}(FV_{f_{t_r}})] \quad (4.53)$$

Hence, we have $U'_{t_r}(n) = U'_2 \circ U'_1(n)$, verifying the two following properties:

(i) $U'_2(0, 0, 0) = 0$;

(ii) $\frac{\partial U'_2(B(s_{t_r}), B(f_{t_r} - s_{t_r}), \mathcal{V}(FV_{f_{t_r}}))}{\partial B(s_{t_r})} < 0$, $\frac{\partial U'_2(B(s_{t_r}), B(f_{t_r} - s_{t_r}), \mathcal{V}(FV_{f_{t_r}}))}{\partial B(f_{t_r} - s_{t_r})} < 0$, and $\frac{\partial U'_2(B(s_{t_r}), B(f_{t_r} - s_{t_r}), \mathcal{V}(FV_{f_{t_r}}))}{\partial \mathcal{V}(FV_{f_{t_r}} - FV_{s_{t_r}})} < 0$.

We assume that a policymaker uses the utility function U'_{t_r} as defined above to maximize the agricultural commodity market efficiency and to minimize its fundamental basis volatility. Properties (i) and (ii) correspond respectively to the following natural insights:

- (i) U'_2 reaches its maximal value on \mathbb{R}^- if and only if the spot price bias, the basis bias and the fundamental futures volatility are null;¹²
- (ii) If either the spot price bias or the fundamental futures volatility increase, the policymaker utility decreases.

The complexity of the maximization does not allow us to extract analytical results. We provide some sensitivity analysis according to main parameters within the CTL. All mechanisms at stake within both time lines are presented in previous section 4.6. Then, we focus here on the impact of the bias in agents expectations, either underestimating the spot price information incoming (i.e. $\xi < 0$) or overestimating the spot price information incoming (i.e. $\xi > 0$).

The set of parameters default value used is given by table 4.4 within an underestimation bias, and by table 4.5 within an overestimation bias.

Parameter	e	a	ω_f	ω_s
Value	5	0,25	15	10

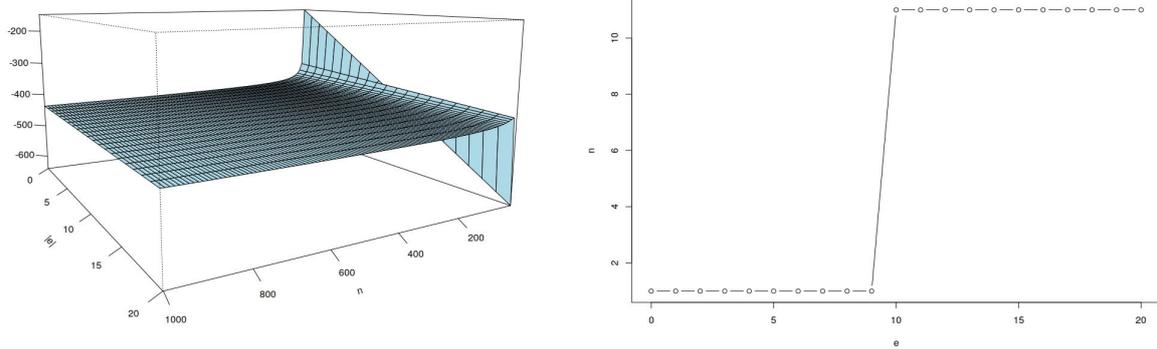
Parameter	t_r	σ_I	Θ	T_r	ϵ	α	ξ
Value	41	0,05	0,05	100	0,5	0,5	-0,6

Table 4.4 – Set of parameters default values used for the sensitivity tests within the CTL, under underestimation bias

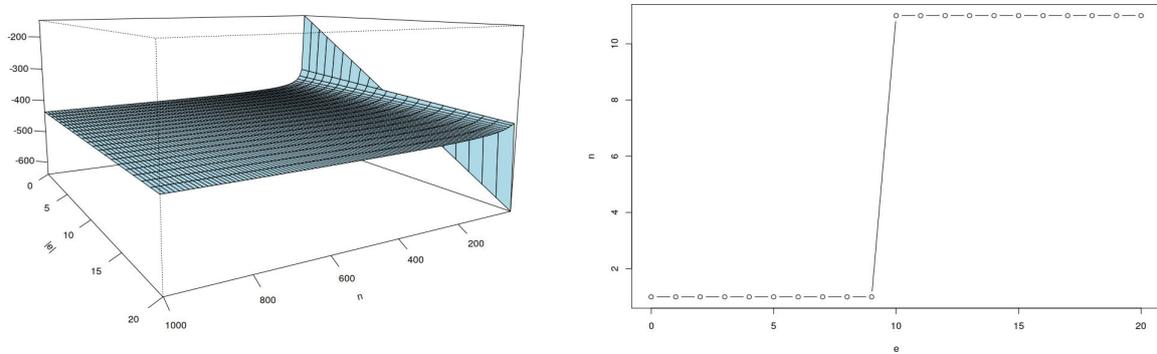
In order to have clear charts, we consider that $n^* \in \{1, 11, 21, \dots, 1001\}$.

Sensitivity analysis to e : We can observe on figure 4.12 that utility is not a monotonous function of n for all tested values of e . The potential reduction of both the spot price bias (cf. theorem 16) and the fundamental basis volatility (cf. theorem 18), and the reduction

¹²This utility function does not verify the axiom of non satiety. Indeed, if the market efficiency is maximized and the fundamental futures volatility is minimized, the utility cannot be improved.



(a) $\xi < 0$



(b) $\xi > 0$

Figure 4.12 – Representation of both utility function $U_{tr}(n)$ (on left side) and n^* (on right side) according to the variation of parameter e ($e = \{0, 1, \dots, 20\}$), with biased expectation.

Parameter	e	a	ω_f	ω_s			
Value	5	0,25	15	10			

Parameter	t_r	σ_I	Θ	T_r	ϵ	α	ξ
Value	41	0,05	0,05	100	0,5	0,5	0,6

Table 4.5 – Set of parameters default values used for the sensitivity tests within the CTL, under overestimation bias

of the fundamental spot price volatility (cf. theorem 17) depends on the n.o.a. In other words, the decrease in the fundamental spot price volatility is not necessarily more than compensated by the evolution in the two other elements. In the end, the spot price bias argument is more and more weighted with increasing values of e such that when e becomes too high, the optimal n.o.a is higher in order to reduce the spot price bias.

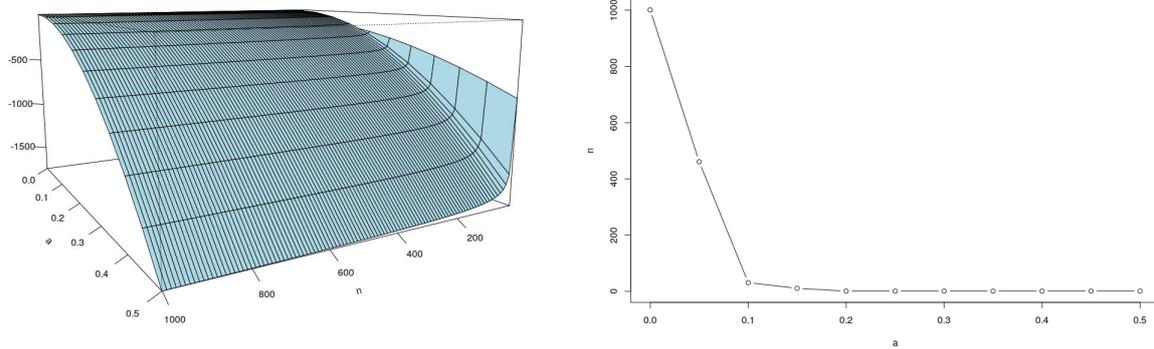
Compared to the no expectation bias situation (cf. figure 4.2), the presence of a bias increases the incentives of the policy maker to allow for the entrance of more agents on the market.

Sensitivity analysis to a : We can observe on figure 4.13 that utility is not a monotonous function of n for all tested values of $a \neq 0$ (contrary to the case when $\xi = 0$, cf. figure 4.3). If the advantage to possess a unit of the storable commodity per unit of clock time is not important and by extension, if the convenience yield is not important, i.e. for low values of a , the utility function depends on the spot price bias, the basis bias and on the fundamental spot price volatility. From remark 23 and first assertion of theorem 17, an infinite n.o.a leads to a biased spot price (but not necessarily maximal) and a maximal fundamental spot price volatility by limit. Hence, according to the risk aversion of the policy maker (i.e. the value of Θ), the increase in the fundamental spot price volatility is less important than the potential effect on the spot price bias (which is ambiguous from theorem 21). Within our set of parameters' default values, the entrance of – a unit of – agents leads to a decrease in the spot price bias. Then the optimal n.o.a is strictly decreasing with a , which increases the fundamental basis volatility.

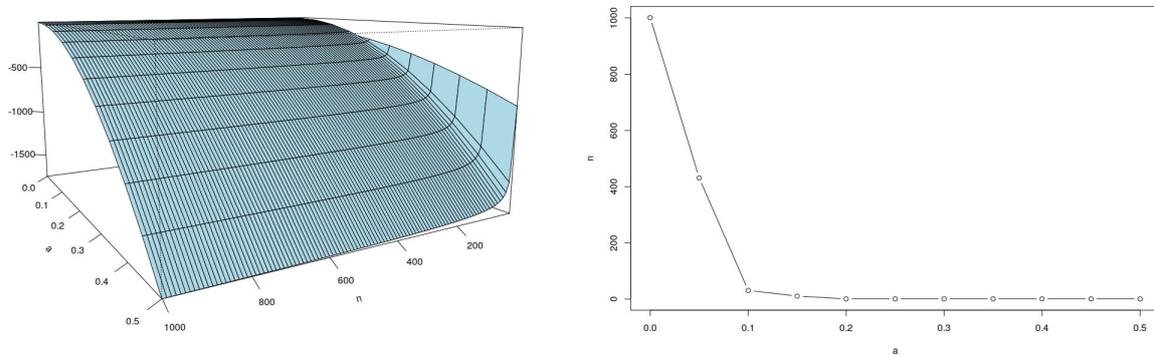
Compared to the no expectation bias situation (cf. figure 4.3), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the market, such that for low values of a , the optimal n.o.a on the market is higher when expectations are biased.

Sensitivity analysis to ω_f : We can observe on figure 4.14 that utility is a strictly decreasing function of n for all tested values of ω_f . As $\alpha \neq 1$, the spot price is biased and an increase in the n.o.a increases the spot price bias in n.o.t. An increase in the n.o.a can secondly lead to an increase in the volatility of occurrence of synchronized transactions. Lastly, the increase in the n.o.a always increases the fundamental spot price volatility (cf. theorem 17).

Within our set of parameters' default values, an entrance of any n.o.a leads to a decrease in the utility. In other words, the policy maker utility is maximized with a minimal n.o.a ($n^* = 1$) and not sensitive to the value of ω_f within our set of parameters' values. In this case, we get:

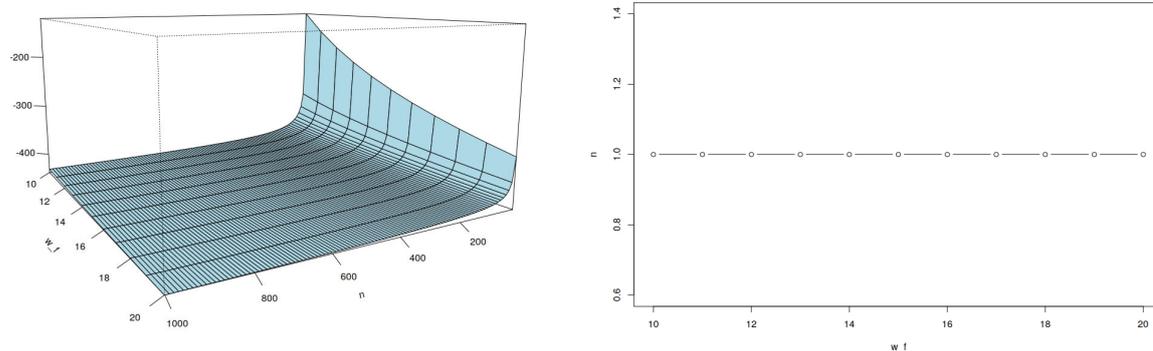


(a) $\xi < 0$

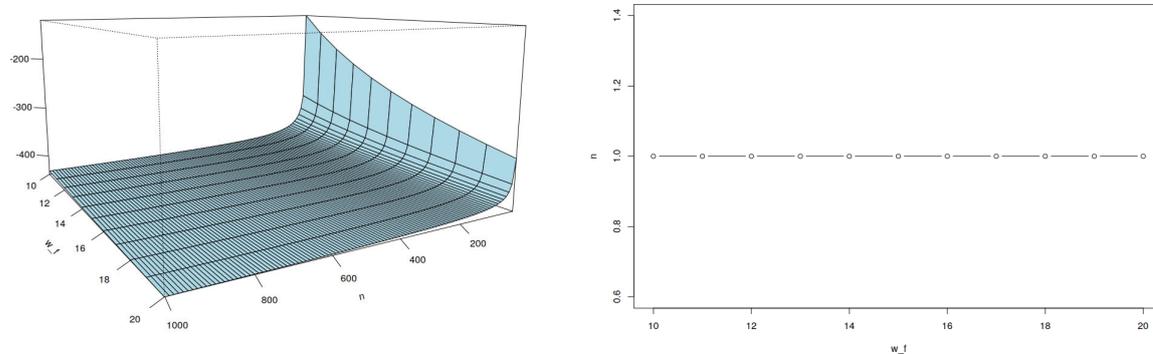


(b) $\xi > 0$

Figure 4.13 – Representation of both utility function $U_{tr}(n)$ (on left side), and n^* (on right side) according to the variation of parameter a ($a = \{0, 0.05, 0.1, \dots, 0.5\}$), with biased expectations.



(a) $\xi < 0$



(b) $\xi > 0$

Figure 4.14 – Representation of both utility function $U_t(n)$ (on left side), and n^* (on right side) according to the variation of parameter ω_f ($\omega_f = \{10, 11, \dots, 20\}$), with biased expectations.

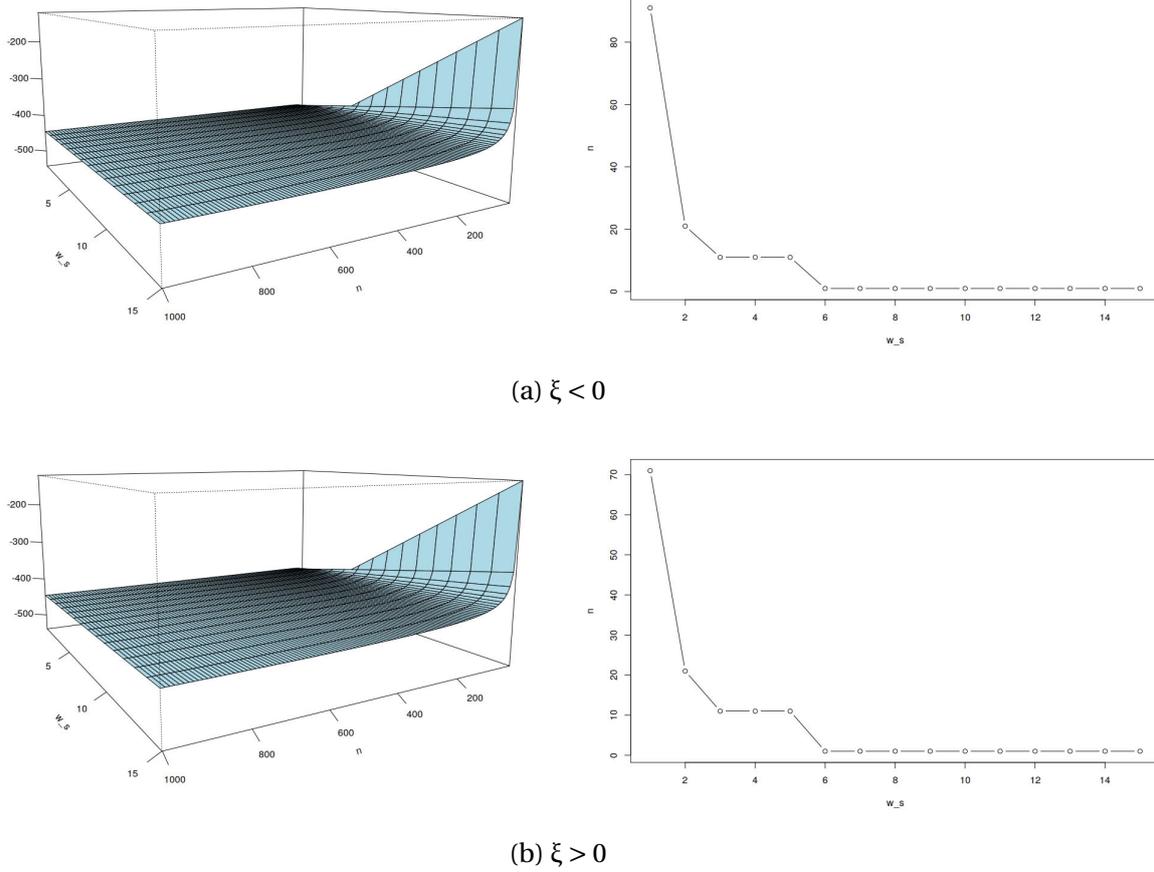


Figure 4.15 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of a parameter of the model (given in figure's subtitle), with biased expectations.

$$\max_n U'_{t_r}(n) = U'_{t_r}(1) = -\frac{|-e|}{\omega_f} \left[\left| (1 + \xi) \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k - t_r \omega_f \xi \right| + (1 + \xi) \sum_{k=1}^{t_r \omega_f} \left(1 - \frac{\omega_s}{\omega_f}\right)^k \right] - \Theta t_r \left[\frac{\sigma_f^2}{\omega_f} \frac{1}{2} + a^2 (T_r - t_r)^2 \omega_s^{1-2\epsilon} \left(1 - \frac{\omega_s}{\omega_f}\right) \right]$$

Compared to the no expectation bias situation (cf. figure 4.4), the presence of a bias does not sufficiently increases the incentives of the policy maker to allow for the entrance of agents on the market, such that the optimal **n.o.a** on the market is not sensitive to agents bias of expectations.

Sensitivity analysis to ω_s : We can observe on figure 4.15 that utility is not a monotonous function of n for all tested values of ω_s . As $\alpha \neq 1$, the spot price is biased and an increase in the **n.o.a** increases the spot price bias in **n.o.t**. An increase in the **n.o.a** can secondly lead to an increase in the volatility of occurrence of synchronized transactions. Lastly, the increase in the **n.o.a** always increases the fundamental spot price volatility (cf. theorem 17).

Within our set of parameters' default values, an entrance of agents has benefits when the intrinsic **RTF** ($\frac{\omega_s}{\omega_f}$) is originally low. In this case, an entrance of agents reduces the volatility of occurrence of synchronized transactions. In other words, the policy maker

utility is maximized with a **n.o.a** decreasing with the intrinsic **RTF**. Then, the optimal **n.o.a** strongly depends on ω_s within our set of parameters' default values.

Compared to the no expectation bias situation (cf. figure 4.5), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the market, such that for low values of ω_s , the optimal **n.o.a** on the market is higher when expectations are biased.

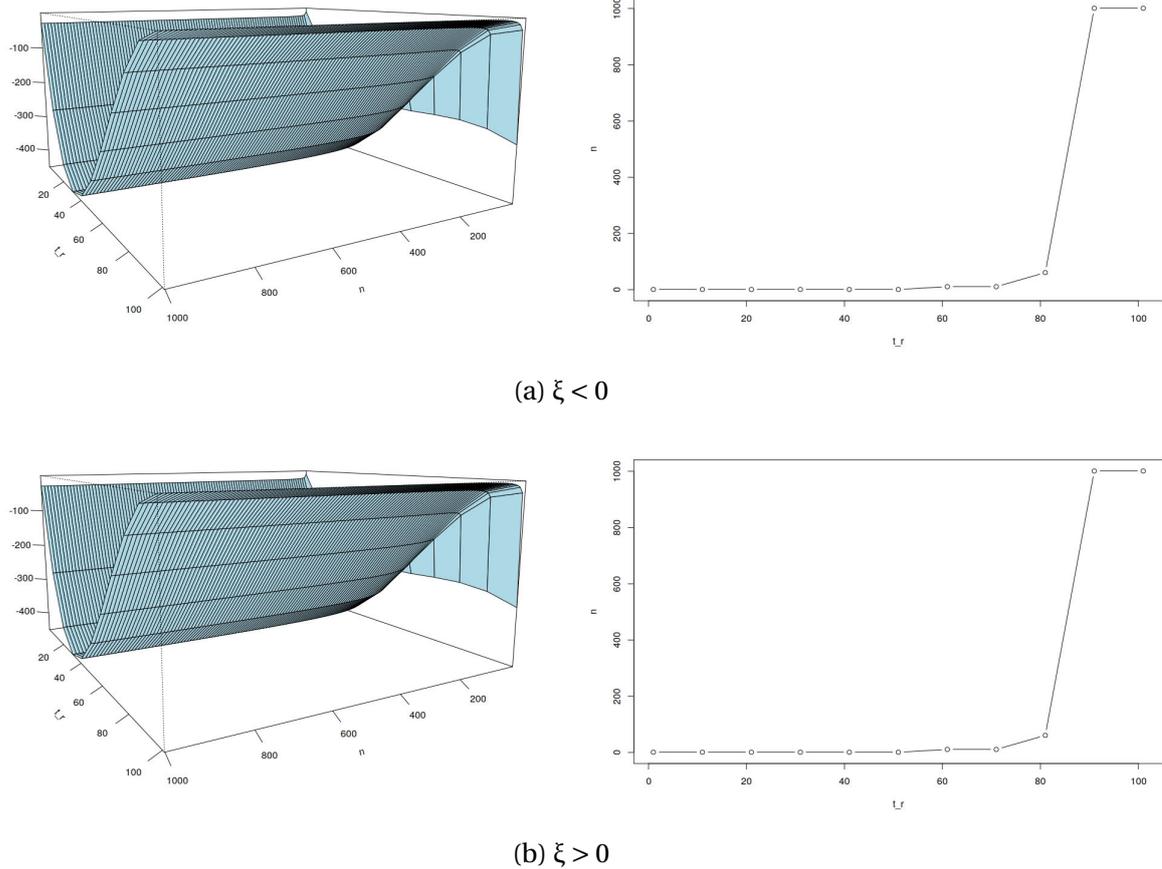


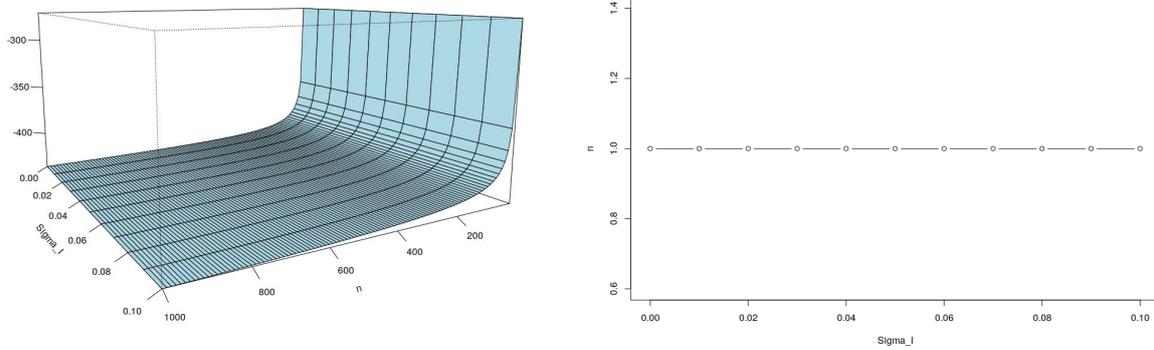
Figure 4.16 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter t_r ($t_r = \{1, 11, 21, \dots, 101\}$), with biased expectations.

Sensitivity analysis to t_r : We can observe on figure 4.16 that utility is not a monotonous function of n for all tested values of t_r . Then, n^* is sensitive to the clock time targeted by the policy maker t_r . The more t_r is far from the maturity, the more the fundamental basis volatility can be strong.¹³ However, the more t_r is far from the maturity, the less clock time is considered, such that the spot price bias and basis bias are limited, as well as the fundamental spot price volatility. The more clock time is considered, the more an entrance of agents has a few impact on the fundamental basis volatility. We extract that the decrease in the bias of expectations for the spot price and the basis increases the incentive of the policy maker to allow for the entrance of agents, despite their increasing impact on the fundamental spot price volatility within our set of parameters' default values.

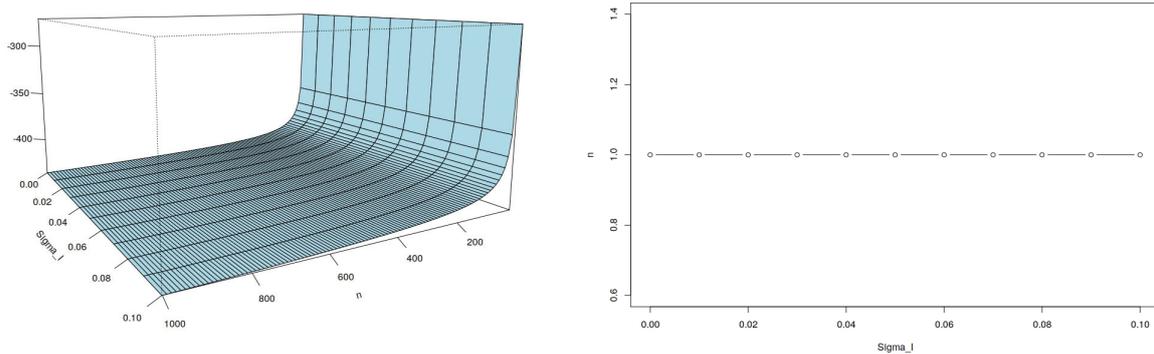
Compared to the no expectation bias situation (cf. figure 4.6), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the

¹³By deriving the fundamental basis volatility $\mathcal{V}(f_{t_r} - s_{t_r})$, it is easy to show that the fundamental basis volatility is maximal for $t_r = \frac{1}{3}T_r$.

market, when the clock time is near the maturity (i.e. the fundamental basis volatility is constrained but the fundamental spot price volatility can be more important, as well as the spot price and basis biases), the optimal **n.o.a** on the market is higher when expectations are biased.



(a) $\xi < 0$



(b) $\xi > 0$

Figure 4.17 – Representation of both utility function $U_r(n)$ (on left side), and n^* (on right side) according to the variation of parameter σ_1 ($\sigma_1 = \{0, 0.01, 0.02, \dots, 0.1\}$), with biased expectations.

Sensitivity analysis to σ_1 : We can observe on figure 4.17 that utility is a strictly decreasing function of n for all tested values of σ_1 , such that n^* is not sensitive to the maximal intrinsic volatility of the spot market information (per unit of real time), σ_1 , under our set of parameters' default values. Then, the optimal **n.o.a** is always minimal ($n^* = 1$).

Compared to the no expectation bias situation (cf. figure 4.7), the presence of a bias do not sufficiently increases the incentives of the policy maker to allow for the entrance of agents on the market, such that the optimal **n.o.a** on the market is not sensitive to agents bias of expectations.

Sensitivity analysis to Θ : We can observe on figure 4.18 that utility is not a monotonous function of n for all tested values of $\Theta \neq 0$ (contrary to the case when $\xi = 0$, cf. figure 4.8). If there is no risk aversion (i.e. $\Theta = 0$), the utility function fully depends on the spot price bias. From remark 23, an infinite **n.o.a** allows for a biased spot price by limit.

If the risk aversion is not important and by extension, i.e. for low values of Θ , the utility function depends mainly on the spot price bias and the basis bias. In this case, an entrance of agents can allow for a decrease of both biases, such that $n^* > 1$.

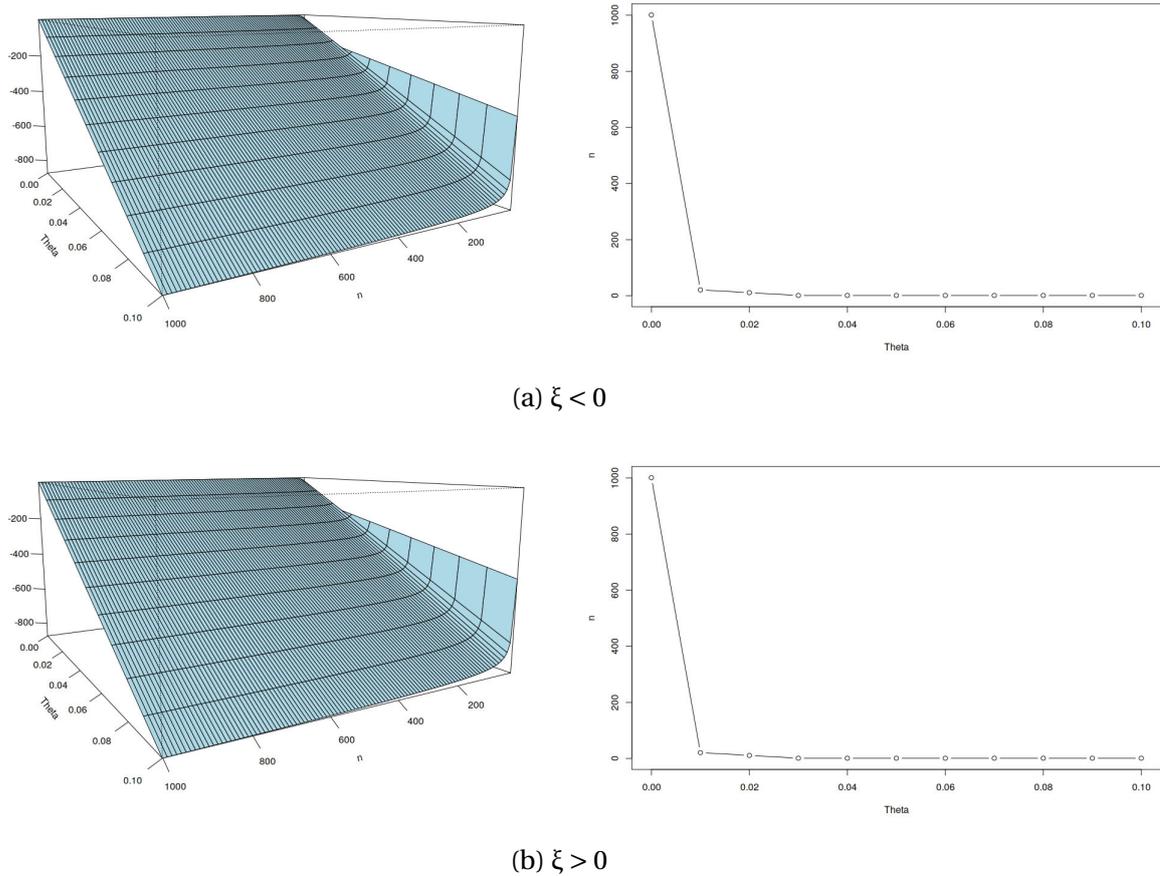


Figure 4.18 – Representation of both utility function $U_{t_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter Θ ($\Theta = \{0, 0.01, 0.02, \dots, 0.1\}$), with biased expectations.

Compared to the no expectation bias situation (cf. figure 4.8), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the market, when the risk aversion is low (i.e. the spot price bias and the basis bias are more weighted), the optimal **n.o.a** on the market is higher when expectations are biased.

Sensitivity analysis to T_r : We can observe on figure 4.19 that utility is not a monotonous function of n for all tested values of T_r . As the maturity reduces, it also reduces the fundamental basis volatility, giving more weight to the fundamental spot price volatility, as well as the spot price bias and the basis bias. Any entrance of agents increases the clock time effect until maturity (cf. theorem 18), by increasing the **n.o.t** on the futures market until maturity. The more the maturity of the futures contract increases, the higher this effect is. Then, the optimal **n.o.a** is always decreasing with the maturity. Hence, if T_r is near from the t_r considered, the same mechanism than in the sensitivity analysis of t_r is at stake.

Compared to the no expectation bias situation (cf. figure 4.9), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the market, when the maturity is near the clock time considered (i.e. the fundamental basis volatility is constrained but the fundamental spot price volatility can be more important, as well as the spot price and basis biases), the optimal **n.o.a** on the market is higher when expectations are biased.

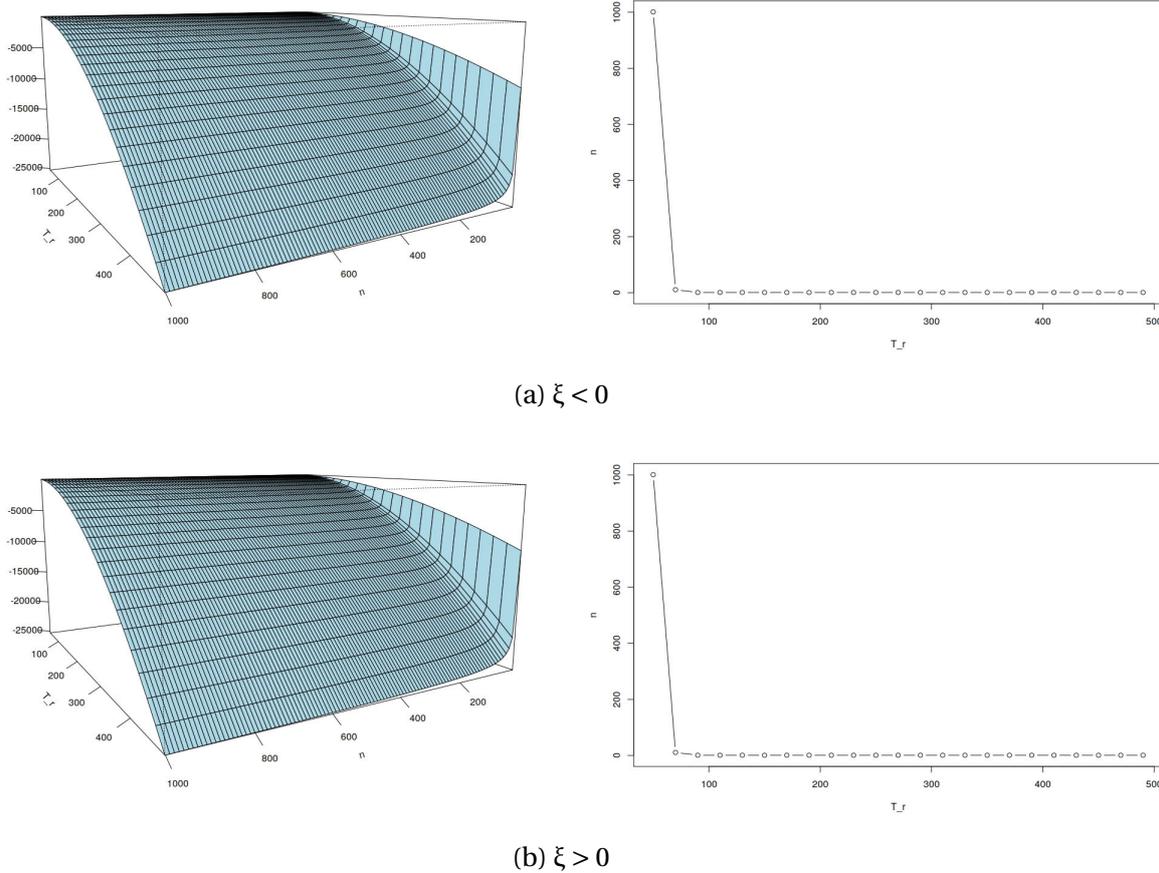
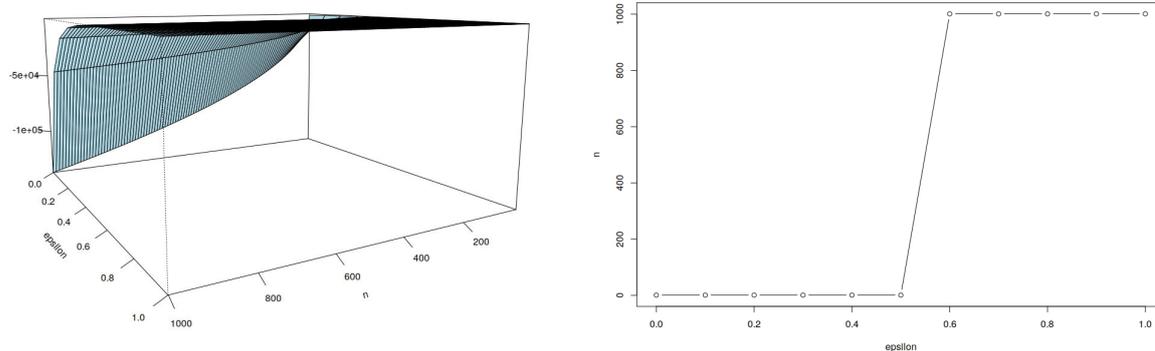


Figure 4.19 – Representation of both utility function $U_{T_r}(n)$ (on left side), and n^* (on right side) according to the variation of parameter T_r ($T_r = \{50, 70, \dots, 500\}$), with biased expectations.

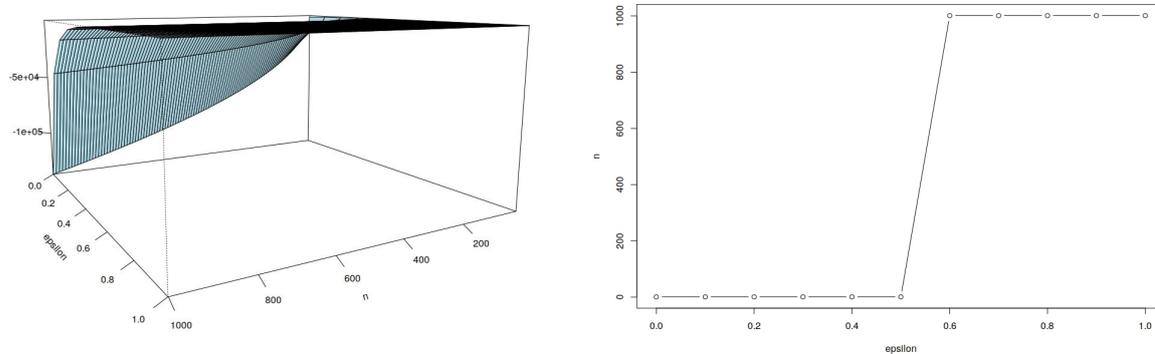
Sensitivity analysis to ϵ : We can observe on figure 4.20 that utility is not a monotonous function of n for all tested values of ϵ . For low values of ϵ , an increase in the n.o.a insufficiently smooth quantities traded at each transaction on the spot market. Then, it does not reduce sufficiently the fundamental basis volatility such that the optimal n.o.a is minimal ($n^* = 1$). However, as soon as ϵ is above a threshold, a maximal n.o.a allows for the fundamental basis volatility to tend to 0. The threshold denoted by $\bar{\epsilon}$, is unchanged and given by equation (4.43). However the values of the market prices biases evolve such that the optimal n.o.a becomes infinite with a lower value of ϵ . This effect overcomes the increasing fundamental spot price volatility and the potential increase in the spot price bias, such that the optimal n.o.a strongly depends on the value of parameter ϵ within the set of our parameters' default values.

Compared to the no expectation bias situation (cf. figure 4.9), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the market, such that an infinite entrance of agents is optimal for lower values of ϵ ($\epsilon = 0,6$ against $\epsilon = 0,7$ without biased expectation cf. figure 4.10).

Sensitivity analysis to α : We can observe on figure 4.21 that utility is not a monotonous function of n for all tested values of α . If the RTF is not sensitive to the n.o.a (i.e. $\alpha = 1$), the spot price bias is constant in n.o.t and the volatility of occurrence of synchronized transaction is unchanged. An infinite n.o.a leads to an unbiased spot price, a maximal fundamental spot price volatility by limit, and a minimal basis volatility by limit (quan-

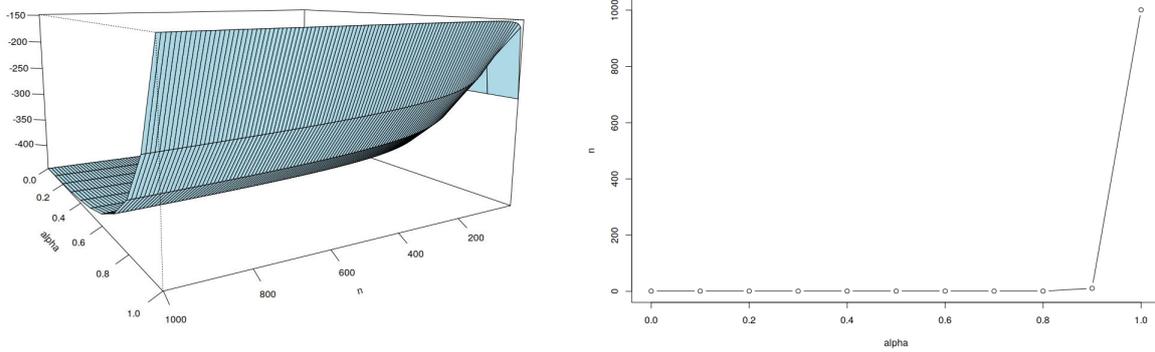


(a) $\xi < 0$

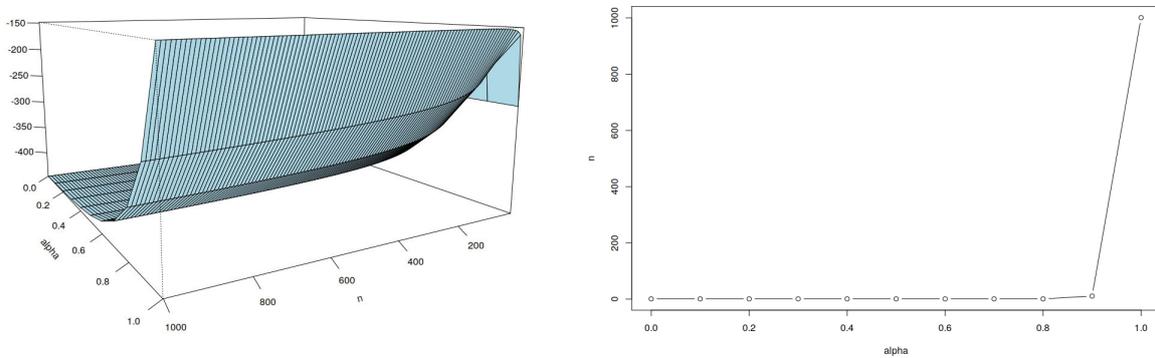


(b) $\xi > 0$

Figure 4.20 – Representation of both utility function $U_t(n)$ (on left side), and n^* (on right side) according to the variation of parameter ϵ ($\epsilon = \{0, 0.1, 0.2, \dots, 1\}$), with biased expectations.



(a) $\xi < 0$



(b) $\xi > 0$

Figure 4.21 – Representation of both utility function $U_{tr}(n)$ (on left side) and n^* (on right side) according to the variation of parameter α ($\alpha = \{0, 0.1, 0.2, \dots, 1\}$), with biased expectations.

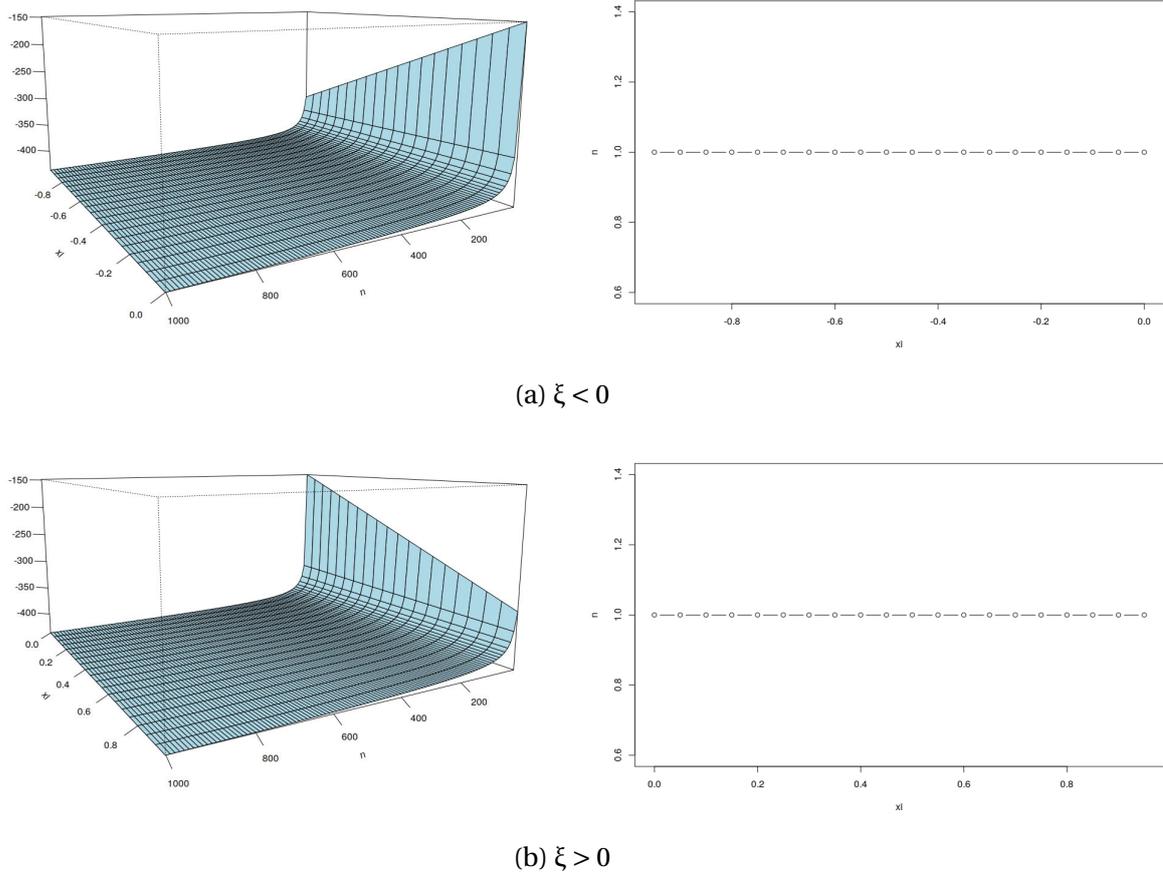


Figure 4.22 – Representation of both utility function $U_t(n)$ (on left side), and n^* (on right side) according to the variation of parameter ξ ($\xi = \{-0.95, -0.9, -0.85, \dots, 0\}$ (on the left side), $\xi = \{0, 0.05, 0.1, \dots, 0.95\}$ (on the right side)), with biased expectations.

tities traded per transaction on the spot market tend to 0 by limit). Hence, according to the sensitivity of the RTF to the n.o.a $(1 - \alpha)$, the increase in the fundamental spot price volatility is compensated by the decrease in the spot price bias, the basis bias and in the fundamental basis volatility, within our set of parameters' default values. The optimal n.o.a is an increasing function of α .

Compared to the no expectation bias situation (cf. figure 4.9), the presence of a bias increases the incentives of the policy maker to allow for the entrance of agents on the market. For high values of α , the policy maker has to authorize the entrance of a few agent to decrease the bias in expectations, despite it is increasing the fundamental spot price volatility and potentially increasing the spot price bias and the fundamental basis volatility.

Sensitivity analysis to ξ : We can observe on figure 4.22 that utility is a strictly decreasing function of n for all tested values of ξ . For the highest value of ξ in absolute value the optimal n.o.a is $n^* = 1$. Then, reducing the bias in expectations does not modify the optimal n.o.a on the market. An expectation bias has the effect of an incentive to let agents enter the market. Our set of parameters' default values is such that with the highest bias of expectations (maximal underestimation of spot market information) $n^* = 1$. Then, the decrease in the bias does not reduce the optimal n.o.a.

In this section, we studied how the introduction of biased expectations of the spot price information does modify the mechanisms at stake considering the market efficiency in sections 4.4, 4.5 and 4.6. We found that underestimating bias and overestimating bias are not equivalent and have asymmetric effects on the spot market efficiency. In the end, having an overestimating bias can allow the spot price to be unbiased for a certain n.o.a and this number is an increasing function of the bias parameter ξ . We also studied what are the conditions to allow an entrance of – a unit of – agents in order to increase the market efficiency. We finally studied the existence of an optimal number of agent but the complexity of the problem prevents us from having tractable analytical solutions. Then, we proceeded with sensitivity analysis to each parameter and show that the introduction of this bias in expectations makes the optimal n.o.a more sensitive to some parameters of the model.

4.8 Conclusion

We have built a simple dynamics pricing model applied to an agricultural commodity market (spot and futures), which is an extension of the model developed in the previous chapter 3. That model is an analytical tool that can be applied to any storable commodity for which the frequency of production is lesser than the frequency of the consumption. We have assumed that agents generate more volatility on the spot market, and that they influence the trade frequencies as well as quantities traded per transaction on the spot market. Considering an objective of market efficiency and prices' stability, we have studied the existence and the level of the optimal number of agents (n.o.a) on an agricultural commodity market. In the model, information that determines the price of the futures is endogenously determined by trades occurring on the spot market and, conversely, the information that determines the price of the spot market is endogenously determined by trades occurring on the futures market. This framework is borrowed from the storage theory and models the links between the two markets. By feedback, the futures market trade frequency (FTF) influences the probability law of incoming spot market information that impacts the spot market. By extension the futures market will also be impacted by the no arbitrage condition. Hence, we considered the specificity of the underlying of a futures contract on an agricultural commodity. Eventually, we also introduced the possibility of biased expectation of the spot market information, the latest being decreasing with the n.o.a. We studied how this incentive to let agents enter the market influence the optimal n.o.a.

We showed that an entrance of more agents not necessarily benefits the market efficiency and the fundamental prices' stability. Our results show that there is a threshold above which, more agents are increasing the market prices' bias and the fundamental prices' volatility. The latest depends on the values of the parameters of the model. From this result, we also extrapolate that a possible trade-off exist between market efficiency (which more agents can bring) and fundamental volatility; As in most of the cases, no analytical solution exist, we conducted some sensitivity tests for all parameters of the model. We find that the optimal n.o.a is strongly sensitive to the value of ϵ , the absolute value of the elasticity of quantities traded per exchange on the spot market to the spot market trade frequency. There is a threshold above which the entrance of agents will decrease quantities traded enough to compensate all the other negative effects.

In the end, we surprisingly find that when considering biased expectations, underestimating and overestimating bias has absolutely not the same consequences. The increasing n.o.a increasing the average delay between two synchronized transactions on both

markets, we showed that when agents overestimate the fundamental price impact of the spot market information, the average delay between the transactions could allow for the fundamental spot price to catch up the real overestimated one, such that on average, the spot price could be unbiased. On the opposite, when agents underestimate the fundamental price impact of the spot market information, the two effects are cumulative.

We are aware that this model presents some limits. Some extensions of this work can be made depending on the form of the q function as well as in the implementation of several commodity markets (or other markets in general) where information is released at different frequencies. Finally, we could also integrate the possibility of speculation on the spot market, such that trades on the spot market would not necessarily imply a commodity consumption, and the possibility of stochastic quantities traded.

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Chapter 5

A synthetic note

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The overall research question of this work was to study the optimal pricing algorithm allowing market efficiency and a minimal prices volatility. It was conducted according to two dimensions of characteristics of information given by table 1.3 (the capability to process it using algorithms and the fact that information is exogenously or endogenously modeled). In this thesis, we highlighted some incomplete points of contemporaneous financial economics regarding the financial markets structure but we also underscored the major potential of these academic works. Within an information-based reasoning, we conceptualized a framework able to unify both Efficient Market Hypothesis (EMH) and behavioral finance paradigms according to the nature of information. During this three years work, we also paid a particular attention to agricultural commodity markets whose information have peculiar characteristics as a frequency of public information known in advance and an underlying asset not frequently traded. We considered that a financial market has an evolutionary dynamics, as in Lo [8]. It lead us to assume that each transaction displays new information. We investigated the impacts of these elements on market efficiency and on prices volatility. Finally, we also investigated the influence of economic agents, as they are a source of information by their actions (cf. Stiglitz [10]). Considering the actual Limit Order Book (LOB) pricing algorithm used by Market Undertakers (MU), we investigated the usually following consensual result: The more agents they are, the more competition on the market there is. The less agents have *market power* and the more they are atomistic, such that market efficiency increases. We also discussed their impacts on prices volatility.

5.1 Investigating the impact of continuous information with a limit order book pricing algorithm

In this part of the thesis, we extended the model of Bretto and Priolon [2]. This approach takes advantage of some properties of the formalism of quantum physics and specifically some of its probabilistic properties. We assume that each independent agent influences the price when s/he sends an order that is aggregated in a central order book and each price influences in turn all independent agents in a never ending process. In this work, information is continuous and we consider that only a part of it can be processed by an algorithm (information is said to be *effective*). This work is compatible with the Adaptive Market Hypothesis developed by Lo [8], whose view is that agents have an evolutionary behavior, trying to reach efficient actions.

The first outcome of this Information-Based model is that it can be a framework allowing the unification of the Efficient Market Hypothesis (EMH) and behavioral finance. Indeed, we highlight that the more information is *effective*, the more the financial market tends to an efficient market. It does not mean that agents have the same expectations of the next price but the outcome of their interactions is as close as possible from the *fundamental value* of the asset.

We also define and develop the implications of the concept of *market system* introduced in Bretto and Priolon [2]. We define the *market system* adequate to a *perfect competition* – with a finite or infinite *n.o.a.*

We introduce the concept of *computational strength* whose closest formulation comes from Hayek [6] to the best of our knowledge. It allows to distinguish between a market fulfilling its informational role (providing in a better way information than any agent separately) or not. In the light of this concept, within a period of *bubble*, the financial market

can keep its computational strength until the bubble turns into a crisis. Above this threshold, within a financial crisis, it is impossible for the market to keep its computational strength. As a consequence, above this threshold, a financial market is neither efficient nor computationally strong. The pricing algorithm of this asset must be modified in this case, in order to fulfill its informational purpose.

We demonstrate that if information is mainly or even fully processed by an algorithms, a high frequency of trades allows the market to be semi-strong efficient in the sense of Fama et al. [5] (information is publicly and freely available in the model). The capability of the market to be semi-strong efficient fully depends on the nature and structure of information. Within this framework, we define a *financial crisis* as being a market whose characteristics are at the opposite of a *semi-strong efficient* market. With the help of these two definitions, we define a *bubble*. We extend the outputs of the model by showing that under a moderate growth with time of *non effective* information compared to *effective* information, a bubble appears and ends up with a financial crisis. This work gives another argument in favor of the implementation of a LOB pricing for assets whose information arrives continuously and also agreed on the presence of Market Makers (MM) allowing transaction's frequency to raise within markets where information is continuously updated. However, financial regulator must limit the sources of *non effective* information to avoid the emergence of a bubble and its transformation into a financial crisis. This work underlines the major influence of information's nature and structure on the capability of a market to be a useful tool (having a computational strength) as well as to be the best tool (being semi-strong efficient). The relation between the two characteristics are summarized in table 2.1.

Aside of these new concepts and their theoretical consequences we also propose indicators of the computational strength of a financial market that can be implemented by the regulator at several given real-time backward (since the computational strength is computed on a given period) using a spectral zeta function. We provide a simulation of the evolution of the computational strength of the market with time, in a period of bubble. We also proceed with simulations to show to what extent a VARMA process estimation can be used to anticipate agents *market power* (i.e. to anticipate the next *market system*)¹. The financial market regulator can use this tool to identify who are the agents with the most *market power* and for instance, study if they have some characteristics in common such as being fast traders, having considerable open positions (or a high volume of the asset) and so on.

There are several possible further steps. Empirical studies using our model should be carried on. One can compute the computational strength of a financial market *a posteriori* and estimate the relative quantity of effective information and non effective information processed during the period. One can also use market data to test how a bubble evolves over time in order to parameterize its (moderate) growth. On the theoretical ground, the main extension of this model would be to consider imperfect information. As we proved it in subsection 2.3.3, the reading (also called the measure) of quantifiable does not commute with time. Having information on quantities and then on prices is not equivalent to the opposite. Asymmetric information could also generate some unexpected effect ac-

¹We recall that the market system at time t gives the probability of each agent to give the transaction price and by extension the market price at time t . These probability are constructed considering the price given by the agent compared to the actual price, but also compared to the distribution of all agents' prices.

according to who is giving the market price first. Within this model we cannot introduce a cost to get the information but we might consider that all agents have a part of the information as for Bayesian equilibrium models (cf. Vives [11]).

5.2 Investigating the impact of trades frequencies on an agricultural commodity market

Agricultural commodity markets have a peculiar structure of information. The latest arrives with a relatively low and fixed frequency. The production is uncertain but not its frequency. However, futures contract are priced with a **LOB** algorithm. However, the relatively low frequency of new information diminishes the cost of losing liquidity. Furthermore, we introduce the fact that a transaction generates, in return, new information. Hence, information's nature and structure is influenced by trades frequencies (a dimension of liquidity) on both spot and futures markets. If trades frequency is infinite on the futures market, it implies that information is also continuous on the spot market. We build an original and simple sequential pricing model in the wake of the ones of Black and Scholes [1], Cox et al. [3] applied to an agricultural commodity market (spot and futures); More generally, that model is an analytical tool that can be applied to any storable commodity for which the frequency of production is lesser than the frequency of the consumption. As in Du and Zhu [4] we have studied the existence and the level of optimal trading frequencies on an agricultural commodity market, and the impact of the degree of liquidity on the market efficiency and on the fundamental basis volatility. In the model, information that have a futures price impact is endogenously influenced by trades occurring on the spot market and, conversely, the information that have a spot price impact is endogenously influenced by trades occurring on the futures market. This framework is borrowed to the storage theory and models the links between the two markets. By feedback, the futures market trade frequency (**FTF**) influences the probability law of incoming spot market information that impacts the spot market. By extension the futures market will also be impacted by the no arbitrage condition. Hence, we consider the specificity of the underlying of a futures contract on an agricultural commodity.

We demonstrate that the fundamental basis volatility can increase even if the increase in liquidity is proportional on both markets. The idea is as simple as it can be: Asynchronous trades on both spot and futures markets generate a risk, in the sense of Knight [7]. We prove and defend that synchronization of transactions on agricultural commodity markets and *a fortiori* of the frequency of these transactions can allow unbiased prices dynamics on spot and futures markets, a *sine qua non* condition for having an efficient agricultural commodity market. Under these conditions, the fundamental volatility of the futures contract is also minimal. Then, a non-proportional increase in liquidity on spot market and futures market can increase the volatility of information on the futures contract if the decrease in quantities traded on the spot market is not sufficient.

We highlight the existence of a trade-off between futures market liquidity and either the fundamental basis volatility or the market efficiency. Indeed, the optimal policy in order to minimize the fundamental basis volatility and maximize the spot and futures market efficiency is to implement a fixing pricing algorithm on the futures market operating at the same frequency than transactions on the spot market. In other words, commodity futures pricing by limit order book is not an optimal policy regarding the objectives of

maximizing market efficiency and of minimizing the fundamental prices' volatility. Previous empirical studies argued that more liquidity allows for better market efficiency; their findings do not necessarily contradict our results, depending on the initial Relative Trade Frequency (RTF) (cf. O'Hara and Ye [9]). Our results recommend the implementation of a fixing pricing operating at the same frequency as trades on the underlying spot market, independently of the time line considered.² However, a way to improve both liquidity and market efficiency of the spot market could be a regulation of the spot market structure to facilitate transactions on this market, allowing an increase in the absolute value of ϵ .

There are several possible further steps. Empirical calibration of the model can be done to estimate the current trade-off between liquidity and market efficiency and fundamental basis volatility as well as its robustness. On the theoretical side, some extensions of this model can be made depending on the form of the q function as well as in the implementation of several commodity markets (or other markets in general) where information is released at different frequencies. We could consider the possibility of asynchronous trades with identical frequencies (which is not currently the case) using Poisson laws for instance. We could try to unify this simple model with the one of Du and Zhu [4] by micro-founding it.

5.3 Investigating the influence of the number of agents on an agricultural commodity market

We have extended the simple dynamics pricing model developed in the previous chapter 3. That extension is an analytical tool that can still be applied to any storable commodity for which the frequency of production is lesser than the frequency of the consumption. We have incorporated the number of agents into the market. Following the work of Stiglitz [10], we consider that agents convey information through their transactions (i.e. through their actions). By assuming that the more agents there are the more information there is (i.e. each new agent conveys a no redundant information), volatility of information on the spot market is an increasing function of the number of agents. We also assume that they influence the trades frequencies as well as quantities traded per transaction on the spot market which is natural. Considering an objective of market efficiency and prices' stability, we have studied the existence and the level of the optimal number of agents (n.o.a) on an agricultural commodity market. In the model, information that determines the price of the futures is endogenously determined by trades occurring on the spot market and, conversely, the information that determines the price of the spot market is endogenously determined by trades occurring on the futures market. By feedback, the futures market trade frequency (FTF) influences the probability law of incoming spot market information that impacts the spot market. By extension the futures market is also impacted by the no arbitrage condition. Hence, we considered the specificity of the underlying of a futures contract on an agricultural commodity. Eventually, we also introduced the possibility of biased expectation of the spot market information, the latest being decreasing with the n.o.a. Then, we introduce on this extension of the model, a *non effective* dimension to information. We studied how this incentive to let agents enter the market influences the optimal n.o.a.

²Otherwise, the liquidity on the futures market must be relatively infinite to ensure the maximization of the policy maker's utility when considering the Transaction Time Line (TTL).

We demonstrated that an entrance of more agents not necessarily benefits the market efficiency and the fundamental prices' stability. Our results show that there is a threshold above which, an entrance of more agents are increasing the market prices' biases and the fundamental prices' volatility. This threshold depends on the values of the parameters of the model and by definition, the market characteristics. From this result, we also extrapolate that a possible trade-off exists between market efficiency (that more agents can bring) and fundamental prices stability; In most of the cases, there is no analytical solution. We conducted some sensitivity tests for all parameters of the model. We find that the optimal **n.o.a** is strongly sensitive to the value of the absolute value of the elasticity of quantities traded per exchange on the spot market to the spot market trade frequency, denoted by ϵ . There is a threshold above which the entrance of agents will decrease quantities traded enough to compensate all the other negative effects.

In the end, we surprisingly find that when considering biased expectations and interpretation of the spot market information (the only one that is partly *non effective* in the model), underestimating and overestimating bias have absolutely not the same consequences. An increasing **n.o.a** increases the average delay between two synchronized transactions on both markets. We showed that when agents overestimate the fundamental price impact of the spot market information, the average delay between the transactions can allow the fundamental spot price to catch up the real overestimated one. On average, the spot price can be unbiased. On the opposite, when agents underestimate the fundamental price impact of the spot market information, the two effects are cumulative and the market efficiency decreases even more.

The overall result of this work corroborates the trade-off between futures market liquidity and either the fundamental values volatility or the market efficiency. The **n.o.a** increases *de facto* the trades frequencies in a **LOB** pricing algorithm. However, a fixing pricing algorithm could benefit the market efficiency and prices stability according to the same mechanisms than in chapter 3. However, another loop of effect is at stake. Agents increase the fundamental volatility of the spot price but decrease in counterpart the spot and basis biases. Hence, the consideration of *non effective* information in our model is an incentive to let agents enter the market. But as they increase the spot market information volatility, a trade-off between market efficiency (increasing with the **n.o.a**), and fundamental prices stability (decreasing with the **n.o.a**) exists. Furthermore, these two trade-offs are linked within a **LOB** pricing where agents influence both trades frequency and information. Within a fixing pricing however, we can dissociate these two trade-offs for a **n.o.a** sufficiently high ensuring that transactions take place at each fixing.³

Empirical calibration of the model can be done to estimate on the first hand the current trade-off between liquidity and markets efficiency and fundamental prices volatility. On the other hand, we can also estimate the trade-off between market efficiency and fundamental prices volatility inherent to the **n.o.a**. Limits and possible extensions of this model are basically identical to the one explained in the previous chapter 3.

Commonly to all this research, the major limit of our theoretical Information-Based Models is that all information is freely available. In chapter 2, assuming asymmetric information or investment to get information will be hard to model. It would change all

³Despite we do not model it, we can assume that an increasing **n.o.a** would increase market depth, another dimension of market liquidity.

the properties of information by definition and would bring a whole new level of complexity. On chapter 3 (and chapter 4 by extension), such properties or mechanisms could be implemented by micro-founding the model and more specifically the occurrence of transactions on both markets. Modeling transaction costs higher on the spot market than on the futures market, transactions on the spot market would be less frequent and information would also be harder to acquire. The introduction of these mechanisms should highlight some public policy but above all, some new mechanisms that could be tested empirically.

That researches create opportunities for calibration of the models, but also for genuinely original empirical tests.

To conclude, in this thesis, we highlight that not only the structure of information (its frequency) but also its nature (effective or not) fundamentally modify the optimal pricing algorithm that should be implemented in financial markets.

When information is continuous and mainly or even fully effective, a **LOB** pricing, allowing very frequent transactions (a dimension of liquidity), is adequate and allows the market to be semi-strong efficient.

However, if information is continuous but not mainly or fully effective, a bubble appears, and a **LOB** pricing favors its growth until it turns into a financial crisis.⁴ It is then very important for the regulator to be careful with the nature of displayed information.

Concerning agricultural commodity market, we find in chapter 3 (where information is assumed to be fully effective) that an infinite **FTF** that requires a **Central Limit Order Book (CLOB)** pricing, would lead to a biased spot price. The spot market trades frequency (**STF**) is inferior in this case, such that fundamental volatility of the futures contract is also not minimal.⁵ Then, the optimal pricing algorithm on the futures market ensures the same frequency of trades on both the spot market and the futures market (for instance a fixing). The latest market structure minimizes the risk on the basis as agents are certain to face a decrease in the available stock at each transaction. It also ensures an unbiased spot price since agents form rational expectations and information is synchronously displayed on both markets. This result is in contradiction with the one of Du and Zhu [4]. Indeed, we do not model inventory costs (guaranteeing a wish to trade back the futures contract as soon as possible), and information is freely available and endogenous to our trades frequencies in our model. In their model, information is exogenous and not freely available such that trades are made respectively to agents private value of the asset. However, the introduction of speculation on the spot market nuances this result. A trade-off between market efficiency and prices volatility becomes mandatory and according to the risk aversion of the policy maker, the optimal **FTF** is higher than previously.

In chapter 4, information can be either effective or non effective or both effective and non effective and is displayed according to trades frequencies.⁶ These frequencies are increasing functions of the **n.o.a** when a **CLOB** algorithm is implemented on the futures market and when transactions on the spot market are not constrained but the **STF** is growing slower than the **FTF**.⁷ In such a market, the spot price, the basis, and the futures price are biased and the fundamental values volatility are not minimal. However, there are two trade-offs at stake. First, a trade-off between trades frequencies (a dimension of liquidity) and market efficiency (strongly dependent to the value of ξ for the spot market efficiency) and fundamental values volatility (strongly dependent to the value of ϵ). Second, a trade-off between market efficiency and fundamental values volatility (dependent to the values of σ_1 , α , and the ratio ω_s/ω_f). In this framework, implementing a fixing algorithm allows to dissociate the two trade-offs. According to the volatility aversion (which is modeled as a risk aversion in the policy maker utility function borrowed from the mean-variance utility functions) of the policy maker, the optimal **FTF** must be compared to the actual ones on

⁴For us, a financial crisis is a period of such uncertainty that the price would reach an infinite volatility because measurement of the price would lead to random values.

⁵On the conclusion, we focus only on the results within the clock time line (**CTL**).

⁶We focus here on the most complete version of the model, with the possibility of biased interpretation and expectation of the spot market information.

⁷We focus on the case $\alpha < 1$, as $\alpha = 1$ is a limit case.

each agricultural commodity market. If inferior, the policy maker has to implement a fixing algorithm calibrated on the optimal FTF . If superior the policy maker has to regulate in order to increase the FTF as possible.

The introduction of speculation on the spot market nuance these two trade-offs, and is an incentive to increase the optimal FTF , but all mechanisms remain valid. This discussion applies more generally to any storable commodity for which the frequency of production is lesser than the frequency of the consumption, where a futures contract exists.

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Appendix A

Annexes

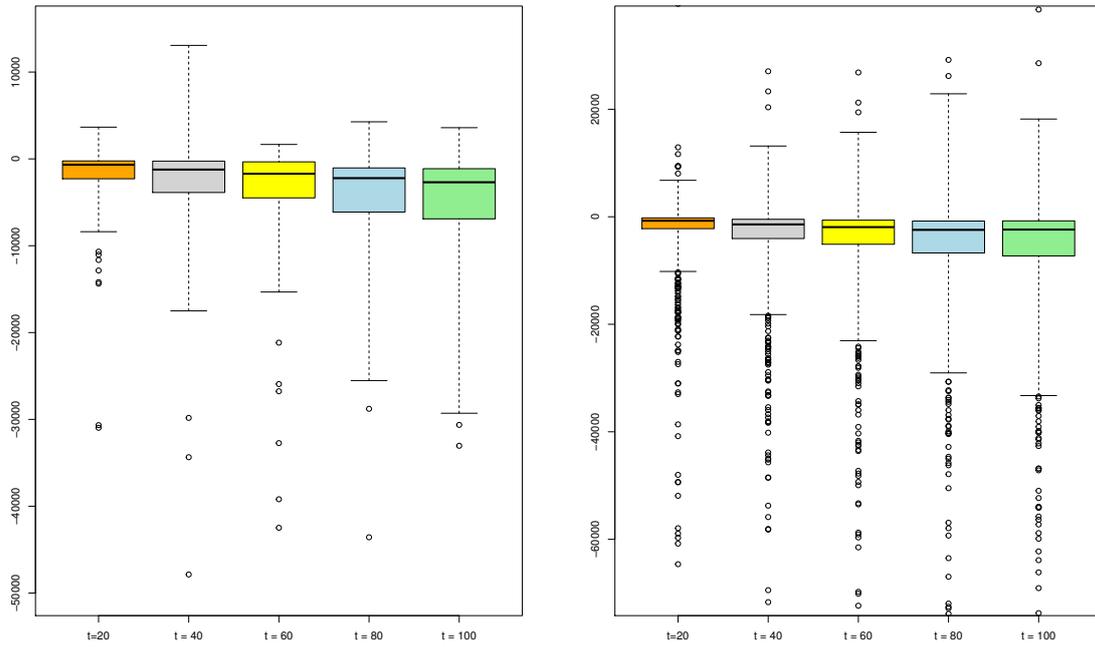
A.1 Simulations of the computational strength of the market within a period of bubble

A.1.1 Set of parameters values used

Parameter	$E(I_{ef} t-1; t)$	σ_{ef}	$E(I_{nef} t-1; t)$	σ_{nef}	n	$E(Y_{i,t}^{ef})$	$\sigma_{Y^{ef}}$	$E(Y_{i,t}^{nef})$	$\sigma_{Y^{nef}}$	σ_{β_i}
Value	$\frac{1,001}{1000}$	0,3	$\frac{1,001^t}{1000}$	0,3	50	1	0,2	0	0,3	0,2

Table A.1 – Set of parameters used for bubbles computational strength simulations

A.1.2 Evolution of the computational strength with bubble duration



(a) A hundred simulations for each bubble duration

(b) A thousand simulations for each bubble duration

Figure A.1 – Computational strength of the market in a bubble of t periods. The box contains 50 percent of the simulations, and the whiskers extend to four times the interquartile range from the box. Some extreme values do not appear for scale sake but are taken into account for the boxes plot

A.2 Simulations of market systems and estimation of a VARMA model

A.2.1 Set of parameters values used

Parameter	T	n	$E(I t-1; t)$	σ_I	$E(\gamma_{i,t})$	σ_γ	σ_{β_i}
Value	300	6	0	1	1	0.02	0.02

Table A.2 – Set of parameters used for market system simulations and its VARMA estimation

A.2.2 Algorithm of the market system simulations and VARMA model estimation (software used: R studio)

```
> setwd("/media/batistasoare/DAVID B-S/ThÃse/Chapitre 0/Documents R/ACP Quantit
```

We operate "nb.simu" simulations in a row.

```
> nb.simu <- 100
```

Simulations are with $n = 6$ agents and $T = 100$ periods.

```
> T <- 300
```

```
> n <- 6
```

Creation of different objects to store simulations data.

```
> clearing.prices <- matrix(0, nrow = nb.simu, ncol = (T+1))
> fundamentals <- matrix(0, nrow = nb.simu, ncol = (T))
> agent.vectorsS <- matrix(0, nrow = nb.simu*(T-1)*n, ncol = n)
> beta.i.carre.tS <- matrix(0, nrow = nb.simu*(T-1), ncol = n)
> sum.beta.i.carre.tS <- matrix(0, nrow = nb.simu, ncol = (T-1))
> sum.probabilities.tS <- matrix(0, nrow = nb.simu, ncol = (T-1))
> order.booksS <- matrix(0, nrow = nb.simu*(T-1), ncol = n)
```

The simulation's loop for the "nb. simu" simulations. At each simulation, we save relevant information.

```
> for(j in 1: nb.simu){
+   source(file='One_price_VARMA_parameters.R')
+   clearing.prices[j,] <- clearing.price
+   fundamentals[j,] <- fundamental.value
+   agent.vectorsS[((T-1)*n*(j-1)+1) : ((T-1)*n*j),] <- agent.vectors
+   beta.i.carre.tS[((T-1)*(j-1)+1):((T-1)*j),] <- beta.i.carre.t
+   sum.beta.i.carre.tS[j, ] <- sum.beta.i.carre.t
+   sum.probabilities.tS[j, ] <- sum.probabilities.t
+   order.booksS[((T-1)*(j-1)+1):((T-1)*j),] <- order.books
+ }
```

Construction of the market system. We have the market system as a - diagonal - matrix, we transform it as a vector. We do not lose information.

```
> marketssystem.matrix <- matrix(0, nrow = nb.simu*(T-1)*n, ncol = n)
> for(s in 1:nb.simu) {
+   for (t in 1:(T-1)) {
```

```

+     for (i in 1:n) {
+       marketsystem.matrix[(1+(n*(t-1))+((s-1)*(T-1)*n)):
+         ((n*t)+((s-1)*(T-1)*n)),i] <- beta.i.carre.tS[(t + (s-1)*(T-1)),i]*
+         c(agent.vectorsS[(1+(n*(t-1))+((s-1)*(T-1)*n)):
+           ((n*t)+((s-1)*(T-1)*n)),i])
+     }
+   }
> marketsystem.matrix.step2 <- matrix(0, ncol = nb.simu*(T-1), nrow = n)
> for (s in 1:nb.simu) {
+   for (t in 1:T-1) {
+     for (i in 1:n) {
+       marketsystem.matrix.step2[i, (s-1)*(T-1)+t] <-
+       sum(t(marketsystem.matrix)[i, (1 + (t-1)*n + (s-1)*(T-1)):
+         (n * t + ((s-1)*(T-1)))] , na.rm = FALSE)
+     }
+   }
> marketsystem.matrix.final <- t(marketsystem.matrix.step2)

```

Lastly, I take randomly one simulation among the "nb.simu" simulations, denoted by "r.s". Then, I isolate the market system of the "random.s"th simulation, denoted by "ms". I multiply all probabilities by 100 to avoid short round from the software.

```

> r.s <- sample(1:nb.simu,1)
> ms <- 100*marketsystem.matrix.final[((1+((r.s - 1)*(T-1))):(r.s*(T-1))),]

```

We make preliminary tests for the VARMA estimation. I need stationary time series for my $\beta_{i,t}$ time series independently, i.e. I need to take out trend and seasonality. For that, I operate augmented dick and fuller test (package urca needed). We accept stationarity.

```

> library("urca")
> for (i in 1:n) {
+ print(summary(ur.df(marketsystem.matrix.step2[i,], type = "drift", lags = 0)))
+ }

```

```

#####
# Augmented Dickey-Fuller Test Unit Root Test #
#####

```

Test regression drift

Call:
lm(formula = z.diff ~ z.lag.1 + 1)

Residuals:
 Min 1Q Median 3Q Max
-0.20154 -0.11444 -0.02453 0.08097 1.00968

Coefficients:
 Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.201540 0.001271 158.6 <2e-16 ***
z.lag.1 -1.204976 0.005661 -212.9 <2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1466 on 29897 degrees of freedom
Multiple R-squared: 0.6025, Adjusted R-squared: 0.6025
F-statistic: 4.531e+04 on 1 and 29897 DF, p-value: < 2.2e-16

Value of test-statistic is: -212.8698 22656.79

Critical values for test statistics:

	1pct	5pct	10pct
tau2	-3.43	-2.86	-2.57
phi1	6.43	4.59	3.78

```
#####  
# Augmented Dickey-Fuller Test Unit Root Test #  
#####
```

Test regression drift

Call:
lm(formula = z.diff ~ z.lag.1 + 1)

Residuals:

Min	1Q	Median	3Q	Max
-0.20213	-0.11481	-0.02610	0.08318	0.96194

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.202128	0.001272	159.0	<2e-16 ***
z.lag.1	-1.202251	0.005664	-212.3	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1457 on 29897 degrees of freedom
Multiple R-squared: 0.6011, Adjusted R-squared: 0.6011
F-statistic: 4.506e+04 on 1 and 29897 DF, p-value: < 2.2e-16

Value of test-statistic is: -212.2651 22528.24

Critical values for test statistics:

	1pct	5pct	10pct
tau2	-3.43	-2.86	-2.57
phi1	6.43	4.59	3.78

```
#####  
# Augmented Dickey-Fuller Test Unit Root Test #  
#####
```

Test regression drift

Call:

lm(formula = z.diff ~ z.lag.1 + 1)

Residuals:

Min	1Q	Median	3Q	Max
-0.20123	-0.11358	-0.02403	0.08097	0.95236

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.201226	0.001274	157.9	<2e-16 ***
z.lag.1	-1.194217	0.005673	-210.5	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1457 on 29897 degrees of freedom
Multiple R-squared: 0.5971, Adjusted R-squared: 0.5971
F-statistic: 4.431e+04 on 1 and 29897 DF, p-value: < 2.2e-16

Value of test-statistic is: -210.4975 22154.6

Critical values for test statistics:

	1pct	5pct	10pct
tau2	-3.43	-2.86	-2.57
phi1	6.43	4.59	3.78

```
#####  
# Augmented Dickey-Fuller Test Unit Root Test #  
#####
```

Test regression drift

Call:

```
lm(formula = z.diff ~ z.lag.1 + 1)
```

Residuals:

Min	1Q	Median	3Q	Max
-0.19527	-0.10905	-0.02647	0.07626	0.94287

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.195266	0.001238	157.8	<2e-16 ***

z.lag.1 -1.197449 0.005670 -211.2 <2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1423 on 29897 degrees of freedom

Multiple R-squared: 0.5987, Adjusted R-squared: 0.5987

F-statistic: 4.461e+04 on 1 and 29897 DF, p-value: < 2.2e-16

Value of test-statistic is: -211.2031 22303.38

Critical values for test statistics:

	1pct	5pct	10pct
tau2	-3.43	-2.86	-2.57
phi1	6.43	4.59	3.78

```
#####  
# Augmented Dickey-Fuller Test Unit Root Test #  
#####
```

Test regression drift

Call:

lm(formula = z.diff ~ z.lag.1 + 1)

Residuals:

	Min	1Q	Median	3Q	Max
	-0.20173	-0.10921	-0.02546	0.07732	0.85544

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.201732	0.001258	160.4	<2e-16 ***
z.lag.1	-1.210419	0.005655	-214.0	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.144 on 29897 degrees of freedom
Multiple R-squared: 0.6051, Adjusted R-squared: 0.6051
F-statistic: 4.581e+04 on 1 and 29897 DF, p-value: < 2.2e-16

Value of test-statistic is: -214.0292 22904.25

Critical values for test statistics:

	1pct	5pct	10pct
tau2	-3.43	-2.86	-2.57
phi1	6.43	4.59	3.78

```
#####  
# Augmented Dickey-Fuller Test Unit Root Test #  
#####
```

Test regression drift

Call:

```
lm(formula = z.diff ~ z.lag.1 + 1)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-0.20207	-0.11027	-0.02557	0.07847	0.86690

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.202068	0.001256	160.9	<2e-16 ***
z.lag.1	-1.214354	0.005649	-215.0	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1439 on 29897 degrees of freedom
Multiple R-squared: 0.6072, Adjusted R-squared: 0.6072
F-statistic: 4.621e+04 on 1 and 29897 DF, p-value: < 2.2e-16

Value of test-statistic is: -214.9706 23106.17

Critical values for test statistics:

	1pct	5pct	10pct
tau2	-3.43	-2.86	-2.57
phi1	6.43	4.59	3.78

If test were not validated, I would have to differentiate the time series until time series are stationary.

```
> #If the first stationary time series are x times lagged, hence,
> #we create the following object and we would have replace "ms" by
> #"ms.lagged.x" in the rest of the code.
> #ms.lagged.x <- matrix(0, nrow = n, ncol = (T-1 - x))
> #for (t in (1 + x):(T-1)) {
> #  ms.lagged.x[, (t-1)] <- ms[,t] - ms[, (t-1)] - ... - ms[, (t-x)]
> #}
```

I choose the smaller orders possible of $AR(p)$ and $MA(q)$ processes first in the VARMA estimation (package MTS needed), with the help of the Extended Cross-Correlation Matrices.

```
> library("MTS")
> Eccm(ms)
```

p-values table of Extended Cross-correlation Matrices:

Column: MA order

Row : AR order

	0	1	2	3	4	5	6
0	0.0000	0.0001	0.0001	0.0007	0.0069	0.0137	0.0376
1	0.0000	0.0106	0.9496	0.5647	0.6686	0.8679	0.9997
2	0.3867	0.9964	0.8797	0.9453	0.9147	0.9965	1.0000
3	0.9908	1.0000	1.0000	1.0000	0.9680	0.9999	1.0000
4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

```
> VARMA(da = ms, p = 0, q = 1)
```

Number of parameters: 42

initial estimates: 17.081 16.1561 16.9631 15.7994 17.2071 16.8025 -0.3342 -0.13 -

Par. lower-bounds: 15.2731 14.4629 15.1875 14.2068 15.4552 15.0531 -0.4779 -0.287

Par. upper-bounds: 18.8889 17.8492 18.7387 17.3921 18.959 18.552 -0.1905 0.0276 0

Final Estimates: 17.08757 16.16229 16.96678 15.76491 17.17211 16.81902 -0.38014

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)
[1,]	1.709e+01	3.121e-04	54751.18	<2e-16 ***
[2,]	1.616e+01	1.825e-02	885.55	<2e-16 ***
[3,]	1.697e+01	2.590e-04	65509.47	<2e-16 ***
[4,]	1.576e+01	2.084e-02	756.51	<2e-16 ***
[5,]	1.717e+01	3.480e-03	4933.86	<2e-16 ***
[6,]	1.682e+01	1.083e-04	155306.24	<2e-16 ***
[7,]	-3.801e-01	NA	NA	NA
[8,]	-1.885e-01	NA	NA	NA
[9,]	-1.246e-01	NA	NA	NA
[10,]	-2.089e-01	NA	NA	NA
[11,]	-1.303e-01	NA	NA	NA
[12,]	-9.494e-02	NA	NA	NA
[13,]	-1.714e-01	NA	NA	NA
[14,]	-4.209e-01	NA	NA	NA
[15,]	-1.884e-01	NA	NA	NA
[16,]	-3.018e-01	NA	NA	NA
[17,]	-1.849e-01	NA	NA	NA
[18,]	-5.377e-02	NA	NA	NA
[19,]	-1.229e-01	NA	NA	NA
[20,]	-1.973e-01	1.780e-02	-11.08	<2e-16 ***
[21,]	-3.665e-01	NA	NA	NA
[22,]	-1.684e-01	NA	NA	NA
[23,]	-5.364e-02	NA	NA	NA
[24,]	-1.624e-01	NA	NA	NA
[25,]	-1.927e-01	NA	NA	NA
[26,]	-7.392e-02	NA	NA	NA
[27,]	-1.739e-01	NA	NA	NA
[28,]	-3.909e-01	NA	NA	NA
[29,]	-1.066e-01	NA	NA	NA
[30,]	-1.370e-01	NA	NA	NA

[31,]	-8.413e-02	NA	NA	NA
[32,]	-1.852e-01	NA	NA	NA
[33,]	-1.576e-01	NA	NA	NA
[34,]	-7.886e-02	NA	NA	NA
[35,]	-4.236e-01	3.299e-06	-128381.68	<2e-16 ***
[36,]	-1.521e-01	NA	NA	NA
[37,]	-8.059e-02	NA	NA	NA
[38,]	1.286e-02	NA	NA	NA
[39,]	-3.552e-02	NA	NA	NA
[40,]	9.122e-02	NA	NA	NA
[41,]	-1.119e-01	NA	NA	NA
[42,]	-3.910e-01	NA	NA	NA

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Estimates in matrix form:

Constant term:

Estimates: 17.08757 16.16229 16.96678 15.76491 17.17211 16.81902

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.3801	0.1885	0.1246	0.2089	0.1303	0.0949
[2,]	0.1714	0.4209	0.1884	0.3018	0.1849	0.0538
[3,]	0.1229	0.1973	0.3665	0.1684	0.0536	0.1624
[4,]	0.1927	0.0739	0.1739	0.3909	0.1066	0.1370
[5,]	0.0841	0.1852	0.1576	0.0789	0.4236	0.1521
[6,]	0.0806	-0.0129	0.0355	-0.0912	0.1119	0.3910

Residuals cov-matrix:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	242.03668	-50.72875	-25.22235	-32.65461	-39.17984	-44.44427
[2,]	-50.72875	206.07691	-39.79410	-31.82979	-38.48864	-19.58631
[3,]	-25.22235	-39.79410	225.36868	-32.32725	-36.82966	-35.70719
[4,]	-32.65461	-31.82979	-32.32725	184.51425	-29.48130	-36.65791
[5,]	-39.17984	-38.48864	-36.82966	-29.48130	220.33443	-52.72253
[6,]	-44.44427	-19.58631	-35.70719	-36.65791	-52.72253	224.39283

aic= 31.47209

bic= 31.99188

> VARMA(da = ms, p = 1, q = 0)

Number of parameters: 42

initial estimates: 24.0828 31.6853 25.727 22.8796 30.2055 20.9837 -0.2524 -0.0274

Par. lower-bounds: 15.3737 23.6401 17.2544 15.2114 21.9815 12.5996 -0.3779 -0.158

Par. upper-bounds: 32.7919 39.7305 34.1995 30.5478 38.4295 29.3678 -0.1269 0.1039

Final Estimates: 24.99067 32.18709 25.4849 22.41207 29.94554 21.09988 -0.254916

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)	
[1,]	24.990668	4.355142	5.738	9.57e-09	***
[2,]	32.187088	3.994743	8.057	8.88e-16	***
[3,]	25.484900	4.202289	6.065	1.32e-09	***
[4,]	22.412072	3.814621	5.875	4.22e-09	***
[5,]	29.945545	4.083350	7.334	2.24e-13	***
[6,]	21.099878	4.162934	5.069	4.01e-07	***
[7,]	-0.254916	0.063115	-4.039	5.37e-05	***
[8,]	-0.031771	0.065942	-0.482	0.629946	
[9,]	-0.007846	0.063971	-0.123	0.902381	
[10,]	-0.093686	0.071001	-1.319	0.187003	
[11,]	-0.031818	0.065325	-0.487	0.626210	
[12,]	-0.041320	0.066458	-0.622	0.534109	
[13,]	-0.095799	0.057889	-1.655	0.097952	.
[14,]	-0.351402	0.060478	-5.810	6.23e-09	***
[15,]	-0.101098	0.058687	-1.723	0.084949	.
[16,]	-0.219760	0.065121	-3.375	0.000739	***
[17,]	-0.135622	0.059915	-2.264	0.023601	*
[18,]	-0.065536	0.060954	-1.075	0.282293	
[19,]	-0.056286	0.060899	-0.924	0.355363	
[20,]	-0.125215	0.063623	-1.968	0.049058	*
[21,]	-0.230151	0.061738	-3.728	0.000193	***
[22,]	-0.053035	0.068507	-0.774	0.438837	
[23,]	0.045753	0.063028	0.726	0.467896	
[24,]	-0.095143	0.064123	-1.484	0.137872	
[25,]	-0.060040	0.055281	-1.086	0.277439	
[26,]	0.046979	0.057751	0.813	0.415943	

[27,]	-0.096365	0.056038	-1.720	0.085499	.
[28,]	-0.241176	0.062183	-3.878	0.000105	***
[29,]	-0.011477	0.057223	-0.201	0.841043	
[30,]	-0.047479	0.058206	-0.816	0.414674	
[31,]	-0.009596	0.059182	-0.162	0.871191	
[32,]	-0.154534	0.061826	-2.500	0.012437	*
[33,]	-0.132045	0.059995	-2.201	0.027742	*
[34,]	0.016818	0.066567	0.253	0.800538	
[35,]	-0.383593	0.061252	-6.263	3.79e-10	***
[36,]	-0.091225	0.062312	-1.464	0.143190	
[37,]	-0.073357	0.060324	-1.216	0.223969	
[38,]	0.056818	0.063019	0.902	0.367271	
[39,]	0.016678	0.061151	0.273	0.785055	
[40,]	0.104661	0.067856	1.542	0.122976	
[41,]	-0.037457	0.062435	-0.600	0.548547	
[42,]	-0.311619	0.063515	-4.906	9.28e-07	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Estimates in matrix form:

Constant term:

Estimates: 24.99067 32.18709 25.4849 22.41207 29.94554 21.09988

AR coefficient matrix

AR(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	-0.2549	-0.0318	-0.00785	-0.0937	-0.0318	-0.0413
[2,]	-0.0958	-0.3514	-0.10110	-0.2198	-0.1356	-0.0655
[3,]	-0.0563	-0.1252	-0.23015	-0.0530	0.0458	-0.0951
[4,]	-0.0600	0.0470	-0.09637	-0.2412	-0.0115	-0.0475
[5,]	-0.0096	-0.1545	-0.13204	0.0168	-0.3836	-0.0912
[6,]	-0.0734	0.0568	0.01668	0.1047	-0.0375	-0.3116

Residuals cov-matrix:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	249.11946	-43.29934	-20.65036	-26.47437	-34.31230	-47.12389
[2,]	-43.29934	209.54588	-35.47388	-25.50914	-39.11067	-23.38615
[3,]	-20.65036	-35.47388	231.91076	-26.78690	-32.36545	-35.79288
[4,]	-26.47437	-25.50914	-26.78690	191.08051	-25.45630	-37.50001

```

[5,] -34.31230 -39.11067 -32.36545 -25.45630 219.01535 -49.52583
[6,] -47.12389 -23.38615 -35.79288 -37.50001 -49.52583 227.51033
----
aic= 31.91982
bic= 32.43962

```

We test again with an increasing order of $AR(p)$ process until AIC and BIC criteria according to the p-values table of Extended Cross-Correlation Matrices. On the other hand, we test again with an increasing order of $MA(q)$ process until AIC and BIC criteria increase.

```
> VARMA(da = ms, p = 0, q = 2)
```

```
Number of parameters: 78
```

```

initial estimates: 17.0608 16.14 16.8686 15.7606 17.3092 16.8482 -0.3385 -0.1328
Par. lower-bounds: 15.2535 14.4622 15.1162 14.1715 15.5835 15.0994 -0.4832 -0.290
Par. upper-bounds: 18.8681 17.8178 18.621 17.3497 19.0349 18.5971 -0.1939 0.025 0
Final Estimates: 17.06261 16.14338 16.87696 15.75287 17.30524 16.85016 -0.37624

```

```
Coefficient(s):
```

	Estimate	Std. Error	t value	Pr(> t)
[1,]	1.706e+01	5.786e-04	29489.073	< 2e-16 ***
[2,]	1.614e+01	5.646e-04	28594.029	< 2e-16 ***
[3,]	1.688e+01	6.262e-04	26952.719	< 2e-16 ***
[4,]	1.575e+01	7.497e-04	21012.952	< 2e-16 ***
[5,]	1.731e+01	4.155e-04	41650.332	< 2e-16 ***
[6,]	1.685e+01	1.318e-04	127803.817	< 2e-16 ***
[7,]	-3.762e-01	1.221e-03	-308.169	< 2e-16 ***
[8,]	-1.777e-01	5.053e-02	-3.516	0.000438 ***
[9,]	-1.180e-01	4.845e-02	-2.436	0.014862 *
[10,]	-1.948e-01	2.369e-03	-82.247	< 2e-16 ***
[11,]	-1.360e-01	3.526e-02	-3.856	0.000115 ***
[12,]	-1.030e-01	3.132e-02	-3.289	0.001006 **
[13,]	-4.513e-02	5.278e-02	-0.855	0.392471
[14,]	-1.002e-01	8.119e-02	-1.234	0.217285
[15,]	-7.532e-02	2.370e-02	-3.178	0.001484 **
[16,]	-1.654e-02	2.298e-02	-0.720	0.471813
[17,]	-7.021e-02	3.121e-02	-2.250	0.024461 *
[18,]	-7.617e-04	1.259e-03	-0.605	0.545275

[19,]	-1.543e-01	NA	NA	NA	
[20,]	-4.222e-01	8.675e-03	-48.674	< 2e-16	***
[21,]	-1.653e-01	2.204e-02	-7.499	6.44e-14	***
[22,]	-2.913e-01	1.152e-03	-252.871	< 2e-16	***
[23,]	-1.479e-01	2.550e-02	-5.800	6.65e-09	***
[24,]	-7.491e-02	2.842e-02	-2.636	0.008395	**
[25,]	2.806e-02	3.567e-02	0.787	0.431485	
[26,]	1.130e-01	5.242e-02	2.155	0.031190	*
[27,]	3.697e-02	4.074e-02	0.908	0.364140	
[28,]	1.003e-01	NA	NA	NA	
[29,]	9.799e-02	2.723e-02	3.599	0.000319	***
[30,]	1.661e-01	5.877e-02	2.826	0.004713	**
[31,]	-1.253e-01	1.614e-02	-7.767	7.99e-15	***
[32,]	-1.923e-01	5.463e-02	-3.520	0.000432	***
[33,]	-3.150e-01	2.545e-03	-123.806	< 2e-16	***
[34,]	-1.659e-01	3.185e-03	-52.079	< 2e-16	***
[35,]	-7.430e-02	2.543e-02	-2.921	0.003485	**
[36,]	-1.662e-01	4.964e-02	-3.348	0.000813	***
[37,]	6.096e-02	5.146e-02	1.185	0.236170	
[38,]	4.330e-02	5.799e-02	0.747	0.455218	
[39,]	-1.097e-01	3.124e-02	-3.512	0.000445	***
[40,]	1.031e-03	1.386e-03	0.744	0.457168	
[41,]	-7.817e-02	3.029e-02	-2.581	0.009845	**
[42,]	-9.012e-03	9.739e-03	-0.925	0.354826	
[43,]	-1.887e-01	NA	NA	NA	
[44,]	-6.495e-02	4.010e-02	-1.620	0.105307	
[45,]	-1.834e-01	3.725e-03	-49.241	< 2e-16	***
[46,]	-3.911e-01	1.141e-03	-342.654	< 2e-16	***
[47,]	-9.685e-02	4.052e-02	-2.390	0.016837	*
[48,]	-1.530e-01	4.764e-02	-3.212	0.001318	**
[49,]	-6.991e-02	3.695e-02	-1.892	0.058494	.
[50,]	-8.884e-02	5.598e-02	-1.587	0.112536	
[51,]	2.736e-02	4.215e-02	0.649	0.516280	
[52,]	-5.608e-02	4.794e-02	-1.170	0.242127	
[53,]	-2.383e-02	2.606e-02	-0.914	0.360500	
[54,]	-7.124e-02	2.318e-02	-3.074	0.002115	**
[55,]	-7.637e-02	3.573e-02	-2.137	0.032578	*
[56,]	-2.083e-01	5.847e-02	-3.562	0.000368	***

[57,]	-1.797e-01	1.696e-02	-10.595	< 2e-16	***
[58,]	-7.456e-02	3.117e-02	-2.392	0.016755	*
[59,]	-4.541e-01	2.549e-05	-17813.074	< 2e-16	***
[60,]	-1.659e-01	4.455e-02	-3.723	0.000196	***
[61,]	2.156e-03	NA	NA	NA	
[62,]	1.351e-01	4.800e-02	2.814	0.004895	**
[63,]	1.230e-01	4.548e-02	2.704	0.006853	**
[64,]	-2.988e-02	3.110e-02	-0.961	0.336608	
[65,]	8.616e-02	2.950e-02	2.920	0.003496	**
[66,]	-1.090e-03	1.825e-03	-0.597	0.550303	
[67,]	-1.163e-01	1.500e-02	-7.752	9.10e-15	***
[68,]	4.948e-04	NA	NA	NA	
[69,]	-6.667e-02	3.334e-02	-1.999	0.045559	*
[70,]	6.035e-02	5.865e-02	1.029	0.303520	
[71,]	-1.028e-01	2.467e-02	-4.166	3.11e-05	***
[72,]	-4.094e-01	2.750e-03	-148.860	< 2e-16	***
[73,]	2.001e-02	2.745e-02	0.729	0.466060	
[74,]	-7.826e-02	5.285e-02	-1.481	0.138682	
[75,]	-2.153e-02	3.560e-02	-0.605	0.545339	
[76,]	1.841e-03	7.033e-03	0.262	0.793491	
[77,]	-9.108e-03	2.605e-02	-0.350	0.726671	
[78,]	1.101e-03	NA	NA	NA	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Estimates in matrix form:

Constant term:

Estimates: 17.06261 16.14338 16.87696 15.75287 17.30524 16.85016

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.3762	0.177684	0.1180	0.1948	0.1360	0.1030
[2,]	0.1543	0.422231	0.1653	0.2913	0.1479	0.0749
[3,]	0.1253	0.192280	0.3150	0.1659	0.0743	0.1662
[4,]	0.1887	0.064954	0.1834	0.3911	0.0969	0.1530
[5,]	0.0764	0.208272	0.1797	0.0746	0.4541	0.1659
[6,]	0.1163	-0.000495	0.0667	-0.0604	0.1028	0.4094

MA(2)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.04513	0.1002	0.0753	0.01654	0.07021	0.000762
[2,]	-0.02806	-0.1130	-0.0370	-0.10030	-0.09799	-0.166073
[3,]	-0.06096	-0.0433	0.1097	-0.00103	0.07817	0.009012
[4,]	0.06991	0.0888	-0.0274	0.05608	0.02383	0.071244
[5,]	-0.00216	-0.1351	-0.1230	0.02988	-0.08616	0.001090
[6,]	-0.02001	0.0783	0.0215	-0.00184	0.00911	-0.001101

Residuals cov-matrix:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	240.63121	-50.16081	-25.86440	-33.34352	-37.51956	-44.44446
[2,]	-50.16081	202.58509	-40.26305	-28.09830	-38.45038	-19.51139
[3,]	-25.86440	-40.26305	223.00077	-30.29289	-34.95261	-35.87696
[4,]	-33.34352	-28.09830	-30.29289	179.63272	-31.66058	-36.93944
[5,]	-37.51956	-38.45038	-34.95261	-31.66058	214.18997	-51.58625
[6,]	-44.44446	-19.51139	-35.87696	-36.93944	-51.58625	223.62178

aic= 31.60759

bic= 32.57292

> VARMA(da = ms, p = 2, q = 0)

Number of parameters: 78

initial estimates: 38.2175 40.4753 28.8049 24.6669 37.8677 31.8855 -0.3095 -0.105

Par. lower-bounds: 19.9562 23.6603 11.212 8.5182 20.5275 14.2935 -0.447 -0.2544 -

Par. upper-bounds: 56.4788 57.2903 46.3978 40.8156 55.2078 49.4776 -0.172 0.0438

Final Estimates: 38.55994 40.69883 28.76388 24.56109 37.8007 31.95405 -0.306867

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)
[1,]	38.5599354	9.0669687	4.253	2.11e-05 ***
[2,]	40.6988257	7.8105772	5.211	1.88e-07 ***
[3,]	28.7638777	8.7234661	3.297	0.000976 ***
[4,]	24.5610930	8.0395650	3.055	0.002250 **
[5,]	37.8006989	8.5259575	4.434	9.27e-06 ***
[6,]	31.9540535	8.7184525	3.665	0.000247 ***
[7,]	-0.3068674	0.0683459	-4.490	7.13e-06 ***
[8,]	-0.1044585	0.0744411	-1.403	0.160546

[9,]	-0.0517584	0.0693059	-0.747	0.455178	
[10,]	-0.1402392	0.0757317	-1.852	0.064056	.
[11,]	-0.0697342	0.0748867	-0.931	0.351752	
[12,]	-0.0481383	0.0739714	-0.651	0.515195	
[13,]	-0.1368381	0.0689886	-1.983	0.047313	*
[14,]	-0.1361312	0.0736214	-1.849	0.064447	.
[15,]	-0.0962955	0.0697753	-1.380	0.167562	
[16,]	-0.1269806	0.0759895	-1.671	0.094716	.
[17,]	-0.0623766	0.0748228	-0.834	0.404474	
[18,]	-0.0014979	0.0748301	-0.020	0.984030	
[19,]	-0.1220442	0.0620110	-1.968	0.049056	*
[20,]	-0.3897511	0.0675731	-5.768	8.03e-09	***
[21,]	-0.1421044	0.0628217	-2.262	0.023696	*
[22,]	-0.2701673	0.0687481	-3.930	8.50e-05	***
[23,]	-0.1381367	0.0677560	-2.039	0.041476	*
[24,]	-0.0615271	0.0661609	-0.930	0.352391	
[25,]	-0.0639516	0.0614643	-1.040	0.298122	
[26,]	-0.0413482	0.0660084	-0.626	0.531047	
[27,]	-0.1384389	0.0625098	-2.215	0.026782	*
[28,]	-0.1361943	0.0682038	-1.997	0.045839	*
[29,]	0.0154765	0.0664439	0.233	0.815819	
[30,]	0.0004761	0.0585910	0.008	0.993517	
[31,]	-0.0483861	0.0656519	-0.737	0.461116	
[32,]	-0.1175617	0.0715092	-1.644	0.100174	
[33,]	-0.2680022	0.0665934	-4.024	5.71e-05	***
[34,]	-0.0741369	0.0727448	-1.019	0.308138	
[35,]	0.0184186	0.0719365	0.256	0.797921	
[36,]	-0.1194046	0.0711519	-1.678	0.093315	.
[37,]	0.0586144	0.0662607	0.885	0.376371	
[38,]	0.0552380	0.0707133	0.781	0.434712	
[39,]	-0.1713669	0.0669880	-2.558	0.010522	*
[40,]	0.0256831	0.0729484	0.352	0.724784	
[41,]	-0.0170904	0.0717784	-0.238	0.811803	
[42,]	-0.0490791	0.0718987	-0.683	0.494851	
[43,]	-0.0803209	0.0602644	-1.333	0.182595	
[44,]	0.0439258	0.0656870	0.669	0.503678	
[45,]	-0.0910322	0.0611453	-1.489	0.136544	
[46,]	-0.2672869	0.0667695	-4.003	6.25e-05	***

[47,]	0.0071605	0.0661665	0.108	0.913822
[48,]	-0.0575473	0.0653458	-0.881	0.378503
[49,]	-0.0561614	0.0608751	-0.923	0.356233
[50,]	0.0392916	0.0650414	0.604	0.545776
[51,]	0.0410027	0.0615456	0.666	0.505273
[52,]	-0.0946342	0.0669787	-1.413	0.157685
[53,]	0.0090893	0.0660883	0.138	0.890610
[54,]	-0.0325365	0.0662728	-0.491	0.623463
[55,]	-0.0344147	0.0646238	-0.533	0.594353
[56,]	-0.1769967	0.0701748	-2.522	0.011662 *
[57,]	-0.1511397	0.0654986	-2.308	0.021025 *
[58,]	-0.0150055	0.0715819	-0.210	0.833959
[59,]	-0.4130061	0.0707585	-5.837	5.32e-09 ***
[60,]	-0.1416117	0.0700241	-2.022	0.043142 *
[61,]	-0.0308284	0.0650550	-0.474	0.635583
[62,]	-0.0015525	0.0683510	-0.023	0.981878
[63,]	-0.0138029	0.0658574	-0.210	0.833989
[64,]	-0.0634274	0.0717013	-0.885	0.376369
[65,]	-0.0618529	0.0705048	-0.877	0.380331
[66,]	-0.1190956	0.0706131	-1.687	0.091682 .
[67,]	-0.1106570	0.0654443	-1.691	0.090864 .
[68,]	0.0089904	0.0713653	0.126	0.899750
[69,]	-0.0235446	0.0664132	-0.355	0.722952
[70,]	0.0943760	0.0725146	1.301	0.193096
[71,]	-0.0985929	0.0717528	-1.374	0.169422
[72,]	-0.3813630	0.0709732	-5.373	7.73e-08 ***
[73,]	-0.0735066	0.0660722	-1.113	0.265915
[74,]	-0.1091607	0.0705167	-1.548	0.121620
[75,]	-0.0349510	0.0667889	-0.523	0.600762
[76,]	0.0589096	0.0727306	0.810	0.417957
[77,]	-0.0833493	0.0716323	-1.164	0.244597
[78,]	-0.1313317	0.0719433	-1.825	0.067927 .

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Estimates in matrix form:

Constant term:

Estimates: 38.55994 40.69883 28.76388 24.56109 37.8007 31.95405

AR coefficient matrix

AR(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	-0.3069	-0.10446	-0.0518	-0.1402	-0.06973	-0.0481
[2,]	-0.1220	-0.38975	-0.1421	-0.2702	-0.13814	-0.0615
[3,]	-0.0484	-0.11756	-0.2680	-0.0741	0.01842	-0.1194
[4,]	-0.0803	0.04393	-0.0910	-0.2673	0.00716	-0.0575
[5,]	-0.0344	-0.17700	-0.1511	-0.0150	-0.41301	-0.1416
[6,]	-0.1107	0.00899	-0.0235	0.0944	-0.09859	-0.3814

AR(2)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	-0.1368	-0.13613	-0.0963	-0.1270	-0.06238	-0.001498
[2,]	-0.0640	-0.04135	-0.1384	-0.1362	0.01548	0.000476
[3,]	0.0586	0.05524	-0.1714	0.0257	-0.01709	-0.049079
[4,]	-0.0562	0.03929	0.0410	-0.0946	0.00909	-0.032536
[5,]	-0.0308	-0.00155	-0.0138	-0.0634	-0.06185	-0.119096
[6,]	-0.0735	-0.10916	-0.0350	0.0589	-0.08335	-0.131332

Residuals cov-matrix:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	242.18248	-48.66776	-18.76413	-26.99190	-33.10314	-46.86434
[2,]	-48.66776	203.14472	-38.22320	-26.55864	-38.54999	-20.77618
[3,]	-18.76413	-38.22320	223.07946	-24.31145	-32.91231	-35.74328
[4,]	-26.99190	-26.55864	-24.31145	187.67570	-27.47700	-36.49828
[5,]	-33.10314	-38.54999	-32.91231	-27.47700	216.67609	-51.46190
[6,]	-46.86434	-20.77618	-35.74328	-36.49828	-51.46190	221.39423

aic= 31.90784

bic= 32.87318

> VARMA(da = ms, p = 0, q = 3)

Number of parameters: 114

initial estimates: 17.0551 16.1475 16.86 15.6173 17.2138 16.8454 -0.3401 -0.1346

Par. lower-bounds: 15.2623 14.501 15.1511 14.0824 15.5005 15.1104 -0.4849 -0.2927

Par. upper-bounds: 18.8479 17.794 18.5689 17.1522 18.927 18.5805 -0.1952 0.0235

Final Estimates: 17.10743 16.19529 16.90949 15.66714 17.2478 16.88176 -0.335934

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)	
[1,]	17.107432	0.027062	632.148	< 2e-16	***
[2,]	16.195288	0.210609	76.897	< 2e-16	***
[3,]	16.909487	0.008058	2098.393	< 2e-16	***
[4,]	15.667140	0.228470	68.574	< 2e-16	***
[5,]	17.247796	0.047923	359.908	< 2e-16	***
[6,]	16.881756	0.001601	10543.538	< 2e-16	***
[7,]	-0.335934	0.014330	-23.443	< 2e-16	***
[8,]	-0.115701	0.073943	-1.565	0.117646	
[9,]	-0.074187	0.084429	-0.879	0.379568	
[10,]	-0.157344	0.057700	-2.727	0.006393	**
[11,]	-0.093724	0.070928	-1.321	0.186372	
[12,]	-0.097785	0.062106	-1.574	0.115377	
[13,]	-0.023364	0.074773	-0.312	0.754682	
[14,]	-0.058420	0.078861	-0.741	0.458812	
[15,]	-0.033768	0.087342	-0.387	0.699037	
[16,]	-0.025505	0.066094	-0.386	0.699575	
[17,]	-0.034084	0.075829	-0.449	0.653086	
[18,]	0.053568	0.082888	0.646	0.518100	
[19,]	-0.012034	0.067922	-0.177	0.859370	
[20,]	0.021436	0.084296	0.254	0.799271	
[21,]	0.003260	0.078046	0.042	0.966680	
[22,]	0.154810	0.078594	1.970	0.048867	*
[23,]	0.031045	0.111387	0.279	0.780465	
[24,]	-0.038684	0.073012	-0.530	0.596229	
[25,]	-0.147674	0.051874	-2.847	0.004416	**
[26,]	-0.417979	NA	NA	NA	
[27,]	-0.155020	0.059466	-2.607	0.009137	**
[28,]	-0.261048	0.013496	-19.343	< 2e-16	***
[29,]	-0.147582	0.057761	-2.555	0.010617	*
[30,]	-0.044659	0.046050	-0.970	0.332156	
[31,]	0.025767	0.068655	0.375	0.707426	
[32,]	0.099073	0.067781	1.462	0.143836	
[33,]	-0.011755	0.052967	-0.222	0.824372	
[34,]	0.067057	0.054756	1.225	0.220709	
[35,]	0.106374	0.061348	1.734	0.082929	.
[36,]	0.113649	0.053298	2.132	0.032981	*

[37,]	-0.008315	0.072397	-0.115	0.908565	
[38,]	-0.029264	0.078396	-0.373	0.708939	
[39,]	0.054599	0.081136	0.673	0.500988	
[40,]	0.075575	0.079031	0.956	0.338939	
[41,]	-0.113768	0.085156	-1.336	0.181551	
[42,]	0.064935	0.079305	0.819	0.412901	
[43,]	-0.103181	0.059241	-1.742	0.081556	.
[44,]	-0.175927	0.064505	-2.727	0.006385	**
[45,]	-0.298460	0.023745	-12.569	< 2e-16	***
[46,]	-0.149048	0.057085	-2.611	0.009028	**
[47,]	-0.053984	0.061124	-0.883	0.377134	
[48,]	-0.178983	0.042166	-4.245	2.19e-05	***
[49,]	0.097418	0.074722	1.304	0.192324	
[50,]	0.094051	0.075417	1.247	0.212367	
[51,]	-0.157135	0.069581	-2.258	0.023926	*
[52,]	0.033068	0.074671	0.443	0.657871	
[53,]	-0.022299	0.070492	-0.316	0.751751	
[54,]	0.039382	0.065796	0.599	0.549481	
[55,]	-0.108302	0.063118	-1.716	0.086188	.
[56,]	-0.066047	0.072268	-0.914	0.360759	
[57,]	0.018804	0.065737	0.286	0.774836	
[58,]	-0.181172	0.051280	-3.533	0.000411	***
[59,]	-0.031643	0.085153	-0.372	0.710187	
[60,]	-0.003576	0.142034	-0.025	0.979912	
[61,]	-0.154906	0.047792	-3.241	0.001190	**
[62,]	-0.047488	0.059374	-0.800	0.423818	
[63,]	-0.170474	0.050062	-3.405	0.000661	***
[64,]	-0.371214	0.003701	-100.295	< 2e-16	***
[65,]	-0.077737	0.055903	-1.391	0.164356	
[66,]	-0.136866	0.049832	-2.747	0.006023	**
[67,]	-0.056000	0.062516	-0.896	0.370382	
[68,]	-0.023583	0.071398	-0.330	0.741166	
[69,]	0.040164	0.069459	0.578	0.563107	
[70,]	-0.054178	0.068944	-0.786	0.431972	
[71,]	0.018652	0.083094	0.224	0.822390	
[72,]	-0.030904	0.082778	-0.373	0.708898	
[73,]	0.039250	0.064352	0.610	0.541914	
[74,]	-0.107638	0.073261	-1.469	0.141768	

[75,]	-0.041052	0.065466	-0.627	0.530612	
[76,]	-0.027104	0.067730	-0.400	0.689023	
[77,]	0.029764	0.070978	0.419	0.674972	
[78,]	-0.059543	0.066925	-0.890	0.373635	
[79,]	-0.097767	0.065245	-1.498	0.134014	
[80,]	-0.223073	0.066536	-3.353	0.000800	***
[81,]	-0.199641	0.056156	-3.555	0.000378	***
[82,]	-0.088210	0.069740	-1.265	0.205925	
[83,]	-0.466032	0.009285	-50.191	< 2e-16	***
[84,]	-0.172216	0.044197	-3.897	9.76e-05	***
[85,]	-0.001019	NA	NA	NA	
[86,]	0.123046	0.073908	1.665	0.095942	.
[87,]	0.125467	0.081518	1.539	0.123774	
[88,]	-0.067321	0.080798	-0.833	0.404733	
[89,]	0.090865	0.061344	1.481	0.138542	
[90,]	-0.031725	0.066014	-0.481	0.630818	
[91,]	-0.067837	0.092026	-0.737	0.461031	
[92,]	-0.066384	0.091184	-0.728	0.466598	
[93,]	0.007481	0.098242	0.076	0.939303	
[94,]	-0.049669	0.108316	-0.459	0.646554	
[95,]	-0.090246	0.094994	-0.950	0.342106	
[96,]	-0.013811	0.064718	-0.213	0.831017	
[97,]	-0.136864	0.062902	-2.176	0.029569	*
[98,]	-0.017236	0.073488	-0.235	0.814566	
[99,]	-0.079726	0.070678	-1.128	0.259313	
[100,]	0.058238	0.078383	0.743	0.457482	
[101,]	-0.128716	0.039636	-3.247	0.001164	**
[102,]	-0.388948	0.010396	-37.414	< 2e-16	***
[103,]	-0.027666	0.096977	-0.285	0.775424	
[104,]	-0.122505	0.105783	-1.158	0.246834	
[105,]	-0.036847	0.117895	-0.313	0.754629	
[106,]	-0.015272	0.094138	-0.162	0.871127	
[107,]	-0.026163	0.115466	-0.227	0.820746	
[108,]	-0.036998	0.094498	-0.392	0.695415	
[109,]	0.089095	0.092990	0.958	0.338004	
[110,]	0.074351	0.080558	0.923	0.356033	
[111,]	-0.038479	0.077146	-0.499	0.617937	
[112,]	-0.011770	0.090333	-0.130	0.896331	

```
[113,] -0.005509    0.071237    -0.077 0.938356
[114,] -0.032641    0.096916    -0.337 0.736272
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Estimates in matrix form:

Constant term:

```
Estimates: 17.10743 16.19529 16.90949 15.66714 17.2478 16.88176
```

MA coefficient matrix

MA(1)-matrix

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.3359 0.1157 0.0742 0.1573 0.0937 0.0978
[2,] 0.1477 0.4180 0.1550 0.2610 0.1476 0.0447
[3,] 0.1032 0.1759 0.2985 0.1490 0.0540 0.1790
[4,] 0.1549 0.0475 0.1705 0.3712 0.0777 0.1369
[5,] 0.0978 0.2231 0.1996 0.0882 0.4660 0.1722
[6,] 0.1369 0.0172 0.0797 -0.0582 0.1287 0.3889
```

MA(2)-matrix

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.02336 0.0584 0.0338 0.0255 0.0341 -0.0536
[2,] -0.02577 -0.0991 0.0118 -0.0671 -0.1064 -0.1136
[3,] -0.09742 -0.0941 0.1571 -0.0331 0.0223 -0.0394
[4,] 0.05600 0.0236 -0.0402 0.0542 -0.0187 0.0309
[5,] 0.00102 -0.1230 -0.1255 0.0673 -0.0909 0.0317
[6,] 0.02767 0.1225 0.0368 0.0153 0.0262 0.0370
```

MA(3)-matrix

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.01203 -0.0214 -0.00326 -0.1548 -0.03104 0.03868
[2,] 0.00831 0.0293 -0.05460 -0.0756 0.11377 -0.06494
[3,] 0.10830 0.0660 -0.01880 0.1812 0.03164 0.00358
[4,] -0.03925 0.1076 0.04105 0.0271 -0.02976 0.05954
[5,] 0.06784 0.0664 -0.00748 0.0497 0.09025 0.01381
[6,] -0.08910 -0.0744 0.03848 0.0118 0.00551 0.03264
```

Residuals cov-matrix:

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 234.33865 -50.51449 -18.50005 -30.31952 -34.19187 -44.94363
[2,] -50.51449 196.61893 -39.82807 -23.89376 -40.23841 -18.52192
```

```

[3,] -18.50005 -39.82807 215.02508 -30.68709 -37.29504 -35.98137
[4,] -30.31952 -23.89376 -30.68709 175.93246 -31.90710 -38.24146
[5,] -34.19187 -40.23841 -37.29504 -31.90710 214.48450 -51.12122
[6,] -44.94363 -18.52192 -35.98137 -38.24146 -51.12122 221.70895

```

aic= 31.73723

bic= 33.1481

> VARMA(da = ms, p = 0, q = 3)

Number of parameters: 114

initial estimates: 17.0551 16.1475 16.86 15.6173 17.2138 16.8454 -0.3401 -0.1346

Par. lower-bounds: 15.2623 14.501 15.1511 14.0824 15.5005 15.1104 -0.4849 -0.2927

Par. upper-bounds: 18.8479 17.794 18.5689 17.1522 18.927 18.5805 -0.1952 0.0235 0

Final Estimates: 17.10743 16.19529 16.90949 15.66714 17.2478 16.88176 -0.335934

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)	
[1,]	17.107432	0.027062	632.148	< 2e-16	***
[2,]	16.195288	0.210609	76.897	< 2e-16	***
[3,]	16.909487	0.008058	2098.393	< 2e-16	***
[4,]	15.667140	0.228470	68.574	< 2e-16	***
[5,]	17.247796	0.047923	359.908	< 2e-16	***
[6,]	16.881756	0.001601	10543.538	< 2e-16	***
[7,]	-0.335934	0.014330	-23.443	< 2e-16	***
[8,]	-0.115701	0.073943	-1.565	0.117646	
[9,]	-0.074187	0.084429	-0.879	0.379568	
[10,]	-0.157344	0.057700	-2.727	0.006393	**
[11,]	-0.093724	0.070928	-1.321	0.186372	
[12,]	-0.097785	0.062106	-1.574	0.115377	
[13,]	-0.023364	0.074773	-0.312	0.754682	
[14,]	-0.058420	0.078861	-0.741	0.458812	
[15,]	-0.033768	0.087342	-0.387	0.699037	
[16,]	-0.025505	0.066094	-0.386	0.699575	
[17,]	-0.034084	0.075829	-0.449	0.653086	
[18,]	0.053568	0.082888	0.646	0.518100	
[19,]	-0.012034	0.067922	-0.177	0.859370	
[20,]	0.021436	0.084296	0.254	0.799271	

[21,]	0.003260	0.078046	0.042	0.966680	
[22,]	0.154810	0.078594	1.970	0.048867	*
[23,]	0.031045	0.111387	0.279	0.780465	
[24,]	-0.038684	0.073012	-0.530	0.596229	
[25,]	-0.147674	0.051874	-2.847	0.004416	**
[26,]	-0.417979	NA	NA	NA	
[27,]	-0.155020	0.059466	-2.607	0.009137	**
[28,]	-0.261048	0.013496	-19.343	< 2e-16	***
[29,]	-0.147582	0.057761	-2.555	0.010617	*
[30,]	-0.044659	0.046050	-0.970	0.332156	
[31,]	0.025767	0.068655	0.375	0.707426	
[32,]	0.099073	0.067781	1.462	0.143836	
[33,]	-0.011755	0.052967	-0.222	0.824372	
[34,]	0.067057	0.054756	1.225	0.220709	
[35,]	0.106374	0.061348	1.734	0.082929	.
[36,]	0.113649	0.053298	2.132	0.032981	*
[37,]	-0.008315	0.072397	-0.115	0.908565	
[38,]	-0.029264	0.078396	-0.373	0.708939	
[39,]	0.054599	0.081136	0.673	0.500988	
[40,]	0.075575	0.079031	0.956	0.338939	
[41,]	-0.113768	0.085156	-1.336	0.181551	
[42,]	0.064935	0.079305	0.819	0.412901	
[43,]	-0.103181	0.059241	-1.742	0.081556	.
[44,]	-0.175927	0.064505	-2.727	0.006385	**
[45,]	-0.298460	0.023745	-12.569	< 2e-16	***
[46,]	-0.149048	0.057085	-2.611	0.009028	**
[47,]	-0.053984	0.061124	-0.883	0.377134	
[48,]	-0.178983	0.042166	-4.245	2.19e-05	***
[49,]	0.097418	0.074722	1.304	0.192324	
[50,]	0.094051	0.075417	1.247	0.212367	
[51,]	-0.157135	0.069581	-2.258	0.023926	*
[52,]	0.033068	0.074671	0.443	0.657871	
[53,]	-0.022299	0.070492	-0.316	0.751751	
[54,]	0.039382	0.065796	0.599	0.549481	
[55,]	-0.108302	0.063118	-1.716	0.086188	.
[56,]	-0.066047	0.072268	-0.914	0.360759	
[57,]	0.018804	0.065737	0.286	0.774836	
[58,]	-0.181172	0.051280	-3.533	0.000411	***

[59,]	-0.031643	0.085153	-0.372	0.710187	
[60,]	-0.003576	0.142034	-0.025	0.979912	
[61,]	-0.154906	0.047792	-3.241	0.001190	**
[62,]	-0.047488	0.059374	-0.800	0.423818	
[63,]	-0.170474	0.050062	-3.405	0.000661	***
[64,]	-0.371214	0.003701	-100.295	< 2e-16	***
[65,]	-0.077737	0.055903	-1.391	0.164356	
[66,]	-0.136866	0.049832	-2.747	0.006023	**
[67,]	-0.056000	0.062516	-0.896	0.370382	
[68,]	-0.023583	0.071398	-0.330	0.741166	
[69,]	0.040164	0.069459	0.578	0.563107	
[70,]	-0.054178	0.068944	-0.786	0.431972	
[71,]	0.018652	0.083094	0.224	0.822390	
[72,]	-0.030904	0.082778	-0.373	0.708898	
[73,]	0.039250	0.064352	0.610	0.541914	
[74,]	-0.107638	0.073261	-1.469	0.141768	
[75,]	-0.041052	0.065466	-0.627	0.530612	
[76,]	-0.027104	0.067730	-0.400	0.689023	
[77,]	0.029764	0.070978	0.419	0.674972	
[78,]	-0.059543	0.066925	-0.890	0.373635	
[79,]	-0.097767	0.065245	-1.498	0.134014	
[80,]	-0.223073	0.066536	-3.353	0.000800	***
[81,]	-0.199641	0.056156	-3.555	0.000378	***
[82,]	-0.088210	0.069740	-1.265	0.205925	
[83,]	-0.466032	0.009285	-50.191	< 2e-16	***
[84,]	-0.172216	0.044197	-3.897	9.76e-05	***
[85,]	-0.001019	NA	NA	NA	
[86,]	0.123046	0.073908	1.665	0.095942	.
[87,]	0.125467	0.081518	1.539	0.123774	
[88,]	-0.067321	0.080798	-0.833	0.404733	
[89,]	0.090865	0.061344	1.481	0.138542	
[90,]	-0.031725	0.066014	-0.481	0.630818	
[91,]	-0.067837	0.092026	-0.737	0.461031	
[92,]	-0.066384	0.091184	-0.728	0.466598	
[93,]	0.007481	0.098242	0.076	0.939303	
[94,]	-0.049669	0.108316	-0.459	0.646554	
[95,]	-0.090246	0.094994	-0.950	0.342106	
[96,]	-0.013811	0.064718	-0.213	0.831017	

[97,]	-0.136864	0.062902	-2.176	0.029569	*
[98,]	-0.017236	0.073488	-0.235	0.814566	
[99,]	-0.079726	0.070678	-1.128	0.259313	
[100,]	0.058238	0.078383	0.743	0.457482	
[101,]	-0.128716	0.039636	-3.247	0.001164	**
[102,]	-0.388948	0.010396	-37.414	< 2e-16	***
[103,]	-0.027666	0.096977	-0.285	0.775424	
[104,]	-0.122505	0.105783	-1.158	0.246834	
[105,]	-0.036847	0.117895	-0.313	0.754629	
[106,]	-0.015272	0.094138	-0.162	0.871127	
[107,]	-0.026163	0.115466	-0.227	0.820746	
[108,]	-0.036998	0.094498	-0.392	0.695415	
[109,]	0.089095	0.092990	0.958	0.338004	
[110,]	0.074351	0.080558	0.923	0.356033	
[111,]	-0.038479	0.077146	-0.499	0.617937	
[112,]	-0.011770	0.090333	-0.130	0.896331	
[113,]	-0.005509	0.071237	-0.077	0.938356	
[114,]	-0.032641	0.096916	-0.337	0.736272	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Estimates in matrix form:

Constant term:

Estimates: 17.10743 16.19529 16.90949 15.66714 17.2478 16.88176

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.3359	0.1157	0.0742	0.1573	0.0937	0.0978
[2,]	0.1477	0.4180	0.1550	0.2610	0.1476	0.0447
[3,]	0.1032	0.1759	0.2985	0.1490	0.0540	0.1790
[4,]	0.1549	0.0475	0.1705	0.3712	0.0777	0.1369
[5,]	0.0978	0.2231	0.1996	0.0882	0.4660	0.1722
[6,]	0.1369	0.0172	0.0797	-0.0582	0.1287	0.3889

MA(2)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.02336	0.0584	0.0338	0.0255	0.0341	-0.0536
[2,]	-0.02577	-0.0991	0.0118	-0.0671	-0.1064	-0.1136
[3,]	-0.09742	-0.0941	0.1571	-0.0331	0.0223	-0.0394

```

[4,] 0.05600 0.0236 -0.0402 0.0542 -0.0187 0.0309
[5,] 0.00102 -0.1230 -0.1255 0.0673 -0.0909 0.0317
[6,] 0.02767 0.1225 0.0368 0.0153 0.0262 0.0370

```

MA(3)-matrix

```

      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.01203 -0.0214 -0.00326 -0.1548 -0.03104 0.03868
[2,] 0.00831 0.0293 -0.05460 -0.0756 0.11377 -0.06494
[3,] 0.10830 0.0660 -0.01880 0.1812 0.03164 0.00358
[4,] -0.03925 0.1076 0.04105 0.0271 -0.02976 0.05954
[5,] 0.06784 0.0664 -0.00748 0.0497 0.09025 0.01381
[6,] -0.08910 -0.0744 0.03848 0.0118 0.00551 0.03264

```

Residuals cov-matrix:

```

      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 234.33865 -50.51449 -18.50005 -30.31952 -34.19187 -44.94363
[2,] -50.51449 196.61893 -39.82807 -23.89376 -40.23841 -18.52192
[3,] -18.50005 -39.82807 215.02508 -30.68709 -37.29504 -35.98137
[4,] -30.31952 -23.89376 -30.68709 175.93246 -31.90710 -38.24146
[5,] -34.19187 -40.23841 -37.29504 -31.90710 214.48450 -51.12122
[6,] -44.94363 -18.52192 -35.98137 -38.24146 -51.12122 221.70895

```

aic= 31.73723

bic= 33.1481

> VARMA(da = ms, p = 0, q = 4)

Number of parameters: 150

initial estimates: 17.0209 16.1175 16.8432 15.6158 17.2289 16.7453 -0.3692 -0.160

Par. lower-bounds: 15.246 14.4809 15.1481 14.096 15.5278 15.0255 -0.5171 -0.321 -

Par. upper-bounds: 18.7957 17.754 18.5383 17.1355 18.9301 18.465 -0.2212 -1e-04 0

Final Estimates: 17.02086 16.11745 16.84315 15.61575 17.22894 16.74528 -0.36918

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)
[1,]	17.0208551	NA	NA	NA
[2,]	16.1174517	NA	NA	NA
[3,]	16.8431513	NA	NA	NA
[4,]	15.6157509	NA	NA	NA

[5,]	17.2289429	NA	NA	NA
[6,]	16.7452829	NA	NA	NA
[7,]	-0.3691800	NA	NA	NA
[8,]	-0.1605305	NA	NA	NA
[9,]	-0.1198674	NA	NA	NA
[10,]	-0.1924934	NA	NA	NA
[11,]	-0.1280118	NA	NA	NA
[12,]	-0.1070864	NA	NA	NA
[13,]	-0.0594227	NA	NA	NA
[14,]	-0.0909700	NA	NA	NA
[15,]	-0.0807205	NA	NA	NA
[16,]	-0.0803675	NA	NA	NA
[17,]	-0.0847654	NA	NA	NA
[18,]	-0.0108178	NA	NA	NA
[19,]	-0.0537088	NA	NA	NA
[20,]	-0.0450547	NA	NA	NA
[21,]	-0.0440276	NA	NA	NA
[22,]	0.1019318	NA	NA	NA
[23,]	-0.0130665	NA	NA	NA
[24,]	-0.1046756	NA	NA	NA
[25,]	-0.1174081	NA	NA	NA
[26,]	-0.0239203	NA	NA	NA
[27,]	0.0294889	NA	NA	NA
[28,]	-0.0821846	NA	NA	NA
[29,]	-0.0861803	NA	NA	NA
[30,]	-0.1702660	NA	NA	NA
[31,]	-0.1254994	NA	NA	NA
[32,]	-0.3790385	NA	NA	NA
[33,]	-0.0972574	NA	NA	NA
[34,]	-0.2295878	NA	NA	NA
[35,]	-0.1127032	NA	NA	NA
[36,]	-0.0446148	NA	NA	NA
[37,]	0.0333602	NA	NA	NA
[38,]	0.1201009	NA	NA	NA
[39,]	-0.0061985	NA	NA	NA
[40,]	0.0764093	NA	NA	NA
[41,]	0.1287304	NA	NA	NA
[42,]	0.1332082	NA	NA	NA

[43,]	-0.0530742	NA	NA	NA
[44,]	-0.0770644	NA	NA	NA
[45,]	0.0874328	NA	NA	NA
[46,]	0.0429783	NA	NA	NA
[47,]	-0.1134411	NA	NA	NA
[48,]	0.0747375	NA	NA	NA
[49,]	0.0457877	NA	NA	NA
[50,]	0.0400902	NA	NA	NA
[51,]	-0.0910974	NA	NA	NA
[52,]	0.0142849	NA	NA	NA
[53,]	0.0044488	NA	NA	NA
[54,]	0.0726181	NA	NA	NA
[55,]	-0.1369927	NA	NA	NA
[56,]	-0.1868080	NA	NA	NA
[57,]	-0.3343318	NA	NA	NA
[58,]	-0.1612709	NA	NA	NA
[59,]	-0.0616532	NA	NA	NA
[60,]	-0.2011680	NA	NA	NA
[61,]	0.0329598	NA	NA	NA
[62,]	0.0143967	NA	NA	NA
[63,]	-0.1686445	NA	NA	NA
[64,]	-0.0169385	NA	NA	NA
[65,]	-0.0779866	NA	NA	NA
[66,]	-0.0219579	NA	NA	NA
[67,]	-0.1818525	NA	NA	NA
[68,]	-0.1451735	NA	NA	NA
[69,]	-0.0146626	NA	NA	NA
[70,]	-0.2794560	NA	NA	NA
[71,]	-0.1143499	NA	NA	NA
[72,]	-0.1267478	NA	NA	NA
[73,]	-0.0637562	NA	NA	NA
[74,]	0.0296499	NA	NA	NA
[75,]	-0.1023721	NA	NA	NA
[76,]	-0.1123319	NA	NA	NA
[77,]	0.0033234	NA	NA	NA
[78,]	-0.0952843	NA	NA	NA
[79,]	-0.1282258	NA	NA	NA
[80,]	-0.0322163	NA	NA	NA

[81,]	-0.1411407	NA	NA	NA
[82,]	-0.3218872	NA	NA	NA
[83,]	-0.0380005	NA	NA	NA
[84,]	-0.1106148	NA	NA	NA
[85,]	-0.0639976	NA	NA	NA
[86,]	-0.0452716	NA	NA	NA
[87,]	0.0412492	NA	NA	NA
[88,]	-0.0529058	NA	NA	NA
[89,]	-0.0024267	NA	NA	NA
[90,]	-0.0509877	NA	NA	NA
[91,]	0.0003873	NA	NA	NA
[92,]	-0.1056882	NA	NA	NA
[93,]	-0.0479571	NA	NA	NA
[94,]	-0.0267506	NA	NA	NA
[95,]	0.0443057	NA	NA	NA
[96,]	-0.1346379	NA	NA	NA
[97,]	0.0539895	NA	NA	NA
[98,]	-0.0409914	NA	NA	NA
[99,]	0.0096585	NA	NA	NA
[100,]	0.0731891	NA	NA	NA
[101,]	0.0092356	NA	NA	NA
[102,]	0.1391648	NA	NA	NA
[103,]	-0.0547291	NA	NA	NA
[104,]	-0.1833405	NA	NA	NA
[105,]	-0.1683858	NA	NA	NA
[106,]	-0.0494505	NA	NA	NA
[107,]	-0.4246831	NA	NA	NA
[108,]	-0.1445278	NA	NA	NA
[109,]	-0.0245223	NA	NA	NA
[110,]	0.1042128	NA	NA	NA
[111,]	0.0935169	NA	NA	NA
[112,]	-0.0568948	NA	NA	NA
[113,]	0.0908082	NA	NA	NA
[114,]	-0.0333148	NA	NA	NA
[115,]	-0.0817539	NA	NA	NA
[116,]	-0.0772539	NA	NA	NA
[117,]	-0.0086847	NA	NA	NA
[118,]	-0.0874917	NA	NA	NA

[119,]	-0.1329053	NA	NA	NA
[120,]	-0.0204599	NA	NA	NA
[121,]	-0.0933581	NA	NA	NA
[122,]	-0.0649545	NA	NA	NA
[123,]	-0.1017486	NA	NA	NA
[124,]	0.0031242	NA	NA	NA
[125,]	-0.0061118	NA	NA	NA
[126,]	-0.1216622	NA	NA	NA
[127,]	-0.1077692	NA	NA	NA
[128,]	-0.0016106	NA	NA	NA
[129,]	-0.0627058	NA	NA	NA
[130,]	0.0728175	NA	NA	NA
[131,]	-0.1270681	NA	NA	NA
[132,]	-0.3930464	NA	NA	NA
[133,]	-0.0340759	NA	NA	NA
[134,]	-0.1119153	NA	NA	NA
[135,]	-0.0490553	NA	NA	NA
[136,]	-0.0213924	NA	NA	NA
[137,]	-0.0345754	NA	NA	NA
[138,]	-0.0270147	NA	NA	NA
[139,]	0.0936256	NA	NA	NA
[140,]	0.0874633	NA	NA	NA
[141,]	-0.0386226	NA	NA	NA
[142,]	-0.0337279	NA	NA	NA
[143,]	-0.0200017	NA	NA	NA
[144,]	-0.0106433	NA	NA	NA
[145,]	-0.1105260	NA	NA	NA
[146,]	-0.1473139	NA	NA	NA
[147,]	-0.0927002	NA	NA	NA
[148,]	-0.0951108	NA	NA	NA
[149,]	-0.1058074	NA	NA	NA
[150,]	-0.0013223	NA	NA	NA

Estimates in matrix form:

Constant term:

Estimates: 17.02086 16.11745 16.84315 15.61575 17.22894 16.74528

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.3692	0.16053	0.1199	0.1925	0.1280	0.1071
[2,]	0.1255	0.37904	0.0973	0.2296	0.1127	0.0446
[3,]	0.1370	0.18681	0.3343	0.1613	0.0617	0.2012
[4,]	0.1282	0.03222	0.1411	0.3219	0.0380	0.1106
[5,]	0.0547	0.18334	0.1684	0.0495	0.4247	0.1445
[6,]	0.1078	0.00161	0.0627	-0.0728	0.1271	0.3930

MA(2)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.0594	0.0910	0.0807	0.0804	0.08477	0.0108
[2,]	-0.0334	-0.1201	0.0062	-0.0764	-0.12873	-0.1332
[3,]	-0.0330	-0.0144	0.1686	0.0169	0.07799	0.0220
[4,]	0.0640	0.0453	-0.0412	0.0529	0.00243	0.0510
[5,]	0.0245	-0.1042	-0.0935	0.0569	-0.09081	0.0333
[6,]	0.0341	0.1119	0.0491	0.0214	0.03458	0.0270

MA(3)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.053709	0.0451	0.04403	-0.1019	0.0131	0.1047
[2,]	0.053074	0.0771	-0.08743	-0.0430	0.1134	-0.0747
[3,]	0.181853	0.1452	0.01466	0.2795	0.1143	0.1267
[4,]	-0.000387	0.1057	0.04796	0.0268	-0.0443	0.1346
[5,]	0.081754	0.0773	0.00868	0.0875	0.1329	0.0205
[6,]	-0.093626	-0.0875	0.03862	0.0337	0.0200	0.0106

MA(4)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.1174	0.0239	-0.02949	0.08218	0.08618	0.17027
[2,]	-0.0458	-0.0401	0.09110	-0.01428	-0.00445	-0.07262
[3,]	0.0638	-0.0296	0.10237	0.11233	-0.00332	0.09528
[4,]	-0.0540	0.0410	-0.00966	-0.07319	-0.00924	-0.13916
[5,]	0.0934	0.0650	0.10175	-0.00312	0.00611	0.12166
[6,]	0.1105	0.1473	0.09270	0.09511	0.10581	0.00132

Residuals cov-matrix:

	[,1]	[,2]	[,3]	[,4]	[,5]
[1,]	3.517474e+58	9.433449e+57	4.142231e+58	1.745148e+58	2.612773e+58
[2,]	9.433449e+57	2.529940e+57	1.110897e+58	4.680282e+57	7.007147e+57
[3,]	4.142231e+58	1.110897e+58	4.877955e+58	2.055114e+58	3.076841e+58
[4,]	1.745148e+58	4.680282e+57	2.055114e+58	8.658324e+57	1.296293e+58

```

[5,] 2.612773e+58 7.007147e+57 3.076841e+58 1.296293e+58 1.940762e+58
[6,] 2.183940e+58 5.857068e+57 2.571841e+58 1.083533e+58 1.622226e+58
      [,6]
[1,] 2.183940e+58
[2,] 5.857068e+57
[3,] 2.571841e+58
[4,] 1.083533e+58
[5,] 1.622226e+58
[6,] 1.355971e+58
----
aic=  NaN
bic=  NaN

```

```
> VARMA(da = ms, p = 0, q = 5)
```

```
Number of parameters: 186
```

```

initial estimates: 17.0284 15.9837 16.8658 15.6064 17.2742 16.7432 -0.3744 -0.185
Par. lower-bounds: 15.2616 14.3845 15.1764 14.1046 15.5931 15.0399 -0.5225 -0.347
Par. upper-bounds: 18.7952 17.5828 18.5552 17.1082 18.9554 18.4465 -0.2263 -0.022
Final Estimates: 17.02844 15.98368 16.8658 15.60638 17.27424 16.74323 -0.374417

```

```
Coefficient(s):
```

	Estimate	Std. Error	t value	Pr(> t)
[1,]	17.0284367	NA	NA	NA
[2,]	15.9836785	NA	NA	NA
[3,]	16.8658044	NA	NA	NA
[4,]	15.6063822	NA	NA	NA
[5,]	17.2742405	NA	NA	NA
[6,]	16.7432293	NA	NA	NA
[7,]	-0.3744170	NA	NA	NA
[8,]	-0.1850377	NA	NA	NA
[9,]	-0.1413030	NA	NA	NA
[10,]	-0.2177214	NA	NA	NA
[11,]	-0.1591193	NA	NA	NA
[12,]	-0.1331454	NA	NA	NA
[13,]	-0.0719351	NA	NA	NA
[14,]	-0.0935207	NA	NA	NA
[15,]	-0.0835662	NA	NA	NA

[16,]	-0.0767819	NA	NA	NA
[17,]	-0.0783845	NA	NA	NA
[18,]	-0.0125157	NA	NA	NA
[19,]	-0.0681742	NA	NA	NA
[20,]	-0.0612584	NA	NA	NA
[21,]	-0.0595493	NA	NA	NA
[22,]	0.0874161	NA	NA	NA
[23,]	-0.0301725	NA	NA	NA
[24,]	-0.1200443	NA	NA	NA
[25,]	-0.1198624	NA	NA	NA
[26,]	-0.0240460	NA	NA	NA
[27,]	0.0230844	NA	NA	NA
[28,]	-0.0975997	NA	NA	NA
[29,]	-0.0957801	NA	NA	NA
[30,]	-0.1778450	NA	NA	NA
[31,]	-0.0506640	NA	NA	NA
[32,]	-0.0936597	NA	NA	NA
[33,]	-0.1331854	NA	NA	NA
[34,]	-0.0222905	NA	NA	NA
[35,]	-0.0145831	NA	NA	NA
[36,]	-0.0136009	NA	NA	NA
[37,]	-0.1271770	NA	NA	NA
[38,]	-0.3920291	NA	NA	NA
[39,]	-0.0863961	NA	NA	NA
[40,]	-0.2566778	NA	NA	NA
[41,]	-0.1022252	NA	NA	NA
[42,]	-0.0616963	NA	NA	NA
[43,]	0.0315711	NA	NA	NA
[44,]	0.0960394	NA	NA	NA
[45,]	-0.0156084	NA	NA	NA
[46,]	0.0531986	NA	NA	NA
[47,]	0.1006035	NA	NA	NA
[48,]	0.1134274	NA	NA	NA
[49,]	-0.0590547	NA	NA	NA
[50,]	-0.0895368	NA	NA	NA
[51,]	0.0740345	NA	NA	NA
[52,]	0.0409164	NA	NA	NA
[53,]	-0.1071139	NA	NA	NA

[54,]	0.0792286	NA	NA	NA
[55,]	0.0259824	NA	NA	NA
[56,]	0.0222540	NA	NA	NA
[57,]	-0.0926326	NA	NA	NA
[58,]	0.0205007	NA	NA	NA
[59,]	0.0038317	NA	NA	NA
[60,]	0.0624852	NA	NA	NA
[61,]	0.0739064	NA	NA	NA
[62,]	-0.0304251	NA	NA	NA
[63,]	0.0724402	NA	NA	NA
[64,]	0.0082186	NA	NA	NA
[65,]	-0.0762422	NA	NA	NA
[66,]	0.1050246	NA	NA	NA
[67,]	-0.1479764	NA	NA	NA
[68,]	-0.1868693	NA	NA	NA
[69,]	-0.3434091	NA	NA	NA
[70,]	-0.1651417	NA	NA	NA
[71,]	-0.0792257	NA	NA	NA
[72,]	-0.2129735	NA	NA	NA
[73,]	0.0170955	NA	NA	NA
[74,]	-0.0037785	NA	NA	NA
[75,]	-0.1864947	NA	NA	NA
[76,]	-0.0175100	NA	NA	NA
[77,]	-0.0870611	NA	NA	NA
[78,]	-0.0226647	NA	NA	NA
[79,]	-0.1965507	NA	NA	NA
[80,]	-0.1598921	NA	NA	NA
[81,]	-0.0271826	NA	NA	NA
[82,]	-0.2931622	NA	NA	NA
[83,]	-0.1355663	NA	NA	NA
[84,]	-0.1449411	NA	NA	NA
[85,]	-0.0695264	NA	NA	NA
[86,]	0.0221660	NA	NA	NA
[87,]	-0.1174761	NA	NA	NA
[88,]	-0.1297208	NA	NA	NA
[89,]	-0.0117509	NA	NA	NA
[90,]	-0.1046712	NA	NA	NA
[91,]	-0.1345552	NA	NA	NA

[92,]	-0.0670988	NA	NA	NA
[93,]	-0.0393914	NA	NA	NA
[94,]	-0.0515222	NA	NA	NA
[95,]	-0.0527838	NA	NA	NA
[96,]	-0.0617041	NA	NA	NA
[97,]	-0.1397763	NA	NA	NA
[98,]	-0.0572289	NA	NA	NA
[99,]	-0.1639064	NA	NA	NA
[100,]	-0.3528426	NA	NA	NA
[101,]	-0.0779397	NA	NA	NA
[102,]	-0.1441470	NA	NA	NA
[103,]	-0.0869822	NA	NA	NA
[104,]	-0.0590553	NA	NA	NA
[105,]	0.0307828	NA	NA	NA
[106,]	-0.0575768	NA	NA	NA
[107,]	-0.0040348	NA	NA	NA
[108,]	-0.0547300	NA	NA	NA
[109,]	-0.0216499	NA	NA	NA
[110,]	-0.1320106	NA	NA	NA
[111,]	-0.0720214	NA	NA	NA
[112,]	-0.0493996	NA	NA	NA
[113,]	0.0186683	NA	NA	NA
[114,]	-0.1579013	NA	NA	NA
[115,]	0.0442016	NA	NA	NA
[116,]	-0.0482646	NA	NA	NA
[117,]	-0.0034273	NA	NA	NA
[118,]	0.0517409	NA	NA	NA
[119,]	-0.0066642	NA	NA	NA
[120,]	0.1256617	NA	NA	NA
[121,]	-0.1106914	NA	NA	NA
[122,]	-0.1362525	NA	NA	NA
[123,]	-0.1338438	NA	NA	NA
[124,]	-0.0185286	NA	NA	NA
[125,]	-0.0349454	NA	NA	NA
[126,]	-0.0331622	NA	NA	NA
[127,]	-0.0571474	NA	NA	NA
[128,]	-0.1572987	NA	NA	NA
[129,]	-0.1434921	NA	NA	NA

[130,]	-0.0430481	NA	NA	NA
[131,]	-0.4167509	NA	NA	NA
[132,]	-0.1325047	NA	NA	NA
[133,]	-0.0218396	NA	NA	NA
[134,]	0.0937434	NA	NA	NA
[135,]	0.0955276	NA	NA	NA
[136,]	-0.0483043	NA	NA	NA
[137,]	0.0932208	NA	NA	NA
[138,]	-0.0153416	NA	NA	NA
[139,]	-0.0772291	NA	NA	NA
[140,]	-0.0750545	NA	NA	NA
[141,]	-0.0020402	NA	NA	NA
[142,]	-0.0817549	NA	NA	NA
[143,]	-0.1345794	NA	NA	NA
[144,]	-0.0204648	NA	NA	NA
[145,]	-0.0882027	NA	NA	NA
[146,]	-0.0662244	NA	NA	NA
[147,]	-0.1066107	NA	NA	NA
[148,]	0.0063220	NA	NA	NA
[149,]	-0.0058521	NA	NA	NA
[150,]	-0.1179475	NA	NA	NA
[151,]	-0.0693235	NA	NA	NA
[152,]	-0.0549225	NA	NA	NA
[153,]	0.1338211	NA	NA	NA
[154,]	0.0434313	NA	NA	NA
[155,]	-0.0228435	NA	NA	NA
[156,]	-0.0132541	NA	NA	NA
[157,]	-0.0975601	NA	NA	NA
[158,]	-0.0039723	NA	NA	NA
[159,]	-0.0600795	NA	NA	NA
[160,]	0.0893678	NA	NA	NA
[161,]	-0.1070519	NA	NA	NA
[162,]	-0.3729925	NA	NA	NA
[163,]	-0.0170690	NA	NA	NA
[164,]	-0.0857892	NA	NA	NA
[165,]	-0.0321930	NA	NA	NA
[166,]	-0.0149533	NA	NA	NA
[167,]	-0.0250271	NA	NA	NA

[168,]	-0.0345515	NA	NA	NA
[169,]	0.1069098	NA	NA	NA
[170,]	0.1081081	NA	NA	NA
[171,]	-0.0216174	NA	NA	NA
[172,]	-0.0197991	NA	NA	NA
[173,]	-0.0005303	NA	NA	NA
[174,]	0.0055084	NA	NA	NA
[175,]	-0.1021724	NA	NA	NA
[176,]	-0.1355792	NA	NA	NA
[177,]	-0.0794440	NA	NA	NA
[178,]	-0.0831417	NA	NA	NA
[179,]	-0.0935676	NA	NA	NA
[180,]	0.0071428	NA	NA	NA
[181,]	0.1485339	NA	NA	NA
[182,]	0.1064247	NA	NA	NA
[183,]	-0.0153134	NA	NA	NA
[184,]	0.0204893	NA	NA	NA
[185,]	-0.0022572	NA	NA	NA
[186,]	-0.0024555	NA	NA	NA

Estimates in matrix form:

Constant term:

Estimates: 17.02844 15.98368 16.8658 15.60638 17.27424 16.74323

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.3744	0.18504	0.1413	0.2177	0.1591	0.1331
[2,]	0.1272	0.39203	0.0864	0.2567	0.1022	0.0617
[3,]	0.1480	0.18687	0.3434	0.1651	0.0792	0.2130
[4,]	0.1398	0.05723	0.1639	0.3528	0.0779	0.1441
[5,]	0.0571	0.15730	0.1435	0.0430	0.4168	0.1325
[6,]	0.0976	0.00397	0.0601	-0.0894	0.1071	0.3730

MA(2)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.0719	0.09352	0.0836	0.0768	0.07838	0.0125
[2,]	-0.0316	-0.09604	0.0156	-0.0532	-0.10060	-0.1134
[3,]	-0.0171	0.00378	0.1865	0.0175	0.08706	0.0227
[4,]	0.0870	0.05906	-0.0308	0.0576	0.00403	0.0547

```
[5,] 0.0218 -0.09374 -0.0955 0.0483 -0.09322 0.0153
[6,] 0.0171 0.08579 0.0322 0.0150 0.02503 0.0346
```

MA(3)-matrix

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.0682 0.0613 0.05955 -0.0874 0.03017 0.12004
[2,] 0.0591 0.0895 -0.07403 -0.0409 0.10711 -0.07923
[3,] 0.1966 0.1599 0.02718 0.2932 0.13557 0.14494
[4,] 0.0216 0.1320 0.07202 0.0494 -0.01867 0.15790
[5,] 0.0772 0.0751 0.00204 0.0818 0.13458 0.02046
[6,] -0.1069 -0.1081 0.02162 0.0198 0.00053 -0.00551
```

MA(4)-matrix

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.1199 0.0240 -0.02308 0.09760 0.09578 0.17785
[2,] -0.0260 -0.0223 0.09263 -0.02050 -0.00383 -0.06249
[3,] 0.0695 -0.0222 0.11748 0.12972 0.01175 0.10467
[4,] -0.0442 0.0483 0.00343 -0.05174 0.00666 -0.12566
[5,] 0.0882 0.0662 0.10661 -0.00632 0.00585 0.11795
[6,] 0.1022 0.1356 0.07944 0.08314 0.09357 -0.00714
```

MA(5)-matrix

```
      [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.0507 0.0937 0.1332 0.02229 0.01458 0.01360
[2,] -0.0739 0.0304 -0.0724 -0.00822 0.07624 -0.10502
[3,] 0.1346 0.0671 0.0394 0.05152 0.05278 0.06170
[4,] 0.1107 0.1363 0.1338 0.01853 0.03495 0.03316
[5,] 0.0693 0.0549 -0.1338 -0.04343 0.02284 0.01325
[6,] -0.1485 -0.1064 0.0153 -0.02049 0.00226 0.00246
```

Residuals cov-matrix:

```
      [,1] [,2] [,3] [,4] [,5]
[1,] 1.737147e+75 5.967243e+74 1.995587e+75 1.231281e+75 8.527933e+74
[2,] 5.967243e+74 2.049797e+74 6.855007e+74 4.229552e+74 2.929415e+74
[3,] 1.995587e+75 6.855007e+74 2.292477e+75 1.414462e+75 9.796658e+74
[4,] 1.231281e+75 4.229552e+74 1.414462e+75 8.727259e+74 6.044556e+74
[5,] 8.527933e+74 2.929415e+74 9.796658e+74 6.044556e+74 4.186499e+74
[6,] 4.775073e+74 1.640277e+74 5.485475e+74 3.384548e+74 2.344160e+74
      [,6]
[1,] 4.775073e+74
[2,] 1.640277e+74
```

```
[3,] 5.485475e+74
[4,] 3.384548e+74
[5,] 2.344160e+74
[6,] 1.312573e+74
```

aic= -Inf

bic= -Inf

```
> VARMA(da = ms, p = 0, q = 6)
```

Number of parameters: 222

initial estimates: 16.9598 15.9602 16.7868 15.6083 17.262 16.8082 -0.3878 -0.1797

Par. lower-bounds: 15.2262 14.3721 15.1032 14.1045 15.5845 15.127 -0.5368 -0.3437

Par. upper-bounds: 18.6934 17.5483 18.4705 17.1121 18.9395 18.4894 -0.2388 -0.015

Final Estimates: 16.95979 15.96019 16.78685 15.60828 17.26203 16.80823 -0.38783

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)
[1,]	1.696e+01	NA	NA	NA
[2,]	1.596e+01	NA	NA	NA
[3,]	1.679e+01	NA	NA	NA
[4,]	1.561e+01	NA	NA	NA
[5,]	1.726e+01	NA	NA	NA
[6,]	1.681e+01	NA	NA	NA
[7,]	-3.878e-01	NA	NA	NA
[8,]	-1.797e-01	NA	NA	NA
[9,]	-1.467e-01	NA	NA	NA
[10,]	-2.396e-01	NA	NA	NA
[11,]	-1.693e-01	NA	NA	NA
[12,]	-1.454e-01	NA	NA	NA
[13,]	-8.714e-02	NA	NA	NA
[14,]	-1.111e-01	NA	NA	NA
[15,]	-9.531e-02	NA	NA	NA
[16,]	-8.293e-02	NA	NA	NA
[17,]	-1.072e-01	NA	NA	NA
[18,]	-1.714e-02	NA	NA	NA
[19,]	-8.876e-02	NA	NA	NA
[20,]	-4.895e-02	NA	NA	NA

[21,]	-3.952e-02	NA	NA	NA
[22,]	7.552e-02	NA	NA	NA
[23,]	-2.503e-02	NA	NA	NA
[24,]	-1.243e-01	NA	NA	NA
[25,]	-1.268e-01	NA	NA	NA
[26,]	-3.405e-02	NA	NA	NA
[27,]	1.785e-02	NA	NA	NA
[28,]	-1.090e-01	NA	NA	NA
[29,]	-9.772e-02	NA	NA	NA
[30,]	-1.881e-01	NA	NA	NA
[31,]	-6.296e-02	NA	NA	NA
[32,]	-9.704e-02	NA	NA	NA
[33,]	-1.324e-01	NA	NA	NA
[34,]	-2.375e-02	NA	NA	NA
[35,]	-1.633e-02	NA	NA	NA
[36,]	-1.741e-02	NA	NA	NA
[37,]	-3.598e-02	NA	NA	NA
[38,]	-4.912e-05	NA	NA	NA
[39,]	-7.475e-02	NA	NA	NA
[40,]	1.850e-01	NA	NA	NA
[41,]	-1.374e-02	NA	NA	NA
[42,]	-8.914e-02	NA	NA	NA
[43,]	-1.189e-01	NA	NA	NA
[44,]	-3.915e-01	NA	NA	NA
[45,]	-8.904e-02	NA	NA	NA
[46,]	-2.468e-01	NA	NA	NA
[47,]	-1.012e-01	NA	NA	NA
[48,]	-6.265e-02	NA	NA	NA
[49,]	2.165e-02	NA	NA	NA
[50,]	9.777e-02	NA	NA	NA
[51,]	-2.130e-02	NA	NA	NA
[52,]	6.586e-02	NA	NA	NA
[53,]	1.067e-01	NA	NA	NA
[54,]	1.148e-01	NA	NA	NA
[55,]	-6.955e-02	NA	NA	NA
[56,]	-1.092e-01	NA	NA	NA
[57,]	4.970e-02	NA	NA	NA
[58,]	3.255e-02	NA	NA	NA

[59,]	-1.349e-01	NA	NA	NA
[60,]	5.931e-02	NA	NA	NA
[61,]	1.380e-02	NA	NA	NA
[62,]	1.661e-02	NA	NA	NA
[63,]	-9.921e-02	NA	NA	NA
[64,]	1.158e-02	NA	NA	NA
[65,]	-8.747e-03	NA	NA	NA
[66,]	5.272e-02	NA	NA	NA
[67,]	5.982e-02	NA	NA	NA
[68,]	-4.267e-02	NA	NA	NA
[69,]	5.666e-02	NA	NA	NA
[70,]	-6.678e-03	NA	NA	NA
[71,]	-9.138e-02	NA	NA	NA
[72,]	9.156e-02	NA	NA	NA
[73,]	-7.192e-02	NA	NA	NA
[74,]	3.964e-02	NA	NA	NA
[75,]	-1.768e-02	NA	NA	NA
[76,]	-8.105e-02	NA	NA	NA
[77,]	-1.265e-01	NA	NA	NA
[78,]	-7.327e-02	NA	NA	NA
[79,]	-1.739e-01	NA	NA	NA
[80,]	-2.277e-01	NA	NA	NA
[81,]	-3.663e-01	NA	NA	NA
[82,]	-1.947e-01	NA	NA	NA
[83,]	-1.031e-01	NA	NA	NA
[84,]	-2.323e-01	NA	NA	NA
[85,]	3.324e-03	NA	NA	NA
[86,]	-3.002e-02	NA	NA	NA
[87,]	-1.981e-01	NA	NA	NA
[88,]	-5.578e-02	NA	NA	NA
[89,]	-1.080e-01	NA	NA	NA
[90,]	-5.786e-02	NA	NA	NA
[91,]	-2.185e-01	NA	NA	NA
[92,]	-1.958e-01	NA	NA	NA
[93,]	-5.180e-02	NA	NA	NA
[94,]	-3.104e-01	NA	NA	NA
[95,]	-1.614e-01	NA	NA	NA
[96,]	-1.658e-01	NA	NA	NA

[97,]	-9.775e-02	NA	NA	NA
[98,]	-9.214e-03	NA	NA	NA
[99,]	-1.446e-01	NA	NA	NA
[100,]	-1.534e-01	NA	NA	NA
[101,]	-3.485e-02	NA	NA	NA
[102,]	-1.268e-01	NA	NA	NA
[103,]	-1.525e-01	NA	NA	NA
[104,]	-8.506e-02	NA	NA	NA
[105,]	-5.763e-02	NA	NA	NA
[106,]	-6.569e-02	NA	NA	NA
[107,]	-6.998e-02	NA	NA	NA
[108,]	-7.647e-02	NA	NA	NA
[109,]	-3.965e-02	NA	NA	NA
[110,]	-1.118e-01	NA	NA	NA
[111,]	-4.127e-02	NA	NA	NA
[112,]	-5.029e-02	NA	NA	NA
[113,]	-9.603e-02	NA	NA	NA
[114,]	3.265e-02	NA	NA	NA
[115,]	-1.377e-01	NA	NA	NA
[116,]	-4.564e-02	NA	NA	NA
[117,]	-1.621e-01	NA	NA	NA
[118,]	-3.514e-01	NA	NA	NA
[119,]	-6.809e-02	NA	NA	NA
[120,]	-1.435e-01	NA	NA	NA
[121,]	-9.009e-02	NA	NA	NA
[122,]	-6.295e-02	NA	NA	NA
[123,]	1.951e-02	NA	NA	NA
[124,]	-5.077e-02	NA	NA	NA
[125,]	-7.603e-03	NA	NA	NA
[126,]	-5.573e-02	NA	NA	NA
[127,]	-2.835e-02	NA	NA	NA
[128,]	-1.337e-01	NA	NA	NA
[129,]	-8.304e-02	NA	NA	NA
[130,]	-5.552e-02	NA	NA	NA
[131,]	1.074e-02	NA	NA	NA
[132,]	-1.694e-01	NA	NA	NA
[133,]	3.825e-02	NA	NA	NA
[134,]	-5.176e-02	NA	NA	NA

[135,]	-9.197e-03	NA	NA	NA
[136,]	4.485e-02	NA	NA	NA
[137,]	-1.326e-02	NA	NA	NA
[138,]	1.174e-01	NA	NA	NA
[139,]	-1.155e-01	NA	NA	NA
[140,]	-1.387e-01	NA	NA	NA
[141,]	-1.380e-01	NA	NA	NA
[142,]	-2.849e-02	NA	NA	NA
[143,]	-4.170e-02	NA	NA	NA
[144,]	-3.897e-02	NA	NA	NA
[145,]	-2.827e-02	NA	NA	NA
[146,]	1.377e-02	NA	NA	NA
[147,]	-6.063e-02	NA	NA	NA
[148,]	-5.293e-02	NA	NA	NA
[149,]	-3.325e-02	NA	NA	NA
[150,]	-5.527e-02	NA	NA	NA
[151,]	-3.333e-02	NA	NA	NA
[152,]	-1.396e-01	NA	NA	NA
[153,]	-1.198e-01	NA	NA	NA
[154,]	-1.235e-02	NA	NA	NA
[155,]	-3.884e-01	NA	NA	NA
[156,]	-1.137e-01	NA	NA	NA
[157,]	-9.513e-03	NA	NA	NA
[158,]	1.052e-01	NA	NA	NA
[159,]	9.974e-02	NA	NA	NA
[160,]	-3.065e-02	NA	NA	NA
[161,]	1.130e-01	NA	NA	NA
[162,]	-3.282e-03	NA	NA	NA
[163,]	-6.342e-02	NA	NA	NA
[164,]	-6.183e-02	NA	NA	NA
[165,]	2.077e-03	NA	NA	NA
[166,]	-8.618e-02	NA	NA	NA
[167,]	-1.327e-01	NA	NA	NA
[168,]	-1.578e-02	NA	NA	NA
[169,]	-7.107e-02	NA	NA	NA
[170,]	-5.120e-02	NA	NA	NA
[171,]	-9.691e-02	NA	NA	NA
[172,]	2.123e-02	NA	NA	NA

[173,]	1.296e-02	NA	NA	NA
[174,]	-9.671e-02	NA	NA	NA
[175,]	-6.567e-02	NA	NA	NA
[176,]	-5.078e-02	NA	NA	NA
[177,]	1.473e-01	NA	NA	NA
[178,]	5.601e-02	NA	NA	NA
[179,]	-1.147e-02	NA	NA	NA
[180,]	-6.529e-03	NA	NA	NA
[181,]	5.098e-02	NA	NA	NA
[182,]	9.167e-02	NA	NA	NA
[183,]	3.215e-02	NA	NA	NA
[184,]	-1.971e-02	NA	NA	NA
[185,]	9.574e-02	NA	NA	NA
[186,]	5.173e-02	NA	NA	NA
[187,]	-1.251e-01	NA	NA	NA
[188,]	-3.727e-02	NA	NA	NA
[189,]	-9.144e-02	NA	NA	NA
[190,]	5.719e-02	NA	NA	NA
[191,]	-1.456e-01	NA	NA	NA
[192,]	-3.887e-01	NA	NA	NA
[193,]	-2.067e-02	NA	NA	NA
[194,]	-8.662e-02	NA	NA	NA
[195,]	-2.434e-02	NA	NA	NA
[196,]	-4.162e-02	NA	NA	NA
[197,]	-3.443e-02	NA	NA	NA
[198,]	-4.969e-02	NA	NA	NA
[199,]	1.038e-01	NA	NA	NA
[200,]	8.377e-02	NA	NA	NA
[201,]	-3.459e-02	NA	NA	NA
[202,]	-7.374e-03	NA	NA	NA
[203,]	8.274e-04	NA	NA	NA
[204,]	1.084e-02	NA	NA	NA
[205,]	-1.175e-01	NA	NA	NA
[206,]	-1.498e-01	NA	NA	NA
[207,]	-8.829e-02	NA	NA	NA
[208,]	-9.433e-02	NA	NA	NA
[209,]	-1.144e-01	NA	NA	NA
[210,]	-1.149e-02	NA	NA	NA

[211,]	1.556e-01	NA	NA	NA
[212,]	1.050e-01	NA	NA	NA
[213,]	-2.986e-02	NA	NA	NA
[214,]	1.131e-02	NA	NA	NA
[215,]	-1.074e-02	NA	NA	NA
[216,]	-4.551e-03	NA	NA	NA
[217,]	-2.893e-02	NA	NA	NA
[218,]	-1.556e-01	NA	NA	NA
[219,]	4.074e-02	NA	NA	NA
[220,]	-6.618e-02	NA	NA	NA
[221,]	-6.039e-02	NA	NA	NA
[222,]	2.917e-02	NA	NA	NA

Estimates in matrix form:

Constant term:

Estimates: 16.95979 15.96019 16.78685 15.60828 17.26203 16.80823

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.3878	0.1797	0.1467	0.2396	0.1693	0.1454
[2,]	0.1189	0.3915	0.0890	0.2468	0.1012	0.0626
[3,]	0.1739	0.2277	0.3663	0.1947	0.1031	0.2323
[4,]	0.1377	0.0456	0.1621	0.3514	0.0681	0.1435
[5,]	0.0333	0.1396	0.1198	0.0124	0.3884	0.1137
[6,]	0.1251	0.0373	0.0914	-0.0572	0.1456	0.3887

MA(2)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.08714	0.1111	0.0953	0.0829	0.1072	0.01714
[2,]	-0.02165	-0.0978	0.0213	-0.0659	-0.1067	-0.11481
[3,]	-0.00332	0.0300	0.1981	0.0558	0.1080	0.05786
[4,]	0.09009	0.0630	-0.0195	0.0508	0.0076	0.05573
[5,]	0.00951	-0.1052	-0.0997	0.0306	-0.1130	0.00328
[6,]	0.02067	0.0866	0.0243	0.0416	0.0344	0.04969

MA(3)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.0888	0.0490	0.03952	-0.07552	0.025026	0.1243
[2,]	0.0696	0.1092	-0.04970	-0.03255	0.134880	-0.0593
[3,]	0.2185	0.1958	0.05180	0.31043	0.161394	0.1658

```

[4,] 0.0284 0.1337 0.08304 0.05552 -0.010744 0.1694
[5,] 0.0634 0.0618 -0.00208 0.08618 0.132684 0.0158
[6,] -0.1038 -0.0838 0.03459 0.00737 -0.000827 -0.0108

```

MA(4)-matrix

```

      [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,] 0.1268 0.03405 -0.0179 0.1090 0.09772 0.1881
[2,] -0.0138 -0.01661 0.0992 -0.0116 0.00875 -0.0527
[3,] 0.0977 0.00921 0.1446 0.1534 0.03485 0.1268
[4,] -0.0383 0.05176 0.0092 -0.0448 0.01326 -0.1174
[5,] 0.0711 0.05120 0.0969 -0.0212 -0.01296 0.0967
[6,] 0.1175 0.14978 0.0883 0.0943 0.11435 0.0115

```

MA(5)-matrix

```

      [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,] 0.0630 0.0970 0.1324 0.02375 0.0163 0.01741
[2,] -0.0598 0.0427 -0.0567 0.00668 0.0914 -0.09156
[3,] 0.1525 0.0851 0.0576 0.06569 0.0700 0.07647
[4,] 0.1155 0.1387 0.1380 0.02849 0.0417 0.03897
[5,] 0.0657 0.0508 -0.1473 -0.05601 0.0115 0.00653
[6,] -0.1556 -0.1050 0.0299 -0.01131 0.0107 0.00455

```

MA(6)-matrix

```

      [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,] 0.0360 4.91e-05 0.0747 -0.1850 0.0137 0.0891
[2,] 0.0719 -3.96e-02 0.0177 0.0810 0.1265 0.0733
[3,] 0.0396 1.12e-01 0.0413 0.0503 0.0960 -0.0327
[4,] 0.0283 -1.38e-02 0.0606 0.0529 0.0332 0.0553
[5,] -0.0510 -9.17e-02 -0.0322 0.0197 -0.0957 -0.0517
[6,] 0.0289 1.56e-01 -0.0407 0.0662 0.0604 -0.0292

```

Residuals cov-matrix:

```

      [,1]      [,2]      [,3]      [,4]      [,5]
[1,] 4.984786e+90 1.955260e+90 6.501544e+90 3.624902e+90 1.404073e+90
[2,] 1.955260e+90 7.669422e+89 2.550202e+90 1.421852e+90 5.507414e+89
[3,] 6.501544e+90 2.550202e+90 8.479817e+90 4.727878e+90 1.831301e+90
[4,] 3.624902e+90 1.421852e+90 4.727878e+90 2.636004e+90 1.021032e+90
[5,] 1.404073e+90 5.507414e+89 1.831301e+90 1.021032e+90 3.954876e+89
[6,] 2.037221e+90 7.990910e+89 2.657102e+90 1.481453e+90 5.738275e+89
      [,6]
[1,] 2.037221e+90

```

```
[2,] 7.990910e+89
[3,] 2.657102e+90
[4,] 1.481453e+90
[5,] 5.738275e+89
[6,] 8.325874e+89
```

aic= -Inf

bic= -Inf

> VARMA(da = ms, p = 1, q = 1)

Number of parameters: 78

```
initial estimates: 15.9825 26.0047 14.8101 12.1987 24.3009 16.3615 0.1206 0.0332
Par. lower-bounds: 1.5615 12.6784 0.733 -0.5254 10.569 2.4713 -0.3457 -0.4185 -0.
Par. upper-bounds: 30.4035 39.331 28.8873 24.9229 38.0328 30.2518 0.5869 0.485 0.
Final Estimates: 15.96561 25.99479 14.79723 12.17174 24.28669 16.34921 0.153974
```

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)	
[1,]	15.965611	0.084240	189.526	< 2e-16	***
[2,]	25.994788	0.024966	1041.196	< 2e-16	***
[3,]	14.797227	0.003828	3865.874	< 2e-16	***
[4,]	12.171742	0.150384	80.938	< 2e-16	***
[5,]	24.286695	0.192989	125.845	< 2e-16	***
[6,]	16.349205	0.004412	3705.523	< 2e-16	***
[7,]	0.153974	0.057383	2.683	0.007290	**
[8,]	0.025391	0.070743	0.359	0.719660	
[9,]	0.202280	0.069671	2.903	0.003692	**
[10,]	-0.232418	0.066943	-3.472	0.000517	***
[11,]	0.049054	0.059700	0.822	0.411269	
[12,]	-0.146935	0.078621	-1.869	0.061637	.
[13,]	0.105147	0.132470	0.794	0.427347	
[14,]	-0.449739	0.088327	-5.092	3.55e-07	***
[15,]	0.004082	0.103217	0.040	0.968454	
[16,]	0.100905	0.141940	0.711	0.477148	
[17,]	-0.179877	0.094943	-1.895	0.058149	.
[18,]	-0.199110	0.112668	-1.767	0.077190	.
[19,]	-0.036066	0.071256	-0.506	0.612750	

[20,]	-0.369212	0.062747	-5.884	4.00e-09	***
[21,]	0.018871	0.091064	0.207	0.835831	
[22,]	0.326887	0.101243	3.229	0.001243	**
[23,]	0.287046	0.050165	5.722	1.05e-08	***
[24,]	-0.116286	0.094476	-1.231	0.218378	
[25,]	0.107525	0.052610	2.044	0.040971	*
[26,]	0.173648	0.024626	7.051	1.77e-12	***
[27,]	-0.224772	0.008015	-28.044	< 2e-16	***
[28,]	0.143812	0.032498	4.425	9.63e-06	***
[29,]	-0.062385	0.027791	-2.245	0.024780	*
[30,]	0.116000	0.042793	2.711	0.006714	**
[31,]	-0.085874	0.141818	-0.606	0.544834	
[32,]	-0.096547	0.094406	-1.023	0.306458	
[33,]	-0.292099	0.094301	-3.098	0.001951	**
[34,]	0.338713	0.113255	2.991	0.002783	**
[35,]	-0.367564	0.051575	-7.127	1.03e-12	***
[36,]	0.089955	0.097906	0.919	0.358206	
[37,]	-0.163547	0.143107	-1.143	0.253109	
[38,]	0.042344	0.112888	0.375	0.707590	
[39,]	0.255040	0.096723	2.637	0.008369	**
[40,]	-0.090423	0.132887	-0.680	0.496220	
[41,]	0.112841	0.134634	0.838	0.401956	
[42,]	-0.125235	0.137182	-0.913	0.361287	
[43,]	-0.529572	0.035950	-14.731	< 2e-16	***
[44,]	-0.239235	0.081114	-2.949	0.003184	**
[45,]	-0.314464	0.050956	-6.171	6.77e-10	***
[46,]	0.054588	0.087801	0.622	0.534125	
[47,]	-0.180111	0.073722	-2.443	0.014561	*
[48,]	0.065576	0.086259	0.760	0.447124	
[49,]	-0.289492	0.152127	-1.903	0.057046	.
[50,]	0.030459	0.123379	0.247	0.805005	
[51,]	-0.154475	0.118773	-1.301	0.193396	
[52,]	-0.443549	0.160490	-2.764	0.005715	**
[53,]	-0.006705	0.106232	-0.063	0.949676	
[54,]	0.121067	0.139101	0.870	0.384108	
[55,]	-0.075806	0.086215	-0.879	0.379253	
[56,]	0.148252	0.078485	1.889	0.058900	.
[57,]	-0.372773	0.103218	-3.611	0.000304	***

[58,]	-0.468896	0.138546	-3.384	0.000713	***
[59,]	-0.336432	0.042401	-7.934	2.22e-15	***
[60,]	-0.054793	0.123818	-0.443	0.658107	
[61,]	-0.253813	0.068982	-3.679	0.000234	***
[62,]	-0.284681	0.045023	-6.323	2.57e-10	***
[63,]	0.004503	0.038807	0.116	0.907619	
[64,]	-0.496605	NA	NA	NA	
[65,]	-0.060338	0.046814	-1.289	0.197440	
[66,]	-0.254747	0.014010	-18.183	< 2e-16	***
[67,]	0.030331	0.153703	0.197	0.843564	
[68,]	-0.118816	0.115834	-1.026	0.305011	
[69,]	0.095987	0.104391	0.919	0.357837	
[70,]	-0.391207	0.145735	-2.684	0.007267	**
[71,]	-0.085271	0.082506	-1.034	0.301365	
[72,]	-0.256792	0.119205	-2.154	0.031225	*
[73,]	0.084741	0.166204	0.510	0.610147	
[74,]	0.001375	0.094617	0.015	0.988409	
[75,]	-0.277294	0.120365	-2.304	0.021235	*
[76,]	0.170491	0.154364	1.104	0.269387	
[77,]	-0.233294	0.152142	-1.533	0.125179	
[78,]	-0.264406	0.139503	-1.895	0.058046	.

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Estimates in matrix form:

Constant term:

Estimates: 15.96561 25.99479 14.79723 12.17174 24.28669 16.34921

AR coefficient matrix

AR(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.1540	0.0254	0.20228	-0.2324	0.0491	-0.147
[2,]	0.1051	-0.4497	0.00408	0.1009	-0.1799	-0.199
[3,]	-0.0361	-0.3692	0.01887	0.3269	0.2870	-0.116
[4,]	0.1075	0.1736	-0.22477	0.1438	-0.0624	0.116
[5,]	-0.0859	-0.0965	-0.29210	0.3387	-0.3676	0.090
[6,]	-0.1635	0.0423	0.25504	-0.0904	0.1128	-0.125

MA coefficient matrix

MA(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	0.5296	0.23923	0.3145	-0.0546	0.1801	-0.0656
[2,]	0.2895	-0.03046	0.1545	0.4435	0.0067	-0.1211
[3,]	0.0758	-0.14825	0.3728	0.4689	0.3364	0.0548
[4,]	0.2538	0.28468	-0.0045	0.4966	0.0603	0.2547
[5,]	-0.0303	0.11882	-0.0960	0.3912	0.0853	0.2568
[6,]	-0.0847	-0.00137	0.2773	-0.1705	0.2333	0.2644

Residuals cov-matrix:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	243.23877	-49.24012	-23.48982	-29.88374	-35.63205	-45.68374
[2,]	-49.24012	201.96323	-40.68012	-32.80408	-40.66410	-17.07555
[3,]	-23.48982	-40.68012	219.13218	-32.09658	-36.10435	-35.47741
[4,]	-29.88374	-32.80408	-32.09658	181.77185	-31.47173	-34.17971
[5,]	-35.63205	-40.66410	-36.10435	-31.47173	215.13551	-49.28117
[6,]	-45.68374	-17.07555	-35.47741	-34.17971	-49.28117	222.49073

aic= 31.64853

bic= 32.61386

> VARMA(da = ms, p = 1, q = 2)

Number of parameters: 114

initial estimates: 43.9872 7.6804 39.1071 31.117 38.8757 23.9737 -0.6919 0.5072 -

Par. lower-bounds: -9.558 -40.9306 -12.1695 -15.5848 -11.873 -27.211 -2.8922 -1.4

Par. upper-bounds: 97.5324 56.2913 90.3837 77.8188 89.6245 75.1584 1.5083 2.4589

Final Estimates: 43.98724 7.680351 39.10711 31.11701 38.87574 23.97373 -0.69190

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)
[1,]	43.987242	NA	NA	NA
[2,]	7.680351	NA	NA	NA
[3,]	39.107105	NA	NA	NA
[4,]	31.117013	NA	NA	NA
[5,]	38.875745	NA	NA	NA
[6,]	23.973728	NA	NA	NA
[7,]	-0.691906	NA	NA	NA
[8,]	0.507174	NA	NA	NA

[9,]	-0.150323	NA	NA	NA
[10,]	-0.821897	NA	NA	NA
[11,]	-0.453084	NA	NA	NA
[12,]	0.001204	NA	NA	NA
[13,]	-1.982724	NA	NA	NA
[14,]	1.346682	NA	NA	NA
[15,]	-0.459552	NA	NA	NA
[16,]	2.917051	NA	NA	NA
[17,]	-2.160462	NA	NA	NA
[18,]	1.168655	NA	NA	NA
[19,]	0.995295	NA	NA	NA
[20,]	-1.331241	NA	NA	NA
[21,]	-0.452626	NA	NA	NA
[22,]	-0.792219	NA	NA	NA
[23,]	0.878476	NA	NA	NA
[24,]	-0.756849	NA	NA	NA
[25,]	1.828113	NA	NA	NA
[26,]	-1.878065	NA	NA	NA
[27,]	0.087478	NA	NA	NA
[28,]	-2.638919	NA	NA	NA
[29,]	2.093380	NA	NA	NA
[30,]	-0.726144	NA	NA	NA
[31,]	0.049439	NA	NA	NA
[32,]	-0.131636	NA	NA	NA
[33,]	-0.200308	NA	NA	NA
[34,]	0.911318	NA	NA	NA
[35,]	-0.956826	NA	NA	NA
[36,]	-0.882031	NA	NA	NA
[37,]	-0.365258	NA	NA	NA
[38,]	-0.340103	NA	NA	NA
[39,]	0.508511	NA	NA	NA
[40,]	0.031863	NA	NA	NA
[41,]	0.353434	NA	NA	NA
[42,]	-0.624847	NA	NA	NA
[43,]	0.343514	NA	NA	NA
[44,]	-0.648815	NA	NA	NA
[45,]	0.056949	NA	NA	NA
[46,]	0.663713	NA	NA	NA

[47,]	0.359205	NA	NA	NA
[48,]	-0.084442	NA	NA	NA
[49,]	-0.277054	NA	NA	NA
[50,]	-0.011942	NA	NA	NA
[51,]	-0.211955	NA	NA	NA
[52,]	-0.254071	NA	NA	NA
[53,]	-0.223939	NA	NA	NA
[54,]	-0.121960	NA	NA	NA
[55,]	1.841380	NA	NA	NA
[56,]	-1.729835	NA	NA	NA
[57,]	0.343328	NA	NA	NA
[58,]	-3.182834	NA	NA	NA
[59,]	2.039950	NA	NA	NA
[60,]	-1.223786	NA	NA	NA
[61,]	-0.164267	NA	NA	NA
[62,]	-0.154987	NA	NA	NA
[63,]	-0.082908	NA	NA	NA
[64,]	0.748203	NA	NA	NA
[65,]	-0.633234	NA	NA	NA
[66,]	0.352861	NA	NA	NA
[67,]	-1.097271	NA	NA	NA
[68,]	1.172962	NA	NA	NA
[69,]	0.170196	NA	NA	NA
[70,]	0.671932	NA	NA	NA
[71,]	-0.892763	NA	NA	NA
[72,]	0.599398	NA	NA	NA
[73,]	0.063331	NA	NA	NA
[74,]	-0.229786	NA	NA	NA
[75,]	-0.350445	NA	NA	NA
[76,]	-0.356777	NA	NA	NA
[77,]	0.143804	NA	NA	NA
[78,]	-0.292484	NA	NA	NA
[79,]	-1.949615	NA	NA	NA
[80,]	1.864218	NA	NA	NA
[81,]	-0.222664	NA	NA	NA
[82,]	2.328118	NA	NA	NA
[83,]	-2.147801	NA	NA	NA
[84,]	0.612962	NA	NA	NA

[85,]	0.084747	NA	NA	NA
[86,]	-0.054767	NA	NA	NA
[87,]	-0.043157	NA	NA	NA
[88,]	-0.903418	NA	NA	NA
[89,]	0.680257	NA	NA	NA
[90,]	-0.191275	NA	NA	NA
[91,]	-0.087316	NA	NA	NA
[92,]	-0.038543	NA	NA	NA
[93,]	0.049849	NA	NA	NA
[94,]	-0.959999	NA	NA	NA
[95,]	0.536281	NA	NA	NA
[96,]	0.737260	NA	NA	NA
[97,]	-0.066417	NA	NA	NA
[98,]	-0.143266	NA	NA	NA
[99,]	-0.041964	NA	NA	NA
[100,]	0.234496	NA	NA	NA
[101,]	-0.388219	NA	NA	NA
[102,]	-0.459267	NA	NA	NA
[103,]	0.270859	NA	NA	NA
[104,]	0.350530	NA	NA	NA
[105,]	-0.558758	NA	NA	NA
[106,]	0.053054	NA	NA	NA
[107,]	-0.465226	NA	NA	NA
[108,]	0.247085	NA	NA	NA
[109,]	-0.205747	NA	NA	NA
[110,]	-0.139116	NA	NA	NA
[111,]	0.081654	NA	NA	NA
[112,]	-0.040261	NA	NA	NA
[113,]	-0.023679	NA	NA	NA
[114,]	-0.181994	NA	NA	NA

Estimates in matrix form:

Constant term:

Estimates: 43.98724 7.680351 39.10711 31.11701 38.87574 23.97373

AR coefficient matrix

AR(1)-matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	-0.6919	0.507	-0.1503	-0.8219	-0.453	0.0012

```

[2,] -1.9827  1.347 -0.4596  2.9171 -2.160  1.1687
[3,]  0.9953 -1.331 -0.4526 -0.7922  0.878 -0.7568
[4,]  1.8281 -1.878  0.0875 -2.6389  2.093 -0.7261
[5,]  0.0494 -0.132 -0.2003  0.9113 -0.957 -0.8820
[6,] -0.3653 -0.340  0.5085  0.0319  0.353 -0.6248

```

MA coefficient matrix

MA(1)-matrix

```

      [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,] -0.3435  0.6488 -0.0569 -0.6637 -0.359  0.0844
[2,] -1.8414  1.7298 -0.3433  3.1828 -2.040  1.2238
[3,]  1.0973 -1.1730 -0.1702 -0.6719  0.893 -0.5994
[4,]  1.9496 -1.8642  0.2227 -2.3281  2.148 -0.6130
[5,]  0.0873  0.0385 -0.0498  0.9600 -0.536 -0.7373
[6,] -0.2709 -0.3505  0.5588 -0.0531  0.465 -0.2471

```

MA(2)-matrix

```

      [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,]  0.2771  0.0119  0.2120  0.2541  0.2239  0.122
[2,]  0.1643  0.1550  0.0829 -0.7482  0.6332 -0.353
[3,] -0.0633  0.2298  0.3504  0.3568 -0.1438  0.292
[4,] -0.0847  0.0548  0.0432  0.9034 -0.6803  0.191
[5,]  0.0664  0.1433  0.0420 -0.2345  0.3882  0.459
[6,]  0.2057  0.1391 -0.0817  0.0403  0.0237  0.182

```

Residuals cov-matrix:

```

      [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,]  2.430951e+161  1.674770e+161 -2.077129e+161 -2.406541e+161 -1.405975e+161
[2,]  1.674770e+161  3.643109e+161 -2.662597e+161 -3.491302e+161 -1.448373e+161
[3,] -2.077129e+161 -2.662597e+161  2.384137e+161  2.963326e+161  1.438692e+161
[4,] -2.406541e+161 -3.491302e+161  2.963326e+161  3.732623e+161  1.745186e+161
[5,] -1.405975e+161 -1.448373e+161  1.438692e+161  1.745186e+161  9.056235e+160
[6,] -1.114076e+161 -2.786513e+160  7.100511e+160  7.428364e+160  5.501238e+160
      [,6]
[1,] -1.114076e+161
[2,] -2.786513e+160
[3,]  7.100511e+160
[4,]  7.428364e+160
[5,]  5.501238e+160
[6,]  6.065788e+160

```

```
----  
aic= Inf  
bic= Inf
```

A.2.3 Summary of the AIC and BIC criteria for all couple (p,q) tested

$p \setminus q$	0	1
0		
1	AIC = 31.91982 BIC = 32.43962	
2	AIC = 31.90784 BIC = 31.90784	

Table A.3 – AIC and BIC for all estimated VARMA(p, q) except when production of NA

Appendix B

List of Acronyms

- AISP** Authorized Investment Services Provider. 4, 26, 29
- AMF** Autorité des Marchés Financiers. 3, 28, 30
- AMIS** Agricultural Markets Information System. 6, 34, 35
- CLOB** Central Limit Order Book. 199
- CME** Chicago Mercantile Exchange. 34
- CTL** Clock Time Line. xi, 91, 94, 100–105, 109, 125, 128, 137, 141, 146, 147, 149, 150, 152–158, 170–173, 175, 199
- D.R.V** Discrete Random Variable. 92, 93, 95, 139
- EGDSM** Expert Group on Agricultural Commodity Derivatives and Spot Markets. 6, 35
- EMH** Efficient Market Hypothesis. 38, 193
- ESMA** European Securities and Markets Authority. 3, 28, 30, 31
- EU** European Union. 35
- FTF** Futures market Trades Frequency. vii, 92, 94, 100–107, 109, 115, 116, 118, 119, 121, 123, 125, 126, 128, 139–141, 144, 146, 147, 149, 152, 153, 155, 169, 172, 186, 195, 196, 199, 200
- LOB** Limit Order Book. 2–4, 6, 8, 12–14, 17, 19, 20, 26, 28–32, 35, 37, 45–47, 50, 193–195, 197, 199
- MM** Market Maker. 8, 29, 37, 194
- MTF** Multilateral Trading Facilities. 26, 30, 31
- MU** Market Undertaking. 26, 29–31, 37, 82, 193
- n.o.a** Number of agents. vii, 40, 42, 43, 46, 49–51, 135–142, 144, 146, 147, 149–152, 154–158, 160, 162–165, 169, 172, 175, 178–182, 185, 186, 193, 196, 197, 199

n.o.t Number of transactions. 4, 29, 30, 89, 93, 96, 100, 106, 107, 109, 115, 118, 119, 123, 128, 135, 139, 141, 146, 147, 152, 153, 155, 160, 164, 175, 178, 181, 182

NYSE New York Stock Exchange. 42, 44

OTC Over The Counter. 26, 33, 34

RTF Relative Trades Frequency. 6, 34, 92, 103, 105–107, 109, 112, 115, 118, 119, 122, 125, 128, 129, 135, 139, 146, 147, 151, 155, 156, 164, 165, 169, 178, 179, 182, 185, 196

STF Spot market Trades Frequency. vii, 92, 94–96, 100–103, 105–107, 109, 112, 113, 115, 118, 119, 123, 126, 128, 139, 141, 146, 147, 152, 154, 155, 164, 165, 169, 199

TTL Transactions Time Line. 91, 93, 100–103, 105, 109, 121, 128, 129, 138, 146, 147, 149–154, 156, 170–172, 196

Appendix C

List of Parameters and Variables

- a Advantage of possessing a unit of the storable agricultural commodity in monetary value, per unit of clock time. [vii](#), [viii](#), [91](#), [122](#), [123](#), [125](#), [128](#), [137](#), [158](#), [159](#), [173](#), [175](#), [176](#)
- α Elasticity of the spot market trade frequency to the number of agents on the market. [viii](#), [137](#), [141](#), [147](#), [154](#), [158](#), [164](#), [165](#), [173](#), [175](#), [182](#), [184](#), [185](#), [199](#)
- C Computational strength of the market. $C(s, t)$ represents the computational strength of the market with parameter s for the spectral zeta function, between time 1 and t . [67](#)
- CY_t Convenience yield (advantage in detaining one unit of stock to face risk or uncertainty) in monetary value at time t until maturity. [91](#), [96](#), [137](#), [138](#), [142](#)
- e Trend of the spot market information delivered in monetary value per unit of clock time. [vii](#), [viii](#), [91](#), [122](#), [123](#), [125](#), [128](#), [137](#), [144](#), [149](#), [158](#), [173–175](#)
- ef Effective information parameter of the market represented by a function capable of changing the market system. [62](#)
- $E_i(\tilde{p}_t)$ Expectation of the asset price at time t by agent i or equivalently measure of the quantifiable price at time t by agent i . [65](#)
- ϵ Absolute value of the elasticity of quantities traded per exchange on the spot market to the spot market trade frequency. [vii](#), [viii](#), [91](#), [95](#), [102](#), [112](#), [114](#), [115](#), [120](#), [122](#), [123](#), [125](#), [126](#), [128](#), [129](#), [137](#), [141](#), [154](#), [158](#), [162–164](#), [173](#), [175](#), [182](#), [183](#), [186](#), [196](#), [197](#), [199](#)
- \widetilde{f}_{t+1} Agents' expectations of the futures price at time t . [91](#), [137](#), [138](#)
- \widetilde{s}_{t+1} Agents' expectations of the spot price at time t . [91](#), [137](#), [138](#), [142](#)
- \widetilde{Q}_t Agents' expectations of the commodity available stock at time t . [91](#), [138](#)
- f_t Observed futures price at time t . [vii](#), [91](#), [96](#), [137](#), [138](#)
- FV_{f_t} Fundamental value of the futures price at time t . [91](#), [137](#), [138](#)
- FV_{s_t} Fundamental value of the spot price at time t . [91](#), [137](#), [138](#)

- Υ_n Capability of agents to process and expect the spot market information. 137, 138, 142, 143
- $I_{]t_1, t_2]}$ Spot market information delivered in monetary value between t_1 and t_2 , t_1 excluded. 91, 137, 138
- \mathcal{Q}_t Quantifiable property of a market system at time t . 64, 71–73, 76
- ms_t Mathematical function which contains all information about market dynamics at time t . 64
- $ms_{t,ef}$ Mathematical function which contains all effective information about market dynamics at time t . 64
- $ms_{t,nef}$ Mathematical function which contains all non effective information about market dynamics at time t . 64
- $\mu_{i,t}$ Price impact expected by agent i of the effective information at time t . If agent m_k gives the measure of quantifiable price at time k , $\mu_{m_k,k}$ is the fundamental value of the asset. 65
- nef Non effective information parameter of the market represented by a function capable of changing the market system. 62
- n Number of agents on the market, $n \in \mathbb{N}^*$. 62, 64, 65, 71, 75, 79–81, 137, 139, 150, 152–156, 158, 160, 162, 164, 167, 170, 171, 173, 175, 178–182, 185
- ω_f Historical trade frequency on the futures market per unit of clock time:
 $\left(\frac{\text{Number of transactions on the futures market}}{t_r} \right)$. vii, viii, 16, 91, 94, 101, 103–105, 110, 111, 115, 118, 137, 146, 155, 156, 158–160, 164, 173, 175, 177, 199
- ω_s Historical trade frequency on the spot market per unit of clock time:
 $\left(\frac{\text{Number of transactions on the spot market}}{t_r} \right)$. vii, viii, 16, 91, 92, 95, 96, 101, 103, 106, 110, 112, 113, 124, 137, 141, 155, 156, 158, 160, 164, 173, 175, 178, 179, 199
- pa Information parameter of the market represented by a function capable of changing the market system. 61
- p_t Asset price at time t or equivalently measure of the quantifiable price at time t . 61, 64, 81
- Q_{op} Optimal stock level of the agricultural commodity for agents. 91, 96, 137, 142
- q_t Quantity of asset traded at time t or equivalently measure of the quantifiable quantity at time t . 61, 64
- q Quantities traded on the spot market at each transaction on the spot market. 91, 94, 97, 102, 119, 129, 137, 138, 141, 187, 196
- Q_t Available stock of the commodity at time t . 91, 137, 138
- σ_I Maximal intrinsic volatility of the spot market information, per unit of clock time. viii, 137, 158, 161, 162, 173, 175, 180, 199

- s_t Observed spot price at time t . [vii](#), [91](#), [94](#), [98](#), [137](#), [138](#), [145](#)
- t Unit of the transactions time line, which represents a transaction on the futures market:
 $t = t_r \times \omega_f$, and $t \in \mathbb{N}$. [vii](#), [viii](#), [91–97](#), [99](#), [101](#), [103](#), [106–108](#), [115](#), [117–119](#), [123](#), [137–143](#), [145](#), [146](#), [149](#), [151](#), [153–155](#), [161](#), [170](#)
- T Maturity expressed in the transactions time line: $T = T_r \times \omega_f$, and $T \in \mathbb{N}$. [91](#), [137](#), [138](#)
- t_r Unit of the clock time line considered, such as a second or a minute. [viii](#), [91](#), [94](#), [103](#), [119](#), [126](#), [137](#), [138](#), [140](#), [146](#), [147](#), [152–155](#), [157](#), [158](#), [160](#), [170](#), [173](#), [175](#), [179](#), [181](#)
- Θ Risk aversion of the policy maker to the commodity fundamental value volatility. [viii](#), [137](#), [158](#), [162](#), [165](#), [173](#), [175](#), [180](#), [181](#)
- t' Latest period when there was a synchronized transaction in the spot and futures markets until t , expressed in the transactions time line. [91–94](#), [96](#), [137–140](#), [142](#)
- T_r Maturity of the futures contract, expressed in the clock time line. [viii](#), [91](#), [137](#), [155](#), [158](#), [162](#), [173](#), [175](#), [181](#), [182](#)
- ξ Bias of agents in the process and the expectation of the spot market information. [ix](#), [137](#), [165](#), [173](#), [175](#), [185](#), [186](#), [199](#)

How do the nature and the structure of information affect the optimal pricing algorithm to guarantee market efficiency and minimize fundamental prices volatility ?

Keywords: Market efficiency – Information – Price volatility – Commodity markets

Version française: Ce travail évalue la capacité des marchés financiers en général puis des marchés financiers de matières premières agricoles standardisés (commodités) à agréger l'information dans le prix de marché et à réduire la volatilité des prix. Cela est fait à l'aide de deux nouveaux modèles basés sur la nature de l'information même. Nous distinguons l'information qui est traitable à l'aide d'algorithme (dite effective) et l'information qui ne l'est pas. L'impact de l'algorithme de détermination du prix majoritairement utilisé qu'est le carnet d'ordres à cours limité est étudié au regard de la nature et de la structure de l'information. Ces modèles soulignent que lorsque l'information est majoritairement effective et continue, le carnet d'ordres à cours limité permet en effet au prix de marché d'être au voisinage de sa valeur fondamentale. Au contraire, quand l'information non effective devient trop importante, ce mécanisme mène à une bulle, voire à une crise. En revanche, nous démontrons que même dans une bulle, le prix de marché peut révéler davantage d'information que n'importe quel agent pris séparément. Lorsque l'information est discontinue en revanche, comme c'est le cas pour les marchés de commodités, la dissociation des algorithmes utilisés sur les marchés physiques et les marchés à termes peut entraîner une diminution de la capacité du prix de marché à véhiculer l'information mais aussi générer un surplus de volatilité. Enfin, nous prenons en compte le nombre d'agents comme variable déterminante de la fréquence des transactions et de l'information arrivant sur le marché d'une part, et la possibilité d'avoir une information non effective d'autre part. L'intuition que plus d'agents permettent au prix de marché de mieux agréger l'information n'est pas toujours valable, et peut même générer davantage de volatilité. Nous montrons également que sous-estimer l'impact de l'information non effective et la surestimer n'ont pas des conséquences identiques sur les propriétés du prix de marché.

English version: This work assesses the capability of financial markets in general and then of standardized agricultural commodity markets to aggregate information in the market price and reduce the price volatility. This work presents two new models based on the nature of the information itself. We split the information that is treatable using an algorithm (effective) and the information that cannot. The impact of the predominantly used price determination algorithm, the limit order book, is examined with regard to the nature and structure of the information. These models underline that when the information is mostly effective and continuous, the limit order book indeed allows the market price to be close to its fundamental value. When non effective information becomes too important, this mechanism leads to a bubble or even a crisis. But even in a bubble, the market price can reveal more information than any agent taken separately. When information is discontinuous, on the other hand, as it is the case in commodity markets, the different pricing algorithms used in physical markets and futures markets can lead to a decrease in the market price efficiency but also to a surplus of price volatility. Finally, we take into account the number of agents as a determining variable of the frequency of transactions and of the incoming information on the commodity market on the first hand, and the possibility of having a non effective information on the other hand. The intuition that more agents increase the market price efficiency is not mandatory in this model. Moreover, it can generate more volatility. We also show that underestimating the impact of the non efficient information and overestimating it do not have identical consequences on the properties of the market price.