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# Unitary representations of mapping class groups 

Biao Ma

## To cite this version:

Biao Ma. Unitary representations of mapping class groups. Group Theory [math.GR]. Université
Côte d'Azur, 2021. English. NNT: 2021COAZ4043 . tel-03377154

HAL Id: tel-03377154
https://theses.hal.science/tel-03377154
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## ÉCOLE DOCTORALE

SCIENCES
FONDAMENTALES
ET APPLIQUÉES


## Représentations Unitaires des Groupes Modulaires

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Laboratoire de Mathématiques J.A. Dieudonné

Présentée en vue de l'obtention
du grade de docteur en Mathématiques
de l'Université Côte d'Azur
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Soutenue le : 29 juin 2021

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# Représentations Unitaires des Groupes Modulaires 

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## Abstract

Mapping class groups of closed surfaces with punctures play important roles as prototypes of current research in geometric group theory. The representation theory of a group is a way to understand both the group structure and dynamic properties of that group. While there are massive literatures on finite dimensional (projective) unitary representations of mapping class groups, not so much on infinite dimensional ones. The aim of this thesis is to investigate mapping class groups from the perspective of infinite dimensional unitary representations based on current understanding of mapping class groups in the context of geometric group theory. It has two parts.

In the first part, for a surface, we introduce a family of unitary representations of its mapping class group based on the space of measured foliations. For this family of representations, we show that none of them has almost invariant vectors. As an application, we obtain an inequality concerning the action of the mapping class group on the Teichmüller space. Moreover, using the same method plus recent results about weak equivalence, we also give a classification, up to weak equivalence, for the unitary quasi-regular representations with respect to geometrical subgroups.

In the second part, for a closed hyperbolic surface, we show that the boundary representation of its mapping class group is ergodic, which generalizes the classical result of Masur on ergodicity of the action of the mapping class group on the projective measured foliation space of the surface. As a corollary, we show that the boundary representation of the mapping class group is irreducible. This confirms a conjecture of Bader-Muchnik in the case of mapping class groups with respect to Thurston measure classes.

Key words: Mapping class group, Unitary representation, Measured foliation, Almost invariant vector, Boundary representation, Irreducibility

## Résumé

Les groupes modulaires de surfaces fermées à points masqués jouent un rôle important comme prototypes par la recherche moderne en théorie géométriques des groupes. La théorie des représentations d'un groupe est un moyen de comprendre à la fois la structure du groupe et ses propriétés dynamiques. Bien qu'il existe beaucoup de littérature sur les représentations unitaires (projectives) de dimension finie des groupes modulaires, il y en a beaucoup moins sur celles de dimension infinie. Le but de cette thèse est d'étudier le groupe modulaire du point de vue des représentations unitaires de dimension infinie, dans le contexte de la théorie géométrique des groupes. Ce mémoire comporte deux parties.

Dans la première partie, nous introduisons pour une surface une famille de représentations unitaires de son groupe modulaire, basée sur l'espace des feuillages mesurés. Pour cette famille de représentations, nous montrons qu'aucune d'elles n'a de vecteurs presque invariants. En corollaire, nous obtenons une inégalité concernant l'action du groupe modulaire sur l'espace de Teichmüller. Nous classifions aussi, à équivalence faible près, les représentations unitaires quasi-régulières par rapport à ses sous-groupes géométriques.

Dans la seconde partie, pour une surface hyperbolique fermée, nous montrons que la représentation au bord de son groupe modulaire est ergodique, ce qui généralise un résultat classique de Masur sur l'ergodicité de l'action du groupe modulaire sur l'espaces projectif des feuillages mesurés de la surface. En corollaire, nous montrons que la représentation au bord du groupe de modulaire est irréductible, ce qui démontre une conjecture de Bader-Muchnik dans le cas du groupe modulaire par rapport à la classe des mesures de Thurston.

Mot clés: Groupe modulaire, Représentation unitaire, Feuillage mesuré, Vecteur presque invariant, Représentation au bord, Irréductibilité

## Acknowledgements

I would like to thank my advisor Indira Chatterji for her constant help, encouragement and guidance over the past four years. For math, she not only encourages me to work on mapping class groups from the viewpoint of unitary representations, but also encourages and supports me to attend infinitely many conferences which broaden my limited knowledge of Geometric Group Theory. Without her constant guidance and patience, this thesis can never be completed. Especially, I would like to thank her for tolerating my very poor English and helping me to correct English errors whenever I need. For life, she is always kindness and helpful. She is a very great advisor and friend.

I would like to thank Ursula Hamenstädt and Dan Margalit for reviewing the thesis. Thanks also goes to François Dahmani, Julien Marché and Barbara Schapira for agreeing to be my thesis jury. I also would like to thank the following professors for discussions that related to works in this thesis: Adrien Boyer, Vincent Delecroix, Ilya Gekhtman, Ursula Hamenstädt, Steve Kerckhoff, François Labourie, Yair Minsky, Alain Valette.

I would like to thank Li Guo at Rutgers for his constant help and encouragement. With Li, I have had the first interesting research experience which inspired me to be a mathematician. Meanwhile I would like to thank my master degree advisor Ruifeng Qiu at ECNU for introducing me to geometric group theory.

It is quite enjoyable to study in Nice. I learned a lot of very interesting math in the past four years through various courses, seminars and reading groups at LJAD. I would like to thank all my colleagues and friends at LJAD for interesting discussion: Kevin Atsou, Erwann Aubry, Bat-Od Battseren, Christophe Cazanave, Yash Chopra, Sorin Dumitrescu, Cécile Gachet, Alexis Garcia, Alexis Gilles, François Gautero, Pengfei Huang, Maxime Ingremeau, Kai Jiang, François Labourie, Mohamed Lamine Messaci, Jie Liu, Emmanuel Militon, Christian Pauly, Carlos Simpson, Chenmin Sun,

Jeremy Toulisse, Léo Vivion, Zhixin Xie, Mehdi Zaidi, Benjamin Zarka, Zhiyan Zhao, Jiqiang Zheng. And I would like to thank Narymen Bouriche, Isabelle De Angelis, Yves D'Angelo, Anita Ibrahim, Jean-Marc Lacroix, Roland Ruelle, Clara Salaun, and Jean-Louis Thomin for their help.

I also would like to thank my friends, with whom I discussed a lot math: Léo Bénard, Lizhi Chen, Xiaoming Du, Dapeng Li, Qing Liu, Jian Wang, Junhua Wang, Zhiqiang Xiao, Xige Yang, Bin Yu, Yanqing Zou.

I also would like to thank my Chinese friends in France who helped me a lot over the past four years: Dongyu Chen, Xi Chen, Jun Du, Siyue Du, Qian Dong, Jiao He, Jing Li, Furong Tang, Dan Wang, Chengfan Yang, Xi Yao, Zhibo Yu, Jiaxing Yue.

I would like to thank China Scholarship Council for their financial support in the past four years.

Finally I owe my family a great debt of thanks for their endless support and encouragement. Without them this thesis would never have come into existence.

## Introduction in English

The main goal of this thesis is to understand some infinite dimensional unitary representations of mapping class groups of a surface. For finite dimensional (projective) unitary representations of mapping class groups, one could refer to, for instance, Roberts [55] or Blanchet-Habegger-Masbaum-Vogel [10] and recent related developments.

Let $S=S_{g, n}$ be a closed, connected, orientable surface of genus $g$ with $n$ punctures. The mapping class group $\operatorname{Mod}(S)$ of $S$ is the group of isotopy classes of orientationpreserving homeomorphisms of $S$. Mapping class groups play important roles in understanding low-dimensional manifolds. For instance, according to the recent solution of the virtually fibered conjecture (see Agol [1], Wise [63] and Thurston's work [50]), mapping class groups essentially allow us to construct all closed hyperbolic manifolds in dimension three.

Associated to $S$, there are two well-known spaces that are equipped with a $\operatorname{Mod}(S)-$ action. First, the space of measured foliations $\mathcal{M F}(S)$ of $S$ which is the set of equivalence classes of measured foliations on $S$ and second, the projective measured foliation space $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ which is the quotient of $\mathcal{M \mathcal { F }}(S)$ by the positive reals $\mathbb{R}_{+}$. Both spaces are important to understand the group structure of $\operatorname{Mod}(S)$, see for example [30]. Our task is to investigate the unitary representations of $\operatorname{Mod}(S)$ associated to
$\mathcal{M \mathcal { F }}(S)$ and $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ with respect to certain natural measures.

### 0.1 Almost invariant vectors and mapping class groups

There is a family of measures on $\mathcal{M \mathcal { F }}(S)$ so that $\operatorname{Mod}(S)$ acts on $\mathcal{M \mathcal { F }}(S)$ ergodically with respect to these measures. Namely the measures classified in LindenstraussMirzakhani [36] and Hamenstädt [28] (see also Section 1.3.1), generalizing the Thurston measure on $\mathcal{M \mathcal { F }}(S)$ (see Masur [39],Masur [41]). We will call these measures generalized Thurston measures. One then obtains a family of unitary representations of $\operatorname{Mod}(S)$ by considering the induced action of $\operatorname{Mod}(S)$ on the associated $L^{2}$-space with respect to these measures. One can check that the family of unitary representations considered in Paris [51] is a special subfamily.

The first chapter investigates these unitary representations of mapping class groups. Recall that a locally compact group $G$ has Kazhdan's Property $(T)$ if every unitary representation of $G$ that has almost invariant vectors also has a non-zero invariant vector (see Definition 1.1.1 for almost invariant vectors). The following question is still widely considered to be open for surfaces of genus at least 3 (for genus at most 2, see Freedman-Krushkal [21] and Taherkhani [59]).

Question 0.1.1 (Ivanov [31]). Does $\operatorname{Mod}(S)$ have Kazhdan's Property (T)?

Note that, becuse of ergodicity, none of these unitary representations defined by generalized Thurston measures on $\mathcal{M} \mathcal{F}(S)$ can have non-zero invariant vectors. Inspired by Ivanov's question, one can ask whether these representations have almost invariant vectors. The first main result of Chapter 1 indicates that they don't. Namely,

Theorem 0.1.2 (Theorem 1.4.1). For a compact surface $S=S_{g, n}$ with $3 g+n \geq 4$ and each generalized Thurston measure $\mu$, the associated representation $\left(\pi^{\mu}, L^{2}(\mathcal{M} \mathcal{F}(S), \mu)\right.$ ) of $\operatorname{Mod}(S)$ does not have almost invariant vectors.

This theorem has two applications both for first cohomology and the action of the mapping class group $\operatorname{Mod}(S)$ on the Teichmüller space $\operatorname{Teich}(S)$ of $S$, see Corollary 1.4.2 and Corollary 1.4.3.

The proof of Theorem 1.4.1 also enables us to give a classification for a family of quasi-regular unitary representations, which is a stronger version of Corollary 5.5 in [51].

Theorem 0.1.3 (Theorem 1.5.4). Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$. Let $\gamma=\sum_{i=1}^{k} \gamma_{i}$ and $\delta=\sum_{i=1}^{l} \delta_{i}$, where $\left\{\gamma_{i}\right\}$ and $\left\{\delta_{i}\right\}$ are two collections of pairwise disjoint, distinct isotopy classes of essential simple closed curves on $S$.

1. If at least one of $k$ and $l$ is not $3 g-3+n$, then the associated unitary representations $\pi_{\gamma}$ and $\pi_{\delta}$ are weakly equivalent if and only if $\gamma$ and $\delta$ are of the same topological type (that is, there is a mapping class $f$ so that $\gamma=f(\delta)$ ).
2. Suppose that $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If $k=3 g-3+n$, then $\pi_{\gamma}$ is weakly equivalent to the regular representation $\lambda_{S}$.
3. Suppose that $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If $k \neq 3 g-3+n$, then $\pi_{\gamma}$ is not weakly contained in $\lambda_{S}$.

### 0.2 Boundary representations of mapping class groups

We now assume that $S=S_{g}$ and $g \geq 2$. As $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ is a quotient of $\mathcal{M F}(S)$, the Thurston measure on $\mathcal{M \mathcal { F }}(S)$ induces a measure $\nu$ on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ by considering a section of the quotient map $\mathcal{M F}(S) \longrightarrow \mathcal{P M} \mathcal{F}(S)$ and the coned-off construction of the measure (Dowdall-Duchin-Masur [16]) (See also Chapter 2 for details). The measure class $[\nu]$ is then $\operatorname{Mod}(S)$-invariant. By a standard construction (Bekka-de la Harpe-Valette [6]), one can construct a unitary representation $\pi=\pi_{\nu}$, called the boundary representation of $\operatorname{Mod}(S)$. The main result of Chapter 2 is the following

Theorem 0.2.1 (Corollary 2.2.15). Let $S=S_{g}$ be a closed surface of genus $g \geq 2$. The boundary representation of $\operatorname{Mod}(S)$ on $L^{2}(\mathcal{P} \mathcal{M F}(S), \nu)$ is irreducible.

This theorem confirms the following conjecture in the case of $\operatorname{Mod}(S)$ with respect to the Thurston measure class $[\nu]$ on $\mathcal{P} \mathcal{M F}(S)$.

Conjecture 0.2.2 (Bader-Muchnik [5]). Let $G$ be a locally compact group and $\mu$ a spread-out probability measure on $G$. The quasi-regular representation associated to the $\mu$-Poisson boundary of $G$ is irreducible.

In fact, we show the following more general ergodic-type theorem. See Definition 2.2.1 for ergodicity of representations.

Theorem 0.2.3 (Theorem 2.2.14). Let $\mathrm{Pr}_{o}$ be the radial projection from the Te ichmüller space $\operatorname{Teich}(S)-\{o\}$ to the Teichmüller boundary $\mathcal{P M} \mathcal{F}(S)$. Then there exists a sequence of finite subsets $E_{n} \subset \operatorname{Mod}(S)$ such that the associated quasi-regular representation $\pi_{\nu}$ is ergodic with respect to $\left(E_{n}, P r_{o}\right)$ and bounded Borel funtions on $\mathcal{P M} \mathcal{F}(S)$.

### 0.3 Questions

We end the introduction by some loose ends. One corollary of Theorem 1.4.1 is that, for any generalized Thurston measure $\mu$ on $\mathcal{M} \mathcal{F}(S)$,

$$
H^{1}\left(\operatorname{Mod}(S), \pi^{\mu}\right)=\overline{H^{1}}\left(\operatorname{Mod}(S), \pi^{\mu}\right)
$$

As one could characterize the Kazhdan's Property ( T ) by the reduced first cohomology (see Shalom [57]), one could ask if $\overline{H^{1}}\left(\operatorname{Mod}(S), \pi^{\mu}\right)=0$ for all generalized Thurston measure $\mu$ on $\mathcal{M} \mathcal{F}(S)$ ? Or more generally,

Question 0.3.1. For $p$ large enough and a generalized Thurston measure $\mu$, does $\operatorname{Mod}(S)$ act on $L^{p}(\mathcal{M} \mathcal{F}(S), \mu)$ via isometries properly?

The above question in fact relates to the following question asked by Hamenstädt [29] in her ICM talk in 2010:

Question 0.3.2. Does $\operatorname{Mod}(S)$ have the Haagerup property?
One motivation for Question 0.3.1 is that, according to a result of Bourdon [11], for large $p$, the $L^{p}$-representation of a finitely generated hyperbolic group $G$ given by the
right action on $G$ gives a proper affine isometric action of $G$. Meanwhile, a special type of generalized Thurston measures is given by the set of vertices in curve graph which is hyperbolic as a graph.

Based on Chapter 2, one could also ask
Question 0.3.3. For $S=S_{g}(g \geq 2)$ and the Thurston measure $\mu$ on $\mathcal{M} \mathcal{F}(S)$, is the associated representation reducible?

The answer to this question in the case of $g=1$ is negative due to the linear structure of $\operatorname{Mod}\left(S_{1}\right)=S L(2, \mathbb{Z})$ and one can not hope it to be true for all generalized Thurston measures, see the examples in Chapter 1. The last question is based on the following observation: the proof of Theorem 1.4.1 uses that some subgroups of $\operatorname{Mod}(S)$ act on $\mathcal{M F}(S)$ essential freely. One then could ask, what happens for dynamics without this property? For instance,

Question 0.3.4. Let $S=S_{g}$ and $X=\operatorname{Rep}\left(\pi_{1}(S), S U(2)\right)$ be the $S U(2)$-character variety of the fundamental group $\pi_{1}(S)$. Let $\mu$ be the symplectic volume on $X$ which is $\operatorname{Mod}(S)$-invariant. Does the representation of $\operatorname{Mod}(S)$, given by the orthogonal complement of $\mathbb{C}_{X}$ in $L^{2}(X, \mu)$, have almost invariant vectors?

It is well-known that $\operatorname{Mod}(S)$ acts on $X$ ergodically with respect to $\mu$ (Goldman [24]). This question actually asks whether the action of $\operatorname{Mod}(S)$ on $X$ is strongly ergodic (Schmidt [56]), or more interestingly, whether the measure $\mu$, which is constructed algebraically, is the unique $\operatorname{Mod}(S)$-invariant mean on $L^{\infty}(X, \mu)$. When $g=1$, the result is known by Schmidt [56], but it is unknown for other cases.

## Introduction en Français

L'objectif principal de cette thèse est de comprendre certaines représentations unitaires de dimension infinie du groupe modulaire d'une surface. Pour les représentations unitaires (projectives) de dimension finie des groupes modulaires, on peut se référer, par exemple, à Roberts [55] ou Blanchet-Habegger-Masbaum-Vogel [10] et aux récents travaux associés.

Soit $S=S_{g, n}$ une surface fermée, connexe, orientable de genre $g$ avec $n$ points marqués. Le groupe modulaire $\operatorname{Mod}(S)$ de $S$ est le groupe des classes d'isotopie d'homéomorphismes préservant l'orientation de $S$. Ces groupes jouent un rôle important dans la compréhension des variétés de petite dimension. La solution récente de la conjecture virtuellement fibrée (voir Agol [1], Wise [63] et le travail de Thurston [50]) dit que, les groupes modulaires nous permettent essentiellement de construire toutes les variétés hyperboliques fermées en dimension trois.

Nous nous intéressons à deux espaces équipés d'une action de $\operatorname{Mod}(S)$. Premièrement, l'espace des feuillages mesurés $\mathcal{M} \mathcal{F}(S)$ de $S$ qui est l'ensemble des classes d'équivalence de feuillages mesurés sur $S$, et deuxièmement l'espace des feuilletages mesurés projecties $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ qui est le quotient de $\mathcal{M} \mathcal{F}(S)$ par les réels positifs $\mathbb{R}_{+}$. Les deux espaces sont importants pour comprendre la structure de groupe de $\operatorname{Mod}(S)$, voir par exemple [30]. Nous étudions dans ce mémoire les représentations unitaires de


### 0.4 Vecteurs presque invariants et groupes modulaires

Il existe une famille de mesures sur $\mathcal{M F}(S)$ par lesquelles l'action de $\operatorname{Mod}(S)$ sur $\mathcal{M} \mathcal{F}(S)$ est ergodiquement. Ce sont les mesures étudiées dans [36] et [28] (voir aussi Section 1.3.1), généralisant la mesure de Thurston sur $\mathcal{M} \mathcal{F}(S)$ (voir [39], [41]), nous les appellerons mesures de Thurston généralisées. On obtient alors une famille de représentations unitaires de $\operatorname{Mod}(S)$ en considérant l'action induite de $\operatorname{Mod}(S)$ sur l'espace des fonctions de carré intégrable. On peut vérifies que la famille de représentations unitaires considérée dans [51] est une sous-famille particulière.

Le premier chapitre étudie ces représentations unitaires du groupe modulaire. Rappelons qu'un groupe localement compact $G$ a la propriété $(T)$ de Kazhdan si toute représentation unitaire de $G$ qui a des vecteurs presque invariants possède un vecteur invariant non nul (voir Définition 1.1.1 pour la notion de vecteurs presque invariants).

La question suivante est encore considérée comme ouverte pour des surfaces de genre de supéreur ou égal à 3 (pour genre au plus 2, voir [21] et [59]).

Question 0.4.1 (Ivanov [31]). Est-ce que $\operatorname{Mod}(S)$ a la propriété (T) de Kazhdan?

Par ergodicité, aucune de représentations unitaires définies par les mesures de Thurston généralisées sur $\mathcal{M} \mathcal{F}(S)$ n'ont de vecteurs invariants non nuls. Inspiré par la question d'Ivanov ci-dessus, on se demande si ces représentations ont des vecteurs presque invariants. Le résultat principal du Chapitre 1 montre qu'elles n'ont pas de vecteurs presque invariants. À savoir,

Théorème 0.4.2 (Theorem 1.4.1). Pour une surface compacte $S=S_{g, n}$ avec $3 g+n \geq$ 4 et une mesure de Thurston généralisée $\mu$, la représentation associée $\pi^{\mu}$ de $\operatorname{Mod}(S)$ n'a pas de vecteur presque invariant.

Ce théorème a deux applications: pour la première cohomologie et l'action du groupe moulaire $\operatorname{Mod}(S)$ sur l'espace Teichmüller $\operatorname{Teich}(S)$ de $S$, voir Corollaire 1.4.2 et Corollaire 1.4.3.

La preuve du Théorème 1.4.1 nous permet également de donner une classification pour une famille de représentations unitaires quasi-régulières, qui est une généralisation du Corollaire 5.5 dans [51].
Théorème 0.4.3 (Theorem 1.5.4). Soit $S=S_{g, n}$ une surface compacte avec $3 g+n \geq$ 4. Soit $\gamma=\sum_{i=1}^{k} \gamma_{i}$ et $\delta=\sum_{i=1}^{l} \delta_{i}$, où $\left\{\gamma_{i}\right\}$ et $\left\{\delta_{i}\right\}$ sont deux collections de deux $\grave{a}$ deux disjontes et de classes d'isotopie distinctes courbes fermées simples essentielles sur $S$,

1. Si $k$ ou $l$ est différent de $3 g-3+n$. Les représentations unitaires $\pi_{\gamma}$ et $\pi_{\delta}$ sont faiblement équivalentes si et seulement si $\gamma$ et $\delta$ sont du même type topologique (c'est-à-dire qu'il existe un element $f$ dans $\operatorname{Mod}(S)$ tel que $\gamma=f(\delta)$ ).
2. Supposons que $S$ ne soit pas $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. Si $k=3 g-3+n$, alors $\pi_{\gamma}$ est faiblement équivalente à la représentation régulière $\lambda_{S}$.
3. Supposons que $S$ ne soit pas $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. Si $k \neq 3 g-3+n$, alors $\pi_{\gamma}$ n'est pas faiblement contenue dans $\lambda_{S}$.

### 0.5 Representations au bord du groupes modulaires

Nous supposons maintenant que $S=S_{g}$ et $g \geq 2$. Comme $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ est un quotient de $\mathcal{M} \mathcal{F}(S)$, la mesure de Thurston sur $\mathcal{M F}(S)$ induit une mesure $\nu$ sur $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ en considérant une section de la projection $\mathcal{M \mathcal { F }}(S) \longrightarrow \mathcal{P} \mathcal{M} \mathcal{F}(S)$ et de la construction naturelle[16]. La classe de la mesure [ $\nu$ ] est alors $\operatorname{Mod}(S)$ - invariante. Par une construction standard [6], on peut construire une représentation unitaire $\left(\pi=\pi_{\nu}, L^{2}(\mathcal{P} \mathcal{M} \mathcal{F}(S), \nu)\right)$, appelée la représentation au bord de $\operatorname{Mod}(S)$. Le résultat principal du Chapitre 2 est le suivant
Théorème 0.5.1 (Corollary 2.2.15). Soit $S=S_{g}$ une surface fermée de genre $g \geq$ 2. La représentation au bord du groupe modulaire $\operatorname{Mod}(S)$ sur $L^{2}(\mathcal{P} \mathcal{M} \mathcal{F}(S), \nu)$ est irréductible.

Ce théorème confirme la conjecture suivante dans le cas des groupes de modulaire

Conjecture 0.5.2 (Bader-Muchnik [5]). Pour un groupe localement compact $G$ et une mesure de probabilité $\mu$ sur $G$, la représentation quasi-régulière associée au $\mu$-bord de Poisson de $G$ est irréductible.

En fait, nous montrons le théorème suivant qui est plus général. Voir Définition 2.2.1 pour l'ergodicité des représentations.
Théorème 0.5.3 ( Theorem 2.2.14). Soit Pr la projection de radiale de l'espace Teichmüller de $S$ à son bord Teichmüller $\mathcal{P} \mathcal{M F}(S)$. Alors il existe une suite de sousensembles finis $E_{n} \subset \operatorname{Mod}(S)$ tels que la représentation quasi-régulière associée $\pi_{\nu}$ soit ergodique par rapport à $\left(E_{n}, \operatorname{Pr}\right)$ et fonctions Borel bornées sur $\mathcal{P M} \mathcal{F}(S)$.

### 0.6 Questions ouvertes

Nous terminons l'introduction par quelques questions connexes. Un corollaire du Théorème 1.4.1 est que, pour toute mesure de Thurston généralisée $\mu \operatorname{sur} \mathcal{M} \mathcal{F}(S)$, On a

$$
H^{1}\left(\operatorname{Mod}(S), \pi^{\mu}\right)=\overline{H^{1}}\left(\operatorname{Mod}(S), \pi^{\mu}\right)
$$

Comme on peut caractériser la propriété ( T ) de Kazhdan par l'annulation de la première cohomologie réduite (voir [57]), il est naturel de se demander si $\overline{H^{1}}\left(\operatorname{Mod}(S), \pi^{\mu}\right)=$ 0 pour tout Thurston généralisé mesure $\mu$ sur $\mathcal{M} \mathcal{F}(S)$ ? Ou plus qénéralement,
Question 0.6.1. Pour un p suffisanent grand et une mesure de Thurston généralisée $\mu$, est-ce que $\operatorname{Mod}(S)$ agit proprement par isomètries affines sur $L^{p}(\mathcal{M} \mathcal{F}(S), \mu)$ via des isométries proprement?

Cette question se rapporte à la question posée par Hamenstaedt[29] dans son discours ICM en 2010,
Question 0.6.2. Est-ce que $\operatorname{Mod}(S)$ a la propriété de Haagerup?
La question 0.6.1 est motivée par un résultat de Bourdon [11] qui prouve que, pour $p$ grand, la représentation $L^{p}$ d'un groupe hyperbolique finiment engendré donné par
l'action à droite sur $G$ donne une action isométrique affine propre de $G$. En même temps, un type spécial de mesures de Thurston généralisées est donné par l'ensemble des sommets du graphe de courbes qui est hyperbolique en tant que graphe.

Basé sur le Chapitre 2, on pourrait aussi se demander
Question 0.6.3. Pour $S=S_{g}(g \geq 2)$ et la mesure de Thurston $\mu$ sur $\mathcal{M} \mathcal{F}(S)$, la représentation associée est-elle réductible?

La réponse à cette question dans le cas de $g=1$ est négative du fait de la structure linéaire de $\operatorname{Mod}\left(S_{1}\right)=S L(2, \mathbb{Z})$ et on ne peut espérer que ce soit vrai pour tous mesures de Thurston généralisées, voir les exemples dans le Chapitre 1. La dernière question est basée sur l'observation suivante: la preuve du Théorème 1.4.1 utilise que certains sous-groupes de $\operatorname{Mod}(S)$ ont une action essentiellent libre sur $\mathcal{M} \mathcal{F}(S)$. Que se passe-t-il pour une dynamique sans cette propriété? Par exemple,

Question 0.6.4. Soit $S=S_{g}$ et $X=\operatorname{Rep}\left(\pi_{1}(S)\right.$, $S U(2)$ ) la $S U(2)$-variété de caractères du groupe de fondamental $\pi_{1}(S)$. Soit $\mu$ le volume symplectique sur $X$ qui est $\operatorname{Mod}(S)$-invariant. La représentation de $\operatorname{Mod}(S)$, donnée par le complément orthogonal de $\mathbb{C}_{X}$ dans $L^{2}(X, \mu)$, a-t-elle des vecteurs presque invariants?

Il est bien connu que $\operatorname{Mod}(S)$ agit ergodiquement sur $X$ par rapport à à [24]. La question est de savoir si l'action de $\operatorname{Mod}(S)$ sur $X$ est fortement ergodique [56], ou encore, si la mesure $\mu$, qui est construite algébriquement, est l'unique moyenne invariante sur l'action de $\operatorname{Mod}(S)$ sur $L^{\infty}(X, \mu)$. Lorsque $g=1$, le résultat est connu par [56], mais il est ouvert dans les autres cas.

## Chapter 1

## Almost invariant vectors and mapping class groups

This chapter is taken from [37].

### 1.1 Introduction

Let $S=S_{g, n}$ be a compact, connected, orientable surface of genus $g$ with $n$ boundaries, the mapping class group $\operatorname{Mod}(S)$ of $S$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of $S$ which preserving each boundary components (without the assumption that it should fix each boundary pointwise). Throughout this paper, $(g, n)$ is assumed to satisfy $3 g+n \geq 4$ and a subsurface of $S$ is allowed to be disconnected.

Given a discrete group $G$, a unitary representation is a pair $(\pi, V)$ where $V$ is a Hilbert space and $\pi: G \rightarrow U(V)$ is a homomorphism from $G$ to the group of all unitary operators of $V$ [6]. Infinite dimensional unitary representations of mapping class groups $\operatorname{Mod}(S)$ received a lot of attention recently. In [51], the author considers unitary representations given by the action of $\operatorname{Mod}(S)$ on the curve complex associated to $S$. See [2],[25] for more topics in this direction.

The group $\operatorname{Mod}(S)$ acts on the space of measured foliations $\mathcal{M \mathcal { F }}(S)$, which is defined as the set of equivalence classes of non-zero measured foliations on $S$. As the action is ergodic with respect to generalized Thurston measures $\mu$ [39],[41],[36], [28] (see Section 1.3.1 for a brief description of the measures), one obtains a family of unitary representations by considering the induced action of $\operatorname{Mod}(S)$ on the space $L^{2}(\mathcal{M F}(S), \mu)$. It is quite easy to see that the family of unitary representations considered in [51] is a special subfamily. However, unlike representations studied in [51],

Example 1.3.4 will show that some of representations considered here are reducible.

Definition 1.1.1. Let $(\pi, V)$ be a unitary representation of a discrete group $G$. The representation $\pi$ is said to have almost invariant vectors if for every finite set $K \subseteq G$ and every $\epsilon>0$, there exists $v \in V$ such that

$$
\max _{g \in K}\|\pi(g) v-v\|<\epsilon\|v\| .
$$

The main result of this paper is about the existence of almost invariant vectors for the representation $\pi^{\mu}$ associated to the action of $\operatorname{Mod}(S)$ on $L^{2}(\mathcal{M} \mathcal{F}(S), \mu)$. The existence of such vectors for other representations of mapping class group has been discussed in [3].

Theorem 1.1.1 (Theorem 1.4.1). For a compact surface $S=S_{g, n}$ with $3 g+n \geq 4$ and each generalized Thurston measure $\mu$, the associated representation $\pi^{\mu}$ of $\operatorname{Mod}(S)$ does not have almost invariant vectors.

The first direct application of this theorem is the following:

Corollary 1.1.2 (Corollary 1.4.2). Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$ and $\mu$ be a generalized Thurston measure, then $H^{1}\left(\operatorname{Mod}(S), \pi^{\mu}\right)=\overline{H^{1}}\left(\operatorname{Mod}(S), \pi^{\mu}\right)$, where $\pi^{\mu}$ is the associated representation of $\operatorname{Mod}(S)$.

For the second application, we will obtain a geometric inequality of independent interest concerning the action of $\operatorname{Mod}(S)$ on the Teichmüller space Teich $(S)$ of $S$.

Corollary 1.1.3 (Corollary 1.4.3). Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$ and $\gamma$ be the isotopy class of an essential simple closed curve on $S$. Then there exists a finite subset $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ of $\operatorname{Mod}(S)$ consisting of pseudo-Anosov mapping classes
and a constant $\epsilon>0$, such that, for every point $\mathcal{X}$ in $\operatorname{Teich}(S)$, we have:

$$
\max _{i \in\{1,2, \ldots, n\}}\left\{\sum_{\alpha \in \operatorname{Mod}(S) \cdot \gamma} e^{-2 \ell_{\mathcal{X}}(\alpha)}\left(e^{\Delta_{\mathcal{X}}^{\phi_{i}}(\alpha)}-1\right)^{2}\right\} \geq \epsilon \sum_{\alpha \in \operatorname{Mod}(S) \cdot \gamma} e^{-2 \ell_{\mathcal{X}}(\alpha)},
$$

where $\Delta_{\mathcal{X}}^{\phi_{i}}(\alpha)=\ell_{\mathcal{X}}(\alpha)-\ell_{\phi_{i} . \mathcal{X}}(\alpha)$ and $\ell_{\mathcal{X}}(\alpha)$ is the geodesic length of $\alpha$.
For unitary representations associated to discrete measures on the space of measured foliations, some of them are irreducible and some are reducible. We will discuss irreducible decompositions (See Proposition 1.5.1). We will also use the same method as in the proof of the main theorem, combined with recent results in [15],[14],[7], to give a classification for a family of quasi-regular unitary representations (with respect to subgroups), which is a stronger version of Corollary 5.5 in [51]. Recall that, given two unitary representations $(\pi, \mathcal{H})$ and $(\phi, \mathcal{K})$ of a discrete group $G, \pi$ is weakly contained in $\phi$ if for every $\xi$ in $\mathcal{H}$, every finite subset $Q$ of $G$ and $\epsilon>0$, there exist $\eta_{1}, \ldots, \eta_{n}$ in $\mathcal{K}$ such that

$$
\max _{g \in Q}\left|\langle\pi(g) \xi, \xi\rangle-\sum_{i=1}^{n}\left\langle\phi(g) \eta_{i}, \eta_{i}\right\rangle\right|<\epsilon .
$$

If $\pi$ is weakly contained in $\phi$ and $\phi$ is weakly contained in $\pi$, then $\phi$ and $\pi$ are said to be weakly equivalent. By Proposition F.1.7 in [6], Definition 1.1.1 is equivalent to say that the trivial representation is weakly contained in the representation $\pi$. We then have the following theorem.

Theorem 1.1.4 (Theorem 1.5.4). Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$. Let $\gamma=\sum_{i=1}^{k} \gamma_{i}$ and $\delta=\sum_{i=1}^{l} \delta_{i}$, where $\left\{\gamma_{i}\right\}$ and $\left\{\delta_{i}\right\}$ are two collections of pairwise disjoint, distinct isotopy classes of essential simple closed curves on $S$.

1. If at least one of $k$ and $l$ is not $3 g-3+n$, then the associated unitary representations $\pi_{\gamma}$ and $\pi_{\delta}$ are weakly equivalent if and only if $\gamma$ and $\delta$ are of the same topological type (that is, there is a mapping class $f$ so that $\gamma=f(\delta)$ ).
2. Suppose $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If $k=3 g-3+n$, then $\pi_{\gamma}$ is weakly equivalent to the regular representation $\lambda_{S}$.
3. Suppose $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If $k \neq 3 g-3+n$, then $\pi_{\gamma}$ is not weakly contained in $\lambda_{S}$.

This paper is organized as follows. Section 1.2 is devoted to preliminary for group cohomology with coefficients in unitary representations. The proof of the main theorem is given in Section 1.4. The proof is divided into two general lemmas: Lemma 1.4.4 and Lemma 1.4.5, and concluded by a technical statement, namely Proposition 1.3.7, concerning actions of subgroups of mapping class groups on $\mathcal{M} \mathcal{F}(S)$. Section 1.3 is mainly devoted to this proposition and Section 1.5 is for irreducible decompositions and the classification up to weak equivalence.

### 1.2 Cohomology with coefficients in representations

Cohomology and reduced cohomology. For a discrete group $G$ and a unitary representation $(V, \pi)$, one can talk about both cohomology and reduced cohomology group of G with coefficients in $\pi$. Definitions of cohomology and reduced cohomology of discrete groups with coefficients in a representation $\pi$ are standard, so we refer to [38],[2],[6]. We briefly recall that one defines the following vector spaces for a unitary representation $(V, \pi)$ :

$$
\begin{gathered}
Z^{1}(G, \pi) \doteq\{b: G \rightarrow V \mid b(g h)=b(g)+\pi(g) b(h), \text { for all } g, h \in G\} \\
B^{1}(G, \pi) \doteq\left\{b \in Z^{1}(G, \pi) \mid \text { there exists } v \in V, \text { such that for all } g \in G\right. \\
b(g)=\pi(g) v-v\} \\
H^{1}(G, \pi) \doteq Z^{1}(G, \pi) / B^{1}(G, \pi) \\
\overline{H^{1}}(G, \pi) \doteq Z^{1}(G, \pi) / \overline{B^{1}(G, \pi)}
\end{gathered}
$$

where the closure in the last one is for uniform convergence. The vector space $H^{1}(G, \pi)\left(\right.$ resp. $\left.\overline{H^{1}}(G, \pi)\right)$ is the first (resp. reduced) cohomology group with coefficients in $\pi$.

Almost invariant vectors. The following Guichardet's theorem provides a way to determine if $H^{1}(G)=\overline{H^{1}}(G)$.

Theorem 1.2.1 ([38]). Let $G$ be a finitely generated discrete group and $(V, \pi)$ be a unitary representation without nonzero invariant vectors. Then the following two are equivalent:

1. The associated first reduced cohomology is the same as the first cohomology, that is, $H^{1}(G, \pi)=\overline{H^{1}}(G, \pi)$;
2. The representation $\pi$ does not have almost invariant vectors.

One observation is that not having almost invariant vectors is closed under taking limit, more precisely, we have the following lemma.

Lemma 1.2.2. Let $(V, \pi)$ be a unitary representation of $G$ and $W$ be a $G$-invariant vector subspace of $V$ such that the closure $\bar{W}=V$. Then $\pi$ does not have almost invariant vectors if and only if the representation $\left.\pi\right|_{W}$ in $W$ does not have almost invariant vectors.

Proof. Suppose that the pair $(K, \epsilon)$, where $K$ is a finite subset of $G$ and $\epsilon>0$, is given by the condition that $\left.\pi\right|_{W}$ does not have almost invariant vector. Given any element $\xi \in V-W$, there is a sequence of elements $\left\{\xi_{n}\right\} \subseteq W$ such that $\xi_{n} \rightarrow \xi$ as $n \rightarrow \infty$. Then, for $n$ large enough, we have:

$$
\begin{gathered}
\max _{g \in K}\|\pi(g) \xi-\xi\|=\max _{g \in K}\left\|\pi(g) \xi-\pi(g) \xi_{n}+\pi(g) \xi_{n}-\xi_{n}+\xi_{n}-\xi\right\| \\
\geq \max _{g \in K}\left\|\pi(g) \xi_{n}-\xi_{n}\right\|-2 \max _{g \in K}\left\|\xi_{n}-\xi\right\| \geq \epsilon\|\xi\|-\delta
\end{gathered}
$$

Now $\delta$ can be small enough, so

$$
\max _{g \in K}\|\pi(g) \xi-\xi\| \geq \epsilon\|\xi\|
$$

Which completes the proof of one direction. The opposite direction is obvious.

Another easy observation is that, in order to show a representation of group does not have almost invariant vectors, one only need to pass to a subgroup. That is,

Lemma 1.2.3. A unitary representation $(\pi, V)$ of a group $G$ does not have almost invariant vectors iff there exists a subgroup $H$ of $G$ such that the unitary representation $\left(\left.\pi\right|_{H}, V\right)$ of $H$ does not have almost invariant vectors.

Amenable groups. A basic strategy in this article is to use the regular representation of the free group $\mathbb{F}_{2}$ of rank 2 , so the following theorem is of fundamental importance.

Theorem 1.2.4 ([18]). For the left regular representation $\lambda_{G}$ of a finitely generated discrete group $G$ on $\ell^{2}(G), \lambda_{G}$ has almost invariant vectors if and only if $G$ is amenable.

Remark 1.2.1. Since $\mathbb{F}_{2}$ is not amenable, the left regular representation of $\mathbb{F}_{2}$ on $\ell^{2}\left(\mathbb{F}_{2}\right)$ does not have almost invariant vectors. We will regard $\ell^{2}\left(\mathbb{F}_{2}\right)$ as $\ell^{2}$-functions on vertices of the Cayley graph of $\mathbb{F}_{2}$ with respect to a chosen generating set, and thus further identify $\ell^{2}\left(\mathbb{F}_{2}\right)$ with the vector space V , where

$$
V=\left\{\sum_{i} \alpha_{i} g_{i}: \sum_{i}\left|\alpha_{i}\right|^{2}<\infty, \alpha_{i} \in \mathbb{C}, g_{i} \in \mathbb{F}_{2}\right\}
$$

### 1.3 Generalized Thurston measures and dynamics on measured foliation spaces

In this section we will describe the integral theory on the space of measured foliations and the action of subgroups of mapping class groups on the space of measured foliations. A subgroup of $\operatorname{Mod}(S)$ in which all elements except the identity are pseudoAnosov mapping classes will be called a pseudo-Anosov subgroup.

### 1.3.1 Measures and $L^{2}$-theory on $\mathcal{M F}(S)$.

Generalized Thurston measures on $\mathcal{M F}(S)$.
The space of measured foliations $\mathcal{M} \mathcal{F}(S)$ of a surface $S$ is the set of equivalence classes of transversely measured (singular) foliations on $S$. Using train tracks, one can show that $\mathcal{M} \mathcal{F}(S)$ has a piecewise linear integral structure such that $\operatorname{Mod}(S)$ acts on it as automorphisms (that is, preserves this piecewise linear integral structure)[60]. Therefore, in such local PL coordinates, $\operatorname{Mod}(S)$ acts as linear transformations.

A consequence of this PL structure is that $\mathcal{M \mathcal { F }}(S)$ can be equipped with a $\operatorname{Mod}(S)$-invariant measure $\mu_{T h}$, called the Thurston measure on $\mathcal{M} \mathcal{F}(S)$. Moreover, this measure can be generalized to obtain a family of locally finite, ergodic $\operatorname{Mod}(S)$-invariant measures $\mu_{T h}^{[(\mathcal{R}, \gamma)]}$ on $\mathcal{M} \mathcal{F}(S)$ for complete pairs $(\mathcal{R}, \gamma)$, which will be called generalized Thurston measures. We present a brief summary of the construction of generalized Thurston measures $\mu_{T h}^{[(\mathcal{R}, \gamma)]}$ according to [36].

Let $\gamma=\sum_{i} c_{i} \gamma_{i}, c_{i}>0$ be a multi-curve on $S$, that is, $\gamma$ is a collection of isotopy classes of pairwise distinct, pairwise disjoint essential simple closed curves $\left\{\gamma_{i}\right\}$ on $S$ so that each curve has been weighted by $c_{i}>0$. After fixing a hyperbolic structure
on $S$, one can think a multi-curve $\gamma=\sum_{i} c_{i} \gamma_{i}, c_{i}>0$ as a collection of simple closed geodesics $\left\{\widetilde{\gamma}_{i}\right\}$ on $S$ with $\widetilde{\gamma}_{i}$ labeled by a positive real number $c_{i}$, where $\widetilde{\gamma}_{i}$ is the unique geodesic representative in $\gamma_{i}$. We will use $\gamma$ to denote both the formal sum $\sum_{i} c_{i} \gamma_{i}$ and the subset $\bigsqcup \widetilde{\gamma}_{i}$ of $S$. Cutting $S$ along $\gamma$, one obtains a decomposition into a disjoint union

$$
\overline{S-\gamma}=\bigsqcup T_{i}
$$

where $\left\{T_{i}\right\}$ is a collection of subsurfaces of $S$ with boundary smoothly embedded in $S$. For

$$
\mathcal{R}=\bigsqcup S_{i}
$$

with $\left\{S_{i}\right\} \subseteq\left\{T_{i}\right\}$, the pair $(\mathcal{R}, \gamma)$ will be called a complete pair. For a complete pair $\left(\mathcal{R}=\bigsqcup S_{i}, \gamma\right)$, define

$$
\mathcal{M F}(\mathcal{R})=\prod_{i} \mathcal{M} \mathcal{F}^{*}\left(S_{i}\right)
$$

where $\mathcal{M} \mathcal{F}^{*}\left(S_{i}\right)=\mathcal{M} \mathcal{F}\left(S_{i}\right) \bigcup 0_{S_{i}}$ in which $0_{S_{i}}$ is the zero foliation on $S_{i}$. The space $\mathcal{M} \mathcal{F}(\mathcal{R})$ can be $\operatorname{Mod}(\mathcal{R}, \gamma)$-embeded on $\mathcal{M} \mathcal{F}(S)$ via enlarging boundary curves [See [20], Exposé 6.6 for enlarging curves]. Denote by $\mathcal{M}(\mathcal{R})$ the image of this embedding. This set is endowed with the product measure $\mu_{\mathcal{R}}=\prod \mu_{T h}^{i}$, where $\mu_{T h}^{i}$ is the Thurston measure of $S_{i}$. Define also

$$
M(\mathcal{R}, \gamma)=\{\mathcal{F}+\gamma: \mathcal{F} \in \mathcal{M}(\mathcal{R})\} \subseteq \mathcal{M} \mathcal{F}(S)
$$

The inclusion induces a measure on $\mathcal{M} \mathcal{F}(S)$, denoted by $\mu_{T h}^{[(\mathcal{R}, \gamma)]}$ and supported on the set of $\operatorname{Mod}(S)$-orbits of $M(\mathcal{R}, \gamma)$, from the product measure $\mu_{\mathcal{R}}$.

Special cases are when $\mathcal{R}=\emptyset$ and $\gamma$ is the isotopy class of a non-separating curve, or when $\mathcal{R}=S$ and $\gamma=\emptyset$. The corresponding measure in the case of $\mathcal{R}=\emptyset$ is a discrete measure, denoted by $\mu_{\gamma}$ and supported on $\operatorname{Mod}(S) \cdot \gamma$ which is regarded as a subset of
$\mathcal{M \mathcal { F }}(S)$, while in the case of $\gamma=\emptyset$ it is exactly the Thurston measure $\mu_{T h}$ on $\mathcal{M} \mathcal{F}(S)$.

The following remarkable theorem indicates that generalized Thurston measures $\mu_{T h}^{[(\mathcal{R}, \gamma)]}$ are exactly the set of all locally finite, $\operatorname{Mod}(S)$-invariant, ergodic measures on $\mathcal{M F}(S)$.

Theorem 1.3.1 (Hamenstädt[28],Lindenstrauss-Mirzakhani[36]). Any locally finite $\operatorname{Mod}(S)$-invariant ergodic measure on $\mathcal{M} \mathcal{F}(S)$, up to a constant multiple, is in the form of $\mu_{T h}^{[(\mathcal{R}, \gamma)]}$, where $(\mathcal{R}, \gamma)$ is a complete pair.

Associated $L^{2}$-theory over $\mathcal{M F}(S)$.
The case of discrete measures. Recall that when $\mathcal{R}=\emptyset, \mu_{T h}^{[(\mathcal{R}, \gamma)]}$ is the discrete measure supported on the set $\operatorname{Mod}(S) \cdot \gamma$, where $\operatorname{Mod}(S) \cdot \gamma$ is regarded as a subset of $\mathcal{M \mathcal { F }}(S)$. We will first deal with the case that $\gamma$ is the isotopy class of an essential simple closed curve on $S$ and denote the measure by $\mu_{\gamma}$.

Let $X_{\gamma}=\mathcal{C}_{\gamma}^{0}(S)$ be the subset of vertices of the curve complex consisting of $\operatorname{Mod}(S) \cdot \gamma$. By considering the Dirac measure supported on $X_{\gamma}$, one can define the Hilbert space $\ell^{2}\left(X_{\gamma}\right)$. It is clear that $\ell^{2}\left(X_{\gamma}\right)$ is $\operatorname{Mod}(S)$-equivariantly isomorphic to $L^{2}\left(\mathcal{M F}(S), \mu_{\gamma}\right)$. On the other hand, let $G_{\gamma}=\operatorname{Mod}(S, \gamma)=\operatorname{Stab}_{\gamma}(\operatorname{Mod}(S))$ be the set of all elements in $\operatorname{Mod}(S)$ that fix $\gamma$, then $\ell^{2}\left(X_{\gamma}\right)$ can be further $\operatorname{Mod}(S)$-equivariantly identified with $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$. These two spaces give the same unitary representation of $\operatorname{Mod}(S)$, actually we have

Theorem 1.3.2 (Paris[51]). The infinite dimensional unitary representation of $\operatorname{Mod}(S)$ given by $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ is irreducible.

Remark 1.3.1. This theorem was proved in a more general setting for 1-multi-curves on $S$, that is, $\gamma=\sum c_{i} \gamma_{i}$ with $c_{i}=1$ for all $i$.

Thus, in particular, this representation does not have non-zero invariant vectors. Meanwhile, the irreducibility also allows us to describe $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ more geometrically.

The first description of $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ is classical. For $f \in \ell^{2}\left(X_{\gamma}\right)$, let $\operatorname{Supp}(f)=$ $\left\{v \in X_{\gamma}: f(v) \neq 0\right\}$. The function $f$ has compactly support if the cardinality of $\operatorname{Supp}(f)$ is finite. Define the subspace $W$ of $\ell^{2}\left(X_{\gamma}\right)$ as the set of elements in $\ell^{2}\left(X_{\gamma}\right)$ which have compact support. As $X_{\gamma}$ is discrete, the following notation will be used to represent $f \in W: f=\sum_{i=1}^{n} k_{i} \alpha_{i}$. Note that $W$ is $\operatorname{Mod}(S)$-invariant and the closure $\bar{W}$ of $W$ in $\ell^{2}\left(X_{\gamma}\right)$ is then $\ell^{2}\left(X_{\gamma}\right)$ itself. This description will be used in the proof of the main theorem in the case of discrete measures.

The second description of $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ needs more explanations. Let Teich $(S)$ be the Teichmüller space of $S$, and for each point $\mathcal{X}$ of Teich $(S)$, define a function on $X_{\gamma}$ by

$$
f_{\mathcal{X}}(\alpha)=e^{-\ell_{\mathcal{X}}(\alpha)}, \alpha \in X_{\gamma}
$$

where $\ell_{\mathcal{X}}(\alpha)$ is the length of the unique geodesic in the isotopy class $\alpha$.
Proposition 1.3.3. The function defined above belongs to $\ell^{2}\left(X_{\gamma}\right)$.
Proof. It amounts to say

$$
\sum_{\alpha \in X_{\gamma}} e^{-2 \ell_{\mathcal{X}}(\alpha)}<\infty
$$

Thus this proposition is a corollary of the result of Birman-Series [8] or Mirzakhani [48] about the polynomial growth of simple closed geodesics.

Let $W^{\prime}$ be the subspace of $\ell^{2}\left(X_{\gamma}\right)$ which consisting of finite linear combinations of elements in $\left\{f_{\mathcal{X}}: \mathcal{X} \in \operatorname{Teich}(S)\right\}$. It is also true to see that this subspace is $\operatorname{Mod}(S)$-invariant. Also by irreducibility, the closure $\overline{W^{\prime}}$ of $W^{\prime}$ is $\ell^{2}\left(X_{\gamma}\right)$.

Remark 1.3.2. The second description gives rise to a parametrization for $\ell^{2}\left(X_{\gamma}\right)$ via the Teichmüller space, thus it can be viewed as a reply to Problem 2.5 in [25] for representations under consideration.

For the case of $\mathcal{R}=\emptyset$ and $\gamma$ is a general integral multicurve $\gamma=\sum k_{i} \gamma_{i}$ with $k_{i} \in \mathbb{N}$, Theorem 1.3.2 is not true in general as shown by the following

Example 1.3.4. Consider the genus 2 closed surface $S$, regarded as a quotient along boundaries of holed sphere with four disjoint open disks deleted. Let $\gamma=2 \gamma_{1}+3 \gamma_{2}, \delta=$ $\gamma_{1}+\gamma_{2}$, where $\gamma_{1}$ and $\gamma_{2}$ are isotopy classes of two distinct images of boundaries. Obviously, there is a mapping class $s$ that permutes the $\gamma_{i}$ 's. Denote $H=\operatorname{Mod}(S, \gamma)$ and $H^{\prime}=\operatorname{Mod}(S, \delta)$, then we have the exact sequence:

$$
1 \rightarrow H \rightarrow H^{\prime} \rightarrow \mathbb{Z}_{2} \rightarrow 1
$$

That is, $H$ is a normal subgroup of $H^{\prime}$ of index 2. This exact sequence allows us to define a self-map of the left cosets $\{f H\}$ as follows. Write $H^{\prime}$ as $H \bigsqcup s H$. There are two $\operatorname{Mod}(S)$-invariant bijections:

$$
\begin{aligned}
& \operatorname{Mod}(S) \cdot \gamma \leftrightarrow\{[g]=g H\} \\
& \operatorname{Mod}(S) \cdot \delta \leftrightarrow\left\{[f]=f H^{\prime}\right\}
\end{aligned}
$$

As $f H^{\prime}=f H \bigsqcup f s H$, the set $\{g H\}$ can be rewritten as $\{f H, f s H\}$, this reformulation induces a well-defined inversion $i: f H=[f] \mapsto[f s]=f s H$.

A function $\phi$ on $G / H=\{g H\}$ is called even if for every $[g] \in G / H, \phi([g])=\phi(i([g]))$ and a function $\varphi$ on $G / H$ is called odd if for every $[g] \in G / H, \varphi([g])=-\phi(i([g]))$.

Define $V_{1}$ to be the subset of $\ell^{2}(G / H)$ consisting of even functions and $V_{2}$ to be the subset of $\ell^{2}(G / H)$ consisting of odd functions. It is easy to see that such two vector spaces are non-empty, closed and $\operatorname{Mod}(S)$-invariant subspaces of $\ell^{2}(G / H)$.

Remark 1.3.3. For any discrete measure mentioned above, the associated unitary representation has no nonzero invariant vectors.

The case of non-discrete measures. For general measures, we mention one remark.

Remark 1.3.4. If $\mathcal{R}$ is nontrivial, ergodicity of the action shows that the associated unitary representation has no nonzero invariant vectors.

### 1.3.2 Actions of subgroups of $\operatorname{Mod}(S)$ on $\mathcal{M F}(S)$.

Train tracks and a construction of pseudo-Anosov mapping classes. For later use, we first recall some facts about train tracks and a construction of pseudoAnosov mapping classes by Thurston. All discussions here are standard and wellknown, we refer to [52],[19],,[[20], Exposé 13],[61] for more details.

A train track $\tau$ in a surface $S$ is an embedded smooth graph with extra conditions on vertices. A train track is called recurrent if it supports a positive transverse measure, that is, a measure assigns a positive number to every edge. A transversely recurrent train track is a train track such that every edge has a nontrivial essential transverse intersection with a simple closed curve. A birecurrent train track is thus a train track that both recurrent and transversely recurrent. A maximal birecurrent train track is a birecurrent train track that cannot be a proper subtrack of any other train track. Any measured foliation is carried by a maximal train track. We only remark here that, for a maximal birecurrent train track $\tau$, the set $E(\tau)$ of all positive transverse
measures on $\tau$ is a positive linear submanifolds, that is, a subset of some Euclidean space defined by a family of linear equations with the condition that all parameters are positive. For the torus $T$, the set $\mathcal{M \mathcal { F }}(T)$ of linear measured foliations can be covered by four affine charts $E\left(\tau_{i}\right)$ associated to four maximal birecurrent train tracks. We fix these four types of train tracks as blocks and denote them by $\left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$. See [[52], Section 2.6, Figure 2.6.1] for such four train tracks in the annulus, thus in the torus.

We now sketch a construction of pseudo-Anosov mapping classes given by Thurston [61]. We only discuss Thurston's construction for closed surfaces. For surfaces with boundaries, one can modify the construction without any difficulty. Let $S=S_{g}(g \geq 2)$ and choose two essential simple closed curves $\alpha$ and $\beta$ on $S$ so that all connected components of $S-\alpha \bigcup \beta$ are open topological disks. For each intersection point $p$ of $\alpha$ and $\beta$, one can assign a rectangle to $p$ so that $S$ has a flat structure $\sigma$ and, with respect to this flat structure, both Dehn twists $T_{\alpha}$ and $T_{\beta}$ act as affine transformations (since we have flat structure, we can talk about affine transformations) with linear parts given by elements in $\operatorname{PSL}(2, \mathbb{R})$. An element in the subgroup of $\operatorname{Mod}(S)$ generated by $T_{\alpha}$ and $T_{\beta}$ is pseudo-Anosov if the associated affine transformation has a hyperbolic linear part.

We now mention some facts about the set $\mathfrak{L}(S, \sigma)$ of linear measured foliations on $S$ induced by the flat structure $\sigma$ above. Note that unstable and stable foliations of pseudo-Anosov mapping classes obtained by Thurstion's construction are in $\mathfrak{L}(S, \sigma)$ and $\mathfrak{L}(S, \sigma)$ is a closed subset of $\mathcal{M} \mathcal{F}(S)$. If we arrange all rectangles mentioned above on the plane such that $\alpha$-sides are horizontal and label the rectangles from left to right by $\left\{\square_{1}, \square_{2}, \ldots, \square_{m}\right\}$, then a linear measured foliation $\mathfrak{F} \in \mathfrak{L}(S, \sigma)$ is given by parallel
lines of the plane and a train track $\tau$ in $S$ carrying $\mathfrak{F}$ has the form that the restriction of $\tau$ in each rectangle $\square_{i}$ is one of $\tau_{i}$ and all such $\tau_{i}$ appearing in $\tau$ are the same. Therefore there are four types of train tracks, denoted also by $\left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$, so that $\mathfrak{L}(S, \sigma) \subseteq \bigcup_{i=1}^{4} E\left(\tau_{i}\right)$. A direct computation shows that linear measured foliations on $S$ induced by this flat structure are determined by weights on two edges of $\tau_{i} \bigcap \square_{1}$, thus each $\mathfrak{L}(S, \sigma) \bigcap E\left(\tau_{i}\right)$ is parameterized by two free independent parameters.

Lemma 1.3.5. Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 5$ and $\alpha, \beta$ be two curves as above, then each $\tau_{i}$ is birecurrent and the set $\mathfrak{L}(S, \sigma)$ of linear measured foliations with respect to a flat structure $\sigma$ constructed as described above is of null $\mu_{T h}$-measure.

Proof. It is obvious that each $\tau_{i}$ is birecurrent. We divide the proof of the rest into two cases according to whether $\tau_{i}$ is maximal or not. If $\tau_{i}$ is not maximal, then any measured foliation carried by $\tau_{i}$ is not maximal [52]. By [[36], Lemma 2.3], $E\left(\tau_{i}\right)$ has null $\mu_{T h}$-measure. If $\tau_{i}$ is maximal, then, as $\tau_{i}$ is a birecurrent train track, $E\left(\tau_{i}\right)$ is an open subset of $\mathcal{M} \mathcal{F}(S)$ and thus every point in $E\left(\tau_{i}\right)$ should be determined by weights on $6 g-6+2 n$ edges of $\tau_{i}$. As remarked above that $E\left(\tau_{i}\right) \bigcap \mathfrak{L}(S, \sigma)$ is determined by weights on two edges of $\tau_{i} \bigcap \square_{1}$ which can be extended to obtain $6 g-6+2 n$ free parameters of $E\left(\tau_{i}\right)$. That is to say, $E\left(\tau_{i}\right) \bigcap \mathfrak{L}(S, \sigma)$ is locally given by $x_{3}=x_{4}=\ldots=$ $x_{6 g-6+2 n}=0$ in $\mathbb{R}^{6 g-6+2 n}$ whose coordinates is given by $\left\{x_{1}, \ldots, x_{6 g-6+2 n}\right\}$. Therefore, $E\left(\tau_{i}\right) \bigcap \mathfrak{L}(S, \sigma)$ is a null set. Since $\mathfrak{L}(S, \sigma) \subseteq \bigcup_{i=1}^{4} E\left(\tau_{i}\right)$, hence $\mathfrak{L}(S, \sigma)$ is a null set as well.

Almost properly discontinuous action. We introduce a concept for a group action on a Borel space (that is, a topological space endowed with a Radon measure) which is weaker than usual properly discontinuous action.

Definition 1.3.1. Let G be a group and $(X, \mu)$ be a Borel space. Suppose that G acts on X by measure-preserving homeomorphisms. We say that G acts on X almost properly discontinuously if there exists a G-invariant subset $K$ with $\mu(K)=0$ such that G acts on $X-K$ properly discontinuously.

Example 1.3.6. Let $H \leq \operatorname{PSL}(2, \mathbb{Z})$ be a Schottky group, then its limit set $\Lambda(H) \subseteq$ $S^{1}$, as a Cantor set, has zero Lebesgue measure, and thus it acts on $\left\{\mathbb{R}^{2}-(0,0)\right\} /\{ \pm 1\}$ almost properly discontinuously.

Although the action of $\operatorname{Mod}(S)$ on $\mathcal{M \mathcal { F }}(S)$ is ergodic with respect to generalized Thurston measures, the action of subgroups of $\operatorname{Mod}(S)$ on $\mathcal{M F}(S)$ is not always ergodic. The following proposition allows us to use properties of the "properly discontinuou" action.

Proposition 1.3.7. For each complete pair $(\mathcal{R}, \gamma)$, there exists a rank 2 free pseudoAnosov subgroup $H$ of $\operatorname{Mod}(S)$ that acts on $\mathcal{M \mathcal { F }}(S)$ almost properly discontinuously with respect to the generalized Thurston measure $\mu_{T h}^{[(\mathcal{R}, \gamma)]}$.

Any such free group will be called a p-rank 2 free subgroup.

The first case is when $\mathcal{R}=\emptyset$ or each component of $\mathcal{R}$ is $S_{0,3}$, then this proposition is obvious by taking $H$ to be any free pseudo-Anosov subgroup generated by two pseudo-Anosov mapping classes (this works the same for non-integral multicurves as for integral multicurves). For other cases, we prove this proposition through two lemmas.

Lemma 1.3.8. There exists a p-rank 2 free subgroup $H$ of $\operatorname{Mod}(S)$ that acts on $\mathcal{M} \mathcal{F}(S)$ almost properly discontinuously with respect to the Thurston measure $\mu_{T h}$.

Proof. If $S=S_{0,4}$ or $S_{1,1}$, then, in both cases, $\mathcal{M F}(S)$ can be identified with $\left\{\mathbb{R}^{2}-(0,0)\right\} /\{ \pm 1\}$ and $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ can be identified with $S^{1}$. Moreover, there is a finite index subgroup of $\operatorname{Mod}(S)$ such that the action of this subgroup on $\mathcal{P \mathcal { M } \mathcal { F } ( S ) , ~}$ is equivalent to the action of $\operatorname{PSL}(2, \mathbb{Z})$ on $S^{1}$, see [[19], Chapter 15] for the case of $S_{0,4}$. By taking $H$ to be any subgroup given in Example 1.3.6 and considering the set $Y=\operatorname{Pr}^{-1}(\Lambda(H))$, where $\operatorname{Pr}: \mathcal{M} \mathcal{F}(S) \rightarrow \mathcal{P} \mathcal{M} \mathcal{F}(S)$ is the projection, the action of $H$ on $\mathcal{M} \mathcal{F}(S)$ is thus almost properly discontinuous and $\mu_{T h}(Y)=0$.

For other $S$, we deduce this lemma by first passing to $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ and then using the result of McCarthy-Papadopoulos[45] on limit sets. Let $\phi$ and $\psi$ be two independent pseudo-Anosov mapping classes obtained by Thurston's construction. By the pingpong lemma, one can construct a free pseudo-Anosov subgroup $H$ generated by some powers of $\phi$ and $\psi$. As remarked before that stable and unstable measured foliations of pseudo-Anosov elements in $H$ are linear measured foliations and $\mathfrak{L}(S, \sigma)$ is a closed subset, therefore, by Lemma 1.3.5, the limit set $\Lambda(H)$ of $H$, which is defined to be the closure of the set of fixed points of non-trivial elements of $H$ with respect to the action on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$, has the property that

$$
\mu_{T H}\left(\operatorname{Pr}^{-1}(\Lambda(H))\right)=0
$$

On the other hand, one can define the zero set $Z(\Lambda(H))(\subseteq \mathcal{P} \mathcal{M} \mathcal{F}(S))$ of $\Lambda(H)$ [45]. By combining with facts [See [45], Proposition 6.1] that $Z(\Lambda(H))-\Lambda(H)$ consists of no uniquely ergodic foliations and uniquely ergodic foliation has full $\mu_{T h}$-measure, we know that $\operatorname{Pr}^{-1}(Z(\Lambda(H)))$ has null $\mu_{T h}$-measure. By [[45],Theorem 7.17], $H$ acts properly discontinuously on $\mathcal{P} \mathcal{M} \mathcal{F}(S)-Z(\Lambda(H))$, thus properly discontinuously on $\mathcal{M F}(S)-\operatorname{Pr}^{-1}(Z(\Lambda(H)))$. Hence $H$ acts almost properly discontinuously on $\mathcal{M F}(S)$.

For $\mathcal{R} \neq S$, a complete pair $(\mathcal{R}, \gamma)$ is called a middle type if $\mathcal{R} \neq \emptyset$ and there is a connected component $\neq S_{0,3}$.

Lemma 1.3.9. For a complete pair $(R, \gamma)$ of middle type, there exists a p-rank 2 free subgroup $H$ of $\operatorname{Mod}(S)$ that acts on $\mathcal{M} \mathcal{F}(S)$ almost properly discontinuously with respect to the measure $\mu_{T h}^{[(R, \gamma)]}$.

Proof. We will follow the idea of [[36], Lemma 3.1] to prove this lemma. Fix any hyperbolic structure $X$ on $S$ and consider the continuous function $\ell_{X}: \mathcal{M} \mathcal{F}(S) \rightarrow \mathbb{R}_{+}$ extending the geodesic length function. Thus

$$
\mathcal{M} \mathcal{F}(S)=\lim _{L_{1} \rightarrow 0, L_{2} \rightarrow \infty} B_{L_{2}}^{L_{1}}(X)
$$

where $B_{L_{2}}^{L_{1}}(X)=\left\{\nu \in \mathcal{M} \mathcal{F}(S): \ell_{X}(\nu) \in\left[L_{1}, L_{2}\right]\right\}$ is a compact set and, as pointed out in the proof of [[36], Lemma 3.1], $B_{L_{2}}^{L_{1}}(X) \bigcap\left(\bigcup_{g \in \operatorname{Mod}(S)} g \cdot M(\mathcal{R}, \gamma)\right)$ is equal to $B_{L_{2}}^{L_{1}}(X) \bigcap\left(\bigcup_{i=1}^{n} g_{i} \cdot M(\mathcal{R}, \gamma)\right)$, for some finite set $\left\{g_{1}, \ldots, g_{n}\right\} \subset \operatorname{Mod}(S)$. Fix a free pseudo-Anosov subgroup $H$ of $\operatorname{Mod}(S)$ and take any compact subset $K \subseteq$ $\bigcup_{g \in \operatorname{Mod}(S)} g \cdot M(\mathcal{R}, \gamma)$. Taking $L_{1}$ small enough and $L_{2}$ large enough, one can assume $K \subseteq B_{L_{2}}^{L_{1}}(X)$. We now claim that

$$
|\{h \in H: h . K \bigcap K \neq \emptyset\}|<\infty
$$

Let $Z=\operatorname{Mod}(S) \cdot \gamma$ and $\ell_{X}: Z \rightarrow R_{+}$. We first claim that there is a finite set $J \subseteq Z$ such that

$$
\{h \in H: h . K \bigcap K \neq \emptyset\} \subseteq\{h \in H: h . J \bigcap J \neq \emptyset\}
$$

For every element in $K$ can be written as $\gamma+\nu$ such that $\ell_{X}(\gamma)$ is bounded. If $h . K \bigcap K \neq \emptyset$, then $h(\gamma)$ also has bounded $\ell_{X}$-length and all bounds can be chosen to be uniform on $K$, say $[a, b]$. Since $\ell_{X}$ is a proper map on $Z$ (that is, the inverse of compact set is also compact $), J=\ell_{X}^{-1}([a, b])$ is then a finite subset of $Z$ containing
both $h(\gamma)$ and $\gamma$. So one has $\{h \in H: h . K \bigcap K \neq \emptyset\} \subseteq\{h \in H: h . J \bigcap J \neq \emptyset\}$. By the discussion of the case $\mathcal{R}=\emptyset$, the set $\{h \in H: h . J \bigcap J \neq \emptyset\}$ is finite which implies that the finiteness of $|\{h \in H: h . K \bigcap K \neq \emptyset\}|$. Now taking the measure zero set to be $Y=\mathcal{M} \mathcal{F}(S)-\bigcup_{g \in \operatorname{Mod}(S)} g \cdot M(\mathcal{R}, \gamma)$ completes the proof.
$H$-related cover. Given a group $H$ and a Borel space $(X, \mu)$. Suppose that $H$ acts on $X$ almost properly discontinuously and freely. Examples for such $(H, X, \mu)$ are given by Proposition 1.3.7. By definition of almost properly discontinuous action, there is a null set $Y$ such that $H$ acts on $X-Y$ properly discontinuously. For any compact subset $K$ of $X-Y$, we will describe a "nice" cover of $K$. Since $X-Y$ is the domain of discontinuity of $H$, for every $p$ in $K$, there is an open neighbourhood $\mathcal{U}_{p}$ of $p$ in $X-Y$ with finite $\mu$-measure such that for all $h \in H$, one has $h . \mathcal{U}_{p} \bigcap \mathcal{U}_{p}=\emptyset$. Thus there is an open cover of $K$. By compactness of $K$, choose a finite sub-cover of this cover. Label the sub-cover by $\mathcal{U}_{1}, \ldots, \mathcal{U}_{n}$ and for each $i \in 1, \ldots, n$, consider $A_{i}=\left\{h . \mathcal{U}_{i} \mid h \in H\right\}$. Starting from $i=1$, form a family $B_{1}=\left\{X_{k} \in A_{1} \mid X_{k} \bigcap K \neq \emptyset\right\}$ as well as $C_{1}=\left\{Y_{k} \mid Y_{k}=X_{k} \bigcap K, X_{k} \in B_{1}\right\}$. Delete $\bigcup_{Y_{k} \in C_{1}} X_{k}$ from $K$ and denote the resulting compact set by $K_{1}$. Then for $K_{1}$, there is a family $B_{2}=\left\{X_{k} \in A_{2} \mid X_{k} \bigcap K_{1} \neq \emptyset\right\}$ as well as $C_{2}=\left\{Y_{k} \mid Y_{k}=X_{k} \bigcap K_{1}, X_{k} \in B_{2}\right\}$. Delete $\bigcup_{Y_{k} \in C_{2}} X_{k}$ from $K_{2}$ and denote the resulting compact set by $K_{3}$. Continuing this process, there is a cover of K which can be written in the following formula:

$$
K \subseteq \bigsqcup_{k=1}^{n} \bigsqcup_{Y_{i} \in C_{k}} Y_{i}
$$

So $K$ can be covered by finitely many pairwise disjoint $\mu$-measurable sets (we allow some of them to be null sets). This will be called an $H$-related cover of $K$, since, for each $k, C_{k}$ is a family of disjoint sets that lie inside the $H$-orbit of some set.

### 1.4 Nonexistence of almost invariant vectors

Let $\mathcal{H}(\mu)=L^{2}(\mathcal{M} \mathcal{F}(S), \mu)$, where $\mu=\mu_{T h}^{[(\mathcal{R}, \gamma)]}$ is a generalized Thurston measure explained in Section 3.1.1, and $\pi^{\mu}$ be the associated unitary representation of $\operatorname{Mod}(S)$. The main result of this section is the following:

Theorem 1.4.1. For a compact surface $S=S_{g, n}$ with $3 g+n \geq 4$ and each generalized Thurston measure $\mu$, the associated representation $\pi^{\mu}$ of $\operatorname{Mod}(S)$ does not have almost invariant vectors.

By using Theorem 1.2.1, Remark 1.3.3 and Remark 1.3.4, we have:

Corollary 1.4.2. Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$ and $\mu$ be $a$ generalized Thurston measure, then $H^{1}\left(\operatorname{Mod}(S), \pi^{\mu}\right)=\overline{H^{1}}\left(\operatorname{Mod}(S), \pi^{\mu}\right)$, where $\pi^{\mu}$ is the associated representation of $\operatorname{Mod}(S)$.

Proof. By Theorem 1.2.1, we only need to show that the representation $\pi^{\mu}$ has no nonzero invariant vectors. The corollary is thus concluded by using Remark 1.3.3 for discrete measures and Remark 1.3.4 for non-discrete measures.

Let $\gamma$ be the isotopy class of an essential simple closed curve on $S, X=\operatorname{Mod}(S) \cdot \gamma$ and $\mathcal{X}$ be a point in the Teichmüller space $\operatorname{Teich}(S)$ of $S$. Denoting $\Delta_{\mathcal{X}}^{\phi_{i}}(\alpha)=\ell_{\mathcal{X}}(\alpha)-$ $\ell_{\phi_{i} \cdot \mathcal{X}}(\alpha)$, where $\alpha \in X$, and using the description of $\ell^{2}(X)$ via $\operatorname{Teich}(S)$ in Section 1.3.1, the following inequality is easy to show:

Corollary 1.4.3. Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$ and $\gamma$ be the isotopy class of an essential simple closed curve on $S$. Then there exists a finite subset $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ of $\operatorname{Mod}(S)$ consisting of pseudo-Anosov mapping classes and a constant
$\epsilon>0$, such that, for every point $\mathcal{X}$ in $\operatorname{Teich}(S)$, we have:

$$
\max _{i \in\{1,2, \ldots, n\}}\left\{\sum_{\alpha \in \operatorname{Mod}(S) \cdot \gamma} e^{-2 \ell_{\mathcal{X}}(\alpha)}\left(e^{\Delta_{\mathcal{X}}^{\phi_{i}}(\alpha)}-1\right)^{2}\right\} \geq \epsilon \sum_{\alpha \in \operatorname{Mod}(S) \cdot \gamma} e^{-2 \ell_{\mathcal{X}}(\alpha)} .
$$

We divide the proof of Theorem 1.4.1 into two lemmas. First we prove a lemma used for discrete measures.

Lemma 1.4.4. Let $G$ be a discrete countable group and $X$ be a discrete set equipped with a $G$-action. Suppose that there is a rank 2 free subgroup $H$ of $G$ such that $H$ acts on $X$ freely. Then the unitary representation $\pi=\ell^{2}(X)$ of $G$ associated to the action of $G$ on $X$ does not have almost invariant vectors.

Remark 1.4.1. This lemma is well-known, we give an elementary proof here mainly for heuristic purposes.

Definition 1.4.1. Let $H$ be a rank 2 free group and $X$ be a space on which $H$ acts. Suppose $x \in X$ such that the stabilizer $\operatorname{Stab}_{H}(x)$ of $x$ is trivial. The image of $H$ under the orbit map $H \rightarrow X, h \mapsto h . x$ is called the 2-tree based at $x$ (with respect to $(H, X))$.

Proof of Lemma 1.4.4. By Lemma 1.2.3, we can pass to subgroups. For the action of the group $H$ on the space $X$ and any point $p \in X$, consider the 2 -tree based at $p$ with respect to $(H, X)$.

Let $W$ be the subspace of $\ell^{2}(X)$ consisting of functions with finite support. As $W$ is $G$-invariant and dense, by Lemma 1.2 .2 , it is enough to show that $\left(\left.\pi\right|_{W}, W\right)$ does not have almost invariant vectors. That is, we have to find $(K, \epsilon)$ with the property that

$$
\max _{g \in K}\|\pi(g) f-f\|^{2} \geq \epsilon\|f\|^{2}, \text { for all } f \in W
$$

Since $H \cong \mathbb{F}_{2}$, as mentioned in Remark 1.2.1, the left regular representation $\ell^{2}(H)$ does not have almost invariant vectors, thus such a pair $(K, \epsilon)$ exists for the regular representation. Fix such pair $(K, \epsilon)$ for the rest of the proof. Here are two facts.

## Facts:

1. For every 2 -tree $\mathbb{T}$ based at a point, $\ell^{2}(\mathbb{T})$ is $H$-equivariantly isomorphic to $\ell^{2}(H)$.
2. Different 2-trees are disjoint and thus, if the support $A_{1}$ of $f_{1} \in \ell^{2}(X)$ and the support $A_{2}$ of $f_{2} \in \ell^{2}(X)$ are located in different 2-trees, then $f_{1}$ and $f_{2}$ are orthogonal.

These two facts imply that we only need to deal with $\ell^{2}$-functions on $X$ whose finite support contained in a single 2 -tree. In fact, for every $f \in W$, if we decompose its support $K_{f}$ as

$$
K_{f}=\bigsqcup_{i=1}^{n} K_{f_{i}}
$$

where $K_{f_{i}}$ lie in different 2-trees and $f_{i}$ is defined to be the restriction of $f$ on such different 2-trees, then

$$
\begin{gathered}
f=\sum_{i=1}^{n} f_{i} \\
\|\pi(g) f-f\|^{2}=\sum_{i=1}^{n}\left\|\pi(g) f_{i}-f_{i}\right\|^{2}, \text { for all } g \in K
\end{gathered}
$$

Note that $K \subseteq H$ is fixed. If the support of $f_{i}$ is contained in a 2 -tree $\mathbb{T}_{i}$, by Remark 1.2.1, there exists $g_{i} \in K$ such that

$$
\left\|\pi\left(g_{i}\right) f_{i}-f_{i}\right\|^{2} \geq \epsilon\left\|f_{i}\right\|^{2}
$$

Now for every $f_{i}$, let $g_{i}$ be an element satisfying the above inequality. If two 2 trees $f_{i}, f_{j}$ correspond to the same $g_{i}=g_{j}$, then $f_{i}+f_{j}$ also satisfies that inequality. As $K$ is finite, denote $\sharp K=m$ and so $f$ can be further decomposed, that is, $f=$ $f_{1}^{\prime}+f_{2}^{\prime}+\cdots+f_{s}^{\prime}(s \leq m)$ such that $f_{k}^{\prime}=\sum_{j} f_{j k}$, where $f_{j k} \in\left\{f_{1}, \ldots, f_{n}\right\}$ and $\left\{f_{j k}\right\}_{j}$
correspond to the same $g_{k} \in K$. We claim that there exists $g_{l} \in K$ such that

$$
\left\|\pi\left(g_{l}\right) f-f\right\|^{2} \geq \frac{\epsilon}{s}\|f\|^{2} \geq \frac{\epsilon}{m}\|f\|^{2}
$$

Otherwise, since for all $g_{i}$ selected, we have

$$
\begin{equation*}
\left\|\pi\left(g_{i}\right) f-f\right\|^{2} \geq\left\|\pi\left(g_{i}\right) f_{i}-f_{i}\right\|^{2} \geq \epsilon\left\|f_{i}\right\|^{2} \tag{1.4.1}
\end{equation*}
$$

then

$$
\begin{aligned}
& \epsilon\|f\|^{2}=\sum_{i=1}^{m} \frac{\epsilon}{m}\|f\|^{2}>\sum_{i=1}^{m}\left\|\pi\left(g_{i}\right) f-f\right\|^{2} \\
& \quad \geq \sum_{i=1}^{s}\left\|\pi\left(g_{i}\right) f-f\right\|^{2} \geq \sum_{i=1}^{s} \epsilon\left\|f_{i}\right\|^{2}=\epsilon\|f\|^{2} .
\end{aligned}
$$

The second inequality is the assumption and the last inequality is inequality (1.4.1). Thus there exists a pair $\left(K, \eta=\frac{\epsilon}{\sharp K}\right)$ such that

$$
\max _{g \in K}\|\pi(g) f-f\|^{2} \geq \eta\|f\|^{2}, \text { for all } f \in W
$$

So the proof of the lemma is completed.

Then we prove a lemma used for non-discrete measures.

Lemma 1.4.5. Let $G$ be a discrete countable group and $(X, \mu)$ be a Borel space. Suppose that $G$ acts on $X$ by measure-preserving homeomorphisms. If there exists a rank 2 free subgroup $H$ of $G$ such that $H$ acts on $X$ almost properly discontinuously and freely, then the unitary representation $\pi=L^{2}(X, \mu)$ of $G$ associated to the action of $G$ on $X$ does not have almost invariant vectors.

Proof of Lemma 1.4.5. Also by Lemma 1.2.3, we can pass to subgroups. Fix a null subset $Y$ of $X$ such that $H$ acts on $X-Y$ properly discontinuously. For any point $p \in X$, consider the image of $H$ under the orbit map, given by

$$
h \longmapsto h . p .
$$

Since the stabilizer $\operatorname{Stab}_{p}(H)$ is trivial, this map is injective. This is the 2-tree based at $p$ with respect to $(H, X)$. Define $W$ to be the $G$-invariant subspace of $L^{2}(X, \mu)$ consisting functions $f \in L^{2}(X, \mu)$ that compactly supported on $X-Y$. Thus $\bar{W}=$ $L^{2}(X, \mu)$ as $\mu$ is a Radon measure. So as before, we only need to prove the theorem in the case of $\left(W,\left.\pi\right|_{W}\right)$. For each $f \in W$ supported on one $H$-orbit of a measurable set $U$, that is,

$$
K_{f} \subseteq \bigsqcup_{h \in H} h . U
$$

where $K_{f}$ is the compact support of $f$ and the union is disjoint indexed by $H$, fix a point $p$ in $U$ and associate an element $A_{f} \in \ell^{2}(\mathbb{T})$, where $\mathbb{T}$ is the 2 -tree based on $p$, via

$$
A_{f}(h \cdot p)=\left(\int_{h \cdot U}|f|^{2} d \mu\right)^{\frac{1}{2}}
$$

Define

$$
K^{\prime}=\left\{g \in H \mid \quad g \quad \text { or } \quad g^{-1} \in K\right\},
$$

where $K$ is the same finite subset of H as in Lemma 1.4.4. For $f$, one has:

$$
\begin{aligned}
& \int_{K_{f}}|\pi(g) f-f|^{2} d \mu \\
& \quad=\sum_{h \in H} \int_{h \cdot U}|\pi(g) f-f|^{2} d \mu \\
& \quad \geq \sum_{h \in H}\left|\left(\int_{h \cdot U}|\pi(g) f|^{2} d \mu\right)^{\frac{1}{2}}-\left(\int_{h \cdot U}|f|^{2} d \mu\right)^{\frac{1}{2}}\right|^{2} \\
&= \sum_{h \in H}\left|A_{\pi(g) f}(h \cdot p)-A_{f}(h \cdot p)\right|^{2} \\
&= \sum_{h \in H}\left|\left(\pi\left(g^{-1}\right) A_{f}\right)(h \cdot p)-A_{f}(h \cdot p)\right|^{2}
\end{aligned}
$$

where the second inequality is the triangle inequality. By Lemma 1.4.4,

$$
\begin{aligned}
& \max _{g \in K^{\prime}}\|\pi(g) f-f\|^{2} \\
& \quad \geq \max _{g \in K^{\prime}} \sum_{h \in H}\left|\left(\pi(g) A_{f}\right)(h \cdot p)-A_{f}(h \cdot p)\right|^{2} \\
& \quad=\max _{g \in K^{\prime}}\left\|\pi(g) A_{f}-A_{f}\right\|^{2} \\
& \quad \geq \eta\left\|A_{f}\right\|^{2} \\
& \quad=\epsilon^{\prime}\|f\|^{2}
\end{aligned}
$$

where $\epsilon^{\prime}$ is a multiple of the constant $\eta$ in Lemma 1.4.4, as in this case we have $\sharp K^{\prime}=2 \sharp K$. If the compact set $K_{f}$ is not contained in one $H$-orbit, one can take an $H$-related cover of $K_{f}$, then by the orthogonality similar to Fact 2 in Lemma 1.4.4 and follow the last few lines in the proof of Lemma 1.4.4, one can also choose the pair $\left(K^{\prime}, \epsilon^{\prime \prime}\right)$, where $\epsilon^{\prime \prime}$ is a suitable multiple of $\epsilon^{\prime}$, to complete the proof.

Proof of Theorem 1.4.1. As any pseudo-Anosov subgroup acts freely on $\mathcal{M F}(S)$, by Lemma 1.4.4 and Proposition 1.3.7, the theorem is true for $\mathcal{R}=\emptyset$. When $\mathcal{R}=S$ or $\mathcal{R}$ is of middle type, it is concluded by Lemma 1.4.5 and Proposition 1.3.7.

Remark 1.4.2. The same trick can be used to show that representations of mapping class groups in the space of $L^{2}$-functions on the Teichmüller spaces with respect to Weil-Petersson volumes also have no almost invariant vectors. As one can show that such representations do not have non-trivial invariant vectors, we have the same conclusion about corresponding cohomology groups.

### 1.5 Classification of quasi-regular representations up to weak containment

### 1.5.1 Irreducible decompositions.

As pointed out in Section 1.3.1, for unitary representations of mapping class groups associated to discrete measures on the space of measured foliations, both reducible and irreducible ones exist. By examining Example 1.3.4 carefully, one sees that, reducible representations have an irreducible decomposition. For any multi-curve $\gamma=\sum_{i=1}^{k} c_{i} \gamma_{i}$ on $S$, where $c_{i}>0$ for all $i$, we form $\tilde{\gamma}=\sum_{i=1}^{k} \gamma_{i}$. Recall that $\left\{\gamma_{i}\right\}$ is a collection of pairwise disjoint isotopy classes of essential simple closed curves on $S$. As before, denote by $G_{\gamma}=\operatorname{Mod}(S, \gamma)$ and $G_{\tilde{\gamma}}=\operatorname{Mod}(S, \tilde{\gamma})$ the corresponding subgroups of $\operatorname{Mod}(S)$. Hence $G_{\gamma}$ is a subgroup of $G_{\tilde{\gamma}}$ of finite index.

Proposition 1.5.1. Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$ and $\gamma, \tilde{\gamma}$ as above.
(1) If the index of $G_{\gamma}$ in $G_{\tilde{\gamma}}$ is one, then the associated representation in $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ of $\operatorname{Mod}(S)$ is irreducible.
(2) If the index of $G_{\gamma}$ in $G_{\tilde{\gamma}}$ is $n>1$, then the associated representation of $\operatorname{Mod}(S)$ in $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ is reducible.

Proof. (1) is obvious, since the representation $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)$ is $\ell^{2}\left(\operatorname{Mod}(S) / G_{\tilde{\gamma}}\right)$ which is irreducible by Remark 1.3.1.

Now assume that $\left[G_{\tilde{\gamma}}: G_{\gamma}\right]=n>1$. Let $X_{\gamma}=\operatorname{Mod}(S) \cdot \gamma$ and $Y_{\tilde{\gamma}}=\operatorname{Mod}(S) \cdot \tilde{\gamma}$, then $X_{\gamma}$ is a $\operatorname{Mod}(S)$-equivariant discrete covering space of $Y_{\tilde{\gamma}}$ of degree $n$. So every $\ell^{2}$-function on $Y_{\tilde{\gamma}}$ defines an $\ell^{2}$-function on $X_{\gamma}$, and such correspondence produces
a proper closed $\operatorname{Mod}(S)$-invariant subspace of $\ell^{2}\left(X_{\gamma}\right)$, which implies the reducibility.

### 1.5.2 Classification up to weak containment.

We first fix some notations. Fix a hyperbolic structure on $S$. Denote by $\gamma=\sum_{i=1}^{k} \gamma_{i}$ and $\delta=\sum_{i=1}^{k} \delta_{i}$, that is, multi-curves on $S$ with coefficients all of 1s. Such multicurves will be called 1 -multi-curves. For any 1 -multi-curve $\gamma=\sum_{i=1}^{k} \gamma_{i}$ on $S$, we will call the union of geodesic representatives of $\gamma$ a geometric multi-curve and, for any $i$, the representative $\alpha_{i}$ a geometric component. Denote by $G_{\gamma}\left(G_{\delta}\right.$, resp.) the corresponding subgroup of $\operatorname{Mod}(S)$, and by $\pi_{\gamma}\left(\pi_{\delta}\right.$, resp $)$ the associated unitary representation on $\ell^{2}\left(\operatorname{Mod}(S) / G_{\gamma}\right)\left(\ell^{2}\left(\operatorname{Mod}(S) / G_{\delta}\right)\right.$, resp.). Let $\lambda_{S}$ be the regular representation of the mapping class group $\operatorname{Mod}(S)$ of $S$ on $\ell^{2}(\operatorname{Mod}(S))$. We first recall some definitions which can be found in [51],[6], [7].

Let $G$ be a countable discrete group and H be a subgroup of $G$, the commensurator of H is defined to be

$$
\operatorname{Com}_{G}(H)=\left\{g \in G: g H g^{-1} \bigcap H \text { has finite index in } H \text { and } g H g^{-1} .\right\}
$$

A discrete group is said to be $C^{*}$-simple if every unitary representation, which is weakly contained in the regular representation of $G$, is weakly equivalent to the regular representation. Let $\gamma$ and $\delta$ be geometric multi-curves, then $\gamma$ and $\delta$ are of the same type if there is an element $f$ in $\operatorname{Mod}(S)$ such that $f(\gamma)=\delta$. We say a subgroup $H$ of $G$ has the spectral gap property if the unitary representation $\ell^{2}(X)$ associated to the action $H \curvearrowright X=G / H-\{H\}$ does not have almost invariant vectors. In this section, we give a classification for unitary representations of $\operatorname{Mod}(S)$ associated to discrete measures.

Lemma 1.5.2. Given a 1 -multi-curve $\gamma$ on $S$ and let $m$ be the number of its geometric components.

1. If $m=3 g-3+n$, then $G_{\gamma}$ is amenable.
2. If $1 \leq m<3 g-3+n$, then $G_{\gamma}$ has the spectral gap property.

Proof. If $m=3 g-3+n$, then $G_{\gamma}$ is virtually abelian, thus it is amenable. For other cases, as $m<3 g-3+n$, one can cut $S$ along geometric components so that the resulting surface has at least one connected component that admits two pseudo-Anosov mapping classes generating a rank 2 pseudo-Anosov subgroup. Assume components admitting pseudo-Anosov mapping classes are labelled as $T_{1}, \ldots, T_{k}$, two pseudoAnosov mapping classes in each $\operatorname{Mod}\left(T_{i}\right)$ and the associated rank 2 pseudo-Anosov subgroup are also denoted by $\varphi_{i}, \psi_{i}, H_{i}$, respectively. Note that pseudo-Anosov homeomorphisms fix boundaries. Then define two maps $\varphi$ and $\psi$ on $S$ (thus their isotopy classes) by extending $\varphi=\prod_{i} \varphi_{i}$ and $\psi=\prod_{i} \psi_{i}$. Hence the subgroup $H$ generated by $\varphi$ and $\psi$ is a rank 2 free group. Moreover the action of $H$ on the set $X_{\gamma}-\{\gamma\}$ has trivial stabilizers. Otherwise, if an element $\phi$ in $H$ fix $\delta \in X_{\gamma}-\{\gamma\}$, then by the construction of $H$, the geometric intersection number of $\delta$ and $\gamma$ is nonzero and thus it intersects one of $T_{i}$. We cut $S$ along $\gamma$ so that $\delta$ becomes a family of isotopy classes of arcs. Since $\phi$ fixes $\delta$, up to some powers of $\phi$, it fixes each resulting isotopy class of arcs. But then it can be shown that, for some $i$, there is an element in $H_{i}$ that fixes the isotopy class of an essential simple closed curve, which contradicts the assumption that $H_{i}$ is a pseudo-Anosov subgroup. By Lemma 1.4.4, we can conclude that $G_{\gamma}$ has the spectral gap property.

Lemma 1.5.3 (Theorem A in [7]). Let $G$ be a countable discrete group and $H$ be $a$ subgroup of $G$ that has the spectral gap property. Let $L$ be a subgroup of $G$ satisfying
$\operatorname{Com}_{G}(L)=L$, then two unitary representations $\ell^{2}(G / H)$ and $\ell^{2}(G / L)$ of $G$ are weakly equivalent if and only if $L$ is conjugate to $H$.

Theorem 1.5.4. Let $S=S_{g, n}$ be a compact surface with $3 g+n \geq 4$. Let $\gamma$ and $\delta$ be two 1-multi-curves on $S$ with $k, l$ geometric components, respectively.
(1) If at least one of $k, l$ is not $3 g-3+n$, then the associated unitary representations $\pi_{\gamma}$ and $\pi_{\delta}$ are weakly equivalent if and only if $\gamma$ and $\delta$ are of the same type.
(2) Suppose $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If the number of geometric components of $\gamma$ is $3 g-3+n$, then $\pi_{\gamma}$ is weakly equivalent to the regular representation $\lambda_{S}$.
(3) Suppose $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$. If the number of geometric components of $\gamma$ is not $3 g-3+n$, then $\pi_{\gamma}$ is not weakly contained in $\lambda_{S}$.

Proof. For any 1-multi-curve $\gamma$ on $S, \operatorname{Com}_{\operatorname{Mod}(S)}\left(G_{\gamma}\right)=G_{\gamma}$ (see [51]). Given two 1 -multi-curves $\gamma$ and $\delta$ with $k, l$ geometric components, respectively, such that at least one of $k$ and $l$ is not $3 g-3+n$, then by Lemma 1.5.2, Lemma 1.5.3 and the fact that $G_{\gamma}$ is conjugate to $G_{\delta}$ if and only if $\gamma$ and $\delta$ are of the same type, we complete the proof for (1). For (2), by [15], if $S$ is not $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$, the mapping class group $\operatorname{Mod}(S)$ is $\mathrm{C}^{*}$-simple. By the result of [14] which states that a discrete group is $\mathrm{C}^{*}$-simple if and only if, for any amenable subgroup $M$ of $G$, the quasi-regular representation $\ell^{2}(G / M)$ is weakly equivalent to the regular one. So combine with Lemma 1.5.2, we complete the proof of (2). The statement (3) is deduced from (2) and the definition of $\mathrm{C}^{*}$-simplicity.

Remark 1.5.1. The "only if" part of (1) is a stronger version of Corollary 5.5 in [51]. Remark 1.5.2. If $S$ is one of $S_{0,4}, S_{1,1}, S_{1,2}, S_{2,0}$, it is easy to show that, if the number of components of $\gamma$ is $3 g-3+n$, then $\pi_{\gamma}$ is weakly contained in the regular representation
$\lambda_{S}$. However, for other types of $\gamma$, we don't know if $\pi_{\gamma}$ is weakly contained in $\lambda_{S}$. And we don't know what can be said about unitary representations corresponding to non-discrete measures on the space of measured foliations.

## Chapter 2

## Boundary representations of mapping class groups

### 2.1 Introduction

Let $S=S_{g}$ be a closed, connected, orientable surface of genus $g$. Recall that the mapping class group $\operatorname{Mod}(S)$ of $S$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of $S$. Throughout this chapter, the genus $g$ is assumed to be at least 2. The space of measured foliations $\operatorname{M\mathcal {F}}(S)$ is the set of equivalence classes of non-zero measured foliations on $S$. The mapping class group $\operatorname{Mod}(S)$ acts on $\mathcal{M} \mathcal{F}(S)$ and preserves a Radon measure $\nu$, called the Thurston measure on $\mathcal{M \mathcal { F }}(S)$. Moreover, the space $\mathcal{M F}(S)$ is equipped with an $\mathbb{R}_{+}$-action that commutes with the $\operatorname{Mod}(S)$-action. Therefore, $\operatorname{Mod}(S)$ acts on the quotient $\mathcal{P} \mathcal{M F}(S)$, called the projective measured foliation space, of $\mathcal{M} \mathcal{F}(S)$ by $\mathbb{R}_{+}$preserving a measure class $[\nu]$, called the Thurston measure (class) on $\mathcal{P} \mathcal{M F}(S)$, defined by the Thurston measure on $\mathcal{M} \mathcal{F}(S)$.

One motivation of this thesis is to use geometric objects, such as $\mathcal{M \mathcal { F }}(S)$ and $\mathcal{P} \mathcal{M F}(S)$ to understand unitary representations of $\operatorname{Mod}(S)$ (see also the first chapter for related topics). Recall that, for a probability measure class-preserving action of $G$ on $(X,[\nu])$, one defines a unitary representation of $G$ on $L^{2}(X, \nu)$, called a quasiregular representation (see Section 2.2.1 for more details and the reader should not confuse this terminology with the one in the first chapter). Hence, for a probability measure class-preserving ergodic action, it is natural to ask that whether the quasi-regular representation is irreducible. Notice that this is not true for a measurepreserving ergodic action as it always has $\mathbb{C}_{X}$ as a nontrivial closed invariant subspace. For the ergodic action of $\operatorname{Mod}(S)$ on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ with respect to [ $\nu$ ], we prove:

Theorem 2.1.1 (See Corollary 2.2.15). Let $S=S_{g}$ be a closed surface of genus $g \geq 2$. The quasi-regular unitary representation of the mapping class group $\operatorname{Mod}(S)$ on $L^{2}(\mathcal{P M} \mathcal{F}(S), \nu)$, the space of square integrable functions on $\mathcal{P M \mathcal { F }}(S)$ with respect to the Thurston measure $\nu$, is irreducible.

In fact, we prove an ergodic-type theorem, namely Theorem 2.2.14, for this quasiregular representation and as a by-product of this ergodic-type theorem, we have the classical result of Masur [41] on ergodicity of the action (see Remark 2.2.1 for the relation between ergocidity of an action and the associated quasi-regular representation). However, since our work uses Masur's result implicitly, so we don't give a new proof to his result.

The main theorem is related to a question of Bader-Muchnik in the context of random walks on groups. Namely, let $G$ be a discrete group and $\mu$ be a probability measure on $G$. Let $(\partial G, \nu)$ be the Poisson boundary of $G$ associated to the $\mu$-random walk on $G$. Then the measure class $[\nu]$ is $G$-invariant, hence defines a quasi-regular representation of $G$ on $L^{2}(\partial G, \nu)$. In [5], inspired by the cases of free groups and lattices in Lie groups, Bader-Muchnik proposed the following conjecture:

Conjecture 2.1.2 (Bader-Muchnik[5]). For a locally compact group $G$ and a spreadout probability measure $\mu$ on $G$, the quasi-regular representation associated to the $\mu$-Poisson boundary of $G$ is irreducible.

Before returning to mapping class groups, we first mention briefly some progress on this conjecture. As mentioned above, this conjecture is true for certain random walks on free groups and lattices in Lie groups (see [5] and references therein). Hence it is true for the mapping class group $\operatorname{Mod}(S)=S L(2, \mathbb{Z})$ of closed surface of genus one acting on $\mathcal{P} \mathcal{M} \mathcal{F}(S)=S^{1}$ with respect to the Lebesgue measure which is also identified
with the Thurston measure on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. All identifications are $\operatorname{Mod}(S)$-equvariant. For lattices in Lie groups, one can also deduce the irreducibility from ergodicity of the associated quasi-regular representation (see [12]). The conjecture is then verified in [5] for the fundamental group of compact negatively curved manifolds with respect to the Patterson-Sullivan measure by Bader-Muchnik. Their result has been further generalized to hyperbolic groups [22] with respect to the Patterson-Sullivan measure by Garncarek and some discrete subgroups of the group of isometries of a $C A T(-1)$ space with non-arithmetic spectrum by Boyer [13]. Note that in all cases above, the Patterson-Sullivan measure on the Gromov boundary coincides with the Poisson boundary of $(G, \mu)$ for some probability measure $\mu$ on $G$. However, Björklund-Hartman-Oppelmayer [9] recently showed that there are random walks on some Lamplighter groups and solvable Baumslag-Solitar groups providing counterexamples to this conjecture.

The relationship between the main theorem and above progress is the following. On the one hand, there is a long history on exploiting similarities between mapping class groups and hyperbolic groups which is quite fruitful. To name very few among massive literatures, we mention [44], [43] and [27]. On the other hand, by [4], the Thurston measure on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ is the Patterson-Sullivan measure on the Teichmüller boundary of the Teichmüller space of $S$ which is in the similar situation with the previous
 the Thurston measure class is the $\mu$-boundary of some random walk on $\operatorname{Mod}(S)$.

Outline of the proof. By regarding $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ as the Teichmüller boundary of the Teichmüller space Teich $(S)$ of $S$, we follow the approach in Boyer-Pittet-Link[12]: irreducibility of a representation is deduced from the ergodicity of the representation
(see Section 2.2.1 for definitions), namely Theorem 2.2.14. To prove Theorem 2.2.14, we adapt a criterion (Theorem 2.2.13) in [12]. The bulk of the work is to construct a family of finite subsets of $\operatorname{Mod}(S)$ and show that they satisfy the conditions in Theorem 2.2.13. The family of subsets is constructed by carefully choosing elements in $\operatorname{Mod}(S)$ with enough hyperbolicity so that the cardinality of these subsets goes to infinity. Actually, we even need the growth to be exponential. The subsets are described before Lemma 2.2.11 relying on Dowdall-Duchin-Masur [16]. Then by results on counting lattices in Athreya-Bufetov-Eskin-Mirzakhani [4], one can show that these sets have exponential growth. Then the next step is to verify the convergence and uniform boundedness in Theorem 2.2.13. As the action of $\operatorname{Mod}(S)$ on $\operatorname{Teich}(S)$ is not homogeneous, the approach in Boyer-Pittet-Link [12] no longer works. We use ideas in Bader-Muchnik [5], Boyer [13]: we first prove the Harish-Chandra estimates in Section 2.4 as in [13] and then use it to deduce convergence and uniform boundedness (Section 2.5.1). Note that the proof of convergence in [12] is purely Lie theoretic and our proof for mapping class groups is based on the Harish-Chandra estimates. The proof of the Harish-Chandra estimates (Section 2.4) is the novelty in this paper. Instead of doing estimations directly, we first relate it to integration on intersection numbers and then use the map considered in Masur-Minsky [43] which relates Teich $(S)$ to the curve complex of $S$ to simplify integrations.

Erratum. The last part of the proof of Lemma 2.3.5 is not correct since by our choice of $E_{n}$, [ $\zeta$ ] might be non uniquely ergodic. But this lemma is correct and Tiozzo-Yang has a proof for it. We also point out that Duchin's proof of Theorem 2.2.7 is unpulished, therefore our proof of Lemma 2.5.4 is not complete.

### 2.2 Quasi-regular unitary representations

### 2.2.1 Quasi-regular representations of discrete groups.

In this section, we will recall ergodic quasi-regular representations and a criterion for showing ergodicity of representations. The reader is referred to [5],[13] and [12] for more details.

Quasi-regular unitary representations. Let $G$ be a locally compact secondcountable group and $X$ be a second-countable Hausdorff topological space. Let $\nu$ be a probability Borel measure on $X$. Assume that $G$ acts on $X$ as homeomorphisms and $G$ preserves the measure class $[\nu]$ of $\nu$, namely, $G$ preserves null $\nu$-measure sets . Choose $\nu \in[\nu]$, thus for every $\gamma \in G$, the measure $\gamma_{*} \nu$ is absolutely continuous with respect to $\nu$ and $\nu$ is absolutely continuous with respect to $\gamma_{*} \nu$. Denote the corresponding Radon-Nikodym derivative by $c(\gamma, \nu)=\frac{d \gamma_{*} \nu}{d \nu}$. One can construct a unitary representation $\pi_{\nu}$ of $G$ on $L^{2}(X, \nu)$ as follows: For every $f \in L^{2}(X, \nu)$, every $x \in X$ and every $\gamma \in G, \pi_{\nu}(\gamma) f(x)$ is defined to be $\pi_{\nu}(\gamma) f(x)=f\left(\gamma^{-1} x\right) c(\gamma, \nu)^{\frac{1}{2}}(x)$. The representation $\pi_{\nu}$ will be called a quasi-regular (unitary) representation of $G$. We remark that if $\nu, \mu$ are in the same measure class, then $\pi_{\nu}, \pi_{\mu}$ are unitary equivalent. Assume that $c(\gamma, \nu)^{\frac{1}{2}}$ is integrable for each $\gamma \in G$ with respect to $\nu$. The HarishChandra function $\Phi$ associated to $\pi_{\nu}$ is then defined to be the integral

$$
\Phi(\gamma)=\left\langle\pi_{\nu}(\gamma) \mathbb{1}_{X}, \mathbb{1}_{X}\right\rangle_{L^{2}(X, \nu)}=\int_{X} c(\gamma, \nu)^{\frac{1}{2}}(x) d \nu(x) .
$$

Ergodic quasi-regular representations. From now on, we always assume that $G$ is a discrete group. Let $(X, \nu), \pi_{\nu}$ as above and $\mathcal{B}\left(L^{2}(X, \nu)\right)$ be the Banach space of bounded operators on $L^{2}(X, \nu)$. Let $e_{K}: K \longrightarrow X$ be a map from a finite subset $K$ of $G$ to $X$ and $f: X \longrightarrow \mathbb{C}$ be a bounded Borel function. Consider the following
elements in $\mathcal{B}\left(L^{2}(X, \nu)\right)$ :

$$
\begin{aligned}
& M_{\left(K, e_{K}\right)}^{f}: L^{2}(X, \nu) \longrightarrow L^{2}(X, \nu), \phi \mapsto \frac{1}{|K|} \sum_{\gamma \in K} f\left(e_{K}(g)\right) \frac{\pi_{\nu}(\gamma) \phi}{\Phi(\gamma)} \\
& P_{\mathbb{1}_{X}}: L^{2}(X, \nu) \longrightarrow L^{2}(X, \nu), \phi \mapsto \int_{X} \phi d \nu \mathbb{1}_{X} \\
& m(f): L^{2}(X, \nu) \longrightarrow L^{2}(X, \nu), \phi \mapsto f \phi
\end{aligned}
$$

We now introduce an ergodicity for quasi-regular representations which generalizes the usual ergodicity for measure class-preserving group actions. Recall that a sequence $F_{n} \in \mathcal{B}\left(L^{2}(X, \nu)\right)$ converges to $F \in \mathcal{B}\left(L^{2}(X, \nu)\right)$, written as $F_{n} \rightarrow F$, in the weak operator topology if, for every $\left.\phi, \psi \in L^{2}(X, \nu), \lim _{n \rightarrow \infty}\left\langle F_{n}(\phi), \psi\right\rangle_{L^{2}}=<F(\phi), \psi\right\rangle_{L^{2}}$.

Definition 2.2.1 (Boyer-Link-Pittet[12]). Let $G,(X, \nu), \pi_{\nu}, f$ as above. Suppose that for every $n \in \mathbb{N}$, there is a pair $\left(K_{n}, e_{n}: K_{n} \longrightarrow X\right)$ such that $K_{n}$ is a finite subset of $G$ and such that $\left|K_{n}\right| \rightarrow \infty$ as $n \rightarrow \infty$. The representation $\pi_{\nu}$ is called ergodic with respect to $\left(K_{n}, e_{n}\right)$ and $f$, if we have the following convergence in the weak operator topology:

$$
M_{\left(K_{n}, e_{n}\right)}^{f} \rightarrow m(f) P_{1_{X}}
$$

Remark 2.2.1. It is easy to see that the ergodicity of a measure class-preserving group action is weaker than the ergodicity of the associated quasi-regular representation. One could refer to [[12], Proposition 2.5] for its proof.

The following criterion for the ergodicity of a quasi-regular representation is essentially contained in [5] and summarized in [12].

Theorem 2.2.1 ([12] Theorem 2.2). Let $G,(X, \nu)$ as above and $\pi_{\nu}$ be the associated quasi-regular representation of $G$ on $L^{2}(X, \nu)$. Let $L$ be a length function on $G$ and let $(X, d)$ be a metric space inducing the topology of $X$. For every $n \in \mathbb{N}$, let $E_{n}$ be a symmetric finite subset of $G$, that is $E_{n}=E_{n}^{-1}$, and $e_{n}: E_{n} \longrightarrow X$ be a map. Assume that the following conditions hold:
(1) for every $g \in G,\left\|\pi_{\nu}(g) \mathbb{1}_{X}\right\|_{L^{\infty}(X, \nu)}<\infty$,
(2) $\lim _{n \rightarrow \infty}\left|E_{n}\right|=\infty$,
(3) for all Borel subsets $W, V \subset X$ such that $\nu(\partial W)=\nu(\partial V)=0$,

$$
\left.\left.\limsup _{n \rightarrow \infty} \frac{1}{\left|E_{n}\right|} \right\rvert\,\left\{\gamma \in E_{n}: e_{n}\left(\gamma^{-1}\right) \in W \text { and } e_{n}(\gamma) \in V\right\} \right\rvert\, \leq \nu(W) \nu(V),
$$

(4) for every $r \geq 0$, there is a non-increasing function $h_{r}:[0, \infty) \longrightarrow[0, \infty)$ such that $\lim _{s \rightarrow \infty} h_{r}(s)=0$ and such that

$$
\forall n \in \mathbb{N}, \forall \gamma \in E_{n}, \frac{\left\langle\pi_{\nu}(\gamma) \mathbb{1}_{X}, \mathbb{1}_{\left\{x \in X: d\left(x, e_{n}(\gamma)\right) \geq r\right\}}\right\rangle_{L^{2}}}{\Phi(\gamma)} \leq h_{r}(L(\gamma)),
$$

$$
\begin{equation*}
\sup _{n}\left\|M_{E_{n}}^{1 \times} \mathbb{1}_{X}\right\|_{L^{\infty}(X, \nu)}<\infty . \tag{5}
\end{equation*}
$$

Then the quasi-regular representation $\pi_{\nu}$ is ergodic with respect to $\left(E_{n}, e_{n}\right)$ and any $f \in \bar{H}^{L^{\infty}(X, \nu)}$, where $H$ is a vector space generated by

$$
\left\{\mathbb{1}_{U}: \nu(\partial U)=0 \text { and } U \text { is a Borel subset of } X\right\} \text {. }
$$

Remark 2.2.2. Thanks to condition (1), the Harish-Chandra function exists for each $\gamma$ in $G$.

Proposition 2.2.2 (Bader-Muchnik[5]). Under the assumptions in the above theorem, if moreover $\nu$ is a Radon measure, then $\pi_{\nu}$ is irreducible.

### 2.2.2 Quasi-regular representations of mapping class groups

We now consider quasi-regular representations of mapping class groups and state our main theorem. For more on mapping class groups and Teichmüller theory, we refer
to [19], [4] and [32].

Mapping class groups and Teichmüller spaces. Let $S=S_{g}$ be a genus $g$, closed, connected, orientable surface. We always assume that $g \geq 2$. All arguments here work for hyperbolic surfaces with punctures as well. The mapping class group $\operatorname{Mod}(S)$ of $S$ is the group of isotopy classes of orientation-preserving homeomorphisms of $S$. Namely, if the group of orientation-preserving homeomorphisms of $S$ is denoted by $\mathrm{Homeo}^{+}(S)$ and the group of homeomorphisms of $S$ that isotopic to the identity is denoted by $\mathrm{Homeo}_{0}(S)$, then

$$
\operatorname{Mod}(S)=\operatorname{Homeo}^{+}(S) / \operatorname{Homeo}_{0}(S)
$$

We remark here that mapping class groups of surfaces are finitely presented and considered to be discrete groups. The Teichmüller space $\operatorname{Teich}(S)$ of $S$ is the space of homotopy classes of hyperbolic structures. The Teichmüller space $\operatorname{Teich}(S)$ is homeomorphic to $\mathbb{R}^{6 g-6}$ and the mapping class group $\operatorname{Mod}(S)$ acts on Teich $(S)$ by changing markings. The quotient $\mathcal{M}(S)=\operatorname{Teich}(S) / \operatorname{Mod}(S)$ is the moduli space of $S$. There are several distances on $\operatorname{Teich}(S)$ so that $\operatorname{Mod}(S)$ acts as isometries, the one that we will use is the Teichmüller distance $d=d_{T}$. It is defined as follows: For $\mathcal{X}=[(X, \phi)], \mathcal{Y}=[(Y, \psi)] \in \operatorname{Teich}(S), d(\mathcal{X}, \mathcal{Y})=\frac{1}{2} \log K_{f}$, where $f: X \longrightarrow Y$ is the Teichmüller mapping, locally in the form of $x+i y \mapsto e^{t} x+i e^{-t} y$, in the isotopy class of $\psi \circ \phi^{-1}$, namely the quasi-conformal homeomorphism with minimal dilatation in the isotopy class of $\psi \circ \phi^{-1}$ and $K_{f}$ is the dilatation of $f$. It is obvious that $\operatorname{Mod}(S) \subset \operatorname{Isom}(\operatorname{Teich}(S), d)$. Neither the Teichmüller space $\operatorname{Teich}(S)$ nor $\mathcal{M}(S)$ is compact.

Measured foliations. The Teichmüller space can be compactified in several ways. The compactification we will use in this paper is the Teichmüller compactification. Fix
a point $o \in \operatorname{Teich}(S)$ that is considered to be a Riemann surface $X$ via uniformization. A holomorphic quadratic differential $q \in \mathrm{H}^{0}\left(\mathrm{X}, \Omega_{\mathrm{X}}^{\otimes 2}\right)$ on $X$ is locally of the form $q(z) d z^{2}$ such that $q(z)$ is a holomorphic function. Define a norm on $q$ by

$$
\|q\|=\int_{X}|q(z)| d x d y
$$

and consider the unit open ball $B^{1}(X)$ with respect to $\|\cdot\|$. The set $Q D(X)$ of holomorphic quadratic differentials is a vector space and can be identified with the cotangent space of $\operatorname{Teich}(S)$ at $o$. There is a homeomorphism $\pi: B^{1}(X) \longrightarrow \operatorname{Teich}(S)$ sending each open unit ray in $Q D(X)$ starting at the origin to a Teichmüller geodesic starting at $o$. The Teichmüller compactification is then the visual compactification by adding ending points in the unit sphere of $Q D(X)$ to each ray. The Teichmüller compactification will be denoted by $\overline{\operatorname{Teich}(S)}$. Thus, the boundary $\partial \overline{\operatorname{Teich}(S)}$ of $\overline{\operatorname{Teich}(S)}$ is the unit sphere $Q D^{1}(X)$.

One could give a geometric description of $\partial \overline{\mathrm{Teich}(S)}$ via projective measured foliations. A measured foliation on $S$ is a singular foliation of $S$ endowed with a transverse measure. The space $\mathcal{M} \mathcal{F}(S)$ of measured foliations is then the set of equivalent classes of measured foliations where the equivalence is given by Whitehead moves and isotopy. The space $\mathcal{M} \mathcal{F}(S)$, endowed with the weak topology on measures, is homeomorphic to $\mathbb{R}^{6 g-6}$. The quotient, called the projective measured foliation space $\mathcal{P} \mathcal{M F}(S)$ of $S$, of $\mathcal{M \mathcal { F }}(S)$ by the nature action of $\mathbb{R}_{+}$is homeomorphic to the $6 g-7$ sphere $S^{6 g-7}$. Both $\mathcal{M F}(S)$ and $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ are equipped with a $\operatorname{Mod}(S)$-action. There is a deep relation between $\mathcal{M} \mathcal{F}(S)$ and $Q D(X)$. Namely, for each holomorphic quadratic differential $q$, the vertical measured foliation $\mathcal{V}(q)$ of $q=q(z) d z^{2}$ is the foliation given by the integral curves of the holomorphic tangent vector field on $S$
such that each vector has a value in negative real numbers under $q$, where the transverse measure is given by integration of $|R e \sqrt{q}|$. By a theorem of Hubbard-Masur, the map $\mathcal{V}$ that assigns each holomorphic quadratic differential $q$ on $X$ to $\mathcal{V}(q)$ is a homeomorphism from $Q D(X)-\{0\}$ onto $\mathcal{M F}(S)$. The composition $\pi \circ \mathcal{V}$ of the $\operatorname{map} \mathcal{V}: Q D(X)-\{0\} \longrightarrow \mathcal{M} \mathcal{F}(S)$ and the quotient map $\pi: \mathcal{M} \mathcal{F}(S) \longrightarrow \mathcal{P} \mathcal{M} \mathcal{F}(S)$ gives the identification of $Q D^{1}(X)$ with $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. Thus, we will regard $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ as the boundary of the Teichmüller compactification of $\operatorname{Teich}(S)$. The equivalent class of $\xi \in \mathcal{M} \mathcal{F}(S)$ in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ will be denoted by $[\xi]$. Any $q \in Q D^{1}(X)$ (hence $[\mathcal{V}(q)] \in \mathcal{P} \mathcal{M} \mathcal{F}(S))$ determines a Teichmüller geodesic ray $g_{t}$ starting from $o$, hence, by abuse of terminologies, we will call $q$ and $[\mathcal{V}(q)]$ the direction of $g_{t}$ and sometimes write $g_{t}$ as $g_{t}^{q}$ or $\mathcal{V}(q)(t)$.

Any isotopy class $\gamma$ of essential simple closed curves on $S$ defines a (topological) foliation $\lambda(\gamma)$. Hence, any weighted isotopy class $c \gamma$ of essential simple closed curves defines a foliation $\lambda \in \mathcal{M \mathcal { F }}(S)$. The measured foliation $\lambda$, as topological foliation, is the same as $\lambda(\gamma)$, but the transverse measure is given by $c$. Therefore, let $\mathcal{C}(S)$ denote the set of isotopy classes of essential simple closed curves, there is an embedding of $\mathcal{C}(S) \times \mathbb{R}_{+}$into $\mathcal{M} \mathcal{F}(S)$. The image is dense (See Thurston [61]). This embedding enable us to define three functions that we will use. The first one is the intersection number on $\mathcal{M \mathcal { F }}(S)$. The intersection number $i: \mathcal{M \mathcal { F }}(S) \times \mathcal{M \mathcal { F }}(S) \longrightarrow \mathbb{R}_{+}$is the unique continuous function on $\mathcal{M F}(S) \times \mathcal{M} \mathcal{F}(S)$ that extends the geometric intersection number of two essential simple closed curves and satisfies $i(c \lambda, \xi)=c i(\lambda, \xi)$ for every $c>0$ (See [54], Corollary 1.11). The second one is the extremal length. Let $o=[(X, \phi)] \in \operatorname{Teich}(S)$ where $X$ is a Riemann surface. Let $\gamma$ be the isotopy class of an essential simple closed curve. The extremal length $\operatorname{Ext}_{X}(\gamma)$ of $\gamma$ in $X$ is defined
to be

$$
\operatorname{Ext}_{X}(\gamma)=\sup _{\rho} \ell_{\rho}(\gamma)^{2},
$$

where $\rho$ runs over all metrics with unit area in the conformal class of $X$ and $\ell_{\rho}(\gamma)$ is the infimum of $\rho$-length of simple closed curves in $\gamma$. Then the extremal length $\operatorname{Ext}_{X}: \mathcal{M} \mathcal{F}(S) \longrightarrow \mathbb{R}_{+}$is the unique continuous function on $\mathcal{M} \mathcal{F}(S)$ that extends the extremal length of $\mathcal{C}(S)$ and satisfies $\operatorname{Ext}_{X}(c \lambda)=c^{2} \operatorname{Ext}_{X}(\lambda)$ for $c \in \mathbb{R}_{+}$(See [32], Proposition 3). Note that the extremal length in fact is defined on Teich $(S) \times \mathcal{M} \mathcal{F}(S)$, namely, if $[(X, \phi)]=[(Y, \psi)] \in \operatorname{Teich}(S)$, then $\operatorname{Ext}_{X}(\cdot)=\operatorname{Ext}_{Y}(\cdot)$. So we will write $\operatorname{Ext}_{o}(\cdot)$ rather than $\operatorname{Ext}_{X}(\cdot)$ for $o=[(X, \phi)]$. The third one is the hyperbolic length $\ell_{o}(\gamma)$ which is defined to be the $X$-length of unique $X$-hyperbolic geodesic $\tilde{\gamma}$ in the isotopy class $\gamma$. The function $\ell_{o}(\cdot)$ can be uniquely extended as well to $\mathcal{M} \mathcal{F}(S)$ to obtain a continuous function $\ell_{o}$ on $\mathcal{M F}(S)$ [33]. We will use the following relation: given a point $o$ in Teich $(S)$, then there exists a constant $C=C(o)$, depending on $o$, such that

$$
\forall \xi \in \mathcal{M F}(S), \frac{1}{C} \ell_{o}(\xi) \leq \sqrt{\operatorname{Ext}_{o}(\xi)} \leq C \ell_{o}(\xi)
$$

Recall that a measured foliation $\lambda$ is called minimal if it has no simple closed leaves. Two measured foliations are said to be topologically equivalent if they, as topological foliations, are differ by isotopies and Whitehead moves. A measured foliation $\xi$ is called uniquely ergodic if it is minimal and any measured foliation $\zeta$ that topologically equivalent to $\xi$ is measure equivalent to $\xi$, that is, $[\xi]=[\zeta]$. When $\xi$ is uniquely ergodic, we will call [ $\xi]$ uniquely ergodic. The following two lemmas are essential to our approach using intersection numbers.

Lemma 2.2.3. (Rees [54], Theorem 1.12 or Masur [40]) Let $\lambda$ be a uniquely ergodic
measured foliation and $\eta$ be any measured foliation. Then $i(\lambda, \eta)=0$ if and only if $[\lambda]=[\eta]$.

Lemma 2.2.4 (Masur's criterion [42]). Given $\epsilon>0$. If a Teichmüller geodesic ray $g_{t}$ starting from o does not leave $\operatorname{Teich}_{\epsilon}(S)$ eventually, then the direction of $g_{t}$ is uniquely ergodic.

One feature of the Teichmüller compactification is that the action of $\operatorname{Mod}(S)$ cannot be extended continuously to $\overline{\operatorname{Teich}(S)}$ [32]. However, uniquely ergodic measured foliations are nice points in terms of $\operatorname{Mod}(S)$-action in the following sense:

Lemma 2.2.5. (Masur [40]) The mapping class group acts continuously on $\overline{\operatorname{Teich}(S)}$ at uniquely ergodic points on the boundary.

The following Kerckhoff's formula concerning the calculation of Teichmüller distances will be frequently used.

## Lemma 2.2.6 (Kerckhoff [32]).

$$
\forall x, y \in \operatorname{Teich}(S), d_{T}(x, y)=\frac{1}{2} \sup _{[\xi] \in \mathcal{P} \mathcal{M} \mathcal{F}(S)} \ln \left(\frac{\operatorname{Ext}_{x}(\xi)}{\operatorname{Ext}_{y}(\xi)}\right) .
$$

Hyperbolicity. It was first proved in [44] that the Teichmüller space $\left(\operatorname{Teich}(S), d_{T}\right)$ is not hyperbolic in the sense of Gromov. However some triangles in $\left(\operatorname{Teich}(S), d_{T}\right)$ are indeed thin. We now collect several related results in order to compare neighborhoods in $\mathcal{P \mathcal { M } \mathcal { F }}(S)$ defined by projections of balls in Teich $(S)$ and the ones defined by intersection numbers.

The first result describes triangles with vertices in a thick part of Teich $(S)$. Recall that, for $\epsilon>0$, the $\epsilon$-thick part $\operatorname{Teich}_{\epsilon}(S)$ of the Teichmüller space $\operatorname{Teich}(S)$ is defined to be

$$
\operatorname{Teich}_{\epsilon}(S)=\left\{y \in \operatorname{Teich}(S): \forall c \in \mathcal{C}(S), \operatorname{Ext}_{y}(c) \geq \epsilon\right\}
$$

Theorem 2.2.7 (Duchin [17]). Let $\epsilon>0$ and $\operatorname{Teich}_{\epsilon}(S)$ be the $\epsilon$-part of $\operatorname{Teich}(S)$. Then there exists $M_{0}=M_{0}(\epsilon)>0$ and $k=k(\epsilon)>0$ such that, for every triangle $\triangle(x, y, z)$ with vertices $x, y, z$ in $\operatorname{Teich}_{\epsilon}(S)$ and side lengths $a=d_{T}(x, y), b=$ $d_{T}(y, z), c=d_{T}(x, z)$ all at least $M_{0}$ and for every $\rho>0$, one has $d \leq k a \rho$, whenever $a+b-c \leq a \rho$, where $w \in T(S)$ be the unique point in the geodesic $[x, z]$ such that $d_{T}(x, w)=d_{T}(x, y)$ and $d=d_{T}(y, w)$.

The next result, generalizing a theorem of Rafi [53], also describes when triangles are thin. We denote $\mathcal{N}_{D}(A)$ for a subset $A$ of $\operatorname{Teich}(S)$ by the $D$-neighborhood of A. Recall that a geodesic segment $I:[a, b] \rightarrow \operatorname{Teich}(S)$ has at least proportion $\theta$ in $\operatorname{Teich}_{\epsilon}(S)$ if

$$
\operatorname{Thk}_{\epsilon}^{\%}[I] \doteq \frac{\left|\left\{a \leq s \leq b: I(s) \in \operatorname{Teich}_{\epsilon}(S)\right\}\right|}{b-a} \geq \theta
$$

Theorem 2.2.8 (Dowdall-Duchin-Masur[16]). Given $\epsilon>0$ and $0<\theta \leq 1$, there exist constants $D=D(\epsilon, \theta), L_{0}=L_{0}(\epsilon, \theta)$ such that if $I \subset[x, y]$ is a geodesic subinterval in $\operatorname{Teich}(S)$ of length at least $L_{0}$ and at least proportion $\theta$ of $I$ is in $\operatorname{Teich}_{\epsilon}(S)$, then for every $z \in \operatorname{Teich}(S)$, we have

$$
I \cap \mathcal{N}_{D}([x, z] \cup[y, z]) \neq \emptyset .
$$

The following result will also be used later. Recall that two parametrized geodesics segment $\delta(t)$ and $\delta^{\prime}(t)$ defined on $[a, b]$ are said to $P$-fellow travel in a parametrized fashion if, for every $t \in[a, b], d_{T}\left(\delta(t), \delta^{\prime}(t)\right) \leq P$.

Theorem 2.2.9 (Rafi [53]). Let $\epsilon>0$. Then there exists $P=P(\epsilon)>0$ such that whenever $x_{1}, x_{2}, y_{1}, y_{2}$ are in $\operatorname{Teich}_{\epsilon}(S)$ with

$$
d_{T}\left(x_{1}, x_{2}\right) \leq 1, d_{T}\left(y_{1}, y_{2}\right) \leq 1,
$$

the geodesic segment $\left[x_{1}, y_{1}\right]$ and $\left[x_{2}, y_{2}\right]$ are $P$-fellow travelling.

Boundary representations of mapping class groups. We are in a position to discuss a special type of quasi-regular unitary representations of mapping class groups. Fix $o \in \operatorname{Teich}(S)$, we first define a Radon measure $\nu_{o}$ on $\mathcal{P M} \mathcal{F}(S)$. Let $\nu_{T h}$ be the Thurston measure on $\mathcal{M} \mathcal{F}(S)$. For any open subset $U \subset \mathcal{P} \mathcal{M} \mathcal{F}(S)$, one defines $\nu_{o}(U)$ to be

$$
\nu_{o}(U)=\nu_{T h}\left(\left\{\xi:[\xi] \in U, \operatorname{Ext}_{o}(\xi) \leq 1\right\}\right)
$$

One could verify that $\forall \gamma \in \operatorname{Mod}(S), \gamma_{*} \nu_{o}=\nu_{\gamma . o}$ and $\left[\nu_{x}\right]=\left[\nu_{y}\right], \forall x, y \in \operatorname{Teich}(S)$. Therefore, one has

$$
\forall x, y \in \operatorname{Teich}(S),[\xi] \in \mathcal{P} \mathcal{M} \mathcal{F}(S), \frac{d \nu_{x}}{d \nu_{y}}([\xi])=\left(\frac{\operatorname{Ext}_{y}(\xi)}{\operatorname{Ext}_{x}(\xi)}\right)^{\frac{6 g-6}{2}}
$$

By the definition of extremal length, the function $[\xi] \mapsto\left(\frac{\operatorname{Ext}_{y}(\xi)}{\operatorname{Ext}_{x}(\xi)}\right)^{\frac{6 g-6}{2}}$ is well-defined on $\mathcal{P M} \mathcal{F}(S)$. We have, in particular,

$$
\forall \gamma \in \operatorname{Mod}(S),[\xi] \in \mathcal{P} \mathcal{M} \mathcal{F}(S), \frac{d \gamma_{*} \nu_{o}}{d \nu_{o}}([\xi])=\left(\frac{\operatorname{Ext}_{o}(\xi)}{\operatorname{Ext}_{\gamma . o}(\xi)}\right)^{\frac{6 g-6}{2}}
$$

Hence one has a quasi-regular unitary representation $\pi_{\nu_{o}}$ of $\operatorname{Mod}(S)$ on the Hilbert space $L^{2}\left(\mathcal{P} \mathcal{M} \mathcal{F}(S), \nu_{o}\right)$. The quasi-regular representation $\pi_{\nu_{o}}$ of $\operatorname{Mod}(S)$ is called the boundary representation of $\operatorname{Mod}(S)$ (with respect to $o$ ).

As intersection numbers will be the main tool, we embed $\mathcal{P} \mathcal{M F}(S)$ into $\mathcal{M F}(S)$. For each $[\xi]$, define $\tau(\xi) \in \mathcal{M} \mathcal{F}(S)$ to be the unique element in $[\xi]$ such that $\operatorname{Ext}_{o}(\tau(\xi))=$ 1. Hence, the map $\tau: \mathcal{P} \mathcal{M} \mathcal{F}(S) \longrightarrow \mathcal{M} \mathcal{F}(S)$ is a section of the projection $\pi$ : $\mathcal{M F}(S) \longrightarrow \mathcal{P} \mathcal{M F}(S)$. When talking about intersection numbers for two points in $\mathcal{P} \mathcal{M F}(S)$, we will always use the image of $\tau$.

Ergodic boundary representation. From now on, let $S=S_{g}(g \geq 2)$ be a genus $g$ closed, orientable surface and fix a point $o=[(X, \phi)] \in \operatorname{Teich}(S)$. Normalize
$\nu_{o}$ to be a probability measure. Denote $h=6 g-6$ and let $\epsilon>0$ and $\theta>0$. Let also $L$ be the length function on $G$ induced by the Teichmüller distance $d_{T}$, namely $L(g)=d_{T}(o, g \cdot o)$. Inspired by [23] and [16], we first describe our choice of $E_{n}$ that fits in Theorem 2.2.13. Let $g_{t}^{q}$ be a Teichmüller geodesic ray starting from $o$ in the direction of $q \in Q D^{1}(X)$. For every $m>0$, recall that

$$
T h k_{\epsilon}^{\%}\left[o, g_{m}\right] \doteq \frac{\left|\left\{0 \leq s \leq m: g_{s} \in \operatorname{Teich}_{\epsilon}(S)\right\}\right|}{d_{T}\left(o, g_{m}\right)}
$$

Theorem 2.2.10 ([16] Proposition 5.5). For all $0<\theta<1$, there exists $\epsilon>0$ such that for all $o=(X, \phi) \in \operatorname{Teich}(S)$

$$
\lim _{R_{0} \rightarrow \infty} \nu_{o}\left(\left\{q \in Q D^{1}(X): T h k_{\epsilon}^{\%}\left[o, g_{m}^{q}\right] \geq \theta, \forall m>R_{0}\right\}\right)=1
$$

We then fix any $\theta \in(0,1)$ and take $\epsilon>0$ by the above theorem. We identify $Q D^{1}(X)$ with $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ and $g_{t}^{q}$ with $\mathcal{V}(q)(t)$. For each $R>0$, we define

$$
U(R, \theta, \epsilon)=\left\{\xi \in \mathcal{P} \mathcal{M} \mathcal{F}(S): T h k_{\epsilon}^{\%}\left[o, \xi_{m}\right] \geq \theta, \forall m>R\right\}
$$

Then if $R_{2} \geq R_{1}>0$, we have

$$
U\left(R_{1}, \theta, \epsilon\right) \subset U\left(R_{2}, \theta, \epsilon\right)
$$

Define

$$
U(\theta, \epsilon)=\cup_{R>0} U(R, \theta, \epsilon),
$$

then, by Theorem 2.2.10, one has

$$
\left.\nu_{o}(U(\theta, \epsilon))\right)=1 .
$$

Furthermore, after a suitable choice of $\theta$, one has $\nu_{o}(\partial(U(\theta, \epsilon)))=0$ and by Masur's criterion (Lemma 2.2.4), the set $U(\epsilon, \theta)$ consists of uniquely ergodic directions. We now fix the choice of $\epsilon$ and $\theta$ and for $\gamma \in \operatorname{Mod}(S)$, denote the direction determined by
the oriented geodesic $[o, \gamma \cdot o]$ by $\xi_{\gamma}$. Now we are in a position to describe $E_{n}$. Fix $\rho>0$ and let $L_{0}=L_{0}(\theta, \epsilon)$ be the constant as Theorem 2.2.8. For $\frac{1}{3 h} \ln \ln n>\max \left\{L_{0}, \rho\right\}$, define the set $\mathcal{E}(\theta, \epsilon, n, o, \rho)$ to be the set of all elements $\gamma$ in $\operatorname{Mod}(S)$ satisfying:
(a) $d(\gamma \cdot o, o) \in(n-\rho, n+\rho)$;
(b) Both $\xi_{\gamma}$ and $\xi_{\gamma^{-1}}$ are in $U(\theta, \epsilon)$;
(c) If $g(t)$ is either the geodesic ray $\xi_{\gamma}(t)$ or $\xi_{\gamma^{-1}}(t)$, then the segment $\left[o, g\left(\frac{1}{3 h} \ln \ln n\right)\right]$ has at least proportion $\theta$ in $\operatorname{Teich}_{\epsilon}(S)$.

Lemma 2.2.11. Let $n$ large enough as before. Then for $\gamma \in \mathcal{E}(\theta, \epsilon, n, o, \rho)$, there exists a geodesic segment $I_{\gamma}$ of length $\frac{1}{3 h} \ln \ln n$ in the geodesic $[o, \gamma \cdot o]$ that has at least proportion $\theta$ in $\operatorname{Teich}_{\epsilon}(S)$ and containing $\gamma \cdot o$.

Proof. Let $\gamma \in \mathcal{E}(\theta, \epsilon, n, o, \rho)$. Since the geodesic ray $\xi_{\gamma^{-1}}(t)$ satisfies (c) in the definition of $\mathcal{E}(\theta, \epsilon, n, o, \rho)$, the first segment $I_{\gamma}$ of $\left[o, \gamma^{-1} \cdot o\right]$ of length $\frac{1}{3 h} \ln \ln n$ has at least proportion $\theta$ in $\operatorname{Teich}_{\epsilon}(S)$. As $\gamma \cdot\left[o, \gamma^{-1} \cdot o\right]=[\gamma \cdot o, o]$, therefore, the geodesic $[o, \gamma \cdot o]$ has a subinterval $\gamma \cdot I_{\gamma^{-1}}$ of length $\frac{1}{3 h} \ln \ln n$ that at least proportion $\theta$ in $\operatorname{Teich}_{\epsilon}(S)$ and starting at point $\gamma \cdot o$.

Later, we will prove that $\mathcal{E}(\theta, \epsilon, n, o, \rho)$ has exponential growth. We first state one obvious property of the boundary representation.

Lemma 2.2.12. Let $\pi_{\nu}$ be the boundary representation of $\operatorname{Mod}(S)$. For every $g \in$ $\operatorname{Mod}(S),\left\|\pi_{\nu_{o}}(g) \mathbb{1}_{\mathcal{P M F}(S)}\right\|_{L^{\infty}\left(\mathcal{P M F}(S), \nu_{o}\right)}<\infty$

Proof. The lemma is an easy consequence of Kerckhoff's formula, namely Lemma 2.2.6, on Teichmüller distances. By Lemma 2.2.6,

$$
\forall x, y \in \operatorname{Teich}(S), \forall[\xi] \in \mathcal{P} \mathcal{M} \mathcal{F}(S),\left(\frac{\operatorname{Ext}_{x}(\xi)}{\operatorname{Ext}_{y}(\xi)}\right)^{\frac{1}{2}} \leq e^{d_{T}(x, y)}
$$

$$
\begin{aligned}
& \text { As } \pi_{\nu_{o}}(g) \mathbb{1}_{\mathcal{P M F}(S)}=\left(\frac{\operatorname{Ext}_{o}(\xi)}{\operatorname{Ext}_{\gamma \cdot o}(\xi)}\right)^{\frac{6 g-6}{4}}, \text { one has } \\
& \qquad\left\|\pi_{\nu_{o}}(g) \mathbb{1}_{\mathcal{P} \mathcal{M F}(S)}\right\|_{L^{\infty}\left(\mathcal{P M \mathcal { F } ( S ) , \nu _ { o } )}\right.} \leq e^{\frac{6 g-6}{2} d_{T}(o, \gamma \cdot o)}<\infty .
\end{aligned}
$$

The following theorem is a slight variant of Theorem 2.2 .1 whose proof is the same as its original proof.

Theorem 2.2.13. Let $\pi_{\nu_{o}}$ be the associated quasi-regular representation of $\operatorname{Mod}(S)$ on $L^{2}\left(\mathcal{P} \mathcal{M} \mathcal{F}(S), \nu_{o}\right)$. Let $i$ be the intersection number function defined on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ induced by the section $\tau: \mathcal{P} \mathcal{M F}(S) \longrightarrow \mathcal{M F}(S)$. Let $n \gg \rho$ and let $E_{n}=E_{n}(\rho) \subset$ $\left\{g \in \operatorname{Mod}(S): d_{T}(o, g \cdot o) \in[n-\rho, n+\rho]\right\}$ be symmetric Let $e_{n}=\operatorname{Pr}: E_{n} \longrightarrow$ $\mathcal{P M F}(S)$ be the radial projection from o. Assume that the following conditions hold:
(1) $\lim _{n \rightarrow \infty}\left|E_{n}\right|=\infty$,
(2) for all Borel subsets $W, V \subset \mathcal{P M \mathcal { F }}(S)$ such that $\nu_{o}(\partial W)=\nu_{o}(\partial V)=0$,

$$
\left.\left.\limsup _{n \rightarrow \infty} \frac{1}{\left|E_{n}\right|} \right\rvert\,\left\{\gamma \in E_{n}: e_{n}\left(\gamma^{-1}\right) \in W \text { and } e_{n}(\gamma) \in V\right\} \right\rvert\, \leq \nu_{o}(W) \nu_{o}(V)
$$

(3) for every $n \gg \rho$, there are two sequences of reals $\left\{h_{r_{n}}(n, \rho)\right\}$ and $\left\{r_{n}\right\}$ such that $\lim _{n \rightarrow \infty} h_{r_{n}}(n, \rho)=\lim _{n \rightarrow \infty} r_{n}=0$ and such that

$$
\forall n \in \mathbb{N}, \forall \gamma \in E_{n}, \frac{\left\langle\pi_{\nu_{o}}(\gamma) \mathbb{1}_{\mathcal{P M F}(S)}, \mathbb{1}_{\left\{x \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(x, e_{n}(\gamma)\right) \geq r_{n}\right\}}\right\rangle}{\Phi(\gamma)} \leq h_{r_{n}}(n, \rho),
$$

$$
\begin{equation*}
\sup _{n}\left\|M_{E_{n}}^{\mathbb{1}_{\mathcal{P} \mathcal{F}(S)}} \mathbb{1}_{\mathcal{P M F}(S)}\right\|_{L^{\infty}\left(\mathcal{P M F}(S), \nu_{o}\right)}<\infty . \tag{4}
\end{equation*}
$$

Then the quasi-regular representation $\pi_{\nu_{o}}$ is ergodic with respect to $\left(E_{n}, e_{n}\right)$ and any $f \in \bar{H}^{L^{\infty}\left(\mathcal{P M} \mathcal{F}(S), \nu_{o}\right)}$, where

$$
H=<\mathbb{1}_{U}: \nu_{o}(\partial U)=0 \text { and } U \text { is a Borel subset of } \mathcal{P} \mathcal{M} \mathcal{F}(S)>
$$

Remark 2.2.3. As Theorem 2.2.13 is slight different from Theorem 2.2.1, it is worth making a few comments. One could easily find the only difference is the point (3) here since the original point (1) has been replaced automatically by Lemma 2.2.12. The assumption (3) in Theorem 2.2.13 is different from the assumption (4) in Theorem 2.2.1. For the proof of Theorem 2.2.13, we modify the proof of Proposition 2.21 and thus Theorem 2.2 in [12] as follows. We use the same notations for convenience. In the proof of the first part of [Proposition 2.21,[12]], first approximate $W$ by subsets $W_{n}$ consisting of uniquely ergodic measured foliations and such that $\nu_{o}\left(W-W_{n}\right) \leq k_{n}$ with $k_{n}$ tends to 0 . Then denote $W_{n}(r)=\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, W_{n}\right) \leq r\right\}$, by Lemma 2.2.3, choose $m=m(n)$ large so that $U \cap W_{n}\left(r_{m}\right)=\emptyset$. Now replace $\left\langle\pi_{\nu}(g) \mathbb{1}_{B}, \mathbb{1}_{W}\right\rangle$ by $\left\langle\pi_{\nu}(g) \mathbb{1}_{B}, \mathbb{1}_{W_{n}}\right\rangle$ and follow essential the same proof. The proof of [Theorem 2.2,[12]] is modified similarly by taking limit simultaneously with respect to $n$ and $r$ rather than taking limit first on $n$ than on $r$ as done in [12].

Our main result is the following theorem.

Theorem 2.2.14. There exists $\theta$ and $\epsilon$ such that, if $E_{n}=\mathcal{E}(\theta, \epsilon, n, o, \rho)$, which is described before Lemma 2.2.11, then the boundary representation $\pi_{\nu_{o}}$ is ergodic with respect to $\left(E_{n}, \operatorname{Pr}\right)$ and any $f \in \bar{H}^{L^{\infty}}$ as above. In other words, the pair $\left(E_{n}=\right.$ $\mathcal{E}(\theta, \epsilon, n, o, \rho), \operatorname{Pr})$ satisfies all conditions listed in Theorem 2.2.13.

As $\nu_{o}$ is a Radon measure, one has immediately the following two corollaries by Proposition 2.2.2 and Remark 2.2.1.

Corollary 2.2.15. The boundary representation $\pi_{\nu_{o}}$ of $\operatorname{Mod}(S)$ is irreducible.
Corollary 2.2.16. The mapping class group $\operatorname{Mod}(S)$ acts ergodically on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ with respect to the measure class $\left[\nu_{o}\right]$.

We then mention a property of the boundary representation $\pi_{\nu_{o}}$. Recall that a unitary
representation of a group $G$ is called tempered if it is weakly contained in the regular representation $L^{2}(G)$.

Proposition 2.2.17. The boundary representation $\pi_{\nu_{o}}$ of $\operatorname{Mod}(S)$ is tempered.
Proof. We argue as Proposition 6.3 in [22]. By the main Theorem in [35], we need to verify that the action of $\operatorname{Mod}(S)$ on $\mathcal{P M} \mathcal{F}(S)$ is amenable. This is Proposition 8.1 in [27] as a corollary of topological amenability of the action of $\operatorname{Mod}(S)$ on $\mathcal{P} \mathcal{M F}(S)$.

Corollary 2.2.18. The trivial representation $\mathbb{1}$ does not weakly contained in the boundary representation $\pi_{\nu_{o}}$. In other words, the boundary representation $\pi_{\nu_{o}}$ does not have almost invariant vectors.

Proof. As $\operatorname{Mod}(S)$ is not amenable for hyperbolic surface $S$, the trivial representation is not contained in the regular representation. According to the above proposition and the fact that weak containment is transitive, we can conclude the proof.

## Notations. We make some conventions for later use.

- $S$ : a genus $g \geq 2$, closed, oriented, connected surface;
- $h=6 g-6$;
- $o$ : the base point in Teich $(S)$ which is chosen to be generic in the sense that $S t a b_{o}(\operatorname{Mod}(S))=i d$. Denote $\nu=\nu_{o}$ and the measure is normalized so that $\nu(\mathcal{P M} \mathcal{F}(S))=1 ;$
- The projective measured foliation space $\mathcal{P M} \mathcal{M}(S)$ is regarded as a subset of $\mathcal{M} \mathcal{F}(S)$ by $\tau$ and an element $[\xi]$ in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ is then written as $\xi$, so both $[\xi]$ and $\xi$ will be called directions when there are no confusions;
- Fix arbitrary $\rho>0$ and assume $n \gg \rho$;
- $P r_{y}: \operatorname{Teich}(S)-\{y\} \longrightarrow \mathcal{P} \mathcal{M} \mathcal{F}(S)$ : the radial projection from Teich $(S)$ to $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ that assigns every point $z \in \operatorname{Teich}(S)-\{y\}$ to the vertical measured foliation of the unit quadratic differential defined by the oriented geodesic $[y, z]$. For $y=o$, we simply denote $\operatorname{Pr}_{o}$ to be $\operatorname{Pr}$;
- $B(y, R)$ : the closed ball in Teich $(S)$ of radius $R$ at $y$ with respect to the Teichmüller distance $d=d_{T}$. $L$ : the length function on $G$ induced by the Teichmüller distance $d_{T}$ through the orbit map: $g \mapsto g \cdot o$;
- $\asymp$ : if $A(t), B(t)$ are two functions, we use the notation $A \asymp B$ to mean $\frac{A(t)}{B(t)} \rightarrow 1$ as $t \rightarrow \infty$ and $A \widetilde{<} B$ to mean $\lim _{t \rightarrow \infty} \frac{A(t)}{B(t)} \leq 1$. The notation $A \widetilde{>} B$ is defined similarly;
- $A \sim_{\theta} B$ : there is multiplicative constants $C_{1}>0, C_{2}>0$ depending on $\theta$ so that

$$
C_{1} A \leq B \leq C_{2} A
$$

$A \prec_{\theta} B$ : there is a multiplicative constant $D=D(\theta)>0$ so that

$$
A \leq D B
$$

And $A \succ_{\theta} B$ is defined similarily;

- Denote $U=U(\theta, \epsilon)$ and $E_{n}=\mathcal{E}(\theta, \epsilon, n, o, \rho)$ in the sequel which is described before Lemma 2.2.11;
- $\xi_{\gamma} \in \mathcal{P M} \mathcal{F}(S)($ for $\gamma \in \operatorname{Mod}(S)-\{i d\})$ : the direction of the oriented geodesic segment $[o, \gamma \cdot o]$.


### 2.3 Exponential growth and shadow lemma

### 2.3.1 Exponential growth.

In this subsection, we will show that $\left|E_{n}\right|$ goes to infinity. In fact, we will show that $\left|E_{n}\right|$ grows exponentially. For any Borel subset $W$ of $\mathcal{P} \mathcal{M F}(S)$, denote by $\operatorname{Sect}_{W}$ the union of geodesics starting from $o$ and ending at $W$. We first recall the following theorem in [4] in our setting. Let

$$
C(n, \rho)=\left\{\gamma \in \operatorname{Mod}(S): d_{T}(\gamma \cdot o, o) \in(n-\rho, n+\rho)\right\}
$$

Theorem 2.3.1 ([4] Theorem 2.10). Let $W$ and $V$ be two Borel subsets of $\mathcal{P M} \mathcal{F}(S)$ with measure zero boundaries. Then as $R$ tends to $\infty$,

$$
\begin{aligned}
& \mid\left\{\gamma \in C(n, \rho): \gamma \cdot o \in \operatorname{Sect}_{W} \text { and } \gamma^{-1} \cdot o \in \operatorname{Sect}_{V}\right\} \mid \\
& \quad \asymp K e^{h n} \nu(W) \nu(V) .
\end{aligned}
$$

where $K$ is a constant depending on $g, \rho$ and $o$. In fact, using the notations in [4], one has $K=\frac{2 \sinh (h \rho)\|\nu(\mathcal{P} \mathcal{M} \mathcal{F}(S))\|^{2}}{h m\left(\mathcal{M}_{g}\right)}$, where $m\left(\mathcal{M}_{g}\right)$ is the push forward of the Masur-Veech volume.

Corollary 2.3.2. Let $n \gg 0$ and $K$ be the constant in Theorem 2.3.1. Then $\left|E_{n}\right| \asymp$ $K e^{h n}$. In particular, $\lim _{n \rightarrow \infty}\left|E_{n}\right|=\infty$.

Proof. As $E_{n} \subset C(n, \rho)$ and, by Theorem 2.3.1, $|C(n, \rho)| \asymp K e^{h n}$, it is obvious that $\left|E_{n}\right| \widetilde{<} K e^{h n}$. We now show that $\left|E_{n}\right| \widetilde{>} K e^{h n}$. Recall that $U(\theta, \epsilon)=\cup_{R>0} U(R, \theta, \epsilon)$ with $\nu(U(\theta, \epsilon))=1$ and $U(S, \theta, \epsilon) \subset U(T, \theta, \epsilon)$ for $T>S$. Let $\delta_{1}>0$ small enough and choose $R \gg 0$ such that

$$
1-\delta_{1} \leq \nu(U(R, \theta, \epsilon)) \leq 1, \nu(\partial U(R, \theta, \epsilon))=0
$$

By Theorem 2.3.1 again, for any $\delta_{2}>0$ small enough, one could choose $n$ large enough so that $\frac{1}{3 h} \ln \ln n>R$ and so that

$$
\begin{aligned}
& \mid\left\{\gamma \in C(n, \rho): \gamma \cdot o \in \operatorname{Sect}_{U(R, \theta, \epsilon)} \text { and } \gamma^{-1} \cdot o \in \operatorname{Sect}_{U(R, \theta, \epsilon)}\right\} \mid \\
& \geq K e^{\delta_{2}} e^{h n}(\nu(U(R, \theta, \epsilon)))^{2} \\
& \geq K e^{\delta_{2}}\left(1-\delta_{1}\right)^{2} e^{h n} .
\end{aligned}
$$

On the other hand, by the choice of $n$ and the definition of $U(R, \theta, \epsilon)$,

$$
\begin{aligned}
& \left\{\gamma \in C(n, \rho): \gamma \cdot o \in \operatorname{Sect}_{U(R, \theta, \epsilon)} \text { and } \gamma^{-1} \cdot o \in \operatorname{Sect}_{U(R, \theta, \epsilon)}\right\} \\
& \subset E_{n} .
\end{aligned}
$$

Therefore, we have $\left|E_{n}\right| \geq K e^{\delta_{2}}\left(1-\delta_{1}\right)^{2} e^{h n}$. As $\delta_{1}$ and $\delta_{2}$ can be arbitrary small, one has $\left|E_{n}\right| \widetilde{>} K e^{h n}$.

Corollary 2.3.3. For all Borel subsets $W, V \subset \mathcal{P M} \mathcal{F}(S)$ such that $\nu(\partial W)=\nu(\partial V)=$ 0 ,

$$
\left.\left.\limsup _{n \rightarrow \infty} \frac{1}{\left|E_{n}\right|} \right\rvert\,\left\{\gamma \in E_{n}: \operatorname{Pr}\left(\gamma^{-1}\right) \in W \text { and } \operatorname{Pr}(\gamma) \in V\right\} \right\rvert\, \leq \nu(W) \nu(V)
$$

Proof. By Corollary 2.3.2 and Theorem 2.3.1, $\left|E_{n}\right| \asymp|C(n, \rho)|$. Notice that $\operatorname{Pr}(\gamma) \in$ $V$ if and only if $\gamma \cdot o \in \operatorname{Sect}_{V}$. Hence,

$$
\begin{aligned}
& \left.\left.\limsup _{n \rightarrow \infty} \frac{1}{\left|E_{n}\right|} \right\rvert\,\left\{\gamma \in E_{n}: \operatorname{Pr}\left(\gamma^{-1}\right) \in W \text { and } \operatorname{Pr}(\gamma) \in V\right\} \right\rvert\, \\
& \left.\left.\leq \limsup _{n \rightarrow \infty} \frac{1}{\left|E_{n}\right|} \right\rvert\,\left\{\gamma \in C(n, \rho): \operatorname{Pr}\left(\gamma^{-1}\right) \in W \text { and } \operatorname{Pr}(\gamma) \in V\right\} \right\rvert\, \\
& \left.\left.=\limsup _{n \rightarrow \infty} \frac{|C(n, \rho)|}{\left|E_{n}\right|} \frac{1}{|C(n, \rho)|} \right\rvert\,\left\{\gamma \in C(n, \rho): \operatorname{Pr}\left(\gamma^{-1}\right) \in W \text { and } \operatorname{Pr}(\gamma) \in V\right\} \right\rvert\, \\
& \leq \nu(W) \nu(V)
\end{aligned}
$$

### 2.3.2 Shadow lemma.

Definition 2.3.1. ( $\mathcal{O}$-points) A point $y$ in $\operatorname{Teich}(S)$ is called an $\mathcal{O}$-point (with respect to $o$ ) if for every $R>0$, there is a real number $C \geq 1$ depending on $R$, such that

$$
\frac{1}{C} \exp (-h d(o, y)) \leq \nu(\operatorname{Pr}(B(y, R))) \leq C \exp (-h d(o, y))
$$

Remark 2.3.1. We call such points $\mathcal{O}$-points because they are the points that satisfy the classic shadow lemma [58].

Definition 2.3.2. An element $g \in \operatorname{Mod}(S)$ is called an $\mathcal{O}$-mapping class (with respect to $o \in \operatorname{Teich}(S))$ if $g \cdot o$ is an $\mathcal{O}$-point.

Recall that $U$ has a full measure. We need a lemma that relates Busemann functions to extremal lengths. Recall that if $\left(X, d_{X}\right)$ is a metric space and $\xi$ is a geodesic ray starting from a point $x_{0} \in X$, then the Busemann function associated to the geodesic ray $\xi$ is the function $b_{\xi}$ on $X$ defined by

$$
b_{\xi}: x \mapsto \lim _{t \rightarrow \infty}\left(d_{X}(x, \xi(t))-t\right)
$$

For $\left(X=\operatorname{Teich}(S), d_{X}=d\right)$ and $\xi$ be a geodesic ray starting from $o$, one has,
Lemma 2.3.4 (Walsh[62]). If [ $\xi$ ] is uniquely ergodic, then the Busemann function associated to the geodesic ray in the direction $[\xi]$ is

$$
\forall x \in \operatorname{Teich}(S), b_{[\xi]}(x)=\frac{1}{2} \ln \left(\frac{\operatorname{Ext}_{x}(\xi)}{\operatorname{Ext}_{o}(\xi)}\right)
$$

The following lemma is contained in [[23], Lemma 5.1] and will be used in the proof of uniform boundedness (Section 2.5.1), we include the proof for the sake of completeness.

Lemma 2.3.5. Let $n \gg \rho$. Then elements in $\bigcup_{n} E_{n}$ are $\mathcal{O}$-mapping classes, where $C$ depends on $R, \theta, \epsilon$.

Proof. Let $g \in E_{n}$ and $R>0$. As

$$
\nu(\operatorname{Pr}(B(g \cdot o, R)))=\int_{\{\eta \in \operatorname{Pr}(B(g \cdot o, R)\}}\left(\frac{\operatorname{Ext}_{g \cdot o}(\eta)}{\operatorname{Ext}_{o}(\eta)}\right)^{\frac{h}{2}} d \nu_{g \cdot o}(\eta)
$$

By Lemma 2.3.4 and the fact that uniquely ergodic measured foliations has a full measure [39], so we have, for almost every $\eta \in \operatorname{Pr}(B(g \cdot o, R)$,

$$
d(o, g \cdot o)-2 R \leq-\frac{1}{2} \ln \left(\frac{\operatorname{Ext}_{g \cdot o}(\eta)}{\operatorname{Ext}_{o}(\eta)}\right) \leq d(o, g \cdot o)
$$

Notice that as such $\eta$ is dense, the above inequality actually holds for all points in $\operatorname{Pr}(B(g \cdot o, R)$. By Lemma 2.2.5, $\operatorname{Mod}(S)$ acts on the Teichmüller compactification continuously at $\nu$-almost every point, thus on one hand

$$
\begin{aligned}
& \nu(\operatorname{Pr}(B(g \cdot o, R))) \\
& \geq \nu_{g \cdot o}(\operatorname{Pr}(B(g \cdot o, R))) e^{-h d(o, g \cdot o)} \\
& =\nu\left(\operatorname{Pr}_{g^{-1} \cdot o}(B(o, R))\right) e^{-h d(o, g \cdot o)} .
\end{aligned}
$$

One the other hand, we have

$$
\begin{aligned}
& \nu( P r \\
&(B(g \cdot o, R))) \leq e^{2 h R-h d_{T}(o, g \cdot o)} \nu_{g \cdot o}(\operatorname{Pr}(B(g \cdot o, R))) \\
& \quad \leq e^{2 h R-h d_{T}(o, g \cdot o)} \nu_{g \cdot o}(\mathcal{P} \mathcal{M F}(S)) \\
& \quad=e^{2 h R-h d_{T}(o, g \cdot o)} \nu(\mathcal{P M} \mathcal{F}(S)) \\
& \quad=C(R) e^{-h d(o, g \cdot o)}
\end{aligned}
$$

We now claim that there exists a constant $D>0$ independent of $g \in \cup_{n} E_{n}$ such that

$$
\nu\left(\operatorname{Pr}_{g^{-1 . o}} B(o, R)\right) \geq D
$$

If not, then there is a sequence $g_{k}^{-1} \cdot o$ converges to a point $[\zeta]$ such that $\nu\left(\operatorname{Pr}_{g_{k}^{-1} . o}(B(o, R))\right)$ tends to 0 . As $\left\{g_{k}\right\} \subset \bigcup_{n} E_{n}$, so by Masur's criterion, namely Lemma 2.2.4, [ $\zeta$ ] is uniquely ergodic, thus $\nu\left(\operatorname{Pr}_{[\zeta]}(B(o, R))\right)$ is zero, which is impossible since $\operatorname{Pr}_{[\zeta]}(B(o, R))$ contains a small open subset in $\mathcal{P} \mathcal{M F}(S)$ and $\nu$ is not atomic.

### 2.4 Harish-Chandra estimates.

This section is devoted to prove the following Harish-Chandra estimates.
Theorem 2.4.1. Given $n \gg \rho$. There exist $a_{1}>0, a_{2}>0, b_{1}, b_{2}, c_{1}>0$ depending on $\epsilon, o, g, \theta, \rho$ such that

$$
\forall \gamma \in E_{n}, \quad\left(a_{1} n-c_{1} \ln \ln n+b_{1}\right) e^{-\frac{h}{2} n} \leq \Phi(\gamma) \leq\left(a_{2} n+b_{2}\right) e^{-\frac{h}{2} n}
$$

Recall that

$$
\begin{aligned}
\Phi(\gamma) & =<\pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{P M F}(S)}, \mathbb{1}_{\mathcal{P M F}(S)}>_{L^{2}(\mathcal{P M F}(S), \nu)} \\
& =\int_{\mathcal{P M F}(S)}\left(\frac{\operatorname{Ext}_{o}(\xi)}{\operatorname{Ext}_{\gamma . o}(\xi)}\right)^{\frac{h}{4}} d \nu([\xi]) .
\end{aligned}
$$

Remark 2.4.1. 1. In [13], the left side is of the form $(a n+b) e^{-\frac{\alpha}{2} n}$. However, some other terms like $\ln \ln n$ should be added for mapping class groups if we require $\lim \frac{\left|E_{n}\right|}{C(n, \rho)}=1$.
2. The following oberservation will be useful, namely $\Phi(\gamma) \asymp n e^{-\frac{h n}{2}}$.

The proof is divided into several steps and will be given at the end of this section.

### 2.4.1 Reduction to intersection numbers.

By our convention, for every $\xi \in \mathcal{P} \mathcal{M F}(S)$, one has $\operatorname{Ext}_{o}(\xi)=1$, we then have

$$
\Phi(\gamma)=\int_{\mathcal{P} \mathcal{M F}(S)}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} d \nu(\zeta)
$$

Let $\xi_{\gamma}$ be the direction of $[o, \gamma \cdot o]$. In order to estimate $\Phi(\gamma)$, we will relate it to the following integrations on intersection numbers:

$$
\Psi(\gamma)=\int_{\mathcal{P} \mathcal{M F}(S)}\left(\frac{1}{i\left(\xi_{\gamma}, \eta\right)}\right)^{\frac{h}{2}} d \nu(\eta)
$$

Denote

$$
\Psi(\gamma)_{\geq A}=\int_{\left\{\eta \in \mathcal{P} \mathcal{M F}(S): i\left(\xi_{\gamma}, \eta\right) \geq A\right\}}\left(\frac{1}{i\left(\xi_{\gamma}, \eta\right)}\right)^{\frac{h}{2}} d \nu(\eta)
$$

The first step is to bound $\Phi^{\prime}(\gamma)$ from above. This can be easily done by Minsky's inequality. Namely,

Lemma 2.4.2 (Minsky's inequality [46]). Let $\xi$ and $\eta$ be two measured foliations on $S$ and $x \in \operatorname{Teich}(S)$, then

$$
i^{2}(\xi, \eta) \leq \operatorname{Ext}_{x}(\xi) \operatorname{Ext}_{x}(\eta)
$$

where the equilty holds if and only if there is a qudratic differential $q$ so that the vertical measured foliation of $q$ on $X$ is $\xi$ and the horizontal measured foliation is $\eta$.

Corollary 2.4.3. There exist constants $C_{3}=C_{3}(g, \rho)>0$ and $C_{4}=C_{4}(g, \rho)>0$ such that, for every $M \in(0,1)$ and every $\gamma \in \operatorname{Mod}(S)$,

$$
\Phi(\gamma) \leq C_{3} e^{-\frac{h}{2} n} \Psi(\gamma) \geq M+C_{4} e^{\frac{h}{2} n} \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, \xi_{\gamma}\right) \leq M\right\}\right)
$$

Proof. Decompose $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ into two subsets $A=\left\{\eta \in \mathcal{P} \mathcal{M F}(S): i\left(\eta, \xi_{\gamma}\right) \leq M\right\}$ and $B=\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, \xi_{\gamma}\right) \geq M\right\}$. Then we have

$$
\begin{align*}
& \Phi^{\prime}(\gamma)=\int_{A}\left(\frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\eta)}\right)^{\frac{h}{4}} d \nu(\eta)+\int_{B}\left(\frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\eta)}\right)^{\frac{h}{4}} d \nu(\eta)  \tag{2.4.1}\\
& =\mathrm{I}+\mathrm{II}
\end{align*}
$$

By Kerckhoff's formula, I $\prec_{g, \rho} e^{\frac{h}{2} n} \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, \xi_{\gamma}\right) \leq M\right\}\right)$. Thanks to Lemma 2.3.4, we can replace $\operatorname{Ext}_{\gamma \cdot o}\left(\xi_{\gamma}\right)$ in Lemma 2.4.2 by $e^{-2 n}$, so one has

$$
\frac{1}{i^{2}\left(\xi_{\gamma}, \eta\right) e^{2 n}} \succ_{g, \rho} \frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\eta)}
$$

which gives the bound for the term II.
In order to bound $\Phi(\gamma)$ from below, we will use the fact that $\gamma \in E_{n}$.
Lemma 2.4.4. There exists a constant $F$ depending on $g, o, \epsilon, \theta, \rho$ such that if $i\left(\xi_{\gamma}, \eta\right) \geq$ $F \ln n e^{-2 n}$, where $\eta \in U(\epsilon, \theta)$ and $\gamma \in E_{n}$, then $i^{2}\left(\xi_{\gamma}, \eta\right) \succ_{g, o, \epsilon, \theta, \rho} \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2 n}$.

Proof. First we remark that, since both $\eta$ and $\xi_{\gamma}$ are uniquely ergodic, by [[34], Proposition 5.1], there is a geodesic whose horizontal and vertical measured foliations are in the projective classes $\xi_{\gamma}$ and $\eta$ respectively. Hence we have a geodesic triangle $\triangle\left(o, \xi_{\gamma}, \eta\right)$. As $\gamma \in E_{n}$, Lemma 2.2.11 implies that there is a geodesic segment $I$ of length $\ell=\frac{1}{3 h} \ln \ln n$ in $[o, \gamma \cdot o]$ ending at $\gamma \cdot o$ that has at least proportion $\theta$ in Teich $_{\epsilon}(S)$. By Theorem 2.2.8,

$$
I \cap \mathcal{N}_{D}\left(\left[o, \xi_{\gamma}\right] \cap[o, \eta]\right) \neq \emptyset
$$

where $D$ comes from Theorem 2.2.8. Choose $q \in I \cap \mathcal{N}_{D}\left(\left[o, \xi_{\gamma}\right] \cap[o, \eta]\right)$. Then there are two possibilities:

Case 1: $d(q, y) \leq D$ with $y \in\left[\xi_{\gamma}, \eta\right]$.

Then we have, by Kerckhoff's formula and Lemma 2.4.2,

$$
\begin{aligned}
& i^{2}\left(\xi_{\gamma}, \eta\right)=\operatorname{Ext}_{y}(\eta) \operatorname{Ext}_{y}\left(\xi_{\gamma}\right) \\
& \succ_{g, o, \theta, \epsilon} \operatorname{Ext}_{q}(\eta) \operatorname{Ext}_{q}\left(\xi_{\gamma}\right) \\
& =\operatorname{Ext}_{q}(\eta) e^{-2 d(o, q)} \\
& \geq \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2 n}
\end{aligned}
$$

which means that, in this case, we always have $i^{2}\left(\xi_{\gamma}, \eta\right) \succ_{g, o, \epsilon, \theta} \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2 n}$.

Case 2: $d(q, y) \leq D$ with $y \in[o, \eta]$.

Then we have

$$
\begin{aligned}
& i^{2}\left(\xi_{\gamma}, \eta\right) \leq \operatorname{Ext}_{y}(\eta) \operatorname{Ext}_{y}\left(\xi_{\gamma}\right) \\
& \sim_{g, o, \theta, \epsilon} \operatorname{Ext}_{y}(\eta) \operatorname{Ext}_{q}\left(\xi_{\gamma}\right) \\
& \sim_{g, o, \theta, \epsilon, \rho} e^{-4 d(o, q)}=e^{-4(d(o, \gamma \cdot o)-d(q, \gamma \cdot o))} \\
& \leq e^{-4 n} e^{4 \ell}
\end{aligned}
$$

Therefore, in this case we have a constant $F_{1}$ depending on $g, o, \epsilon, \theta, \rho$ such that

$$
i\left(\xi_{\gamma}, \eta\right) \leq F_{1} e^{-2 n} e^{2 \ell} \leq F_{1} e^{-2 n} e^{\ln \ln n}=F_{1} \ln n e^{-2 n}
$$

Thus if we take $F \gg F_{1}$ and require $i\left(\xi_{\gamma}, \eta\right) \geq F \ln n e^{-2 n}$, it forces us in the Case 1 which implies the conclusion that

$$
i^{2}\left(\xi_{\gamma}, \eta\right) \succ_{g, o, \epsilon, \theta} \operatorname{Ext}_{\gamma \cdot o}(\eta) e^{-2 n}
$$

Corollary 2.4.5. For every $\gamma \in E_{n}$, take $\bar{M}=F \ln n e^{-2 n}$ where $F$ is the constant in Lemma 2.4.4. Then $\Phi(\gamma) \succ_{g, o, \epsilon, \theta, \rho} e^{-\frac{h}{2} n} \Psi(\gamma)_{\geq \bar{M}}$.

Proof. Note that $U(\epsilon, \theta)$ has a full measure. Hence, by Lemma 2.4.4

$$
\begin{aligned}
& \Phi(\gamma)=\int_{U(\epsilon, \theta)}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\eta)}\right)^{\frac{h}{4}} d \nu(\eta) \\
& \geq \int_{\left\{\eta \in U(\epsilon, \theta): i\left(\eta, \xi_{\gamma}\right) \geq \bar{M}\right\}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\eta)}\right)^{\frac{h}{4}} d \nu(\eta) \\
& \succ_{g, o, \epsilon, \theta} e^{-\frac{h n}{2}} \Psi(\gamma)_{\geq \bar{M}} .
\end{aligned}
$$

Lemma 2.4.6. Assume that there exist $N_{0}>0, a>0$ and $b>0$ such that

$$
\forall N \leq N_{0}, \quad a N^{\frac{h}{2}} \leq \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, \xi_{\gamma}\right) \leq N\right\}\right) \leq b N^{\frac{h}{2}}
$$

then there exist $A, B, D_{1}, D_{2}$ such that

$$
-A \ln N+D_{1} \leq \Psi(\gamma)_{\geq N} \leq-B \ln N+D_{2}
$$

Proof. The proof is same as the one in [13]. We repeat here for completeness. Namely,

$$
\begin{align*}
& \Psi(\gamma)_{\geq N}=\int_{\left\{\eta \in \mathcal{P M} \mathcal{F}(S): i\left(\xi_{\gamma}, \eta\right) \geq N\right\}}\left(\frac{1}{i\left(\xi_{\gamma}, \eta\right)}\right)^{\frac{h}{2}} d \nu(\eta) \\
& =\int_{\mathbb{R}} \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S):\left(\frac{1}{i\left(\xi_{\gamma}, \eta\right)}\right)^{\frac{h}{2}} \geq t\right\}\right) d t \\
& =\int_{1}^{\frac{1}{N^{\frac{h}{2}}}} \nu\left(\left\{\eta \in \mathcal{P M} \mathcal{F}(S): i\left(\xi_{\gamma}, \eta\right) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) d t  \tag{2.4.2}\\
& =\int_{1}^{N_{0}} \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\xi_{\gamma}, \eta\right) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) d t \\
& +\int_{N_{0}}^{\frac{1}{N^{\frac{h}{2}}}} \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\xi_{\gamma}, \eta\right) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) d t
\end{align*}
$$

By the assumption and $\nu$ is a probability measure, one can easily have the conclusion.

### 2.4.2 A basic example.

Before continuing our discussions, we digress for the case of once-punctured torus $S_{1,1}$. Some standard facts are taken from [[49], 7.2 Examples].

Let $S=S_{1,1}$. Then $\operatorname{Mod}(S)=S L(2, \mathbb{Z})$ and $\operatorname{Teich}(S)=\mathbb{H}^{2}$, the upper half plane. Take $o$ to be $i \in \mathbb{H}^{2}$. The space $\mathcal{M F}(S)$ of measured foliations can be identified with the real plane module the inversion, namely $\left\{\mathbb{R}^{2}-(0,0)\right\} /\{I,-I\}$. By the ergodicity of the Thurston measure $\nu_{T h}$, up to a constant multiple, the measure $\nu_{T h}$, which is defined by the weak limit of counting measures on $\mathcal{M} \mathcal{F}(S)$, can be identified with the Lebesgue measure on $\mathbb{R}^{2}$. Rays in $\left.\left\{\mathbb{R}^{2}-(0,0)\right\} /\{I,-I\}\right\}$ are then identified with points in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. It implies that $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ can be identified with $\mathbb{R} P^{1}$. Notice that all identifications here are $\operatorname{Mod}(S)$-equivariant. Hence $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ can be
represented as $\left\{[x: y]: x^{2}+y^{2} \neq 0, x, y \in \mathbb{R}\right\}$, or $\mathbb{R} \cup\{\infty\}$. $\overline{\operatorname{Teich}(S)}$ is then the usual compactification of $\mathbb{H}^{2}$. In this case, $\operatorname{Mod}(S)$ acts on $\overline{\text { Teich }(S)}$ via linear fractional transformations. For $(x, y) \in \mathbb{R}^{2}$, the extremal length at $o$ is

$$
\operatorname{Ext}_{o}((x, y))=x^{2}+y^{2}
$$

hence the image of $\mathcal{P M} \mathcal{F}(S)$ under $\tau$ is the circle. We will ignore the difference between $\mathbb{R}^{2}$ and $\mathbb{R}^{2} /\{I,-I\}$. For two points $(x, y),(p, q) \in \mathcal{M F}(S)$, the intersection number is $|q x-p y|$. Write the image of $\mathcal{P M \mathcal { F }}(S)$ in the form of $(\sin (\theta), \cos (\theta))$, and fix any $\xi=\left(\sin \left(\theta_{0}\right), \cos \left(\theta_{0}\right)\right) \in \mathcal{P} \mathcal{M} \mathcal{F}(S)$. Let $M$ to be small enough, then

$$
\begin{align*}
& \{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i(\xi, \eta) \leq M\} \\
& =\left\{\theta \in[0,2 \pi]:\left|\sin (\theta) \cos \left(\theta_{0}\right)-\cos (\theta) \sin \left(\theta_{0}\right)\right| \leq M\right\}  \tag{2.4.3}\\
& =\left\{\theta \in[0,2 \pi]:\left|\sin \left(\theta-\theta_{0}\right)\right| \leq M\right\} \\
& =\left\{\theta \in[0,2 \pi]:-M \leq \sin \left(\theta-\theta_{0}\right) \leq M\right\}
\end{align*}
$$

As $M$ is enough small, $\sin (\theta)$ is almost the same as $\theta$, so there exist constants $A$ and $B$, so that

$$
A M \leq \nu(\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i(\xi, \eta) \leq M\}) \leq B M
$$

Notice that, when $S$ is $S_{1,1}$, we have $h=6 g-6+n=6 \times 1-6+2 \times 1=2$, hence $\frac{h}{2}=1$.

### 2.4.3 Approximation by pants curves.

Now we want to prove that the assumption in Lemma 2.4.6 holds, however, instead of proving it directly, we will approximate by pants curves using the map considered in [43].

Recall that, thanks to Bers' theorem, there is a constant $C_{1}=C(g)$, depending only on the genus $g$, such that for every point $x \in \operatorname{Teich}(S)$, there exists a pants
decomposition, namely a collection of $3 g-3$ essential simple closed curves $\mathcal{P}=$ $\left\{\alpha_{1}, \cdots, \alpha_{3 g-3}\right\}$, such that

$$
\forall 1 \leq i \leq 3 g-3, \operatorname{Ext}_{x}\left(\alpha_{i}\right) \leq C^{2}
$$

If $x$ is further assumed to be in $\operatorname{Teich}_{\epsilon}(S)$, we can choose a collection of $3 g-3$ essential simple closed curves $\left\{\alpha_{1}(x), \cdots, \alpha_{3 g-3}(x)\right\}$ on $S$ such that

$$
\forall 1 \leq i \leq 3 g-3, \epsilon \leq \operatorname{Ext}_{x}\left(\alpha_{i}(x)\right) \leq C_{1}^{2}
$$

Denote $\alpha(x) \in M F(S)$ to be the measured foliation $\alpha(x)=\sum_{i=1}^{3 g-3} \alpha_{i}(x)$ and $[\alpha(x)]$ be its projective class in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. By the Jenkins-Strebel theorem (see also [32] Theorem 2.1), there is a unit holomorphic quadratic differential $q=q(x, \alpha(x))$ on $o \in \operatorname{Teich}(S)$ whose projective class of the vertical measured foliation is $[\alpha(x)]$.

Given $\gamma \in E_{n}$, by our construction of $E_{n}$, the ending point $\xi_{\gamma}$ of the geodesic ray $g_{t}^{\gamma}$ determined by $[o, \gamma \cdot o]$ is in $U(\epsilon, \theta)$. Hence, there is a sequence of points $y(k, \gamma) \in \operatorname{Teich}_{\epsilon}(S)$ in $g_{t}^{\gamma}$ tends to $\xi_{\gamma}$ in $\overline{\operatorname{Teich}(S)}$. By the above discussion, there is a sequence of pants curves $\alpha(y(k, \gamma))$ and a sequence of points $[\alpha(y(k, \gamma))]$ in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. By Minsky's inequality, $[\alpha(y(k, \gamma))]$ converges to $\left[\xi_{\gamma}\right]$ in $\mathcal{P M} \mathcal{F}(S)$, which means using the map $\tau, \frac{\alpha(y(k, \gamma))}{\sqrt{\operatorname{Ext}_{o}(\alpha(y(k, \gamma)))}}$ converges to $\xi_{\gamma}$. We first estimate $\operatorname{Ext}_{o}(\alpha(y(k, \gamma)))$.

Now let $y \in \operatorname{Teich}_{\epsilon}(S)$ and denote $\alpha(y)$ and $q(y)=q(y, \alpha(y))$ as above. Let $g_{t}$ be the Teichmüller geodesic ray starting from $o$ in the direction of $q(y)$. Let $t_{y}$ be the unique point in $g_{t}$ such that $t_{y}$ has maximal distance with $o$ and

$$
\forall 1 \leq i \leq 3 g-3, \epsilon \leq \operatorname{Ext}_{t_{y}}\left(\alpha_{i}(y)\right) \leq C_{1}^{2}
$$

Lemma 2.4.7. There is a constant $C_{2}$ depending on $g$ and $\epsilon$ such that

$$
\forall y \in \operatorname{Teich}_{\epsilon}(S), d\left(y, t_{y}\right) \leq C_{2}
$$

The proof is based on the following theorem. We use the theorem in the form of Theorem 5.3 in [4]. For the definition of twist numbers $t w(\alpha, \beta)$, the reader is referred to [52].

Theorem 2.4.8 (Minsky[47]). Let $x \in \operatorname{Teich}(S)$ and $\mathcal{P}=\left\{\alpha_{1}, \cdots, \alpha_{3 g-3}\right\}$ be a pants decomposition produced by the Bers' theorem mentioned above. Then for any simple closed curve $\beta$,

$$
\begin{equation*}
\operatorname{Ext}_{x}(\beta) \sim_{g} \max _{1 \leq i \leq 3 g-3}\left(\frac{i^{2}\left(\beta, \alpha_{i}\right)}{\operatorname{Ext}_{x}\left(\alpha_{i}\right)}+t w^{2}\left(\beta, \alpha_{i}\right) \operatorname{Ext}_{x}\left(\alpha_{i}\right)\right) \tag{2.4.4}
\end{equation*}
$$

Proof of Lemma 2.4.7. By Kerckhoff's formula, we only need to bound the ratio $\frac{\operatorname{Ext}_{y}(\beta)}{\operatorname{Ext}_{t y}(\beta)}$ for any essential simple closed curve $\beta$ on $S$. However, by the construction of $y(x)$, the two hyperbolic surfaces $t_{y}$ and $y$ have the same pants decomposition which satisfies the condition in Theorem 2.4.8, namely $\alpha(\gamma)=\left\{\alpha_{1}(y), \cdots, \alpha_{3 g-3}(y)\right\}$. As, for $1 \leq i \leq 3 g-3$, both extremal lengths $\operatorname{Ext}_{y}\left(\alpha_{i}\right)$ and $\operatorname{Ext}_{t_{y}}\left(\alpha_{i}\right)$ are bounded below by the constant $\epsilon$ and above by a constant $C_{1}^{2}$ depending only on $g$, we can conclude the proof of the lemma by using Equation (2.4.4).

Corollary 2.4.9. We have $\left|d_{T}(o, y)-d_{T}\left(o, t_{y}\right)\right| \leq C_{2}$, where $C_{2}$ is the constant in Lemma 2.4.7. Hence, $\operatorname{Ext}_{o}\left(\alpha_{i}(y(k, \gamma))\right) \sim_{g, \epsilon, o} e^{2 d(y(k, \gamma), o)}$.

Proof. We only need to show the second statement. Let $y=y(k, \gamma)$ and $t_{y}=t_{y(k, \gamma)}$. Let $T=d\left(o, t_{y}\right)$ and $f: o \rightarrow t_{y}$ be the Teichmüller mapping with dilatation $e^{2 T}$ between $o$ and $t_{y}$. We know that $T \sim_{g, \epsilon} d(o, y(k, \gamma))$. We want to show that $\operatorname{Ext}_{o}\left(\alpha_{i}(y)\right) \sim_{g, \epsilon, o} e^{2 T}$, for each $1 \leq i \leq 3 g-3$. Notice that $\operatorname{Ext}_{t_{y}}\left(\alpha_{i}(y)\right) \sim_{g, \epsilon} 1$. On the one hand, by Kerckhoff's formula, one has

$$
\operatorname{Ext}_{o}\left(\alpha_{i}(y)\right) \prec_{g, \epsilon} e^{2 T}
$$

In order to bound $\operatorname{Ext}\left(\alpha_{i}(y)\right)$ from below, we construct a metric and use the analytic definition of extremal length in [32]. Fix $1 \leq i \leq 3 g-3$. Let $q$ be the unit quadratic
differential on $o$ in the direction of $\left[o, t_{y}\right]$, that is, under the above notations, $q=$ $q(y, \alpha(y))$. Let $m_{i}$ be the modulus of the cylinder $C_{i}$ determined by $q$, where $C_{i}$ has the same core curve with $\alpha_{i}(y)$. According to the proof of Proposition 2 in [32], there is a metric $\sigma$ in the conformal class of $t_{y}$ such that the core curve of $\alpha_{i}(y)$ has $\sigma$-length 1 and area $e^{T} m_{i}+A$, where $A$ is a constant depending on $o$. One also has

$$
\frac{1}{e^{T} m_{i}} \geq \operatorname{Ext}_{t_{y}}\left(\alpha_{i}(y)\right) \geq \frac{1}{e^{T} m_{i}+A} .
$$

Now consider the metric $\Sigma=f^{*} \sigma$ on $o$ defined by the pullback of $\sigma$ via the Teichmüller mapping $f$. As $f$ preserves the area but shrink the vertical length, the core curve of $\alpha_{i}(y)$ has $\Sigma$-length $e^{T}$. Thus

$$
\operatorname{Ext}_{o}\left(\alpha_{i}(y)\right) \geq \frac{e^{2 T}}{e^{T} m_{i}+A}
$$

As

$$
e^{T} m_{i} \prec_{\epsilon, g} 1,
$$

one then further has

$$
\operatorname{Ext}_{o}\left(\alpha_{i}(y)\right) \succ_{g, \epsilon, o} e^{2 T}
$$

We now summarize all discussions above. There are $A>0, B>0$, depending on $\epsilon, g, o$, such that, for every $\gamma \in E_{n}$, there is a sequence $\left\{\xi_{k}(\gamma) \in \mathcal{P} \mathcal{M} \mathcal{F}(S)\right\}$ satisfying
(a) $\forall k, \xi_{k}(\gamma)=\left[x_{k}(\gamma)=\sum_{i=1}^{3 g-3} \alpha_{i}(\gamma)\right]$ where $\left\{\alpha_{i}(\gamma)\right\}_{i=1}^{3 g-3}$ is a pants decomposition of $S$;
(b) For each $i$, there is $t_{i}$ such that $A e^{2 t_{i}} \leq \operatorname{Ext}_{o}\left(\alpha_{i}(\gamma)\right) \leq B e^{2 t_{i}}$ and $\lim _{i} t_{i}=\infty$;
(c) The limit of $\left\{\xi_{k}(\gamma)=\frac{x_{k}(\gamma)}{\sqrt{\operatorname{Ext}_{o}\left(x_{k}(\gamma)\right)}}\right\}$ in $\mathcal{P M} \mathcal{F}(S)$ is $\xi_{\gamma}$.

Lemma 2.4.10. If there exists $N_{0}>0$ small enough such that, for every $\gamma \in E_{n}$ and $\xi_{k}(\gamma)$, one has

$$
\forall N \leq N_{0}, a N^{\frac{h}{2}} \leq \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, \xi_{k}(\gamma)\right) \leq N\right\}\right) \leq b N^{\frac{h}{2}}
$$

then

$$
\forall N \leq N_{0}, a N^{\frac{h}{2}} \leq \nu\left(\left\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i\left(\eta, \xi_{\gamma}\right) \leq N\right\}\right) \leq b N^{\frac{h}{2}}
$$

Proof. In general, convergence in $\mathcal{P M \mathcal { F }}(S)$, which is defined by weak convergence of measures, is hard to understand. However, in our case, the convergence $\xi_{k}(\gamma) \rightarrow \xi_{\gamma}$ can be understood as follows (see [26] for backgrounds). Recall that the projective measured foliation space $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ can be identified with the projective measured lamination space $\mathcal{P} \mathcal{M} \mathcal{L}(S)$ on $S$. Regarding both $\xi_{k}(\gamma)$ and $\xi_{\gamma}$ as measured laminations, then according to [26], $\xi_{k}(\gamma)$ converges to $\xi_{\gamma}$ in the sense of coarse Hausdorff topology. That is, as subsets, the limit $\zeta$ of $\xi_{k}(\gamma)$ contains $\xi_{\gamma}$ as a sublamination and the complement $\zeta-\xi_{\gamma}$ consists of isolated leaves. For each $\epsilon>0$, take $k$ large enough, then we could find a subsurface $F$ of $S$, so that, on $F$, any $\eta \in \mathcal{P} \mathcal{M} \mathcal{L}(S)$ with $i\left(\eta, \xi_{\gamma}\right)$ is uniformly approximated by $\xi_{k}(\gamma)$ and outside $F, i\left(\eta, \xi_{k} \gamma\right) \leq \epsilon$. Therefore such $\eta$ has intersection number less than $N+\epsilon$ with $\xi_{k}(\gamma)$, we then have proved the upper bound for the measure. The left inequality can be proved similarly.

### 2.4.4 Regularity at pants curves.

We are now in a position to prove that the assumption in Lemma 2.4.10 holds. We first summarize all properties of $\xi_{k}(\gamma)$ that we really need.

More conventions: From now on, we will use the hyperbolic length function $\ell_{o}(\cdot)$. Since $\ell_{o}^{2}(\cdot) \sim_{o} \operatorname{Ext}_{o}(\cdot)$, we can use $\ell_{o}(\cdot)$ to replace $\operatorname{Ext}_{o}(\cdot)$ without affecting the
result when we defining the measure $\nu_{o}$, the embedding $\tau: \mathcal{P \mathcal { M } \mathcal { F }}(S) \longrightarrow \mathcal{M F}(S)$ and $\xi_{k}(\gamma)$. For instance, for a measurable subset $U \in \mathcal{P} \mathcal{M F}(S)$, we have

$$
\nu_{0}(U)=\mu\left(\left\{\eta:[\eta] \in U, \ell_{o}(\eta) \leq 1\right\}\right)
$$

Set-up 0: Let $\alpha=\left\{\alpha_{1}, \cdots, \alpha_{3 g-3}\right\}$ be a pants decomposition of $S$ and consider it to be a measured foliation still denoted by $\alpha$. Then $[\alpha]$ defines a unit holomorphic quadratic differential $q$ on $o$, namely the unique $q$ such that $[\mathcal{V}(q)]=[\alpha]$. Let $\xi=\frac{\alpha}{\ell_{o}(\alpha)}$, then $\xi$ is the image of $[\alpha]$ under $\tau$. We denote $g_{t}$ the Teichmüller geodesic defined by $q$. We assume that for all $i \in\{1, \cdots, 3 g-3\}, \ell_{o}\left(\alpha_{i}\right)$ is bounded bleow and above, up to multiplicative constants depending only on $g, o, \epsilon$, by $e^{T}$ for some $T$.

Theorem 2.4.11. Under the above Set-up 0, there exist $M_{0}>0, C>0$ and $D>0$, depending on $g, o, \epsilon$ such that when $M<M_{0}$, we have

$$
C M^{\frac{h}{2}} \leq \nu(\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i(\eta, \xi) \leq M\}) \leq D M^{\frac{h}{2}}
$$

The main tool to prove the above theorem is the following Dehn-Thurston theorem. Let $P=\left\{\alpha_{k}\right\}$ be a pants decomposition. For each $\alpha_{k}$, let $m_{k}: \mathcal{M F}(S) \longrightarrow \mathbb{R}_{\geq 0}, \xi \mapsto$ $i\left(\alpha_{k}, \xi\right)$ be the intersection function defined by $\alpha_{k}$ and $t_{k}=t w_{k}$ be the twist function associated to $\alpha_{k}$.

Theorem 2.4.12 (The Dehn-Thurston theorem [52], Theorem 3.1.1). Let $S=S_{g}$ and $\alpha=\left\{\alpha_{1}, \cdots, \alpha_{3 g-3}\right\}$ be a pants decomposition of $S$. Then the map

$$
\begin{align*}
& \varpi: \mathcal{M F}(S) \longrightarrow \mathbb{R}^{6 g-6}  \tag{2.4.5}\\
& \mathcal{F} \mapsto\left(m_{1}(\mathcal{F}), \cdots m_{3 g-3}(\mathcal{F}), t_{1}(\mathcal{F}), \cdots, t_{3 g-3}(\mathcal{F})\right)
\end{align*}
$$

gives a global coordinate for $\mathcal{M F}(S)$.

Corollary 2.4.13. The symplectic form $\omega=d m_{1} \wedge d t_{1}+\cdots+d m_{3 g-3} \wedge d t_{3 g-3}$ gives a $\operatorname{Mod}(S)$-invariant measure $\mu=\frac{1}{n!} \omega^{n}$ on $\mathcal{M} \mathcal{F}(S)$, hence up to a multiplicative constant, the measure $\mu^{\prime}$ coincides with the Thurston measure $\nu_{\text {Th }}$ on $\mathcal{M} \mathcal{F}(S)$.

Proof. First we note that, different pants decompositions give rise to different train tracks charts which locally differ by an element in $S L(k, \mathbb{Z})$, for some $k$, keeping the volume form invariant, hence $\mu^{\prime}$ is $\operatorname{Mod}(S)$-invariant and independent of the choice of the pants decomposition $P$. As $\nu_{T h}$ is ergodic and both $\mu$ and $\nu_{T h}$ are in the Lebesgue measure class, the conclusion follows.

We now use $\mu$ to replace $\nu_{T h}$ with $\alpha$ is fixed to the one given in Set-up 0. Note that $\frac{h}{2}=3 g-3$. We now prove the theorem.

Proof of Theorem 2.4.11. By Lemma 2.4.2, for every two elements $\xi$ and $\eta$ in $\mathcal{P M} \mathcal{F}(S)$, the intersection number $i(\eta, \xi) \leq 1$ and 1 is achievable. So we take $M_{0}=\frac{1}{4}$ and let $M \leq M_{0}$. The proof is then divided into two parts. In the sequel, we will denote $a=\frac{1}{\sum_{i} \ell_{o}\left(\alpha_{i}\right)}$ and $\ell$ to be

$$
\ell=\frac{1}{a \prod\left(\ell_{o}\left(\alpha_{i}\right)\right)^{\frac{2}{h}}}=\frac{\sum_{i} \ell_{o}\left(\alpha_{i}\right)}{\prod\left(\ell_{o}\left(\alpha_{i}\right)\right)^{\frac{2}{h}}}
$$

Then by our assumption, there exists $A_{1}>0$ and $B_{1}>0$ depending on $g, o, \epsilon$, such that $B_{1} \leq \ell \leq A_{1}$.

Upper bound: $\nu(\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i(\eta, \xi) \leq M\}) \leq D M^{\frac{h}{2}}$ :

By the definition of $\nu$ and $i(\eta, \xi)=a \sum_{k=1}^{3 g-3} i\left(\eta, \alpha_{k}\right)$, we have,

$$
\begin{align*}
& \nu(\{\eta \in \mathcal{P M} \mathcal{F}(S): i(\eta, \xi) \leq M\}) \\
& =\mu\left(\left\{t \eta \in \mathcal{M} \mathcal{F}(S): i(\eta, \xi) \leq M, \ell_{o}(\eta)=1,0 \leq t \leq 1\right\}\right) \text { (by definition) } \\
& =\mu\left(\left\{t \eta \in \mathcal{M} \mathcal{F}(S): a \sum_{k} m_{k}(\eta) \leq M, \ell_{o}(\eta)=1,0 \leq t \leq 1\right\}\right)  \tag{2.4.6}\\
& \leq \mu\left(\left\{\eta: \forall k, m_{k}(\eta) \leq \frac{M}{a}, t_{i}(\eta) \ell_{o}\left(\alpha_{i}\right) \leq A_{2}\right\}\right)
\end{align*}
$$

where $A_{2}$ is a constant depending only on $o$. In fact, $A_{2}$ depends on the diameter of $X$. The last step comes from the fact that large twists will make the length to be large. Thus we further have

$$
\begin{align*}
& \nu\left(\left\{\eta \in \mathcal{P \mathcal { M } \mathcal { F } ( S ) : i ( \eta , \xi ) \leq M \} )} \begin{array}{l}
\leq \mu\left(\left\{\left(m_{1}, \cdots, m_{3 g-3}, t_{1}, \cdots, t_{3 g-3}\right): \forall k, m_{k} \leq \frac{M}{a}, t_{k} \leq \frac{A_{2}}{\ell_{o}\left(\alpha_{k}\right)}\right\}\right)(\text { by }(2.4 .6)) \\
\leq A_{3} M^{3 g-3} \frac{1}{a^{3 g-3} \prod_{k=1}^{3 g-3} \ell_{0}\left(\alpha_{k}\right)}(\text { by definition of } \mu) \\
=A_{3} M^{3 g-3} \ell^{3 g-3} \\
\leq A_{3} M^{3 g-3} A_{1}^{3 g-3}\left(\text { since } B_{1} \leq \ell \leq A_{1}\right) \\
\leq D M^{\frac{h}{2}}
\end{array}\right.\right.
\end{align*}
$$

Lower bound: $C M^{\frac{h}{2}} \leq \nu(\{\eta \in \mathcal{P} \mathcal{M F}(S): i(\eta, \xi) \leq M\})$ :
In order to bound the measure from below, we will construct a subset contained in the set. We first fix, for each $i$, a positive orientation for $\alpha_{i}$. Let

$$
\begin{equation*}
V=\left\{t \eta \in \mathcal{M} \mathcal{F}(S): i(\eta, \xi) \leq M, \ell_{o}(\eta)=1,0 \leq t \leq 1\right\} \tag{2.4.8}
\end{equation*}
$$

Then $\eta^{0}=\frac{1}{3 g-3} \xi$ is in $V$. Let $a$ and $M$ as above, and $\delta>0$ be a positive number.

Define a set of $6 g-6$-tuples by

$$
\begin{aligned}
& W_{0}(a, M)= \\
& \left\{\left(x_{1}, \cdots, x_{3 g-3}, y_{1}, \cdots, y_{3 g-3}\right): \forall i, 0 \leq a x_{i} \leq \frac{1}{3 g-3} M, 0 \leq\left(y_{i}-x_{i}\right) \ell_{o}\left(\alpha_{i}\right) \leq \delta\right\}
\end{aligned}
$$

Let also $\varpi$ be the coordinate map in Theorem 2.4.12. Then, on the one hand, by hyperbolic geometry, there is $\delta_{0}>0$ and $M_{0}>0$, depending on $o$, such that for all $\delta \leq \delta_{0}$ and $M \leq M_{0}$, one has

$$
\varpi^{-1}\left(\varpi\left(\eta^{0}\right)+W_{0}(a, M)\right) \subset V .
$$

Notice that one could choose $a$ large enough such that $\varpi^{-1}$ is a homeomorphism on $\varpi\left(\eta^{0}\right)+W_{0}(a, M)$. Therefore $\mu(V) \geq \nu\left(\varpi^{-1}\left(\varpi\left(\eta^{0}\right)+W_{0}(a, M)\right)\right)$. On the other hand,

$$
\begin{aligned}
& \nu\left(\varpi^{-1}\left(\varpi\left(\eta^{0}\right)+W_{0}(a, M)\right)\right) \\
& =\nu\left(\varpi^{-1}\left(W_{0}(a, M)\right)\right) .
\end{aligned}
$$

The last measure is easy to see to be at least $C M^{\frac{h}{2}}$. Hence the proof is finished.

Proof of Theorem 2.4.1. Ler $\gamma \in E_{n}$ and take $M=e^{-2 n}$ in Corollary 2.4.3 and $\bar{M}=$ $F \ln n e^{-2 n}$ in Corollary 2.4.5 where $F$ is the constant in Lemma 2.4.4. Then Theorem 2.4.11 implies the assumption in Lemma 2.4.10, hence Lemma 2.4.6 is true and hence $\Psi(\gamma)_{\geq M} \sim_{g, o, \epsilon} n$ and $\Psi(\gamma)_{\geq \bar{M}} \sim_{g, o, \epsilon} a_{1} n-c_{1} \ln n$. Then by Corollary 2.4.3, Corollary 2.4.5 and Theorem 2.4.11 again, the proof is finished.

### 2.5 Ergodicity of boundary representation

In this section, we will prove the main theorem: Theorem 2.2.14, namely,
Theorem 2.5.1. Let $S=S_{g}(g \geq 2)$ and $\pi_{\nu}$ be the associated quasi-regular representation of the mapping class group $\operatorname{Mod}(S)$ on $L^{2}(\mathcal{P} \mathcal{M F}(S), \nu)$. Let $\tilde{d}$ be a metric on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ which is compatible with the topology of $\mathcal{P M \mathcal { F }}(S)$. Let $n>\rho$
and $E_{n}=\mathcal{E}(\theta, \epsilon, n, o, \rho)$. Let $e_{n}=\operatorname{Pr}: E_{n} \longrightarrow \mathcal{P} \mathcal{M F}(S)$ be the radial projection which assigns $g \in E_{n}$ to the direction $\xi_{g}$ of the oriented geodesic $[o, g \cdot o]$. Then the quasi-regular representation $\pi_{\nu}$ is ergodic with respect to $\left(E_{n}, e_{n}\right)$ and any $f \in \bar{H}^{L^{\infty}(\mathcal{P M F}(S), \nu)}$, where

$$
H=<\mathbb{1}_{U}: \nu(\partial U)=0 \text { and } U \text { is a Borel subset of } \mathcal{P} \mathcal{M} \mathcal{F}(S)>
$$

Proof. The proof consists of verifying all assumptions in Theorem 2.2.13 for $E_{n}$. The first two will be verified by showing $E_{n}$ is of exponential growth (namely, Corollary 2.3.2, Corollary 2.3.3). The third one is by Proposition 2.5.2. The last one is Theorem 2.5.6.

Proposition 2.5.2. For every $n \gg \rho$, there are two sequences of real numbers $\left\{h_{r_{n}}(n, \rho)\right\}$ and $\left\{r_{n}\right\}$ such that $\lim _{n \rightarrow \infty} h_{r_{n}}(n, \rho)=\lim _{n \rightarrow \infty} r_{n}=0$ and such that

$$
\forall n \in \mathbb{N}, \forall \gamma \in E_{n}, \frac{\left\langle\pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{P M} \mathcal{F}(S)}, \mathbb{1}_{\left\{x \in \mathcal{P M} \mathcal{F}(S): i\left(x, e_{n}(\gamma)\right) \geq r_{n}\right\}}\right\rangle}{\Phi(\gamma)} \leq h_{r_{n}}(n, \rho)
$$

Proof. Let $n \gg \rho$ and $\gamma \in E_{n}$. Let $x_{\gamma}$ as before. Consider the intersection function on $\mathcal{P M} \mathcal{F}(S)$ defined by $\xi_{\gamma}$, namely, $i\left(\xi_{\gamma}, \cdot\right)$. Let $r_{n}=\frac{1}{n}$. By Corollary 2.4.3, Lemma 2.4.6 and the proof of its assumption (Theorem 2.4.11),

$$
\frac{\left\langle\pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{P M} \mathcal{F}(S)}, \mathbb{1}_{\left\{x \in \mathcal{P M F}(S): i\left(x, \xi_{\gamma}\right) \geq \frac{1}{n}\right\}}\right\rangle}{\Phi(\gamma)} \leq c(g, o, \rho) \frac{\ln n-D}{a_{1} n-c_{1} \ln \ln n+b_{1}}
$$

Take $h(n, \rho)=c(g, o, \rho) \frac{\ln n-D}{a_{1} n-c_{1} \ln \ln n+b_{1}}$, we complete the proof.

### 2.5.1 Uniform boundedness.

In this section, we complete our proof of the main theorem by proving the uniform boundedness. We start by some lemmas comparing of two types of neighborhoods.

Lemma 2.5.3. Using notations as Corollary 2.4.9. Let $\xi_{\gamma} \in \mathcal{P M \mathcal { F }}(S)$ be the direction of $[o, y=\gamma \cdot o]$ and $\xi^{\gamma} \in \mathcal{P} \mathcal{M F}(S)$ be the direction of $\left[o, x_{\gamma}=t_{y}\right]$. Then there exists a constant $C$ such that $i\left(\xi_{\gamma}, \xi^{\gamma}\right) \leq C e^{-2 L(\gamma)}$.

Proof. By Lemma 2.4.2,

$$
i^{2}\left(\xi_{\gamma}, \xi^{\gamma}\right) \leq \operatorname{Ext}_{x_{\gamma}}\left(\xi_{\gamma}\right) \operatorname{Ext}_{x_{\gamma}}\left(\xi^{\gamma}\right)
$$

Let $\alpha=\alpha(\gamma)$ as before. Then $\xi^{\gamma}=\frac{\alpha}{\sqrt{\operatorname{Ext}_{o}(\alpha)}} \sim_{g, o} e^{-L(\gamma)} \alpha$. As $\operatorname{Ext}_{x_{\gamma}}(\alpha) \sim_{g} 1$, we have $\operatorname{Ext}_{x_{\gamma}}\left(\xi^{\gamma}\right)=\frac{1}{\operatorname{Ext}_{o}(\alpha)} \operatorname{Ext}_{x_{\gamma}}(\alpha) \prec_{g, o} e^{-2 L(\gamma)}$. On the other hand, by Lemma 2.4.7, up to a multiplicative constant, one could replace $\operatorname{Ext}_{x_{\gamma}}\left(\xi_{\gamma}\right)$ by $\operatorname{Ext}_{\gamma . o}\left(\xi_{\gamma}\right)=e^{-2 L(\gamma)}$. Collect all discussions together, one can finish the proof.

Let $\xi \in \mathcal{P} \mathcal{M} \mathcal{F}(S)$ and $x \in[o, \xi]$, denote $\mathcal{I}_{C}(\xi, x)=\{\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i(\eta, \xi) \leq$ $\left.C e^{-2 d(x, o)}\right\}$. Let $M_{0}, L, \theta, \epsilon$ as in Theorem 2.2.7 and Theorem 2.2.8.

Lemma 2.5.4. Let $\eta \in \mathcal{P M \mathcal { F }}(S)$ and $\gamma \in E_{n}$. Suppose that $\eta$ does not leave $\operatorname{Teich}_{\epsilon}(S)$ eventually. Let $x \in[o, \eta]$ such that $d(x, o)=n$. Let $C>0$ and n large enough. Then if $\gamma \in E_{n}$ such that $i\left(\xi_{\gamma}, \eta\right) \leq C e^{-2 n}$, then $d(x, \gamma \cdot o) \leq \frac{1}{h} \ln \ln n$.

Proof. We argue as Lemma 2.4.4. Denote $\xi=\xi_{\gamma}$, hence by assumption $i(\xi, \eta) \leq$ $C e^{-2 n}$. First we remark that, since both $\eta$ and $\xi$ are uniquely ergodic, we have a geodesic triangle $\triangle(o, \xi, \eta)$. As $\gamma \in E_{n}$, there is also a geodesic segment $I$ of length $\ell=\frac{1}{3 h} \ln \ln n$ in $[o, \gamma \cdot o]$ ending at $p=\gamma \cdot o$ that has at least proportion $\theta$ in $\operatorname{Teich}_{\epsilon}(S)$. By Theorem 2.2.8,

$$
I \cap \mathcal{N}_{D}([o, \xi] \cap[o, \eta]) \neq \emptyset
$$

where $D$ as in Theorem 2.2.8. Choose $q \in I \cap \mathcal{N}_{D}([o, \xi] \cap[o, \eta])$. Then there are two possibilities:

Case 1: $d(q, y) \leq D$ with $y \in[o, \eta]$.

Then

$$
d(q, o)-D \leq d(o, y) \leq d(q, o)+D
$$

Since

$$
n-\ell-\rho \leq d(q, o) \leq n+\rho,
$$

we have

$$
0 \leq d(x, y) \leq \ell+D+\rho
$$

Hence,

$$
\begin{aligned}
& d(x, \gamma \cdot o) \leq d(x, y)+d(y, q)+d(q, p) \\
& \leq \ell+D+D+\ell+\rho \\
& \leq 2(\ell+D+\rho) \\
& \leq 3 \ell
\end{aligned}
$$

Case 2: $d(q, y) \leq D$ with $y \in[\xi, \eta]$.
Then by Lemma 2.4.2, one has

$$
i^{2}(\eta, \xi)=\operatorname{Ext}_{y}(\xi) \operatorname{Ext}_{y}(\eta)
$$

Now, since $d(q, y) \leq D$, by Kerckhoff's formula, we have

$$
e^{-2 D} \operatorname{Ext}_{q}(\xi) \leq \operatorname{Ext}_{y}(\xi), e^{-2 D} \operatorname{Ext}_{q}(\eta) \leq \operatorname{Ext}_{y}(\eta)
$$

Therefore,

$$
e^{-4 D} \operatorname{Ext}_{q}(\xi) \operatorname{Ext}_{q}(\eta) \leq i^{2}(\xi, \eta)
$$

On the other hand, we have

$$
\operatorname{Ext}_{q}(\xi)=e^{-2 d(o, q)}, i(\xi, \eta) \leq C e^{-2 n}
$$

which implies that

$$
e^{-4 D} \operatorname{Ext}_{q}(\eta) e^{-2 d(o, q)} \leq C^{2} e^{-4 n}
$$

That is,

$$
e^{-4 D} \operatorname{Ext}_{q}(\eta) e^{2(n-d(o, q))} \leq C^{2} e^{-2 n}
$$

By Kerckhoff's formula again,

$$
\operatorname{Ext}_{p}(\eta) \leq C^{2} e^{2 \rho+4 D} e^{-2 n}
$$

or

$$
\frac{1}{2} \ln \operatorname{Ext}_{p}(\eta) \leq \ln \left(C e^{\rho+2 D}\right)-n
$$

Apply Lemma 2.3.4, one could choose $z \in[o, \eta] \cap \operatorname{Teich}_{\epsilon}(S)$ so that, if denote $d(o, p)=$ $t, d(p, z)=a$ and $d(z, o)=b$, then $a-b \leq-n+\ln \left(C e^{2 \rho+4 D}\right)+1$. Therefore, we have

$$
0 \leq t+a-b \leq \ln \left(C e^{2 \rho+4 D}\right)+1+\rho
$$

By Theorem 2.2.7, we have $d(p, y) \leq k\left(\ln \left(C e^{2 \rho+4 D}\right)+1+\rho\right) \leq 3 \ell$. We complete the proof.

Corollary 2.5.5. Let $\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S)$ and suppose that $\eta$ does not leave Teich $\epsilon_{\epsilon}$ eventually. Let $x \in[o, \eta]$ such that $d(x, o)=n$. Let further $C>0$ and $n$ large enough. Then

$$
\begin{aligned}
& \left|\left\{\gamma \in E_{n}: \gamma \cdot o \in \operatorname{Sec}_{\mathcal{I}_{C}(\eta, x)}\right\}\right| \\
& \prec_{g, o, \rho} \ln n .
\end{aligned}
$$

Proof. By Theorem 1.2 in [4] (note that $\Lambda$ in the theorem is a constant function), when $n$ is large enough, there exists a constant $N_{0}>0$, such that $|B(x, R) \cap \operatorname{Mod}(S) \cdot o| \leq$ $N_{0} e^{h R}$. Apply Lemma 2.5.4, we have the conclusion.

Theorem 2.5.6. Under the notations used in Theorem 2.5.1, we have

$$
\sup _{n}\left\|M_{E_{n}}^{\mathbb{1}_{\mathcal{P M F}(S)}} \mathbb{1}_{\mathcal{P M F}(S)}\right\|_{L^{\infty}(\mathcal{P} \mathcal{M F}(S), \nu)}<\infty
$$

Recall that

$$
\begin{aligned}
& M_{E_{n}}^{\mathbb{1}_{\mathcal{M F}(S)}} \mathbb{1}_{\mathcal{P M \mathcal { M } ( S )}}([\xi]) \\
& \quad=\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in E_{n}} \frac{\pi_{\nu}(\gamma) \mathbb{1}_{\mathcal{P M F}(S)}([\xi])}{\Phi(\gamma)} \\
& \quad=\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in E_{n}}\left(\frac{\operatorname{Ext}_{o}(\xi)}{\operatorname{Ext}_{\gamma . o}(\xi)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} .
\end{aligned}
$$

By using the embedding map $\tau$ of $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ into $\mathcal{M \mathcal { F }}(S)$. One can rewrite the above formula to be

$$
\begin{aligned}
& M_{E_{n}}^{\mathbb{1}_{\mathcal{P F F}(S)}} \mathbb{1}_{\mathcal{P M \mathcal { M } ( S )}}([\xi]) \\
& \quad=\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in E_{n}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\xi)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} .
\end{aligned}
$$

We first introduce a type of open sets $\mathcal{I N}$ in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ defined by intersection numbers. For every $\eta \in \mathcal{P} \mathcal{M} \mathcal{F}(S), C>0, t>0$, we define

$$
\mathcal{I N}(\eta, t, C)=\left\{\xi \in \mathcal{P} \mathcal{M} \mathcal{F}(S): i(\xi, \eta) \leq C e^{-2 t}\right\}
$$

Proof of Theorem 2.5.6. Let $U(\epsilon, \theta)$ the subset of $\mathcal{P M \mathcal { F }}(S)$ of full measure. We shall give a bound independent on $n \gg \rho$ for $M_{E_{n}}^{\mathbb{1}_{\mathcal{M} \mathcal{F}(S)}} \mathbb{1}_{\mathcal{P} \mathcal{M F}(S)}(\zeta)$ for every point $\zeta \in U(\epsilon, \theta)$. Fix $R>0$. As this stage, $R$ is arbitrary, but it will be carefully chosen at the end of the proof. As usual, for $\gamma \in E_{n}$, denote $\xi_{\gamma}$ to be the direction corresponding to $[o, \gamma \cdot o]$, hence a point in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. For each point $\gamma \cdot o$, consider the open ball $B(\gamma, R)$ of radius $R$ at $\gamma \cdot o$. Denote the projection of $B(\gamma, R)$ to $\mathcal{P} \mathcal{M} \mathcal{F}(S)$ by $\mathcal{O}(\gamma \cdot o, R)$. Then by Lemma 2.3.5, the measure $\nu(\mathcal{O}(\gamma \cdot o, R)) \sim_{g, R, \rho} e^{-h n}$. Fix any $C>0$, for instance $C=1$. Dividing $E_{n}$ to be two sets $E_{n}^{1}$ and $E_{n}^{2}=E_{n}-E_{n}^{1}$ where
$E_{n}^{1}$ consists of $\gamma \in E_{n}$ so that $\xi_{\gamma} \notin \mathcal{I N}(\zeta, n, C)$. We then have, for each $\zeta \in U(\epsilon, \theta)$,

$$
\begin{align*}
& M_{E_{n}}^{1_{\mathcal{P} \mathcal{M}(S)}} \mathbb{1}_{\mathcal{P M} \mathcal{M}(S)}(\zeta) \\
& \quad=\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in E_{n}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\
& \quad=\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in E_{n}^{1}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}+\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in E_{n}^{2}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}  \tag{2.5.1}\\
& =\mathrm{I}+\mathrm{II} .
\end{align*}
$$

First we want to bound term I in Equation (2.5.1). The set $E_{n}^{1}$ can be further decomposed as two sets: $F_{n}^{1}$ and $F_{n}^{2}=E_{n}^{1}-F_{n}^{1}$, where $F_{n}^{1}=\left\{\gamma \in E_{n}^{1}: \operatorname{Pr}(B(\gamma, R)) \cap\right.$ $\mathcal{I N}(\zeta, n, C)=\emptyset\}$. One then has,

$$
\begin{align*}
\mathrm{I}= & \frac{1}{\left|E_{n}\right|} \sum_{\gamma \in F_{n}^{1}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}+\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in F_{n}^{2}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}  \tag{2.5.2}\\
& =\mathrm{III}+\mathrm{IV}
\end{align*}
$$

We first deal with term III. First notice that

$$
\forall y \in B(\gamma, R), \frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\zeta)} \sim_{R} \frac{1}{\operatorname{Ext}_{y}(\zeta)}
$$

on the other hand, by Lemma 2.4.2, for $\nu$-almost every $\xi_{y} \in \mathcal{O}(\gamma \cdot o, R)$,

$$
\left(\frac{1}{\operatorname{Ext}_{y}(\zeta)}\right)^{\frac{h}{4}} \prec_{\rho, R} \frac{1}{e^{\frac{h n}{2}}\left(i\left(\xi_{y}, \zeta\right)\right)^{\frac{h}{2}}}
$$

Hence, for $\nu$-almost every $\xi_{y} \in \mathcal{O}(\gamma \cdot o, R)$,

$$
\begin{aligned}
& \left(\frac{1}{\operatorname{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}} \\
& \prec_{R, \rho} e^{-\frac{h n}{2}} \frac{1}{\left(i\left(\xi_{y}, \zeta\right)\right)^{\frac{h}{2}}} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \text { III }=\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in F_{n}^{1}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\
& \prec_{R} \frac{1}{\left|E_{n}\right|} \sum_{\gamma \in F_{n}^{1}} \frac{e^{-\frac{h n}{2}}}{\nu(\mathcal{O}(\gamma \cdot o, R))} \int_{\mathcal{O}(\gamma \cdot o, R)} \frac{1}{(i(\eta, \zeta))^{\frac{h}{2}}} d \nu(\eta) \frac{1}{\Phi(\gamma)} .
\end{aligned}
$$

Note that there are bounded number intersections of open sets on the form $\mathcal{O}(\gamma$. $o, R)$ and the bound depends on $R$ and $\rho$. Thus, since $\left|E_{n}\right| \asymp e^{h n}$ (Corollary 2.3.2) and $\Phi(\gamma) \succ_{g, o, \rho}\left(a_{1} n-c_{1} \ln \ln n+b_{1}\right) e^{-\frac{h n}{2}}$ (Harish-Chandra estimates), substitute all these together, one has,

$$
\begin{align*}
& \mathrm{III} \prec_{g, o, \rho, R} \frac{1}{a_{1} n-c_{1} \ln \ln n+b_{1}} \int_{\left\{\eta \in \mathcal{P} \mathcal{M F}(S): i(\eta, \zeta)>C e^{-2 n}\right\}}\left(\frac{1}{i(\eta, \zeta)}\right)^{\frac{h}{2}} d \nu(\eta)  \tag{2.5.3}\\
& \prec_{g, o, \rho, R} 1 .
\end{align*}
$$

The last inequality follows from the fact that $\zeta \in U(\epsilon, \theta)$ and the proof of HarishChandra estimats.

We now deal with terms IV and II. Take $H_{n}=E_{n}^{2} \cup F_{n}^{2}$. These two terms can be put together to obtain:

$$
\begin{align*}
& \mathrm{IV}+\mathrm{II} \\
& =\frac{1}{\left|E_{n}\right|} \sum_{\gamma \in H_{n}}\left(\frac{1}{\operatorname{Ext}_{\gamma . o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\
& \leq \frac{1}{\left|E_{n}\right|} \sum_{\gamma \in H_{n}} \frac{e^{\frac{h L(\gamma)}{2}}}{\Phi(\gamma)}  \tag{2.5.4}\\
& \sim_{g, \rho, o} e^{-h n} \sum_{\gamma \in H_{n}} \frac{e^{\frac{h n}{2}}}{\left(a_{1} n-c_{1} \ln \ln n+b_{1}\right) e^{\frac{-h n}{2}}} \\
& =\frac{1}{a_{1} n-c_{1} \ln \ln n+b_{1}}\left|H_{n}\right| .
\end{align*}
$$

We now CLAIM that $\left|H_{n}\right| \prec_{g, o, \rho, \epsilon, \theta} \ln n$, thus the sum IV + II tends to 0 when $n \rightarrow \infty$ which finishes the proof of the theorem.

It remains to prove the above CLAIM.

Proof of the CLAIM. By Corollary 2.5.5, the number $\left|E_{n}^{2}\right| \prec \ln n$. We now show that so is $\left|F_{n}^{2}\right|$. Choose $R \leq \min \left\{1, S_{0}\right\}$ where $S_{0}$ is the injective radius of $o$ in the $\epsilon$ - thick part of the moduli space $\mathcal{M}(S)=\operatorname{Teich}(S) / \operatorname{Mod}(S)$. Hence for every $\gamma \in \operatorname{Mod}(S)$ and every point $q \in B(\gamma \cdot o, R), q \in \operatorname{Teich}_{\epsilon}(S)$ and $d_{T}(\gamma \cdot o, q) \leq 1$. Fix such $R$ a priori. Assume now that $\gamma \in F_{n}^{2}$, namely $\operatorname{Pr}(B(\gamma \cdot o, R)) \cap \mathcal{I N}(\zeta, n, C) \neq \emptyset$. As $U(\epsilon, \theta)$ has full measure, in particular, it is dense in $\mathcal{P} \mathcal{M} \mathcal{F}(S)$, thus one can choose $q \in B(\gamma \cdot o, R)$ so that the direction $\xi_{q}$ of $[o, q]$ is in $U(\epsilon, \theta) \cap \mathcal{I N}(\zeta, n, C)$. By Theorem 2.2.9, there is a $P=P(\epsilon)$, so that the two geodesics $[o, \gamma \cdot o]$ and $[o, q]$ are $P$-fellow travelling in a parametrized fashsion. Now consider the $P$-neighborhood $\mathcal{N}_{P}$ of $\operatorname{Teich}_{\epsilon}(S)$, namely the union of points in Teich $(S)$ that has distance at most $P$ with a point in $\operatorname{Teich}_{\epsilon}(S)$. As $\operatorname{Mod}(S)$ acts as isometries on Teich $(S)$ and $\operatorname{Teich}_{\epsilon}(S)$ is $\operatorname{Mod}(S)$-invariant and cocompact, the neighborhood $\mathcal{N}_{P}$ is $\operatorname{Mod}(S)$-invariant and cocompact. By Mumford's compactness, there is a small $\epsilon^{\prime}$ so that $\mathcal{N}_{P} \subset \operatorname{Teich}_{\epsilon^{\prime}}(S)$. Then as $\gamma \in E_{n}$, the geodesic segment $[o, q]$ has the property that it contains a segment $I=[a, q]$ of length $\frac{1}{3 h} \ln \ln n$ such that $I$ has at least $\theta$ in $\operatorname{Teich}_{\epsilon^{\prime}}(S)$. Note that $\epsilon$ is fixed, hence $C$ depends on $g$ and $o$. Hence by Theorem 2.2.8, there are two constants $D^{\prime}=D^{\prime}\left(\epsilon^{\prime}, \theta\right)$ and $L_{0}^{\prime}=L_{0}^{\prime}\left(\epsilon^{\prime}, \theta\right)$ satisfy Theorem 2.2.8. Take $n$ large enough and follow the proof Lemma 2.5.4 and Corollary 2.5.5, one has that $\left|F_{n}^{2}\right| \prec \ln n$.

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