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$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \nabla \cdot T + f$$

$$e^{i\pi} + 1 = 0$$

THÈSE DE DOCTORAT

Groupes aux coupures modérées

Présentée par

BAT-OD BATTSEREN

Présentée en vue de l'obtention du grade de

Docteur en Mathématiques de l'Université Côte d'Azur

Thèse dirigée par

INDIRA CHATTERJI

Soutenue le 30 juin 2021

Devant le jury composé de

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Résumé

Dans cette thèse, nous étudierons quatre types de suites de fonctions continues à support compact sur un groupe localement compact, à savoir les coupures modérées [caractéristiques] (complètement bornées), et leurs croissances dans l'algèbre de Banach des multiplicateurs de Fourier (complètement bornés). Cette nouvelle notion étend la moyennabilité faible et la propriété de décroissance rapide. L'objectif principal est de fournir des exemples de groupes admettant ou pas ce type de suites, à l'aide d'outils analytiques, algébriques et géométriques.

Nous démontrons que les groupes de Baumslag-Solitar $BS(p, q)$ pour $p, q \in \mathbb{N}$ et certains groupes métabéliens de type fini, dont $\mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ pour $p, q \in \mathbb{N}$ premiers entre eux, $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ pour $d \in \mathbb{N}$, et le groupe de l'allumeur de réverbères $\mathbb{Z}_2 \wr \mathbb{Z}$, admettent des coupures modérées caractéristiques complètement bornées. Ceci est réalisé en montrant que l'existence de coupures modérées [caractéristiques] (complètement bornées) est stable par extension par un groupe à croissance polynomiale. Nous proposerons également une méthode pour construire un groupe de type fini sans coupures modérées en utilisant la propriété (T_{Schur}, G, K) .

De plus, nous proposerons deux résultats comme applications de coupures modérées. Le premier résultat montre que tout réseau uniforme dans $SL_3(\mathbb{R})$ admet un multiplicateur de Fourier qui n'est pas complètement borné. Ceci fournit un exemple à l'appui de la question ouverte: "La moyennabilité d'un groupe discret Γ est-elle caractérisée par le fait que tous les multiplicateurs de Fourier de Γ sont complètement bornés?" Le deuxième résultat est lié à l'application d'induction $M_0A(\Gamma) \rightarrow M_0A(G)$ qui est contractante pour tout groupe localement compact G et son réseau Γ . En particulier, lorsque G (ou Γ) est moyennable, l'application d'induction $MA(\Gamma) \rightarrow MA(G)$ est contractante. Nous démontrerons que la moyennabilité de G est essentielle pour la continuité de cette dernière application.

Mots-clefs: Groupes localement compacts, algèbres de groupe, multiplicateurs de Fourier, moyennabilité faible, propriété de décroissance rapide, coupures modérées.

Abstract

In this thesis, we will study four types of sequences of compactly supported continuous functions on a locally compact group, namely (completely bounded) [characteristic] tame cuts, and their growth in the Banach algebra of (completely bounded) Fourier multipliers. This new notion extends weak amenability and Rapid Decay property. The main goal is to provide examples of groups admitting or not-admitting such kind of sequences using analytic, algebraic, and geometric tools.

We will prove in particular that the Baumslag-Solitar groups $BS(p, q)$ for $p, q \in \mathbb{N}$ and some finitely generated metabelian groups, including $\mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ for coprime $p, q \in \mathbb{N}$, $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ for $d \in \mathbb{N}$, and the Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$, admit completely bounded characteristic tame cuts. This is achieved by showing that the existence of (completely bounded) [characteristic] tame cuts is stable under extension by a group with polynomial growth. We will also propose a method to construct a finitely generated group without tame cuts using property (T_{Schur}, G, K) .

In addition, we will propose two results as an application of tame cuts. The first one states that any uniform lattice in $SL_3(\mathbb{R})$ admits a Fourier multiplier that is not completely bounded. This provides a supporting example to the following open question: “Is amenability of a discrete group Γ characterized by the fact that all Fourier multipliers of Γ are completely bounded?” The second application is related to the induction mapping $M_0A(\Gamma) \rightarrow M_0A(G)$ which is known to be norm decreasing for any locally compact group G and any of its lattice Γ . In particular, when G (or Γ) is amenable, the induction mapping $MA(\Gamma) \rightarrow MA(G)$ is continuous. We will show that the amenability of G is essential for the continuity of the latter mapping.

Key words: Locally compact groups, group algebras, Fourier multipliers, weak amenability, Rapid Decay property, tame cuts.

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Introduction en français

Dans cette thèse, nous étudions quatre types de suites de fonctions continues à support compact sur un groupe localement compact, à savoir les coupures modérées [caractéristiques] (complètement bornées), et leur croissance dans l'algèbre des multiplicateurs de Fourier (complètement bornés). Les définitions précises sont données dans le Chapitre 3. Dans ce qui suit, nous décrirons comment les coupures modérées sont liées à certaines propriétés bien connues et exposerons nos principaux résultats.

Analyse harmonique

Rappelons un peu d'analyse harmonique classique. Pour chaque fonction intégrable $f \in L^1(\mathbb{T})$, on lui associe des polynômes trigonométriques

$$S_N[f](t) = \sum_{n=-N}^N \widehat{f}(n)e^{int}, \quad \text{où} \quad \widehat{f}(n) = \int_0^{2\pi} f(t)e^{-int} dt.$$

On peut aussi écrire $S_N[f] = D_N * f$ en utilisant le produit de convolution où

$$D_N = \sum_{n=-N}^N e^{int} = \frac{\sin((N + 1/2)x)}{\sin(x/2)}$$

est le noyau de Dirichlet. La suite $(S_N[f])_{N \in \mathbb{N}}$ ne converge pas nécessairement vers f pour tout $f \in L^1(\mathbb{T})$ dans la topologie de la norme L^1 (ou de manière équivalente, pour tout $f \in C(\mathbb{T})$ dans la topologie de convergence uniforme) [Kol23]. En effet, si $\|D_N * f - f\|_1 \rightarrow 0$ pour tout $f \in L^1(\mathbb{T})$, la norme $\|D_N\|_1$ de l'opérateur de convolution doit être uniformément bornée pour tout $N \in \mathbb{N}$, mais nous avons $\|D_N\|_1 = \frac{4}{\pi} \log(N) + O(1)$, qui sont appelées les constantes de Lebesgue. Plus tard, il a été prouvé dans [MPS81] que l'estimation $O(\log(N))$ est asymptotiquement minimale, dans le sens que pour toute suite de noyaux caractéristiques $(k_n)_{n \in \mathbb{N}}$ telle

que pour chaque $n \in \mathbb{N}$ le spectre de k_n contient l'ensemble $\{-n, -n + 1, \dots, n\}$, on a $\|k_n\|_1 \geq C \log(N)$ pour tout $n \in \mathbb{N}$. En étudiant les coupures modérées caractéristiques (complètement bornées), nous obtiendrons des informations asymptotiques pour un groupe discret et dénombrable. Pour cela, nous utilisons l'algèbre de Fourier $A(G)$ et les algèbres des multiplicateurs de Fourier, $MA(G)$ et $M_0A(G)$. On note que lorsque G est abélien, l'algèbre de Fourier $A(G)$ est isomorphe à $L^1(\widehat{G})$, et les algèbres $MA(G)$ et $M_0A(G)$ sont isomorphes à l'algèbre des mesures $M(\widehat{G})$ via la transformation de Fourier.

L'idée de généraliser des propriétés d'analyse harmonique classique n'est pas nouvelle. La plus connue est la moyennabilité de John von Neumann. Rappelons qu'un groupe G localement compact est moyennable s'il existe une approximation de l'unité dans l'algèbre de Fourier $A(G)$ constituée de fonctions définies positives, c'est-à-dire une suite généralisée $(\varphi_n)_{n \in I}$ de fonctions définies positives dans $A(G)$ telles que $\|\varphi_n f - f\|_A \rightarrow 0$ pour tout $f \in A(G)$. Dans le cas de $G = \mathbb{Z}$, cela équivaut à l'existence d'une suite $(\psi_n)_{n \in \mathbb{N}}$ de polynômes trigonométriques normalisés positifs sur \mathbb{T} telle que $\|\psi_n * f - f\|_1 \rightarrow 0$ pour tout $f \in L^1(\mathbb{T})$. Un exemple d'une telle suite est donné par les noyaux de Fejér définis par

$$F_N = \frac{1}{N} \sum_{n=0}^{N-1} D_n = \frac{1}{N} \left(\frac{\sin(Nx/2)}{\sin(x/2)} \right)^2.$$

Dans la définition de moyennabilité, si on remplace la condition de positivité définie par $\sup_n \|\varphi_n\|_{M_0A} < \infty$, on obtient la définition de moyennabilité faible. Si G est faiblement moyennable, la valeur minimale possible $\Lambda(G)$ de $\sup_n \|\varphi_n\|_{M_0A}$ existe et s'appelle la constante de Cowling-Haagerup. Si G n'est pas faiblement moyennable, pour toute suite $(\varphi_n)_{n \in \mathbb{N}}$ dans $A(G)$ avec $\varphi_n \rightarrow 1$ uniformément sur les compacts, la norme $\|\varphi_n\|_{M_0A}$ tend vers l'infini. Dans cette thèse, nous proposons d'étudier la croissance de $\|\varphi_n\|_{M_0A}$ et $\|\varphi_n\|_{MA}$, qui est l'essence des coupures modérées (complètement bornées). Remarquons qu'en répétant les mêmes fonctions dans la suite, nous pouvons rendre la croissance de $\|\varphi_n\|_{M_0A}$ arbitrairement lente. Afin de remédier à cet inconvénient, nous utilisons une fonction de longueur propre $\ell : G \rightarrow \mathbb{R}_+$, et nous ajoutons la condition que φ_n prend la valeur 1 sur la boule $B_n = \{x \in G : \ell(x) \leq n\}$. Maintenant, il est possible de parler de croissance asymptotiquement minimale de $\|\varphi_n\|_{M_0A}$ qui serait une information dépendant du groupe G et de la fonction de longueur ℓ . Lorsque G est faiblement moyennable, une telle croissance est $O(1)$ pour toute fonction de longueur propre, donc la con-

stante de Cowling-Haagerup est une information plus fine que les coupures modérées complètement bornées.

Algèbres de groupe

Parmi les groupes topologiques, les groupes localement compacts sont particulièrement intéressants car ils admettent une mesure de Haar. Cette mesure est l'un des principaux outils pour étudier le groupe de manière analytique, même si le groupe n'est initialement équipé que d'une structure topologique et algébrique. Les premiers espaces que nous pouvons construire en utilisant la mesure de Haar sont les espaces classiques $L^p(G)$. C'est une amélioration par rapport aux espaces $C(G)$, $C_c(G)$ et $C_b(G)$ qui peuvent être construits pour n'importe quel groupe topologique. L'espace de Banach $L^1(G)$ a une structure $*$ -algèbre avec le produit de convolution et l'involution. Une structure de C^* -algèbre offre une théorie plus riche et mieux comprise. Malheureusement, $L^1(G)$ ne peut pas être une C^* -algèbre à l'exception du groupe trivial. Nous considérons les $*$ -représentations de $L^1(G)$ pour extraire une C^* -algèbre. La plus naturelle est la représentation régulière $(\lambda, L^2(G))$ où $\lambda(f)$ pour $f \in L^1(G)$ agit sur l'espace de Hilbert $L^2(G)$ par convolution: $\lambda(f)g = f * g$, $g \in L^2(G)$. La complétion $C_\lambda^*(G)$ (resp. $L(G)$) de $\lambda(L^1(G))$ dans la topologie de norme (resp. topologie d'opérateur fort) est appelée C^* -algèbre réduite (resp. algèbre de von Neumann) du groupe. De nombreuses propriétés de groupe importantes telles que moyennabilité, propriété de Haagerup, moyennabilité faible, propriété A et propriété (T) de Kazhdan sont formulées en termes de ces algèbres. L'unique prédual Banach $A(G)$ de $L(G)$ est appelé l'algèbre de Fourier de G . Il coïncide avec l'espace de tous les coefficients matriciels associés à la représentation régulière, équipé de la norme $\varphi \mapsto \inf\{\|\xi\|_2\|\eta\|_2 : \xi, \eta \in L^2(G), \varphi = \langle \lambda(\cdot)\xi, \eta \rangle\}$. Avec la multiplication ponctuelle, $A(G)$ devient une algèbre de Banach commutative. Nous renvoyons les lecteurs à [Eym64] pour les principales propriétés de ces algèbres.

Une autre algèbre importante provenant du groupe localement compact est donnée par les multiplicateurs de Fourier $MA(G)$, constituée des fonctions continues bornées qui laisse l'algèbre de Fourier $A(G)$ invariante par multiplication ponctuelle. Par le théorème du graphe fermé, chaque fonction dans $MA(G)$ définit une application linéaire continue sur $A(G)$, et avec la norme d'opérateur correspondante, $MA(G)$ devient une algèbre de Banach. Dans [DCH85], cet espace a été caractérisé comme des multiplicateurs de $C_\lambda^*(G)$ et $L(G)$. Nous pouvons donc

considérer des multiplicateurs de Fourier complètement bornés, $M_0A(G)$. Cet espace a de meilleures caractérisations que les multiplicateurs de Fourier habituels, et bénéficie en outre de quelques propriétés de stabilité de base [CH89, DCH85]. Nous voulons étudier la croissance des coupures modérées dans ces deux espaces. Si G est moyennable, ces deux espaces coïncident isométriquement, et l'inclusion $A(G) \rightarrow MA(G)$ est isométrique, donc les calculs sont facilités. Dans le cas où G est abélien, on a même $A(G) \cong L^1(\widehat{G})$ et $MA(G) = M_0A(G) \cong M(\widehat{G})$ comme algèbres de Banach commutatives.

Dans le Chapitre 1, nous discuterons de ces algèbres et leur relation avec les représentations du groupe plus en détail.

Opérateur de troncature

Les comportements de la $C_\lambda^*(G)$ -norme sont notoirement difficiles à comprendre. Par exemple, la conjecture de Valette sur la décroissance rapide pour les réseaux uniformes dans les groupes de Lie de rang supérieur est toujours ouverte [Val02]. Différentes approches pour comprendre les normes d'opérateurs sont étudiées dans des cas particuliers: lorsque G est moyennable, on a $\|\lambda(f)\| = \|f\|_1$ pour toute fonction non négative $f \in C_c(G)$, et lorsque G est un groupe de Lie simple connexe, la fonction sphérique de Harish-Chandra ϕ_0 satisfait $\|\lambda(f)\| = \int_G \phi_0(x)f(x)dx$ pour tout $f \in C_c(G)_+$. D'autre part, lorsqu'un groupe discret Γ admet des coupures modérées caractéristiques $(\varphi_n)_{n \in \mathbb{N}}$, nous obtenons des informations asymptotiques sur la façon dont la norme $\|\lambda(f)\|$ change après avoir tronqué (ou coupé) la fonction $f \in C_c(G)$ par le support de φ_n . Dans de nombreux espaces, le processus de troncature définit un opérateur contractant. Par exemple, les espace $\ell^p(G)$, $1 \leq p \leq \infty$ ou plus généralement l'espace $C_c(\Gamma)$ équipé d'une norme inconditionnelle N , à savoir $N(f) \leq N(g)$ si $|f| \leq |g|$. La situation est très différente pour $C_\lambda^*(\Gamma)$. Par exemple, la norme du multiplicateur de Fourier de la fonction caractéristique $\varphi_n = \mathbb{1}_{\mathbb{Z} \cap [-n, n]} \in \ell^\infty(\mathbb{Z})$ est donnée par la constante de Lebesgue

$$\|M_{\varphi_n} : C_\lambda^*(\mathbb{Z}) \rightarrow C_\lambda^*(\mathbb{Z})\| = \|\varphi_n\|_{MA} = \|D_n\|_1 = \frac{4}{\pi} \log n + O(1), \quad (1)$$

et elle n'est pas borné. Les coupures modérées caractéristiques (complètement bornées) nous indiquent à quel point la norme $C_\lambda^*(\Gamma)$ est loin d'être inconditionnelle.

La propriété de Décroissance Rapide

La propriété de Décroissance Rapide a été initialement observée dans [Haa79] pour le groupe libre F_2 à deux générateurs et développée dans [Jol90], qui lui a donné son nom. En établissant la propriété de Décroissance Rapide pour F_2 , Haagerup a prouvé qu'il y a une approximation n -positive de l'unité dans l'algèbre de Fourier $A(F_2)$. Cela montre que $C_\lambda^*(F_2)$ a la propriété d'approximation n -positive pour toute $n \in \mathbb{N}$ mais pas la propriété d'approximation complètement positive.

Une autre contribution importante de la propriété de Décroissance Rapide est son application dans la théorie- K . Le calcul des groupes $K_*(C_\lambda^*(G))$ peut être très difficile, mais lorsque le group G a la propriété de Décroissance Rapide, pour un paramètre suffisamment grand $s \in \mathbb{R}_+$, l'espace s -Sobolev $H_\ell^s(G)$ devient une sous-algèbre dense de $C_\lambda^*(G)$ avec les mêmes K -groupes. Par rapport à la norme d'opérateur compliquée de $C_\lambda^*(G)$, la norme d'espace s -Sobolev a une formule explicite, ce qui fait de $H_\ell^s(G)$ une cible plus facile à étudier. En utilisant cette idée, Lafforgue a montré dans [Laf98] que la conjecture de Baum-Connes a une réponse affirmative pour tout réseau uniforme dans $SL_3(\mathbb{R})$. Ce fut le premier groupe discret et infini satisfaisant la conjecture de Baum-Connes et la propriété (T) de Kazhdan.

Dans cette thèse, la propriété de Décroissance Rapide joue un rôle important pour fournir les premiers exemples de groupes discrets avec des coupures modérées caractéristiques: la discreture assure que $C_\lambda^*(\Gamma)$ -norme domine toujours $\ell^2(\Gamma)$ -norme, et la propriété de Décroissance Rapide fournit l'inégalité inverse avec un facteur polynomial. Certains résultats de stabilité de la propriété de Décroissance Rapide (par exemple [Jol90, Gar15]) sont partiellement adaptables aux coupures modérées. Nous utiliserons également le résultat de Lafforgue sur les réseaux uniformes de $SL_3(\mathbb{R})$ pour fournir deux applications de coupures modérées.

La moyennabilité faible

Comme mentionné précédemment, la notion de coupures modérées (complètement bornées) est un analogue de la moyennabilité faible. Cette similitude nous permet d'adapter certaines idées et techniques issues du développement de moyennabilité faible. Par exemple, les inégalités de [LDis11], [dL13] et [dLdIS18] impliquent directement qu'aucun groupe de Lie simple de rang supérieur avec un centre fini ne peut admettre de coupures modérées, et aucun de ses réseaux uniformes n'admet

coupures modérées complètement bornées.

La constante de Cowling Haagerup est un invariant d'équivalence mesuré (EM), illustré dans [Oza12]. Ceci a été réalisé en considérant l'application d'induction $\Phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Lambda)$ dont les restrictions $M_0A(\Gamma) \rightarrow M_0A(\Lambda)$ et $A(\Gamma) \rightarrow A(\Lambda)$ sont décroissantes. On pourrait penser que le taux de croissance ou le degré des coupures modérées (complètement bornées) est invariant par EM. Cependant, cela ne semble pas évident car l'équivalence n'a pas grand-chose à voir avec la croissance du groupe. En effet, tous les groupes discrets, dénombrables et moyennables sont dans la même classe d'EM, et leur croissance peut être de tout type. Un autre problème est que l'application d'induction ne préserve pas le support fini. Nous avons néanmoins prouvé le résultat suivant.

Théorème 1. *Soit Λ et Γ deux groupes dénombrables, (X, d) un espace métrique et μ une mesure de Borel sur X . Supposons qu'il existe deux actions $\Lambda \curvearrowright X \curvearrowright \Gamma$ commutantes, libres et propres qui préservent la mesure et la distance et tel que l'action Γ admette un domaine fondamental compact. Fixons un point base $v_0 \in X$ et définissons les fonctions de longueur $\ell_\Gamma : \delta \in \Gamma \mapsto d(v_0 \cdot \delta, v_0)$ et $\ell_\Lambda : s \in \Lambda \mapsto d(s \cdot v_0, v_0)$. Si (Γ, ℓ_Γ) a des coupures modérées complètement bornées, il en va de même pour (Λ, ℓ_Λ) .*

En utilisant l'application d'induction contractante $M_0A(\Gamma) \rightarrow M_0A(G)$, Haagerup a montré qu'un réseau Γ dans un groupe localement compact G est faiblement moyennable si et seulement si G l'est [Haa16]. En particulier, si G ou Γ est moyennable, nous avons $MA(\Gamma) \rightarrow MA(G)$ continue. Le théorème ci-dessous montre que moyennabilité est essentielle pour que cette dernière application soit continue. La preuve utilise des coupures modérées et sera donnée dans la Section 3.6.

Théorème 2. *Soit Γ un réseau uniforme dans $G = SL_3(\mathbb{R})$. Alors l'application*

$$\Phi : MA(\Gamma) \rightarrow MA(G), \quad \varphi \mapsto \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi \mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega$$

n'est pas continue.

Considérez la question ouverte suivante.

Conjecture. Soit Γ un groupe discret. Si tous les multiplicateurs de Fourier de Γ sont complètement bornés, c'est-à-dire $M_0A(\Gamma) = MA(\Gamma)$, alors Γ est moyennable.

La réciproque est démontrée dans [Neb82, Los84] dans le cas des groupes localement compacts. Dans [HSS10], un multiplicateur de Fourier non complètement

borné explicite du groupe libre F_2 à générateurs a été construit. Le même résultat a été fait dans [Boz82] en utilisant des ensembles lacunaires infinis. Il s'ensuit que la conjecture a une réponse positive pour tout groupe discret contenant une copie de F_2 (par exemple, des groupes linéaires discrets finis sur un corps). Nous montrerons le théorème suivant. L'idée de la preuve est très proche de celle du Théorème 2, et nous n'utiliserons pas le fait que Γ contient F_2 comme un sous-groupe.

Théorème 3. *Soit Γ un réseau uniforme dans $G = SL_3(\mathbb{R})$. Alors, il existe un multiplicateur de Fourier de Γ qui n'est pas complètement borné. En d'autres termes, $M_0A(\Gamma) \neq MA(\Gamma)$.*

Propriété (T_{Schur}, G, K)

La propriété (T_{Schur}, G, K) a été introduite dans [Lia16] comme un analogue de la propriété (T_{Schur}) . Supposons que G est un groupe réductif sur un corp local, K est son sous-groupe compact maximal, et Γ est un réseau de G satisfaisant la propriété (T_{Schur}, G, K) . Alors, toutes les méthodes connues pour prouver la conjecture de Baum-Connes échouent pour Γ . Dans notre contexte, la propriété (T_{Schur}, G, K) s'oppose à l'existence de coupures modérées de la manière suivante.

Théorème 4. *Soit H un sous-groupe fermé d'un groupe G localement compact non-borné. Supposons que H satisfait la propriété (T_{Schur}, G, K, ℓ) pour un sous-groupe compact K de G et une fonction de longueur propre ℓ de G . Alors $(H, \ell|_H)$ n'admet pas de coupures modérées K -bi-invariantes.*

Corollaire 1. *Si G est non-borné et a la propriété (T_{Schur}, G, K, ℓ) , alors (G, ℓ) n'a pas de coupures modérées.*

Corollaire 2. *Supposons que G soit un groupe infini de type fini et que H soit un sous-groupe de type fini de G . Supposons que H soit au plus polynomialement déformé dans G , c'est-à-dire qu'il existe $k \geq 0$ tel que $\ell_H(x) \leq k\ell_G(x)^k + k$ pour tout $x \in H$, où ℓ_G et ℓ_H sont les fonctions de longueur de mot de G et H , respectivement. Si H a la propriété $(T_{Schur}, G, \{e\}, \ell_G)$, alors (H, ℓ_H) n'a pas de coupures modérées.*

Il est à noter que le seul exemple connu satisfaisant la propriété (T_{Schur}, G, K) est le groupe H (ou tout sous-groupe discret de $Sp_4(\mathbb{F}_q((\pi)))$ contenant H) des matrices triangulaires supérieures dans $Sp_4(\mathbb{F}_q((\pi)))$ dont les entrées sont dans $\mathbb{F}_q[\pi^{-1}]$, et 1 sur la diagonale.

Plus d'exemples

Les groupes hyperboliques [Oza08], les groupes CAT(0) cubiques de dimension finie [Miz08], les groupes de Coxeter [Fen02], et les groupes agissant proprement sur un produit des graphes hyperboliques à géométrie bornée [Ver19] ont tous des coupures modérées caractéristiques complètement bornées. En outre, les groupes discrets satisfaisants la propriété de Décroissance Rapide ont des coupures modérées caractéristiques. Nous fournissons en plus les exemples suivants.

Théorème 5. *Les groupes suivants ont des coupures modérées caractéristiques complètement bornées.*

(i) $\Gamma_A = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ pour tout $d \in \mathbb{N}$ et $A \in SL_d(\mathbb{Z})$.

(ii) $\Gamma_{p,q} = \mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{1}{q}} \mathbb{Z}$ pour tout $p, q \in \mathbb{N}$ premiers entre eux.

(iii) Groupe d'allumeur de réverbères $\mathbb{Z}_p \wr \mathbb{Z}$ pour tout $p \in \mathbb{N}$.

(iv) Groupe de Baumslag-Solitar $BS(p, q) = \langle a, t \mid ta^pta^{-1} = a^q \rangle$ pour tout $p, q \in \mathbb{N}$.

Les exemples (i) – (iii) sont obtenus en utilisant Proposition 3.3.9 qui dit que coupures modérées [caractéristiques] (complètement bornés) sont stables sous extension par un groupe à croissance polynomiale, et (ii) est utilisé pour montrer (iv).

Organisation du manuscrit

Dans le Chapitre 1, nous rapellons les notions, telles que les représentations de groupe et les algèbres de groupe. Dans le Chapitre 2, nous discuterons plus en détail de la moyennabilité faible et de la propriété de Décroissance Rapide. Le Chapitre 3 est entièrement dédié aux coupures modérées [caractéristiques] (complètement bornées). Dans la Section 3.1, nous donnerons les définitions des coupures modérées et fournirons les premiers exemples en explorant les connexions à la moyennabilité faible et à la propriété de Décroissance Rapide. Dans la Section 3.2, nous proposerons une condition suffisante pour admettre des coupures modérées qui étend la propriété de Décroissance Rapide. Dans la Section 3.3, nous étudierons les propriétés de stabilité des coupures modérées. Dans la Section 3.4 et 3.5 nous prouverons les Théorème 4 et 5. Dans la Section 3.6, nous prouverons les Théorème 2 et 3.

Introduction

In this thesis, we will study four types of sequences of compactly supported continuous functions on a locally compact group, namely (completely bounded) [characteristic] tame cuts, and their growth in the algebra of (completely bounded) Fourier multipliers. The precise definitions are given in Chapter 3. In what follows, we will describe how tame cuts relate to some well-known properties and state our main results.

Harmonic analysis

Let us recall some facts from classical harmonic analysis. For each integrable function $f \in L^1(\mathbb{T})$, we associate the trigonometric polynomials

$$S_N[f](t) = \sum_{n=-N}^N \widehat{f}(n)e^{int}, \quad \text{where} \quad \widehat{f}(n) = \int_0^{2\pi} f(t)e^{-int} dt.$$

One can also write $S_N[f] = D_N * f$ using the convolution product, where

$$D_N = \sum_{n=-N}^N e^{int} = \frac{\sin((N + 1/2)x)}{\sin(x/2)}$$

is the Dirichlet kernel. It is a well-known fact that the sequence $(S_N[f])_{N \in \mathbb{N}}$ does not necessarily converge to f for any $f \in L^1(\mathbb{T})$ in the $L^1(\mathbb{T})$ -topology (or equivalently in the topology of uniform convergence) [Kol23]. Indeed, should $\|D_N * f - f\|_1 \rightarrow 0$ for all $f \in L^1(\mathbb{T})$, the convolution operator norms $\|D_N\|_1$, $N \in \mathbb{N}$ must be bounded, yet we have $\|D_N\|_1 = \frac{4}{\pi} \log(N) + O(1)$ which are called the Lebesgue constants. Later, it was proved in [MPS81] that such estimation $O(\log(N))$ is asymptotically minimal in the sense that for any sequence of characteristic kernels $(k_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$ the spectrum of k_n contains the set $\{-n, -n + 1, \dots, n\}$, we have

$\|k_n\|_1 \geq C \log(N)$ for all $n \in \mathbb{N}$. In our study of (completely bounded) characteristic tame cuts, we aim to understand such asymptotic information for any discrete, countable group. For that, we use the Fourier algebra $A(G)$ and Fourier multiplier algebras $MA(G)$ and $M_0A(G)$. We note that when G is abelian, the Fourier algebra $A(G)$ is isomorphic to $L^1(\widehat{G})$, and the algebras $MA(G)$ and $M_0A(G)$ are isomorphic to the measure group algebra $M(\widehat{G})$ via Fourier transform.

This idea to generalize properties of harmonic analysis on \mathbb{T} is not new. The most important one is John von Neumann's amenability. Recall that a locally compact group G is amenable if there is an approximate unit in the Fourier algebra $A(G)$, consisting of positive definite functions, that is a net $(\varphi_n)_{n \in I}$ of positive definite functions in $A(G)$ such that $\|\varphi_n f - f\|_A \rightarrow 0$ for all $f \in A(G)$. In the case of $G = \mathbb{Z}$, this is equivalent to the existence of a sequence $(\psi_n)_{n \in \mathbb{N}}$ of normalized, positive, trigonometric polynomials on \mathbb{T} such that $\|\psi_n * f - f\|_1 \rightarrow 0$ for all $f \in L^1(\mathbb{T})$. An example of such sequence is given by the Fejér kernels, defined by

$$F_N = \frac{1}{N} \sum_{n=0}^{N-1} D_n = \frac{1}{N} \left(\frac{\sin(Nx/2)}{\sin(x/2)} \right)^2.$$

In the definition of amenability, if we replace positive definite with $\sup_n \|\varphi_n\|_{M_0A} < \infty$, we get the definition of weak amenability. The smallest possible value $\Lambda(G)$ of $\sup_n \|\varphi_n\|_{M_0A}$ exists and is called the Cowling Haagerup constant. Not all groups are weakly amenable. If G is not weakly amenable, for any sequence $(\varphi_n)_{n \in \mathbb{N}}$ of $A(G)$ -functions with $\varphi_n \rightarrow 1$ uniformly on compact sets, the norm $\|\varphi_n\|_{M_0A}$ goes to infinity. In this thesis, we propose to study the growth of $\|\varphi_n\|_{M_0A}$ and $\|\varphi_n\|_{MA}$ which is the essence of (completely bounded) tame cuts. Note that by repeating the same functions in the sequence, we can make the growth of $\|\varphi_n\|_{M_0A}$ arbitrarily slow, so we need an additional condition to control for that. To this end, we use a proper length function $\ell : G \rightarrow \mathbb{R}_+$, and we require that φ_n takes value 1 on the ball $B_n = \{x \in G : \ell(x) \leq n\}$. Now, it is possible to talk about asymptotically minimal growth of $\|\varphi_n\|_{M_0A}$ which depends on the group G and the length function ℓ . When G is weakly amenable, such growth is $O(1)$ for any proper length function, so the Cowling-Haagerup constant is finer information than completely bounded tame cuts.

Group algebras

Among topological groups, locally compact groups admit a unique (up to a constant factor) left invariant, regular, Borel measure, called Haar measure. This measure is one of the main tools to study the group via analysis, even though the group was initially endowed only with a topological and algebraic structure. The first things that we can construct using the Haar measure are the classical $L^p(G)$ spaces. This is an improvement compared to the spaces $C(G)$, $C_c(G)$, and $C_b(G)$ of continuous functions because the latter spaces can be constructed for any topological groups regardless of the condition of local compactness. The Banach space $L^1(G)$ has a $*$ -algebra structure with the convolution product that comes from the group law. We want to have C^* -algebras because the theory is richer on these particular $*$ -algebras. Unfortunately, $L^1(G)$ fails to be C^* -algebra with an exception for the trivial group. We consider the $*$ -representations of $L^1(G)$ to extract a C^* -algebra. The most natural one is the (left) regular representation $(\lambda, L^2(G))$, where $\lambda(f)$ for $f \in L^1(G)$ acts on the Hilbert space $L^2(G)$ by left convolution: $\lambda(f)g = f * g$, $g \in L^2(G)$. The completion $C_\lambda^*(G)$ (resp. $L(G)$) of $\lambda(L^1(G))$ in the norm topology (resp. strong operator topology) is called the reduced group C^* -algebra (resp. group von Neumann algebra). Many important group properties such as amenability, a-T-menability, weak amenability, Yu's property A, and Kazhdan's property (T) are formulated in terms of these algebras. The unique Banach predual $A(G)$ of $L(G)$ is called the Fourier algebra of G . It coincides with the space of all matrix coefficients associated to the regular representation, endowed with the norm $\varphi \mapsto \inf\{\|\xi\|_2\|\eta\|_2 : \xi, \eta \in L^2(G), \varphi = \langle \lambda(\cdot)\xi, \eta \rangle\}$. With the pointwise multiplication, $A(G)$ becomes a commutative Banach algebra. We refer readers to [Eym64] for the main properties of these algebras.

Another important algebra coming from the group is the Fourier multipliers $MA(G)$ consisting of the bounded continuous functions that leave the Fourier algebra $A(G)$ invariant under pointwise multiplication. By the closed graph theorem, each function in $MA(G)$ defines a continuous linear map on $A(G)$, and with the corresponding operator norm, $MA(G)$ becomes a Banach algebra. In [DCH85], this space was characterized as multipliers of $C_\lambda^*(G)$ and $L(G)$, so we can consider completely bounded Fourier multipliers, $M_0A(G)$. This space has better characterizations than the usual Fourier multipliers and enjoys some basic stability properties (see [CH89, DCH85]). We want to see how tame cuts grow in these two spaces.

When G is amenable, these two spaces are isometrically isomorphic, and the inclusion map $A(G) \rightarrow MA(G)$ is isometric as well, so the calculations are slightly eased. When G is abelian, we even have $A(G) \cong L^1(\widehat{G})$ and $MA(G) = M_0A(G) \cong M(\widehat{G})$ as commutative Banach algebras. We will discuss these algebras and their relation to group representations more in Chapter 1, .

Truncation operator

The behaviors of the operator norm in the reduced group C^* -algebra are notoriously difficult to understand. For instance, Valette's conjecture on Rapid Decay for uniform lattices in higher rank Lie groups is still open [Val02]. Different approaches to understand the operator norm are studied for some particular cases: when G is amenable, we have $\|\lambda(f)\| = \|f\|_1$ for all non-negative functions $f \in C_c(G)$, and when G is a connected simple Lie group, the Harish-Chandra spherical function ϕ_0 satisfies $\|\lambda(f)\| = \int_G \phi_0(x)f(x)dx$ for non-negative $f \in C_c(G)$. On the other hand, when a discrete group Γ admits characteristic tame cuts $(\varphi_n)_{n \in \mathbb{N}}$, we get asymptotic information of how the operator norm $\|\lambda(f)\|$ is changed after truncating (or cutting off) the function $f \in C_\lambda^*(\Gamma)$ to the support of φ_n . In many spaces, the truncation process defines a norm decreasing operators. For instance, $\ell^p(G)$ for $1 \leq p \leq \infty$ and more generally the space $C_c(\Gamma)$ endowed with an unconditional norm N , that is $N(f) \leq N(g)$ whenever $|f| \leq |g|$. However, the situation is very different in $C_\lambda^*(\Gamma)$. For instance, the Fourier multiplier norm of the characteristic function $\varphi_n = \mathbb{1}_{\mathbb{Z} \cap [-n, n]} \in \ell^\infty(\mathbb{Z})$ is given by the Lebesgue constant

$$\|M_{\varphi_n} : C_\lambda^*(\mathbb{Z}) \rightarrow C_\lambda^*(\mathbb{Z})\| = \|\varphi_n\|_{MA} = \|D_n\|_1 = \frac{4}{\pi} \log n + O(1), \quad (2)$$

and it is not bounded. From this, one can see that the (completely bounded) characteristic tame cuts tell how far the $C_\lambda^*(\Gamma)$ -norm is from being unconditional.

Rapid Decay property

The Rapid Decay property was initially observed in [Haa79] for the free group F_2 of two generators and developed in [Jol90] where a proper name was given by Jolissaint. Haagerup, by establishing the Rapid Decay property for F_2 , proved that there are n -positive approximate units in the Fourier algebra $A(F_2)$. This shows that $C_\lambda^*(F_2)$

has the n -positive approximation property for all $n \in \mathbb{N}$ but does not have the completely positive approximation property.

An important application of the Rapid Decay property is in K -theory. Calculating the groups $K_*(C_\lambda^*(G))$ of a given group G can be very difficult, but when a group has the Rapid Decay property, for a sufficiently large parameter $s \in \mathbb{R}_+$, the s -Sobolev space $H_\ell^s(G)$ becomes a dense subalgebra of $C_\lambda^*(G)$ with the same K -groups. Compared to the complicated operator norm of $C_\lambda^*(G)$, s -Sobolev space norm has an explicit formula, which makes $H_\ell^s(G)$ an easier target to study. Using this idea, Lafforgue proved in [Laf98] that the Baum-Connes conjecture has an affirmative answer for any uniform lattices in $SL_3(\mathbb{R})$. This was a breakthrough result for discovering the first discrete infinite group satisfying Baum-Connes conjecture and Kazhdan's property (T).

In this thesis, the Rapid Decay property plays an important role for providing the first examples of discrete groups with characteristic tame cuts: discreteness assures that the $C_\lambda^*(\Gamma)$ -norm always dominates the $\ell^2(\Gamma)$ -norm, and the Rapid Decay property provides the converse inequality with polynomial factor. Some stability results of the Rapid Decay property (e.g. [Jol90, Gar15]) are partially adaptable to tame cuts. In addition, we will use Lafforgue's result of uniform lattices in $SL_3(\mathbb{R})$ to provide two applications of tame cuts.

Weak amenability

As previously mentioned, the notion of (completely bounded) tame cuts is an analogue of weak amenability. Such similarity allows us to adapt ideas and techniques from weak amenability. For example, the inequalities of [Ldis11], [dL13], and [dLdis18] directly imply that any higher rank simple Lie group with finite center does not admit tame cuts, and none of its uniform lattices admits completely bounded tame cuts.

The Cowling Haagerup constant is a Measure Equivalence (ME) invariant, shown in [Oza12]. This was achieved by constructing an induction map $\Phi : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Lambda)$ whose restrictions $M_0A(\Gamma) \rightarrow M_0A(\Lambda)$ and $A(\Gamma) \rightarrow A(\Lambda)$ are norm decreasing. One could think that the growth rate or degree of (completely bounded) tame cuts might be ME-invariant. However, it seems not to be the case basically because the ME has not much to do with group growth. Indeed, all discrete, countable, amenable groups are in the same ME-class, and their growth can be quite arbitrary. Another

issue is that the induction map does not preserve finite support. Nevertheless, we managed to show the following result.

Theorem 1. *Let Λ and Γ be two countable groups, (X, d) a metric space, and μ a Borel measure on X . Suppose that there exist commuting, measure preserving, distance preserving, proper, free actions $\Lambda \curvearrowright X \curvearrowright \Gamma$ such that the Γ action admits a precompact fundamental domain. Fix a base point $v_0 \in X$ and define the length functions $\ell_\Gamma : \delta \in \Gamma \mapsto d(v_0, \delta, v_0)$ and $\ell_\Lambda : s \in \Lambda \mapsto d(s.v_0, v_0)$. If (Γ, ℓ_Γ) has completely bounded tame cuts, so does (Λ, ℓ_Λ) .*

A lattice Γ of a locally compact group G is weakly amenable if and only if G is weakly amenable [Haa16]. This result is achieved by using a norm decreasing induction map $M_0A(\Gamma) \rightarrow M_0A(G)$. In particular, if G or Γ is amenable, we have $MA(\Gamma) \rightarrow MA(G)$ continuous. The theorem below shows that amenability is essential for the latter map to be continuous. The proof uses tame cuts and will be given in Section 3.6.

Theorem 2. *Let Γ be a lattice in $G = SL_3(\mathbb{R})$ with a compact fundamental domain Ω . Then the map*

$$\Phi : MA(\Gamma) \rightarrow MA(G), \quad \varphi \mapsto \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi \mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega$$

is not bounded (possibly not well defined).

Consider the following open question.

Conjecture. Let Γ be a discrete group. If all Fourier multipliers of Γ are completely bounded, that is $M_0A(\Gamma) = MA(\Gamma)$, then Γ is amenable.

The converse is already known by [Neb82, Los84] even for locally compact groups. In [HSS10], an explicit non-completely bounded Fourier multiplier of the free group F_2 of two generators was constructed. The same result was made in [Bož82] using infinite lacunary sets. It follows that the conjecture has a positive answer for any discrete group containing a copy of F_2 (e.g. finitely generated discrete linear groups over a field). We will show the following theorem. The idea of the proof is very close to that of Theorem 2, and we will not use the fact that Γ contains F_2 as a subgroup.

Theorem 3. *Let Γ be a uniform lattice in $G = SL_3(\mathbb{R})$. Then there is a Fourier multiplier of Γ which is not completely bounded. In other words, $M_0A(\Gamma) \neq MA(\Gamma)$.*

Property (T_{Schur}, G, K)

Property (T_{Schur}, G, K) was introduced in [Lia16] as an analogue of property (T_{Schur}) . Suppose that G is a reductive group over a local field, K is its maximal compact subgroup, and Γ is a lattice of G satisfying property (T_{Schur}, G, K) . Then, all known methods to prove the Baum-Connes conjecture fail for Γ . In our context, property (T_{Schur}, G, K) opposes the existence of tame cuts as follows.

Theorem 4. *Let H be a closed subgroup of an unbounded locally compact group G . Suppose that H satisfies property (T_{Schur}, G, K, ℓ) for some compact subgroup K and a proper length function ℓ of G . Then $(H, \ell|_H)$ does not admit K -bi-invariant tame cuts.*

Corollary 0.0.1. *If G is unbounded and has property (T_{Schur}, G, K, ℓ) , then (G, ℓ) does not have tame cuts.*

Corollary 0.0.2. *Suppose that G is a finitely generated infinite group and H is a finitely generated subgroup of G . Suppose that H is at most polynomially distorted in G . Recall that H is polynomially distorted in G there exists $k \geq 0$ such that $\ell_H(x) \leq k\ell_G(x)^k + k$ for all $x \in H$, where ℓ_G and ℓ_H are the word length functions of G and H , respectively. If H has property $(T_{Schur}, G, \{e\}, \ell_G)$, then (H, ℓ_H) does not have tame cuts.*

It is worth noting that the only known example satisfying property (T_{Schur}, G, K) is the group H (or any discrete subgroup of $Sp_4(\mathbb{F}_q((\pi)))$ containing H) of upper triangular matrices in $Sp_4(\mathbb{F}_q((\pi)))$ whose entries are in $\mathbb{F}_q[\pi^{-1}]$ and 1 on diagonal.

More examples

Hyperbolic groups [Oza08], finite dimensional cubical groups [Miz08], Coxeter groups [Fen02], and groups acting properly on a product of finitely many hyperbolic graphs with bounded geometry [Ver19], all have completely bounded characteristic tame cuts. Also, the discrete groups satisfying Rapid Decay property have characteristic tame cuts. We will provide the following examples.

Theorem 5. *The following groups have completely bounded characteristic tame cuts.*

- (i) $\Gamma_A = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ for any $d \in \mathbb{N}$ and $A \in SL_d(\mathbb{Z})$.

(ii) $\Gamma_{p,q} = \mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ for any coprime $p, q \in \mathbb{N}$.

(iii) Lamplighter groups $\mathbb{Z}_p \wr \mathbb{Z}$ for any $p \in \mathbb{N}$.

(iv) Baumslag-Solitar groups $BS(p, q) = \langle a, t \mid ta^pt^{-1} = a^q \rangle$ for any $p, q \in \mathbb{N}$.

The examples (i) – (iii) are achieved by proving Proposition 3.3.9 which states (completely bounded) [characteristic] tame cuts are stable under extension by a group with polynomial growth, and (ii) is used to prove (iv).

Organization of the manuscript

In Chapter 1, we will recall basic notions such as group representations and group algebras. In Chapter 2, we will discuss weak amenability and Rapid Decay property more. Chapter 3 is dedicated to (completely bounded) [characteristic] tame cuts. In Section 3.1, we give the definitions of tame cuts and provide the first examples by exploring connections to weak amenability and Rapid Decay property. In Section 3.2, we will propose a sufficient condition to admit tame cuts that extends the Rapid Decay property. In Section 3.3, we will investigate the stability properties of tame cuts. In Section 3.4 and 3.5, we will prove Theorem 4 and 5. In Section 3.6, we will prove Theorem 2 and 3.

Chapter 1

Preliminaries

Throughout the paper, G denotes a locally compact group and Γ denotes a discrete countable group. We also denote by dx a left invariant Haar measure of G . Let f be a measurable function on G . The L^p -norm of f is given by

$$\|f\|_p = \left(\int_G |f|^p dx \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\|f\|_\infty = \text{ess sup}_G |f| = \inf \{ t \in \mathbb{R}_+ : \int \mathbb{1}_{\{|f|>t\}} = 0 \}$$

if $p = \infty$. Denote by $\mathcal{L}^p(G)$ the space of all measurable functions on G with finite L^p -norm. We say two measurable functions are *almost everywhere (a.e) equivalent* if their difference is 0 except on a null set. The Banach space $L^p(G)$ is the quotient of $\mathcal{L}^p(G)$ by a.e-equivalence endowed with the induced L^p -norm.

1.1 Group representations

Representations play an important role in the study of group theory. There are many different kind of representations. The most popular ones are the unitary representations. Some other representations, such as uniformly bounded or even unbounded representations, can be used for certain problems. We give a short introduction to these representations.

In general, by a representation of a group G , we understand a group homomorphism $\pi : G \rightarrow GL(V)$, where V is a vector space and $GL(V)$ is the group

of linear automorphisms of V . As far as this paper is concerned, we only consider representations on Hilbert spaces over the complex field. When $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space, we denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded operators, by $GL(\mathcal{H})$ the group of invertible operators, and by $\mathcal{U}(\mathcal{H})$ the group of all unitary operators on \mathcal{H} . To make use of the topological structure of the group, we usually consider continuous representations. The space $\mathcal{B}(\mathcal{H})$ has its natural topology given by the operator norm

$$\|T\| = \sup\{\|Tv\| : v \in \mathcal{H}, \|v\| = 1\}, \quad (\forall T \in \mathcal{B}(\mathcal{H})).$$

In some situations, the norm topology could be not convenient to work with. For example, the left regular representation $(\lambda, L^2(G))$, one of the most natural representations, is usually not continuous when $\mathcal{U}(L^2(G))$ is endowed with the norm topology. On the other hand, it is always continuous when $\mathcal{U}(L^2(G))$ is endowed with the strong operator topology (SOT). Recall that SOT on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms

$$N_v : T \in \mathcal{B}(\mathcal{H}) \mapsto \|Tv\|, \quad v \in \mathcal{H},$$

and the weak operator topology (WOT) on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms

$$N_{v,w} : T \in \mathcal{B}(\mathcal{H}) \mapsto |\langle Tv, w \rangle|, \quad v, w \in \mathcal{H}.$$

Definition 1.1.1. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and π a group homomorphism from a locally compact group G into $GL(\mathcal{H})$. We say that the couple (π, \mathcal{H}) is a *continuous representation* of G if π is continuous when $GL(\mathcal{H})$ is endowed with the SOT, equivalently if the maps

$$x \in G \mapsto \pi(x)v \in \mathcal{H}$$

are continuous for all $v \in \mathcal{H}$. If (π, \mathcal{H}) is a continuous representation of G , and v, w are two vectors in \mathcal{H} , the function

$$c_{v,w} : x \in G \mapsto \langle \pi(x)v, w \rangle$$

is continuous. This function is called the matrix coefficient of π associated to the

vectors $v, w \in \mathcal{H}$.

Throughout the paper, all representations are continuous unless otherwise stated.

Definition 1.1.2. Two representations (π_i, \mathcal{H}_i) , $i = 1, 2$ of a group G are *similar* if there is an invertible bounded operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$T^{-1}\pi_2(x)T = \pi_1(x), \quad (\forall x \in G).$$

Furthermore, if T is unitary, we say that the two representations are *unitarily equivalent* and write $\pi_1 \simeq \pi_2$.

Definition 1.1.3. Suppose that (π, \mathcal{H}) is a representation of a group G . A closed subspace \mathcal{K} of \mathcal{H} is said *G-invariant* if the set $\{\pi(x)v : x \in G, v \in \mathcal{K}\}$ sits inside \mathcal{K} . The representation (π, \mathcal{H}) is *irreducible* if there is no closed G -invariant subspace different from $\{0\}$ and \mathcal{H} .

1.1.1 Unitary representations

Definition 1.1.4. A representation (π, \mathcal{H}) of a group G is called *unitary representation* if it takes image in the unitary group $\mathcal{U}(\mathcal{H})$.

Example 1.1.5. The simplest example of a unitary representation is the *trivial representation*:

$$x \in G \mapsto \text{id}_{\mathcal{H}} \in \mathcal{U}(\mathcal{H}), \quad (\forall x \in G).$$

Example 1.1.6. The *left regular representation* defined as

$$\lambda : G \rightarrow \mathcal{U}(L^2(G)), \quad \lambda(x)f(y) = f(x^{-1}y)$$

can be obtained for any locally compact group. Similarly, we can define the right regular representation as

$$\rho : G \rightarrow \mathcal{U}(L^2(G)), \quad \rho(x)f(y) = f(yx)\Delta(x)^{1/2},$$

where $\Delta : G \rightarrow \mathbb{R}_{>0}$ is the modular function of G . It is worth noting that the left and right regular representations are unitarily equivalent.

The left regular representation is a particular case of the classical way to induce a unitary representation from a group action.

Proposition 1.1.7 (Proposition A.6.1 of [BdlHV08]). *Let G be a σ -compact locally compact group, let (Y, μ) be a σ -finite measurable space such that $L^2(Y, \mu)$ is separable, and let $G \times Y \rightarrow Y$ be a measurable action such that μ is quasi-invariant, that is, for each measurable set $A \subseteq Y$, we have $\mu(A) = 0$ if and only if $\mu(xA) = 0$ for all $x \in G$. Then the group homomorphism $\lambda_Y : G \rightarrow \mathcal{B}(L^2(Y, \mu))$ defined by*

$$[\lambda_Y(x)f](y) = f(x^{-1} \cdot y) \left(\frac{dx_*\mu}{d\mu} \right)^{1/2}(y), \quad (\forall f \in L^2(Y, \mu), \forall x \in G, \forall y \in Y)$$

is a unitary representation.

The representation $(\lambda_Y, L^2(Y, \mu))$ is called the *Koopman representation* associated to the measurable action $G \curvearrowright (Y, \mu)$. When G is discrete, we can omit the σ -compactness, σ -finiteness, and separability conditions.

Lemma 1.1.8. *Suppose that a discrete countable group Γ acts freely on a measure space (X, μ) by measure preserving transformations. Suppose that the action $\Gamma \curvearrowright X$ has a measurable fundamental domain, that is a measurable set $\Omega \subseteq X$ such that $\{s \cdot \Omega : s \in \Gamma\}$ is a partition of X . Then the Koopman representation $(\lambda_X, L^2(X, \mu))$ of Γ is unitarily equivalent to $\lambda_\Gamma \otimes \text{id}_{L^2(\Omega, \mu)}$.*

Proof. By hypothesis, each element $x \in X$ can be written as $x = \gamma(x) \cdot \omega(x)$ for a unique $\gamma(x) \in \Gamma$ and $\omega(x) \in \Omega$. With this notation, the action $\Gamma \curvearrowright (\Omega, \mu)$, $(s, w) \mapsto \omega(s \cdot w)$ is measure preserving and free. Let μ_Γ be the counting measure on Γ . Then the action $\Gamma \curvearrowright (X, \mu)$ can be seen as the measure preserving action $\Gamma \curvearrowright (\Gamma \times \Omega, \mu_\Gamma \otimes \mu)$ defined by $s \cdot (t, w) = (\gamma(st \cdot w), \omega(st \cdot w))$ for all $s, t \in \Gamma$ and $w \in \Omega$. We have unitary operator $V : \ell^2(\Gamma) \otimes L^2(\Omega) \rightarrow L^2(X)$, $f \otimes g \mapsto F_{f,g}$, where $F_{f,g}(t \cdot w) = f(t)g(w)$ for all $t \in \Gamma, w \in \Omega$, and via this map, the Koopman representation $(\lambda_X, L^2(X))$ is unitarily equivalent to the representation σ on $\ell^2(\Gamma) \otimes L^2(\Omega)$ defined by

$$\sigma(s)[f \otimes g] = V^* \lambda_X(s) V F_{f,g} = [\lambda_\Gamma(s)f] \otimes g, \quad (\forall f \in \ell^2(\Gamma), g \in L^2(\Omega)),$$

which is nothing but $\lambda_\Gamma \otimes \text{id}_{L^2(\Omega)}$. □

Definition 1.1.9. We denote by $\Sigma = \Sigma(G)$ all unitary representations of G up to unitary equivalence and by \widehat{G} the irreducible ones.

The family \widehat{G} for a locally compact abelian group G , in which case \widehat{G} is called the *Pontryagin dual* of G , is special for having a structure of locally compact abelian group. Indeed, by Schur's lemma, all irreducible representations of G are one dimensional, so the family \widehat{G} is identified with $\text{Hom}(G, \mathbb{T})$, where $\mathbb{T} \cong \mathcal{U}(\mathbb{C})$ is the unit sphere in the complex plane, and the $\text{Hom}(G, \mathbb{T})$ is a locally compact group equipped with the pointwise multiplication and the topology of uniform convergence on compact sets. For example, when $G = \mathbb{Z}$, the Pontryagin dual is $\widehat{G} \cong \text{Hom}(\mathbb{Z}, \mathbb{T}) \cong \mathbb{T}$. One of the nicest facts of the Pontryagin duality is that it is reflexive: $\widehat{\widehat{G}}$ is isomorphic to G as locally compact groups. Another important fact is that the *Fourier transform*

$$\mathcal{F} : L^2(G, dx) \rightarrow L^2(\widehat{G}, d\chi), \quad [\mathcal{F}(f)](\chi) = \int_G f(x) \overline{\chi(x)} dx, \quad (\forall f \in L^2(G), \forall \chi \in \widehat{G})$$

gives a Hilbert space isomorphism for a suitable choice of Haar measure $d\chi$ on \widehat{G} .

1.1.2 Uniformly bounded representations

Definition 1.1.10. A representation (π, \mathcal{H}) is said *uniformly bounded* if the image of π is bounded in $\mathcal{B}(\mathcal{H})$, that is the quantity $\|\pi\| = \sup_{x \in G} \|\pi(x)\|$ is finite.

It is clear that any unitary representation is uniformly bounded by 1. Moreover, unitary representations can be used to construct many non-trivial uniformly bounded representations.

Proposition 1.1.11. *Any representation similar to a unitary representation is uniformly bounded. More precisely, if (π, \mathcal{H}) is a unitary representation of a locally compact group G , and if T is an invertible operator on \mathcal{H} , then the map*

$$\pi_T : G \rightarrow GL(\mathcal{H}), \quad \pi_T(x) = T^{-1}\pi(x)T, \quad (\forall x \in G)$$

defines a uniformly bounded representation with $\|\pi_T\| \leq \|T\|\|T^{-1}\|$.

Proof. Since the map $x \in G \mapsto \pi_T(x)v \in \mathcal{H}$ is a composition of the two continuous maps

$$x \in G \mapsto \pi(x)Tv \in \mathcal{H} \quad \text{and} \quad T^{-1} : \mathcal{H} \rightarrow \mathcal{H},$$

it follows that π_T is continuous. The calculation

$$\pi_T(xy) = T^{-1}\pi(xy)T = T^{-1}\pi(x)TT^{-1}\pi(y)T = \pi_T(x)\pi_T(y)$$

shows that π_T is a representation. Since

$$\|\pi_T(x)\| = \|T^{-1}\pi(x)T\| \leq \|T^{-1}\| \|T\|,$$

π_T is uniformly bounded. □

It is interesting to know if any uniformly bounded representation is similar to a unitary representation. In that direction, a positive answer was given for amenable groups in [Dix50, Day50]. However, this fails for some non-amenable groups. For example, any discrete group containing a copy of the free group of two generators admits a uniformly bounded representation which is not similar to any unitary representation. We refer to [Pis01] where the problem is exclusively interpreted.

1.1.3 Positive definite functions

Definition 1.1.12. Let X be a set. A function of the form $k : X \times X \rightarrow \mathbb{C}$ is called a *kernel* on X . A kernel $k : X \times X \rightarrow \mathbb{C}$ is *positive definite* if for any elements x_1, \dots, x_n of X , the matrix $(k(x_i, x_j))_{i,j} \in M_n(\mathbb{C})$ defines a positive matrix: for any $a_1, \dots, a_n \in \mathbb{C}$, we have

$$\sum_{1 \leq i, j \leq n} a_i \bar{a}_j k(x_i, x_j) \geq 0.$$

A function $\varphi : G \rightarrow \mathbb{C}$ on a group G is *positive definite* if the kernel $k_\varphi : G \times G \rightarrow \mathbb{C}$ defined by $k_\varphi(x, y) = \varphi(y^{-1}x)$ is positive definite.

Example 1.1.13. Let \mathcal{H} be a Hilbert space, X a set, and $f : X \rightarrow \mathcal{H}$ a map. The map

$$k : X \times X \rightarrow \mathcal{H}, \quad k(x, y) = \langle f(x), f(y) \rangle \tag{1.1}$$

defines a positive definite kernel. Indeed, if $x_1, \dots, x_n \in X$ and $a_1, \dots, a_n \in \mathbb{C}$, then

$$\sum_{1 \leq i, j \leq n} a_i \bar{a}_j k(x_i, x_j) = \left\| \sum_{i=1}^n a_i f(x_i) \right\|^2 \geq 0.$$

Example 1.1.14. Let (π, \mathcal{H}) be a unitary representation of G . The matrix coefficient $c_v(x) = \langle \pi(x)v, v \rangle$ defines a positive definite function on G for all vectors $v \in \mathcal{H}$. To see that, we could apply the function $f(x) = \pi(x)v$ on (1.1):

$$c_v(y^{-1}x) = \langle \pi(x)v, \pi(y)v \rangle.$$

This function is called the positive definite function *associated* to the unitary representation (π, \mathcal{H}) .

Theorem 1.1.15. (*GNS construction*) *Every positive definite kernel comes from a function as in (1.1). Moreover, if $\varphi : G \rightarrow \mathbb{C}$ is a continuous positive definite function, then there is a unitary representation (π, \mathcal{H}) and a vector $v \in \mathcal{H}$ such that $\varphi(x) = \langle \pi(x)v, v \rangle$ for all $x \in G$.*

Remark 1.1.1. A positive definite function φ is uniformly bounded by $\varphi(e)$. Indeed, $|\varphi(x)| = |\langle \pi(x)v, v \rangle| \leq \|\pi(x)\| \|v\|^2 = \langle \pi(e)v, v \rangle = \varphi(e)$ by the theorem above. Furthermore, if φ is continuous, it is also uniformly continuous because we have

$$\begin{aligned} |\varphi(x) - \varphi(y)| &= |\langle (\pi(x) - \pi(y))v, v \rangle| \\ &= |\langle (\pi(e) - \pi(y^{-1}x))v, \pi(y)v \rangle| \\ &\leq \|v - \pi(y^{-1}x)v\| \|v\| \end{aligned}$$

for all $x, y \in G$.

Definition 1.1.16. A kernel $k : X \times X \rightarrow \mathbb{R}$ is *conditionally negative definite* if

- (i) $k(x, x) = 0$ and $k(x, y) = k(y, x)$ for all $x, y \in X$,
- (ii) $\sum_{1 \leq i, j \leq n} a_i a_j k(x_i, x_j) \leq 0$ for all $a_i \in \mathbb{R}$, $x_i \in G$, and $n \in \mathbb{N}$ with $\sum_{i=1}^n a_i = 0$.

A function $\varphi : G \rightarrow \mathbb{C}$ on a group G is *conditionally negative definite* if the kernel $k_\varphi : G \times G \rightarrow \mathbb{C}$ is conditionally negative definite.

Example 1.1.17. Let \mathcal{H} be a Hilbert space and $f : X \rightarrow \mathcal{H}$ a map. The map

$$\varphi : X \times X \rightarrow \mathcal{H}, \quad k(x, y) = \|f(x) - f(y)\|^2 \quad (1.2)$$

defines a conditionally negative definite kernel. Indeed, if $x_1, \dots, x_n \in X$ and $a_1, \dots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = 0$, then

$$\sum_{1 \leq i, j \leq n} a_i a_j k(x_i, x_j) = -2 \left\| \sum_{i=1}^n a_i f(x_i) \right\|^2 \leq 0.$$

Theorem 1.1.18 (GNS construction). *Every conditionally negative definite kernel comes from a function as in (1.2).*

Theorem 1.1.19 (Schoenberg's theorem). *A kernel $k : X \times X \rightarrow \mathbb{R}_+$ is conditionally negative definite if and only if the kernel $\exp(-tk) : X \times X \rightarrow \mathbb{C}$ is positive definite for all $t \in \mathbb{R}_+$.*

Example 1.1.20 (Lemma 1.2 of [Haa79]). The combinatorial distance d of a tree $T = (V, E)$ is conditionally negative definite. Consequently, the kernel $(v, w) \in V \times V \mapsto \exp(-td(v, w))$ is positive definite for any $t > 0$.

Proof. Fix a base point $v_0 \in V$. We equip the set E of edges with directions so that each edge points toward v_0 . For any vertex $v \in V$, denote $\gamma_v : \{0, 1, \dots, d(v, v_0)\} \rightarrow V$ the geodesic from v to v_0 . Define the map $f : V \rightarrow \ell^2(E)$ by

$$f(v) = \sum_{i=0}^{d(v, v_0)-1} \delta_{(\gamma_v(i), \gamma_v(i+1))}, \quad (\forall v \in V).$$

Then we have

$$d(v, w) = \|f(v) - f(w)\|_{\ell^2(E)}^2, \quad (\forall v, w \in V).$$

Now, the statement directly follows from Example 1.1.17 and Schoenberg's theorem. \square

1.2 Group algebras

In this section, we will recall group C^* -algebras, group von Neumann algebra, Fourier and Fourier-Stieltjes algebras, and Fourier multiplier algebras.

1.2.1 Group measure algebra, $M(G)$

Definition 1.2.1. Let X be a locally compact Hausdorff space. Denote by \mathcal{B} the Borel space of X , that is the σ -algebra generated by the topology of X . A (*complex*) *measure* is a map $\mu : \mathcal{B} \rightarrow \mathbb{C}$ such that

$$\mu \left(\bigcup_{i \in \mathbb{N}} E_i \right) = \sum_{i \in \mathbb{N}} \mu(E_i)$$

for any disjoint measurable subsets $E_i \in \mathcal{B}$, $i \in \mathbb{N}$. A measure is *positive* if it takes non-negative values. Given a complex measure μ , its *total variation* $|\mu|$ is a positive measure defined as

$$|\mu|(E) = \sup_{\mathcal{F}} \sum_{F \in \mathcal{F}} |\mu(F)|, \quad (\forall E \in \mathcal{B}),$$

where the supremum is taken over all measurable partitions \mathcal{F} of E . A positive measure ν is *inner regular* if

$$\nu(E) = \sup \{ \nu(K) : K \subseteq E, K \text{ compact} \}, \quad (\forall E \in \mathcal{B}),$$

outer regular if

$$\nu(E) = \inf \{ \nu(U) : E \subseteq U, U \text{ open} \}, \quad (\forall E \in \mathcal{B}),$$

and *regular* if ν is both inner and outer regular. A complex measure μ is *regular* (resp. *finite*) if its total variation $|\mu|$ is regular (resp. $|\mu|(X)$ is finite). We denote by $M(X)$ the Banach space of all complex regular finite measures on X endowed with the total variation norm $\mu \mapsto |\mu|(X)$.

By Riesz representation theorem, $M(X)$ is isometrically isomorphic to the dual Banach space $(C_0(X))^*$ via $\mu \mapsto \int_G d\mu$ (cf. [Rud87, Theorem 6.19]). The Dirac measures $\delta_x, x \in X$ are typical elements of $M(G)$. When X is a locally compact group, say $X = G$, the Haar measure dx is in $M(G)$ if and only if dx is finite if and

only if G is compact. Nonetheless, $f dx$ is finite and regular for any $f \in L^1(G)$. This gives an isometric identification of $L^1(G)$ with a closed subspace of $M(G)$. Recall that a $*$ -algebra is an algebra A endowed with an involution operator $x \mapsto x^*$, $x \in A$. In the following, we use the group law to make $M(G)$ a $*$ -algebra. Moreover, $L^1(G)$ becomes a closed two-sided ideal in $M(G)$.

Definition 1.2.2. The *measure algebra* of a locally compact group G is the Banach space $M(G)$ of complex regular finite measures on G . It has a Banach $*$ -algebra structure with

- Total variation norm: $|\mu|(G) = \sup \{ \sum_i |\mu(E_i)| : (E_i)_{i \in I} \text{ is a partition of } G \}$
- Convolution product: $(\mu * \nu)(E) = \int_G \int_G \mathbb{1}_E(xy) d\mu(x) d\nu(y)$
- Involution: $\mu^*(E) = \overline{\mu(E^{-1})}$

for all $\mu, \nu \in M(G)$ and $E \in \mathcal{B}$.

Theorem 1.2.3. Let G be a locally compact group. The map $\iota : f \in L^1(G) \mapsto f dx \in M(G)$ is well defined and identifies $L^1(G)$ with a two-sided symmetric closed ideal of $M(G)$.

Proof. To see that ι is well defined, we only need to check that $g \in C_0(G) \mapsto \int_G f g dx$ defines a continuous functional for any $f \in L^1(G)$, which is obviously true. Moreover, one easily checks that ι is isometric, and thus $\iota(L^1(G))$ is closed in $M(G)$ since $L^1(G)$ is complete.

Take any $\mu \in M(G)$, $f \in L^1(G)_+$ and $E \in \mathcal{B}$ with $\int_G \mathbb{1}_E dx = 0$. By Radon-Nikodym's theorem, since

$$\begin{aligned} (\mu * f dx)(E) &= \int_G \int_G \mathbb{1}_E(xy) f(y) d\mu(x) dy \\ (\text{Fubini + left invariance}) &= \int_G \mathbb{1}_E(y) \left(\int_G f(x^{-1}y) d\mu(x) \right) dy \\ (\text{conventionally } 0 \cdot \infty = 0) &= 0, \end{aligned}$$

and

$$(f dx)^*(E) = \overline{(f dx)(E^{-1})} = \int_G \overline{f(x)} \mathbb{1}_E(x^{-1}) dx = \int_G \overline{f(x^{-1})} \Delta(x^{-1}) \mathbb{1}_E(x) dx = 0,$$

we deduce that $\iota(L^1(G))$ is a two-sided symmetric ideal of $M(G)$. \square

Remark 1.2.1. We do not distinguish $f \in L^1(G)$ and $\iota(f) = f dx \in M(G)$. From the proof above, we can extract the convolution product and the involution on $L^1(G)$:

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \quad \text{and} \quad f^*(x) = f(x^{-1})\Delta(x^{-1})$$

for all $f, g \in L^1(G)$ and $x \in G$. Two more notations will be useful:

$$\check{f}(x) = f(x^{-1}) \quad \text{and} \quad \tilde{f}(x) = \overline{f(x^{-1})}.$$

1.2.2 Full and reduced group C^* -algebras, $C^*(G)$ and $C_\lambda^*(G)$

A C^* -algebra is a Banach $*$ -algebra A such that the C^* -condition $\|x^*x\| = \|x\|^2$ is satisfied for all $x \in A$. If \mathcal{H} is a Hilbert space, the space $\mathcal{B}(\mathcal{H})$ of bounded operators has a natural C^* -algebra structure. It follows that any closed $*$ -subalgebra A of $\mathcal{B}(\mathcal{H})$ inherits the C^* -algebra structure. The converse statement is known as Gelfand-Neimark's theorem: any C^* -algebra can be identified as a closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} .

The space $C_0(G)$ of continuous functions vanishing at infinity becomes a C^* -algebra with pointwise multiplication and complex conjugation. The same is true for the space $C_b(G)$ of bounded continuous functions. However, these C^* -algebras have nothing to do with the group law. This downside is not the case for $L^1(G)$ thanks to the convolution product. However, $L^1(G)$ is a C^* -algebra if and only if $G = \{e\}$.

Proposition 1.2.4. *Let G be a locally compact group. The $*$ -algebra $L^1(G)$ is a C^* -algebra if and only if G is a trivial group.*

Proof. We will construct an integrable function $g \in L^1(G)$ that does not satisfy the C^* -condition. When $G = \mathbb{Z}$, we can choose $g = \delta_0 + i\delta_1 + \delta_2 \in \ell^1(\mathbb{Z})$. Our proof for general locally compact groups is an extension of this idea.

Take any element $x \in G$ different from the identity e . As we assume that all topological groups are Hausdorff, there exist disjoint compact neighborhoods U and V of e and x , respectively. The neighborhood

$$W = (U \cap x^{-1}V) \cap (U \cap x^{-1}V)^{-1}$$

of the identity is compact and symmetric and satisfies

$$W \cap xW = xW \cap x^2W = \emptyset.$$

The function $f = \chi_W + \chi_W^* = \chi_W(1 + \Delta^{-1})$ defines a self-adjoint integrable positive function on G . We claim that the function

$$g = f + \delta_{x^2} * f + i\delta_x * f$$

does not satisfy the C^* -condition. Since the supports $W \cup x^2W$ and W of the functions $f + \delta_{x^2} * f$ and $i\delta_x * f$ are disjoint, and since the function $f + \delta_{x^2} * f$ is positive, we can calculate

$$\begin{aligned} \|g\|_1 &= \|f + \delta_{x^2} * f + i\delta_x * f\|_1 \\ (\text{disjoint supports}) &= \|f + \delta_{x^2} * f\|_1 + \|i\delta_x * f\|_1 \\ (\text{positivity}) &= \|f\|_1 + \|\delta_{x^2} * f\|_1 + \|i\delta_x * f\|_1 \\ (\text{left invariance}) &= 3\|f\|_1 \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|g^* * g\|_1 &= \|(f - if * \delta_{x^{-1}} + f * \delta_{x^{-2}})(f + i\delta_x * f + \delta_{x^2} * f)\|_1 \\ &= \|3f * f + f * \delta_{x^2} * f + f * \delta_{x^{-2}} * f\|_1 \\ &\leq 5\|f\|_1^2 < \|g\|_1^2, \end{aligned}$$

where the “ \leq ” inequality is valid because the Banach algebra norm is submultiplicative and the Haar measure is left invariant. This shows that $g \in L^1(G)$ does not satisfy the C^* -condition. \square

To obtain a C^* -algebra from $L^1(G)$, we use unitary representations of G .

Proposition 1.2.5. *Let (π, \mathcal{H}) be a unitary representation of a locally compact group G . Then there is a unique $*$ -homomorphism $\tilde{\pi} : L^1(G) \rightarrow \mathcal{B}(\mathcal{H})$ satisfying*

$$\langle \tilde{\pi}(f)v, w \rangle = \int_G f(x) \langle \pi(x)v, w \rangle dx$$

for all $f \in L^1(G)$ and $v, w \in \mathcal{H}$.

Proof. Note that the expression $\langle \tilde{\pi}(f)v, w \rangle = \int_G f(x) \langle \pi(x)v, w \rangle dx$ makes sense for any $f \in L^1(G)$ and $v, w \in \mathcal{H}$ since the matrix coefficients are bounded and continuous. Also, we have

$$\|\tilde{\pi}(f)\| = \sup |\langle \tilde{\pi}(f)v, w \rangle| \leq \sup \int_G |f(x) \langle \pi(x)v, w \rangle| dx \leq \|f\|_1,$$

where the supremums are taken over $v, w \in \mathcal{H}$ with $\|v\| = \|w\| = 1$. It is left to prove that $\tilde{\pi}$ is a $*$ -homomorphism. Linearity is obvious. Take any $f, g \in L^1(G)$. Involution preservation:

$$\begin{aligned} \langle \tilde{\pi}(f^*)v, w \rangle &= \int_G \overline{f(x^{-1})} \Delta(x^{-1}) \langle \pi(x)v, w \rangle dx \\ &= \overline{\int_G f(x) \langle \pi(x)w, v \rangle dx} = \langle \tilde{\pi}(f)^*v, w \rangle. \end{aligned}$$

Multiplicativity:

$$\begin{aligned} \langle \tilde{\pi}(f * g)v, w \rangle &= \int_G (f * g)(x) \langle \pi(x)v, w \rangle dx \\ &= \int_G \int_G f(y) g(y^{-1}x) \langle \pi(x)v, w \rangle dy dx \\ \text{(Fubini)} &= \int_G f(y) \left(\int_G g(y^{-1}x) \langle \pi(x)v, w \rangle dx \right) dy \\ \text{(left invariance)} &= \int_G f(y) \left(\int_G g(x) \langle \pi(yx)v, w \rangle dx \right) dy \\ &= \int_G f(y) \langle \pi(y) \tilde{\pi}(g)v, w \rangle dy \\ &= \langle \tilde{\pi}(f) \tilde{\pi}(g)v, w \rangle. \end{aligned}$$

□

By abuse of notation, we will use π for both π and $\tilde{\pi}$.

Example 1.2.6. Consider the regular representation $(\lambda, L^2(G))$. The operator $\lambda(f)$ for $f \in L^1(G)$ is nothing but a left convolution operator: $\lambda(f)\xi = f * \xi$ for all $\xi \in L^2(G)$.

Definition 1.2.7. Let (π, \mathcal{H}) be a unitary representation of G . The norm closure $C_\pi^*(G)$ of $\pi(L^1(G))$ in $\mathcal{B}(\mathcal{H})$ is called the *group C^* -algebra* of G associated to the representation (π, \mathcal{H}) . The group C^* -algebra $C_\lambda^*(G)$ associated to the (left) regular

representation $(\lambda, L^2(G))$ is called the (*left*) *reduced group C^* -algebra*. The group C^* -algebra $C^*(G)$ associated to the representation $(\pi_{\max}, \bigoplus_{\pi \in \Sigma} \mathcal{H}_\pi)$ defined by

$$\pi_{\max}(f) [\bigoplus_{\pi \in \Sigma} v_\pi] = \bigoplus_{\pi \in \Sigma} \pi(f)v_\pi, \quad (\forall f \in L^1(G), \forall v_\pi \in \mathcal{H}_\pi, \forall \pi \in \Sigma)$$

is called the *full group C^* -algebra* of G .

These C^* -algebras are extensively used to study the group G .

Definition 1.2.8. Let (π_i, \mathcal{H}_i) , $i = 1, 2$ be two unitary representations of a locally compact group G . We say that π_1 is *weakly contained* in π_2 and denote $\pi_1 \prec \pi_2$ if we have

$$\|\pi_1(f)\| \leq \|\pi_2(f)\|, \quad (\forall f \in L^1(G)).$$

If $\pi_1 \prec \pi_2$ and $\pi_2 \prec \pi_1$, we say that π_1 and π_2 are weakly equivalent and write $\pi_1 \sim \pi_2$.

In other words, we say that π_1 is weakly contained in π_2 if the identity map on $L^1(G)$ gives rise to a norm decreasing surjection $C_{\pi_2}^*(G) \twoheadrightarrow C_{\pi_1}^*(G)$. In particular, we always have the surjection $C^*(G) \twoheadrightarrow C_\lambda^*(G)$. The injectivity of this map characterizes the amenability of G , which is known as Hulanicki's theorem [Hul64].

Let us end this subsection by illustrating the reduced group C^* -algebra of abelian groups, which is also the full group C^* -algebra since all abelian groups are amenable.

Proposition 1.2.9. *If G is a locally compact abelian group, we have the $*$ -isomorphism*

$$C_\lambda^*(G) \rightarrow C_0(\widehat{G}), \quad \lambda(f) \mapsto \mathcal{F}(f), \quad (\forall f \in C_c(G)),$$

where \mathcal{F} is the Fourier transform.

Proof. Since the Fourier transform gives an isomorphism between the Hilbert spaces $L^2(G)$ and $L^2(\widehat{G})$, we have a $*$ -isomorphism

$$\mathcal{G} : \mathcal{B}(L^2(G)) \rightarrow \mathcal{B}(L^2(\widehat{G})), \quad T \mapsto \mathcal{F}T\mathcal{F}^{-1}.$$

The algebra $C_0(\widehat{G})$ can be seen as a concrete C^* -subalgebra of $\mathcal{B}(L^2(\widehat{G}))$, where $\phi \in C_0(\widehat{G})$ acts on $L^2(\widehat{G})$ by pointwise multiplication: $M_\phi f = \phi f$, $f \in L^2(G)$. A

routine calculation shows that

$$\mathcal{G}(\lambda(f))\xi = \mathcal{F}\lambda(f)\mathcal{F}^{-1}\xi = \mathcal{F}(f)\xi, \quad (\forall f \in C_c(G), \forall \xi \in L^2(\widehat{G})),$$

so $\mathcal{G}(\lambda(f))$ is a pointwise multiplication operator on $L^2(\widehat{G})$. To see that $\mathcal{F}(f)$ is continuous, take any net (χ_i) in \widehat{G} converging to $\chi \in \widehat{G}$. Since f is compactly supported,

$$|\mathcal{F}(f)(\chi_i) - \mathcal{F}(f)(\chi)| = \left| \int_G f(x) \overline{\chi_i(x)} - \overline{\chi(x)} dx \right| \leq \|f\|_1 \|\chi_i - \chi\|_{L^\infty(\text{supp}(f))} \rightarrow 0.$$

Therefore, we have $\mathcal{F}(C_c(G)) \subseteq L^2(\widehat{G}) \cap C(\widehat{G}) \subseteq C_0(\widehat{G})$ and thus the map

$$C_c^*(G) \rightarrow C_0(\widehat{G}), \quad \lambda(f) \mapsto \mathcal{F}(f), \quad (\forall f \in C_c(G))$$

is well defined. The surjectivity follows from Stone–Weierstrass’ theorem. \square

1.2.3 Group von Neumann Algebra, $L(G)$

In [vN30], John von Neumann introduced a special kind of C^* -algebras under the name “rings of operators” that today we call von Neumann algebras. Originally, a von Neumann algebra is defined as a WOT-closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} , and, in the same article, von Neumann algebras were characterized by being equal to their bicommutant. Another characterization was given in [Sak71], namely a C^* -algebra admits a (unique) Banach predual if and only if it is a von Neumann algebra. For that spirit, von Neumann algebras are also called W^* -algebras. This characterization proves that the first two definitions do not depend on the faithful $*$ -representation.

Definition 1.2.10. Let G be a locally compact group. The group von Neumann algebra $L(G)$ is the smallest von Neumann subalgebra of $\mathcal{B}(L^2(G))$ that contains the left translation operators $\lambda(G) = \{\lambda(x) : x \in G\}$.

By von Neumann’s bicommutant theorem, the group von Neumann algebra also can be defined as $L(G) = \lambda(G)''$. Any operator $T \in \mathcal{B}(L^2(G))$ commuting with the left translations also commute with $\lambda(f)$ for any $f \in L^1(G)$ because

$$\langle T\lambda(f)\xi, \eta \rangle = \int_G f(x) \langle \lambda(x)\xi, T^*\eta \rangle dx = \int_G f(x) \langle \lambda(x)T\xi, \eta \rangle dx = \langle \lambda(f)T\xi, \eta \rangle.$$

Therefore, $L(G)$ is also WOT-completion of $\lambda(L^1(G))$. The density argument shows that $L(G) = \lambda(G)'' = \lambda(L^1(G))'' = \lambda(C_c(G))'' = C_\lambda^*(G)''$.

Proposition 1.2.11. *When G is a locally compact abelian group, the Fourier transform identifies $L(G) \cong L^\infty(\widehat{G})$ as Banach *-algebras.*

Proof. We have seen that the reduced group C^* -algebra of G can be identified with the pointwise multiplication operators $M(C_0(\widehat{G})) = \{M_\phi \in \mathcal{B}(L^2(\widehat{G})) : \phi \in C_0(\widehat{G})\}$. The multiplication operators $M(L^\infty(\widehat{G})) = \{M_\phi \in \mathcal{B}(L^2(\widehat{G})) : \phi \in L^\infty(\widehat{G})\}$ commute with $M(C_0(\widehat{G}))$. Taking into account that the commutant A' of a commutative subalgebra A contains A , we deduce that $M(C_0(\widehat{G}))''$ contains $M(L^\infty(\widehat{G}))$. Since $M(L^\infty(\widehat{G}))$ is maximal abelian in $\mathcal{B}(L^2(\widehat{G}))$, we have $L(G) \cong L^\infty(\widehat{G})$. \square

Interestingly, the group von Neumann algebra $L(G)$ is not sufficient to restore the group G . The following theorem shows that there are many different groups whose von Neumann algebras are isomorphic.

Theorem 1.2.12 ([Con76]). *Let Γ be a countable amenable group with infinite conjugacy classes except for the trivial one. Then the group von Neumann algebra $L(\Gamma)$ is isomorphic to the hyperfinite II_1 factor.*

1.2.4 Fourier and Fourier-Stieltjes algebras, $A(G)$ and $B(G)$

In this subsection, we will describe the Fourier algebra $A(G)$ – the unique Banach predual of $L(G)$. We will see its connection to the dual space $B(G)$ of the full group C^* -algebra, also known as the Fourier-Stieltjes algebra.

Definition 1.2.13. Let G be a locally compact group and let (π, \mathcal{H}) be a unitary representation of G . We denote by $B_\pi(G)$ the space of all matrix coefficients associated to a representation that is weakly contained in π :

$$B_\pi(G) = \{\langle \pi'(\cdot)v, w \rangle \in C_b(G) : (\pi', \mathcal{H}') \prec (\pi, \mathcal{H}) \text{ and } v, w \in \mathcal{H}'\}.$$

We endow $B_\pi(G)$ with the norm

$$\|f\|_{B_\pi} = \inf\{\|v\|\|w\| : (\pi', \mathcal{H}') \prec (\pi, \mathcal{H}), v, w \in \mathcal{H}', \text{ and } f = \langle \pi'(\cdot)v, w \rangle\}.$$

For example, the space of all matrix coefficients of the unitary representations of G is exactly $B_{\pi_{max}}(G)$. This space is called the *Fourier-Stieltjes algebra* of G , and

we simply write $B(G)$ instead of $B_{\pi_{max}}(G)$. The Fourier-Stieltjes algebra is indeed an algebra under the pointwise multiplication because a product of two matrix coefficients is again a matrix coefficient:

$$\langle \pi_1(\cdot)v_1, w_1 \rangle_{\mathcal{H}_1} \langle \pi_2(\cdot)v_2, w_2 \rangle_{\mathcal{H}_2} = \langle (\pi_1 \otimes \pi_2)(\cdot)v_1 \otimes v_2, w_1 \otimes w_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}.$$

The Banach space $B_\lambda(G)$ is isometrically identified as a closed ideal of Fourier-Stieltjes algebra, which follows from the Fell's absorption principle. We, however, note that not all spaces $B_\pi(G)$ are closed under the pointwise multiplication.

Theorem 1.2.14 ([Eym64]). *Let G be a locally compact group and let (π, \mathcal{H}) be a unitary representation of G . The space $B_\pi(G)$ is isometrically isomorphic to the dual space of $C_\pi^*(G)$. The duality is given by*

$$\langle c_{v,w}, T \rangle = \langle Tv, w \rangle, \quad (\forall v, w \in \mathcal{H}, \forall T \in C_\pi^*(G)).$$

When T is of the form $\pi(f)$ for some $f \in L^1(G)$, the duality formula is simplified as

$$\langle c_{v,w}, \pi(f) \rangle = \langle \pi(f)v, w \rangle = \int_G c_{v,w}(x)f(x)dx.$$

Here, the functions $c_{v,w}$ are the matrix coefficients defined in Definition (1.1.1).

Definition 1.2.15. The Fourier algebra $A(G)$ is the subspace of $B(G)$ consisting of the matrix coefficients of the left regular representation.

In the following theorem, we collect the main properties of the Fourier algebra. We refer to [Eym64] for the proof.

Theorem 1.2.16. *Let G be a locally compact group.*

- (i) $A(G)$ is a closed ideal of $B(G)$.
- (ii) The space $C_c(G) \cap B(G)$ is a dense subspace of $A(G)$.
- (iii) We have isometric embeddings $A(G) \subseteq B_\lambda(G) \subseteq B(G)$.
- (iv) For $\varphi \in A(G)$, we have $\|\varphi\|_A = \inf\{\|\xi\|_2\|\eta\|_2 : \varphi = \xi * \tilde{\eta}, \xi, \eta \in L^2(G)\}$

(v) $A(G)$ is the unique predual of the group von Neumann algebra $L(G)$. The duality is given by

$$\langle \varphi, \lambda(f) \rangle = \int_G \varphi f dx, \quad (\forall \varphi \in A(G), \forall f \in L^1(G))$$

or equivalently

$$\langle \langle \lambda(\cdot)\xi, \eta \rangle, T \rangle = \langle T\xi, \eta \rangle, \quad (\forall \xi, \eta \in L^2(G), \forall T \in L(G)).$$

Proposition 1.2.17. *If G is a locally compact abelian group, the Fourier algebra $A(G)$ (resp. $B(G)$) is isometrically isomorphic to $L^1(\widehat{G})$ (resp. $M(\widehat{G})$) as Banach algebras.*

Proof. Since $L^1(\widehat{G})$ is the unique Banach predual of $L^\infty(\widehat{G}) \cong L(G)$, the Fourier algebra $A(G)$ is indeed isometrically isomorphic to $L^1(\widehat{G})$ as Banach spaces, so we only need to show the multiplication correspondence. The identification $A(G) \cong L^1(\widehat{G})$ is given by the transpose of the Fourier transform

$$\mathcal{G}^*|_{L^1(\widehat{G})} : L^1(\widehat{G}) \rightarrow A(G)$$

which coincides with the Fourier transform of \widehat{G} . Moreover, for any $\phi_1, \phi_2 \in L^1(\widehat{G})$ and $f \in L^1(G)$, we have

$$\begin{aligned} \langle \mathcal{G}^*(\phi_1 * \phi_2), \lambda(f) \rangle &= \langle \phi_1 * \phi_2, \mathcal{F}(f) \rangle \\ &= \int_{\widehat{G}} (\phi_1 * \phi_2)(\chi) \int_G f(x) \overline{\chi(x)} dx d\chi \\ (\text{Fubini}) &= \int_G f(x) \int_{\widehat{G}} (\phi_1 * \phi_2)(\chi) \overline{\chi(x)} d\chi dx \\ &= \int_G f(x) \mathcal{F}(\phi_1 * \phi_2)(x) dx \\ &= \int_G f(x) \mathcal{F}(\phi_1)(x) \mathcal{F}(\phi_2)(x) dx \\ &= \langle \mathcal{G}^*(\phi_1) \mathcal{G}^*(\phi_2), \lambda(f) \rangle. \end{aligned}$$

The assertion $B(G) \cong M(\widehat{G})$ is proven similarly. \square

The following lemma shows that the Fourier algebra contains sufficiently many functions to separate the points of G .

Lemma 1.2.18 (Eymard's trick). *Let G be a locally compact group, K a compact subset, and U an open subset containing K . Then there exists a positive function $f \in A(G)$ such that $f|_K = 1$ and $f|_{G \setminus U} = 0$.*

Proof. For each point $k \in K$, we can find a precompact symmetric open neighborhood V_k of the identity such that $kV_k^3 \subseteq U$ by considering the continuity of the multiplication. Since K is compact, there exist finite number of elements $k_1, \dots, k_n \in K$ such that $K \subseteq \bigcup_{i=1}^n k_i V_{k_i}$. Note that the set $V = \bigcap_{i=1}^n V_{k_i}$ is precompact and open. It follows that

$$KV^2 \subseteq \bigcup_{i=1}^n k_i V_{k_i}^3 \subseteq U.$$

Now, the normalized matrix coefficient

$$f : x \in G \mapsto \frac{1}{|V|} \langle \lambda(x) \mathbb{1}_V, \mathbb{1}_{KV} \rangle = \frac{|xV \cap KV|}{|V|}$$

satisfies all the necessary conditions. □

1.2.5 Fourier multipliers, $MA(G)$ and $M_0A(G)$

This subsection is prepared essentially from [DCH85] and [CH89].

Proposition 1.2.19 ([DCH85]). *Let G be a locally compact group and let φ be a continuous function on G . Then the following statements are equivalent:*

- (i) *The map $m_\varphi : A(G) \rightarrow A(G)$, $m_\varphi(f) = \varphi f$ is a well defined (bounded) operator.*
- (ii) *The map $M_\varphi : \lambda(x) \mapsto \overline{\varphi(x)} \lambda(x)$, $\forall x \in G$ extends to a (unique) weak*-continuous bounded operator on $L(G)$.*
- (iii) *The map $\overline{M}_\varphi : C_c(G) \rightarrow C_c(G)$, $\overline{M}_\varphi(f) = \varphi f$ extends to a (unique) bounded operator on $C_\lambda^*(G)$.*
- (iv) *The map $\overline{M}_\varphi^* : B_\lambda(G) \rightarrow B_\lambda(G)$, $\overline{M}_\varphi^*(f) = \varphi f$ is a well defined (bounded) operator.*

Moreover, if these conditions are satisfied, we have

$$\|m_\varphi\| = \|M_\varphi\| = \|\overline{M}_\varphi\| = \|\overline{M}_\varphi^*\|.$$

The duality argument on $A(G)^* = L(G)$ and $C_\lambda^*(G)^* = B_\lambda(G)$ shows the equivalences (i) \Leftrightarrow (ii) and (iii) \Leftrightarrow (iv). To show (ii) \Rightarrow (iii) and (iv) \Rightarrow (i), we can use the density argument on $\lambda(C_c(G)) \subseteq C_\lambda^*(G)$ and $B(G) \cap C_c(G) \subseteq A(G)$.

Definition 1.2.20. We say that a bounded continuous function φ on a locally compact group G defines a *multiplier of the Fourier algebra* (or simply φ is a *Fourier multiplier* of G) if one of the equivalent statements in Proposition 1.2.19 is satisfied. Denote by $MA(G)$ the space of all Fourier multipliers of G endowed with the norm $\|\varphi\|_{MA} = \|M_\varphi\|$. A Fourier multiplier $\varphi \in MA(G)$ is said *completely bounded* if M_φ defines a completely bounded operator on $L(G)$, that is

$$\|M_\varphi\|_{cb} = \sup_{n \in \mathbb{N}} \|M_\varphi \otimes \text{id}_n : L(G) \otimes M_n \rightarrow L(G) \otimes M_n\| < \infty.$$

Denote by $M_0A(G)$ the space of all completely bounded Fourier multipliers of G endowed with the norm $\|\varphi\|_{M_0A} = \|M_\varphi\|_{cb}$.

One can show that $MA(G)$ and $M_0A(G)$ are commutative Banach algebras. In general, calculating a completely bounded norm can be very difficult. However, we have the following various characterizations from [Gil74], [DCH85], [BF84], [Haa16], and [Jol92] which give a way of estimating completely bounded multiplier norms.

Theorem 1.2.21. *Let φ be a bounded continuous function on a locally compact group G . The following statements are equivalent:*

- (i) $\varphi \in M_0A(G)$.
- (ii) $\varphi \otimes \text{id} \in MA(G \times SU(2))$.
- (iii) $\varphi \otimes \text{id} \in MA(G \times H)$ for any locally compact group H .
- (iv) There is a Hilbert space \mathcal{H} and bounded continuous maps $\xi, \eta : G \rightarrow \mathcal{H}$ such that $\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$ for all $x, y \in G$.
- (v) φ defines a multiplier on the projective tensor product space $L^2(G) \otimes_\gamma L^2(G)$ in the sense that for all $u, v \in L^2(G)$, there exist $h_i, k_i \in L^2(G)$ such that $\sum_i \|h_i\|_2 \|k_i\|_2 < \infty$ and

$$\varphi(y^{-1}x)u(x)v(y) = \sum_i h_i(x)k_i(y), \quad (\forall x, y \in G).$$

(vi) There is a constant $C > 0$ such that for any $n \in \mathbb{N}$, $A = (a_{ij})_{i,j} \in M_n(\mathbb{C})$, and distinct elements $x_1, \dots, x_n \in G$, we have

$$\|(\varphi(x_j^{-1}x_i)a_{ij})\|_{M_n} \leq C\|(a_{ij})\|_{M_n}.$$

(vii) There exist a Hilbert space \mathcal{H} and bounded operators $P, Q : L^1(G) \rightarrow \mathcal{H}$ such that

$$\int_G \int_G \varphi(y^{-1}x)f(x)g(y)dxdy = \langle P(f), Q(g) \rangle, \quad (\forall f, g \in L^1(G)).$$

Moreover, we have

$$\begin{aligned} \|\varphi\|_{M_0A} &= \|\varphi\|_{MA(G \times SU(2))} = \sup_H \|\varphi\|_{MA(G \times H)} = \inf \|\xi\|_\infty \|\eta\|_\infty \\ &= \|\varphi\|_{L^2 \otimes_\gamma L^2 \rightarrow L^2 \otimes_\gamma L^2} = \inf C = \inf \|P\| \|Q\|. \end{aligned}$$

So, for example, if $\varphi \in C_b(G)$ is positive definite, or equivalently if there is a unitary representation (π, \mathcal{H}) and a vector $v \in \mathcal{H}$ such that $\varphi(y^{-1}x) = \langle \pi(x)v, \pi(y)v \rangle$ for all $x, y \in G$, then the M_0A -norm is bounded by $\|v\|^2 = \varphi(e)$. Conversely, MA -norm is never less than the uniform norm, hence

$$\|\varphi\|_{MA} = \|\varphi\|_{M_0A} = \varphi(e). \tag{1.3}$$

Similarly, any matrix coefficient $x \mapsto \langle \pi(x)\xi, \eta \rangle$ of a uniformly bounded representation π is a completely bounded Fourier multiplier whose norm is bounded by $\|\pi\|^2 \|\xi\| \|\eta\|$ and by $\sup_{x,y \in G} \|\pi(x)\xi\| \|\pi(y^{-1})^* \eta\|$. The space of such coefficients is usually denoted by $UB(G)$ or $\bigcup_{c \geq 1} B_c(G)$, where $B_c(G)$ is the Banach space of matrix coefficients of the representations uniformly bounded by c . All these algebras in one picture, we have the inclusions

$$\begin{array}{ccccccc} A(G) & \subseteq & B_\lambda(G) & \subseteq & B(G) & \subseteq & UB(G) \subseteq M_0A(G) \subseteq MA(G) \\ \parallel & & \parallel & & \parallel & & \\ L(G)_* & & C_\lambda^*(G)^* & & C^*(G)^* & & \end{array} \tag{1.4}$$

of which the first two inclusions are isometric. The equality $B(G) = MA(G)$ characterizes the amenability of G [Neb82, Los84, Boz85], so estimating MA -norm is eased for amenable groups. We also have the following theorem for groups that

differ from being amenable by a compact subgroup.

Theorem 1.2.22 ([CH89]). *Let G be a locally compact group and K a compact subgroup of G . Then the $K \times K$ double averaging map*

$$\varphi \mapsto \dot{\varphi}, \quad \dot{\varphi}(x) = \int_K \int_K \varphi(kxk') dk dk', \quad (\forall x \in G)$$

defines norm decreasing maps $A(G) \rightarrow A(G)$, $MA(G) \rightarrow MA(G)$, and $M_0A(G) \rightarrow M_0A(G)$. Moreover, suppose that S is an amenable closed subgroup of G such that $G = SK$ set theoretically. Then for any K -bi-invariant function $\varphi \in C_0(G)$, we have

$$\|\varphi\|_{M_0A(G)} = \|\varphi\|_{MA(G)} = \|\varphi|_S\|_{B(S)}.$$

Another important Fourier multipliers are positive Fourier multipliers. Recall that an element a of a C^* -algebra A is *positive* if $a = b^*b$ for some $b \in A$. A linear map $T : A \rightarrow B$ between C^* -algebras is *positive* if it maps positive elements to positive elements, *n -positive* if the map $T \otimes \text{id}_n : A \otimes M_n(\mathbb{C}) \rightarrow B \otimes M_n(\mathbb{C})$ is positive, and *completely positive* if T is n -positive for any $n \in \mathbb{N}$. One of the nicest feature of a positive linear map $T : A \rightarrow B$ is that its norm is equal to $\|T(1_A)\|$ provided A is unital. Thus, for example, when $M_\varphi : L(G) \rightarrow L(G)$ is positive, the Fourier multiplier norm is simply $\|\varphi\|_{MA} = \|M_\varphi\| = \|\varphi(e)\lambda(e)\| = \varphi(e)$, and when M_φ is completely positive, we have

$$\|\varphi\|_{M_0A} = \sup_n \|\varphi(e) \text{id}_{L(G) \otimes M_n}\| = \varphi(e). \quad (1.5)$$

It is a fact that a bounded continuous function on a locally compact group is positive definite if and only if it is a completely bounded Fourier multiplier [DCH85, Proposition 4.2]. This means that the norm calculations in (1.3) and (1.5) are actually the same result, yet the obtaining methods are different.

Chapter 2

Two group properties

2.1 Weak-amenability

Amenable groups play an important role in group theory. As amenability is characterized in many different ways, many problems tend to soften for amenable groups. The most commonly used definition would be the very first one given in [Neu29]: a discrete group Γ is amenable if there exists a non-trivial Γ -invariant positive linear functional on $\ell^\infty(\Gamma)$. This definition can easily be extended to locally compact groups by replacing the space $\ell^\infty(\Gamma)$ with $L^\infty(G)$. For our use, the following definition is more suitable.

Definition 2.1.1. Let G be a locally compact group. An *approximate unit* in the Fourier algebra $A(G)$ is a net $(\varphi_i)_{i \in I}$ of functions in $A(G)$ such that $\|\varphi_i \psi - \psi\|_A \rightarrow 0$ for all $\psi \in A(G)$. We say that G is *amenable* if there is an approximate unit in $A(G)$ consisting of positive definite functions.

Some interesting weaker versions of this definition are studied. For example, Haagerup property (also known as a-T-menability), weak amenability, and property A. In this section, we treat some view points of weak amenability.

Theorem 2.1.2 (Lemma 2.2 of [Haa16]). *Let G be a locally compact group and let $k \geq 1$. The following statements are equivalent:*

- (i) *There exists a net (φ_i) in $A(G)$ such that $\sup_i \|\varphi_i\|_{M_0A} \leq k$ and $\varphi_i \rightarrow 1$ in $\sigma(L^\infty, L^1)$ -topology.*
- (ii) *There exists a net (φ_i) in $A(G)$ such that $\sup_i \|\varphi_i\|_{M_0A} \leq k$ and $\varphi_i \rightarrow 1$ uniformly on compact sets.*

(iii) There exists an approximate unit (φ_i) in $A(G)$ such that $\sup_i \|\varphi_i\|_{M_0A} \leq k$.

Definition 2.1.3. We say that a locally compact group G is *weakly amenable* if one of the equivalent statements in Theorem 2.1.2 is true for some $k \geq 1$. The minimum possible value of k is called the *Cowling-Haagerup constant* and denoted by $\Lambda(G)$. If no such k exists, we conventionally write $\Lambda(G) = \infty$.

For some technical purpose, we sometimes want the functions φ_i to be compactly supported. Thanks to [CH89, Proposition 1.1], if G is weakly amenable, we can assume that all φ_i are compactly supported after replacing k by $k + \varepsilon$, where $\varepsilon > 0$ can be chosen arbitrarily small.

2.1.1 Groups acting on a tree

Amenable groups are trivial examples of weakly amenable groups. The first non-trivial weakly amenable group is the free group F_2 of two generators, or more generally, any group acting properly on a tree by isometries [Szw91, PS86, BP93]. The idea of the proof is very important because it extends to Gromov's hyperbolic groups [Oza08] and finite dimensional CAT(0) cubical groups [Miz08, GH10], so here we illustrate the proof. We will also use it to prove that Baumslag-Solitar groups have completely bounded characteristic tame cuts.

Theorem 2.1.4. *Suppose that a discrete group Γ acts on an infinite tree $T = (V, E)$ by isometries, where T is endowed with its combinatorial distance d . Fix a base point $v_0 \in V$. For $n \in \mathbb{N}$, define the function $\phi_n : x \in \Gamma \mapsto \mathbb{1}_{\{0,1,\dots,n\}}(d(v_0, xv_0))$. We have the following:*

(i) $\|\phi_n\|_{M_0A} \leq 2n + 1$.

(ii) The length function $\ell : x \in \Gamma \mapsto d(v_0, xv_0)$ is conditionally negative definite.

(iii) For any $t > 0$, the function $\rho_t : x \mapsto e^{-t\ell(x)}$ on Γ is positive definite.

(iv) If the action $\Gamma \curvearrowright T$ is proper, Γ is weakly amenable with $\Lambda(G) = 1$.

Proof. To prove (i), we fix an infinite geodesic ray $\gamma : \mathbb{N}_0 \rightarrow V$ in T . Each $v \in V$ admits a unique geodesic ray γ_v beginning from v that eventually merges with γ , that is $\gamma_v(0) = v$ and there are constants $a, b \in \mathbb{N}_0$ with $\gamma_v(a + t) = \gamma(b + t)$ for all

$t \in \mathbb{N}_0$. Note that being $d(v, w) \in \{n, n-2, \dots\} \cap \mathbb{N}_0$ is equivalent to the existence of $k \in \{0, 1, \dots, n\}$ with $\gamma_v(k) = \gamma_w(n-k)$. It follows that

$$\begin{aligned} \mathbb{1}_{\{n, n-2, \dots\} \cap \mathbb{N}_0}(y^{-1}x) &= \sum_{k=0}^n \langle \delta_{\gamma_{xv_0}(k)}, \delta_{\gamma_{yv_0}(n-k)} \rangle_{\ell^2(V)} \\ &= \langle \bigoplus_{k=0}^n \delta_{\gamma_{xv_0}(k)}, \bigoplus_{k=0}^n \delta_{\gamma_{yv_0}(n-k)} \rangle_{\bigoplus_{k=0}^n \ell^2(V)}, \end{aligned}$$

for all $x, y \in \Gamma$, and $\|\mathbb{1}_{\{n, n-2, \dots\} \cap \mathbb{N}_0}\|_{M_0A} \leq n+1$ by Theorem 1.2.21. Thus, we have $\|\phi_n\|_{M_0A} \leq \|\mathbb{1}_{\{n, n-2, \dots\} \cap \mathbb{N}_0}\|_{M_0A} + \|\mathbb{1}_{\{n-1, n-3, \dots\} \cap \mathbb{N}_0}\|_{M_0A} \leq 2n+1$.

Next two assertions are dealt in Example 1.1.20. Let us prove the last assertion. The normalized positive definite functions ρ_t are bounded in $M_0A(\Gamma)$ and goes to 1 uniformly on compact sets as t goes to 0, but we do not know if ρ_t is in $A(G)$. We will approximate ρ_t by a compactly supported function in the space $M_0A(\Gamma)$. As the action is proper, the functions ϕ_k are compactly supported as well as

$$\varphi_{n,t} = \phi_0 + \sum_{k=1}^n e^{-tk}(\phi_k - \phi_{k-1}), \quad (\forall n \in \mathbb{N}, \forall t > 0).$$

Moreover, we have

$$\|\rho_t - \varphi_{n,t}\|_{M_0A} \leq \sum_{k=n+1}^{\infty} e^{-tk} \|\phi_k - \phi_{k-1}\|_{M_0A} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof. □

2.1.2 Induction map

The methods to establish weak amenability for discrete groups and for continuous groups vary. In [Haa16], Haagerup proposed a method that allows one to work on a lattice instead of the ambient group. More precisely, if a lattice Γ in a locally compact group G is weakly amenable, then so is G with the same Cowling-Haagerup constants. The main tool used to prove this result is the induction map $\Phi : \ell^\infty(\Gamma) \rightarrow C_b(G)$ that sends a bounded function φ on Γ to the function

$$\Phi(\varphi) = \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi \mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega \tag{2.1}$$

on G , where Ω is a Borel fundamental domain for Γ and μ_Γ is the measure on G that counts the elements of Γ . We assume that the Haar measure dx on G is normalized so that $\int_\Omega dx = 1$. Then we have the following result.

Proposition 2.1.5 ([Haa16]).

(i) If $\varphi \in M_0A(\Gamma)$, then $\widehat{\varphi} \in M_0A(G)$ with $\|\widehat{\varphi}\|_{M_0A(G)} \leq \|\varphi\|_{M_0A(\Gamma)}$.

(ii) If $\varphi \in A(\Gamma)$, then $\widehat{\varphi} \in A(G)$ with $\|\widehat{\varphi}\|_{A(G)} \leq \|\varphi\|_{A(\Gamma)}$.

Corollary 2.1.6 ([Haa16]). A lattice Γ in G is weakly amenable if and only if the ambient group G is weakly amenable. In this case, $\Lambda(\Gamma) = \Lambda(G)$.

Remark 2.1.1. The induction map can be used to prove that some other properties such as amenability, a-T-menability, Kazhdan's property (T), and Yu's property A are inherited by lattices.

The induction map $\Phi : M_0A(\Gamma) \rightarrow M_0A(G)$ is always continuous. In particular, if G (equivalently Γ) is amenable, we have $\Phi : MA(\Gamma) \rightarrow MA(G)$ continuous. In Section 3.6, we will show that amenability is essential for the latter map to be continuous.

Corollary 2.1.6 gives the choice to work on the ambient group G or on a lattice when calculating the Cowling-Haagerup constant. Working on discrete groups has its advantage of characterizations in terms of the group algebras, $C_\lambda^*(\Gamma)$ and $L(G)$.

Definition 2.1.7. A C^* -algebra A has the *completely bounded approximation property* (CBAP) if there is a net $(T_i)_{i \in I}$ of continuous finite rank operators on A such that

$$\sup_i \|T_i\|_{cb} < \infty \quad \text{and} \quad \|T_i x - x\| \rightarrow 0 \quad \text{for all } x \in A.$$

The minimum value (when it exists) of $\sup_i \|T_i\|_{cb}$ while the net $(T_i)_{i \in I}$ runs through all possible nets satisfying the above conditions is denoted by $\Lambda(A)$.

Definition 2.1.8. A von Neumann algebra M has the *weak* completely bounded approximation property* (w^* -CBAP) if there is a net $(T_i)_{i \in I}$ of w^* -continuous finite rank operators on M such that

$$\sup_i \|T_i\|_{cb} < \infty \quad \text{and} \quad \langle T_i x, v \rangle \rightarrow 0 \quad \text{for all } x \in M, v \in M_*.$$

The minimum value (when it exists) of $\sup_i \|T_i\|_{cb}$ while the net $(T_i)_{i \in I}$ runs through all possible nets satisfying the above conditions is denoted by $\Lambda^*(M)$.

From the characterizations of Fourier multipliers in Proposition 1.2.19, if Γ is a weakly amenable discrete group, then the multipliers $(M_{\varphi_i})_{i \in I}$ and $(\overline{M_{\varphi_i}})_{i \in I}$ satisfy the above conditions, and consequently $C_\lambda^*(\Gamma)$ has CBAP and $L(\Gamma)$ has w^* -CBAP. The converse statement happens to be true.

Theorem 2.1.9 ([CH89]). *Let Γ be a discrete group. Then the following statements are equivalent:*

- (i) Γ is weakly amenable.
- (ii) $C_\lambda^*(\Gamma)$ has CBAP.
- (iii) $L(\Gamma)$ has w^* -CBAP.

In this case, we have $\Lambda(\Gamma) = \Lambda(C_\lambda^*(\Gamma)) = \Lambda^*(L(\Gamma))$.

2.1.3 Non-examples

The linear groups $SL_n(\mathbb{R})$ and $SL_n(\mathbb{Z})$ for $n \geq 3$ are known to satisfy a property that opposes amenability in a very strong way, namely these groups satisfy Kazhdan's property (T). Therefore, it is naturally interesting to know if these groups are weakly amenable. There are three different approaches giving a negative answer to this question. Chronologically, the first proof is in [Haa16] where Haagerup constructed a distribution $D \in L(SL_2(\mathbb{R}) \times \mathbb{R}^2)$ that has a simplified formulation for $SO(2)$ -bi-invariant functions.

Theorem 2.1.10 ([Haa16]). *Suppose that $\varphi \in C_c^\infty(SL_2(\mathbb{R}) \times \mathbb{R}^2)$ is a $SO(2)$ -bi-invariant compactly supported smooth function on $SL_2(\mathbb{R}) \times \mathbb{R}^2$. Then we have*

$$|D(\varphi)| = \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^2/4}} \varphi \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dx \right| \leq 4\pi \|\varphi\|_{M_0A}. \quad (2.2)$$

In [Haa16], the inequality (2.2) was written for the norm $\|\varphi\|_{H_3} \|A(H_3)$, where

$$H_3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

is the Heisenberg group, but this is equal to $\|\varphi\|_{M_0A}$ by Theorem 1.2.22. When φ is positive and takes value 1 on the ball B_n with respect to the length function $x \mapsto \log \|x\| + \log \|x^{-1}\|$, the inequality (2.2) gives

$$n \leq \int_{-2^n}^{2^n} \frac{1}{\sqrt{1+x^2/4}} dx \leq 4\pi \|\varphi\|_{M_0A}. \quad (2.3)$$

This shows that $SL_2(\mathbb{R}) \rtimes \mathbb{R}^2$ is not weakly amenable since the lower bound goes to infinity. The same idea works to prove that $Sp_4(\mathbb{R})$ is not weakly amenable. These two results imply that any simple Lie group G of real rank at least 2 with finite center is not weakly amenable.

The second proof is due to Lafforgue-de la Salle. They came up with even stronger lower bound than (2.2) for $SO(3)$ -bi-invariant functions on $SL_3(\mathbb{R})$.

Theorem 2.1.11 ([LDIS11]). *For any $SO(3)$ -bi-invariant function $\varphi \in C_0(SL_3(\mathbb{R}))$, and any $t > 0$, we have*

$$\left| \varphi \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \right| \leq 100e^{-t/2} \|\varphi\|_{M_0A(SL_3(\mathbb{R}))}.$$

In their original statement, the inequality above was given for multipliers of Schatten p -class, and Theorem 2.1.11 is exactly the case when $p = \infty$. We will use this inequality in Section 3.6. Similar inequalities are made for the groups $Sp(4, \mathbb{R})$ and $SL_{2n-3}(\mathbb{R})$, $n \geq 3$ in [dL13] and [dLdIS18]. This lower bound of course proves again that $SL_3(\mathbb{R})$ is not weakly amenable, but it is not applicable to prove that $SL_2(\mathbb{R}) \rtimes \mathbb{R}^2$ is not weakly amenable.

These two proofs both rely on Theorem 1.2.22. On the contrary, the third known proof given in [Oza12] works on the discrete group $SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$ and its action on \mathbb{T}^2 . This discrete group can not be written as a set theoretic product of an amenable subgroup and a compact subgroup, thus no use of Theorem 1.2.22. Rather, the proof relies on the fact that there is no $SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$ -invariant measure on \mathbb{T}^2 , equivalently the action $SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2 \curvearrowright \mathbb{Z}^2$ is not amenable.

2.1.4 ME-subgroup

Closed subgroups inherit the weak amenability and the Cowling-Haagerup constant becomes smaller. This is also observed for ME-subgroups and Measure Equivalent

groups. We recall the definition from [Sak09].

Definition 2.1.12. Let Γ and Λ be two countable groups. We say that Λ is *ME-subgroup* of Γ and write $\Lambda \stackrel{ME}{\leq} \Gamma$ if there exist commuting, measure preserving, free actions of Γ and Λ on a Lebesgue measure space (X, μ) such that each of Λ and Γ actions admits a Borel fundamental domain and the induced measure of $X_\Gamma = X/\Gamma$ is finite. Furthermore, if the induced measure of $X_\Lambda = \Lambda \backslash X$ is finite, we say that Γ and Λ are *Measure Equivalent (ME)* and write $\Gamma \stackrel{ME}{\sim} \Lambda$.

Lattices in a locally compact second countable group are ME to each other. Another interesting example is that the ME-class containing the infinite cyclic group is exactly the family of all countable amenable groups.

Let us go back to the fact stating that weak amenability passes to ME-subgroups. We explain in detail the proof given in [Oza12]. The main tool is the induction map constructed as follows. Suppose that the actions $\Lambda \curvearrowright (X, \mu) \curvearrowright \Gamma$ establish Λ as a ME-subgroup of Γ . Denote by Ω a Borel fundamental domain of the right action $X \curvearrowright \Gamma$ and normalize the measure μ so that $\mu(\Omega) = 1$. The disjoint union $X = \bigsqcup_{\gamma \in \Gamma} \Omega\gamma$ yields well defined measurable maps

$$\omega : X \rightarrow \Omega, \quad \gamma : X \rightarrow \Gamma \quad (2.4)$$

satisfying $x = \omega(x).\gamma(x)$ for all $x \in X$. From the identity $(ts).x = t.(s.x) = (t.\omega(s.x)).\gamma(s.x)$, we can extract the formula

$$\gamma(ts.x) = \gamma(t.\omega(s.x))\gamma(s.x), \quad (\forall t, s \in \Lambda, \forall x \in X),$$

or equivalently, by the change of variables $(s, t) \mapsto (t^{-1}s, t)$,

$$\gamma(t^{-1}s.x)^{-1} = \gamma(s.x)^{-1}\gamma(t.\omega(t^{-1}s.x)), \quad (\forall t, s \in \Lambda, \forall x \in X). \quad (2.5)$$

The induction map is defined as

$$\Phi : \varphi \in \ell^\infty(\Gamma) \mapsto \widehat{\varphi} \in \ell^\infty(\Lambda), \quad \widehat{\varphi}(s) = \int_{\Omega} \varphi(\gamma(s.\omega)^{-1})d\mu(\omega). \quad (2.6)$$

Lemma 2.1.13 ([Oza12]). *Suppose that Λ is a ME-subgroup of Γ .*

- (i) *If $\varphi \in M_0A(\Gamma)$, then $\widehat{\varphi} \in M_0A(\Lambda)$ and $\|\widehat{\varphi}\|_{M_0A} \leq \|\varphi\|_{M_0A}$.*
- (ii) *If $\varphi \in A(\Gamma)$, then $\widehat{\varphi} \in A(\Lambda)$ and $\|\widehat{\varphi}\|_A \leq \|\varphi\|_A$.*

Proof. Suppose that there are some bounded maps P, Q from Γ to a Hilbert space \mathcal{H} such that $\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle_{\mathcal{H}}$ for all $x, y \in \Gamma$. Then by (2.5)

$$\begin{aligned} \widehat{\varphi}(t^{-1}s) &= \int_{\Omega} \varphi(\gamma(s.w)^{-1}\gamma(t.\omega(t^{-1}s.w))) d\mu(w) \\ &= \int_{\Omega} \varphi(\gamma(s.\omega(s^{-1}.w))^{-1}\gamma(t.\omega(t^{-1}.w))) d\mu(w) \\ &= \int_{\Omega} \langle P(\gamma(t.\omega(t^{-1}.w))), Q(\gamma(s.\omega(s^{-1}.w))) \rangle_{\mathcal{H}} d\mu(w) \\ &= \langle P'(t), Q'(s) \rangle_{L^2(\Omega, \mathcal{H}, \mu)} \end{aligned}$$

where the maps

$$P'(t) = \int_{\Omega} P(\gamma(t.\omega(t^{-1}.w))) d\mu(w) \quad \text{and} \quad Q'(s) = \int_{\Omega} Q(\gamma(s.\omega(s^{-1}.w))) d\mu(w)$$

have uniform norms at most $\|P\|_{\infty}$ and $\|Q\|_{\infty}$, respectively. Now, (i) follows from Theorem 1.2.21 (iv).

To prove the second assertion, take a matrix coefficient $\varphi = \langle \lambda_{\Gamma}(\cdot)\xi, \eta \rangle \in A(\Gamma)$. Recall that the Koopman representation $(\lambda_X, L^2(X))$ associated to the action $\Lambda \curvearrowright (X, \mu)$ is unitarily equivalent to $\lambda_{\Lambda} \otimes \text{id}$ by Lemma 1.1.8. Therefore, to prove $\widehat{\varphi} \in A(\Lambda)$, it is enough to prove that $\widetilde{\varphi}$ is a matrix coefficient of λ_X . Choose the vectors $\xi', \eta' \in L^2(X)$ such that $\xi'(x) = \xi(\gamma(x))$ and $\eta'(x) = \eta(\gamma(x))$ for all $x \in X$. Observe that

$$\lambda_X(s)\xi'(x) = \xi'(s^{-1}.x) = \xi(\gamma(s^{-1}.x)) = \xi(\gamma(s^{-1}.w)\delta)$$

for all $x = w.\delta \in X$ with $w \in \Omega$ and $\delta \in \Gamma$. The following calculation concludes the proof.

$$\begin{aligned} \widetilde{\varphi}(s) &= \int_{\Omega} \langle \lambda_{\Gamma}(\gamma(s^{-1}.w)^{-1})\xi, \eta \rangle d\mu(w) \\ &= \sum_{\delta \in \Gamma} \int_{\Omega} \xi(\gamma(s^{-1}.w)\delta) \overline{\eta(\delta)} d\mu(w) \\ &= \int_X \xi(\gamma(s^{-1}.w)\delta) \overline{\eta(\delta)} d\mu(x) \\ &= \langle \lambda_X(s)\xi', \eta' \rangle_{L^2(X)}. \end{aligned}$$

□

Lemma 2.1.14. *Suppose that Λ is a ME-subgroup of Γ . The induction map in (2.6) is the transpose of the norm decreasing map $\psi \in \ell^1(\Lambda) \mapsto \psi^* \in \ell^1(\Gamma)$ defined by*

$$\psi^*(\delta) = \sum_{s \in \Lambda} \psi(s) \mu(w \in \Omega : \gamma(s.w)^{-1} = \delta), \quad (\forall \delta \in \Gamma).$$

Proof. The continuity of $\psi \mapsto \psi^*$ follows from

$$\|\psi^*\|_1 \leq \sum_{s \in \Lambda} |\psi(s)| \underbrace{\sum_{\delta \in \Gamma} \mu(w \in \Omega : \gamma(s.w)^{-1} = \delta)}_{\leq 1} \leq \|\psi\|_1.$$

The last assertion follows from

$$\begin{aligned} \langle \widehat{\varphi}, \psi \rangle &= \sum_{s \in \Lambda} \widehat{\varphi}(s) \psi(s) = \sum_{s \in \Lambda} \int_{\Omega} \varphi(\gamma(s.w)^{-1}) \psi(s) d\mu(w) \\ &= \sum_{\delta \in \Gamma} \varphi(\delta) \sum_{s \in \Lambda} \psi(s) \mu(w \in \Omega : \gamma(s.w)^{-1} = \delta) \\ &= \sum_{\delta \in \Gamma} \varphi(\delta) \psi^*(\delta) = \langle \varphi, \psi^* \rangle \end{aligned}$$

for all $\varphi \in \ell^\infty(\Gamma)$ and $\psi \in \ell^1(\Lambda)$. □

Corollary 2.1.15. *Suppose that Λ is a ME-subgroup of Γ and that $(\varphi_i)_{i \in I}$ is a net in $\ell^\infty(\Gamma)$ such that $\varphi_i \rightarrow \mathbb{1}_\Gamma$ in the $\sigma(\ell^\infty, \ell^1)$ -topology. Then $\widehat{\varphi}_i \rightarrow \mathbb{1}_\Lambda$ in the $\sigma(\ell^\infty, \ell^1)$ -topology.*

Proof. By Lemma 2.1.14, $\langle \widehat{\varphi}_i, \psi \rangle = \langle \varphi_i, \psi^* \rangle \rightarrow 1$ for all $\psi \in \ell^1(\Lambda)$. □

The following theorem is now straightforward.

Theorem 2.1.16 ([Oza12]). *ME-subgroups inherit weak amenability. Moreover, if Λ is a ME-subgroup of Γ , then $\Lambda(\Lambda) \leq \Lambda(\Gamma)$.*

Remark 2.1.2. When the groups Λ and Γ are two lattices in a group, the map in (2.6) is more or less the induction map in (2.1). Beware that $\widehat{\varphi}$ might be not finitely supported even if φ is.

2.2 Rapid Decay property

In this section, we give a short introduction to the Rapid Decay property. Our first example of groups with characteristic tame cuts came from this property.

2.2.1 Length function

Definition 2.2.1. Let G be a group. A function $\ell : G \rightarrow \mathbb{R}_+$ is a *length function* of G if the following conditions are satisfied:

(i) $\ell(e) = 0$.

(ii) $\ell(x^{-1}) = \ell(x)$ for all $x \in G$.

(iii) $\ell(xy) \leq \ell(x) + \ell(y)$ for all $x, y \in G$.

The ball of radius $n \in \mathbb{N}$ with respect to ℓ is the set $B_{n,\ell} = \{x \in G : \ell(x) \leq n\}$. When there is no confusion, we just write B_n instead of $B_{n,\ell}$. When G is a locally compact group and the length function ℓ is Borel measurable, the corresponding *growth function* is given by

$$\gamma_\ell(n) : G \rightarrow \mathbb{R}_+, \quad \gamma_\ell(n) = \int_G \mathbb{1}_{B_n}(x) dx.$$

Remark 2.2.1. Recall that a map $f : X \rightarrow Y$ between two topological spaces is proper if the preimage of every compact set in Y is compact in X . We usually consider proper length functions. Note that any proper function $\ell : G \rightarrow \mathbb{R}$ is automatically Borel measurable. Also, note that a locally compact group G admitting a proper length function is σ -compact as $G = \bigcup_{n \in \mathbb{N}} \ell^{-1}([0, n])$.

Example 2.2.2. The most classical example of proper length functions comes from a compact generating set. Suppose that G is generated by a compact symmetric subset S , that is $S = S^{-1}$ and $G = \bigcup_{n \in \mathbb{N}} S^n$. Then the function

$$\ell_S : x \in G \mapsto \min\{n \in \mathbb{N} : x \in S^n\}$$

is a locally finite, proper length function of G . We call ℓ_S *the word length function* associated to the generating set S .

Example 2.2.3. Let G be a connected Lie group. Endow G with a left-invariant Riemannian structure and denote d the associated distance. The distance topology on G agrees with the manifold topology on G . The function $\ell(x) = d(1, x)$ is a continuous (hence locally bounded), proper length function of G (see [Roe03, Theorem 1.5] for properness). Indeed, the compact ball of radius 1 generates G , and ℓ is Lipschitz-equivalent to the word length function.

Example 2.2.4. Let Γ be a finitely generated lattice in G . If Γ is a uniform lattice, then the word length function of Γ is equivalent to the restriction of the word length function of G by quasi-isometry. This is in general not true for non-uniform lattices, but we know when G is a semisimple Lie group of rank at least 2 and Γ is an irreducible lattice, the restriction of ℓ_G onto Γ is Lipschitz-equivalent to ℓ_Γ [LMR93, LMR00]. The condition on the rank is essential. For example, the length of the element $u_n = \begin{pmatrix} 1 & n \\ 1 & 1 \end{pmatrix}$ in $SL_2(\mathbb{Z})$ is $O(n)$ while it is $O(\log n)$ in $SL_2(\mathbb{R})$.

Example 2.2.5. When G acts on a metric space (Y, d) by isometries, for any base point $y_0 \in Y$, one can associate a length function $\ell_{d, y_0} : x \in G \mapsto d(x.y_0, y_0)$. Conversely, every length function is obtained in this way. Suppose that ℓ is a length function of G . We want to construct a metric space (Y, d) and a base point y_0 such that $\ell = \ell_{d, y_0}$. To this end, define the equivalence relation \sim on G by $x \sim y$ if and only if $\ell(y^{-1}x) = 0$. The quotient space $Y = G / \sim = \{[x] : x \in G\}$ is a metric space with the distance $d([x], [y]) = \ell(y^{-1}x)$, and G naturally acts on Y by isometries. Now, it is enough to choose $y_0 = [e]$ to have $\ell = \ell_{d, y_0}$.

Definition 2.2.6. Let $f, g : X \rightarrow \mathbb{R}_+$ be two functions. We say that f *dominates* g if there is a constant $k > 0$ such that

$$g(x) \leq kf(x) + k, \quad (\forall x \in X).$$

We say that f and g are *equivalent* if they dominate each other.

It turns out that all measurable length functions are locally finite (see [Sch93, Theorem 1.2.11]), hence the following proposition.

Proposition 2.2.7. *The word length function associated to a compact generating set (when it exists) dominates all measurable length functions.*

Proof. Suppose that G is generated by a compact subset $S = S^{-1}$. Let ℓ be a measurable length function of G . Put $k = \sup\{\ell(s) : s \in S\}$. Then for any element $x \in G$ with $\ell_S(x) = n$ and $x = s_1 \dots s_n$ for some $s_1, \dots, s_n \in S$, we have

$$\ell(x) \leq \sum_{i=1}^n \ell(s_i) \leq nk = k\ell_S(x).$$

□

The proposition implies that all length functions associated to a compact generating set are equivalent. That is why we say *the* word length function.

Proposition 2.2.8. *Let K be a compact subgroup of G , and ℓ a length function of G . Then there exists an integral valued length function ℓ' equivalent to ℓ . Furthermore, if ℓ is Borel measurable, we can assume that ℓ' satisfies the following properties:*

(i) $\ell'(k_1 x k_2) = \ell'(x)$ for all $x \in G$ and $k_1, k_2 \in K$.

(ii) $K = \{x \in G : \ell'(x) = 0\}$.

Proof. Jolissaint gave a proof for discrete groups in [Jol90]. His proof still works for the general case, and we provide it here for completeness. Define the functions L_1 , L_2 , L_3 , and L_4 on G as follows:

$$L_1(x) = \begin{cases} 0, & \text{if } \ell(x) = 0 \\ \lfloor \ell(x) \rfloor + 1, & \text{otherwise} \end{cases}$$

$$L_2(x) = \int_K L_1(k x k^{-1}) dk$$

$$L_3(x) = \min\{L_2(k_1 x k_2) : k_1, k_2 \in K\}$$

$$L_4(x) = \begin{cases} 0, & \text{if } x \in K \\ 1, & \text{otherwise.} \end{cases}$$

For the first part, it is enough to take $\ell' = L_1$. For the second part, it is enough to take $\ell' = L_3 + L_4$. □

In the sequel, all length functions are proper unless otherwise stated.

2.2.2 s-Sobolev completion

Definition 2.2.9. Let G be a locally compact group and let ℓ be a length function of G . For $s \in \mathbb{R}$, we consider the following weighted- L^2 -norm of $f \in C_c(G)$:

$$\|f\|_{s,\ell} = \left(\int_G |f(x)|^2 (1 + \ell(x))^{2s} dx \right)^{1/2}.$$

The s -Sobolev completion $H_\ell^s(G)$ is the completion of $C_c(G)$ with respect to the norm $\|\cdot\|_{s,\ell}$. The functions of Rapid Decay are given by

$$H_\ell^\infty(G) = \bigcap_{s \in \mathbb{R}_+} H_\ell^s(G).$$

Remark 2.2.2. The isometry $f \in (C_c(G), \|\cdot\|_{s,\ell}) \mapsto f(1+\ell)^s \in L^2(G)$ extends to the isometry $H_\ell^s(G) \rightarrow L^2(G)$. This allows us to see $H_\ell^s(G)$ as a space of functions. Observe that for any $s \in \mathbb{R}$, $f \in H_\ell^s(G)$, and $g \in H_\ell^{-s}(G)$, by Cauchy–Schwarz inequality, we have

$$\left| \int_G fg dx \right| \leq \|f\|_{s,\ell} \|g\|_{-s,\ell}$$

which gives rise to the duality between $H_\ell^s(G)$ and $H_\ell^{-s}(G)$ given by

$$\langle f, g \rangle_s = \int_G fg dx. \quad (2.7)$$

Proposition 2.2.10. *Let $s \in \mathbb{R}$. The duality (2.7) isometrically identifies the space $H_\ell^{-s}(G)$ with the dual space $H_\ell^s(G)^*$.*

Proof. Take any continuous linear functional $\omega \in H_\ell^s(G)^*$. We only need to prove that ω can be represented as $\langle \cdot, g \rangle_s$ for some $g \in H_\ell^{-s}(G)$. Since $H_\ell^s(G)$ isometrically embeds in $L^2(G)$, by Hahn-Banach extension theorem and Riesz representation theorem, there is a square integrable function $\xi_\omega \in L^2(G)$ such that

$$\omega(f) = \langle f(1+\ell)^s, \xi_\omega \rangle = \langle f, \xi_\omega(1+\ell)^s \rangle, \quad (\forall f \in H_\ell^s(G)).$$

Note that $g = \overline{\xi_\omega}(1+\ell)^s \in H_\ell^{-s}(G)$ and $\omega = \langle \cdot, g \rangle_s$. This completes the proof. \square

2.2.3 Equivalent definitions of the Rapid Decay property

Definition 2.2.11. A locally compact group G endowed with a length function ℓ has the *Rapid Decay property* if the identity map on $C_c(G)$ extends to a continuous map $H_\ell^s(G) \rightarrow C_\lambda^*(G)$ for some $s \in \mathbb{R}_+$.

Theorem 2.2.12 ([Jol90]). *Let G be a locally compact group, and ℓ a measurable length function. The following statements are equivalent:*

- (i) (G, ℓ) has the Rapid Decay property.

(ii) The identity map on $C_c(G)$ extends to a continuous map $B_\lambda(G) \rightarrow H_\ell^{-s}(G)$ for some $s \in \mathbb{R}_+$.

(iii) The identity map on $C_c(G)$ extends to a continuous map $A(G) \rightarrow H_\ell^{-s}(G)$ for some $s \in \mathbb{R}_+$.

(iv) There are constants $C, s \geq 0$ such that for all $\xi, \eta \in L^2(G)$,

$$\left(\int_G \frac{|\langle \lambda(x)\xi, \eta \rangle|}{(1 + \ell(x))^s} dx \right)^{1/2} \leq C \|\xi\|_2 \|\eta\|_2.$$

(v) There are constants $C, s \geq 0$ such that $\|\lambda(f)\| \leq Cn^s \|f\|_2$ for all $f \in C_c(G)$ with $\text{supp}(f) \subseteq B_n$.

Proof. The equivalence (i) \Leftrightarrow (ii) is trivial from the dualities $B_\lambda(G) = C_\lambda^*(G)^*$ and $H_\ell^{-s}(G) = H_\ell^s(G)^*$. Also (ii) \Rightarrow (iii) since $A(G)$ isometrically embeds in $B_\lambda(G)$. (iv) is just a verbatim of (iii). Let us prove (iv) \Rightarrow (i). Assume (iv). For any $f \in C_c(G)$, we have

$$\begin{aligned} \|\lambda(f)\| &= \sup \{ |\langle \lambda(f), \varphi \rangle| : \|\varphi\|_A = 1 \} \\ &= \sup \left\{ \left| \int_G f(x) \langle \lambda(x)\xi, \eta \rangle dx \right| : \|\xi\|_2 = \|\eta\|_2 = 1 \right\} \\ &\leq \sup \{ \|f(x)\|_{s,\ell} \|\langle \lambda(\cdot)\xi, \eta \rangle\|_{-s,\ell} : \|\xi\|_2 = \|\eta\|_2 = 1 \} \\ &\text{(apply (iv))} \leq C \|f\|_{s,\ell}, \end{aligned}$$

hence (i). At this point, we have equivalence between the first four statements. Let us prove (i) \Rightarrow (v). Take any $f \in C_c(G)$ with $\text{supp}(f) \subseteq B_n$. By (i), we have

$$\|\lambda(f)\|^2 \leq C^2 \|f\|_{s,\ell}^2 = C^2 \int_{B_n} |f(x)|^2 (1 + \ell(x))^{2s} dx \leq C^2 (1 + n)^{2s} \|f\|_2^2$$

for some constants $C, s \geq 0$, hence (v). Conversely, assume that (v) is satisfied for some constants $C, s \geq 0$. Take any $f \in C_c(G)$. The following calculation completes

the proof:

$$\begin{aligned} \|\lambda(f)\| &\leq \sum_{n=0}^{\infty} \|\lambda(f \mathbb{1}_{B_n})\| \leq \sum_{n=0}^{\infty} Cn^s \|f \mathbb{1}_{B_n}\|_2 \leq \sum_{n=0}^{\infty} C(1+n)^{-1} \|(1+n)^{s+1} f \mathbb{1}_{B_n}\|_2 \\ &\leq C \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=0}^{\infty} \|(1+n)^{s+1} f \mathbb{1}_{B_n}\|_2^2 \right)^{1/2} = C \frac{\pi}{\sqrt{6}} \|f\|_{s+1, \ell}. \end{aligned}$$

□

Many Lie groups, including the semisimple ones with a finite center, have the Rapid Decay property (see [CPSC07] and [Boy13]). Non-uniform lattices in higher rank Lie groups tend not to satisfy the Rapid Decay property for having a cyclic subgroup with exponential relative growth. For uniform lattices, we have the following conjecture which is one of the main problems in this area.

Conjecture 2.2.1 ([Val02]). Uniform lattices in a semisimple Lie group have the Rapid Decay property.

The conjecture is supported by the following important result.

Theorem 2.2.13 ([Laf00, RRS98, Cha03]). *Uniform lattices in $SL_3(\mathbb{R})$, $SL_3(\mathbb{C})$, $SL_3(\mathbb{H})$, $SL_3(\mathbb{Q}_p)$, $E_{6(-26)}$, or a direct product of these Lie groups satisfy the Rapid Decay property.*

To name more discrete groups with Rapid Decay property, we have groups of polynomial growth, Gromov's hyperbolic groups, cocompact cubical CAT(0) groups, mapping class groups, 3-manifold groups not containing *Sol*, large type Artin groups, Wise non-Hopfian group, and some small cancellation groups. See the survey [Cha17] for more.

2.2.4 Obstruction to the Rapid Decay property

The following theorem gives the main obstruction to the Rapid Decay property.

Theorem 2.2.14 ([Jol90]). *Let Γ be a discrete amenable group endowed with a proper length function ℓ . Then (Γ, ℓ) has the Rapid Decay property if and only if Γ has polynomial growth with respect to ℓ .*

Taking into account that the Rapid Decay property is inherited to open subgroups for the restricted length function, one can easily deduce that if (Γ, ℓ) contains

an amenable subgroup with superpolynomial relative growth, then (Γ, ℓ) does not satisfy Rapid Decay property. As far as we know, all known discrete non-examples are explained using this obstruction. In what follows, we use this obstruction to illustrate some non-examples.

Example 2.2.15. For $A \in SL_d(\mathbb{Z})$, the group $\Gamma_A = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ has exponential growth (hence, does not satisfy the Rapid Decay property) if the matrix A admits an eigenvalue with an absolute value greater than 1. Here, we sketch the proof given in [dlH00, Proposition 5, p. 189] (see also [Š55]). Suppose that the matrix A has an eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda| \neq 1$. Since $\det(A) = 1$, we can assume that $|\lambda| > 1$. Also, since Γ_A contains Γ_{A^k} as a subgroup for all $k \in \mathbb{N}$, we can assume that $|\lambda| > 2$. Let $u \in \mathbb{C}^d$ be a $\bar{\lambda}$ -eigenvector of A^t and let $P \in M_d(\mathbb{C})$ be the projection onto $\mathbb{C}u$. Note that $PA^k = \lambda^k P$ for all $k \in \mathbb{N}$. Since P is a non-zero operator, there is a non-zero vector $v \in \mathbb{Z}^d$ such that $Pv \neq 0$. We choose the word length function associated to some finite generating set containing the elements $(0, 1)$ and $(v, 1)$. Fix $n \in \mathbb{N}$. Let us show that the ball B_n contains at least 2^n distinct elements. This can be achieved considering the following elements

$$E_\varepsilon = (\varepsilon_1 v, 1)(\varepsilon_2 v, 1) \dots (\varepsilon_n v, 1) = (\varepsilon_1 A v + \varepsilon_2 A^2 v + \dots + \varepsilon_n A^n v, n) \in \Gamma_A$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ runs through $\{0, 1\}^n$. Indeed, since

$$P(\varepsilon_1 A v + \varepsilon_2 A^2 v + \dots + \varepsilon_n A^n v) = (\varepsilon_1 \lambda + \varepsilon_2 \lambda^2 + \dots + \varepsilon_n \lambda^n) P(v)$$

are all distinct, so are the elements E_ε for $\varepsilon \in \{0, 1\}^n$.

A concrete example of such matrix is $A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as its eigenvalues are $\lambda_{1,2} = (3 \pm \sqrt{5})/2$. It follows that the groups $SL_n(\mathbb{Z})$, $n \geq 3$ do not satisfy the Rapid Decay property since these groups contain Γ_{A_0} :

$$\Gamma_{A_0} \cong \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in SL_3(\mathbb{Z}) : A \in SL_2(\mathbb{Z}), v \in \mathbb{Z}^2 \right\} < SL_3(\mathbb{Z}) < SL_n(\mathbb{Z}).$$

Being virtually free, the group $SL_2(\mathbb{Z})$ has Rapid Decay property.

Example 2.2.16. Let P be a finitely generated infinite group, and F a non-trivial finite group. The wreath product $F \wr P = (\oplus_{s \in P} F) \rtimes P$ (e.g. the Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$) has exponential growth and thus it does not have the Rapid Decay property.

To see that, fix a finite generating subset S_P of P . Then the finite subset

$$S = \{E_f = (s_f, e_P) : f \in F\} \cup (\{\delta_{e_F}\} \times S_P)$$

generates $F \wr P$, where s_f is identified with the element of $\bigoplus_{s \in P} F$ that takes f on the place indexed with the identity e_P , and takes identity e_F elsewhere. Fix $n \in \mathbb{N}$ and a geodesic $C_n = \{e = x_0, x_1, \dots, x_n\}$ on the Cayley graph of $F \wr P$. Note that the subset

$$A_n = \{(v, e_P) : v \in \bigoplus_{s \in P} F, \text{supp}(v) \subseteq C_n\}$$

contains exactly $|F|^{|C_n|} \geq 2^n$ distinct elements, so it is enough to prove that B_{4n} contains A_n . That is true because any element $(v, e_P) \in A_n$ can be written as a product of at most $3n + 1$ many elements of S :

$$(v, e_P) = [E_{f_0}] [(e_F, x_1)E_{f_1}(e_F, x_1^{-1})] \dots [(e_F, x_1)E_{f_n}(e_F, x_1^{-n})].$$

Example 2.2.17. Let $p, q \in \mathbb{N}$. The action

$$\alpha : \mathbb{Z} \times \mathbb{Z}\left[\frac{1}{pq}\right] \rightarrow \mathbb{Z}\left[\frac{1}{pq}\right], \quad n.P = \frac{p^n P}{q^n}, \quad \left(\forall n \in \mathbb{Z}, \forall P \in \mathbb{Z}\left[\frac{1}{pq}\right]\right)$$

defines the semidirect product $\Gamma_{p,q} = \mathbb{Z}\left[\frac{1}{pq}\right] \rtimes_{\frac{p}{q}} \mathbb{Z}$. We suppose that p and q are coprime so that $\Gamma_{p,q}$ is finitely generated. The finite subset

$$S = \{a^{\pm 1} = (\pm 1, 0), t^{\pm 1} = (0, \pm 1)\}.$$

generates $\Gamma_{p,q}$. The simplest case $p = q = 1$ gives $\Gamma_{1,1} \cong \mathbb{Z}^2$, which has quadratic growth. We claim the other cases have exponential growth. We can assume that $q < p$. Fix a constant $r \in \mathbb{N}$ such that $2q^r \leq p^r$. Fix $n \in \mathbb{N}$. The elements

$$E_\varepsilon = \left(\sum_{i=1}^n \varepsilon_i \left(\frac{p}{q}\right)^{ri}, 0 \right) \in \Gamma_{p,q}, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1\}^n \quad (2.8)$$

are all distinct. It is enough to show that these 2^n elements are in the ball B_{2rn} ,

where the length function is $\ell = \ell_S$. To see that, observe

$$\ell((1 + P, k)) = \ell((1, 0)(P, k)) \leq 1 + \ell(P, k) \quad (2.9)$$

$$\ell(((p/q)^r P, k)) = \ell((0, r)(P, k)(0, -r)) \leq 2r + \ell(P, k) \quad (2.10)$$

for any $(P, k) \in \Gamma_{p,q}$. Since

$$\sum_{i=1}^n \varepsilon_i (p/q)^{ri} = \sum_{j=1}^h (p/q)^{rj} = (p/q)^{rm_1} [1 + (p/q)^{rm_2} [\dots [1 + (p/q)^{rm_h} \dots]],$$

by applying (2.9) on E_ε multiple times, we get $\ell(E_\varepsilon) \leq 2rn$ for all $\varepsilon \in \{0, 1\}^n$. This shows that $\Gamma_{p,q}$ has exponential growth. Therefore, being amenable group with exponential growth, $\Gamma_{p,q}$ does not satisfy the Rapid Decay property for any coprime $p, q \in \mathbb{N}$ with $p \neq 1$ or $q \neq 1$.

Example 2.2.18. Let $p, q \in \mathbb{N}$ be distinct integers. We claim the Baumslag-Solitar group defined by the presentation

$$BS(p, q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$$

does not satisfy the Rapid Decay property. The automorphism $a \mapsto a, t \mapsto t^{-1}$ of $BS(p, q)$ allows us to assume that $p < q$. When $p = 1$, we have $BS(1, q) = \Gamma_{1,q}$ so this case falls to the previous example. When $p > 1$, the set $\{a^{-1}ta, t\}$ generates a free subgroup and $BS(p, q)$ is not amenable, so we should find an amenable proper subgroup with superpolynomial relative growth. From the case of $p = 1$, one gets an intuition to check the cyclic subgroup $\langle a \rangle$. Fix $n \in \mathbb{N}$ and take any integer $k \in \{1, \dots, \lfloor (q/p)^n \rfloor + 1\}$. Write $k = dq + r$ for some $d, r \in \mathbb{N}_0$ and $r < q$. We have

$$\ell(a^k) = \ell((a^q)^N a^r) = \ell((t^{-1}a^p t)^d a^r) = \ell(t^{-1}a^{dp} t a^r) \leq (q + 1) + \ell(a^{p \lfloor k/q \rfloor}).$$

Repeat the same inequality on the last terms until its power becomes 1. This will be done in at most n steps, thus we have

$$\ell(a^k) \leq (q + 1)n + 1,$$

which shows the ball $B_{(q+1)n+1}$ contains at least $(q/r)^n$ elements. Therefore, the cyclic subgroup $\langle a \rangle$ has exponential relative growth, and $BS(p, q)$ does not satisfy

the Rapid Decay property.

For $p = q \in \mathbb{N}$, the group $BS(p, p)$ contains a finite index subgroup isomorphic to $F_p \times \mathbb{Z}$. This shows that $BS(p, p)$ has the Rapid Decay property.

Chapter 3

Tame cuts

In this chapter, we will introduce (completely bounded) [characteristic] tame cuts for locally compact groups.

In Section 3.1, we will provide the first examples admitting (completely bounded) [characteristic] tame cuts using weakly amenable groups, groups with Rapid Decay property, and the estimation of Lebesgue constants. Asymptotically minimal property of the Lebesgue constants provides the first example without characteristic tame cuts: $(\mathbb{Z}, \log(1 + \log(1 + |\cdot|)))$.

In Section 3.2, we will introduce groups with RD_p where $1 \leq p < \infty$. These groups extend the Rapid Decay property and admit characteristic tame cuts. Interestingly, for $1 \leq p < q \leq 2$, the groups with both RD_p and RD_q are exactly the groups with polynomial growth.

In Section 3.3, some stability properties are studied. Recall that ME-subgroups inherit weak amenability. We will formulate a similar result for completely bounded tame cuts. Also, inheritance from polynomial co-growth subgroup is studied. This will be used in Section 3.4 to construct groups with completely bounded characteristic tame cuts and without Rapid Decay property: some semidirect products, $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ and $\mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$, Lamplighter groups $\mathbb{Z}_p \wr \mathbb{Z}$, and Baumslag-Solitar groups $BS(p, q)$.

In Section 3.5, we will see that property (T_{Schur}, G, K) is an obstruction to the existence of K -bi-invariant tame cuts.

In Section 3.6, we will provide two applications of tame cuts.

3.1 Definition and the first examples

Definition 3.1.1. Let G be a locally compact group equipped with a proper length function ℓ . A sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C_c(G)$ is called

(i) *tame cuts* for (G, ℓ) if there are constants $C, a \geq 0$ such that

$$\|\varphi_n\|_{MA} \leq Cn^a \quad \text{and} \quad \varphi_n|_{B_n} \equiv 1 \quad \text{for all } n \in \mathbb{N}.$$

(ii) *completely bounded tame cuts* for (G, ℓ) if there are constants $C, a \geq 0$ such that

$$\|\varphi_n\|_{M_0A} \leq Cn^a \quad \text{and} \quad \varphi_n|_{B_n} \equiv 1 \quad \text{for all } n \in \mathbb{N}.$$

As it connects to the Rapid Decay property, when G is discrete, we also consider (completely bounded) tame cuts composed of characteristic functions. We define the intervals $I(G, \ell)$, $I^{char}(G, \ell)$, $I_{cb}(G, \ell)$, and $I_{cb}^{char}(G, \ell)$ in \mathbb{R}_+ containing all possible degrees a occurring in the definition above. For example, $a \in I_{cb}^{char}(G, \ell)$ if and only if there exists a sequence of characteristic functions $(\varphi_n)_{n \in \mathbb{N}}$ in $C_c(G)$ such that $B_n \subseteq \text{supp}(\varphi_n)$ and $\|\varphi_n\|_{M_0A} \leq Cn^a$ for some constant $C \geq 0$ and for all $n \in \mathbb{N}$, and we call the sequence $(\varphi_n)_{n \in \mathbb{N}}$ *completely bounded characteristic tame cuts* for (G, ℓ) . These intervals might be either (a_0, ∞) , $[a_0, \infty)$, or empty. When there is no ambiguity, we just write I , I_{cb} , I^{char} , and I_{cb}^{char} . We have the following obvious inclusions

$$I_{cb}^{char} \subseteq I_{cb} \subseteq I \quad \text{and} \quad I_{cb}^{char} \subseteq I^{char} \subseteq I,$$

and when G is amenable, we have

$$I = I_{cb} \quad \text{and} \quad I^{char} = I_{cb}^{char}.$$

One can see that the property of having tame cuts is an analogue of weak amenability. Indeed, we replaced the condition of boundedness by polynomial growth and the condition of uniform convergence on compact sets by fixed values on the balls. The main difference is that the tame cuts strongly depend on the chosen length functions while weak amenability does not. The following proposition shows a direct relation between these two properties.

Proposition 3.1.2. *A locally compact group (G, ℓ) endowed with a proper length function has completely bounded tame cuts with $I_{cb} = [0, \infty)$ if and only if G is weakly amenable. Similarly, (G, ℓ) has tame cuts with $I = [0, \infty)$ if and only if there is an approximate unit $(\varphi_n)_{n \in \mathbb{N}}$ in $A(G)$ with $\sup_{n \in \mathbb{N}} \|\varphi_n\|_{MA} < \infty$.*

Proof. The “only if” parts are obvious from the definitions. For the “if” part, see [CH89, Proposition 1.1]. Although the last part is not much different from the first part, we provide the proof for convenience.

Suppose that we have an approximate unit $(\varphi_n)_{n \in \mathbb{N}}$ in $A(G)$ with $\sup_{n \in \mathbb{N}} \|\varphi_n\|_{MA} < \infty$. Choose non-negative functions $f_n \in C_c(G) \cap A(G)$ with $f_n|_{B_n} = 1$. This is doable using Lemma 1.2.18. Since $A(G) \cap C_c(G)$ is dense in $A(G)$, we can assume that φ_n is compactly supported. By passing to a subsequence, we can assume that $\|\varphi_n f_n - f_n\|_A \leq 1$ for all $n \in \mathbb{N}$. Now, the functions defined by $\psi_n = \varphi_n - \varphi_n f_n + f_n$ are compactly supported and takes value 1 on the ball B_n . Moreover, we have

$$\|\psi_n\|_{MA} \leq \|\varphi_n\|_{MA} + \|\varphi_n f_n - f_n\|_A = O(1).$$

This proves the statement. □

Another point of view for tame cuts, especially the characteristic ones, comes from classical harmonic analysis. For any integrable function $f : [0, 2\pi] \rightarrow \mathbb{R}$, one can associate a trigonometric series

$$S[f](t) = \sum_{k \in \mathbb{N}} \mathcal{F}(f)(k) e^{ikt}.$$

Under some regularity conditions, the series $S[f]$ pointwise converges to f . Would the partial sums $S_n[f](t) = \sum_{k=-n}^n \mathcal{F}(f)(k) e^{ikt}$ converge uniformly to f for all $f \in C(\mathbb{T})$? To answer this question, firstly we need to verify if the Dirichlet kernels $D_n : [0, 2\pi] \rightarrow \mathbb{R}$ defined by

$$D_n(t) = \sum_{k=-n}^n e^{ikt} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

are uniformly bounded in the space $L^1(\mathbb{T})$. However, we have the following well-known estimation (see e.g. [Zyg02]).

Lemma 3.1.3. *Dirichlet kernels D_n , $n \in \mathbb{N}$ satisfy $\|D_n\|_{L^1([0, 2\pi])} = \frac{4}{\pi} \log n + O(1)$.*

Even though the sequence of Dirichlet kernels fails to be bounded, the growth rate still keeps some information about the underlying groups, \mathbb{T} and $\mathbb{Z} = \widehat{\mathbb{T}}$. The notion of (completely bounded) characteristic tame cuts then can be seen as a non-commutative version of the Dirichlet kernels. Observe that since $L^1(\mathbb{T})$ is identified with $A(\mathbb{Z})$ by Fourier transform, the estimation in Lemma 3.1.3 just says $\|\mathcal{F}(D_n)\|_{A(\mathbb{Z})} = \frac{4}{\pi} \log n + O(1)$ where $\mathcal{F}(D_n) = \sum_{k=-n}^n \delta_k = \mathbb{1}_{[-n,n] \cap \mathbb{Z}} \in A(\mathbb{Z})$. In other words, the sequence $(\mathcal{F}(D_n))_{n \in \mathbb{N}}$ forms tame cuts for \mathbb{Z} with respect to the logarithmic length function $k \in \mathbb{Z} \mapsto \log(1 + |k|)$. On the other hand, we have the following result.

Theorem 3.1.4 ([MPS81]). *For any trigonometric polynomial $p(t) = \sum_{k=1}^N c_k e^{in_k t}$ on \mathbb{T} where the n_k are distinct integers and $|c_k| \geq 1$ for all $k \in \mathbb{N}$, we have*

$$\left\| \sum_{k=1}^N c_k e^{in_k t} \right\|_{L^1(\mathbb{T})} > K \log N$$

where $K > 0$ is a constant independent from N .

The following result is now straightforward.

Proposition 3.1.5. *The infinite cyclic group $\Gamma = \mathbb{Z}$ has characteristic tame cuts with respect to the logarithmic length function $\log(1 + |\cdot|)$ but does not with respect to the double logarithmic length function $\log(1 + \log(1 + |\cdot|))$. More precisely, we have*

$$I^{char}(\mathbb{Z}, \log(1 + |\cdot|)) = [1, \infty) \quad \text{and} \quad I^{char}(\mathbb{Z}, \log(1 + \log(1 + |\cdot|))) = \emptyset.$$

Proof. Lemma 3.1.3 shows that the sequence $(D_{3^n})_{n \in \mathbb{N}}$ of Dirichlet kernels forms characteristic tame cuts for $(\mathbb{Z}, \log(1 + |\cdot|))$. Theorem 3.1.4 shows that the linear growth of characteristic tame cuts is optimal. For the second part, assume that there exists characteristic tame cuts $(\varphi_n)_{n \in \mathbb{N}}$. Then the support of φ_n would contain at least $e^{e^n} \approx |B_n|$ elements, and Theorem 3.1.4 implies

$$\|\varphi_n\|_{M_0 A} = \|\varphi_n\|_A = \|\mathcal{F}(\varphi_n)\|_{L^1(\mathbb{T})} \geq C e^n,$$

which contradicts the assumption. □

Remark 3.1.1. We have $I(\mathbb{Z}, \ell) = [0, \infty)$ for any proper length function ℓ since \mathbb{Z} is

amenable, and $I^{char}(\mathbb{Z}, |\cdot|) = (0, \infty)$ since the logarithmic function is slower than any polynomial.

The following example is the very first class of discrete groups with characteristic tame cuts.

Proposition 3.1.6. *Let Γ be a discrete group satisfying the Rapid Decay property with respect to a proper length function ℓ . Then (Γ, ℓ) has characteristic tame cuts and $I^{char}(\Gamma)$ contains all Rapid Decay degrees in Theorem 2.2.12 (v).*

Proof. To show that, first note that for any non-zero function $f \in C_c(\Gamma)$, we have

$$\|\lambda(f)\| \geq \frac{\langle \lambda(f)\delta_e, f \rangle}{\|f\|_2} = \|f\|_2.$$

Thus, if we put $\varphi_n = \mathbb{1}_{B_n}$, by Rapid Decay property, we have

$$\begin{aligned} \|\lambda(\varphi_n f)\| &\leq Cn^a \|\varphi_n f\|_2 \\ &\leq Cn^a \|f\|_2 \\ &\leq Cn^a \|\lambda(f)\|, \end{aligned}$$

for all $f \in C_c(\Gamma)$, hence $\|\varphi_n\|_{MA} \leq Cn^a$. \square

Second proof for the first part. Suppose that there are constants $C, s > 0$ satisfying $\|\lambda(f)\| \leq C\|f\|_{s,\ell}$ for all $f \in C_c(\Gamma)$. The function $\psi_n = \mathbb{1}_{B_n} + \lceil C^2(2+n)^{2s+2} \rceil \delta_e$ satisfies all conditions of Lemma 3.1.7 below. Thus, ψ_n is a positive Fourier multiplier and $\|\psi_n\|_{MA} = \psi_n(e) = 1 + \lceil C^2(2+n)^{2s+2} \rceil$. Now, the inequality

$$\begin{aligned} \|\mathbb{1}_{B_n}\|_{MA} &\leq \|\psi_n\|_{MA} - \|\lceil C^2(2+n)^{2s+2} \rceil \delta_e\|_{MA} \\ &\leq 1 + \lceil C^2(2+n)^{2s+2} \rceil + C^2(2+n)^{2s+2} \\ &\leq C'n^{2s+2} \end{aligned}$$

proves the statement. \square

Lemma 3.1.7 ([JV91]). *Let Γ be a discrete countable group, ℓ a proper length function of Γ with $\ell(x) > 0$ for any $x \in \Gamma \setminus \{e\}$, and $n \in \mathbb{N}$. Assume that there are constants $C, s > 0$ such that*

$$\|\lambda(f)\| \leq C\|f\|_{s,\ell}, \quad (\forall f \in C_c(\Gamma)).$$

Suppose that $\psi : \Gamma \rightarrow \mathbb{C}$ is a function such that $\psi = \psi^*$ and

$$nC^2|\psi(x)|(2 + \ell(x))^{2s+2} \leq \psi(e), \quad (\forall x \in \Gamma \setminus \{e\}).$$

Then ψ defines a n -positive Fourier multiplier.

3.2 ℓ -comparable norms

Here we state a sufficient condition for a group to admit characteristic tame cuts.

Definition 3.2.1. We say that a norm N on $C_c(G)$ is *unconditional* if for any $f, g \in C_c(G)$ with $|f| \leq |g|$, we have $N(f) \leq N(g)$.

Example 3.2.2. (Any weighted) L^p -norm is unconditional for $1 \leq p \leq \infty$.

Definition 3.2.3. Let ℓ be a proper length function of a locally compact group G . Two norms N_1 and N_2 on $C_c(G)$ are *ℓ -comparable* if there are constants $C, a \geq 0$ such that the inequalities

$$N_1(f) \leq Cn^a N_2(f) \quad \text{and} \quad N_2(f) \leq Cn^a N_1(f)$$

hold for all $f \in C_c(G)$ and $n \in \mathbb{N}$ with $\text{supp}(f) \subseteq B_n$.

Example 3.2.4. Suppose that ℓ is a proper length function of G . For $s \in \mathbb{R}_+$ and $1 \leq p < \infty$, define $L^{p,s}$ -norm on $C_c(G)$ by

$$\|f\|_{p,s,\ell} = \left(\int_G |f(x)|^p (1 + \ell(x))^{sp} dx \right)^{1/p}, \quad (\forall f \in C_c(G)).$$

Observe that if $\text{supp}(f) \subseteq B_n$, then

$$\int_G |f(x)|^p dx \leq \int_G |f(x)|^p (1 + \ell(x))^{sp} dx \leq (1 + n)^{sp} \int_G |f(x)|^p dx.$$

Thus, $L^{p,s}$ -norm is ℓ -comparable with L^p -norm. It is also true for $s < 0$.

Example 3.2.5. For a discrete group Γ , the ℓ^2 -norm is ℓ -comparable with the $C_\lambda^*(\Gamma)$ -norm if and only if (Γ, ℓ) has the Rapid Decay property. Note that it is not always true for general locally compact groups. For example, consider the group $G = \mathbb{T}$. Being compact, this group has Rapid Decay property for any proper length

function. Take the length function $\ell : \mathbb{T} \mapsto \{0\}$. The $L^2(\mathbb{T})$ -norm is ℓ -comparable with the $C_\lambda^*(\mathbb{T})$ -norm if and only if they are Lipschitz equivalent, which would imply that the Banach spaces $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T}) \subseteq C_\lambda^*(\mathbb{T})$ coincide. This is false since the function $f(x) = x^{-1/2}$ on $[0, 1]$ is integrable but not square integrable.

Proposition 3.2.6. *Assume that G is a locally compact group, ℓ is a proper length function of G , and N is an unconditional norm on $C_c(G)$ that is ℓ -comparable with the $C_\lambda^*(G)$ -norm (or with the $A(G)$ -norm). Then G has tame cuts. Moreover, if G is discrete, then G has characteristic tame cuts.*

Proof. The main point is that if N is unconditional, any pointwise multiplier of $(C_c(G), N)$ corresponding to a function $\varphi \in C_c(G)$ with $|\varphi| \leq 1$ is contracting.

Take a large enough $r \in \mathbb{N}$ such that B_r has positive measure (equivalently the interior B_r^0 of the ball B_r is non-empty). We can find a non-negative function $f \in A(G)$ with $\int_G f dx = 1$ and $\text{supp}(f) \subseteq B_{2r}$. For example, apply $K = B_r$ and $U = B_r B_r^0$ on Proposition 1.2.18 and normalize the resulting function. If G is discrete, we choose $f = \delta_e$. Put $\varphi_n = f * \mathbb{1}_{B_{n+3r}} \in A(G)$. Note that $\varphi_n|_{B_n} \equiv 1$, $0 \leq \varphi_n \leq 1$, $\text{supp}(\varphi_n) \subseteq B_{n+5r}$, and φ_n is continuous. Using the ℓ -comparison condition, we get

$$\|\lambda(\varphi_n g)\| \leq C(n + 5r)^a N(\varphi_n g) \leq C(n + 5r)^a N(g) \leq C(n + 5r)^a \|\lambda(g)\|,$$

for all $g \in C_c(G)$ and for some constants $C, a \geq 0$. This proves that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ forms tame cuts for (G, ℓ) . \square

Inspired by the Rapid Decay property, the following classes of discrete groups are also interesting.

Definition 3.2.7. Let $1 \leq p \leq \infty$. A discrete group (Γ, ℓ) endowed with a length function has RD_p if the ℓ^p -norm on $C_c(\Gamma)$ is ℓ -comparable with the $C_\lambda^*(\Gamma)$ -norm. In particular, RD_2 coincides with the Rapid Decay property.

Proposition 3.2.8. *Let (Γ, ℓ) be a discrete group endowed with a proper length function.*

- (i) *If (Γ, ℓ) has RD_p for some $1 \leq p \leq \infty$, then (Γ, ℓ) has characteristic tame cuts.*
- (ii) *If (Γ, ℓ) has polynomial growth, then it has RD_p for all $1 \leq p < \infty$.*

(iii) If (Γ, ℓ) has RD_p and RD_q for some $1 \leq p < q \leq 2$, then it has polynomial growth.

Proof.

(i) Clearly, ℓ^p -norm is unconditional, and since ℓ^p -norm is ℓ -comparable with the $C_\lambda^*(\Gamma)$ -norm by assumption, it follows from Proposition 3.2.6 that (Γ, ℓ) has characteristic tame cuts.

(ii) Take any $f \in C_c(\Gamma)$ with $\text{supp}(f) \subseteq B_n$. Observe that

$$\|f\|_p = \left(\sum_{x \in B_n} |f(x)|^p \right)^{1/p} \leq |B_n|^{\frac{2-p}{2}} \|f\|_2 \leq |B_n|^{\frac{2-p}{2}} \|\lambda(f)\|,$$

where the first inequality is Hölder's inequality, and the second one is because $\|\lambda(f)\| \geq \frac{\langle \lambda(f)\delta_e, f \rangle}{\|f\|_2} = \|f\|_2$. Similarly, we have

$$\|\lambda(f)\| \leq \|f\|_1 \leq |B_n|^{\frac{p-1}{p}} \|f\|_p.$$

(iii) By transitivity of ℓ -comparison, ℓ^p -norm is ℓ -comparable to ℓ^q -norm, so there exist constants $C, a \geq 0$ such that $\|f\|_p \leq Cn^a \|f\|_q$ for all $f \in C_c(\Gamma)$ with $\text{supp}(f) \subseteq B_n$. In particular, if we put $f = \mathbb{1}_{B_n}$, we get

$$|B_n|^{1/p} \leq Cn^a |B_n|^{1/q}.$$

Thus, the growth of (Γ, ℓ) is bounded by $(Cn^a)^{\frac{pq}{q-p}}$.

□

The point (iii) asserts that the classes RD_p for $1 \leq p \leq 2$ give a priori very different classes of groups admitting tame cuts. We do not know any example of RD_1 groups besides those with polynomial growth.

3.3 Stability properties

3.3.1 Subgroups

Proposition 3.3.1. *Let G be a locally compact group, H a closed subgroup of G , and ℓ a proper length function of G . If $(\varphi_n)_{n \in \mathbb{N}}$ forms (completely bounded)*

[characteristic] tame cuts for (G, ℓ) , then $(\varphi_n|_H)_{n \in \mathbb{N}}$ forms (completely bounded) [characteristic] tame cuts for $(H, \ell|_H)$. Moreover, we have $I(G, \ell) \subseteq I(H, \ell|_H)$, $I^{char}(G, \ell) \subseteq I^{char}(H, \ell|_H)$, $I_{cb}(G, \ell) \subseteq I_{cb}(H, \ell|_H)$, and $I_{cb}^{char}(G, \ell) \subseteq I_{cb}^{char}(H, \ell|_H)$.

This proposition is a direct consequence of the following lemma.

Lemma 3.3.2 (Proposition 1.12 of [DCH85]). *Let H be a closed subgroup of a locally compact group G .*

(i) *If $\varphi \in MA(G)$, then $\varphi|_H \in MA(H)$ and $\|\varphi|_H\|_{MA(H)} \leq \|\varphi\|_{MA(G)}$.*

(ii) *If $\varphi \in M_0A(G)$, then $\varphi|_H \in M_0A(H)$ and $\|\varphi|_H\|_{M_0A(H)} \leq \|\varphi\|_{M_0A(G)}$.*

Proof. By the characterization (iii) in Theorem 1.2.21, the restriction map from $M_0A(G)$ to $M_0A(H)$ is clearly norm decreasing. For the restriction map from $MA(G)$ to $MA(H)$, we provide an elementary proof for the case when H is an open subgroup of G . For the general case, see [DCH85, Proposition 1.12].

Suppose that $\varphi \in MA(G)$. Equivalently, $\varphi \in C_b(G)$ and there is a constant $C < \infty$ such that

$$\|\lambda_G(\varphi f)\| \leq C\|\lambda_G(f)\|, \quad (\forall f \in C_c(G)). \quad (3.1)$$

Since H is open in G , any function $f \in C_c(H)$ can be seen as a function on G supported on H . Using (3.1) and Lemma 3.3.3 below, we get

$$\|\lambda_H(\varphi f)\| = \|\lambda_G(\varphi f)\| \leq C\|\lambda_G(f)\| = C\|\lambda_H(f)\|, \quad (\forall f \in C_c(H)).$$

This proves $\|\varphi|_H\|_{MA(H)} \leq C$. □

Lemma 3.3.3. *Let G be a locally compact group, H an open subgroup of G , and $x \in G$. If $f \in C_c(G)$ and $\text{supp}(f) \subseteq H$, then f acts on $L^2(Hx)$ by (left) convolution and the norm*

$$\|f\|_{\mathcal{B}(L^2(Hx))} = \sup\{\|f * \eta\| \mid \eta \in L^2(Hx), \|\eta\|_{L^2(Hx)} = 1\}$$

is equal to $\|\lambda_G(f)\|$. In particular, $\|\lambda_H(f|_H)\| = \|\lambda_G(f)\|$.

Proof. Suppose that $f \in C_c(G)$ is as in the statement. Let us first prove that the norm $\|f\|_{\mathcal{B}(L^2(Hx))}$ does not depend on x . Consider the map $U : L^2(H) \rightarrow L^2(Hx)$,

$\eta \mapsto \eta(\cdot x^{-1})$. It is clearly a surjective isometry. Moreover, we have $U(f * \eta) = f * U(\eta)$ for all $\eta \in L^2(H)$. Indeed, for all $hx \in Hx$, we have

$$\begin{aligned} [U(f * \eta)](hx) &= (f * \eta)(h) = \int_H f(s)\eta(s^{-1}h)ds = \int_H f(s)\eta(s^{-1}hxx^{-1})ds \\ &= \int_H f(s)(U\eta)(s^{-1}hx)ds = [f * (U\eta)](hx). \end{aligned}$$

It follows that $\|f\|_{\mathcal{B}(L^2(H))} = \|f\|_{\mathcal{B}(L^2(Hx))}$.

Let us prove $\|\lambda_H(f|_H)\| = \|\lambda_G(f)\|$. Take any $\xi \in L^2(H)$. We identify ξ with the element of $L^2(G)$ that coincides with ξ on H and takes value 0 on H^c . From the calculation

$$\begin{aligned} \|f * \xi\|_{L^2(G)}^2 &= \int_G \left| \int_G f(x)\xi(x^{-1}k)dx \right|^2 dk = \int_G \left| \int_H f(x)\xi(x^{-1}k)dx \right|^2 dk \\ &\geq \int_H \left| \int_H f(x)\xi(x^{-1}k)dx \right|^2 dk = \|f * \xi\|_{L^2(H)}^2, \end{aligned}$$

it is clear that $\|\lambda_G(f)\| \geq \|\lambda_H(f|_H)\|$. Now, we will prove the converse inequality. Let $\{Hx_i \mid i \in I\}$ be the right cosets and take any $\xi \in L^2(G)$. For each $i \in I$, let ξ_i be the orthogonal projection of ξ onto $L^2(Hx_i)$. Then we have

$$\begin{aligned} \|f * \xi\|_{L^2(G)}^2 &= \sum_{i \in I} \int_{Hx_i} \left| \int_H f(x)\xi(x^{-1}k)dx \right|^2 dk = \sum_{i \in I} \|f * \xi_i\|_{L^2(Hx_i)}^2 \\ &\leq \sum_{i \in I} \|f\|_{\mathcal{B}(L^2(Hx_i))}^2 \|\xi_i\|_{L^2(Hx_i)}^2 = \|\lambda_H(f|_H)\| \|\xi\|_{L^2(G)}^2, \end{aligned}$$

and we conclude. □

3.3.2 From a uniform lattice

The following result is almost direct from Proposition 2.1.5, but we provide the proof for completeness.

Proposition 3.3.4. *Suppose that Γ is a uniform lattice in a locally compact group G . Let ℓ be a proper length function of G . Then $(\Gamma, \ell|_\Gamma)$ has completely bounded tame cuts if and only if so does (G, ℓ) .*

Proof. The “if” part is dealt in the previous subsection. Let us prove the “only if” part. Take a large enough number $r \in \mathbb{N}$ such that B_r contains a compact

fundamental domain Ω of Γ . Take any function $\varphi \in C_c(\Gamma)$ with $\varphi|_{B_{n+2r}} \equiv 1$ for some $n \in \mathbb{N}$. We will use the induction map $\Phi : M_0A(\Gamma) \rightarrow M_0A(G)$ defined in (2.1). For each $x \in G$, we can write $x = \gamma(x)\omega(x)$ for a unique $\gamma(x) \in \Gamma$ and $\omega(x) \in \Omega$. With this notation, if ϕ is in $M_0A(\Gamma)$, then we can write

$$\Phi(\phi)(x) = \widehat{\phi}(x) = \int_{\Omega} \phi(\gamma(xw))dw, \quad (\forall x \in G).$$

Since the induction map is norm decreasing, to prove the statement, it is enough to check that $\widehat{\varphi}$ is compactly supported and takes 1 on the ball B_n . Since φ is compactly supported, there exists a large enough $R \geq 0$ such that $\text{supp}(\varphi) \subseteq B_R$. If $x \in G$ is such that $\ell(x) > R + 2r$, then

$$\ell(\gamma(xw)) = \ell(\omega(xw)^{-1}xw) \geq \ell(x) - \ell(\omega(xw)) - \ell(w) > R,$$

and $\varphi(\gamma(xw)) = 0$ for all $w \in \Omega$. This shows that $\widehat{\varphi}$ is supported on the compact ball B_{R+2r} . Now, take any $x \in B_n$. Since

$$\ell(\gamma(xw)) = \ell(\omega(xw)^{-1}xw) \leq \ell(x) + \ell(\omega(xw)) + \ell(w) \leq n + 2r, \quad (\forall w \in \Omega),$$

we have $\widehat{\varphi}(x) = \int_{\Omega} \varphi(\gamma(xw))dw = \int_{\Omega} 1dw = 1$. This completes the proof. \square

We have seen that ME-subgroups inherit the weak amenability. It is quite intuitive that completely bounded tame cuts are transferred to ME-subgroups via the induction map defined in (2.6). However, the problem occurs as the induction map might not preserve the compact support. We can avoid this inconvenience as follows.

Theorem 3.3.5. *Let Λ and Γ be two countable groups, (X, d) a metric space, and μ a Borel measure on X . Suppose that there exist commuting, measure preserving, distance preserving, proper, free actions $\Lambda \curvearrowright X \curvearrowright \Gamma$ such that the Γ action admits a precompact fundamental domain Ω . Fix a base point $v_0 \in X$ and define the length functions $\ell_{\Gamma} : \delta \in \Gamma \mapsto d(v_0, \delta, v_0)$ and $\ell_{\Lambda} : s \in \Lambda \mapsto d(s.v_0, v_0)$. If (Γ, ℓ_{Γ}) has completely bounded tame cuts, so does $(\Lambda, \ell_{\Lambda})$.*

Proof. We use the notations $\gamma : X \rightarrow \Gamma$ and $\omega : X \rightarrow \Omega$ from (2.4). Recall the induction map sends $\varphi \in C_c(\Gamma)$ to $\widehat{\varphi} : \Lambda \rightarrow \mathbb{C}$ defined by

$$\widehat{\varphi}(s) = \int_{\Omega} \varphi(\gamma(s.w)^{-1})d\mu(w), \quad (\forall s \in \Lambda).$$

Let us prove that $\widehat{\varphi}$ is finitely supported. Choose a large enough $R \geq 0$ and $R' \geq 0$ such that $\Omega \subseteq B_R(v_0)$ and $\text{supp}(\varphi) \subseteq B_{R'}$. If $s \in \Lambda$ is such that $\ell_\Lambda(s) > 2R + R'$, then by the triangle inequality

$$\begin{aligned} \ell_\Gamma(\gamma(s.w)) &= d(v_0, v_0.\gamma(s.w)) \\ &\geq d(v_0, s.v_0) - d(s.v_0, s.w) - d(\omega(s.w).\gamma(s.w), v_0.\gamma(s.w)) \\ &\geq \ell_\Lambda(s) - 2R > R', \end{aligned}$$

and $\varphi(\gamma(s.w)^{-1}) = 0$ for all $w \in \Omega$. This shows that $\widehat{\varphi}$ is supported on the finite set $B_{2R+R'}$.

Suppose that $\varphi|_{B_{n+2R}} \equiv 1$ for some $n \in \mathbb{N}$. Let us prove that $\widehat{\varphi}|_{B_n} \equiv 1$. Take any $s \in \Lambda$ such that $\ell_\Lambda(s) \leq n$. Then we have

$$\begin{aligned} \ell_\Gamma(\gamma(s.w)) &= d(v_0, v_0.\gamma(s.w)) \\ &\leq d(v_0, s.v_0) + d(s.v_0, s.w) + d(\omega(s.w).\gamma(s.w), v_0.\gamma(s.w)) \\ &\leq n + 2R \end{aligned}$$

for all $w \in \Omega$, hence $\widehat{\varphi}(s) = \int_\Omega \varphi(\gamma(s.w)^{-1})d\mu(w) = \int_\Omega 1d\mu(w) = 1$. Now, the statement follows from Lemma 2.1.13. \square

Remark 3.3.1. As you can see, we do not require the Λ -action to have a Borel fundamental domain. This condition was used only in Lemma 2.1.13 to prove that the map $A(\Gamma) \rightarrow A(\Lambda)$ is norm decreasing.

Corollary 3.3.6. *Suppose that Γ is a uniform lattice in G , and Λ is a discrete subgroup of G . If Γ has completely bounded tame cuts, so does Λ .*

Remark 3.3.2. The corollary also follows from Proposition 3.3.4.

3.3.3 Polynomial co-growth

The main result of this subsection is Proposition 3.3.9 which states that the property of admitting (completely bounded) [characteristic] tame cuts is inherited by a normal open subgroup with polynomial co-growth. This result is used to construct groups with completely bounded characteristic tame cuts that do not satisfy the Rapid Decay property. The proof is inspired by [Jol90] where the stability of Rapid Decay property under certain extensions was studied.

Definition 3.3.7. Let (G, ℓ) be a locally compact group endowed with an integral valued length function and let H be a normal open subgroup of G . Let $\pi : G \rightarrow H \backslash G$ be the natural projection. Choose a cross-section $\sigma : Q \rightarrow G$ (a map such that $\pi \circ \sigma = \text{id}_Q$ and $\sigma(H) = e_G$) in a way that $\ell(\sigma(q)) = \min\{\ell(h\sigma(q)) : h \in H\}$ for all $q \in Q$. In other words, we want $\sigma(q)$ to be inside the smallest ball intersecting with the corresponding coset. We call this cross section *ℓ -optimal*. It is not difficult to see that the function $\ell_Q : q \in Q \mapsto \ell(\sigma(q))$ is a length function of Q . If Q has polynomial growth with respect to ℓ_Q , we say H has *polynomial ℓ -co-growth* in G .

Example 3.3.8. If G is generated by a compact subset S , then the quotient group Q is generated by the compact subset $\pi(S)$. Moreover, the word length function $\ell_{\pi(S)}$ of Q is equivalent to ℓ_Q for any ℓ -optimal cross section. Let us prove it. We already know that the word length function dominates ℓ_Q . For the converse, take any $q \in Q$ and suppose $\ell_Q(q) = n$. It follows that there exist $h \in H$ and $s_1, \dots, s_n \in S$ such that $h\sigma(q) = s_1 \dots s_n$. By applying π on both side, we get $q = \pi(s_1 \dots s_n) = \pi(s_1) \dots \pi(s_n)$. In other words, $\ell_{\pi(S)}(q) \leq n = \ell_Q(q)$ for all $q \in Q$.

Proposition 3.3.9. *Let G be a unimodular locally compact group, ℓ a proper length function of G , and H a normal open subgroup of G with polynomial ℓ -co-growth. The group $(H, \ell|_H)$ has (completely bounded) tame cuts if and only if so does (G, ℓ) . Moreover, the statement is valid for (completely bounded) characteristic tame cuts when G is discrete.*

Corollary 3.3.10. *Let Γ be a finitely generated group, ℓ the word length function of Γ , and H a normal subgroup of Γ . Assume Γ/H has polynomial growth. If $(H, \ell|_H)$ has (completely bounded) [characteristic] tame cuts, then so does (Γ, ℓ) .*

We need some preparation before the proof. Every element $x \in G$ can be written as $x = h\sigma(q)$ for a unique $h \in H$ and $q \in Q$. For $f \in C_c(G)$, $\xi \in L^2(G)$, $h \in H$, and $q, r \in Q$, we write $f_q(h) = f(h\sigma(q))$ and $\xi_{q,r}(h) = \xi(\sigma(q)^{-1}h\sigma(r))$. Note that $f_q \in C_c(H)$ and $\xi_{q,r} \in L^2(H)$. Indeed, for a fixed $q \in Q$, we have

$$\begin{aligned} \|\xi\|_{L^2(G)}^2 &= \int_G |\xi(x)|^2 dx = \int_G |\xi(\sigma(q)^{-1}x)|^2 dx \\ &= \sum_{r \in Q} \int_H |\xi(\sigma(q)^{-1}h\sigma(r))|^2 dh = \sum_{r \in Q} \|\xi_{q,r}\|_{L^2(H)}^2. \end{aligned} \tag{3.2}$$

We need the following two lemmas.

Lemma 3.3.11. *Assume the notations above. The linear map $C_\lambda^*(G) \rightarrow C_\lambda^*(H)$, $f \mapsto f_q$ is norm decreasing for all $q \in Q$. Moreover, if $f \in C_c(G)$ is supported on $H\sigma(q)$, then we have $\|\lambda_G(f)\| = \|\lambda_H(f_q)\|$.*

Proof. For every $\eta \in L^2(H)$ with $\|\eta\|_{L^2(H)} = 1$, we choose $\xi \in L^2(G)$ to be

$$\xi(h\sigma(p)) = \begin{cases} \eta(\sigma(q)^{-1}h\sigma(q)), & p = H \\ 0, & p \neq H. \end{cases}$$

for all $h \in H$ and $p \in Q$. Note that $\|\xi\|_{L^2(G)} = \|\eta\|_{L^2(H)} = 1$ and $\text{supp}(\xi) \subseteq H$. Further, we have

$$\begin{aligned} \|\lambda_G(f)\|^2 &\geq \|f * \xi\|_{L^2(G)}^2 = \sum_{r \in Q} \int_H \left| \sum_{p \in Q} \int_H f(k\sigma(p)) \xi(\sigma(p)^{-1}k^{-1}h\sigma(r)) dk \right|^2 dh \\ &\quad (r = q) \geq \int_H \left| \sum_{p \in Q} \int_H f(k\sigma(p)) \xi(\sigma(p)^{-1}k^{-1}h\sigma(q)) dk \right|^2 dh \\ &\quad (\text{supp}(\xi) \subseteq H) = \int_H \left| \int_H f(k\sigma(q)) \xi(\sigma(q)^{-1}k^{-1}h\sigma(q)) dk \right|^2 dh \\ &\quad = \int_H \left| \int_H f_q(k) \eta(k^{-1}h) dk \right|^2 dh \\ &\quad = \|f_q * \eta\|_{L^2(H)}^2. \end{aligned}$$

It follows that $\|\lambda_H(f_q)\| \leq \|\lambda_G(f)\|$.

Suppose $\text{supp}(f) \subseteq H\sigma(q)$. Take any $\xi \in L^2(G)$. We have

$$\begin{aligned}
\|f * \xi\|_{L^2(G)}^2 &= \sum_{r \in Q} \int_H \left| \sum_{p \in Q} \int_H f(k\sigma(p)) \xi(\sigma(p)^{-1} k^{-1} h \sigma(r)) dk \right|^2 dh \\
(\text{supp}(f) \subseteq H\sigma(q)) &= \sum_{r \in Q} \int_H \left| \int_H f(k\sigma(q)) \xi(\sigma(q)^{-1} k^{-1} h \sigma(r)) dk \right|^2 dh \\
&= \sum_{r \in Q} \int_H \left| \int_H f_q(k) \xi_{q,r}(k^{-1} h) dk \right|^2 dh \\
&= \sum_{r \in Q} \|f_q * \xi_{q,r}\|_{L^2(H)}^2 \\
&\leq \|\lambda_H(f_q)\|^2 \sum_{r \in Q} \|\xi_{q,r}\|_{L^2(H)}^2 \\
&= \|\lambda_H(f_q)\|^2 \|\xi\|_{L^2(G)}^2,
\end{aligned}$$

where the last equality is from (3.2). Thus, $\|\lambda_G(f)\| = \|\lambda_H(f_q)\|$. \square

Lemma 3.3.12. *Assume the notations above. Suppose that H has polynomial ℓ -growth. Then the map*

$$N : f \in C_c(G) \mapsto \max_{q \in Q} \|\lambda_H(f_q)\|$$

defines a norm on $C_c(G)$ that is ℓ -comparable with the $C_\lambda^(G)$ -norm.*

Proof. Thanks to Lemma 3.3.11, we have $N(f) = \max_{q \in Q} \|\lambda_H(f_q)\| \leq \|\lambda_G(f)\|$ for all $f \in C_c(G)$, so N is well defined. It is routine to check that N is a norm on $C_c(G)$. Take any function $f \in C_c(G)$ with $\text{supp}(f) \subseteq B_n$. Noting that $\text{supp}(f) \subseteq$

$H\sigma(\pi(B_n))$ implies $f_q \equiv 0$ for all $q \in \pi(B_n)^c$, we have

$$\begin{aligned}
\|f * \xi\|_{L^2(G)}^2 &= \sum_{r \in Q} \int_H \left| \sum_{q \in Q} \int_H f(k\sigma(q)) \xi(\sigma(q)^{-1}k^{-1}h\sigma(r)) dk \right|^2 dh \\
&= \sum_{r \in Q} \int_H \left| \sum_{q \in Q} \int_H f_q(k) \xi_{q,r}(k^{-1}h) dk \right|^2 dh \\
&= \sum_{r \in Q} \int_H \left| \left(\sum_{q \in Q} f_q *_{H} \xi_{q,r} \right) (h) \right|^2 dh \\
&= \sum_{r \in Q} \left\| \sum_{q \in \pi(B_n)} f_q *_{H} \xi_{q,r} \right\|_{L^2(H)}^2.
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}
\|f * \xi\|_{L^2(G)}^2 &\leq \sum_{r \in Q} \left(\sum_{q \in \pi(B_n)} \|f_q *_{H} \xi_{q,r}\|_{L^2(H)} \right)^2 \\
&\leq N(f)^2 \sum_{r \in Q} \left(\sum_{q \in \pi(B_n)} \|\xi_{q,r}\|_{L^2(H)} \right)^2 \\
&\leq N(f)^2 \sum_{r \in Q} |\pi(B_n)| \sum_{q \in \pi(B_n)} \|\xi_{q,r}\|_{L^2(H)}^2 \\
&\leq N(f)^2 |\pi(B_n)| \sum_{q \in \pi(B_n)} \sum_{r \in Q} \|\xi_{q,r}\|_{L^2(H)}^2 \\
&\text{(Apply (3.2))} = N(f)^2 |\pi(B_n)|^2 \|\xi\|_{L^2(G)}^2,
\end{aligned}$$

and it follows that N is ℓ -comparable with the $C_\lambda^*(G)$ -norm since the growth of $|\pi(B_n)|$ is polynomial. \square

Proof of Proposition 3.3.9. Let us prove first the non-completely bounded case. The “only if” part is due to Proposition 3.3.1. For the converse, assume that there exists tame cuts $(\psi_n)_{n \in \mathbb{N}}$ for $(H, \ell|_H)$. By assumption, we have

$$\|\psi_n\|_{MA(H)} \leq Cn^a, \quad \psi_n|_{B_n \cap H} \equiv 1, \quad \text{and} \quad |\pi(B_n)| \leq Dn^b$$

for some constants $C, D, a, b \geq 0$ and for all $n \in \mathbb{N}$. Define $\varphi_n : G \rightarrow \mathbb{C}$ as

$$\varphi_n(h\sigma(q)) = \begin{cases} \psi_{2n}(h), & q \in \pi(B_n) \\ 0, & \text{otherwise} \end{cases}, \quad (\forall h \in H, q \in Q).$$

Notice that $\varphi_n \in C_c(G)$ since it is a sum of finitely many compactly supported continuous functions whose supports are in disjoint open subsets of G . Conventionally, we chose the cross-section σ such that $\ell(\sigma(q)) \leq \ell(h\sigma(q))$ for any $h \in H$. Now, it is easy to see $\varphi_n|_{B_n} \equiv 1$. Indeed, if $h\sigma(q) \in B_n$, then $q \in \pi(B_n)$ and

$$\ell(h) \leq \ell(h\sigma(q)) + \ell(\sigma(q)) \leq 2\ell(h\sigma(q)) \leq 2n,$$

hence $\varphi_n(h\sigma(q)) = \psi_{2n}(h) = 1$ by construction. Note also that if ψ_{2n} is a characteristic function, then so is φ_n . Since $\text{supp}(\varphi_n) \subseteq H\sigma(\pi(B_n))$, by Lemma 3.3.12,

$$\begin{aligned} \|\lambda_G(\varphi_n f)\| &\leq Dn^b N(\varphi_n f) \\ &= Dn^b \max_{q \in \pi(B_n)} \|\lambda_H([\varphi_n f]_q)\| \\ &= Dn^b \max_{q \in \pi(B_n)} \|\lambda_H(\psi_{2n} f_q)\| \\ &\leq Dn^b \|\psi_{2n}\|_{MA(H)} \max_{q \in \pi(B_n)} \|\lambda_H(f_q)\| \\ &\leq Dn^b \|\psi_{2n}\|_{MA(H)} N(f) \\ &\leq CDn^{a+b} \|\lambda_G(f)\| \end{aligned}$$

for all $f \in C_c(G)$. This shows that $(\varphi_n)_{n \in \mathbb{N}}$ forms tame cuts for (G, ℓ) .

The completely bounded case can be seen by considering the open subgroup $H \times SO(3)$ of $G \times SO(3)$. \square

We end this subsection by proposing another sufficient condition for an extension to have (characteristic) tame cuts.

Proposition 3.3.13. *Let Γ be a finitely generated group, ℓ the word length function of Γ , H a normal subgroup of Γ , and $\sigma : Q = H \backslash \Gamma \rightarrow \Gamma$ an ℓ -optimal cross section. Define the map $f \in C_c(\Gamma) \rightarrow f' \in C_c(Q)$, $f'(q) = \|\lambda_H(f_q)\|$, where $f_q \in C_c(H)$ is defined by $f_q(h) = f(h\sigma(q))$ for all $q \in Q$ and $h \in H$. If $(H, \ell|_H)$ has (characteristic) tame cuts and $\|\lambda_Q(f')\| \leq \|\lambda_\Gamma(f)\|$ for all $f \in C_c(\Gamma)$, then (Γ, ℓ) has (characteristic) tame cuts.*

Proof. Take any function $\psi \in C_c(H)$ with $\psi|_{B_{2n} \cap H} \equiv 1$. Define $\varphi \in C_c(\Gamma)$ as

$$\varphi(h\sigma(q)) = \begin{cases} \psi(h), & \ell(h\sigma(q)) \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Since σ is ℓ -optimal, we have $\varphi|_{B_n} \equiv 1$. Also, note that φ is characteristic if so is ψ . To prove the statement, it is enough to see that $\|\varphi\|_{MA} \leq \|\psi\|_{MA}$. Recall the notations

$$\begin{aligned} f_q(h) &= f(h\sigma(q)) \\ \xi_{p,r}(h) &= \xi(\sigma(p)^{-1}h\sigma(r)) \\ f'(p) &= \|\lambda_H(f_p)\| \\ \xi_1(q) &= \left(\sum_{h \in H} |\xi(h\sigma(q))|^2 \right)^{1/2} = \|\xi_q\|_{\ell^2(H)} \end{aligned}$$

for all $h \in H$, $p, q, r \in Q$, and $f, \xi \in C_c(\Gamma)$. It is easily seen that

$$\|\xi_{p,r}\|_{\ell^2(H)} = \xi_1(p^{-1}r) \quad (3.3)$$

$$\|\xi\|_{\ell^2(\Gamma)} = \|\xi_1\|_{\ell^2(Q)}. \quad (3.4)$$

It follows that

$$\begin{aligned} \|\varphi f *_{\Gamma} \xi\|_{\ell^2(\Gamma)}^2 &= \sum_{r \in Q} \left\| \sum_{p \in Q} \varphi_p f_p *_{H} \xi_{p,r} \right\|_{\ell^2(H)}^2 \\ (\text{triangle inequality}) &\leq \sum_{r \in Q} \left(\sum_{p \in Q} \|\varphi_p f_p *_{H} \xi_{p,r}\|_{\ell^2(H)} \right)^2 \\ (\text{operator norms}) &\leq \|\psi\|_{MA(H)}^2 \sum_{r \in Q} \left(\sum_{p \in Q} \|\lambda_H(f_p)\| \|\xi_{p,r}\|_{\ell^2(H)} \right)^2 \\ (\text{apply (3.3)}) &= \|\psi\|_{MA(H)}^2 \|f' *_{Q} \xi_1\|_{\ell^2(Q)}^2 \\ (\text{operator norms}) &\leq \|\psi\|_{MA(H)}^2 \|\lambda_Q(f')\|^2 \|\xi_1\|_{\ell^2(Q)}^2 \\ (\text{apply the hypothesis and (3.4)}) &\leq \|\psi\|_{MA(H)}^2 \|\lambda_{\Gamma}(f)\|^2 \|\xi\|_{\ell^2(\Gamma)}^2. \end{aligned}$$

This proves $\|\varphi\|_{MA} \leq \|\psi\|_{MA}$. □

3.3.4 Quasi-direct product

It seems the stability of admitting (characteristic) tame cuts under the direct product is not clear. The completely bounded case, on the other hand, is not difficult to see. In this subsection, we will discuss the stability of completely bounded tame cuts under a version of the direct product, namely quasi-direct product.

Definition 3.3.14. Let G be a locally compact group, $\{H_1, \dots, H_k\}$ a collection of some normal closed subgroups of G . We say that G is a *quasi-direct product* of $\{H_1, \dots, H_k\}$, or the collection is *quasi-orthogonal*, if the intersection $\bigcap_{i=1}^k KH_i$ is compact for any compact subset $K \subseteq G$.

For example, if G is equal to the direct product $H_1 \times \dots \times H_k$, then the collection $\{H_1, \dots, H_k\}$ is quasi-orthogonal. Indeed, suppose that $K \subseteq G$ is a compact subset. Note that

$$KH_i \subseteq p_1(K) \times \dots \times p_{i-1}(K) \times H_i \times p_{i+1}(K) \times \dots \times p_k(K),$$

where $p_i : G \rightarrow H_i$ are the canonical projections. Thus,

$$\bigcap_{i=1}^k KH_i \subseteq \prod_{i=1}^k p_i(K).$$

The right hand side is compact, and the left hand side is closed, hence also compact.

Lemma 3.3.15. *Let H be a normal closed subgroup of a locally compact group G . If ψ is a completely bounded Fourier multiplier on $Q = G/H$, then the periodic extension*

$$\varphi : G \rightarrow \mathbb{C}, \quad \varphi(x) = \psi(xH), \quad x \in G$$

of ψ defines a completely bounded Fourier multiplier on G with $\|\varphi\|_{M_0A(G)} \leq \|\psi\|_{M_0A(Q)}$.

Proof. By hypothesis, there is a Hilbert space \mathcal{H} and two bounded continuous maps $\xi, \eta : Q \rightarrow \mathcal{H}$ such that

$$\psi(p^{-1}q) = \langle \xi(q), \eta(p) \rangle, \quad (\forall p, q \in Q)$$

and $\|\psi\|_{M_0A} = \|\xi\|_\infty \|\eta\|_\infty$. Define $\xi', \eta' : G \rightarrow \mathcal{H}$ as $\xi'(x) = \xi(xH)$ and $\eta'(y) =$

$\eta(yH)$ for all $x, y \in G$. Then we have

$$\varphi(y^{-1}x) = \psi(y^{-1}xH) = \psi((yH)^{-1}xH) = \langle \xi(xH), \eta(yH) \rangle = \langle \xi'(x), \eta'(y) \rangle,$$

which follows that $\varphi \in M_0A(G)$ and $\|\varphi\|_{M_0A} \leq \|\xi\|_\infty \|\eta\|_\infty = \|\psi\|_{M_0A}$. \square

Proposition 3.3.16. *Let G be a locally compact group that is a quasi-direct product of subgroups $\{H_1, \dots, H_k\}$, and ℓ a proper length function of G . For each $i \in \{1, \dots, k\}$, denote by $\sigma_i : Q_i = G/H_i \rightarrow G$ an ℓ -optimal cross section. If each quotient group $(Q_i = G/H_i, \ell \circ \sigma_i)$ admits completely bounded tame cuts, then so does G . If G is discrete and each quotient group $(Q_i = G/H_i, \ell \circ \sigma_i)$ admits completely bounded characteristic tame cuts, then so does G .*

Proof. Let $(\psi_{i,n} : Q_i \rightarrow \mathbb{C})_{n \in \mathbb{N}}$ be tame cuts for $(Q_i, \ell \circ \sigma_i)$. Put $\varphi_n = \prod_{i=1}^k \varphi_{i,n}$, where each $\varphi_{i,n}$ is the periodic extension of $\psi_{i,n}$. We have

$$\|\varphi_n\|_{M_0A} \leq \prod_{i=1}^k \|\varphi_{i,n}\|_{M_0A}$$

since $M_0A(G)$ is Banach algebra with pointwise multiplication. Also, note that the right hand side is at most a polynomial of n .

The functions φ_n have compact support. Indeed, since H_1, \dots, H_k are quasi-orthogonal, the support

$$\text{supp}(\varphi_n) = \bigcap_{i=1}^k \text{supp}(\varphi_{i,n}) \subseteq \bigcap_{i=1}^k \underbrace{\left(\bigcup_{i=1}^k \text{supp}(\varphi_{i,n}) \right)}_{\text{compact}} H_i$$

is compact. It remains to show that φ_n take 1 on the ball B_n of radius n . By assumption, the periodic extension $\varphi_{i,n}$ takes value 1 on $\ell^{-1}([0, n])H_i$. It follows that φ_n takes value 1 on the intersection $\bigcap_{i=1}^k \ell^{-1}([0, n])H_i$, which obviously contains the ball B_n . \square

Corollary 3.3.17. *Admitting completely bounded (characteristic) tame cuts is stable under direct products.*

Corollary 3.3.18. *Suppose that H is a compact normal subgroup of a locally compact group G . If the quotient group $Q = G/H$ has completely bounded tame cuts,*

then so does G . If G is discrete and $Q = G/H$ has completely bounded (characteristic) tame cuts, then so does G .

Proof. Use the proposition for the quasi-orthogonal collection $\{G, H\}$. \square

3.3.5 Quotients by a normal compact subgroup

Let G be a locally compact group, H a normal compact subgroup of G . Denote by $\pi : G \rightarrow G/H = Q$ the natural quotient map. If G is generated by a compact subset S with $S = S^{-1}$ and such that $H \subseteq S$, then Q is compactly generated by $\pi(S)$. Denote by ℓ_G and ℓ_Q the corresponding word length functions. By [Gar15, Remark 4.2], we can choose a Borel measurable cross-section $\sigma : Q \rightarrow G$ such that $\ell_Q(q) = \ell_G(\sigma(q))$ for all $q \in Q$. We write elements $x \in G$ in its unique form $x = h\sigma(q)$, $h \in H$, $q \in Q$. We normalize the Haar measures of G and Q so that the Haar measure dh of H corresponding to the formula (cf. [BdlHV08, Theorem B.1.4])

$$\int_G f(x)dx = \int_Q \int_H f(h\sigma(q))dhdq$$

is probability.

Proposition 3.3.19. *Under the assumptions above, if (G, ℓ_G) has tame cuts, so does (Q, ℓ_Q) .*

The proof is inspired by [Jol90]. We need the following lemma.

Lemma 3.3.20. *The linear map $C_\lambda^*(Q) \rightarrow C_\lambda^*(G)$, $f \mapsto f' = f \circ \pi$ is norm decreasing.*

Proof. For $\xi \in L^2(G)$, $k, h \in H$, and $p, q \in Q$, denote $\xi^k(p) = \xi(k\sigma(p))$. Noting $h\sigma(q)\sigma(p)^{-1}k^{-1}H = \sigma(qp^{-1})H$, we have

$$\begin{aligned} \|f' * \xi\|_{L^2(G)}^2 &= \int_Q \int_H \left| \int_Q \int_H f'(h\sigma(q)\sigma(p)^{-1}k^{-1})\xi(k\sigma(p))dkdp \right|^2 dhdq \\ &= \int_Q \int_H \left| \int_Q \int_H f(qp^{-1})\xi^k(p)dkdp \right|^2 dhdq \\ &= \left\| f *_{\mathcal{Q}} \left(\int_H \xi^k dk \right) \right\|_{L^2(Q)}^2 \leq \|\lambda_Q(f)\|^2 \left(\int_H \|\xi^k\|_{L^2(Q)} dk \right)^2 \\ &\leq \|\lambda_Q(f)\|^2 \int_H \|\xi^k\|_{L^2(Q)}^2 = \|\lambda_Q(f)\|^2 \|\xi\|_{L^2(G)}^2 \end{aligned}$$

for all $f \in C_c(Q)$ and $\xi \in C_c(G)$. \square

Proof of Proposition 3.3.19. Suppose that $\psi \in C_c(G) \cap MA(G)$ takes value 1 on B_{n+1} . By [BdlHV08, Lemma B.1.2], the function $\varphi : Q \rightarrow \mathbb{C}$ defined by

$$\varphi(q) = \int_H \psi(h\sigma(q))dh \quad (3.5)$$

is compactly supported and continuous. We claim $\varphi|_{B_n} \equiv 1$. Indeed, if $\ell_Q(q) \leq n$, we have $\ell_G(h\sigma(q)) \leq 1 + n$ for all $h \in H$ since $H \subseteq S$ and $\ell_G(\sigma(q)) \leq n$ by the choice of σ . Recall that dh is probability on H and $\psi|_{B_{n+1}} \equiv 1$. It follows that $\varphi_n|_{B_n} \equiv 1$ from its definition in (3.5).

It is enough to prove $\|\varphi\|_{MA} \leq \|\psi\|_{MA}$. Let us observe two useful formulas in advance. For any $f, \xi \in C_c(Q)$, we have

$$\begin{aligned} \|f' *_G \xi'\|_{L^2(G)}^2 &= \int_Q \int_H \left| \int_Q \int_H f'(k\sigma(p))\xi'(\sigma(p)^{-1}k^{-1}h\sigma(q))dkdp \right|^2 dhdq \\ &= \int_Q \int_H \left| \int_Q \int_H f(p)\xi(p^{-1}q)dkdp \right|^2 dhdq \\ &= \int_Q \left| \int_Q f(p)\xi(p^{-1}q)dp \right|^2 dq \\ &= \|f *_Q \xi\|_{L^2(Q)}^2 \end{aligned}$$

and

$$\begin{aligned} (\varphi f)' *_G \xi'(h\sigma(q)) &= \int_Q \varphi(p)f(p)\xi(p^{-1}q)dp \\ &= \int_Q \left(\int_H \psi(k\sigma(p))dk \right) f(p)\xi(p^{-1}q)dp \\ &= \psi f' *_G \xi'(h\sigma(q)). \end{aligned}$$

It follows that

$$\begin{aligned} \|\varphi f *_Q \xi\|_{L^2(Q)}^2 &= \|(\varphi f)' *_G \xi'\|_{L^2(G)} \\ &= \|\psi f' *_G \xi'\|_{L^2(G)}^2 \\ &\leq \|\psi\|_{MA(G)}^2 \|\lambda_G(f')\|^2 \|\xi'\|_{L^2(G)}^2 \\ &\leq \|\psi\|_{MA(G)}^2 \|\lambda_Q(f)\|^2 \|\xi\|_{L^2(Q)}^2. \end{aligned}$$

The last inequality follows from Lemma 3.3.20. This completes the proof. \square

3.4 More examples

In this section, we will provide examples of groups with completely bounded characteristic tame cuts. Some geometric groups are already known to have this property. For example, hyperbolic groups [Oza08], groups acting properly by isometries on ℓ^1 -product of geometrically finite hyperbolic graphs [Ver19], Coxeter groups [Fen02], and finite dimensional CAT(0) cubical groups [Miz08]. These examples are all based on the groups acting on a tree that we illustrated in Theorem 2.1.4. Our examples will be less geometric, and we will use the stability properties from the previous section and the following lemma.

Lemma 3.4.1. *Let G be a locally compact group, H an open subgroup. Then $\|\varphi\|_{A(G)} = \|\varphi|_H\|_{A(H)}$ for every $\varphi \in A(G)$ with $\text{supp}(\varphi) \subseteq H$.*

Proof. As H is open, the space of square integrable functions on G with support on H can be identified with $L^2(H)$, and when $f, g \in L^2(G)$ are supported on H , we have $f|_H *_H \tilde{g}|_H = (f *_G \tilde{g})|_H$. It follows that

$$\begin{aligned} \|\varphi\|_{A(G)} &= \inf\{\|f\|_2\|g\|_2 : f, g \in L^2(G), \varphi = f *_G \tilde{g}\} \\ &\leq \inf\{\|f\|_2\|g\|_2 : f, g \in L^2(G), \varphi = f *_G \tilde{g}, \text{supp}(f) \cup \text{supp}(g) \subseteq H\} \\ &= \inf\{\|f\|_2\|g\|_2 : f, g \in L^2(H), \varphi|_H = f *_H \tilde{g}\} \\ &= \|\varphi|_H\|_{A(H)}. \end{aligned}$$

To show the converse inequality, take any $\psi \in A(H)$ and $f, g \in L^2(G)$ such that $\varphi = f *_H \tilde{g}$ and $\|\varphi\|_{A(H)} = \|f\|_2\|g\|_2$. Then their trivial extensions on G satisfy $\varphi = f *_G \tilde{g}$ and $\|\psi\|_{A(G)} = \|f\|_2\|g\|_2$. \square

Proposition 3.4.2. *Let P be a finitely generated group with polynomial growth and let F be a finite group. Then the wreath product $\Gamma = F \wr P$ has characteristic tame cuts. In particular, the Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ has characteristic tame cuts.*

Proof. By definition, Γ is an extension of $H = \bigoplus_{x \in P} F$ by the polynomial growth group P . According to Proposition 3.3.9, it is enough to prove that H has characteristic tame cuts with respect to the restriction of the word length function of Γ . Suppose that $S \subseteq P$ is a finite generating set of P . For $t \in F$, denote by $\tilde{t} : P \rightarrow F$

the map taking value t on e_P , and identity e_F elsewhere. Then Γ is generated by the finite set

$$T = \{(\tilde{t}, e_P) \mid t \in F\} \cup \{(\tilde{e}_F, s) \mid s \in S\}.$$

Endow Γ with the word length function $\ell = \ell_T$. For $n \in \mathbb{N}$, denote

$$G_n = \bigoplus_{x \in P, \ell(x) \leq n} F \quad \text{and} \quad H_n = \bigoplus_{x \in P, \ell(x) > n} F$$

so that $H = G_n \oplus H_n$. Notice that G_n is finite and contains the relative ball $B_n \cap H$. Put $\varphi_n = \mathbb{1}_{G_n}$, the characteristic function of G_n . We want to estimate the norm $\|\varphi_n\|_{MA(H)}$. Recall that $\|\varphi_n\|_{MA(H)} = \|\varphi_n\|_{A(H)}$ because H is amenable and φ_n is finitely supported. Since the Fourier multiplier $\mathbb{1}_{G_n}$ acts on $A(G_n)$ trivially, and since $\text{supp}(\varphi_n) = G_n$, by Lemma 3.4.1, we have

$$\|\varphi_n\|_{MA(H)} = \|\varphi_n\|_{A(H)} = \|\varphi_n|_{G_n}\|_{A(G_n)} = \|\mathbb{1}_{G_n}\|_{MA(G_n)} = 1.$$

This completes the proof. \square

Proposition 3.4.3. *For any coprime integers $p, q \in \mathbb{N}$, the group $\Gamma_{p,q} = \mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ has characteristic tame cuts.*

Proof. Recall that $\Gamma = \mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ is isomorphic to the subgroup of $GL_2(\mathbb{R})$

$$\Gamma \cong \left\{ \begin{pmatrix} \left(\frac{p}{q}\right)^k & P \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}, P \in \mathbb{Z}[\frac{1}{pq}] \right\}$$

generated by the finite set

$$S = \left\{ s^{\pm 1} = \begin{pmatrix} \frac{p}{q} & 0 \\ 0 & 1 \end{pmatrix}^{\pm 1}, t^{\pm 1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\pm 1} \right\}.$$

Let $\ell = \ell_S$ be the word length function of Γ . According to Proposition 3.3.9, we only need to prove that the subgroup

$$H = \left\{ \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix} : P \in \mathbb{Z}[\frac{1}{pq}] \right\} \cong \mathbb{Z}[\frac{1}{pq}]$$

has characteristic tame cuts with respect to the restricted length function $\ell|_H$. Suppose that $x = \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix} \in H$ and $\ell(x) \leq n$. Then we can write $x = s_1 \dots s_n$ for some $s_i = \begin{pmatrix} \left(\frac{p}{q}\right)^{\varepsilon_i} & \delta_i \\ 0 & 1 \end{pmatrix} \in S$, $\varepsilon_i, \delta_i \in \{-1, 0, 1\}$, $1 \leq i \leq n$ with $\sum \varepsilon_i = 0$. Moreover, we have

$$\begin{aligned} P &= \delta_1 + \delta_2 \left(\frac{p}{q}\right)^{\varepsilon_1} + \delta_3 \left(\frac{p}{q}\right)^{\varepsilon_1 + \varepsilon_2} + \dots + \delta_n \left(\frac{p}{q}\right)^{\varepsilon_1 + \dots + \varepsilon_{n-1}} \\ &= \frac{\sum_{i=1}^n \delta_i p^{n + \sum_{j=1}^{i-1} \varepsilon_j} q^{n - \sum_{j=1}^{i-1} \varepsilon_j}}{q^n p^n}. \end{aligned}$$

From this, it is easy to see that the cyclic subgroup H_n of H generated by the element

$$x_n = \begin{pmatrix} 1 & \frac{1}{q^n p^n} \\ 0 & 1 \end{pmatrix} \in H$$

contains the relative ball $B_n \cap H = \{x \in H \mid \ell(x) \leq n\}$, and for any element $x \in B_n \cap H$, its absolute power $|x|$ with respect to the element x_n has an upper bound

$$|x| = \left| \sum_{i=1}^n \delta_i p^{n + \sum_{j=1}^{i-1} \varepsilon_j} q^{n - \sum_{j=1}^{i-1} \varepsilon_j} \right| \leq n q^{2n} p^{2n}.$$

Denote by A_n the subset of H_n containing the elements with absolute power less than $n q^{2n} p^{2n}$. Note that A_n is finite set containing $B_n \cap H$. Combining the facts that H is amenable, Lemma 3.4.1, and the estimation of L^1 -norms of Dirichlet kernels, we get

$$\begin{aligned} \|\mathbb{1}_{A_n}\|_{M_0 A(H)} &= \|\mathbb{1}_{A_n}\|_{A(H)} = \|\mathbb{1}_{A_n}\|_{A(H_n)} \\ &= \|\mathcal{F}(\mathbb{1}_{A_n})\|_{L^1(\mathbb{T})} = \frac{4}{\pi} \log(n q^{2n} p^{2n}) + O(1). \end{aligned} \tag{3.6}$$

This completes the proof since (3.6) is at most polynomial. \square

Proposition 3.4.4. *Let $d \in \mathbb{N}$ and let $P < SL_d(\mathbb{Z})$ be a finitely generated group with polynomial growth. Then the semidirect product $\Gamma = \mathbb{Z}^d \rtimes P$ has characteristic tame cuts.*

Proof. It is practical to use the following unique canonical form

$$x = (v, p), \quad v \in \mathbb{Z}^d, p \in P$$

for every element $x \in \Gamma$. In this form, the group law is given by

$$(v, p)(w, q) = (v + pw, pq).$$

Suppose that T is a finite generating set of P with $T = T^{-1}$. Let $\{e_1, \dots, e_d\} \subset \mathbb{Z}^d$ be the usual bases. Then the finite set

$$S = \{(\pm e_i, t) : 1 \leq i \leq d, t \in T\}$$

generates Γ . We endow Γ with the word length function ℓ associated to S . Denote

$$H = \langle e_1, \dots, e_d \rangle = \{(v, 0) \in \Gamma : v \in \mathbb{Z}^d\} < \Gamma.$$

Since P has polynomial growth, by Proposition 3.3.9, it is enough to show that $(H, \ell|_H)$ has characteristic tame cuts. Suppose $\ell(v, 0) \leq n$. Then we can write

$$(v, 0) = s_1 \dots s_n$$

for some $s_i = (\varepsilon_i, t_i) \in S$, $1 \leq i \leq n$ with $t_1 \dots t_n = I_d$ and

$$v = \varepsilon_1 + t_1 \varepsilon_2 + \dots + t_1 \dots t_{n-1} \varepsilon_n.$$

Notice that

$$\|v\|_\infty \leq \|v\|_2 \leq 1 + \|t_1\| + \dots + \|t_1 \dots t_{n-1}\| \leq n (\max \{\|t\| : t \in T\})^n.$$

This can be bounded by C^n for some constant $C \in \mathbb{N}$ depending only on the set T . Thus, the finite set $A_n = \{(v, 0) \in H \mid v \in \mathbb{Z}^d, \|v\|_\infty \leq C^n\}$ contains the relative ball $B_n \cap H$ of radius n . Moreover, by the estimation of L^1 -norms of Dirichlet kernels

$$\begin{aligned} \|\mathbb{1}_{A_n}\|_{M_0A(G)} &= \|\mathcal{F}(\mathbb{1}_{A_n})\|_{L^1(\mathbb{T}^d)} = \|D_{C^n} \otimes \dots \otimes D_{C^n}\|_{L^1(\mathbb{T}^d)} \\ &= \|D_{C^n}\|_{L^1(\mathbb{T})}^d = \left(\frac{4n \log(C)}{\pi} + O(1) \right)^d. \end{aligned} \tag{3.7}$$

Since (3.7) is a polynomial, we conclude. \square

Our next example is the Baumslag-Solitar groups. The idea is based on the work [GJ03]. The proof also works for N -BS groups, which generalizes the classical Baumslag-Solitar groups. Let us explain what N -BS group is.

Definition 3.4.5. Let N be a locally compact group, G a closed subgroup of N , and H_1 and H_2 two closed subgroups of G such that there is an automorphism $\alpha \in \text{Aut}(N)$ with $\alpha(H_1) = H_2$. If the subgroups H_1 and H_2 have finite indices in G , the group N -BS(G, H_1, H_2, α) is defined by the presentation

$$HNN(G, H_1, H_2, \alpha) \cong \langle G, t \mid R, tht^{-1} = \alpha(h), \forall h \in H_1 \rangle,$$

where $G = \langle G \mid R \rangle$. There is a unique translation invariant topology on N -BS(G, H_1, H_2, α) such that the obvious homomorphism $G \rightarrow N$ -BS(G, H_1, H_2, α) is an embedding onto an open subset. With this topology, the group N -BS(G, H_1, H_2, α) becomes a locally compact group.

Example 3.4.6. Let $p, q \in \mathbb{N}$. Put $d = \gcd(p, q)$, $p' = p/d$, and $q' = q/d$. The Baumslag-Solitar group $BS(p, q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$ is exactly the group $\mathbb{Z}[\frac{1}{p'q'}]$ -BS $\left(\mathbb{Z}, p\mathbb{Z}, q\mathbb{Z}, \alpha : x \mapsto \frac{qx}{p} \right)$.

Proposition 3.4.7. Let N be a discrete group. Consider a group of the form $\Gamma = N$ -BS(G, H_1, H_2, α). If the group $N \rtimes_{\alpha} \mathbb{Z}$ has completely bounded characteristic tame cuts, then Γ has completely bounded characteristic tame cuts.

Before giving the proof, we prepare some ingredients. Recall that the group $\Gamma = HNN(G, H_1, H_2, \alpha)$ acts by isometries on the Bass-Serre tree T , that is a $([G : H_1] + [G : H_2])$ -regular tree (cf. [Ser03, Section 5.1]). This gives rise to a continuous homomorphism $j_T : \Gamma \rightarrow \text{Aut}(T)$, where the group $\text{Aut}(T)$ of isometries on T is endowed with the compact-open topology. Since each of H_1 and H_2 has a finite index in G , the Bass-Serre tree T is locally finite, and $\text{Aut}(T)$ is locally compact and acts properly on T (cf. [CGK01]). Let $v_0 \in T$ be the vertex fixed by G -action. Endow $\text{Aut}(T)$ with the proper length function $\ell_T(x) = d(x.v_0, v_0)$, where d is the combinatorial distance on T . We also have another homomorphism $j_N : \Gamma \rightarrow N \rtimes_{\alpha} \mathbb{Z}$ such that $j_N|_G \equiv \text{id}_G$ and $j_N(t) = (e_N, 1)$. Let us denote $G_T = \text{Aut}(T)$ and $G_N = N \rtimes_{\alpha} \mathbb{Z}$. The following theorem is crucial to the proof.

Theorem 3.4.8 ([GJ03]). *Let N, G, H_1, H_2 , and α be as in Definition 3.4.5. Let $\mathcal{G} = N\text{-BS}(G, H_1, H_2, \alpha)$, and let j_T and j_N be the homomorphisms defined above. Then the homomorphism*

$$j = (j_T, j_N) : \mathcal{G} \rightarrow G_T \times G_N$$

is an embedding onto a closed subgroup.

We also need the following lemma.

Lemma 3.4.9 (Lemma 1.4 of [CH89]). *Suppose that G_1 and G_2 are locally compact groups. If $\varphi \in M_0A(G_1)$ and $\psi \in M_0A(G_2)$, then the function $\varphi \otimes \psi : G_1 \times G_2 \rightarrow \mathbb{C}$, $(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y)$ is in $M_0A(G_1 \times G_2)$ and $\|\varphi \otimes \psi\|_{M_0A} = \|\varphi\|_{M_0A}\|\psi\|_{M_0A}$.*

Proof of Proposition 3.4.11. Theorem 3.4.8 allows us to see Γ as a discrete subgroup of $G_T \times G_N$. By hypothesis, there are characteristic tame cuts $(g_n)_{n \in \mathbb{N}}$ for (G_N, ℓ_N) , where ℓ_N a proper length function of G_N . The function $\ell : (x, y) \in G_T \times G_N \mapsto \max\{\ell_T(x), \ell_N(y)\}$ gives a proper length function of $G_T \times G_N$. Denote by $f_n : G_T \rightarrow \mathbb{C}$ the characteristic function of the ball $\{x \in G_T : \ell_T(x) \leq n\}$. Beware that f_n might be not in $M_0A(G)$ as the continuity is not guaranteed, but we have $f_n \in M_0A((G_T)_d)$ and $\|f_n\|_{M_0A((G_T)_d)} \leq 2n + 1$ by Theorem 2.1.4. Here, $(G_T)_d$ is the discrete realization of G_T . The new function $\varphi_n = f_n \otimes g_n$ on $G_T \times G_N$ defined by $\varphi_n(x, y) = f_n(x)g_n(y)$ is a compactly supported, characteristic function and obviously takes value 1 on the ball $B_n = \{(x, y) \in G_T \times G_N : \ell(x, y) \leq n\}$. By Lemma 3.3.2 and 3.4.9, we have

$$\|\varphi_n|_\Gamma\|_{M_0A(\Gamma)} \leq \|\varphi_n\|_{M_0A((G_T)_d \times G_N)} = \|f_n\|_{M_0A((G_T)_d)} \|g_n\|_{M_0A(G_N)}$$

which is bounded by a polynomial. This proves that the sequence $(\varphi_n|_\Gamma)_{n \in \mathbb{N}}$ gives completely bounded characteristic tame cuts for $(\Gamma, \ell|_\Gamma)$. \square

Corollary 3.4.10. *The Baumslag-Solitar group $BS(p, q)$ has completely bounded characteristic tame cuts for any $p, q \in \mathbb{N}$.*

Proof. Let $p' = p/\gcd(p, q)$ and $q' = q/\gcd(p, q)$. We note that that $BS(p, q)$ is a $\mathbb{Z}[\frac{1}{p'q'}]$ -BS group and that $G_N = \mathbb{Z}[\frac{1}{p'q'}] \rtimes_{\frac{p'}{q'}} \mathbb{Z} = \Gamma_{p', q'}$ has completely bounded characteristic tame cuts. \square

Proposition 3.4.11. *Let N , G , H_1 , H_2 , and α be as in Definition 3.4.5 and let $\mathcal{G} = N\text{-}BS(G, H_1, H_2, \alpha)$. If the group $N \rtimes_{\alpha} \mathbb{Z}$ has completely bounded tame cuts, then \mathcal{G} has completely bounded tame cuts.*

Proof. The proof is essentially the same as Proposition 3.4.7. The only problem is that f_n might be not continuous. To regulate that, we repeat the same technique used to prove Proposition 3.2.6. Find a large enough number $r \in \mathbb{R}_+$ such that the ball $\{x \in G_T : \ell_T(x) \leq r\}$ has a non-empty interior. Find a non-negative function $f \in A(G)$ with $\text{supp}(f) \subseteq B_{2r}$ and $\int_{G_T} f = 1$. Now the functions $(f * f_{n+2r}) \otimes g_n$ give completely bounded tame cuts for \mathcal{G} . \square

3.5 Property (T_{Schur}, G, K)

Concerning non-examples, the rigidity inequality of Theorem 2.1.11 shows that higher rank simple Lie groups with finite center do not have tame cuts, and their uniform lattices do not have completely bounded tame cuts. On the other hand, so far we have only $(\mathbb{Z}, \log(1 + \log(1 + |\cdot|)))$ as an example of discrete group without characteristic tame cuts, but this length function is not as natural as the word length function. We do not know any finitely generated group without tame cuts with respect to the word length function. As far as we investigated, property (T_{Schur}, G, K) is the closest to be an obstruction for the existence of tame cuts.

Definition 3.5.1. Let G be a locally compact group, K a compact subgroup, H a closed subgroup of G , and ℓ a proper length function of G . For $n \in \mathbb{N}$ and $f \in C(G)$, define the quantity

$$\|f\|_{MA(H,G,\ell,n)} = \sup \{ \|\lambda_H(f|_H\varphi)\| : \varphi \in C_c(H), \text{supp}(\varphi) \subseteq B_n, \|\lambda_H(\varphi)\| \leq 1 \}.$$

When G and ℓ are already fixed, we also write $\|f\|_{MA(H,G,\ell,n)} = \|f\|_{MA(H,n)}$. We say that H has *property (T_{Schur}, G, K, ℓ)* if there exist a positive constant $s > 0$ and a function $\phi \in C_0(G)$ vanishing at infinity such that for any $D > 0$ and K -bi-invariant function $\varphi \in C(G)$ with the following condition

$$\|\varphi\|_{MA(H,n)} \leq De^{sn}, \quad (\forall n \in \mathbb{N}),$$

there exists a limit $\varphi_\infty \in \mathbb{C}$ to which φ tends uniformly rapidly

$$|\varphi(x) - \varphi_\infty| \leq D\phi(x), \quad (\forall x \in G).$$

Lemma 3.5.2. *Let G be an unbounded locally compact group endowed with a proper length function ℓ . Sup K and H are compact subgroups of G . Then H does not have property (T_{Schur}, G, K, ℓ) .*

Proof. Assume, by contradiction, that H has property (T_{Schur}, G, K, ℓ) and let $s > 0$ and $\phi \in C_0(G)$ be as in Definition 3.5.1. By Proposition 2.2.8, we can assume ℓ takes 0 on K so that the balls are K -bi-invariant. Choose a large enough $r \in \mathbb{N}$ such that B_r has a non-empty interior and contains H . For each $m \geq r$, construct the function f_m by applying $\mathcal{U} = B_{2m}^0$ and $K = B_m$ on Lemma 1.2.18. Then $\psi_m = f_{4m} - f_m$ is a non-negative compactly supported function in $A(G)$ such that $\psi_m|_H = 0$ and $\psi_m(x) \neq 0$ for some $x \in G$ with $\ell(x) \geq m$. We use $K \times K$ double averaging and normalization on ψ_m in order to have a K -bi-invariant function $\varphi_m \in C_c(G)$ such that $\varphi_m|_H = 0$ and $\varphi_m(x) = 1$ for some $x \in G$ with $\ell(x) \geq m$. Now, we have $\|\varphi_m\|_{MA(H,n)} = 0 \leq e^{sn}$ for all $n \in \mathbb{N}$ and $m \geq r$. It follows that $|\varphi_m(x)| \leq \phi(x)$ for all $x \in G$ and $m \geq r$. Taking $\lim_{x \rightarrow \infty} \sup_{m \geq r}$, we get a desired contradiction. \square

Definition 3.5.3. With the above notations, a function $\varphi \in C(H)$ is said *K -bi-invariant* if there is a K -bi-invariant continuous function on G whose restriction on H is exactly φ .

In [Lia16, Proposition 2.3], it was proved that if a discrete subgroup Γ of G has property (T_{Schur}, G, K, ℓ) , then (Γ, ℓ) does not have Rapid Decay property. We give its analogue for tame cuts in the following theorem.

Theorem 3.5.4. *Let H be closed subgroup of an unbounded locally compact group G . Suppose that H satisfies property (T_{Schur}, G, K, ℓ) for a compact subgroup K and a proper length function ℓ of G . Then $(H, \ell|_H)$ does not admit any K -bi-invariant tame cuts.*

Proof. We prove by contradiction. Assume that there exists K -bi-invariant tame cuts $(\varphi_m)_{m \in \mathbb{N}}$ for $(H, \ell|_H)$. There are constants $C, a \geq 0$ such that

$$\|\varphi_m\|_{MA(H)} \leq Cm^a \tag{3.8}$$

$$\varphi_m|_{B_m} \equiv 1 \tag{3.9}$$

for all $m \in \mathbb{N}$. Take any $f \in C_c(H)$ with $\text{supp}(f) \subseteq B_n$. If $m \geq n$, the multiplier M_{φ_m} acts trivially on f , so $\|\lambda_H(\varphi_m f)\| = \|\lambda_H(f)\|$. If $m < n$, then we have

$$\|\lambda_H(\varphi_m f)\| \leq C m^a \|\lambda_H(f)\| \leq C n^a \|\lambda_H(f)\|$$

by (3.8). Unifying these two cases, if we denote by $\varphi'_m \in C(G)$ a K -bi-invariant extension of φ_m , we get the inequality $\|\varphi'_m\|_{MA(H,n)} \leq C n^a$ for all $n, m \in \mathbb{N}$. Let $s > 0$ and $\phi \in C_0(G)$ be from property (T_{Schur}, G, K, ℓ) . Put $D = \sup_{n \in \mathbb{N}} C n^a e^{-sn}$ so that we get

$$\|\varphi'_m\|_{MA(H,n)} \leq D e^{sn}, \quad (\forall m, n \in \mathbb{N}).$$

By property (T_{Schur}, G, K, ℓ) , we get

$$|\varphi_m(x)| = |\varphi'_m(x)| \leq D \phi(x), \quad (\forall x \in H, \forall m \in \mathbb{N}). \quad (3.10)$$

Now, if we take the sequential limits $\lim_{x \rightarrow \infty} \lim_{m \rightarrow \infty}$ on (3.10), the left hand side goes to 1 whereas the right hand side goes to 0. This gives a desired contradiction. \square

Corollary 3.5.5. *If G is unbounded and has property (T_{Schur}, G, K, ℓ) , then (G, ℓ) does not have tame cuts.*

Corollary 3.5.6. *Suppose that G is a finitely generated infinite group and H is a finitely generated subgroup of G . Suppose that H is at most polynomially distorted in G . Recall that H is polynomially distorted in G there exists $k \geq 0$ such that $\ell_H(x) \leq k \ell_G(x)^k + k$ for all $x \in H$, where ℓ_G and ℓ_H are the word length functions of G and H , respectively. If H has property $(T_{Schur}, G, \{e\}, \ell_G)$, then (H, ℓ_H) does not have tame cuts.*

Let \mathbb{F}_q be a finite field of characteristic different from 2 with cardinality q . Let G be the symplectic group $Sp_4(\mathbb{F}_q((\pi)))$ over the local field $\mathbb{F}_q((\pi))$ and $K = Sp_4(\mathbb{F}_q[[\pi]])$ the maximal compact subgroup of G . Let Γ be the non-uniform lattice $Sp_4(\mathbb{F}_q[\pi^{-1}])$ in G . Let $H < \Gamma$ be the subgroup consisting of the elements of the

form

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \in \Gamma.$$

For $i, j \in \mathbb{N}_0$, denote $D(i, j) = \text{diag}(\pi^{-i}, \pi^{-j}, \pi^j, \pi^i)$. By Cartan's decomposition theorem, every element $x \in G$ can be written as $x = kD(i, j)k'$ for some $k, k' \in K$ and a unique $(i, j) \in \mathbb{N}_0^2$ with $i \geq j$. Moreover the length function $\ell : kD(i, j)k' \mapsto i + j$ is equivalent to the word length function of G , and even its restriction to the lattice Γ is equivalent to the word length function of Γ . In [Lia16, Theorem 3.1], it was proved that H and Γ have property (T_{Schur}, G, K, ℓ) , hence the following corollary.

Corollary 3.5.7. *The lattice $Sp_4(\mathbb{F}_q[\pi^{-1}])$ of $Sp_4(\mathbb{F}_q((\pi)))$ does not have $Sp_4(\mathbb{F}_q[[\pi]])$ -bi-invariant tame cuts.*

3.6 Application of tame cuts

In this section, we provide two applications of tame cuts. The first application is related to the norm decreasing induction map $\Phi : M_0A(\Gamma) \rightarrow M_0A(G)$, where Γ is a lattice of a locally compact group G . When G or Γ is amenable, we also have $\Phi : MA(\Gamma) \rightarrow MA(G)$ norm decreasing. The following result shows that the amenability is essential for the latter map to be continuous.

Theorem 3.6.1. *Let Γ be a lattice in $G = SL_3(\mathbb{R})$ with a compact fundamental domain Ω . Then the induction map*

$$\Phi : MA(\Gamma) \rightarrow MA(G), \quad \varphi \mapsto \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi \mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega$$

is not continuous (possibly not well defined).

Proof. Recall that the word length function ℓ_S of Γ , the restriction $\ell_C|_\Gamma$ of the word length function ℓ_C of G , and the restriction $L|_\Gamma$ of the length function $L : x \in G \mapsto \log \|x\| + \log \|x^{-1}\|$ are all equivalent on Γ (cf. Example 2.2.4). We choose L as the main length function of both Γ and G .

Let us prove by contradiction. Suppose that the map $\Phi : MA(\Gamma) \rightarrow MA(G)$ is bounded. By Theorem 2.2.13, Γ has the Rapid Decay property, and a fortiori characteristic tame cuts, so there is a finitely supported characteristic functions φ_n such that

$$\|\varphi_n\|_{MA(\Gamma)} \leq Cn^a \quad \text{and} \quad \varphi_n|_{B_n} \equiv 1 \quad (\forall n \in \mathbb{N})$$

for some constants $C, a \geq 0$. By continuity of Φ , there is a constant $C' > 0$ such that

$$\|\hat{\varphi}_n\|_{MA(G)} \leq C'n^a, \quad (\forall n \in \mathbb{N}).$$

By Theorem 1.2.22, the $K \times K$ double averaging $\dot{\hat{\varphi}}_n$ of $\hat{\varphi}_n$ is a K -bi-invariant Fourier multiplier of G with

$$\|\dot{\hat{\varphi}}_n\|_{MA(G)} \leq \|\hat{\varphi}_n\|_{MA(G)} \leq C'n^a, \quad (\forall n \in \mathbb{N}).$$

Applying the rigidity inequality of Theorem 2.1.11 on $\dot{\hat{\varphi}}_n$, we get

$$C''e^{t/3} \left| \dot{\hat{\varphi}}_n \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \right| \leq \|\dot{\hat{\varphi}}_n\|_{MA} \leq C'n^a, \quad (\forall n \in \mathbb{N}, \forall t \in \mathbb{R}_+). \quad (3.11)$$

Put $c = \max\{L(\omega) : \omega \in \Omega\}$ and choose $t = n/4 - c$ so that $L(\text{diag}(e^t, 1, e^{-t})) = 2t < n - 2c$. By construction, we have

$$\dot{\hat{\varphi}}_n(x) = \int_K \int_K \int_\Omega \varphi_n(\gamma(k_1 x k_2 \omega)) d\omega dk_1 dk_2.$$

Note that

$$L(\gamma(k_1 x k_2 \omega)) = L(\omega' k_1 x k_2 \omega) \leq L(k_1 x k_2) + L(\omega') + L(\omega) \leq 2c + L(x).$$

Thus, if $L(x) \leq n - 2c$, we have $\dot{\hat{\varphi}}_n(x) = 1$. Therefore, the left hand side of (3.11) grows exponentially while the right hand side grows polynomially, which gives a desired contradiction. \square

Remark 3.6.1. There is no difficulty to extend the theorem to uniform lattices of $SL_3(\mathbb{C})$, $SL_3(\mathbb{H})$, $SL_3(\mathbb{Q}_p)$, and $E_{6(-26)}$.

We mentioned in subsection 1.2.5 that all Fourier multipliers of an amenable group are completely bounded. The following theorem provides a non-amenable group admitting a non-completely bounded Fourier multiplier.

Theorem 3.6.2. *Let Γ be a uniform lattice in $G = SL_3(\mathbb{R})$. Then there is a Fourier multiplier of Γ which is not completely bounded. In other words, $M_0A(\Gamma) \neq MA(\Gamma)$.*

Proof. Suppose by contradiction $M_0A(\Gamma) = MA(\Gamma)$. Since the inclusion $M_0A(\Gamma) \rightarrow MA(\Gamma)$ is a contraction between two Banach spaces, the norms $\|\cdot\|_{MA}$ and $\|\cdot\|_{M_0A}$ are equivalent by the closed graph theorem applied to the inverse map.

Let us use the functions φ_n , $\hat{\varphi}_n$, and $\dot{\hat{\varphi}}_n$ from the proof of the previous theorem. By equivalence of two norms, we have

$$\|\varphi_n\|_{M_0A} \leq Cn^a, \quad (\forall n \in \mathbb{N})$$

for some constants $C, a \geq 0$. By Theorem 1.2.22, we have $\|\dot{\hat{\varphi}}_n\|_{M_0A} \leq \|\hat{\varphi}_n\|_{M_0A} \leq Cn^a$. Again, the rigidity inequality on $\dot{\hat{\varphi}}_n$ gives a desired contradiction for the same choice of t as in Theorem 3.6.1. \square

Remark 3.6.2. Proposition 3.6.2 is also a direct consequence of [HSS10, Boz82], where it is proved $M_0A(F_2) \neq MA(F_2)$. More precisely, by Tits alternative theorem, Γ contains a copy of F_2 , and the trivial extension gives isometric maps

$$MA(F_2) \rightarrow MA(\Gamma) \quad \text{and} \quad M_0A(F_2) \rightarrow M_0A(\Gamma),$$

so the trivial extension of any function $\varphi \in MA(F_2) \setminus M_0A(F_2)$ is in $MA(\Gamma) \setminus M_0A(\Gamma)$.

3.7 Summary

To summarize the examples:

- (1) Groups with completely bounded characteristic tame cuts:
 - (a) Polynomial growth discrete groups.
 - (b) Groups acting properly by isometries on a fine hyperbolic graph, on a product of geometrically finite hyperbolic graphs, or on a finite dimensional CAT(0) cube complex.
 - (c) $(\mathbb{Z}, \log(1 + |\cdot|))$.

- (d) Semidirect product $\mathbb{Z}^d \rtimes P$, where $P < SL_d(\mathbb{Z})$ is a finitely generated group with polynomial growth.
 - (e) Semidirect product $\mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$, where $p, q \in \mathbb{N}$ are coprime.
 - (f) Wreath product $F \wr P$, where F is a finite group and P is a group with polynomial growth. In particular, the Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.
 - (g) Baumslag-Solitar group $BS(p, q)$ for $p, q \in \mathbb{N}$.
- (2) Groups with completely bounded tame cuts:
- (a) Groups in (1).
 - (b) Weakly amenable groups.
- (3) Groups with characteristic tame cuts:
- (a) Groups in (1).
 - (b) Groups satisfying Rapid Decay property or RD_p for some $1 \leq p \leq \infty$.
- (4) Groups with tame cuts:
- (a) Groups in (1), (2), and (3).
 - (b) Groups admitting MA -bounded approximation property.
- (5) Groups without tame cuts.
- (a) Simple Lie groups with finite center and rank at least 2.
- (6) Groups without completely bounded tame cuts:
- (a) Groups in (5) and their uniform lattices.
- (7) Groups without (completely bounded) characteristic tame cuts:
- (a) $(\mathbb{Z}, \log(1 + \log(1 + |\cdot|)))$.

Bibliography

- [BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.
- [BF84] Marek Bożejko and Gero Fendler. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital. A (6)*, 3(2):297–302, 1984.
- [Boy13] Adrien Boyer. Semisimple Lie groups satisfy property RD, a short proof. *C. R. Math. Acad. Sci. Paris*, 351(9-10):335–338, 2013.
- [Bož85] Marek Bożejko. Positive definite bounded matrices and a characterization of amenable groups. *Proc. Amer. Math. Soc.*, 95(3):357–360, 1985.
- [Bož82] Marek Bożejko. Remark on Herz-Schur multipliers on free groups. *Math. Ann.*, 258(1):11–15, 1981/82.
- [BP93] Marek Bożejko and Massimo A. Picardello. Weakly amenable groups and amalgamated products. *Proc. Amer. Math. Soc.*, 117(4):1039–1046, 1993.
- [CGK01] John D. Clemens, Su Gao, and Alexander S. Kechris. Polish metric spaces: their classification and isometry groups. *Bull. Symbolic Logic*, 7(3):361–375, 2001.
- [CH89] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.
- [Cha03] Indira Chatterji. Property (RD) for cocompact lattices in a finite product of rank one Lie groups with some rank two Lie groups. *Geom. Dedicata*, 96:161–177, 2003.
- [Cha17] Indira Chatterji. Introduction to the rapid decay property. In *Around Langlands correspondences*, volume 691 of *Contemp. Math.*, pages 53–72. Amer. Math. Soc., Providence, RI, 2017.

- [Con76] A. Connes. Classification of injective factors. Cases II_1 , II_∞ , III_λ , $\lambda \neq 1$. *Ann. of Math. (2)*, 104(1):73–115, 1976.
- [CPSC07] I. Chatterji, C. Pittet, and L. Saloff-Coste. Connected Lie groups and property RD. *Duke Math. J.*, 137(3):511–536, 2007.
- [Day50] Mahlon M. Day. Means for the bounded functions and ergodicity of the bounded representations of semi-groups. *Trans. Amer. Math. Soc.*, 69:276–291, 1950.
- [DCH85] Jean De Cannière and Uffe Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.*, 107(2):455–500, 1985.
- [Dix50] Jacques Dixmier. Les moyennes invariantes dans les semi-groupes et leurs applications. *Acta Sci. Math. (Szeged)*, 12:213–227, 1950.
- [dL13] Tim de Laat. Approximation properties for noncommutative L^p -spaces associated with lattices in Lie groups. *J. Funct. Anal.*, 264(10):2300–2322, 2013.
- [dLdS18] Tim de Laat and Mikael de la Salle. Approximation properties for non-commutative L^p -spaces of high rank lattices and nonembeddability of expanders. *J. Reine Angew. Math.*, 737:49–69, 2018.
- [dlH00] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [Eym64] Pierre Eymard. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.
- [Fen02] Gero Fendler. Weak amenability of coxeter groups. *arXiv preprint math/0203052*, 2002.
- [Gar15] Łukasz Garncarek. Property of rapid decay for extensions of compactly generated groups. *Publ. Mat.*, 59(2):301–312, 2015.
- [GH10] Erik Guentner and Nigel Higson. Weak amenability of CAT(0)-cubical groups. *Geom. Dedicata*, 148:137–156, 2010.
- [Gil74] John E. Gilbert. L^p -convolution operators and tensor products of Banach spaces. *Bull. Amer. Math. Soc.*, 80:1127–1132, 1974.
- [GJ03] Światosław R. Gal and Tadeusz Januszkiewicz. New a-T-menable HNN-extensions. *J. Lie Theory*, 13(2):383–385, 2003.
- [Haa16] Uffe Haagerup. Group C^* -algebras without the completely bounded approximation property. *J. Lie Theory*, 26(3):861–887, 2016.

- [Haa79] Uffe Haagerup. An example of a nonnuclear C^* -algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.
- [HSS10] U. Haagerup, T. Steenstrup, and R. Szwarc. Schur multipliers and spherical functions on homogeneous trees. *Internat. J. Math.*, 21(10):1337–1382, 2010.
- [Hul64] A. Hulanicki. Groups whose regular representation weakly contains all unitary representations. *Studia Math.*, 24:37–59, 1964.
- [Jol90] Paul Jolissaint. Rapidly decreasing functions in reduced C^* -algebras of groups. *Trans. Amer. Math. Soc.*, 317(1):167–196, 1990.
- [Jol92] Paul Jolissaint. A characterization of completely bounded multipliers of Fourier algebras. *Colloq. Math.*, 63(2):311–313, 1992.
- [JV91] P. Jolissaint and A. Valette. Normes de Sobolev et convoluteurs bornés sur $L^2(G)$. *Ann. Inst. Fourier (Grenoble)*, 41(4):797–822, 1991.
- [Kol23] Andrey Kolmogoroff. Une série de fourier-lebesgue divergente presque partout. *Fundamenta Mathematicae*, 4(1):324–328, 1923.
- [Laf98] Vincent Lafforgue. Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p -adique et pour certains groupes discrets possédant la propriété (T). *C. R. Acad. Sci. Paris Sér. I Math.*, 327(5):439–444, 1998.
- [Laf00] Vincent Lafforgue. A proof of property (RD) for cocompact lattices of $SL(3, \mathbf{R})$ and $SL(3, \mathbf{C})$. *J. Lie Theory*, 10(2):255–267, 2000.
- [LDis11] Vincent Lafforgue and Mikael De la Salle. Noncommutative L^p -spaces without the completely bounded approximation property. *Duke Math. J.*, 160(1):71–116, 2011.
- [Lia16] Benben Liao. About the obstacle to proving the Baum-Connes conjecture without coefficient for a non-cocompact lattice in Sp_4 in a local field. *J. Noncommut. Geom.*, 10(4):1243–1268, 2016.
- [LMR93] Alexander Lubotzky, Shahar Mozes, and M. S. Raghunathan. Cyclic subgroups of exponential growth and metrics on discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(8):735–740, 1993.
- [LMR00] Alexander Lubotzky, Shahar Mozes, and M. S. Raghunathan. The word and Riemannian metrics on lattices of semisimple groups. *Inst. Hautes Études Sci. Publ. Math.*, (91):5–53 (2001), 2000.
- [Los84] Viktor Losert. Properties of the Fourier algebra that are equivalent to amenability. *Proc. Amer. Math. Soc.*, 92(3):347–354, 1984.

- [Miz08] Naokazu Mizuta. A Bożejko-Picardello type inequality for finite-dimensional CAT(0) cube complexes. *J. Funct. Anal.*, 254(3):760–772, 2008.
- [MPS81] O. Carruth McGehee, Louis Pigno, and Brent Smith. Hardy’s inequality and the L^1 norm of exponential sums. *Ann. of Math. (2)*, 113(3):613–618, 1981.
- [Neb82] Claudio Nebbia. Multipliers and asymptotic behaviour of the Fourier algebra of nonamenable groups. *Proc. Amer. Math. Soc.*, 84(4):549–554, 1982.
- [Neu29] J. Neumann. Zur allgemeinen theorie des masses. *Fundamenta Mathematicae*, 13(1):73–116, 1929.
- [Oza08] Narutaka Ozawa. Weak amenability of hyperbolic groups. *Groups Geom. Dyn.*, 2(2):271–280, 2008.
- [Oza12] Narutaka Ozawa. Examples of groups which are not weakly amenable. *Kyoto J. Math.*, 52(2):333–344, 2012.
- [Pis01] Gilles Pisier. *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 2001. Includes the solution to “The Halmos problem”.
- [PS86] T. Pytlik and R. Szwarc. An analytic family of uniformly bounded representations of free groups. *Acta Math.*, 157(3-4):287–309, 1986.
- [Roe03] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.
- [RRS98] J. Ramagge, G. Robertson, and T. Steger. A Haagerup inequality for $\tilde{A}_1 \times \tilde{A}_1$ and \tilde{A}_2 buildings. *Geom. Funct. Anal.*, 8(4):702–731, 1998.
- [Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [Sak71] Shôichirô Sakai. *C^* -algebras and W^* -algebras*. Springer-Verlag, New York-Heidelberg, 1971. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 60.
- [Sak09] Hiroki Sako. The class S as an ME invariant. *Int. Math. Res. Not. IMRN*, (15):2749–2759, 2009.
- [Sch93] Larry B. Schweitzer. Dense m -convex Fréchet subalgebras of operator algebra crossed products by Lie groups. *Internat. J. Math.*, 4(4):601–673, 1993.

- [Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
- [Szw91] Ryszard Szwarc. Groups acting on trees and approximation properties of the Fourier algebra. *J. Funct. Anal.*, 95(2):320–343, 1991.
- [Val02] Alain Valette. *Introduction to the Baum-Connes conjecture*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.
- [Ver19] Ignacio Vergara. Radial Schur multipliers on some generalisations of trees. *Studia Math.*, 249(1):59–109, 2019.
- [vN30] J. v. Neumann. Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math. Ann.*, 102(1):370–427, 1930.
- [Š55] A. S. Švarc. A volume invariant of coverings. *Dokl. Akad. Nauk SSSR (N.S.)*, 105:32–34, 1955.
- [Zyg02] A. Zygmund. *Trigonometric series. Vol. I, II*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2002. With a foreword by Robert A. Fefferman.