Galois cohomology of p-adic fields and (phi,tau)-modules

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Cohomologie galoisienne des corps p-adiques et (φ, τ)-modules.

sous la direction d'Olivier BRINON

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In memory of my grandmother.

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Résumé

Soient p un nombre premier et K un corps de valuation discrète de caractéristique 0 complet, à corps résiduel parfait de caractéristique p. Le but de cette thèse est de construire des complexes, définis en termes d'invariants attachés à une représentation p-adique du groupe de Galois absolu de K, et dont l'homologie est isomorphe à la cohomologie galoisienne de la représentation. Dans sa thèse, Herr a construit un tel complexe, à trois termes, à partir du (φ,Γ)-module associé à la représentation (défini à partir de l'extension cyclotomique de K). Pour de nombreuses questions, il est cependant utile de travailler avec une extension de Breuil-Kisin, obtenue à partir de K en lui adjoignant un système compatible de racines p^n-ièmes d'une uniformisante de K. Une différence essentielle (et une difficulté notable) par rapport à la théorie cyclotomique est que l'extension obtenue n'est pas galoisienne. Une solution naturelle, apportée par Tavares Ribeiro dans sa thèse, est de travailler avec l'extension composée de l'extension cyclotomique avec une extension de Breuil-Kisin et le groupe de Galois correspondant, ce qui fournit un complexe à quatre termes. Depuis, Caruso a développé la théorie des (φ,τ)-modules, qui sont à une extension de Breuil-Kisin ce que les (φ,Γ)-modules sont à l'extension cyclotomique : ils fournissent une classification complète des représentations p-adiques (entières ou non). Notre premier résultat est la construction d'un complexe à trois termes, défini à partir du (φ,τ)-module d'une représentation p-adique, et dont l'homologie est isomorphe à la cohomologie galoisienne de la représentation. Nous prouvons qu'il raffine celui de Tavares Ribeiro lorsque le corps résiduel est fini, en construisant un quasi-isomorphisme entre les deux. Ensuite, nous construisons un opérateur ψ (analogue à celui existant dans la théorie cyclotomique), et montrons que dans notre complexe, on peut le substituer à l'opérateur de Frobenius. En s'appuyant sur la surconvergence des (φ,τ)-modules (démontrée par Gao-Poyeton, et que nous raffinons pour les représentations entières), nous définissons des versions surconvergentes de nos complexes, et prouvons qu'ils calculent les bons H^0 et H^1. Par ailleurs, en utilisant des résultats de Poyeton, nous construisons un complexe sur l'anneau de Robba, plus simple que les précédents (l'opérateur τ est remplacé par une dérivation), et dont le H^0 et le H^1 sont isomorphes à la limite inductive des H^0 et H^1 galoisiens le long d'une extension de Breuil-Kisin. Enfin, nous appliquons ce qui précède au calcul de la cohomologie galoisienne du module de Tate d'un groupe p-divisible sur l'anneau des entiers de K, en termes du module de Breuil-Kisin associé.

Mots-clefs

cohomologie galoisienne;
(φ,τ)-modules;
groupes p-divisibles;
corps locaux.

Abstract

Let p be a prime number and K a complete discrete valuation field of characteristic 0, with perfect residue field of characteristic p. The goal of this thesis is to build complexes, defined in terms of invariants attached to a p-adic representation of the absolute Galois group of K, and whose homology is isomorphic to the Galois cohomology of the representation. In his thesis, Herr constructed such a three-term complex using the (φ, Γ)-module associated to the representation (defined from the cyclotomic extension of K). For many questions, however, it is useful to work with a Breuil-Kisin extension, obtained from K by adding to it a compatible system of p^n-th roots of a uniformizer of K. An essential difference (and a notable difficulty) compared to the cyclotomic theory is that the extension obtained is not Galois. A natural solution, provided by Tavares Ribeiro in his thesis, is to work with the composite extension of the cyclotomic extension with a Breuil-Kisin extension and the corresponding Galois group, which provides a four-term complex. Since then, Caruso has developed the theory of (φ, τ)-modules, which are to a Breuil-Kisin extension what (φ, Γ)-modules are to the cyclotomic extension: they provide a complete classification of p-adic representations (integral or not). Our first result is the construction of a three-term complex, defined in terms of the (φ, τ)-module of a p-adic representation, and whose homology is isomorphic to the Galois cohomology of the representation. We prove that it refines that of Tavares Ribeiro in the finite residue field case, by building a quasi-isomorphism between the two. Then, we construct an operator ψ (analogous to the one existing in the cyclotomic theory), and show that in our complex, we can substitute the Frobenius operator with it. Using the overconvergence of (φ, τ)-modules (proved by Gao-Poyeton, and which we refine for integral representations), we define overconvergent versions of our complexes, and prove that they calculate the correct H^0 and H^1. Moreover, using Poyeton's results, we construct a complex over the Robba ring, simpler than the previous ones (the τ operator is replaced by a derivation), and whose H^0 and H^1 are isomorphic to the inductive limits of the Galois cohomology H^0 and H^1 along a Breuil-Kisin extension. Finally, we apply the above to the computation of Galois cohomology of the Tate modules of a p-divisible group over the ring of integers of K, in terms of the associated Breuil-Kisin module.

Keywords

Galois cohomology;
(φ, τ)-modules;
p-divisible groups;
local fields.

Contents

Introduction

Soient p un nombre premier et K un corps de valuation discrète complet, de caractéristique 0, dont le corps résiduel k est parfait de caractéristique p. Fixons une clôture algébrique $\overline{K}$ de K et posons $\mathscr{G}_K = \mathsf{Gal}(\overline{K}/K)$. À une sous-extension strictement arithmétiquement profinie K_∞/Kde $\overline{K}/K$, Fontaine et Wintenberger ont associé des corps de normes parfait (le tilt du complété de K_∞) et imparfait (*cf* [52]). Ce dernier est isomorphe à un corps de séries formelles en une variable à coefficients dans une extension finie de k, et son groupe de Galois absolu est isomorphe à $\mathsf{Gal}(\overline{K}/K_\infty)$.

Le cas le plus étudié est celui où K_∞ est l'extension cyclotomique : fixons $\varepsilon = (\zeta_{p^n})_{n\in\mathbf{N}}$ un système compatible de racines primitives p^n-ièmes de l'unité, et supposons $K_\infty = K_\zeta := \bigcup_{n=0}^{\infty} K(\zeta_{p^n})$. L'extension K_∞/K est alors galoisienne et son groupe de Galois Γ s'identifie, via le caractère cyclotomique, à un sous-groupe ouvert de $\mathbf{Z}_p^\times$. Dans [24], Fontaine a relevé le corps des normes imparfait en caractéristique 0 : il a construit un anneau de Cohen $\mathbf{A}_K$ muni d'un relèvement de Frobenius et d'une action du groupe Γ qui commutent. Il s'agit d'un sous-anneau de l'anneau des séries formelles en une variable à coefficients dans une extension finie de l'anneau des vecteurs de Witt $\mathsf{W}(k)$, et qui convergent sur la couronne d'épaisseur nulle et de rayon 1. Cela lui a permis de construire une équivalence entre la catégorie des représentations p-adiques entières de $\mathscr{G}_K$ (constituée des $\mathbf{Z}_p$-modules de type fini munis d'une action linéaire et continue de $\mathscr{G}_K$) et la catégorie des (φ,Γ)-modules étales sur $\mathbf{A}_K$: ce sont des $\mathbf{A}_K$-modules de type fini munis d'un endomorphisme de Frobenius semi-linéaire, et d'une action semi-linéaire de Γ qui commutent (« étale » signifiant que le linéarisé du Frobenius est un isomorphisme).

Cette théorie a été raffinée par Cherbonnier-Colmez dans [20], dans lequel ils montrent la surconvergence des représentations p-adiques. Cela signifie que dans la théorie des (φ,Γ)-modules qui précède, on peut remplacer l'anneau $\mathbf{A}_K$ par le sous-anneau $\mathbf{A}_K^\dagger$ constitué des éléments qui surconvergent, c'est-à-dire qui convergent sur une couronne d'épaisseur non nulle et de rayon extérieur 1 (et dont $\mathbf{A}_K$ est le complété p-adique). Un des intérêts de ce raffinement est qu'il permet de relier le (φ,Γ)-module associé à une representation p-adique à ses invariants issus de la théorie de Hodge p-adique (*cf* [5], [26], [7] et [10]). L'action infinitésimale de Γ permet en particulier de munir le (φ,Γ)-module sur l'anneau de Robba d'une connexion, grâce à laquelle Berger a prouvé que la conjecture de Crew (démontrée indépendamment par André, Kedlaya et Mebkhout) implique la conjecture de monodromie p-adique de Fontaine (*cf* [5]).

La théorie des (φ,Γ)-modules a de très nombreuses applications : la correspondance de Langlands pour $\mathsf{GL}_2(\mathbf{Q}_p)$ (construite par Colmez, *cf* [22]), et dont la généralisation est l'une des motivations principales de développements récents de la théorie, les lois de réciprocité, la théorie des fonctions L p-adiques et d'Iwasawa (*cf* [2], [6], [3]). Bien logiquement, on peut calculer la cohomologie galoisienne d'une représentation p-adique directement à partir de son (φ,Γ)-module au moyen d'un complexe à trois termes très simple :

Théorème. *(Herr, [29]). Soient T une représentation p-adique entière de $\mathscr{G}_K$ et D le (φ,Γ)-module associé. Supposons que $\Gamma = \mathsf{Gal}(K_\zeta/K)$ soit topologiquement engendré par un élément γ. Alors l'homologie du*

complexe

$$0 \longrightarrow D \longrightarrow D \oplus D \longrightarrow D \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x),(\gamma-1)(x))$$

$$(y,z) \longmapsto (\gamma-1)(y)-(\varphi-1)(z)$$

est canoniquement isomorphe à la cohomologie galoisienne de T.

Ce résultat a permis à Herr de redémontrer les théorèmes de dualité de Tate (*cf* [30]). L'analogue surconvergent et sur l'anneau de Robba sont également valables (*cf* [4], [33]).

La théorie des (φ,Γ)-modules a été généralisée dans de nombreuses directions : dans le cas relatif par Andreatta (*cf* [1]), pour les extensions de Lubin-Tate (*cf* [8], [9]), par Schneider *et al.* (ces dernières fortement motivées par la correspondance de Langlands p-adique).

Depuis les travaux de Breuil (*cf* [11]) et Kisin (*cf* [32]), il est apparu que pour de nombreuses questions (applications aux groupes p-divisibles, théorie de Hodge p-adique « entière », étude des déformations de représentations p-adiques), il est judicieux de travailler avec d'autres extensions profondément ramifiées : les extension de Breuil-Kisin. Pour les construire, on se donne une uniformisante π de K, et un système compatible $\widetilde{\pi}=(\pi_n)_{n\in\mathbf{N}}$ de racines p^n-èmes de π (*i.e.* tel que $\pi_0=\pi$ et $\pi_{n+1}^p=\pi_n$ pour tout $n\in\mathbf{N}$) : l'extension associée est alors $K_\infty=K_\pi:=\bigcup_{n=0}^{\infty}K(\pi_n)$. Là encore, on peut construire un anneau de Cohen (que nous noterons $\mathcal{O}_\mathcal{E}$) pour les corps de normes afférents, mais contrairement au cas cyclotomique, l'extension K_π/K n'est pas galoisienne, ce qui fait qu'il « manque » une action galoisienne sur les φ-modules étales que l'on associe aux représentations p-adiques dans ce cadre. Cela a néanmoins permis à Kisin d'associer des invariants (des fibrés sur des domaines du disque unité ouvert) à certaines représentations semi-stables, qui lui ont permis de classifier les représentations cristallines, les groupes p-divisibles et les schémas en groupes finis et plats sur $\mathcal{O}_K$, et de montrer que les représentations cristallines de poids de Hodge-Tate 0 et 1 proviennent toutes de groupes p-divisibles. Depuis, cette théorie a donné naissance à une littérature abondante, notamment sous l'impulsion de Caruso et T. Liu (*cf* [34], [35], [16], [36], [17], [18], [37], [19], [38], [39], [40], [23], [14], [15], [27]).

Dans ce contexte, il est naturel de chercher à calculer la cohomologie galoisienne d'une représentation p-adique à partir de ses invariants attachés à une extension de Breuil-Kisin, au moyen d'un complexe analogue au complexe de Herr mentionné plus haut. C'est le premier objectif de cette thèse. Comme on l'a vu, l'obstacle principal est que l'extension K_π/K n'est pas galoisienne. Il est naturel de considérer la clôture galoisienne de K_π dans $\overline{K}$: c'est le compositum $L=K_\pi K_\zeta$ de K_π avec l'extension cyclotomique. Le groupe de Galois $\mathsf{Gal}(L/K)$ est alors un produit semi-direct de $\mathbf{Z}_p(1)$ par un sous-groupe ouvert de $\mathbf{Z}_p^\times$ (l'extension L/K est dite métabélienne). Ce point de vue a été utilisé par Tavares Ribeiro dans sa thèse (*cf* [50]), dans laquelle il a construit une théorie analogue à celle des (φ,Γ)-modules, et un complexe à quatre termes calculant la cohomologie de la représentation.

Théorème. *(Tavares Ribeiro, [50, §1.5]) Supposons que $\Gamma=\mathsf{Gal}(K_\zeta/K)$ soit topologiquement engendré par un élément γ et soit τ un générateur topologique de $\mathsf{Gal}(L/K_\zeta)$. Soit T une représentation p-adique entière de $\mathscr{G}_K$ et posons $M=D_L(T)=\left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\otimes_{\mathbf{Z}_p}T\right)^{\mathscr{G}_L}$. Alors l'homologie du complexe*

$$0\to M\xrightarrow{\tilde{\alpha}}M\oplus M\oplus M\xrightarrow{\tilde{\beta}}M\oplus M\oplus M\xrightarrow{\tilde{\eta}}M\to 0$$

où

$$\tilde{\alpha}=\begin{pmatrix}\varphi-1\\ \gamma-1\\ \tau-1\end{pmatrix},\ \tilde{\beta}=\begin{pmatrix}\gamma-1 & 1-\varphi & 0\\ \tau-1 & 0 & 1-\varphi\\ 0 & \tau^{\chi(\gamma)}-1 & \delta-\gamma\end{pmatrix}$$

$$\tilde{\eta} = \left(\tau^{\chi(\gamma)} - 1, \delta - \gamma, \varphi - 1\right)$$

et $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1} \in \mathbf{Z}_p[\![\tau - 1]\!]$ *est canoniquement isomorphe à la cohomologie galoisienne de* T.

Cela lui a permis de prouver la loi de réciprocité de Brückner-Vostokov pour un groupe formel.

Cela dit, la théorie des (φ, τ)-modules de Caruso (*cf* [19]) fournit un avatar de la théorie des (φ, Γ)-modules dans le cadre des extensions de Breuil-Kisin. Étant donnée une représentation p-adique (disons entière) T de $\mathscr{G}_K$, l'idée est de considérer non pas seulement le φ-module étale $\mathscr{D}(T)$ associé sur $\mathscr{O}_{\mathscr{E}}$, mais aussi l'action d'un générateur topologique τ de $\mathsf{Gal}(L/K_\zeta)$ sur $\mathcal{D}(T)_\tau := \mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T)$ (où $\mathcal{E}_\tau$ est une extension convenable du corps des fraction $\mathcal{E}$ de $\mathcal{O}_\mathcal{E}$). Explicitement on a :

Théorème. *(Caruso, [19, §1.3], cf section 1.1) Le foncteur*

$$\begin{aligned} \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) &\to \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau) \\ T &\mapsto \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T\right)^{\mathscr{G}_{K_\pi}} \end{aligned}$$

est une équivalence de catégories entre la catégorie des représentations p-adiques entières et celle des (φ, τ)*-modules sur* $\mathcal{O}_\mathcal{E}$.

La première contribution de cette thèse est la construction d'un complexe à trois termes proche de celui de Herr, construit à partir du (φ, τ)-module $\mathcal{D}(T)$, et qui calcule la cohomologie galoisienne de T.

Plus précisément, on a

Théorème. *(cf théorème 1.1.13) Soient* T *une représentation p-adique entière de* $\mathscr{G}_K$ *et* (D, D_τ) *le* (φ, τ)*-module associé. Supposons* Γ *topologiquement engendré par un élément. Posons*

$$D_{\tau,0} = \left\{x \in D_\tau,\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow (g \otimes 1)(x) = x + \tau_D(x) + \cdots + \tau_D^{\chi(g)-1}(x)\right\}.$$

Alors l'homologie du complexe

$$0 \longrightarrow D \longrightarrow D \oplus D_{\tau,0} \longrightarrow D_{\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longmapsto (\tau_D - 1)(y) - (\varphi - 1)(z)$$

est canoniquement isomorphe à la cohomologie galoisienne de T.

Les techniques employées dans la preuve sont standard (effaçabilité, dévissage et passage à la limite), mais cette dernière est rendue un peu plus délicate par le fait que le complexe fait intervenir un sous-groupe $\mathcal{D}(T)_{\tau,0}$ de $\mathcal{D}(T)$ qui n'est pas facile d'appréhender (mais qui est nécessaire, le complexe « naïf » ne fournissant pas les bons groupes de cohomologie, *cf* section 1.5).

Dans la deuxième partie, nous construisons un morphisme entre le complexe de Tavares Ribeiro et le nôtre, et prouvons que c'est un quasi-isomorphisme lorsque le corps résiduel est fini, ce qui montre que ce dernier est un raffinement du premier dans ce cas (*cf* theorem 2.1.14).

Dans la troisième partie, nous construisons un opérateur ψ (un inverse à gauche du Frobenius) et l'utilisons pour construire d'autres complexes calculant la cohomologie galoisienne. Le corps résiduel de $\mathcal{E}_\tau$ étant parfait, un tel opérateur ψ ne peut pas être construit sur ce dernier : il est nécessaire d'utiliser un raffinement de la théorie de (φ, τ)-modules, plus précisément, il faut travailler avec des coefficients partiellement déperfectisés $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ (*cf* [19, §1.2.2], section 3.1). On montre que les complexes analogues à celui du théorème 1.1.13, mais à coefficients dans $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ et utilisant le Frobenius ou l'opérateur ψ calculent la cohomologie galoisienne (*cf* theorem 3.2.4 et 3.3.11).

Dans la quatrième partie, on construit des avatars surconvergents des complexes précédents, utilisant soit l'opérateur φ de Frobenius, soit l'opérateur ψ. Bien entendu, on s'appuie de façon cruciale sur la surconvergence des (φ,τ)-modules, démontrée par Gao et Poyeton (*cf* [28]), et que nous la raffinons au cas des représentations entières (*cf* proposition 4.1.9). Cela nous permet de prouver que les complexes surconvergents calculent les bons H^0 et H^1. Le cas du H^2 reste hélas non démontré. Signalons qu'à notre connaissance, le fait que le complexe de Herr surconvergent « classique » (correspondant au cas de l'extension cyclotomique) calcule la cohomologie galoisienne n'est démontré que dans le cas où k est fini (*i.e.* où K est une extension finie de $\mathbf{Q}_p$, *cf* [33], [4]).

Dans la cinquième partie, nous construisons un complexe à trois termes à partir du (φ,N_∇)-module sur l'anneau de Robba $\mathcal{R}$ associé à une représentation (*i.e.* à son (φ,τ)-module). Plus précisément, si V est une représentation p-adique de $\mathscr{G}_K$ et $D^\dagger_{\mathsf{rig}}$ le (φ,N_∇)-module associé, le complexe en question est de la forme

$$0 \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \oplus D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow 0$$

$$x \longmapsto \big((\varphi-1)(x), N_\nabla(x)\big)$$

$$(y,z) \longmapsto N_\nabla(y) - (c\varphi - 1)(z)$$

où $c \in \mathcal{R}$ est un élément explicite.

Il s'agit, après extension des scalaires à un anneau de Robba convenable, de remplacer l'opérateur τ_D par l'action infinitésimale (convenablement normalisée) de $\mathsf{Gal}(L/K_\zeta)$ (*cf* [45, §2.2]). Pour cette raison, le complexe ne peut bien entendu pas calculer la cohomologie galoisienne de la représentation galoisienne V dont on est parti, mais « presque »: nous montrons (*cf* proposition 5.3.17) que son H^i est isomorphe à $\varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n}, V)$ pour $i \in \{0,1\}$. Là encore, le cas du H^2 pose encore problème. Dans le cas où le corps résiduel k est fini, nous construisons un accouplement analogue à celui qui donne lieu à la dualité de Tate (*cf* [30]), mais sa non dégénérescence (qui permetterait d'étendre l'énoncé évoqué ci-dessus au cas $i=2$) n'est pas démontrée. Observons que, bien qu'un peu moins fin que les complexes construits précédemment et qui calculent la cohomologie galoisienne, le complexe considéré dans cette partie a le gros avantage de ne pas faire intervenir un groupe du type $D_{\tau,0}$ (qui comme on l'a dit, est assez peu tangible), mais seulement le (φ,N_∇)-module : il est donc *a priori* plus maniable.

Dans la sixième et dernière partie, nous appliquons ce qui précède aux représentations provenant des groupes p-divisibles sur $\mathcal{O}_K$. Depuis les travaux de Breuil et Kisin (*cf* [32]), on sait qu'ils sont classifiés par certains φ-modules sur $\mathfrak{S} := W[\![u]\!]$. L'idée est alors de relier le (φ,N_∇)-module associé au module de Tate d'un groupe p-divisible avec le $\mathfrak{S}$-module associé. Signalons que d'après Caruso, on est même capable de retrouver l'action de τ à partir de celle de N_∇ dans ce cas, ce qui permet de calculer la cohomologie du dual du module de Tate du groupe p-divisible à partir du module de Breuil-Kisin (*cf* corollary 6.5.2).

L'objectif initial de cette thèse était de calculer la cohomologie galoisienne du module de Tate d'un groupe p-divisible sur une base affine assez générale à partir de son display de Zink (*cf* [53], [54]). La complexité des coefficients mis en jeu et l'intérêt du cas des groupes p-divisibles sur l'anneau des entiers d'un corps p-adique, en lien avec la théorie des (φ,τ)-modules de Caruso, ont fait qu'il ne sera pas question des displays dans cette thèse. Nous espérons revenir sur ces derniers ultérieurement.

Introduction

Let p be a prime number and K a complete discrete valuation field, of characteristic 0, whose residue field k is perfect of characteristic p. Fix an algebraic closure $\overline{K}$ of K and put $\mathscr{G}_K = \mathsf{Gal}(\overline{K}/K)$. For any strictly arithmetic profinite subextension K_∞/K of $\overline{K}/K$, Fontaine and Wintenberger have associated perfect (the tilt of the completion of K_∞) and imperfect (*cf* [52]) fields of norms. The latter is isomorphic to a field of formal Laurent series in one variable with coefficients in a finite extension of k, and its absolute Galois group is isomorphic to $\mathsf{Gal}(\overline{K}/K_\infty)$.

The most studied case is the one where K_∞ is the cyclotomic extension: fix $\varepsilon = (\zeta_{p^n})_{n\in\mathbf{N}}$ a compatible system of primitive p^n-th roots of unity, and assume $K_\infty = K_\zeta := \bigcup_{n=0}^{\infty} K(\zeta_{p^n})$. The extension K_∞/K is then Galois and its Galois group Γ is identified, via the cyclotomic character, with an open subgroup of $\mathbf{Z}_p^\times$. In [24], Fontaine lifted the imperfect field of norms to characteristic 0: he constructed a Cohen ring $\mathbf{A}_K$ endowed with a lifted Frobenius map and an action of the group Γ which commutes with Frobenius. It is a sub-ring of the ring of formal Laurent series in one variable with coefficients in a finite extension of the ring of Witt vectors $\mathsf{W}(k)$, and which converges on the annulus of thickness 0 and radius 1. This allowed him to build an equivalence of categories between the category of integral p-adic representations of $\mathscr{G}_K$ (constituted by $\mathbf{Z}_p$-modules of finite type with a continuous linear action of $\mathscr{G}_K$) and the category of étale (φ,Γ)-modules on $\mathbf{A}_K$: these are $\mathbf{A}_K$-modules of finite type endowed with a semi-linear Frobenius endomorphism, and a semi-linear action of Γ which commutes with Frobenius ("étale" meaning that the linearization of Frobenius is an isomorphism).

This theory has been refined by Cherbonnier-Colmez in [20], in which they showed the overconvergence of p-adic representations. This means that in the theory of (φ,Γ)-modules which precedes, one can replace the ring $\mathbf{A}_K$ by the sub-ring $\mathbf{A}_K^\dagger$ constituted by overconvergent elements, that is to say, which converge on an annulus of non-zero thickness and outer radius 1 (and $\mathbf{A}_K$ is its p-adic completion). One of the interests of this refinement is that it allows to relate the (φ,Γ)-module associated to a p-adic representation to its invariants coming from p-adic Hodge theory (*cf* [5], [26], [7] and [10]). The infinitesimal action of Γ allows in particular to endow the (φ,Γ)-module over the Robba ring with a connection, thanks to which Berger proved that Crew's conjecture (proved independently by André, Kedlaya and Mebkhout) implies Fontaine's p-adic monodromy conjecture (*cf* [5]).

The theory of (φ,Γ)-modules has many applications: the Langlands correspondence for $\mathsf{GL}_2(\mathbf{Q}_p)$ (constructed by Colmez, *cf* [22]), and whose generalization is one of the main motivations of recent developments of the theory, reciprocity laws, p-adic L functions theory and Iwasawa theory (*cf* [2], [6], [3]). Logically, one can compute the Galois cohomology of a p-adic representation directly from its (φ,Γ)-module using a simple three-terms complex:

Theorem. *(Herr, [29]). Let T be an integral p-adic representation of $\mathscr{G}_K$ and D the associated (φ,Γ)-module.*

Assume Γ is topologically generated by an element γ. Then the homology of the complex

$$0 \longrightarrow D \longrightarrow D \oplus D \longrightarrow D \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x),(\gamma-1)(x))$$

$$(y,z) \longmapsto (\gamma-1)(y)-(\varphi-1)(z)$$

is canonically isomorphic to the Galois cohomology of T.

This result allowed him to reprove Tate's duality theorem (*cf* [30]). The analogues with (φ,Γ)-modules over overconvergent rings and over the Robba ring are also valid (see [4], [33]).

The theory of (φ,Γ)-modules has been developed in many directions: in the relative case by Andreatta (*cf* [1]), for Lubin-Tate extensions (*cf* [8], [9]), by Schneider *et al.* (the latter is strongly motivated by the Langlands p-adic correspondence).

Since the work of Breuil (*cf* [11]) and Kisin (*cf* [32]), it has been clear that for many questions (applications to p-divisible groups, "integral" p-adic Hodge theory, study of the deformations of p-adic representations), it is judicious to work with other deeply ramified extensions: the Breuil-Kisin extensions. To construct them, we fix a uniformizer π of K, and a compatible system $\widetilde{\pi}=(\pi_n)_{n\in\mathbf{N}}$ of p^n-roots of π (*i.e.* such that $\pi_0=\pi$ and $\pi_{n+1}^p=\pi_n$ for all $n\in\mathbf{N}$): the associated extension is then $K_\infty=K_\pi:=\bigcup_{n=0}^{\infty}K(\pi_n)$. Again, one can construct a Cohen ring (which we will note $\mathcal{O}_{\mathcal{E}}$) for the corresponding field of norms, but unlike the cyclotomic case, the extension K_π/K is not Galois, so that a Galois action on the étale φ-modules is "missing", which is needed to associate p-adic representations in this framework. Nevertheless, this allowed Kisin to associate invariants (bundles over domains in the open unit disk) to certain semi-stable representations, which allowed him to classify crystalline representations, p-divisible groups and finite flat group schemes over $\mathcal{O}_K$, and to show that crystalline representations of Hodge-Tate weights 0 and 1 all come from p-divisible groups. Since then, this theory has given rise to an abundant literature, notably under the impulse of Caruso and T. Liu (*cf* [34], [35], [16], [36], [17], [18], [37], [19], [38], [39], [40], [23], [14], [15], [27]).

In this context, it is natural to try to compute the Galois cohomology of a p-adic representation from its invariants attached to a Breuil-Kisin extension, by means of a complex similar to the Herr complex mentioned above. This is the first objective of this thesis. As we have seen, the main obstacle is that the extension K_π/K is not Galois. It is natural to consider the Galois closure of K_π in $\overline{K}$: it is the compositum $L=K_\pi K_\zeta$ of K_π with the cyclotomic extension. The Galois group $\mathsf{Gal}(L/K)$ is then a semi-direct product of $\mathbf{Z}_p(1)$ by an open subgroup of $\mathbf{Z}_p^\times$ (the extension L/K is said to be métabélian). This point of view is used by Tavares Ribeiro in his thesis (*cf* [50]), in which he constructed a theory analogous to that of (φ,Γ)-modules, and a four-term complex computing the Galois cohomology of the representation.

Theorem. *(Tavares Ribeiro, [50, §1.5]) Suppose that Γ is topologically generated by an element γ and let τ be a topological generator of $\mathsf{Gal}(L/K_\zeta)$. Let T be an integral p-adic representation of $\mathscr{G}_K$ and let $M=D_L(T)=\left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\otimes_{\mathbf{Z}_p}T\right)^{\mathscr{G}_L}$. Then the homology of the complex*

$$0\to M\xrightarrow{\tilde{\alpha}}M\oplus M\oplus M\xrightarrow{\tilde{\beta}}M\oplus M\oplus M\xrightarrow{\tilde{\eta}}M\to 0$$

where

$$\tilde{\alpha}=\begin{pmatrix}\varphi-1\\ \gamma-1\\ \tau-1\end{pmatrix},\ \tilde{\beta}=\begin{pmatrix}\gamma-1 & 1-\varphi & 0\\ \tau-1 & 0 & 1-\varphi\\ 0 & \tau^{\chi(\gamma)}-1 & \delta-\gamma\end{pmatrix}$$

$$\tilde{\eta}=\left(\tau^{\chi(\gamma)}-1,\delta-\gamma,\varphi-1\right)$$

and $\delta=(\tau^{\chi(\gamma)}-1)(\tau-1)^{-1}\in\mathbf{Z}_p[\![\tau-1]\!]$ is canonically isomorphic to the Galois cohomology of T.

This allowed him to prove the Brückner-Vostokov reciprocity law for a formal group.

This said, Caruso's (φ,τ)-module theory (*cf* [19]) provides an avatar of (φ,Γ)-module theory in the context of Breuil-Kisin extensions. Given a p-adic (let's say integral) representation T of $\mathscr{G}_K$, the idea is to consider not only the associated φ-module $\mathscr{D}(T)$ over $\mathscr{O}_{\mathscr{E}}$, but also the action of a topological generator τ of $\mathsf{Gal}(L/K_\zeta)$ on $\mathcal{D}(T)_\tau := \mathcal{O}_{\mathcal{E}_\tau}\otimes_{\mathcal{O}_\mathcal{E}}\mathcal{D}(T)$ (where $\mathcal{E}_\tau$ is a suitable extension of the fraction field $\mathcal{E}$ of $\mathcal{O}_\mathcal{E}$ and the action is usually denoted τ_D.) Explicitly we have:

Theorem. *(Caruso, [19, §1.3], cf section 1.1) The functor*

$$\begin{aligned}\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) &\to \mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)\\ T &\mapsto \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\otimes_{\mathbf{Z}_p}T\right)^{\mathscr{G}_{K_\pi}}\end{aligned}$$

is an equivalence between the category of integral p-adic representations and that of (φ,τ)-modules over $\mathcal{O}_\mathcal{E}$.

The first contribution of this thesis is the construction of a three term complex close to Herr's, built from the (φ,τ)-module $\mathcal{D}(T)$, and which computes the Galois cohomology of T

More precisely, we have

Theorem. *(cf theorem 1.1.13). Let T be an integral p-adic representation of $\mathscr{G}_K$ and (D,D_τ) the associated (φ,τ)-module. Assume that Γ is topologically generated by an element. Put*

$$D_{\tau,0}=\left\{x\in D_\tau,\ (\forall g\in\mathscr{G}_{K_\pi})\ \chi(g)\in\mathbf{Z}_{>0}\Rightarrow(g\otimes 1)(x)=x+\tau_D(x)+\cdots+\tau_D^{\chi(g)-1}(x)\right\}.$$

Then the homology of the complex

$$0\longrightarrow D\longrightarrow D\oplus D_{\tau,0}\longrightarrow D_{\tau,0}\longrightarrow 0$$

$$x\longmapsto((\varphi-1)(x),(\tau_D-1)(x))$$

$$(y,z)\longmapsto(\tau_D-1)(y)-(\varphi-1)(z)$$

is canonically isomorphic to the Galois cohomology of T.

The techniques used in the proof are standard (effaceability, dévissage and passing to limit), but a bit more difficult (compared to the cyclotomic case) by the fact that the complex involves a subgroup $\mathcal{D}(T)_{\tau,0}$ of $\mathcal{D}(T)_\tau$ which is not easy to understand (which is necessary, as the "naive" complex does not provide the correct cohomology groups, *cf* section 1.5).

In the second part, we construct a morphism between the Tavares Ribeiro complex and ours, and prove that it is a quasi-isomorphism when the residue field is finite, which shows that the latter is a refinement of the former in this case (*cf* theorem 2.1.14).

In the third part, we construct a ψ operator (a left inverse of the Frobenius) and use it to construct other complexes that compute Galois cohomology. Since the residue field of $\mathcal{E}_\tau$ is perfect, such a ψ operator cannot be constructed on it: it is necessary to use a refinement of (φ,τ)-module theory, more precisely, it is necessary to work with partially unperfected coefficients $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ (*cf* [19, §1.2.2], section 3.1). It is shown that complexes analogous to the one in theorem 1.1.13, but with coefficients in $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ and using operator φ or ψ compute the Galois cohomology (*cf* theorems 3.2.4 and 3.3.11).

In the fourth part, we build overconvergent avatars of the previous complexes, using either the Frobenius operator φ, or the ψ operator. Of course, we rely crucially on the overconvergence of (φ,τ)-modules, proved by Gao and Poyeton (*cf* [28]), which we refine to the case of integral representations (*cf* proposition 4.1.9). This allows us to prove that the overconvergent complexes compute the correct H^0 and H^1. The case of

H^2 remains unfortunately open. Let us point out that, as far as we know, the fact that the "classical" overconvergent Herr complex (corresponding to the case of the cyclotomic extension) is shown to compute the Galois cohomology only in the case where k is finite (*i.e.* where K is a finite extension of $\mathbf{Q}_p$, *cf* [33], [4]).

In the fifth part, we construct a three-term complex from the (φ, N_∇)-module over the Robba ring $\mathcal{R}$ associated to a representation (*i.e.* associated to its (φ, τ)-module).

More precisely, if V is a p-adic representation of $\mathscr{G}_K$ and $D^\dagger_{\mathsf{rig}}$ the associated (φ, N_∇)-module, the complex in question is of the form

$$0 \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \oplus D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow 0$$

$$x \longmapsto \big((\varphi - 1)(x), N_\nabla(x)\big)$$

$$(y, z) \longmapsto N_\nabla(y) - (c\varphi - 1)(z)$$

where $c \in \mathcal{R}$ is an explicit element. It is a matter, after extension of the scalars to a suitable Robba ring, of replacing the operator τ_D by the (suitably normalized) infinitesimal action of $\mathsf{Gal}(L/K_\zeta)$ (*cf* [45, §2.2]). For this reason, the complex of course cannot compute Galois cohomology of the Galois representation V from which we started, but it can almost be computed: we show (*cf* proposition 5.3.17) that its H^i is isomorphic to $\varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n}, V)$ for $i \in \{0, 1\}$. Again, the case of H^2 is still problematic. In the case where the residue field k is finite, we construct a pairing analogous to the one which gives rise to the Tate duality (*cf* [30]), but its non-degeneracy (which would allow to extend the above mentioned assertion to the case $i = 2$) is not demonstrated. Let us observe that, although a little less fine than the complexes constructed previously (which compute the Galois cohomology), the complex considered in this part has the great advantage of not involving a group of the type $D_{\tau,0}$ (which as we said, is rather intangible), but only the (φ, N_∇)-module: it is thus *a priori* easier to deal with.

In the sixth and last part, we apply the above to representations coming from p-divisible groups over $\mathcal{O}_K$. From the work of Breuil and Kisin (*cf* [32]), we know that they are classified by some φ-modules on $\mathfrak{S} := W[\![u]\!]$. The idea is then to relate the (φ, N_∇)-module associated to the Tate module of a p-divisible group with the associated $\mathfrak{S}$-module. Let us point out that according to Caruso, we are even able to find the action of τ from that of N_∇ in this case, which allows us to compute the Galois cohomology of the dual of the Tate module of a p-divisible group, using its Breuil-Kisin module (*cf* corollary 6.5.2).

The initial goal of this thesis was to compute the Galois cohomology of the Tate module of a p-divisible group over a general affine base in terms of its Zink's display (*cf* [53], [54]). The complexity of the coefficients involved and the great interest of the case of p-divisible groups on the ring of integers of a p-adic field, in connection with Caruso's (φ, τ)-module theory, have led to the fact that displays will not be discussed in this thesis. We hope to come back to them later.

Notations

Extensions of fields

Let p be a prime, K a complete discretly valued extension of $\mathbf{Q}_p$, with perfect residue field k. Let v be the normalized valuation on K. Fix an algebraic closure $\overline{K}$ of K : the valuation v extends to $\overline{K}$, denote by C its completion. The group $\mathscr{G}_K = \mathsf{Gal}(\overline{K}/K)$ acts by continuity on C.

Let

$$C^\flat = \varprojlim_{x \mapsto x^p} C = \left\{ (x_n)_{n\in\mathbf{N}} \in C^{\mathbf{N}} \,;\, (\forall n \in \mathbf{N})\, x_{n+1}^p = x_n \right\}$$

be the tilt of C (*cf* [48, §3]): products and sums are

$$(x_n)_{n\in\mathbf{N}}.(y_n)_{n\in\mathbf{N}} = (x_n y_n)_{n\in\mathbf{N}},$$

and

$$(x_n)_{n\in\mathbf{N}} + (y_n)_{n\in\mathbf{N}} = (z_n)_{n\in\mathbf{N}},$$

where

$$z_n = \lim_{m\to\infty} \left(x_{n+m} + y_{n+m}\right)^{p^m}.$$

The field $C^\flat$ is algebraically closed of characteristic p and complete for the valuation given by

$$\begin{aligned} v^\flat \colon C^\flat &\to \mathbf{R} \cup \{\infty\} \\ \left(x_n\right)_{n\in\mathbf{N}} &\mapsto v\left(x_0\right). \end{aligned}$$

We denote $\mathcal{O}_{C^\flat}$ as the ring of integers of $C^\flat$ and recall that the natural map

$$\begin{aligned} \mathcal{O}_{C^\flat} &\to \varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \\ \left(x_n\right)_{n\in\mathbf{N}} &\mapsto \left(x_n \bmod p\right)_{n\in\mathbf{N}} \end{aligned}$$

is an isomorphism.

Fix $\tilde{\pi} = (\pi_n) \in \mathcal{O}_{C^\flat}$ with $\pi_0 = \pi$ and $\varepsilon = (\zeta_{p^n})_{n\in\mathbf{N}} \in \mathcal{O}_{C^\flat}$ such that ζ_p is a primitive p-th root of unity. Let

$$K_\pi := \bigcup_n K(\pi_n), \quad K_\zeta := \bigcup_n K(\zeta_{p^n}),$$

and

$$L := K_\pi(\zeta_{p^\infty}) = K_\pi K_\zeta.$$

We denote the corresponding Galois groups as follows:

$$\mathscr{G}_{K_\pi} = \mathsf{Gal}(\overline{K}/K_\pi), \quad \mathscr{G}_{K_\zeta} = \mathsf{Gal}(\overline{K}/K_\zeta), \quad \mathscr{G}_L = \mathsf{Gal}(\overline{K}/L).$$

Let $\chi \colon \mathscr{G}_K \to \mathbf{Z}_p^\times$ be the cyclotomic character. It induces an injective morphism with open image:

$$\Gamma := \mathsf{Gal}(K_\zeta/K) \hookrightarrow \mathbf{Z}_p^\times .$$

In what follows, *assume* that Γ has a topological generator.

Lemma 0.0.1. *There exists a topological generator $\gamma \in \Gamma$ with $\chi(\gamma) \in \mathbf{Z}_{>0}$.*

Proof. Let γ_0 be a topological generator of Γ. As $\chi(\Gamma)$ is an open subgroup of $\mathbf{Z}_p$, there exists $N \in \mathbf{Z}_{>0}$ such that $1 + p^N \mathbf{Z}_p \subset \chi(\Gamma)$, hence $\chi(\gamma_0) + p^N \mathbf{Z}_p \subset \chi(\Gamma)$. As $\mathbf{Z}_{\geq 0}$ is dense in $\mathbf{Z}_p$, we can choose $\gamma \in \Gamma$ such that

$$\chi(\gamma) \in \mathbf{Z}_{\geq 0} \cap (\chi(\gamma_0) + p^N \mathbf{Z}_p).$$

We have $\overline{\langle \chi(\gamma) \rangle} \subset \chi(\Gamma)$ and both have the same image under $\mathbf{Z}_p^\times \to (\mathbf{Z}/p^N \mathbf{Z})^\times$: we have

$$\chi(\gamma_0)\chi(\gamma)^{-1} \in \chi(\Gamma) \cap (1 + p^N \mathbf{Z}_p),$$

so we can write $\chi(\gamma_0)\chi(\gamma)^{-1} = \chi(\gamma_0)^z$ with $z \in p\mathbf{Z}_p$ (if $N \gg 0$). We have $\chi(\gamma_0) = \chi(\gamma)\chi(\gamma_0)^z$, so

$$\chi(\gamma_0) = \chi(\gamma)^{1+z}\chi(\gamma_0)^{z^2},$$

and by induction

$$\chi(\gamma_0) = \chi(\gamma)^{(1+z+z^2+\cdots+z^r)}\chi(\gamma_0)^{z^{r+1}},$$

and passing to the limit gives

$$\chi(\gamma_0) = \chi(\gamma)^{(1+z+z^{2\cdots})}$$

so that $\gamma_0 = \gamma^{\frac{1}{1-z}} \in \overline{\langle \gamma \rangle}$. □

The element τ

If $g \in \mathscr{G}_K$, we denote by $c(g)$ the unique element in $\mathbf{Z}_p$ such that $g(\tilde{\pi}) = \varepsilon^{c(g)}\tilde{\pi}$. Indeed, for any $n \in \mathbf{N}$, there exists a unique element $c_n(g) \in \mathbf{Z}/p^n\mathbf{Z}$ such that $g(\pi_n) = \zeta_{p^n}^{c_n(g)}\pi_n$. As $c_{n+1}(g) \equiv c_n(g) \bmod p^n$, the sequence $\big(c_n(g)\big)_n$ defines the element $c(g) \in \mathbf{Z}_p$. Notice that the map $c\colon \mathscr{G}_K \to \mathbf{Z}_p(1)$ is a 1-cocycle (with $c^{-1}(0) = \mathscr{G}_{K_\pi}$), *i.e.* for any $g, h \in \mathscr{G}_K$:

$$c(gh) = c(g) + \chi(g)c(h).$$

Fix an element $\tau \in \mathsf{Gal}(\overline{K}/K_\zeta)$ such that $\tau(\tilde{\pi}) = \varepsilon\tilde{\pi}$ (*i.e.* $c(\tau) = 1$). As $\mathscr{G}_L \subset c^{-1}(0)$, we still denote τ for its image in $\mathsf{Gal}(L/K_\zeta)$ (*cf* diagram below).

Remark 0.0.2. The sequence

$$1 \to \mathscr{G}_L \to \mathscr{G}_K \xrightarrow{\chi_\infty} \mathbf{Z}_p \rtimes \Gamma \to 1$$

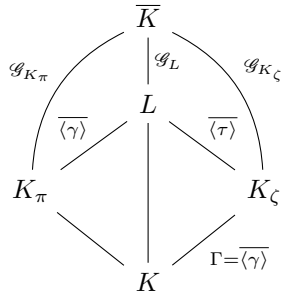
is exact with $\chi_\infty = (c, \chi)$ and we have $\mathbf{Z}_p \rtimes \Gamma \simeq \mathscr{G}_K/\mathscr{G}_L = \mathsf{Gal}(L/K)$. Moreover, $\mathsf{Gal}(L/K) \simeq \overline{\langle \tau \rangle} \rtimes \overline{\langle \gamma \rangle}$ with $\gamma\tau\gamma^{-1} = \tau^{\chi(\gamma)}$.

Remark 0.0.3. If $g \in \mathscr{G}_{K_\pi}$, then

$$\begin{cases} g\tau g^{-1}(\tilde{\pi}) = \varepsilon^{\chi(g)}\tilde{\pi} = \tau^{\chi(g)}(\tilde{\pi}) \\ g\tau g^{-1}(\varepsilon) = \varepsilon = \tau^{\chi(g)}(\varepsilon) \end{cases}$$

so that $g\tau g^{-1}$ and $\tau^{\chi(g)}$ have the same image in $\mathscr{G}_K/\mathscr{G}_L$.

We summarize these notations in a diagram:



Topologies on $\mathsf{W}(C^\flat)$

Let $\mathsf{W}(C^\flat)$ be the ring of Witt vectors with coefficients in $C^\flat$. The group $\mathscr{G}_K$ naturally acts on $\mathsf{W}(C^\flat)$ and we now describe some topologies on $\mathsf{W}(C^\flat)$.

Definition 0.0.4. We define the *weak topology* on $\mathsf{W}(C^\flat)$ as follows. Write

$$\mathsf{W}(C^\flat) = \varprojlim_n \mathsf{W}_n(C^\flat)$$

and endow $\mathsf{W}(C^\flat)$ with the inverse limit topology (*i.e.* the topology induced by the product topology on $\prod_n \mathsf{W}_n(C^\flat)$) where we endow $\mathsf{W}_n(C^\flat) = (C^\flat)^n$ with the topology induced by the valuation topology on $C^\flat$.

Remark 0.0.5.

(1) The p-adic topology on $\mathsf{W}(C^\flat)$ induces the discrete topology on $C^\flat$.

(2) The weak topology induces the valuation topology on $C^\flat$.

(3) The $\mathscr{G}_K$-action is not continuous on $C^\flat$ with the discrete topology, while it is continuous on $C^\flat$ with the valuation topology. Indeed we have $\lim_{g\to \mathrm{id}} \varepsilon^{\chi(g)} = \varepsilon$ for the valuation topology, but not for the discrete topology.

Chapter 1

The complex $\mathcal{C}_{\varphi,\tau}$

In this chapter, we introduce the category $\mathbf{Mod}_{\mathcal{E},\mathcal{E}_\tau}(\varphi,\tau)$ of (φ,τ)-modules over $(\mathcal{E},\mathcal{E}_\tau)$ (introduced by Caruso *cf* [19]), which is categorically equivalent to the category of p-adic representations. Then for any p-adic representation V, we construct a complex $\mathcal{C}_{\varphi,\tau}(V)$ (which is functorial) using the (φ,τ)-module associated to V, and we show that this complex computes the continuous Galois cohomology of V. Hence, this is indeed a variant of Herr complex for p-adic representations (*cf* [29]).

1.1 Construction of the complex

Notation 1.1.1. Let $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ (resp. $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, resp. $\mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$) be the category of $\mathbf{Z}_p$-representations (resp. p-adic representations, resp. $\mathbf{F}_p$-representations), whose objects are $\mathbf{Z}_p$-modules (resp. $\mathbf{Q}_p$-vector spaces, resp. $\mathbf{F}_p$-vector spaces) of finite type endowed with a linear and continuous action of $\mathscr{G}_K$. Let $\mathbf{Rep}_{\mathbf{Z}_p,\mathrm{tors}}(\mathscr{G}_K)$ be the subcategory of $\mathbf{Z}_p$-representations that are killed by some power of p. Moreover, for a fixed $r \in \mathbf{N}$, let $\mathbf{Rep}_{\mathbf{Z}_p,p^r\text{-tors}}(\mathscr{G}_K)$ be the subcategory of $\mathbf{Z}_p$-representations that are killed by p^r.

Notation 1.1.2. Let $F_\tau = (C^\flat)^{\mathscr{G}_L}$, $F_0 = k(\!(\tilde{\pi})\!)$ and F_0^{sep} be the separable closure of F_0 in $C^\flat$. We then have $F_0 \subset (F_0^{\mathrm{sep}})^{\mathscr{G}_L} \subset F_\tau$. Put

$$\mathcal{O}_\mathcal{E} = \Big\{ \sum_{i\in\mathbf{Z}} a_i u^i;\ a_i \in W(k),\ \lim_{i\to-\infty} a_i = 0 \Big\},$$

and embed it into $\mathsf{W}(C^\flat)$ by sending u to $[\tilde{\pi}]$. Endow $\mathcal{O}_\mathcal{E}$ with the p-adic valuation: it is a discrete valuation ring with residue field F_0. Put $\mathcal{E} = \mathsf{Frac}(\mathcal{O}_\mathcal{E})$. Let $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ be the unique ind-étale sub-algebra of $\mathsf{W}(C^\flat)$ whose residue field is $F_0^{\mathrm{sep}} \subset C^\flat$ and $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ its p-adic completion. We put $\mathcal{O}_{\mathcal{E}_\tau} = \mathsf{W}(F_\tau)$ and $\mathcal{E}_\tau = \mathsf{Frac}\,\mathcal{O}_{\mathcal{E}_\tau}$.

Remark 1.1.3. We have $\mathsf{Gal}(\mathsf{Frac}\,\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} / \mathsf{Frac}\,\mathcal{O}_\mathcal{E}) \simeq \mathsf{Gal}(F_0^{\mathrm{sep}}/F_0) \simeq \mathscr{G}_{K_\pi}$ (*cf* [52, Théorème 3.2.2]).

Definition 1.1.4. (*cf* [19, Définition 1.17]) A (φ,τ)-*module over* $(\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau})$ consists of

(i) an étale φ-module D over $\mathcal{O}_\mathcal{E}$ (this means that the linearization $1\otimes\varphi\colon\ \mathcal{O}_\mathcal{E}\otimes_{\varphi,\mathcal{O}_\mathcal{E}} D \to D$ is an isomorphism);

(ii) a τ-semi-linear endomorphism τ_D on $D_\tau := \mathcal{O}_{\mathcal{E}_\tau}\otimes_{\mathcal{O}_\mathcal{E}} D$ which commutes with $\varphi_{\mathcal{O}_{\mathcal{E}_\tau}} \otimes \varphi_D$ (where $\varphi_{\mathcal{O}_{\mathcal{E}_\tau}}$ is the Frobenius map on $\mathcal{O}_{\mathcal{E}_\tau}$ and φ_D the Frobenius map on D) and such that:

$$(\forall x \in D)\ (g\otimes 1)\circ\tau_D(x) = \tau_D^{\chi(g)}(x),$$

for all $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g) \in \mathbf{Z}_{>0}$.

Let $\mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)$ be the corresponding category. One defines similarly the notion of (φ,τ)-module over $(\mathcal{E},\mathcal{E}_\tau)$, and the corresponding category $\mathbf{Mod}_{\mathcal{E},\mathcal{E}_\tau}(\varphi,\tau)$.

Remark 1.1.5. (1) In the following we will use light notations, take $\mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)$ for example: we will simply denote D or (D,D_τ) for objects of this category.

(2) The condition for (φ,τ)-modules can be rewritten as follows (*cf* [19, §1.2.3]): if $g\in\mathscr{G}_K/\mathscr{G}_L$ is such that $\chi(g)\in\mathbf{Z}_{>0}$ then

$$(g\otimes 1)\circ\tau_D=\tau_D^{\chi(g)}\circ(\tau^{-\chi(g)}g\tau\otimes 1)=\tau_D^{\chi(g)}\circ(g\otimes 1).$$

Notice that the first equality is hold on D_τ, while the second equality is hold on D.

Theorem 1.1.6. *The functors*

$$\begin{aligned}\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)&\simeq\mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)\\ T&\mapsto\mathcal{D}(T)=(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\otimes T)^{\mathscr{G}_{K_\pi}}\\ \mathcal{T}(D)=(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\otimes_{\mathcal{O}_{\mathcal{E}}}D)^{\varphi=1}&\hookleftarrow D\end{aligned}$$

establish quasi-inverse equivalences of categories.

Similarly, we have quasi-inverse equivalences of categories

$$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)\simeq\mathbf{Mod}_{\mathcal{E},\mathcal{E}_\tau}(\varphi,\tau).$$

Proof. *cf* [19, §1.3]. □

Notation 1.1.7. Let $(D,D_\tau)\in\mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)$, then we put

$$D_{\tau,0}:=\big\{x\in D_\tau;\ (\forall g\in\mathscr{G}_{K_\pi})\ \chi(g)\in\mathbf{Z}_{>0}\Rightarrow(g\otimes 1)(x)=x+\tau_D(x)+\cdots+\tau_D^{\chi(g)-1}(x)\big\}.$$

Recall that we have the following lemma:

Lemma 1.1.8. *Let $T\in\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_{K_\pi})$, we then have an isomorphism of Γ-modules:*

$$\mathcal{D}(T)_\tau=\mathcal{O}_{\mathcal{E}_\tau}\otimes_{\mathcal{O}_{\mathcal{E}}}\mathcal{D}(T)\simeq(\mathsf{W}(C^\flat)\otimes_{\mathbf{Z}_p}T)^{\mathscr{G}_L}.$$

Proof. *cf* [19, Lemma 1.18]. □

Notation 1.1.9. If $z\in\mathbf{Z}_p$ and $n\in\mathbf{Z}_{\geq 0}$, let

$$\binom{z}{n}=\frac{z(z-1)\cdots(z-n+1)}{n!}\in\mathbf{Z}_p\,.$$

As $\lim\limits_{m\to\infty}\tau_D^{p^m}=1$ (*cf* [19, Proposition 1.3]), the operator τ_D-1 is topologically nilpotent on D_τ, and we may define

$$\tau_D^z=\sum_{n=0}^{\infty}\binom{z}{n}(\tau_D-1)^n$$

for all $z\in\mathbf{Z}_p$. In particular, we have

$$\delta:=1+\tau_D+\cdots+\tau_D^{\chi(\gamma)-1}=\frac{\tau_D^{\chi(\gamma)}-1}{\tau_D-1}=\sum_{n=1}^{\infty}\binom{\chi(\gamma)}{n}(\tau_D-1)^{n-1}.$$

Lemma 1.1.10. *We have* $\mathcal{D}(T)_{\tau,0}=\big\{x\in\mathcal{D}(T)_\tau;\ (\gamma\otimes 1)(x)=x+\tau_D(x)+\cdots+\tau_D^{\chi(\gamma)-1}(x)\big\}$.

Proof. Let $x \in \mathcal{D}_\tau$ and suppose

$$(\gamma \otimes 1)(x) = x + \tau_D(x) + \cdots + \tau_D^{\chi(\gamma)-1}(x).$$

Then for any $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ with $\chi(g) \in \mathbf{Z}_{>0}$, we show that

$$(g \otimes 1)(x) = x + \tau_D(x) + \cdots + \tau_D^{\chi(g)-1}(x).$$

Let $g \in \mathscr{G}_{K_\pi}$. Under the isomorphism of lemma 1.1.8, the action of $g \otimes 1$ on $\mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T)$ corresponds to the diagonal action $g \otimes g$ on $\mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T$. Similarly, lemma 1.1.8 also tells that the action τ_D corresponds to $\tau \otimes \tau$ on $\mathcal{D}(T)_\tau \subset \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T$. We prove by induction on $n \in \mathbf{Z}_{>0}$ that

$$(g \otimes 1)(x) = (1 + \tau_D + \cdots + \tau_D^{\chi(g)-1})(x)$$

when $g = \gamma^n$ (this is true for $n = 1$ by hypothesis). Assume $n > 1$. Seeing x as an element of $\mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T)$, we have

$$\begin{aligned}
(g \otimes g)(x) &= (\gamma \otimes \gamma)\big((\gamma^{n-1} \otimes \gamma^{n-1})(x)\big) \\
&= (\gamma \otimes \gamma)\Big(\sum_{i=0}^{\chi(\gamma)^{n-1}-1} \tau^i \otimes \tau^i\Big)(x) \\
&= \Big(\sum_{i=0}^{\chi(\gamma)^{n-1}-1} \tau^{i\chi(\gamma)} \otimes \tau^{i\chi(\gamma)}\Big)(\gamma \otimes \gamma)(x) \\
&= \Big(\sum_{i=0}^{\chi(\gamma)^{n-1}-1} \tau^{i\chi(\gamma)} \otimes \tau^{i\chi(\gamma)}\Big)\Big(\sum_{j=0}^{\chi(\gamma)-1} \tau^j \otimes \tau^j\Big)(x) \\
&= \Big(\sum_{k=0}^{\chi(\gamma)^n-1} \tau^k \otimes \tau^k\Big)(x).
\end{aligned}$$

Let $g \in \mathscr{G}_{K_\pi}$ be such that $\chi(g) \in \mathbf{Z}_{>0}$. As γ is a topological generator of Γ, there exists a sequence $(b_m)_{m \in \mathbf{Z}_{\geq 0}}$ in $\mathbf{Z}_p$ such that $\lim_{m\to\infty} \gamma^{b_m} = g$. As $\mathbf{Z}_{\geq 0}$ is dense in $\mathbf{Z}_p$, there exists $a_m \in \mathbf{Z}_{\geq 0}$ such that $b_m - a_m \in p^n \mathbf{Z}_p$. Then $\lim_{m\to\infty} \gamma^{a_m} = g$. We have

$$(\gamma^{a_m} \otimes 1)(x) = (1 + \tau_D + \cdots + \tau_D^{\chi(\gamma)^{a_m}-1})(x) = \sum_{n=1}^{\infty} \binom{\chi(\gamma)^{a_m}}{n} (\tau_D - 1)^n(x)$$

for all m. Passing to the limit as $m \to \infty$, we get

$$(g \otimes 1)(x) = \sum_{n=1}^{\infty} \binom{\chi(g)}{n} (\tau_D - 1)^n(x) = (1 + \tau_D + \cdots + \tau_D^{\chi(g)-1})(x).$$

□

Lemma 1.1.11. *The maps $\varphi - 1 \colon D_\tau \to D_\tau$ and $\tau_D - 1 \colon D_\tau \to D_\tau$ induce maps $\varphi - 1 \colon D_{\tau,0} \to D_{\tau,0}$ and $\tau_D - 1 \colon D \to D_{\tau,0}$.*

Proof. For $\varphi - 1$, this results from the fact that $\varphi \colon D_\tau \to D_\tau$ commutes with τ_D and the action of $\mathscr{G}_{K_\pi}$.

Let $x \in D$ and $y := (\tau_D - 1)(1 \otimes x) \in D_\tau$, we then claim that $y \in D_{\tau,0}$. Indeed, if $g \in \mathscr{G}_{K_\pi}$ satisfies $\chi(g) = n \in \mathbf{Z}_{>0}$, then

$$\begin{aligned}(1 + \tau_D + \cdots + \tau_D^{n-1})(y) &= (1 + \tau_D + \cdots + \tau_D^{n-1})((\tau_D - 1)(1 \otimes x)) \\ &= (\tau_D^n - 1)(1 \otimes x) \\ &= (g \otimes 1)(\tau_D(1 \otimes x)) - 1 \otimes x \\ &= (g \otimes 1)((\tau_D - 1)(1 \otimes x)) \\ &= g(y).\end{aligned}$$

Indeed, $\tau_D^n(1 \otimes x) = \tau_D^n \circ (g \otimes 1)(1 \otimes x) = (g \otimes 1)\tau_D(1 \otimes x)$, where the last equality follows from remark 1.1.5 (2). □

Definition 1.1.12. Let $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau)$. We define a complex $\mathcal{C}_{\varphi,\tau}(D)$ as follows:

$$0 \longrightarrow D \longrightarrow D \bigoplus D_{\tau,0} \longrightarrow D_{\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longmapsto (\tau_D - 1)(y) - (\varphi - 1)(z)$$

where the first term is of degree -1.

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}_{\varphi,\tau}(\mathcal{D}(T))$, which will also be simply denoted $\mathcal{C}_{\varphi,\tau}(T)$.

Similarly, we can attach a complex $\mathcal{C}_{\varphi,\tau}(D)$ to any $D \in \mathbf{Mod}_{\mathcal{E},\mathcal{E}_\tau}(\varphi, \tau)$, and in particular attach a complex $\mathcal{C}_{\varphi,\tau}(V)$ to any $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$.

Theorem 1.1.13. *For any $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and natural integer i, there is a canonical and functorial isomorphism*

$$\mathsf{H}^i(\mathcal{C}_{\varphi,\tau}(T)) \simeq \mathsf{H}^i(\mathscr{G}_K, T).$$

Similarly, if $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have a canonical and functorial isomorphism $\mathsf{H}^i(\mathcal{C}_{\varphi,\tau}(V)) \simeq \mathsf{H}^i(\mathscr{G}_K, V)$ for all i.

Notation 1.1.14. If $\mathscr{C}$ is a category, we denote by $\mathsf{Ind}\,\mathscr{C}$ the associated ind-category, whose objects are inductive systems of objects of $\mathscr{C}$ indexed by some filtered category.

Remark 1.1.15. (1) The category $\mathsf{Ind}\,\mathbf{Rep}_{\mathbf{Z}_p, p^r\text{-tors}}(\mathscr{G}_K)$ coincides with the category of discrete $\mathscr{G}_K$-modules killed by p^r, *i.e.* of discrete $(\mathbf{Z}/p^r\,\mathbf{Z})[\![\mathscr{G}_K]\!]$-modules.

(2) The equivalences of theorem 1.1.6 extend into equivalences

$$\mathsf{Ind}\,\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) \simeq \mathsf{Ind}\,\mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau).$$

Definition 1.1.16. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p, p^r\text{-tors}}(\mathscr{G}_K)$. The *induced module* of T is $\mathsf{Ind}_{\mathscr{G}_K}(T) := C^0(\mathscr{G}_K, T)$, the set of all continuous maps from $\mathscr{G}_K$ to T. Endow $\mathsf{Ind}_{\mathscr{G}_K} T$ with the discrete topology and the action of $\mathscr{G}_K$ given by

$$\begin{aligned}\mathscr{G}_K \times \mathsf{Ind}_{\mathscr{G}_K}(T) &\to \mathsf{Ind}_{\mathscr{G}_K}(T) \\ (g, \eta) &\mapsto [x \mapsto \eta(xg)].\end{aligned}$$

Then $\mathsf{Ind}_{\mathscr{G}_K}(T) \in \mathsf{Ind}\,\mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$, and T canonically injects into $\mathsf{Ind}_{\mathscr{G}_K}(T)$ by sending $v \in T$ to η_v, where $\eta_v(g) = g(v)$ for any $g \in \mathscr{G}_K$.

Lemma 1.1.17. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p,\mathrm{tors}}(\mathscr{G}_K)$ and K' be any subfield of $\overline{K}$ containing K. We then have*

$$\mathsf{H}^1(\mathscr{G}_{K'}, \mathsf{Ind}_{\mathscr{G}_K} T) = 0.$$

Proof. Put $U = \mathsf{Ind}_{\mathscr{G}_K} T$. Let $I_{K'}$ be the set of finite subextensions of K'/K. Put $H = \bigcap_{M \in I_{K'}} \mathscr{G}_M$: this is a closed subgroup of $\mathscr{G}_K$ (since all $\mathscr{G}_M$ are) and $\mathscr{G}_{K'} \subset H$. Hence we have $\overline{K}^H \subset \overline{K}^{\mathscr{G}_{K'}} = K'$. On the other hand, for each $M \in I_{K'}$, we have $H \leq \mathscr{G}_M$, whence $M = \overline{K}^{\mathscr{G}_M} \subset \overline{K}^H$. As this holds for each M, we get $K' = \bigcup_{M \in I_{K'}} M \subset \overline{K}^H$, *i.e.* $\overline{K}^H = K' = \overline{K}^{\mathscr{G}_{K'}}$: by Galois correspondence we have $H = \mathscr{G}_{K'}$. We have

$$\mathsf{H}^1(\mathscr{G}_{K'}, U) = \varinjlim_M \mathsf{H}^1(\mathscr{G}_M, U) \tag{1.1}$$

by [47, Chapitre I, Proposition 8]. Indeed, we saw that the groups $\{\mathscr{G}_M\}_{M \in I_{K'}}$ form, for inclusion, a projective system with limit $\bigcap \mathscr{G}_M = \mathscr{G}_{K'}$ and this system is compatible with the inductive system formed by U, seen as a $\mathscr{G}_M$-module by restriction. While the limit is U, seen as a $\mathscr{G}_{K'}$-module by restriction again. Now we claim that $\mathsf{H}^1(\mathscr{G}_{K'}, U) = 0$. Indeed, $\mathscr{G}_M$ being open in $\mathscr{G}_K$, we have the finite decomposition $\mathscr{G}_K = \bigsqcup_{t \in \mathscr{G}_K/\mathscr{G}_M} t\mathscr{G}_M$ from which we deduce that, as a $\mathscr{G}_M$-module, U admits a decomposition

$$U = \bigoplus_{t \in \mathscr{G}_K/\mathscr{G}_M} C^0(t\mathscr{G}_M, T) \simeq \bigoplus_{t \in \mathscr{G}_K/\mathscr{G}_M} C^0(\mathscr{G}_M, T) \simeq \bigoplus_{t \in \mathscr{G}_K/\mathscr{G}_M} \mathsf{Ind}_{\mathscr{G}_M} T.$$

Indeed, the homeomorphism is given by

$$\begin{aligned} C^0(t\mathscr{G}_M, T) &\simeq C^0(\mathscr{G}_M, T) \\ f &\mapsto \tilde{f}, \end{aligned}$$

where $\tilde{f}(g) = f(tg)$ for all $g \in \mathscr{G}_M$. It is an isomorphism of $\mathscr{G}_M$-modules: if $f \in C^0(t\mathscr{G}_M, T)$ and $g, h \in \mathscr{G}_M$, we have

$$(g \cdot \tilde{f})(h) = \tilde{f}(hg) = f(thg) = (g \cdot f)(th) = \widetilde{g \cdot f}(h),$$

i.e. $g \cdot \tilde{f} = \widetilde{g \cdot f}$.

Thus, we have

$$\mathsf{H}^1(\mathscr{G}_M, U) \simeq \bigoplus_{\mathscr{G}_K/\mathscr{G}_M} \mathsf{H}^1(\mathscr{G}_M, \mathsf{Ind}_{\mathscr{G}_M} T),$$

and the summands of the right-hand side are zero by classical results, refering to [47, VII, Proposition 1]. Now (1.1) implies that $\mathsf{H}^1(\mathscr{G}_{K'}, U) = 0$. □

Lemma 1.1.18. *(cf [50, Lemma 1.8]) The following two maps*

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \xrightarrow{\varphi - 1} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$$

$$\mathsf{W}(C^\flat) \xrightarrow{\varphi - 1} \mathsf{W}(C^\flat)$$

admit continuous sections. [1]

Proof. For any element y of F_0^{sep}, there exists a finite field extension F_1/F_0 such that $y \in F_1$. If $v^\flat(y) > 0$, then we have $-\sum_{k=0}^{\infty} \varphi^k(y) \in F_1$ and $y = (\varphi - 1)(-\sum_{k=0}^{\infty} \varphi^k(y))$. This defines a continuous section $\overline{s}\colon \mathfrak{m}_{F_0^{\mathrm{sep}}} \to \mathfrak{m}_{F_0^{\mathrm{sep}}}$.

[1] Remark that the sections are not ring morphisms.

We have $F_0^{\mathrm{sep}} = \bigsqcup_{i\in I}(i+\mathfrak{m}_{F_0^{\mathrm{sep}}})$: this is a union of disjoint opens, where I is a fixed set of representatives of the quotient $F_0^{\mathrm{sep}}/\mathfrak{m}_{F_0^{\mathrm{sep}}}$. For any $i\in I$ choose a $y_i\in F_0^{\mathrm{sep}}$ such that $(\varphi-1)(y_i)=i$. Indeed, we can solve the equations $T^p-T=i$ in F_0^{sep}. Observe that $y_i\in\mathcal{O}_{F_0^{\mathrm{sep}}}$ when $i\in\mathcal{O}_{F_0^{\mathrm{sep}}}$. We define a continuous section on F_0^{sep} by sending $i+x$ to $y_i+\overline{s}(x)$ (note that $\overline{s}$ maps $\mathcal{O}_{F_0^{\mathrm{sep}}}$ into itself). As F_0^{sep} is dense in $C^\flat$, this section extends by continuity into a continuous section $\overline{s}$ of $\varphi-1\colon C^\flat\to C^\flat$ that maps $\mathcal{O}_{C^\flat}$ into itself.
The map $s_1\colon \mathsf{W}(C^\flat)\to\mathsf{W}(C^\flat)$ defined by $s_1(x)=[\overline{s}(\overline{x})]$ is a continuous (for the weak topology) section of $\varphi-1$ modulo p, that maps $\mathsf{W}(\mathcal{O}_{C^\flat})$ into itself. We deform s_1 inductively into maps $s_n\colon \mathsf{W}(C^\flat)\to\mathsf{W}(C^\flat)$ which are sections of $\varphi-1$ modulo p^n and map $\mathsf{W}(\mathcal{O}_{C^\flat})$ into itself: if s_n is constructed, we have

$$(\varphi-1)\circ s_n=\mathsf{Id}+p^n f_n,$$

where $f_n\colon \mathsf{W}(C^\flat)\to\mathsf{W}(C^\flat)$ is a continuous map that maps $\mathsf{W}(\mathcal{O}_{C^\flat})$ into itself. Then

$$s_{n+1}=s_n-p^n s_1\circ f_n$$

has the required properties. The sequence $(s_n)_{n\in\mathbf{Z}_{>0}}$ converges to a section s of $\varphi-1$ on $\mathsf{W}(C^\flat)$ that maps $\mathsf{W}(\mathcal{O}_{C^\flat})$ into itself.

Let $y\in\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$: there exists $x\in\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ such that $(\varphi-1)(x)=y$, so that $(\varphi-1)(s(y)-x)=0$, *i.e.* $s(y)-x\in\mathbf{Z}_p$, so that $s(y)\in\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$. This proves that $s(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})\subset\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$, so that s induces a continuous section of $\varphi-1\colon \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\to\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$.

□

Corollary 1.1.19. *Let $T\in\mathbf{Rep}_{\mathbf{Z}_p\text{-tors}}(\mathscr{G}_K)$ and $U=\mathsf{Ind}_{\mathscr{G}_K}T$. We have*

$$\mathsf{H}^1(\mathscr{G}_L,U)=0$$
$$\mathsf{H}^1(\mathscr{G}_{K_\pi},U)=0$$

and hence the following exact sequences:

$$0\to U^{\mathscr{G}_L}\to\mathcal{D}(U)_\tau\xrightarrow{\varphi-1}\mathcal{D}(U)_\tau\to 0$$
$$0\to U^{\mathscr{G}_{K_\pi}}\to\mathcal{D}(U)\xrightarrow{\varphi-1}\mathcal{D}(U)\to 0.$$

Proof. Let $K'=L$, then lemma 1.1.17 gives $\mathsf{H}^1(\mathscr{G}_L,U)=0$. Consider the following exact sequence

$$0\to\mathbf{Z}_p\to\mathsf{W}(C^\flat)\xrightarrow{\varphi-1}\mathsf{W}(C^\flat)\to 0.$$

Tensorize with the injective U and then, by lemma 1.1.18, we can take Galois invariants to get a long exact sequence

$$0\to U^{\mathscr{G}_L}\to\mathcal{D}(U)_\tau\xrightarrow{\varphi-1}\mathcal{D}(U)_\tau\to\mathsf{H}^1(\mathscr{G}_L,U)=0.$$

The case $K'=K_\pi$ can be proved similarly. □

1.1.20 Tensor product and internal Hom

Remark 1.1.21. Let $T\in\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we then have a $\mathscr{G}_{K_\pi}$-equivariant isomorphism:

$$\mathcal{D}(T)\otimes_{\mathcal{O}_\mathcal{E}}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\simeq T\otimes_{\mathbf{Z}_p}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}.$$

Proposition 1.1.22. *Let $T_1,T_2\in\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then we have*

$$\mathcal{D}(T_1\otimes_{\mathbf{Z}_p}T_2)\simeq\mathcal{D}(T_1)\otimes_{\mathcal{O}_\mathcal{E}}\mathcal{D}(T_2)$$
$$\mathcal{D}(\mathsf{Hom}_{\mathbf{Z}_p}(T_1,T_2))\simeq\mathsf{Hom}_{\mathcal{O}_\mathcal{E}}(\mathcal{D}(T_1),\mathcal{D}(T_2)).$$

Proof. We have the following $\mathscr{G}_{K_\pi}$-equivariant isomorphisms:

$$\begin{aligned}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T_1 \otimes_{\mathbf{Z}_p} T_2) &\simeq T_1 \otimes_{\mathbf{Z}_p} T_2 \otimes_{\mathbf{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \\ &\simeq (T_1 \otimes_{\mathbf{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \otimes_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}} (T_2 \otimes_{\mathbf{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \\ &\simeq (\mathcal{D}(T_1) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \otimes_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}} (\mathcal{D}(T_2) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \\ &\simeq \mathcal{D}(T_1) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T_2) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}.\end{aligned}$$

Recall that $\mathscr{G}_{K_\pi}$ acts trivially on $\mathcal{D}(T)$. Taking fixed points under $\mathscr{G}_{K_\pi}$ on both sides of $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T_1 \otimes_{\mathbf{Z}_p} T_2) \simeq \mathcal{D}(T_1) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T_2) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ we have

$$\mathcal{D}(T_1 \otimes_{\mathbf{Z}_p} T_2) \simeq \mathcal{D}(T_1) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T_2).$$

We also have the following $\mathscr{G}_{K_\pi}$-equivariant isomorphisms:

$$\begin{aligned}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(\mathsf{Hom}_{\mathbf{Z}_p}(T_1, T_2)) &\simeq \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} \mathsf{Hom}_{\mathbf{Z}_p}(T_1, T_2) \\ &\simeq \mathsf{Hom}_{\mathbf{Z}_p}(T_1, T_2 \otimes_{\mathbf{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \\ &\simeq \mathsf{Hom}_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}(T_1 \otimes_{\mathbf{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}, T_2 \otimes_{\mathbf{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \\ &\simeq \mathsf{Hom}_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}(\mathcal{D}(T_1) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}, \mathcal{D}(T_2) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \\ &\simeq \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathsf{Hom}_{\mathcal{O}_\mathcal{E}}(\mathcal{D}(T_1), \mathcal{D}(T_2)).\end{aligned}$$

Recall the Galois action on $\mathsf{Hom}_{\mathcal{O}_\mathcal{E}}(\mathcal{D}(T_1), \mathcal{D}(T_2))$ is defined as $g(f) = g \circ f \circ g^{-1}$. Taking fixed points under $\mathscr{G}_{K_\pi}$ on both sides of

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(\mathsf{Hom}_{\mathbf{Z}_p}(T_1, T_2)) \simeq \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathsf{Hom}_{\mathcal{O}_\mathcal{E}}(\mathcal{D}(T_1), \mathcal{D}(T_2)),$$

and we have

$$\mathcal{D}(\mathsf{Hom}_{\mathbf{Z}_p}(T_1, T_2)) \simeq \mathsf{Hom}_{\mathcal{O}_\mathcal{E}}(\mathcal{D}(T_1), \mathcal{D}(T_2)).$$

□

1.2 Cohomological functoriality

Definition 1.2.1. For $i \in \mathbf{Z}_{\geq 0}$, we put

$$\begin{aligned}\mathcal{F}^i \colon \mathsf{Ind}\,\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) &\to \mathbf{Ab} \\ T &\mapsto \mathsf{H}^i(\mathcal{C}_{\varphi,\tau}(T)).\end{aligned}$$

To prove theorem 1.1.13, we use the strategy of [50, 1.5.2]: we show that $\{\mathcal{F}^i\}_{i \in \mathbf{Z}_{\geq 0}}$ forms a δ-functor that coincides with invariants under $\mathscr{G}_K$ in degree 0 (*cf* proposition 1.2.12), and we prove its effaceability (*cf* section 1.4). As in *loc. cit.*, we firstly show the result for torsion representations, then pass to the limit. For torsion representations, it is necessary to work in a category with sufficiently many injectives: we have to embed $\mathbf{Rep}_{\mathbf{Z}_p,\mathrm{tors}}(\mathscr{G}_K)$ in its ind-category.

Notation 1.2.2. For $T \in \mathbf{Rep}_{\mathbf{Z}_p,\mathrm{tors}}(\mathscr{G}_K)$, we put $U := \mathsf{Ind}_{\mathscr{G}_K}(T) \in \mathsf{Ind}\,\mathbf{Rep}_{\mathbf{Z}_p,\mathrm{tors}}(\mathscr{G}_K)$.

Lemma 1.2.3. *If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we have $\mathcal{F}^0(T) = \mathsf{H}^0(\mathscr{G}_K, T)$.*

Proof. By definition we have:

$$\mathcal{F}^0(T) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\pi}, \varphi=1, \tau_D=1} = ((\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\varphi=1} \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\pi}, \tau_D=1} = T^{\langle \mathscr{G}_{K_\pi}, \tau \rangle} = T^{\mathscr{G}_K} = \mathsf{H}^0(\mathscr{G}_K, T).$$

□

Lemma 1.2.4. If $T \in \mathbf{Rep}_{\mathbf{Z}_p,\mathrm{tors}}(\mathscr{G}_K)$, then

$$\mathsf{H}^i(\mathscr{G}_L, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$$
$$\mathsf{H}^i(\mathscr{G}_{K_\pi}, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$$
$$\mathsf{H}^i(\mathscr{G}_{K_\zeta}, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$$

for all $i > 0$.

Proof. Recall that if M is a subextension of $\overline{K}/K$ whose completion is perfectoid (this is the case for K_π, K_ζ and L) and $i \in \mathbf{Z}_{>0}$, then $\mathsf{H}^i(\mathscr{G}_M, \mathcal{O}_{C^\flat} \otimes_{\mathbf{F}_p} T)$ is almost zero (this follows from the almost vanishing of $\mathsf{H}^i(\mathscr{G}_M, \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$), so that $\mathsf{H}^i(\mathscr{G}_M, C^\flat \otimes_{\mathbf{F}_p} T) = 0$. Then we proceed by induction on $r \in \mathbf{Z}_{>0}$ such that $p^r T = 0$: the exact sequence

$$0 \to pT \to T \to T/pT \to 0$$

induces the exact sequence

$$\mathsf{H}^i\big(\mathscr{G}_M, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} pT\big) \to \mathsf{H}^i\big(\mathscr{G}_M, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T\big) \to \mathsf{H}^i\big(\mathscr{G}_M, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} (T/pT)\big),$$

so that $\mathsf{H}^i\big(\mathscr{G}_M, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T\big) = 0$ since $\mathsf{H}^i\big(\mathscr{G}_M, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} pT\big) = 0$ (because pT is killed by p^{r-1}) and $\mathsf{H}^i\big(\mathscr{G}_M, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} (T/pT)\big) = 0$ (because $T/pT \in \mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$). □

Corollary 1.2.5. *If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then*

$$\mathsf{H}^i(\mathscr{G}_L, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$$
$$\mathsf{H}^i(\mathscr{G}_{K_\pi}, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$$
$$\mathsf{H}^i(\mathscr{G}_{K_\zeta}, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$$

for all $i > 0$.

Proof. By [43, Theorem 2.3.4], we have the exact sequence

$$0 \to \mathsf{R}^1 \varprojlim_n \mathsf{H}^{i-1}(\mathscr{G}_L, \mathsf{W}_n(C^\flat) \otimes_{\mathbf{Z}_p} T) \to \mathsf{H}^i(\mathscr{G}_L, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) \to \varprojlim_n \mathsf{H}^i(\mathscr{G}_L, \mathsf{W}_n(C^\flat) \otimes_{\mathbf{Z}_p} T) \to 0.$$

By lemma 1.2.4 we have

$$\mathsf{H}^{i-1}(\mathscr{G}_L, \mathsf{W}_n(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0, \text{ if } i > 1,$$

while when $i = 1$ we know that $\{\mathsf{H}^0(\mathscr{G}_L, \mathsf{W}_n(C^\flat) \otimes_{\mathbf{Z}_p} T)\}_n$ has the Mittag-Leffler property. This implies

$$\mathsf{H}^i(\mathscr{G}_L, \mathsf{W}_n(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0, \text{ if } i > 0$$

and

$$\mathsf{R}^1 \varprojlim_n \mathsf{H}^{i-1}(\mathscr{G}_L, \mathsf{W}_n(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0, \text{ if } i > 0,$$

hence

$$\mathsf{H}^i(\mathscr{G}_L, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0, \text{ if } i > 0.$$

The proofs of the other statements are similar. □

Notation 1.2.6. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we will put $D_\tau = \mathcal{D}(T)_\tau$ in the rest of this section to keep light notations.

Lemma 1.2.7. *Let H be a profinite group homeomorphic to $\mathbf{Z}_p$ with σ a topological generator, and M be a p-torsion H-module. Then the continuous cohomology $\mathsf{H}^i(H, M)$ is computed by the complex*

$$[M \xrightarrow{\sigma-1} M]$$

in which the first term is in degree 0.

Proof. *cf* [43, Proposition 1.6.13]. □

Lemma 1.2.8. If $n \in \mathbf{Z}_{>0}$, the operators $\gamma - 1$ and $\tau_D^n - 1$ are surjective on D_τ.

Proof. As $\mathsf{H}^i(\mathscr{G}_L, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = 0$ for all $i > 0$, together with lemma 1.2.7 we have

$$0 = \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = \mathsf{H}^1(\mathsf{Gal}(L/K_\pi), D_\tau) = \mathsf{Coker}\left(D_\tau \xrightarrow{\gamma-1} D_\tau\right)$$

since $\mathsf{Gal}(L/K_\pi) = \overline{\langle\gamma\rangle}$. Similarly, put $K_{\zeta,n} = L^{\overline{\langle\tau^n\rangle}}$: this is a finite extension of K_ζ and we have

$$0 = \mathsf{H}^1(\mathscr{G}_{K_{\zeta,n}}, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T) = \mathsf{H}^1(\mathsf{Gal}(L/K_{\zeta,n}), D_\tau) = \mathsf{Coker}\left(D_\tau \xrightarrow{\tau_D^n-1} D_\tau\right)$$

since $\mathsf{Gal}(L/K_{\zeta,n}) = \overline{\langle\tau^n\rangle}$. □

Lemma 1.2.9. We have $(\delta - \gamma \otimes 1) \circ (\tau_D - 1) = \left(1 - \tau_D^{\chi(\gamma)}\right) \circ (\gamma \otimes 1 - 1)$ on D_τ.

Proof. As $(\gamma \otimes 1) \circ \tau_D = \tau_D^{\chi(\gamma)} \circ (\gamma \otimes 1)$, we have

$$\begin{aligned}(\delta - \gamma \otimes 1) \circ (\tau_D - 1) &= \delta \circ (\tau_D - 1) - \gamma \circ \tau_D + \gamma \otimes 1 \\ &= \tau_D^{\chi(\gamma)} - 1 - \tau_D^{\chi(\gamma)} \circ (\gamma \otimes 1) + \gamma \otimes 1 \\ &= \left(1 - \tau_D^{\chi(\gamma)}\right) \circ (\gamma \otimes 1 - 1).\end{aligned}$$

□

Proposition 1.2.10. The map $\delta - \gamma \otimes 1$ is surjective on D_τ.

Proof. As $\gamma - 1$ and $\tau_D^{\chi(\gamma)} - 1$ are surjective by lemma 1.2.8, so is $(\delta - \gamma \otimes 1) \circ (\tau_D - 1)$. Hence $\delta - \gamma$ is surjective. □

Corollary 1.2.11. If $0 \to T' \to T \to T'' \to 0$ is an exact sequence in $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then the sequences

$$0 \to \mathcal{D}(T') \to \mathcal{D}(T) \to \mathcal{D}(T'') \to 0$$
$$0 \to \mathcal{D}(T')_\tau \to \mathcal{D}(T)_\tau \to \mathcal{D}(T'')_\tau \to 0$$
$$0 \to \mathcal{D}(T')_{\tau,0} \to \mathcal{D}(T)_{\tau,0} \to \mathcal{D}(T'')_{\tau,0} \to 0$$

are exact. In particular, the functor $T \mapsto \mathcal{C}_{\varphi,\tau}(T)$ is exact.

Proof. As $\mathsf{W}(C^\flat)$ is torsion-free, we have the exact sequence

$$0 \to \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T' \to \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T \to \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T'' \to 0$$

which induces the exact sequence

$$0 \to \mathcal{D}(T')_\tau \to \mathcal{D}(T)_\tau \to \mathcal{D}(T'')_\tau \to \mathsf{H}^1(\mathscr{G}_L, \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T').$$

By corollary 1.2.5, we get the second exact sequence. Similarly, by the observation $\mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T') = 0$ (where $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is endowed with the p-adic topology, *cf* lemma 2.1.5) we have the first exact sequence. Moreover, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T')_\tau & \longrightarrow & \mathcal{D}(T)_\tau & \longrightarrow & \mathcal{D}(T'')_\tau & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \\ 0 & \longrightarrow & \mathcal{D}(T')_\tau & \longrightarrow & \mathcal{D}(T)_\tau & \longrightarrow & \mathcal{D}(T'')_\tau & \longrightarrow & 0 \end{array}$$

The snake lemma and proposition 1.2.10 provide the last exact sequence. □

Proposition 1.2.12. *The functors $\{\mathcal{F}^i\}_{i\in\mathbf{N}}$ form a δ-functor.*

Proof. Let $0 \to T' \to T \to T'' \to 0$ be a short exact sequence in $\mathsf{Ind}\,\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$. We have a short exact sequence of complexes $0 \to \mathcal{C}_{\varphi,\tau}(T') \to \mathcal{C}_{\varphi,\tau}(T) \to \mathcal{C}_{\varphi,\tau}(T'') \to 0$ by corollary 1.2.11. Classical result (*cf* [51, Theorem 1.3.1]) gives the desired long exact sequence of cohomological groups. □

1.3 Computation of $\mathsf{H}^1(\mathcal{C}_{\varphi,\tau})$

Notation 1.3.1. For any (φ,τ)-module D, we denote $\mathsf{Ext}_{\mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)}(\mathcal{O}_\mathcal{E}, D)$ the group of extensions of $\mathcal{O}_\mathcal{E}$ (the unit object in $\mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)$) by D. More precisely, the group of exact sequences

$$0 \to D \to E \to \mathcal{O}_\mathcal{E} \to 0$$

in $\mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)$ modulo equivalence. Two extensions E_1, E_2 are equivalent when there exists a map f making the diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & E_1 & \longrightarrow & \mathcal{O}_\mathcal{E} & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & E_2 & \longrightarrow & \mathcal{O}_\mathcal{E} & \longrightarrow & 0. \end{array}$$

Lemma 1.3.2. *Define two submodules of $D \oplus D_\tau$ as follows:*

$$M := \Big\{(\lambda,\mu) \in D \oplus D_\tau;\ \begin{cases} (\varphi-1)\mu = (\tau_D - 1)\lambda \\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow g(\mu) = \mu + \tau_D(\mu) + \tau_D^2(\mu) + \cdots + \tau_D^{\chi(g)-1}(\mu) \end{cases}\Big\},$$

$$N := \Big\{\big((\varphi-1)d, (\tau_D-1)d\big);\ d \in D\Big\}.$$

Then there is a group isomorphism

$$\mathsf{Ext}(\mathcal{O}_\mathcal{E}, D) \simeq M/N.$$

Proof. Consider an extension of $\mathcal{O}_\mathcal{E}$ by D :

$$0 \to D \to E \xrightarrow{\epsilon} \mathcal{O}_\mathcal{E} \to 0.$$

It is equivalent to giving a (φ,τ)-module structure to the $\mathcal{O}_\mathcal{E}$-module $E = D \oplus \mathcal{O}_\mathcal{E} \cdot x$, where $x \in E$ is a preimage of 1 under ϵ. Since D is already a (φ,τ)-module, it suffices to specify the images of x by φ and τ_D. Since $\epsilon(\varphi(x) - x) = 0$, we must have $\varphi(x) - x \in D$ and hence we put

$$\varphi(x) = x + \lambda \text{ with } \lambda \in D.$$

For τ_D, we tensorize the original exact sequence with $\mathcal{O}_{\mathcal{E}_\tau}$ as follows

$$0 \to D_\tau \to \mathcal{O}_{\mathcal{E}_\tau} \otimes E \xrightarrow{\epsilon} \mathcal{O}_{\mathcal{E}_\tau} \to 0$$

then similarly we have $\epsilon((\tau_D - 1)(1 \otimes x)) = 0$ and hence $(\tau_D - 1)(1 \otimes x) \in D_\tau$. We then put

$$\tau_D(1 \otimes x) = 1 \otimes x + \mu \text{ with } \mu \in D_\tau.$$

Such a pair (λ,μ) has to satisfy two conditions to give an extension E. The first condition is that φ commutes with τ_D, *i.e.* $\varphi(\tau_D(x)) = \tau_D(\varphi(x))$. Notice

$$\varphi(\tau_D(1\otimes x)) = \tau_D(\varphi(1\otimes x)) \iff \varphi(1\otimes x + \mu) = \tau_D(1\otimes x + \lambda) \iff 1\otimes x + \lambda + \varphi(\mu) = 1\otimes x + \mu + \tau_D(\lambda).$$

This is equivalent to:

$$\varphi(\mu) - \mu = \tau_D(\lambda) - \lambda. \tag{1.2}$$

The second condition is: $(g \otimes 1)(\tau_D(1\otimes x)) = \tau_D^{\chi(g)}(1\otimes x)$ whenever $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ is such that $\chi(g) \in \mathbf{Z}_{>0}$. By induction we have $\tau_D^{\chi(g)}(1\otimes x) = 1\otimes x + \mu + \tau_D(\mu) + \cdots + \tau_D^{\chi(g)-1}(\mu)$. Thus the second condition rewrites as

$$g(\mu) = \mu + \tau_D(\mu) + \tau_D^2(\mu) + \cdots + \tau_D^{\chi(g)-1}(\mu). \tag{1.3}$$

Indeed

$$(g\otimes 1)(\mu)=(g\otimes 1)(\tau_D(1\otimes x)-1\otimes x)=\tau_D^{\chi(g)}(1\otimes x)-1\otimes x=\mu+\tau_D(\mu)+\tau_D^2(\mu)+\cdots+\tau_D^{\chi(g)-1}(\mu).$$

Hence we are left to show that an extension E arising from the pair (λ,μ) is trivial if and only if there exists $d\in D$ such that $\lambda=(\varphi-1)(d)$ and $\mu=(\tau_D-1)(d)$. Indeed, E is trivial if and only if there exists $d\in D$ such that $\mathcal{O}_{\mathcal{E}}\cdot(x-d)$ is a sub-(φ,τ)-module of E. In other words, $E=D\oplus\mathcal{O}_{\mathcal{E}}\cdot(x-d)$ as (φ,τ)-modules. This is equivalent to the existence of $\alpha\in\mathcal{O}_{\mathcal{E}}$ and $\beta\in\mathcal{O}_{\mathcal{E}_\tau}$ such that $\varphi(x-d)=\alpha(x-d)$ and $\tau_D(x-d)=\beta(x-d)$. More precisely, it is $x+\lambda-\varphi(d)=\alpha(x-d)$ and after applying the map ϵ we have $\alpha=1$ and $\beta=1$. We now have $\varphi(x-d)=x-d$ and it gives directly $(\varphi-1)(d)=\lambda$. Similarly we have $\mu=(\tau_D-1)(d)$. Hence we find the desired d. □

Proposition 1.3.3. *Let $T\in\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then we have*

$$\mathsf{H}^1(\mathcal{C}_{\varphi,\tau}(T))=\mathsf{H}^1(\mathscr{G}_K,T).$$

Proof. As $\mathcal{D}$ establishes the equivalence of categories, we have

$$\mathsf{H}^1(\mathscr{G}_K,T)\simeq\mathsf{Ext}_{\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)}(\mathbf{Z}_p,T)\simeq\mathsf{Ext}_{\mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)}(\mathcal{O}_{\mathcal{E}},\mathcal{D}(T))$$

(since $\mathcal{D}(\mathbf{Z}_p)=\mathcal{O}_{\mathcal{E}}$ is the trivial (φ,τ)-module). Consider the complex

$$0\to\mathcal{D}\xrightarrow{\alpha}\mathcal{D}\oplus\mathcal{D}_{\tau,0}\xrightarrow{\beta}\mathcal{D}_{\tau,0}\to 0.$$

We firstly describe $\mathsf{H}^1(\mathcal{C}_{\varphi,\tau}(T))$. A pair $(\lambda,\mu)\in\mathcal{D}\oplus\mathcal{D}_\tau$ is in $\mathsf{Ker}\,\beta$ if and only if it satisfies the following two conditions:

$$\varphi(\mu)-\mu=\tau_D(\lambda)-\lambda \tag{1.4}$$

$$\mu\in D_{\tau,0} \tag{1.5}$$

We see that (1.4) and (1.5) correspond to (1.2) and (1.3) respectively. It suffices to show that two pairs (λ_1,μ_1) and (λ_2,μ_2) give equivalent extensions if and only if $(\lambda_2,\mu_2)-(\lambda_1,\mu_1)\in\mathsf{Im}\,\alpha$, but this is clear. □

1.4 Effaceability

Lemma 1.4.1. *The map $\delta-\gamma\otimes 1\colon U^{\mathscr{G}_L}\to U^{\mathscr{G}_L}$ is surjective.*

Proof. By lemma 1.2.9, it is sufficient to prove that $\gamma-1$ and $\tau_D^{\chi(\gamma)}-1$ are surjective over $U^{\mathscr{G}_L}$, which is equivalent to the vanishing of $\mathsf{H}^1(\overline{\langle\gamma\rangle},U^{\mathscr{G}_L})$ and $\mathsf{H}^1(\overline{\langle\tau^{\chi(\gamma)}\rangle},U^{\mathscr{G}_L})$ according to lemma 1.2.7. Indeed, by lemma 1.1.17 we have $\mathsf{H}^1(\mathscr{G}_L,U)=0$, which implies that

$$\mathsf{H}^1(\overline{\langle\gamma\rangle},U^{\mathscr{G}_L})=\mathsf{H}^1(\overline{\langle\gamma\rangle},\mathsf{H}^0(\mathscr{G}_L,U))=\mathsf{H}^1(\mathscr{G}_{K_\pi},U).$$

The group $\mathsf{H}^1(\mathscr{G}_{K_\pi},U)$ vanishes by lemma 1.1.17. Assume $\chi(\gamma)=n$ and put $K_{\zeta,n}=L^{\overline{\langle\tau^n\rangle}}$. Then similarly we have

$$\mathsf{H}^1(\overline{\langle\tau^{\chi(\gamma)}\rangle},U^{\mathscr{G}_L})=\mathsf{H}^1(\overline{\langle\tau^{\chi(\gamma)}\rangle},\mathsf{H}^0(\mathscr{G}_L,U))=\mathsf{H}^1(\mathscr{G}_{K_{\zeta,n}},U)=0,$$

where the last equality follows from lemma 1.1.17. □

Lemma 1.4.2. *The map $\varphi-1\colon\mathcal{D}(U)_{\tau,0}\to\mathcal{D}(U)_{\tau,0}$ is surjective.*

Proof. By corollary 1.1.19 we have the following exact sequence:

$$0 \to U^{\mathscr{G}_L} \to \mathcal{D}(U)_\tau \xrightarrow{\varphi-1} \mathcal{D}(U)_\tau \to 0.$$

Consider the following map of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^{\mathscr{G}_L} & \longrightarrow & \mathcal{D}(U)_\tau & \xrightarrow{\varphi-1} & \mathcal{D}(U)_\tau & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \\ 0 & \longrightarrow & U^{\mathscr{G}_L} & \longrightarrow & \mathcal{D}(U)_\tau & \xrightarrow{\varphi-1} & \mathcal{D}(U)_\tau & \longrightarrow & 0. \end{array}$$

By snake lemma we have the exact sequence:

$$\mathcal{D}(U)_{\tau,0} \xrightarrow{\varphi-1} \mathcal{D}(U)_{\tau,0} \to \mathsf{Coker}(\delta - \gamma \otimes 1 \colon U^{\mathscr{G}_L} \to U^{\mathscr{G}_L}) = 0$$

where the last equality is lemma 1.4.1. □

Proof of theorem 1.1.13. Write $\mathcal{C}_{\varphi,\tau}(\mathcal{D}(U))$ as follows:

$$0 \longrightarrow \mathcal{D}(U) \xrightarrow{\alpha} \mathcal{D}(U) \bigoplus \mathcal{D}(U)_{\tau,0} \xrightarrow{\beta} \mathcal{D}(U)_{\tau,0} \longrightarrow 0$$

$$x \longrightarrow ((\varphi-1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longrightarrow (\tau_D - 1)(y) - (\varphi - 1)(z).$$

As remarked in section 1.2, it suffices to show the effaceability of $\mathcal{F}^i$, more precisely that $\mathcal{F}^i(U) = 0$ for $i \in \{1, 2\}$. For $i = 1$, by proposition 1.3.3 we have $\mathsf{H}^1(\mathcal{C}_{\varphi,\tau}(U)) = \mathsf{H}^1(\mathscr{G}_K, U)$, which is zero by lemma 1.1.17. For $i = 2$, it follows from lemma 1.4.2. □

Remark 1.4.3. We recover the fact that $\mathsf{H}^i(\mathscr{G}_K, T) = 0$ for $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and $i \geq 3$.

1.5 A counter example

A more natural complex can be defined as follows.

Definition 1.5.1. Let $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau)$. We define a complex $\mathcal{C}^{\mathsf{naive}}_{\varphi,\tau}(D)$ as follows:

$$0 \longrightarrow D \longrightarrow D \bigoplus D_\tau \longrightarrow D_\tau \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longrightarrow (\tau_D - 1)(y) - (\varphi - 1)(z).$$

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}^{\mathsf{naive}}_{\varphi,\tau}(\mathcal{D}(T))$, which will also be simply denoted $\mathcal{C}^{\mathsf{naive}}_{\varphi,\tau}(T)$.

However, this complex *does not* compute the continuous Galois cohomology in general, as showed in the following example.

Example 1.5.2. *Let $T = \mathbf{F}_p$ be the trivial representation: then $D = F_0$ and $D_\tau = F_\tau$. Now we look at the following diagram, which embeds the complex $\mathcal{C}_{\varphi,\tau}(\mathbf{F}_p)$ into $\mathcal{C}^{\mathsf{naive}}_{\varphi,\tau}(\mathbf{F}_p)$ and we denote the cokernel complex by $\mathcal{C}(\mathbf{F}_p)$.*

$$\begin{array}{lccccccc}
\mathcal{C}_{\varphi,\tau}(\mathbf{F}_p) & 0 \longrightarrow & F_0 & \xrightarrow{(\varphi-1,\tau-1)} & F_0 \oplus F_{\tau,0} & \xrightarrow{(\tau-1)\ominus(\varphi-1)} & F_{\tau,0} & \longrightarrow 0 \\
 & & \| & & \downarrow & & \downarrow & \\
\mathcal{C}^{\mathsf{naive}}_{\varphi,\tau}(\mathbf{F}_p) & 0 \longrightarrow & F_0 & \xrightarrow{(\varphi-1,\tau-1)} & F_0 \oplus F_\tau & \xrightarrow{(\tau-1)\ominus(\varphi-1)} & F_\tau & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{C}(\mathbf{F}_p) & 0 \longrightarrow & 0 & \longrightarrow & F_\tau/F_{\tau,0} & \xrightarrow{\varphi-1} & F_\tau/F_{\tau,0} & \longrightarrow 0
\end{array}$$

The associated long exact sequence is

$$\begin{aligned}
0 \to \mathsf{H}^0(\mathscr{G}_K, \mathbf{F}_p) \to \mathsf{H}^0_{\mathsf{naive}}(\mathscr{G}_K, \mathbf{F}_p) \to 0 \to \mathsf{H}^1(\mathscr{G}_K, \mathbf{F}_p) \to \mathsf{H}^1_{\mathsf{naive}}(\mathscr{G}_K, \mathbf{F}_p) \\
\to \mathsf{Ker}(\varphi - 1) \to \mathsf{H}^2(\mathscr{G}_K, \mathbf{F}_p) \to \mathsf{H}^2_{\mathsf{naive}}(\mathscr{G}_K, \mathbf{F}_p) \to \mathsf{Coker}(\varphi - 1) \to 0
\end{aligned}$$

where the subscript "naive" refers to the complex $\mathcal{C}^{\mathsf{naive}}_{\varphi,\tau}(\mathbf{F}_p)$ in the middle of the diagram, while $\mathsf{Ker}(\varphi-1)$ and $\mathsf{Coker}(\varphi-1)$ refer to the last complex $F_\tau/F_{\tau,0} \xrightarrow{\varphi-1} F_\tau/F_{\tau,0}$.

Notice that $1 \notin F_{\tau,0}$ and $1 \in \mathsf{Ker}(F_\tau \xrightarrow{\varphi-1} F_\tau)$, this implies that $\mathsf{Ker}(\varphi-1)$ is not trivial. In particular, this implies that

$$\mathsf{H}^1(\mathscr{G}_K, \mathbf{F}_p) \subsetneq \mathsf{H}^1_{\mathsf{naive}}(\mathscr{G}_K, \mathbf{F}_p).$$

More precisely, take any $\alpha \in k$, then $(\alpha, 1) \in \mathsf{Ker}((\tau-1)\ominus(\varphi-1)) \setminus (F_0 \oplus F_{\tau,0})$ induces an element of $\mathsf{H}^1_{\mathsf{naive}}(\mathscr{G}_K, \mathbf{F}_p) \setminus \mathsf{H}^1(\mathscr{G}_K, \mathbf{F}_p)$.

Remark 1.5.3. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ with $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau)$ its (φ,τ)-module. Then we have

$$(\tau_D - 1)D \subset D_{\tau,0} \subset D_\tau$$

(*cf* lemma 1.1.11), and it is natural to consider the following complex:

$$0 \longrightarrow D \longrightarrow D \bigoplus (\tau_D - 1)D \longrightarrow (\tau_D - 1)D \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longrightarrow (\tau_D - 1)(y) - (\varphi - 1)(z).$$

However, this complex *does not* compute the continuous Galois cohomology of T in general as it has trivial H^2.

Chapter 2

Relation with Tavares Ribeiro's complex

In this chapter, we show that the complex $\mathcal{C}_{\varphi,\tau}$ defined in chapter 1 is a refinement of Tavares Ribeiro's complex introduced in [50, §1.5], at least in the finite residue field case.

Theorem 2.0.1. *Let T be an integral p-adic representation of $\mathscr{G}_K$ and let $M = D_L(T) = \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T\right)^{\mathscr{G}_L}$. Then the homology of the complex $\mathcal{C}_{\varphi,\gamma,\tau}(M)$ defined as follows:*

$$0 \to M \xrightarrow{\tilde{\alpha}} M \oplus M \oplus M \xrightarrow{\tilde{\beta}} M \oplus M \oplus M \xrightarrow{\tilde{\eta}} M \to 0$$

where

$$\tilde{\alpha} = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \tau - 1 \end{pmatrix}, \ \tilde{\beta} = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \tau - 1 & 0 & 1 - \varphi \\ 0 & \tau^{\chi(\gamma)} - 1 & \delta - \gamma \end{pmatrix}$$

$$\tilde{\eta} = \left(\tau^{\chi(\gamma)} - 1, \delta - \gamma, \varphi - 1\right)$$

with $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1} \in \mathbf{Z}_p[\![\tau - 1]\!]$, identifies canonically and functorially with the continuous Galois cohomology of T.

Proof. *cf* [50, Theorem 1.5]. □

2.1 Tavares Ribeiro's complex with D_τ

By replacing $D_L(T)$ in [50, Lemma 1.9] with $\mathcal{D}(T)_\tau$, we have:

Lemma 2.1.1. *For any $T \in \mathsf{Ind}\,\mathbf{Rep}_{\mathbf{Z}_p, p^r - tor}(\mathscr{G}_K)$ and $\alpha \in \mathbf{Z}_p^\times$, we have:*

$$0 \to \mathsf{Ind}_{\widehat{G}}(T) \to \mathcal{D}(\mathsf{Ind}_{\mathscr{G}_K}(T))_\tau \xrightarrow{\varphi - 1} \mathcal{D}(\mathsf{Ind}_{\mathscr{G}_K}(T))_\tau \to 0$$

$$0 \to \mathsf{Ind}_\Gamma(T) \to \mathsf{Ind}_{\widehat{G}}(T) \xrightarrow{\tau^\alpha - 1} \mathsf{Ind}_{\widehat{G}}(T) \to 0$$

$$0 \to \mathsf{Ind}_{\mathscr{G}_K}(T)^{\mathscr{G}_K} \to \mathsf{Ind}_\Gamma(T) \xrightarrow{\gamma - 1} \mathsf{Ind}_\Gamma(T) \to 0.$$

Proof. The first exact sequence follows from corollary 1.1.19, while the proof for the rest ones are the same as in [50, Lemma 1.9]. □

Definition 2.1.2. For $i \in \mathbf{N}$, we put

$$\begin{aligned} \mathcal{F}^i \colon \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) &\to \mathbf{Ab} \\ T &\to \mathsf{H}^i(\mathcal{C}_{\varphi,\gamma,\tau}(\mathcal{D}(T)_\tau)). \end{aligned}$$

Remark 2.1.3. The functor $\{\mathcal{F}^i\}_i$ just defined is different from that of the previous chapter (*cf* definition 1.2.1).

Lemma 2.1.4. *We have*

$$\mathcal{F}^0(T) \simeq \mathsf{H}^0(\mathscr{G}_K, T).$$

Proof. We have $\mathcal{F}^0(T) = (\mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_L, \varphi=1, \gamma=1, \tau=1} = (\mathsf{W}(C^\flat)^{\varphi=1} \otimes_{\mathbf{Z}_p} T)^{\langle \mathscr{G}_L, \tau, \gamma\rangle} = T^{\langle \mathscr{G}_L, \tau, \gamma\rangle} = T^{\mathscr{G}_K}$. □

Lemma 2.1.5. *Let $\mathcal{M}$ be an $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$-module of finite type with a continuous (for the p-adic topology) semi-linear action of $\mathscr{G}_{K_\pi}$, i.e. $(\forall g \in \mathscr{G}_{K_\pi})(\forall \lambda \in \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})(\forall m \in \mathcal{M})\ g(\lambda m) = g(\lambda)g(m)$, then*

$$\mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}) = 0.$$

Proof. (1) We first assume that $\mathcal{M}$ is a F_0^{sep}-vector space of finite dimension, and hence $p\mathcal{M} = 0$. Denote $d = \dim_{F_0^{\mathrm{sep}}} \mathcal{M}$ and denote $\underline{e} = (e_1, \ldots, e_d)$ a basis of $\mathcal{M}$ over F_0^{sep}. For any $g \in \mathscr{G}_{K_\pi}$, we denote by $U_g \in \mathsf{GL}_d(F_0^{\mathrm{sep}})$ the matrix of the action of g under the fixed basis $\underline{e}$. There is a bijection between the classes of F_0^{sep}-representations of $\mathscr{G}_{K_\pi}$ and the set $\mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathsf{GL}_d(F_0^{\mathrm{sep}}))$, which has one point by Hilbert 90 theorem (here F_0^{sep} is endowed with the discrete topology). This implies that $\mathcal{M} \simeq (F_0^{\mathrm{sep}})^d$ as a $\mathscr{G}_{K_\pi}$-module and hence $\mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}) \simeq \mathsf{H}^1(\mathscr{G}_{K_\pi}, F_0^{\mathrm{sep}})^d = 0$.

(2) Now we assume that $\mathcal{M}$ is killed by a power of p. We use induction as follows: if $\mathcal{M}$ is killed by p^n, then we have an exact sequence

$$0 \to p^{n-1}\mathcal{M} \to \mathcal{M} \to \mathcal{M}/p^{n-1}\mathcal{M} \to 0,$$

hence an exact sequence

$$\cdots \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, p^{n-1}\mathcal{M}) \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}) \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}/p^{n-1}) \to \cdots.$$

As $p^{n-1}\mathcal{M}$ is killed by p and $\mathcal{M}/p^{n-1}$ by p^{n-1}, the induction hypothesis and (1) implies the vanishing of two sides, hence the vanishing in the middle.

(3) If $\mathcal{M}$ is an $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$-module of finite type, we have $\mathcal{M} = \varprojlim_n \mathcal{M}/p^n\mathcal{M}$ and hence we have the following exact sequence

$$\mathsf{R}^1 \varprojlim_n \mathsf{H}^0(\mathscr{G}_{K_\pi}, \mathcal{M}/p^n) \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}) \to \varprojlim_n \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}/p^n).$$

The first term vanishes since the short exact sequence

$$0 \to p^n\mathcal{M}/p^{n+1} \to \mathcal{M}/p^{n+1} \to \mathcal{M}/p^n \to 0$$

gives the exact sequence

$$\mathsf{H}^0(\mathscr{G}_{K_\pi}, \mathcal{M}/p^{n+1}) \to \mathsf{H}^0(\mathscr{G}_{K_\pi}, \mathcal{M}/p^n) \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, p^n\mathcal{M}/p^{n+1}),$$

and we know the right term vanishes by situation (2) we discussed. Hence we have Mittag-Leffler condition for the system of H^0 and then $\mathsf{R}^1 \varprojlim_n \mathsf{H}^0(\mathscr{G}_{K_\pi}, \mathcal{M}/p^n) = 0$. Notice that $\varprojlim_n \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}/p^n) = 0$ by case (2), and hence $\mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathcal{M}) = 0$.

□

Lemma 2.1.6. *The group $\mathscr{G}_{K_\pi}$ acts continuously on the ring $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$, where the latter is endowed with the p-adic topology.*

Proof. The extension $\mathcal{E}^{ur}/\mathcal{E}$ is Galois with group $\mathscr{G}_{K_\pi}$.The p-adic valuation is the unique valuation that extends the p-adic valuation on $\mathcal{E}$: this implies that $\mathscr{G}_{K_\pi}$ acts by isometries hence continuously on $\mathcal{E}^{\mathrm{ur}}$. In particular, the action over $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is continuous for the p-adic topology. □

Corollary 2.1.7. *The functors $D(-)$ and $D(-)_\tau$ are exact.*

Proof. The functor $D(-)$ is left exact by construction. If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_{K_\pi})$, the semi-linear action on $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T$ is continuous for the p-adic topology (because it is continuous on T and $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$) : by lemma 2.1.5, the functor $D(-)$ is also right exact. Recall $D(T)_\tau = \mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T)$, and hence it is exact. □

Lemma 2.1.8. *The functors* $\{\mathcal{F}^i\}_{i\in\mathbf{N}}$: $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) \to \mathbf{Ab}$ *form a* δ*-functor.*

Proof. This follows directly from corollary 2.1.7. □

Theorem 2.1.9. *Let* $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ *and* $i \in \mathbf{N}$*, we then have*

$$\mathsf{H}^i(\mathscr{G}_K, T) \simeq \mathcal{F}^i(T).$$

Proof. By lemmas 2.1.4 and 2.1.8, it suffices to prove the effaceability of $\{\mathcal{F}^i\}_{i\in\mathbf{N}}$. Similar proof as [50, Proposition 1.7] works, as it is based on [50, Lemma 1.9]. □

Notation 2.1.10. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$. To make light notations, in the rest of this chapter we denote $\mathcal{C}_{\varphi,\tau}$ for the complex $\mathcal{C}_{\varphi,\tau}(T)$, and $\mathcal{C}_{\mathsf{TR}}$ for Tavares Ribeiro's complex attached to $\mathcal{D}(T)_\tau$, *i.e.* $\mathcal{C}_{\varphi,\gamma,\tau}(\mathcal{D}(T)_\tau)$.

Using the morphisms introduced by Tavares Ribeiro, we have another description of $\mathcal{D}(T)_{\tau,0}$ as follows:

Lemma 2.1.11. *Let* $x \in \mathcal{D}(T)_\tau$*, then* $x \in \mathcal{D}(T)_{\tau,0} \iff (\delta-\gamma)x = 0$.

Proof. This is a translation of lemma 1.1.10 with new notations. □

Notation 2.1.12. Let F_0^{rad} be the radical closure of F_0 in $C^\flat$, and $\widehat{F_0^{\mathrm{rad}}}$ the closure (for the valuation topology) of F_0^{rad} in $C^\flat$: this is a complete perfect subfield of $C^\flat$. Let $\mathsf{W}(\widehat{F_0^{\mathrm{rad}}})$ be the ring of Witt vectors of $\widehat{F_0^{\mathrm{rad}}}$.

Lemma 2.1.13. *In* $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$*, we have* $(\mathcal{D}(T)_\tau)^{\gamma=1} = \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes \mathcal{D}(T)$.

Proof. $(\mathcal{D}(T)_\tau)^{\gamma=1} = (\mathcal{D}(T)_\tau)^{\mathscr{G}_{K_\pi}} = (\mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T))^{\mathscr{G}_{K_\pi}} = (\mathcal{O}_{\mathcal{E}_\tau})^{\mathscr{G}_{K_\pi}} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{D}(T) = \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes \mathcal{D}(T)$. Indeed, $F_\tau^{\gamma=1} = F_\tau^{\mathscr{G}_{K_\pi}} = \widehat{F_0^{\mathrm{rad}}}$. □

Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and put $D = \mathcal{D}(T)$. We have the following morphism of complexes from $\mathcal{C}_{\varphi,\tau}$ to $\mathcal{C}_{\mathsf{TR}}$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & D & \xrightarrow{\alpha} & D \oplus D_{\tau,0} & \xrightarrow{\beta} & D_{\tau,0} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow & & \\
0 & \longrightarrow & D_\tau & \xrightarrow{\widetilde{\alpha}} & D_\tau \oplus D_\tau \oplus D_\tau & \xrightarrow{\widetilde{\beta}} & D_\tau \oplus D_\tau \oplus D_\tau & \xrightarrow{\widetilde{\eta}} & D_\tau & \longrightarrow & 0.
\end{array}$$

where

$$\begin{aligned}
\alpha(d) &= ((\varphi-1)d, (\tau_D-1)d), \\
\beta(x,z) &= (\tau_D-1)x - (\varphi-1)z, \\
\widetilde{\alpha} &= \begin{pmatrix} \varphi-1 \\ \gamma-1 \\ \tau_D-1 \end{pmatrix}, \\
\widetilde{\beta} &= \begin{pmatrix} \gamma-1 & 1-\varphi & 0 \\ \tau_D-1 & 0 & 1-\varphi \\ 0 & \tau_D^{\chi(\gamma)}-1 & \delta-\gamma \end{pmatrix}, \\
\widetilde{\eta} &= (\tau_D^{\chi(\gamma)}-1, \delta-\gamma, \varphi-1)
\end{aligned}$$

and

$$\begin{array}{ccccccccc}
d & \xmapsto{\alpha} & (x=(\varphi-1)d, z=(\tau_D-1)d) & \xmapsto{\beta} & t=(\tau_D-1)x-(\varphi-1)z & \longmapsto & 0 \\
\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow \\
d & \xmapsto{\widetilde{\alpha}} & (x,0,z) & \xmapsto{\widetilde{\beta}} & (0,t,0) & \xmapsto{\widetilde{\eta}} & 0
\end{array}$$

Theorem 2.1.14. *This morphism of complexes is a quasi-isomorphism when the residue field of* K *is finite.*

2.2 The intermediate complex $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$

Definition 2.2.1. Let $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau)$. We define a complex $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}(D)$ as follows:

$$0 \longrightarrow \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D \xrightarrow{\alpha} \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D \bigoplus D_{\tau,0} \xrightarrow{\beta} D_{\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x), (\tau_D-1)(x))$$

$$(y,z) \longrightarrow (\tau_D-1)(y) - (\varphi-1)(z).$$

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}(\mathcal{D}(T))$, which will also be simply denoted $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}(T)$.

Remark 2.2.2. The morphism $\tau_D - 1\colon \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D \to D_{\tau,0}$ is well-defined. Indeed, for any $x \in \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D$ (*i.e.* $x \in D_\tau$ with $(\gamma-1)x = 0$ *cf* lemma 2.1.13), we have by lemma 1.2.9

$$(\delta-\gamma)(\tau_D-1)x = (1-\tau_D^{\chi(\gamma)})(\gamma-1)x = 0.$$

This exactly tells that $(\tau_D-1)x \in D_{\tau,0}$. Hence the complex $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ is well defined.

Remark 2.2.3. We have a morphism from $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ to $\mathcal{C}_{\mathsf{TR}}$ as follows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D & \xrightarrow{\alpha} & \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D \bigoplus D_{\tau,0} & \xrightarrow{\beta} & D_{\tau,0} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow & & \\
0 & \longrightarrow & D_\tau & \xrightarrow{\widetilde{\alpha}} & D_\tau \oplus D_\tau \oplus D_\tau & \xrightarrow{\widetilde{\beta}} & D_\tau \oplus D_\tau \oplus D_\tau & \xrightarrow{\widetilde{\eta}} & D_\tau & \longrightarrow & 0
\end{array}$$

where the maps are defined similarly as in section 2.1.

Theorem 2.2.4. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, the homology of the complex $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}(T)$ computes $\mathsf{H}^i(\mathscr{G}_K, T)$.*

To prove the theorem, we need the following results.

Lemma 2.2.5. *The constructed map from $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ to $\mathcal{C}_{\mathsf{TR}}$ induces an isomorphism on H^1.*

Proof. For injectivity: suppose a pair $(x,z) \in \mathsf{Ker}\,\beta$ is mapped to 0 in $\mathsf{H}^1(\mathcal{C}_{\mathsf{TR}})$ by u, *i.e.* there exists $d \in D_\tau$ such that

$$\begin{cases} (\varphi-1)d = x \\ (\gamma-1)d = 0 \\ (\tau_D-1)d = z. \end{cases}$$

Then $d \in (D_\tau)^{\gamma=1} = \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D$ and $\alpha(d) = (x,z) \in \mathsf{Im}\,\alpha$ and hence $[(x,z)] = 0 \in \mathsf{H}^1(\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau})$.

For surjectivity: given $(a,b,c) \in \mathsf{Ker}\,\tilde{\beta}$, *i.e.*

$$\begin{cases} (\gamma-1)a + (1-\varphi)b = 0 \\ (\tau_D-1)a + (1-\varphi)c = 0 \\ (\tau_D^{\chi(\gamma)}-1)b + (\delta-\gamma)c = 0. \end{cases}$$

By lemma 1.2.8, we can fix an element $s \in D_\tau$ such that $(\gamma-1)s = b$. Denote $x' := (\varphi-1)s - a$ and $z' := c - (\tau_D-1)s \in D_\tau$. We have $(x',z') \in \mathsf{Ker}\,\beta$ so that $[u(x',z')] = [(a,b,c)] \in \mathsf{H}^1(\mathcal{C}_{\mathsf{TR}})$. Indeed, we have

$$(\gamma-1)x' = (\gamma-1)((\varphi-1)s - a) = (\varphi-1)b - (\gamma-1)a = 0$$

and

$$\begin{aligned}(\delta-\gamma)z' &= (\delta-\gamma)(c-(\tau_D-1)s)\\ &= -(\delta-\gamma)c+(\delta-\gamma)((\tau_D-1)s)\\ &= -(\delta-\gamma)c+(\tau_D^{\chi(\gamma)}-1)s-\gamma(\tau_D-1)s\\ &= -(\delta-\gamma)c+(\tau_D^{\chi(\gamma)}-1)s-(\tau_D^{\chi(\gamma)}\gamma-\gamma)s\\ &= -(\delta-\gamma)c+(\tau_D^{\chi(\gamma)}-1)(1-\gamma)s\\ &= -(\delta-\gamma)c+(\tau_D^{\chi(\gamma)}-1)b\\ &= 0\end{aligned}$$

hence $x' \in (D_\tau)^{\gamma=1} = \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes D$ and $z' \in D_{\tau,0}$. Finally we have

$$\begin{aligned}&(\tau_D-1)x'-(\varphi-1)z'\\ =&(\tau_D-1)((\varphi-1)s-a)-(\varphi-1)(c-(\tau_D-1)s)\\ =&(\tau_D-1)a-(\varphi-1)c\\ =&0\end{aligned}$$

hence $(x', z') \in \mathsf{Ker}\,\beta$ is as required. □

Lemma 2.2.6. *The constructed map from $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ to $\mathcal{C}_{\mathsf{TR}}$ induces an injective map on H^2.*

Proof. Take $t \in D_{\tau,0}$ and suppose $v(t)$ vanishes in $\mathsf{H}^2(\mathcal{C}_{\varphi,\tau})$. Hence there exists $(a,b,c) \in D_\tau^{\oplus 3}$ such that $\tilde{\beta}(a,b,c) = (0,t,0)$, *i.e.* the following relations holds:

$$(\gamma-1)a+(1-\varphi)b = 0 \tag{2.1}$$

$$(\tau_D-1)a+(1-\varphi)c = t \tag{2.2}$$

$$(\tau_D^{\chi(\gamma)}-1)b+(\delta-\gamma)c = 0. \tag{2.3}$$

It suffices to show there exist $a' \in \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$ and $c' \in D_{\tau,0}$ such that $(\tau_D-1)a'+(\varphi-1)c' = t$. By lemma 1.2.8, we can fix an element $b' \in D_\tau$ such that $(\gamma-1)b' = b$. From (2.1) we have

$$(\gamma-1)(a+(1-\varphi)b') = 0 \tag{2.4}$$

and from (2.3) we have $(\tau_D^{\chi(\gamma)}-1)(\gamma-1)b'+(\delta-\gamma)c = 0$. Lemma 1.2.9 then tells

$$(\delta-\gamma)(c-(\tau_D-1)b') = 0. \tag{2.5}$$

Now (2.2) implies $(\tau_D-1)(a+(1-\varphi)b')+(1-\varphi)(c-(\tau_D-1)b') = t$. Hence $a' := a+(1-\varphi)b'$ and $c' := -c+(\tau_D-1)b'$ are good candidates, as they satisfy $(\tau_D-1)a'+(\varphi-1)c' = t$. Indeed, we have now

$$\begin{aligned}(\gamma-1)a' &= 0\\ (\delta-\gamma)c' &= 0\end{aligned}$$

which tells exactly $a' \in D_\tau^{\mathscr{G}_{K_\pi}} = \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{F_0} D$ and $c' \in D_{\tau,0}$. □

Lemma 2.2.7. *The constructed map from $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ to $\mathcal{C}_{\mathsf{TR}}$ induces a surjective map on H^2.*

Proof. Let $(\tilde{a},\tilde{b},\tilde{c}) \in \mathsf{Ker}\,\tilde{\eta}$, *i.e.*

$$(\tau_D^{\chi(\gamma)}-1)\tilde{a}+(\delta-\gamma)\tilde{b}+(\varphi-1)\tilde{c}=0. \tag{2.6}$$

It suffices to find $t \in D_{\tau,0}$ such that $[v(t)] = [(\tilde{a},\tilde{b},\tilde{c})] \in \mathsf{H}^2(\mathcal{C}_{\mathsf{TR}})$. More precisely, to find $(a,b,c) \in D_\tau^{\oplus 3}$ and $t \in D_{\tau,0}$ that satisfy the following system of equations:

$$\begin{cases}(\gamma-1)a+(1-\varphi)b=\tilde{a}\\(\tau_D-1)a+(1-\varphi)c+t=\tilde{b}\\(\tau_D^{\chi(\gamma)}-1)b+(\delta-\gamma)c=\tilde{c}.\end{cases}$$

Take $b=0$: it suffices to solve

$$\begin{cases}(\gamma-1)a=\tilde{a}\\(\tau_D-1)a+(1-\varphi)c+t=\tilde{b}\\(\delta-\gamma)c=\tilde{c}.\end{cases}$$

By lemma 1.2.8, we can find $a,\ c \in D_\tau$ such that $(\gamma-1)a=\tilde{a},\ (\delta-\gamma)c=\tilde{c}$. Now we have

$$\begin{aligned}(\delta-\gamma)((\tau_D-1)a+(1-\varphi)c) =&(1-\tau_D^{\chi(\gamma)})(\gamma-1)a+(1-\varphi)\tilde{c}\\ =&(1-\tau_D^{\chi(\gamma)})\tilde{a}+(1-\varphi)\tilde{c}\\ =&(\delta-\gamma)\tilde{b}\end{aligned}$$

where the first equality follows from lemma 1.2.9 and the last one is from (2.6). This shows that

$$t:=\tilde{b}-(\tau_D-1)a-(1-\varphi)c$$

belongs to $D_{\tau,0}$, as required. □

Remark 2.2.8. We hence proved theorem 2.2.4 without refering to proposition 1.3.3.

2.3 Proof of Theorem 2.1.14

Lemma 2.3.1. *If $\delta \in \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$ is such that $(\varphi-1)(\delta) \in D$, then $\delta \in D$.*

Proof. (1) First consider the case when D is killed by p : we have to prove that if $\delta \in \widehat{F_0^{\mathrm{rad}}} \otimes_{F_0} D$ satisfies $(\varphi-1)\delta \in D$, then $\delta \in D$.

We have

$$F_0^{\mathrm{rad}}=\bigoplus_{\lambda\in\Lambda} F_0\pi^\lambda$$

where $\Lambda = \mathbf{Z}\left[p^{-1}\right] \cap [0,1)$. This shows that

$$F_\tau^{\gamma=1}=\Big\{\sum_{\lambda\in\Lambda}\alpha_\lambda\pi^\lambda\,;\,(\alpha_\lambda)_{\lambda\in\Lambda}\in F_0^\Lambda,\ \lim_\lambda \alpha_\lambda=0\Big\}$$

where the limit is taken for the filter complement of finite subsets in Λ (this means that for every $C \in \mathbf{R}$, the set $\{\lambda \in \Lambda\,;\, v_\pi(a_\lambda) < C\}$ is finite). This implies that

$$D_\tau^{\gamma=1}=F_\tau^{\gamma=1}\otimes_{F_0} D=\Big\{\sum_{\lambda\in\Lambda}\pi^\lambda\otimes d_\lambda\,;\,(d_\lambda)_{\lambda\in\Lambda}\in D^\Lambda,\ \lim_\lambda d_\lambda=0\Big\}.$$

Here we use a basis $\mathfrak{B}=(e_1,\dots,e_r)$ of D over F_0, and endow D with the "valuation"

$$v_{\mathfrak{B},\pi}(x_1e_1+\cdots+x_re_r)=\min_{1\le i\le r} v_\pi(x_i).$$

Put $\mathcal{D} = \mathcal{D}_{\mathfrak{B}} := \bigoplus_{i=1}^{r} k[\![\pi]\!]e_i$, this is a lattice in D.

Note that the writting $\sum_{\lambda\in\Lambda} \pi^\lambda \otimes d_\lambda$ of an element in $F_\tau^{\gamma=1} \otimes_{F_0} D$ as above is unique.

Write $\delta = \sum_{\lambda\in\Lambda} \pi^\lambda \otimes d_\lambda$ as above: we have

$$\begin{aligned}(\varphi-1)\delta &= \sum_{\lambda\in\Lambda} (\pi^{p\lambda}\otimes\varphi(d_\lambda) - \pi^\lambda\otimes d_\lambda)\\ &= \sum_{\lambda\in\Lambda} \pi^\lambda \otimes \Big(\sum_{m=0}^{p-1} \pi^m \varphi\Big(d_{\frac{m+\lambda}{p}}\Big) - d_\lambda\Big)\end{aligned}$$

so that $d_\lambda = \sum_{m=0}^{p-1} \pi^m \varphi\Big(d_{\frac{m+\lambda}{p}}\Big)$ for all $\lambda \in \Lambda\setminus\{0\}$.

For $k \in \mathbf{Z}_{>0}$, put $c_k = \min_{\substack{\lambda\in\Lambda\\ p^{k-1}\lambda\notin\mathbf{Z}}} v_{\mathfrak{B},\pi}(d_\lambda)$. We have $\lim_{k\to\infty} c_k = +\infty$.

Let $k \in \mathbf{Z}_{>0}$ and $\lambda \in \Lambda$ such that $p^k\lambda \in \mathbf{Z}$ and $p^{k-1}\lambda \notin \mathbf{Z}$. If $m \in \{0,\dots,p-1\}$ and $\mu = \frac{m+\lambda}{p}$, we have $p^k\mu \notin \mathbf{Z}$, so $v_{\mathfrak{B},\pi}(d_\mu) \geq c_{k+1}$. Assume $c_{k+1} > 0$: we have $v_{\mathfrak{B},\pi}(\varphi(d_\mu)) \geq pc_{k+1}$. This implies that $v_{\mathfrak{B},\pi}(d_\lambda) \geq pc_{k+1} \geq c_{k+1}$. Thus we have

$$c_{k+1} > 0 \Rightarrow c_k \geq c_{k+1} > 0.$$

Let $c \in \mathbf{R}_{>0}$: we have $c_k \geq c$ for $k \gg 0$, and the above shows that $c_k \geq c$ for all $k \in \mathbf{Z}_{>0}$. As this holds for all $c > 0$, this means that $c_1 = +\infty$, *i.e.* $d_\lambda = 0$ whenever $\lambda \neq 0$, so that

$$\delta = d_0 \in D.$$

(2) In the general case, let $\delta \in \mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$ be such that $(\varphi-1)\delta \in D$. We show that

$$\delta \in D + p^n \mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$$

by induction on $n \in \mathbf{Z}_{>0}$. The case $n = 1$ follows from the special case above. Assume $n \geq 1$ and that we have $\delta \in D + p^n \mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$: write $\delta = \delta_n + \delta'_n$ with $\delta_n \in D$ and $\delta'_n \in p^n \mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$. Then

$$(\varphi-1)\delta'_n = (\varphi-1)\delta_n - (\varphi-1)\delta \in D \cap p^n \mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D = p^n D.$$

If we apply the special case to the image of δ'_n in $p^nD/p^{n+1}D$ (which is a φ-module over F_0)[1], we deduce that $\delta'_n \in p^nD + p^{n+1}\mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$, which shows that

$$\delta \in D + p^{n+1} \mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D.$$

As this holds for all $n \in \mathbf{Z}_{>0}$, this shows that δ belongs to the closure of D in $\mathrm{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$ for the p-adic topology: as D is closed, we deduce that $\delta \in D$.

□

[1] Observe that the proof of the particular case (D killed by p) uses nothing but the φ-module structure of D : it holds with $p^nD/p^{n+1}D$.

2.3.2 Morphism from $\mathcal{C}_{\varphi,\tau}$ to $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$

We have a morphism from $\mathcal{C}_{\varphi,\tau}$ to $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\alpha} & D \oplus D_{\tau,0} & \xrightarrow{\beta} & D_{\tau,0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} & & \| & & \\
0 & \longrightarrow & \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D & \xrightarrow{\alpha^{\mathrm{rad}}} & \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D \bigoplus D_{\tau,0} & \xrightarrow{\beta^{\mathrm{rad}}} & D_{\tau,0} & \longrightarrow & 0.
\end{array}$$

Proposition 2.3.3. *The above morphism is a quasi-isomorphism when the residue field of K is finite.*

Proof. For H^0 : We have $\mathsf{H}^0(\mathcal{C}_{\varphi,\tau}) = \mathsf{Ker}(\alpha) \subset \mathsf{Ker}(\alpha^{\mathrm{rad}}) = \mathsf{H}^0(\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau})$. We have $(\mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D)^{\varphi=1} = D^{\varphi=1}$ (by lemma 2.3.1). Thus $\mathsf{Ker}(\alpha^{\mathrm{rad}}) \subset \mathsf{Ker}(\alpha) = \mathsf{H}^0(\mathcal{C}_{\varphi,\tau})$.

Injectivity of H^1 : Let $(x, y) \in \mathsf{Ker}(\beta)$ whose image in $\mathsf{H}^1(\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau})$ is zero: there exists $\delta \in \mathsf{W}(\widehat{F_0^{\mathrm{rad}}}) \otimes_{\mathcal{O}_\mathcal{E}} D$ such that $(x, y) = ((\varphi - 1)\delta, (\tau_D - 1)\delta)$. As $(\varphi - 1)\delta = x \in D$: this implies $\delta \in D$ by lemma 2.3.1, so $(x, y) \in \mathsf{Im}(\alpha)$, and so that the class of (x, y) is zero in $\mathsf{H}^1(\mathcal{C}_{\varphi,\tau})$. This proves the injectivity of $\mathsf{H}^1(\mathcal{C}_{\varphi,\tau}) \to \mathsf{H}^1(\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau})$.

Surjectivity of H^1 : By theorems 1.1.13 and 2.2.4, the homology groups H^1 of both complexes are isomorphic to $\mathsf{H}^1(\mathscr{G}_K, T)$, hence they have the same finite dimension. Notice the map is a linear transformation, hence injectivity implies surjectivity.

For H^2 : Since we proved that both $\mathcal{C}_{\varphi,\tau}$ and $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ have the correct H^2 by theorems 1.1.13 and 2.2.4, hence have the same finite dimension. It is enough to show either injectivity or surjectivity. But the surjectivity is direct. □

Corollary 2.3.4. *The complexes $\mathcal{C}_{\varphi,\tau}$, $\mathcal{C}^{\mathrm{rad}}_{\varphi,\tau}$ and $\mathcal{C}_{\mathsf{TR}}$ are all quasi-isomorphic when the residue field of K is finite.*

In particular, we have proved theorem 2.1.14.

Chapter 3

Complexes with ψ-operator

In this chapter, we construct a complex with ψ operator (similar as in [29, §3]) that computes the continuous Galois cohomology.

As F_τ is perfect, we cannot have such an operator on (φ,τ)-modules over $(\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau})$ (*cf* remark 3.3.7): we have to use a refinement of (φ,τ)-module theory developped in [19, 1.2.2]. More precisely, we will work with coefficients $\mathcal{O}_{\mathcal{E}_{u,\tau}}$, whose residue field is not perfect (*cf* notation 4.1.1).

We construct a complex $\mathcal{C}^u_{\varphi,\tau}$ in section 3.2 and we show that it computes the continuous Galois cohomology. Replacing the operator φ with ψ in $\mathcal{C}^u_{\varphi,\tau}$ provides another complex $\mathcal{C}^u_{\psi,\tau}$ and we show that these two complexes are quasi-isomorphic. Hence $\mathcal{C}^u_{\psi,\tau}$ computes the continuous Galois cohomology.

To prove the results mentioned above, we will as usual start with $\mathbf{F}_p$-representations and then pass to $\mathbf{Z}_p$-representations by dévissage.

3.1 The (φ,τ)-modules over partially unperfected coefficients

For simplicity, we will denote by u and η the elements $\widetilde{\pi}$ and $\varepsilon-1$ in $C^\flat$ (this is a little abuse of notation since, strictly speaking, u is a variable that maps to $\widetilde{\pi}$ under the injective map $F_0\to C^\flat$ and similarly for η) or the elements $[\widetilde{\pi}]$ and $[\varepsilon]-1$ in $\mathsf{W}(C^\flat)$ under the injective map $\mathcal{O}_\mathcal{E}\to\mathsf{W}(C^\flat)$.

Notation 3.1.1. (*cf* [19, §1.2.2])

(1) We put

$$F_{u,\tau}:=k(\!(u,\eta^{1/p^\infty})\!)=k(\!(u,\eta)\!)\big[\eta^{1/p^\infty}\big]=\bigcup_{n\in\mathbf{N}}k(\!(u,\eta)\!)[\eta^{1/p^n}]\subset C^\flat.$$

(2) By an abuse of notation we denote

$$C^\flat_{u-\mathsf{np}}=F^{\mathrm{sep}}_{u,\tau}\subset C^\flat$$

the separable closure of $F_{u,\tau}$ in $C^\flat$.

Note that $C^\flat_{u-\mathsf{np}}$ is *not* the tilt of a perfectoid field, though ambiguously it carries a superscript $\flat$ in the notation.

(3) We put

$$\mathsf{G}:=\mathsf{Gal}(F^{\mathrm{sep}}_{u,\tau}/F_{u,\tau}).$$

Lemma 3.1.2. *The group* G *acts isometrically over* $F^{\mathrm{sep}}_{u,\tau}$.

Proof. Denote v the valuation of $C^\flat$ that is normalized by $v(\widetilde{\pi}) = 1/e$. We show that $v_{|_{F^{\mathrm{sep}}_{u,\tau}}}$ is the unique valuation of $F^{\mathrm{sep}}_{u,\tau}$ that extends $v_{|_{F_{u,\tau}}}$.

If $\alpha \in F^{\mathrm{sep}}_{u,\tau}$, then α is algebraic over $F_{u,\tau} = \bigcup_n k(\!(u, \eta^{1/p^n})\!)$. This implies that there exists $n \in \mathbf{N}$ such that α is algebraic over $k(\!(u, \eta^{1/p^n})\!)$. Notice that $k(\!(u, \eta^{1/p^n})\!)$ is complete for the valuation v and hence there exists a unique valuation over $k(\!(u, \eta^{1/p^n})\!)[\alpha]$ that extends $v_{|_{F_{u,\tau}}}$ (*cf* [44, Chapter II, §4, Theorem 4.8]). This implies that for any $g \in \mathsf{G}$, $v \circ g = v$ over $F^{\mathrm{sep}}_{u,\tau}$. □

Remark 3.1.3. Notice that the action of $\mathsf{G} \simeq \mathsf{Gal}(F^{\mathrm{sep}}_{u,\tau}/F_{u,\tau})$ on $F^{\mathrm{sep}}_{u,\tau}$ is continuous for the discrete topology.

Proposition 3.1.4. *We have*

$$\mathsf{G} \simeq \mathscr{G}_L.$$

Proof. We first prove that there are injective maps $\mathscr{G}_L \to \mathsf{G} \to \mathscr{G}_{K_\pi}$ whose composite is the inclusion.

Let $\alpha \in F^{\mathrm{sep}}_{u,\tau}$ and $P(X) \in F_{u,\tau}[X]$ its minimal polynomial of α over $F_{u,\tau}$. Then for any $g \in \mathscr{G}_L$, we have

$$g(P(X)) = P(X) \quad \Longrightarrow \quad P(g(\alpha)) = g(P(\alpha)) = 0 \quad \Longrightarrow \quad g(\alpha) \in F^{\mathrm{sep}}_{u,\tau}.$$

This shows that $F^{\mathrm{sep}}_{u,\tau}$ is stable under $\mathscr{G}_L$. As $\mathscr{G}_L$ fixes $F_{u,\tau}$, this implies that we have a morphism of groups:

$$\mathscr{G}_L \xrightarrow{\ \rho\ } \mathsf{G}.$$

Similarly, let $\alpha \in F^{\mathrm{sep}}_0$ and $Q(X) \in F_0[X]$ its minimal polynomial over F_0. Then for any $g \in \mathsf{G}$, we have

$$g(Q(X)) = Q(X) \quad \Longrightarrow \quad Q(g(\alpha)) = g(Q(\alpha)) = 0 \quad \Longrightarrow g(\alpha) \in F^{\mathrm{sep}}_0.$$

This shows that F^{sep}_0 is stable under G. As G fixes F_0, this implies that we have a morphism of groups:

$$\mathsf{G} \xrightarrow{\ \lambda\ } \mathsf{Gal}(F^{\mathrm{sep}}_0/F_0) \simeq \mathscr{G}_{K_\pi}.$$

Hence finally we have the diagram:

$$\begin{array}{ccc} \mathscr{G}_L & \xrightarrow{\ \rho\ } & \mathsf{G} \\ & \searrow & \downarrow \lambda \\ & & \mathscr{G}_{K_\pi}. \end{array}$$

The map ρ is thus injective and we are left to prove that λ is injective. Take any $g \in \mathsf{G}$ that acts trivially on F^{sep}_0. Recall that F^{sep}_0 is dense in $C^\flat$ for the valuation topology (*cf* [19, Proof of Proposition 1.8]) and hence also dense in $F^{\mathrm{sep}}_{u,\tau}$. By lemma 3.1.2, we have $g = \mathrm{id}_{F^{\mathrm{sep}}_{u,\tau}}$ and hence λ is injective.

Now we can see $\mathsf{G}/\rho(\mathscr{G}_L)$ as a subgroup of $\mathscr{G}_{K_\pi}/\mathscr{G}_L \simeq \overline{\langle\gamma\rangle}$. Suppose $z \in \mathbf{Z}_p$ is such that γ^z acts trivially on $F_{u,\tau}$, then we have

$$\varepsilon - 1 = \gamma^z(\varepsilon - 1) = \varepsilon^{\chi(\gamma)^z} - 1, \ \textit{i.e.}\ \varepsilon^{\chi(\gamma)^z} = \varepsilon, \quad \text{thus } \gamma^z = \mathrm{id}\,.$$

This shows that ρ is surjective, and we conclude that $\mathsf{G} \simeq \mathscr{G}_L$. □

Corollary 3.1.5. *We have* $F_{u,\tau} = (C^\flat_{u-\mathsf{np}})^{\mathscr{G}_L}$.

Proof. By definition we have $F_{u,\tau} = (C^\flat_{u-\mathsf{np}})^{\mathsf{G}}$, which is $(C^\flat_{u-\mathsf{np}})^{\mathscr{G}_L}$ by proposition 3.1.4. □

Remark 3.1.6. (1) Recall that

$$\mathcal{O}_{\mathcal{E}} = \Big\{ \sum_{i\in\mathbf{Z}} a_i u^i;\ a_i \in W(k),\ \lim_{i\to-\infty} a_i = 0 \Big\}$$

is a Cohen ring for $k(\!(u)\!)$, and is equipped with the lift φ of the Frobenius of $k(\!(u)\!)$, such that $\varphi(u) = u^p$. It embeds in $\mathsf{W}(C^\flat)$ by sending u to $[\widetilde{\pi}]$. Similarly we have a Cohen ring $\mathcal{O}_{\mathcal{F}_u}$ for $k(\!(u, \eta^{1/p^\infty})\!)$ which is endowed with a Frobenius φ and embeds in $\mathsf{W}(C^\flat)$ so that

$$\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{F}_u} \to \mathsf{W}(C^\flat)$$

are compatible with Frobeniuses (*cf* [19, 1.3.3]).

(2) Note that $F_{u,\tau}$ is stable by the action of τ on $\big(C^\flat\big)^{\mathscr{G}_L}$, because $\tau(u) = u(\eta+1)$ and $\tau(\eta^{1/p^n}) = \eta^{1/p^n}$ for all $n \in \mathbf{N}$. Similarly, $\mathcal{O}_{\mathcal{F}_u}$ is stable under the action of τ on $\mathsf{W}(C^\flat)^{\mathscr{G}_L}$.

Notation 3.1.7. Let $\mathcal{F}_u = \mathsf{Frac}(\mathcal{O}_{\mathcal{F}_u})$ and $\mathcal{F}_u^{\mathrm{ur}}$ the maximal unramified extension in $\mathsf{W}(C^\flat)[1/p]$ and $\mathcal{O}_{\mathcal{F}_u^{\mathrm{ur}}}$ its ring of integers. We denote $\mathcal{O}_{\widehat{\mathcal{F}_u^{\mathrm{ur}}}}$ its p-adic completion and put $\widehat{\mathcal{F}_u^{\mathrm{ur}}} = \mathcal{O}_{\widehat{\mathcal{F}_u^{\mathrm{ur}}}}[1/p]$.

The ring $\mathcal{O}_{\mathcal{F}_u^{\mathrm{ur}}}$ is endowed with a Frobenius map that is compatible with the Frobenius map in $\mathsf{W}(C^\flat)$ by our construction above. By continuity, it extends into a Frobenius map on $\mathcal{O}_{\widehat{\mathcal{F}_u^{\mathrm{ur}}}}$ and $\widehat{\mathcal{F}_u^{\mathrm{ur}}}$.

We put

$$\mathcal{O}_{\mathcal{E}_{u,\tau}} := (\mathcal{O}_{\widehat{\mathcal{F}_u^{\mathrm{ur}}}})^{\mathscr{G}_L}, \quad \mathcal{E}_{u,\tau} = \mathcal{O}_{\mathcal{E}_{u,\tau}}[1/p].$$

Remark 3.1.8. We summarize the notations by the following diagram:

$$\begin{array}{ccc}
\mathsf{W}(C^\flat) & \twoheadrightarrow & C^\flat \\
\uparrow & & \uparrow \\
\mathcal{O}_{\mathcal{F}_u^{\mathrm{ur}}} & \twoheadrightarrow & F_{u,\tau}^{\mathrm{sep}} \\
\uparrow & & \uparrow \\
\mathcal{O}_{\mathcal{F}_u} & \twoheadrightarrow & F_{u,\tau} \\
\uparrow & & \uparrow \\
\mathbf{Z}_p & \twoheadrightarrow & \mathbf{F}_p\,.
\end{array}$$

Theorem 3.1.9. *We have* $\mathcal{O}_{\mathcal{F}_u} = \mathcal{O}_{\mathcal{E}_{u,\tau}}$.

Proof. We have the following diagram:

$$\begin{array}{ccccc}
 & & \mathcal{O}_{\widehat{\mathcal{F}_u^{\mathrm{ur}}}} & & \\
 & \swarrow & & \searrow & \\
\mathcal{O}_{\mathcal{F}_u} & & & & \mathcal{O}_{\mathcal{E}_{u,\tau}} \\
\downarrow & & & & \downarrow \\
F_{u,\tau} & & \overset{\text{Proposition 3.1.4}}{=\!=\!=} & & (F_{u,\tau}^{\mathrm{sep}})^{\mathscr{G}_L}.
\end{array}$$

Notice that $\mathcal{O}_{\mathcal{F}_u}$ is fixed by $\mathscr{G}_L$, hence $\mathcal{O}_{\mathcal{F}_u} \subset \mathcal{O}_{\mathcal{E}_{u,\tau}}$. Both $\mathcal{O}_{\mathcal{F}_u}$ and $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ have the same residue field $F_{u,\tau}$. Indeed, $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ has residue field $(F_{u,\tau}^{\mathrm{sep}})^{\mathscr{G}_L}$, which is $F_{u,\tau}$ by proposition 3.1.4. Hence $\mathcal{O}_{\mathcal{F}_u} \subset \mathcal{O}_{\mathcal{E}_{u,\tau}}$ are two Cohen rings for $F_{u,\tau}$ and they must be equal as $\mathcal{O}_{\mathcal{F}_u}$ is dense and closed inside $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ for the p-adic topology. □

Remark 3.1.10. The theorem 3.1.9 can be rewritten as $\mathcal{F}^{\mathsf{int}}_{u-\mathsf{np}} = \mathcal{E}^{\mathsf{int}}_{u-\mathsf{np},\tau}$ with Caruso's notations in [19, 1.3.3].

Definition 3.1.11. A (φ,τ)-*module over* $(F_0, F_{u,\tau})$ is the data:

(1) an étale φ-module D over F_0;

(2) a τ-semi-linear endomorphism τ_D over $D_{u,\tau} := F_{u,\tau} \otimes_{F_0} D$ which commutes with $\varphi_{F_{u,\tau}} \otimes \varphi_D$ (where $\varphi_{F_{u,\tau}}$ is the Frobenius map on $F_{u,\tau}$ and φ_D the Frobenius map on D) such that

$$\big(\forall x \in D\big)\ (g \otimes 1) \circ \tau_D(x) = \tau_D^{\chi(g)}(x),$$

for all $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g) \in \mathbf{N}$.

We denote $\mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau)$ the corresponding category.

Theorem 3.1.12. *The functors*

$$\begin{aligned} \mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K) &\to \mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau) \\ T &\mapsto \mathcal{D}(T) = (F_0^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T)^{\mathscr{G}_{K_\pi}} \\ \mathcal{T}(D) = (F_0^{\mathrm{sep}} \otimes_{F_0} D)^{\varphi=1} &\hookleftarrow D \end{aligned}$$

establish quasi-inverse equivalences of categories, where the τ-semilinear endomorphism τ_D over $\mathcal{D}(T)_{u,\tau} := F_{u,\tau} \otimes_{F_0} \mathcal{D}(T)$ is induced by $\tau \otimes \tau$ on $C^{\flat}_{u-\mathsf{np}} \otimes T$ using the following lemma 3.1.13.

Proof. cf [19, Théorème 1.14]. □

Lemma 3.1.13. *The natural map $F_{u,\tau} \otimes_{F_0} \mathcal{D}(T) \to (C^{\flat}_{u-\mathsf{np}} \otimes T)^{\mathscr{G}_L}$ is an isomorphism.*

Proof. cf [19, Lemma 1.12]. □

More generally, we have the integral analogue of theorem 3.1.12.

Definition 3.1.14. A (φ,τ)-*module over* $(\mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}_{u,\tau}})$ is the data:

(1) an étale φ-module D over $\mathcal{O}_{\mathcal{E}}$;

(2) a τ-semi-linear endomorphism τ_D over $D_{u,\tau} := \mathcal{O}_{\mathcal{E}_{u,\tau}} \otimes_{\mathcal{O}_{\mathcal{E}}} D$ which commutes with $\varphi_{\mathcal{O}_{\mathcal{E}_{u,\tau}}} \otimes \varphi_D$ (where $\varphi_{\mathcal{O}_{\mathcal{E}_{u,\tau}}}$ is the Frobenius map on $\mathcal{O}_{\mathcal{E}_{u,\tau}}$ and φ_D the Frobenius map on D) such that

$$(\forall x \in D)\ (g \otimes 1) \circ \tau_D(x) = \tau_D^{\chi(g)}(x),$$

for all $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g) \in \mathbf{N}$.

We denote $\mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau)$ the corresponding category.

Theorem 3.1.15. *The functors*

$$\begin{aligned} \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) &\to \mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau) \\ T &\mapsto \mathcal{D}(T) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\pi}} \\ \mathcal{T}(D) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_{\mathcal{E}}} D)^{\varphi=1} &\hookleftarrow D \end{aligned}$$

establish quasi-inverse equivalences of categories, where the τ-semilinear endomorphism τ_D over $\mathcal{D}(T)_{u,\tau} := (\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_L} \simeq \mathcal{O}_{\mathcal{E}_{u,\tau}} \otimes \mathcal{D}(T)$ is induced by $\tau \otimes \tau$ on $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T$.

Remark 3.1.16. For $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we can similarly define $\mathbf{Mod}_{\mathcal{E},\mathcal{E}_{u,\tau}}(\varphi,\tau)$: the category of (φ,τ)-modules over $(\mathcal{E}, \mathcal{E}_{u,\tau})$ and establish an equivalence of category.

3.2 The complex $\mathcal{C}^u_{\varphi,\tau}$

Notation 3.2.1. Let $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau)$, we put

$$D_{u,\tau,0} := \big\{x \in D_{u,\tau};\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow (g\otimes 1)(x) = x + \tau_D(x) + \tau_D^2(x) + \cdots + \tau_D^{\chi(g)-1}(x)\big\}.$$

By similar arguments as that of lemma 1.1.10, we have

$$D_{u,\tau,0} = \big\{x \in D_{u,\tau};\ (\gamma\otimes 1)x = (1 + \tau_D + \tau_D^2 + \cdots + \tau_D^{\chi(\gamma)-1})(x)\big\}.$$

Lemma 3.2.2. *Let $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau)$, then $\varphi-1$ and τ_D-1 induce maps $\varphi-1 : D_{u,\tau,0} \to D_{u,\tau,0}$ and $\tau_D - 1 : D \to D_{u,\tau,0}$.*

Proof. cf lemma 1.1.11. □

Definition 3.2.3. Let $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau)$ (resp. $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{E},\mathcal{E}_{u,\tau}}(\varphi,\tau)$). We define a complex $\mathcal{C}^u_{\varphi,\tau}(D)$ as follows:

$$0 \longrightarrow D \longrightarrow D \bigoplus D_{u,\tau,0} \longrightarrow D_{u,\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x), (\tau_D-1)(x))$$

$$(y,z) \longmapsto (\tau_D-1)(y) - (\varphi-1)(z).$$

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ (resp. $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$), we have in particular the complex $\mathcal{C}^u_{\varphi,\tau}(\mathcal{D}(T))$ (resp. $\mathcal{C}^u_{\varphi,\tau}(\mathcal{D}(V))$), which will also be simply denoted $\mathcal{C}^u_{\varphi,\tau}(T)$ (resp. $\mathcal{C}^u_{\varphi,\tau}(V)$).

Theorem 3.2.4. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then the complex $\mathcal{C}^u_{\varphi,\tau}(T)$ computes the continuous Galois cohomology of T, i.e. $\mathsf{H}^i(\mathscr{G}_K, \mathcal{C}^u_{\varphi,\tau}(T)) \simeq \mathsf{H}^i(\mathscr{G}_K, T)$ for $i \in \mathbf{N}$.*

Remark 3.2.5. (1) Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and $(D, D_{u,\tau})$ be its (φ,τ)-module over $(\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_{u,\tau}})$. We have the following diagram of complexes

$$\begin{array}{lccccccc}
\mathcal{C}^u_{\varphi,\tau}(D) & 0 \longrightarrow & D & \xrightarrow{(\varphi-1,\tau_D-1)} & D \oplus D_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} & D_{u,\tau,0} & \longrightarrow 0 \\
 & & \| & & \downarrow & & \downarrow & \\
\mathcal{C}_{\varphi,\tau}(D) & 0 \longrightarrow & D & \xrightarrow{(\varphi-1,\tau_D-1)} & D \oplus D_{\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} & D_{\tau,0} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{C}(D) & 0 \longrightarrow & 0 & \longrightarrow & D_{\tau,0}/D_{u,\tau,0} & \xrightarrow{\varphi-1} & D_{\tau,0}/D_{u,\tau,0} & \longrightarrow 0.
\end{array}$$

(2) Recall that for $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, the complex $\mathcal{C}_{\varphi,\tau}(D)$ computes the continuous Galois cohomology by theorem 1.1.13. We will show in the following that $\mathcal{C}^u_{\varphi,\tau}(D)$ and $\mathcal{C}_{\varphi,\tau}(D)$ are quasi-isomorphic, and hence the complex $\mathcal{C}^u_{\varphi,\tau}(D)$ also computes the continuous Galois cohomology.

3.2.6 Proof of theorem 3.2.4: the quasi-isomorphism

Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, and denote by $(D, D_{u,\tau})$ its (φ,τ)-module over $(\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_{u,\tau}})$.

Lemma 3.2.7. *The map $D_\tau/D_{u,\tau} \xrightarrow{\varphi-1} D_\tau/D_{u,\tau}$ is injective.*

Proof. For any $x \in D_\tau = (\mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_L}$ such that $(\varphi-1)x \in D_{u,\tau} = (\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_L}$, we have to show that $x \in D_{u,\tau}$. Obviously it suffices to show that for any element $x \in \mathsf{W}(C^\flat) \otimes_{\mathbf{Z}_p} T$, the relation $(\varphi-1)x \in \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T$ implies $x \in \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T$. By dévissage we can reduce to the case $pT = 0$. For any $x \in C^\flat \otimes_{\mathbf{F}_p} T$, we claim that $(\varphi-1)x \in k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T$ implies $x \in k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T$. We have $C^\flat \otimes T \simeq (C^\flat)^d$ and

$$k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes T \simeq (k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}})^d$$

as φ-modules. Hence it suffices to show that for any $x \in C^\flat$, $x^p - x \in k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}}$ will imply that $x \in k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}}$. Put $P(X) = X^p - X - (x^p - x) \in k(\!(u, \eta^{1/p^\infty})\!)[X]$: it is separable as $P'(X) = -1$ so that x is separable over $k(\!(u, \eta^{1/p^\infty})\!)$ and hence $x \in k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}}$. □

Lemma 3.2.8. *Assume T is killed by p, then there are exact sequences of abelian groups*

$$0 \to T^{\mathscr{G}_L} \to D_{u,\tau} \xrightarrow{\varphi-1} D_{u,\tau} \to \mathsf{H}^1(\mathscr{G}_L, T) \to 0$$
$$0 \to T^{\mathscr{G}_L} \to D_\tau \xrightarrow{\varphi-1} D_\tau \to \mathsf{H}^1(\mathscr{G}_L, T) \to 0.$$

Proof. The sequence of $\mathscr{G}_L$-modules

$$0 \to \mathbf{F}_p \to k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \xrightarrow{\varphi-1} k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \to 0$$

and

$$0 \to \mathbf{F}_p \to C^\flat \xrightarrow{\varphi-1} C^\flat \to 0$$

are exact (here we endow $k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}}$ with the discrete topology and $C^\flat$ with its valuation topology). Tensoring with T and taking continuous cohomology (the first for the discrete topology, the second for valuation topology) gives exact sequences:

$$0 \to T^{\mathscr{G}_L} \to D_{u,\tau} \xrightarrow{\varphi-1} D_{u,\tau} \to \mathsf{H}^1(\mathscr{G}_L, T) \to \mathsf{H}^1(\mathscr{G}_L, k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T)$$
$$0 \to T^{\mathscr{G}_L} \to D_\tau \xrightarrow{\varphi-1} D_\tau \to \mathsf{H}^1(\mathscr{G}_L, T) \to \mathsf{H}^1(\mathscr{G}_L, C^\flat \otimes_{\mathbf{F}_p} T).$$

The lemma follows from the vanishing of $\mathsf{H}^1(\mathscr{G}_L, k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T)$ (by Hilbert 90) and the vanishing of $\mathsf{H}^1(\mathscr{G}_L, C^\flat \otimes_{\mathbf{F}_p} T)$ (that follows from the fact that $\widehat{L}$ is perfectoid.) □

Lemma 3.2.9. If $T \in \mathbf{Rep}_{\mathbf{Z}_p, \mathrm{tors}}(\mathscr{G}_K)$, then

$$\mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T) = 0$$

where $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ is endowed with the p-adic topology.

Proof. By dévissage we may assume that T is killed by p, in which case this reduces to the vanishing of $\mathsf{H}^1(\mathscr{G}_L, C^\flat_{u-\mathsf{np}} \otimes_{\mathbf{F}_p} T)$, which follows from Hilbert 90 since $\mathscr{G}_L$ acts continuously on $C^\flat_{u-\mathsf{np}} = k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}}$ (endowed with the discrete topology). □

Corollary 3.2.10. *If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then*

$$\mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T) = 0,$$

where $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ is endowed with the p-adic topology.

Proof. Denote $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u},n} := \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} / p^n$. By [43, Theorem 2.3.4], we have the exact sequence

$$0 \to \mathsf{R}^1 \varprojlim_n \mathsf{H}^0(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u},n} \otimes_{\mathbf{Z}_p} T) \to \mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T) \to \varprojlim_n \mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u},n} \otimes_{\mathbf{Z}_p} T).$$

We have $\mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u},n} \otimes_{\mathbf{Z}_p} T) = 0$ by lemma 3.2.9 and $\mathsf{R}^1 \varprojlim_n \mathsf{H}^0(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u},n} \otimes_{\mathbf{Z}_p} T) = 0$ from the observation that $\{\mathsf{H}^0(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u},n} \otimes_{\mathbf{Z}_p} T)\}_n$ has the Mittag-Leffler property. Hence we get $\mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T) = 0$. □

Corollary 3.2.11. If $0 \to T' \to T \to T'' \to 0$ is an exact sequence in $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then the sequence

$$0 \to \mathcal{D}(T')_{u,\tau} \to \mathcal{D}(T)_{u,\tau} \to \mathcal{D}(T'')_{u,\tau} \to 0$$

is exact.

Proof. As $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ is torsion-free, we have the exact sequence

$$0 \to \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T' \to \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T \to \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T'' \to 0$$

which induces the exact sequences

$$0 \to \mathcal{D}(T')_{u,\tau} \to \mathcal{D}(T)_{u,\tau} \to \mathcal{D}(T'')_{u,\tau} \to \mathsf{H}^1(\mathscr{G}_L, \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \otimes_{\mathbf{Z}_p} T').$$

By corollary 3.2.10, we get the exact sequence. □

Proposition 3.2.12. *The map $D_\tau/D_{u,\tau} \xrightarrow{\varphi-1} D_\tau/D_{u,\tau}$ is bijective.*

Proof. (1) We first assume T is killed by p. By lemma 3.2.7, it suffices to show the surjectivity. By lemma 3.2.8, the inclusion $k(\!(u, \eta^{1/p^\infty})\!)^{\mathrm{sep}} \subset C^\flat$ induces a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^{\mathscr{G}_L} & \longrightarrow & D_{u,\tau} & \xrightarrow{\varphi-1} & D_{u,\tau} & \longrightarrow & \mathsf{H}^1(\mathscr{G}_L, T) \longrightarrow 0 \\
 & & \| & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & T^{\mathscr{G}_L} & \longrightarrow & D_\tau & \xrightarrow{\varphi-1} & D_\tau & \longrightarrow & \mathsf{H}^1(\mathscr{G}_L, T) \longrightarrow 0.
\end{array}$$

If $y \in D_\tau$, there exists $z \in D_{u,\tau}$ having the same image in $\mathsf{H}^1(\mathscr{G}_L, T)$: hence $y - z$ maps to 0 in $\mathsf{H}^1(\mathscr{G}_L, T)$, so there exists $x \in D_\tau$ such that $(\varphi - 1)x = y - z$ and thus $(\varphi - 1)(x + D_{u,\tau}) = y + D_{u,\tau}$. This finishes the proof.

(2) We now use dévissage for $\mathbf{Z}_p$-representations. Notice that $\mathcal{D}(T/p^n) = \mathcal{D}(T)/p^n$ and $T, \mathcal{D}(T)$ are both p-adically complete. Hence it suffices to deal with the case where T is killed by p^n with $n \in \mathbf{N}_{\geq 0}$. We use induction over n. Suppose T is killed by p^n, we put $T^{'} = pT, T^{''} = T/pT$ and consider the following exact sequence

$$0 \to T^{'} \to T \to T^{''} \to 0$$

in $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$. Then we have the following diagram of exact sequences by corollary 3.2.11

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{D}(T')_{u,\tau} & \longrightarrow & \mathcal{D}(T)_{u,\tau} & \longrightarrow & \mathcal{D}(T'')_{u,\tau} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}(T')_\tau & \longrightarrow & \mathcal{D}(T)_\tau & \longrightarrow & \mathcal{D}(T'')_\tau & \longrightarrow & 0.
\end{array}$$

By snake lemma we have the exact sequence

$$0 \to \mathcal{D}(T')_\tau/\mathcal{D}(T')_{u,\tau} \to \mathcal{D}(T)_\tau/\mathcal{D}(T)_{u,\tau} \to \mathcal{D}(T'')_\tau/\mathcal{D}(T'')_{u,\tau} \to 0$$

and we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{D}(T')_\tau/\mathcal{D}(T')_{u,\tau} & \longrightarrow & \mathcal{D}(T)_\tau/\mathcal{D}(T)_{u,\tau} & \longrightarrow & \mathcal{D}(T'')_\tau/\mathcal{D}(T'')_{u,\tau} & \longrightarrow & 0 \\
 & & \varphi-1 \downarrow & & \varphi-1 \downarrow & & \varphi-1 \downarrow & & \\
0 & \longrightarrow & \mathcal{D}(T')_\tau/\mathcal{D}(T')_{u,\tau} & \longrightarrow & \mathcal{D}(T)_\tau/\mathcal{D}(T)_{u,\tau} & \longrightarrow & \mathcal{D}(T'')_\tau/\mathcal{D}(T'')_{u,\tau} & \longrightarrow & 0.
\end{array}$$

Notice that the first and the third vertical maps are isomorphisms by induction hypothesis, hence the middle one is an isomorphism. We then conclude by passing to the limit.

□

Corollary 3.2.13. *The map $D_{\tau,0}/D_{u,\tau,0} \xrightarrow{\varphi-1} D_{\tau,0}/D_{u,\tau,0}$ is bijective.*

Proof. Consider the following morphism of short exact sequences:

$$\begin{array}{ccccccccc} & & & & D_{\tau,0}/D_{u,\tau,0} & \xrightarrow{\varphi-1} & D_{\tau,0}/D_{u,\tau,0} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_\tau/D_{u,\tau} & \xrightarrow{\varphi-1} & D_\tau/D_{u,\tau} & \longrightarrow & D_\tau/D_{u,\tau} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta-\gamma} & & \downarrow{\scriptstyle \delta-\gamma} & & \downarrow{\scriptstyle \delta-\gamma} & & \\ 0 & \longrightarrow & D_\tau/D_{u,\tau} & \xrightarrow{\varphi-1} & D_\tau/D_{u,\tau} & \longrightarrow & D_\tau/D_{u,\tau} & \longrightarrow & 0. \end{array}$$

We then have the claimed isomorphism using the snake lemma. □

Remark 3.2.14. By remark 3.2.5, this finishes the proof of theorem 3.2.4.

3.3 The complex $\mathcal{C}^u_{\psi,\tau}$

In this section we will define a ψ operator for (φ,τ)-modules over $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$, and then construct a complex $\mathcal{C}^u_{\psi,\tau}$. At the end of this section, we will show that this complex $\mathcal{C}^u_{\psi,\tau}$ computes the continuous Galois cohomology: we will first prove the case of $\mathbf{F}_p$-representations and then pass to the case of $\mathbf{Z}_p$-representations by dévissage.

Recall that we have defined $\mathcal{O}_{\mathcal{F}^{\mathrm{ur}}_u}$ in definition 3.1.7 and denote $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ its p-adic completion. The ring $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ has a Frobenius map that lifts that of the residue field $k(\!(u,\eta^{1/p^\infty})\!)^{\mathrm{sep}}$ (*cf* [42, Theorem 29.2]).

Lemma 3.3.1. *Let M be a field of characteristic p, such that $[M:\varphi(M)]<\infty$, then $\varphi(M^{\mathrm{sep}})\otimes_{\varphi(M)} M \simeq M^{\mathrm{sep}}$ and in particular $[M^{\mathrm{sep}}:\varphi(M^{\mathrm{sep}})] = [M:\varphi(M)]$.*

Proof. It suffices to show that for any separable and algebraic extension L/M (we pass to inductive limit for the case of separable extension) we have an isomorphism $M\otimes_{\varphi(M)}\varphi(L)\simeq L$. By [42, Theorem 26.4], the natural map $M\otimes_{\varphi(M)}\varphi(L)\to M\varphi(L)$ is an isomorphism. This implies that $[M\varphi(L):M] = [\varphi(L):\varphi(M)] = [L:M]$. As $M\subset M\varphi(L)\subset L$, then $[L:M] = [L:M\varphi(L)][M\varphi(L):M]$ and hence $[L:M\varphi(L)]=1$, *i.e.* $M\varphi(L)=L$. □

Corollary 3.3.2. *The extension $C^\flat_{u-\mathsf{np}}/\varphi(C^\flat_{u-\mathsf{np}})$ has degree p.*

Proof. By definition $C^\flat_{u-\mathsf{np}} = k(\!(u,\eta^{1/p^\infty})\!)^{\mathrm{sep}}$, and we apply lemma 3.3.1. □

Lemma 3.3.3. *The extension $\widehat{\mathcal{F}^{\mathrm{ur}}_u}/\varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u})$ has degree p.*

Proof. We have the diagram

$$\begin{array}{ccc} \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} & \twoheadrightarrow & C^\flat_{u-\mathsf{np}} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} & \twoheadrightarrow & C^\flat_{u-\mathsf{np}} \end{array}$$

with $[C^\flat_{u-\mathsf{np}}:\varphi(C^\flat_{u-\mathsf{np}})] = p$. Hence there exist $a_1,\dots,a_p\in\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ such that whose image $(\overline{a_1},\dots,\overline{a_p})$ modulo p forms a basis of $C^\flat_{u-\mathsf{np}}$ over $\varphi(C^\flat_{u-\mathsf{np}})$. The following map is surjective

$$\rho\colon \left(\varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})\right)^p \to \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$$
$$(\lambda_1,\dots,\lambda_p) \mapsto \sum_{i=1}^p \lambda_i a_i.$$

Indeed, for any $a \in \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ there exists $(\lambda_1^{(1)},\dots,\lambda_p^{(1)}) \in \left(\varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})\right)^p$ such that

$$a - \sum_{i=1}^p \lambda_i^{(1)} a_i \in p\,\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}.$$

Hence there exists $(\lambda_1^{(2)},\dots,\lambda_p^{(2)}) \in \left(\varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})\right)^p$ such that

$$a - \sum_{i=1}^p \lambda_i^{(1)} a_i - p\sum_{i=1}^p \lambda_i^{(2)} a_i \in p^2\,\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}.$$

By induction, for any $n \in \mathbf{N}$ there exists $(\lambda_1^{(n)},\dots,\lambda_p^{(n)}) \in \left(\varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})\right)^p$ such that

$$a - \sum_{i=1}^p \left(\sum_{j=1}^n p^{j-1}\lambda_i^{(j)}\right) a_i \in p^n\,\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}.$$

As $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ is p-adically complete, we have

$$a = \sum_{i=1}^p \left(\sum_{j=1}^\infty p^{j-1}\lambda_i^{(j)}\right) a_i.$$

Notice that $\sum_{j=1}^\infty p^{j-1}\lambda_i^{(j)} \in \varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})$ and hence we proved the surjectivity of the map ρ defined above. Now $\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ is a $\varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})$-module of finite type and we can apply Nakayama's lemma: the map is an isomorphism since it is so modulo p by corollary 3.3.2. □

Definition 3.3.4. For any $x \in \widehat{\mathcal{F}^{\mathrm{ur}}_u}$, we put

$$\psi(x) = \frac{1}{p}\varphi^{-1}(\mathrm{Tr}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}/\varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u})}(x)).$$

In particular we have $\psi\circ\varphi = \mathrm{id}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ on $\widehat{\mathcal{F}^{\mathrm{ur}}_u}$. Applying lemma 3.3.3 to $\widehat{\mathcal{E}^{\mathrm{ur}}}$, we see that the operator ψ induces an operator $\psi\colon \widehat{\mathcal{E}^{\mathrm{ur}}} \to \widehat{\mathcal{E}^{\mathrm{ur}}}$. Note that $\psi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}) \subset \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}$ and $\psi(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \subset \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$.

Remark 3.3.5. We have $\psi\circ g = g\circ\psi$ for all $g \in \mathscr{G}_K$. Indeed, we have the following commutative square

$$\begin{array}{ccc}
\widehat{\mathcal{F}^{\mathrm{ur}}_u} & \xrightarrow{\ g\ } & \widehat{\mathcal{F}^{\mathrm{ur}}_u} \\
{\scriptstyle\varphi^{-1}}\uparrow\downarrow{\scriptstyle\varphi} & & {\scriptstyle\varphi}\downarrow\uparrow{\scriptstyle\varphi^{-1}} \\
\varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u}) & \xrightarrow{\ g\ } & \varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u}).
\end{array}$$

This implies φ^{-1} commutes with $g \in \mathscr{G}_K$ over $\varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u})$. As $\mathrm{Tr}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}/\varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u})}\colon \widehat{\mathcal{F}^{\mathrm{ur}}_u} \to \varphi(\widehat{\mathcal{F}^{\mathrm{ur}}_u})$ commutes with $g \in \mathscr{G}_K$, so does ψ on $\widehat{\mathcal{F}^{\mathrm{ur}}_u}$.

Proposition 3.3.6. *Let $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau)$. There exists a unique additive map*

$$\psi_D \colon D \to D,$$

satisfying

(1)

$$(\forall a \in \mathcal{O}_\mathcal{E})(\forall x \in D) \quad \psi_D(a\varphi_D(x)) = \psi_{\mathcal{O}_\mathcal{E}}(a)x,$$

(2)

$$(\forall a \in \mathcal{O}_\mathcal{E})(\forall x \in D) \quad \psi_D(\varphi_{\mathcal{O}_\mathcal{E}}(a)x) = a\psi_D(x).$$

This map is surjective and satisfies $\psi_D \circ \varphi_D = \mathrm{id}_D$.

There also exists a unique additive map $\psi_{D_{u,\tau}} : D_{u,\tau} \to D_{u,\tau}$ that satisfies similar conditions as above and extends the additive map ψ_D.

Proof. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ be such that $D = \mathcal{D}(T)$. We have defined ψ on $\widehat{\mathcal{F}_u^{\mathrm{ur}}}$, hence it is defined over $\mathcal{E}_{u,\tau} = (\widehat{\mathcal{F}_u^{\mathrm{ur}}})^{\mathscr{G}_L}$. The operator $\psi \otimes 1$ on $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T$ and $\mathcal{O}_{\widehat{\mathcal{F}_u^{\mathrm{ur}}}} \otimes_{\mathbf{Z}_p} T$ induces operators ψ on D and $D_{u,\tau}$. One easily verifies the above properties: this shows the existence. The unicity follows from the fact that D is étale, *i.e.* $\varphi(D)$ generates D as an $\mathcal{O}_\mathcal{E}$-module. □

Remark 3.3.7. (1) Let $(D, D_\tau) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_\tau}}(\varphi, \tau)$. Suppose there exists an operator ψ_{D_τ} over D_τ that extends ψ_D and $\psi_{D_\tau} \circ \varphi_{D_\tau} = \mathrm{id}_{D_\tau}$, then φ_{D_τ} being bijective over D_τ (as D is étale and $\varphi_{\mathcal{O}_{\mathcal{E}_\tau}}$ is bijective over $\mathcal{O}_{\mathcal{E}_\tau}$) will imply that ψ_{D_τ} is bijective, which contradicts the fact ψ_D is not injective.

(2) When there is no confusion, we will simply denote ψ instead of $\psi_{\mathcal{O}_\mathcal{E}}, \psi_D$ and $\psi_{D_{u,\tau}}$.

Lemma 3.3.8. *Let $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi, \tau)$. We have a map $\psi \colon D_{u,\tau,0} \to D_{u,\tau,0}$.*

Proof. We have the ψ operator on $D_{u,\tau}$ by proposition 3.3.6. Notice that ψ commutes with $g \in \mathscr{G}_K$ by remark 3.3.5, hence ψ induces a $\mathbf{Z}_p$-linear endomorphism on $D_{u,\tau,0}$. Indeed, if $x \in D_{u,\tau,0}$, then we have

$$(\gamma \otimes 1)x = (1 + \tau_D + \cdots + \tau_D^{\chi(\gamma)-1})x.$$

Applying ψ to both sides and by the commutativity we have

$$(\gamma \otimes 1)(\psi(x)) = (1 + \tau_D + \cdots + \tau_D^{\chi(\gamma)-1})\psi(x).$$

This implies $\psi(x) \in D_{u,\tau,0}$. □

Remark 3.3.9. By proposition 3.3.6 and lemma 3.3.8, ψ is surjective on D and $D_{u,\tau,0}$.

We now define the following complex:

Definition 3.3.10. Let $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi, \tau)$. We define a complex $\mathcal{C}^u_{\psi,\tau}(D)$ as follows:

$$0 \longrightarrow D \longrightarrow D \bigoplus D_{u,\tau,0} \longrightarrow D_{u,\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\psi - 1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longmapsto (\tau_D - 1)(y) - (\psi - 1)(z).$$

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}^u_{\psi,\tau}(\mathcal{D}(T))$, which will also be simply denoted $\mathcal{C}^u_{\psi,\tau}(T)$.

Theorem 3.3.11. *The morphism of complexes*

$$\begin{array}{lccccccccc}
\mathcal{C}^u_{\varphi,\tau}: & 0 & \longrightarrow & D & \xrightarrow{(\varphi-1,\tau_D-1)} & D\oplus D_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} & D_{u,\tau,0} & \longrightarrow & 0\\
& & & \| & & \downarrow{\scriptstyle -\psi\ominus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\
\mathcal{C}^u_{\psi,\tau}: & 0 & \longrightarrow & D & \xrightarrow{(\psi-1,\tau_D-1)} & D\oplus D_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\psi-1)} & D_{u,\tau,0} & \longrightarrow & 0
\end{array}$$

is a quasi-isomorphism.

Remark 3.3.12. (1) The diagram in theorem 3.3.11 is indeed a morphism of complexes.

Proof. As ψ commutes with the Galois action, it induces a map $\psi\colon D_{u,\tau,0}\to D_{u,\tau,0}$. We claim that $\psi_{D_{u,\tau}|_D}=\psi_D$, and hence the diagram in theorem 3.3.11 commutes. Indeed, we have the following commutative square

$$\begin{array}{ccc}
\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} & \hookrightarrow & \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}\\
{\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi}\\
\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} & \hookrightarrow & \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}.
\end{array}$$

By lemma 3.3.3, $[\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}:\varphi(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})]=[\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}}:\varphi(\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}})]=p$, hence we conclude by the construction of ψ.

□

(2) We have diagram

$$\begin{array}{lccccccccc}
 & 0 & \longrightarrow & 0 & \longrightarrow & D^{\psi=0} & \xrightarrow{\tau_D-1} & D^{\psi=0}_{u,\tau,0} & \longrightarrow & 0\\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}^u_{\varphi,\tau}: & 0 & \longrightarrow & D & \xrightarrow{(\varphi-1,\tau_D-1)} & D\oplus D_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} & D_{u,\tau,0} & \longrightarrow & 0\\
 & & & \| & & \downarrow{\scriptstyle -\psi\ominus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\
\mathcal{C}^u_{\psi,\tau}: & 0 & \longrightarrow & D & \xrightarrow{(\psi-1,\tau_D-1)} & D\oplus D_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\psi-1)} & D_{u,\tau,0} & \longrightarrow & 0\\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

As ψ is surjective, the cokernel complex is trivial and it suffices to show that the kernel complex is acyclic, *i.e.* the map

$$D^{\psi=0}\xrightarrow{\tau_D-1}D^{\psi=0}_{u,\tau,0}$$

is an isomorphism.

To prove the theorem 3.3.11, we will start with the case of $\mathbf{F}_p$-representations and then pass to the $\mathbf{Z}_p$-representations by dévissage.

3.3.13 The case of $\mathbf{F}_p$-representations

We assume in this subsection that $T\in\mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$.

Lemma 3.3.14. *For all $r\in\mathbf{N}$, we have $(C^\flat_{u-\mathsf{np}})^{\mathscr{G}_{K_\zeta}}=F^{\tau^{p^r}}_{u,\tau}=k(\!(\eta^{1/p^\infty})\!)$.*

Proof. Let $z \in F_{u,\tau}$, write $z = \sum_{i=m}^{\infty} f_i(\eta)u^i$, where $m \in \mathbf{Z}$ and $f_i(X) \in k(\!(X^{1/p^{n_i}})\!)$ for some $n_i \in \mathbf{N}$. We have $\tau^{p^n}(z) = \sum_{i=n}^{\infty} \varepsilon^{ip^n} f_i(\eta)u^i$ so that $\tau^{p^n}(z) = z$ implies that $\varepsilon^{ip^n} f_i(\eta) = f_i(\eta)$ and hence $f_i = 0$ for $i \neq 0$. We conclude $z = f_0(\eta) \subset \bigcup_{n\in\mathbf{N}} k(\!(\eta^{1/p^n})\!)$ and hence $F_{u,\tau}^{\tau^{p^r}} \subset k(\!(\eta^{1/p^\infty})\!)$. Conversely, we have $k(\!(\eta^{1/p^\infty})\!) \subset F_{u,\tau}^{\tau^{p^r}}$, as $k(\!(\eta^{1/p^\infty})\!) \subset F_{u,\tau}$ and τ acts trivially over K_ζ. Notice that $\mathscr{G}_{K_\zeta} = \langle \mathscr{G}_L, \tau\rangle$ and $F_{u,\tau} = (C_{u-\mathsf{np}}^\flat)^{\mathscr{G}_L}$ by corollary 3.1.5, so that $(C_{u-\mathsf{np}}^\flat)^{\mathscr{G}_{K_\zeta}} = F_{u,\tau}^{\tau}$, which is $k(\!(\eta^{1/p^\infty})\!)$ by similar computations as above. □

Lemma 3.3.15. *Put $\mathbf{D}(T) = (C_{u-\mathsf{np}}^\flat \otimes T)^{\mathscr{G}_{K_\zeta}}$. We have a $\mathscr{G}_{K_\zeta}$-equivariant isomorphism*

$$C_{u-\mathsf{np}}^\flat \otimes_{\mathbf{F}_p} T \simeq C_{u-\mathsf{np}}^\flat \otimes_{k(\!(\eta^{1/p^\infty})\!)} \mathbf{D}(T).$$

In particular, we have $\dim_{k(\!(\eta^{1/p^\infty})\!)} \mathbf{D}(T) = \dim_{\mathbf{F}_p} T$.

Proof. Denote $\mathbb{D}(T) = \big(k(\!(\eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T\big)^{\mathscr{G}_{K_\zeta}}$, then from the field of norm theory (of perfect fields) (*cf* [24, Proposition 1.2.4]) we have the following $\mathscr{G}_{K_\zeta}$-equivariant isomorphism

$$k(\!(\eta^{1/p^\infty})\!)^{\mathrm{sep}} \otimes_{k(\!(\eta^{1/p^\infty})\!)} \mathbb{D}(T) \simeq k(\!(\eta^{1/p^\infty})\!) \otimes_{\mathbf{F}_p} T.$$

Tensoring with $C_{u-\mathsf{np}}^\flat$ over $k(\!(\eta^{1/p^\infty})\!)$, we have

$$C_{u-\mathsf{np}}^\flat \otimes_{k(\!(\eta^{1/p^\infty})\!)} \mathbb{D}(T) \simeq C_{u-\mathsf{np}}^\flat \otimes_{\mathbf{F}_p} T. \tag{3.1}$$

Taking the points fixed by $\mathscr{G}_{K_\zeta}$ on both sides gives

$$\big(C_{u-\mathsf{np}}^\flat \otimes_{k(\!(\eta^{1/p^\infty})\!)} \mathbb{D}(T)\big)^{\mathscr{G}_{K_\zeta}} \simeq \big(C_{u-\mathsf{np}}^\flat \otimes_{\mathbf{F}_p} T\big)^{\mathscr{G}_{K_\zeta}} = \mathbf{D}(T).$$

As $\mathbb{D}(T)$ is fixed by $\mathscr{G}_{K_\zeta}$ from definition, the left hand side is

$$\big(C_{u-\mathsf{np}}^\flat \otimes_{k(\!(\eta^{1/p^\infty})\!)} \mathbb{D}(T)\big)^{\mathscr{G}_{K_\zeta}} = (C_{u-\mathsf{np}}^\flat)^{\mathscr{G}_{K_\zeta}} \otimes_{k(\!(\eta^{1/p^\infty})\!)} \mathbb{D}(T) = \mathbb{D}(T)$$

by lemma 3.3.14. This proves that $\mathbb{D}(T) = \mathbf{D}(T)$, hence equation 3.1 gives what we want. □

Lemma 3.3.16. *Let $r \in \mathbf{N}_{>0}$, we then have $\mathcal{D}(T)_{u,\tau}^{\tau_D^{p^r}} \subset \mathbf{D}(T)$.*

Proof. We have $\mathcal{D}(T)_{u,\tau}^{\tau_D^{p^r}} = (C_{u-\mathsf{np}}^\flat \otimes_{\mathbf{F}_p} T)^{\mathscr{G}_{K_\zeta(\pi^{1/p^r})}}$. Notice that by lemma 3.3.15, we have

$$\begin{aligned}
(C_{u-\mathsf{np}}^\flat \otimes_{\mathbf{F}_p} T)^{\mathscr{G}_{K_\zeta(\pi^{1/p^r})}} &= (C_{u-\mathsf{np}}^\flat)^{\mathscr{G}_{K_\zeta(\pi^{1/p^r})}} \otimes \mathbf{D}(T) \\
&= ((C_{u-\mathsf{np}}^\flat)^{\mathscr{G}_L})^{\tau^{p^r}} \otimes \mathbf{D}(T) \\
&= F_{u,\tau}^{\tau^{p^r}} \otimes \mathbf{D}(T) \\
&= \mathbf{D}(T).
\end{aligned}$$

The last step follows from lemma 3.3.14, as $\bigcup_{n\in\mathbf{N}} k(\!(\eta)\!)[\eta^{1/p^n}] \subset (C_{u-\mathsf{np}}^\flat)^{\mathscr{G}_{K_\zeta}}$.

□

Proposition 3.3.17. *The map $\frac{\tau_D^{p^r}-1}{\tau_D-1}\colon D_{u,\tau,0}^{\psi=0} \to D_{u,\tau^{p^r},0}^{\psi=0}$ is injective.*

Proof. Take any $x \in D_{u,\tau,0}^{\psi=0}$ with $\frac{\tau_D^{p^r}-1}{\tau_D-1}(x) = 0$, then in particular $(\tau_D^{p^r} - 1)x = 0$. Hence $x \in D_{u,\tau}^{\tau_D^{p^r},\psi=0}$ and $x \in \mathbf{D}(T)^{\psi=0}$ by lemma 3.3.16. Lemma 3.3.14 and lemma 3.3.15 imply that $x = \varphi(x')$ for some $x' \in \mathbf{D}(T)$ (as $\mathbf{D}(T)$ is étale and the base field $k(\!(\eta^{1/p^\infty})\!)$ is perfect) and hence $0 = \psi(x) = \psi(\varphi(x')) = x'$ implies $x = 0$. □

Recall that by remark 3.3.12 (2), to prove theorem 3.3.11 it suffices to prove $D^{\psi=0} \xrightarrow{\tau_D-1} D_{u,\tau,0}^{\psi=0}$ being isomorphic: we firstly prove the case of $\mathbf{F}_p$-representations, in several steps.

The injectivity

Recall that $F_0 = k(\!(u)\!)$ embeds into $C^\flat$ by $u \mapsto \tilde{\pi}$.

Lemma 3.3.18. *We have* $F_0^{\tau=1} = k$.

Proof. Recall that $\tau(u) = \varepsilon u = (\eta+1)u$ and $\tau(u^i) = (\eta+1)^i u^i$. Suppose $x = \sum_{i=i_0}^{\infty} \lambda_i u^i \in F_0^{\tau=1}$ with $\lambda_i \in k$, then $\tau(x) = \sum_{i=i_0}^{\infty} \lambda_i (\eta+1)^i u^i$. Hence $\lambda_i = \lambda_i(\eta+1)^i$, so that $\lambda_i((\eta+1)^i - 1) = 0$ for all $i \geq i_0$. If $i \neq 0$, then $(\eta+1)^i \neq 1$, hence $(\eta+1)^i - 1 \in k(\!(\eta)\!)^\times$ so that $\lambda_i = 0$. In particular, $x = \lambda_0 \in k$. □

Lemma 3.3.19. *We have* $k(\!(\eta^{1/p^\infty})\!)^{\gamma=1} = k$.

Proof. We have $k(\!(\eta^{1/p^\infty})\!) \subset (C^\flat)^{\mathscr{G}_K \varsigma}$, hence $k(\!(\eta^{1/p^\infty})\!)^{\gamma=1} \subset (C^\flat)^{\mathscr{G}_K}$. As $C^\flat = \varprojlim_{x \mapsto x^p} C$, we have

$$(C^\flat)^{\mathscr{G}_K} = \varprojlim_{x \mapsto x^p} K.$$

Let $x = (x_n)_n \in \varprojlim_{x \mapsto x^p} K$, we have $x_0 = x_n^{p^n}$ for all $n \in \mathbf{N}$. In particular, we have $v(x_0) = p^n v(x_n)$. If $x \neq 0$, this implies $v(x_n) = 0$ for all $n \in \mathbf{N}$. Let $\overline{x_0}$ be the image of x_0 in $k = \mathcal{O}_K/(\pi)$ and $y = \left([\overline{x_0}^{p^{-n}}]\right)_n \in \varprojlim_{x \mapsto x^p} K$, then $y^{-1}x = (z_n)_n \in \varprojlim_{x \mapsto x^p} K$ satisfies $z_n \equiv 1 \bmod \pi\,\mathcal{O}_K$ for all $n \in \mathbf{N}$. This implies $z_n = z_{n+m}^{p^m} \equiv 1 \bmod \pi^{m+1}\,\mathcal{O}_K$ for all $m \in \mathbf{N}$, hence $z_n = 1$ for all $n \in \mathbf{N}$. This shows that $x = y$, and that the map

$$\begin{aligned} k &\to \varprojlim_{x \mapsto x^p} K = (C^\flat)^{\mathscr{G}_K} \\ \alpha &\mapsto \left([\alpha^{p^{-n}}]\right)_n \end{aligned}$$

is a ring isomorphism. As $k \subset k(\!(\eta^{1/p^\infty})\!)^{\gamma=1} \subset (C^\flat)^{\mathscr{G}_K}$, this shows that $k = k(\!(\eta^{1/p^\infty})\!)^{\gamma=1} = (C^\flat)^{\mathscr{G}_K}$. □

Proposition 3.3.20. *The natural map* $k(\!(\eta^{1/p^\infty})\!) \otimes_k \mathbf{D}(T)^{\gamma=1} \to \mathbf{D}(T)$ *is injective.*

Proof. We use the standard argument: assume it is not and let $x = \sum_{i=1}^{r} \lambda_i \otimes \alpha_i$, with $\lambda_i \in k(\!(\eta^{1/p^\infty})\!)$ and $\alpha_i \in \mathbf{D}(T)^{\gamma=1}$ for all i, be a nonzero element in the kernel such that r is minimal. Deviding by λ_r we may assume that $\lambda_r = 1$. As $\sum_{i=1}^{r} \gamma(\lambda_i) \otimes \alpha_i$ maps to $\gamma(\sum_{i=1}^{r} \lambda_i \alpha_i) = 0$, the element $\sum_{i=1}^{r-1} (\gamma - 1)(\lambda_i) \otimes \alpha_i$ lies in the kernel, by the minimality of r, we have $\gamma(\lambda_i) = \lambda_i$, *i.e.* $\lambda_i \in k$ for all $i \in \{1, \ldots, r\}$ (*cf* lemma 3.3.19). Then $x = 1 \otimes (\sum_{i=1}^{r} \lambda_i \alpha_i) = 0$, which is a contradiction to the assumption. Hence the map is injective. □

Corollary 3.3.21. *We have* $\dim_k \mathbf{D}(T)^{\gamma=1} \leq \dim_{\mathbf{F}_p} T$.

Lemma 3.3.22. *If* $x \in D^{\psi=0}$ *is such that* $\tau_D(x) = x$, *then* $x = 0$.

Proof. Let $x \in D^{\psi=0}$ be such that $\tau_D(x) = x$. This shows that x, seen as an element of $C^\flat_{u-\mathsf{np}} \otimes_{\mathbf{F}_p} T$, is fixed by $\mathscr{G}_K$: it belongs to $\mathbf{D}(T)^{\gamma=1}$. By corollary 3.3.21, the latter is a finite dimensional k-vector space. It is endowed with the restriction φ of the Frobenius map, and this restriction is injective. As k is perfect, this implies that $\varphi\colon \mathbf{D}(T)^{\gamma=1} \to \mathbf{D}(T)^{\gamma=1}$ is bijective: there exists $y \in \mathbf{D}(T)^{\gamma=1}$ such that $x = \varphi(y)$. Then $y = \psi\varphi(y) = \psi(x) = 0$, hence $x = \varphi(y) = 0$. □

Corollary 3.3.23. *The map* $D^{\psi=0} \xrightarrow{\tau_D - 1} D^{\psi=0}_{u,\tau,0}$ *is injective.*

The trivial case

Recall that $\psi\colon F_0 \to F_0$ is given by the formula $\psi(x) = \frac{1}{p}\varphi^{-1}(\mathrm{Tr}_{F_0/\varphi(F_0)}(x))$. The elements $1, u, \dots, u^{p-1}$ form a basis of F_0 over $\varphi(F_0)$.

Lemma 3.3.24. *Let $x = \sum_{i=0}^{p-1} x_i u^i$ be an element of F_0 with $x_i \in \varphi(F_0)$. Then we have $\psi(x) = \varphi^{-1}(x_0)$.*

Proof. For $1 \leq i < p$, we have $\psi(u^i) = 0$ as $\mathrm{Tr}_{F_0/\varphi(F_0)}(u^i) = 0$. Indeed, the minimal polynomial of u^i is $f(X) = X^p - u^{pi} \in \varphi(F_0)[X]$ when $1 \leq i < p$. □

Corollary 3.3.25. *An element $x = \sum_{i=0}^{p-1} x_i u^i \in F_0$ with $x_i \in \varphi(F_0)$ is killed by ψ if and only if $x_0 = 0$.*

Proof. By the lemma 3.3.24 and notice that φ is injective. □

Let $z = \sum_{i=1}^{p-1} u^i z_i \in {F_0}^{\psi=0}$ with $z_i \in \varphi(F_0) = k(\!(u^p)\!)$. More precisely, write $z_i = \sum_{j=n_i}^{+\infty} b_{ij} u^{pj}$ with $b_{ij} \in k$. Then $z = \sum_{i=1}^{p-1} \sum_{j=n_i}^{+\infty} b_{ij} u^{i+pj}$ with $b_{ij} \in k$ so that

$$(\tau - 1)z = \sum_{i=1}^{p-1} \sum_{j=n_i}^{\infty} b_{ij} u^{i+pj} (\varepsilon^{i+pj} - 1). \tag{3.2}$$

Lemma 3.3.26. *The map $\tau - 1\colon F_0^{\psi=0} \to F_{u,\tau,0}^{\psi=0}$ is surjective.*

Proof. Let $x \in F_{u,\tau,0}^{\psi=0} \subset k(\!(u, \eta^{1/p^\infty})\!)^{\psi=0}$: by corollary 3.3.25 we can write uniquely $x = \sum_{i=1}^{p-1} u^i x_i$ with

$$x_i \in \varphi\big(k(\!(u, \eta^{1/p^\infty})\!)^{\psi=0}\big).$$

We can write

$$x_i = \sum_{j=m_i}^{\infty} u^{pj} f_{ij}(\eta)$$

with $m_i \in \mathbf{Z}$ and $f_{ij}(X) \in k(\!(X^{1/p^{n_{ij}}})\!)$ for some $n_{ij} \in \mathbf{N}$, hence

$$x = \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} u^{i+pj} f_{ij}(\eta).$$

By definition, $x \in F_{u,\tau,0}$ implies

$$\gamma(x) = (1 + \tau + \cdots + \tau^{\chi(\gamma)-1})(x).$$

The left hand side is (recall that $\eta = \varepsilon - 1$)

$$\gamma(x) = \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} u^{i+pj} f_{ij}(\gamma(\eta)) = \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} u^{i+pj} f_{ij}(\varepsilon^{\chi(\gamma)} - 1),$$

and the right hand side is

$$(1 + \tau + \cdots + \tau^{\chi(\gamma)-1})(x) = \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} u^{i+pj} \cdot (1 + \varepsilon^{i+pj} + \varepsilon^{2(i+pj)} + \cdots + \varepsilon^{(\chi(\gamma)-1)(i+pj)}) \cdot f_{ij}(\eta).$$

This implies that for all i, j, we have

$$f_{ij}(\gamma(\eta)) = f_{ij}(\varepsilon^{\chi(\gamma)} - 1) = (1 + \varepsilon^{i+pj} + \varepsilon^{2(i+pj)} + \cdots + \varepsilon^{(\chi(\gamma)-1)(i+pj)}) \cdot f_{ij}(\eta).$$

Put $l = \chi(\gamma)$ and $m = i + pj$ (notice $m \neq 0$ as $p \nmid i$), the condition translates into

$$\gamma(f_{ij}(\eta)) = \frac{\varepsilon^{lm} - 1}{\varepsilon^m - 1} f_{ij}(\eta),$$

i.e.

$$\gamma\Big(\frac{f_{ij}(\eta)}{\varepsilon^m - 1}\Big) = \frac{f_{ij}(\eta)}{\varepsilon^m - 1}.$$

As $\frac{f_{ij}(\eta)}{\varepsilon^m - 1} \in k(\!(\eta^{1/p^\infty})\!)$, we have $\frac{f_{ij}(\eta)}{\varepsilon^m - 1} \in k$ by lemma 3.3.19: there exist $b_{i,j} \in k$ such that

$$x = \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} b_{ij} u^{i+pj} (\varepsilon^{i+pj} - 1).$$

By equation (3.2), an inverse image of $x = \sum\limits_{i=1}^{p-1} \sum\limits_{j=m_i}^{\infty} b_{ij} u^{i+pj} (\varepsilon^{i+pj} - 1)$ is

$$(\tau - 1)^{-1}(x) := \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} b_{ij} u^{i+pj}.$$

□

Corollary 3.3.27. *An element $x \in \mathcal{O}^{\psi=0}_{\mathcal{E}_{u,\tau,0}}$ can be written in the form*

$$x = \sum_{i=1}^{p-1} \sum_{j \in \mathbf{Z}} c_{ij} u^{i+pj} ([\varepsilon]^{i+pj} - 1)$$

with $c_{ij} \in \mathsf{W}(k)$ such that $\lim\limits_{j \to -\infty} c_{ij} = 0$ for all $i \in \{1, \ldots, p-1\}$.

Proof. By lemma 3.3.26, elements of $F^{\psi=0}_{u,\tau,0}$ can be written in the form

$$x = \sum_{i=1}^{p-1} \sum_{j=m_i}^{\infty} b_{ij} u^{i+pj} (\varepsilon^{i+pj} - 1) \text{ with } b_{ij} \in k.$$

We then conclude by dévissage. □

Corollary 3.3.28. *The map $\tau - 1 \colon F^{\psi=0}_0 \to F^{\psi=0}_{u,\tau,0}$ is bijective.*

Proof. This follows from corollary 3.3.23 and lemma 3.3.26. □

We have proved the bijection for trivial representations, and we now prove the general case.

The general case

Recall that $D = \mathcal{D}(T)$ is an étale (φ, τ)-module; the natural map $\varphi^* \colon F_0 \otimes_{\varphi, F_0} D \to D$ is an isomorphism. Let $(e_1, \ldots, e_d)$ be a basis of D over F_0, then $(\varphi(e_1), \ldots, \varphi(e_d))$ is again a basis, *i.e.* $D = \oplus_{i=1}^d F_0 \varphi(e_i)$.

Lemma 3.3.29. If $x = \sum\limits_{i=1}^{d} \lambda_i \varphi(e_i) \in D$, then $\psi(x) = \sum\limits_{i=1}^{d} \psi(\lambda_i) e_i$.

Proof. We have $D = (F_0^{\mathrm{sep}} \otimes_{\mathbf{F}_p} T)^{\mathscr{G}_{K_\pi}}$. Let $(v_1, \dots, v_d)$ be a basis of T over $\mathbf{F}_p$: we can write $e_i = \sum_{j=1}^{d} \alpha_j \otimes v_j$, so that $\varphi(e_i) = \sum_{j=1}^{d} \alpha_j^p \otimes v_j$, hence $\psi(\lambda_i \varphi(e_i)) = \sum_{j=1}^{d} \psi(\lambda_i)\alpha_j \otimes v_j = \psi(\lambda_i)e_i$ for all $i \in \{1, \dots, d\}$. □

Corollary 3.3.30. We have $\sum_{i=1}^{d} \lambda_i \varphi(e_i) \in D^{\psi=0}$ if and only if $\lambda_i \in F_0^{\psi=0}$ for all $i \in \{1, \dots, d\}$.

Proof. This follows from lemma 3.3.29. □

Corollary 3.3.31. If $\mu_1, \dots, \mu_d \in F_{u,\tau}$, we have $\sum_{i=1}^{d} \mu_i \varphi(e_i) \in D_{u,\tau}^{\psi=0}$ if and only if $\mu_i \in F_{u,\tau}^{\psi=0}$ for all $i \in \{1, \dots, d\}$.

Proof. The proof is similar to that of lemma 3.3.29. □

Lemma 3.3.32. Let $n \in \mathbf{Z}$ and $f = \sum_{j=n}^{\infty} \lambda_j u^j \in F_{u,\tau}$, where $\lambda_j \in k(\!(\eta^{1/p^\infty})\!) := \bigcup_{n=0}^{\infty} k(\!(\eta^{1/p^n})\!)$. Then $\psi(f) = 0$ if and only if $p \mid j \Rightarrow \lambda_j = 0$.

Proof. If $\lambda \in k(\!(\eta^{1/p^\infty})\!)$ and $i \in \mathbf{Z}$, we have

$$\psi(\lambda u^i) = \psi\big(\varphi(\varphi^{-1}(\lambda))u^i\big) = \varphi^{-1}\psi(u^i),$$

and

$$\psi(u^i) = \begin{cases} u^{i/p}, & \text{if } p|i \\ 0, & \text{else.} \end{cases}$$

We have thus

$$\psi(f) = \sum_{j \geq n,\, p|j} \varphi^{-1}(\lambda_j) u^{j/p},$$

so that $\psi(f) = 0$ if and only if $\varphi^{-1}(\lambda_j) = 0$, *i.e.* $\lambda_j = 0$ for all j such that $p|j$. □

Notation 3.3.33. We have $k[\![u, \eta]\!] \subset k(\!(\eta)\!)[\![u]\!]$, so that there is an inclusion $k(\!(u, \eta)\!) \subset k(\!(\eta)\!)(\!(u)\!)$, hence an inclusion $F_{u,\tau} = k(\!(u, \eta^{1/p^\infty})\!) \subset \bigcup_{n=0}^{\infty} k(\!(\eta)\!)(\!(u)\!)[\eta^{1/p^n}]$ (*cf* notation 3.1.1). Put

$$\mathscr{D} = \bigoplus_{i=1}^{d} k[\![u]\!]\varphi(e_i)$$

and

$$\mathscr{D}_{u,\tau} = \bigoplus_{i=1}^{d} k(\!(\eta)\!)[\![u]\!]\varphi(e_i).$$

By construction, we have $D_{u,\tau} \subset \bigcup_{n=0}^{\infty} \mathscr{D}_{u,\tau}\big[\eta^{1/p^n}, \frac{1}{u}\big]$ (*cf* definition 3.1.11) and $\mathscr{D}_{u,\tau}$ is u-adically separated and complete.

Lemma 3.3.34. *Assume $n, k \in \mathbf{N}$ are such that $\tau_D^{p^k}(\varphi(e_i)) \in \mathscr{D}_{u,\tau}[\eta^{1/p^n}]$. Then for any $m \in \mathbf{N}$, $y \in \frac{1}{u^m}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$ implies $\tau_D^{p^k}(y) \in \frac{1}{u^m}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$.*

Proof. Write $y = \sum_{i=1}^{d} \lambda_i \varphi(e_i)$ with $\lambda_i \in \frac{1}{u^m} k(\!(\eta)\!)[\![u]\!]$. Then $\tau_D^{p^k}(y) = \sum_{i=1}^{d} \tau^{p^k}(\lambda_i)\tau_D^{p^k}(\varphi(e_i))$. By assumption we know that $\tau_D^{p^k}(\varphi(e_i)) \in \mathscr{D}_{u,\tau}[\eta^{1/p^n}]$: it remains to control $\tau^{p^k}(\lambda_i)$. Recall that $\tau(u) = \varepsilon u = u + \eta u$. Write $\lambda_i = \frac{1}{u^m}\mu_i$ with $\mu_i \in k(\!(\eta)\!)[\![u]\!]$: we have $\tau^{p^k}(\lambda_i) = \frac{1}{u^{m\varepsilon^{p^k m}}}\tau^{p^k}(\mu_i) \in \frac{1}{u^m} k(\!(\eta)\!)[\![u]\!]$ as $\varepsilon = 1 + \eta \in k[\![\eta]\!]^\times$. □

Proof of theorem 3.3.11 for $\mathbf{F}_p$-representations: Let $y \in D^{\psi=0}_{u,\tau,0}$: there exist $n, m \in \mathbf{N}$ such that $y \in \frac{1}{u^m}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$. By continuity, there exists $r \in \mathbf{N}$ such that $\tau_D^{p^r}(e_i) \equiv e_i \mod u^m\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$ (making n larger if necessary), whence $\tau_D^{p^r}(\varphi(e_i)) \equiv \varphi(e_i) \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$ (recall that τ_D and φ commute). Put [1]

$$z = \tfrac{\tau_D^{p^r}-1}{\tau_D-1}(y) = (1+\tau_D+\cdots+\tau_D^{p^r-1})(y) \in D^{\psi=0}_{u,\tau^{p^r},0}$$

We have $z \in \frac{1}{u^m}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]^{\psi=0}$ by lemma 3.3.34 and the fact that ψ commutes with τ_D. By lemma 3.3.32, we can write $z = \sum_{i=1}^{d}\sum_{\substack{j\geq -m\\ p\nmid j}} f_{i,j}(\eta)u^j\varphi(e_i)$ with $f_{i,j}(\eta) \in k(\!(\eta^{1/p^n})\!)$. An argument similar to the proof of lemma 3.3.26 shows that for all i, j, there exists $c_{i,j} \in k$ such that $f_{i,j}(\eta) = c_{i,j}(\varepsilon^{jp^r}-1) \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$.

Indeed $z \in D^{\psi=0}_{u,\tau^{p^r},0}$ implies that

$$(\gamma\otimes 1)z = (1+\tau_D^{p^r}+\tau_D^{2p^r}+\cdots+\tau_D^{(\chi(\gamma)-1)p^r})(z).$$

The left hand side is

$$(\gamma\otimes 1)z = (\gamma\otimes 1)\big(\sum_{i=1}^{d}\sum_{\substack{j\geq -m\\ p\nmid j}} f_{i,j}(\eta)u^j\varphi(e_i)\big) = \sum_{i=1}^{d}\sum_{\substack{j\geq -m\\ p\nmid j}} f_{i,j}(\eta^{\chi(\gamma)})u^j\varphi(e_i)$$

as $\varphi(e_i)$ are all fixed by $\mathscr{G}_{K_\pi}$, hence in particular fixed by γ. For the right hand side, we have

$$\begin{aligned}
&(1+\tau_D^{p^r}+\tau_D^{2p^r}+\cdots+\tau_D^{(\chi(\gamma)-1)p^r})(z)\\
\equiv&\sum_{i=1}^{d}\sum_{\substack{j\geq -m\\ p\nmid j}} f_{i,j}(\eta)u^j(1+\varepsilon^{jp^r}+\varepsilon^{2jp^r}+\cdots+\varepsilon^{(\chi(\gamma)-1)jp^r})\varphi(e_i) \quad \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]\\
=&\sum_{i=1}^{d}\sum_{\substack{j\geq -m\\ p\nmid j}} f_{i,j}(\eta)u^j\big(\tfrac{\varepsilon^{\chi(\gamma)jp^r}-1}{\varepsilon^{jp^r}-1}\big)\varphi(e_i)
\end{aligned}$$

as $\tau_D^{p^r}(\varphi(e_i)) \equiv \varphi(e_i) \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$. Hence for all $i \in \{1,\dots,d\}$ and $j < pm$, there exists $c_{i,j} \in k$ such that $f_{i,j}(\eta) \equiv c_{i,j}(\varepsilon^{jp^r}-1) \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}]$ (*cf* lemma 3.3.26). Hence

$$z \equiv \sum_{i=1}^{d}\sum_{\substack{p\nmid j\\ -m\leq j\leq pm}} c_{i,j}(\varepsilon^{jp^r}-1)u^j\varphi(e_i) \quad \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}].$$

Put

$$x_0 = \sum_{i=1}^{d}\Big(\sum_{\substack{p\nmid j\\ -m\leq j\leq pm}} c_{i,j}u^j\Big)\varphi(e_i) \in \tfrac{1}{u^m}\mathscr{D}.$$

For all $i \in \{1,\dots,d\}$, we can write $\tau_D^{p^r}(\varphi(e_i)) = \varphi(e_i) + u^{pm}g_i$ with $g_i \in \mathscr{D}_{u,\tau}[\eta^{1/p^n}]$: we have

$$(\tau_D^{p^r}-1)(x_0) = \sum_{i=1}^{d}\sum_{\substack{p\nmid j\\ -m\leq j\leq pm}} c_{i,j}\Big(\varepsilon^{jp^r}u^j(\varphi(e_i)+u^{pm}g_i) - u^j\varphi(e_i)\Big) \equiv z - z_1 \quad \mod u^{pm}\mathscr{D}_{u,\tau}[\eta^{1/p^n}],$$

[1] *cf* corollary 3.4.6 for more details.

where

$$z_1 = \sum_{i=1}^{d} \sum_{\substack{j \geq pm \\ p \nmid j}} f_{i,j}(\eta) u^j \varphi(e_i) - \sum_{i=1}^{d} \sum_{\substack{p \nmid j \\ -m \leq j \leq pm}} c_{i,j} \varepsilon^{jp^r} u^{j+pm} g_i \in u^{(p-1)m} \mathscr{D}_{u,\tau}[\eta^{1/p^n}].$$

By construction, we have $\psi(x_0) = 0$, which implies $\psi\big((\tau_D^{p^r} - 1)(x_0)\big) = 0$: as $\psi(z) = 0$, we have $\psi(z_1) = 0$ as well. Note also that $\tau_D^{p^r} - 1$ maps D into $D_{u,\tau^{p^r},0}$: as $z \in D_{u,\tau^{p^r},0}$, we also have $z_1 \in D_{u,\tau^{p^r},0}$. This shows that we can carry on the preceding construction, and build sequences $(z_\ell)_{\ell \in \mathbf{N}}$ in $D_{u,\tau^{p^r},0}$ and $(x_\ell)_{\ell \in \mathbf{N}}$ in D such that $z_0 = z$, $z_\ell \in u^{(\ell p-1)m} \mathscr{D}_{u,\tau}[\eta^{1/p^n}]$, $x_\ell \in u^{(\ell p-1)m} \mathscr{D}[\eta^{1/p^n}]$ and $(\tau_D^{p^r} - 1)(x_\ell) = z_\ell - z_{\ell+1}$ for all $\ell \in \mathbf{N}$. The series $x = \sum_{\ell=0}^{\infty} x_\ell$ converges in $\frac{1}{u^m}\mathscr{D}$, and summing all the equalities gives $(\tau_D^{p^r} - 1)(x) = z$.

This shows that $\frac{\tau_D^{p^r}-1}{\tau_D - 1}\big((\tau_D - 1)(x)\big) = (\tau_D^{p^r} - 1)(x) = \frac{\tau_D^{p^r}-1}{\tau_D-1}(y)$: as $\frac{\tau_D^{p^r}-1}{\tau_D-1} \colon D_{u,\tau,0}^{\psi=0} \to D_{u,\tau^{p^r},0}^{\psi=0}$ is injective by proposition 3.3.17, we get $y = (\tau_D - 1)(x)$, showing that y belongs to the image of $\tau_D - 1$. As this holds for all $y \in D_{u,\tau,0}^{\psi=0}$, this proves the surjectivity. Together with corollary 3.3.23, we finish the proof of theorem 3.3.11 for $\mathbf{F}_p$-representations. □

Corollary 3.3.35. *For all $r \in \mathbf{N}_{\geq 0}$, the maps $\tau_D^{p^r} - 1 \colon D^{\psi=0} \to D_{u,\tau^{p^r},0}^{\psi=0}$ and $\frac{\tau_D^{p^r}-1}{\tau_D-1} \colon D_{u,\tau,0}^{\psi=0} \to D_{u,\tau^{p^r},0}^{\psi=0}$ are bijective.*

Proof. Replacing T by its restriction to $\mathscr{G}_{K_r}$ and replacing its (φ,τ)-module by the corresponding (φ,τ^{p^r})-module, *i.e.* τ_D by $\tau_D^{p^r}$, the $\mathbf{F}_p$-case of theorem 3.3.11 thus implies that the map $\tau_D^{p^r} - 1 \colon D^{\psi=0} \to D_{u,\tau^{p^r},0}^{\psi=0}$ is bijective. The statement about $\frac{\tau_D^{p^r}-1}{\tau_D-1}$ follows. □

3.3.36 Proof of theorem 3.3.11: the quasi-isomophism

Lemma 3.3.37. *Let $0 \to T' \to T \to T'' \to 0$ be a short exact sequence in $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we then have a short exact sequence*

$$0 \to \mathcal{D}(T')^{\psi=0} \to \mathcal{D}(T)^{\psi=0} \to \mathcal{D}(T'')^{\psi=0} \to 0.$$

Proof. We know by lemma 1.2.11 that $\mathcal{D}(-)$ is an exact functor and hence we have the short exact sequence $0 \to \mathcal{D}(T') \to \mathcal{D}(T) \to \mathcal{D}(T'') \to 0$. Now we consider the following diagram of complexes:

$$\begin{array}{ccccccccc}
 & & \mathcal{D}(T')^{\psi=0} & \longrightarrow & \mathcal{D}(T)^{\psi=0} & \longrightarrow & \mathcal{D}(T'')^{\psi=0} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}(T') & \longrightarrow & \mathcal{D}(T) & \longrightarrow & \mathcal{D}(T'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & \mathcal{D}(T') & \longrightarrow & \mathcal{D}(T) & \longrightarrow & \mathcal{D}(T'') & \longrightarrow & 0.
\end{array}$$

Since ψ is surjective, by snake lemma we get the following short exact sequence:

$$0 \to \mathcal{D}(T')^{\psi=0} \to \mathcal{D}(T)^{\psi=0} \to \mathcal{D}(T'')^{\psi=0} \to 0.$$

□

Proposition 3.3.38. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we then have a bijection $\mathcal{D}(T)^{\psi=0} \xrightarrow{\tau_D - 1} \mathcal{D}(T)_{u,\tau,0}^{\psi=0}$.*

Proof. Notice that $\mathcal{D}(T/(p^n)) = \mathcal{D}(T)/(p^n)$ and T, $\mathcal{D}(T)$ are both p-adically complete. Hence it suffices to prove the cases when T is killed by p^n with $n \in \mathbf{N}$. We use induction on n, the case $n = 1$ being corollary 3.3.35 with $r = 1$. Suppose T is killed by p^n. Put $T' = p^{n-1}T$, $T'' = T/T'$ and consider the exact sequence in $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$

$$0 \to T' \to T \to T'' \to 0.$$

We then have the following commutative diagram

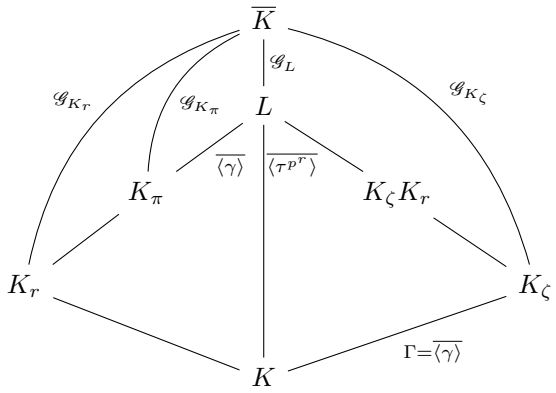
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T')^{\psi=0} & \longrightarrow & \mathcal{D}(T)^{\psi=0} & \longrightarrow & \mathcal{D}(T'')^{\psi=0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_D - 1} & & \downarrow{\scriptstyle \tau_D - 1} & & \downarrow{\scriptstyle \tau_D - 1} & & \\ 0 & \longrightarrow & \mathcal{D}(T')^{\psi=0}_{u,\tau,0} & \longrightarrow & \mathcal{D}(T)^{\psi=0}_{u,\tau,0} & \longrightarrow & \mathcal{D}(T'')^{\psi=0}_{u,\tau,0} & & \end{array}$$

The first line is exact by lemma 3.3.37 and the second from the fact $\mathcal{D}(-)_{u,\tau,0}$ is left exact and then we apply the functor $(-)^{\psi=0}$ which is also left exact. As the first and third vertical maps are isomorphisms by induction hypothesis, so is that in the middle. We then finish the proof by passing to the limit. □

Remark 3.3.39. The theorem 3.3.11 follows from proposition 3.3.38.

3.4 The (φ, τ^{p^r})-modules

Notation 3.4.1. We put $K_r = K(\pi_r)$ for $r \in \mathbf{N}$ and we have the following diagram:



Definition 3.4.2. Let $r \in \mathbf{N}$ and then a (φ, τ^{p^r})-*module over* (F_0, F_τ) is the data:

(1) an étale φ-module D over F_0;

(2) a τ^{p^r}-semi-linear map $\tau_D^{p^r}$ on $D_\tau := F_\tau \otimes_{F_0} D$ which commutes with $\varphi_{F_\tau} \otimes \varphi_D$ (where φ_{F_τ} is the Frobenius map on F_τ and φ_D the Frobenius map on D) and such that

$$(\forall x \in M)\,(g \otimes 1) \circ \tau_D^{p^r}(x) = (\tau_D^{p^r})^{\chi(g)}(x),$$

for all $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g) \in \mathbf{N}$.

We denote $\mathbf{Mod}_{F_0,F_\tau}(\varphi, \tau^{p^r})$ the corresponding category.

By [19, Remark 1.15], we have an equivalence of categories between $\mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_{K_r})$ and the category of (φ, τ^{p^r})-modules over (F_0, F_τ).

Notation 3.4.3. Let $(D, D_\tau) \in \mathbf{Mod}_{F_0,F_\tau}(\varphi, \tau^{p^r})$. We put

$$D_{\tau^{p^r},0} := \big\{x \in D_\tau;\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow (g\otimes 1)(x) = x + \tau_D^{p^r}(x) + \tau_D^{2p^r}(x) + \cdots + \tau_D^{p^r(\chi(g)-1)}(x)\big\}.$$

By similar arguments as that of lemma 1.1.10, we have

$$D_{\tau^{p^r},0} = \big\{x \in D_\tau;\ (\gamma\otimes 1)x = (1 + \tau_D^{p^r} + \tau_D^{2p^r} + \cdots + \tau_D^{p^r(\chi(\gamma)-1)})(x)\big\}.$$

Definition 3.4.4. Let $(D, D_\tau) \in \mathbf{Mod}_{F_0,F_\tau}(\varphi, \tau^{p^r})$. We define a complex $\mathcal{C}_{\varphi,\tau^{p^r}}(D)$ as follows:

$$0 \longrightarrow D \longrightarrow D \oplus D_{\tau^{p^r},0} \longrightarrow D_{\tau^{p^r},0} \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x), (\tau^{p^r}-1)(x))$$

$$(y,z) \longmapsto (\tau^{p^r}-1)(y) - (\varphi-1)(z).$$

If $T \in \mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}_{\varphi,\tau^{p^r}}(\mathcal{D}(T))$, which will also be simply denoted $\mathcal{C}_{\varphi,\tau^{p^r}}(T)$.

Proposition 3.4.5. *The complex $\mathcal{C}_{\varphi,\tau^{p^r}}(T)$ computes the continuous Galois cohomology $\mathsf{H}^i(\mathscr{G}_{K_r}, T)$ for $i \in \mathbf{N}$.*

Proof. This follows from theorem 1.1.13 by replacing K by K_r (*cf* [19, Remarque 1.15]). □

Corollary 3.4.6. *For any $r \in \mathbf{N}$, we have the following morphism of complexes:*

$$\begin{array}{lccccccc}
\mathcal{C}_{\varphi,\tau^{p^r}}: & 0 \longrightarrow & D & \longrightarrow & D \oplus D_{\tau^{p^r},0} & \longrightarrow & D_{\tau^{p^r},0} & \longrightarrow 0 \\
& & \| & & \downarrow{\scriptstyle \mathrm{id}\oplus\frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}} & & \downarrow{\scriptstyle \frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}} & \\
\mathcal{C}_{\varphi,\tau^{p^{r+1}}}: & 0 \longrightarrow & D & \longrightarrow & D \oplus D_{\tau^{p^{r+1}},0} & \longrightarrow & D_{\tau^{p^{r+1}},0} & \longrightarrow 0.
\end{array}$$

Proof. This follows from direct computations. Recall that for any $g \in \mathscr{G}_K$ and $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, the action $g\otimes 1$ over $\mathcal{D}(T)_\tau$ is induced by that of $g\otimes g$ over $\mathsf{W}(C^\flat)\otimes_{\mathbf{Z}_p} T$. For any $x \in D_{\tau^{p^r},0}$: we have $(\gamma\otimes 1)x = \big(1+\tau_D^{p^r}+\tau_D^{2p^r}+\cdots+\tau_D^{p^r(\chi(\gamma)-1)}\big)(x)$. Now we verify that $y := \frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}(x)$ is in $D_{\tau^{p^{r+1}},0}$. Indeed, by the relation $\gamma\tau = \tau^{\chi(\gamma)}\gamma$ we have

$$\begin{aligned}
\gamma(y) &= \gamma\Big(\frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}\Big)(x) \\
&= \frac{\tau_D^{\chi(\gamma)p^{r+1}}-1}{\tau_D^{\chi(\gamma)p^r}-1}(\gamma(x)) \\
&= \frac{\tau_D^{\chi(\gamma)p^{r+1}}-1}{\tau_D^{\chi(\gamma)p^r}-1}\big(\big(1+\tau_D^{p^r}+\tau_D^{2p^r}+\cdots+\tau_D^{p^r(\chi(\gamma)-1)}\big)(x)\big) \\
&= \frac{\tau_D^{\chi(\gamma)p^{r+1}}-1}{\tau_D^{\chi(\gamma)p^r}-1}\cdot\frac{\tau_D^{\chi(\gamma)p^r}-1}{\tau_D^{p^r}-1}(x) \\
&= \frac{\tau_D^{\chi(\gamma)p^{r+1}}-1}{\tau_D^{p^r}-1}(x).
\end{aligned}$$

On the other hand

$$\begin{aligned}&\big(1+\tau_D^{p^{r+1}}+\tau_D^{2p^{r+1}}+\cdots+\tau_D^{p^{r+1}(\chi(\gamma)-1)}\big)(y)\\ =&\big(1+\tau_D^{p^{r+1}}+\tau_D^{2p^{r+1}}+\cdots+\tau_D^{p^{r+1}(\chi(\gamma)-1)}\big)\Big(\frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}\Big)(x)\\ =&\frac{\tau_D^{\chi(\gamma)p^{r+1}}-1}{\tau_D^{p^{r+1}}-1}\cdot\frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}(x)\\ =&\frac{\tau_D^{\chi(\gamma)p^{r+1}}-1}{\tau_D^{p^r}-1}(x).\end{aligned}$$

This shows that

$$\gamma(y)=\big(1+\tau_D^{p^{r+1}}+\tau_D^{2p^{r+1}}+\cdots+\tau_D^{p^{r+1}(\chi(\gamma)-1)}\big)(y),$$

and we conclude that $y\in D_{\tau^{p^{r+1}},0}$. □

Lemma 3.4.7. *Let $n,m\in\mathbf{N}$ with $n\le m$. There is a natural map $\frac{\tau_D^{p^m}-1}{\tau_D^{p^n}-1}\colon D_{\tau^{p^n},0}\to D_{\tau^{p^m},0}$.*

Proof. By direct computations as in the proof of corollary 3.4.6. □

The (φ,τ^{p^r})-modules over partially unperfected coefficients

Similarly, we have results for (φ,τ^{p^r})-modules over partially unperfected coefficients.

Definition 3.4.8. Let $r\in\mathbf{N}$ and then a (φ,τ^{p^r})-*module over* $(F_0,F_{u,\tau})$ is the data:

(1) an étale φ-module D over F_0;

(2) a τ^{p^r}-semilinear map $\tau_D^{p^r}$ on $D_{u,\tau}:=F_{u,\tau}\otimes_{F_0}D$ which commutes with $\varphi_{F_{u,\tau}}\otimes\varphi_D$ (where $\varphi_{F_{u,\tau}}$ is the Frobenius map on $F_{u,\tau}$ and φ_D the Frobenius map on D) and such that

$$(\forall x\in M)\quad(g\otimes 1)\circ\tau_D^{p^r}(x)=(\tau_D^{p^r})^{\chi(g)}(x),$$

for all $g\in\mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g)\in\mathbf{N}$.

We denote $\mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau^{p^r})$ the corresponding category.

Remark 3.4.9. We have an equivalence of categories between $\mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_{K_r})$ and $\mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau^{p^r})$ (*cf* [19, Remark 1.15]).

Notation 3.4.10. Let $(D,D_{u,\tau})\in\mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau^{p^r})$. We put

$$D_{u,\tau^{p^r},0}:=\big\{x\in D_{u,\tau};\ (\forall g\in\mathscr{G}_{K_\pi})\ \chi(g)\in\mathbf{Z}_{>0}\Rightarrow(g\otimes 1)(x)=x+\tau_D^{p^r}(x)+\tau_D^{2p^r}(x)+\cdots+\tau_D^{p^r(\chi(g)-1)}(x)\big\}.$$

By similar arguments as that of lemma 1.1.10, we see that

$$D_{u,\tau^{p^r},0}=\big\{x\in D_{u,\tau};\ (\gamma\otimes 1)x=(1+\tau_D^{p^r}+\tau_D^{2p^r}+\cdots+\tau_D^{p^r(\chi(\gamma)-1)})(x)\big\}.$$

Definition 3.4.11. Let $(D,D_{u,\tau})\in\mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau^{p^r})$. We define a complex $\mathcal{C}^u_{\varphi,\tau^{p^r}}(D)$ as follows:

$$0\longrightarrow D\longrightarrow D\oplus D_{u,\tau^{p^r},0}\longrightarrow D_{u,\tau^{p^r},0}\longrightarrow 0$$

$$x\longmapsto((\varphi-1)(x),(\tau^{p^r}-1)(x))$$

$$(y,z)\longmapsto(\tau^{p^r}-1)(y)-(\varphi-1)(z).$$

If $T \in \mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}^u_{\varphi,\tau^{p^r}}(\mathcal{D}(T))$, which will also be simply denoted $\mathcal{C}^u_{\varphi,\tau^{p^r}}(T)$.

Proposition 3.4.12. *The complex $\mathcal{C}^u_{\varphi,\tau^{p^r}}(T)$ computes the continuous Galois cohomology $\mathsf{H}^i(\mathscr{G}_{K_r}, T)$ for $i \in \mathbf{N}$.*

Proof. This follows from theorem 3.2.4 replacing K by K_r. □

Lemma 3.4.13. *Let $n, m \in \mathbf{N}$ with $n < m$. There is a natural map $D_{u,\tau^{p^n},0} \xrightarrow{\frac{\tau_D^{p^m}-1}{\tau_D^{p^n}-1}} D_{u,\tau^{p^m},0}$.*

Proof. By direct computations similar as in corollary 3.4.6. □

Corollary 3.4.14. *There is a morphism between complexes:*

$$\begin{array}{lccccccc}
\mathcal{C}^u_{\varphi,\tau^{p^r}}: & 0 \longrightarrow & D & \longrightarrow & D \oplus D_{u,\tau^{p^r},0} & \longrightarrow & D_{u,\tau^{p^r},0} & \longrightarrow 0 \\
& & \| & & \downarrow{\scriptstyle \mathrm{id} \oplus \frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}} & & \downarrow{\scriptstyle \frac{\tau_D^{p^{r+1}}-1}{\tau_D^{p^r}-1}} & \\
\mathcal{C}^u_{\varphi,\tau^{p^{r+1}}}: & 0 \longrightarrow & D & \longrightarrow & D \oplus D_{u,\tau^{p^{r+1}},0} & \longrightarrow & D_{u,\tau^{p^{r+1}},0} & \longrightarrow 0.
\end{array}$$

Proof. This follows from direct computations similar as in corollary 3.4.6. □

Remark 3.4.15. In subsection 3.4, (φ, τ^{p^r})-modules with $r \in \mathbf{N}$ are only defined for $\mathbf{Rep}_{\mathbf{F}_p}(\mathscr{G}_K)$. One can easily define the categories $\mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau}}(\varphi,\tau^{p^r})$, $\mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau^{p^r})$, $\mathbf{Mod}_{\mathcal{E},\mathcal{E}_\tau}(\varphi,\tau^{p^r})$, $\mathbf{Mod}_{\mathcal{E},\mathcal{E}_{u,\tau}}(\varphi,\tau^{p^r})$, and generalize the related notions to $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$. The results given for $\mathbf{Mod}_{F_0,F_\tau}(\varphi,\tau^{p^r})$ and $\mathbf{Mod}_{F_0,F_{u,\tau}}(\varphi,\tau^{p^r})$ hold similarly for the other categories.

Chapter 4

Complexes over overconvergent rings

In this chapter, we introduce (φ,τ)-modules over overconvergent rings $(\mathcal{E}^\dagger,\mathcal{E}^\dagger_\tau)$ and $(\mathcal{E}^\dagger,\mathcal{E}^\dagger_{u,\tau})$. Then we will define complexes $\mathcal{C}^u_{\varphi,\tau}(D^\dagger)$ and $\mathcal{C}^u_{\psi,\tau}(D^\dagger)$, which embed respectively into the complexes $\mathcal{C}^u_{\varphi,\tau}(D)$ and $\mathcal{C}^u_{\psi,\tau}(D)$ (defined in chapter 3). We will show that they are quasi-isomorphic and calculate the correct H^0 and H^1.

4.1 Locally analytic vectors

In this section, we will use results of Poyeton (*cf* [46]), hence also some notations of *loc. cit.* However we made some modifications for the sake of consistency with our notations.

Since we concentrate on (φ,τ)-modules, we remove subscripts τ in the notations of Poyeton when it is used to distinguish (φ,τ)-modules from (φ,Γ)-modules. We also replace subscripts K in the notations of Poyeton with K_π when it corresponds to invariants under $\mathscr{G}_{K_\pi}$ (this is also consistent with our subscripts K_ζ, *cf* notation 4.1.1 for details).

Notation 4.1.1. We have the following dictionary between our notations used in the first three chapters and the notations from Poyeton (*cf* [46]).

$\widetilde{\mathbf{E}}^+=\mathcal{O}_{C^\flat},\ \widetilde{\mathbf{E}}=C^\flat,\ \widetilde{\mathbf{E}}_u=C^\flat_{u-\mathsf{np}},\ \mathbf{E}_{K_\pi}=F_0,\ \mathbf{E}=F_0^{\mathrm{sep}},\ \widetilde{\mathbf{E}}_L=F_\tau,\ \widetilde{\mathbf{E}}_{u,L}=F_{u,\tau}$

$\widetilde{\mathbf{A}}^+=\mathsf{W}(\mathcal{O}_{C^\flat}),\ \widetilde{\mathbf{A}}=\mathsf{W}(C^\flat),\ \widetilde{\mathbf{A}}_u=\mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}},\ \mathbf{A}^+_{K_\pi}=\mathsf{W}(k)[\![u]\!],\ \mathbf{A}_{K_\pi}=\mathcal{O}_{\mathcal{E}},\ \mathbf{A}=\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}},\ \widetilde{\mathbf{A}}_L=\mathcal{O}_{\mathcal{E}_\tau},\ \widetilde{\mathbf{A}}_{u,L}=\mathcal{O}_{\mathcal{E}_{u,\tau}}$

$\widetilde{\mathbf{B}}^+=\mathsf{W}(\mathcal{O}_{C^\flat})[1/p],\ \widetilde{\mathbf{B}}=\mathsf{W}(C^\flat)[1/p],\ \widetilde{\mathbf{B}}_u=\widehat{\mathcal{F}^{\mathrm{ur}}_u},\ \mathbf{B}^+_{K_\pi}=\mathbf{A}^+_{K_\pi}[1/p],\ \mathbf{B}_{K_\pi}=\mathcal{E},\ \mathbf{B}=\widehat{\mathcal{E}^{\mathrm{ur}}},\ \widetilde{\mathbf{B}}_L=\mathcal{E}_\tau,\ \widetilde{\mathbf{B}}_{u,L}=\mathcal{E}_{u,\tau}$

Notice that we can read information from the notation itself: if A is an algebra endowed with an action of $\mathscr{G}_L$ (resp. $\mathscr{G}_{K_\pi}$), we put $A_L=A^{\mathscr{G}_L}$ (resp. $A_{K_\pi}=A^{\mathscr{G}_{K_\pi}}$). If A is a perfect ring of characteristic p, A_u (or $A_{u-\mathsf{np}}$) is the partially unperfected subring of it, and we also put subscript u for a Cohen ring of A_u and also the fraction field of this Cohen ring (for example $\widetilde{\mathbf{E}}_u$, $\widetilde{\mathbf{A}}_u$ and $\widetilde{\mathbf{B}}_u$). Notice that the above rule works also for double-subscripts, for example $\widetilde{\mathbf{A}}_{u,L}$ is an unperfected version of $\widetilde{\mathbf{A}}_L$, while the latter is $\widetilde{\mathbf{A}}^{\mathscr{G}_L}$.

Definition 4.1.2. (*cf* [45, §1.1]) For $r>0$, we define the set of *overconvergent elements of* $\widetilde{\mathbf{B}}$ *of radius* r by

$$\widetilde{\mathbf{B}}^{\dagger,r}=\Big\{\sum_{n\gg-\infty}p^n[x_n]\in\widetilde{\mathbf{B}},\quad \lim_{n\to+\infty}v^\flat(x_n)+\frac{pr}{p-1}n=+\infty\Big\}$$

and we denote $\widetilde{\mathbf{B}}^\dagger=\bigcup_{r>0}\widetilde{\mathbf{B}}^{\dagger,r}\subset\widetilde{\mathbf{B}}$ the set of overconvergent elements (we have $r_1\leq r_2\Rightarrow\widetilde{\mathbf{B}}^{\dagger,r_1}\subset\widetilde{\mathbf{B}}^{\dagger,r_2}$). We put $\mathbf{B}^\dagger=\mathbf{B}\cap\widetilde{\mathbf{B}}^\dagger$, and similarly $\widetilde{\mathbf{A}}^\dagger=\widetilde{\mathbf{B}}^\dagger\cap\widetilde{\mathbf{A}}$ and $\mathbf{A}^\dagger=\widetilde{\mathbf{A}}^\dagger\cap\mathbf{A}$. We put $\mathcal{E}^{\mathrm{ur},\dagger}:=\mathbf{B}^\dagger$ and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur},\dagger}}:=\mathbf{A}^\dagger$.

Remark 4.1.3. The subrings $\widetilde{\mathbf{B}}^{\dagger,r}$ and $\widetilde{\mathbf{B}}^{\dagger}$ of $\widetilde{\mathbf{B}}$ are stable by $\mathscr{G}_K$, and $\varphi(\widetilde{\mathbf{B}}^{\dagger,r}) \subset \widetilde{\mathbf{B}}^{\dagger,pr}$ for all $r > 0$, so that $\widetilde{\mathbf{B}}^{\dagger}$ is stable by φ in $\widetilde{\mathbf{B}}$. Recall that $\widetilde{\mathbf{B}}^{\dagger}$ and $\mathbf{B}^{\dagger}$ are fields (*cf* [41, Proposition 3.2]).

We recall that (*cf* definition 1.1.4) for $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, the (φ,τ)-module associated to V over $(\mathbf{B}_{K_\pi}, \widetilde{\mathbf{B}}_L)$ (*i.e.* $(\mathcal{E}, \mathcal{E}_\tau)$) is the φ-module

$$\mathcal{D}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_{K_\pi}}$$

and a semi-linear τ-action over

$$\mathcal{D}(V)_\tau = (\widetilde{\mathbf{B}} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L}.$$

Definition 4.1.4. (*cf* [46, Définition 4.1.16]) For $r > 0$, we define

$$\mathcal{D}^{\dagger,r}(V)_\tau = (\widetilde{\mathbf{B}}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L}$$

and

$$\mathcal{D}^{\dagger,r}(V) = \mathcal{D}(V) \cap \mathcal{D}^{\dagger,r}(V)_\tau = (\mathbf{B}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_{K_\pi}},$$

where $\mathbf{B}^{\dagger,r} = \widetilde{\mathbf{B}}^{\dagger,r} \cap \mathbf{B}$. We put

$$\mathcal{D}^{\dagger}(V)_\tau = (\widetilde{\mathbf{B}}^{\dagger} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L} = \bigcup_{r>0} \mathcal{D}^{\dagger,r}(V)_\tau$$

and

$$\mathcal{D}^{\dagger}(V) = (\mathbf{B}^{\dagger} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_{K_\pi}} = \mathcal{D}(V) \cap \mathcal{D}^{\dagger}(V)_\tau = \bigcup_{r>0} \mathcal{D}^{\dagger,r}(V).$$

We say that a (φ,τ)-module D associated to V is *overconvergent* if there exists $r > 0$ such that we have

$$\mathcal{D}(V) = \mathbf{B}_{K_\pi} \otimes_{\mathbf{B}^{\dagger,r}_{K_\pi}} \mathcal{D}^{\dagger,r}(V)$$

and

$$\mathcal{D}(V)_\tau = \widetilde{\mathbf{B}}_L \otimes_{\widetilde{\mathbf{B}}^{\dagger,r}_L} \mathcal{D}^{\dagger,r}(V)_\tau$$

where $\mathbf{B}^{\dagger,r}_{K_\pi} = \mathbf{B}_{K_\pi} \cap \widetilde{\mathbf{B}}^{\dagger,r}$ and $\widetilde{\mathbf{B}}^{\dagger,r}_L = \widetilde{\mathbf{B}}_L \cap \widetilde{\mathbf{B}}^{\dagger,r}$.

When this holds, we have in particular

$$\mathcal{D}(V) = \mathbf{B}_{K_\pi} \otimes_{\mathbf{B}^{\dagger}_{K_\pi}} \mathcal{D}^{\dagger}(V)$$

and

$$\mathcal{D}(V)_\tau = \widetilde{\mathbf{B}}_L \otimes_{\widetilde{\mathbf{B}}^{\dagger}_L} \mathcal{D}^{\dagger}(V)_\tau,$$

where $\mathbf{B}^{\dagger}_{K_\pi} = \bigcup_{r>0} \mathbf{B}^{\dagger,r}_{K_\pi}$ and $\widetilde{\mathbf{B}}^{\dagger}_L = \bigcup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}_L$. We put $\mathcal{E}^{\dagger} := \mathbf{B}^{\dagger}_{K_\pi}$ and $\mathcal{E}^{\dagger}_\tau := \widetilde{\mathbf{B}}^{\dagger}_L$ to match to the notations used in the first three chapters (*cf* notation 4.1.1).

Remark 4.1.5. (1) As we mentioned, our notations are slightly different from those of Poyeton in [46] as we do not work with (φ,Γ)-modules: we denote by $\mathcal{D}(V)$ and $\mathcal{D}^{\dagger}(V)$ for what are denoted by $\mathcal{D}_\tau(V)$ and $\mathcal{D}^{\dagger}_\tau(V)$ in *loc. cit., while our* $\mathcal{D}(V)_\tau$ *comes from the pair* $(\mathcal{D}(V), \mathcal{D}(V)_\tau)$, *which is part of the data of a* (φ,τ)*-module.*

(2) We have $\mathcal{D}^{\dagger}(V)_\tau = \widetilde{\mathbf{B}}^{\dagger}_L \otimes \mathcal{D}^{\dagger}(V) \simeq (\widetilde{\mathbf{B}}^{\dagger} \otimes V)^{\mathscr{G}_L}$.

Theorem 4.1.6. *Let* V *be a* $\mathbf{Q}_p$*-representation of* $\mathscr{G}_K$. *Then the associated* (φ,τ)*-module over* $(\mathbf{B}_{K_\pi}, \widetilde{\mathbf{B}}_L)$ *is overconvergent.*

Proof. cf [46, Théorème 4.3.29]. □

Definition 4.1.7. A (φ,τ)-*module over* $(\mathbf{B}^\dagger_{K_\pi},\widetilde{\mathbf{B}}^\dagger_L)$ consists of

(i) an étale φ-module $D^\dagger$ over $\mathbf{B}^\dagger_{K_\pi}$;

(ii) a τ-semi-linear endomorphism τ_D on $D^\dagger_\tau := \widetilde{\mathbf{B}}^\dagger_L \otimes_{\mathbf{B}^\dagger_{K_\pi}} D^\dagger$ which commutes with $\varphi_{\widetilde{\mathbf{B}}^\dagger_L}\otimes\varphi_D$ (where $\varphi_{\widetilde{\mathbf{B}}^\dagger_L}$ is the Frobenius map on $\widetilde{\mathbf{B}}^\dagger_L$ and φ_D is the Frobenius map on D) and that satisfies

$$(\forall x\in D^\dagger)\ (g\otimes 1)\circ\tau_D(x)=\tau_D^{\chi(g)}(x),$$

for all $g\in\mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g)\in\mathbf{Z}_{>0}$. The corresponding category is denoted $\mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi},\widetilde{\mathbf{B}}^\dagger_L}(\varphi,\tau)$.

One defines similarly the notion of (φ,τ)-module over $(\mathbf{A}^\dagger_{K_\pi},\widetilde{\mathbf{A}}^\dagger_L) =: (\mathcal{O}_{\mathcal{E}^\dagger},\mathcal{O}_{\mathcal{E}^\dagger_\tau})$, and the corresponding category is denoted $\mathbf{Mod}_{\mathbf{A}^\dagger_{K_\pi},\widetilde{\mathbf{A}}^\dagger_L}(\varphi,\tau)$.

Theorem 4.1.8. *The functors*

$$\begin{aligned}\mathcal{D}^\dagger\colon \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K) &\simeq \mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi},\widetilde{\mathbf{B}}^\dagger_L}(\varphi,\tau)\\ V &\mapsto \mathcal{D}^\dagger(V)=(\mathbf{B}^\dagger\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_{K_\pi}}\\ \mathcal{V}(D^\dagger)=(\mathbf{B}^\dagger\otimes_{\mathbf{B}^\dagger_{K_\pi}}D^\dagger)^{\varphi=1} &\hookleftarrow D^\dagger\end{aligned}$$

(*with the natural* τ*-semi-linear endomorphism* τ_D *over* $\mathcal{D}^\dagger(V)_\tau=\widetilde{\mathbf{B}}^\dagger_L\otimes_{\mathbf{B}^\dagger_{K_\pi}}\mathcal{D}^\dagger(V)$) *establish quasi-inverse equivalences of categories, which are refinements of the equivalences of [19, Théorème 1.14].*

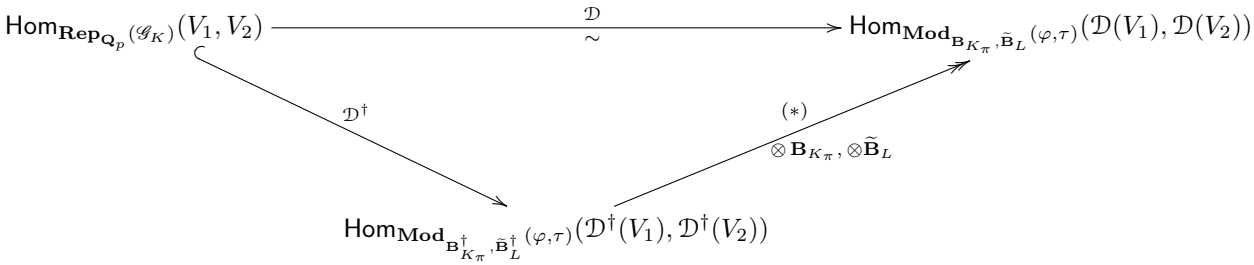
Proof. Recall that we have the category equivalence

$$\mathcal{D}\colon \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)\simeq\mathbf{Mod}_{\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_L}(\varphi,\tau).$$

For any $V_1,V_2\in\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have $\mathsf{Hom}_{\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)}(V_1,V_2)\simeq\mathsf{Hom}_{\mathbf{Mod}_{\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_L}(\varphi,\tau)}(\mathcal{D}(V_1),\mathcal{D}(V_2))$ by [19, Théorème 1.14]. By theorem 4.1.6 we have

$$\mathsf{Hom}_{\mathbf{Mod}_{\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_L}(\varphi,\tau)}(\mathcal{D}(V_1),\mathcal{D}(V_2))=\mathsf{Hom}_{\mathbf{Mod}_{\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_L}(\varphi,\tau)}(\mathbf{B}_{K_\pi}\otimes_{\mathbf{B}^\dagger_{K_\pi}}\mathcal{D}^\dagger(V_1),\mathbf{B}_{K_\pi}\otimes_{\mathbf{B}^\dagger_{K_\pi}}\mathcal{D}^\dagger(V_2)).$$

We prove this is isomorphic to $\mathsf{Hom}_{\mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi},\widetilde{\mathbf{B}}^\dagger_L}(\varphi,\tau)}(\mathcal{D}^\dagger(V_1),\mathcal{D}^\dagger(V_2))$, whence the fully-faithfullness. We have the commutative diagram



It is enough to show the map $(*)$ is injective, and hence an isomorphism. For any map $f\colon D_1^\dagger\to D_2^\dagger$ in $\mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi},\widetilde{\mathbf{B}}^\dagger_L}(\varphi,\tau)$, we have the commutative square:

$$\begin{array}{ccc} D_1^\dagger & \xrightarrow{f} & D_2{}^\dagger \\ \downarrow & & \downarrow \\ \mathbf{B}_{K_\pi} \otimes_{\mathbf{B}_{K_\pi}^\dagger} D_1^\dagger & \xrightarrow{1\otimes f} & \mathbf{B}_{K_\pi} \otimes_{\mathbf{B}_{K_\pi}^\dagger} D_2^\dagger \end{array}$$

which shows that $1 \otimes f = 0$ implies that $f = 0$.

Let $(D^\dagger, D_\tau^\dagger) \in \mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_L^\dagger}(\varphi, \tau)$. Tensoring with $\mathbf{B}_{K_\pi}$ and $\widetilde{\mathbf{B}}_L$ respectively, we have

$$(\mathbf{B}_{K_\pi} \otimes_{\mathbf{B}_{K_\pi}^\dagger} D^\dagger, \widetilde{\mathbf{B}}_L \otimes_{\widetilde{\mathbf{B}}_L^\dagger} D_\tau^\dagger) \in \mathbf{Mod}_{\mathbf{B}_{K_\pi}, \widetilde{\mathbf{B}}_L}(\varphi, \tau)$$

by theorem 4.1.6. By the equivalence induced by the functor $\mathcal{D}$, the (φ, τ) module $(\mathbf{B}_{K_\pi} \otimes_{\mathbf{B}_{K_\pi}^\dagger} D^\dagger, \widetilde{\mathbf{B}}_L \otimes_{\widetilde{\mathbf{B}}_L^\dagger} D_\tau^\dagger)$ corresponds to a p-adic representation $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$. We show that $\mathcal{D}^\dagger(V)$ (more precisely $(\mathcal{D}^\dagger(V), \mathcal{D}^\dagger(V)_\tau)$) equals the original object $(D^\dagger, D_\tau^\dagger) \in \mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_L^\dagger}(\varphi, \tau)$. Indeed, $\mathcal{D}^\dagger(V)$ and $(D^\dagger, D_\tau^\dagger)$ map to the same (φ, τ)-module in $\mathbf{Mod}_{\mathbf{B}_{K_\pi}, \widetilde{\mathbf{B}}_L}(\varphi, \tau)$, and hence they are isomorphic in $\mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_L^\dagger}(\varphi, \tau)$ as the map $(*)$ in the diagram above is an isomorphism. This shows that $\mathcal{D}^\dagger$ is essentially surjective and hence estabilishes the equivalence of categories. □

Proposition 4.1.9. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then its (φ, τ)-module $(D, D_\tau) \in \mathbf{Mod}_{\mathbf{A}_{K_\pi}, \widetilde{\mathbf{A}}_L}(\varphi, \tau)$ is overconvergent.*

Proof. (1) We first suppose T is p-torsion free. Recall that $\mathcal{D}^\dagger(T) = (\mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\pi}}$ and we have a natural map

$$\mathbf{A}_{K_\pi} \otimes_{\mathbf{A}_{K_\pi}^\dagger} \mathcal{D}^\dagger(T) \xrightarrow{f} \mathcal{D}(T),$$

such that tensoring $\otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is an isomorphism (*cf* [46, Théorème 4.3.29]). This implies that f is injective and $\mathcal{D}^\dagger(T)$ is free of rank $d := \mathrm{rank}_{\mathbf{Z}_p} T$ over $\mathbf{A}_{K_\pi}^\dagger$. In particular, there are integers $(n_i)_{1 \le i \le d}$ such that

$$\mathsf{Coker}(f) \simeq \bigoplus_{i=1}^{d} \mathbf{A}_{K_\pi}/(p^{n_i}).$$

We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}_{K_\pi}^\dagger} \mathcal{D}^\dagger(T) & \xrightarrow{p} & \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}_{K_\pi}^\dagger} \mathcal{D}^\dagger(T) & \longrightarrow & \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}_{K_\pi}^\dagger} \mathcal{D}^\dagger(T/pT) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow \sim & & \\ 0 & \longrightarrow & \mathcal{D}(T) & \xrightarrow{p} & \mathcal{D}(T) & \longrightarrow & \mathcal{D}(T/pT) & \longrightarrow & 0 \end{array}$$

which shows that multiplication by p is bijective over $\mathsf{Coker}(f)$ and hence bijective over $\mathbf{A}_{K_\pi}/(p^{n_i})$ for $i \in \{1, \ldots, d\}$. This implies $n_i = 0$ for all $i \in \{1, \ldots, d\}$, and hence $\mathsf{Coker}(f) = 0$. Thus

$$\mathbf{A}_{K_\pi} \otimes_{\mathbf{A}_{K_\pi}^\dagger} \mathcal{D}^\dagger(T) \xrightarrow{\simeq} \mathcal{D}(T)$$

is an isomorphism and the φ-module $\mathcal{D}^\dagger(T)$ is overconvergent. Recall that $\mathcal{D}(T)_\tau = \widetilde{\mathbf{A}}_L \otimes_{\mathbf{A}_{K_\pi}} \mathcal{D}(T)$ and $\mathcal{D}^\dagger(T)_\tau = \widetilde{\mathbf{A}}_L^\dagger \otimes_{\mathbf{A}_{K_\pi}^\dagger} \mathcal{D}^\dagger(T)$: we have

$$\mathcal{D}(T)_\tau = \widetilde{\mathbf{A}}_L \otimes_{\widetilde{\mathbf{A}}_L^\dagger} \mathcal{D}^\dagger(T)_\tau.$$

(2) In general, for any $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we have the short exact sequence:

$$0 \to T' \to T \to T'' \to 0,$$

where T' is the p-torsion submodule of T and $T'' = T/T'$ is p-torsion free. Notice that $\mathcal{D}^\dagger(T') = \mathcal{D}(T')$, and we have the exact sequence

$$0 \to \mathcal{D}(T') \to \mathcal{D}^\dagger(T) \to \mathcal{D}^\dagger(T'') \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathbf{A} \otimes T') = 0,$$

where the last equality follows from Hilbert 90 ($\mathbf{A}$ is endowed with p-adic topology). We thus have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T') & \longrightarrow & \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}^\dagger_{K_\pi}} \mathcal{D}^\dagger(T) & \longrightarrow & \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}^\dagger_{K_\pi}} \mathcal{D}^\dagger(T'') & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow \sim & & \\ 0 & \longrightarrow & \mathcal{D}(T') & \longrightarrow & \mathcal{D}(T) & \longrightarrow & \mathcal{D}(T'') & \longrightarrow & 0 \end{array}$$

which shows that f is an isomorphism.

□

Corollary 4.1.10. *The functors*

$$\begin{aligned} \mathcal{D}^\dagger \colon \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) &\simeq \mathbf{Mod}_{\mathbf{A}^\dagger_{K_\pi}, \widetilde{\mathbf{A}}^\dagger_L}(\varphi, \tau) \\ T &\mapsto \mathcal{D}^\dagger(T) = (\mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\pi}} \\ \mathcal{V}(D^\dagger) = (\mathbf{A}^\dagger \otimes_{\mathbf{A}^\dagger_{K_\pi}} D^\dagger)^{\varphi=1} &\hookleftarrow D^\dagger \end{aligned}$$

(*with the natural τ-semi-linear endomorphism τ_D over $\mathcal{D}^\dagger(T)_\tau = \widetilde{\mathbf{A}}^\dagger_L \otimes_{\mathbf{A}^\dagger_{K_\pi}} \mathcal{D}^\dagger(T)$) establish quasi-inverse equivalences of categories.*

Proof. The proof is almost the same as that of theorem 4.1.8, with the following modifications. For any map $f \colon D_1^\dagger \to D_2^\dagger$ in $\mathbf{Mod}_{\mathbf{A}^\dagger_{K_\pi}, \widetilde{\mathbf{A}}^\dagger_L}(\varphi, \tau)$ we consider the commutative square:

$$\begin{array}{ccc} D_1^\dagger & \xrightarrow{\quad f \quad} & D_2{}^\dagger \\ \downarrow & & \downarrow \\ \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}^\dagger_{K_\pi}} D_1^\dagger & \xrightarrow{1 \otimes f} & \mathbf{A}_{K_\pi} \otimes_{\mathbf{A}^\dagger_{K_\pi}} D_2^\dagger. \end{array}$$

As $\mathbf{A}_{K_\pi}$ is faithfully flat over $\mathbf{A}^\dagger_{K_\pi}$ and $\widetilde{\mathbf{A}}_L$ is faithfully flat over $\widetilde{\mathbf{A}}^\dagger_L$ (*cf* [42, Theorem 7.2 (3)]), $1 \otimes f = 0$ implies $f = 0$. For essential surjectivity, we use proposition 4.1.9 instead of theorem 4.1.6. □

Notation 4.1.11. Let $(D^\dagger, D^\dagger_\tau) \in \mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi}, \widetilde{\mathbf{B}}^\dagger_L}(\varphi, \tau)$. Put

$$D^\dagger_{\tau,0} := \big\{x \in D^\dagger_\tau;\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow (g \otimes 1)(x) = \big(1 + \tau_D + \cdots + \tau_D^{\chi(g)-1}\big)(x)\big\}.$$

By similar arguments as that of lemma 1.1.10, we see that

$$D^\dagger_{\tau,0} := \big\{x \in D^\dagger_\tau;\ (\gamma \otimes 1)x = \big(1 + \tau_D + \cdots + \tau_D^{\chi(g)-1}\big)(x)\big\}.$$

4.1.12 Locally analytic vectors with partially unperfected coefficients

Definition 4.1.13. (*cf* [45, §1.1]) We define for $r>0$, $\widetilde{\mathbf{B}}_u^{\dagger,r}=\widetilde{\mathbf{B}}^{\dagger,r}\cap\widetilde{\mathbf{B}}_u$ and we denote $\widetilde{\mathbf{B}}_u^{\dagger}=\bigcup\limits_{r>0}\widetilde{\mathbf{B}}_u^{\dagger,r}\subset\widetilde{\mathbf{B}}_u$ the set of overconvergent elements (we have $r_1\leq r_2\Rightarrow\widetilde{\mathbf{B}}_u^{\dagger,r_1}\subset\widetilde{\mathbf{B}}_u^{\dagger,r_2}$). We put $\mathbf{B}_u^{\dagger}=\mathbf{B}\cap\widetilde{\mathbf{B}}_u^{\dagger}$.

Remark 4.1.14. The subrings $\widetilde{\mathbf{B}}_u^{\dagger,r}$ and $\widetilde{\mathbf{B}}_u^{\dagger}$ of $\widetilde{\mathbf{B}}_u$ are stable by $\mathscr{G}_K$, and $\varphi(\widetilde{\mathbf{B}}_u^{\dagger,r})\subset\widetilde{\mathbf{B}}_u^{\dagger,pr}$ for all $r>0$, so that $\widetilde{\mathbf{B}}_u^{\dagger}$ is stable by φ in $\widetilde{\mathbf{B}}_u$. Remark that $\widetilde{\mathbf{B}}_u^{\dagger}$ and $\mathbf{B}_u^{\dagger}$ are fields (*cf* [41, Proposition 3.2]).

Recall that (*cf* remark 3.1.16) if V is a $\mathbf{Q}_p$-representation of $\mathscr{G}_K$, then its associated (φ,τ)-module over $(\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_{u,L})$ (*i.e.* $(\mathcal{E},\mathcal{E}_{u,\tau})$) is the φ-module $\mathcal{D}(V)=(\mathbf{B}\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_{K_\pi}}$ together with a τ-semi-linear action over $\mathcal{D}(V)_{u,\tau}=\widetilde{\mathbf{B}}_{u,L}\otimes_{\mathbf{B}_{K_\pi}}\mathcal{D}(V)$.

Definition 4.1.15. (*cf* [46, Définition 4.1.16])

For $r>0$, we define

$$\mathcal{D}^{\dagger,r}(V)_{u,\tau}=(\widetilde{\mathbf{B}}_u^{\dagger,r}\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_L}$$

and

$$\mathcal{D}^{\dagger,r}(V)=\mathcal{D}(V)\cap\mathcal{D}^{\dagger,r}(V)_{u,\tau}=(\mathbf{B}^{\dagger,r}\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_{K_\pi}},$$

observing that $\mathbf{B}^{\dagger,r}=\widetilde{\mathbf{B}}_u^{\dagger,r}\cap\mathbf{B}$. We put

$$\mathcal{D}^{\dagger}(V)_{u,\tau}=(\widetilde{\mathbf{B}}_u^{\dagger}\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_L}=\bigcup_{r>0}\mathcal{D}^{\dagger,r}(V)_{u,\tau}$$

and

$$\mathcal{D}^{\dagger}(V)=(\mathbf{B}^{\dagger}\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_{K_\pi}}=\mathcal{D}(V)\cap\mathcal{D}^{\dagger}(V)_{u,\tau}=\bigcup_{r>0}\mathcal{D}^{\dagger,r}(V).$$

We say that a (φ,τ)-module D associated to V is *overconvergent* if there exists $r>0$ such that we have

$$\mathcal{D}(V)=\mathbf{B}_{K_\pi}\otimes_{\mathbf{B}_{K_\pi}^{\dagger,r}}\mathcal{D}^{\dagger,r}(V)$$

and

$$\mathcal{D}(V)_{u,\tau}=\widetilde{\mathbf{B}}_{u,L}\otimes_{\widetilde{\mathbf{B}}_{u,L}^{\dagger,r}}\mathcal{D}^{\dagger,r}(V)_{u,\tau}$$

where $\widetilde{\mathbf{B}}_{u,L}^{\dagger,r}=\widetilde{\mathbf{B}}_{u,L}\cap\widetilde{\mathbf{B}}_u^{\dagger,r}$.

When this holds, we have in particular

$$\mathcal{D}(V)=\mathbf{B}_{K_\pi}\otimes_{\mathbf{B}_{K_\pi}^{\dagger}}\mathcal{D}^{\dagger}(V)$$

and

$$\mathcal{D}(V)_{u,\tau}=\widetilde{\mathbf{B}}_{u,L}\otimes_{\widetilde{\mathbf{B}}_{u,L}^{\dagger}}\mathcal{D}^{\dagger}(V)_{u,\tau},$$

where $\widetilde{\mathbf{B}}_{u,L}^{\dagger}=\bigcup\limits_{r>0}\widetilde{\mathbf{B}}_{u,L}^{\dagger,r}$.

Remark 4.1.16. (1) We have $\mathcal{D}^{\dagger}(V)_{u,\tau}=(\widetilde{\mathbf{B}}_u^{\dagger}\otimes_{\mathbf{Q}_p}V)^{\mathscr{G}_L}\simeq\widetilde{\mathbf{B}}_{u,L}^{\dagger}\otimes_{\mathbf{B}_{K_\pi}^{\dagger}}\mathcal{D}^{\dagger}(V)$.

(2) Notice that for any $r>0$ we have:

$$\mathbf{B}^{\dagger,r}=\widetilde{\mathbf{B}}_u^{\dagger,r}\cap\mathbf{B}=\widetilde{\mathbf{B}}^{\dagger,r}\cap\mathbf{B},$$

and hence

$$\mathbf{B}^{\dagger}=\widetilde{\mathbf{B}}_u^{\dagger}\cap\mathbf{B}=\widetilde{\mathbf{B}}^{\dagger}\cap\mathbf{B}.$$

This is simply because we have

$$\mathbf{B}\subset\widetilde{\mathbf{B}}_u\subset\widetilde{\mathbf{B}}.$$

Definition 4.1.17. A (φ,τ)-*module over* $(\mathbf{B}^{\dagger}_{K_\pi},\widetilde{\mathbf{B}}^{\dagger}_{u,L}) =: (\mathcal{E}^{\dagger},\mathcal{E}^{\dagger}_{u,\tau})$ consists of

(i) an étale φ-module $D^{\dagger}$ over $\mathbf{B}^{\dagger}_{K_\pi}$;

(ii) a τ-semi-linear endomorphism τ_D on $D^{\dagger}_{u,\tau} := \widetilde{\mathbf{B}}^{\dagger}_{u,L}\otimes_{\mathbf{B}^{\dagger}_{K_\pi}} D^{\dagger}$ which commutes with $\varphi_{\widetilde{\mathbf{B}}^{\dagger}_{u,L}}\otimes\varphi_{D^{\dagger}}$ (where $\varphi_{\widetilde{\mathbf{B}}^{\dagger}_{u,L}}$ is the Frobenius map on $\widetilde{\mathbf{B}}^{\dagger}_{u,L}$ and $\varphi_{D^{\dagger}}$ is the Frobenius map on $D^{\dagger}$) and which satisfies

$$(\forall x\in D^{\dagger})\ (g\otimes 1)\circ\tau_D(x)=\tau_D^{\chi(g)}(x),$$

for all $g\in\mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g)\in\mathbf{Z}_{>0}$. The corresponding category is denoted $\mathbf{Mod}_{\mathbf{B}^{\dagger}_{K_\pi},\widetilde{\mathbf{B}}^{\dagger}_{u,L}}(\varphi,\tau)$.

One defines similarly the notion of (φ,τ)-module over $(\mathbf{A}^{\dagger}_{K_\pi},\widetilde{\mathbf{A}}^{\dagger}_{u,L}) =: (\mathcal{O}_{\mathcal{E}^{\dagger}},\mathcal{O}_{\mathcal{E}^{\dagger}_{u,\tau}})$, and the corresponding category is denoted $\mathbf{Mod}_{\mathbf{A}^{\dagger}_{K},\widetilde{\mathbf{A}}^{\dagger}_{u,L}}(\varphi,\tau)$.

4.1.18 The overconvergence for (φ,τ)-modules over partially unperfected coefficients

For (φ,τ)-modules over $(\mathcal{E},\mathcal{E}_\tau)$, we have seen the overconvergence result in theorem 4.1.6. In this subsection we will prove the overconvergence result for (φ,τ)-modules over $(\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_{u,L})$(*i.e.* $(\mathcal{E},\mathcal{E}_{u,\tau})$).

Theorem 4.1.19. *Let* $V\in\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, *then its* (φ,τ)-*module* $(D,D_{u,\tau})\in\mathbf{Mod}_{\mathbf{B}_{K_\pi},\widetilde{\mathbf{B}}_{u,L}}(\varphi,\tau)$ *is overconvergent.*

Proof. Let $(D,D_{u,\tau})\in\mathbf{Mod}_{\mathcal{E},\mathcal{E}_{u,\tau}}(\varphi,\tau)$ be the (φ,τ)-module over $(\mathcal{E},\mathcal{E}_{u,\tau})$ associated to $V\in\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$. By definition, it suffices to show that

$$D=\mathcal{E}\otimes_{\mathcal{E}^{\dagger}}D^{\dagger} \tag{4.1}$$

and

$$D_{u,\tau}=\mathcal{E}_{u,\tau}\otimes_{\mathcal{E}^{\dagger}_{u,\tau}}D^{\dagger}_{u,\tau}. \tag{4.2}$$

The first equality (4.1) follows directly from theorem 4.1.6. For the second equality (4.2), it suffices to show that the following map is an isomorphism

$$f\colon\ \mathcal{E}^{\dagger}_{u,\tau}\otimes_{\mathcal{E}^{\dagger}}D^{\dagger}\to D^{\dagger}_{u,\tau}. \tag{4.3}$$

Indeed, this impies that

$$\mathcal{E}_{u,\tau}\otimes_{\mathcal{E}^{\dagger}}D^{\dagger}\simeq\mathcal{E}_{u,\tau}\otimes_{\mathcal{E}^{\dagger}_{u,\tau}}D^{\dagger}_{u,\tau},$$

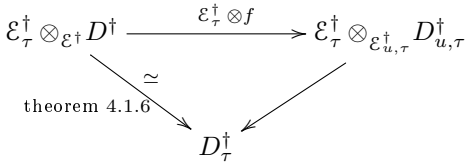
i.e.

$$\mathcal{E}_{u,\tau}\otimes_{\mathcal{E}}(\mathcal{E}\otimes_{\mathcal{E}^{\dagger}}D^{\dagger})\simeq\mathcal{E}_{u,\tau}\otimes_{\mathcal{E}^{\dagger}_{u,\tau}}D^{\dagger}_{u,\tau}.$$

By equality (4.1) the left hand side is exactly $\mathcal{E}_{u,\tau}\otimes_{\mathcal{E}}D=D_{u,\tau}$, hence this gives equality (4.2). To prove that (4.3) is an isomorphism, it suffices to prove it is so after tensoring $\mathcal{E}^{\dagger}_{\tau}\otimes_{\mathcal{E}^{\dagger}_{u,\tau}}$, *i.e.* that

$$\mathcal{E}^{\dagger}_{\tau}\otimes_{\mathcal{E}^{\dagger}_{u,\tau}}f\colon\ \mathcal{E}^{\dagger}_{\tau}\otimes_{\mathcal{E}^{\dagger}}D^{\dagger}\to\mathcal{E}^{\dagger}_{\tau}\otimes_{\mathcal{E}^{\dagger}_{u,\tau}}D^{\dagger}_{u,\tau}$$

is an isomorphic. We consider the commutative diagram:



Notice that this is a digram of linear-transformations of $\mathcal{E}_\tau^\dagger$-vector spaces: it suffices to use dimensions. Notice that by theorem 4.1.6 the arrow on the left is an isomorphism, and

$$\dim \mathcal{E}_\tau^\dagger \otimes_{\mathcal{E}_{u,\tau}^\dagger} D_{u,\tau}^\dagger \leq \dim \mathcal{E}_\tau^\dagger \otimes D^\dagger.$$

We hence conclude that $\mathcal{E}_\tau^\dagger \otimes f$ is an isomoprhism. □

Proposition 4.1.20. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then its (φ,τ)-module $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathbf{A}_{K_\pi}, \widetilde{\mathbf{A}}_{u,L}}(\varphi,\tau)$ is overconvergent.*

Proof. A similar proof as that of proposition 4.1.9 works. □

Remark 4.1.21. One might also prove proposition 4.1.9 using similar arguments as Gao and Poyeton (*cf* [28]), based on Tate-Sen's method (*cf* [10, §3]).

4.1.22 The categorical equivalence

Proposition 4.1.23. *The functors*

$$\begin{aligned} \mathcal{D}^\dagger \colon \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K) &\simeq \mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_{u,L}^\dagger}(\varphi,\tau) \\ V &\mapsto \mathcal{D}^\dagger(V) = (\mathbf{B}^\dagger \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_{K_\pi}} \\ \mathcal{V}(D^\dagger) = (\mathbf{B}^\dagger \otimes_{\mathbf{B}_{K_\pi}^\dagger} D^\dagger)^{\varphi=1} &\hookleftarrow D^\dagger \end{aligned}$$

(with the τ-semi-linear endomorphism $\tau_D = \tau \otimes \tau$ over $\mathcal{D}^\dagger(V)_{u,\tau} = (\widetilde{\mathbf{B}}_u^\dagger \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L}$) establish quasi-inverse equivalences of categories, which refines the equivalence of [19, Théorème 1.14].

Proof. The same proof as that of theorem 4.1.8 works, except that for the overconvergence result we have to use theorem 4.1.19. □

Notation 4.1.24. Let $(D^\dagger, D_{u,\tau}^\dagger) \in \mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_{u,L}^\dagger}(\varphi,\tau)$. We put

$$D_{u,\tau,0}^\dagger := \left\{x \in D_{u,\tau}^\dagger;\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow (g\otimes 1)(x) = x + \tau_D(x) + \cdots + \tau_D^{\chi(g)-1}(x)\right\}.$$

Let $n \in \mathbf{N}$ and we put

$$D_{u,\tau^{p^n},0}^\dagger := \left\{x \in D_{u,\tau}^\dagger;\ (\forall g \in \mathscr{G}_{K_\pi})\ \chi(g) \in \mathbf{Z}_{>0} \Rightarrow (g\otimes 1)(x) = x + \tau_D^{p^n}(x) + \tau_D^{2p^n}(x) + \cdots + \tau_D^{(\chi(g)-1)p^n}(x)\right\}.$$

Notice that these are all subgroups of $D_{u,\tau}^\dagger$.

4.2 The complexes $\mathcal{C}_{\varphi,\tau}^{u,\dagger}$ and $\mathcal{C}_{\psi,\tau}^{u,\dagger}$

Lemma 4.2.1. *Let $(D^\dagger, D_{u,\tau}^\dagger) \in \mathbf{Mod}_{\mathbf{A}_{K_\pi}^\dagger, \widetilde{\mathbf{A}}_{u,L}^\dagger}(\varphi,\tau)$ (resp. $\mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_{u,L}^\dagger}(\varphi,\tau)$). Then there are maps $\tau_D - 1 \colon D^\dagger \to D_{u,\tau,0}^\dagger$, $\varphi - 1 \colon D_{u,\tau,0}^\dagger \to D_{u,\tau,0}^\dagger$ and $\psi \colon D_{u,\tau,0}^\dagger \to D_{u,\tau,0}^\dagger$.*

Proof. Notice that for any $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathbf{B}_u, \widetilde{\mathbf{B}}_{u,L}}(\varphi,\tau)$, we have $\tau_D - 1 \colon D \to D_{u,\tau,0}$, $\varphi - 1 \colon D_{u,\tau,0} \to D_{u,\tau,0}$ (*cf* lemma 3.2.2), and $\psi \colon D_{u,\tau,0} \to D_{u,\tau,0}$ (*cf* lemma 3.3.8). The lemma then follows from the fact that operator τ_D on D_τ preserves overconvergence in $D_{u,\tau}$. Same arguments work for φ and ψ. □

Definition 4.2.2. Let $(D^\dagger, D_{u,\tau}^\dagger) \in \mathbf{Mod}_{\mathbf{A}_{K_\pi}^\dagger, \widetilde{\mathbf{A}}_{u,L}^\dagger}(\varphi,\tau)$ (resp. $\mathbf{Mod}_{\mathbf{B}_{K_\pi}^\dagger, \widetilde{\mathbf{B}}_{u,L}^\dagger}(\varphi,\tau)$).

(1) We define the complex $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(D^\dagger)$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^\dagger & \longrightarrow & D^\dagger \bigoplus D^\dagger_{u,\tau,0} & \longrightarrow & D^\dagger_{u,\tau,0} & \longrightarrow & 0 \\ & & x & \longmapsto & ((\varphi-1)(x),(\tau_D-1)(x)) & & & & \\ & & & & (y,z) & \longmapsto & (\tau_D-1)(y)-(\varphi-1)(z). & & \end{array}$$

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ (resp. $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$), we have in particular the complex $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(\mathcal{D}^\dagger(T))$ (resp. $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(\mathcal{D}^\dagger(V))$), which will also be simply denoted $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)$ (resp. $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(V)$).

(2) Similarly, we define the complex $\mathcal{C}^{u,\dagger}_{\psi,\tau}(D^\dagger)$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^\dagger & \longrightarrow & D^\dagger \bigoplus D^\dagger_{u,\tau,0} & \longrightarrow & D^\dagger_{u,\tau,0} & \longrightarrow & 0 \\ & & x & \longmapsto & ((\psi-1)(x),(\tau_D-1)(x)) & & & & \\ & & & & (y,z) & \longmapsto & (\tau_D-1)(y)-(\psi-1)(z). & & \end{array}$$

If $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ (resp. $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$), we have in particular the complex $\mathcal{C}^{u,\dagger}_{\psi,\tau}(\mathcal{D}^\dagger(T))$ (resp. $\mathcal{C}^{u,\dagger}_{\psi,\tau}(\mathcal{D}^\dagger(V))$), which will also be simply denoted $\mathcal{C}^{u,\dagger}_{\psi,\tau}(T)$ (resp. $\mathcal{C}^{u,\dagger}_{\psi,\tau}(V)$).

Remark 4.2.3. Notice that for $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we use superscripts $\dagger$ in $\mathcal{C}^{u,\dagger}_{\psi,\tau}(T)$ and $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)$ to distinguish from $\mathcal{C}^{u}_{\psi,\tau}(T)$ and $\mathcal{C}^{u}_{\varphi,\tau}(T)$ (*cf* definition 3.2.3 and definition 3.3.10).

4.2.4 Quasi-isomorphism between $\mathcal{C}^{u,\dagger}_{\psi,\tau}$ and $\mathcal{C}^{u,\dagger}_{\varphi,\tau}$

Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and $(D, D_{u,\tau})$ (resp. $(D^\dagger, D^\dagger_{u,\tau})$) the associated (φ,τ)-module over $(\mathcal{O}_\mathcal{E}, \mathcal{O}_{\mathcal{E}_{u,\tau}})$ (resp. $(\mathcal{O}_{\mathcal{E}^\dagger}, \mathcal{O}_{\mathcal{E}^\dagger_{u,\tau}})$). We consider the following morphism of complexes:

$$\begin{array}{lccccccccc} \mathcal{C}^{u,\dagger}_{\varphi,\tau}(T): & 0 & \longrightarrow & D^\dagger & \xrightarrow{(\varphi-1,\tau_D-1)} & D^\dagger \oplus D^\dagger_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} & D^\dagger_{u,\tau,0} & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle -\psi\ominus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\ \mathcal{C}^{u,\dagger}_{\psi,\tau}(T): & 0 & \longrightarrow & D^\dagger & \xrightarrow{(\psi-1,\tau_D-1)} & D^\dagger \oplus D^\dagger_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\psi-1)} & D^\dagger_{u,\tau,0} & \longrightarrow & 0. \end{array}$$

Theorem 4.2.5. *The morphism just defined is a quasi-isomorphism.*

Remark 4.2.6. We have the following diagram:

$$\begin{array}{lccccccccc}
 & 0 & \longrightarrow & 0 & \longrightarrow & (D^\dagger)^{\psi=0} & \xrightarrow{\tau_D-1} & (D^\dagger_{u,\tau,0})^{\psi=0} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T): & 0 & \longrightarrow & D^\dagger & \xrightarrow{(\varphi-1,\tau_D-1)} & D^\dagger \oplus D^\dagger_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} & D^\dagger_{u,\tau,0} & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle -\psi\ominus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\
\mathcal{C}^{u,\dagger}_{\psi,\tau}(T): & 0 & \longrightarrow & D^\dagger & \xrightarrow{(\psi-1,\tau_D-1)} & D^\dagger \oplus D^\dagger_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\psi-1)} & D^\dagger_{u,\tau,0} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Hence to prove theorem 4.2.5, it suffices to show that $\tau_D - 1\colon (D^\dagger)^{\psi=0} \to (D^\dagger_{u,\tau,0})^{\psi=0}$ is bijective.

Proof of theorem 4.2.5

As the operator ψ is surjective (since $\psi \circ \varphi = \mathsf{Id}$), there is an exact sequence of complexes

$$0 \to \mathcal{C}^{u,\dagger}(T) \to \mathcal{C}^{u,\dagger}_{\varphi,\tau}(T) \to \mathcal{C}^{u,\dagger}_{\psi,\tau}(T) \to 0$$

where $\mathcal{C}^{u,\dagger}(T)$ is the complex $[(D^\dagger)^{\psi=0} \xrightarrow{\tau_D-1} (D^\dagger_{u,\tau,0})^{\psi=0}]$ in which the first term is in degree 1.

Similarly, we denote $\mathcal{C}(D)$ the complex $[D^{\psi=0} \xrightarrow{\tau_D-1} D^{\psi=0}_{u,\tau,0}]$, which is acyclic by proposition 3.3.38. There is a natural morphism of complexes $\mathcal{C}^{u,\dagger}(T) \to \mathcal{C}(D)$ induced by the commutative square

$$\begin{array}{ccc}
(D^\dagger)^{\psi=0} & \xrightarrow{\tau_D-1} & (D^\dagger_{u,\tau,0})^{\psi=0} \\
\downarrow & & \downarrow \\
D^{\psi=0} & \xrightarrow[\sim]{\tau_D-1} & D^{\psi=0}_{u,\tau,0}
\end{array}$$

which implies the injectivity of $\tau_D - 1\colon (D^\dagger)^{\psi=0} \to (D^\dagger_{u,\tau,0})^{\psi=0}$.

If $x = \sum_{m=0}^{\infty} p^m[x_m] \in \mathsf{W}(C^\flat)$ and $n \in \mathbf{N}$, we put $w_n(x) = \inf_{0\le m\le n} v^\flat(x_m)$. Reacll that if $x, y \in \mathsf{W}(C^\flat)$, we have $w_n(x+y) \ge \inf\{w_n(x), w_n(y)\}$ and $w_n(xy) \ge \inf_{n_1+n_2\le n} \big(w_{n_1}(x) + w_{n_2}(y)\big)$ (*cf* [20, p. 584]). Also, an element $x \in \mathsf{W}(C^\flat)$ is overconvergent if there exists $r \in \mathbf{R}_{\ge 0}$ such that the sequence $\big(\big(w_n(x) + \frac{pr}{p-1}n\big)\big)_{n\in\mathbf{N}}$ is bounded below.

Lemma 4.2.7. *The map $\tau-1\colon \widetilde{\mathbf{A}}^{\psi=0}_{u,L} \to \widetilde{\mathbf{A}}^{\psi=0}_{u,L}$ is bijective. If $y \in \big(\widetilde{\mathbf{A}}^\dagger_{u,L}\big)^{\psi=0}$, then $(\tau-1)^{-1}(y) \in \big(\widetilde{\mathbf{A}}^\dagger_{u,L}\big)^{\psi=0}$.*

Proof. By dévissage, the first assertion can be checked by modulo p: we have $\widetilde{\mathbf{A}}_{u,L}/(p) \simeq k(\!(u, \eta^{1/p^\infty})\!)$, so that $\big(\widetilde{\mathbf{A}}_{u,L}\big)^{\psi=0}/(p) \simeq k(\!(u,\eta^{1/p^\infty})\!)^{\psi=0} = \bigoplus_{i=1}^{p-1} u^i k(\!(u^p, \eta^{1/p^\infty})\!)$. As element $x \in k(\!(u,\eta^{1/p^\infty})\!)^{\psi=0}$ can be written in a unique way

$$x = \sum_{i=1}^{p-1}\sum_{j\in\mathbf{Z}} u^{i+pj} x_{i,j}$$

where $x_{i,j} \in k(\!(\eta^{1/p^\infty})\!)$ is zero when $j \ll 0$. Then we have

$$(\tau-1)(x) = \sum_{i=1}^{p-1}\sum_{j\in\mathbf{Z}} u^{i+pj}(\varepsilon^{i+pj}-1)x_{i,j}.$$

As $\varepsilon^{i+pj} \neq 1$ for all $i \in \{1,\dots,p-1\}$ and $j \in \mathbf{Z}$, this implies that $\tau - 1$ is injective. If $y = \sum_{i=1}^{p-1} \sum_{j\in\mathbf{Z}} u^{i+pj} y_{i,j}$ is an element in $k(\!(u, \eta^{1/p^\infty})\!)^{\psi=0}$, then

$$(\tau-1)^{-1}(y) = \sum_{i=1}^{p-1} \sum_{j\in\mathbf{Z}} u^{i+pj} \tfrac{y_{i,j}}{\varepsilon^{i+pj}-1}.$$

For the second assertion, let $y \in \big(\widetilde{\mathbf{A}}^\dagger_{u,L}\big)^{\psi=0}$. We can write

$$y = \sum_{i=1}^{p-1} \sum_{j\in\mathbf{Z}} u^{i+pj} y_{i,j}$$

(where $u = [\tilde{\pi}]$ here) with $y_{i,j} \in \mathsf{W}\big(k(\!(\eta^{1/p^\infty})\!)\big)$, where $\lim_{j\to-\infty} y_{i,j} = 0$ (for the p-adic topology) for all $i \in \{1,\dots,p-1\}$. Then $(\tau-1)^{-1}(y) = \sum_{i=1}^{p-1} \sum_{j\in\mathbf{Z}} u^{i+pj} \frac{y_{i,j}}{[\varepsilon]^{i+pj}-1}$. As y is overconvergent, there exists $r \in \mathbf{R}_{\geq 0}$ and $c \in \mathbf{R}$ such that $w_n(u^{i+pj} y_{i,j}) + \frac{pr}{p-1} n \geq c$ for all $i \in \{1,\dots,p-1\}$, $j \in \mathbf{Z}$ and $n \in \mathbf{N}$. If $n \in \mathbf{N}$ we have

$$\begin{aligned}
w_n\big(u^{i+pj}\tfrac{y_{i,j}}{[\varepsilon]^{i+pj}-1}\big) + \tfrac{pr}{p-1}n &= w_n\big(\tfrac{u^{i+pj}}{[\varepsilon^{i+pj}-1]} y_{i,j} \tfrac{[\varepsilon^{i+pj}-1]}{[\varepsilon]^{i+pj}-1}\big) + \tfrac{pr}{p-1}n \\
&= -\tfrac{p}{p-1} + w_n\big(u^{i+pj} y_{i,j} \tfrac{[\varepsilon^{i+pj}-1]}{[\varepsilon]^{i+pj}-1}\big) + \tfrac{pr}{p-1}n \\
&\geq -\tfrac{p}{p-1} + \inf_{n_1+n_2\leq n} \Big(w_{n_1}\big(u^{i+pj} y_{i,j}\big) + w_{n_2}\big(\tfrac{[\varepsilon^{i+pj}-1]}{[\varepsilon]^{i+pj}-1}\big)\Big) + \tfrac{pr}{p-1}n \\
&\geq -\tfrac{p}{p-1} + \inf_{n_1+n_2\leq n} \Big(w_{n_1}\big(u^{i+pj} y_{i,j}\big) + \tfrac{pr}{p-1} n_1 + \big(\tfrac{pr}{p-1} - 1\big) n_2\Big)
\end{aligned}$$

since $v^\flat(\varepsilon^{i+pj}-1) = \frac{p}{p-1}$ (because $p \nmid i+pj$) and $w_n\big(\frac{[\varepsilon^{i+pj}-1]}{[\varepsilon]^{i+pj}-1}\big) \geq -n$ (by [20, Corollaire II.1.5]). Taking r larger if necessary, we may assume that $\frac{pr}{p-1} - 1 \geq 0$. Then $w_n\big(u^{i+pj}\frac{y_{i,j}}{[\varepsilon]^{i+pj}-1}\big) + \frac{pr}{p-1} n \geq -\frac{p}{p-1} + c$ for all $i \in \{1,\dots,p-1\}$, $j \in \mathbf{Z}$ and $n \in \mathbf{N}$. This shows that $(\tau-1)^{-1}(y)$ is overconvergent. □

Proposition 4.2.8. *The map $\tau_D - 1\colon (D^\dagger)^{\psi=0} \to (D^\dagger_{u,\tau,0})^{\psi=0}$ is bijective.*

Proof. Put $\mathbb{D}^\dagger(T) = (\mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\zeta}}$ (this is the overconvergent (φ,Γ)-module associated to T). We have a $\mathscr{G}_K$-equivariant isomorphism $\mathbf{A}^\dagger \otimes_{\mathbf{A}^\dagger_{K_\zeta}} \mathbb{D}^\dagger(T) \simeq \mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T$: extending the scalars to $\widetilde{\mathbf{A}}^\dagger_u := \mathcal{O}_{\widehat{\mathcal{F}^{\mathrm{ur}}_u}} \cap \widetilde{\mathbf{A}}^\dagger$, we deduce a $\mathscr{G}_K$-equivariant isomorphism $\widetilde{\mathbf{A}}^\dagger_u \otimes_{\mathbf{A}^\dagger_{K_\zeta}} \mathbb{D}^\dagger(T) \simeq \widetilde{\mathbf{A}}^\dagger_u \otimes_{\mathbf{Z}_p} T$. Taking $\mathscr{G}_L$ invariants provides a $\mathscr{G}_K/\mathscr{G}_L$-equivariant isomorphism

$$D^\dagger_{u,\tau} = \big(\widetilde{\mathbf{A}}^\dagger_u \otimes_{\mathbf{Z}_p} T\big)^{\mathscr{G}_L} \simeq \widetilde{\mathbf{A}}^\dagger_{u,L} \otimes_{\mathbf{A}^\dagger_{K_\zeta}} \mathbb{D}^\dagger(T).$$

(1) Assume T is torsion-free: so is $\mathbb{D}^\dagger(T)$. Let $(e_1,\dots,e_d)$ be a basis of $\mathbb{D}^\dagger(T)$ so that $(\varphi(e_1),\dots,\varphi(e_d))$ is a basis of $\mathbb{D}^\dagger(T)$ over $\mathbf{A}^\dagger_{K_\zeta}$. Then $D^\dagger_{u,\tau} = \bigoplus_{i=1}^d \widetilde{\mathbf{A}}^\dagger_{u,L} \varphi(e_i)$. If $x = \sum_{i=1}^d \lambda_i \varphi(e_i) \in D^\dagger_{u,\tau}$ with $\lambda_1,\dots,\lambda_d \in \widetilde{\mathbf{A}}^\dagger_{u,L}$, we have $\psi(x) = \sum_{i=1}^d \psi(\lambda_i) e_i$. This implies that $x \in (D^\dagger_{u,\tau})^{\psi=0}$ if and only if $\psi(\lambda_i) = 0$ *i.e.* $\lambda_i \in (\widetilde{\mathbf{A}}^\dagger_{u,L})^{\psi=0}$. Lemma 4.2.7 implies that $\mu_i := (\tau-1)^{-1}(\lambda_i) \in (\widetilde{\mathbf{A}}^\dagger_{u,L})^{\psi=0}$. As $\tau_D(\varphi(e_i)) = \varphi(e_i)$, this implies that $y = (\tau_D - 1)(x)$ with $x = \sum_{i=1}^d \mu_i \varphi(e_i) \in D^\dagger_{u,\tau}$. As we know that $\tau_D - 1\colon D^{\psi=0} \to D^{\psi=0}_{u,\tau,0}$ is bijective, this implies that in fact $x \in D \cap D^\dagger_{u,\tau} = D^\dagger$, finishing the proof in that case.

(2) In the general case, there is an exact sequence of representations

$$0 \to T' \to T \to T'' \to 0$$

where T' is finite and T'' torsion-free. Tensoring with $\mathbf{A}^\dagger$ and taking $\mathscr{G}_{K_\pi}$ invariants provides the exact sequence

$$0 \to \mathcal{D}(T') \to \mathcal{D}^\dagger(T) \to \mathcal{D}^\dagger(T'') \to \mathsf{H}^1(\mathscr{G}_{K_\pi}, \mathbf{A} \otimes_{\mathbf{Z}_p} T') = 0$$

(because $\mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T' = \mathbf{A} \otimes_{\mathbf{Z}_p} T'$ since T' is torsion). As the maps ψ are surjective, the snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T') & \longrightarrow & \mathcal{D}^\dagger(T) & \longrightarrow & \mathcal{D}^\dagger(T'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \mathcal{D}(T') & \longrightarrow & \mathcal{D}^\dagger(T) & \longrightarrow & \mathcal{D}^\dagger(T'') & \longrightarrow & 0 \end{array}$$

shows that the sequence

$$0 \to \mathcal{D}(T')^{\psi=0} \to \mathcal{D}^\dagger(T)^{\psi=0} \to \mathcal{D}^\dagger(T'')^{\psi=0} \to 0$$

is exact. Similarly, tensoring the short exact sequence $0 \to \mathcal{D}(T') \to \mathcal{D}^\dagger(T) \to \mathcal{D}^\dagger(T'') \to 0$ by $\widetilde{\mathbf{A}}^\dagger_{u,L}$ provides the short exact sequence

$$0 \to \mathcal{D}(T')_{u,\tau} \to \mathcal{D}^\dagger(T)_{u,\tau} \to \mathcal{D}^\dagger(T'')_{u,\tau} \to 0$$

(by flatness of $\widetilde{\mathbf{A}}^\dagger_{u,L}$ over the discrete valuation ring $\mathbf{A}^\dagger_{K_\pi}$). Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T')_{u,\tau} & \longrightarrow & \mathcal{D}^\dagger(T)_{u,\tau} & \longrightarrow & \mathcal{D}^\dagger(T'')_{u,\tau} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \downarrow{\scriptstyle \delta-\gamma\otimes 1} & & \\ 0 & \longrightarrow & \mathcal{D}(T')_{u,\tau} & \longrightarrow & \mathcal{D}^\dagger(T)_{u,\tau} & \longrightarrow & \mathcal{D}^\dagger(T'')_{u,\tau} & \longrightarrow & 0, \end{array}$$

and we have the exact sequence

$$0 \to \mathcal{D}(T')_{u,\tau,0} \to \mathcal{D}^\dagger(T)_{u,\tau,0} \to \mathcal{D}^\dagger(T'')_{u,\tau,0}.$$

Now consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T')_{u,\tau,0} & \longrightarrow & \mathcal{D}^\dagger(T)_{u,\tau,0} & \xrightarrow{f} & \mathcal{D}^\dagger(T'')_{u,\tau,0} & . & \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \mathcal{D}(T')_{u,\tau,0} & \longrightarrow & \mathcal{D}^\dagger(T)_{u,\tau,0} & \xrightarrow{f} & \mathcal{D}^\dagger(T'')_{u,\tau,0} & & \end{array}$$

The composition $\mathcal{D}(T')^{\psi=0}_{u,\tau,0} \subset \mathcal{D}(T')_{u,\tau,0} \to \mathcal{D}^\dagger(T)_{u,\tau,0}$ is injective, hence $\mathcal{D}(T')^{\psi=0}_{u,\tau,0} \to \mathcal{D}^\dagger(T)^{\psi=0}_{u,\tau,0}$ is injective. Suppose $x \in \mathcal{D}^\dagger(T)^{\psi=0}_{u,\tau,0}$ is mapped to 0 in $\mathcal{D}^\dagger(T'')^{\psi=0}_{u,\tau,0}$, then $x \in \mathsf{Ker}(f) \cap \mathcal{D}^\dagger(T)^{\psi=0}_{u,\tau,0}$ and hence x is in the image of $\mathcal{D}(T')^{\psi=0}_{u,\tau,0}$. This implies that the sequence

$$0 \to \mathcal{D}(T')^{\psi=0}_{u,\tau,0} \to \mathcal{D}^\dagger(T)^{\psi=0}_{u,\tau,0} \to \mathcal{D}^\dagger(T'')^{\psi=0}_{u,\tau,0}$$

is exact. Now we consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(T')^{\psi=0} & \longrightarrow & \mathcal{D}^\dagger(T)^{\psi=0} & \longrightarrow & \mathcal{D}^\dagger(T'')^{\psi=0} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \tau_{D'}-1} & & \downarrow{\scriptstyle \tau_D-1} & & \downarrow{\scriptstyle \tau_{D''}-1} & & \\ 0 & \longrightarrow & \mathcal{D}(T')^{\psi=0}_{u,\tau,0} & \longrightarrow & \mathcal{D}^\dagger(T)^{\psi=0}_{u,\tau,0} & \longrightarrow & \mathcal{D}^\dagger(T'')^{\psi=0}_{u,\tau,0} & & \end{array}$$

As the maps $\tau_{D'} - 1$ and $\tau_{D''} - 1$ are bijective (by proposition 3.3.38 for the first, and by the special case above for the second), so is $\tau_D - 1$ in the middle.

□

This shows in particular that the complex $\mathcal{C}^{u,\dagger}(T)$ is acyclic, so that the complexes $\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)$ and $\mathcal{C}^{u,\dagger}_{\psi,\tau}(T)$ are quasi-isomorphic.

4.3 $\mathsf{H}^i(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T))$ and $\mathsf{H}^i(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T))$ for $i \in \{0,1\}$

Proposition 4.3.1. *Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_k)$. We have $\mathsf{H}^i(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)) \simeq \mathsf{H}^i(\mathscr{G}_K, T)$ for $i \in \{0,1\}$.*

Proof. We have $\mathsf{H}^0(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)) = (\mathbf{A}^\dagger \otimes_{\mathbf{Z}_p} T)^{\mathscr{G}_{K_\pi}, \varphi=1, \tau_D=1} = T^{\langle \mathscr{G}_{K_\pi}, \tau\rangle} = T^{\mathscr{G}_K} = \mathsf{H}^0(\mathscr{G}_K, T)$.

We claim that the natural $\mathbf{Z}_p$-linear map $\mathsf{H}^1(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)) \to \mathsf{H}^1(\mathcal{C}_{\varphi,\tau}(T))$ is an isomorphism. Recall that $\mathsf{H}^1(\mathcal{C}_{\varphi,\tau}(T))$ classifies all extensions of $\mathcal{D}(T)$ by $\mathbf{A}_{K_\pi}$ in the category $\mathbf{Mod}_{\mathbf{A}_{K_\pi}, \widetilde{\mathbf{A}}_L}(\varphi,\tau)$. Similarly, $\mathsf{H}^1(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T))$ classifies all extensions of $\mathcal{D}^\dagger(T)$ by $\mathbf{A}^\dagger_{K_\pi}$ in the category $\mathbf{Mod}_{\mathbf{A}^\dagger_{K_\pi}, \widetilde{\mathbf{A}}^\dagger_{u,L}}(\varphi,\tau)$. Since both categories are equivalent to $\mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, we conclude that $\mathsf{H}^1(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T)) \simeq \mathsf{H}^1(\mathcal{C}_{\varphi,\tau}(T))$, which is isomorphic to $\mathsf{H}^1(\mathscr{G}_K, T)$ by proposition 1.3.3. □

Corollary 4.3.2. *We have $\mathsf{H}^i(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T)) \simeq \mathsf{H}^i(\mathscr{G}_K, T)$ for $i \in \{0,1\}$.*

Proof. By theorem 4.2.5, we have $\mathsf{H}^i(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T)) \simeq \mathsf{H}^i(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(V))$ for $i \in \mathbf{N}$: the result follows from proposition 4.3.1. □

4.4 Some remarks on the morphism between $\mathcal{C}^{u,\dagger}_{\psi,\tau}$ and $\mathcal{C}^{u}_{\psi,\tau}$

Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathbf{A}_{K_\pi}, \widetilde{\mathbf{A}}_{u,L}}(\varphi,\tau)$ (resp. $(D^\dagger, D^\dagger_{u,\tau}) \in \mathbf{Mod}_{\mathbf{A}^\dagger_{K_\pi}, \widetilde{\mathbf{A}}^\dagger_{u,L}}(\varphi,\tau)$) its associated (φ,τ)-modules. We then have a natural map from $\mathcal{C}^{u,\dagger}_{\psi,\tau}(T)$ to $\mathcal{C}^{u}_{\psi,\tau}(T)$:

$$\begin{array}{lccccccccc}
\mathcal{C}^{u,\dagger}_{\psi,\tau}(T): & 0 \longrightarrow & D^\dagger & \xrightarrow{(\psi-1,\tau_D-1)} & D^\dagger \oplus D^\dagger_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\psi-1)} & D^\dagger_{u,\tau,0} & \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{C}^{u}_{\psi,\tau}(T): & 0 \longrightarrow & D & \xrightarrow{(\psi-1,\tau_D-1)} & D \oplus D_{u,\tau,0} & \xrightarrow{(\tau_D-1)\ominus(\psi-1)} & D_{u,\tau,0} & \longrightarrow 0.
\end{array}$$

Remark 4.4.1. (1) We summarize the relations between the different complexes:

$$\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T) \xleftarrow[\text{quasi-iso}]{\text{theorem 4.2.5}} \mathcal{C}^{u,\dagger}_{\psi,\tau}(T) \longrightarrow \mathcal{C}^{u}_{\psi,\tau}(T) \xleftarrow[\text{quasi-iso}]{\text{theorem 3.3.11}} \mathcal{C}^{u}_{\varphi,\tau}(T) \xleftarrow[\text{quasi-iso}]{\text{theorem 3.2.4}} \mathcal{C}_{\varphi,\tau}(T)\ .$$

In the first chapter we have showed that $\mathcal{C}_{\varphi,\tau}(T)$ computes the continuous Galois cohomology of T (*cf* theorem 1.1.13). It is hence natural to study whether the morphism between $\mathcal{C}^{u,\dagger}_{\psi,\tau}$ and $\mathcal{C}^{u}_{\psi,\tau}$ is a quasi-isomorphism (we expect it is).

(2) The complex $\mathcal{C}^{u}_{\psi,\tau}(D)$ is the total complex of the double complex

$$\begin{array}{ccc}
D & \xrightarrow{\psi-1} & D \\
{\scriptstyle \tau_D-1}\downarrow & & \downarrow{\scriptstyle \tau_D-1} \\
D_{u,\tau,0} & \xrightarrow[\psi-1]{} & D_{u,\tau,0}
\end{array}$$

Similarly, the complex $\mathcal{C}^{u,\dagger}_{\psi,\tau}(D^\dagger)$ is the total complex of the double complex

$$\begin{array}{ccc} D^\dagger & \xrightarrow{\psi-1} & D^\dagger \\ {\scriptstyle \tau_D-1}\downarrow & & \downarrow{\scriptstyle \tau_D-1} \\ D^\dagger_{u,\tau,0} & \xrightarrow[\psi-1]{} & D^\dagger_{u,\tau,0} \end{array}$$

Lemma 4.4.2. *The complex $\mathcal{C}^{u}_{\psi,\tau}(D)$ is quasi-isomorphic to $\mathcal{C}^{u,\dagger}_{\psi,\tau}(D^\dagger)$ if the following two morphisms of complexes are both quasi-isomorphisms:*

$$\begin{array}{ccc} [D^\dagger & \xrightarrow{\psi-1} & D^\dagger] \\ \downarrow & & \downarrow \\ [D & \xrightarrow{\psi-1} & D] \end{array} \qquad\qquad \begin{array}{ccc} [D^\dagger_{u,\tau,0} & \xrightarrow{\psi-1} & D^\dagger_{u,\tau,0}] \\ \downarrow & & \downarrow \\ [D_{u,\tau,0} & \xrightarrow{\psi-1} & D_{u,\tau,0}]. \end{array}$$

Proof. *cf* [51, Exercise 1.3.4]. □

Remark 4.4.3. (1) When the residue field k is finite. The morphism

$$\begin{array}{ccc} [D^\dagger & \xrightarrow{\psi-1} & D^\dagger] \\ \downarrow & & \downarrow \\ [D & \xrightarrow{\psi-1} & D] \end{array}$$

is a quasi-isomorphism. Indeed, this can be proved using classical methods: for h^0, it suffices to show that $x \in D^{\psi=1}$ implies $x \in D^\dagger$ (*cf* [21, Lemma I.6.4; Proposition III.2.1 (ii)]); for h^1, it suffices to show that $D^\dagger/(\psi-1) \simeq D/(\psi-1)$ (*cf* [33, Lemma 2.6], [21, Corollaire I.7.4] and [29, Proposition 3.6 (2)]).

(2) As $k(\!(\eta)\!)$ is never finite, it seems hard to use the same method to show that

$$\begin{array}{ccc} [D^\dagger_{u,\tau,0} & \xrightarrow{\psi-1} & D^\dagger_{u,\tau,0}] \\ \downarrow & & \downarrow \\ [D_{u,\tau,0} & \xrightarrow{\psi-1} & D_{u,\tau,0}] \end{array}$$

is a quasi-isomorphism. In particular the structure of $D_{u,\tau,0}/(\psi-1)$ is not clear.

Some remarks on $\mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T))$ and $\mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(T))$

Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ and $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau)$ its associated (φ,τ)-module. Notice that by theorem 4.2.5, $\mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\psi,\tau}(V)) \simeq \mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\varphi,\tau}(V))$. We have

$$\mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\psi,\tau}(V)) \simeq D^\dagger_{u,\tau,0}\big/\big((\psi-1)D^\dagger_{u,\tau,0} + (\tau_D-1)D^\dagger\big),$$

and

$$\mathsf{H}^2(\mathcal{C}_{\psi,\tau}(V)) \simeq D_{u,\tau,0}\big/\big((\psi-1)D_{u,\tau,0} + (\tau_D-1)D\big) \simeq \mathsf{H}^2(\mathscr{G}_K, V).$$

Lemma 4.4.4. *Let* $(D, D_{u,\tau}) \in \mathbf{Mod}_{\mathcal{O}_{\mathcal{E}},\mathcal{O}_{\mathcal{E}_{u,\tau}}}(\varphi,\tau)$ (*resp.* $\mathbf{Mod}_{\mathcal{E},\mathcal{E}_{u,\tau}}(\varphi,\tau)$), *we then have*

$$\begin{aligned} D &= \varphi(D) \oplus D^{\psi=0}, \\ D_{u,\tau} &= \varphi(D_{u,\tau}) \oplus D^{\psi=0}_{u,\tau}, \\ D_{u,\tau,0} &= \varphi(D_{u,\tau,0}) \oplus D^{\psi=0}_{u,\tau,0}. \end{aligned}$$

There are similar equalities with $D^{\dagger}, D^{\dagger}_{u,\tau}$ *and* $D^{\dagger}_{u,\tau,0}$.

Proof. For any $z \in D_{u,\tau}$, put $z_0 = \varphi(\psi(z))$ and write $z = z_0 + (z - z_0)$. Notice that $z_0 \in \varphi(D_{u,\tau})$ and $z - z_0 \in D^{\psi=0}_{u,\tau}$. Suppose $x \in \varphi(D_{u,\tau}) \cap D^{\psi=0}_{u,\tau}$, then $x = \varphi(y)$ and $0 = \psi(z) = y$, hence $x = 0$. This proves $D_{u,\tau} = \varphi(D_{u,\tau}) + D^{\psi=0}_{u,\tau}$, and similarly we have $D = \varphi(D) \oplus D^{\psi=0}$.

For any $z \in D_{u,\tau,0}$, again put $z_0 = \varphi(\psi(z))$ and write $z = z_0 + (z - z_0)$. Then $z_0 \in \varphi(D_{u,\tau,0})$ and $z - z_0 \in D^{\psi=0}_{u,\tau,0}$. Indeed, as ψ commutes with the action of $\mathscr{G}_K$, $z \in D_{u,\tau,0}$ implies $\psi(z) \in D_{u,\tau,0}$.

The last statement follows from the fact that the operator φ and ψ respect overconvergence. □

Remark 4.4.5. Let $f\colon D_{u,\tau,0} \to \mathsf{H}^2(\mathcal{C}^{u}_{\psi,\tau}(T))$ and $f^{\dagger}\colon D^{\dagger}_{u,\tau,0} \to \mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T))$ be the canonical surjections. If $x \in D_{u,\tau,0}$, we can write $x = \varphi(y) + z$ with $y \in D_{u,\tau,0}$ and $z \in D^{\psi=0}_{u,\tau,0}$ by lemma 4.4.4. By proposition 3.3.38, we have $D^{\psi=0}_{u,\tau,0} = (\tau_D - 1)(D^{\psi=0})$, so that the image of z in $\mathsf{H}^2(\mathcal{C}^{u}_{\psi,\tau}(T))$ is zero, *i.e.* $f(x) = f(\varphi(y))$. Iterating, this shows that f induces a surjective map

$$f\colon \bigcap_{n=0}^{\infty} \varphi^n(D_{u,\tau,0}) \to \mathsf{H}^2(\mathcal{C}^{u}_{\psi,\tau}(T)).$$

We have a similar statement for $f^{\dagger}\colon \bigcap_{n=0}^{\infty} \varphi^n(D^{\dagger}_{u,\tau,0}) \to \mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T))$. A strategy to prove the bijectivity of

$$\mathsf{H}^2(\mathcal{C}^{u}_{\psi,\tau}(T)) \to \mathsf{H}^2(\mathcal{C}^{u,\dagger}_{\psi,\tau}(T))$$

could be to compare $\bigcap_{n=0}^{\infty} \varphi^n(D_{u,\tau,0})$ and $\bigcap_{n=0}^{\infty} \varphi^n(D^{\dagger}_{u,\tau,0})$, or more precisely the cokernels of $\psi - 1$ on these.

Chapter 5

Complex over the Robba ring with (φ, N_∇)-modules

In this chapter, for any Galois representation $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we construct a three-term complex $\mathcal{C}_{\varphi,N_\nabla}(V)$, using its corresponding (φ, N_∇)-module over the Robba ring. We show (*cf* proposition 5.3.17) that its H^i is isomorphic to $\varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n}, V)$ for $i \in \{0,1\}$, and we construct a pairing analogous to the one which gives rise to the Tate duality when k is finite.

If V is a crystalline representation with Hodge-Tate weights $\mathsf{HT}(V) \subset \{2,\dots,h\}$ for some $h \in \mathbf{N}_{\geq 2}$, we construct similarly a three-term complex $\mathcal{C}_{\varphi,\partial_\tau}(V)$ from its corresponding (φ,∂_τ)-module over the Robba ring, which has similar results for $\mathsf{H}^i, i \in \{0,1\}$ (*cf* proposition 5.4.13) and construct similarly a pairing when k is finite.

5.1 The Robba ring

5.1.1 The rings $\widetilde{\mathbf{B}}^I$ and $\mathbf{B}^I$

Definition 5.1.2. (*cf* [5, §2]) When A is a p-adic complete ring, we use $A\{X,Y\}$ to denote the p-adic completion of $A[X,Y]$. Recall that $\eta = \varepsilon - 1$ and we put

(1)

$$\begin{aligned}\widetilde{\mathbf{A}}^{[r,s]} &:= \widetilde{\mathbf{A}}^+\left\{\frac{p}{[\eta]^r}, \frac{[\eta]^s}{p}\right\}, \text{ when } r \leq s \in \mathbf{Z}_{\geq 0}[1/p], s > 0;\\ \widetilde{\mathbf{A}}^{[r,+\infty]} &:= \widetilde{\mathbf{A}}^+\left\{\frac{p}{[\eta]^r}\right\}, \text{ when } r \in \mathbf{Z}_{\geq 0}[1/p];\\ \widetilde{\mathbf{A}}^{[+\infty,+\infty]} &:= \widetilde{\mathbf{A}}.\end{aligned}$$

(2) If I is one of the closed intervals mentioned above, we put $\widetilde{\mathbf{B}}^I := \widetilde{\mathbf{A}}^I[1/p]$.

Remark 5.1.3. More precisely, in the above definition

$$\widetilde{\mathbf{A}}^+\left\{\frac{p}{[\eta]^r}, \frac{[\eta]^s}{p}\right\} = \widetilde{\mathbf{A}}^+\{X,Y\}/\big([\eta]^r X - p, pY - [\eta]^s, XY - [\eta]^{s-r}\big),$$

and similarly for others.

Definition 5.1.4. When $r \in \mathbf{Z}_{\geq 0}[1/p]$, put

$$\widetilde{\mathbf{B}}^{[r,+\infty)} := \bigcap_{n\geq 0} \widetilde{\mathbf{B}}^{[r,s_n]},$$

where $s_n \in \mathbf{Z}_{>0}[1/p]$ and $\lim\limits_{n\to+\infty} s_n = +\infty$. We endow $\widetilde{\mathbf{B}}^{[r,+\infty)}$ with its natural Fréchet topology (*cf* [5, Définition 2.16]).

Remark 5.1.5. (*cf* [28, §2])

(1) If I is one of the closed intervals above, then $\widetilde{\mathbf{A}}^I$ is p-adically separated and complete.

(2) For $r \in \mathbf{Z}_{\geq 0}[1/p]$, notice the difference between $\widetilde{\mathbf{B}}^{[r,+\infty)}$ and $\widetilde{\mathbf{B}}^{[r,+\infty]}$. Indeed, $\widetilde{\mathbf{B}}^{[r,+\infty)}$ is complete with respect to Fréchet topology and contains $\widetilde{\mathbf{B}}^{[r,+\infty]}$ as a dense subring.

Definition 5.1.6. (*cf* [28, Definition 2.1.8]) Suppose $r \in \mathbf{Z}_{\geq 0}[1/p]$, and let $x = \sum\limits_{i\geq i_0} p^i[x_i] \in \widetilde{\mathbf{B}}^{[r,+\infty]}$ with $i_0 \in \mathbf{Z}$ and $x_i \in C^\flat$ for all $i \geq i_0$. Denote $w_k(x) = \inf\limits_{i<k}\{v^\flat(x_i)\}$. Put

$$W^{[s,s]}(x) = \inf_{k\geq k_0}\Big\{k, \frac{p-1}{ps}\cdot v^\flat(x_k)\Big\} = \inf_{k\geq k_0}\Big\{k, \frac{p-1}{ps}\cdot w_k(x)\Big\};$$

this is a well-defined valuation (*cf* [13, Proposition 5.4]). For $I \subset [r,+\infty)$ a non-empty closed interval such that $I \neq [0,0]$, let

$$W^I(x) := \inf_{\alpha\in I, \alpha\neq 0}\big\{W^{[\alpha,\alpha]}(x)\big\}.$$

Remark 5.1.7. Notice that $W^{[s,s]}$ is a variant of p-adic valuation, as the latter is $\inf\limits_{\{k\in\mathbf{Z};x_k\neq 0\}}\{k\}$.

Definition 5.1.8. (1) When $r \in \mathbf{Z}_{\geq 0}[1/p]$, we put

$$\mathbf{A}^{[r,+\infty]} := \mathbf{A} \cap \widetilde{\mathbf{A}}^{[r,+\infty]},$$

$$\mathbf{B}^{[r,+\infty]} := \mathbf{B} \cap \widetilde{\mathbf{B}}^{[r,+\infty]}.$$

(2) When $r, s \in \mathbf{Z}_{\geq 0}[1/p]$ and $s \neq 0$, put $\mathbf{B}^{[r,s]}$ the closure of $\mathbf{B}^{[r,+\infty]}$ in $\widetilde{\mathbf{B}}^{[r,s]}$ with respect to the topology given by the valuation $W^{[r,s]}$. Put $\mathbf{A}^{[r,s]} := \mathbf{B}^{[r,s]} \cap \widetilde{\mathbf{A}}^{[r,s]}$, this is the ring of integers of $\mathbf{B}^{[r,s]}$.

(3) When $r \in \mathbf{Z}_{\geq 0}[1/p]$, put

$$\mathbf{B}^{[r,+\infty)} = \bigcap_{n\geq 0} \mathbf{B}^{[r,s_n]},$$

where $s_n \in \mathbf{Z}_{>0}[1/p]$ and $\lim\limits_{n\to+\infty} s_n = +\infty$ (*cf* [28, Definition 2.1.4]).

Definition 5.1.9. (*cf* [45, §1.3]) If $r \geq 0$, we define a valuation $V(\cdot, r)$ on $\widetilde{\mathbf{B}}^+[1/[\widetilde{\pi}]]$ by

$$V(x,r) = \inf_{k\in\mathbf{Z}}\Big(k + \frac{p-1}{pr}v^\flat(x_k)\Big)$$

for $x = \sum\limits_{k\gg -\infty} p^k[x_k]$. If now I is a closed subinterval of $[0,+\infty)$, we put $V(x,I) = \inf\limits_{r\in I} V(x,r)$. We then define the ring $\widetilde{\mathbf{B}}^I$ as the completion of $\widetilde{\mathbf{B}}^+[1/[\widetilde{\pi}]]$ for the valuation $V(\cdot, I)$ if $0 \notin I$, and as the completion of $\widetilde{\mathbf{B}}^+$ for $V(\cdot, I)$ if $I = [0, r]$.

Notation 5.1.10. (*cf* [45, §1.3] and [28, Definition 3.4.6])

(1) Let I be an interval. When $\widetilde{\mathbf{B}}^I$ (resp. $\widetilde{\mathbf{A}}^I$) is defined, put $\widetilde{\mathbf{B}}^I_{K_\pi} := (\widetilde{\mathbf{B}}^I)^{\mathscr{G}_{K_\pi}}$ (resp. $\widetilde{\mathbf{A}}^I_{K_\pi} := (\widetilde{\mathbf{A}}^I)^{\mathscr{G}_{K_\pi}}$). Similarly, put $\mathbf{B}^I_{K_\pi} := (\mathbf{B}^I)^{\mathscr{G}_{K_\pi}}$ (resp. $\mathbf{A}^I_{K_\pi} := (\mathbf{A}^I)^{\mathscr{G}_{K_\pi}}$).

(2) We put

$$\widetilde{\mathbf{B}}^{\dagger,r}_{\mathsf{rig}} = \widetilde{\mathbf{B}}^{[r,+\infty)}, \quad \widetilde{\mathbf{B}}^{\dagger}_{\mathsf{rig}} = \bigcup_{r\geq 0} \widetilde{\mathbf{B}}^{\dagger,r}_{\mathsf{rig}},$$

$$\mathbf{B}^{\dagger,r}_{\mathsf{rig}} := \mathbf{B}^{[r,+\infty)}, \quad \mathbf{B}^{\dagger}_{\mathsf{rig}} := \bigcup_{r\geq 0} \mathbf{B}^{\dagger,r}_{\mathsf{rig}},$$

$$\mathbf{B}^{\dagger,r}_{\mathsf{rig},K_\pi} = \mathbf{B}^{[r,+\infty)}_{K_\pi}, \quad \mathbf{B}^{\dagger}_{\mathsf{rig},K_\pi} = \bigcup_{r\geq 0} \mathbf{B}^{\dagger,r}_{\mathsf{rig},K_\pi}.$$

Remark 5.1.11. We have

$$\mathbf{B}^{\dagger}_{\mathsf{rig},K_\pi} = \bigcup_{r\geq 0} \bigcap_{\substack{s>r \\ s\in \mathbf{Z}[1/p]}} \mathbf{B}^{[r,s]}_{K_\pi}.$$

5.1.12 Relation with Laurent series

Notation 5.1.13. (*cf* [32]) Denote by $D[0,1)$ the open rigid analytic disc of outer radius 1 with co-ordinate u. For any subinterval $I \subset [0,1)$, we denote by $D(I) \subset D[0,1)$ the admissible open subspace whose $\overline{K}$-points correspond to $x \in \overline{K}$ with $|x| \in I$. We set $\mathcal{O}_I = \Gamma(D(I), \mathcal{O}_{D(I)})$, and $\mathcal{O} = \mathcal{O}_{[0,1)}$.

Remark 5.1.14. We have $\mathcal{O}_{[0,1)} = \mathbf{B}^{[0,+\infty)}_{K_\pi}$.

Definition 5.1.15. (1) Let I be as in the last section. We put

$$\mathbf{B}^I(K_\pi) = \Big\{ \sum_{k\in\mathbf{Z}} a_k T^k;\ a_k \in \mathsf{W}(k)[1/p],\ \big(\forall \rho \in I\big)\ \lim_{|k|\to\infty} v_p(a_k) + \frac{p-1}{pe}\cdot\frac{k}{\rho} = +\infty \Big\}.$$

(2) Let $r \geq 0$ and we put

$$\mathcal{R}_r = \Big\{ \sum_{i\in\mathbf{Z}} a_i u^i;\ a_i \in W(k)[1/p],\ \big(\forall \rho \in [p^{-r},1)\big)\ \lim_{i\to\pm\infty} |a_i|\rho^i = 0 \Big\}.$$

In other words, elements of $\mathcal{R}_r$ are Laurent series $\sum_{i\in\mathbf{Z}} a_i u^i$ that converge on the annulus $p^{-r} \leq |u| < 1$. We define *Robba ring* to be

$$\mathcal{R} = \bigcup_{r>0} \mathcal{R}_r.$$

In other words, elements of $\mathcal{R}$ are Laurent series $\sum_{i\in\mathbf{Z}} a_i u^i$ that converge on the annulus $p^{-r} \leq |u| < 1$ for some $r \geq 0$.

(3) For any $0 < \rho < 1$ and $x = \sum_{i\in\mathbf{Z}} a_i u^i \in \mathcal{R}$, we define the *$\rho$-Gauss norm* over $\mathcal{R}$ as follows:

$$\big|x\big|_\rho := \sup_i \big\{|a_i|\rho^i\big\}.$$

Remark 5.1.16.

(1) The ring $\mathcal{R}_r$ carries a Fréchet topology, in which a sequence converges if and only if it converges under the ρ-Gauss norm for all $\rho \in [p^{-r}, 1)$. (It is complete for this topology.)

(2) The ring $\mathcal{R}$ carries a *limit-of-Fréchet topology*, or *LF topology*. This topology is defined on $\mathcal{R}$ by taking the locally convex direct limit of the $\mathcal{R}_r$ (each equipped with the Fréchet topology). In particular, a sequence converges in $\mathcal{R}$ if it is a convergent sequence in $\mathcal{R}_r$ for some $r > 0$.

Proposition 5.1.17. *If the endpoints of an interval I lie in $\mathbf{Z}_{>0}[1/p]$, the map $T \mapsto u$ induces an isomorphism*

$$\mathbf{B}^I(K_\pi) \simeq \mathbf{B}^I_{K_\pi}$$

hence we have an equality $\mathcal{R} = \mathbf{B}^\dagger_{\mathsf{rig},K_\pi}$ by remark 5.1.11.

Proof. *cf* [28, Lemma 2.2.7 (2)]. □

Remark 5.1.18. (*cf* [28, Proposition 2.2.10]) For $n \in \mathbf{Z}_{\geq 0}$, put $r_n := (p-1)p^n - 1$. We have

$$\mathbf{A}^{[0,+\infty]}_{K_\pi} = \mathbf{A}^+_{K_\pi},$$

$$\mathbf{A}^{[0,r_k]}_{K_\pi} = \mathbf{A}^+_{K_\pi}\Big\{\frac{u^{ep^k}}{p}\Big\},$$

$$\mathbf{A}^{[r_l,+\infty]}_{K_\pi} = \mathbf{A}^+_{K_\pi}\Big\{\frac{p}{u^{ep^l}}\Big\},$$

$$\mathbf{A}^{[r_l,r_k]}_{K_\pi} = \mathbf{A}^+_{K_\pi}\Big\{\frac{u^{ep^k}}{p}, \frac{p}{u^{ep^l}}\Big\}.$$

Notation 5.1.19. (*cf* [45, Proposition 2.2.5]) Put $\mathcal{R}_\tau = (\widetilde{\mathbf{B}}^\dagger_{\mathsf{rig}})^{\mathscr{G}_L} = \widetilde{\mathbf{B}}^\dagger_{\mathsf{rig},L}$. Then we have $\mathbf{B}^\dagger_{K_\pi} \subset \mathcal{R} \subset \mathcal{R}_\tau \subset \widetilde{\mathbf{B}}^\dagger_{\mathsf{rig}}$.

5.2 The (φ, τ)-modules over $(\mathcal{R}, \mathcal{R}_\tau)$

Definition 5.2.1. A (φ, τ)-*module over* $(\mathcal{R}, \mathcal{R}_\tau)$ consists of

(i) an étale φ-module $D^\dagger_{\mathsf{rig}}$ over $\mathcal{R}$;

(ii) a τ-semi-linear endomorphism τ_D on $D^\dagger_{\mathsf{rig},\tau} := \mathcal{R}_\tau \otimes_{\mathcal{R}} D^\dagger_{\mathsf{rig}}$ which commutes with $\varphi_{\mathcal{R}_\tau} \otimes \varphi_{D^\dagger_{\mathsf{rig}}}$ (where $\varphi_{\mathcal{R}_\tau}$ is the Frobenius map on $\mathcal{R}_\tau$ and $\varphi_{D^\dagger_{\mathsf{rig}}}$ is the Frobenius map on $D^\dagger_{\mathsf{rig}}$) and which satisfies:

$$(\forall x \in D^\dagger_{\mathsf{rig}}) \quad (g \otimes 1) \circ \tau_D(x) = \tau_D^{\chi(g)}(x),$$

for all $g \in \mathscr{G}_{K_\pi}/\mathscr{G}_L$ such that $\chi(g) \in \mathbf{Z}_{>0}$. The corresponding category is denoted $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau)$.

Theorem 5.2.2. *The functors*

$$\begin{aligned} \mathcal{D}^\dagger_{\mathsf{rig}}\colon \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K) &\to \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau) \\ V &\mapsto \mathcal{D}^\dagger_{\mathsf{rig}}(V) = \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V) \\ \mathcal{V}(D^\dagger_{\mathsf{rig}}) = (\mathbf{B}^\dagger_{\mathsf{rig}} \otimes_{\mathcal{R}} D^\dagger_{\mathsf{rig}})^{\varphi=1} &\hookleftarrow D^\dagger_{\mathsf{rig}} \end{aligned}$$

(*with the natural τ-semi-linear endomorphism τ_D over $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_\tau = \mathcal{R}_\tau \otimes_{\mathcal{R}} \mathcal{D}^\dagger_{\mathsf{rig}}(V)$*) *establish quasi-inverse equivalences of categories.*

Proof. By theorem 4.1.8, it is enough to show that the functor

$$\begin{aligned}\mathbf{Mod}_{\mathcal{E}^\dagger,\mathcal{E}^\dagger_\tau}(\varphi,\tau) &\to \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau)\\ (D^\dagger, D^\dagger_\tau) &\mapsto (\mathcal{R}\otimes_{\mathcal{E}^\dagger} D^\dagger, \mathcal{R}_\tau\otimes_{\mathcal{E}^\dagger} D^\dagger)\end{aligned}$$

is an equivalence of category. By [31, Theorem 6.33], the functor

$$\begin{aligned}\mathbf{Mod}^{\text{ét}}_{\mathcal{E}^\dagger}(\varphi) &\xrightarrow{(*)} \mathbf{Mod}^{\text{ét}}_{\mathcal{R}}(\varphi)\\ D^\dagger &\mapsto \mathcal{R}\otimes_{\mathcal{E}^\dagger} D^\dagger\end{aligned}$$

is an equivalence of categories, where $\mathbf{Mod}^{\text{ét}}_{\mathcal{E}^\dagger}(\varphi)$ is the category of étale φ-modules over $\mathcal{E}^\dagger$ and $\mathbf{Mod}^{\text{ét}}_{\mathcal{R}}(\varphi)$ is the category of étale φ-modules over $\mathcal{R}$ (*i.e.* pure of slope 0). Let $(D_1^\dagger, D_{1,\tau}^\dagger), (D_2^\dagger, D_{2,\tau}^\dagger) \in \mathbf{Mod}_{\mathcal{E}^\dagger,\mathcal{E}^\dagger_\tau}(\varphi,\tau)$ and

$$f : \mathcal{R}\otimes_{\mathcal{E}^\dagger} D_1^\dagger \to \mathcal{R}\otimes_{\mathcal{E}^\dagger} D_2^\dagger$$

be a morphism in $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau)$. By fully-faithfullness of $(*)$, f comes from a unique map $f_0 : D_1^\dagger \to D_2^\dagger$ in $\mathbf{Mod}_{\mathcal{E}^\dagger,\mathcal{E}^\dagger_\tau}(\varphi,\tau)$ by extension of scalars. The latter induces a map $1\otimes f_0 : D_{1,\tau}^\dagger \to D_{2,\tau}^\dagger$ that inserts in the commutative square:

$$\begin{array}{ccc} D_{1,\tau}^\dagger & \xrightarrow{1\otimes f_0} & D_{2,\tau}^\dagger \\ \downarrow & & \downarrow \\ \mathcal{R}_\tau\otimes_{\mathcal{E}^\dagger} D_1^\dagger & \xrightarrow{1\otimes f} & \mathcal{R}_\tau\otimes_{\mathcal{E}^\dagger} D_2^\dagger, \end{array}$$

which shows that $1\otimes f_0$ is compatible with the operator τ_{D_1} and τ_{D_2}, *i.e.* that f_0 is a morphism in $\mathbf{Mod}_{\mathcal{E}^\dagger,\mathcal{E}^\dagger_\tau}(\varphi,\tau)$.

Let $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau)$. By the essential surjectivity of $(*)$, there exists $D^\dagger \in \mathbf{Mod}^{\text{ét}}_{\mathcal{E}^\dagger}(\varphi)$, such that $D^\dagger_{\mathsf{rig}} \simeq \mathcal{R}\otimes_{\mathcal{E}^\dagger} D^\dagger$ as a φ-module. We have

$$D^\dagger_\tau = \mathcal{E}^\dagger_\tau \otimes_{\mathcal{E}^\dagger} D^\dagger \subset D^\dagger_{\mathsf{rig},\tau}.$$

Notice that $D^\dagger_\tau$ is the unique φ-stable $\mathcal{E}^\dagger_\tau$-lattice of $D^\dagger_{\mathsf{rig},\tau}$ (as $\mathcal{R}_\tau$ is faithfully flat over $\mathcal{E}^\dagger$): this implies that the τ_D map over $D^\dagger_{\mathsf{rig},\tau}$ maps $D^\dagger_\tau$ into itself, so that $(D^\dagger, D^\dagger_\tau) \in \mathbf{Mod}_{\mathcal{E}^\dagger,\mathcal{E}^\dagger_\tau}(\varphi,\tau)$. □

Remark 5.2.3. We can define similarly the category $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^r})$ for $r \in \mathbf{N}$. Notice that for any $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^r})$, there is a τ^{p^r}-semilinear endomorphism $\tau_D^{p^r}$ on $D^\dagger_{\mathsf{rig},\tau}$ rather than a τ-semilinear endomorphism τ_D (which is the case in $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau)$).

Definition 5.2.4. (*cf* [45, §1.2]) Let G be a p-adic Lie group and let W be a Banach $\mathbf{Q}_p$-representation of G. Let H be an open subgroup of G that admits a system of coordinates that induces an analytic bijection $H \to \mathbf{Z}_p^d$.

(1) We say that $x \in W$ is *H-analytic* if the orbit map $H \to W$; $g \mapsto g(x)$ is analytic (*cf* [9, §2]).

(2) We say that $x \in W$ is *locally analytic* (for G), if the orbit map $G \to W$; $g \mapsto g(x)$ is locally analytic, *i.e.* there exists some open $H \leqslant G$ such that x is H-analytic. We denote by W^{la} the set of locally analytic elements in W.

(3) Let W be a Fréchet space, whose topology is defined by a sequence of semi-norms $(p_i)_{i\geq 1}$. We then denote W_i the completion of W for p_i, hence we have $W = \varprojlim_{i\geq 1} W_i$. We say that $w \in W$ is *pro-analytic* if its image $\pi_i(w) \in W_i$ is locally analytic for all i. We denote by W^{pa} the set of pro-analytic elements in W.

Definition 5.2.5. (1) Let $E(u) \in \mathsf{W}(k)[u]$ be the Eisenstein polynomial of π. We put (*cf* [32])

$$c := p\frac{E(u)}{E(0)} \in \mathsf{W}(k)[u]$$

$$\lambda := \prod_{n=0}^{\infty} \varphi^n(E(u)/E(0)) \in \mathcal{O}$$

$$t = \log([\varepsilon]) := \sum_{n=1}^{\infty}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n} \in \widetilde{\mathbf{B}}^{\dagger}_{\mathsf{rig},L}.$$

(2) For any $\mathscr{G}_{K_\pi}$-representation, we define the following operators (*cf* [45, Définition 1.2.12]):

$$\log(\tau^{p^n}) := -\sum_{k=1}^{\infty}\frac{(1-\tau^{p^n})^k}{k}, \text{ for } n \in \mathbf{N}$$

$$\nabla_\tau := \frac{1}{p^n}\log(\tau^{p^n}), \text{ for } n \gg 0.$$

By [45, §2.2], the operator ∇_τ is defined on $(\widetilde{\mathbf{B}}^{\dagger}_{\mathsf{rig},L})^{\mathrm{pa}}$.

(3) We put (*cf* [45, §2.2])

$$N_\nabla := \frac{-\lambda}{t}\nabla_\tau.$$

It is showed in *loc. cit.* that N_∇ induces an operator on $\mathcal{R}$ (whereas ∇_τ does not). Remark that later we will extend it to certain $\mathcal{R}$-modules, and then to $\mathcal{D}^{\dagger}_{\mathsf{rig}}(V)$ for $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$.

Remark 5.2.6. (1) We have $\lambda = \frac{E(u)}{E(0)}\varphi(\lambda)$.

(2) The element c we just defined is different from that in [19], there $c = \frac{E(0)}{p} \in \mathsf{W}(k)^\times$.

(3) For any $n \in \mathbf{N}$, the element $\prod_{i=0}^{n}\varphi^i(\frac{E(u)}{E(0)})$ is invertible in $\mathcal{R}$.

Proof. Notice that $\frac{E(u)}{E(0)} = \frac{E(u)}{pa}$ with $a \in W(k)^\times$: it is hence in $\mathcal{E}^\dagger$. Hence $\frac{E(u)}{E(0)} \in \mathcal{R}$ is a unit of the Robba ring (*cf* [12, diagram (10.4.3)]), thus $\varphi^n(\frac{E(u)}{E(0)})$ are units of the Robba ring for all $n \in \mathbf{N}$, and so is $\prod_{i=0}^{n}\varphi^i(\frac{E(u)}{E(0)})$. □

(4) (*cf* [45, §2.2]) We have $N_\nabla = \frac{-\lambda}{t}\nabla_\tau = -u\lambda\frac{d}{du}$ as operators over $\mathcal{R}$: the operator N_∇ we defined coincides with that in [32, §1].

Proof. By direct computation we have

$$-u\lambda\frac{d}{du}(u^n) = -n\lambda u^n.$$

On the other hand $\exp(p^r\nabla_\tau)(u^n) = \tau^{p^r}(u^n)$ for $r \gg 0$ and $\tau^{p^r}(u^n) = [\varepsilon]^{np^r}u^n$. Using the expansion $\exp(p^r\nabla_\tau)(u^n) = u^n + p^r\nabla_\tau(u^n) + O(p^{2r})$ for $r \gg 0$, we have

$$\begin{aligned}\nabla_\tau(u^n) &= \lim_{r\to\infty}\frac{[\varepsilon]^{np^r}-1}{p^r}u^n \\ &= \lim_{r\to\infty}\frac{e^{tnp^r}-1}{p^r}u^n \\ &= ntu^n.\end{aligned}$$

Hence we have

$$\frac{-\lambda}{t}\nabla_\tau(u^n) = -\frac{\lambda}{t}ntu^n = -n\lambda u^n.$$

This finishes the proof. □

5.3 The complex $\mathcal{C}_{\varphi,N_\nabla}$

Definition 5.3.1. (*cf* [45, 2.2.1]) We define a (φ, N_∇)-*module over* $\mathcal{R}$ to be a free $\mathcal{R}$-module D endowed with a Frobenius map φ and a connection $N_\nabla\colon D \to D$ over $N_\nabla\colon \mathcal{R} \to \mathcal{R}$, *i.e.* an additive map such that

$$(\forall m \in D)\ (\forall x \in \mathcal{R}) \quad N_\nabla(x \cdot m) = N_\nabla(x) \cdot m + x \cdot N_\nabla(m),$$

that satisfies $N_\nabla \circ \varphi = c\varphi N_\nabla$. The corresponding category is denoted $\mathbf{Mod}_{\mathcal{R}}(\varphi, N_\nabla)$.

Proposition 5.3.2. *(cf [45, Proposition 2.2.2]) Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, then*

$$N_\nabla(\mathcal{D}^\dagger_{\mathsf{rig}}(V)) \subset \mathcal{D}^\dagger_{\mathsf{rig}}(V).$$

This implies that if V is in $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, the operator N_∇ associated to its (φ,τ)-module $\mathcal{D}^\dagger_{\mathsf{rig}}(V)$ provides a (φ, N_∇)-module structure.

Definition 5.3.3. Let $D^\dagger_{\mathsf{rig}} \in \mathbf{Mod}_{\mathcal{R}}(\varphi, N_\nabla)$, we define the complex $\mathcal{C}_{\varphi,N_\nabla}(D^\dagger_{\mathsf{rig}})$ as follows:

$$0 \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \bigoplus D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), N_\nabla(x))$$

$$(y, z) \longmapsto N_\nabla(y) - (c\varphi - 1)(z).$$

If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}_{\varphi,N_\nabla}(\mathcal{D}^\dagger_{\mathsf{rig}}(V))$, which will also be simply denoted $\mathcal{C}_{\varphi,N_\nabla}(V)$.

Remark 5.3.4. The complex $\mathcal{C}_{\varphi,N_\nabla}(V)$ above is well-defined. Indeed we have

$$\Big(N_\nabla \ominus (c\varphi - 1)\Big) \circ (\varphi - 1, N_\nabla) = N_\nabla \circ (\varphi - 1) - (c\varphi - 1) \circ N_\nabla = 0,$$

as $N_\nabla \circ \varphi = c\varphi N_\nabla$ by definition of a (φ, N_∇)-module over $\mathcal{R}$.

Lemma 5.3.5. *Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ and $D = \mathcal{D}^\dagger_{\mathsf{rig}}(V)$. We have $\mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,N_\nabla)}(D, \mathcal{R}) \simeq \mathsf{H}^1(\mathcal{C}_{\varphi,N_\nabla}(D))$.*

Proof. To give an extension

$$0 \to D \to E \xrightarrow{\epsilon} \mathcal{R} \to 0$$

of $\mathcal{R}$ by D in $\mathbf{Mod}_{\mathcal{R}}(\varphi, N_\nabla)$ is equivalent to giving a (φ, N_∇)-module structure to the $\mathcal{R}$-module

$$E = D \oplus \mathcal{R} \cdot e,$$

where $e \in E$ is a preimage of 1 under ϵ. Since D is already a (φ, N_∇)-module, it suffices to specify the image of e by φ and N_∇. Since $\epsilon(\varphi(e) - e) = 0$, we must have $\varphi(e) - e \in D$ and write:

$$\varphi(e) = e + \lambda \text{ with } \lambda \in D.$$

For N_∇, we have $\epsilon(N_\nabla(e)) = N_\nabla(1) = 0$: put $\mu = N_\nabla(e) \in D$ (this completely defines N_∇ on E). In order to have an extension, the only condition that $(\lambda, \mu) \in D \oplus D$ has to satisfy is $N_\nabla \circ \varphi(e) = c\varphi \circ N_\nabla(e)$, which by construction is $N_\nabla(\lambda) + \mu = c\varphi(\mu)$, *i.e.*

$$N_\nabla(\lambda) = (c\varphi - 1)(\mu).$$

Notice that the submodule of $D \oplus D$

$$M := \big\{(\lambda, \mu) \in D \oplus D; N_\nabla(\lambda) = (c\varphi - 1)(\mu)\big\}$$

is exactly $\mathsf{Ker}\,\beta$ in the following complex (*cf* definition 5.3.3)

$$\mathcal{C}_{\varphi, N_\nabla}(D) \,:\, 0 \to D \xrightarrow{\alpha} D \oplus D \xrightarrow{\beta} D \to 0.$$

One checks that (λ, μ) corresponds to the trivial extension if and only if it lies in the $\mathsf{Im}\,\alpha$, *i.e.* there exists $d \in D$ such that $\lambda = (\varphi - 1)(d)$ and $\mu = N_\nabla(d)$. We define another submodule of M

$$N := \Big\{\big((\varphi - 1)d, (\tau_D - 1)d\big);\ d \in D\Big\}.$$

Then there is an isomorphism

$$\mathsf{Ext}(\mathcal{O}_\mathcal{E}, D) \simeq M/N \simeq \mathsf{Ker}\,\beta/\,\mathsf{Im}\,\alpha \simeq \mathsf{H}^1(\mathcal{C}_{\varphi, N_\nabla}(D)).$$

□

Notation 5.3.6. Let $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau)$. We put

$$D^\dagger_{\mathsf{rig},\tau^{p^n},0} = \big\{x \in D^\dagger_{\mathsf{rig},\tau}; (\gamma \otimes 1)x = (1 + \tau_D^{p^n} + \tau_D^{2p^n} \cdots + \tau_D^{(\chi(\gamma)-1)p^n})(x)\big\}$$

for any $n \in \mathbf{N}$.

Definition 5.3.7. Let $n \in \mathbf{N}$ and $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau^{p^n})$. We define the complex $\mathcal{C}^{\mathsf{rig}}_{\varphi,\tau^{p^n}}(D^\dagger_{\mathsf{rig}})$ as follows:

$$0 \longrightarrow D^\dagger_{\mathsf{rig}} \longrightarrow D^\dagger_{\mathsf{rig}} \bigoplus D^\dagger_{\mathsf{rig},\tau^{p^n},0} \longrightarrow D^\dagger_{\mathsf{rig},\tau^{p^n},0} \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), (\tau_D^{p^n} - 1)(x))$$

$$(y, z) \longmapsto (\tau_D^{p^n} - 1)(y) - (\varphi - 1)(z).$$

If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}^{\mathsf{rig}}_{\varphi,\tau^{p^n}}(\mathcal{D}^\dagger_{\mathsf{rig}}(V))$, which will also be simply denoted $\mathcal{C}^{\mathsf{rig}}_{\varphi,\tau^{p^n}}(V)$.

Lemma 5.3.8. *Let $r \in \mathbf{N}$ and $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, then $\mathsf{H}^i(\mathcal{C}^{\mathsf{rig}}_{\varphi,\tau^{p^r}}(V)) \simeq \mathsf{H}^i(\mathscr{G}_{K_r}, V)$ for $i \in \{0, 1\}$.*

Proof. For $i = 0$ this follows by direct computation. For $i = 1$, similar argument as in proposition 1.3.3 works, as we have an equivalence of categories between $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau^{p^r})$ and $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{K_r})$ (the proof is similar as that of theorem 5.2.2). □

Proposition 5.3.9. *Let $D \in \mathbf{Mod}_{\mathcal{R}}(\varphi, N_\nabla)$. Then there exists $n \geq 0$ and $D^\dagger_{\mathsf{rig}} \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau^{p^n})$ such that $D \simeq D^\dagger_{\mathsf{rig}}$ as (φ, N_∇)-modules over $\mathcal{R}$.*

Proof. *cf* [45, Proposition 2.2.5]. □

Lemma 5.3.10. *Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$. We have an isomorphism of groups*

$$\varinjlim_n \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R}) \to \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,N_\nabla)}(\mathcal{D}(V),\mathcal{R}).$$

Proof. Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, and we associate its (φ,N_∇)-module $\mathcal{D}(V)$ and (φ,τ^{p^n})-module $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n$ (notice that $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n = \mathcal{D}^\dagger_{\mathsf{rig}}(V)$ as φ-modules, here we just use the subscript n to indicate it is an object of $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})$). If $n \in \mathbf{N}$, an extension of its associated (φ,τ^{p^n})-module over $\mathcal{R}$ is naturally equipped with a (φ,N_∇)-module structure. This provides a map from $\mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})}(\mathcal{D}(V)_n,\mathcal{R})$ to $\mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,N_\nabla)}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R})$. Those maps are compatible as n grows: this provides the map

$$\varinjlim_n \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n,\mathcal{R}) \to \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,N_\nabla)}(\mathcal{D}(V),\mathcal{R}).$$

An extension E of $\mathcal{D}(V)$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi,N_\nabla)$ has a (φ,τ^{p^n})-module structure for some $n \gg 0$ by proposition 5.3.9. This shows that the extension E comes from an extension of $\mathcal{D}^\dagger_{\mathsf{rig}}(V)$ in the category of $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})$ for some $n \gg 0$, which implies the map is surjective. Moreover, it is also injective by [45, remark 2.2.4]. □

Lemma 5.3.11. *Let $n \in \mathbf{N}$ and $(D^\dagger_{\mathsf{rig}},D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau)$. We have an operator*

$$\frac{N_\nabla}{\tau_D^{p^n}-1} : (D^\dagger_{\mathsf{rig},\tau})^{\mathsf{pa}} \to (D^\dagger_{\mathsf{rig},\tau})^{\mathsf{pa}},$$

such that $N_\nabla = \frac{N_\nabla}{\tau_D^{p^n}-1} \circ (\tau_D^{p^n}-1) = (\tau_D^{p^n}-1) \circ \frac{N_\nabla}{\tau_D^{p^n}-1}$ on $(D^\dagger_{\mathsf{rig},\tau})^{\mathsf{pa}}$.

Proof. We have

$$\begin{aligned}
\frac{N_\nabla}{\tau_D^{p^n}-1} &= -\frac{\lambda}{t}\cdot\frac{\nabla_\tau}{\tau_D^{p^n}-1}\\
&= -\frac{\lambda}{p^m t}\cdot\frac{\log(\tau_D^{p^m})}{\tau_D^{p^n}-1} \quad (\text{for } m \gg 0)\\
&= \frac{\lambda}{p^m t}\cdot\sum_{i=1}^{\infty}\frac{(1-\tau_D^{p^m})^i}{i(\tau_D^{p^n}-1)}\\
&= -\frac{\lambda}{p^m t}\cdot\sum_{i=1}^{\infty}\frac{(1-\tau_D^{p^m})^{i-1}}{i}\cdot\left(\frac{\tau_D^{p^m}-1}{\tau_D^{p^n}-1}\right) \quad (\text{we can assume } n|m)\\
&= -\frac{\lambda}{p^m t}\cdot\sum_{i=1}^{\infty}\frac{(1-\tau_D^{p^m})^{i-1}}{i}\cdot(1+\tau_D^{p^n}+\cdots+\tau_D^{(\frac{m}{n}-1)p^n})
\end{aligned}$$

which shows it is well defined over $(D^\dagger_{\mathsf{rig},\tau})^{\mathsf{pa}}$ (*cf* [45, §2.2]). □

Definition 5.3.12. Let $(D^\dagger_{\mathsf{rig}},D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau)$. For any $n \in \mathbf{N}$, we put

$$D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0} = D^\dagger_{\mathsf{rig},\tau^{p^n},0} \cap (D^\dagger_{\mathsf{rig},\tau})^{\mathsf{pa}}.$$

We then define a complex

$$\mathcal{C}^{\mathsf{pa}}_{\varphi,\tau^{p^n}}(D^\dagger_{\mathsf{rig}}):\quad 0 \to D^\dagger_{\mathsf{rig}} \xrightarrow{(\varphi-1,\tau_D^{p^n}-1)} D^\dagger_{\mathsf{rig}} \bigoplus D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0} \xrightarrow{(\tau_D^{p^n}-1)\ominus(\varphi-1)} D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0} \to 0.$$

If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}^{\mathsf{pa}}_{\varphi,\tau^{p^n}}(\mathcal{D}(V)^\dagger_{\mathsf{rig}})$, which will also be simply denoted $\mathcal{C}^{\mathsf{pa}}_{\varphi,\tau^{p^n}}(V)$.

Remark 5.3.13. Notice that $D^\dagger_{\mathsf{rig}} \subset (D^\dagger_{\mathsf{rig},\tau})^{\mathsf{pa}}$ (*cf* [45, Lemma 1.3.4]), and the complex $\mathcal{C}^{\mathsf{pa}}_{\varphi,\tau^{p^n}}(D^\dagger_{\mathsf{rig}})$ is well-defined as φ and τ_D preserve pro-analyticity.

Lemma 5.3.14. *We have* $\gamma \circ N_\nabla = N_\nabla \circ \gamma$.

Proof. We have $\gamma \circ \tau = \tau^{\chi(\gamma)} \circ \gamma$, hence $\gamma \circ \tau^{p^m} = \tau^{\chi(\gamma)p^m} \circ \gamma$. We have

$$\begin{aligned}\gamma \circ \log(\tau^{p^m}) &= \log(\tau^{\chi(\gamma)p^m}) \circ \gamma \\ &= \chi(\gamma)\log(\tau^{p^m}) \circ \gamma,\end{aligned}$$

hence $\gamma \circ \nabla_\tau = \chi(\gamma)\nabla_\tau \circ \gamma$. We then have

$$\begin{aligned}\gamma \circ N_\nabla &= \gamma\big(-\frac{\lambda}{t}\nabla_\tau\big) \\ &= -\frac{\lambda}{\gamma(t)}\gamma \circ \nabla_\tau \\ &= -\frac{\lambda}{\chi(\gamma)t}\chi(\gamma)\nabla_\tau \circ \gamma \\ &= N_\nabla \circ \gamma.\end{aligned}$$

□

Proposition 5.3.15. *Let* $n \in \mathbf{N}$, *then the operator* $\frac{N_\nabla}{\tau_D^{p^n}-1}$ *induces a map* $\frac{N_\nabla}{\tau_D^{p^n}-1} : D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0} \to D^\dagger_{\mathsf{rig}}$.

Proof. Let $x \in D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0}$, then by lemma 5.3.14 we have

$$\begin{aligned}\gamma\bigg(\frac{N_\nabla}{\tau_D^{p^n}-1}(x)\bigg) &= \frac{N_\nabla}{\tau_D^{p^n\chi(\gamma)}-1}(\gamma(x)) \\ &= \frac{N_\nabla}{\tau_D^{p^n\chi(\gamma)}-1}\big(1+\tau_D^{p^n}+\cdots+\tau_D^{p^n(\chi(\gamma)-1)}\big)(x) \\ &= \frac{N_\nabla}{\tau_D^{p^n\chi(\gamma)}-1}\cdot\frac{\tau_D^{p^n\chi(\gamma)}-1}{\tau_D^{p^n}-1}(x) \\ &= \frac{N_\nabla}{\tau_D^{p^n}-1}(x).\end{aligned}$$

Hence $\gamma\Big(\frac{N_\nabla}{\tau_D^{p^n}-1}(x)\Big) \in (D^\dagger_{\mathsf{rig},\tau})^{\gamma=1} = D^\dagger_{\mathsf{rig}}$. □

Remark 5.3.16. Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ and $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau})$ be its (φ,τ^{p^n})-module over $(\mathcal{R},\mathcal{R}_\tau)$, then in particular $D^\dagger_{\mathsf{rig}}$ is equipped with a (φ, N_∇)-module structure. When $n \gg 0$, we have the following morphism of complexes:

$$\begin{array}{lccccccc}
\mathcal{C}^{\mathsf{pa}}_{\mathsf{rig},\tau^{p^n}}(V): & 0 \longrightarrow & D^\dagger_{\mathsf{rig}} & \xrightarrow{(\varphi-1,\tau_D^{p^n}-1)} & D^\dagger_{\mathsf{rig}} \bigoplus D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0} & \xrightarrow{(\tau_D^{p^n}-1)\ominus(\varphi-1)} & D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n},0} & \longrightarrow 0 \\
& & \| & & \downarrow{\scriptstyle \mathrm{id}\oplus\frac{N_\nabla}{\tau_D^{p^n}-1}} & & \downarrow{\scriptstyle \frac{N_\nabla}{\tau_D^{p^n}-1}} & \\
\mathcal{C}_{\varphi,N_\nabla}(V): & 0 \longrightarrow & D^\dagger_{\mathsf{rig}} & \xrightarrow{(\varphi-1,N_\nabla)} & D^\dagger_{\mathsf{rig}} \bigoplus D^\dagger_{\mathsf{rig}} & \xrightarrow{N_\nabla\ominus(\frac{pE(u)}{E(0)}\varphi-1)} & D^\dagger_{\mathsf{rig}} & \longrightarrow 0.
\end{array}$$

Proposition 5.3.17. *We have a group isomorphism* $\mathsf{H}^i(\mathcal{C}_{\varphi,N_\nabla}(V)) \simeq \varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n}, V)$ *for* $i \in \{0,1\}$.

Proof. When $i=0$, it suffices to show that $N_\nabla(x)=0$ if and only if $\tau^{p^n}(x)=x$ for some $n\gg 0$. This is indeed the case as $\exp(p^n\frac{t}{-\lambda}N_\nabla)=\tau^{p^n}$ for $n\gg 0$. When $i=1$, this follows from the bijectivity of $\varinjlim_m \mathsf{Ext}^1_{(\varphi,\tau^{p^m})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R})\to \mathsf{Ext}^1_{(\varphi,N_\nabla)}(\mathcal{D}(V),\mathcal{R})$ (*cf* lemmas 5.3.5 and 5.3.10), noting that $\mathsf{Ext}^1_{(\varphi,\tau^{p^n})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R})\simeq \mathsf{H}^1(K_n,V)$ for all $n\in\mathbf{N}$. □

Proposition 5.3.18. *If $D^\dagger_{\mathsf{rig},\tau^{p^n},0}\subset D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n}}$ for all $n\in\mathbf{N}$, then the map in proposition 5.3.17 is a natural $\mathbf{Q}_p$-linear isomorphism.*

Proof. If $D^\dagger_{\mathsf{rig},\tau^{p^n},0}\subset D^{\dagger,\mathsf{pa}}_{\mathsf{rig},\tau^{p^n}}$, then we have $\frac{N_\nabla}{\tau_D^{p^n}-1}: D^\dagger_{\mathsf{rig},\tau^{p^n},0}\to D^\dagger_{\mathsf{rig}}$ by proposition 5.3.15. We then have the following map of complexes

$$\begin{array}{lccccccc}
\mathcal{C}^{\mathsf{rig}}_{\varphi,\tau^{p^n}}(V): & 0\longrightarrow & D^\dagger_{\mathsf{rig}} & \xrightarrow{(\varphi-1,\tau_D^{p^n}-1)} & D^\dagger_{\mathsf{rig}}\bigoplus D^\dagger_{\mathsf{rig},\tau^{p^n},0} & \xrightarrow{(\tau_D^{p^n}-1)\ominus(\varphi-1)} & D^\dagger_{\mathsf{rig},\tau^{p^n},0} & \longrightarrow 0\\
& & \| & & \downarrow{\scriptstyle \mathrm{id}\oplus\frac{N_\nabla}{\tau_D^{p^n}-1}} & & \downarrow{\scriptstyle \frac{N_\nabla}{\tau_D^{p^n}-1}} & \\
\mathcal{C}_{\varphi,N_\nabla}(V): & 0\longrightarrow & D^\dagger_{\mathsf{rig}} & \xrightarrow{(\varphi-1,N_\nabla)} & D^\dagger_{\mathsf{rig}}\bigoplus D^\dagger_{\mathsf{rig}} & \xrightarrow{N_\nabla\ominus(\frac{pE(u)}{E(0)}\varphi-1)} & D^\dagger_{\mathsf{rig}} & \longrightarrow 0
\end{array}$$

this induces a natural $\mathbf{Q}_p$-linear map $\varinjlim_m \mathsf{H}^1(\mathscr{G}_{K_m},V)\to \mathsf{H}^1(\mathcal{C}_{\varphi,N_\nabla}(V))$, which identifies with the bijective map $\varinjlim_m \mathsf{Ext}^1_{(\varphi,\tau^{p^m})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R})\to \mathsf{Ext}^1_{(\varphi,N_\nabla)}(\mathcal{D}(V),\mathcal{R})$ mentioned in proposition 5.3.17. □

Remark 5.3.19. As V has finite dimension, the increasing sequence $(\mathsf{H}^0(\mathscr{G}_{K_n},V))_{n\in\mathbf{N}}$ is stationary, so that $\mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V))=\mathsf{H}^0(\mathscr{G}_{K_n},V)$ for $n\gg 0$. The analogue is not true for H^1. For instance, (suppose k is finite) if $V=\mathbf{Q}_p(1)$, we have $\dim_{\mathbf{Q}_p}(\mathsf{H}^1(\mathscr{G}_{K_n},V))\geq 1+[K_n:\mathbf{Q}_p]=1+p^n[K:\mathbf{Q}_p]$, so that the sequence $(\mathsf{H}^1(\mathscr{G}_{K_n},V))_{n\in\mathbf{N}}$ is not stationary.

5.3.20 Examples: the (φ,τ)-module of $\mathbf{Z}_p(n)$

Let $\mathfrak{t}\in \mathsf{W}(\mathcal{O}_{C^\flat})$ be an element that is not divisible by p and satisfies $\varphi(\mathfrak{t})=c\mathfrak{t}$. In fact $\mathfrak{t}$ is unique up to multiplication by an element in $\mathbf{Z}_p^\times$. By [45, Propositions 1.3.5, 1.3.6], we can normalize $\mathfrak{t}$ such that $\mathfrak{t}=\frac{t}{p\lambda}\in(\widetilde{\mathbf{A}}^+)^{\mathscr{G}_L}$ (*cf* also [35, Example 3.2.3], this element is denoted b_γ in *loc. cit.*).

By [19, 1.3.4] we see that the (φ,τ)-module structure over $(\mathcal{O}_\mathcal{E},\mathcal{O}_{\mathcal{E}_\tau})$ associated to the representation $\mathbf{Z}_p(n)$ is given by:

$$\begin{aligned}
&\mathcal{D}(\mathbf{Z}_p(n))=\mathcal{O}_\mathcal{E}\,\mathfrak{t}^{-n}\\
&\varphi(\mathfrak{t}^{-n})=c^{-n}\mathfrak{t}^{-n}\\
&\tau(\mathfrak{t}^{-n})=\Big(\frac{\lambda}{\tau(\lambda)}\Big)^{-n}\mathfrak{t}^{-n}.
\end{aligned}$$

Remark that the minus sign in the first equality follows from the fact that our constructions are covariant (compare with [19, 1.3.4]).

Remark 5.3.21. We have $g(\mathfrak{t})=\chi(g)\frac{\lambda}{g(\lambda)}\mathfrak{t}$ for $g\in\mathscr{G}_K$, and in particular $g(\mathfrak{t})=\chi(g)\mathfrak{t}$ for $g\in\mathscr{G}_{K_\pi}$.

Proof. *cf* [45, Proposition 1.3.6]. □

Lemma 5.3.22. *We have $N_\nabla(t)=0$.*

Proof. This follows directly from the definition of the element t and the operator N_∇ over $(\widetilde{\mathbf{B}}^\dagger_{\mathsf{rig},L})^{\mathsf{pa}}$. □

Lemma 5.3.23. *Let $n\in\mathbf{N}$, then we have $N_\nabla(\mathfrak{t}^{-n})=n\mathfrak{t}^{-n}\frac{N_\nabla(\lambda)}{\lambda}$.*

Proof. Recall that $t = p\lambda\mathfrak{t}$. By lemma 5.3.22 we have $N_\nabla(t) = 0$. By Leibniz's rule we have

$$N_\nabla(\lambda)\mathfrak{t} + \lambda N_\nabla(\mathfrak{t}) = 0,$$

i.e.

$$N_\nabla(\mathfrak{t}) = -\frac{\mathfrak{t}}{\lambda} N_\nabla(\lambda).$$

Again, by Leibniz's rule we have

$$\begin{aligned} N_\nabla(\mathfrak{t}^{-n}) &= -n\mathfrak{t}^{-n-1} N_\nabla(\mathfrak{t}) \\ &= -n\mathfrak{t}^{-n-1}\big(-\frac{\mathfrak{t}}{\lambda}\big) N_\nabla(\lambda) \\ &= n\mathfrak{t}^{-n}\frac{N_\nabla(\lambda)}{\lambda}. \end{aligned}$$

□

Remark 5.3.24. Similarly, the (φ, τ)-module structure over $(\mathcal{R}, \mathcal{R}_\tau)$ associated to the representation $\mathbf{Q}_p(n)$ is given by:

$$\mathcal{D}^\dagger_{\mathsf{rig}}(\mathbf{Q}_p(n)) = \mathcal{R}\,\mathfrak{t}^{-n},$$

with the operators φ and N_∇ as mentioned above.

5.3.25 Construction of a pairing

Lemma 5.3.26. *Let $T_1, T_2 \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then we have*

$$\begin{aligned} &\mathcal{D}^\dagger(T_1 \otimes_{\mathbf{Z}_p} T_2) \simeq \mathcal{D}^\dagger(T_1) \otimes_{\mathcal{O}^\dagger_\mathcal{E}} \mathcal{D}^\dagger(T_2) \\ &\mathcal{D}^\dagger(\mathsf{Hom}_{\mathbf{Z}_p}(T_1, T_2)) \simeq \mathsf{Hom}_{\mathcal{O}^\dagger_\mathcal{E}}(\mathcal{D}^\dagger(T_1), \mathcal{D}^\dagger(T_2)). \end{aligned}$$

Proof. The proof is similar to that of proposition 1.1.22. □

Proposition 5.3.27. *Suppose the residue field k is finite. We have a pairing of groups*

$$\mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) \times \mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) \to \mathbf{Q}_p\,.$$

Proof. By lemma 5.3.26, we have $\mathcal{D}^\dagger(V^\vee(1)) = \mathsf{Hom}_{\mathcal{E}^\dagger}(\mathcal{D}^\dagger(V), \mathcal{D}^\dagger(\mathbf{Q}_p(1)))$. Hence we have

$$\begin{aligned} \mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) &= (\mathcal{R} \otimes_{\mathcal{E}^\dagger} (\mathcal{D}^\dagger(V^\vee(1)))^{\varphi=1, N_\nabla=0} \\ &= (\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathsf{Hom}_{\mathcal{E}^\dagger}(\mathcal{D}^\dagger(V), \mathcal{D}^\dagger(\mathbf{Q}_p(1))))^{\varphi=1, N_\nabla=0} \\ &= (\mathsf{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V), \mathcal{R}\,\mathfrak{t}^{-1}))^{\varphi=1, N_\nabla=0}. \end{aligned}$$

Indeed, we used the $\mathscr{G}_K$-equivariant isomorphism of $\mathcal{R}$-modules $\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(\mathbf{Q}_p(1)) \simeq \mathcal{R}\,\mathfrak{t}^{-1}$ (*cf* section 5.3.20). We start with the pairing:

$$\begin{aligned} \mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) \times (\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)) &\to \mathcal{R}\,\mathfrak{t}^{-1} \\ (f,\ x) &\mapsto f(x). \end{aligned}$$

Recall that elements in $\mathcal{R}$ can be written uniquely as series $\sum_{n\in\mathbf{Z}} a_n u^n$ with $a_n \in \mathsf{W}(k)[1/p]$. We define a residue map over $\mathcal{R}\,\mathfrak{t}^{-1}$ as follows:

$$\begin{aligned} \mathrm{res}\colon \mathcal{R}\,\mathfrak{t}^{-1} &\to \mathsf{W}(k)[1/p] \\ \mathfrak{t}^{-1}\sum_{n\in\mathbf{Z}} a_n u^n &\mapsto a_0. \end{aligned}$$

Composing with the trace map $\mathrm{Tr} = \mathrm{Tr}_{\mathsf{W}(k)[1/p]/\mathbf{Q}_p} \colon \mathsf{W}(k)[1/p] \to \mathbf{Q}_p$, we obtain a map

$$\begin{aligned} \mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) \times (\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)) &\to \mathbf{Q}_p \\ (f,\ x) &\mapsto \mathrm{Tr}(\mathrm{res}\, f(x)). \end{aligned}$$

Recall that $\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) \simeq (\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V))/\,\mathsf{Im}((c\varphi - 1) \oplus N_\nabla))$. To construct the claimed pairing, we must show that the above map factors through $\mathsf{Im}((c\varphi - 1) \oplus N_\nabla) \subset \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)$. In other words, for any $x \in \mathsf{Im}((c\varphi - 1) \oplus N_\nabla)$, we have to show that $\mathrm{Tr}(\mathrm{res}\, f(x)) = 0$.

Notice that for any $f \in \mathsf{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V), \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(\mathbf{Q}_p(1)))^{\varphi=1, N_\nabla=0}$, the condition $N_\nabla(f) = 0$ means that $N_\nabla \circ f = f \circ N_\nabla$, and $\varphi(f) = f$ means that $\varphi \circ f = f \circ \varphi$. Indeed, the actions are $N_\nabla(f) = N_\nabla \circ f - f \circ N_\nabla$ and $\varphi(f) = \varphi \circ f \circ \varphi^{-1}$.

For any $x \in \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)$ and $f \in \mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1)))$, write $f(x) = \mathfrak{t}^{-1} r = \mathfrak{t}^{-1} \sum\limits_{n \in \mathbf{Z}} a_n u^n \in \mathcal{R}\,\mathfrak{t}^{-1}$ with $a_n \in \mathsf{W}(k)[1/p]$. By remark 5.2.6 and lemma 5.3.23, we have

$$\begin{aligned} \mathrm{Tr} \circ \mathrm{res}(f(N_\nabla(x))) &= \mathrm{Tr} \circ \mathrm{res}\left(N_\nabla(f(x))\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(N_\nabla(\mathfrak{t}^{-1} r)\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(N_\nabla(\mathfrak{t}^{-1}) r + N_\nabla(r)\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{N_\nabla(\lambda)}{\lambda}\mathfrak{t}^{-1} r + N_\nabla(r)\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{N_\nabla(\lambda r)}{\lambda}\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(-u\frac{d}{du}(\lambda r)\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr}(0) \\ &= 0. \end{aligned}$$

By section 5.3.20 we have $\varphi(\mathfrak{t}^{-1}) = c^{-1}\mathfrak{t}^{-1}$, hence

$$\begin{aligned} \mathrm{Tr} \circ \mathrm{res}(f((c\varphi - 1)x)) &= \mathrm{Tr} \circ \mathrm{res}((c\varphi - 1)f(x)) \\ &= \mathrm{Tr} \circ \mathrm{res}\left((c\varphi - 1)(\mathfrak{t}^{-1} \sum_{n \in \mathbf{Z}} a_n u^n)\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(cc^{-1}\mathfrak{t}^{-1}\varphi(\sum_{n \in \mathbf{Z}} a_n u^n) - \mathfrak{t}^{-1} \sum_{n \in \mathbf{Z}} a_n u^n\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\mathfrak{t}^{-1}(\varphi - 1)(\sum_{n \in \mathbf{Z}} a_n u^n)\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\mathfrak{t}^{-1}(\sum_{n \in \mathbf{Z}} \varphi(a_n) u^{pn} - \sum_{n \in \mathbf{Z}} a_n u^n)\right) \\ &= \mathrm{Tr}(\varphi(a_0) - a_0) \\ &= (\varphi - 1)\,\mathrm{Tr}(a_0) \\ &= 0. \end{aligned}$$

Hence we have the desired pairing

$$\begin{aligned} \mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) \times \mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) &\to \mathbf{Q}_p \\ (f, x) &\mapsto B(f,x) = \mathrm{Tr}(\mathrm{res}\, f(x)). \end{aligned}$$

□

Proposition 5.3.28. *The pairing just constructed is nondegenerate on the left.*

Proof. Recall that $\mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) = \mathsf{Hom}_{\mathcal{R},\varphi,N_\nabla}(\mathcal{R}\otimes\mathcal{D}^\dagger(V), \mathcal{R}\mathfrak{t}^{-1})$ (*cf* the proof of proposition 5.3.27). Suppose $\mathcal{R}\otimes\mathcal{D}^\dagger(V)$ has rank $d \in \mathbf{N}_{>0}$, and fix a $\mathcal{R}$-basis $\{e_1, \ldots, e_d\}$. Then $f \in \mathsf{Hom}_{\mathcal{R},\varphi,N_\nabla}(\mathcal{R}\otimes\mathcal{D}^\dagger(V), \mathcal{R}\mathfrak{t}^{-1})$ is determined by the image of the basis (by $\mathcal{R}$-linearity), *i.e.* it is determined by a $d\times 1$ matrix with coefficients in $\mathcal{R}\mathfrak{t}^{-1}$ (it also satisfies certain conditions arising from the compatibility with φ and N_∇). Denote

$$\begin{aligned} B\colon \mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) \times \mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) &\to \mathbf{Q}_p \\ (f, x) &\mapsto B(f,x) = \mathrm{Tr}(\mathrm{res}\, f(x)) \end{aligned}$$

the pairing constructed in proposition 5.3.27. Let $f \in \mathsf{Hom}_{\mathcal{R},\varphi,N_\nabla}(\mathcal{R}\otimes\mathcal{D}^\dagger(V), \mathcal{R}\mathfrak{t}^{-1})$ be such that $B(f,z) = 0$ for all $z \in \mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V))$, we claim that $f = 0$.

Suppose f corresponds to $(r_1\mathfrak{t}^{-1}, \ldots, r_d\mathfrak{t}^{-1}) \in \mathsf{M}_{d\times 1}(\mathcal{R}\mathfrak{t}^{-1})$ under the fixed basis with $r_i \in \mathcal{R}$ for $i \in \{1, \ldots, d\}$. Write $r_i = \sum\limits_{j\in\mathbf{Z}} a_{i,j}u^j$ with $a_{i,j} \in \mathsf{W}(k)[1/p]$. We first prove $r_1 = 0$ and the others follow similarly. For any $x \in \mathcal{R}\otimes_{\mathcal{E}^\dagger}\mathcal{D}^\dagger(V)$ with its image $\overline{x} \in \mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) \simeq (\mathcal{R}\otimes_{\mathcal{E}^\dagger}\mathcal{D}^\dagger(V))/\,\mathsf{Im}((c\varphi - 1)\oplus N_\nabla)$, we have $B(f,\overline{x}) = 0$ by assumption. To prove $r_1 = 0$, it suffices to prove $a_{1,j} = 0$ for any $j \in \mathbf{Z}$. For any $j \in \mathbf{Z}$, consider x whose coordinates under the fixed basis is $(\alpha u^{-j}, 0, \ldots, 0) \in \mathcal{R}^d$, with $\alpha \in \mathsf{W}(k)[1/p]$. Then we have

$$f(\overline{x}) = \alpha u^{-j}\mathfrak{t}^{-1}r_1 = \mathfrak{t}^{-1}\alpha u^{-j}\sum_{m\in\mathbf{Z}} a_{1,m}u^m = \mathfrak{t}^{-1}\sum_{m\in\mathbf{Z}} \alpha a_{1,m}u^{m-j}.$$

We have $\mathrm{res}(f(\overline{x})) = \alpha a_{1,j}$. If $a_{1,j} \neq 0$, then we can always find $\alpha \in \mathsf{W}(k)[1/p]$ such that $\mathrm{Tr}(\alpha a_{1,j}) \neq 0$, which then contradicts the hypothesis that $B(f,\overline{x}) = 0$ for any $x \in \mathcal{R}\otimes_{\mathcal{E}^\dagger}\mathcal{D}^\dagger(V)$. Hence $a_j^1 = 0$ for all $j \in \mathbf{Z}$ and hence $r_1 = 0$. Similarly we can show that $r_i = 0$ for all $i \in \{1, \ldots, d\}$ and hence $f = 0$. This shows that B is nondegenerate on the left. □

Remark 5.3.29. Assume the previous pairing is perfect, then for any $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have

$$\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) = \mathsf{H}^2(\mathscr{G}_{K_n}, V),\ n \gg 0.$$

Proof. The perfect pairing

$$\mathsf{H}^0(\mathcal{C}_{\varphi,N_\nabla}(V^\vee(1))) \times \mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) \to \mathbf{Q}_p$$

implies

$$\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) = \mathsf{Hom}_{\mathbf{Q}_p}(\varinjlim_m \mathsf{H}^0(\mathscr{G}_{K_m}, V^\vee(1)), \mathbf{Q}_p).$$

Notice that the sequence $\left(\mathsf{H}^0(\mathscr{G}_{K_m}, V^\vee(1))\right)_{m\geq 0}$ is stationary (it is an increasing sequence of sub $\mathbf{Q}_p$-vector spaces of $V^\vee(1)$), hence for $n \gg 0$, we have

$$\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V)) = \mathsf{Hom}_{\mathbf{Q}_p}(\mathsf{H}^0(\mathscr{G}_{K_n}, V^\vee(1)), \mathbf{Q}_p) = \mathsf{H}^0(\mathscr{G}_{K_n}, V^\vee(1))^\vee.$$

Since $\mathsf{H}^0(\mathscr{G}_{K_n}, V^\vee(1))^\vee$ is isomorphic to $\mathsf{H}^2(\mathscr{G}_{K_n}, V)$ by Tate duality, we hence conclude. □

5.3.30 Examples with $\mathbf{Q}_p(n)$

Assume k is finite.

Remark 5.3.31. Let $n, r \in \mathbf{N}$, then by Tate duality we have

$$\mathsf{H}^2(\mathscr{G}_{K_r}, \mathbf{Q}_p(n)) \simeq \mathsf{H}^0(\mathscr{G}_{K_r}, \mathbf{Q}_p(1-n)).$$

As

$$\mathsf{H}^0(\mathscr{G}_{K_r}, \mathbf{Q}_p(n)) \simeq \begin{cases} \mathbf{Q}_p & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases}$$

we have

$$\mathsf{H}^2(\mathscr{G}_{K_r},\mathbf{Q}_p(n))\simeq\begin{cases}\mathbf{Q}_p & \text{if } n=1\\ 0 & \text{if } n\neq 1.\end{cases}$$

We have computed $\mathsf{H}^i(\mathcal{C}_{\varphi,N_\nabla}(V))$ for $i\in\{0,1\}$ in proposition 5.3.17. Let's see some examples for $\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(V))$.

Example 5.3.32. *(1) For $V=\mathbf{Q}_p(n)$ we have $\mathcal{D}^\dagger_{\mathsf{rig}}(\mathbf{Q}_p(n))\simeq\mathcal{R}\,\mathfrak{t}^{-n}$ and the complex $\mathcal{C}_{\varphi,N_\nabla}(\mathbf{Q}_p)$ is*

$$\begin{array}{ccccccccc}0&\longrightarrow&\mathcal{R}\,\mathfrak{t}^{-n}&\longrightarrow&\mathcal{R}\,\mathfrak{t}^{-n}\bigoplus\mathcal{R}\,\mathfrak{t}^{-n}&\longrightarrow&\mathcal{R}\,\mathfrak{t}^{-n}&\longrightarrow&0\\ &&x&\longmapsto&((\varphi-1)(x),N_\nabla(x))&&&&\\ &&&&(y,z)&\longmapsto&N_\nabla(y)-(c\varphi-1)(z).&&\end{array}$$

For any $f,g\in\mathcal{R}$, we have

$$\begin{aligned}N_\nabla(f\mathfrak{t}^{-n})&=N_\nabla(f)\mathfrak{t}^{-n}+fn\mathfrak{t}^{-n}\frac{N_\nabla(\lambda)}{\lambda}=(N_\nabla+n\frac{N_\nabla(\lambda)}{\lambda})(f)\cdot\mathfrak{t}^{-n},\\ (c\varphi-1)(g\mathfrak{t}^{-n})&=c\varphi(g)c^{-n}\mathfrak{t}^{-n}-g\mathfrak{t}^{-n}=(c^{1-n}\varphi-1)(g)\cdot\mathfrak{t}^{-n}.\end{aligned}$$

Hence

$$\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(\mathbf{Q}_p(n)))\simeq\mathcal{R}\,\big/\big(\mathsf{Im}(N_\nabla+n\frac{N_\nabla(\lambda)}{\lambda})+\mathsf{Im}(c^{1-n}\varphi-1)\big).$$

(2) In particular, for $V=\mathbf{Q}_p$, the complex $\mathcal{C}_{\varphi,N_\nabla}(\mathbf{Q}_p)$ is

$$\begin{array}{ccccccccc}0&\longrightarrow&\mathcal{R}&\longrightarrow&\mathcal{R}\bigoplus\mathcal{R}&\longrightarrow&\mathcal{R}&\longrightarrow&0\\ &&x&\longmapsto&((\varphi-1)(x),N_\nabla(x))&&&&\\ &&&&(y,z)&\longmapsto&N_\nabla(y)-(c\varphi-1)(z),&&\end{array}$$

and we have $\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(\mathbf{Q}_p))\simeq\mathcal{R}\,\big/\big(\mathsf{Im}(N_\nabla)+\mathsf{Im}(c\varphi-1)\big)$.

Notation 5.3.33. By definition 5.1.15, any element of $\mathcal{R}$ can be written in the form $f(u)=\sum\limits_{i\in\mathbf{Z}}a_iu^i$ with $a_i\in\mathsf{W}(k)[1/p]$. We put $f^+(u)=\sum\limits_{i\geq 0}a_iu^i$ and $f^-(u)=\sum\limits_{i<0}a_iu^i$, then $f(u)=f^+(u)+f^-(u)$.

Lemma 5.3.34. *With notations as above, $\sum\limits_{n=0}^{\infty}(c\varphi)^n(f^+(u))$ converges in $\mathcal{O}$.*

Proof. Recall that any $h(u)=\sum\limits_{i=0}^{\infty}\lambda_iu^i\in\mathsf{W}(k)[1/p][\![u]\!]$ belongs to $\mathcal{O}$ if and only if for any $r\in[0,1)$. $\lim\limits_{i\to\infty}|\lambda_i|r^i=0$ ($|\cdot|$ denotes the p-adic absolute value), which is equivalent to $|h|_r:=\sup_i|\lambda_i|r^i$ being finite for any $r\in[0,1)$. Recall $\frac{p}{E(0)}\in\mathsf{W}(k)^\times$ and $c=\frac{pE(u)}{E(0)}$, we hence have $|c|_r=\max\{1/p,r^e\}$ and $|\varphi^k(c)|=\max\{1/p,r^{p^ke}\}$, which is $1/p$ when $k\gg 0$. Notice also that $|\varphi(h(u))|_r=|h(u)|_{r^p}$. We have (for $k\gg 0$)

$$\big|c\varphi(c)\varphi^2(c)\cdots\varphi^{k-1}(c)\varphi^k(f^+(u))\big|_r\leq O_r(\frac{1}{p^k})\big|f^+(u)\big|_{r^{p^k}}=O_r(\frac{1}{p^k}).$$

Hence $\sum_{n=0}^{\infty}(c\varphi)^n(f^+(u))$ converges in $\mathcal{O}$ as its k-th term $c\varphi(c)\varphi^2(c)\cdots\varphi^k(c)\varphi^{k+1}(f^+(u))$ tends to 0 when k tends to $+\infty$. □

Remark 5.3.35. To check $\mathsf{H}^2(\mathcal{C}_{\varphi,N_\nabla}(\mathbf{Q}_p)) = 0$ (hence $\mathcal{C}_{\varphi,N_\nabla}(\mathbf{Q}_p)$ computes the Galois cohomology by remark 5.3.31), it is equivalent to showing that $\mathcal{R}\oplus\mathcal{R} \xrightarrow{c\varphi-1\oplus N_\nabla} \mathcal{R}$ is surjective. Although we cannot prove it, we can see from the following lemma that the image contains a lot of elements.

Lemma 5.3.36. *We have*

$$\Big\{\sum_{i\geq m}^{+\infty} a_i u^i \in \mathcal{R};\ a_i \in \mathsf{W}(k)[1/p], m\in\mathbf{Z}\Big\} \subset \mathsf{Im}(c\varphi-1\oplus N_\nabla).$$

Proof. Let $f(u)=\sum_{i\in\mathbf{Z}} a_i u^i \in \mathcal{R}$, then we claim $f^+(u)\in\mathsf{Im}(c\varphi-1)$. Put $f_1(u)=\sum_{m\geq 0}(c\varphi)^m(f^+(u))$, it is well-defined in $\mathcal{R}$ by lemma 5.3.34 and satisfies $(c\varphi-1)(f_1(u))=f^+(u)$.

We now study $f^-(u)$. Observe that $a_j u^{-j}\in\mathsf{Im}(c\varphi-1\oplus N_\nabla)$ for any $j\in\mathbf{Z}_+$ and $a_j\in\mathsf{W}(k)[1/p]$. We prove by induction as follows. As $\lambda\in\mathcal{O}$, we can write $\lambda=\sum_{i=0}^{\infty}\lambda_i u^i$ and then

$$N_\nabla(a_1u^{-1}) = a_1\lambda u^{-1} = a_1\sum_{i=0}^{\infty}\lambda_i u^{i-1} = a_1\lambda_0 u^{-1} + a_1\sum_{i=1}^{\infty}\lambda_i u^{i-1}.$$

Observe that $\lambda_0=1$ and $a_1\sum_{i=1}^{\infty}\lambda_i u^{i-1}\in\mathcal{O}\subset\mathsf{Im}(c\varphi-1)$, hence $a_1u^{-1}\in\mathsf{Im}(c\varphi-1\oplus N_\nabla)$. For any $j\geq 2$, we have

$$\begin{aligned} N_\nabla(a_ju^{-j}) &= a_jj\lambda u^{-j}\\ &= a_jj\sum_{i=0}^{\infty}\lambda_i u^{i-j}\\ &= a_jju^{-j} + a_jj\lambda_1u^{1-j} + \cdots + a_jj\lambda_{j-1}u^{-1} + a_jj\sum_{i=j}^{\infty}\lambda_i u^{i-j}. \end{aligned}$$

We have $a_jj\lambda_1u^{1-j}+\cdots+a_jj\lambda_{j-1}u^{-1}\in\mathsf{Im}(c\varphi-1\oplus N_\nabla)$ by induction hypothesis and $a_jj\sum_{i=j}^{\infty}\lambda_iu^{i-j}\in\mathcal{O}$, hence in $\mathsf{Im}(c\varphi-1\oplus N_\nabla)$. Hence $a_jju^{-j}\in\mathsf{Im}(c\varphi-1\oplus N_\nabla)$ and then $a_ju^{-j}\in\mathsf{Im}(c\varphi-1\oplus N_\nabla)$. □

5.4 The complex $\mathcal{C}_{\varphi,\partial_\tau}$

Lemma 5.4.1. *For any $x=\sum_{n\in\mathbf{Z}\setminus\{0\}} x_nu^n$ with $x_n\in\mathsf{W}(k)[1/p]$, we have $u\frac{d}{du}(x)\in\mathcal{R}$ if and only if $x\in\mathcal{R}$.*

Proof. For light notation, we put $\partial_\tau = u\frac{d}{du}$ and we have $\partial_\tau(x)=\sum_{n\in\mathbf{Z}\setminus\{0\}} nx_nu^n$.

Assume $\partial_\tau(x)\in\mathcal{R}$: we show that $x\in\mathcal{R}$. For any $0<r<1$, we have $|nx_n|r^n \xrightarrow[n\to+\infty]{} 0$ and in particular for $1>\rho'>\rho$ we have

$$|nx_n|(\rho')^n = |nx_n|\big(\frac{\rho'}{\rho}\big)^n\rho^n \xrightarrow[n\to+\infty]{} 0.$$

Notice that $|\frac{1}{n}| \leq \left(\frac{\rho'}{\rho}\right)^n$ for $n \gg 0$, so that $|x_n|\rho^n \leq |nx_n|(\rho')^n$ for $n \gg 0$. Hence $|x_n|\rho^n \to 0$ as $n \to +\infty$. On the other hand, as $\partial_\tau(x) \in \mathcal{R}$, there exists $0 < r_0 < 1$ such that $|nx_n|r^n \xrightarrow[n\to-\infty]{} 0$ for any $r \in (r_0, 1)$. We claim that $|x_n|\rho^n \xrightarrow[n\to-\infty]{} 0$ for any $\rho \in (r_0, 1)$. There exists ρ' such that $r_0 < \rho' < \rho$ and

$$|nx_n|(\rho')^n = |nx_n|\left(\frac{\rho'}{\rho}\right)^n(\rho)^n \xrightarrow[n\to-\infty]{} 0.$$

We then conclude similarly from the observation $|\frac{1}{n}| \leq \left(\frac{\rho'}{\rho}\right)^n$ for $n \ll 0$.

To show $x \in \mathcal{R}$ implies $\partial_\tau(x) \in \mathcal{R}$ is direct, as $|nx_n| \leq |x_n|$ for $n \in \mathbf{Z}$. □

Notation 5.4.2. Put $\partial_\tau = \frac{1}{t}\nabla_\tau = -\frac{1}{\lambda}N_\nabla = u\frac{d}{du}$ (*cf* remark 5.2.6). Remark that this is an operator over $\mathcal{R}$ by lemma 5.4.1, and we will extend it to certain $\mathcal{R}$-modules.

Definition 5.4.3. We define a (φ, ∂_τ)-*module over* $\mathcal{R}$ to be a free $\mathcal{R}$-module D endowed with a Frobenius map φ and a connection $\partial_\tau \colon D \to D$ over $\partial_\tau \colon \mathcal{R} \to \mathcal{R}$, *i.e.* an additive map such that

$$(\forall m \in D)\ (\forall x \in \mathcal{R}) \quad \partial_\tau(x \cdot m) = \partial_\tau(x) \cdot m + x \cdot \partial_\tau(m),$$

that satisfies $\partial_\tau \circ \varphi = p\varphi\partial_\tau$. The corresponding category is denoted $\mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$.

Definition 5.4.4. (*cf* [45, §1.3])
(1) Let I be a sub-interval of $[0, +\infty)$ and $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we define $\widetilde{D}_L^I(V)$ by

$$\widetilde{D}_L^I(V) = (\widetilde{\mathbf{B}}^I \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L}.$$

(2) Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we put

$$\widetilde{D}^{\dagger,r}_{\mathsf{rig},L}(V) = (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathsf{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L}.$$

Lemma 5.4.5. *Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ with $(D^\dagger, D^\dagger_\tau) \in \mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi}, \widetilde{\mathbf{B}}^\dagger_L}(\varphi, \tau)$ its associated (φ, τ)-module over $(\mathbf{B}^\dagger_{K_\pi}, \widetilde{\mathbf{B}}^\dagger_L)$, and $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau)$ its associated (φ, τ)-module over $(\mathcal{R}, \mathcal{R}_\tau)$. Suppose $r \geq 0$ is such that $D^\dagger = \mathbf{B}^\dagger_{K_\pi} \otimes_{\mathbf{B}^{\dagger,r}_{K_\pi}} (\mathbf{B}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_{K_\pi}}$. Then for any compact interval I such that $r \leq \min(I)$, the elements of $D^\dagger$ and $D^\dagger_{\mathsf{rig}}$, seen as elements of $\widetilde{D}^I_L(V)$ are locally analytic for the group $\mathsf{Gal}(L/K)$.*

Proof. Notice that for $s \geq r$ we have

$$D^{\dagger,r} \subset D^{\dagger,r}_{\mathsf{rig}} \subset (\widetilde{D}^{\dagger,r}_{\mathsf{rig},L})^{\mathsf{la}} \subset (\widetilde{D}^{[r,s]}_L)^{\mathsf{la}}$$

(*cf* [45, Lemma 1.3.4]). □

Theorem 5.4.6. *Let M be a (φ, τ)-module over $\mathcal{R}$ whose N_∇-action is locally trivial (cf [45, Définition 3.4.1]), then there exists a unique (φ, τ)-module $D \subset M[1/\lambda]$ such that $D[1/\lambda] \subset M[1/\lambda]$ and such that $\partial_\tau(D) \subset D$.*

Proof. *cf* [45, Théorème 3.4.10]. □

Corollary 5.4.7. *Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ be a semi-stable representation and $(D^\dagger_{\mathsf{rig}}, D^\dagger_{\mathsf{rig},\tau}) \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau)$ the corresponding (φ, τ)-module over $(\mathcal{R}, \mathcal{R}_\tau)$. Then we have*

$$\partial_\tau(D^\dagger_{\mathsf{rig}}) \subset D^\dagger_{\mathsf{rig}} \subset D^\dagger_{\mathsf{rig}}[1/\lambda],$$

hence there is a (φ, ∂_τ)-module structure on $D^\dagger_{\mathsf{rig}}$.

Proof. Notice that the φ-module $D^\dagger_{\mathsf{rig}}$ of a semi-stable representation has locally trivial N_∇-action by [45, Lemma 3.4.5, Théorèm 3.4.12], hence it has a connection ∂_τ by theorem 5.4.6. □

Definition 5.4.8. Let $D \in \mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$. We define a complex $\mathcal{C}_{\varphi,\partial_\tau}(D)$ as follows:

$$0 \longrightarrow D \longrightarrow D \bigoplus D \longrightarrow D \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), \partial_\tau(x))$$

$$(y, z) \longmapsto \partial_\tau(y) - (p\varphi - 1)(z).$$

If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we have in particular the complex $\mathcal{C}_{\varphi,\partial_\tau}(\mathcal{D}^\dagger_{\mathsf{rig}}(V))$, which will be simply denoted $\mathcal{C}_{\varphi,\partial_\tau}(V)$.

5.4.9 $\mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V))$ and $\mathsf{H}^1(\mathcal{C}_{\varphi,\partial_\tau}(V))$

Proposition 5.4.10. *(cf [45, Proposition 2.2.3]) Let $V, V' \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ be crystalline representations, then $\mathcal{D}^\dagger_{\mathsf{rig}}(V)$ and $\mathcal{D}^\dagger_{\mathsf{rig}}(V')$ define the same (φ, ∂_τ)-module in $\mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$ if and only if there exist some $n \geq 0$ such that V and V' are isomorphic in $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{K_n})$.*

Proof. The proof is basically the same as that of [45, Proposition 2.2.3]. If there exists $n \geq 0$ such that $V \simeq V'$ in $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{K_n})$, then in particular $V \simeq V'$ in $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{K_\pi})$, and we have $\mathcal{D}^\dagger_{\mathsf{rig}}(V) = \mathcal{D}^\dagger_{\mathsf{rig}}(V')$ as φ-modules. Moreover, the action τ^{p^n} is the same over $(\widetilde{\mathbf{B}}^\dagger \otimes_{\mathbf{Q}_p} V)^{\mathscr{G}_L}$ and $(\widetilde{\mathbf{B}}^\dagger \otimes_{\mathbf{Q}_p} V')^{\mathscr{G}_L}$, hence $(\mathcal{D}^\dagger(V), \mathcal{D}^\dagger(V)_\tau)$ and $(\mathcal{D}^\dagger(V'), \mathcal{D}^\dagger(V')_\tau)$ are isomorphic in $\mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi}, \widetilde{\mathbf{B}}^\dagger_L}(\varphi, \tau^{p^n})$. By the definition of the operator ∂_τ, we have $\mathcal{D}^\dagger_{\mathsf{rig}}(V) = \mathcal{D}^\dagger_{\mathsf{rig}}(V')$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$.

Conversely, if V and V' are two representations such that $\mathcal{D}^\dagger_{\mathsf{rig}}(V) = \mathcal{D}^\dagger_{\mathsf{rig}}(V')$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$, we prove that there exists $n \geq 0$ such that $(\mathcal{D}^\dagger(V), \mathcal{D}^\dagger(V)_\tau) = (\mathcal{D}^\dagger(V'), \mathcal{D}^\dagger(V')_\tau)$ in $\mathbf{Mod}_{\mathbf{B}^\dagger_{K_\pi}, \widetilde{\mathbf{B}}^\dagger_L}(\varphi, \tau^{p^n})$. Let $r \geq 0$ and $(e_1, \ldots, e_d) \subset \mathcal{D}^{\dagger,r}(V) \cap \mathcal{D}^{\dagger,r}(V')$ be such that $(e_1, \ldots, e_d)$ is a basis of the φ-module $\mathcal{D}^\dagger(V) = \mathcal{D}^\dagger(V')$. Let $s \geq r$ and let $I = [r, s]$. By lemma 5.4.5, the e_i are locally analytic vectors in $\widetilde{D}^I_L(V) = \widetilde{\mathbf{B}}^I_L \otimes_{\mathbf{B}^{\dagger,r}} \mathcal{D}^{\dagger,r}(V)$ and $\widetilde{D}^I_L(V') = \widetilde{\mathbf{B}}^I_L \otimes_{\mathbf{B}^{\dagger,r}} \mathcal{D}^{\dagger,r}(V')$ (remark that $\widetilde{D}^I_L(V) = \widetilde{D}^I_L(V')$ as φ-modules by hypothesis). In particular, there exists $n \geq 0$ such that for all i, $\exp(p^n t\partial_\tau)(e_i)$ converges in $(\widetilde{D}^I_L(V))^{\mathsf{la}}$ and $(\widetilde{D}^I_L(V'))^{\mathsf{la}}$. Hence for all i, the action of τ^{p^n} on e_i inside $\widetilde{D}^I_L(V)$ and $\widetilde{D}^I_L(V')$ are the same. As we have an injection $\widetilde{D}^{\dagger,r}_L(V) \hookrightarrow \widetilde{D}^I_L(V)$ (*cf* [5, Lemma 2.7]), we conclude that the τ^{p^n}-actions over $\widetilde{D}^\dagger_{\mathsf{rig},L}(V)$ and $\widetilde{D}^\dagger_{\mathsf{rig},L}(V')$ coincide (hence the $\tau^{p^n}_D$ endomorphisms over $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_\tau$ and $\mathcal{D}^\dagger_{\mathsf{rig}}(V')_\tau$ coincide by the proof of 5.4.5), which finishes the proof. □

We now show how to construct a (φ, τ^{p^n})-module (with $n \gg 0$) from a given (φ, ∂_τ)-module.

Proposition 5.4.11. *Let $D \in \mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$. Then there exist $n \geq 0$ and $D' \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau^{p^n})$ such that $D \simeq D'$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$.*

Proof. For any $D \in \mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$, it has a natural (φ, N_∇)-module structure (as $N_\nabla = -\lambda\partial_\tau$). Then there exist $n \gg 0$ and $D' \in \mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi, \tau^{p^n})$ such that $D \simeq D'$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi, N_\nabla)$ (*cf* [45, Proposition 2.2.5]). Hence the operator N_∇ over D' is also divisible by λ (since it is for D), so that D' has a (φ, ∂_τ)-module structure and $D \simeq D'$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi, \partial_\tau)$. □

Lemma 5.4.12. *If $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ is crystalline with Hodge-Tate weights $\mathsf{HT}(V) \subset \{2, \ldots, h\}$ for some $h \in \mathbf{N}_{\geq 2}$, then we have an isomorphism of groups*

$$\varinjlim_n \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V), \mathcal{R}) \to \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,\partial_\tau)}(\mathcal{D}^\dagger_{\mathsf{rig}}(V), \mathcal{R}).$$

Proof. We associate V with its (φ,∂_τ)-module $\mathcal{D}^\dagger_{\mathsf{rig}}(V)$ and (φ,τ^{p^n})-module $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n$ (notice that $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n = \mathcal{D}^\dagger_{\mathsf{rig}}(V)$ as φ-modules, here we just use the subscript n to indicate it is an object of $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})$). If $n\in\mathbf{N}$, an extension of $\mathcal{R}$ by $\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n$ is associated to an extension of $\mathbf{Q}_p$ by V (as representations of $\mathscr{G}_{K_n}$), which is semi-stable by [5, Lemmas 6.5 and 6.6], whence is equipped with an operator ∂_τ by corollary 5.4.7. This provides a map from $\mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n,\mathcal{R})$ to $\mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,\partial_\tau)}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R})$. Those maps are compatible as n grows: this provides the map

$$\varinjlim_n \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})}(\mathcal{D}^\dagger_{\mathsf{rig}}(V)_n,\mathcal{R})\to \mathsf{Ext}^1_{\mathbf{Mod}_{\mathcal{R}}(\varphi,\partial_\tau)}(\mathcal{D}^\dagger_{\mathsf{rig}}(V),\mathcal{R}).$$

By proposition 5.4.11, an extension E of $\mathcal{R}$ by $\mathcal{D}^\dagger_{\mathsf{rig}}(V)$ in $\mathbf{Mod}_{\mathcal{R}}(\varphi,\partial_\tau)$ has a (φ,τ^{p^n})-module structure for some $n\gg 0$. This shows that the extension E comes from an extension of $\mathcal{R}$ by $\mathcal{D}^\dagger_{\mathsf{rig}}(V)$ in the category $\mathbf{Mod}_{\mathcal{R},\mathcal{R}_\tau}(\varphi,\tau^{p^n})$ for some $n\gg 0$. This implies the map is surjective. Moreover, it is also injective by proposition 5.4.10. □

Proposition 5.4.13. *If $V\in\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ is crystalline with Hodge-Tate weights $\mathsf{HT}(V)\subset\{2,\dots,h\}$ for some $h\in\mathbf{N}_{\geq 2}$, we have a group isomorphism*

$$\mathsf{H}^i(\mathcal{C}_{\varphi,\partial_\tau}(V))\simeq\varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n},V) \text{ for } i\in\{0,1\}.$$

Proof. When $i=0$, it suffices to show that $\partial_\tau(x)=0$ is equivalent to $\tau^{p^n}(x)=x$ for some $n\gg 0$. This is indeed the case as $\exp(p^n t\partial_\tau)=\tau^{p^n}$ for $n\gg 0$. When $i=1$, the isomorphism of lemma 5.4.12 identifies with an isomorphism $\mathsf{H}^1(\mathcal{C}_{\varphi,\partial_\tau}(V))\simeq\varinjlim_n \mathsf{H}^1(\mathscr{G}_{K_n},V)$. □

5.4.14 $\mathsf{H}^2(\mathcal{C}_{\varphi,\partial_\tau}(V))$

We assume k is finite.

Proposition 5.4.15. *Let $V\in\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, then we have a pairing of groups*

$$\mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V^\vee(1)))\times\mathsf{H}^2(\mathcal{C}_{\varphi,\partial_\tau}(V))\to\mathbf{Q}_p\,.$$

Proof. We have $\mathcal{D}^\dagger(V^\vee(1))=\mathsf{Hom}_{\mathcal{E}^\dagger}(\mathcal{D}^\dagger(V),\mathcal{D}^\dagger(\mathbf{Q}_p(1)))$. Hence we have

$$\begin{aligned}\mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V^\vee(1))) &= (\mathcal{D}^\dagger(V^\vee(1)))^{\varphi=1,\partial_\tau=0}\\ &= (\mathsf{Hom}_{\mathcal{E}^\dagger}(\mathcal{D}^\dagger(V),\mathcal{D}^\dagger(\mathbf{Q}_p(1))))^{\varphi=1,\partial_\tau=0}\\ &= (\mathsf{Hom}_{\mathcal{R}}(\mathcal{D}^\dagger(V),\mathcal{E}^\dagger\,\mathfrak{t}^{-1}))^{\varphi=1,\partial_\tau=0}\end{aligned}$$

We start with the following pairing:

$$\begin{aligned}\mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V^\vee(1)))\times(\mathcal{R}\otimes_{\mathcal{E}^\dagger}\mathcal{D}^\dagger(V))&\to\mathcal{R}\,\mathfrak{t}^{-1}\\ (f,\ x)&\mapsto f(x).\end{aligned}$$

Recall that elements in $\mathcal{R}$ can be written uniquely as series $\sum_{n\in\mathbf{Z}} a_n u^n$ with $a_n\in\mathsf{W}(k)[1/p]$. We define a residue map over $\mathcal{R}\,\mathfrak{t}^{-1}$ as follows: for any $z=\mathfrak{t}^{-1}\sum_{n\in\mathbf{Z}} a_n u^n=\frac{p\lambda}{t}\sum_{n\in\mathbf{Z}} a_n u^n=\frac{1}{t}\sum_{n\in\mathbf{Z}} b_n u^n$,

$$\begin{aligned}\mathrm{res}:\ \mathcal{R}\,\mathfrak{t}^{-1}&\to\mathsf{W}(k)[1/p]\\ z&\mapsto b_0.\end{aligned}$$

Compose with the trace map $\mathrm{Tr} = \mathrm{Tr}_{\mathsf{W}(k)[1/p]/\mathbf{Q}_p} \colon \mathsf{W}(k)[1/p] \to \mathbf{Q}_p$, we have the following map

$$\begin{aligned}\mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V^\vee(1))) \times (\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)) &\to \mathbf{Q}_p \\ (f,\ x) &\mapsto \mathrm{Tr}(\mathrm{res}\, f(x)).\end{aligned}$$

Recall that $\mathsf{H}^2(\mathcal{C}_{\varphi,\partial_\tau}(V)) \simeq (\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V))/\,\mathsf{Im}((p\varphi - 1) \oplus \partial_\tau))$. To construct the claimed pairing, we must show that the above map factors through $\mathsf{Im}((p\varphi - 1) \oplus \partial_\tau) \subset \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)$. In other words, for any $x \in \mathsf{Im}((p\varphi - 1) \oplus \partial_\tau$, we have to show that $\mathrm{Tr}(\mathrm{res}\, f(x)) = 0$.

Notice that for any $f \in \mathsf{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V), \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(\mathbf{Q}_p(1)))^{\varphi=1,\partial_\tau=0}$, the condition $\partial_\tau(f) = 0$ means that $\partial_\tau \circ f = f \circ \partial_\tau$, and $\varphi(f) = f$ means that $\varphi \circ f = f \circ \varphi$. Indeed, the actions are $\partial_\tau(f) = \partial_\tau \circ f - f \circ \partial_\tau$ and $\varphi(f) = \varphi \circ f \circ \varphi^{-1}$.

For any $x \in \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^\dagger(V)$ and $f \in \mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V^\vee(1)))$, write $f(x) = \mathfrak{t}^{-1}r = \mathfrak{t}^{-1} \sum\limits_{n\in\mathbf{Z}} a_n u^n \in \mathcal{R}\,\mathfrak{t}^{-1}$ with $a_n \in \mathsf{W}(k)[1/p]$. We have

$$\begin{aligned}\mathrm{Tr} \circ \mathrm{res}(f(\partial_\tau(x))) &= \mathrm{Tr} \circ \mathrm{res}\left(\partial_\tau(f(x))\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\partial_\tau(\mathfrak{t}^{-1}r)\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\partial_\tau(\mathfrak{t}^{-1})r + \partial_\tau(r)\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{\partial_\tau(\lambda)}{\lambda}\mathfrak{t}^{-1}r + \partial_\tau(r)\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{\partial_\tau(\lambda r)}{\lambda}\mathfrak{t}^{-1}\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{p}{t}\partial_\tau(\lambda r)\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{p}{t} \cdot u\frac{d}{du}(\lambda r)\right) \\ &= \mathrm{Tr}(0) \\ &= 0.\end{aligned}$$

We have $\varphi(\mathfrak{t}^{-1}) = c^{-1}\mathfrak{t}^{-1}$, hence

$$\begin{aligned}\mathrm{Tr} \circ \mathrm{res}(f((p\varphi - 1)x)) &= \mathrm{Tr} \circ \mathrm{res}((p\varphi - 1)f(x)) \\ &= \mathrm{Tr} \circ \mathrm{res}\left((p\varphi - 1)(\mathfrak{t}^{-1}\sum_{n\in\mathbf{Z}} a_n u^n)\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(pc^{-1}\mathfrak{t}^{-1}\varphi(\sum_{n\in\mathbf{Z}} a_n u^n) - \mathfrak{t}^{-1}\sum_{n\in\mathbf{Z}} a_n u^n\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(pc^{-1}\frac{p\lambda}{t}\varphi(\sum_{n\in\mathbf{Z}} a_n u^n) - \frac{p\lambda}{t}\sum_{n\in\mathbf{Z}} a_n u^n\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(pc^{-1}\frac{p\lambda}{t}\varphi(\sum_{n\in\mathbf{Z}} a_n u^n) - \frac{p\lambda}{t}\sum_{n\in\mathbf{Z}} a_n u^n\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{p}{t}\varphi(\lambda)\varphi(\sum_{n\in\mathbf{Z}} a_n u^n) - \frac{p}{t}\lambda\sum_{n\in\mathbf{Z}} a_n u^n\right) \\ &= \mathrm{Tr} \circ \mathrm{res}\left(\frac{p}{t} \cdot (\varphi - 1)\big(\lambda \cdot \sum_{n\in\mathbf{Z}} a_n u^n\big)\right) \\ &= \mathrm{Tr}(\varphi(b_0) - b_0) \\ &= (\varphi - 1)\,\mathrm{Tr}(b_0) \\ &= 0.\end{aligned}$$

where b_0 is the constant term of $p \cdot \lambda \cdot \sum_{n\in\mathbf{Z}} a_n u^n$.

Hence we have the desired pairing

$$\begin{aligned}\mathsf{H}^0(\mathcal{C}_{\varphi,\partial_\tau}(V^\vee(1))) \times \mathsf{H}^2(\mathcal{C}_{\varphi,\partial_\tau}(V)) &\to \mathbf{Q}_p \\ (f,x) &\mapsto B(f,x) = \mathrm{Tr}(\mathrm{res}\, f(x)).\end{aligned}$$

□

Remark 5.4.16. As $t = p\lambda\mathfrak{t}$ and $\partial_\tau(t) = 0$, we have

$$\partial_\tau(\lambda)\mathfrak{t} + \lambda\partial_\tau(\mathfrak{t}) = 0.$$

Hence

$$\partial_\tau(\mathfrak{t}) = -\frac{\mathfrak{t}}{\lambda}\partial_\tau(\lambda)$$

and

$$\partial_\tau(\mathfrak{t}^n) = n\mathfrak{t}^{n-1}\partial_\tau(\mathfrak{t}) = -n\mathfrak{t}^n\frac{\partial_\tau(\lambda)}{\lambda}.$$

Example 5.4.17. *For $V = \mathbf{Q}_p$, we have $\mathcal{D}_{\varphi,\partial_\tau}(\mathbf{Q}_p) \simeq \mathcal{R}$ and the complex $\mathcal{C}_{\varphi,\partial_\tau}(\mathbf{Q}_p)$ is*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{R}\bigoplus\mathcal{R} & \longrightarrow & \mathcal{R} & \longrightarrow & 0 \\ & & x & \longmapsto & ((\varphi-1)(x), \partial_\tau(x)) & & & & \\ & & & & (y,z) & \longmapsto & \partial_\tau(y) - (p\varphi-1)(z). & & \end{array}$$

Let's check that $\mathsf{H}^2(\mathcal{C}_{\varphi,\partial_\tau}(\mathbf{Q}_p)) = 0$. For any $x = \sum_{n\in\mathbf{Z}} x_n u^n \in \mathcal{R}$, we have $a_0 \in \mathsf{Im}(c\varphi - 1)$ by lemma 5.3.34. Hence it suffices to show that

$$x' := \sum_{n\in\mathbf{Z}\setminus\{0\}} x_n u^n \in \mathsf{Im}(\partial_\tau).$$

As $y := \sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{x_n}{n} u^n$ satisfies $\partial_\tau(y) = x'$, it is enough to show that $y \in \mathcal{R}$, which follows from lemma 5.4.1.

Remark 5.4.18. Let $n \in \mathbf{Z}$, we then have

$$\mathsf{H}^2(\mathcal{C}_{\varphi,\partial_\tau}(\mathbf{Q}_p)(n)) = \mathcal{R}\,\mathfrak{t}^{-n}\big/(p\varphi-1)(\mathcal{R}\,\mathfrak{t}^{-n}) + \partial_\tau(\mathcal{R}\,\mathfrak{t}^{-n}) \simeq \mathcal{R}\big/\big(\mathsf{Im}(pc^{-n}\varphi - 1) + \mathsf{Im}(\partial_\tau + n\frac{\partial_\tau(\lambda)}{\lambda})\big).$$

Chapter 6

Applications to p-divisible groups

In this chapter, we apply the results in the previous chapters to p-divisible groups, which allows us to compute the Galois cohomology of (the dual of) the Tate module of some p-divisible group using its associated Breuil-Kisin module.

6.1 p-divisible groups and Tate modules

Definition 6.1.1. (*cf* [49]) A *Barsotti-Tate group* (or *p-divisible group*) of *height* h over a commutative ring R is an inductive system $(G_n, i_n)_{i\geq 1}$ in which:

(1) G_n is a finite, commutative group scheme over R of order p^{nh};

(2) for each n, we have an exact sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1}$$

(that is, i_n is a closed immersion and identifies G_n with the kernel of multiplication by p^n on G_{n+1}).

The corresponding category of Barsotti-Tate groups over R is denoted $\mathbf{BT}_R$.

Remark 6.1.2. In particular, $\mathbf{BT}_{\mathcal{O}_K}$ is the category of Barsotti-Tate groups over the ring of integers of K.

Definition 6.1.3. Let $G = \varinjlim_n G_n$ be a p-divisible group. The *Tate module* of G is $\varprojlim_n G_n(\overline{K})$, denoted $\mathsf{T}_p G$.

It carries a natural continuous $\mathscr{G}_K$-action on it and hence $\mathsf{T}_p G \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, inducing a functor:

$$\begin{aligned} \mathsf{T}_p \colon \mathbf{BT}_{\mathcal{O}_K} &\to \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K) \\ G &\mapsto \mathsf{T}_p G. \end{aligned}$$

Theorem 6.1.4. *The functor T_p induces an equivalence of categories between $\mathbf{BT}_{\mathcal{O}_K}$ and $\mathbf{Rep}_{\mathbf{Z}_p}^{\mathrm{cris},\{0,1\}}(\mathscr{G}_K)$ (the category of crystalline representations of $\mathscr{G}_K$ over $\mathbf{Z}_p$ with Hodge-Tate weights in $\{0,1\}$).*

Proof. *cf* [32, Corollary 2.2.6]. □

6.2 Some inputs from Kisin's work

Definition 6.2.1. (1) We define a (φ, N)-*module* over $\mathfrak{S} = \mathsf{W}(k)[\![u]\!]$ to be a finite free $\mathfrak{S}$-module $\mathfrak{M}$, equipped with a φ-similinear Frobenius $\varphi\colon \mathfrak{M} \to \mathfrak{M}$, and a linear endomorphism

$$N\colon (\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \to (\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

such that $N\varphi = p\varphi N$ on $(\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

We say that $\mathfrak{M}$ is of *finite E-height* if the cokernel of $1 \otimes \varphi \colon \varphi^* \mathfrak{M} \to \mathfrak{M}$ is killed by some power of $E(u)$. We denote by $\mathbf{Mod}_{\mathfrak{S}}(\varphi, N)$ the category of (φ, N)-modules over $\mathfrak{S}$ of finite E-height.

(2) (*cf* [32, §2.1.3]) We denote by $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ the category of finite free $\mathfrak{S}$-modules equipped with an $\mathfrak{S}$-linear map $1 \otimes \varphi \colon \varphi^* \mathfrak{M} \to \mathfrak{M}$ whose cokernel is killed by some power of $E(u)$.

Remark 6.2.2. The category $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ is a full subcategory of $\mathbf{Mod}_{\mathfrak{S}}(\varphi, N)$ by taking the operator N to be 0 on an object of $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$.

Definition 6.2.3. (*cf* [32, §2.2.1]) We denote by $\mathbf{BT}^{\varphi}_{\mathfrak{S}}$ the category consisting of objects $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ such that $\mathfrak{M}/\varphi^*(\mathfrak{M})$ is killed by $E(u)$, where $\varphi^* = 1 \otimes \varphi \colon \varphi^* \mathfrak{M} \to \mathfrak{M}$. Objects of $\mathbf{BT}^{\varphi}_{\mathfrak{S}}$ are also called *Breuil-Kisin modules* (or *Kisin modules of height 1*).

Remark 6.2.4. The category $\mathbf{BT}^{\varphi}_{\mathfrak{S}}$ is a full subcategory of $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$.

Proposition 6.2.5. *There exists a functor from* $\mathbf{BT}^{\varphi}_{\mathfrak{S}}$ *to the category* $\mathbf{Rep}^{\mathrm{cris},\{0,1\}}_{\mathbf{Q}_p}(\mathscr{G}_K)$ (*the category of crystalline representations of* $\mathscr{G}_K$ *with Hodge-Tate weights in* $\{0,1\}$) *and it induces an anti-equivalence on the corresponding isogeny category.*

Proof. *cf* [32, Proposition 2.2.2]. □

Definition 6.2.6. (1) (*cf* [32, §1.1.4]) A *φ-module over* $\mathcal{O}$ is a finite free $\mathcal{O}$-module $\mathcal{M}$ endowed with φ-semilinear injective map $\varphi \colon \mathcal{M} \to \mathcal{M}$. We denote $\mathbf{Mod}_{\mathcal{O}}(\varphi)$ the corresponding category.

(2) A *(φ, N_∇)-module over* $\mathcal{O}$ is a φ-module $\mathcal{M}$ equipped with a differential operator $N_\nabla \colon \mathcal{M} \to \mathcal{M}$ over $N_\nabla \colon \mathcal{O} \to \mathcal{O}$, *i.e.* a map such that

$$(\forall f \in \mathcal{O})(\forall m \in \mathcal{M}) \quad N_\nabla(fm) = N_\nabla(f)m + fN_\nabla(m),$$

satisfying the condition $N_\nabla \circ \varphi = c\varphi \circ N_\nabla$. We denote $\mathbf{Mod}_{\mathcal{O}}(\varphi, N_\nabla)$ the corresponding category.

(3) (*cf* [32, §1.3.9]) A *(φ, N)-module over* $\mathcal{O}$ is a φ-module $\mathcal{M}$ over $\mathcal{O}$ together with a $\mathsf{W}(k)[1/p]$-linear map

$$N \colon \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$$

which satisfies $N\varphi = p\varphi N$, where we have written φ for the endomorphism of $\mathcal{M}/u\mathcal{M}$ obtained by reducing $\varphi \colon \mathcal{M} \to \mathcal{M}$ modulo u. We denote by $\mathbf{Mod}_{\mathcal{O}}(\varphi, N)$ the category of (φ, N)-modules over $\mathcal{O}$ of finite E-height.

(4) (*cf* [32, §1.2.5]) Let $\mathcal{M} \in \mathbf{Mod}_{\mathcal{O}}(\varphi)$. The associated *filtered φ-module* $\mathcal{D}(\mathcal{M})$ is defined to be the $\mathsf{W}(k)[1/p]$-vector space $\mathcal{M}/u\mathcal{M}$ together with the operator φ induced from $\mathcal{M}$ (the filtration is described in [32, §1.2.7]).

Lemma 6.2.7. *Let $\mathcal{M}$ be a φ-module over $\mathcal{O}$. There is a unique $\mathcal{O}$-linear, φ-equivariant morphism*

$$\xi \colon \mathcal{D}(\mathcal{M}) \otimes_{\mathsf{W}(k)[1/p]} \mathcal{O} \to \mathcal{M},$$

whose reduction modulo u induces the identity on $\mathcal{D}(\mathcal{M})$. The map ξ is injective and its cokernel is killed by a finite power of λ. If $r \in (|\pi|, |\pi|^{1/p})$, then the image of the map $\xi_{[0,r)}$ induced by ξ over $D([0,r])$ coincides with the image of $1 \otimes \varphi \colon (\varphi^*\mathcal{M})_{[0,r)} \to \mathcal{M}_{[0,r)}$.

Proof. *cf* [32, Lemma 1.2.6]. □

Lemma 6.2.8. *Let $\mathcal{M} \in \mathbf{Mod}_{\mathcal{O}}(\varphi, N_\nabla)$, the morphism*

$$\xi \colon \mathcal{D}(\mathcal{M}) \otimes_{\mathsf{W}(k)[1/p]} \mathcal{O} \to \mathcal{M}$$

is compatible with the differential operator N_∇ (the one on the left hand side being given by $1 \otimes N_\nabla$).

Proof. This is [32, Lemma 1.2.12 (3)], noting that we have $N = 0$ thus $\eta = \mathrm{id}$ in our case. □

Remark 6.2.9. (1) Let $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ and $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ the corresponding object in $\mathbf{Mod}_{\mathcal{O}}(\varphi, N)$. Then $\mathcal{M}$ is also an object in $\mathbf{Mod}_{\mathcal{O}}(\varphi)$. By lemma 6.2.8, we have a map $\mathcal{D}(\mathcal{M}) \otimes_{\mathsf{W}(k)[1/p]} \mathcal{O} \to \mathcal{M}$ that lifts the identity of $\mathcal{D}(\mathcal{M}) \simeq \mathcal{M}/u\mathcal{M}$, and is compatible with φ and N_∇. Here N_∇ acts on $\mathcal{D}(\mathcal{M}) \otimes_{\mathsf{W}(k)[1/p]} \mathcal{O}$ as $1 \otimes -u\lambda\frac{d}{du}$ (*cf* [32, 1.2]).

(2) As $\mathcal{M}/\varphi^*\mathcal{M}$ is killed by some power of $E(u)$, say $E(u)^n$ with $n \in \mathbf{N}$, then $\mathcal{M}/(\mathcal{D}(\mathcal{M}) \otimes_{\mathsf{W}(k)[1/p]} \mathcal{O})$ is killed by λ^n (*cf* lemma 6.2.7 and the proof of [32, Lemma 1.2.6]). Let $m \in \mathcal{M}$, and write $m = \sum_{i=1}^{r} d_i \otimes \lambda^{-n} f_i$ with $d_i \in \mathcal{D}(\mathcal{M})$ and $f_i \in \mathcal{O}$, then we have

$$N_\nabla(m) = \sum_{i=1}^{r} d_i \otimes -u\lambda\frac{d}{du}(\lambda^{-n} f_i) = \sum_{i=1}^{r} d_i \otimes (un\lambda^{-n} f_i \frac{d\lambda}{du} - u\lambda^{1-n}\frac{df_i}{du}).$$

6.3 Recover τ-action from the N_∇-action

Definition 6.3.1. (1) (*cf* [19, Définition 2.1]) Let D be an étale φ-module over $\mathcal{O}_{\mathcal{E}}$. A *$\varphi$-lattice* in D is a finite type sub $\mathfrak{S}$-module $\mathfrak{M} \subset D$ which is stable by φ and such that $D = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$.

(2) (*cf* [19, Définition 2.19]) Let (D, D_τ) be a (φ, τ)-module over $(\mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}_\tau})$. A *$(\varphi, \tau)$-lattice* in (D, D_τ) is a φ-lattice $\mathfrak{M}$ of D such that $\mathfrak{S}_\tau \otimes_{\mathfrak{S}} \mathfrak{M} \subset \mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathcal{O}_{\mathcal{E}}} D = D_\tau$ is stable by τ, where $\mathfrak{S}_\tau = \mathcal{O}_{\mathcal{E}_\tau} \cap \mathsf{W}(\mathcal{O}_{C^\flat})$.

Remark 6.3.2. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ be such that $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T$ is semistable with non negative Hodge-Tate weights. Denote by $\mathfrak{M}$ the corresponding (φ, τ)-lattice in the (contravariant) (φ, τ)-module (*cf* [19, Proposition 3.1]):

$$\mathcal{D}^*(T[1/p]) := \mathsf{Hom}_{\mathbf{Q}_p[\mathscr{G}_{K_\pi}]}(T[1/p], \widehat{\mathcal{E}^{\mathrm{ur}}}) = \mathcal{D}(T^\vee[1/p]).$$

(1) The underlying φ-lattice is the object in $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ associated to T by Kisin in [32] (*cf* [19, §3.1.1]).

(2) Put $\mathcal{M} = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$, equipped with its operator N_∇. By [19, Proposition 2.23], we have

$$N_\nabla(\mathfrak{M}) \subset \mathfrak{S}_\nabla \otimes_{\mathfrak{S}} \mathfrak{M},$$

where $\mathfrak{S}_\nabla$ is the set of sums of the form $\sum_{n \geq 0} \frac{P_n(u)}{p^{n+1}} u^{e\frac{p^n-1}{p-1}}$ and the $P_n(u)$ are polynomials with coefficients in $\mathsf{W}(k)$.

Definition 6.3.3. (*cf* [19, §3.3.2]) For any nonnegative integer i, we define

$$\mathcal{R}_i^{\mathsf{int}} = \Big\{ \sum_{n \geq 0} a_n u^n;\ v_p(a_n) + i\log_p(n) \text{ are bounded below for } n \geq 1 \Big\}.$$

Remark 6.3.4. (*cf* [19, §3.3.2]) Remark that $\mathcal{R}_i^{\mathsf{int}} \subset \mathcal{O}$. If i and j are two integers, the product of one function of $\mathcal{R}_i^{\mathsf{int}}$ by another function of $\mathcal{R}_j^{\mathsf{int}}$ falls into $\mathcal{R}_{i+j}^{\mathsf{int}}$.

Definition 6.3.5. (*cf* [19, §3.3.2]) We define $N_\nabla^{(0)} = \mathrm{id}$ and

$$N_\nabla^{(i+1)} = iu \cdot \frac{d\lambda}{du} \cdot N_\nabla^{(i)} + N_\nabla \circ N_\nabla^{(i)},$$

which then define $\mathfrak{S}$-linear maps $N_\nabla^{(i)} \colon \mathfrak{M} \to \mathcal{R}_i^{\mathsf{int}} \otimes_{\mathfrak{S}} \mathfrak{M}$. The operator $\sum_{i \geq 0} \frac{(p\mathfrak{t})^i}{i!} \cdot N_\nabla^{(i)}$ converges to a $\mathsf{W}(k)$-linear map

$$\tau \colon \mathfrak{M} \to \mathbf{B}_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \mathfrak{M}$$

(recall that $\mathbf{B}_{\mathrm{cris}}^+ = \mathbf{A}_{\mathrm{cris}}[1/p]$ where $\mathbf{A}_{\mathrm{cris}}$ is the p-adic completion of the divided power envelope of $\mathsf{W}(\mathcal{O}_{C^\flat})$ with respect to the ideal generated by $1 + [\varepsilon^{1/p}] + [\varepsilon^{1/p}]^2 + \cdots + [\varepsilon^{1/p}]^{p-1}$, *cf* [25]).

Remark 6.3.6. (1) By [19, Remarque 3.25], we have in fact

$$\tau(\mathfrak{M}) \subset \mathsf{A} \otimes_{\mathfrak{S}} \mathfrak{M},$$

where $\mathsf{A} = \mathsf{W}(\mathcal{O}_{C^\flat})[\mathfrak{t}/p]^\wedge$ is the p-adic completion of $\mathsf{W}(\mathcal{O}_{C^\flat})[\mathfrak{t}/p]$.

(2) By [19, Proposition 3.27], we have $\tau(\mathfrak{M}) \subset \mathsf{W}(\mathcal{O}_{C^\flat}) \otimes_{\mathfrak{S}} \mathfrak{M}$, which inserts in the following commutative square

$$\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\tau} & \mathsf{W}(\mathcal{O}_{C^\flat}) \otimes_{\mathfrak{S}} \mathfrak{M} \\
\downarrow & & \uparrow \\
\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} & & \mathcal{O}_{\mathcal{E}_\tau} \otimes_{\mathfrak{S}} \mathfrak{M} \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{D}^*(T) & \xrightarrow{\tau_D} & \mathcal{D}^*(T)_\tau.
\end{array}$$

(3) We have $\tau(ux) = [\varepsilon] \cdot \tau(x)$ for any $x \in \mathfrak{M}$, which extends τ to $\mathbf{B}^+_{\mathrm{cris}} \otimes_{\mathfrak{S}} \mathfrak{M}$ by semi-linearity (*cf* [19, §3.3.2]).

6.4 Applications to p-divisible groups

Let G be a p-divisible group over $\mathcal{O}_K$, denote $T = \mathsf{T}_p G$ its Tate module and $\mathfrak{M}$ its associated Breuil-Kisin module. Recall that $\mathcal{D}^*(T) := \mathsf{Hom}_{\mathbf{Z}_p[\mathscr{G}_{K_\pi}]}(T, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \simeq \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $\mathcal{D}^*(T[1/p]) \simeq \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{E}$.

Notation 6.4.1. Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$, then we put

$$\mathcal{D}^{*,\dagger}(T) = \mathsf{Hom}_{\mathbf{Z}_p[\mathscr{G}_{K_\pi}]}(T, \mathcal{O}_{\mathcal{E}^{\mathrm{ur},\dagger}}).$$

Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, then we put

$$\mathcal{D}^{*,\dagger}(V) = \mathsf{Hom}_{\mathbf{Q}_p[\mathscr{G}_{K_\pi}]}(V, \mathcal{E}^{\mathrm{ur},\dagger}).$$

Lemma 6.4.2. *We have*

$$\mathcal{D}^{*,\dagger}(T[1/p]) \simeq \mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M},$$

hence

$$\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^{*,\dagger}(T[1/p]) \simeq \mathcal{R} \otimes_{\mathfrak{S}} \mathfrak{M}.$$

Proof. By [32, Theorem 2.2.7], we have

$$T = \mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$$

(where $\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \cap \mathsf{W}(\mathcal{O}_{C^\flat})$). Consider the following diagram

$$\begin{array}{ccc}
T = \mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) & \longrightarrow & \mathsf{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \simeq \mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \\
& \searrow & \uparrow \\
& & \mathsf{Hom}_{\mathcal{O}_{\mathcal{E}^\dagger},\varphi}(\mathcal{O}_{\mathcal{E}^\dagger} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur},\dagger}}) \simeq \mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur},\dagger}}).
\end{array}$$

The horizontal map is an isomorphism (*cf* [32, 2.1.4]), and the vertical map is injective (as $\mathcal{O}_{\mathcal{E}^{\mathrm{ur},\dagger}} \subset \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$), hence all maps in the diagram are isomorphisms. We then have

$$T[1/p] \simeq \mathsf{Hom}_{\mathcal{E}^\dagger,\varphi}(\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{E}^{\mathrm{ur},\dagger}) = \big((\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M})^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E}^{\mathrm{ur},\dagger}\big)^{\varphi=1}.$$

In particular $T[1/p] \subset (\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M})^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E}^{\mathrm{ur},\dagger}$, hence we have a $\mathscr{G}_{K_\pi}$-equivariant map

$$\mathcal{E}^{\mathrm{ur},\dagger} \otimes_{\mathbf{Q}_p} T[1/p] \to (\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M})^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E}^{\mathrm{ur},\dagger} \tag{6.1}$$

by $\mathcal{E}^{\mathrm{ur},\dagger}$-linearity. Notice that (6.1) is an isomorphism after tensoring with $\widehat{\mathcal{E}^{\mathrm{ur}}}$ (as $\mathcal{D}^*(T[1/p]) \simeq \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$), hence (6.1) is an isomorphism. We then have

$$\begin{aligned}
\mathcal{D}^{*,\dagger}(T[1/p]) &= \mathsf{Hom}_{\mathbf{Q}_p[\mathscr{G}_{K_\pi}]}(T[1/p], \mathcal{E}^{\mathrm{ur},\dagger}) \\
&= \mathsf{Hom}_{\mathcal{E}^{\mathrm{ur},\dagger}}(\mathcal{E}^{\mathrm{ur},\dagger} \otimes_{\mathbf{Q}_p} T[1/p], \mathcal{E}^{\mathrm{ur},\dagger})^{\mathscr{G}_{K_\pi}} \\
&= \mathsf{Hom}_{\mathcal{E}^{\mathrm{ur},\dagger}}((\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M})^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E}^{\mathrm{ur},\dagger}, \mathcal{E}^{\mathrm{ur},\dagger})^{\mathscr{G}_{K_\pi}} \\
&= \mathsf{Hom}_{\mathcal{E}^\dagger}((\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M})^\vee, \mathcal{E}^{\mathrm{ur},\dagger})^{\mathscr{G}_{K_\pi}} \\
&= \left((\mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M}) \otimes_{\mathcal{E}^\dagger} \mathcal{E}^{\mathrm{ur},\dagger}\right)^{\mathscr{G}_{K_\pi}} \\
&= \mathcal{E}^\dagger \otimes_{\mathfrak{S}} \mathfrak{M}.
\end{aligned}$$

□

Recall that we have the following equivalences of categories:

Theorem 6.4.3. *There exists an equivalence of categories between* $\mathbf{BT}^{\varphi}_{\mathfrak{S}}$ *and* $\mathbf{BT}_{\mathcal{O}_K}$.

Proof. This is [32, Theorem 2.2.7] if $p > 2$ and [38, Theorem 1.0.1] when $p = 2$. □

Remark 6.4.4. (*cf* [32, Proposition 2.2.2]) Let $\mathfrak{M} \in \mathbf{BT}^{\varphi}_{\mathfrak{S}}$ and $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ be the corresponding object in $\mathbf{Mod}_{\mathcal{O}}(\varphi, N)$, then in particular $\mathcal{M} \in \mathbf{Mod}_{\mathcal{O}}(\varphi)$. We have an operator N_∇ on $\mathfrak{M}$ by remark 6.2.9, which extends into a derivation on $\mathcal{R} \otimes_{\mathcal{O}} (\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}) \simeq \mathcal{R} \otimes_{\mathfrak{S}} \mathfrak{M}$.

6.5 Complexes with Breuil-Kisin modules

For any $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, recall that we have the following complex $\mathcal{C}_{\varphi,\tau}(V)$, which computes the continuous Galois cohomology of V by theorem 1.1.13.

$$0 \longrightarrow \mathcal{D}(V) \longrightarrow \mathcal{D}(V) \bigoplus \mathcal{D}(V)_{\tau,0} \longrightarrow \mathcal{D}(V)_{\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longmapsto (\tau_D - 1)(y) - (\varphi - 1)(z).$$

In particular, let $G \in \mathbf{BT}_{\mathcal{O}_K}$ and $\mathsf{T}_p G \in \mathbf{Rep}_{\mathbf{Z}_p}(\mathscr{G}_K)$ be its Tate module, we then consider the representation $V = (\mathsf{T}_p G)^\vee[1/p]$. We have the complex $\mathcal{C}_{\varphi,\tau}((\mathsf{T}_p G)^\vee[1/p])$ as follows:

$$0 \longrightarrow \mathcal{D}((\mathsf{T}_p G)^\vee[1/p]) \longrightarrow \mathcal{D}((\mathsf{T}_p G)^\vee[1/p]) \bigoplus \mathcal{D}((\mathsf{T}_p G)^\vee[1/p])_{\tau,0} \longrightarrow \mathcal{D}((\mathsf{T}_p G)^\vee[1/p])_{\tau,0} \longrightarrow 0$$

$$x \longmapsto ((\varphi - 1)(x), (\tau_D - 1)(x))$$

$$(y, z) \longmapsto (\tau_D - 1)(y) - (\varphi - 1)(z).$$

Recall that we have $\mathcal{D}((\mathsf{T}_p G)^\vee[1/p]) \simeq \mathcal{D}^*(\mathsf{T}_p G[1/p]) \simeq \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$, hence we can rewrite the complex $\mathcal{C}_{\varphi,\tau}(V)$ using the Breuil-Kisin module $\mathfrak{M}$ associated to $\mathsf{T}_p G$.

We have the formula of N_∇-action over $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ by remark 6.4.4, and in particular a formula of N_∇-action restricted on $\mathfrak{M}$. By remark 6.3.6, we have the following formula of τ-action over $\mathfrak{M}$

$$\tau := \sum_{i \geq 0} \frac{(p\mathfrak{t})^i}{i!} \cdot N_\nabla^{(i)} \colon \mathfrak{M} \to \mathbf{B}_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \mathfrak{M}.$$

As we have seen, it takes values in $\mathsf{W}(\mathcal{O}_{C^\flat}) \otimes_{\mathfrak{S}} \mathfrak{M}$, and factors through a map

$$\tau_D \colon \mathfrak{M} \to \mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M},$$

which extends into

$$\tau_D \colon \mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M} \to \mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M}.$$

Definition 6.5.1. Let $\mathfrak{M} \in \mathbf{BT}_{\mathfrak{S}}^{\varphi}$, we define the complex $\mathcal{C}_{\varphi,\tau}(\mathfrak{M})$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} & \longrightarrow & \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} \bigoplus (\mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M})_0 & \longrightarrow & (\mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M})_0 & \longrightarrow & 0 \\ & & x & \longmapsto & ((\varphi-1)(x), (\tau_D - 1)(x)) & & & & \\ & & & & (y,z) & \longmapsto & (\tau_D - 1)(y) - (\varphi - 1)(z) & & \end{array}$$

where $(\mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M})_0 := \left\{ x \in \mathcal{E}_\tau \otimes_{\mathfrak{S}} \mathfrak{M};\ (\gamma \otimes 1)x = (1 + \tau_D + \tau_D^2 + \cdots + \tau_D^{(\chi(\gamma)-1)})(x) \right\}$.

Corollary 6.5.2. *Let $G \in \mathbf{BT}_{\mathcal{O}_K}$, denote $\mathsf{T}_p G$ its Tate module and $\mathfrak{M}$ its corresponding Breuil-Kisin module. Then we have*

$$\mathsf{H}^i(\mathscr{G}_K, (\mathsf{T}_p G)^\vee[1/p]) \simeq \mathsf{H}^i(\mathcal{C}_{\varphi,\tau}(\mathfrak{M}))$$

for all $i \in \mathbf{N}$.

Proof. This follows from theorem 1.1.13. □

Let $V \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$ and $\mathcal{D}_{\mathsf{rig}}^\dagger(V) \in \mathbf{Mod}_{\mathcal{R}}(\varphi, N_\nabla)$ the corresponding (φ, N_∇)-module. We have the complex $\mathcal{C}_{\varphi,N_\nabla}(V)$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}_{\mathsf{rig}}^\dagger(V) & \longrightarrow & \mathcal{D}_{\mathsf{rig}}^\dagger(V) \bigoplus \mathcal{D}_{\mathsf{rig}}^\dagger(V) & \longrightarrow & \mathcal{D}_{\mathsf{rig}}^\dagger(V) & \longrightarrow & 0 \\ & & x & \longmapsto & ((\varphi-1)(x), N_\nabla(x)) & & & & \\ & & & & (y,z) & \longmapsto & N_\nabla(y) - (c\varphi - 1)(z). & & \end{array}$$

By proposition 5.3.17 we have $\mathsf{H}^i(\mathcal{C}_{\varphi,N_\nabla}(V)) \simeq \varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n}, V)$ for $i \in \{0,1\}$. By lemma 6.4.2, for $G \in \mathbf{BT}_{\mathcal{O}_K}$ we have

$$\mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{D}^{*,\dagger}(\mathsf{T}_p G[1/p]) \simeq \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{R}.$$

Let $V = (\mathsf{T}_p G)^\vee[1/p] \in \mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_K)$, we can rewrite the complex with corresponding Breuil-Kisin modules.

Definition 6.5.3. For any $\mathfrak{M} \in \mathbf{BT}^{\varphi}_{\mathfrak{S}}$, we define the complex $\mathcal{C}_{\varphi,N_\nabla}(\mathfrak{M})$ as follows:

$$0 \longrightarrow \mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{R} \longrightarrow \mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{R}\bigoplus\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{R} \longrightarrow \mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{R} \longrightarrow 0$$

$$x \longmapsto ((\varphi-1)(x), N_\nabla(x))$$

$$(y,z) \longmapsto N_\nabla(y) - (c\varphi-1)(z)$$

where the action of N_∇ is given by the formula in remark 6.4.4.

Corollary 6.5.4. *Let $G \in \mathbf{BT}_{\mathcal{O}_K}$, denote $\mathsf{T}_p G$ its Tate module and $\mathfrak{M}$ its corresponding Breuil-Kisin module. Then we have*

$$\mathsf{H}^i(\mathcal{C}_{\varphi,N_\nabla}(\mathfrak{M})) \simeq \varinjlim_n \mathsf{H}^i(\mathscr{G}_{K_n}, (\mathsf{T}_p G)^\vee[1/p]) \text{ for } i \in \{0,1\}.$$

Proof. This follows from proposition 5.3.17. □

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