

Unitary representations of mapping class groups Biao Ma

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ÉCOLE DOCTORALE

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SCIENCES FONDAMENTALES ET APPLIQUÉES

THÈSE DE DOCTORAT

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Représentations Unitaires des Groupes Modulaires

BIAO MA

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Représentations Unitaires des Groupes Modulaires

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Abstract

Mapping class groups of closed surfaces with punctures play important roles as prototypes of current research in geometric group theory. The representation theory of a group is a way to understand both the group structure and dynamic properties of that group. While there are massive literatures on finite dimensional (projective) unitary representations of mapping class groups, not so much on infinite dimensional ones. The aim of this thesis is to investigate mapping class groups from the perspective of infinite dimensional unitary representations based on current understanding of mapping class groups in the context of geometric group theory. It has two parts.

In the first part, for a surface, we introduce a family of unitary representations of its mapping class group based on the space of measured foliations. For this family of representations, we show that none of them has *almost invariant vectors*. As an application, we obtain an inequality concerning the action of the mapping class group on the Teichmüller space. Moreover, using the same method plus recent results about weak equivalence, we also give a classification, up to *weak equivalence*, for the unitary quasi-regular representations with respect to geometrical subgroups.

In the second part, for a closed hyperbolic surface, we show that *the boundary representation* of its mapping class group is *ergodic*, which generalizes the classical result of Masur on ergodicity of the action of the mapping class group on the projective measured foliation space of the surface. As a corollary, we show that the boundary representation of the mapping class group is irreducible. This confirms a conjecture of Bader-Muchnik in the case of mapping class groups with respect to Thurston measure classes.

Key words: Mapping class group, Unitary representation, Measured foliation, Almost invariant vector, Boundary representation, Irreducibility

Résumé

Les groupes modulaires de surfaces fermées à points masqués jouent un rôle important comme prototypes par la recherche moderne en théorie géométriques des groupes. La théorie des représentations d'un groupe est un moyen de comprendre à la fois la structure du groupe et ses propriétés dynamiques. Bien qu'il existe beaucoup de littérature sur les représentations unitaires (projectives) de dimension finie des groupes modulaires, il y en a beaucoup moins sur celles de dimension infinie. Le but de cette thèse est d'étudier le groupe modulaire du point de vue des représentations unitaires de dimension infinie, dans le contexte de la théorie géométrique des groupes. Ce mémoire comporte deux parties.

Dans la première partie, nous introduisons pour une surface une famille de représentations unitaires de son groupe modulaire, basée sur l'espace des feuillages mesurés. Pour cette famille de représentations, nous montrons qu'aucune d'elles n'a *de vecteurs presque invariants*. En corollaire, nous obtenons une inégalité concernant l'action du groupe modulaire sur l'espace de Teichmüller. Nous classifions aussi, à *équivalence faible* près, les représentations unitaires quasi-régulières par rapport à ses sous-groupes géométriques.

Dans la seconde partie, pour une surface hyperbolique fermée, nous montrons que *la représentation au bord* de son groupe modulaire est *ergodique*, ce qui généralise un résultat classique de Masur sur l'ergodicité de l'action du groupe modulaire sur l'espaces projectif des feuillages mesurés de la surface. En corollaire, nous montrons que la représentation au bord du groupe de modulaire est irréductible, ce qui démontre une conjecture de Bader-Muchnik dans le cas du groupe modulaire par rapport à la classe des mesures de Thurston.

Mot clés: Groupe modulaire, Représentation unitaire, Feuillage mesuré, Vecteur presque invariant, Représentation au bord, Irréductibilité

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Introduction in English

The main goal of this thesis is to understand some infinite dimensional unitary representations of mapping class groups of a surface. For finite dimensional (projective) unitary representations of mapping class groups, one could refer to, for instance, Roberts [55] or Blanchet-Habegger-Masbaum-Vogel [10] and recent related developments.

Let $S = S_{g,n}$ be a closed, connected, orientable surface of genus g with n punctures. The mapping class group Mod(S) of S is the group of isotopy classes of orientationpreserving homeomorphisms of S. Mapping class groups play important roles in understanding low-dimensional manifolds. For instance, according to the recent solution of the virtually fibered conjecture (see Agol [1], Wise [63] and Thurston's work [50]), mapping class groups essentially allow us to construct all closed hyperbolic manifolds in dimension three.

Associated to S, there are two well-known spaces that are equipped with a Mod(S)action. First, the space of measured foliations $\mathcal{MF}(S)$ of S which is the set of equivalence classes of measured foliations on S and second, the projective measured foliation space $\mathcal{PMF}(S)$ which is the quotient of $\mathcal{MF}(S)$ by the positive reals \mathbb{R}_+ . Both spaces are important to understand the group structure of Mod(S), see for example [30]. Our task is to investigate the unitary representations of Mod(S) associated to $\mathcal{MF}(S)$ and $\mathcal{PMF}(S)$ with respect to certain natural measures.

0.1 Almost invariant vectors and mapping class groups

There is a family of measures on $\mathcal{MF}(S)$ so that Mod(S) acts on $\mathcal{MF}(S)$ ergodically with respect to these measures. Namely the measures classified in Lindenstrauss-Mirzakhani [36] and Hamenstädt [28] (see also Section 1.3.1), generalizing the Thurston measure on $\mathcal{MF}(S)$ (see Masur [39], Masur [41]). We will call these measures generalized Thurston measures. One then obtains a family of unitary representations of Mod(S) by considering the induced action of Mod(S) on the associated L^2 -space with respect to these measures. One can check that the family of unitary representations considered in Paris [51] is a special subfamily.

The first chapter investigates these unitary representations of mapping class groups. Recall that a locally compact group G has Kazhdan's Property (T) if every unitary representation of G that has almost invariant vectors also has a non-zero invariant vector (see Definition 1.1.1 for almost invariant vectors). The following question is still widely considered to be open for surfaces of genus at least 3 (for genus at most 2, see Freedman-Krushkal [21] and Taherkhani [59]).

Question 0.1.1 (Ivanov [31]). Does Mod(S) have Kazhdan's Property (T)?

Note that, becuse of ergodicity, none of these unitary representations defined by generalized Thurston measures on $\mathcal{MF}(S)$ can have non-zero invariant vectors. Inspired by Ivanov's question, one can ask whether these representations have almost invariant vectors. The first main result of Chapter 1 indicates that they don't. Namely,

Theorem 0.1.2 (Theorem 1.4.1). For a compact surface $S = S_{g,n}$ with $3g+n \ge 4$ and each generalized Thurston measure μ , the associated representation $(\pi^{\mu}, L^2(\mathcal{MF}(S), \mu))$ of Mod(S) does not have almost invariant vectors. This theorem has two applications both for first cohomology and the action of the mapping class group Mod(S) on the Teichmüller space Teich(S) of S, see Corollary 1.4.2 and Corollary 1.4.3.

The proof of Theorem 1.4.1 also enables us to give a classification for a family of quasi-regular unitary representations, which is a stronger version of Corollary 5.5 in [51].

Theorem 0.1.3 (Theorem 1.5.4). Let $S = S_{g,n}$ be a compact surface with $3g+n \ge 4$. Let $\gamma = \sum_{i=1}^{k} \gamma_i$ and $\delta = \sum_{i=1}^{l} \delta_i$, where $\{\gamma_i\}$ and $\{\delta_i\}$ are two collections of pairwise disjoint, distinct isotopy classes of essential simple closed curves on S.

- 1. If at least one of k and l is not 3g 3 + n, then the associated unitary representations π_{γ} and π_{δ} are weakly equivalent if and only if γ and δ are of the same topological type (that is, there is a mapping class f so that $\gamma = f(\delta)$).
- 2. Suppose that S is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If k = 3g 3 + n, then π_{γ} is weakly equivalent to the regular representation λ_S .
- 3. Suppose that S is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If $k \neq 3g-3+n$, then π_{γ} is not weakly contained in λ_S .

0.2 Boundary representations of mapping class groups

We now assume that $S = S_g$ and $g \ge 2$. As $\mathcal{PMF}(S)$ is a quotient of $\mathcal{MF}(S)$, the Thurston measure on $\mathcal{MF}(S)$ induces a measure ν on $\mathcal{PMF}(S)$ by considering a section of the quotient map $\mathcal{MF}(S) \longrightarrow \mathcal{PMF}(S)$ and the coned-off construction of the measure (Dowdall-Duchin-Masur [16]) (See also Chapter 2 for details). The measure class $[\nu]$ is then Mod(S)-invariant. By a standard construction (Bekka-de la Harpe-Valette [6]), one can construct a unitary representation $\pi = \pi_{\nu}$, called the boundary representation of Mod(S). The main result of Chapter 2 is the following

Theorem 0.2.1 (Corollary 2.2.15). Let $S = S_g$ be a closed surface of genus $g \ge 2$. The boundary representation of Mod(S) on $L^2(\mathcal{PMF}(S), \nu)$ is irreducible. This theorem confirms the following conjecture in the case of Mod(S) with respect

to the Thurston measure class $[\nu]$ on $\mathcal{PMF}(S)$.

Conjecture 0.2.2 (Bader-Muchnik [5]). Let G be a locally compact group and μ a spread-out probability measure on G. The quasi-regular representation associated to the μ -Poisson boundary of G is irreducible.

In fact, we show the following more general ergodic-type theorem. See Definition

2.2.1 for ergodicity of representations.

Theorem 0.2.3 (Theorem 2.2.14). Let Pr_o be the radial projection from the Teichmüller space $\operatorname{Teich}(S) - \{o\}$ to the Teichmüller boundary $\mathcal{PMF}(S)$. Then there exists a sequence of finite subsets $E_n \subset \operatorname{Mod}(S)$ such that the associated quasi-regular representation π_{ν} is ergodic with respect to (E_n, Pr_o) and bounded Borel functions on $\mathcal{PMF}(S)$.

0.3 Questions

We end the introduction by some loose ends. One corollary of Theorem 1.4.1 is that, for any generalized Thurston measure μ on $\mathcal{MF}(S)$,

$$H^1(\operatorname{Mod}(S), \pi^{\mu}) = \overline{H^1}(\operatorname{Mod}(S), \pi^{\mu}).$$

As one could characterize the Kazhdan's Property (T) by the reduced first cohomology (see Shalom [57]), one could ask if $\overline{H^1}(Mod(S), \pi^{\mu}) = 0$ for all generalized Thurston measure μ on $\mathcal{MF}(S)$? Or more generally,

Question 0.3.1. For p large enough and a generalized Thurston measure μ , does Mod(S) act on $L^p(\mathcal{MF}(S),\mu)$ via isometries properly?

The above question in fact relates to the following question asked by Hamenstädt [29] in her ICM talk in 2010:

Question 0.3.2. Does Mod(S) have the Haagerup property?

One motivation for Question 0.3.1 is that, according to a result of Bourdon [11], for large p, the L^p -representation of a finitely generated hyperbolic group G given by the right action on G gives a proper affine isometric action of G. Meanwhile, a special type of generalized Thurston measures is given by the set of vertices in curve graph which is hyperbolic as a graph.

Based on Chapter 2, one could also ask

Question 0.3.3. For $S = S_g(g \ge 2)$ and the Thurston measure μ on $\mathcal{MF}(S)$, is the associated representation reducible?

The answer to this question in the case of g = 1 is negative due to the linear structure of $Mod(S_1) = SL(2,\mathbb{Z})$ and one can not hope it to be true for all generalized Thurston measures, see the examples in Chapter 1. The last question is based on the following observation: the proof of Theorem 1.4.1 uses that some subgroups of Mod(S) act on $\mathcal{MF}(S)$ essential freely. One then could ask, what happens for dynamics without this property? For instance,

Question 0.3.4. Let $S = S_g$ and $X = Rep(\pi_1(S), SU(2))$ be the SU(2)-character variety of the fundamental group $\pi_1(S)$. Let μ be the symplectic volume on X which is Mod(S)-invariant. Does the representation of Mod(S), given by the orthogonal complement of \mathbb{Cl}_X in $L^2(X, \mu)$, have almost invariant vectors?

It is well-known that Mod(S) acts on X ergodically with respect to μ (Goldman [24]). This question actually asks whether the action of Mod(S) on X is strongly ergodic (Schmidt [56]), or more interestingly, whether the measure μ , which is constructed algebraically, is the unique Mod(S)-invariant mean on $L^{\infty}(X, \mu)$. When g = 1, the result is known by Schmidt [56], but it is unknown for other cases.

Introduction en Français

L'objectif principal de cette thèse est de comprendre certaines représentations unitaires de dimension infinie du groupe modulaire d'une surface. Pour les représentations unitaires (projectives) de dimension finie des groupes modulaires, on peut se référer, par exemple, à Roberts [55] ou Blanchet-Habegger-Masbaum-Vogel [10] et aux récents travaux associés.

Soit $S = S_{g,n}$ une surface fermée, connexe, orientable de genre g avec n points marqués. Le groupe modulaire Mod(S) de S est le groupe des classes d'isotopie d'homéomorphismes préservant l'orientation de S. Ces groupes jouent un rôle important dans la compréhension des variétés de petite dimension. La solution récente de la conjecture virtuellement fibrée (voir Agol [1], Wise [63] et le travail de Thurston [50]) dit que, les groupes modulaires nous permettent essentiellement de construire toutes les variétés hyperboliques fermées en dimension trois.

Nous nous intéressons à deux espaces équipés d'une action de Mod(S). Premièrement, l'espace des feuillages mesurés $\mathcal{MF}(S)$ de S qui est l'ensemble des classes d'équivalence de feuillages mesurés sur S, et deuxièmement l'espace des feuilletages mesurés projecties $\mathcal{PMF}(S)$ qui est le quotient de $\mathcal{MF}(S)$ par les réels positifs \mathbb{R}_+ . Les deux espaces sont importants pour comprendre la structure de groupe de Mod(S), voir par exemple [30]. Nous étudions dans ce mémoire les représentations unitaires de Mod(S) associées à $\mathcal{MF}(S)$ et $\mathcal{PMF}(S)$ par rapport à certaines mesures naturelles.

0.4 Vecteurs presque invariants et groupes modulaires

Il existe une famille de mesures sur $\mathcal{MF}(S)$ par lesquelles l'action de Mod(S) sur $\mathcal{MF}(S)$ est ergodiquement. Ce sont les mesures étudiées dans [36] et [28] (voir aussi Section 1.3.1), généralisant la mesure de Thurston sur $\mathcal{MF}(S)$ (voir [39], [41]), nous les appellerons mesures de Thurston généralisées. On obtient alors une famille de représentations unitaires de Mod(S) en considérant l'action induite de Mod(S) sur l'espace des fonctions de carré intégrable. On peut vérifies que la famille de représentations unitaires considérée dans [51] est une sous-famille particulière.

Le premier chapitre étudie ces représentations unitaires du groupe modulaire. Rappelons qu'un groupe localement compact G a *la propriété* (T) *de Kazhdan* si toute représentation unitaire de G qui a des vecteurs presque invariants possède un vecteur invariant non nul (voir Définition 1.1.1 pour la notion de *vecteurs presque invariants*). La question suivante est encore considérée comme ouverte pour des surfaces de genre de supéreur ou égal à 3 (pour genre au plus 2, voir [21] et [59]).

Question 0.4.1 (Ivanov [31]). Est-ce que Mod(S) a la propriété (T) de Kazhdan ?

Par ergodicité, aucune de représentations unitaires définies par les mesures de Thurston généralisées sur $\mathcal{MF}(S)$ n'ont de vecteurs invariants non nuls. Inspiré par la question d'Ivanov ci-dessus, on se demande si ces représentations ont des vecteurs presque invariants. Le résultat principal du Chapitre 1 montre qu'elles n'ont pas de vecteurs presque invariants. À savoir,

Théorème 0.4.2 (Theorem 1.4.1). Pour une surface compacte $S = S_{g,n}$ avec $3g+n \ge 4$ et une mesure de Thurston généralisée μ , la représentation associée π^{μ} de Mod(S) n'a pas de vecteur presque invariant.

Ce théorème a deux applications: pour la première cohomologie et l'action du groupe moulaire Mod(S) sur l'espace Teichmüller Teich(S) de S, voir Corollaire 1.4.2 et Corollaire 1.4.3.

La preuve du Théorème 1.4.1 nous permet également de donner une classification pour

une famille de représentations unitaires quasi-régulières, qui est une généralisation du

Corollaire 5.5 dans [51].

Théorème 0.4.3 (Theorem 1.5.4). Soit $S = S_{g,n}$ une surface compacte avec $3g+n \ge 4$. Soit $\gamma = \sum_{i=1}^{k} \gamma_i$ et $\delta = \sum_{i=1}^{l} \delta_i$, où $\{\gamma_i\}$ et $\{\delta_i\}$ sont deux collections de deux à deux disjontes et de classes d'isotopie distinctes courbes fermées simples essentielles sur S,

- 1. Si k ou l est différent de 3g 3 + n. Les représentations unitaires π_{γ} et π_{δ} sont faiblement équivalentes si et seulement si γ et δ sont du même type topologique (c'est-à-dire qu'il existe un element f dans Mod(S) tel que $\gamma = f(\delta)$).
- 2. Supposons que S ne soit pas $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. Si k = 3g 3 + n, alors π_{γ} est faiblement équivalente à la représentation régulière λ_S .
- 3. Supposons que S ne soit pas $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. Si $k \neq 3g-3+n$, alors π_{γ} n'est pas faiblement contenue dans λ_S .

0.5 Representations au bord du groupes modulaires

Nous supposons maintenant que $S = S_g$ et $g \ge 2$. Comme $\mathcal{PMF}(S)$ est un quotient de $\mathcal{MF}(S)$, la mesure de Thurston sur $\mathcal{MF}(S)$ induit une mesure ν sur $\mathcal{PMF}(S)$ en considérant une section de la projection $\mathcal{MF}(S) \longrightarrow \mathcal{PMF}(S)$ et de la construction naturelle[16]. La classe de la mesure $[\nu]$ est alors Mod(S)- invariante. Par une construction standard [6], on peut construire une représentation unitaire $(\pi = \pi_{\nu}, L^2(\mathcal{PMF}(S), \nu))$, appelée *la représentation au bord* de Mod(S). Le résultat principal du Chapitre 2 est le suivant

Théorème 0.5.1 (Corollary 2.2.15). Soit $S = S_g$ une surface fermée de genre $g \ge 2$. La représentation au bord du groupe modulaire Mod(S) sur $L^2(\mathcal{PMF}(S), \nu)$ est irréductible.

Ce théorème confirme la conjecture suivante dans le cas des groupes de modulaire

par rapport à la classe de mesure de Thurston $[\nu]$ sur $\mathcal{PMF}(S)$.

Conjecture 0.5.2 (Bader-Muchnik [5]). Pour un groupe localement compact G et une mesure de probabilité μ sur G, la représentation quasi-régulière associée au μ -bord de Poisson de G est irréductible.

En fait, nous montrons le théorème suivant qui est plus général. Voir Définition 2.2.1

pour l'ergodicité des représentations.

Théorème 0.5.3 (Theorem 2.2.14). Soit Pr la projection de radiale de l'espace Teichmüller de S à son bord Teichmüller $\mathcal{PMF}(S)$. Alors il existe une suite de sousensembles finis $E_n \subset \operatorname{Mod}(S)$ tels que la représentation quasi-régulière associée π_{ν} soit ergodique par rapport à (E_n, Pr) et fonctions Borel bornées sur $\mathcal{PMF}(S)$.

0.6 Questions ouvertes

Nous terminons l'introduction par quelques questions connexes. Un corollaire du Théorème 1.4.1 est que, pour toute mesure de Thurston généralisée μ sur $\mathcal{MF}(S)$, On a

$$H^1(\operatorname{Mod}(S), \pi^{\mu}) = \overline{H^1}(\operatorname{Mod}(S), \pi^{\mu}).$$

Comme on peut caractériser la propriété (T) de Kazhdan par l'annulation de la première cohomologie réduite (voir [57]), il est naturel de se demander si $\overline{H^1}(Mod(S), \pi^{\mu}) =$

0 pour tout Thurston généralisé mesure μ sur $\mathcal{MF}(S)?$ Ou plus généralement,

Question 0.6.1. Pour un p suffisanent grand et une mesure de Thurston généralisée μ , est-ce que Mod(S) agit proprement par isomètries affines sur $L^p(\mathcal{MF}(S),\mu)$ via des isométries proprement?

Cette question se rapporte à la question posée par Hamenstaedt[29] dans son discours ICM en 2010,

Question 0.6.2. *Est-ce que* Mod(S) *a la propriété de Haagerup?*

La question 0.6.1 est motivée par un résultat de Bourdon [11] qui prouve que, pour p grand, la représentation L^p d'un groupe hyperbolique finiment engendré donné par

l'action à droite sur G donne une action isométrique affine propre de G. En même temps, un type spécial de mesures de Thurston généralisées est donné par l'ensemble des sommets du graphe de courbes qui est hyperbolique en tant que graphe.

Basé sur le Chapitre 2, on pourrait aussi se demander

Question 0.6.3. Pour $S = S_g(g \ge 2)$ et la mesure de Thurston μ sur $\mathcal{MF}(S)$, la représentation associée est-elle réductible?

La réponse à cette question dans le cas de g = 1 est négative du fait de la structure linéaire de $Mod(S_1) = SL(2,\mathbb{Z})$ et on ne peut espérer que ce soit vrai pour tous mesures de Thurston généralisées, voir les exemples dans le Chapitre 1. La dernière question est basée sur l'observation suivante: la preuve du Théorème 1.4.1 utilise que certains sous-groupes de Mod(S) ont une action essentiellent libre sur $\mathcal{MF}(S)$. Que se passe-t-il pour une dynamique sans cette propriété? Par exemple,

Question 0.6.4. Soit $S = S_g$ et $X = Rep(\pi_1(S), SU(2))$ la SU(2)-variété de caractères du groupe de fondamental $\pi_1(S)$. Soit μ le volume symplectique sur X qui est Mod(S)-invariant. La représentation de Mod(S), donnée par le complément orthogonal de $\mathbb{C}1_X$ dans $L^2(X, \mu)$, a-t-elle des vecteurs presque invariants?

Il est bien connu que Mod(S) agit ergodiquement sur X par rapport à μ [24]. La question est de savoir si l'action de Mod(S) sur X est fortement ergodique [56], ou encore, si la mesure μ , qui est construite algébriquement, est l'unique moyenne invariante sur l'action de Mod(S) sur $L^{\infty}(X, \mu)$. Lorsque g = 1, le résultat est connu par [56], mais il est ouvert dans les autres cas.

Chapter 1

Almost invariant vectors and mapping class groups

This chapter is taken from [37].

1.1 Introduction

Let $S = S_{g,n}$ be a compact, connected, orientable surface of genus g with n boundaries, the mapping class group Mod(S) of S is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of S which preserving each boundary components (without the assumption that it should fix each boundary pointwise). Throughout this paper, (g, n) is assumed to satisfy $3g + n \ge 4$ and a subsurface of Sis allowed to be disconnected.

Given a discrete group G, a unitary representation is a pair (π, V) where V is a Hilbert space and $\pi : G \to U(V)$ is a homomorphism from G to the group of all unitary operators of V [6]. Infinite dimensional unitary representations of mapping class groups Mod(S) received a lot of attention recently. In [51], the author considers unitary representations given by the action of Mod(S) on the curve complex associated to S. See [2],[25] for more topics in this direction.

The group $\operatorname{Mod}(S)$ acts on the space of measured foliations $\mathcal{MF}(S)$, which is defined as the set of equivalence classes of non-zero measured foliations on S. As the action is ergodic with respect to generalized Thurston measures μ [39],[41],[36], [28] (see Section 1.3.1 for a brief description of the measures), one obtains a family of unitary representations by considering the induced action of $\operatorname{Mod}(S)$ on the space $L^2(\mathcal{MF}(S),\mu)$. It is quite easy to see that the family of unitary representations considered in [51] is a special subfamily. However, unlike representations studied in [51], Example 1.3.4 will show that some of representations considered here are reducible.

Definition 1.1.1. Let (π, V) be a unitary representation of a discrete group G. The representation π is said to have almost invariant vectors if for every finite set $K \subseteq G$ and every $\epsilon > 0$, there exists $v \in V$ such that

$$\max_{g \in K} \|\pi(g)v - v\| < \epsilon \|v\|.$$

The main result of this paper is about the existence of almost invariant vectors for the representation π^{μ} associated to the action of Mod(S) on $L^2(\mathcal{MF}(S),\mu)$. The existence of such vectors for other representations of mapping class group has been discussed in [3].

Theorem 1.1.1 (Theorem 1.4.1). For a compact surface $S = S_{g,n}$ with $3g + n \ge 4$ and each generalized Thurston measure μ , the associated representation π^{μ} of Mod(S) does not have almost invariant vectors.

The first direct application of this theorem is the following:

Corollary 1.1.2 (Corollary 1.4.2). Let $S = S_{g,n}$ be a compact surface with $3g+n \ge 4$ and μ be a generalized Thurston measure, then $H^1(Mod(S), \pi^{\mu}) = \overline{H^1}(Mod(S), \pi^{\mu})$, where π^{μ} is the associated representation of Mod(S).

For the second application, we will obtain a geometric inequality of independent interest concerning the action of Mod(S) on the Teichmüller space Teich(S) of S.

Corollary 1.1.3 (Corollary 1.4.3). Let $S = S_{g,n}$ be a compact surface with $3g+n \ge 4$ and γ be the isotopy class of an essential simple closed curve on S. Then there exists a finite subset $\{\phi_1, ..., \phi_n\}$ of Mod(S) consisting of pseudo-Anosov mapping classes and a constant $\epsilon > 0$, such that, for every point \mathcal{X} in $\operatorname{Teich}(S)$, we have:

$$\max_{i \in \{1,2,\dots,n\}} \left\{ \sum_{\alpha \in \operatorname{Mod}(S).\gamma} e^{-2\ell_{\mathcal{X}}(\alpha)} (e^{\Delta_{\mathcal{X}}^{\phi_i}(\alpha)} - 1)^2 \right\} \ge \epsilon \sum_{\alpha \in \operatorname{Mod}(S).\gamma} e^{-2\ell_{\mathcal{X}}(\alpha)},$$

where $\Delta_{\mathcal{X}}^{\phi_i}(\alpha) = \ell_{\mathcal{X}}(\alpha) - \ell_{\phi_i,\mathcal{X}}(\alpha)$ and $\ell_{\mathcal{X}}(\alpha)$ is the geodesic length of α .

For unitary representations associated to discrete measures on the space of measured foliations, some of them are irreducible and some are reducible. We will discuss irreducible decompositions (See Proposition 1.5.1). We will also use the same method as in the proof of the main theorem, combined with recent results in [15],[14],[7], to give a classification for a family of quasi-regular unitary representations (with respect to subgroups), which is a stronger version of Corollary 5.5 in [51]. Recall that, given two unitary representations (π , \mathcal{H}) and (ϕ , \mathcal{K}) of a discrete group G, π is weakly contained in ϕ if for every ξ in \mathcal{H} , every finite subset Q of G and $\epsilon > 0$, there exist $\eta_1, ..., \eta_n$ in \mathcal{K} such that

$$\max_{g \in Q} \left| \langle \pi(g)\xi, \xi \rangle - \sum_{i=1}^{n} \langle \phi(g)\eta_i, \eta_i \rangle \right| < \epsilon.$$

If π is weakly contained in ϕ and ϕ is weakly contained in π , then ϕ and π are said to be *weakly equivalent*. By Proposition F.1.7 in [6], Definition 1.1.1 is equivalent to say that the trivial representation is weakly contained in the representation π . We then have the following theorem.

Theorem 1.1.4 (Theorem 1.5.4). Let $S = S_{g,n}$ be a compact surface with $3g + n \ge 4$. Let $\gamma = \sum_{i=1}^{k} \gamma_i$ and $\delta = \sum_{i=1}^{l} \delta_i$, where $\{\gamma_i\}$ and $\{\delta_i\}$ are two collections of pairwise disjoint, distinct isotopy classes of essential simple closed curves on S.

1. If at least one of k and l is not 3g - 3 + n, then the associated unitary representations π_{γ} and π_{δ} are weakly equivalent if and only if γ and δ are of the same topological type (that is, there is a mapping class f so that $\gamma = f(\delta)$).

- 2. Suppose S is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If k = 3g-3+n, then π_{γ} is weakly equivalent to the regular representation λ_S .
- 3. Suppose S is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If $k \neq 3g 3 + n$, then π_{γ} is not weakly contained in λ_S .

This paper is organized as follows. Section 1.2 is devoted to preliminary for group cohomology with coefficients in unitary representations. The proof of the main theorem is given in Section 1.4. The proof is divided into two general lemmas: Lemma 1.4.4 and Lemma 1.4.5, and concluded by a technical statement, namely Proposition 1.3.7, concerning actions of subgroups of mapping class groups on $\mathcal{MF}(S)$. Section 1.3 is mainly devoted to this proposition and Section 1.5 is for irreducible decompositions and the classification up to weak equivalence.

1.2 Cohomology with coefficients in representations

Cohomology and reduced cohomology. For a discrete group G and a unitary representation (V, π) , one can talk about both cohomology and reduced cohomology group of G with coefficients in π . Definitions of cohomology and reduced cohomology of discrete groups with coefficients in a representation π are standard, so we refer to [38],[2],[6]. We briefly recall that one defines the following vector spaces for a unitary representation (V, π) :

$$Z^{1}(G,\pi) \doteq \{b: G \to V | b(gh) = b(g) + \pi(g)b(h), \text{ for all } g, h \in G\};$$

$$B^{1}(G,\pi) \doteq \{b \in Z^{1}(G,\pi) | \text{there exists } v \in V, \text{ such that for all } g \in G,$$

$$b(g) = \pi(g)v - v\};$$

$$H^{1}(G,\pi) \doteq Z^{1}(G,\pi)/B^{1}(G,\pi);$$

$$\overline{H^{1}}(G,\pi) \doteq Z^{1}(G,\pi)/\overline{B^{1}(G,\pi)},$$

where the closure in the last one is for uniform convergence. The vector space $H^1(G,\pi)(\text{resp. }\overline{H^1}(G,\pi))$ is the first (resp. reduced) cohomology group with coefficients in π .

Almost invariant vectors. The following Guichardet's theorem provides a way to determine if $H^1(G) = \overline{H^1}(G)$.

Theorem 1.2.1 ([38]). Let G be a finitely generated discrete group and (V, π) be a unitary representation without nonzero invariant vectors. Then the following two are equivalent:

- 1. The associated first reduced cohomology is the same as the first cohomology, that is, $H^1(G,\pi) = \overline{H^1}(G,\pi);$
- 2. The representation π does not have almost invariant vectors.

One observation is that not having almost invariant vectors is closed under taking limit, more precisely, we have the following lemma.

Lemma 1.2.2. Let (V, π) be a unitary representation of G and W be a G-invariant vector subspace of V such that the closure $\overline{W} = V$. Then π does not have almost invariant vectors if and only if the representation $\pi \mid_W$ in W does not have almost invariant vectors.

Proof. Suppose that the pair (K, ϵ) , where K is a finite subset of G and $\epsilon > 0$, is given by the condition that $\pi|_W$ does not have almost invariant vector. Given any element $\xi \in V - W$, there is a sequence of elements $\{\xi_n\} \subseteq W$ such that $\xi_n \to \xi$ as $n \to \infty$. Then, for n large enough , we have:

$$\max_{g \in K} \| \pi(g)\xi - \xi \| = \max_{g \in K} \| \pi(g)\xi - \pi(g)\xi_n + \pi(g)\xi_n - \xi_n + \xi_n - \xi \|$$
$$\geq \max_{g \in K} \| \pi(g)\xi_n - \xi_n \| - 2\max_{g \in K} \| \xi_n - \xi \| \ge \epsilon \| \xi \| - \delta.$$

Now δ can be small enough , so

$$\max_{g \in K} \parallel \pi(g)\xi - \xi \parallel \geq \epsilon \parallel \xi \parallel,$$

Which completes the proof of one direction. The opposite direction is obvious. \Box

Another easy observation is that, in order to show a representation of group does not have almost invariant vectors, one only need to pass to a subgroup. That is,

Lemma 1.2.3. A unitary representation (π, V) of a group G does not have almost invariant vectors iff there exists a subgroup H of G such that the unitary representation $(\pi|_H, V)$ of H does not have almost invariant vectors.

Amenable groups. A basic strategy in this article is to use the regular representation of the free group \mathbb{F}_2 of rank 2, so the following theorem is of fundamental importance.

Theorem 1.2.4 ([18]). For the left regular representation λ_G of a finitely generated discrete group G on $\ell^2(G)$, λ_G has almost invariant vectors if and only if G is amenable.

Remark 1.2.1. Since \mathbb{F}_2 is not amenable, the left regular representation of \mathbb{F}_2 on $\ell^2(\mathbb{F}_2)$ does not have almost invariant vectors. We will regard $\ell^2(\mathbb{F}_2)$ as ℓ^2 -functions on vertices of the Cayley graph of \mathbb{F}_2 with respect to a chosen generating set, and thus further identify $\ell^2(\mathbb{F}_2)$ with the vector space V, where

$$V = \left\{ \sum_{i} \alpha_{i} g_{i} : \sum_{i} |\alpha_{i}|^{2} < \infty, \alpha_{i} \in \mathbb{C}, g_{i} \in \mathbb{F}_{2} \right\}.$$

1.3 Generalized Thurston measures and dynamics on measured foliation spaces

In this section we will describe the integral theory on the space of measured foliations and the action of subgroups of mapping class groups on the space of measured foliations. A subgroup of Mod(S) in which all elements except the identity are pseudo-Anosov mapping classes will be called a pseudo-Anosov subgroup.

1.3.1 Measures and L^2 -theory on $\mathcal{MF}(S)$.

Generalized Thurston measures on $\mathcal{MF}(S)$.

The space of measured foliations $\mathcal{MF}(S)$ of a surface S is the set of equivalence classes of transversely measured (singular) foliations on S. Using train tracks, one can show that $\mathcal{MF}(S)$ has a piecewise linear integral structure such that Mod(S) acts on it as automorphisms (that is, preserves this piecewise linear integral structure)[60]. Therefore, in such local PL coordinates, Mod(S) acts as linear transformations.

A consequence of this PL structure is that $\mathcal{MF}(S)$ can be equipped with a Mod(S)-invariant measure μ_{Th} , called the Thurston measure on $\mathcal{MF}(S)$. Moreover, this measure can be generalized to obtain a family of locally finite, ergodic Mod(S)-invariant measures $\mu_{Th}^{[(\mathcal{R},\gamma)]}$ on $\mathcal{MF}(S)$ for complete pairs (\mathcal{R},γ) , which will be called generalized Thurston measures. We present a brief summary of the construction of generalized Thurston measures $\mu_{Th}^{[(\mathcal{R},\gamma)]}$ according to [36].

Let $\gamma = \sum_{i} c_i \gamma_i, c_i > 0$ be a multi-curve on S, that is, γ is a collection of isotopy classes of pairwise distinct, pairwise disjoint essential simple closed curves $\{\gamma_i\}$ on Sso that each curve has been weighted by $c_i > 0$. After fixing a hyperbolic structure on S, one can think a multi-curve $\gamma = \sum_i c_i \gamma_i, c_i > 0$ as a collection of simple closed geodesics $\{\tilde{\gamma}_i\}$ on S with $\tilde{\gamma}_i$ labeled by a positive real number c_i , where $\tilde{\gamma}_i$ is the unique geodesic representative in γ_i . We will use γ to denote both the formal sum $\sum_i c_i \gamma_i$ and the subset $\prod \tilde{\gamma}_i$ of S. Cutting S along γ , one obtains a decomposition into a disjoint union

$$\overline{S-\gamma} = \bigsqcup T_i,$$

where $\{T_i\}$ is a collection of subsurfaces of S with boundary smoothly embedded in S. For

$$\mathcal{R} = \bigsqcup S_i$$

with $\{S_i\} \subseteq \{T_i\}$, the pair (\mathcal{R}, γ) will be called *a complete pair*. For a complete pair $(\mathcal{R} = \bigsqcup S_i, \gamma)$, define

$$\mathcal{MF}(\mathcal{R}) = \prod_i \mathcal{MF}^*(S_i)$$

where $\mathcal{MF}^*(S_i) = \mathcal{MF}(S_i) \bigcup 0_{S_i}$ in which 0_{S_i} is the zero foliation on S_i . The space $\mathcal{MF}(\mathcal{R})$ can be $Mod(\mathcal{R}, \gamma)$ -embedded on $\mathcal{MF}(S)$ via enlarging boundary curves [See [20], Exposé 6.6 for enlarging curves]. Denote by $\mathcal{M}(\mathcal{R})$ the image of this embedding. This set is endowed with the product measure $\mu_{\mathcal{R}} = \prod \mu_{Th}^i$, where μ_{Th}^i is the Thurston measure of S_i . Define also

$$M(\mathcal{R},\gamma) = \{\mathcal{F} + \gamma : \mathcal{F} \in \mathcal{M}(\mathcal{R})\} \subseteq \mathcal{MF}(S).$$

The inclusion induces a measure on $\mathcal{MF}(S)$, denoted by $\mu_{Th}^{[(\mathcal{R},\gamma)]}$ and supported on the set of Mod(S)-orbits of $M(\mathcal{R},\gamma)$, from the product measure $\mu_{\mathcal{R}}$.

Special cases are when $\mathcal{R} = \emptyset$ and γ is the isotopy class of a non-separating curve, or when $\mathcal{R} = S$ and $\gamma = \emptyset$. The corresponding measure in the case of $\mathcal{R} = \emptyset$ is a discrete measure, denoted by μ_{γ} and supported on Mod(S). γ which is regarded as a subset of $\mathcal{MF}(S)$, while in the case of $\gamma = \emptyset$ it is exactly the Thurston measure μ_{Th} on $\mathcal{MF}(S)$.

The following remarkable theorem indicates that generalized Thurston measures $\mu_{Th}^{[(\mathcal{R},\gamma)]}$ are exactly the set of all locally finite, Mod(S)-invariant, ergodic measures on $\mathcal{MF}(S)$.

Theorem 1.3.1 (Hamenstädt[28],Lindenstrauss-Mirzakhani[36]). Any locally finite Mod(S)-invariant ergodic measure on $\mathcal{MF}(S)$, up to a constant multiple, is in the form of $\mu_{Th}^{[(\mathcal{R},\gamma)]}$, where (\mathcal{R},γ) is a complete pair.

Associated L^2 -theory over $\mathcal{MF}(S)$.

The case of discrete measures. Recall that when $\mathcal{R} = \emptyset$, $\mu_{Th}^{[(\mathcal{R},\gamma)]}$ is the discrete measure supported on the set $Mod(S).\gamma$, where $Mod(S).\gamma$ is regarded as a subset of $\mathcal{MF}(S)$. We will first deal with the case that γ is the isotopy class of an essential simple closed curve on S and denote the measure by μ_{γ} .

Let $X_{\gamma} = C_{\gamma}^{0}(S)$ be the subset of vertices of the curve complex consisting of $\operatorname{Mod}(S) \cdot \gamma$. By considering the Dirac measure supported on X_{γ} , one can define the Hilbert space $\ell^{2}(X_{\gamma})$. It is clear that $\ell^{2}(X_{\gamma})$ is $\operatorname{Mod}(S)$ -equivariantly isomorphic to $L^{2}(\mathcal{MF}(S), \mu_{\gamma})$. On the other hand, let $G_{\gamma} = \operatorname{Mod}(S, \gamma) = Stab_{\gamma}(\operatorname{Mod}(S))$ be the set of all elements in $\operatorname{Mod}(S)$ that fix γ , then $\ell^{2}(X_{\gamma})$ can be further $\operatorname{Mod}(S)$ -equivariantly identified with $\ell^{2}(\operatorname{Mod}(S)/G_{\gamma})$. These two spaces give the same unitary representation of $\operatorname{Mod}(S)$, actually we have

Theorem 1.3.2 (Paris[51]). The infinite dimensional unitary representation of Mod(S)given by $\ell^2(Mod(S)/G_{\gamma})$ is irreducible.

Remark 1.3.1. This theorem was proved in a more general setting for 1-multi-curves on S, that is, $\gamma = \sum c_i \gamma_i$ with $c_i = 1$ for all i. Thus, in particular, this representation does not have non-zero invariant vectors. Meanwhile, the irreducibility also allows us to describe $\ell^2(\operatorname{Mod}(S)/G_{\gamma})$ more geometrically.

The first description of $\ell^2(\operatorname{Mod}(S)/G_{\gamma})$ is classical. For $f \in \ell^2(X_{\gamma})$, let $Supp(f) = \{v \in X_{\gamma} : f(v) \neq 0\}$. The function f has compactly support if the cardinality of Supp(f) is finite. Define the subspace W of $\ell^2(X_{\gamma})$ as the set of elements in $\ell^2(X_{\gamma})$ which have compact support. As X_{γ} is discrete, the following notation will be used to represent $f \in W$: $f = \sum_{i=1}^n k_i \alpha_i$. Note that W is $\operatorname{Mod}(S)$ -invariant and the closure \overline{W} of W in $\ell^2(X_{\gamma})$ is then $\ell^2(X_{\gamma})$ itself. This description will be used in the proof of the main theorem in the case of discrete measures.

The second description of $\ell^2(\operatorname{Mod}(S)/G_{\gamma})$ needs more explanations. Let $\operatorname{Teich}(S)$ be the Teichmüller space of S, and for each point \mathcal{X} of $\operatorname{Teich}(S)$, define a function on X_{γ} by

$$f_{\mathcal{X}}(\alpha) = e^{-\ell_{\mathcal{X}}(\alpha)}, \alpha \in X_{\gamma}$$

where $\ell_{\mathcal{X}}(\alpha)$ is the length of the unique geodesic in the isotopy class α .

Proposition 1.3.3. The function defined above belongs to $\ell^2(X_{\gamma})$.

Proof. It amounts to say

$$\sum_{\alpha \in X_{\gamma}} e^{-2\ell_{\mathcal{X}}(\alpha)} < \infty.$$

Thus this proposition is a corollary of the result of Birman-Series [8] or Mirzakhani [48] about the polynomial growth of simple closed geodesics. \Box

Let W' be the subspace of $\ell^2(X_{\gamma})$ which consisting of finite linear combinations of elements in $\{f_{\mathcal{X}} : \mathcal{X} \in \operatorname{Teich}(S)\}$. It is also true to see that this subspace is $\operatorname{Mod}(S)$ -invariant. Also by irreducibility, the closure $\overline{W'}$ of W' is $\ell^2(X_{\gamma})$. *Remark* 1.3.2. The second description gives rise to a parametrization for $\ell^2(X_{\gamma})$ via the Teichmüller space, thus it can be viewed as a reply to Problem 2.5 in [25] for representations under consideration.

For the case of $\mathcal{R} = \emptyset$ and γ is a general integral multicurve $\gamma = \sum k_i \gamma_i$ with $k_i \in \mathbb{N}$, Theorem 1.3.2 is not true in general as shown by the following

Example 1.3.4. Consider the genus 2 closed surface S, regarded as a quotient along boundaries of holed sphere with four disjoint open disks deleted. Let $\gamma = 2\gamma_1 + 3\gamma_2$, $\delta = \gamma_1 + \gamma_2$, where γ_1 and γ_2 are isotopy classes of two distinct images of boundaries. Obviously, there is a mapping class s that permutes the γ_i 's. Denote $H = \text{Mod}(S, \gamma)$ and $H' = \text{Mod}(S, \delta)$, then we have the exact sequence:

$$1 \to H \to H' \to \mathbb{Z}_2 \to 1.$$

That is, H is a normal subgroup of H' of index 2. This exact sequence allows us to define a self-map of the left cosets $\{fH\}$ as follows. Write H' as $H \bigsqcup sH$. There are two Mod(S)-invariant bijections:

$$Mod(S) \cdot \gamma \leftrightarrow \{[g] = gH\},$$
$$Mod(S) \cdot \delta \leftrightarrow \{[f] = fH'\}.$$

As $fH' = fH \bigsqcup fsH$, the set $\{gH\}$ can be rewritten as $\{fH, fsH\}$, this reformulation induces a well-defined inversion $i: fH = [f] \mapsto [fs] = fsH$.

A function ϕ on $G/H = \{gH\}$ is called even if for every $[g] \in G/H$, $\phi([g]) = \phi(i([g]))$ and a function φ on G/H is called odd if for every $[g] \in G/H$, $\varphi([g]) = -\phi(i([g]))$. Define V_1 to be the subset of $\ell^2(G/H)$ consisting of even functions and V_2 to be the subset of $\ell^2(G/H)$ consisting of odd functions. It is easy to see that such two vector spaces are non-empty, closed and Mod(S)-invariant subspaces of $\ell^2(G/H)$.

Remark 1.3.3. For any discrete measure mentioned above, the associated unitary representation has no nonzero invariant vectors.

The case of non-discrete measures. For general measures, we mention one remark.

Remark 1.3.4. If \mathcal{R} is nontrivial, ergodicity of the action shows that the associated unitary representation has no nonzero invariant vectors.

1.3.2 Actions of subgroups of Mod(S) on $\mathcal{MF}(S)$.

Train tracks and a construction of pseudo-Anosov mapping classes. For later use, we first recall some facts about train tracks and a construction of pseudo-Anosov mapping classes by Thurston. All discussions here are standard and wellknown, we refer to [52],[19],[[20], Exposé 13],[61] for more details.

A train track τ in a surface S is an embedded smooth graph with extra conditions on vertices. A train track is called *recurrent* if it supports a positive transverse measure, that is, a measure assigns a positive number to every edge. A *transversely recurrent* train track is a train track such that every edge has a nontrivial essential transverse intersection with a simple closed curve. A *birecurrent* train track is thus a train track that both recurrent and transversely recurrent. A *maximal* birecurrent train track is a birecurrent train track that cannot be a proper subtrack of any other train track. Any measured foliation is carried by a maximal train track. We only remark here that, for a maximal birecurrent train track τ , the set $E(\tau)$ of all positive transverse measures on τ is a positive linear submanifolds, that is, a subset of some Euclidean space defined by a family of linear equations with the condition that all parameters are positive. For the torus T, the set $\mathcal{MF}(T)$ of linear measured foliations can be covered by four affine charts $E(\tau_i)$ associated to four maximal birecurrent train tracks. We fix these four types of train tracks as blocks and denote them by $\{\tau_1, \tau_2, \tau_3, \tau_4\}$. See [[52], Section 2.6, Figure 2.6.1] for such four train tracks in the annulus, thus in the torus.

We now sketch a construction of pseudo-Anosov mapping classes given by Thurston [61]. We only discuss Thurston's construction for closed surfaces. For surfaces with boundaries, one can modify the construction without any difficulty. Let $S = S_g(g \ge 2)$ and choose two essential simple closed curves α and β on S so that all connected components of $S - \alpha \bigcup \beta$ are open topological disks. For each intersection point p of α and β , one can assign a rectangle to p so that S has a flat structure σ and, with respect to this flat structure, both Dehn twists T_{α} and T_{β} act as affine transformations (since we have flat structure, we can talk about affine transformations) with linear parts given by elements in $PSL(2, \mathbb{R})$. An element in the subgroup of Mod(S) generated by T_{α} and T_{β} is pseudo-Anosov if the associated affine transformation has a hyperbolic linear part.

We now mention some facts about the set $\mathfrak{L}(S, \sigma)$ of linear measured foliations on S induced by the flat structure σ above. Note that unstable and stable foliations of pseudo-Anosov mapping classes obtained by Thurstion's construction are in $\mathfrak{L}(S, \sigma)$ and $\mathfrak{L}(S, \sigma)$ is a closed subset of $\mathcal{MF}(S)$. If we arrange all rectangles mentioned above on the plane such that α -sides are horizontal and label the rectangles from left to right by $\{\Box_1, \Box_2, ..., \Box_m\}$, then a linear measured foliation $\mathfrak{F} \in \mathfrak{L}(S, \sigma)$ is given by parallel

lines of the plane and a train track τ in S carrying \mathfrak{F} has the form that the restriction of τ in each rectangle \Box_i is one of τ_i and all such τ_i appearing in τ are the same. Therefore there are four types of train tracks, denoted also by $\{\tau_1, \tau_2, \tau_3, \tau_4\}$, so that $\mathfrak{L}(S, \sigma) \subseteq \bigcup_{i=1}^4 E(\tau_i)$. A direct computation shows that linear measured foliations on S induced by this flat structure are determined by weights on two edges of $\tau_i \cap \Box_1$, thus each $\mathfrak{L}(S, \sigma) \cap E(\tau_i)$ is parameterized by two free independent parameters.

Lemma 1.3.5. Let $S = S_{g,n}$ be a compact surface with $3g + n \ge 5$ and α, β be two curves as above, then each τ_i is birecurrent and the set $\mathfrak{L}(S, \sigma)$ of linear measured foliations with respect to a flat structure σ constructed as described above is of null μ_{Th} -measure.

Proof. It is obvious that each τ_i is birecurrent. We divide the proof of the rest into two cases according to whether τ_i is maximal or not. If τ_i is not maximal, then any measured foliation carried by τ_i is not maximal [52]. By [[36], Lemma 2.3], $E(\tau_i)$ has null μ_{Th} -measure. If τ_i is maximal, then, as τ_i is a birecurrent train track, $E(\tau_i)$ is an open subset of $\mathcal{MF}(S)$ and thus every point in $E(\tau_i)$ should be determined by weights on 6g - 6 + 2n edges of τ_i . As remarked above that $E(\tau_i) \cap \mathfrak{L}(S, \sigma)$ is determined by weights on two edges of $\tau_i \cap \Box_1$ which can be extended to obtain 6g - 6 + 2n free parameters of $E(\tau_i)$. That is to say, $E(\tau_i) \cap \mathfrak{L}(S, \sigma)$ is locally given by $x_3 = x_4 = ... =$ $x_{6g-6+2n} = 0$ in $\mathbb{R}^{6g-6+2n}$ whose coordinates is given by $\{x_1, ..., x_{6g-6+2n}\}$. Therefore, $E(\tau_i) \cap \mathfrak{L}(S, \sigma)$ is a null set. Since $\mathfrak{L}(S, \sigma) \subseteq \bigcup_{i=1}^4 E(\tau_i)$, hence $\mathfrak{L}(S, \sigma)$ is a null set as well. \Box

Almost properly discontinuous action. We introduce a concept for a group action on a Borel space (that is, a topological space endowed with a Radon measure) which is weaker than usual properly discontinuous action. **Definition 1.3.1.** Let G be a group and (X, μ) be a Borel space. Suppose that G acts on X by measure-preserving homeomorphisms. We say that G acts on X almost properly discontinuously if there exists a G-invariant subset K with $\mu(K) = 0$ such that G acts on X - K properly discontinuously.

Example 1.3.6. Let $H \leq PSL(2, \mathbb{Z})$ be a Schottky group, then its limit set $\Lambda(H) \subseteq S^1$, as a Cantor set, has zero Lebesgue measure, and thus it acts on $\{\mathbb{R}^2 - (0,0)\}/\{\pm 1\}$ almost properly discontinuously.

Although the action of Mod(S) on $\mathcal{MF}(S)$ is ergodic with respect to generalized Thurston measures, the action of subgroups of Mod(S) on $\mathcal{MF}(S)$ is not always ergodic. The following proposition allows us to use properties of the "properly discontinuou" action.

Proposition 1.3.7. For each complete pair (\mathcal{R}, γ) , there exists a rank 2 free pseudo-Anosov subgroup H of Mod(S) that acts on $\mathcal{MF}(S)$ almost properly discontinuously with respect to the generalized Thurston measure $\mu_{Th}^{[(\mathcal{R},\gamma)]}$.

Any such free group will be called a *p*-rank 2 free subgroup.

The first case is when $\mathcal{R} = \emptyset$ or each component of \mathcal{R} is $S_{0,3}$, then this proposition is obvious by taking H to be any free pseudo-Anosov subgroup generated by two pseudo-Anosov mapping classes (this works the same for non-integral multicurves as for integral multicurves). For other cases, we prove this proposition through two lemmas.

Lemma 1.3.8. There exists a p-rank 2 free subgroup H of Mod(S) that acts on $\mathcal{MF}(S)$ almost properly discontinuously with respect to the Thurston measure μ_{Th} .

Proof. If $S = S_{0,4}$ or $S_{1,1}$, then, in both cases, $\mathcal{MF}(S)$ can be identified with $\{\mathbb{R}^2 - (0,0)\}/\{\pm 1\}$ and $\mathcal{PMF}(S)$ can be identified with S^1 . Moreover, there is a finite index subgroup of Mod(S) such that the action of this subgroup on $\mathcal{PMF}(S)$ is equivalent to the action of $PSL(2,\mathbb{Z})$ on S^1 , see [[19],Chapter 15] for the case of $S_{0,4}$. By taking H to be any subgroup given in Example 1.3.6 and considering the set $Y = Pr^{-1}(\Lambda(H))$, where $Pr : \mathcal{MF}(S) \to \mathcal{PMF}(S)$ is the projection, the action of H on $\mathcal{MF}(S)$ is thus almost properly discontinuous and $\mu_{Th}(Y) = 0$.

For other S, we deduce this lemma by first passing to $\mathcal{PMF}(S)$ and then using the result of McCarthy-Papadopoulos[45] on limit sets. Let ϕ and ψ be two independent pseudo-Anosov mapping classes obtained by Thurston's construction. By the pingpong lemma, one can construct a free pseudo-Anosov subgroup H generated by some powers of ϕ and ψ . As remarked before that stable and unstable measured foliations of pseudo-Anosov elements in H are linear measured foliations and $\mathfrak{L}(S, \sigma)$ is a closed subset, therefore, by Lemma 1.3.5, the limit set $\Lambda(H)$ of H, which is defined to be the closure of the set of fixed points of non-trivial elements of H with respect to the action on $\mathcal{PMF}(S)$, has the property that

$$\mu_{TH}(Pr^{-1}(\Lambda(H))) = 0.$$

On the other hand, one can define the zero set $Z(\Lambda(H))(\subseteq \mathcal{PMF}(S))$ of $\Lambda(H)$ [45]. By combining with facts [See [45], Proposition 6.1] that $Z(\Lambda(H)) - \Lambda(H)$ consists of no uniquely ergodic foliations and uniquely ergodic foliation has full μ_{Th} -measure, we know that $Pr^{-1}(Z(\Lambda(H)))$ has null μ_{Th} -measure. By [[45], Theorem 7.17], Hacts properly discontinuously on $\mathcal{PMF}(S) - Z(\Lambda(H))$, thus properly discontinuously on $\mathcal{MF}(S) - Pr^{-1}(Z(\Lambda(H)))$. Hence H acts almost properly discontinuously on $\mathcal{MF}(S)$. For $\mathcal{R} \neq S$, a complete pair (\mathcal{R}, γ) is called *a middle type* if $\mathcal{R} \neq \emptyset$ and there is a connected component $\neq S_{0,3}$.

Lemma 1.3.9. For a complete pair (R, γ) of middle type, there exists a p-rank 2 free subgroup H of Mod(S) that acts on $\mathcal{MF}(S)$ almost properly discontinuously with respect to the measure $\mu_{Th}^{[(R,\gamma)]}$.

Proof. We will follow the idea of [[36], Lemma 3.1] to prove this lemma. Fix any hyperbolic structure X on S and consider the continuous function $\ell_X : \mathcal{MF}(S) \to \mathbb{R}_+$ extending the geodesic length function. Thus

$$\mathcal{MF}(S) = \lim_{L_1 \to 0, L_2 \to \infty} B^{L_1}_{L_2}(X),$$

where $B_{L_2}^{L_1}(X) = \{\nu \in \mathcal{MF}(S) : \ell_X(\nu) \in [L_1, L_2]\}$ is a compact set and, as pointed out in the proof of [[36], Lemma 3.1], $B_{L_2}^{L_1}(X) \bigcap (\bigcup_{g \in Mod(S)} g.M(\mathcal{R}, \gamma))$ is equal to $B_{L_2}^{L_1}(X) \bigcap (\bigcup_{i=1}^n g_i.M(\mathcal{R}, \gamma))$, for some finite set $\{g_1, ..., g_n\} \subset Mod(S)$. Fix a free pseudo-Anosov subgroup H of Mod(S) and take any compact subset $K \subseteq \bigcup_{g \in Mod(S)} g.M(\mathcal{R}, \gamma)$. Taking L_1 small enough and L_2 large enough, one can assume $K \subseteq B_{L_2}^{L_1}(X)$. We now claim that

$$|\{h \in H : h.K \bigcap K \neq \emptyset\}| < \infty.$$

Let Z = Mod(S). γ and $\ell_X : Z \to R_+$. We first claim that there is a finite set $J \subseteq Z$ such that

$$\{h \in H : h.K \bigcap K \neq \emptyset\} \subseteq \{h \in H : h.J \bigcap J \neq \emptyset\}.$$

For every element in K can be written as $\gamma + \nu$ such that $\ell_X(\gamma)$ is bounded. If $h.K \bigcap K \neq \emptyset$, then $h(\gamma)$ also has bounded ℓ_X -length and all bounds can be chosen to be uniform on K, say [a, b]. Since ℓ_X is a proper map on Z (that is, the inverse of compact set is also compact), $J = \ell_X^{-1}([a, b])$ is then a finite subset of Z containing

both $h(\gamma)$ and γ . So one has $\{h \in H : h.K \cap K \neq \emptyset\} \subseteq \{h \in H : h.J \cap J \neq \emptyset\}$. By the discussion of the case $\mathcal{R} = \emptyset$, the set $\{h \in H : h.J \cap J \neq \emptyset\}$ is finite which implies that the finiteness of $|\{h \in H : h.K \cap K \neq \emptyset\}|$. Now taking the measure zero set to be $Y = \mathcal{MF}(S) - \bigcup_{g \in Mod(S)} g.M(\mathcal{R}, \gamma)$ completes the proof. \Box

H-related cover. Given a group H and a Borel space (X, μ) . Suppose that H acts on X almost properly discontinuously and freely. Examples for such (H, X, μ) are given by Proposition 1.3.7. By definition of almost properly discontinuous action, there is a null set Y such that H acts on X - Y properly discontinuously. For any compact subset K of X - Y, we will describe a "nice" cover of K. Since X - Y is the domain of discontinuity of H, for every p in K, there is an open neighbourhood \mathcal{U}_p of p in X - Y with finite μ -measure such that for all $h \in H$, one has $h \mathcal{U}_p \cap \mathcal{U}_p = \emptyset$. Thus there is an open cover of K. By compactness of K, choose a finite sub-cover of this cover. Label the sub-cover by $\mathcal{U}_1, ..., \mathcal{U}_n$ and for each $i \in 1, ..., n$, consider $A_i = \{h : \mathcal{U}_i | h \in H\}$. Starting from i = 1, form a family $B_1 = \{X_k \in A_1 | X_k \cap K \neq \emptyset\}$ as well as $C_1 = \{Y_k | Y_k = X_k \cap K, X_k \in B_1\}$. Delete $\bigcup_{Y_k \in C_1} X_k$ from K and denote the resulting compact set by K_1 . Then for K_1 , there is a family $B_2 = \{X_k \in A_2 | X_k \bigcap K_1 \neq \emptyset\}$ as well as $C_2 = \{Y_k | Y_k = X_k \bigcap K_1, X_k \in B_2\}.$ Delete $\bigcup_{Y_k \in C_2} X_k$ from K_2 and denote the resulting compact set by K_3 . Continuing this process, there is a cover of K which can be written in the following formula:

$$K \subseteq \bigsqcup_{k=1}^{n} \bigsqcup_{Y_i \in C_k} Y_i$$

So K can be covered by finitely many pairwise disjoint μ -measurable sets (we allow some of them to be null sets). This will be called an H-related cover of K, since, for each k, C_k is a family of disjoint sets that lie inside the H-orbit of some set.

1.4 Nonexistence of almost invariant vectors

Let $\mathcal{H}(\mu) = L^2(\mathcal{MF}(S), \mu)$, where $\mu = \mu_{Th}^{[(\mathcal{R},\gamma)]}$ is a generalized Thurston measure explained in Section 3.1.1, and π^{μ} be the associated unitary representation of Mod(S). The main result of this section is the following:

Theorem 1.4.1. For a compact surface $S = S_{g,n}$ with $3g+n \ge 4$ and each generalized Thurston measure μ , the associated representation π^{μ} of Mod(S) does not have almost invariant vectors.

By using Theorem 1.2.1, Remark 1.3.3 and Remark 1.3.4, we have:

Corollary 1.4.2. Let $S = S_{g,n}$ be a compact surface with $3g + n \ge 4$ and μ be a generalized Thurston measure, then $H^1(Mod(S), \pi^{\mu}) = \overline{H^1}(Mod(S), \pi^{\mu})$, where π^{μ} is the associated representation of Mod(S).

Proof. By Theorem 1.2.1, we only need to show that the representation π^{μ} has no nonzero invariant vectors. The corollary is thus concluded by using Remark 1.3.3 for discrete measures and Remark 1.3.4 for non-discrete measures.

Let γ be the isotopy class of an essential simple closed curve on S, $X = \text{Mod}(S).\gamma$ and \mathcal{X} be a point in the Teichmüller space Teich(S) of S. Denoting $\Delta_{\mathcal{X}}^{\phi_i}(\alpha) = \ell_{\mathcal{X}}(\alpha) - \ell_{\phi_i,\mathcal{X}}(\alpha)$, where $\alpha \in X$, and using the description of $\ell^2(X)$ via Teich(S) in Section 1.3.1, the following inequality is easy to show:

Corollary 1.4.3. Let $S = S_{g,n}$ be a compact surface with $3g + n \ge 4$ and γ be the isotopy class of an essential simple closed curve on S. Then there exists a finite subset $\{\phi_1, ..., \phi_n\}$ of Mod(S) consisting of pseudo-Anosov mapping classes and a constant

 $\epsilon > 0$, such that, for every point \mathcal{X} in Teich(S), we have:

$$\max_{i \in \{1,2,\dots,n\}} \left\{ \sum_{\alpha \in \operatorname{Mod}(S).\gamma} e^{-2\ell_{\mathcal{X}}(\alpha)} (e^{\Delta_{\mathcal{X}}^{\phi_i}(\alpha)} - 1)^2 \right\} \ge \epsilon \sum_{\alpha \in \operatorname{Mod}(S).\gamma} e^{-2\ell_{\mathcal{X}}(\alpha)}.$$

We divide the proof of Theorem 1.4.1 into two lemmas. First we prove a lemma used for discrete measures.

Lemma 1.4.4. Let G be a discrete countable group and X be a discrete set equipped with a G-action. Suppose that there is a rank 2 free subgroup H of G such that H acts on X freely. Then the unitary representation $\pi = \ell^2(X)$ of G associated to the action of G on X does not have almost invariant vectors.

Remark 1.4.1. This lemma is well-known, we give an elementary proof here mainly for heuristic purposes.

Definition 1.4.1. Let H be a rank 2 free group and X be a space on which H acts. Suppose $x \in X$ such that the stabilizer $Stab_H(x)$ of x is trivial. The image of H under the orbit map $H \to X, h \mapsto h.x$ is called the 2-tree based at x (with respect to (H, X)).

Proof of Lemma 1.4.4. By Lemma 1.2.3, we can pass to subgroups. For the action of the group H on the space X and any point $p \in X$, consider the 2-tree based at p with respect to (H, X).

Let W be the subspace of $\ell^2(X)$ consisting of functions with finite support. As W is G-invariant and dense, by Lemma 1.2.2, it is enough to show that $(\pi|_W, W)$ does not have almost invariant vectors. That is, we have to find (K, ϵ) with the property that

$$\max_{g \in K} \|\pi(g)f - f\|^2 \ge \epsilon \|f\|^2, \text{ for all } f \in W.$$

Since $H \cong \mathbb{F}_2$, as mentioned in Remark 1.2.1, the left regular representation $\ell^2(H)$ does not have almost invariant vectors, thus such a pair (K, ϵ) exists for the regular representation. Fix such pair (K, ϵ) for the rest of the proof. Here are two facts.

Facts:

1. For every 2-tree \mathbb{T} based at a point, $\ell^2(\mathbb{T})$ is H-equivariantly isomorphic to $\ell^2(H)$.

2. Different 2-trees are disjoint and thus, if the support A_1 of $f_1 \in \ell^2(X)$ and the support A_2 of $f_2 \in \ell^2(X)$ are located in different 2-trees, then f_1 and f_2 are orthogonal. These two facts imply that we only need to deal with ℓ^2 -functions on X whose finite support contained in a single 2-tree. In fact, for every $f \in W$, if we decompose its support K_f as

$$K_f = \bigsqcup_{i=1}^n K_{f_i},$$

where K_{f_i} lie in different 2-trees and f_i is defined to be the restriction of f on such different 2-trees, then

$$f = \sum_{i=1}^{n} f_i,$$
$$\|\pi(g)f - f\|^2 = \sum_{i=1}^{n} \|\pi(g)f_i - f_i\|^2, \text{ for all } g \in K$$

Note that $K \subseteq H$ is fixed. If the support of f_i is contained in a 2-tree \mathbb{T}_i , by Remark 1.2.1, there exists $g_i \in K$ such that

$$\|\pi(g_i)f_i - f_i\|^2 \ge \epsilon \|f_i\|^2.$$

Now for every f_i , let g_i be an element satisfying the above inequality. If two 2trees f_i, f_j correspond to the same $g_i = g_j$, then $f_i + f_j$ also satisfies that inequality. As K is finite, denote #K = m and so f can be further decomposed, that is, $f = f'_1 + f'_2 + \cdots + f'_s (s \leq m)$ such that $f'_k = \sum_j f_{jk}$, where $f_{jk} \in \{f_1, \dots, f_n\}$ and $\{f_{jk}\}_j$ correspond to the same $g_k \in K$. We claim that there exists $g_l \in K$ such that

$$\|\pi(g_l)f - f\|^2 \ge \frac{\epsilon}{s} \|f\|^2 \ge \frac{\epsilon}{m} \|f\|^2$$

Otherwise, since for all g_i selected, we have

$$\|\pi(g_i)f - f\|^2 \ge \|\pi(g_i)f_i - f_i\|^2 \ge \epsilon \|f_i\|^2,$$
(1.4.1)

then

$$\epsilon \|f\|^{2} = \sum_{i=1}^{m} \frac{\epsilon}{m} \|f\|^{2} > \sum_{i=1}^{m} \|\pi(g_{i})f - f\|^{2}$$
$$\geq \sum_{i=1}^{s} \|\pi(g_{i})f - f\|^{2} \ge \sum_{i=1}^{s} \epsilon \|f_{i}\|^{2} = \epsilon \|f\|^{2}.$$

The second inequality is the assumption and the last inequality is inequality (1.4.1). Thus there exists a pair $(K, \eta = \frac{\epsilon}{\sharp K})$ such that

$$\max_{g \in K} \|\pi(g)f - f\|^2 \ge \eta \|f\|^2, \text{ for all } f \in W.$$

So the proof of the lemma is completed.

Then we prove a lemma used for non-discrete measures.

Lemma 1.4.5. Let G be a discrete countable group and (X, μ) be a Borel space. Suppose that G acts on X by measure-preserving homeomorphisms. If there exists a rank 2 free subgroup H of G such that H acts on X almost properly discontinuously and freely, then the unitary representation $\pi = L^2(X, \mu)$ of G associated to the action of G on X does not have almost invariant vectors.

Proof of Lemma 1.4.5. Also by Lemma 1.2.3, we can pass to subgroups. Fix a null subset Y of X such that H acts on X - Y properly discontinuously. For any point $p \in X$, consider the image of H under the orbit map, given by

$$h \mapsto h.p.$$

Since the stabilizer $Stab_p(H)$ is trivial, this map is injective. This is the 2-tree based at p with respect to (H, X). Define W to be the G-invariant subspace of $L^2(X, \mu)$ consisting functions $f \in L^2(X, \mu)$ that compactly supported on X - Y. Thus $\overline{W} = L^2(X, \mu)$ as μ is a Radon measure. So as before, we only need to prove the theorem in the case of $(W, \pi|_W)$. For each $f \in W$ supported on one H-orbit of a measurable set U, that is,

$$K_f \subseteq \bigsqcup_{h \in H} h.U,$$

where K_f is the compact support of f and the union is disjoint indexed by H, fix a point p in U and associate an element $A_f \in \ell^2(\mathbb{T})$, where \mathbb{T} is the 2-tree based on p, via

$$A_f(h.p) = \left(\int_{h.U} |f|^2 d\mu\right)^{\frac{1}{2}}.$$

Define

$$K' = \left\{ g \in H | g \text{ or } g^{-1} \in K \right\},\$$

where K is the same finite subset of H as in Lemma 1.4.4. For f, one has:

$$\begin{split} \int_{K_f} |\pi(g)f - f|^2 d\mu \\ &= \sum_{h \in H} \int_{h.U} |\pi(g)f - f|^2 d\mu \\ &\geq \sum_{h \in H} \left| \left(\int_{h.U} |\pi(g)f|^2 d\mu \right)^{\frac{1}{2}} - \left(\int_{h.U} |f|^2 d\mu \right)^{\frac{1}{2}} \right|^2 \\ &= \sum_{h \in H} \left| A_{\pi(g)f}(h.p) - A_f(h.p) \right|^2 \\ &= \sum_{h \in H} \left| \left(\pi(g^{-1})A_f \right) (h.p) - A_f(h.p) \right|^2, \end{split}$$

where the second inequality is the triangle inequality. By Lemma 1.4.4,

$$\max_{g \in K'} \|\pi(g)f - f\|^{2}$$

$$\geq \max_{g \in K'} \sum_{h \in H} |(\pi(g)A_{f})(h.p) - A_{f}(h.p)|^{2}$$

$$= \max_{g \in K'} \|\pi(g)A_{f} - A_{f}\|^{2}$$

$$\geq \eta \|A_{f}\|^{2}$$

$$= \epsilon' \|f\|^{2},$$

where ϵ' is a multiple of the constant η in Lemma 1.4.4, as in this case we have #K' = 2#K. If the compact set K_f is not contained in one H-orbit, one can take an H-related cover of K_f , then by the orthogonality similar to Fact 2 in Lemma 1.4.4 and follow the last few lines in the proof of Lemma 1.4.4, one can also choose the pair (K', ϵ'') , where ϵ'' is a suitable multiple of ϵ' , to complete the proof.

Proof of Theorem 1.4.1. As any pseudo-Anosov subgroup acts freely on $\mathcal{MF}(S)$, by Lemma 1.4.4 and Proposition 1.3.7, the theorem is true for $\mathcal{R} = \emptyset$. When $\mathcal{R} = S$ or \mathcal{R} is of middle type, it is concluded by Lemma 1.4.5 and Proposition 1.3.7.

Remark 1.4.2. The same trick can be used to show that representations of mapping class groups in the space of L^2 -functions on the Teichmüller spaces with respect to Weil-Petersson volumes also have no almost invariant vectors. As one can show that such representations do not have non-trivial invariant vectors, we have the same conclusion about corresponding cohomology groups.

1.5 Classification of quasi-regular representations up to weak containment

1.5.1 Irreducible decompositions.

As pointed out in Section 1.3.1, for unitary representations of mapping class groups associated to discrete measures on the space of measured foliations, both reducible and irreducible ones exist. By examining Example 1.3.4 carefully, one sees that, reducible representations have an irreducible decomposition. For any multi-curve $\gamma = \sum_{i=1}^{k} c_i \gamma_i$ on S, where $c_i > 0$ for all i, we form $\tilde{\gamma} = \sum_{i=1}^{k} \gamma_i$. Recall that $\{\gamma_i\}$ is a collection of pairwise disjoint isotopy classes of essential simple closed curves on S. As before, denote by $G_{\gamma} = Mod(S, \gamma)$ and $G_{\tilde{\gamma}} = Mod(S, \tilde{\gamma})$ the corresponding subgroups of Mod(S). Hence G_{γ} is a subgroup of $G_{\tilde{\gamma}}$ of finite index.

Proposition 1.5.1. Let $S = S_{g,n}$ be a compact surface with $3g + n \ge 4$ and $\gamma, \tilde{\gamma}$ as above.

- (1) If the index of G_γ in G_γ is one, then the associated representation in l²(Mod(S)/G_γ) of Mod(S) is irreducible.
- (2) If the index of G_γ in G_γ is n > 1, then the associated representation of Mod(S) in l²(Mod(S)/G_γ) is reducible.

Proof. (1) is obvious, since the representation $\ell^2(\operatorname{Mod}(S)/G_{\gamma})$ is $\ell^2(\operatorname{Mod}(S)/G_{\tilde{\gamma}})$ which is irreducible by Remark 1.3.1.

Now assume that $[G_{\tilde{\gamma}} : G_{\gamma}] = n > 1$. Let $X_{\gamma} = \text{Mod}(S).\gamma$ and $Y_{\tilde{\gamma}} = \text{Mod}(S).\tilde{\gamma}$, then X_{γ} is a Mod(S)-equivariant discrete covering space of $Y_{\tilde{\gamma}}$ of degree n. So every ℓ^2 -function on $Y_{\tilde{\gamma}}$ defines an ℓ^2 -function on X_{γ} , and such correspondence produces a proper closed Mod(S)-invariant subspace of $\ell^2(X_{\gamma})$, which implies the reducibility.

1.5.2 Classification up to weak containment.

We first fix some notations. Fix a hyperbolic structure on S. Denote by $\gamma = \sum_{i=1}^{k} \gamma_i$ and $\delta = \sum_{i=1}^{k} \delta_i$, that is, multi-curves on S with coefficients all of 1s. Such multicurves will be called 1-multi-curves. For any 1-multi-curve $\gamma = \sum_{i=1}^{k} \gamma_i$ on S, we will call the union of geodesic representatives of γ a geometric multi-curve and, for any i, the representative α_i a geometric component. Denote by $G_{\gamma}(G_{\delta}, resp.)$ the corresponding subgroup of Mod(S), and by $\pi_{\gamma}(\pi_{\delta}, resp)$ the associated unitary representation on $\ell^2(Mod(S)/G_{\gamma})(\ell^2(Mod(S)/G_{\delta}), resp.)$. Let λ_S be the regular representation of the mapping class group Mod(S) of S on $\ell^2(Mod(S))$. We first recall some definitions which can be found in [51], [6], [7].

Let G be a countable discrete group and H be a subgroup of G, the *commensurator* of H is defined to be

$$Com_G(H) = \left\{ g \in G : gHg^{-1} \bigcap H \text{ has finite index in } H \text{ and } gHg^{-1} \right\}$$

A discrete group is said to be C^* -simple if every unitary representation, which is weakly contained in the regular representation of G, is weakly equivalent to the regular representation. Let γ and δ be geometric multi-curves, then γ and δ are of the same type if there is an element f in Mod(S) such that $f(\gamma) = \delta$. We say a subgroup H of G has the spectral gap property if the unitary representation $\ell^2(X)$ associated to the action $H \curvearrowright X = G/H - \{H\}$ does not have almost invariant vectors. In this section, we give a classification for unitary representations of Mod(S) associated to discrete measures. **Lemma 1.5.2.** Given a 1-multi-curve γ on S and let m be the number of its geometric components.

- 1. If m = 3g 3 + n, then G_{γ} is amenable.
- 2. If $1 \le m < 3g 3 + n$, then G_{γ} has the spectral gap property.

Proof. If m = 3g - 3 + n, then G_{γ} is virtually abelian, thus it is amenable. For other cases, as m < 3g - 3 + n, one can cut S along geometric components so that the resulting surface has at least one connected component that admits two pseudo-Anosov mapping classes generating a rank 2 pseudo-Anosov subgroup. Assume components admitting pseudo-Anosov mapping classes are labelled as $T_1, ..., T_k$, two pseudo-Anosov mapping classes in each $Mod(T_i)$ and the associated rank 2 pseudo-Anosov subgroup are also denoted by φ_i, ψ_i, H_i , respectively. Note that pseudo-Anosov homeomorphisms fix boundaries. Then define two maps φ and ψ on S (thus their isotopy classes) by extending $\varphi = \prod_i \varphi_i$ and $\psi = \prod_i \psi_i$. Hence the subgroup H generated by φ and ψ is a rank 2 free group. Moreover the action of H on the set $X_{\gamma} - \{\gamma\}$ has trivial stabilizers. Otherwise, if an element ϕ in H fix $\delta \in X_{\gamma} - \{\gamma\}$, then by the construction of H, the geometric intersection number of δ and γ is nonzero and thus it intersects one of T_i . We cut S along γ so that δ becomes a family of isotopy classes of arcs. Since ϕ fixes δ , up to some powers of ϕ , it fixes each resulting isotopy class of arcs. But then it can be shown that, for some i, there is an element in H_i that fixes the isotopy class of an essential simple closed curve, which contradicts the assumption that H_i is a pseudo-Anosov subgroup. By Lemma 1.4.4, we can conclude that G_{γ} has the spectral gap property.

Lemma 1.5.3 (Theorem A in [7]). Let G be a countable discrete group and H be a subgroup of G that has the spectral gap property. Let L be a subgroup of G satisfying

 $Com_G(L) = L$, then two unitary representations $\ell^2(G/H)$ and $\ell^2(G/L)$ of G are weakly equivalent if and only if L is conjugate to H.

Theorem 1.5.4. Let $S = S_{g,n}$ be a compact surface with $3g + n \ge 4$. Let γ and δ be two 1-multi-curves on S with k, l geometric components, respectively.

- (1) If at least one of k, l is not 3g-3+n, then the associated unitary representations π_{γ} and π_{δ} are weakly equivalent if and only if γ and δ are of the same type.
- (2) Suppose S is not S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}. If the number of geometric components of γ is 3g 3 + n, then π_γ is weakly equivalent to the regular representation λ_S.
- (3) Suppose S is not S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}. If the number of geometric components of γ is not 3g 3 + n, then π_γ is not weakly contained in λ_S.

Proof. For any 1-multi-curve γ on S, $Com_{Mod(S)}(G_{\gamma}) = G_{\gamma}$ (see [51]). Given two 1-multi-curves γ and δ with k, l geometric components, respectively, such that at least one of k and l is not 3g - 3 + n, then by Lemma 1.5.2, Lemma 1.5.3 and the fact that G_{γ} is conjugate to G_{δ} if and only if γ and δ are of the same type, we complete the proof for (1). For (2), by [15], if S is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$, the mapping class group Mod(S) is C*-simple. By the result of [14] which states that a discrete group is C*-simple if and only if, for any amenable subgroup M of G, the quasi-regular representation $\ell^2(G/M)$ is weakly equivalent to the regular one. So combine with Lemma 1.5.2, we complete the proof of (2). The statement (3) is deduced from (2) and the definition of C*-simplicity.

Remark 1.5.1. The "only if" part of (1) is a stronger version of Corollary 5.5 in [51]. Remark 1.5.2. If S is one of $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$, it is easy to show that, if the number of components of γ is 3g-3+n, then π_{γ} is weakly contained in the regular representation λ_S . However, for other types of γ , we don't know if π_{γ} is weakly contained in λ_S . And we don't know what can be said about unitary representations corresponding to non-discrete measures on the space of measured foliations. Chapter 2

Boundary representations of mapping class groups

2.1 Introduction

Let $S = S_g$ be a closed, connected, orientable surface of genus g. Recall that the mapping class group Mod(S) of S is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of S. Throughout this chapter, the genus gis assumed to be at least 2. The space of measured foliations $\mathcal{MF}(S)$ is the set of equivalence classes of non-zero measured foliations on S. The mapping class group Mod(S) acts on $\mathcal{MF}(S)$ and preserves a Radon measure ν , called the Thurston measure on $\mathcal{MF}(S)$. Moreover, the space $\mathcal{MF}(S)$ is equipped with an \mathbb{R}_+ -action that commutes with the Mod(S)-action. Therefore, Mod(S) acts on the quotient $\mathcal{PMF}(S)$, called the projective measured foliation space, of $\mathcal{MF}(S)$ by \mathbb{R}_+ preserving a measure class $[\nu]$, called the Thurston measure (class) on $\mathcal{PMF}(S)$, defined by the Thurston measure on $\mathcal{MF}(S)$.

One motivation of this thesis is to use geometric objects, such as $\mathcal{MF}(S)$ and $\mathcal{PMF}(S)$ to understand unitary representations of Mod(S) (see also the first chapter for related topics). Recall that, for a probability measure class-preserving action of G on $(X, [\nu])$, one defines a unitary representation of G on $L^2(X, \nu)$, called a *quasiregular representation* (see Section 2.2.1 for more details and the reader should not confuse this terminology with the one in the first chapter). Hence, for a probability measure class-preserving ergodic action, it is natural to ask that whether the quasi-regular representation is irreducible. Notice that this is not true for a measurepreserving ergodic action as it always has $\mathbb{C}1_X$ as a nontrivial closed invariant subspace. For the ergodic action of Mod(S) on $\mathcal{PMF}(S)$ with respect to $[\nu]$, we prove: **Theorem 2.1.1** (See Corollary 2.2.15). Let $S = S_g$ be a closed surface of genus $g \ge 2$. The quasi-regular unitary representation of the mapping class group Mod(S) on $L^2(\mathcal{PMF}(S), \nu)$, the space of square integrable functions on $\mathcal{PMF}(S)$ with respect to the Thurston measure ν , is irreducible.

In fact, we prove an ergodic-type theorem, namely Theorem 2.2.14, for this quasiregular representation and as a by-product of this ergodic-type theorem, we have the classical result of Masur [41] on ergodicity of the action (see Remark 2.2.1 for the relation between ergocidity of an action and the associated quasi-regular representation). However, since our work uses Masur's result implicitly, so we don't give a new proof to his result.

The main theorem is related to a question of Bader-Muchnik in the context of random walks on groups. Namely, let G be a discrete group and μ be a probability measure on G. Let $(\partial G, \nu)$ be the Poisson boundary of G associated to the μ -random walk on G. Then the measure class $[\nu]$ is G-invariant, hence defines a quasi-regular representation of G on $L^2(\partial G, \nu)$. In [5], inspired by the cases of free groups and lattices in Lie groups, Bader-Muchnik proposed the following conjecture:

Conjecture 2.1.2 (Bader-Muchnik[5]). For a locally compact group G and a spreadout probability measure μ on G, the quasi-regular representation associated to the μ -Poisson boundary of G is irreducible.

Before returning to mapping class groups, we first mention briefly some progress on this conjecture. As mentioned above, this conjecture is true for certain random walks on free groups and lattices in Lie groups (see [5] and references therein). Hence it is true for the mapping class group $Mod(S) = SL(2,\mathbb{Z})$ of closed surface of genus one acting on $\mathcal{PMF}(S) = S^1$ with respect to the Lebesgue measure which is also identified with the Thurston measure on $\mathcal{PMF}(S)$. All identifications are Mod(S)-equvariant. For lattices in Lie groups, one can also deduce the irreducibility from ergodicity of the associated quasi-regular representation (see [12]). The conjecture is then verified in [5] for the fundamental group of compact negatively curved manifolds with respect to the Patterson-Sullivan measure by Bader-Muchnik. Their result has been further generalized to hyperbolic groups [22] with respect to the Patterson-Sullivan measure by Garncarek and some discrete subgroups of the group of isometries of a CAT(-1) space with non-arithmetic spectrum by Boyer [13]. Note that in all cases above, the Patterson-Sullivan measure on the Gromov boundary coincides with the Poisson boundary of (G, μ) for some probability measure μ on G. However, Björklund-Hartman-Oppelmayer [9] recently showed that there are random walks on some Lamplighter groups and solvable Baumslag-Solitar groups providing counterexamples to this conjecture.

The relationship between the main theorem and above progress is the following. On the one hand, there is a long history on exploiting similarities between mapping class groups and hyperbolic groups which is quite fruitful. To name very few among massive literatures, we mention [44], [43] and [27]. On the other hand, by [4], the Thurston measure on $\mathcal{PMF}(S)$ is the Patterson-Sullivan measure on the Teichmüller boundary of the Teichmüller space of S which is in the similar situation with the previous known cases. We also mention that Rafi recently announced that $\mathcal{PMF}(S)$ equipped the Thurston measure class is the μ -boundary of some random walk on Mod(S).

Outline of the proof. By regarding $\mathcal{PMF}(S)$ as the Teichmüller boundary of the Teichmüller space Teich(S) of S, we follow the approach in Boyer-Pittet-Link[12]: irreducibility of a representation is deduced from the ergodicity of the representation

(see Section 2.2.1 for definitions), namely Theorem 2.2.14. To prove Theorem 2.2.14, we adapt a criterion (Theorem 2.2.13) in [12]. The bulk of the work is to construct a family of finite subsets of Mod(S) and show that they satisfy the conditions in Theorem 2.2.13. The family of subsets is constructed by carefully choosing elements in Mod(S) with enough hyperbolicity so that the cardinality of these subsets goes to infinity. Actually, we even need the growth to be exponential. The subsets are described before Lemma 2.2.11 relying on Dowdall-Duchin-Masur [16]. Then by results on counting lattices in Athreya-Bufetov-Eskin-Mirzakhani [4], one can show that these sets have exponential growth. Then the next step is to verify the convergence and uniform boundedness in Theorem 2.2.13. As the action of Mod(S) on Teich(S)is not homogeneous, the approach in Boyer-Pittet-Link [12] no longer works. We use ideas in Bader-Muchnik [5], Boyer [13]: we first prove the Harish-Chandra estimates in Section 2.4 as in [13] and then use it to deduce convergence and uniform boundedness (Section 2.5.1). Note that the proof of convergence in [12] is purely Lie theoretic and our proof for mapping class groups is based on the Harish-Chandra estimates. The proof of the Harish-Chandra estimates (Section 2.4) is the novelty in this paper. Instead of doing estimations directly, we first relate it to integration on intersection numbers and then use the map considered in Masur-Minsky [43] which relates $\operatorname{Teich}(S)$ to the curve complex of S to simplify integrations.

Erratum. The last part of the proof of Lemma 2.3.5 is not correct since by our choice of E_n , $[\zeta]$ might be non uniquely ergodic. But this lemma is correct and Tiozzo-Yang has a proof for it. We also point out that Duchin's proof of Theorem 2.2.7 is unpulshed, therefore our proof of Lemma 2.5.4 is not complete.

2.2 Quasi-regular unitary representations

2.2.1 Quasi-regular representations of discrete groups.

In this section, we will recall ergodic quasi-regular representations and a criterion for showing ergodicity of representations. The reader is referred to [5],[13] and [12] for more details.

Quasi-regular unitary representations. Let G be a locally compact secondcountable group and X be a second-countable Hausdorff topological space. Let ν be a probability Borel measure on X. Assume that G acts on X as homeomorphisms and G preserves the measure class $[\nu]$ of ν , namely, G preserves null ν -measure sets . Choose $\nu \in [\nu]$, thus for every $\gamma \in G$, the measure $\gamma_*\nu$ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to $\gamma_*\nu$. Denote the corresponding Radon-Nikodym derivative by $c(\gamma,\nu) = \frac{d\gamma_*\nu}{d\nu}$. One can construct a unitary representation π_{ν} of G on $L^2(X,\nu)$ as follows: For every $f \in L^2(X,\nu)$, every $x \in X$ and every $\gamma \in G$, $\pi_{\nu}(\gamma)f(x)$ is defined to be $\pi_{\nu}(\gamma)f(x) = f(\gamma^{-1}x)c(\gamma,\nu)^{\frac{1}{2}}(x)$. The representation π_{ν} will be called a *quasi-regular (unitary) representation* of G. We remark that if ν, μ are in the same measure class, then π_{ν}, π_{μ} are unitary equivalent. Assume that $c(\gamma, \nu)^{\frac{1}{2}}$ is integrable for each $\gamma \in G$ with respect to ν . The Harish-Chandra function Φ associated to π_{ν} is then defined to be the integral

$$\Phi(\gamma) = \langle \pi_{\nu}(\gamma) \mathbb{1}_X, \mathbb{1}_X \rangle_{L^2(X,\nu)} = \int_X c(\gamma,\nu)^{\frac{1}{2}}(x) d\nu(x).$$

Ergodic quasi-regular representations. From now on, we always assume that G is a discrete group. Let $(X, \nu), \pi_{\nu}$ as above and $\mathcal{B}(L^2(X, \nu))$ be the Banach space of bounded operators on $L^2(X, \nu)$. Let $e_K : K \longrightarrow X$ be a map from a finite subset K of G to X and $f : X \longrightarrow \mathbb{C}$ be a bounded Borel function. Consider the following

elements in $\mathcal{B}(L^2(X,\nu))$:

$$\begin{split} M^f_{(K,e_K)} &: L^2(X,\nu) \longrightarrow L^2(X,\nu), \phi \mapsto \frac{1}{|K|} \sum_{\gamma \in K} f(e_K(g)) \frac{\pi_\nu(\gamma)\phi}{\Phi(\gamma)}, \\ P_{\mathbb{1}_X} &: L^2(X,\nu) \longrightarrow L^2(X,\nu), \phi \mapsto \int_X \phi d\nu \mathbb{1}_X, \\ m(f) &: L^2(X,\nu) \longrightarrow L^2(X,\nu), \phi \mapsto f\phi. \end{split}$$

We now introduce an ergodicity for quasi-regular representations which generalizes the usual ergodicity for measure class-preserving group actions. Recall that a sequence $F_n \in \mathcal{B}(L^2(X,\nu))$ converges to $F \in \mathcal{B}(L^2(X,\nu))$, written as $F_n \to F$, in the weak operator topology if, for every $\phi, \psi \in L^2(X,\nu)$, $\lim_{n\to\infty} \langle F_n(\phi), \psi \rangle_{L^2} = \langle F(\phi), \psi \rangle_{L^2}$.

Definition 2.2.1 (Boyer-Link-Pittet[12]). Let $G, (X, \nu), \pi_{\nu}, f$ as above. Suppose that for every $n \in \mathbb{N}$, there is a pair $(K_n, e_n : K_n \longrightarrow X)$ such that K_n is a finite subset of G and such that $|K_n| \to \infty$ as $n \to \infty$. The representation π_{ν} is called ergodic with respect to (K_n, e_n) and f, if we have the following convergence in the weak operator topology:

$$M^f_{(K_n,e_n)} \to m(f)P_{\mathbb{1}_X}.$$

Remark 2.2.1. It is easy to see that the ergodicity of a measure class-preserving group action is weaker than the ergodicity of the associated quasi-regular representation. One could refer to [[12]], Proposition 2.5] for its proof.

The following criterion for the ergodicity of a quasi-regular representation is essentially contained in [5] and summarized in [12].

Theorem 2.2.1 ([12] Theorem 2.2). Let $G, (X, \nu)$ as above and π_{ν} be the associated quasi-regular representation of G on $L^2(X, \nu)$. Let L be a length function on G and let (X, d) be a metric space inducing the topology of X. For every $n \in \mathbb{N}$, let E_n be a symmetric finite subset of G, that is $E_n = E_n^{-1}$, and $e_n : E_n \longrightarrow X$ be a map. Assume that the following conditions hold:

- (1) for every $g \in G$, $\|\pi_{\nu}(g)\mathbb{1}_X\|_{L^{\infty}(X,\nu)} < \infty$,
- (2) $\lim_{n\to\infty} |E_n| = \infty$,
- (3) for all Borel subsets $W, V \subset X$ such that $\nu(\partial W) = \nu(\partial V) = 0$,

$$\limsup_{n \to \infty} \frac{1}{|E_n|} \left| \left\{ \gamma \in E_n : e_n(\gamma^{-1}) \in W \text{ and } e_n(\gamma) \in V \right\} \right| \le \nu(W)\nu(V).$$

(4) for every $r \ge 0$, there is a non-increasing function $h_r : [0, \infty) \longrightarrow [0, \infty)$ such that $\lim_{s\to\infty} h_r(s) = 0$ and such that

$$\forall n \in \mathbb{N}, \forall \gamma \in E_n, \frac{\langle \pi_{\nu}(\gamma) \mathbb{1}_X, \mathbb{1}_{\{x \in X : d(x, e_n(\gamma)) \ge r\}} \rangle_{L^2}}{\Phi(\gamma)} \le h_r(L(\gamma)),$$

(5)

$$\sup_{n} \left\| M_{E_n}^{\mathbb{1}_X} \mathbb{1}_X \right\|_{L^{\infty}(X,\nu)} < \infty.$$

Then the quasi-regular representation π_{ν} is ergodic with respect to (E_n, e_n) and any $f \in \overline{H}^{L^{\infty}(X,\nu)}$, where H is a vector space generated by

$$\{\mathbb{1}_U : \nu(\partial U) = 0 \text{ and } U \text{ is a Borel subset of } X\}.$$

Remark 2.2.2. Thanks to condition (1), the Harish-Chandra function exists for each γ in G.

Proposition 2.2.2 (Bader-Muchnik[5]). Under the assumptions in the above theorem, if moreover ν is a Radon measure, then π_{ν} is irreducible.

2.2.2 Quasi-regular representations of mapping class groups

We now consider quasi-regular representations of mapping class groups and state our main theorem. For more on mapping class groups and Teichmüller theory, we refer to [19], [4] and [32].

Mapping class groups and Teichmüller spaces. Let $S = S_g$ be a genus g, closed, connected, orientable surface. We always assume that $g \ge 2$. All arguments here work for hyperbolic surfaces with punctures as well. The mapping class group Mod(S) of S is the group of isotopy classes of orientation-preserving homeomorphisms of S. Namely, if the group of orientation-preserving homeomorphisms of S is denoted by $Homeo^+(S)$ and the group of homeomorphisms of S that isotopic to the identity is denoted by $Homeo_0(S)$, then

$$Mod(S) = Homeo^+(S)/Homeo_0(S).$$

We remark here that mapping class groups of surfaces are finitely presented and considered to be discrete groups. The *Teichmüller space* Teich(S) of S is the space of homotopy classes of hyperbolic structures. The Teichmüller space Teich(S) is homeomorphic to \mathbb{R}^{6g-6} and the mapping class group Mod(S) acts on Teich(S) by changing markings. The quotient $\mathcal{M}(S) = \text{Teich}(S)/\text{Mod}(S)$ is the *moduli space* of S. There are several distances on Teich(S) so that Mod(S) acts as isometries, the one that we will use is the *Teichmüller distance* $d = d_T$. It is defined as follows: For $\mathcal{X} = [(X, \phi)], \mathcal{Y} = [(Y, \psi)] \in \text{Teich}(S), d(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \log K_f$, where $f : X \longrightarrow Y$ is the Teichmüller mapping, locally in the form of $x + iy \mapsto e^t x + ie^{-t}y$, in the isotopy class of $\psi \circ \phi^{-1}$, namely the quasi-conformal homeomorphism with minimal dilatation in the isotopy class of $\psi \circ \phi^{-1}$ and K_f is the dilatation of f. It is obvious that Mod(S) $\subset Isom(\text{Teich}(S), d)$. Neither the Teichmüller space Teich(S) nor $\mathcal{M}(S)$ is compact.

Measured foliations. The Teichmüller space can be compactified in several ways. The compactification we will use in this paper is the Teichmüller compactification. Fix a point $o \in \text{Teich}(S)$ that is considered to be a Riemann surface X via uniformization. A holomorphic quadratic differential $q \in \text{H}^0(X, \Omega_X^{\otimes 2})$ on X is locally of the form $q(z)dz^2$ such that q(z) is a holomorphic function. Define a norm on q by

$$\|q\| = \int_X |q(z)| dx dy$$

and consider the unit open ball $B^1(X)$ with respect to $\|\cdot\|$. The set QD(X) of holomorphic quadratic differentials is a vector space and can be identified with the cotangent space of $\operatorname{Teich}(S)$ at o. There is a homeomorphism $\pi: B^1(X) \longrightarrow \operatorname{Teich}(S)$ sending each open unit ray in QD(X) starting at the origin to a Teichmüller geodesic starting at o. The *Teichmüller compactification* is then the visual compactification by adding ending points in the unit sphere of QD(X) to each ray. The Teichmüller compactification will be denoted by $\overline{\operatorname{Teich}(S)}$. Thus, the boundary $\partial \overline{\operatorname{Teich}(S)}$ of $\overline{\operatorname{Teich}(S)}$ is the unit sphere $QD^1(X)$.

One could give a geometric description of $\partial \overline{\text{Teich}(S)}$ via projective measured foliations. A measured foliation on S is a singular foliation of S endowed with a transverse measure. The space $\mathcal{MF}(S)$ of measured foliations is then the set of equivalent classes of measured foliations where the equivalence is given by Whitehead moves and isotopy. The space $\mathcal{MF}(S)$, endowed with the weak topology on measures, is homeomorphic to \mathbb{R}^{6g-6} . The quotient, called *the projective measured foliation* space $\mathcal{PMF}(S)$ of S, of $\mathcal{MF}(S)$ by the nature action of \mathbb{R}_+ is homeomorphic to the 6g-7sphere S^{6g-7} . Both $\mathcal{MF}(S)$ and $\mathcal{PMF}(S)$ are equipped with a Mod(S)-action. There is a deep relation between $\mathcal{MF}(S)$ and QD(X). Namely, for each holomorphic quadratic differential q, the vertical measured foliation $\mathcal{V}(q)$ of $q = q(z)dz^2$ is the foliation given by the integral curves of the holomorphic tangent vector field on S such that each vector has a value in negative real numbers under q, where the transverse measure is given by integration of $|Re\sqrt{q}|$. By a theorem of Hubbard-Masur, the map \mathcal{V} that assigns each holomorphic quadratic differential q on X to $\mathcal{V}(q)$ is a homeomorphism from $QD(X) - \{0\}$ onto $\mathcal{MF}(S)$. The composition $\pi \circ \mathcal{V}$ of the map $\mathcal{V}: QD(X) - \{0\} \longrightarrow \mathcal{MF}(S)$ and the quotient map $\pi: \mathcal{MF}(S) \longrightarrow \mathcal{PMF}(S)$ gives the identification of $QD^1(X)$ with $\mathcal{PMF}(S)$. Thus, we will regard $\mathcal{PMF}(S)$ as the boundary of the Teichmüller compactification of Teich(S). The equivalent class of $\xi \in \mathcal{MF}(S)$ in $\mathcal{PMF}(S)$ will be denoted by [ξ]. Any $q \in QD^1(X)$ (hence $[\mathcal{V}(q)] \in \mathcal{PMF}(S)$) determines a Teichmüller geodesic ray g_t starting from o, hence, by abuse of terminologies, we will call q and $[\mathcal{V}(q)]$ the *direction* of g_t and sometimes write g_t as g_t^q or $\mathcal{V}(q)(t)$.

Any isotopy class γ of essential simple closed curves on S defines a (topological) foliation $\lambda(\gamma)$. Hence, any weighted isotopy class $c\gamma$ of essential simple closed curves defines a foliation $\lambda \in \mathcal{MF}(S)$. The measured foliation λ , as topological foliation, is the same as $\lambda(\gamma)$, but the transverse measure is given by c. Therefore, let $\mathcal{C}(S)$ denote the set of isotopy classes of essential simple closed curves, there is an embedding of $\mathcal{C}(S) \times \mathbb{R}_+$ into $\mathcal{MF}(S)$. The image is dense (See Thurston [61]). This embedding enable us to define three functions that we will use. The first one is the intersection number on $\mathcal{MF}(S)$. The intersection number $i : \mathcal{MF}(S) \times \mathcal{MF}(S) \longrightarrow \mathbb{R}_+$ is the unique continuous function on $\mathcal{MF}(S) \times \mathcal{MF}(S)$ that extends the geometric intersection number of two essential simple closed curves and satisfies $i(c\lambda, \xi) = ci(\lambda, \xi)$ for every c > 0 (See [54], Corollary 1.11). The second one is the extremal length. Let $o = [(X, \phi)] \in \operatorname{Teich}(S)$ where X is a Riemann surface. Let γ be the isotopy class of an essential simple closed curve. The extremal length $\operatorname{Ext}_X(\gamma)$ of γ in X is defined to be

$$\operatorname{Ext}_X(\gamma) = \sup_{\rho} \ell_{\rho}(\gamma)^2,$$

where ρ runs over all metrics with unit area in the conformal class of X and $\ell_{\rho}(\gamma)$ is the infimum of ρ -length of simple closed curves in γ . Then the extremal length $\operatorname{Ext}_X : \mathcal{MF}(S) \longrightarrow \mathbb{R}_+$ is the unique continuous function on $\mathcal{MF}(S)$ that extends the extremal length of $\mathcal{C}(S)$ and satisfies $\operatorname{Ext}_X(c\lambda) = c^2 \operatorname{Ext}_X(\lambda)$ for $c \in \mathbb{R}_+$ (See [32], Proposition 3). Note that the extremal length in fact is defined on $\operatorname{Teich}(S) \times \mathcal{MF}(S)$, namely, if $[(X, \phi)] = [(Y, \psi)] \in \operatorname{Teich}(S)$, then $\operatorname{Ext}_X(\cdot) = \operatorname{Ext}_Y(\cdot)$. So we will write $\operatorname{Ext}_o(\cdot)$ rather than $\operatorname{Ext}_X(\cdot)$ for $o = [(X, \phi)]$. The third one is the hyperbolic length $\ell_o(\gamma)$ which is defined to be the X-length of unique X-hyperbolic geodesic $\tilde{\gamma}$ in the isotopy class γ . The function $\ell_o(\cdot)$ can be uniquely extended as well to $\mathcal{MF}(S)$ to obtain a continuous function ℓ_o on $\mathcal{MF}(S)$ [33]. We will use the following relation: given a point o in $\operatorname{Teich}(S)$, then there exists a constant C = C(o), depending on o, such that

$$\forall \xi \in \mathcal{MF}(S), \frac{1}{C}\ell_o(\xi) \le \sqrt{\operatorname{Ext}_o(\xi)} \le C\ell_o(\xi).$$

Recall that a measured foliation λ is called *minimal* if it has no simple closed leaves. Two measured foliations are said to be *topologically equivalent* if they, as topological foliations, are differ by isotopies and Whitehead moves. A measured foliation ξ is called *uniquely ergodic* if it is minimal and any measured foliation ζ that topologically equivalent to ξ is measure equivalent to ξ , that is, $[\xi] = [\zeta]$. When ξ is uniquely ergodic, we will call $[\xi]$ uniquely ergodic. The following two lemmas are essential to our approach using intersection numbers.

Lemma 2.2.3. (Rees [54], Theorem 1.12 or Masur [40]) Let λ be a uniquely ergodic

measured foliation and η be any measured foliation. Then $i(\lambda, \eta) = 0$ if and only if $[\lambda] = [\eta]$.

Lemma 2.2.4 (Masur's criterion [42]). Given $\epsilon > 0$. If a Teichmüller geodesic ray g_t starting from o does not leave $\operatorname{Teich}_{\epsilon}(S)$ eventually, then the direction of g_t is uniquely ergodic.

One feature of the Teichmüller compactification is that the action of Mod(S) cannot be extended continuously to $\overline{Teich(S)}$ [32]. However, uniquely ergodic measured foliations are nice points in terms of Mod(S)-action in the following sense:

Lemma 2.2.5. (Masur [40]) The mapping class group acts continuously on $\overline{\text{Teich}(S)}$ at uniquely ergodic points on the boundary.

The following Kerckhoff's formula concerning the calculation of Teichmüller distances will be frequently used.

Lemma 2.2.6 (Kerckhoff [32]).

$$\forall x, y \in \operatorname{Teich}(S), d_T(x, y) = \frac{1}{2} \sup_{[\xi] \in \mathcal{PMF}(S)} \ln\left(\frac{\operatorname{Ext}_x(\xi)}{\operatorname{Ext}_y(\xi)}\right)$$

Hyperbolicity. It was first proved in [44] that the Teichmüller space (Teich(S), d_T) is not hyperbolic in the sense of Gromov. However some triangles in (Teich(S), d_T) are indeed thin. We now collect several related results in order to compare neighborhoods in $\mathcal{PMF}(S)$ defined by projections of balls in Teich(S) and the ones defined by intersection numbers.

The first result describes triangles with vertices in a thick part of $\operatorname{Teich}(S)$. Recall that, for $\epsilon > 0$, the ϵ -thick part $\operatorname{Teich}_{\epsilon}(S)$ of the $\operatorname{Teichmüller}$ space $\operatorname{Teich}(S)$ is defined to be

$$\operatorname{Teich}_{\epsilon}(S) = \{ y \in \operatorname{Teich}(S) : \forall c \in \mathcal{C}(S), \operatorname{Ext}_{y}(c) \ge \epsilon \}.$$

Theorem 2.2.7 (Duchin [17]). Let $\epsilon > 0$ and $\operatorname{Teich}_{\epsilon}(S)$ be the ϵ -part of $\operatorname{Teich}(S)$. Then there exists $M_0 = M_0(\epsilon) > 0$ and $k = k(\epsilon) > 0$ such that, for every triangle $\Delta(x, y, z)$ with vertices x, y, z in $\operatorname{Teich}_{\epsilon}(S)$ and side lengths $a = d_T(x, y), b =$ $d_T(y, z), c = d_T(x, z)$ all at least M_0 and for every $\rho > 0$, one has $d \leq ka\rho$, whenever $a + b - c \leq a\rho$, where $w \in T(S)$ be the unique point in the geodesic [x, z] such that $d_T(x, w) = d_T(x, y)$ and $d = d_T(y, w)$.

The next result, generalizing a theorem of Rafi [53], also describes when triangles are thin. We denote $\mathcal{N}_D(A)$ for a subset A of Teich(S) by the D-neighborhood of A. Recall that a geodesic segment $I : [a, b] \to \text{Teich}(S)$ has at least proportion θ in $\text{Teich}_{\epsilon}(S)$ if

$$Thk_{\epsilon}^{\%}[I] \doteq \frac{|\{a \le s \le b : I(s) \in \operatorname{Teich}_{\epsilon}(S)\}|}{b-a} \ge \theta$$

Theorem 2.2.8 (Dowdall-Duchin-Masur[16]). Given $\epsilon > 0$ and $0 < \theta \leq 1$, there exist constants $D = D(\epsilon, \theta), L_0 = L_0(\epsilon, \theta)$ such that if $I \subset [x, y]$ is a geodesic subinterval in Teich(S) of length at least L_0 and at least proportion θ of I is in Teich_{ϵ}(S), then for every $z \in \text{Teich}(S)$, we have

$$I \cap \mathcal{N}_D([x, z] \cup [y, z]) \neq \emptyset.$$

The following result will also be used later. Recall that two parametrized geodesics segment $\delta(t)$ and $\delta'(t)$ defined on [a, b] are said to P-fellow travel in a parametrized fashion if, for every $t \in [a, b]$, $d_T(\delta(t), \delta'(t)) \leq P$.

Theorem 2.2.9 (Rafi [53]). Let $\epsilon > 0$. Then there exists $P = P(\epsilon) > 0$ such that whenever x_1, x_2, y_1, y_2 are in $\operatorname{Teich}_{\epsilon}(S)$ with

$$d_T(x_1, x_2) \le 1, d_T(y_1, y_2) \le 1,$$

the geodesic segment $[x_1, y_1]$ and $[x_2, y_2]$ are *P*-fellow travelling.

Boundary representations of mapping class groups. We are in a position to discuss a special type of quasi-regular unitary representations of mapping class groups. Fix $o \in \text{Teich}(S)$, we first define a Radon measure ν_o on $\mathcal{PMF}(S)$. Let ν_{Th} be the Thurston measure on $\mathcal{MF}(S)$. For any open subset $U \subset \mathcal{PMF}(S)$, one defines $\nu_o(U)$ to be

$$\nu_o(U) = \nu_{Th} \left(\{ \xi : [\xi] \in U, \operatorname{Ext}_o(\xi) \le 1 \} \right).$$

One could verify that $\forall \gamma \in \text{Mod}(S), \gamma_*\nu_o = \nu_{\gamma,o} \text{ and } [\nu_x] = [\nu_y], \forall x, y \in \text{Teich}(S).$ Therefore, one has

$$\forall x, y \in \operatorname{Teich}(S), [\xi] \in \mathcal{PMF}(S), \frac{d\nu_x}{d\nu_y}([\xi]) = \left(\frac{\operatorname{Ext}_y(\xi)}{\operatorname{Ext}_x(\xi)}\right)^{\frac{6g-6}{2}}$$

By the definition of extremal length, the function $[\xi] \mapsto \left(\frac{\operatorname{Ext}_y(\xi)}{\operatorname{Ext}_x(\xi)}\right)^{\frac{6g-6}{2}}$ is well-defined on $\mathcal{PMF}(S)$. We have, in particular,

$$\forall \gamma \in \operatorname{Mod}(S), [\xi] \in \mathcal{PMF}(S), \frac{d\gamma_*\nu_o}{d\nu_o}([\xi]) = \left(\frac{\operatorname{Ext}_o(\xi)}{\operatorname{Ext}_{\gamma.o}(\xi)}\right)^{\frac{6g-6}{2}}.$$

Hence one has a quasi-regular unitary representation π_{ν_o} of Mod(S) on the Hilbert space $L^2(\mathcal{PMF}(S), \nu_o)$. The quasi-regular representation π_{ν_o} of Mod(S) is called the *boundary representation* of Mod(S) (with respect to o).

As intersection numbers will be the main tool, we embed $\mathcal{PMF}(S)$ into $\mathcal{MF}(S)$. For each [ξ], define $\tau(\xi) \in \mathcal{MF}(S)$ to be the unique element in [ξ] such that $\operatorname{Ext}_o(\tau(\xi)) =$ 1. Hence, the map $\tau : \mathcal{PMF}(S) \longrightarrow \mathcal{MF}(S)$ is a section of the projection $\pi :$ $\mathcal{MF}(S) \longrightarrow \mathcal{PMF}(S)$. When talking about intersection numbers for two points in $\mathcal{PMF}(S)$, we will always use the image of τ .

Ergodic boundary representation. From now on, let $S = S_g(g \ge 2)$ be a genus g closed, orientable surface and fix a point $o = [(X, \phi)] \in \text{Teich}(S)$. Normalize

 ν_o to be a probability measure. Denote h = 6g - 6 and let $\epsilon > 0$ and $\theta > 0$. Let also L be the length function on G induced by the Teichmüller distance d_T , namely $L(g) = d_T(o, g \cdot o)$. Inspired by [23] and [16], we first describe our choice of E_n that fits in Theorem 2.2.13. Let g_t^q be a Teichmüller geodesic ray starting from o in the direction of $q \in QD^1(X)$. For every m > 0, recall that

$$Thk_{\epsilon}^{\%}[o, g_m] \doteq \frac{|\{0 \le s \le m : g_s \in \operatorname{Teich}_{\epsilon}(S)\}|}{d_T(o, g_m)}.$$

Theorem 2.2.10 ([16] Proposition 5.5). For all $0 < \theta < 1$, there exists $\epsilon > 0$ such that for all $o = (X, \phi) \in \text{Teich}(S)$

$$\lim_{R_0 \to \infty} \nu_o \left(\{ q \in QD^1(X) : Thk_{\epsilon}^{\%}[o, g_m^q] \ge \theta, \forall m > R_0 \} \right) = 1.$$

We then fix any $\theta \in (0, 1)$ and take $\epsilon > 0$ by the above theorem. We identify $QD^1(X)$ with $\mathcal{PMF}(S)$ and g_t^q with $\mathcal{V}(q)(t)$. For each R > 0, we define

$$U(R,\theta,\epsilon) = \{\xi \in \mathcal{PMF}(S) : Thk_{\epsilon}^{\mathscr{H}}[o,\xi_m] \ge \theta, \forall m > R\}.$$

Then if $R_2 \ge R_1 > 0$, we have

$$U(R_1, \theta, \epsilon) \subset U(R_2, \theta, \epsilon).$$

Define

$$U(\theta,\epsilon) = \bigcup_{R>0} U(R,\theta,\epsilon),$$

then, by Theorem 2.2.10, one has

$$\nu_o(U(\theta, \epsilon))) = 1.$$

Furthermore, after a suitable choice of θ , one has $\nu_o(\partial(U(\theta, \epsilon))) = 0$ and by Masur's criterion (Lemma 2.2.4), the set $U(\epsilon, \theta)$ consists of uniquely ergodic directions. We now fix the choice of ϵ and θ and for $\gamma \in Mod(S)$, denote the direction determined by

the oriented geodesic $[o, \gamma \cdot o]$ by ξ_{γ} . Now we are in a position to describe E_n . Fix $\rho > 0$ and let $L_0 = L_0(\theta, \epsilon)$ be the constant as Theorem 2.2.8. For $\frac{1}{3h} \ln \ln n > \max \{L_0, \rho\}$, define the set $\mathcal{E}(\theta, \epsilon, n, o, \rho)$ to be the set of all elements γ in Mod(S) satisfying:

- (a) $d(\gamma \cdot o, o) \in (n \rho, n + \rho);$
- (b) Both ξ_{γ} and $\xi_{\gamma^{-1}}$ are in $U(\theta, \epsilon)$;
- (c) If g(t) is either the geodesic ray $\xi_{\gamma}(t)$ or $\xi_{\gamma^{-1}}(t)$, then the segment $[o, g(\frac{1}{3h} \ln \ln n)]$ has at least proportion θ in $\operatorname{Teich}_{\epsilon}(S)$.

Lemma 2.2.11. Let n large enough as before. Then for $\gamma \in \mathcal{E}(\theta, \epsilon, n, o, \rho)$, there exists a geodesic segment I_{γ} of length $\frac{1}{3h} \ln \ln n$ in the geodesic $[o, \gamma \cdot o]$ that has at least proportion θ in $\operatorname{Teich}_{\epsilon}(S)$ and containing $\gamma \cdot o$.

Proof. Let $\gamma \in \mathcal{E}(\theta, \epsilon, n, o, \rho)$. Since the geodesic ray $\xi_{\gamma^{-1}}(t)$ satisfies (c) in the definition of $\mathcal{E}(\theta, \epsilon, n, o, \rho)$, the first segment I_{γ} of $[o, \gamma^{-1} \cdot o]$ of length $\frac{1}{3h} \ln \ln n$ has at least proportion θ in Teich_{ϵ}(S). As $\gamma \cdot [o, \gamma^{-1} \cdot o] = [\gamma \cdot o, o]$, therefore, the geodesic $[o, \gamma \cdot o]$ has a subinterval $\gamma \cdot I_{\gamma^{-1}}$ of length $\frac{1}{3h} \ln \ln n$ that at least proportion θ in Teich_{ϵ}(S) and starting at point $\gamma \cdot o$.

Later, we will prove that $\mathcal{E}(\theta, \epsilon, n, o, \rho)$ has exponential growth. We first state one obvious property of the boundary representation.

Lemma 2.2.12. Let π_{ν} be the boundary representation of Mod(S). For every $g \in Mod(S)$, $\|\pi_{\nu_o}(g) \mathbb{1}_{\mathcal{PMF}(S)}\|_{L^{\infty}(\mathcal{PMF}(S),\nu_o)} < \infty$

Proof. The lemma is an easy consequence of Kerckhoff's formula, namely Lemma 2.2.6, on Teichmüller distances. By Lemma 2.2.6,

$$\forall x, y \in \operatorname{Teich}(S), \forall [\xi] \in \mathcal{PMF}(S), \left(\frac{\operatorname{Ext}_x(\xi)}{\operatorname{Ext}_y(\xi)}\right)^{\frac{1}{2}} \leq e^{d_T(x,y)}$$

As
$$\pi_{\nu_o}(g) \mathbb{1}_{\mathcal{PMF}(S)} = \left(\frac{\operatorname{Ext}_o(\xi)}{\operatorname{Ext}_{\gamma \cdot o}(\xi)}\right)^{\frac{6g-6}{4}}$$
, one has
 $\|\pi_{\nu_o}(g) \mathbb{1}_{\mathcal{PMF}(S)}\|_{L^{\infty}(\mathcal{PMF}(S),\nu_o)} \le e^{\frac{6g-6}{2}d_T(o,\gamma \cdot o)} < \infty.$

The following theorem is a slight variant of Theorem 2.2.1 whose proof is the same as its original proof.

Theorem 2.2.13. Let π_{ν_o} be the associated quasi-regular representation of Mod(S)on $L^2(\mathcal{PMF}(S), \nu_o)$. Let *i* be the intersection number function defined on $\mathcal{PMF}(S)$ induced by the section $\tau : \mathcal{PMF}(S) \longrightarrow \mathcal{MF}(S)$. Let $n \gg \rho$ and let $E_n = E_n(\rho) \subset$ $\{g \in Mod(S) : d_T(o, g \cdot o) \in [n - \rho, n + \rho]\}$ be symmetric Let $e_n = Pr : E_n \longrightarrow$ $\mathcal{PMF}(S)$ be the radial projection from *o*. Assume that the following conditions hold:

- (1) $\lim_{n\to\infty} |E_n| = \infty$,
- (2) for all Borel subsets $W, V \subset \mathcal{PMF}(S)$ such that $\nu_o(\partial W) = \nu_o(\partial V) = 0$,

$$\limsup_{n \to \infty} \frac{1}{|E_n|} | \left\{ \gamma \in E_n : e_n(\gamma^{-1}) \in W \text{ and } e_n(\gamma) \in V \right\} | \le \nu_o(W)\nu_o(V),$$

(3) for every $n \gg \rho$, there are two sequences of reals $\{h_{r_n}(n,\rho)\}$ and $\{r_n\}$ such that $\lim_{n\to\infty} h_{r_n}(n,\rho) = \lim_{n\to\infty} r_n = 0$ and such that

$$\forall n \in \mathbb{N}, \forall \gamma \in E_n, \frac{\langle \pi_{\nu_o}(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\{x \in \mathcal{PMF}(S): i(x, e_n(\gamma)) \ge r_n\}} \rangle}{\Phi(\gamma)} \le h_{r_n}(n, \rho),$$

$$\sup_{n} \left\| M_{E_{n}}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)} \right\|_{L^{\infty}(\mathcal{PMF}(S),\nu_{o})} < \infty.$$

Then the quasi-regular representation π_{ν_o} is ergodic with respect to (E_n, e_n) and any $f \in \overline{H}^{L^{\infty}(\mathcal{PMF}(S),\nu_o)}$, where

$$H = < \mathbb{1}_U : \nu_o(\partial U) = 0 \text{ and } U \text{ is a Borel subset of } \mathcal{PMF}(S) > 0$$

Remark 2.2.3. As Theorem 2.2.13 is slight different from Theorem 2.2.1, it is worth making a few comments. One could easily find the only difference is the point (3) here since the original point (1) has been replaced automatically by Lemma 2.2.12. The assumption (3) in Theorem 2.2.13 is different from the assumption (4) in Theorem 2.2.1. For the proof of Theorem 2.2.13, we modify the proof of Proposition 2.21 and thus Theorem 2.2 in [12] as follows. We use the same notations for convenience. In the proof of the first part of [Proposition 2.21,[12]], first approximate W by subsets W_n consisting of uniquely ergodic measured foliations and such that $\nu_o(W-W_n) \leq k_n$ with k_n tends to 0. Then denote $W_n(r) = \{\eta \in \mathcal{PMF}(S) : i(\eta, W_n) \leq r\}$, by Lemma 2.2.3, choose m = m(n) large so that $U \cap W_n(r_m) = \emptyset$. Now replace $\langle \pi_{\nu}(g) \mathbb{1}_B, \mathbb{1}_W \rangle$ by $\langle \pi_{\nu}(g) \mathbb{1}_B, \mathbb{1}_{W_n} \rangle$ and follow essential the same proof. The proof of [Theorem 2.2,[12]] is modified similarly by taking limit simultaneously with respect to n and r rather than taking limit first on n than on r as done in [12].

Our main result is the following theorem.

Theorem 2.2.14. There exists θ and ϵ such that, if $E_n = \mathcal{E}(\theta, \epsilon, n, o, \rho)$, which is described before Lemma 2.2.11, then the boundary representation π_{ν_o} is ergodic with respect to (E_n, Pr) and any $f \in \overline{H}^{L^{\infty}}$ as above. In other words, the pair $(E_n = \mathcal{E}(\theta, \epsilon, n, o, \rho), Pr)$ satisfies all conditions listed in Theorem 2.2.13.

As ν_o is a Radon measure, one has immediately the following two corollaries by Proposition 2.2.2 and Remark 2.2.1.

Corollary 2.2.15. The boundary representation π_{ν_o} of Mod(S) is irreducible.

Corollary 2.2.16. The mapping class group Mod(S) acts ergodically on $\mathcal{PMF}(S)$ with respect to the measure class $[\nu_o]$.

We then mention a property of the boundary representation π_{ν_o} . Recall that a unitary

representation of a group G is called *tempered* if it is weakly contained in the regular representation $L^2(G)$.

Proposition 2.2.17. The boundary representation π_{ν_o} of Mod(S) is tempered.

Proof. We argue as Proposition 6.3 in [22]. By the main Theorem in [35], we need to verify that the action of Mod(S) on $\mathcal{PMF}(S)$ is amenable. This is Proposition 8.1 in [27] as a corollary of topological amenability of the action of Mod(S) on $\mathcal{PMF}(S)$. \Box

Corollary 2.2.18. The trivial representation 1 does not weakly contained in the boundary representation π_{ν_o} . In other words, the boundary representation π_{ν_o} does not have almost invariant vectors.

Proof. As Mod(S) is not amenable for hyperbolic surface S, the trivial representation is not contained in the regular representation. According to the above proposition and the fact that weak containment is transitive, we can conclude the proof.

Notations. We make some conventions for later use.

- S: a genus $g \ge 2$, closed, oriented, connected surface;
- h = 6g 6;
- o: the base point in $\operatorname{Teich}(S)$ which is chosen to be generic in the sense that $Stab_o(\operatorname{Mod}(S)) = id$. Denote $\nu = \nu_o$ and the measure is normalized so that $\nu(\mathcal{PMF}(S)) = 1;$
- The projective measured foliation space *PMF(S)* is regarded as a subset of *MF(S)* by *τ* and an element [ξ] in *PMF(S)* is then written as ξ, so both [ξ] and ξ will be called directions when there are no confusions;
- Fix arbitrary $\rho > 0$ and assume $n \gg \rho$;

- Pr_y: Teich(S) {y} → PMF(S): the radial projection from Teich(S) to PMF(S) that assigns every point z ∈ Teich(S) {y} to the vertical measured foliation of the unit quadratic differential defined by the oriented geodesic [y, z]. For y = o, we simply denote Pr_o to be Pr;
- B(y, R): the closed ball in Teich(S) of radius R at y with respect to the Teichmüller distance $d = d_T$. L: the length function on G induced by the Teichmüller distance d_T through the orbit map: $g \mapsto g \cdot o$;
- \approx : if A(t), B(t) are two functions, we use the notation $A \approx B$ to mean $\frac{A(t)}{B(t)} \to 1$ as $t \to \infty$ and $A \approx B$ to mean $\lim_{t\to\infty} \frac{A(t)}{B(t)} \leq 1$. The notation $A \approx B$ is defined similarly;
- $A \sim_{\theta} B$: there is multiplicative constants $C_1 > 0, C_2 > 0$ depending on θ so that

$$C_1 A \le B \le C_2 A$$

 $A \prec_{\theta} B$: there is a multiplicative constant $D = D(\theta) > 0$ so that

$$A \leq DB.$$

And $A \succ_{\theta} B$ is defined similarly;

- Denote $U = U(\theta, \epsilon)$ and $E_n = \mathcal{E}(\theta, \epsilon, n, o, \rho)$ in the sequel which is described before Lemma 2.2.11;
- $\xi_{\gamma} \in \mathcal{PMF}(S)$ (for $\gamma \in Mod(S) \{id\}$): the direction of the oriented geodesic segment $[o, \gamma \cdot o]$.

2.3 Exponential growth and shadow lemma

2.3.1 Exponential growth.

In this subsection, we will show that $|E_n|$ goes to infinity. In fact, we will show that $|E_n|$ grows exponentially. For any Borel subset W of $\mathcal{PMF}(S)$, denote by $Sect_W$ the union of geodesics starting from o and ending at W. We first recall the following theorem in [4] in our setting. Let

$$C(n,\rho) = \{\gamma \in \operatorname{Mod}(S) : d_T(\gamma \cdot o, o) \in (n-\rho, n+\rho)\}.$$

Theorem 2.3.1 ([4] Theorem 2.10). Let W and V be two Borel subsets of $\mathcal{PMF}(S)$ with measure zero boundaries. Then as R tends to ∞ ,

$$\left| \left\{ \gamma \in C(n,\rho) : \gamma \cdot o \in Sect_W \text{ and } \gamma^{-1} \cdot o \in Sect_V \right\} \right| \\ \asymp Ke^{hn} \nu(W) \nu(V).$$

where K is a constant depending on g, ρ and o. In fact, using the notations in [4], one has $K = \frac{2sinh(h\rho) ||\nu(\mathcal{PMF}(S))||^2}{hm(\mathcal{M}_g)}$, where $m(\mathcal{M}_g)$ is the push forward of the Masur-Veech volume.

Corollary 2.3.2. Let $n \gg 0$ and K be the constant in Theorem 2.3.1. Then $|E_n| \asymp Ke^{hn}$. In particular, $\lim_{n\to\infty} |E_n| = \infty$.

Proof. As $E_n \subset C(n, \rho)$ and, by Theorem 2.3.1, $|C(n, \rho)| \simeq Ke^{hn}$, it is obvious that $|E_n| \approx Ke^{hn}$. We now show that $|E_n| \approx Ke^{hn}$. Recall that $U(\theta, \epsilon) = \bigcup_{R>0} U(R, \theta, \epsilon)$ with $\nu(U(\theta, \epsilon)) = 1$ and $U(S, \theta, \epsilon) \subset U(T, \theta, \epsilon)$ for T > S. Let $\delta_1 > 0$ small enough and choose $R \gg 0$ such that

$$1 - \delta_1 \le \nu(U(R, \theta, \epsilon)) \le 1, \ \nu(\partial U(R, \theta, \epsilon)) = 0.$$

By Theorem 2.3.1 again, for any $\delta_2 > 0$ small enough, one could choose n large enough so that $\frac{1}{3h} \ln \ln n > R$ and so that

$$\begin{split} \left| \left\{ \gamma \in C(n,\rho) : \gamma \cdot o \in Sect_{U(R,\theta,\epsilon)} \text{ and } \gamma^{-1} \cdot o \in Sect_{U(R,\theta,\epsilon)} \right\} \right| \\ \geq K e^{\delta_2} e^{hn} \left(\nu(U(R,\theta,\epsilon)) \right)^2 \\ \geq K e^{\delta_2} (1-\delta_1)^2 e^{hn}. \end{split}$$

On the other hand, by the choice of n and the definition of $U(R, \theta, \epsilon)$,

$$\{\gamma \in C(n,\rho) : \gamma \cdot o \in Sect_{U(R,\theta,\epsilon)} \text{ and } \gamma^{-1} \cdot o \in Sect_{U(R,\theta,\epsilon)}\}$$

 $\subset E_n.$

Therefore, we have $|E_n| \ge Ke^{\delta_2}(1-\delta_1)^2 e^{hn}$. As δ_1 and δ_2 can be arbitrary small, one has $|E_n| \ge Ke^{hn}$.

Corollary 2.3.3. For all Borel subsets $W, V \subset \mathcal{PMF}(S)$ such that $\nu(\partial W) = \nu(\partial V) = 0$,

$$\limsup_{n \to \infty} \frac{1}{|E_n|} \left| \left\{ \gamma \in E_n : Pr(\gamma^{-1}) \in W \text{ and } Pr(\gamma) \in V \right\} \right| \le \nu(W)\nu(V).$$

Proof. By Corollary 2.3.2 and Theorem 2.3.1, $|E_n| \simeq |C(n, \rho)|$. Notice that $Pr(\gamma) \in V$ if and only if $\gamma \cdot o \in Sect_V$. Hence,

$$\begin{split} &\limsup_{n \to \infty} \frac{1}{|E_n|} \left| \left\{ \gamma \in E_n : Pr(\gamma^{-1}) \in W \text{ and } Pr(\gamma) \in V \right\} \right| \\ &\leq \limsup_{n \to \infty} \frac{1}{|E_n|} \left| \left\{ \gamma \in C(n,\rho) : Pr(\gamma^{-1}) \in W \text{ and } Pr(\gamma) \in V \right\} \right| \\ &= \limsup_{n \to \infty} \frac{|C(n,\rho)|}{|E_n|} \frac{1}{|C(n,\rho)|} \left| \left\{ \gamma \in C(n,\rho) : Pr(\gamma^{-1}) \in W \text{ and } Pr(\gamma) \in V \right\} \right| \\ &\leq \nu(W)\nu(V). \end{split}$$

2.3.2 Shadow lemma.

Definition 2.3.1. (\mathcal{O} -points) A point y in Teich(S) is called an \mathcal{O} -point (with respect to o) if for every R > 0, there is a real number $C \ge 1$ depending on R, such that

$$\frac{1}{C}\exp\left(-hd(o,y)\right) \le \nu(\Pr(B(y,R))) \le C\exp\left(-hd(o,y)\right)$$

Remark 2.3.1. We call such points \mathcal{O} -points because they are the points that satisfy the classic shadow lemma [58].

Definition 2.3.2. An element $g \in Mod(S)$ is called an \mathcal{O} -mapping class (with respect to $o \in Teich(S)$) if $g \cdot o$ is an \mathcal{O} -point.

Recall that U has a full measure. We need a lemma that relates Busemann functions to extremal lengths. Recall that if (X, d_X) is a metric space and ξ is a geodesic ray starting from a point $x_0 \in X$, then the Busemann function associated to the geodesic ray ξ is the function b_{ξ} on X defined by

$$b_{\xi}: x \mapsto \lim_{t \to \infty} \left(d_X(x, \xi(t)) - t \right).$$

For $(X = \text{Teich}(S), d_X = d)$ and ξ be a geodesic ray starting from o, one has,

Lemma 2.3.4 (Walsh[62]). If $[\xi]$ is uniquely ergodic, then the Busemann function associated to the geodesic ray in the direction $[\xi]$ is

$$\forall x \in \operatorname{Teich}(S), b_{[\xi]}(x) = \frac{1}{2} \ln \left(\frac{\operatorname{Ext}_x(\xi)}{\operatorname{Ext}_o(\xi)} \right).$$

The following lemma is contained in [[23], Lemma 5.1] and will be used in the proof of uniform boundedness (Section 2.5.1), we include the proof for the sake of completeness.

Lemma 2.3.5. Let $n \gg \rho$. Then elements in $\bigcup_n E_n$ are \mathcal{O} -mapping classes, where C depends on R, θ, ϵ .

Proof. Let $g \in E_n$ and R > 0. As

$$\nu(Pr(B(g \cdot o, R))) = \int_{\{\eta \in Pr(B(g \cdot o, R))\}} \left(\frac{\operatorname{Ext}_{g \cdot o}(\eta)}{\operatorname{Ext}_o(\eta)}\right)^{\frac{\mu}{2}} d\nu_{g \cdot o}(\eta)$$

By Lemma 2.3.4 and the fact that uniquely ergodic measured foliations has a full measure [39], so we have, for almost every $\eta \in Pr(B(g \cdot o, R))$,

$$d(o, g \cdot o) - 2R \le -\frac{1}{2} \ln \left(\frac{\operatorname{Ext}_{g \cdot o}(\eta)}{\operatorname{Ext}_o(\eta)} \right) \le d(o, g \cdot o).$$

Notice that as such η is dense, the above inequality actually holds for all points in $Pr(B(g \cdot o, R))$. By Lemma 2.2.5, Mod(S) acts on the Teichmüller compactification continuously at ν -almost every point, thus on one hand

$$\nu(Pr(B(g \cdot o, R)))$$

$$\geq \nu_{g \cdot o}(Pr(B(g \cdot o, R)))e^{-hd(o, g \cdot o)}$$

$$= \nu(Pr_{g^{-1} \cdot o}(B(o, R)))e^{-hd(o, g \cdot o)}.$$

One the other hand, we have

$$\nu(Pr(B(g \cdot o, R))) \leq e^{2hR - hd_T(o,g \cdot o)} \nu_{g \cdot o}(Pr(B(g \cdot o, R)))$$
$$\leq e^{2hR - hd_T(o,g \cdot o)} \nu_{g \cdot o}(\mathcal{PMF}(S))$$
$$= e^{2hR - hd_T(o,g \cdot o)} \nu(\mathcal{PMF}(S))$$
$$= C(R)e^{-hd(o,g \cdot o)}.$$

We now claim that there exists a constant D > 0 independent of $g \in \bigcup_n E_n$ such that

$$\nu(Pr_{q^{-1} \cdot o}B(o, R)) \ge D$$

If not, then there is a sequence $g_k^{-1} \cdot o$ converges to a point $[\zeta]$ such that $\nu(Pr_{g_k^{-1} \cdot o}(B(o, R)))$ tends to 0. As $\{g_k\} \subset \bigcup_n E_n$, so by Masur's criterion, namely Lemma 2.2.4, $[\zeta]$ is uniquely ergodic, thus $\nu(Pr_{[\zeta]}(B(o, R)))$ is zero, which is impossible since $Pr_{[\zeta]}(B(o, R))$ contains a small open subset in $\mathcal{PMF}(S)$ and ν is not atomic. \Box

2.4 Harish-Chandra estimates.

This section is devoted to prove the following Harish-Chandra estimates.

Theorem 2.4.1. Given $n \gg \rho$. There exist $a_1 > 0, a_2 > 0, b_1, b_2, c_1 > 0$ depending on $\epsilon, o, g, \theta, \rho$ such that

$$\forall \gamma \in E_n, \ (a_1n - c_1 \ln \ln n + b_1)e^{-\frac{h}{2}n} \le \Phi(\gamma) \le (a_2n + b_2)e^{-\frac{h}{2}n}.$$

Recall that

$$\Phi(\gamma) = <\pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\mathcal{PMF}(S)} >_{L^{2}(\mathcal{PMF}(S),\nu)}$$
$$= \int_{\mathcal{PMF}(S)} \left(\frac{\operatorname{Ext}_{o}(\xi)}{\operatorname{Ext}_{\gamma.o}(\xi)}\right)^{\frac{h}{4}} d\nu([\xi]).$$

Remark 2.4.1. 1. In [13], the left side is of the form $(an + b)e^{-\frac{\alpha}{2}n}$. However, some other terms like $\ln \ln n$ should be added for mapping class groups if we require $\lim \frac{|E_n|}{C(n,\rho)} = 1.$

2. The following oberservation will be useful, namely $\Phi(\gamma) \asymp n e^{-\frac{hn}{2}}$.

The proof is divided into several steps and will be given at the end of this section.

2.4.1 Reduction to intersection numbers.

By our convention, for every $\xi \in \mathcal{PMF}(S)$, one has $\operatorname{Ext}_o(\xi) = 1$, we then have

$$\Phi(\gamma) = \int_{\mathcal{PMF}(S)} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\zeta)}\right)^{\frac{n}{4}} d\nu(\zeta).$$

Let ξ_{γ} be the direction of $[o, \gamma \cdot o]$. In order to estimate $\Phi(\gamma)$, we will relate it to the following integrations on intersection numbers:

$$\Psi(\gamma) = \int_{\mathcal{PMF}(S)} \left(\frac{1}{i(\xi_{\gamma},\eta)}\right)^{\frac{n}{2}} d\nu(\eta).$$

Denote

$$\Psi(\gamma)_{\geq A} = \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_{\gamma}, \eta) \geq A\}} \left(\frac{1}{i(\xi_{\gamma}, \eta)}\right)^{\frac{h}{2}} d\nu(\eta).$$

Lemma 2.4.2 (Minsky's inequality [46]). Let ξ and η be two measured foliations on S and $x \in \text{Teich}(S)$, then

$$i^2(\xi,\eta) \le \operatorname{Ext}_x(\xi) \operatorname{Ext}_x(\eta),$$

where the equilty holds if and only if there is a quartic differential q so that the vertical measured foliation of q on X is ξ and the horizontal measured foliation is η .

Corollary 2.4.3. There exist constants $C_3 = C_3(g, \rho) > 0$ and $C_4 = C_4(g, \rho) > 0$ such that, for every $M \in (0, 1)$ and every $\gamma \in Mod(S)$,

$$\Phi(\gamma) \le C_3 e^{-\frac{h}{2}n} \Psi(\gamma)_{\ge M} + C_4 e^{\frac{h}{2}n} \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_\gamma) \le M\}).$$

Proof. Decompose $\mathcal{PMF}(S)$ into two subsets $A = \{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_{\gamma}) \leq M\}$ and $B = \{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_{\gamma}) \geq M\}$. Then we have

$$\Phi'(\gamma) = \int_{A} \left(\frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\eta)}\right)^{\frac{h}{4}} d\nu(\eta) + \int_{B} \left(\frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\eta)}\right)^{\frac{h}{4}} d\nu(\eta)$$

$$= \mathrm{I} + \mathrm{II}.$$
(2.4.1)

By Kerckhoff's formula, I $\prec_{g,\rho} e^{\frac{h}{2}n} \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_{\gamma}) \leq M\})$. Thanks to Lemma 2.3.4, we can replace $\operatorname{Ext}_{\gamma \cdot o}(\xi_{\gamma})$ in Lemma 2.4.2 by e^{-2n} , so one has

$$\frac{1}{i^2(\xi_{\gamma},\eta)e^{2n}} \succ_{g,\rho} \frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\eta)},$$

which gives the bound for the term II.

In order to bound $\Phi(\gamma)$ from below, we will use the fact that $\gamma \in E_n$.

Lemma 2.4.4. There exists a constant F depending on $g, o, \epsilon, \theta, \rho$ such that if $i(\xi_{\gamma}, \eta) \geq F \ln n e^{-2n}$, where $\eta \in U(\epsilon, \theta)$ and $\gamma \in E_n$, then $i^2(\xi_{\gamma}, \eta) \succ_{g, o, \epsilon, \theta, \rho} \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2n}$.

Proof. First we remark that, since both η and ξ_{γ} are uniquely ergodic, by [[34], Proposition 5.1], there is a geodesic whose horizontal and vertical measured foliations are in the projective classes ξ_{γ} and η respectively. Hence we have a geodesic triangle $\Delta(o, \xi_{\gamma}, \eta)$. As $\gamma \in E_n$, Lemma 2.2.11 implies that there is a geodesic segment Iof length $\ell = \frac{1}{3h} \ln \ln n$ in $[o, \gamma \cdot o]$ ending at $\gamma \cdot o$ that has at least proportion θ in Teich_{ϵ}(S). By Theorem 2.2.8,

$$I \cap \mathcal{N}_D([o,\xi_\gamma] \cap [o,\eta]) \neq \emptyset,$$

where D comes from Theorem 2.2.8. Choose $q \in I \cap \mathcal{N}_D([o, \xi_{\gamma}] \cap [o, \eta])$. Then there are two possibilities:

Case 1: $d(q, y) \leq D$ with $y \in [\xi_{\gamma}, \eta]$.

Then we have, by Kerckhoff's formula and Lemma 2.4.2,

$$i^{2}(\xi_{\gamma}, \eta) = \operatorname{Ext}_{y}(\eta) \operatorname{Ext}_{y}(\xi_{\gamma})$$
$$\succ_{g,o,\theta,\epsilon} \operatorname{Ext}_{q}(\eta) \operatorname{Ext}_{q}(\xi_{\gamma})$$
$$= \operatorname{Ext}_{q}(\eta) e^{-2d(o,q)}$$
$$\geq \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2n},$$

which means that, in this case, we always have $i^2(\xi_{\gamma}, \eta) \succ_{g,o,\epsilon,\theta} \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2n}$.

Case 2: $d(q, y) \leq D$ with $y \in [o, \eta]$.

Then we have

$$i^{2}(\xi_{\gamma},\eta) \leq \operatorname{Ext}_{y}(\eta) \operatorname{Ext}_{y}(\xi_{\gamma})$$
$$\sim_{g,o,\theta,\epsilon} \operatorname{Ext}_{y}(\eta) \operatorname{Ext}_{q}(\xi_{\gamma})$$
$$\sim_{g,o,\theta,\epsilon,\rho} e^{-4d(o,q)} = e^{-4(d(o,\gamma \cdot o) - d(q,\gamma \cdot o))}$$
$$\leq e^{-4n} e^{4\ell}.$$

Therefore, in this case we have a constant F_1 depending on $g, o, \epsilon, \theta, \rho$ such that

$$i(\xi_{\gamma},\eta) \le F_1 e^{-2n} e^{2\ell} \le F_1 e^{-2n} e^{\ln \ln n} = F_1 \ln n e^{-2n}.$$

Thus if we take $F \gg F_1$ and require $i(\xi_{\gamma}, \eta) \ge F \ln n e^{-2n}$, it forces us in the Case 1 which implies the conclusion that

$$i^{2}(\xi_{\gamma},\eta) \succ_{g,o,\epsilon,\theta} \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2n}.$$

Corollary 2.4.5. For every $\gamma \in E_n$, take $\overline{M} = F \ln n e^{-2n}$ where F is the constant in Lemma 2.4.4. Then $\Phi(\gamma) \succ_{g,o,\epsilon,\theta,\rho} e^{-\frac{h}{2}n} \Psi(\gamma)_{\geq \overline{M}}$.

Proof. Note that $U(\epsilon, \theta)$ has a full measure. Hence, by Lemma 2.4.4

$$\Phi(\gamma) = \int_{U(\epsilon,\theta)} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\eta)}\right)^{\frac{h}{4}} d\nu(\eta)$$

$$\geq \int_{\{\eta \in U(\epsilon,\theta): i(\eta,\xi_{\gamma}) \ge \overline{M}\}} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\eta)}\right)^{\frac{h}{4}} d\nu(\eta)$$

$$\succ_{g,o,\epsilon,\theta} e^{-\frac{hn}{2}} \Psi(\gamma)_{\ge \overline{M}}.$$

Lemma 2.4.6. Assume that there exist $N_0 > 0, a > 0$ and b > 0 such that

$$\forall N \le N_0, \quad aN^{\frac{h}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_{\gamma}) \le N\}) \le bN^{\frac{h}{2}},$$

then there exist A, B, D_1, D_2 such that

$$-A\ln N + D_1 \le \Psi(\gamma)_{\ge N} \le -B\ln N + D_2$$

Proof. The proof is same as the one in [13]. We repeat here for completeness. Namely,

$$\begin{split} \Psi(\gamma)_{\geq N} &= \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_{\gamma}, \eta) \geq N\}} \left(\frac{1}{i(\xi_{\gamma}, \eta)}\right)^{\frac{h}{2}} d\nu(\eta) \\ &= \int_{\mathbb{R}} \nu\left(\left\{\eta \in \mathcal{PMF}(S): \left(\frac{1}{i(\xi_{\gamma}, \eta)}\right)^{\frac{h}{2}} \geq t\right\}\right) dt \\ &= \int_{1}^{\frac{1}{N^{\frac{h}{2}}}} \nu\left(\left\{\eta \in \mathcal{PMF}(S): i(\xi_{\gamma}, \eta) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) dt \\ &= \int_{1}^{N_{0}} \nu\left(\left\{\eta \in \mathcal{PMF}(S): i(\xi_{\gamma}, \eta) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) dt \\ &+ \int_{N_{0}}^{\frac{1}{N^{\frac{h}{2}}}} \nu\left(\left\{\eta \in \mathcal{PMF}(S): i(\xi_{\gamma}, \eta) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) dt. \end{split}$$
(2.4.2)

By the assumption and ν is a probability measure, one can easily have the conclusion.

2.4.2 A basic example.

Before continuing our discussions, we digress for the case of once-punctured torus $S_{1,1}$. Some standard facts are taken from [[49], 7.2 Examples].

Let $S = S_{1,1}$. Then $Mod(S) = SL(2,\mathbb{Z})$ and $Teich(S) = \mathbb{H}^2$, the upper half plane. Take *o* to be $i \in \mathbb{H}^2$. The space $\mathcal{MF}(S)$ of measured foliations can be identified with the real plane module the inversion, namely $\{\mathbb{R}^2 - (0,0)\}/\{I,-I\}$. By the ergodicity of the Thurston measure ν_{Th} , up to a constant multiple, the measure ν_{Th} , which is defined by the weak limit of counting measures on $\mathcal{MF}(S)$, can be identified with the Lebesgue measure on \mathbb{R}^2 . Rays in $\{\mathbb{R}^2 - (0,0)\}/\{I,-I\}\}$ are then identified with points in $\mathcal{PMF}(S)$. It implies that $\mathcal{PMF}(S)$ can be identified with \mathbb{RP}^1 . Notice that all identifications here are Mod(S)-equivariant. Hence $\mathcal{PMF}(S)$ can be represented as $\{[x:y]: x^2 + y^2 \neq 0, x, y \in \mathbb{R}\}$, or $\mathbb{R} \cup \{\infty\}$. Teich(S) is then the usual compactification of \mathbb{H}^2 . In this case, Mod(S) acts on $\overline{Teich(S)}$ via linear fractional transformations. For $(x, y) \in \mathbb{R}^2$, the extremal length at o is

$$\operatorname{Ext}_o((x,y)) = x^2 + y^2,$$

hence the image of $\mathcal{PMF}(S)$ under τ is the circle. We will ignore the difference between \mathbb{R}^2 and $\mathbb{R}^2/\{I, -I\}$. For two points $(x, y), (p, q) \in \mathcal{MF}(S)$, the intersection number is |qx - py|. Write the image of $\mathcal{PMF}(S)$ in the form of $(\sin(\theta), \cos(\theta))$, and fix any $\xi = (\sin(\theta_0), \cos(\theta_0)) \in \mathcal{PMF}(S)$. Let M to be small enough, then

$$\{\eta \in \mathcal{PMF}(S) : i(\xi, \eta) \leq M\}$$

$$= \{\theta \in [0, 2\pi] : |\sin(\theta) \cos(\theta_0) - \cos(\theta) \sin(\theta_0)| \leq M\}$$

$$= \{\theta \in [0, 2\pi] : |\sin(\theta - \theta_0)| \leq M\}$$

$$= \{\theta \in [0, 2\pi] : -M \leq \sin(\theta - \theta_0) \leq M\}.$$
(2.4.3)

As M is enough small, $\sin(\theta)$ is almost the same as θ , so there exist constants A and B, so that

$$AM \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\xi, \eta) \le M\}) \le BM,$$

Notice that, when S is $S_{1,1}$, we have $h = 6g - 6 + n = 6 \times 1 - 6 + 2 \times 1 = 2$, hence $\frac{h}{2} = 1$.

2.4.3 Approximation by pants curves.

Now we want to prove that the assumption in Lemma 2.4.6 holds, however, instead of proving it directly, we will approximate by pants curves using the map considered in [43].

Recall that, thanks to Bers' theorem, there is a constant $C_1 = C(g)$, depending only on the genus g, such that for every point $x \in \text{Teich}(S)$, there exists a pants decomposition, namely a collection of 3g - 3 essential simple closed curves $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3}\}$, such that

$$\forall 1 \leq i \leq 3g - 3, \ \operatorname{Ext}_x(\alpha_i) \leq C^2.$$

If x is further assumed to be in $\operatorname{Teich}_{\epsilon}(S)$, we can choose a collection of 3g-3 essential simple closed curves $\{\alpha_1(x), \cdots, \alpha_{3g-3}(x)\}$ on S such that

$$\forall 1 \le i \le 3g - 3, \epsilon \le \operatorname{Ext}_x(\alpha_i(x)) \le C_1^2.$$

Denote $\alpha(x) \in MF(S)$ to be the measured foliation $\alpha(x) = \sum_{i=1}^{3g-3} \alpha_i(x)$ and $[\alpha(x)]$ be its projective class in $\mathcal{PMF}(S)$. By the Jenkins-Strebel theorem (see also [32] Theorem 2.1), there is a unit holomorphic quadratic differential $q = q(x, \alpha(x))$ on $o \in \operatorname{Teich}(S)$ whose projective class of the vertical measured foliation is $[\alpha(x)]$.

Given $\gamma \in E_n$, by our construction of E_n , the ending point ξ_{γ} of the geodesic ray g_t^{γ} determined by $[o, \gamma \cdot o]$ is in $U(\epsilon, \theta)$. Hence, there is a sequence of points $y(k, \gamma) \in \operatorname{Teich}_{\epsilon}(S)$ in g_t^{γ} tends to ξ_{γ} in $\overline{\operatorname{Teich}(S)}$. By the above discussion, there is a sequence of pants curves $\alpha(y(k, \gamma))$ and a sequence of points $[\alpha(y(k, \gamma))]$ in $\mathcal{PMF}(S)$. By Minsky's inequality, $[\alpha(y(k, \gamma))]$ converges to $[\xi_{\gamma}]$ in $\mathcal{PMF}(S)$, which means using the map τ , $\frac{\alpha(y(k, \gamma))}{\sqrt{\operatorname{Ext}_o(\alpha(y(k, \gamma)))}}$ converges to ξ_{γ} . We first estimate $\operatorname{Ext}_o(\alpha(y(k, \gamma)))$.

Now let $y \in \text{Teich}_{\epsilon}(S)$ and denote $\alpha(y)$ and $q(y) = q(y, \alpha(y))$ as above. Let g_t be the Teichmüller geodesic ray starting from o in the direction of q(y). Let t_y be the unique point in g_t such that t_y has maximal distance with o and

$$\forall 1 \leq i \leq 3g-3, \ \epsilon \leq \operatorname{Ext}_{t_u}(\alpha_i(y)) \leq C_1^2.$$

Lemma 2.4.7. There is a constant C_2 depending on g and ϵ such that

$$\forall y \in \operatorname{Teich}_{\epsilon}(S), d(y, t_y) \leq C_2$$

The proof is based on the following theorem. We use the theorem in the form of Theorem 5.3 in [4]. For the definition of twist numbers $tw(\alpha, \beta)$, the reader is referred to [52].

Theorem 2.4.8 (Minsky[47]). Let $x \in \text{Teich}(S)$ and $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition produced by the Bers' theorem mentioned above. Then for any simple closed curve β ,

$$\operatorname{Ext}_{x}(\beta) \sim_{g} \max_{1 \leq i \leq 3g-3} \left(\frac{i^{2}(\beta, \alpha_{i})}{\operatorname{Ext}_{x}(\alpha_{i})} + tw^{2}(\beta, \alpha_{i}) \operatorname{Ext}_{x}(\alpha_{i}) \right).$$
(2.4.4)

Proof of Lemma 2.4.7. By Kerckhoff's formula, we only need to bound the ratio $\frac{\operatorname{Ext}_y(\beta)}{\operatorname{Ext}_{t_y}(\beta)}$ for any essential simple closed curve β on S. However, by the construction of y(x), the two hyperbolic surfaces t_y and y have the same pants decomposition which satisfies the condition in Theorem 2.4.8, namely $\alpha(\gamma) = \{\alpha_1(y), \dots, \alpha_{3g-3}(y)\}$. As, for $1 \leq i \leq 3g - 3$, both extremal lengths $\operatorname{Ext}_y(\alpha_i)$ and $\operatorname{Ext}_{t_y}(\alpha_i)$ are bounded below by the constant ϵ and above by a constant C_1^2 depending only on g, we can conclude the proof of the lemma by using Equation (2.4.4).

Corollary 2.4.9. We have $|d_T(o, y) - d_T(o, t_y)| \leq C_2$, where C_2 is the constant in Lemma 2.4.7. Hence, $\operatorname{Ext}_o(\alpha_i(y(k, \gamma))) \sim_{g,\epsilon,o} e^{2d(y(k,\gamma),o)}$.

Proof. We only need to show the second statement. Let $y = y(k, \gamma)$ and $t_y = t_{y(k,\gamma)}$. Let $T = d(o, t_y)$ and $f : o \to t_y$ be the Teichmüller mapping with dilatation e^{2T} between o and t_y . We know that $T \sim_{g,\epsilon} d(o, y(k, \gamma))$. We want to show that $\operatorname{Ext}_o(\alpha_i(y)) \sim_{g,\epsilon,o} e^{2T}$, for each $1 \leq i \leq 3g - 3$. Notice that $\operatorname{Ext}_{t_y}(\alpha_i(y)) \sim_{g,\epsilon} 1$. On the one hand, by Kerckhoff's formula, one has

$$\operatorname{Ext}_o(\alpha_i(y)) \prec_{g,\epsilon} e^{2T}$$

In order to bound $\operatorname{Ext}(\alpha_i(y))$ from below, we construct a metric and use the analytic definition of extremal length in [32]. Fix $1 \leq i \leq 3g - 3$. Let q be the unit quadratic

differential on o in the direction of $[o, t_y]$, that is, under the above notations, $q = q(y, \alpha(y))$. Let m_i be the modulus of the cylinder C_i determined by q, where C_i has the same core curve with $\alpha_i(y)$. According to the proof of Proposition 2 in [32], there is a metric σ in the conformal class of t_y such that the core curve of $\alpha_i(y)$ has σ -length 1 and area $e^T m_i + A$, where A is a constant depending on o. One also has

$$\frac{1}{e^T m_i} \ge \operatorname{Ext}_{t_y}(\alpha_i(y)) \ge \frac{1}{e^T m_i + A}$$

Now consider the metric $\Sigma = f^* \sigma$ on o defined by the pullback of σ via the Teichmüller mapping f. As f preserves the area but shrink the vertical length, the core curve of $\alpha_i(y)$ has Σ -length e^T . Thus

$$\operatorname{Ext}_o(\alpha_i(y)) \ge \frac{e^{2T}}{e^T m_i + A}.$$

As

$$e^T m_i \prec_{\epsilon,g} 1$$

one then further has

$$\operatorname{Ext}_o(\alpha_i(y)) \succ_{g,\epsilon,o} e^{2T}$$

We now summarize all discussions above. There are A > 0, B > 0, depending on ϵ, g, o , such that, for every $\gamma \in E_n$, there is a sequence $\{\xi_k(\gamma) \in \mathcal{PMF}(S)\}$ satisfying

- (a) $\forall k, \xi_k(\gamma) = [x_k(\gamma) = \sum_{i=1}^{3g-3} \alpha_i(\gamma)]$ where $\{\alpha_i(\gamma)\}_{i=1}^{3g-3}$ is a pants decomposition of S;
- (b) For each *i*, there is t_i such that $Ae^{2t_i} \leq \operatorname{Ext}_o(\alpha_i(\gamma)) \leq Be^{2t_i}$ and $\lim_i t_i = \infty$;
- (c) The limit of $\{\xi_k(\gamma) = \frac{x_k(\gamma)}{\sqrt{\operatorname{Ext}_o(x_k(\gamma))}}\}$ in $\mathcal{PMF}(S)$ is ξ_{γ} .

Lemma 2.4.10. If there exists $N_0 > 0$ small enough such that, for every $\gamma \in E_n$ and $\xi_k(\gamma)$, one has

$$\forall N \le N_0, \ aN^{\frac{h}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_k(\gamma)) \le N\}) \le bN^{\frac{h}{2}},$$

then

$$\forall N \le N_0, \ aN^{\frac{h}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_{\gamma}) \le N\}) \le bN^{\frac{h}{2}}.$$

Proof. In general, convergence in $\mathcal{PMF}(S)$, which is defined by weak convergence of measures, is hard to understand. However, in our case, the convergence $\xi_k(\gamma) \to \xi_{\gamma}$ can be understood as follows (see [26] for backgrounds). Recall that the projective measured foliation space $\mathcal{PMF}(S)$ can be identified with the projective measured lamination space $\mathcal{PML}(S)$ on S. Regarding both $\xi_k(\gamma)$ and ξ_{γ} as measured laminations, then according to [26], $\xi_k(\gamma)$ converges to ξ_{γ} in the sense of coarse Hausdorff topology. That is, as subsets, the limit ζ of $\xi_k(\gamma)$ contains ξ_{γ} as a sublamination and the complement $\zeta - \xi_{\gamma}$ consists of isolated leaves. For each $\epsilon > 0$, take k large enough, then we could find a subsurface F of S, so that, on F, any $\eta \in \mathcal{PML}(S)$ with $i(\eta, \xi_{\gamma})$ is uniformly approximated by $\xi_k(\gamma)$ and outside F, $i(\eta, \xi_k \gamma) \leq \epsilon$. Therefore such η has intersection number less than $N + \epsilon$ with $\xi_k(\gamma)$, we then have proved the upper bound for the measure. The left inequality can be proved similarly.

2.4.4 Regularity at pants curves.

We are now in a position to prove that the assumption in Lemma 2.4.10 holds. We first summarize all properties of $\xi_k(\gamma)$ that we really need.

More conventions: From now on, we will use the hyperbolic length function $\ell_o(\cdot)$. Since $\ell_o^2(\cdot) \sim_o \operatorname{Ext}_o(\cdot)$, we can use $\ell_o(\cdot)$ to replace $\operatorname{Ext}_o(\cdot)$ without affecting the

result when we defining the measure ν_o , the embedding $\tau : \mathcal{PMF}(S) \longrightarrow \mathcal{MF}(S)$ and $\xi_k(\gamma)$. For instance, for a measurable subset $U \in \mathcal{PMF}(S)$, we have

$$\nu_0(U) = \mu(\{\eta : [\eta] \in U, \ell_o(\eta) \le 1\})$$

Set-up 0: Let $\alpha = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition of S and consider it to be a measured foliation still denoted by α . Then $[\alpha]$ defines a unit holomorphic quadratic differential q on o, namely the unique q such that $[\mathcal{V}(q)] = [\alpha]$. Let $\xi = \frac{\alpha}{\ell_o(\alpha)}$, then ξ is the image of $[\alpha]$ under τ . We denote g_t the Teichmüller geodesic defined by q. We assume that for all $i \in \{1, \dots, 3g-3\}, \ell_o(\alpha_i)$ is bounded bleow and above, up to multiplicative constants depending only on g, o, ϵ , by e^T for some T.

Theorem 2.4.11. Under the above **Set-up** 0, there exist $M_0 > 0, C > 0$ and D > 0, depending on g, o, ϵ such that when $M < M_0$, we have

$$CM^{\frac{n}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \le M\}) \le DM^{\frac{n}{2}}.$$

The main tool to prove the above theorem is the following Dehn-Thurston theorem. Let $P = \{\alpha_k\}$ be a pants decomposition. For each α_k , let $m_k : \mathcal{MF}(S) \longrightarrow \mathbb{R}_{\geq 0}, \xi \mapsto i(\alpha_k, \xi)$ be the intersection function defined by α_k and $t_k = tw_k$ be the twist function associated to α_k .

Theorem 2.4.12 (The Dehn-Thurston theorem [52], Theorem 3.1.1). Let $S = S_g$ and $\alpha = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition of S. Then the map

$$\varpi : \mathcal{MF}(S) \longrightarrow \mathbb{R}^{6g-6}$$

$$\mathcal{F} \mapsto (m_1(\mathcal{F}), \cdots m_{3g-3}(\mathcal{F}), t_1(\mathcal{F}), \cdots, t_{3g-3}(\mathcal{F})).$$
(2.4.5)

gives a global coordinate for $\mathcal{MF}(S)$.

Corollary 2.4.13. The symplectic form $\omega = dm_1 \wedge dt_1 + \cdots + dm_{3g-3} \wedge dt_{3g-3}$ gives a Mod(S)-invariant measure $\mu = \frac{1}{n!}\omega^n$ on $\mathcal{MF}(S)$, hence up to a multiplicative constant, the measure μ' coincides with the Thurston measure ν_{Th} on $\mathcal{MF}(S)$.

Proof. First we note that, different pants decompositions give rise to different train tracks charts which locally differ by an element in $SL(k, \mathbb{Z})$, for some k, keeping the volume form invariant, hence μ' is Mod(S)-invariant and independent of the choice of the pants decomposition P. As ν_{Th} is ergodic and both μ and ν_{Th} are in the Lebesgue measure class, the conclusion follows.

We now use μ to replace ν_{Th} with α is fixed to the one given in **Set-up 0**. Note that $\frac{h}{2} = 3g - 3$. We now prove the theorem.

Proof of Theorem 2.4.11. By Lemma 2.4.2, for every two elements ξ and η in $\mathcal{PMF}(S)$, the intersection number $i(\eta, \xi) \leq 1$ and 1 is achievable. So we take $M_0 = \frac{1}{4}$ and let $M \leq M_0$. The proof is then divided into two parts. In the sequel, we will denote $a = \frac{1}{\sum_i \ell_o(\alpha_i)}$ and ℓ to be

$$\ell = \frac{1}{a \prod \left(\ell_o(\alpha_i)\right)^{\frac{2}{h}}} = \frac{\sum_i \ell_o(\alpha_i)}{\prod \left(\ell_o(\alpha_i)\right)^{\frac{2}{h}}}$$

Then by our assumption, there exists $A_1 > 0$ and $B_1 > 0$ depending on g, o, ϵ , such that $B_1 \leq \ell \leq A_1$.

Upper bound: $\nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \le M\}) \le DM^{\frac{h}{2}}$:

By the definition of ν and $i(\eta, \xi) = a \sum_{k=1}^{3g-3} i(\eta, \alpha_k)$, we have,

$$\nu\left(\left\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \leq M\right\}\right)$$

$$= \mu\left(\left\{t\eta \in \mathcal{MF}(S) : i(\eta, \xi) \leq M, \ell_o(\eta) = 1, 0 \leq t \leq 1\right\}\right) \text{ (by definition)}$$

$$= \mu\left(\left\{t\eta \in \mathcal{MF}(S) : a\sum_k m_k(\eta) \leq M, \ell_o(\eta) = 1, 0 \leq t \leq 1\right\}\right)$$

$$\leq \mu\left(\left\{\eta : \forall k, m_k(\eta) \leq \frac{M}{a}, t_i(\eta)\ell_o(\alpha_i) \leq A_2\right\}\right),$$
(2.4.6)

where A_2 is a constant depending only on o. In fact, A_2 depends on the diameter of X. The last step comes from the fact that large twists will make the length to be large. Thus we further have

$$\nu \left(\left\{ \eta \in \mathcal{PMF}(S) : i(\eta, \xi) \leq M \right\} \right) \\
\leq \mu \left(\left\{ (m_1, \cdots, m_{3g-3}, t_1, \cdots, t_{3g-3}) : \forall k, m_k \leq \frac{M}{a}, t_k \leq \frac{A_2}{\ell_o(\alpha_k)} \right\} \right) (\text{by } (2.4.6)) \\
\leq A_3 M^{3g-3} \frac{1}{\prod_{k=1}^{3g-3} \ell_0(\alpha_k)} (\text{by definition of } \mu) \\
= A_3 M^{3g-3} \ell^{3g-3} \\
\leq A_3 M^{3g-3} A_1^{3g-3} (\text{since } B_1 \leq \ell \leq A_1) \\
\leq DM^{\frac{h}{2}}.$$
(2.4.7)

Lower bound: $CM^{\frac{h}{2}} \leq \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \leq M\}):$

In order to bound the measure from below, we will construct a subset contained in the set. We first fix, for each i, a positive orientation for α_i . Let

$$V = \{ t\eta \in \mathcal{MF}(S) : i(\eta, \xi) \le M, \ell_o(\eta) = 1, 0 \le t \le 1 \}.$$
 (2.4.8)

Then $\eta^0 = \frac{1}{3g-3}\xi$ is in V. Let a and M as above, and $\delta > 0$ be a positive number.

Define a set of 6g - 6-tuples by

$$W_0(a, M) = \{(x_1, \cdots, x_{3g-3}, y_1, \cdots, y_{3g-3}) : \forall i, 0 \le ax_i \le \frac{1}{3g-3} M, 0 \le (y_i - x_i)\ell_o(\alpha_i) \le \delta\}.$$

Let also ϖ be the coordinate map in Theorem 2.4.12. Then, on the one hand, by hyperbolic geometry, there is $\delta_0 > 0$ and $M_0 > 0$, depending on o, such that for all $\delta \leq \delta_0$ and $M \leq M_0$, one has

$$\varpi^{-1}(\varpi(\eta^0) + W_0(a, M)) \subset V.$$

Notice that one could choose a large enough such that ϖ^{-1} is a homeomorphism on $\varpi(\eta^0) + W_0(a, M)$. Therefore $\mu(V) \ge \nu(\varpi^{-1}(\varpi(\eta^0) + W_0(a, M)))$. On the other hand, $\nu(\varpi^{-1}(\varpi(\eta^0) + W_0(a, M)))$ $= \nu(\varpi^{-1}(W_0(a, M))).$

The last measure is easy to see to be at least $CM^{\frac{h}{2}}$. Hence the proof is finished. \Box

Proof of Theorem 2.4.1. Let $\gamma \in E_n$ and take $M = e^{-2n}$ in Corollary 2.4.3 and $\overline{M} = F \ln n e^{-2n}$ in Corollary 2.4.5 where F is the constant in Lemma 2.4.4. Then Theorem 2.4.11 implies the assumption in Lemma 2.4.10, hence Lemma 2.4.6 is true and hence $\Psi(\gamma)_{\geq M} \sim_{g,o,\epsilon} n$ and $\Psi(\gamma)_{\geq \overline{M}} \sim_{g,o,\epsilon} a_1 n - c_1 \ln n$. Then by Corollary 2.4.3, Corollary 2.4.5 and Theorem 2.4.11 again, the proof is finished.

2.5 Ergodicity of boundary representation

In this section, we will prove the main theorem: Theorem 2.2.14, namely,

Theorem 2.5.1. Let $S = S_g(g \ge 2)$ and π_{ν} be the associated quasi-regular representation of the mapping class group Mod(S) on $L^2(\mathcal{PMF}(S), \nu)$. Let \tilde{d} be a metric on $\mathcal{PMF}(S)$ which is compatible with the topology of $\mathcal{PMF}(S)$. Let $n \gg \rho$ and $E_n = \mathcal{E}(\theta, \epsilon, n, o, \rho)$. Let $e_n = Pr : E_n \longrightarrow \mathcal{PMF}(S)$ be the radial projection which assigns $g \in E_n$ to the direction ξ_g of the oriented geodesic $[o, g \cdot o]$. Then the quasi-regular representation π_{ν} is ergodic with respect to (E_n, e_n) and any $f \in \overline{H}^{L^{\infty}(\mathcal{PMF}(S),\nu)}$, where

$$H = < \mathbb{1}_U : \nu(\partial U) = 0 \text{ and } U \text{ is a Borel subset of } \mathcal{PMF}(S) > .$$

Proof. The proof consists of verifying all assumptions in Theorem 2.2.13 for E_n . The first two will be verified by showing E_n is of exponential growth (namely, Corollary 2.3.2, Corollary 2.3.3). The third one is by Proposition 2.5.2. The last one is Theorem 2.5.6.

Proposition 2.5.2. For every $n \gg \rho$, there are two sequences of real numbers $\{h_{r_n}(n,\rho)\}$ and $\{r_n\}$ such that $\lim_{n\to\infty} h_{r_n}(n,\rho) = \lim_{n\to\infty} r_n = 0$ and such that

$$\forall n \in \mathbb{N}, \forall \gamma \in E_n, \frac{\langle \pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\{x \in \mathcal{PMF}(S): i(x,e_n(\gamma)) \ge r_n\}} \rangle}{\Phi(\gamma)} \le h_{r_n}(n,\rho).$$

Proof. Let $n \gg \rho$ and $\gamma \in E_n$. Let x_{γ} as before. Consider the intersection function on $\mathcal{PMF}(S)$ defined by ξ_{γ} , namely, $i(\xi_{\gamma}, \cdot)$. Let $r_n = \frac{1}{n}$. By Corollary 2.4.3, Lemma 2.4.6 and the proof of its assumption (Theorem 2.4.11),

$$\frac{\langle \pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\{x \in \mathcal{PMF}(S): i(x,\xi_{\gamma}) \geq \frac{1}{n}\}}\rangle}{\Phi(\gamma)} \leq c(g,o,\rho) \frac{\ln n - D}{a_1 n - c_1 \ln \ln n + b_1}.$$

Take $h(n,\rho) = c(g,o,\rho) \frac{\ln n - D}{a_1 n - c_1 \ln \ln n + b_1}$, we complete the proof.

2.5.1 Uniform boundedness.

In this section, we complete our proof of the main theorem by proving the uniform boundedness. We start by some lemmas comparing of two types of neighborhoods.

Lemma 2.5.3. Using notations as Corollary 2.4.9. Let $\xi_{\gamma} \in \mathcal{PMF}(S)$ be the direction of $[o, y = \gamma \cdot o]$ and $\xi^{\gamma} \in \mathcal{PMF}(S)$ be the direction of $[o, x_{\gamma} = t_y]$. Then there exists a constant C such that $i(\xi_{\gamma}, \xi^{\gamma}) \leq Ce^{-2L(\gamma)}$.

Proof. By Lemma 2.4.2,

$$i^2(\xi_{\gamma},\xi^{\gamma}) \leq \operatorname{Ext}_{x_{\gamma}}(\xi_{\gamma})\operatorname{Ext}_{x_{\gamma}}(\xi^{\gamma}).$$

Let $\alpha = \alpha(\gamma)$ as before. Then $\xi^{\gamma} = \frac{\alpha}{\sqrt{\operatorname{Ext}_o(\alpha)}} \sim_{g,o} e^{-L(\gamma)} \alpha$. As $\operatorname{Ext}_{x_{\gamma}}(\alpha) \sim_g 1$, we have $\operatorname{Ext}_{x_{\gamma}}(\xi^{\gamma}) = \frac{1}{\operatorname{Ext}_o(\alpha)} \operatorname{Ext}_{x_{\gamma}}(\alpha) \prec_{g,o} e^{-2L(\gamma)}$. On the other hand, by Lemma 2.4.7, up to a multiplicative constant, one could replace $\operatorname{Ext}_{x_{\gamma}}(\xi_{\gamma})$ by $\operatorname{Ext}_{\gamma \cdot o}(\xi_{\gamma}) = e^{-2L(\gamma)}$. Collect all discussions together, one can finish the proof.

Let $\xi \in \mathcal{PMF}(S)$ and $x \in [o,\xi]$, denote $\mathcal{I}_C(\xi,x) = \{\eta \in \mathcal{PMF}(S) : i(\eta,\xi) \leq Ce^{-2d(x,o)}\}$. Let M_0, L, θ, ϵ as in Theorem 2.2.7 and Theorem 2.2.8.

Lemma 2.5.4. Let $\eta \in \mathcal{PMF}(S)$ and $\gamma \in E_n$. Suppose that η does not leave Teich_{ϵ}(S) eventually. Let $x \in [o, \eta]$ such that d(x, o) = n. Let C > 0 and n large enough. Then if $\gamma \in E_n$ such that $i(\xi_{\gamma}, \eta) \leq Ce^{-2n}$, then $d(x, \gamma \cdot o) \leq \frac{1}{h} \ln \ln n$.

Proof. We argue as Lemma 2.4.4. Denote $\xi = \xi_{\gamma}$, hence by assumption $i(\xi, \eta) \leq Ce^{-2n}$. First we remark that, since both η and ξ are uniquely ergodic, we have a geodesic triangle $\Delta(o, \xi, \eta)$. As $\gamma \in E_n$, there is also a geodesic segment I of length $\ell = \frac{1}{3h} \ln \ln n$ in $[o, \gamma \cdot o]$ ending at $p = \gamma \cdot o$ that has at least proportion θ in $\text{Teich}_{\epsilon}(S)$. By Theorem 2.2.8,

$$I \cap \mathcal{N}_D([o,\xi] \cap [o,\eta]) \neq \emptyset,$$

where D as in Theorem 2.2.8. Choose $q \in I \cap \mathcal{N}_D([o, \xi] \cap [o, \eta])$. Then there are two possibilities:

Case 1: $d(q, y) \leq D$ with $y \in [o, \eta]$.

Then

$$d(q, o) - D \le d(o, y) \le d(q, o) + D.$$

Since

$$n - \ell - \rho \le d(q, o) \le n + \rho,$$

we have

$$0 \le d(x, y) \le \ell + D + \rho.$$

Hence,

$$\begin{aligned} d(x, \gamma \cdot o) &\leq d(x, y) + d(y, q) + d(q, p) \\ &\leq \ell + D + D + \ell + \rho \\ &\leq 2(\ell + D + \rho) \\ &\leq 3\ell. \end{aligned}$$

Case 2: $d(q, y) \leq D$ with $y \in [\xi, \eta]$.

Then by Lemma 2.4.2, one has

$$i^2(\eta, \xi) = \operatorname{Ext}_y(\xi) \operatorname{Ext}_y(\eta).$$

Now, since $d(q, y) \leq D$, by Kerckhoff's formula, we have

$$e^{-2D}\operatorname{Ext}_q(\xi) \le \operatorname{Ext}_y(\xi), e^{-2D}\operatorname{Ext}_q(\eta) \le \operatorname{Ext}_y(\eta).$$

Therefore,

$$e^{-4D} \operatorname{Ext}_q(\xi) \operatorname{Ext}_q(\eta) \le i^2(\xi, \eta).$$

On the other hand, we have

$$\operatorname{Ext}_{q}(\xi) = e^{-2d(o,q)}, i(\xi,\eta) \le Ce^{-2n},$$

which implies that

$$e^{-4D} \operatorname{Ext}_q(\eta) e^{-2d(o,q)} \le C^2 e^{-4n}.$$

That is,

$$e^{-4D} \operatorname{Ext}_{q}(\eta) e^{2(n-d(o,q))} \leq C^{2} e^{-2n}$$

By Kerckhoff's formula again,

$$\operatorname{Ext}_p(\eta) \le C^2 e^{2\rho + 4D} e^{-2n},$$

or

$$\frac{1}{2}\ln \operatorname{Ext}_p(\eta) \le \ln(Ce^{\rho+2D}) - n.$$

Apply Lemma 2.3.4, one could choose $z \in [o, \eta] \cap \operatorname{Teich}_{\epsilon}(S)$ so that, if denote d(o, p) = t, d(p, z) = a and d(z, o) = b, then $a - b \leq -n + \ln(Ce^{2\rho + 4D}) + 1$. Therefore, we have

$$0 \le t + a - b \le \ln(Ce^{2\rho + 4D}) + 1 + \rho.$$

By Theorem 2.2.7, we have $d(p, y) \le k(\ln(Ce^{2\rho+4D}) + 1 + \rho) \le 3\ell$. We complete the proof.

Corollary 2.5.5. Let $\eta \in \mathcal{PMF}(S)$ and suppose that η does not leave $\operatorname{Teich}_{\epsilon}$ eventually. Let $x \in [o, \eta]$ such that d(x, o) = n. Let further C > 0 and n large enough. Then

$$\left| \{ \gamma \in E_n : \gamma \cdot o \in Sec_{\mathcal{I}_C(\eta, x)} \} \right|$$
$$\prec_{g, o, \rho} \ln n.$$

Proof. By Theorem 1.2 in [4] (note that Λ in the theorem is a constant function), when n is large enough, there exists a constant $N_0 > 0$, such that $|B(x, R) \cap \operatorname{Mod}(S) \cdot o| \leq N_0 e^{hR}$. Apply Lemma 2.5.4, we have the conclusion.

Theorem 2.5.6. Under the notations used in Theorem 2.5.1, we have

$$\sup_{n} \left\| M_{E_{n}}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)} \right\|_{L^{\infty}(\mathcal{PMF}(S),\nu)} < \infty.$$

Recall that

$$M_{E_n}^{\mathbb{1PMF}(S)} \mathbb{1}_{\mathcal{PMF}(S)}([\xi]) = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \frac{\pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}([\xi])}{\Phi(\gamma)} = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \left(\frac{\operatorname{Ext}_o(\xi)}{\operatorname{Ext}_{\gamma \cdot o}(\xi)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}.$$

By using the embedding map τ of $\mathcal{PMF}(S)$ into $\mathcal{MF}(S)$. One can rewrite the above formula to be

$$M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)}([\xi]) = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\xi)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}$$

We first introduce a type of open sets \mathcal{IN} in $\mathcal{PMF}(S)$ defined by intersection numbers. For every $\eta \in \mathcal{PMF}(S), C > 0, t > 0$, we define

$$\mathcal{IN}(\eta, t, C) = \{\xi \in \mathcal{PMF}(S) : i(\xi, \eta) \le Ce^{-2t}\}.$$

Proof of Theorem 2.5.6. Let $U(\epsilon, \theta)$ the subset of $\mathcal{PMF}(S)$ of full measure. We shall give a bound independent on $n \gg \rho$ for $M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}}\mathbb{1}_{\mathcal{PMF}(S)}(\zeta)$ for every point $\zeta \in U(\epsilon, \theta)$. Fix R > 0. As this stage, R is arbitrary, but it will be carefully chosen at the end of the proof. As usual, for $\gamma \in E_n$, denote ξ_{γ} to be the direction corresponding to $[o, \gamma \cdot o]$, hence a point in $\mathcal{PMF}(S)$. For each point $\gamma \cdot o$, consider the open ball $B(\gamma, R)$ of radius R at $\gamma \cdot o$. Denote the projection of $B(\gamma, R)$ to $\mathcal{PMF}(S)$ by $\mathcal{O}(\gamma \cdot o, R)$. Then by Lemma 2.3.5, the measure $\nu(\mathcal{O}(\gamma \cdot o, R)) \sim_{g,R,\rho} e^{-hn}$. Fix any C > 0, for instance C = 1. Dividing E_n to be two sets E_n^1 and $E_n^2 = E_n - E_n^1$ where E_n^1 consists of $\gamma \in E_n$ so that $\xi_{\gamma} \notin \mathcal{IN}(\zeta, n, C)$. We then have, for each $\zeta \in U(\epsilon, \theta)$,

$$M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)}(\zeta) = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} = \frac{1}{|E_n|} \sum_{\gamma \in E_n^1} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} + \frac{1}{|E_n|} \sum_{\gamma \in E_n^2} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} = \mathrm{I} + \mathrm{II}.$$

$$(2.5.1)$$

First we want to bound term I in Equation (2.5.1). The set E_n^1 can be further decomposed as two sets: F_n^1 and $F_n^2 = E_n^1 - F_n^1$, where $F_n^1 = \{\gamma \in E_n^1 : Pr(B(\gamma, R)) \cap \mathcal{IN}(\zeta, n, C) = \emptyset\}$. One then has,

$$I = \frac{1}{|E_n|} \sum_{\gamma \in F_n^1} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\zeta)} \right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} + \frac{1}{|E_n|} \sum_{\gamma \in F_n^2} \left(\frac{1}{\operatorname{Ext}_{\gamma.o}(\zeta)} \right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}$$

$$= \operatorname{III} + \operatorname{IV}.$$
(2.5.2)

We first deal with term III. First notice that

$$\forall y \in B(\gamma, R), \frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\zeta)} \sim_R \frac{1}{\operatorname{Ext}_y(\zeta)},$$

on the other hand, by Lemma 2.4.2, for ν -almost every $\xi_y \in \mathcal{O}(\gamma \cdot o, R)$,

$$\left(\frac{1}{\operatorname{Ext}_{y}(\zeta)}\right)^{\frac{h}{4}} \prec_{\rho,R} \frac{1}{e^{\frac{hn}{2}}(i(\xi_{y},\zeta))^{\frac{h}{2}}}.$$

Hence, for ν -almost every $\xi_y \in \mathcal{O}(\gamma \cdot o, R)$,

$$\left(\frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}}$$
$$\prec_{R,\rho} e^{-\frac{hn}{2}} \frac{1}{(i(\xi_y,\zeta))^{\frac{h}{2}}}$$

Therefore,

$$\begin{aligned} \text{III} &= \frac{1}{|E_n|} \sum_{\gamma \in F_n^1} \left(\frac{1}{\text{Ext}_{\gamma.o}(\zeta)} \right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\ &\prec_R \frac{1}{|E_n|} \sum_{\gamma \in F_n^1} \frac{e^{-\frac{hn}{2}}}{\nu(\mathcal{O}(\gamma \cdot o, R))} \int_{\mathcal{O}(\gamma \cdot o, R)} \frac{1}{(i(\eta, \zeta))^{\frac{h}{2}}} d\nu(\eta) \frac{1}{\Phi(\gamma)}. \end{aligned}$$

Note that there are bounded number intersections of open sets on the form $\mathcal{O}(\gamma \cdot o, R)$ and the bound depends on R and ρ . Thus, since $|E_n| \simeq e^{hn}$ (Corollary 2.3.2) and $\Phi(\gamma) \succ_{g,o,\rho} (a_1n - c_1 \ln \ln n + b_1)e^{-\frac{hn}{2}}$ (Harish-Chandra estimates), substitute all these together, one has,

$$\begin{aligned} \text{III} \prec_{g,o,\rho,R} \frac{1}{a_1 n - c_1 \ln \ln n + b_1} \int_{\{\eta \in \mathcal{PMF}(S): i(\eta,\zeta) > Ce^{-2n}\}} \left(\frac{1}{i(\eta,\zeta)}\right)^{\frac{h}{2}} d\nu(\eta) \\ \prec_{g,o,\rho,R} 1. \end{aligned}$$
(2.5.3)

The last inequality follows from the fact that $\zeta \in U(\epsilon, \theta)$ and the proof of Harish-Chandra estimats.

We now deal with terms IV and II. Take $H_n = E_n^2 \cup F_n^2$. These two terms can be put together to obtain:

$$IV + II$$

$$= \frac{1}{|E_n|} \sum_{\gamma \in H_n} \left(\frac{1}{\text{Ext}_{\gamma.o}(\zeta)} \right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}$$

$$\leq \frac{1}{|E_n|} \sum_{\gamma \in H_n} \frac{e^{\frac{hL(\gamma)}{2}}}{\Phi(\gamma)}$$

$$\sim_{g,\rho,o} e^{-hn} \sum_{\gamma \in H_n} \frac{e^{\frac{hn}{2}}}{(a_1n - c_1 \ln \ln n + b_1)e^{\frac{-hn}{2}}}$$

$$= \frac{1}{a_1n - c_1 \ln \ln n + b_1} |H_n|.$$
(2.5.4)

We now *CLAIM* that $|H_n| \prec_{g,o,\rho,\epsilon,\theta} \ln n$, thus the sum IV + II tends to 0 when $n \to \infty$ which finishes the proof of the theorem.

It remains to prove the above *CLAIM*.

Proof of the CLAIM. By Corollary 2.5.5, the number $|E_n^2| \prec \ln n$. We now show that so is $|F_n^2|$. Choose $R \leq \min\{1, S_0\}$ where S_0 is the injective radius of o in the ϵ - thick part of the moduli space $\mathcal{M}(S) = \operatorname{Teich}(S) / \operatorname{Mod}(S)$. Hence for every $\gamma \in \operatorname{Mod}(S)$ and every point $q \in B(\gamma \cdot o, R), q \in \operatorname{Teich}_{\epsilon}(S)$ and $d_T(\gamma \cdot o, q) \leq 1$. Fix such R a priori. Assume now that $\gamma \in F_n^2$, namely $Pr(B(\gamma \cdot o, R)) \cap \mathcal{IN}(\zeta, n, C) \neq \emptyset$. As $U(\epsilon, \theta)$ has full measure, in particular, it is dense in $\mathcal{PMF}(S)$, thus one can choose $q \in B(\gamma \cdot o, R)$ so that the direction ξ_q of [o, q] is in $U(\epsilon, \theta) \cap \mathcal{IN}(\zeta, n, C)$. By Theorem 2.2.9, there is a $P = P(\epsilon)$, so that the two geodesics $[o, \gamma \cdot o]$ and [o, q] are P-fellow travelling in a parametrized fashsion. Now consider the P-neighborhood \mathcal{N}_P of Teich_e(S), namely the union of points in Teich(S) that has distance at most P with a point in $\operatorname{Teich}_{\epsilon}(S)$. As $\operatorname{Mod}(S)$ acts as isometries on $\operatorname{Teich}(S)$ and $\operatorname{Teich}_{\epsilon}(S)$ is Mod(S)-invariant and cocompact, the neighborhood \mathcal{N}_P is Mod(S)-invariant and cocompact. By Mumford's compactness, there is a small ϵ' so that $\mathcal{N}_P \subset \operatorname{Teich}_{\epsilon'}(S)$. Then as $\gamma \in E_n$, the geodesic segment [o, q] has the property that it contains a segment I = [a,q] of length $\frac{1}{3h} \ln \ln n$ such that I has at least θ in Teich_{ϵ'}(S). Note that ϵ is fixed, hence C depends on g and o. Hence by Theorem 2.2.8, there are two constants $D' = D'(\epsilon', \theta)$ and $L'_0 = L'_0(\epsilon', \theta)$ satisfy Theorem 2.2.8. Take *n* large enough and follow the proof Lemma 2.5.4 and Corollary 2.5.5, one has that $|F_n^2| \prec \ln n$.

Bibliography

- Ian Agol, The virtual Haken conjecture, Doc. Math. 18 (2013), 1045–1087, With an appendix by Agol, Daniel Groves, and Jason Manning. MR 3104553
- Jørgen Ellegaard Andersen and Rasmus Villemoes, Cohomology of mapping class groups and the abelian moduli space, Quantum Topol. 3 (2012), no. 3-4, 359–376. MR 2928089
- [3] Jorgen Ellegaard Andersen, Mapping class groups do not have kazhdan's property (t), arXiv preprint arXiv:0706.2184 (2007).
- [4] Jayadev Athreya, Alexander Bufetov, Alex Eskin, and Maryam Mirzakhani, Lattice point asymptotics and volume growth on Teichmüller space, Duke Math. J. 161 (2012), no. 6, 1055–1111. MR 2913101
- [5] Uri Bader and Roman Muchnik, Boundary unitary representations irreducibility and rigidity, J. Mod. Dyn. 5 (2011), no. 1, 49–69. MR 2787597
- [6] Bachir Bekka, Pierre de la Harpe, and Alain Valette, Kazhdan's property (T), New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. MR 2415834
- Bachir Bekka and Mehrdad Kalantar, Quasi-regular representations of discrete groups and associated C*-algebras, Trans. Amer. Math. Soc. 373 (2020), no. 3, 2105–2133. MR 4068291

- [8] Joan S. Birman and Caroline Series, Geodesics with bounded intersection number on surfaces are sparsely distributed, Topology 24 (1985), no. 2, 217–225. MR 793185
- [9] Michael Björklund, Yair Hartman, and Hanna Oppelmayer, Random walks on dense subgroups of locally compact groups, 2020.
- [10] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, Topology **31** (1992), no. 4, 685–699. MR 1191373
- [11] Marc Bourdon, Cohomologie et actions isométriques propres sur les espaces L_p, Geometry, topology, and dynamics in negative curvature, London Math. Soc. Lecture Note Ser., vol. 425, Cambridge Univ. Press, Cambridge, 2016, pp. 84– 109. MR 3497258
- [12] A. Boyer, G. Link, and Ch. Pittet, Ergodic boundary representations, Ergodic Theory Dynam. Systems 39 (2019), no. 8, 2017–2047. MR 3977334
- [13] Adrien Boyer, Equidistribution, ergodicity, and irreducibility in CAT(-1) spaces, Groups Geom. Dyn. 11 (2017), no. 3, 777–818. MR 3692899
- [14] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa, C*-simplicity and the unique trace property for discrete groups, Publ. Math. Inst. Hautes Études Sci. 126 (2017), 35–71. MR 3735864
- [15] Martin R. Bridson and Pierre de la Harpe, Mapping class groups and outer automorphism groups of free groups are C*-simple, J. Funct. Anal. 212 (2004), no. 1, 195–205. MR 2065242
- [16] Spencer Dowdall, Moon Duchin, and Howard Masur, Statistical hyperbolicity in Teichmüller space, Geom. Funct. Anal. 24 (2014), no. 3, 748–795. MR 3213829
- [17] Moon Duchin, Curvature, stretchiness, and dynamics, In the tradition of Ahlfors-Bers. IV, Contemp. Math., vol. 432, Amer. Math. Soc., Providence, RI, 2007, pp. 19–30. MR 2342803

- [18] Pierre Eymard, Moyennes invariantes et représentations unitaires, Lecture Notes in Mathematics, Vol. 300, Springer-Verlag, Berlin-New York, 1972. MR 0447969
- [19] Benson Farb and Dan Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR 2850125
- [20] Albert Fathi, François Laudenbach, and Valentin Poénaru, Thurston's work on surfaces, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012, Translated from the 1979 French original by Djun M. Kim and Dan Margalit. MR 3053012
- [21] Michael Freedman and Vyacheslav Krushkal, On the asymptotics of quantum SU(2) representations of mapping class groups, Forum Math. 18 (2006), no. 2, 293–304. MR 2218422
- [22] Lukasz Garncarek, Boundary representations of hyperbolic groups, 2016.
- [23] Ilya Gekhtman, Stable type of the mapping class group, 2013.
- [24] William M. Goldman, Ergodic theory on moduli spaces, Ann. of Math. (2) 146 (1997), no. 3, 475–507. MR 1491446
- [25] William M Goldman, Mapping class group dynamics on surface group representations, arXiv preprint math/0509114 (2005).
- [26] Ursula Hamenstädt, Train tracks and the Gromov boundary of the complex of curves, Spaces of Kleinian groups, London Math. Soc. Lecture Note Ser., vol. 329, Cambridge Univ. Press, Cambridge, 2006, pp. 187–207. MR 2258749
- [27] _____, Geometry of the mapping class groups. I. Boundary amenability, Invent. Math. 175 (2009), no. 3, 545–609. MR 2471596
- [28] _____, Invariant Radon measures on measured lamination space, Invent. Math. 176 (2009), no. 2, 223–273. MR 2495764

- [29] _____, Actions of the mappings class group, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 1002–1021. MR 2827829
- [30] Nikolai V. Ivanov, Subgroups of Teichmüller modular groups, Translations of Mathematical Monographs, vol. 115, American Mathematical Society, Providence, RI, 1992, Translated from the Russian by E. J. F. Primrose and revised by the author. MR 1195787
- [31] _____, Fifteen problems about the mapping class groups, Problems on mapping class groups and related topics, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., Providence, RI, 2006, pp. 71–80. MR 2264532
- [32] Steven P. Kerckhoff, The asymptotic geometry of Teichmüller space, Topology 19 (1980), no. 1, 23–41. MR 559474
- [33] _____, Earthquakes are analytic, Comment. Math. Helv. 60 (1985), no. 1, 17–30. MR 787659
- [34] Erica Klarreich, The boundary at infinity of the curve complex and the relative teichmüller space, 2018.
- [35] M. Gabriella Kuhn, Amenable actions and weak containment of certain representations of discrete groups, Proc. Amer. Math. Soc. 122 (1994), no. 3, 751–757.
 MR 1209424
- [36] Elon Lindenstrauss and Maryam Mirzakhani, Ergodic theory of the space of measured laminations, Int. Math. Res. Not. IMRN (2008), no. 4, Art. ID rnm126, 49. MR 2424174
- [37] Biao Ma, On a family of unitary representations of mapping class groups, 2021.
- [38] Florian Martin and Alain Valette, On the first L^p-cohomology of discrete groups, Groups Geom. Dyn. 1 (2007), no. 1, 81–100. MR 2294249

- [39] Howard Masur, Interval exchange transformations and measured foliations, Ann. of Math. (2) 115 (1982), no. 1, 169–200. MR 644018
- [40] _____, Two boundaries of Teichmüller space, Duke Math. J. 49 (1982), no. 1, 183–190. MR 650376
- [41] _____, Ergodic actions of the mapping class group, Proc. Amer. Math. Soc. 94 (1985), no. 3, 455–459. MR 787893
- [42] _____, Hausdorff dimension of the set of nonergodic foliations of a quadratic differential, Duke Math. J. 66 (1992), no. 3, 387–442. MR 1167101
- [43] Howard A. Masur and Yair N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, Invent. Math. 138 (1999), no. 1, 103–149. MR 1714338
- [44] Howard A. Masur and Michael Wolf, Teichmüller space is not Gromov hyperbolic, Ann. Acad. Sci. Fenn. Ser. A I Math. 20 (1995), no. 2, 259–267. MR 1346811
- [45] John McCarthy and Athanase Papadopoulos, Dynamics on Thurston's sphere of projective measured foliations, Comment. Math. Helv. 64 (1989), no. 1, 133–166.
 MR 982564
- [46] Yair N. Minsky, Teichmüller geodesics and ends of hyperbolic 3-manifolds, Topology 32 (1993), no. 3, 625–647. MR 1231968
- [47] _____, Extremal length estimates and product regions in Teichmüller space, Duke Math. J. 83 (1996), no. 2, 249–286. MR 1390649
- [48] Maryam Mirzakhani, Growth of the number of simple closed geodesics on hyperbolic surfaces, Ann. of Math. (2) 168 (2008), no. 1, 97–125. MR 2415399
- [49] Hideki Miyachi, Extremal length functions are log-plurisubharmonic, In the tradition of Ahlfors-Bers. VII, Contemp. Math., vol. 696, Amer. Math. Soc., Providence, RI, 2017, pp. 225–250. MR 3715450
- [50] Jean-Pierre Otal, Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3, Astérisque (1996), no. 235, x+159. MR 1402300

- [51] Luis Paris, Actions and irreducible representations of the mapping class group, Math. Ann. 322 (2002), no. 2, 301–315. MR 1893918
- [52] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992. MR 1144770
- [53] Kasra Rafi, Hyperbolicity in Teichmüller space, Geom. Topol. 18 (2014), no. 5, 3025–3053. MR 3285228
- [54] Mary Rees, An alternative approach to the ergodic theory of measured foliations on surfaces, Ergodic Theory Dynam. Systems 1 (1981), no. 4, 461–488 (1982). MR 662738
- [55] Justin Roberts, Skeins and mapping class groups, Math. Proc. Cambridge Philos. Soc. 115 (1994), no. 1, 53–77. MR 1253282
- [56] Klaus Schmidt, Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group-actions, Ergodic Theory Dynam. Systems 1 (1981), no. 2, 223–236. MR 661821
- [57] Yehuda Shalom, Rigidity of commensurators and irreducible lattices, Invent. Math. 141 (2000), no. 1, 1–54. MR 1767270
- [58] Dennis Sullivan, The density at infinity of a discrete group of hyperbolic motions, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171–202. MR 556586
- [59] Feraydoun Taherkhani, The Kazhdan property of the mapping class group of closed surfaces and the first cohomology group of its cofinite subgroups, Experiment. Math. 9 (2000), no. 2, 261–274. MR 1780211
- [60] William P Thurston, The geometry and topology of three-manifolds, Princeton University Princeton, NJ, 1979.
- [61] _____, On the geometry and dynamics of diffeomorphisms of surfaces, Bulletin (new series) of the american mathematical society 19 (1988), no. 2, 417–431.

- [62] Cormac Walsh, The asymptotic geometry of the Teichmüller metric, Geom. Dedicata 200 (2019), 115–152. MR 3956189
- [63] Daniel T. Wise, Research announcement: the structure of groups with a quasiconvex hierarchy, Electron. Res. Announc. Math. Sci. 16 (2009), 44–55. MR 2558631