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Zhixin Xie

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THÈSE DE DOCTORAT

Géométrie des diviseurs anticanoniques pour certaines variétés
rationnellement connexes de petite dimension

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Géométrie des diviseurs anticanoniques pour certaines variétés rationnellement connexes de petite dimension

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Résumé

Cette thèse est consacrée à l'étude de la géométrie des variétés complexes projectives à fibré anticanonique positif. Un des objectifs principaux est la classification de ces variétés par l'analyse de leur système anticanonique et de leur structure géométrique.

La première partie a pour but de classifier les variétés à fibré anticanonique nef, qui constituent une généralisation naturelle des variétés de Fano. La classification de ces variétés est plus compliquée car des nouveaux phénomènes se produisent et de nombreux résultats pour les variétés de Fano ne restent plus valables dans ce cas. On considère les variétés rationnellement connexes de dimension 3 à fibré anticanonique nef. Plus précisément, on se concentre sur le cas difficile : celui où le fibré anticanonique est nef, mais pas semiample. Nous établissons les résultats suivants :

- Une classification complète dans le cas où le système anticanonique n'a pas de diviseur fixe.
- Une description géométrique dans le cas où le système anticanonique a un diviseur fixe : après une suite finie de flops, la variété admet une fibration au-dessus de \mathbb{P}^1 . Selon la géométrie de l'unique courbe dans le système anticanonique de la fibre générale, on obtient une classification complète dans certains cas.

Dans la deuxième partie, nous nous intéressons aux espaces de modules Y de faisceaux semistables, sans torsion et de rang 2 sur une surface polarisée $(S, -K_S)$ de del Pezzo de degré 1, tels que $c_1 = -K_S$, $c_2 = 2$. En effet, ces espaces sont des exemples de variétés de Fano de dimension 4, et possèdent des propriétés géométriques remarquables. L'étude de cette famille de variétés est motivée par le fait que les variétés de Fano de dimension 4 ne sont pas classifiées, et qu'en dehors des produits de surfaces de del Pezzo, peu d'exemples de variétés de Fano de dimension 4 de grand nombre de Picard sont connus. Nous décrivons :

- Le schéma de base du système linéaire anticanonique $|-K_Y|$.
- Une description de l'action de l'involution de Bertini sur Y restreinte à une surface particulière incluse dans Y .

Mots-clés Variétés de Fano, positivité du fibré anticanonique, théorie de Mori, système anticanonique, espaces projectifs, surfaces de del Pezzo, involution de Bertini

Abstract

This thesis is devoted to the study of the geometry of complex projective manifolds with positive anticanonical bundle. One of the main goals is the classification of these manifolds via an analysis of their anticanonical systems and their geometric structure.

The first part of the thesis aims to classify a class of manifolds which appears as a natural generalisation of Fano manifolds, namely manifolds with nef anticanonical bundle. The classification of this class of manifolds is more complicated, as new phenomena arise and many results for Fano varieties no longer hold in this case. We focus on rationally connected threefolds with nef anticanonical divisor, and more precisely on the delicate case where the anticanonical divisor is not semi-ample. Our main results are the following:

- A complete classification when the anticanonical system has no fixed divisor.
- A geometric description when the anticanonical system has a fixed divisor: after a finite sequence of flops, the threefold admits a fibration over \mathbb{P}^1 . Certain cases are classified according to the geometry of the anticanonical divisor of the general fibre.

The second part of the thesis addresses a family of Fano fourfolds with remarkable geometric properties, namely the moduli spaces Y of semi-stable rank-two torsion-free sheaves with $c_1 = -K_S$, $c_2 = 2$ on a polarised degree-one del Pezzo surface $(S, -K_S)$. As Fano manifolds have been classified only up to dimension three, and few examples of Fano fourfolds with large Picard number which are not products of del Pezzo surfaces are known, the geometry of such an example is worth studying. Below are the main results of this part:

- The base scheme of the anticanonical system $| -K_Y |$.
- A description of the action of the Bertini involution of Y on a special surface in Y .

Keywords Fano varieties, positivity of anticanonical bundle, Mori theory, anticanonical system, projective spaces, del Pezzo surfaces, Bertini involution

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Résumé

L'objet d'étude principal de cette thèse est les variétés complexes projectives rationnellement connexes à fibré anticanonique positif, en particulier, les variétés de Fano et plus généralement les variétés à fibré anticanonique nef.

Les variétés de Fano apparaissent comme une composante élémentaire dans la classification birationnelle des variétés : pour les variétés de dimension de Kodaira négative, on s'attend, à la fin du programme du modèle minimal (MMP), à un morphisme de type fibré dont les fibres sont des variétés de Fano (singulières).

Notons ici que la notion de connexité rationnelle joue un rôle important. Les variétés de Fano sont rationnellement connexes, tandis qu'en général, une variété à fibré anticanonique nef n'est pas nécessairement rationnellement connexe. Cependant, au vu du théorème de décomposition pour les variétés complexes projectives à fibré anticanonique nef, démontré par Cao et Höring dans [CH19], il est essentiel d'étudier les variétés à fibré anticanonique nef rationnellement connexes.

Les questions abordées s'incrivent dans le contexte du problème de la classification des variétés projectives. Les variétés de Fano de dimension au plus 3 sont complètement déterminées : en dimension 1, c'est la droite projective ; en dimension 2, ce sont les surfaces de del Pezzo (c'est-à-dire $\mathbb{P}^1 \times \mathbb{P}^1$ et les éclatements de \mathbb{P}^2 en au plus 8 points en position générale, soit 10 familles) ; en dimension 3, elles se répartissent en 105 familles, grâce aux travaux de Mori-Mukai (cf. [MM82]) et de Iskovskikh (cf. [Isk77] [Isk78]). Cependant, pour les classes de variétés plus générales, par exemple les variétés de dimension 3 à fibré anticanonique nef ou les variétés de Fano de dimension 4, on est loin d'obtenir une liste complète de familles de variétés.

Avant d'aborder le problème de la classification des variétés projectives rationnellement connexes à fibré anticanonique positif, il faudrait vérifier qu'il y a un nombre fini de possibilités dans la liste de classification. Bien que notre classification ne concerne que des variétés lisses, la discussion sur les singularités intervient naturellement dans les preuves lorsque l'on fait tourner le MMP. Par conséquent, nous énonçons au paragraphe suivant les résultats de finitude dans le cas singulier.

On dit qu'un ensemble χ de variétés de même dimension forme une famille bornée s'il existe des schémas de type fini \mathbb{X}, \mathbb{B} , et un morphisme $\phi: \mathbb{X} \rightarrow \mathbb{B}$ tel que toute fibre géométrique de ϕ soit une variété dans χ , et que toute variété $X \in \chi$ soit une fibre géométrique de ϕ . Suite à des avancées récentes en géométrie birationnelles (notamment dues à Birkar, Cascini, Hacon, McKernan et Shokurov), Birkar a confirmé la conjecture de Borisov-Alexeev-Borisov (BAB) dans [Bir21] : les variétés de Fano à singularités ϵ -lc forment une famille bornée, à dimension et ϵ donnés. McKernan

et Prokhorov ont énoncé une conjecture de BAB généralisée ([MP04]) : les variétés projectives normales rationnellement connexes à fibré anticanonique nef et à singularités ϵ -lc forment une famille bornée, à dimension et ϵ donnés. Notons encore une fois que l’hypothèse de la connexité rationnelle est indispensable ici. Par exemple, les surfaces K3 projectives à singularités canoniques ne forment pas une famille bornée ; elles satisfont les hypothèses de la conjecture en dimension 2 pour $\epsilon = 1$, sauf qu’elles ne sont pas rationnellement connexes.

Revenons au problème de la classification. L’approche commune pour mieux comprendre la géométrie d’une variété à fibré anticanonique positif consiste à étudier son système linéaire anticanonique, qui est l’un des premiers objets intrinsèquement attachés à une telle variété.

Dans toute cette thèse, on travaille sur le corps des complexes.

Variétés rationnellement connexes de fibré anticanonique nef

La classification des surfaces projectives lisses à fibré anticanonique nef, non big et non numériquement trivial est une conséquence de la classification de Kodaira-Enriques (*cf.* [BP04, p. 318] ou section 2.4). En particulier, si la surface S est rationnelle (la connexité rationnelle est équivalente à la rationalité pour les variétés propres et lisses de dimension au plus 2), alors elle est obtenue par éclatement de \mathbb{P}^2 en 9 points. De plus, si la position des 9 points éclatés est suffisamment générale, alors $-K_S$ est nef et non semiample. Dans ce cas, la géométrie du système anticanonique $| -K_S |$ pourrait déjà être pathologique : l’unique élément dans $| -K_S |$ peut être réductible et non réduit. En revanche, pour une surface de del Pezzo de degré 1 (c’est-à-dire l’éclatement de \mathbb{P}^2 en 8 points en position générale), tout élément du système anticanonique est irréductible et réduit, et un élément général est une courbe elliptique lisse.

En dimension supérieure, de nombreux efforts ont été faits pour étudier la finitude de la famille de variétés projectives rationnellement connexes à fibré anticanonique nef. Récemment, Birkar, Di Cerbo et Svaldi ont montré ([BCS20, Theorem 1.6]) une version plus faible de la conjecture de BAB généralisée en dimension 3. En particulier, leur résultat nous dit que les variétés projectives lisses rationnellement connexes de dimension 3 à fibré anticanonique nef forment une famille bornée à flops près. Par conséquent, il est en principe possible de classifier cette classe de variétés, comme l’ont fait par Mori et Mukai pour les variétés de Fano de dimension 3 à l’aide du MMP.

Soit X une variété projective lisse rationnellement connexe de dimension 3 à fibré anticanonique $-K_X$ nef. Si $-K_X$ est semiample, on trouve une classification partielle dans [BP04, sections 5 et 6]. On s’intéresse ainsi au cas plus délicat où le fibré anticanonique est nef et non semiample. Bauer et Peternell ont montré que la non semiampleté de $-K_X$ implique que sa dimension nef (*cf.* définition 2.4.2) $n(-K_X)$ est 3 et que sa dimension numérique (*cf.* notation 2.4.1) $\nu(-K_X)$ est 2. Cela est dû au résultat suivant :

Théorème ([BP04], Theorem 2.1). *Soit X une variété projective lisse rationnellement connexe de dimension 3 à fibré anticanonique $-K_X$ nef. Alors sa dimension d’Itaka vérifie :*

$$\kappa(-K_X) \geq 1.$$

Si $n(-K_X) = 1$ ou 2, alors $-K_X$ est semiample et l’application de réduction nef associée à $-K_X$

peut être prise comme (la factorisation de Stein de) l'application définie par un certain multiple globalement engendré de $-K_X$.

Esquissons la preuve de Bauer-Peternell pour montrer que $\kappa(-K_X) \geq 1$: dans le cas où $\nu(-K_X) = 2$, cela est une conséquence directe du théorème d'annulation de Kawamata-Viehweg et de la formule de Riemann-Roch ; dans le cas où $\nu(-K_X) = 1$, la preuve est plus compliquée et nécessite une discussion sur les contractions de Mori possibles de X .

Au vu de la non semiampleté du diviseur anticanonique, une première approche est d'étudier le lieu de base du système anticanonique. Commençons par le cas où le système anticanonique n'admet pas de diviseur fixe (cf. chapitre 3). On obtient dans ce cas une classification complète.

Théorème (= Theorem 3.0.1). *Soit X une variété projective lisse rationnellement connexe de dimension 3, dont le fibré anticanonique est nef avec $n(-K_X) = 3$ et $\nu(-K_X) = 2$. Supposons que le système anticanonique n'a pas de diviseur fixe. Alors $-K_X$ est divisible par 2 dans $\text{Pic}(X)$, et on est dans l'un des cas suivants :*

- (1) $X \rightarrow \mathbb{P}^1$ est une fibration de del Pezzo à fibre générale isomorphe à $\mathbb{P}^1 \times \mathbb{P}^1$. Alors X est un élément du système linéaire $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$, où $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, et F est une fibre générale de $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$.
- (2) $X = \mathbb{P}(\mathcal{E})$ est un fibré en \mathbb{P}^1 sur une surface rationnelle lisse Y avec $-K_Y$ nef, où \mathcal{E} est un fibré vectoriel sur Y de rang 2 avec $c_1(\mathcal{E}) = -K_Y$, $c_2(\mathcal{E}) = K_Y^2$ et donné par l'extension suivante :

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \mathcal{J}_Z \otimes \mathcal{O}_Y(-K_Y) \rightarrow 0,$$

où \mathcal{J}_Z est le faisceau d'idéaux d'un sous-schéma Z de Y de longueur $c_2(\mathcal{E})$.

- (3) $X = \text{Bl}_p(Y)$ est l'éclatement d'une variété Y de dimension 3 presque de del Pezzo de degré 1 (c'est-à-dire $-K_Y \sim 2H_Y$, où H_Y est un diviseur premier sur Y qui est nef et vérifie $H_Y^3 = 1$) en un point p qui n'est pas l'unique point de base du système linéaire $|H_Y|$.

Réciproquement, soit X une variété qui apparaît dans l'un des trois cas ci-dessus, satisfaisant respectivement l'une des conditions suivantes :

- (1') l'élément X de $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$ est très général.
- (2') les points de l'ensemble fini Z sont en position suffisamment générale sur Y .
- (3') le point éclaté $p \in Y$ est général.

Alors $-K_X$ est nef, non semiample et divisible par 2 dans $\text{Pic}(X)$.

Considérons ensuite le cas où le système anticanonique admet un diviseur fixe (cf. chapitre 4). On trouve alors une description géométrique de la variété :

Théorème (= Theorem 4.0.1). *Soit X une variété projective lisse rationnellement connexe de dimension 3, dont le fibré anticanonique est nef et vérifie $n(-K_X) = 3$ et $\nu(-K_X) = 2$. Supposons que le système anticanonique a un diviseur fixe non nul. Alors il existe une suite finie de flops $\psi: X \dashrightarrow X'$ telle que :*

- X' est lisse,
- $-K_{X'}$ est nef,
- la partie mobile $|B'|$ de $|-K_{X'}|$ est nef.

Dans ce cas, on a $B'^2 = 0$, de sorte que $|B'|$ est sans point base. Cette partie mobile induit une fibration $f: X' \rightarrow \mathbb{P}^1$. De plus, la fibre générale F de f est une surface lisse avec $-K_F$ nef, $\nu(-K_F) = 1$ et $n(-K_F) = 2$.

Revenons au problème de la finitude birationnelle de la famille de variétés projectives lisses rationnellement connexes de dimension 3 à fibré anticanonique nef. Dans le cas où le diviseur fixe du système anticanonique d'une telle variété est non nul, la finitude ne se déduit pas directement du théorème ci-dessus. Cependant, on s'attend à une classification complète dans ce cas comme dans le premier.

Notons que, dans le théorème ci-dessus, la structure géométrique de la fibration $f: X' \rightarrow \mathbb{P}^1$ pourrait être compliquée (X' n'est pas nécessairement un produit). Pour étudier la structure géométrique de cette fibration, on considèrera la configuration suivante dans la suite de cette section et on notera désormais X au lieu de X' .

Hypothèses (= Setup 4.2.5). *Soit X une variété projective lisse rationnellement connexe de dimension 3, dont le fibré anticanonique est nef avec $n(-K_X) = 3$ et $\nu(-K_X) = 2$. Supposons que le système anticanonique a un diviseur fixe non nul et que sa partie mobile est nef, de sorte que cette dernière est sans point base et qu'elle induit une fibration $f: X \rightarrow \mathbb{P}^1$. Dans ce cas, la fibre générale F de f est une surface lisse avec $-K_F$ nef, $\nu(-K_F) = 1$, $n(-K_F) = 2$, et $|-K_X| = A + |kF|$ avec $k \geq 2$.*

Traisons d'abord le cas où la fibre générale F est rationnelle (cf. chapitres 5 et 6). Énonçons explicitement le cas particulier suivant, qui est complètement classifié dans le chapitre 5.

Proposition (= Proposition 5.1.1). *Sous les hypothèses ci-dessus, supposons de plus que F est rationnelle et que l'unique élément $B \in |-K_X|$ est une courbe elliptique lisse. Alors $k = 2$, $A \simeq B \times \mathbb{P}^1$, la restriction de f sur A est la seconde projection, et on est dans l'un des cas suivants :*

- (1) A est nef. Le diviseur anticanonique relatif $-K_{X/\mathbb{P}^1}$ est nef, et la fibration f est localement triviale.
- (2) A est le diviseur exceptionnel d'une contraction de Mori. La variété X est obtenue comme l'éclatement $\varphi: X \rightarrow X'$ le long d'une courbe elliptique lisse R sur une variété lisse X' de dimension 3 avec $-K_{X'}$ nef, $n(-K_{X'}) = 3$ et $\nu(-K_{X'}) = 2$. De plus, le système anticanonique $|-K_{X'}| = |2G|$ avec $G := \varphi(F) \simeq F$ n'a pas de diviseur fixe et $\text{Bs}|G| = R$.
- (3) A est le diviseur exceptionnel d'une contraction crépante extrême. Il existe une contraction de Mori birationnelle $\varphi: X \rightarrow X'$ telle que la fibration f se factorise comme $f = f' \circ \varphi$ avec $f': X' \rightarrow \mathbb{P}^1$ et on est dans l'un des cas suivants :

- (i) X' satisfait les mêmes hypothèses sur X : $-K_{X'}$ est nef tel que $|-K_{X'}| = A' + |2F'|$ avec $A' := \varphi(A) \simeq A$, $F' := \varphi(F) \simeq F$ et A' est le diviseur exceptionnel d'une contraction de Mori. Dans ce cas, φ est l'éclatement de X' le long d'une courbe elliptique lisse contenue dans une fibre de $f'|_{A'}$.
- (ii) $-K_{X'}$ est nef et big tel que $|-K_{X'}| = |A' + 2F'|$, où $A' := \varphi(A) \simeq A$, et la restriction $\varphi|_F: F \rightarrow F' := \varphi(F)$ est une contraction de certaines (-1) -courbes dans F de sorte que $-K_{F'}$ est nef et big. Dans ce cas, φ est l'éclatement de X' le long d'une courbe lisse contenue dans A' que f' envoie surjectivement sur \mathbb{P}^1 .

En outre, chaque cas dans la proposition ci-dessus existe (cf. section 5.2).

On donne ici un exemple qui n'est pas un produit. Prenons $X_0 := \mathbb{P}^1 \times S$, où S est l'éclatement de \mathbb{P}^2 en 9 points en position suffisamment générale, de sorte que $-K_S$ est nef et non semiample, et que l'unique élément $B \in |-K_S|$ est une courbe elliptique lisse. Soient $f_0: X_0 \rightarrow \mathbb{P}^1$ la première projection de fibre générale $F_0 \simeq S$ et $\pi_S: X_0 \rightarrow S$ la seconde projection. Alors $|-K_{X_0}| = A_0 + |2F_0|$, avec $A_0 := \pi_S^*(B)$ isomorphe à $\mathbb{P}^1 \times B$, et $-K_{X_0}$ est nef et non semiample. On fixe deux points $p_1, p_2 \in \mathbb{P}^1$. Soit $\mu: X := \text{Bl}_{p_1 \times B, p_2 \times B} X_0 \rightarrow X_0$ l'éclatement de X_0 le long de deux fibres elliptiques de A_0 . Alors,

- $-K_X$ est nef et non semiample,
- $|-K_X| = A + |2F|$, où A est la transformée stricte de A_0 et F est le transformé stricte de F_0 ,
- $A \simeq B \times \mathbb{P}^1$ est le diviseur exceptionnel d'une contraction crépante extrême, et $F \simeq S$.

En particulier, X est un exemple du cas (3)(i) de la proposition ci-dessus. On a $\mu^* \pi_S^*(B) = A + E_1 + E_2$, où E_1 et E_2 sont les deux diviseurs exceptionnels au-dessus des deux fibres elliptiques éclatées sur A_0 . Alors $A|_A = -E_1|_A - E_2|_A$. Soit $f := f_0 \circ \mu: X \rightarrow \mathbb{P}^1$, alors f admet exactement deux fibres réductibles. Donc X n'est pas un produit.

Néanmoins, la variété a dans certains cas une structure géométrique très particulière. On analyse dans le chapitre 6 la situation plus générale où l'unique élément dans $|-K_F|$ n'est pas lisse.

Proposition (= Theorem 6.2.1). *Sous les hypothèses ci-dessus, supposons de plus que F soit rationnelle et que l'unique élément dans $|-K_F|$ soit une courbe cubique cuspidale. Alors A est nef, et la fibration $f: X \rightarrow \mathbb{P}^1$ est localement triviale.*

On conclut ce chapitre par la conjecture suivante :

Conjecture (= Conjecture 6.4.6). *Sous les hypothèses ci-dessus, supposons de plus que F est rationnelle et que l'unique élément dans $|-K_F|$ est réduit. Alors $|-K_X| = A + |2F|$, et A n'a pas de composante f -verticale.*

En particulier, la conjecture sera démontrée dans le cas où la paire (X, A) est log canonique au-dessus du point générique de \mathbb{P}^1 (cf. chapitre 6), et on évoquera certaines pistes de recherche pour les autres cas dans le même chapitre.

Le dernier chapitre de la première partie est consacré au cas où la surface F est non rationnelle. Dans ce cas, f ne peut pas être un produit puisque X est rationnellement connexe. On montrera que cette situation est très particulière (*cf.* proposition 7.1.3) et que ce cas existe (*cf.* exemple 7.1.4).

Modèle de Fano de l'éclatement de \mathbb{P}^4 en 8 points généraux

En dimension 4, les travaux de Casagrande (notamment [Cas12, Cas13]) donnent de nombreux résultats sur la classification de certaines variétés de Fano, selon leur nombre de Picard et leur défaut de Lefschetz.

La seconde partie de cette thèse est consacrée à l'étude d'une famille de variétés de Fano de dimension 4 dont la géométrie est très riche, à savoir les espaces de modules Y de faisceaux semistables, sans torsion et de rang 2 sur une surface polarisée $(S, -K_S)$ de del Pezzo de degré 1, tels que $c_1 = -K_S$, $c_2 = 2$. Le nombre de Picard d'un tel espace de modules Y est 9. L'étude de cette famille de variétés est motivée par le fait que peu d'exemples de variétés de Fano de dimension 4 de grand nombre de Picard (par exemple au moins 7) sont connus, en dehors des produits des surfaces de del Pezzo. Comme indiqué dans l'article [CCF19], cette famille d'espaces de modules Y est le seul exemple connu de variété de Fano de dimension 4, qui a comme nombre de Picard au moins 9 et qui n'est pas un produit.

Pour étudier la géométrie de Y , Casagrande, Codogni et Fanelli établissent, dans l'article [CCF19], le lien explicite entre la variété de Fano Y et l'éclatement X de 8 points sur \mathbb{P}^4 : la variété Y est obtenue à partir de X en flippant les transformées strictes des droites passant par 2 points éclatés et des courbes quartiques normales passant par 7 points éclatés dans \mathbb{P}^4 . Grâce à ce lien explicite, ils montrent dans le même article que le lieu de base de $|-K_Y|$ contient la transformée stricte R_Y d'une courbe quintique rationnelle lisse passant par les 8 points éclatés dans \mathbb{P}^4 , et que le système bianticanonique $|-2K_Y|$ est sans point base. À l'aide de leurs résultats et d'une analyse plus fine du système anticanonique $|-K_Y|$, on détermine le schéma de base de ce dernier :

Proposition (=Proposition 10.1.12). *Le schéma de base de $|-K_Y|$ est la courbe lisse R_Y .*

On obtient comme corollaire :

Corollaire (=Corollary 10.1.13). *Soit $D \in |-K_Y|$ un diviseur général. Alors D est lisse.*

Par ailleurs, l'analyse du système anticanonique de X est essentielle pour comprendre un automorphisme particulier de la variété Y , à savoir l'involution de Bertini ι_Y . Cette involution de Bertini ι_Y est induite par l'involution classique sur la surface de del Pezzo S de degré 1, qui est définie comme suit : rappelons ici que $|-2K_S|$ est sans point base et définit un morphisme $\phi: S \rightarrow Q \subset \mathbb{P}^3$ de degré 2, dont l'image est un cône quadrique. L'involution de Bertini ι_S provient alors du revêtement double $S \rightarrow Q$.

Pour comprendre l'action de ι_Y , on la décrit sur une surface particulière W_Y incluse dans Y . Cette surface W_Y est la transformée stricte de la surface cubique réglée engendrée par le pinceau des courbes elliptiques quintiques passant par les 8 points éclatés dans \mathbb{P}^4 . Par analogie entre S

et Y , on étudie la restriction à W_Y du morphisme défini par le système bianticanonique de Y , et on obtient la description suivante :

Proposition (=Proposition 10.2.1). *L'involution de Bertini ι_Y préserve la surface W_Y , et la restriction $\iota_Y|_{W_Y}$ à W_Y est une involution birégulière définie par le revêtement double*

$$\phi_{|-2K_Y|_{W_Y}} : W_Y \rightarrow V_{2,4} \subset \mathbb{P}^7,$$

où $V_{2,4} \simeq \mathbb{F}_2$ est une surface réglée normale rationnelle de bidegré $(2,4)$. En particulier, l'involution de Bertini ι_Y est l'identité sur la courbe R_Y et induit une involution sur chaque fibre elliptique F_Y de $W_Y \rightarrow \mathbb{P}^1$.

De plus, il existe une courbe lisse $R' \sim 3(R_Y + F_Y)$ de genre 4 sur la surface W_Y , telle que R' est disjointe de R_Y et contenue dans le lieu fixe de ι_Y .

Dans la suite de ce travail en cours, il est envisagé de décrire complètement le lieu fixe de l'involution ι_Y , ainsi que son action sur le système anticanonique $|-K_Y|$ et le système bianticanonique $|-2K_Y|$.

Part I

Rationally connected threefolds with nef anticanonical divisor

Chapter 1

Introduction

1.1 Complex projective manifolds with nef anticanonical divisor

Let X be a complex projective manifold. We say that X is a Fano manifold if the anticanonical divisor $-K_X$ is ample. The classification of three-dimensional Fano manifolds by Mori and Mukai (for $\rho > 1$, [MM82]) and by Iskovskikh (for $\rho = 1$, [Isk77] [Isk78]) is one of the first achievements of the Minimal Model Program (MMP) with an impressive number of applications. Projective manifolds with nef anticanonical divisor $-K_X$ are a natural generalisation of Fano manifolds, and one hopes to similarly fulfil a complete classification for this class of manifolds.

One of the most important methods to study a manifold with nef anticanonical bundle is to analyse its Albanese map.

Theorem 1.1.1 ([Cao19], Theorem 1.2). *Let X be a projective manifold with $-K_X$ nef, and let $\alpha: X \rightarrow Y$ be its Albanese map. Then α is locally trivial, i.e. for any small open set $U \subset Y$, $\alpha^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fibre of α .*

Thanks to the above result, Cao and Höring [CH19] showed a decomposition theorem for projective manifolds with nef anticanonical bundle: for such a manifold X , its universal cover \tilde{X} decomposes as a product

$$\tilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times Z,$$

where Y_j are irreducible projective Calabi-Yau manifolds, S_k are irreducible projective hyperkähler manifolds (so that Y_j and S_k have trivial canonical bundle), and Z is a projective rationally connected manifold with $-K_Z$ nef (and non trivial as Z is rationally connected). In view of this result, it is important to study the case when X is rationally connected, and it is also the most difficult one.

1.2 Rationally connected threefolds with nef anticanonical divisor

Recently, Birkar, Di Cerbo and Svaldi proved in [BCS20, Theorem 1.6] that birationally, there are only finitely many deformation families of projective rationally connected threefolds with

ϵ -lc singularities and nef anticanonical divisor. Thus it is in principle possible to classify these varieties.

Let X be a smooth projective rationally connected threefold with $-K_X$ nef. If $-K_X$ is semi-ample, i.e. some multiple of it is globally generated, we refer to [BP04, Section 5, Section 6] for a partial classification. Another approach to boundedness and a classification in this case stems from its similarity with weak Fano threefolds, i.e. threefolds with nef and big anticanonical bundle. One may analyse the (pluri-)anticanonical morphism

$$\phi_{|-mK_X|}: X \rightarrow Y$$

for m sufficiently large, as did in weak Fano case, which led to boundedness of weak Fano threefolds (see [Bor01, KMMT00, Mck02]). Together with a discussion of the Mori contractions, one may obtain a classification by following the strategy in [JPR05, JPR11], where the authors gave a complete classification of weak Fano threefolds.

We thus focus on the case where $-K_X$ is not semi-ample. In [BP04], Bauer and Peternell gave the following criterion for verifying the non semi-ampleness.

Proposition 1.2.1 ([BP04], Theorem 2.1). *Let X be a smooth projective rationally connected threefold with $-K_X$ nef. Then the Iitaka dimension $\kappa(-K_X)$ is at least 1.*

If the nef dimension (see Definition 2.4.2) $n(-K_X)$ is 1 or 2, then $-K_X$ is semi-ample and the nef reduction map associated to $-K_X$ can be taken as (the Stein factorisation of) the map defined by some positive multiple of $-K_X$ which is globally generated.

By a result of Kawamata [Kaw85, Theorem 6.1], if $\nu(-K_X) = \kappa(-K_X)$, where $\nu(-K_X)$ is the numerical dimension of $-K_X$ (see 2.4.1 for definition), then $-K_X$ is semi-ample. Thus in practice, the proposition (together with Theorem 2.4.4) implies that the non semi-ampleness of $-K_X$ is equivalent to $n(-K_X) = 3$ and $\nu(-K_X) = 2$, which is also equivalent to $\nu(-K_X) = 2$ and $\kappa(-K_X) = 1$.

We start the investigation with the base locus of the anticanonical system as the latter one is not semi-ample. We can write

$$|-K_X| = A + |B|,$$

where A is the fixed divisor (which may be zero) and $|B|$ is the mobile part.

Lemma 1.2.2 (see Corollary 2.5.4). *Let X be a smooth projective rationally connected threefold with $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. Let $|B|$ be the mobile part of the anticanonical system $|-K_X|$. Then*

$$B \sim mH,$$

where $m \geq 2$ and H is a prime divisor.

A natural approach to the classification problem stems from the MMP. The above lemma imposes special restrictions on the geometry of X and leads to only a few possibilities when running the MMP.

When the anticanonical system has no fixed divisor, which is a missing case in [BP04, Section 7], running the MMP leads to the following complete classification.

Theorem 1.2.3 (= Theorem 3.0.1). *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system has no fixed divisor. Then $-K_X$ is divisible by 2 in $\text{Pic}(X)$ and X is one of the following:*

- (1) $X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration with general fibre isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then X is an element of the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$, where $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and F is a general fibre of $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$.
- (2) $X = \mathbb{P}(\mathcal{E})$ is a \mathbb{P}^1 -bundle over a smooth rational surface Y with $-K_Y$ nef, where \mathcal{E} is a nef rank-2 vector bundle with $c_1(\mathcal{E}) = -K_Y$ and $c_2(\mathcal{E}) = K_Y^2$, given by an extension

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_Y(-K_Y) \rightarrow 0,$$

where \mathcal{I}_Z is the ideal sheaf of a length- $c_2(\mathcal{E})$ subscheme Z of Y .

- (3) $X = \text{Bl}_p(Y)$ is the blow-up in a point p of a smooth almost del Pezzo threefold Y of degree 1 (i.e. $-K_Y \sim 2H_Y$ for some prime divisor H_Y on Y with H_Y nef and $H_Y^3 = 1$) such that p is not the unique base point of $|H_Y|$.

Conversely, let X be a variety that appears in one of the above cases with respectively the following conditions:

- (1') $X \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$ is a very general member;
- (2') the points of the finite set Z are in sufficiently general position on Y ;
- (3') the blown up point $p \in Y$ is general.

Then $-K_X$ is nef, not semi-ample and divisible by 2 in $\text{Pic}(X)$.

The type of varieties in case (3) also appeared in [LO16, Section 2], where an explicit example of this case is constructed. For a complete list of smooth almost del Pezzo threefolds of degree 1, we refer to [JP08]. Notice here that we deduce boundedness for the family of smooth projective rationally connected threefolds with nef (and not semi-ample) anticanonical divisor from the above theorem (and Proposition 3.2.1) in the case where the anticanonical system has no fixed divisor.

When the anticanonical system has a non-zero fixed divisor, it turns out that, after a finite sequence of flops, one can assume that the mobile part is always nef. We will further show that the mobile part is base-point-free when it is nef.

Theorem 1.2.4 (= Theorem 4.0.1). *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system has a non-zero fixed divisor. Then there exists a finite sequence of flops $\psi: X \dashrightarrow X'$ such that the following holds:*

- X' is smooth,
- $-K_{X'}$ is nef,

- the mobile part $|B'|$ of $|-K_{X'}|$ is nef.

In this case, $B^2 = 0$ so that $|B'|$ is base-point-free and induces a fibration $f: X' \rightarrow \mathbb{P}^1$. Furthermore, a general fibre F of f is a smooth surface with $-K_F$ nef and effective, $(-K_F)^2 = 0$, and $n(-K_F) = 2$.

Notice here that in the case where the anticanonical system has a non-zero fixed divisor, birational boundedness does not follow from this theorem. However, we expect a complete classification in this case as in the previous one.

The main obstacle towards a complete classification in this case is that the property of having nef anticanonical divisor is not stable under the MMP. It is shown in [PS98] that the smallest class of varieties appearing in the MMP of threefolds with nef anticanonical divisor is the class of terminal threefolds X with $-K_X$ almost nef, i.e. $-K_X \cdot C \geq 0$ for all curves C with only finitely many exceptions, and these exceptions are all rational curves. By the work of Demailly, Peternell and Schneider ([DPS93, Proposition 3.3]), there are two types of birational Mori contractions in dimension three which do not preserve the nefness of the anticanonical bundle (see Section 2.4). Let us remark that these two types of Mori contractions do not appear when $-K_X$ is divisible by 2 in $\text{Pic}(X)$.

In order to study the geometry of the fibration $X' \rightarrow \mathbb{P}^1$ in Theorem 1.2.4, we consider the following setup where we denote X' by X for simplicity of notation in the rest of our discussion.

Setup 1.2.5 (= Setup 4.2.5). *Let X be a smooth projective rationally connected threefold with anticanonical bundle $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. We suppose that the anticanonical system $|-K_X|$ has a non-zero fixed divisor A and that its mobile part $|B|$ is nef so that $B^2 = 0$ by Theorem 1.2.4. Hence the mobile part induces a fibration $f: X \rightarrow \mathbb{P}^1$.*

If F is a fibre of f , then $|-K_X| = A + |kF|$ with $k \geq 2$. Furthermore, $A^3 = A^2 \cdot F = 0$, and $-K_F$ is nef with $n(-K_F) = 2$, $\nu(-K_F) = 1$.

Now we write $A = A_h + A_v$, where A_h and A_v are effective divisors such that $A_h|_F = -K_F$ and $A_v|_F = 0$ for a general fibre F .

Our approach to the classification problem is exploring the geometry of X in Setup 1.2.5 according to the geometry of the unique member D in $|-K_F|$. We first consider the case where F is rational and D is a smooth elliptic curve. It is shown in [BP04, Proposition 7.7] that in this case, $|-K_X| = A + |2F|$, and $A \simeq D \times \mathbb{P}^1$. In Chapter 5, we will show that the property of being smooth and having nef anticanonical divisor is stable under the MMP, which leads to a complete classification in this case (see Proposition 5.1.1 and Proposition 5.1.3). Notice that even though A is isomorphic to a product, the structure of X can be complicated. Examples of such X which are not isomorphic to a product can be found (see Section 5.2).

More generally, under Setup 1.2.5, we consider the case where F is rational and the unique member D in $|-K_F|$ is reduced. We first show that if the pair (X, A_h) is log canonical over the generic point of \mathbb{P}^1 , then (X, A_h) is log canonical, and $|-K_X| = A_h + |2F|$ (Theorem 6.1.1). This concerns most of the cases when F is rational and D is reduced.

We then examine more precisely the geometry of X when F is rational and D is either a cuspidal or a nodal cubic curve. In order to avoid the two types of Mori contractions which spoil the nefness of the anticanonical bundle, our strategy is to choose particular extremal rays

when running a (log-)MMP. By a discussion on the classification of three-dimensional flops and extremal divisorial crepant contractions, we show that the structure of X is very special in this case: when D is a cuspidal curve, the fibration $f: X \rightarrow \mathbb{P}^1$ is locally trivial (Theorem 6.2.1); when D is a nodal curve (and thus the pair (X, A_h) is generically log canonical), we give an explicit description of all the possible singular fibres of $f|_{A_h}: A_h \rightarrow \mathbb{P}^1$ (Proposition 6.3.1).

We conclude this case by analysing the remaining cases where the pair (X, A_h) is not generically log canonical (see Section 6.4), which brings to the following conjecture.

Conjecture 1.2.6 (= Conjecture 6.4.6). *In Setup 1.2.5, assume moreover that F is rational and that the unique member in $| -K_F|$ is reduced. Then $| -K_X| = A_h + |2F|$.*

Let us point out that the case where the general fibre F is non-rational in Setup 1.2.5 might happen only in very few situations (see Proposition 7.1.3). We also construct an example in this case (see Example 7.1.4). Notice that $X \not\cong F \times \mathbb{P}^1$ when F is non-rational, since X is rationally connected.

Plan of Part I. This part is organised as follows. In Chapter 2, we review relevant material and results from singularities of pairs and the Mori contractions in dimension three. We also collect some basic notions and properties about varieties with nef anticanonical bundle.

In Chapter 3, we prove Theorem 1.2.3 thanks to Lemma 1.2.2 and the Mori contractions in dimension three.

In Chapter 4, we prove Theorem 1.2.4. The main part of this chapter is devoted to the proof by contradiction of the statement $B^2 = 0$.

The rest of Part I is under Setup 1.2.5. In Chapter 5, we give a complete classification when F is rational and the unique member in $| -K_F|$ is a smooth elliptic curve, and we construct examples in each case of the classification. In Chapter 6, we study the more general case where F is rational and the unique member in $| -K_F|$ is reduced. Finally, in Chapter 7, we analyse the case when F is non-rational.

Chapter 2

Preliminaries

2.1 Notation and conventions

We work over \mathbb{C} .

- Let X be a projective variety. Let C_1, C_2 be two algebraic cycles on X .

We note $C_1 \sim C_2$ if the two cycles are rationally equivalent. In particular, if X is normal, then rational equivalence for cycles of codimension 1 coincides with linear equivalence of Weil divisors.

We note $C_1 \equiv C_2$ or $C_1 \equiv_{num} C_2$ if the two cycles are numerically equivalent.

- Given a variety X , a point $x \in X$ is called *general* if it is in the complement of a proper algebraic subset of X , and *very general* if it is in the complement of a countable union of proper algebraic subsets of X .
- In order to describe the configurations of blowing up points on \mathbb{P}^2 , we adopt the following conventions.

Let Σ be a set of $r \leq 8$ distinct points on \mathbb{P}^2 . We say that Σ is in *general position* if no 3 of them are collinear, no 6 of them are on a conic, and when $r = 8$, there exists no singular cubic which passes through all the points of Σ and has one of them as the singular point. Note that we obtain a del Pezzo of degree $9 - r$ by blowing up a set of points in general position.

Let Σ be a set of $r \leq 8$ points on \mathbb{P}^2 (infinitely close points are allowed). We say that Σ is in *almost general position* if no 4 of them are collinear, no 7 of them are on a conic, and no 3 of them are infinitely close. Note that we obtain an almost del Pezzo of degree $9 - r$ by blowing up a set of points in almost general position.

Let Σ be a set of 9 points on \mathbb{P}^2 (infinitely close points are allowed). We say that Σ is in *sufficiently general position* if by blowing up the points of Σ , we obtain a surface with nef and non semi-ample anticanonical divisor. We refer to [Sak01, Section B] for precise configurations of points. Notice that the condition on Σ of being in sufficiently general position is much weaker than the two previous conditions; in particular, all the 9 points of Σ can be infinitely close (see [Sak01, p. 222, $E_8^{(1)}$ -surface]).

2.2 Singularities of pairs

We recall in this section some definitions and results of singularities of pairs. The references for this part are [Kol97, Sect. 2] and [Fuj11, Sect. 4].

Definition 2.2.1. *Let X be a normal variety. Let Δ be a \mathbb{Q} -Weil divisor (or \mathbb{Q} -divisor) on X , i.e. $\Delta = \sum_i d_i \Delta_i$ where $d_i \in \mathbb{Q}$ and Δ_i are distinct prime divisors. We say that Δ is \mathbb{Q} -Cartier if $m\Delta$ is Cartier for some non-zero $m \in \mathbb{Z}$. We say that X is \mathbb{Q} -factorial if every \mathbb{Q} -divisor is \mathbb{Q} -Cartier. We call (X, Δ) a pair if $K_X + \Delta$ is \mathbb{Q} -Cartier.*

Definition 2.2.2 (Log resolution). *A log resolution of a pair (X, Δ) with $\Delta = \sum_i d_i \Delta_i$ an effective \mathbb{Q} -divisor is a proper birational morphism $\mu: Y \rightarrow X$ satisfying that Y is smooth, and that the exceptional locus $\text{Exc}(\mu) \subset Y$ is a divisor such that $\text{Exc}(\mu) \cup \mu_*^{-1}(\Delta)$ has a simple normal crossing support.*

The existence of log resolution was shown by Hironaka, and generalised by Szabó (see [Sza94, Resolution Lemma]).

In order to get a global measure of the singularities of the pair (X, Δ) , we introduce the following.

Definition 2.2.3 (Singularities of pairs). *Let (X, Δ) be a pair with $\Delta = \sum_i d_i \Delta_i$ an effective \mathbb{Q} -divisor. Let $\mu: Y \rightarrow X$ be a proper birational morphism with Y normal. We can write*

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta) E_i,$$

where $E_i \subset Y$ are distinct prime divisors and $a(E_i, X, \Delta) \in \mathbb{Q}$. We adopt the convention that a non-exceptional divisor E appears in the sum of the right hand side if and only if $E = \mu_*^{-1} \Delta_i$ for some i , and then its coefficient $a(E_i, X, \Delta) = -d_i$.

- We call $a(E_i, X, \Delta)$ the discrepancy of E_i with respect to (X, Δ) . Then we say that (X, Δ) is log canonical or lc for abbreviation (resp. Kawamata log terminal or klt for abbreviation) if $a(E_i, X, \Delta) \geq -1$ (resp. $a(E_i, X, \Delta) > -1$) for every i . Note that the discrepancy $a(E, X, \Delta)$ can be defined for every prime divisor E over X .
- There exists a largest Zariski open subset U (resp. U') of X such that (X, Δ) is log canonical (resp. Kawamata log terminal) on U (resp. U'). We note

$$\text{Nlc}(X, \Delta) = X \setminus U$$

resp.

$$\text{Nklt}(X, \Delta) = X \setminus U'$$

and call it the non-lc locus (resp. non-klt locus) of the pair (X, Δ) .

Note that by blowing up along smooth centres on a given Y , we create new exceptional divisors. Nevertheless, only additional positive coefficients for new exceptional divisors will appear when we continue this process. Therefore, the above definition for lc (resp. klt) pair is consistent.

Now we introduce the notion of log canonical threshold and log canonical centre.

Definition 2.2.4 (Log canonical threshold). *Let (X, Δ) be a log canonical pair and let B be an effective \mathbb{Q} -Cartier divisor on X . The log canonical threshold of (X, Δ) with respect to B is defined by*

$$lct((X, \Delta), B) = \sup\{t \mid (X, \Delta + tB) \text{ is lc}\}$$

Definition 2.2.5 (Log canonical centre). *Let (X, Δ) be a pair with Δ an effective \mathbb{Q} -divisor. A place for (X, Δ) is a prime divisor E on some proper birational model $\mu: Y \rightarrow X$ of X such that $a(E, X, \Delta) = -1$. If $\mu(E)$ is not contained in $\text{Nlc}(X, \Delta)$, then $\mu(E)$ is called a log canonical centre of (X, Δ) . There are at most finitely many lc centres.*

2.3 Mori contractions in dimension three

Let X be a smooth projective threefold and let Γ be an extremal ray of $\overline{\text{NE}}(X)$. Consider a Mori contraction $\varphi: X \rightarrow Y$ corresponding to Γ . Recall that the length of Γ is defined as

$$l(\Gamma) = \min\{-K_X \cdot Z \mid [Z] \in \Gamma\}.$$

Remark 2.3.1. *An extremal ray Γ of $\overline{\text{NE}}(X)$ is not necessarily K_X -negative. We say that the corresponding contraction φ_Γ is a Mori contraction if Γ is K_X -negative; we say that φ_Γ is an extremal crepant contraction if Γ is K_X -trivial.*

We list here the Mori contractions in dimension three (see [MM83, Sect. 3]).

Case $\dim Y = 3$. In this case, φ is a divisorial contraction. Denote by E the exceptional divisor of φ . Then φ is the blow-up of Y in the ideal defining $\varphi(E)$ (given the reduced structure).

Type of Γ	φ and E	$l(\Gamma)$
E_1	$\varphi(E)$ is a smooth curve, Y is smooth and $\varphi _E: E \rightarrow \varphi(E)$ is a \mathbb{P}^1 -bundle	1
E_2	$\varphi(E)$ is a point, Y is smooth, $E \simeq \mathbb{P}^2$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$	2
E_3	$\varphi(E)$ is an ordinary double point, $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$	1
E_4	$\varphi(E)$ is a double point, E is a quadric cone in \mathbb{P}^3 and $\mathcal{O}_E(E) \simeq \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$	1
E_5	$\varphi(E)$ is a quadruple non Gorenstein point, $E \simeq \mathbb{P}^2$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$	1

Case $\dim Y = 2$. In this case, Y is a smooth projective surface and $\varphi: X \rightarrow Y$ is a conic bundle. Moreover, $\varphi^{-1}(C)$ is irreducible for every irreducible curve C on Y .

Type of Γ	φ	$l(\Gamma)$
C_1	φ has a singular fibre	1
C_2	φ is a \mathbb{P}^1 -bundle	2

Case $\dim Y = 1$. In this case, Y is a smooth curve. Every fibre of φ is irreducible and reduced, and the generic fibre is a del Pezzo surface. In particular, $Y \simeq \mathbb{P}^1$ if $q(X) = 0$.

Type of Γ	φ	$l(\Gamma)$
D_1	the general fibre of φ is a del Pezzo surface of degree d , with $1 \leq d \leq 6$	1
D_2	φ is a quadric bundle, i.e. the general fibre is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and a singular fibre is isomorphic to a quadric cone in \mathbb{P}^3	2
D_3	φ is a \mathbb{P}^2 -bundle	3

Case $\dim Y = 0$. Then X is a Fano threefold with Picard number one.

2.4 Varieties with nef and effective anticanonical divisor

In order to measure the positivity of an effective divisor L on a smooth projective variety, we study the asymptotic behaviour of the linear system $|mL|$ as m increases. The two most important features of the asymptotic behaviour are the following:

Notation 2.4.1 ([Laz04], Def. 2.1.3, Rem. 2.3.17). *Let X be a normal projective variety and L a Cartier divisor on X . We denote by*

- $\kappa(L)$ the Iitaka (Kodaira) dimension of L defined as follows.

Let m be a positive integer such that $H^0(X, L^{\otimes m}) \neq 0$. Consider the rational map

$$\phi_m := \phi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes m})$$

associated to the complete linear system $|L^{\otimes m}|$. Then the Iitaka dimension of L is defined to be

$$\kappa(L) = \kappa(X, L) = \max_{m > 0} \{\dim \phi_m(X)\}.$$

If $H^0(X, L^{\otimes m}) = 0$ for all $m > 0$, one puts $\kappa(L) = -\infty$.

- $\nu(L) := \max\{n \mid L^n \neq 0\}$ the numerical dimension of L when L is nef.

Note that the numerical dimension can also be defined for a pseudo-effective divisor, see for example [Leh14, Def. 2.14].

For a nef divisor L on a variety X , one would like to describe the maximal "quotient" of X contracting all the curves C with $L \cdot C = 0$. This is done in [BCE⁺02], where the authors construct a reduction map for nef divisors.

Theorem-Definition 2.4.2 ([BCE⁺02], Thm. 2.1). *Let L be a nef line bundle on a normal projective variety X . Then there exists an almost holomorphic dominant meromorphic map $f : X \dashrightarrow B$ with connected fibres such that*

- (1) L is numerically trivial on all compact fibres of f of dimension $\dim X - \dim B$,
- (2) for a general point $x \in X$ and every irreducible curve C passing through x such that $\dim f(C) > 0$, we have $L \cdot C > 0$.

The map f is unique up to birational equivalence of B . In particular $\dim B$ is an invariant of L and we set $n(L) = \dim B$, the nef dimension of L .

Recall that a meromorphic map $f: X \dashrightarrow B$ is almost holomorphic if there exists a Zariski dense open subset $U \subset X$ such that $f|_U$ is holomorphic and proper. So the fibre over a point $b \in f(U)$ is well defined, which is $f|_U^{-1}(b)$.

We have the following inequalities which relate the above three invariants:

Theorem-Definition 2.4.3 ([Kaw85], Prop. 2.2, Thm. 6.1 [BCE⁺02], Prop. 2.8). *Let X be a smooth projective variety. Let L be a nef divisor on X . Then we have*

$$\kappa(L) \leq \nu(L) \leq n(L).$$

We say that L is good if $\kappa(L) = \nu(L)$, otherwise we say that it is bad. If $L = \pm K_X$ is good, then it is semi-ample.

By the Abundance Conjecture, the canonical divisor is expected to be always good, whereas the anticanonical divisor can be bad.

More generally, the concept of reduction map can be generalised to a pseudo-effective divisor (see [Leh14, Thm. 1.3]). We mention the following equivalent conditions for the goodness of a nef divisor, which is a particular case of a more general result in [Leh14, Thm. 6.1].

Theorem 2.4.4. *Let X be a normal projective variety. Let L a divisor on X with $\kappa(L) \geq 0$. The following are equivalent:*

- (1) $\kappa(L) = \nu(L)$.
- (2) $\kappa(L) = n(L)$.

Surfaces with nef anticanonical divisor

Complex projective surfaces with nef and not numerically trivial anticanonical bundle are completely classified. This is a consequence of the Kodaira-Enriques classification for compact complex surfaces.

Lemma 2.4.5 ([BP04], Prop. 1.5, Prop. 1.6). *Let S be a smooth projective surface with $-K_S$ nef and $\nu(-K_S) = 1$. Then S is one of the following:*

1. $n(-K_S) = 1$: S admits an elliptic fibration and $-K_S$ is semi-ample;
2. $n(-K_S) = 2$: we have $\kappa(-K_S) = 0$ and either
 - (A) S is \mathbb{P}^2 blown up in 9 points in sufficiently general position or
 - (B) $S = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a rank-2 vector bundle over an elliptic curve which is an extension

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

with \mathcal{L} a line bundle of degree 0 and either

- (i) $\mathcal{L} = \mathcal{O}$ and the extension is non-split or
- (ii) \mathcal{L} is not torsion.

The structure of the unique element D in $|-K_S|$ is as follows:

- (i) either $D = 2C$ and C is a smooth elliptic curve,
- (ii) or $D = C_1 + C_2$ where C_1 and C_2 are smooth elliptic curves which do not meet.

In the case where S is \mathbb{P}^2 blown up in 9 points and $-K_S$ is not semi-ample, the unique member $D \in |-K_S|$ is one of the types in Kodaira's table of singular fibres for an elliptic fibration. We list here all the possible configurations for D by Sakai (see [Sak01, p. 181, Table 2]).

Type	Intersection matrix and Kodaira's symbol for D
Elliptic type	$A_0^{(1)} (= I_0)$
Multiplicative type	$A_0^{(1)*} (= I_1), A_1^{(1)} (= I_2), A_2^{(1)} (= I_3), \dots, A_7^{(1)}, A_7^{(1)'} = (I_8), A_8^{(1)} (= I_9)$
Additive type	$A_0^{(1)**} (= II), A_1^{(1)*} (= III), A_2^{(1)*} (= IV),$ $D_4^{(1)} (= I_0^*), \dots, D_8^{(1)} (= I_4^*),$ $E_6^{(1)} (= IV^*), E_7^{(1)} (= III^*), E_8^{(1)} (= II^*)$

Table 2.1 – Sakai's Table

We refer the readers to [Sak01, Sect. B] for the realisation of each configuration as the blow-up of \mathbb{P}^2 in 9 points. Let us point out that for elliptic type, D is a smooth elliptic curve; for every configuration of multiplicative type, D is reduced; for additive type, D is reduced only in the 3 configurations $A_0^{(1)**} (= II), A_1^{(1)*} (= III)$ and $A_2^{(1)*} (= IV)$. In particular, D is a cubic nodal curve in configuration $A_0^{(1)*} (= I_1)$, and D is a cubic cuspidal curve in configuration $A_0^{(1)**} (= II)$.

Remark 2.4.6. Let S be a smooth rational projective surface. Suppose that $-K_S$ is nef, effective and not big. Then by Lemma 2.4.5, the surface S is obtained by blowing up 9 points on \mathbb{P}^2 . The case where $-K_S$ is not semi-ample is discussed as above. Now suppose that $-K_S$ is semi-ample. Let $m \geq 1$ be the minimal integer such that $|-mK_S|$ is base-point-free. Then there exists an irreducible pencil of curves of degree $3m$ with 9 base points of multiplicity m in \mathbb{P}^2 , such that S is the blow-up of the 9 base points and $|-mK_S|$ is the proper transform of this pencil (the set of base points may contain infinitely close points), see for example [CD12, Prop. 2.2].

Threefolds with nef anticanonical divisor

Let X be a smooth projective threefold with $-K_X$ nef and $\kappa(X) = -\infty$. Let $\varphi: X \rightarrow Y$ be a Mori contraction. There are exactly two types of birational Mori contractions which do not preserve the nefness of the anticanonical bundle:

Proposition 2.4.7 ([DPS93], Prop. 3.3). *Assume that $\dim Y = 3$, in which case φ is a birational divisorial contraction and the exceptional divisor E is irreducible. Then we have one of the following.*

- (1) If $\dim \varphi(E) = 0$, then $-K_Y$ is nef and big, and $q(X) = 0$.
- (2) If $\dim \varphi(E) = 1$, then Y is smooth, and φ is the blow-up of the smooth curve $C_0 := \varphi(E)$ and $-K_Y$ is again nef except in the case where C_0 is rational, and either

- (A) $\mathcal{N}_{C_0/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, or
- (B) $\mathcal{N}_{C_0/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

For a non-birational Mori contraction, the structure of φ is as follows.

Proposition 2.4.8 ([DPS93], Prop. 3.1). *Assume that $\dim Y \leq 2$. Then one of the following cases occurs.*

- (1) *If Y is a point, then X is Fano.*
- (2) *If Y is a smooth curve, then φ is a del Pezzo fibration, and $g(Y) \leq 1$. In the case when $g(Y) = 1$, we have $\varphi = \alpha$, where $\alpha: X \rightarrow \text{Alb}(X)$ is the Albanese map; in the case when $g(Y) = 0$, we have $q(X) = 0$.*
- (3) *If Y is a smooth surface, then either*
 - (a) *φ is a \mathbb{P}^1 -bundle and $-K_Y$ is nef, or*
 - (b) *φ is a proper conic bundle with discriminant locus Δ such that $-(4K_Y + \Delta)$ is nef, and $q(Y) \leq 1$.*

For case (3)(b) above, a more precise result on the positivity of direct image sheaves states that the anticanonical bundle of the surface Y is nef:

Proposition 2.4.9 ([Hör10], Cor. 3.31). *Let X be a smooth projective threefold which is a conic bundle $\varphi: X \rightarrow Y$ over a surface Y . Let L be a line bundle over X that is nef and φ -big. Then the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ is nef.*

2.5 Basic results

In this section, we prove some fundamental geometric properties of rationally connected threefolds with nef and not semi-ample anticanonical divisor, which are useful throughout Part I.

We start by showing some results which are valid in a more general setting.

Lemma 2.5.1. *Let X be a normal projective \mathbb{Q} -factorial variety with $-K_X$ nef. Let D be an effective \mathbb{Q} -divisor such that the pair (X, D) is log canonical. If D is not nef, then there exists a $(K_X + D)$ -negative extremal ray Γ such that $D \cdot \Gamma < 0$.*

Proof. Suppose that there is no such extremal ray. Since D is not nef, there exists an irreducible curve $l \subset X$ such that $D \cdot l < 0$. Then we can write

$$l = \sum_i \lambda_i \Gamma_i + R,$$

where

- $\lambda_i \geq 0$;
- the Γ_i are $(K_X + D)$ -negative extremal rays. By assumption they all satisfy $D \cdot \Gamma_i \geq 0$;

- $(K_X + D) \cdot R \geq 0$.

Therefore,

$$0 > D \cdot l = \sum_i \lambda_i D \cdot \Gamma_i + D \cdot R \geq D \cdot R,$$

i.e. $D \cdot R < 0$.

Since $(K_X + D) \cdot R \geq 0$, we have

$$K_X \cdot R \geq -D \cdot R > 0,$$

which contradicts the fact that $-K_X$ is nef. \square

Lemma 2.5.2. *Let X be a smooth projective rationally connected threefold. Let D be a divisor with $\kappa(D) = 1$. Suppose that the linear system $|D|$ has no fixed divisor and the general member in $|D|$ is reducible. Then D is linearly equivalent to mH , where H is a prime divisor and $m \geq 2$. Furthermore, $h^0(X, \mathcal{O}_X(H)) = 2$ and $h^0(H, \mathcal{O}_H(H)) = 1$.*

Proof. Let $\phi: X \dashrightarrow C$ be the rational map determined by the linear system $|D|$. Then $C \simeq \mathbb{P}^1$ as $\kappa(D) = 1$ and $H^1(X, \mathcal{O}_X) = 0$.

Let $\mu: \tilde{X} \rightarrow X$ be a birational modification which resolves ϕ . Let F be a general fibre of the induced morphism $\tilde{\phi}: \tilde{X} \rightarrow C$. Since $|D|$ has no fixed divisor, the pushforward $\mu_*(F)$ is a general member of $|D|$. Furthermore, the general fibre F is not connected as the general member in $|D|$ is reducible.

Let $\tilde{X} \xrightarrow{\tilde{\phi}'} C' \xrightarrow{\nu} C$ be the Stein factorisation of the morphism $\tilde{\phi}$. Then $\tilde{\phi}'$ has smooth connected general fibres and $C' \simeq \mathbb{P}^1$ as $H^1(X, \mathcal{O}_X) = 0$.

For a point $p \in C$, we have

$$\nu^*(p) \simeq \mathcal{O}_{\mathbb{P}^1}(m)$$

with $m \geq 2$ the number of connected components of $\tilde{\phi}^*(p)$, and thus

$$\tilde{\phi}^*(p) = \tilde{\phi}'^*(\nu^*(p)) \simeq \tilde{\phi}'^*(\mathcal{O}_{\mathbb{P}^1}(m)).$$

Let F' be a general fibre of $\tilde{\phi}': \tilde{X} \rightarrow \mathbb{P}^1$. Then $F \sim mF'$, and thus a general member in $|D|$ is linearly equivalent to mH where $H := \mu_*(F')$. Hence,

$$h^0(X, \mathcal{O}_X(H)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2.$$

Now the exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(H) \rightarrow \mathcal{O}_H(H) \rightarrow 0$$

gives $h^0(H, \mathcal{O}_H(H)) = 1$. \square

For the rest of this section, X will be a smooth projective rationally connected threefold with $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$.

By [BP04, Thm. 2.1], the conditions $n(-K_X) = 3$ and $\nu(-K_X) = 2$ on $-K_X$ are equivalent to $\nu(-K_X) = 2$ and $\kappa(-K_X) = 1$. The latter one is more useful since it is in practice easier to compute the Iitaka dimension than the nef dimension.

Since X is rationally connected, we have $\chi(\mathcal{O}_X) = 1$. Together with $K_X^3 = 0$, we deduce by the Riemann-Roch theorem that $\chi(-K_X) = 3$. Moreover, as $-K_X$ is nef and $\nu(-K_X) = 2$, by the Kawamata-Viehweg vanishing theorem [Kaw82, Corollary], one has $H^1(X, \mathcal{O}_X(2K_X)) = 0$. Hence we deduce

$$H^2(X, \mathcal{O}_X(-K_X)) = 0$$

by Serre duality. Therefore,

$$h^0(X, \mathcal{O}_X(-K_X)) \geq 3.$$

Lemma 2.5.3. *Let X be a smooth projective rationally connected threefold with $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. Let $|B|$ be the mobile part of the anticanonical system $|-K_X|$ and D be a general member in $|B|$. Then D has at least two irreducible components.*

Proof. We can write $|-K_X| = A + |B|$ with A the fixed divisor (which can be zero) and $|B|$ the mobile part. For a general member D of $|B|$, we have the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-K_X - D) \rightarrow \mathcal{O}_X(-K_X) \rightarrow \mathcal{O}_D(-K_X) \rightarrow 0.$$

Since

$$h^0(X, \mathcal{O}_X(-K_X - D)) = h^0(X, \mathcal{O}_X(A)) = 1,$$

together with $h^0(X, \mathcal{O}_X(-K_X)) \geq 3$, we have $h^0(D, \mathcal{O}_D(-K_X)) \geq 2$.

Now suppose by contradiction that D is irreducible. Let $\nu: \bar{D} \rightarrow D$ be the normalisation of the surface D . Then for the pullback of the Cartier divisor $-K_X|_D$, we have

$$h^0(\bar{D}, \nu^*(-K_X|_D)) \geq h^0(D, -K_X|_D) \geq 2.$$

Hence the linear system $|\nu^*(-K_X|_D)|$ on \bar{D} has a mobile part M . On the other hand, since $-K_X$ is nef and $(-K_X)^3 = 0$, one has $(-K_X)^2 \cdot D = 0$, i.e. $(-K_X|_D)^2 = 0$. Since $\nu^*(-K_X|_D)$ is nef and $\nu^*(-K_X|_D)^2 = (-K_X|_D)^2 = 0$, we deduce that

$$\nu^*(-K_X|_D) \cdot M = 0.$$

Therefore, \bar{D} is covered by $\nu^*(-K_X|_D)$ -trivial curves, from which we deduce that D is covered by $(-K_X)$ -trivial curves. As D moves, this contradicts the fact that $n(-K_X) = 3$. \square

Now Lemmas 2.5.2 and 2.5.3 give the following:

Corollary 2.5.4. *Let X be a smooth projective rationally connected threefold with $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. Let $|B|$ be the mobile part of the anticanonical system $|-K_X|$. Then*

$$B \sim mH,$$

where $m \geq 2$ and H is a prime divisor such that $h^0(H, \mathcal{O}_H(-K_X)) = 1$, $h^0(X, \mathcal{O}_X(H)) = 2$ and $h^0(H, \mathcal{O}_H(H)) = 1$.

Proof. By Lemma 2.5.3 and 2.5.2, it remains to show that

$$h^0(H, \mathcal{O}_H(-K_X)) = 1.$$

By contradiction, suppose that $h^0(H, \mathcal{O}_H(-K_X)) \geq 2$. In Lemma 2.5.3, we may repeat the same argument in the second part of the proof with H playing the role of D , then the argument following from the normalisation of the surface leads to a contradiction. \square

We close this section with the following lemma which is convenient for verifying the nefness and the non semi-ampleness of the anticanonical divisor.

Lemma 2.5.5. *Let X be a smooth projective rationally connected threefold with $-K_X$ non-zero effective, divisible by 2 in $\text{Pic}(X)$, and $K_X^3 = 0$. Suppose that there exists an irreducible normal surface $H \in |-\frac{1}{2}K_X|$ such that $-K_H$ is nef, non-zero, effective, and not semi-ample. Then $-K_X$ is nef and not semi-ample, i.e. $\nu(-K_X) = 2$ and $\kappa(-K_X) = 1$.*

Proof. We have $-K_X \sim 2H$. The adjunction formula gives $-K_H \sim H|_H$. We first show that H (and thus $-K_X$) is nef. Indeed, it is enough to show that the restriction of H on itself is nef: let $C \subset H$ be an integral curve, then

$$H \cdot C = H|_H \cdot C = -K_H \cdot C \geq 0$$

as $-K_H$ is nef.

Since there exists a non-zero effective divisor in $| -K_H |$, we deduce that

$$\nu(-K_X) = \nu(H) = 2.$$

As $-K_H$ is not semi-ample, we have $0 \leq \kappa(H, -K_H) < \nu(H, -K_H) < n(H, -K_H) \leq 2$ by Definition-Theorem 2.4.3 and Theorem 2.4.4. Thus $\kappa(H, -K_H) = 0$. Then for any $m \geq 1$,

$$h^0(H, \mathcal{O}_H(mH)) = h^0(H, \mathcal{O}_H(-mK_H)) = 1.$$

Now the short exact sequence

$$0 \rightarrow \mathcal{O}_X((m-1)H) \rightarrow \mathcal{O}_X(mH) \rightarrow \mathcal{O}_H(mH) \rightarrow 0$$

gives $h^0(X, \mathcal{O}_X(mH)) \leq h^0(X, \mathcal{O}_X((m-1)H)) + 1$, and thus

$$\kappa(-K_X) = \kappa(H) \leq 1. \quad \square$$

Together with $\kappa(-K_X) \geq 1$ by Proposition 1.2.1, one has $\kappa(-K_X) = 1$.

Chapter 3

Anticanonical system without fixed divisor

The main result of this chapter is the following classification:

Theorem 3.0.1. *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system has no fixed divisor. Then $-K_X$ is divisible by 2 in $\text{Pic}(X)$ and X is one of the following:*

- (1) $X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration with general fibre isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then X is an element of the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$, where $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and F is a general fibre of $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$.
- (2) $X = \mathbb{P}(\mathcal{E})$ is a \mathbb{P}^1 -bundle over a smooth rational surface Y with $-K_Y$ nef, where \mathcal{E} is a nef rank-2 vector bundle with $c_1(\mathcal{E}) = -K_Y$ and $c_2(\mathcal{E}) = K_Y^2$, given by an extension

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_Y(-K_Y) \rightarrow 0,$$

where \mathcal{I}_Z is the ideal sheaf of a length- $c_2(\mathcal{E})$ subscheme Z of Y .

- (3) $X = \text{Bl}_p(Y)$ is the blow-up in a point p of a smooth almost del Pezzo threefold Y of degree 1 such that p is not the unique base point of $|-\frac{1}{2}K_Y|$.

Conversely, let X be a variety that appears in one of the above cases with respectively the following conditions:

- (1') $X \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$ is a very general member;
- (2') the finite set Z of $c_2(\mathcal{E})$ points are in sufficiently general position on Y ;
- (3') the blown up point $p \in Y$ is general.

Then $-K_X$ is nef, not semi-ample and divisible by 2 in $\text{Pic}(X)$.

Let us point out the Picard number of X in each case. In case (1), X has $\rho(X) = 2$. In case (2), X has $2 \leq \rho(X) \leq 11$, since $1 \leq \rho(Y) \leq 10$ (this is because Y is isomorphic to \mathbb{P}^2 blown up in at most 9 points). In case (3), X has $2 \leq \rho(X) \leq 11$, since $1 \leq \rho(Y) \leq 10$ (see [CJR08, Thm. 1.1]). Therefore, $2 \leq \rho(X) \leq 11$, and $\rho(X) = 11$ if and only if we are in one of the following cases:

- (a) either $X = X_1 := \mathbb{P}(\mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_1}(-K_{Y_1}))$, where Y_1 is \mathbb{P}^2 blown up in 9 points in sufficiently general position such that $-K_{Y_1}$ is nef and not semi-ample,
- (b) or $X = X_2 := \text{Bl}_p(Y_2)$ is the blow-up of Y_2 at a point p which is not the unique base point of $|-\frac{1}{2}K_{Y_2}|$, where Y_2 is a smooth almost del Pezzo threefold of degree 1 such that there exists a finite sequence of flops $\chi: Y_2 \dashrightarrow \text{Bl}_{p_1, \dots, p_8}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3)))$ (see [CJR08, Thm. 1.1]).

Remark that in case (a), by [CJR08, Lem. 2.6] (where the bigness assumption on the anticanonical divisor is not used in their proof), there exists a finite sequence of flops

$$\chi_1: X_1 \dashrightarrow \text{Bl}_{p_1, \dots, p_9}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))).$$

Note that in case (b), the blow-up point $p \in Y_2$ is not contained in the flopping locus of χ , as $-K_X$ is nef. Therefore, we can lift χ to obtain a finite sequence of flops

$$\chi_2: X_2 \dashrightarrow \text{Bl}_{p_1, \dots, p_9}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))).$$

Therefore, $\rho(X) = 11$ if and only if there exists a finite sequence of flops $X \dashrightarrow \text{Bl}_{p_1, \dots, p_9}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3)))$.

We consider the following setup throughout this chapter:

Setup 3.0.2. *Let X be a smooth projective rationally connected threefold with anticanonical bundle $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. We suppose that the anticanonical system $| -K_X |$ has no fixed divisor so that by Corollary 2.5.4, we can write*

$$-K_X \sim mH$$

with $m \geq 2$ and H some prime divisor.

We may now run the Minimal Model Program.

3.1 Del Pezzo fibrations

Proposition 3.1.1. *In Setup 3.0.2, suppose that there exists a Mori contraction $\varphi: X \rightarrow \mathbb{P}^1$. Then $X \subset \mathbb{P}(\mathcal{E})$ such that $X \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) + 4F|$, where F is a general fibre of $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$, and*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Since $-K_X$ is divisible by $m \geq 2$ in $\text{Pic}(X)$, we deduce from the classification of Mori (see Section 2.3) that $m = 2$ or 3 .

Case 1. If $m = 3$, then φ is a \mathbb{P}^2 -bundle and we can write $X = \mathbb{P}(\mathcal{E})$ where \mathcal{E} is a vector bundle over \mathbb{P}^1 of rank 3. Denote the tautological line bundle by $\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then the Grothendieck relation gives

$$\xi^3 - \xi^2 \cdot \varphi^*(c_1(\mathcal{E})) = 0.$$

Hence

$$\begin{aligned} (-K_X)^3 &= (3\xi + \varphi^*(-K_{\mathbb{P}^1} - c_1(\mathcal{E})))^3 \\ &= 27\xi^3 + 27\xi^2 \cdot \varphi^*(\mathcal{O}_{\mathbb{P}^1}(2) - c_1(\mathcal{E})) \\ &= 27\xi^2 \cdot \varphi^*(\mathcal{O}_{\mathbb{P}^1}(2)) \\ &= 54 \end{aligned}$$

which contradicts the fact that $K_X^3 = 0$.

Case 2. If $m = 2$, then $\varphi: X \rightarrow \mathbb{P}^1$ is a quadric bundle with general fibre $F_X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and every fibre is a smooth quadric or a quadric cone in \mathbb{P}^3 . Define $\mathcal{E} := \varphi_*(\mathcal{O}_X(H))$ which is a vector bundle on \mathbb{P}^1 of rank

$$r = h^0(F_X, H|_{F_X}) = h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)) = 4.$$

Now the morphism $\varphi^*\mathcal{E} \rightarrow \mathcal{O}_X(H)$ is surjective as it is the evaluation map on each fibre and the restriction of H on each fibre is base-point-free. Hence it gives an embedding $X \subset \mathbb{P}(\mathcal{E})$ such that $H = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_X$. Let $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$ such that $\varphi = \pi|_X$.

We write $\mathcal{E} = \bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_1 \geq a_2 \geq a_3 \geq a_4$. Denote a tautological divisor associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ by ξ and a general fibre of π by F . Since

$$K_{\mathbb{P}(\mathcal{E})} \sim -4\xi + \pi^*(K_{\mathbb{P}^1} + c_1(\mathcal{E}))$$

and $K_X \sim -2\xi|_X$, we deduce from the adjunction formula that

$$X \in |2\xi + \alpha F|$$

with $\alpha = -c_1(\mathcal{E}) + 2$, because the morphism $\text{Pic}(\mathbb{P}(\mathcal{E})) \rightarrow \text{Pic}(X)$ is injective (indeed $\text{Pic}(\mathbb{P}(\mathcal{E})) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and both ξ , and F are non-trivial and linearly independent on X).

On the other hand, by the Grothendieck relation, we have

$$\xi^4 - \xi^3 \cdot \pi^*(c_1(\mathcal{E})) = 0.$$

Hence

$$0 = H^3 = (\xi|_X)^3 = \xi^3 \cdot (2\xi + \alpha F) = 2c_1(\mathcal{E}) + \alpha.$$

Therefore, $c_1(\mathcal{E}) = -2$ and $\alpha = 4$.

Since $h^0(\mathbb{P}^1, \mathcal{E}) = h^0(X, \mathcal{O}_X(H)) = 2$ by Corollary 2.5.4, there are two possibilities: either

$$a_1 = 1, a_2 = a_3 = a_4 = -1,$$

or

$$a_1 = a_2 = 0, a_3 = a_4 = -1.$$

Now suppose that $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(-1)^{\oplus 3}$. Then $\text{Bs } |\xi| = \mathbb{P}(\mathcal{O}(-1)^{\oplus 3}) =: D_0$ and $\xi \sim D_0 + F$.

Since $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi - X)) = H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-\xi - 4F)) = 0$, we deduce from the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi - X) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi) \rightarrow \mathcal{O}_X(\xi) \rightarrow 0$$

that the restriction morphism $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi)) \rightarrow H^0(X, \mathcal{O}_X(H))$ is injective, hence surjective as $h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi)) = h^0(X, \mathcal{O}_X(H))$.

Therefore, when we restrict the base locus D_0 of $|\xi|$ to X , we have

$$D_0 \cap X \subset \text{Bs } |H|.$$

But this implies that the base locus of $|H|$ on X has a divisorial part, which contradicts the fact that $|H|$ is mobile on X . \square

Remark 3.1.2. *In the setting of Proposition 3.1.1, $\varphi: X \rightarrow \mathbb{P}^1$ is a quadric bundle with general fibre $F_X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $|-K_X| = |2H|$. Let D be a general member of $|H|$, then $\mathcal{O}_{F_X}(D) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. Hence, a general fibre of $\varphi: D \rightarrow \mathbb{P}^1$ is isomorphic to \mathbb{P}^1 or two \mathbb{P}^1 's intersecting transversally at one point.*

Proof of Theorem 3.0.1 (1'). Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ and $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$ be the projection morphism. Denote a tautological divisor associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ by ξ , and a general fibre of π by F . Let X be a very general member in $|2\xi + 4F|$. Since $S^2\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(4)$ is globally generated, a general member in $|2\xi + 4F|$ is smooth. As

$$K_{\mathbb{P}(\mathcal{E})} \sim -4\xi + \pi^*(K_{\mathbb{P}^1} + c_1(\mathcal{E})),$$

the adjunction formula gives $K_X \sim -2\xi|_X$. Let $H := \xi|_X$, then $-K_X \sim 2H$.

Let $\mathcal{E}_0 := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ and $D_0 := \mathbb{P}(\mathcal{E}_0)$. Then $D_0 \in |\xi|$ and we have the projection morphism $\pi_0 := \pi|_{D_0}: D_0 \rightarrow \mathbb{P}^1$ and a tautological divisor $\xi_0 \sim \xi|_{D_0}$ associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E}_0)}(1)$. Let $H_0 := X \cap D_0$. Then $H_0 \in |H|$. Let $S_0 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Then $S_0 \in |\xi_0|$.

Since $R^i\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = 0$ for all $i > 0$,

$$\begin{aligned} H^1(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(X - D_0)) &\simeq H^1(\mathbb{P}^1, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(4)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 2}) \\ &= 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} H^1(D_0, \mathcal{O}_{D_0}(H_0 - S_0)) &\simeq H^1(\mathbb{P}^1, \mathcal{E}_0 \otimes \mathcal{O}_{\mathbb{P}^1}(4)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 2}) \\ &= 0. \end{aligned}$$

Therefore, the evaluation maps

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(X)) \rightarrow H^0(D_0, \mathcal{O}_{D_0}(X))$$

and

$$H^0(D_0, \mathcal{O}_{D_0}(H_0)) \rightarrow H^0(S_0, \mathcal{O}_{S_0}(H_0))$$

are surjective. On the other hand, since $2\xi + 4F$ is globally generated, its restriction to D_0 (resp. to S_0) is globally generated. Hence, by the surjectivity of the above evaluation maps, we deduce that $H_0 = X \cap D_0$ (resp. $C_0 := X \cap S_0$) is smooth for a general $X \in |2\xi + 4F|$.

Claim. H (and thus $-K_X$) is nef.

For any curve $C \subset \mathbb{P}(\mathcal{E})$ such that $\xi \cdot C < 0$, we have $C \subset S_0$.

Denote the two rulings of S by f_1 and f_2 , where $f_1 := F|_S$ and f_2 is mapped surjective to \mathbb{P}^1 by π . Then

$$\xi|_{S_0} = \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})}(1) \sim -f_1 + f_2.$$

Therefore,

$$X|_{S_0} \sim (2\xi + 4F)|_{S_0} \sim 2(-f_1 + f_2) + 4f_1 = 2(f_1 + f_2).$$

Now suppose by contradiction that there exists an integral curve $C \subset X$ such that $-K_X \cdot C < 0$. Then $\xi \cdot C = \xi|_X \cdot C < 0$, and thus $C \subset X \cap S_0$. But $C_0 = X \cap S_0$ is a smooth irreducible curve (it is a smooth elliptic curve), we deduce that

$$[C] = [2(f_1 + f_2)],$$

which implies that

$$\xi \cdot C = \xi|_{S_0} \cdot C = (-f_1 + f_2) \cdot 2(f_1 + f_2) = 0.$$

This contradicts the fact that $\xi \cdot C < 0$. Hence H is nef, and this proves the claim.

By the adjunction formula, one has that $-K_{H_0} \sim H|_{H_0}$ is nef with $(-K_{H_0})^2 = 0$ and $C_0 \in |-K_{H_0}|$. Furthermore, π induces a fibration on H_0 over \mathbb{P}^1 with general fibre isomorphic to \mathbb{P}^1 .

Now let S be the blow-up of \mathbb{P}^2 at 9 points in very general position such that $-K_S$ is nef, not semi-ample, and the unique member $D \in |-K_S|$ is a smooth elliptic curve. Denote the blow-up by $\sigma: S \rightarrow \mathbb{P}^2$. Let $h = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and C_i be a conic on S , i.e. a smooth rational curve such that $-K_S \cdot C_i = 2$ and $C_i^2 = 0$. Here, for example, we take C_i the strict transform of a general line through a blown up point $p_i \in \mathbb{P}^2$ such that $C_i \sim h - e_i$, where e_i is the exceptional curve over p_i . Then the class of C_i induces a conic bundle $\tau: S \rightarrow \mathbb{P}^1$.

Since $\tau: S \rightarrow \mathbb{P}^1$ is a regular conic bundle, one has

$$R^i \tau_*(\mathcal{O}_S(-K_S)) = 0$$

for all $i > 0$ and $\tau_*(\mathcal{O}_S(-K_S))$ is a locally free sheaf of rank 3 that we denote by \mathcal{V} . Therefore,

$$H^k(\mathbb{P}^1, \mathcal{V}) \simeq H^k(S, \mathcal{O}_S(-K_S))$$

for all $k \geq 0$ and thus $\chi(\mathbb{P}^1, \mathcal{V}) = \chi(S, \mathcal{O}_S(-K_S)) = 1$. By the Grothendieck-Riemann-Roch theorem, one has

$$\chi(\mathbb{P}^1, \mathcal{V}) = \deg(\mathcal{V}) + 3,$$

and thus $c_1(\mathcal{V}) = -2$.

Since $h^0(\mathbb{P}^1, \mathcal{V}) = h^0(S, \mathcal{O}_S(-K_S)) = 1$, we can write

$$\mathcal{V} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

with $a, b < 0$. As $a + b = -2$, we deduce that $a = b = -1$. Therefore,

$$S \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) = D_0,$$

and we have $S \in |(2\xi + 4F)|_{D_0}|$ by the adjunction formula. Hence by semicontinuity of cohomology, the surface $H_0 = X \cap D_0$ has nef and not semi-ample anticanonical divisor for a very general element $X \in |2\xi + 4F|$. Thus $-K_X$ is not semi-ample by Lemma 2.5.5. \square

3.2 Conic bundles

Proposition 3.2.1. *In Setup 3.0.2, suppose that there exists a Mori contraction $\varphi: X \rightarrow Y$ to a surface Y . Then $X = \mathbb{P}(\mathcal{E})$ is a \mathbb{P}^1 -bundle over Y with $-K_Y$ nef, \mathcal{E} is a nef rank-2 vector bundle with $c_1(\mathcal{E}) = -K_Y$ and $c_2(\mathcal{E}) = K_Y^2$, given by an extension*

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_Y(-K_Y) \rightarrow 0,$$

where \mathcal{I}_Z is the ideal sheaf of a length- $c_2(\mathcal{E})$ subscheme Z of Y . Furthermore, the set of such X forms a bounded family.

Proof. By the classification of Mori (see Section 2.3), φ is a conic bundle and Y is a smooth rational surface. Since $-K_X$ is divisible by $m \geq 2$ in $\text{Pic}(X)$, we deduce from the classification that $m = 2$ and φ is a \mathbb{P}^1 -bundle. By Proposition 2.4.8, the anticanonical bundle $-K_Y$ is nef. Let $d := (-K_Y)^2$, thus we have that $0 \leq d \leq 9$, and that Y is either $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{P}^2 blown up in $(9 - d)$ points.

We write $X = \mathbb{P}(\mathcal{E})$ with $\mathcal{E} = \varphi_*(\mathcal{O}_X(H))$. Then $H \in |\xi|$, where ξ is a tautological divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. As $-K_X = 2H$ and

$$-K_X \sim \varphi^*(-K_Y - \det(\mathcal{E})) + 2\xi,$$

one has $c_1(\mathcal{E}) = \det(\mathcal{E}) = -K_Y$.

On the other hand, since $(-K_X)^3 = 0$, one has

$$0 = \xi^3 = c_1^2(\mathcal{E}) - c_2(\mathcal{E}),$$

from which we deduce that $c_2(\mathcal{E}) = K_Y^2 = d$.

Claim. \mathcal{E} has a section which vanishes in codimension at least 2.

Suppose by contradiction that every non-zero section in $H^0(Y, \mathcal{E})$ vanishes in codimension 1. Let $s \in H^0(Y, \mathcal{E})$ be a non-zero section and H_s the element in $|H|$ associated to s . Let D be the one-dimensional component of the vanishing locus of s taken with multiplicity. Now consider the vector bundle $\mathcal{E}' := \mathcal{E} \otimes \mathcal{O}_Y(-D)$. Then it has a non-zero section $s' \in H^0(Y, \mathcal{E}')$ which vanishes in codimension at least 2. We denote the element associated to s' in $|\mathcal{O}_{\mathbb{P}(\mathcal{E}')}|$ by $H_{s'}$. Then

one has an isomorphism $X \simeq \mathbb{P}(\mathcal{E}')$ under which $H_{s'}$ corresponds to $H_s \otimes \varphi^*(-D)$. Hence, there exists an effective divisor R on X (which corresponds to $H_{s'}$) such that

$$H_s = \varphi^*(D) + R.$$

Notice that R is non-zero as the restriction of H_s to a general fibre is $\mathcal{O}_{\mathbb{P}^1}(1)$. Since this holds for every non-zero section $s \in H^0(Y, \mathcal{E})$, it contradicts the fact that H is irreducible and reduced. This proves the claim.

Therefore, following [Br96, Sect. 4.1, p. 85–87], we have an exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \mathcal{J}_Z \otimes \mathcal{O}_Y(-K_Y) \rightarrow 0, \quad (3.1)$$

where Z is the zero locus of a general section of \mathcal{E} with length $c_2(\mathcal{E}) = d$.

If $d = 0$, then we have $Z = \emptyset$, and (3.1) must split as

$$\mathrm{Ext}^1(\mathcal{O}_Y(-K_Y), \mathcal{O}_Y) \simeq H^1(Y, \mathcal{O}_Y(K_Y)) \simeq H^1(Y, \mathcal{O}_Y) = 0.$$

Thus $\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-K_Y)$. Consider the case when $d > 0$. For a fixed smooth rational surface Y such that $-K_Y$ is nef, Z is a finite subscheme of length $d = K_Y^2$ on Y . Hence it is parameterised by the Hilbert scheme $Y^{[d]}$. Furthermore, the extensions (3.1) are parameterised by the vector space $\mathrm{Ext}^1(\mathcal{J}_Z \otimes \mathcal{O}_Y(-K_Y), \mathcal{O}_Y)$ of finite dimension. Therefore, the varieties $\mathbb{P}(\mathcal{E})$ such that \mathcal{E} is a vector bundle of rank 2 over Y satisfying (3.1) form a bounded family.

Now since smooth rational surfaces Y with $-K_Y$ nef form a bounded family, we deduce that $X = \mathbb{P}(\mathcal{E})$ form a bounded family as well. \square

Remark 3.2.2. *In the setting of Proposition 3.2.1, one has $X = \mathbb{P}(\mathcal{E})$, where \mathcal{E} is a rank-2 vector bundle on the surface Y , and $-K_X \sim 2H$, where H is a tautological divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.*

Let D be a general member in $|H|$. Since \mathcal{E} is given by the short exact sequence (3.1), one has

$$D = \mathrm{Bl}_Z(Y).$$

Example 3.2.3. Let S be \mathbb{P}^2 blown up at 9 points in sufficiently general position, such that $-K_S$ is nef and not semi-ample. Then there exists a unique element $D \in |-K_S|$. We have $\kappa(-K_S) = 0$ and $K_S^2 = 0$.

Now define $\mathcal{E} := \mathcal{O}_S \oplus \mathcal{O}_S(-K_S)$ and $\pi: X := \mathbb{P}(\mathcal{E}) \rightarrow S$. Then \mathcal{E} is nef, and thus $-K_X \sim 2\xi$ is nef, where ξ is a tautological divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Furthermore, we have $c_1(\mathcal{E}) = D$ and $c_2(\mathcal{E}) = 0$.

For $n \in \mathbb{N}^*$, we have

$$h^0(X, \mathcal{O}_X(-nK_X)) = h^0(S, S^{2n}(\mathcal{E})) = 2n + 1.$$

Hence $\kappa(-K_X) = 1$.

Now we consider the sections associated to π . Notice that for any extension

$$0 \rightarrow L \rightarrow \mathcal{E} \rightarrow Q \rightarrow 0,$$

where L and Q are line bundles on S , we have

$$\mathbb{P}(Q) = \xi - \pi^*(L).$$

Hence there are two types of sections: either it corresponds to the quotient $\mathcal{E} \rightarrow \mathcal{O}_S(-K_S) \rightarrow 0$ and thus gives an element $D_1 \simeq S$ such that $D_1 \in |\xi|$, or it corresponds to the quotient $\mathcal{E} \rightarrow \mathcal{O}_S \rightarrow 0$ and thus gives an element $D_2 \simeq S$ such that $D_2 \in |\xi - \pi^*D|$. Therefore, there are two types of elements in $|\xi|$: one of the form D_1 , and the other of the form $D_2 + \pi^*D$, where D_1 and D_2 are two disjoint sections of π .

Since $D_1 \in |\xi|$ moves, D_1^2 is an effective 1-cycle. By the Grothendieck relation, one has

$$\xi^2 - \xi \cdot \pi^*c_1(\mathcal{E}) \sim 0.$$

Hence $\xi^2 \sim D_1^2 \sim D_1 \cdot \pi^*D$ is a non-zero effective 1-cycle represented by $D_1 \cap \pi^*D$ which is a curve isomorphic to D . Furthermore,

$$\xi^3 = \xi \cdot (\pi^*D)^2 = 0$$

as $D^2 = 0$. Therefore, $\nu(-K_X) = 2$.

Proof of Theorem 3.0.1 (2'). Let \mathcal{E} be a nef rank-2 vector bundle on a smooth rational surface Y with nef anticanonical divisor such that $c_1(\mathcal{E}) = -K_Y$, $c_2(\mathcal{E}) = (-K_Y)^2$, fitting into a sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{E} \rightarrow \mathcal{J}_Z \otimes \mathcal{O}_Y(-K_Y) \rightarrow 0, \quad (3.2)$$

where \mathcal{J}_Z is the ideal sheaf of a subscheme Z of $c_2(\mathcal{E})$ points in sufficiently general position on Y . Let $X = \mathbb{P}(\mathcal{E})$ and ξ be a tautological divisor. Let H be a general member in $|\xi|$. Then

$$-K_X \sim 2H,$$

and $(-K_X)^3 = 8\xi^3 = 8(c_1(\mathcal{E})^2 - c_2(\mathcal{E})) = 0$. Furthermore, the sequence (3.2) gives

$$H \simeq \text{Bl}_Z(Y), \quad N_{H/X} = -K_H.$$

Since Y is a smooth rational surface with $-K_Y$ nef (and thus Y is the blow-up of \mathbb{P}^2 at $9 - (-K_Y)^2$ points in almost general position), and the $c_2(\mathcal{E})$ points are in sufficiently general position, we deduce that $-K_H$ is nef and not semi-ample. Hence, $-K_X$ is nef and not semi-ample by Lemma 2.5.5. \square

3.3 Birational contractions

Proposition 3.3.1. *In Setup 3.0.2, suppose that there exists a birational Mori contraction $\varphi: X \rightarrow Y$. Then $-K_X$ is divisible by 2 in $\text{Pic}(X)$, Y is a smooth almost del Pezzo threefold of degree 1, and φ is the blow-up at a point $p \in Y$. Furthermore, if we write $|-K_Y| = |2H_Y|$, then $p \notin \text{Bs}|H_Y|$.*

Proof. Since $-K_X$ is divisible by $m \geq 2$, by the classification of Mori contractions on smooth threefolds (see Section 2.3), one has $m = 2$ and φ is the blow-up at a smooth point p on Y with exceptional divisor $E \simeq \mathbb{P}^2$, and $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence, $-K_Y$ is nef by Proposition 2.4.7, and

$$(-K_Y)^3 = (-K_X)^3 + (2E)^3 = 8,$$

i.e. $-K_Y$ is big.

On the other hand,

$$-K_Y = \varphi_*(-K_X) = 2\varphi_*(H) =: 2H_Y$$

with $H_Y \in \text{Pic}(Y)$. Then H_Y is nef and big with $(H_Y)^3 = 1$. We conclude that Y is an almost del Pezzo threefold of degree 1, and the base scheme of $|H_Y|$ is one point by [JP08, Sect. 2].

If p is the base point of $|H_Y|$, then $\text{Bs}|H| = \emptyset$ as the base scheme of $|H_Y|$ is one point. This is absurd because $|-K_X|$ is not semi-ample. \square

Proof of Theorem 3.0.1 (3'). Let Y be a smooth almost del Pezzo threefold of degree one. Then a general member in $|-\frac{1}{2}K_Y|$ is a smooth almost del Pezzo surface of degree one. Now fix a general member $D_Y \in |-\frac{1}{2}K_Y|$. Since D_Y is the blow-up of \mathbb{P}^2 at 8 points in almost general position, by choosing a sufficiently general point $p \in D_Y \subset Y$, the blow-up $D := \text{Bl}_p D_Y$ of D_Y at the point p has nef and not semi-ample anticanonical divisor. Let $\varphi: X \rightarrow Y$ be the blow-up of Y at p . Then

$$-K_X = 2(\varphi^*D_Y - E) =: 2H,$$

and $D \in |H|$. Therefore, we deduce by Lemma 2.5.5 that $-K_X$ is nef and not semi-ample. \square

Remark 3.3.2. *In the setting of Proposition 3.3.1, let D_Y be a general member in $|H_Y|$ and D be a general member in $|H|$. Then D_Y is a smooth almost del Pezzo surface of degree one, i.e. D_Y is isomorphic to \mathbb{P}^2 blown up at 8 points, and thus D is isomorphic to \mathbb{P}^2 blown up at 9 points.*

Chapter 4

Anticanonical system with a non-zero fixed divisor

In this chapter, we will prove the following main result:

Theorem 4.0.1. *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system has a non-zero fixed divisor. Then there exists a finite sequence of flops $\psi: X \dashrightarrow X'$ such that the following holds:*

- X' is smooth,
- $-K_{X'}$ is nef,
- the mobile part $|B'|$ of $|-K_{X'}|$ is nef.

In this case, $B'^2 = 0$ so that $|B'|$ is base-point-free and induces a fibration $f: X' \rightarrow \mathbb{P}^1$. Furthermore, a general fibre F of f is a smooth surface with $-K_F$ nef and effective, and $(-K_F)^2 = 0$, $n(-K_F) = 2$.

Throughout this chapter, we consider the case when the anticanonical system $|-K_X|$ has a non-zero fixed divisor, and we can write $|-K_X| = A + |B|$ with A the fixed divisor and $|B|$ the mobile part. By Corollary 2.5.4, we have $|B| = |mH|$ where $m \geq 2$ and H is some prime divisor.

Proposition 4.0.2. *Let X be a smooth projective rationally connected threefold with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. If the anticanonical system*

$$|-K_X| = A + |mH|, \quad m \geq 2$$

has a non-zero fixed divisor A , then there exists a finite sequence of flops $\psi: X \dashrightarrow X'$ such that X' is smooth with $-K_{X'}$ nef and $H' := \psi_(H)$ is nef.*

Proof. Fix a general member $F \in |H|$. Since X is smooth, for sufficiently small $\epsilon > 0$, the pair $(X, \epsilon F)$ is log canonical. It follows from Lemma 2.5.1 that if F is not nef, then there exists a $(K_X + \epsilon F)$ -negative extremal ray Γ such that $\epsilon F \cdot \Gamma < 0$. Let c_Γ be the contraction of the extremal ray Γ and l a contracted curve. Thus $F \cdot l < 0$, which implies $l \subset \text{Bs}(|H|)$. But $|H|$

is mobile, it follows that c_Γ is small. This implies that $K_X \cdot l = 0$ since there is no flipping contraction for smooth threefolds. Hence there exists a flop of c_Γ and the flopped threefold X^+ is smooth by [Kol89, Thm. 2.4].

By repeating the above argument and by the termination of three-dimensional flops (see [KM98, Cor. 6.19]), we deduce that there exists a sequence of flops $\psi : X \dashrightarrow X'$ such that $H' := \psi_*(H)$ is nef. \square

Lemma 4.0.3. *In the setting of Proposition 4.0.2, if H is nef, then*

$$A^3 = A^2 \cdot H = A \cdot H^2 = H^3 = 0.$$

Proof. As $-K_X$ is nef, one has $K_X^2 \cdot A \geq 0$ and $K_X^2 \cdot H \geq 0$. Then

$$0 = (-K_X)^3 = K_X^2 \cdot (A + mH)$$

gives $K_X^2 \cdot A = K_X^2 \cdot H = 0$. From this we further conclude that

$$0 = -K_X \cdot (A + mH) \cdot H = -K_X \cdot (A \cdot H + mH^2).$$

Since H moves, one has that $A \cdot H$ and H^2 are effective 1-cycles. This implies that

$$-K_X \cdot A \cdot H = -K_X \cdot H^2 = 0.$$

Hence, $A^2 \cdot H + mA \cdot H^2 = 0$ and $A \cdot H^2 + mH^3 = 0$. As H is nef, and $A \cdot H, H^2$ are effective 1-cycles, we deduce that

$$A \cdot H^2 = H^3 = 0.$$

This implies $A^2 \cdot H = 0$. Together with $K_X^2 \cdot A = 0$, we conclude that $A^3 = 0$. \square

After performing possibly a sequence of flops, the mobile part $|B| = |mH|$ of the anticanonical system $|-K_X|$ becomes nef. In this case, either $B^2 = 0$ and we are in the case described in Theorem 4.0.1, or B^2 is a non-zero effective 1-cycle which is the case that we will study in the rest of the chapter. We will show that this latter case does not occur.

4.1 Description of the anticanonical system

First we study the anticanonical system in this case, and we will give a description of the geometry of its fixed divisor, as well as a general member of its mobile part.

Proposition 4.1.1. *Consider as above a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system $|-K_X| = A + |mH|$ with $m \geq 2$ has a non-zero fixed divisor A , and that H is nef such that H^2 is a non-zero effective 1-cycle. Let F be a general member of $|H|$. Then $-K_F$ is nef, effective and divisible by $r \geq 2$ in $\text{NS}(F)$. Furthermore, $\kappa(F, -K_F) = 0$, $K_F^2 = 0$, and F is not covered by $(-K_F)$ -trivial curves.*

Proof. By the adjunction formula, we have

$$-K_F \sim -(K_X + F)|_F \sim A|_F + (m-1)F|_F.$$

As F is nef, it suffices to show that $A|_F$ is nef to obtain the nefness of $-K_F$: suppose that there exists an irreducible curve $l \subset F$ such that $A|_F \cdot l < 0$. Then l is an irreducible component of $C := A \cap F$. On the other hand, F is nef, and $F \cdot C = 0$ as $A \cdot F^2 = 0$, from which we deduce that $F \cdot l = 0$. Hence

$$-K_X \cdot l = A \cdot l + mF \cdot l = A \cdot l < 0,$$

which contradicts the fact that $-K_X$ is nef. Therefore, the restriction $A|_F$ is nef.

Note that $A|_F$ cannot be zero: since $-K_X$ is nef with numerical dimension two, the support of a divisor $D \in \lfloor -K_X \rfloor$ is connected in codimension one by [Sha99, Lem. 2.3.9].

Now let $\nu: \tilde{F} \rightarrow F$ be a desingularisation of the surface F . Since $A|_F$ and $F|_F$ are nef Cartier divisors such that $A|_F \cdot F|_F = 0$, their pullbacks to the desingularisation \tilde{F} remain nef and orthogonal to each other. Let

$$V := \langle \nu^*(A|_F), \nu^*(F|_F) \rangle \subset \text{NS}(\tilde{F}).$$

Let H be an ample divisor on \tilde{F} , then $\text{NS}(\tilde{F}) = \mathbb{R}H \oplus (\mathbb{R}H)^\perp$. If $\dim V \geq 2$, then $\dim(V \cap (\mathbb{R}H)^\perp) \geq 1$. Hence, there exists $v \in V \cap (\mathbb{R}H)^\perp$ which is non zero, and $v^2 < 0$ by the Hodge index theorem. But $v = \lambda\nu^*(A|_F) + \mu\nu^*(F|_F)$ with $\lambda, \mu \in \mathbb{R}$, which implies that $v^2 \geq 0$. This is absurd. Hence $\dim V = 1$, i.e. $\nu^*(A|_F)$ and $\nu^*(F|_F)$ are non-zero and numerically proportional. Hence $-K_F$ is divisible by $r \geq 2$ with $r \in \mathbb{N}$.

The surface F is not covered by $(-K_F)$ -trivial curves: otherwise, F is covered by $(-K_X)$ -trivial curves as $-K_F \sim -K_X|_F - F|_F$ and $-K_F$ is numerically proportional to $F|_F$. As F moves in X , this implies that X is covered by $(-K_X)$ -trivial curves. This is absurd because $n(-K_X) = 3$.

Furthermore, as $A^2 \cdot F = A \cdot F^2 = F^3 = 0$, we have $K_F^2 = 0$.

It remains to show that $\kappa(F, -K_F) = 0$. Indeed, for any $n \in \mathbb{N}$, we have

$$1 \leq h^0(F, \mathcal{O}_F(-nK_F)) \leq h^0(\tilde{F}, \nu^*(-nK_F)).$$

If $h^0(\tilde{F}, \nu^*(-nK_F)) \geq 2$ for some n , then the linear system $|\nu^*(-nK_F)|$ has some non-zero mobile part M on \tilde{F} , and $\nu^*(-K_F) \cdot M = 0$ as $(-K_F)^2 = 0$ and $-K_F$ is nef. Thus \tilde{F} is covered by $\nu^*(-K_F)$ -trivial curves, from which we deduce that F is covered by $(-K_F)$ -trivial curves. This is absurd. \square

In order to get a more precise description of the geometric structure of A and F , we need the following lemma:

Lemma 4.1.2. *Let S be a projective Gorenstein surface such that the anticanonical divisor $-K_S$ is of the form:*

$$-K_S \sim D_1 + D_2,$$

where D_1 is effective and D_2 is a non-zero effective Cartier divisor which is nef and divisible by $r \geq 2$ in $\text{NS}(S)$.

Suppose that $D_2^2 = 0$, and that one of the following assertions holds:

- (i) S is not covered by D_2 -trivial curves;
- (ii) D_2 contains a smooth curve of positive genus.

Then $D_1 = 0$, and S is normal with at most rational singularities. Furthermore, the surface \tilde{S} obtained by the minimal resolution of S is relatively minimal.

Proof. Special case. Assume that S is smooth. Suppose by contradiction that D_1 is not zero. Since D_2 is divisible by $r \geq 2$ in $\text{NS}(S)$, we put $rL \equiv_{\text{num}} D_2$, with L nef and $L^2 = 0$. Then

$$-(K_S + rL) \equiv_{\text{num}} D_1,$$

and we deduce that the adjoint bundle $K_S + rL$ is not nef as D_1 is effective.

Now for every irreducible reduced curve $l \subset S$ such that $(K_S + rL) \cdot l < 0$, one has $K_S \cdot l < 0$ since L is nef. Then by the cone theorem, there exists a K_S -negative extremal ray R which is $(K_S + rL)$ -negative. We denote the contraction of the extremal ray R by $\phi: S \rightarrow Z$.

1. If $\dim Z = 1$, then $\phi: S \rightarrow Z$ is a \mathbb{P}^1 -bundle over a smooth algebraic curve Z . Let f be a fibre of ϕ , then $f^2 = 0$ and $(K_S + rL) \cdot f < 0$.
Since $K_S \cdot f = -2$ by the adjunction formula, together with $L \cdot f \geq 0$ and $r \geq 2$, we have $L \cdot f = 0$.
2. If Z is a point, then $S = \mathbb{P}^2$. But L is nef, not ample and not numerically trivial, this is absurd.

If ϕ is birational, let $l \in R$ be an integral contracted curve, then l is actually a (-1) -curve since we contract a K_S -negative extremal ray. Hence $L \cdot l = 0$ as $(K_S + rL) \cdot l < 0$ and L is nef.

Now we put $L' := \phi_*(L)$ and $D'_1 := \phi_*(D_1)$. Then

$$-K_Z \equiv_{\text{num}} rL' + D'_1.$$

Since $L \cdot l = 0$, we know by the contraction theorem that $L \simeq \phi^*(L')$. Hence L' is nef and $L'^2 = 0$.

Notice that the two assertions in the lemma are preserved by the contraction ϕ . More precisely,

- (i) if Z is covered by $\phi_*(D_2)$ -trivial curves, then S is covered by D_2 -trivial curves as $L = \phi^*(L')$;
- (ii) $\phi_*(D_2)$ contains a curve of positive genus, as ϕ does not contract any curve of positive genus.

Moreover, Z cannot be a minimal surface. Indeed if K_Z is nef, then

$$K_S + rL = \phi^*(K_Z) + C + rL$$

is pseudo-effective, where C is an effective divisor supported on the exceptional locus. Therefore, $K_S + rL$ is zero as it is anti-effective. This is absurd because D_1 is not zero.

Therefore, by running a $(K_S + rL)$ -Minimal Model Program, we can suppose that S is a \mathbb{P}^1 -bundle as described in the first case above. Now we show that this will lead to a contradiction:

- (i) We first consider the case when S is not covered by D_2 -trivial curves: since $L \cdot f = 0$ for every fibre f of ϕ , the surface S is covered by L -trivial curves. Hence S is covered by D_2 -trivial curve, which is absurd.
- (ii) For the case when D_2 contains a smooth curve of positive genus: since

$$D_2 \cdot f = rL \cdot f = 0$$

for a general fibre f of ϕ , D_2 is contained in some special fibre of the \mathbb{P}^1 -bundle. This is absurd because D_2 contains a curve of positive genus.

Therefore, $D_1 = 0$. Furthermore, since $-K_S \sim D_2$ is divisible by $r \geq 2$ in $\text{NS}(S)$, the surface S does not contain any (-1) -curve, i.e. S is relatively minimal.

General case. Let $\nu: \bar{S} \rightarrow S$ be the normalisation of S , and $\mu: \tilde{S} \rightarrow \bar{S}$ the minimal resolution of \bar{S} . We put $\pi := \nu \circ \mu: \tilde{S} \rightarrow S$. Computing the anticanonical bundles, we get

$$-K_{\bar{S}} = \nu^*(-K_S) + E_1$$

with some effective Weil divisor E_1 supported on the zero locus of the conductor ideal and

$$-K_{\tilde{S}} = \pi^*(-K_S) + \tilde{E}_1 + E_2$$

with \tilde{E}_1 the proper transform of E_1 in \tilde{S} and E_2 some effective divisor supported on the exceptional locus.

Now \tilde{S} is a smooth surface such that

$$-K_{\tilde{S}} \sim \tilde{D}_1 + \tilde{D}_2$$

with $\tilde{D}_1 := \tilde{E}_1 + E_2 + \pi^*(D_1)$ effective divisor, $\tilde{D}_2 := \pi^*(D_2)$ non-zero, effective, nef and divisible by $r \geq 2$ in $\text{NS}(\tilde{S})$.

Furthermore, one has $\tilde{D}_2^2 = D_2^2 = 0$ and \tilde{D}_2 satisfies one of the two assertions in the lemma if D_2 does:

- (i) if \tilde{S} is covered by $\pi^*(D_2)$ -trivial curves, then S is covered by D_2 -trivial curves;
- (ii) $\pi^*(D_2)$ contains a smooth curve of positive genus which is mapped surjectively onto the one contained in D_2 .

Hence by the previous smooth case, we deduce that $\tilde{D}_1 = 0$. This implies that $D_1 = 0$ and S is normal as it is Cohen-Macaulay, with at worst rational singularities.

Let $\mu: \tilde{S} \rightarrow S$ be the minimal resolution of S . Then $-K_{\tilde{S}} = \mu^*(-K_S) \sim \mu^*(D_2)$ is divisible by $r \geq 2$ in $\text{NS}(\tilde{S})$. Thus \tilde{S} does not contain any (-1) -curve, i.e. \tilde{S} is relatively minimal. \square

Corollary 4.1.3. *In the setting of Lemma 4.1.2, the surface S is smooth. It is a \mathbb{P}^1 -bundle over a smooth elliptic curve.*

Proof. Let $\mu: \tilde{S} \rightarrow S$ be the minimal resolution of S . Then by Lemma 4.1.2 $-K_{\tilde{S}} = \mu^*(-K_S)$ is non-zero, effective and nef. Hence \tilde{S} is uniruled and thus it admits a Mori fibration. Furthermore, since \tilde{S} is relatively minimal by Lemma 4.1.2, we deduce that \tilde{S} is a \mathbb{P}^1 -bundle over a smooth curve.

Now by the classification in Lemma 2.4.5, the surface \tilde{S} is either an elliptic fibration or a \mathbb{P}^1 -bundle over a smooth elliptic curve. In both of the two cases, we deduce that \tilde{S} is a \mathbb{P}^1 -bundle over a smooth elliptic curve and $S = \tilde{S}$. \square

Lemma 4.1.4. *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system $|-K_X| = A + |mH|$ with $m \geq 2$ has a non-zero fixed divisor A , and H is nef such that H^2 is a non-zero effective 1-cycle. Let F be a general member of $|H|$. Then F is a smooth surface such that $-K_F$ is nef and divisible by 2 in $\text{NS}(F)$ with $\nu(-K_F) = 1$, $\kappa(-K_F) = 0$. More precisely, $F = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a rank-2 vector bundle over an elliptic curve as described in Lemma 2.4.5, 2.(B). Furthermore, we have $m = 2$, and $A \cap F$ (resp. the intersection of two general members in $|H|$) is a smooth elliptic curve.*

Proof. By Proposition 4.1.1, we have that $-K_F$ is non-zero, effective, nef and divisible by $r \geq 2$. Furthermore, $(-K_F)^2 = 0$ and F is not covered by $(-K_F)$ -trivial curves. Hence we can apply Lemma 4.1.2 and Corollary 4.1.3 to obtain that the surface F is a \mathbb{P}^1 -bundle over a smooth elliptic curve. Now since F is not covered by $(-K_F)$ -trivial curves, i.e. $n(-K_F) = 2$, we deduce from the classification in Lemma 2.4.5 that $F = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a rank-2 vector bundle over an elliptic curve defined as in the case (B) of the same lemma.

Since $-K_F \sim A|_F + (m-1)F|_F$ with $F|_F$ and $A|_F$ non-zero effective, we deduce from the structure of the unique element in $|-K_F|$ that $m = 2$, and $A \cap F$ (resp. the intersection of two general members in $|H|$) is a smooth elliptic curve. \square

Lemma 4.1.5. *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system $|-K_X| = A + |mH|$ with $m \geq 2$ has a non-zero fixed divisor A , and H is nef such that H^2 is a non-zero effective 1-cycle. Then A is an integral smooth surface such that $-K_A$ is nef and divisible by 2 in $\text{Pic}(A)$ with $\nu(-K_A) = 1$. More precisely, the surface A is a \mathbb{P}^1 -bundle over a smooth elliptic curve.*

Proof. Let F be a general member in $|H|$. As $A|_F$ is an irreducible reduced curve by Lemma 4.1.4, we can find a divisor A_1 which occurs in A with multiplicity one, and the rest A' does not meet F . Since $m = 2$, and $A \cap F$ is a smooth elliptic curve by the Lemma 4.1.4, the adjunction formula gives

$$-K_{A_1} \sim (A' + 2F)|_{A_1} = A'|_{A_1} + 2C_0,$$

where C_0 is a smooth elliptic curve and $A'|_{A_1}$ is an effective divisor on A_1 .

Moreover, since F is nef and $A \cdot F^2 = 0$, C_0 is nef and $C_0^2 = 0$ on A_1 .

Now we can apply Lemma 4.1.2 and Corollary 4.1.3 to the surface A_1 , which gives $A'|_{A_1} = 0$ and A_1 is a \mathbb{P}^1 -bundle over a smooth elliptic curve. Moreover, the support of a divisor $D \in |-K_X|$ is connected in codimension one by [Sha99, Lem. 2.3.9]. As A' does not meet F , and $A'|_{A_1} = 0$, we obtain $A' = 0$. Thus $A = A_1$ and $-K_A \sim 2F|_A$. \square

In conclusion, we have the following description of the anticanonical system:

Proposition 4.1.6. *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system has a non-zero fixed divisor and that its mobile part $|B|$ is nef. If $B^2 \neq 0$, then $|-K_X| = A + |2H|$ where H is a prime divisor. In this case, let F be a general member in $|H|$. Then both A and F are \mathbb{P}^1 -bundles over a smooth elliptic curve, such that their anticanonical divisors $-K_A$ (resp. $-K_F$) are nef and divisible by 2 in $\text{Pic}(A)$ (resp. $\text{NS}(F)$). Furthermore, $A \cap F$ (resp. the intersection of two general members in $|H|$) is a smooth elliptic curve.*

Proof. It follows from Lemma 4.1.4 and Lemma 4.1.5. □

4.2 Running the Minimal Model Program

In this subsection, we consider the following setup:

Setup 4.2.1. *Let X be a smooth projective rationally connected threefold X with $-K_X$ nef, $n(-K_X) = 3$, $\nu(-K_X) = 2$. Suppose that the anticanonical system $|-K_X| = A + |mH|$ with $m \geq 2$ has a non-zero fixed divisor A , and H is nef such that H^2 is a non-zero effective 1-cycle.*

Remind that in this setup, one has $|-K_X| = A + |2H|$, both A and a general member F in $|H|$ are \mathbb{P}^1 -bundles over a smooth elliptic curve, such that their anticanonical divisors are nef and divisible by two in $\text{Pic}(A)$ (resp. in $\text{NS}(F)$). Furthermore, $A \cap F$ (resp. the intersection of two general members in $|H|$) is a smooth elliptic curve.

Consider a Mori contraction $\varphi: X \rightarrow Y$. Let Γ be the extremal ray contracted by φ . Let l be a rational curve such that $[l] \in \Gamma$ and that $-K_X \cdot l = l(\Gamma)$, where $l(\Gamma)$ is the length of Γ . In the birational case, we denote the exceptional divisor of φ by E .

Now we are in the position to run the Minimal Model Program in the setting of Prop 4.1.6.

4.2.1 Non-birational cases

In this subsection, we will show that the contraction $\varphi: X \rightarrow Y$ cannot be of Mori fibre type.

Case $\dim Y = 1$. In this case, $-K_X \cdot l = 1, 2$ or 3 . Recall that for a Mori contraction $\varphi: X \rightarrow \mathbb{P}^1$, all the fibres are irreducible. Since A is the fixed divisor of $|-K_X|$, it cannot be a fibre of φ . As for H , since H^2 is a non-zero effective cycle, it cannot be a fibre of φ . We deduce that $A \cdot l > 0$ and $H \cdot l > 0$, as $\text{Pic}(X)$ is generated by a fibre of φ and another element which has positive intersection with l . Therefore, $-K_X \cdot l = 3$, $A \cdot l = H \cdot l = 1$, and φ is a \mathbb{P}^2 -bundle over \mathbb{P}^1 .

Now we can write $X = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a rank-3 vector bundle over $Y = \mathbb{P}^1$. After twisting \mathcal{E} by some line bundle, we can suppose that $\mathcal{E} = \varphi_* \mathcal{O}_X(H)$ with H a tautological divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Since H is nef, the vector bundle \mathcal{E} is nef. From the fact that a vector bundle on \mathbb{P}^1 is nef if and only if it is generated by its global sections, we deduce that \mathcal{E} is generated by its global sections. Therefore, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is also generated by its global sections. Since $h^0(X, \mathcal{O}_X(H)) = 2$ by Corollary 2.5.4, and $H^2 \neq 0$, we have $\text{Bs}|H| \neq \emptyset$. This leads to a contradiction.

Case $\dim Y = 2$. In this case, $\varphi: X \rightarrow Y$ is a conic bundle and we have $-K_X \cdot l = 1$ or 2 .

- (i) If $F \cdot l = 0$, then we have $F = \varphi^*(C)$, where C is an irreducible curve on Y . Hence the intersection of two general members in $|H|$ is contained in some fibres of φ . This is absurd because such an intersection is a smooth elliptic curve, but φ only contracts rational curves.
- (ii) If $F \cdot l = 1$, then φ is a \mathbb{P}^1 -bundle and induces a birational morphism from F to Y . This is impossible since $q(F) = 1$ and $q(Y) = 0$.

4.2.2 Birational contractions

Since X is a smooth threefold, the contraction φ is divisorial.

Case $A \cdot l = 0$. In this case, we have $F \cdot l = 1$ and $-K_X \cdot l = A \cdot l + 2F \cdot l = 2$. Hence φ is the blow-up of Y at a smooth point, with exceptional divisor $E \simeq \mathbb{P}^2$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. Now the adjunction formula $K_E \sim (K_X + E)|_E$ gives

$$\mathcal{O}_E(A) \otimes \mathcal{O}_E(2F) = \mathcal{O}_E(2).$$

As $E \cdot F$ is a non-zero effective 1-cycle, we deduce that $A \cdot E = 0$ and $F|_E \sim l$. Hence, $E \cap F$ is a line on $E \simeq \mathbb{P}^2$. On the other hand, we have

$$(E|_F)^2 = F \cdot E^2 = F \cdot (-l) = -1.$$

Hence $E \cap F$ is a (-1) -curve on the surface F , which contradicts the fact that F is relatively minimal.

Case $A \cdot l < 0$. Since the contraction is divisorial, we have $E = A$ in this case. Since A is a ruled surface over a smooth elliptic curve, we know that l is a fibre of A and $F \cdot l = 1$. Therefore, φ is the blow-up along an elliptic curve in Y , and Y is smooth with $-K_Y$ nef by Proposition 2.4.7. Furthermore, as we contract the curves meeting F transversally, we conclude that $G := \varphi(F) \simeq F$. Since

$$-K_Y = \varphi_*(-K_X) = \varphi_*(A + 2F) = 2\varphi_*(F) = 2G,$$

we see that $|-K_Y| = |2G|$ has no fixed divisor.

We can compute the Kodaira dimension and the numerical dimension for $-K_Y$:

$$\kappa(-K_Y) = \kappa(\varphi^*(-K_Y)) = \kappa(-K_X + E),$$

and similarly, for the numerical dimension we have

$$\nu(-K_Y) = \nu(-K_X + E).$$

On the other hand, since $E = A$, we have

$$\kappa(-K_X) \leq \kappa(-K_X + E) \leq \kappa(-2K_X) = \kappa(-K_X),$$

and similarly,

$$\nu(-K_X) \leq \nu(-K_X + E) \leq \nu(-2K_X) = \nu(-K_X).$$

Hence, $\kappa(-K_Y) = \kappa(-K_X) = 1$ and $\nu(-K_Y) = \nu(-K_X) = 2$.

Case $A \cdot l > 0$. In this case, $F \cdot l = 0$ since otherwise $-K_X \cdot l > 2$, which contradicts the classification of Mori (see Section 2.3). Furthermore, $E \neq A$ and thus $A \cdot E$ is an effective 1-cycle. We will show that the only possible case is when φ contract E to a smooth curve of positive genus.

By the classification of Mori (see Section 2.3), we are in one of the following cases:

(1) If E is contracted to a point, then one of the following cases occurs:

(i) $E \simeq \mathbb{P}^2$, $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. In this case, we have $A \cdot l = 2$ and the adjunction formula $K_E \sim (K_X + E)|_E$ gives

$$\mathcal{O}_E(A) \otimes \mathcal{O}_E(2F) \simeq \mathcal{O}_E(2).$$

As $A \cdot E$ is a non-zero effective 1-cycle, we deduce $F|_E = 0$ and $\mathcal{O}_E(A) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$.

(ii) $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. In this case, we have $A \cdot l = 1$ and the adjunction formula gives

$$\mathcal{O}_E(A) \otimes \mathcal{O}_E(2F) \simeq \mathcal{O}_E(1, 1).$$

As $A \cdot E$ is a non-zero effective 1-cycle, we deduce $F|_E = 0$ and $\mathcal{O}_E(A) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.

(iii) E is a quadric cone in \mathbb{P}^3 with $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_E$. In this case, we have $A \cdot l = 1$ and the adjunction formula gives

$$\mathcal{O}_E(A) \otimes \mathcal{O}_E(2F) \simeq \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_E = \mathcal{O}_E(2l).$$

But since $F|_E$ is Cartier, one cannot have $F|_E = l$ which is 2-Cartier. Hence, $\mathcal{O}_E(A) \simeq \mathcal{O}_E(2l)$, $F|_E = 0$.

(iv) $E \simeq \mathbb{P}^2$, $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$. In this case, we have $A \cdot l = 1$ and the adjunction formula gives

$$\mathcal{O}_E(A) \otimes \mathcal{O}_E(2F) \simeq \mathcal{O}_E(1).$$

As $A \cdot E$ is a non-zero effective 1-cycle, we deduce $F|_E = 0$ and $\mathcal{O}_E(A) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$.

We now show that E cannot be contracted to a point. Suppose that we are in one of the above cases, then $F \cdot E = 0$, and $A \cap E$ is some rational curve(s). On the other hand, as A is a ruled surface over an elliptic curve, this implies that $E|_A$ consists of some fibres on A . But $F|_A$ is an elliptic curve which is a section, hence $E|_A \cdot F|_A > 0$. This contradicts $F \cdot E = 0$.

(2) If φ contracts E to a smooth curve $C \subset Y$ of genus g , then $E \simeq \mathbb{P}(\mathcal{N}_{C/Y}^*)$. Let $\mathcal{V} = \mathcal{N}_{C/Y}^* \otimes \mathcal{L}$ (with \mathcal{L} some line bundle on C) be the normalisation of the conormal bundle [Har77, Chapter. V, Prop. 2.8]. Then $\mathcal{N}_{E/X} = \mathcal{O}_E(-C_1 + \mu l)$ where C_1 is the minimal section satisfying $C_1^2 = c_1(V) =: -d$ and $\mu := \deg \mathcal{L}$.

In this case, one has $-K_X \cdot l = 1$, $F \cdot l = 0$ and $A \cdot l = 1$. Hence $F|_E \sim bl$ with $b \geq 0$ and the adjunction formula gives

$$-K_E \sim (A + 2F)|_E - E|_E,$$

i.e. $A|_E \sim C_1 + (d + \mu + 2(1 - g - b))l$.

Since F (resp. A) is a \mathbb{P}^1 -bundle over a smooth elliptic curve, we deduce that the intersection $F \cap E$ (resp. $A \cap E$) does not contain the curve l or any of its deformations, since otherwise l moves on the surface F (resp. A). Therefore, $F \cdot E = 0$ and $A \cap E$ is a section of $\varphi|_E: E \rightarrow C$. In particular, $\varphi(F) \simeq F$ as $E \cdot F = 0$, and all the curves l meet A transversally in one point, which implies that $\varphi|_A$ is an isomorphism.

Now by the same argument as in the case (1), we deduce that the integral curve $A \cap E$ cannot be a rational curve. Hence C is of genus $g > 0$. By Proposition 2.4.7, $-K_Y$ is again nef.

Hence we have the following proposition:

Proposition 4.2.2. *In Setup 4.2.1, let $\varphi: X \rightarrow Y$ be a Mori contraction. Then φ is the blow-up of a smooth curve C of positive genus in the smooth threefold Y with $-K_Y$ nef, $\kappa(-K_Y) = 1$, $\nu(-K_Y) = 2$. Let E be the exceptional divisor of φ . Then one of the following two cases occurs:*

- (1) $E = A$ and we have $|-K_Y| = |2G|$ with $G := \varphi(F) \simeq F$. Furthermore, the blown up curve C is a smooth elliptic curve contained in $\text{Bs}|G|$.
- (2) $E \neq A$ and $E \cdot F = 0$. We have $|-K_Y| = A_Y + |2F_Y|$, where $A_Y := \varphi(A) \simeq A$, $F_Y := \varphi(F) \simeq F$, and the intersection of two general members in $|F_Y|$ is a smooth elliptic curve. In particular, Y satisfies again Setup 4.2.1.

Proof. (1) It remains to prove the last sentence of the first case. Since $|-K_X| = A + |2H|$, one has $h^0(X, \mathcal{O}_X(H)) = 2$ by Corollary 2.5.4.

Now consider the threefold Y , since the anticanonical system $|-K_Y| = |2G|$ has no fixed divisor, and again $-K_Y$ is nef with $n(-K_Y) = 3$, $\nu(-K_Y) = 2$, one has that $h^0(Y, \mathcal{O}_Y(G)) = 2$ by the Corollary 2.5.4.

Since F is the strict transform of G by φ , we deduce from $h^0(X, \mathcal{O}_X(F)) = h^0(Y, \mathcal{O}_Y(G))$ that the blown up elliptic curve C must be contained in the base locus of $|G|$.

(2) Since $E \cdot F = 0$, we have that $\varphi^*(F_Y) \sim F$. We deduce that F_Y is nef as $\varphi^*(F_Y) \sim F$ is nef.

We first show that $-K_Y$ is not semi-ample, which implies $\kappa(-K_Y) = 1$ and $\nu(-K_Y) = 2$.

Since F^2 is a non-zero effective 1-cycle and $E \cdot F = 0$, we deduce that $F_Y^2 = \varphi(F)^2$ is also a non-zero effective 1-cycle. Since F_Y moves, $A_Y \cdot F_Y$ is an effective 1-cycle. By the adjunction formula, we get

$$-K_{F_Y} \sim (-K_Y - F_Y)|_{F_Y} \sim (A_Y + F_Y)|_{F_Y},$$

and thus

$$-K_Y|_{F_Y} \sim -K_{F_Y} + F_Y|_{F_Y}$$

is a non-zero effective divisor on F_Y such that $-K_Y|_{F_Y} \leq -2K_{F_Y}$, i.e.

$$h^0(F_Y, \mathcal{O}_{F_Y}(-2K_{F_Y} - (-K_Y))) > 0.$$

Suppose by contradiction that $-K_Y$ is semi-ample, then $|-mK_Y|$ is base-point-free for $m \gg 0$. Hence $|-mK_Y|_{F_Y}$ is also base-point-free.

On the other hand, since $F_Y \simeq F$, we have $\kappa(F_Y, -K_{F_Y}) = 0$. Hence,

$$1 \leq h^0(F_Y, \mathcal{O}_{F_Y}(-mK_Y)) \leq h^0(F_Y, \mathcal{O}_{F_Y}(-2mK_{F_Y})) = 1.$$

Therefore, the linear system $|-mK_Y|_{F_Y}$ is fixed, which contradicts the fact that $|-mK_Y|_{F_Y}$ is base-point-free.

Now we show that the anticanonical system $|-K_Y|$ has a fixed divisor. Since F_Y is mobile, it is then clear that A_Y is the fixed divisor of $|-K_Y|$.

Suppose by contradiction that $|-K_Y|$ has no fixed divisor, then $-K_Y$ is divisible by two in $\text{Pic}(Y)$ by Theorem 3.0.1. As $-K_Y \sim A_Y + 2F_Y$, this implies $A_Y \sim 2L$ for some $L \in \text{Pic}(Y)$. Hence

$$-K_{F_Y} \sim (A_Y + F_Y)|_{F_Y} \sim (2L + F_Y)|_{F_Y}.$$

Since $F_Y \simeq F$, the surface F_Y is a \mathbb{P}^1 -bundle over a smooth elliptic curve such that $-K_{F_Y} \cdot f = 2$, where f is a fibre. Since $F \cap F'$ (where F' is another general member in $|F|$) a smooth elliptic curve (i.e. a section of the \mathbb{P}^1 -bundle F), and $E \cdot F = 0$, we deduce that $F_Y \cap F'_Y$ (where F'_Y is another general member in $|F_Y|$) is also a smooth elliptic curve (i.e. a section of the \mathbb{P}^1 -bundle F_Y). Thus

$$F_Y|_{F_Y} \cdot f = 1.$$

This implies that $2L|_{F_Y} \cdot f = 1$, which contradicts the fact that $L|_{F_Y}$ is a Cartier divisor. \square

Remark 4.2.3. *In the setting of Proposition 4.2.2 (2), we deduce by the same proposition that there exists a finite sequence*

$$X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_k} X_k$$

where

- φ_i is the blow-up of X_i along a smooth curve C_i of positive genus;
- X_i satisfies again Setup 4.2.1;
- X_k has a birational Mori contraction which contracts the fixed divisor A_k of $|-K_{X_k}|$.

Furthermore, the curve C_i is contained in A_i , where A_i is the fixed divisor of $|-K_{X_i}|$ and is a \mathbb{P}^1 -bundle over a smooth elliptic curve D_i . Then C_k is an elliptic curve and $k = 1$.

Proof. For $1 \leq i \leq k$, let g_i be a fibre of the \mathbb{P}^1 -bundle A_i .

Since C_i has positive genus and it is contained in the \mathbb{P}^1 -bundle A_i , it must be surjective to the curve D_i . Let α_i be the degree of C_i onto the elliptic curve D_i . Then g_i meets C_i at α_i point(s). Hence in X_{i-1} , we have

$$E_{i-1} \cdot g_{i-1} = \alpha_i,$$

where E_{i-1} is the exceptional divisor of φ_i , and g_{i-1} is the strict transform of g_i . Therefore,

$$-K_{X_{i-1}} \cdot g_{i-1} = \varphi_k^*(-K_{X_i}) \cdot g_{i-1} - E_{i-1} \cdot g_{i-1} = -K_{X_i} \cdot g_i - \alpha_i.$$

Since $-K_{X_{i-1}}$ is nef, we deduce that $-K_{X_i} \cdot g_i - \alpha_i \geq 0$, and thus $-K_{X_i} \cdot g_i \geq 1$.

For $i = k$, since A_k is the exceptional divisor of a Mori contraction, we have that $-K_{X_k} \cdot g_k = 1$. Hence, $\alpha_k = 1$ (which implies $C_k \simeq D_k$ is a smooth elliptic curve), and $-K_{X_{k-1}} \cdot g_{k-1} = 0$ (which implies $k = 1$). \square

To summarise, we have the following:

Proposition 4.2.4. *In the setting of Proposition 4.1.6, one has*

$$X \xrightarrow{\varphi_1} Y \quad \text{or} \quad X \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} Y,$$

where X_1 is a smooth threefold with $-K_{X_1}$ nef such that $|-K_{X_1}|$ has a non-zero fixed divisor, φ_i (for $i = 1, 2$) is the blow-up along a smooth elliptic curve, and Y is a smooth threefold with $-K_Y$ nef, $n(-K_Y) = 3$, $\nu(-K_Y) = 2$ such that $|-K_Y| = |2G|$ has no fixed divisor and a general member in $|G|$ is isomorphic to F .

Proof. It follows from Proposition 4.2.2 and Remark 4.2.3. \square

Proof of Theorem 4.0.1. Let $|B|$ be the mobile part of the anticanonical system $|-K_X|$. By Proposition 4.0.2, there exists a finite sequence of flops $\psi: X \dashrightarrow X'$ such that $-K_{X'}$ is nef and the mobile part $|B'|$ of $|-K_{X'}|$ is nef.

Now we consider the case when B is nef and suppose by contradiction that B^2 is a non-zero effective 1-cycle. Then by Proposition 4.2.4, one has

$$X \xrightarrow{\varphi_1} Y \quad \text{or} \quad X \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} Y,$$

where φ_i (for $i = 1, 2$) is the blow-up along a smooth elliptic curve, and Y is one of the cases described in Theorem 3.0.1 with $|-K_Y| = |2G|$. Moreover, a general member $D \in |G|$ is isomorphic to F , where F is a general member in $|H|$. Hence, D is a \mathbb{P}^1 -bundle over a smooth elliptic curve as described in Lemma 2.4.5, 2.(B).

On the other hand, D is in one of the following cases:

1. If Y is a del Pezzo fibration: $\phi: Y \rightarrow \mathbb{P}^1$, then by Remark 3.1.2, $\phi: D \rightarrow \mathbb{P}^1$ induces a fibration on D with general fibre isomorphic to \mathbb{P}^1 or two \mathbb{P}^1 's intersecting at one point.
2. If Y is a \mathbb{P}^1 -bundle over a smooth rational surface, then by Remark 3.2.2, D is a rational surface.
3. If Y has a birational Mori contraction, then D is a rational surface by Remark 3.3.2.

Hence $D \not\cong F$, which gives a contradiction.

The last statement on the general fibre of f follows from [BP04, Lem. 7.4]. \square

In view of Theorem 4.0.1 and Lemma 4.0.3, we will consider the following setup in the rest of Part I and discuss the geometry of the threefold X according to different geometry of the general fibre:

Setup 4.2.5. *Let X be a smooth projective rationally connected threefold with anticanonical bundle $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. We suppose that the anticanonical system $|-K_X|$ has a non-zero fixed divisor A , and that its mobile part $|B|$ is nef so that $B^2 = 0$. Hence, the mobile part induces a fibration $f: X \rightarrow \mathbb{P}^1$.*

If F is a fibre of f , then $|-K_X| = A + |kF|$ with $k \geq 2$. Furthermore, $A^3 = A^2 \cdot F = 0$, and $-K_F$ is nef with $n(-K_F) = 2$, $\nu(-K_F) = 1$.

Now we write $A = A_h + A_v$, where A_h and A_v are effective divisors such that $A_h|_F = -K_F$ and $A_v|_F = 0$ for a general fibre F .

Let us point out an important special case under the setup. Suppose that the relative anticanonical divisor $-K_{X/\mathbb{P}^1}$ is nef. Then the fibration f is locally trivial (isomorphic to the product family) in the Euclidean topology, see [PZ19, Thm. A12] and [Cao19, Prop. 2.8]. Applying the latter proposition, we further obtain $X \simeq F \times \mathbb{P}^1$; we explain how to apply the proposition in our case, as follows.

Note that in [Cao19, Prop. 2.8], the assumption is different. Instead of assuming (H1): *the relative anticanonical divisor is nef*, they assume (H2): *there exists an f -very ample line bundle L such that $f_*(mL)$ is a numerically flat vector bundle for every integer $m \leq 1$* . However, when the fibration is over a smooth curve, [PZ19, Prop. A11] shows that (H1) implies (H2). As the fibration is over \mathbb{P}^1 (which is simply connected) in our case, [Cao19, Prop. 2.8] gives $X \simeq F \times \mathbb{P}^1$.

Chapter 5

Rational general fibre: smooth elliptic case

In this chapter, we consider the following setting:

Setup 5.0.1. *Under Setup 4.2.5, we further assume that F is rational (so F is \mathbb{P}^2 blown up in 9 points) and that there exists a unique member B in $|-K_F|$, which is a smooth elliptic curve. By [BP04, Prop. 7.7], one has $k = 2$, $A \simeq B \times \mathbb{P}^1$, and f restricted to A is the second projection.*

5.1 Running the Minimal Model Program

Here is the main result of this section:

Proposition 5.1.1. *Assume that X is as Setup 5.0.1. Then one of the following cases occurs:*

- (1) *A is nef. Then the relative anticanonical divisor $-K_{X/\mathbb{P}^1}$ is nef, and the fibration f is locally trivial.*
- (2) *A is the exceptional divisor of a Mori contraction. Then X is obtained by the blow-up $\varphi: X \rightarrow X'$ along a smooth elliptic curve R of a smooth threefold X' with $-K_{X'}$ nef, $n(-K_{X'}) = 3$, and $\nu(-K_{X'}) = 2$. Furthermore, $|-K_{X'}| = |2G|$ with $G := \varphi(F) \simeq F$ has no fixed divisor, and $\text{Bs}|G| = R$.*
- (3) *A is the exceptional divisor of a crepant extremal contraction. Then there exists a birational Mori contraction $\varphi: X \rightarrow X'$ such that the fibration f factors as $f = f' \circ \varphi$ with $f': X' \rightarrow \mathbb{P}^1$, and we are in one of the following cases:*
 - (i) *X' again satisfies the setup: $-K_{X'}$ is nef such that $|-K_{X'}| = |A' + 2F'|$ with $A' := \varphi(A) \simeq A$, $F' := \varphi(F) \simeq F$, and A' is the exceptional divisor of a Mori contraction. In this case, φ is the blow-up along an elliptic curve contained in some fibre of $f'|_{A'}$.*
 - (ii) *$-K_{X'}$ is nef and big such that $|-K_{X'}| = |A' + 2F'|$, where $A' := \varphi(A) \simeq A$, and the restriction $\varphi|_F: F \rightarrow F' := \varphi(F)$ is the blow-down of some (-1) -curves in F with $-K_{F'}$ nef and big. In this case, φ is the blow-up along a smooth curve in A' , which is mapped surjectively onto \mathbb{P}^1 by f' .*

Definition 5.1.2. Let X be a smooth projective threefold with $-K_X$ nef. We say that X has structure (I_0) if the following conditions are satisfied:

- there exists a fibration $f: X \rightarrow \mathbb{P}^1$ such that $|-K_X| = |A + 2F|$ with F a general fibre;
- F is a smooth rational surface with $-K_F$ nef such that $|-K_F|$ contains a smooth elliptic curve B ;
- $A \simeq B \times C$ with $C = \mathbb{P}^1$ such that $f|_A$ is the second projection.

Proposition 5.1.3. In case (3)(ii) of Proposition 5.1.1, by running the Minimal Model Program, either X' becomes a smooth almost del Pezzo threefold, or it stays weak Fano with the structure (I_0) , and the outcome is a Mori fibre space $\varphi': X' \rightarrow Y$ which is one of the following:

- (a) Y is smooth rational surface, either $Y = F'$, $X' = \mathbb{P}^1 \times Y$ and φ' is the second projection, or $Y = \mathbb{P}^1 \times \mathbb{P}^1$, F' is a ruled surface over \mathbb{P}^1 and $\varphi': X' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a \mathbb{P}^1 -bundle.
- (b) $\varphi' = f'$ is a del Pezzo fibration over \mathbb{P}^1 .

Proof of Prop. 5.1.1 (1)(2). (1) If the relative anticanonical divisor $A = -K_{X/\mathbb{P}^1}$ is nef, then the fibration f is locally trivial by [PZ19, Thm. A.12].

(2) Suppose that A is the exceptional divisor of a Mori contraction $\varphi: X \rightarrow X'$. Let l be a rational curve which generates the extremal ray Γ corresponding to the contraction φ and such that $-K_X \cdot l = l(\Gamma)$, where $l(\Gamma)$ is the length of the extremal ray Γ .

Since $A \simeq B \times \mathbb{P}^1$, we know that l is a fibre of the first projection of A , and that $F \cdot l = 1$ as $A|_F = B$. As we contract the curves meeting F transversally, we conclude that $G := \varphi(F) \simeq F$. Therefore, φ is the blow-up along a smooth elliptic curve R in X' , and X' is smooth with $-K_{X'}$ nef by Proposition 2.4.7. Furthermore, we have that the anticanonical system $|-K_{X'}| = |2G|$ has no fixed divisor.

Now we can compute the Kodaira dimension and the numerical dimension for $-K_{X'}$:

$$\kappa(-K_{X'}) = \kappa(\varphi^*(-K_{X'})) = \kappa(-K_X + A),$$

and similarly, for the numerical dimension, we have

$$\nu(-K_{X'}) = \nu(-K_X + A).$$

On the other hand,

$$\kappa(-K_X) \leq \kappa(-K_X + A) \leq \kappa(-2K_X) = \kappa(-K_X)$$

and similarly,

$$\nu(-K_X) \leq \nu(-K_X + A) \leq \nu(-2K_X) = \nu(-K_X),$$

we deduce that $\kappa(-K_{X'}) = \kappa(-K_X) = 1$ and $\nu(-K_{X'}) = \nu(-K_X) = 2$. This implies that $-K_{X'}$ is not semi-ample and thus $n(-K_{X'}) = 3$. Therefore, $\text{Bs } |G|$ has dimension at least one. As $\text{Bs } |G| \subset R$ by construction, one has that $\text{Bs } |G| = R$. \square

The rest of this section is devoted to proving Propositions 5.1.1 and 5.1.3.

Lemma 5.1.4. *Assume that X has structure (I_0) . If $-K_X$ is semi-ample, then $-K_F$ is semi-ample.*

Proof. Suppose that $-K_X$ is semi-ample, then $\mathcal{O}_X(-mK_X)$ is globally generated for $m \gg 0$. Hence its restriction $\mathcal{O}_F(-mK_X)$ to F is also globally generated. As

$$-K_F \sim (-K_X - F)|_F \sim -K_X|_F,$$

we deduce that $-K_F$ is semi-ample. \square

Lemma 5.1.5. *Assume that X has structure (I_0) . If A is not nef, then A is the exceptional divisor of an extremal contraction ϕ which contracts A to a smooth curve and one of the following two cases occurs:*

- (i) $A \cdot C = -1$ and ϕ is a Mori contraction.
- (ii) $A \cdot C = -2$ and ϕ is a crepant extremal contraction.

Proof. If A is not nef, then by Lemma 2.5.1 there exists a $(K_X + A)$ -negative extremal ray Γ such that $A \cdot \Gamma < 0$. Let l be an integral curve such that the class $[l]$ generates Γ . Then $2F \cdot l = -(K_X + A) \cdot l > 0$ and $A \cdot l < 0$, i.e. l is an f -horizontal curve contained in A . Therefore, $\Gamma = \mathbb{R}_+[C]$.

Since C moves in A , the contraction ϕ of the extremal ray Γ is divisorial. Moreover, as the image of $N_1(A)$ in $N_1(X)$ has dimension two (this is because $F \cdot B = 0$ and $F \cdot C = 1$, we deduce that numerical classes $[B]$ and $[C]$ are not proportional in $N_1(X)$), ϕ contracts A to a curve. Hence either ϕ is a Mori contraction and $A \cdot C = -1$, or ϕ is a crepant contraction and $A \cdot C = -2$. \square

5.1.1 Birational case

Proposition 5.1.6. *Assume that X has structure (I_0) and $A \cdot C = -2$. Let $\varphi: X \rightarrow X'$ be a birational Mori contraction with E the exceptional divisor. Then E is mapped to a curve, and the fibration f factors as $f = f' \circ \varphi$ such that $f': X' \rightarrow \mathbb{P}^1$ gives X' the structure (I_0) : $-K_{X'} \sim A' + 2F'$ is nef, where $A' := \varphi(A) \simeq A$ and $F' := \varphi(F)$. We are in one of the following two cases:*

- (i) E is contained in some fibre of f , $F' \simeq F$ and

$$(-K_{X'})^3 = (-K_X)^3 + 2(-K_X)^2 \cdot F \geq (-K_X)^3.$$

Moreover, A' is the exceptional divisor of a Mori contraction.

- (ii) $\varphi|_F$ is the blow-down of some (-1) -curves in F , $-K_{X'}$ is big with

$$(-K_{X'})^3 > (-K_X)^3.$$

Proof. Step 1. Since $A \simeq B \times C$, one has $A^2 = (A \cdot B)C + (A \cdot C)B$ and thus

$$A^3 = 2(A \cdot B)(A \cdot C).$$

Hence

$$(-K_X)^3 = A^3 + 6A^2 \cdot F = 2(A \cdot B)(A \cdot C) + 6A \cdot B.$$

Since $A \cdot C = -2$ by assumption, one has $(-K_X)^3 = 2A \cdot B$.

Step 2. Let l be a rational curve which generates the extremal ray Γ corresponding to the contraction φ and such that $-K_X \cdot l = l(\Gamma)$, with $l(\Gamma)$ the length of Γ .

By [BP04, Prop. 7.11], E cannot be mapped to a point. Hence φ contracts E to a smooth curve D of genus g and X' is smooth. Notice that $E \neq A$ since $A \cdot C = -2$ and $E \cdot l = -1$. As $-K_X \cdot l = 1$, we obtain $A \cdot l = 1$ and $F \cdot l = 0$. Hence $F \cdot E \sim bl$ with $b \geq 0$.

Notice that $A \cap E$ does not contain the curve l or any of its deformations, since otherwise, the curve l moves on A as A is a product. As $A \cdot l = 1$, we deduce that $A \cap E$ is a section of $\varphi|_E: E \rightarrow D$. We denote this section by D_0 .

Case 1. If $f(E)$ is a point, then E is contained in a special fibre F_0 of f . Hence the fibration f factors as $f = f' \circ \varphi$ such that $f': X' \rightarrow \mathbb{P}^1$, φ is an isomorphism outside F_0 , and $E|_A = aB_0$, where $a > 0$ and B_0 is a special fibre of the second projection $f|_A: B \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

On the other hand, as $D_0 = A \cap E$ is a section of $\varphi|_E: E \rightarrow D$, we deduce that $D_0 = A \cap E = B_0$, and that D is a smooth elliptic curve.

Therefore, $F' := \varphi(F) \simeq F$, $A' = \varphi(A) \simeq A$, and φ contracts E to a smooth elliptic curve in X' , which implies that $-K_{X'}$ is again nef by Proposition 2.4.7. In particular, we have $A' \simeq B' \times C'$, where B' is a smooth elliptic curve and $C' \simeq \mathbb{P}^1$. Since $E \cdot C = E|_A \cdot C = B_0 \cdot C = 1$ and $E \cdot B = E|_A \cdot B = B_0 \cdot B = 0$ on the surface A , one has

$$A' \cdot C' = \varphi^*(A') \cdot C = (A + E) \cdot C = A \cdot C + 1$$

(in particular, $A' \cdot C' = -1$ as $A \cdot C = -2$, and thus A' is the exceptional divisor of a Mori contraction by Lemma 5.1.5), and

$$A' \cdot B' = \varphi^*(A') \cdot B = (A + E) \cdot B = A \cdot B.$$

This implies $A'^2 = (A \cdot B)C' + (A \cdot C + 1)B'$, and $A'^3 = 2(A \cdot B)(A \cdot C + 1)$.

Hence,

$$(-K_{X'})^3 = A'^3 + 6A'^2 \cdot F' = 2(A \cdot B)(A \cdot C + 1) + 6A \cdot B = (-K_X)^3 + 2A \cdot B.$$

As $A \cdot B = (-K_X) \cdot B = (-K_X)^2 \cdot F \geq 0$, we have

$$(-K_{X'})^3 = (-K_X)^3 + 2(-K_X)^2 \cdot F \geq (-K_X)^3.$$

Case 2. If $f|_E$ maps onto \mathbb{P}^1 , then $E \cdot F$ is a non-zero effective 1-cycle. Hence, $E \cdot F \sim bl$ with $b > 0$, and a fibre l_0 of φ must be contained in some fibre F_0 of f . In particular, by the rigidity lemma [Deb01, Lem. 1.15], the fibration f factors as $f = f' \circ \varphi$ such that $f': X' \rightarrow \mathbb{P}^1$,

and that φ contracts E to a smooth curve $D \subset X'$ which is mapped surjectively onto \mathbb{P}^1 by f' . Restricted to a general fibre F , $\varphi|_F$ blows down some (-1) -curves in F and each of them meets $B \in |-K_F|$ transversally in one point. In particular, every curve l meets A transversally in one point, which implies that $\varphi|_A$ is an isomorphism.

Claim. $-K_{X'}$ is nef.

If $g \geq 1$, then $-K_{X'}$ is again nef by Proposition 2.4.7. Now suppose that D is a smooth rational curve. Then $A \cap E = D_0$ is a smooth rational curve. Since B is not rational, we see that D_0 is a fibre of the first projection $A \simeq B \times C \rightarrow B$.

Suppose by contradiction that $-K_{X'}$ is not nef. Then $\mathbb{R}_+[D]$ contains only a finite number of curves, i.e. D is a very rigid curve. On the other hand, since $\varphi(D_0) = D$, we have $\varphi_*([C]) = \varphi_*([D_0]) = m[D]$ for some positive integer m . As C deforms in A and A is not contracted, we deduce that D is not rigid, which leads to a contradiction. This proves the claim.

On the surface A , we can write

$$D_0 \sim \alpha C + \beta B$$

with $\alpha > 0, \beta \geq 0$ integers.

Now we write $A' \simeq B' \times C'$, where B' is a smooth elliptic curve and $C' \simeq \mathbb{P}^1$. Since $E \cdot C = E|_A \cdot C = D_0 \cdot C = \beta$, one has

$$A' \cdot C' = \varphi^*(A') \cdot C = (A + E) \cdot C = A \cdot C + \beta.$$

Since $E \cdot B = E|_A \cdot B = D_0 \cdot B = \alpha$, one has

$$A' \cdot B' = \varphi^*(A') \cdot B = (A + E) \cdot B = A \cdot B + \alpha.$$

Therefore,

$$A'^2 = (A \cdot C + \beta)B' + (A \cdot B + \alpha)C',$$

and $A'^3 = 2(A \cdot B + \alpha)(A \cdot C + \beta)$. Moreover, we have

$$A'^2 \cdot F' = A' \cdot B' = (A \cdot B + \alpha).$$

Hence,

$$(-K_{X'})^3 = A'^3 + 3A'^2 \cdot (2F') = 2(A \cdot B + \alpha)(A \cdot C + \beta) + 6(A \cdot B + \alpha).$$

Since $A \cdot C = -2$, $A \cdot B = (-K_X) \cdot B \geq 0$, and $\alpha > 0, \beta \geq 0$, one has

$$(-K_{X'})^3 = (-K_X)^3 + 2\alpha + 2\alpha\beta + 2\beta A \cdot B > (-K_X)^3 \geq 0.$$

Hence, $-K_{X'}$ is big. □

Proof of Proposition 5.1.1. It follows from Lemma 5.1.5 and Proposition 5.1.6. □

5.1.2 Mori fibre spaces

This subsection is a rewriting of [BP04, Prop. 7.16, Prop. 7.18]. Assume that X has structure (I_0) . Consider a Mori contraction $\varphi: X \rightarrow Y$ of fibre type. Let Γ be an extremal ray contracted by φ . Let l be a rational curve such that $[l]$ generates Γ and that $-K_X \cdot l = l(\Gamma)$ with $l(\Gamma)$ the length of Γ .

Case $\dim Y = 2$. In this case, $-K_X \cdot l = 1$ or 2 and $\varphi: X \rightarrow Y$ is a conic bundle. Since $[l]$ is a movable class, we have $A \cdot l \geq 0$.

- (i) We first consider the case $F \cdot l > 0$. Then $F \cdot l = 1$ and $-K_X \cdot l = A \cdot l + 2F \cdot l = 2$. We deduce that φ is a \mathbb{P}^1 -bundle and $\varphi|_F$ is birational. Consider the product map $p := f \times \varphi: X \rightarrow \mathbb{P}^1 \times Y$ which is generically one to one.

Claim. p is an isomorphism.

Suppose that there exists a curve $D \subset X$ which is contracted by p , then D is also contracted by φ . Hence D is a fibre $\varphi^{-1}(y_0)$ of φ , where $y_0 \in Y$. By the rigidity lemma [Deb01, Lem. 1.15], there exists a neighbourhood $Y_0 \subset Y$ of y_0 and a factorization $f|_{\varphi^{-1}(Y_0)}: \varphi^{-1}(Y_0) \xrightarrow{\varphi} Y_0 \rightarrow \mathbb{P}^1$, which implies $f(D)$ is a point. This is absurd because $F \cdot D = 1$, i.e. D is a f -horizontal curve. This proves the claim.

Therefore, $X \simeq \mathbb{P}^1 \times Y \simeq \mathbb{P}^1 \times F$ and φ is the second projection.

- (ii) The other case is $F \cdot l = 0$. This implies $F = \varphi^*(D)$ for some irreducible curve D on Y , which gives a factorization $f: X \xrightarrow{\varphi} Y \xrightarrow{pr_2} \mathbb{P}^1$, where Y is a smooth rational surface. Notice that $\varphi|_A$ is finite as $A \simeq B \times \mathbb{P}^1$ (where B is a smooth elliptic curve) has no contractible curve. Restricted to a general fibre F , $\varphi|_F$ maps the smooth elliptic curve $A|_F = B$ onto $pr_2^{-1}(f(F)) \simeq \mathbb{P}^1$. Hence, $\varphi|_A$ is not birational and thus $A \cdot \ell \geq 2$, where ℓ is a general conic of φ (and we have that either $[\ell] = [l]$ and $-K_X \cdot \ell = 2$, or $[\ell] = 2[l]$ and $-K_X \cdot \ell = 1$). Since $2 \geq -K_X \cdot \ell = A \cdot \ell \geq 2$, we get equality. Hence, φ restricted to A is generically $2:1$ and $A \cdot \ell = 2$. As $pr_2 \circ \varphi|_A$ is the second projection of A , the ramification locus of $\varphi|_A$ must be equal to some fibre of the first projection of A , which gives $Y = \mathbb{P}^1 \times \mathbb{P}^1$.

Now we show that the discriminant locus Δ of the conic bundle φ is empty, so that φ is a \mathbb{P}^1 -bundle. Indeed, let $Q = \{p\} \times \mathbb{P}^1$ be a general fibre of the first projection $Y \rightarrow \mathbb{P}^1$ and let $X_Q = \varphi^{-1}(Q)$. Then $A|_{X_Q}$ consists of two disjoint sections Q_1 and Q_2 of $\varphi|_{X_Q}$. Since $A \cdot \ell = 2$, we deduce that $Q_i \cdot \ell = 1$ for $i = 1, 2$. Moreover, as Q_i is a Cartier divisor on X_Q , we deduce that $\varphi|_{X_Q}$ is a smooth conic bundle, i.e. X_Q is a ruled surface. Hence Δ is contained in some special fibres of the first projection of Y . On the other hand, as φ is an extremal contraction, by [Miy83, p. 83, Remark], every non singular rational curve in Δ must meet the other components of Δ in at least two points. This implies that Δ is empty.

Therefore, φ is a \mathbb{P}^1 -bundle and $\varphi|_F$ exhibits $F = f^{-1}(p)$ with $p \in \mathbb{P}^1$ as a ruled surface over $pr_2^{-1}(p) \simeq \mathbb{P}^1$.

Case $\dim Y = 1$. In this case, $Y = \mathbb{P}^1$, $-K_X \cdot l = 1, 2$ or 3 and a general fibre of φ is a del Pezzo surface. Since A is not del Pezzo and thus not a fibre of φ , one has $A \cdot l > 0$. If $-K_X \cdot l = 1$

or 2, this implies $F \cdot l = 0$ and thus $F = \varphi^*(p)$ with $p \in Y$. Therefore, f and φ coincide. If $-K_X \cdot l = 3$, then φ is a \mathbb{P}^2 -bundle. As \mathbb{P}^2 is not fibred, F restricted to a \mathbb{P}^2 is trivial. Hence again, the two fibrations φ and f coincide.

Proof of Proposition 5.1.3. It follows from Proposition 5.1.6 and the above discussion. \square

5.2 Examples

Example 5.2.1. (*Case (2) of Prop. 5.1.1*)

Let S be \mathbb{P}^2 blown up in 9 points in sufficiently general position such that $-K_S$ is nef and not semi-ample, and the unique member in $|-K_S|$ is a smooth elliptic curve D . Define $\mathcal{E} := \mathcal{O}_S \oplus \mathcal{O}_S(-K_S)$ and $\pi: Y := \mathbb{P}(\mathcal{E}) \rightarrow S$. Thus \mathcal{E} is nef and thus $-K_Y \sim 2\xi$ is nef, where ξ is a tautological divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Furthermore, $(-K_Y)^3 = 0$, $\kappa(-K_Y) = 1$ and $\nu(-K_Y) = 2$.

As discussed in Example 3.2.3, there are two types of sections associated to π : one of the form $D_1 \in |\xi|$, and the other of the form $D_2 \in |\xi - \pi^*D|$, where D_1 and D_2 are disjoint. Furthermore, $D_1 \cdot (D_2 + \pi^*(D)) = D_1 \cdot \pi^*D$, $D_1^2 \sim D_1 \cdot \pi^*D$, and $R := D_1 \cap \pi^*D$ is a smooth elliptic curve. Since $|\xi|$ has non-empty base locus, we have $\text{Bs}|\xi| \subset R$. Moreover, as ξ is not semi-ample, we deduce that $\text{Bs}|\xi|$ has dimension at least one. Thus $\text{Bs}|\xi| = R$.

Now let $\mu: X := \text{Bl}_R(Y) \rightarrow Y$ be the blow-up of Y along the curve R . Let E be the exceptional divisor and F be the strict transform of D_1 . Then

$$-K_X = \mu^*(-K_Y) - E \sim 2F + E.$$

and the linear system $|F|$ is base-point-free.

Claim. $-K_X$ is nef.

Suppose by contradiction that $-K_X$ is not nef. Then there exists some integral curve $l \subset X$ such that $-K_X \cdot l < 0$. Thus

$$(2F + E) \cdot l < 0,$$

and

$$(2F + 2E) \cdot l = \mu^*(-K_Y) \cdot l \geq 0,$$

from which we deduce $E \cdot l > 0$ and $F \cdot l < 0$. Therefore, $l \subset \text{Bs}|F|$, which contradicts the fact that $|F|$ is base-point-free. This proves the claim.

Since F is mobile, one has $\kappa(X, F) \geq 1$. As

$$\kappa(X, F) \leq \kappa(X, \mu^*(D_1)) = \kappa(Y, \xi),$$

we deduce that $\kappa(X, F) = 1$. Therefore, the linear system $|F|$ induces a fibration $f: X \rightarrow \mathbb{P}^1$. Now since the exceptional divisor $E = \mathbb{P}(\mathcal{N}_{R/Y}^*)$ has two contractions $\mu|_E: E \rightarrow R$ and $f|_E: E \rightarrow \mathbb{P}^1$, we deduce that E is a product that we denote by $B \times C$, where B is a smooth elliptic curve and $C \simeq \mathbb{P}^1$. Therefore, X has structure (I_0) .

Since $F \simeq D_1 \simeq S$, we see that $-K_F$ is not semi-ample. Therefore $-K_X$ is not semi-ample by Lemma 5.1.4. Hence $A := E$ (which is the exceptional locus of a Mori contraction) is the fixed divisor of the anticanonical system $|-K_X|$.

Example 5.2.2. (Case (3)(i) of Prop. 5.1.1)

Let X be as in Example 5.2.1. Let B_0 be a special fibre of $f|_A: A \rightarrow \mathbb{P}^1$.

Let $\mu_1: X_1 := \text{Bl}_{B_0}(X) \rightarrow X$ be the blow-up of X along the smooth elliptic curve B_0 . Let E_1 be the exceptional divisor. Let F_1 be the strict transform of F , and let A_1 be the strict transform of A . Then $\mu_1^*(F) = F_1$, $\mu_1^*(A) = A_1 + E_1$, and thus

$$-K_{X_1} = \mu_1^*(-K_X) - E_1 = 2F_1 + A_1.$$

Claim. $-K_{X_1}$ is nef.

Suppose by contradiction that $-K_{X_1}$ is not nef. Then there exists some integral curve $l \subset X_1$ such that $-K_{X_1} \cdot l < 0$. Thus we have

$$(2F_1 + A_1) \cdot l < 0.$$

As $F_1 = \mu_1^*(F)$ is nef, we have $A_1 \cdot l < 0$. Hence $l \subset A_1$.

On the other hand, since $A_1 \simeq A \simeq B \times C$ with B a general fibre of $f|_A: A \rightarrow \mathbb{P}^1$ and $C \simeq \mathbb{P}^1$ a general fibre of the first projection, we have $A_1 \simeq B_1 \times C_1$ where B_1 (resp. C_1) is the strict transform of B (resp. C). Since $-K_X \cdot B = 0$ and B does not meet B_0 , one has

$$-K_{X_1} \cdot B_1 = \mu_1^*(-K_X) \cdot B_1 = 0.$$

Since $-K_X \cdot C = 1$, and C meets B_0 transversally at one point, one has $\mu_1^*(-K_X) \cdot C_1 = 1$, and $E_1 \cdot C_1 = 1$. Hence,

$$-K_{X_1} \cdot C_1 = \mu_1^*(-K_X) \cdot C_1 - E_1 \cdot C_1 = 0.$$

Since $\overline{\text{NE}}(A_1)$ is generated by B_1 and C_1 , this shows that there is no $(-K_{X_1})$ -negative curve in A_1 , which gives a contradiction. This proves the claim.

Since $F_1 \simeq F$, we deduce that $-K_{F_1}$ is not semi-ample. Therefore, $-K_{X_1}$ is not semi-ample by Lemma 5.1.4. Hence A_1 (which is the exceptional locus of a divisorial crepant extremal contraction) is the fixed divisor of $|-K_{X_1}|$.

For the last two examples of this section, we will need the following lemma:

Lemma 5.2.3. *Let S be a ruled surface over a smooth elliptic B . Suppose that S admits an elliptic fibration $\tau: S \rightarrow \mathbb{P}^1$. If $h^0(S, \mathcal{O}_S(-K_S)) \geq 3$, then $S \simeq B \times \mathbb{P}^1$.*

Proof. Since S is a \mathbb{P}^1 -bundle over a smooth elliptic curve B , and S admits an elliptic fibration, by [Suw69, Thm. 5] we deduce that $S = \mathbb{P}(\mathcal{V})$, where \mathcal{V} is one of the following:

- (a) \mathcal{V} is the unique indecomposable rank-2 vector bundle of degree 1 on B ;
- (b) $\mathcal{V} = \mathcal{O}_B \oplus \mathcal{L}$, where \mathcal{L} is a (possibly trivial) torsion line bundle.

In the first case, let ℓ be a fibre of the ruling, and let θ_i be a section with minimal self-intersection, i.e. $\theta_i^2 = 1$. Then $-K_S \sim 2\theta_i - \ell$. By [Suw69, Thm. 5(iii)], the elliptic fibration is given by the linear system $|4\theta_i - 2\ell|$. Hence,

$$h^0(S, \mathcal{O}_S(-K_S)) = 1.$$

In the second case, we have that

$$h^0(S, \mathcal{O}_S(-K_S)) = h^0(B, S^2\mathcal{V} \otimes \mathcal{L}^*) = h^0(B, \mathcal{L} \oplus \mathcal{L}^* \oplus \mathcal{O}_B) \leq 3,$$

with equality if and only if $\mathcal{L} = \mathcal{O}_B$.

Since $h^0(S, \mathcal{O}_S(-K_S)) \geq 3$ by assumption, we deduce that $\mathcal{V} = \mathcal{O}_B \oplus \mathcal{O}_B$. Thus $S \simeq B \times \mathbb{P}^1$. \square

Example 5.2.4. (*Case (3)(ii) of Prop. 5.1.1*) Consider V a del Pezzo fibration of degree 2 with nef and big anticanonical divisor $-K_V$, which is a two-sheeted cover of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$ (as described in [JPR05, Thm. 2.6]). Then $(-K_V)^3 = 4$.

Let $\varphi: V \rightarrow \mathbb{P}^1$ be the del Pezzo fibration with general fibre F_V , and let $\psi: V \rightarrow Y$ be the anticanonical model. Then Y is a degree-two covering over the weighted projective space $\mathbb{P}(1^2, 2^2)$, and the unique prime divisor $D \sim -K_V - 2F_V$ contracted by ψ is a \mathbb{P}^1 -bundle (induced by $\psi|_D$) over a smooth elliptic curve. Moreover, by the adjunction formula, $-K_D \sim (-K_V - D)|_D \sim 2F_V|_D$. Thus

$$h^0(D, \mathcal{O}_D(-K_D)) = h^0(D, (\varphi|_D)^*(\mathcal{O}_{\mathbb{P}^1}(2))) = 3.$$

Since D is ruled over a smooth elliptic curve and admits an elliptic fibration (induced by $\varphi|_D$), by Lemma 5.2.3, we obtain that $D \simeq B_0 \times C_0$ with B_0 a smooth elliptic curve, and $C_0 \simeq \mathbb{P}^1$. Furthermore, $D|_{F_V} \sim -K_V|_{F_V} \sim -K_{F_V}$, hence $D|_{F_V}$ is an elliptic curve in the anticanonical system $|-K_{F_V}|$.

Let $\phi: X \rightarrow V$ be the blow-up of two rational curves C_1 and C_2 , where $C_i = \{p_i\} \times \mathbb{P}^1$ is a general fibre of $\psi|_D$ for $i = 1, 2$, and p_1, p_2 are two sufficiently general distinct points on a general fibre of $\varphi|_D$. Let E_1 (resp. E_2) be the exceptional divisor over C_1 (resp. over C_2). Let A be the strict transform of D , and let F be the strict transform of F_V . Then $-K_X \sim A + 2F$ with $A \simeq D$. Restricted to a fibre F , the morphism $\phi: F \rightarrow F_V$ is the blow-up at two sufficiently general points on an elliptic curve in $|-K_{F_V}|$. Hence $-K_F$ is nef and not semi-ample.

We can write $A \simeq B \times C$ with B (resp. C) the strict transform of B_0 (resp. C_0). Then

$$-K_X \cdot B = \phi^*(-K_V) \cdot B - E_1 \cdot B - E_2 \cdot B = -K_V \cdot B_0 - 2 = 2 - 2 = 0,$$

and

$$-K_X \cdot C = \phi^*(-K_V) \cdot C - E_1 \cdot C - E_2 \cdot C = -K_V \cdot C_0 = 0.$$

Hence by the same arguments as in Example 5.2.2, we deduce that $-K_X$ is nef. Furthermore, $-K_X$ is not semi-ample by Lemma 5.1.4.

In particular, $A \cdot C = -K_X \cdot C - 2F \cdot C = 0 - 2 = -2$. Hence, A is the exceptional locus of a divisorial crepant extremal contraction.

Example 5.2.5. (*Case (3)(ii) of Prop. 5.1.1*) Consider V a del Pezzo fibration of degree 2 over \mathbb{P}^1 which is a double covering of $Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ branched along $\Gamma \in |4H|$ with H a tautological divisor of $\pi: Y \rightarrow \mathbb{P}^1$ (as described in [Tak09, (5.6.1)]). Then $-K_V$ is ample with $(-K_V)^3 = 8$, and we have the following diagram:

$$\begin{array}{ccc}
V & \xrightarrow[2:1]{\psi} & Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \\
\downarrow \varphi & & \swarrow \pi \\
\mathbb{P}^1 & &
\end{array}$$

where $\varphi: V \rightarrow \mathbb{P}^1$ is the del Pezzo fibration.

Let F_Y be a fibre of $\pi: Y \rightarrow \mathbb{P}^1$. Let $H_V := \psi^*(H)$ and $F_V := \psi^*(F_Y)$. We have $-K_V \sim H_V + F_V$ and $\rho(V) = 2$. The extremal ray not corresponding to $\varphi: V \rightarrow \mathbb{P}^1$ is generated by the strict transform of the section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1})$ of $\pi: Y \rightarrow \mathbb{P}^1$ and the corresponding contraction morphism is defined by $|H_V|$ which maps to \mathbb{P}^3 with degree 2. By taking the Stein factorization, this morphism factors through the double covering V_2 of \mathbb{P}^3 branched along a quartic surface and the exceptional divisor E of the morphism is the unique member of $|H_V - F_V|$. Therefore, there exists a contraction $\mu: V \rightarrow V_2$ which is the blow-up of V_2 along a smooth elliptic curve R_2 such that $-K_{V_2} \cdot R_2 = 4$, and V_2 is a Fano threefold of index 2.

By the adjunction formula, $-K_E \sim (-K_V - E)|_E \sim 2F_V|_E$. Hence,

$$h^0(E, \mathcal{O}_E(-K_E)) = h^0(E, (\varphi|_E)^*(\mathcal{O}_{\mathbb{P}^1}(2))) = 3.$$

Since the exceptional divisor E is ruled over a smooth elliptic curve (induced by $\mu|_E$) and admits an elliptic fibration (induced by $\varphi|_E$), we deduce from Lemma 5.2.3 that $E \simeq B_0 \times C_0$ with B_0 a smooth elliptic curve, and $C_0 \simeq \mathbb{P}^1$. Moreover,

$$E|_{F_V} \sim (H_V - F_V)|_{F_V} \sim H_V|_{F_V} \sim -K_V|_{F_V} \sim -K_{F_V},$$

thus $E|_{F_V}$ is a smooth elliptic curve in $|-K_{F_V}|$.

Let $R \in |2C_0 + B_0|$ be a smooth elliptic curve on E . Let $\phi: X \rightarrow V$ be the blow-up of the curve R and let E_1 be the exceptional divisor. Let F (resp. A) be the strict transform of F_V (resp. E). Then $-K_X \sim A + 2F$ with $A \simeq E$. Restricted to a fibre F , the morphism $\phi: F \rightarrow F_V$ is the blow-up at two points on an elliptic curve in $|-K_{F_V}|$. Hence by choosing R sufficiently general, we obtain that $-K_F$ is nef and not semi-ample.

We can write $A \simeq B \times C$ with B (resp. C) the strict transform of B_0 (resp. C_0). Then

$$-K_X \cdot B = \phi^*(-K_V) \cdot B - E_1 \cdot B = -K_V \cdot B_0 - 2 = 2 - 2 = 0$$

and

$$-K_X \cdot C = \phi^*(-K_V) \cdot C - E_1 \cdot C = -K_V \cdot C_0 - 1 = 1 - 1 = 0.$$

Hence by the same arguments as in Example 5.2.2, we deduce that $-K_X$ is nef.

In particular, $A \cdot C = -K_X \cdot C - 2F \cdot C = -2$. Hence, A is the exceptional locus of a divisorial crepant extremal contraction.

Chapter 6

Rational general fibre: general case

In this chapter, we first consider the general Setup 4.2.5.

Remark 6.0.1. *Remind that under the general setup, if the general fibre F of f is a rational surface, i.e. F is \mathbb{P}^2 blown up at 9 points in sufficiently general position, then the unique member in $|-K_F|$ is one of the types in Kodaira's table of singular fibres for an elliptic fibration (see Table 2.1).*

We first prove the following lemma in a more general situation.

Lemma 6.0.2. *Let X be a complex projective smooth threefold such that $|-K_X| = A + |kF|$, where A is the non-zero fixed divisor of the anticanonical system, F is a prime divisor, and $k \geq 2$ an integer. Suppose that there exists an ϵA -flop (with $\epsilon > 0$ such that the pair $(X, \epsilon A)$ is lc), then A has multiplicity at least k along the flopping curve.*

Proof. By assumption, there exists an ϵA -flop:

$$\psi: X \dashrightarrow X^+,$$

where X^+ is again smooth (by [Kol89, Thm. 2.4]). Since ψ induces an isomorphism in codimension one, the anticanonical system $|-K_{X^+}|$ has non-empty fixed divisor $A^+ := \psi_*(A)$ and we can write

$$|-K_{X^+}| = A^+ + |kF^+|,$$

where $F^+ := \psi_*(F)$ and $|kF^+|$ is the mobile part of the anticanonical system.

Since ψ is a flop, there exists a common resolution:

$$\begin{array}{ccc} & \tilde{X} & \\ g \swarrow & & \searrow h \\ X & \overset{\psi}{\dashrightarrow} & X^+ \end{array}$$

such that $g^*(K_X) = h^*(K_{X^+})$. Moreover, by [KMM87, Prop. 5-1-11], one has

$$K_{\tilde{X}} = g^*(K_X + \epsilon A) + \sum_i a_i E_i = h^*(K_{X^+} + \epsilon A^+) + \sum_i a_i^+ E_i,$$

where $a_i^+ \geq a_i$, and $a_i^+ > a_i$ if and only if $g(E_i)$ is contained in the flopping locus.

Hence,

$$h^*(kF^+) - g^*(kF) = (1 - \epsilon)(g^*(A) - h^*(A^+)) + \sum_i (a_i^+ - a_i)E_i = \frac{1}{\epsilon} \sum_i (a_i^+ - a_i)E_i$$

is effective. Since F and F^+ are Cartier divisors, we can write $h^*(kF^+) - g^*(kF) = \sum_i kn_i E_i$ with $n_i \in \mathbb{N}$.

Since $g^*(A) = h^*(A^+) + h^*(kF^+) - g^*(kF)$, one has that $g^*(A) - \tilde{A} - \sum_i kn_i E_i$ is effective, where $\tilde{A} := g_*^{-1}(A)$ is the strict transform of A in \tilde{X} . Therefore, A has multiplicity at least k along the flopping curve. \square

Recall that in Setup 4.2.5, we write $A = A_h + A_v$, where A_h and A_v are effective divisors such that $A_h|_F = -K_F$ and $A_v|_F = 0$ for a general fibre F of $f: X \rightarrow \mathbb{P}^1$. Applying the above lemma to our case, we prove the following.

Lemma 6.0.3. *In Setup 4.2.5, the divisor A is f -nef. If A is not nef, then we are in one of the following cases:*

- (i) *there is an extremal contraction which contracts horizontally an irreducible component of A_h to a curve;*
- (ii) *there is an ϵA -flop (with $\epsilon > 0$ such that the pair $(X, \epsilon A)$ is lc), and A_h has multiplicity at least k along the f -horizontal flopping curve.*

Proof. Suppose that A is not nef. Then applying Lemma 2.5.1 to the pair $(X, \epsilon A)$ for $\epsilon > 0$ small enough such that $(X, \epsilon A)$ is lc, there exists a $(K_X + \epsilon A)$ -negative extremal ray Γ such that $A \cdot \Gamma < 0$. Let l be an integral curve such that the class $[l]$ generates Γ . Then $l \subset A$ as $A \cdot l < 0$.

We first notice that l is f -horizontal, i.e. $\dim f(l) = 1$. This is because otherwise $F \cdot l = 0$ and thus $-K_X \cdot l = A \cdot l + kF \cdot l < 0$, which contradicts to the fact that $-K_X$ is nef. As $\dim f(A_v) = 0$, we deduce that $l \subset A_h$.

(i) If the contraction φ_Γ of Γ is divisorial, then it contracts an irreducible component B_h of A_h . Since any f -vertical curve $b \subset B_h$ satisfies $F \cdot b = 0$, we have $[b] \notin \Gamma$. Hence, φ_Γ contracts B_h to a curve.

(ii) If the contraction φ_Γ of Γ is small, then $K_X \cdot l = 0$ because otherwise, l is not rigid. Hence, there exists an ϵA -flop:

$$\psi: X \dashrightarrow X^+,$$

where X^+ is again smooth (by [Kol89, Thm. 2.4]) with nef anticanonical divisor. By Lemma 6.0.2, we obtain that A (and thus A_h) has multiplicity at least k along the flopping curve. \square

Corollary 6.0.4. *In Setup 4.2.5, suppose that the general fibre F is rational and that the unique member in $|-K_F|$ is a nodal (resp. cuspidal) cubic curve. Let C be the closure of the rational section of f formed by the nodes (resp. cusps). If A is not nef, then $k = 2$, and C is the flopping curve of the unique $(K_X + \epsilon A)$ -negative extremal ray Γ such that $A \cdot \Gamma < 0$.*

Proof. In the case where the unique member in $|-K_F|$ is a nodal or cuspidal cubic curve, the surface A_h is irreducible and non-normal.

Suppose that A is not nef. Let Γ be a $(K_X + \epsilon A)$ -negative extremal ray such that $A \cdot \Gamma < 0$. Denote by φ_Γ the contraction of Γ . We apply Lemma 6.0.3:

(i) We first show that φ_Γ cannot be divisorial. If φ_Γ is a Mori contraction, then A_h is a smooth ruled surface by the classification of Mori (see Section 2.3), which contradicts the fact that A_h is non-normal. If φ_Γ is a crepant contraction, then A_h is a conic bundle by the classification of Wilson (that we cite in Prop 6.0.5 for readers' convenience). As A_h is non-normal, a general fibre of φ_Γ is two lines denoted by l_1 and l_2 intersecting at one point. In particular, $A_h \cdot l_1 = A_h \cdot l_2 = -1$. Since l_i is f -horizontal, one has $F \cdot l_i \geq 1$ and $A_v \cdot l_i \geq 0$ for $i = 1, 2$. Hence,

$$0 = -K_X \cdot l_i = A_h \cdot l_i + A_v \cdot l_i + kF \cdot l_i \geq k - 1,$$

i.e. $k \leq 1$, which contradicts the fact that $k \geq 2$.

(ii) If φ_Γ is a small crepant contraction, then A_h has multiplicity at least $k \geq 2$ along the flopping curve which is f -horizontal. Since the general fibre of $f|_{A_h}: A_h \rightarrow \mathbb{P}^1$ is a nodal or a cuspidal curve, the only possible flopping curve is C . Thus, we have that $k = 2$, and the unique possible $(K_X + \epsilon A)$ -negative extremal ray is generated by $[C]$. \square

Proposition 6.0.5 ([Wil92], Thm.2.2, [Wil93], [Wil97], Prop.3.1). *Let X be a smooth complex projective threefold and let $\phi: X \rightarrow Y$ be a crepant contraction of an extremal ray, contracting some irreducible surface $E \subset X$ down to a curve $C \subset Y$. Then C is a smooth curve and $\phi: E \rightarrow C$ is a conic bundle over C such that one of the following holds:*

- (i) E is normal and a general fibre of $\phi: E \rightarrow C$ is a smooth conic;
- (ii) E is non-normal and a general fibre of $\phi: E \rightarrow C$ is two lines meeting at one point.

For a general fibre l of $\phi: E \rightarrow C$, one has $E \cdot l = -2$. A singular fibre of $\phi: E \rightarrow C$ is either two \mathbb{P}^1 's intersecting at one point, or a double line.

Furthermore, if E is normal, then the possible singularities of E are A_n singularities at the point where distinct components of a singular fibre meet, or A_1 singularities appearing as a pair on some double fibre.

6.1 Generically log canonical case

Theorem 6.1.1. *In Setup 4.2.5, if the pair (X, A_h) is lc over the generic point of \mathbb{P}^1 , then $k = 2$, $A_v = 0$ and the pair (X, A_h) is lc.*

Proof. We follow the strategy of the proof in [AD13, Thm. 5.1]. Let $\pi: \tilde{X} \rightarrow X$ be a log resolution of singularities of (X, A_h) . Then

$$K_{\tilde{X}} + \tilde{A}_h \sim_{\mathbb{Q}} \pi^*(K_X + A_h) + E_+ - E_-,$$

where $\tilde{A}_h := \pi_*^{-1}(A_h)$ is the strict transform of A_h , E_+ and E_- are effective π -exceptional divisors with no common components such that the support of $\pi_*^{-1}(A_h) + E_+ + E_-$ is a snc divisor.

Since the pair (X, A_h) is lc over the generic point of \mathbb{P}^1 , we can write E_- as

$$E_- = E + E',$$

where $[E]$ is reduced and E_- coincides with E over the generic point of \mathbb{P}^1 . Furthermore,

$$\pi(\text{Supp}(E')) = \text{Nlc}(X, A_h).$$

In particular, E' is not dominant onto \mathbb{P}^1 .

Let $\tilde{f} := f \circ \pi$. Observe that $\tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E)$ is weakly positive by [Cam04, Thm. 4.13].

Since E_+ is effective and π -exceptional, $\pi_* \mathcal{O}_{\tilde{X}}(E_+) \simeq \mathcal{O}_X$. Hence

$$\tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E_-) = \tilde{f}_* \mathcal{O}_{\tilde{X}}(\pi^*(K_{X/\mathbb{P}^1} + A_h) + E_+) \simeq f_* \mathcal{O}_X(K_{X/\mathbb{P}^1} + A_h).$$

Together with

$$K_{X/\mathbb{P}^1} + A_h \sim -(k-2)F - A_v,$$

one has

$$\begin{aligned} \tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E) &\xrightarrow{i} \tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E + E') \\ &= \tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E_-) \\ &\simeq \mathcal{O}_{\mathbb{P}^1}(-(k-2)) \otimes f_* \mathcal{O}_X(-A_v) \\ &\xrightarrow{i'} \mathcal{O}_{\mathbb{P}^1}(-(k-2)). \end{aligned}$$

However, $\tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E)$ is pseudo-effective and $k \geq 2$. This implies that $k = 2$ and i, i' are isomorphisms. Hence, $A_v = 0$ and

$$\tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E_-) \otimes \tilde{f}_* \mathcal{O}_{\tilde{X}}(-E') \simeq \tilde{f}_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/\mathbb{P}^1} + \tilde{A}_h + E_-) \simeq \mathcal{O}_{\mathbb{P}^1}.$$

Therefore, $E' = 0$, and we deduce that the pair (X, A_h) is lc. \square

In the two following sections, we describe the geometric structure of X explicitly in two particular cases.

6.2 Cuspidal case

Theorem 6.2.1. *In Setup 4.2.5, assume moreover that the general fibre F is rational and that the unique member in $|-K_F|$ is a cuspidal cubic curve. Then A is nef and thus the fibration $f: X \rightarrow \mathbb{P}^1$ is locally trivial.*

Proof. Suppose by contradiction that A is not nef. Then by Corollary 6.0.4, the curve C (where C is the closure of the rational section of f formed by the cusps) is a flopping curve and $-K_X \sim A_h + A_v + 2F$.

Let $\mu: Y \rightarrow X$ be the blow-up of X along the curve C and let \tilde{A}_h be the strict transform of A_h in Y . Let E be the exceptional divisor of μ . Since C is an f -horizontal curve and A_h has multiplicity 2 along C , one has

$$-K_Y = \mu^*(-K_X) - E \sim \tilde{A}_h + 2E + \tilde{A}_v + 2\tilde{F} - E = \tilde{A}_h + \tilde{A}_v + 2\tilde{F} + E,$$

where \tilde{A}_v (resp. \tilde{F}) is the strict transform of A_v (resp. F).

Since E is a ruled surface, one has

$$-K_E \sim 2C_1 + (d+2)f_E,$$

where

- case $d > 0$: C_1 is the minimal section with $C_1^2 = -d$, and f_E is a fibre of $\mu|_E$;
- case $d = 0$, i.e. $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$: C_1 is a fibre of the second projection, and f_E is a fibre of the first projection $\mu|_E$.

Moreover, the fibration $(f \circ \mu)|_E$ coincides with the ruling.

On the other hand, by the adjunction formula, one has

$$-K_E \sim (-K_Y - E)|_E \sim (\tilde{A}_h + \tilde{A}_v + 2\tilde{F})|_E \sim (\tilde{A}_h + \tilde{A}_v)|_E + 2f_E.$$

As \tilde{A}_v is contained in some fibres of $f \circ \mu$, one has $\tilde{A}_v|_E \sim \alpha f_E$ with $0 \leq \alpha \leq d$, and

$$\tilde{A}_h|_E \sim 2C_1 + (d - \alpha)f_E.$$

Therefore,

- If $d > 0$, then on the surface E one has

$$(2C_1 + (d - \alpha)f_E) \cdot C_1 = -2d + d - \alpha = -d - \alpha < 0$$

and thus, C_1 is contained in the fixed divisor of the linear system $|2C_1 + (d - \alpha)f_E|$ on E . Hence C_1 is an irreducible component of the intersection $\tilde{A}_h \cap E$.

- If $d = 0$, then $\tilde{A}_h|_E \sim 2C_1$. Since the general fibre of $f|_{A_h}: A_h \rightarrow \mathbb{P}^1$ is a cuspidal curve, the surface E is tangent to \tilde{A}_h along an $(f \circ \mu)$ -horizontal curve. From now on, we define C_1 as the reduction of the intersection $\tilde{A}_h \cap E$ when $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Since E is smooth, one has

$$0 \rightarrow \mathcal{N}_{C_1/E} \rightarrow \mathcal{N}_{C_1/Y} \rightarrow \mathcal{N}_{E/Y}|_{C_1} \rightarrow 0. \quad (6.1)$$

Since C is a flopping curve, by [Pin83, Thm. 4] we are in one of the following cases:

(a) $\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{N}_{C_1/Y} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

In this case, we have $\alpha = d = 0$, $\tilde{A}_h \cdot E \sim 2C_1$, and the short exact sequence (6.1) gives $E \cdot C_1 = -1$.

On the other hand, we have $\tilde{A}_h \cap E = 2C_1$, i.e. E and \tilde{A}_h intersect along a smooth curve without embedded point. Hence, \tilde{A}_h is smooth near C_1 , and thus C_1 is a Cartier divisor on \tilde{A}_h , i.e. $C_1^2 \in \mathbb{Z}$ on \tilde{A}_h . Therefore,

$$E \cdot C_1 = E|_{\tilde{A}_h} \cdot C_1 = 2C_1^2,$$

where the last self-intersection number is computed on \tilde{A}_h . Hence, $E \cdot C_1$ is divisible by 2, which contradicts the fact that $E \cdot C_1 = -1$.

(b) $\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, $E \simeq \mathbb{F}_2$, and either $\mathcal{N}_{C_1/Y} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ or $\mathcal{N}_{C_1/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

In this case, the short exact sequence (6.1) gives $E \cdot C_1 = 0$. By the adjunction formula, one has

$$-K_{\tilde{A}_h} \cdot C_1 = (-K_Y - \tilde{A}_h)|_{\tilde{A}_h} \cdot C_1 = \tilde{A}_v \cdot C_1 + 2\tilde{F} \cdot C_1 + E \cdot C_1 \geq 2.$$

This is because $2\tilde{F} \cdot C_1 = 2$, and $C_1 \not\subset \tilde{A}_v$ by construction, which implies $\tilde{A}_v \cdot C_1 \geq 0$.

(c) $\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$, $E \simeq \mathbb{F}_4$, and either $\mathcal{N}_{C_1/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ or $\mathcal{N}_{C_1/Y} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$.

In this case, the short exact sequence (6.1) gives $E \cdot C_1 = 1$. By the adjunction formula, one has

$$-K_{\tilde{A}_h} \cdot C_1 = (-K_Y - \tilde{A}_h)|_{\tilde{A}_h} \cdot C_1 = \tilde{A}_v \cdot C_1 + 2\tilde{F} \cdot C_1 + E \cdot C_1 \geq 3$$

by the same argument as in case (b).

For cases (b) and (c), we will obtain a contradiction as follows. Let $\lambda: \hat{A}_h \rightarrow \tilde{A}_h$ be the normalisation of the surface \hat{A}_h , and let $\pi: \bar{A}_h \rightarrow \hat{A}_h$ be the minimal resolution of the surface \hat{A}_h . Denote by $\nu := \lambda \circ \pi$ the composition map. Then

$$-K_{\hat{A}_h} = \lambda^*(-K_{\tilde{A}_h}) + D_1,$$

where D_1 is an effective Weil divisor supported on the zero locus of the conductor ideal of the normalisation λ , and

$$-K_{\bar{A}_h} = \nu^*(-K_{\tilde{A}_h}) + \bar{D}_1 + D_2 = \nu^*(-K_{\tilde{A}_h}) + D,$$

where \bar{D}_1 is the strict transform of D_1 , D_2 is an effective Weil divisor supported on the exceptional locus of π , and $D := \bar{D}_1 + D_2$ is an effective Weil divisor.

Since C_1 is not contained in the singular locus of \hat{A}_h (because \tilde{A}_h is generically smooth along C_1), one has $D \cdot \bar{C}_1 \geq 0$, where $\bar{C}_1 \subset \bar{A}_h$ is the strict transform of C_1 . Therefore,

$$-K_{\bar{A}_h} \cdot \bar{C}_1 \geq \nu^*(-K_{\tilde{A}_h}) \cdot \bar{C}_1 \geq -K_{\tilde{A}_h} \cdot C_1 \geq 2.$$

By [Kol96, Chap. 2, Thm. 1.14], the rational curve $\overline{C_1}$ deforms on the surface $\overline{A_h}$. Hence, C_1 deforms on the surface \tilde{A}_h . Since \tilde{A}_h is not contracted by the morphism μ , we deduce that $C \subset X$ is not rigid. This contradicts the fact that the extremal ray $\mathbb{R}_+[C]$ contains only a finite number of curves.

Therefore, the divisor A is nef. Since $-K_X \sim A + kF$ with $k \geq 2$ and F nef, the relative anticanonical divisor $-K_{X/\mathbb{P}^1}$ is nef. Hence, the fibration f is locally trivial by [PZ19, Thm. A.12]. \square

6.3 Nodal case

Proposition 6.3.1. *In Setup 4.2.5, assume moreover that the general fibre F is rational and that the unique member in $| -K_F |$ is a nodal cubic curve. Let C be the closure of the rational section of f formed by the nodes. If A is not nef, then we are in one of the following cases:*

- (i) *A is smooth outside C . There are exactly two degenerated fibres of $f|_A$ and both are of type I_2 .*
- (ii) *A has exactly one degenerated fibre F_0 of $f|_A$, and F_0 is of the type I_2 . The surface A has one A_1 singularity at the intersection point of the two irreducible components of F_0 , which is not contained in C .*
- (iii) *A is smooth outside C . There is exactly one degenerated fibre of $f|_A$ and it is of type I_3 .*

Here the types of degenerated fibres are described in terms of the Kodaira's table of singular elliptic fibres.

Proof. By Theorem 6.1.1, one has that $A = A_h$,

$$-K_X \sim A + 2F,$$

and the pair (X, A) is log canonical.

Suppose that A is not nef. Let $\mu: Y \rightarrow X$ be the blow-up of X along C with exceptional divisor E . Denote by \tilde{A} the strict transform of A . Then by repeating the argument in the proof of Theorem 6.2.1, the only possible case is when $\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

In this case, $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\mu|_E: E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow C \simeq \mathbb{P}^1$ is the first projection, and $A \cap E = C_1 + C_2$, where C_1 and C_2 are two fibres of the second projection $E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, which are mapped surjectively onto C by μ . Notice that C_1 and C_2 are distinct fibres: the general fibre of $f|_A: A \rightarrow \mathbb{P}^1$ is a nodal curve, so \tilde{A} and E are not tangent over the generic point of \mathbb{P}^1 . As the intersection $\tilde{A} \cap E$ is smooth, we deduce that \tilde{A} is smooth near C_1 and C_2 . Thus C_1, C_2 are Cartier divisors on \tilde{A} . Moreover,

$$-1 = E \cdot C_1 = E|_{\tilde{A}} \cdot C_1 = C_1^2,$$

where the last self-intersection number is computed on \tilde{A} . Similarly, we have $C_2^2 = -1$ on \tilde{A} .

Let $\psi: X \dashrightarrow X^+$ be the A -flop which flops the curve C to a curve C^+ . We have the following commutative diagram:

$$\begin{array}{ccc}
& Y & \\
\mu \swarrow & & \searrow \rho \\
X & \overset{\psi}{\dashrightarrow} & X^+
\end{array}$$

where $\rho: Y \rightarrow X^+$ is the blow-up of X^+ along the curve C^+ with exceptional divisor E , and $\rho|_E: E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow C^+ \simeq \mathbb{P}^1$ is the second projection. Then X^+ is again smooth (by [Kol89, Thm. 2.4]) with nef anticanonical divisor. Since ψ induces an isomorphism in codimension one, the anticanonical system $|-K_{X^+}|$ has a non-zero fixed divisor $A^+ := \psi_*(A)$ and we can write

$$|-K_{X^+}| = A^+ + |2F^+|,$$

where $F^+ := \psi_*(F)$ and $|2F^+|$ is the mobile part of the anticanonical system.

Let $\ell := A \cap F$ and let ℓ^+ be its strict transform on X^+ . Then ℓ^+ is a rational curve. Since ψ is an isomorphism outside the indeterminacy curves (with ℓ not contained in the flopping locus), and $\tilde{A} \cap E = C_1 + C_2$ (so that C^+ is not contained in A^+), we deduce that $A^+ \cap F^+ = \ell^+$. Thus ℓ^+ is a Cartier divisor on A^+ . By the adjunction formula,

$$-K_{A^+} \sim (-K_{X^+} - A^+)|_{A^+} \sim 2F^+|_{A^+} = 2\ell^+.$$

By the adjunction formula on the surface A^+ ,

$$K_{A^+} \cdot \ell^+ + (\ell^+)^2 = -2,$$

and thus $-2\ell^+ \cdot \ell^+ + (\ell^+)^2 = -2$. Therefore, $(\ell^+)^2 = 2$ and $K_{A^+} \cdot \ell^+ = -4$ on the surface A^+ . Since ψ is a flop, one has $K_{X^+} \cdot \ell^+ = K_X \cdot \ell = 0$, which gives $A^+ \cdot \ell^+ = K_{A^+} \cdot \ell^+ - K_{X^+} \cdot \ell^+ = -4$.

Claim. A^+ is a normal surface.

Since $\rho|_{\tilde{A}}: \tilde{A} \rightarrow A^+$ is the blow-down of two (-1) -curves C_1 and C_2 contained in the smooth locus of \tilde{A} , it is enough to show that \tilde{A} is normal.

Suppose by contradiction that \tilde{A} is non-normal. Let $\tilde{f} := (f \circ \mu)|_{\tilde{A}}: \tilde{A} \rightarrow \mathbb{P}^1$ and let $f_{\tilde{A}}$ be a general fibre. By the adjunction formula, one has

$$-K_{\tilde{A}} \sim (-K_Y - \tilde{A})|_{\tilde{A}} \sim C_1 + C_2 + 2f_{\tilde{A}}.$$

Let $\lambda: \hat{A} \rightarrow \tilde{A}$ be the normalisation of the surface \tilde{A} , and let $\pi: \bar{A} \rightarrow \hat{A}$ be the minimal resolution of the surface \hat{A} . Denote by $\nu := \lambda \circ \pi$ the composition map. Then

$$-K_{\hat{A}} = \lambda^*(-K_{\tilde{A}}) + D_1,$$

where D_1 is an effective Weil divisor supported on the zero locus of the conductor ideal of the normalisation λ , and

$$-K_{\bar{A}} = \nu^*(-K_{\tilde{A}}) + \bar{D}_1 + D_2 = \nu^*(-K_{\tilde{A}}) + D,$$

where \bar{D}_1 is the strict transform of D_1 , D_2 is an effective Weil divisor supported on the exceptional locus of π , and $D := \bar{D}_1 + D_2$ is an effective Weil divisor. Then D is non-zero by assumption.

Note that D_1 is $(\tilde{f} \circ \lambda)$ -vertical, as the general fibre of $f|_A: A \rightarrow \mathbb{P}^1$ is a nodal curve. Hence, \overline{D}_1 is $(\tilde{f} \circ \nu)$ -vertical. As D_2 is also $(\tilde{f} \circ \nu)$ -vertical, we deduce that D is $(\tilde{f} \circ \nu)$ -vertical. Since C_1 and C_2 are contained in the smooth locus of \tilde{A} , and $f_{\tilde{A}}$ is a general fibre (so that we can choose a general fibre contained in the smooth locus of \tilde{A}), we have

$$-K_{\tilde{A}} \sim \overline{C}_1 + \overline{C}_2 + 2f_{\tilde{A}} + D,$$

where \overline{C}_1 (resp. \overline{C}_2) is the strict transform of C_1 (resp. C_2), and $f_{\tilde{A}}$ is a general fibre of $\tilde{f} \circ \nu$; moreover, D is disjoint from $\overline{C}_1, \overline{C}_2$ and $f_{\tilde{A}}$.

On the other hand, let $M \in |-K_{\tilde{A}}|$. By the short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{A}}(K_{\tilde{A}}) \rightarrow \mathcal{O}_{\tilde{A}} \rightarrow \mathcal{O}_M \rightarrow 0,$$

one has $H^0(M, \mathcal{O}_M) \simeq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}) \simeq \mathbb{C}$ as $h^1(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}})) = h^1(\tilde{A}, \mathcal{O}_{\tilde{A}}) = 0$ by Serre duality. Therefore, M is connected. This contradicts the fact that $\overline{C}_1 + \overline{C}_2 + 2f_{\tilde{A}} + D \in |-K_{\tilde{A}}|$ and that D is a non-zero component disjoint from $\overline{C}_1 + \overline{C}_2 + 2f_{\tilde{A}}$. This proves the claim.

Claim. A^+ is the exceptional divisor of an extremal crepant contraction denoted by ϕ .

Since X^+ is smooth, for sufficiently small $\epsilon > 0$, the pair $(X^+, \epsilon A^+)$ is lc. Since A^+ is not nef, it follows from Lemma 2.5.1 that there exists a $(K_X + \epsilon A^+)$ -negative extremal ray which is A^+ -negative; denote by ϕ the associated contraction. Since A^+ is a normal surface, ϕ cannot be a small contraction (namely an ϵA^+ -flop) by Lemma 6.0.2. Since $-K_{X^+} \cdot \ell^+ = 0$, we deduce that ϕ is an extremal crepant contraction which contracts the divisor A^+ . This proves the claim.

Since $\tilde{f} = (f \circ \mu)|_{\tilde{A}}: \tilde{A} \rightarrow \mathbb{P}^1$ induces a fibration with general fibre isomorphic to \mathbb{P}^1 such that the two exceptional (-1) -curves C_1, C_2 of $\rho|_{\tilde{A}}$ are two sections, we deduce that a fibre of \tilde{f} is the strict transform of some curve (not necessarily integral) on A^+ with self-intersection number 2 passing through the two blown up points denoted by p_1, p_2 of $\rho|_{\tilde{A}}$ (with C_i contracted to p_i , for $i = 1, 2$).

By the classification of the extremal crepant contractions, we are in one of the following cases.

The surface A^+ is contracted to a point. Since $A^+ \cdot \ell^+ = -4$, we deduce that A^+ is a Gorenstein del Pezzo surface of degree 8 by [Tam04, 2.2]. Hence, A^+ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or \mathbb{F}_1 , or a quadric cone $Q \subset \mathbb{P}^3$ by [Tam04, 2.4]. Since $-K_{A^+}$ is divisible by 2 in $\text{Pic}(A^+)$, we deduce that $A^+ \not\cong \mathbb{F}_1$.

- (A) *Case $A^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1$.* Let l_1 (resp. l_2) be a general fibre of the first projection (resp. second projection) of $A^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Then $\tilde{f}: \tilde{A} \rightarrow \mathbb{P}^1$ is the conic bundle defined by the linear system $|(\rho|_{\tilde{A}})^*(l_1 + l_2) - C_1 - C_2|$.

Note that p_1, p_2 cannot lie on the same fibre of $A^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ as $(l_1 + l_2) \cdot l_i = 1$ for $i = 1, 2$.

There are exactly two singular fibres of \tilde{f} . Indeed, let l_{11} (resp. l_{12}) be the fibre through p_1 (resp. through p_2) of the first projection $A^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Let l_{21} (resp. l_{22}) be the fibre through p_1 (resp. through p_2) of the second projection $A^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Then $\tilde{l}_{11} + \tilde{l}_{22}$ and $\tilde{l}_{21} + \tilde{l}_{12}$ are the two singular fibres of the conic bundle $\tilde{f}: \tilde{A} \rightarrow \mathbb{P}^1$, where $\tilde{l}_{ij} \subset \tilde{A}$ is the strict transform of l_{ij} for $i, j = 1, 2$, and the two singular fibres are two \mathbb{P}^1 's intersecting

transversally at one point. Moreover, each component of a singular fibre intersects one of the two sections C_1, C_2 , and these two intersection points are distinct from the intersection point of the two components.

- (B) *Case* $A \simeq Q \subset \mathbb{P}^3$. Let l be a ruling of the cone Q . Then $\tilde{f}: \tilde{A} \rightarrow \mathbb{P}^1$ is the conic bundle defined by the linear system $|(\rho|_{\tilde{A}})^*(2l) - C_1 - C_2|$.

Note that p_1, p_2 are smooth points of Q and thus distinct from the vertex. Moreover, p_1 and p_2 cannot lie on the same ruling of Q , as $2l \cdot l = 1$.

There is exactly one singular fibre of \tilde{f} . Indeed, let l_1 (resp. l_2) be the ruling through p_1 (resp. p_2). Then $\tilde{l}_1 + \tilde{l}_2$ is the singular fibre of \tilde{f} , where \tilde{l}_i is the strict transform of l_i for $i = 1, 2$. There is exactly one singular point on \tilde{A} , which is an A_1 singularity at the point where \tilde{l}_1 and \tilde{l}_2 meet, and \tilde{l}_i intersects C_i at one point distinct from the singular point of \tilde{A} .

The surface A^+ is contracted to a curve. By Proposition 6.0.5, the crepant contraction ϕ induces a conic bundle on A^+ over a smooth curve. Since $A^+ \cdot \ell^+ = -4$, the curve ℓ^+ is not contracted by ϕ . Let R be a general fibre of the conic bundle $\phi|_{A^+}$. Since A^+ is normal, we deduce that R is a smooth conic with $A^+ \cdot R = -2$.

On the surface A^+ , one has

$$2\ell^+ \cdot R = -K_{A^+} \cdot R = -K_{X^+} \cdot R - A^+ \cdot R = 2,$$

i.e. $\ell^+ \cdot R = 1$. Hence, ℓ^+ is a section of the conic bundle $\phi|_{A^+}$, and

$$F^+ \cdot R = F^+|_{A^+} \cdot R = \ell^+ \cdot R = 1.$$

Then we can show that $\phi|_{A^+}$ has no singular fibre. Indeed, suppose that there exists a singular fibre, and let R' be an irreducible reduced component of the singular fibre. Then $[R'] = \frac{1}{2}[R]$ by Proposition 6.0.5. Hence, $F^+ \cdot R' = \frac{1}{2}$, which contradicts the fact that F^+ is Cartier.

Therefore, A^+ is a ruled surface over a smooth rational curve. Since $-K_{A^+}$ is divisible by two in $\text{Pic}(A^+)$, we deduce that $A^+ \simeq \mathbb{F}_{2d}$ with $d \in \mathbb{N}$. If $d > 0$, let C_0 be the minimal section of $\phi|_{A^+}: A^+ \rightarrow \mathbb{P}^1$. Since $-K_{A^+} \sim 2\ell^+$, one has

$$\ell^+ \sim C_0 + (d+1)R.$$

As ℓ^+ is a section of $\phi|_{A^+}$, one has $C_0 \cdot \ell^+ \geq 0$, i.e. $d \leq 1$ and thus $d = 1$.

- If $A^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then the same argument in the case (A) can be applied to describe the singular fibres of \tilde{A} .
- If $A^+ \simeq \mathbb{F}_2$, then $\ell^+ \sim C_0 + 2R$, and $\tilde{f}: \tilde{A} \rightarrow \mathbb{P}^1$ is defined by the linear system $|(\rho|_{\tilde{A}})^*(C_0 + 2R) - C_1 - C_2|$.

Note that p_1, p_2 cannot lie on the same fibre of $\phi|_{A^+}$ as $R \cdot \ell^+ = 1$. Moreover, $p_1, p_2 \notin C_0$, as $C_0 \cdot \ell^+ = 0$.

There is exactly one singular fibre of \tilde{f} . Indeed, let R_1 (resp. R_2) be the fibre of $\phi|_{A^+}$ through p_1 (resp. p_2). Then $\tilde{C}_0 + \tilde{R}_1 + \tilde{R}_2$ is the singular fibre of \tilde{f} , where \tilde{C}_0 is the strict transform of C_0 , and \tilde{R}_i is the strict transform of R_i for $i = 1, 2$. Moreover, \tilde{R}_i intersects C_i at one point, and \tilde{C}_0 is disjoint from C_i . \square

In view of the above discussion on the nodal case, if A is not nef, we have a precise description on the geometric structure of the smooth threefold X^+ . As A^+ is the exceptional locus of an extremal crepant contraction, by contracting A^+ , we obtain a threefold X' with \mathbb{Q} -factorial Gorenstein canonical singularities. Moreover, $-K_{X'}$ is nef and divisible by two as Weil divisor (but not as Cartier divisor). One may expect to classify this kind of varieties X' in order to obtain a complete classification of X in the nodal case.

6.4 Other cases

In this section, we discuss the remaining cases when the general fibre F is rational and the unique member in $|-K_F|$ is reduced in Setup 4.2.5.

Setup 6.4.1. *In Setup 4.2.5, assume moreover that the general fibre F is rational and that the unique member in $|-K_F|$ is of the type $A_1^{(1)*}$ (= III) or $A_2^{(1)*}$ (= IV). Then the pair (X, cA_h) (with $c = \frac{3}{4}$ for the type $A_1^{(1)*}$ (= III), and $c = \frac{2}{3}$ for the type $A_2^{(1)*}$ (= IV)) is lc over the generic point of \mathbb{P}^1 , and there is a corresponding rational section formed by the singular points of the general fibres of the morphism f whose closure is denoted by C . Hence, the pair (X, cA_h) is lc over the generic point of C .*

Definition 6.4.2. ([OX12, Def. 2.1]) *Let X be a normal variety and let $\Delta = \sum_i a_i \Delta_i$ be a \mathbb{Q} -divisor with distinct prime divisors Δ_i on X and rational numbers a_i . Assume $0 \leq a_i \leq 1$ for all i . We say that a birational projective morphism $\mu: (\tilde{X}, \tilde{\Delta}) \rightarrow (X, \Delta)$ gives a log canonical model over (X, Δ) if with the divisor $\tilde{\Delta} = \mu_*^{-1}(\Delta) + E^{=1}$ on \tilde{X} , where $E^{=1}$ denotes the sum of μ -exceptional prime divisors with coefficients 1, the pair $(\tilde{X}, \tilde{\Delta})$ satisfies*

- (1) $(\tilde{X}, \tilde{\Delta})$ is log canonical,
- (2) $K_{\tilde{X}} + \tilde{\Delta}$ is μ -ample.

Remark 6.4.3. *From the negativity lemma, we know that $\mu: \tilde{X} \rightarrow X$ is isomorphic over the maximal open locus $X^{lc} \subset X$ on which (X, Δ) is log canonical.*

Theorem 6.4.4. ([BHN15, Lem. 3.1]) *Let (X, Δ) be a projective log canonical pair, and let $W \subset X$ be a log canonical centre. Let $\nu: W^n \rightarrow W$ be the normalisation. Then there exists an effective divisor Δ_{W^n} on W^n such that*

$$K_{W^n} + \Delta_{W^n} \sim_{\mathbb{Q}} \nu^*(K_X + \Delta)|_W.$$

Suppose that $Z \subset W$ is a log canonical centre such that $\dim Z = \dim W - 1$. Then we have a set-theoretic inclusion

$$\nu^{-1}(Z) \subset \Delta_{W^n}.$$

Proposition 6.4.5. *In Setup 6.4.1, one has that $k + A_v.C \leq 3$. For the type $A_1^{(1)*}$ (= III), one has that $k = 2$, $A_v.C = 0$, and that the pair $(X, \frac{3}{4}A_h)$ is lc in a neighbourhood of C .*

Proof. Let $D := cA_h$ and let $\mu: (\tilde{X}, \tilde{D}) \rightarrow (X, D)$ be a log canonical model over the pair (X, D) . Let $\tilde{f} = f \circ \mu: \tilde{X} \rightarrow \mathbb{P}^1$. Then

$$K_{\tilde{X}} + \tilde{D} \sim_{\mathbb{Q}} \mu^*(K_X + D) - E$$

with E effective μ -exceptional divisor.

Let $\tilde{C} := \mu_*^{-1}(C)$ be the strict transform of C . By the construction of C , we know that the curve \tilde{C} is a section of \tilde{f} and that it is the log canonical centre of the pair (\tilde{X}, \tilde{D}) . Moreover, as (X, D) is lc over the generic point of C , the curve \tilde{C} is not contained in the support of E . We deduce that $E \cdot \tilde{C} \geq 0$.

By Theorem 6.4.4, there exists an effective divisor $D_{\tilde{C}}$ on \tilde{C} such that

$$(K_{\tilde{X}} + \tilde{D})|_{\tilde{C}} \sim_{\mathbb{Q}} K_{\tilde{C}} + D_{\tilde{C}}.$$

Hence,

$$\mu^*(K_X + D)|_{\tilde{C}} \sim_{\mathbb{Q}} K_{\tilde{C}} + D_{\tilde{C}} + E|_{\tilde{C}}$$

and thus

$$-2 = \deg(K_{\tilde{C}}) \leq \deg(\mathcal{O}_{\tilde{C}}(\mu^*(K_X + D))).$$

Since $-K_X$ is nef, $\mu^*(-K_X)$ is nef and thus one has $\mu^*(K_X) \cdot \tilde{C} \leq 0$. Hence,

$$\mu^*(-A_h) \cdot \tilde{C} = \mu^*(K_X + kF + A_v) \cdot \tilde{C} \leq k + \mu^*(A_v) \cdot \tilde{C}.$$

Therefore,

$$\begin{aligned} \deg(\mathcal{O}_{\tilde{C}}(\mu^*(K_X + D))) &= \deg(\mathcal{O}_{\tilde{C}}(\mu^*(-A_h - A_v - kF + cA_h))) \\ &= \deg(\mathcal{O}_{\tilde{C}}(\mu^*(-A_v - kF - (1-c)A_h))) \\ &\leq -\mu^*(A_v) \cdot \tilde{C} - k + (1-c)(k + \mu^*(A_v) \cdot \tilde{C}) \\ &= -c(k + \mu^*(A_v) \cdot \tilde{C}). \end{aligned}$$

Since $\mu^*(A_v)$ is \tilde{f} -vertical and \tilde{C} is a section, one has $q := \mu^*(A_v) \cdot \tilde{C} \geq 0$. Hence,

$$-2 \leq -c(k + l),$$

i.e.

- $k + q \leq \frac{8}{3}$ for the type $A_1^{(1)*}(= III)$,
- $k + q \leq 3$ for the type $A_2^{(1)*}(= IV)$.

In particular, for the type $A_1^{(1)*}(= III)$, as k, q are integers such that $k \geq 2$ and $q \geq 0$, we deduce that $k = 2$ and $q = 0$. Hence, $A_v \cdot C = 0$. Moreover,

$$-2 + \deg D_{\tilde{C}} + E \cdot \tilde{C} \leq -\frac{3}{2}.$$

Since $\deg D_{\tilde{C}} \geq 0$, and $E \cdot \tilde{C}$ is a positive integer, we deduce that $E \cdot \tilde{C} = 0$. Therefore, μ induces an isomorphism over a neighbourhood of C . \square

We finish this chapter by proposing a conjecture on the structure of X :

Conjecture 6.4.6. *In Setup 1.2.5, assume moreover that F is rational and that the unique member in $|-K_F|$ is reduced. Then $|-K_X| = A_h + |2F|$.*

Most of the cases are already proved in this chapter, except for the two cases where the unique member in $|-K_F|$ is of the type $A_1^{(1)*}$ (= III) or $A_2^{(1)*}$ (= IV). However, Proposition 6.4.5 gives evidence for the conjecture to hold true in these two cases.

Chapter 7

Non-rational general fibre

In this chapter, we study the case when a general fibre is non-rational:

Setup 7.0.1. *Under Setup 4.2.5, assume that the general fibre F is non-rational.*

We will show that this case exists, and we give an example at the end of this chapter (see Example 7.1.4). Remark that when F is non-rational, X cannot be a product (i.e. $X \not\cong F \times \mathbb{P}^1$), since otherwise X is not rationally connected.

By Lemma 2.4.5, we have that $F = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a rank-2 vector bundle over an elliptic curve which is an extension

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

with \mathcal{L} a line bundle of degree 0 and either

- (i) $\mathcal{L} = \mathcal{O}$ and the extension is non-split or
- (ii) \mathcal{L} is not torsion.

The structure of the unique element D in $|-K_F|$ is as follows:

- (i) either $D = 2C$ and C is a smooth elliptic curve,
- (ii) or $D = C_1 + C_2$, where C_1 and C_2 are smooth elliptic curves which do not meet.

Lemma 7.0.2. *In Setup 7.0.1, A_h is not a prime divisor and one of the following cases occurs:*

- (i) $-K_F = 2C$ and C is a smooth elliptic curve, then $A_h = 2D$ where the restriction of f to D is an elliptic fibration.
- (ii) $-K_F = C_1 + C_2$ where C_1 and C_2 are smooth elliptic curves which do not meet, then $A_h = D_1 + D_2$ where the restriction of f to D_i is an elliptic fibration, $i = 1, 2$ and $D_1 \cap D_2$ is contained in some fibres of f .

Proof. Suppose by contradiction that A_h is a prime divisor. By the adjunction formula, we have

$$-K_{A_h} \sim (-K_X - A_h)|_{A_h} \sim (A_v + kF)|_{A_h}.$$

Since $F|_{A_h}$ is nef and contains a smooth elliptic curve, $(F|_{A_h})^2 = 0$ and $k \geq 2$, we can apply Lemma 4.1.2 and Corollary 4.1.3 to the surface A_h , which implies that $A_v|_{A_h} = 0$, $-K_{A_h} \sim kF|_{A_h}$ and A_h is a \mathbb{P}^1 -bundle over a smooth elliptic curve.

On the other hand, the restriction of f to A_h induces a fibration on A_h such that the general fibre $F|_{A_h}$ is either

- $2C$, or
- $C_1 + C_2$

with C , C_1 and C_2 smooth elliptic curves. The first case is impossible by the generic smoothness of the \mathbb{P}^1 -bundle A_h . In the second case, $-K_{A_h} \sim kF|_{A_h}$ with $k \geq 2$ contains at least 4 elliptic curves (counted with multiplicity). Let $\ell \subset A_h$ be a fibre of the ruling. Then $-K_{A_h} \cdot \ell \geq 4$, which contradicts $-K_{A_h} \cdot \ell = 2$ for a \mathbb{P}^1 -bundle over a smooth curve. \square

Lemma 7.0.3. *In the case (ii) of Lemma 7.0.2, we have $k = 2$, $A_v = 0$ and $A = D_1 + D_2$, where D_1 and D_2 are disjoint with $D_i \simeq C_i \times \mathbb{P}^1$, $i = 1, 2$.*

Proof. For $i, j = 1, 2$ with $i \neq j$, the adjunction formula gives

$$-K_{D_i} \sim (-K_X - D_i)|_{D_i} \sim (A_v + D_j + kF)|_{D_i}.$$

Recall that $F|_{D_i}$ is an elliptic curve, $(F|_{D_i})^2 = 0$ and $k \geq 2$, then by Lemma 4.1.2 and Corollary 4.1.3 we have $A_v|_{D_i} = D_j|_{D_i} = 0$, $-K_{D_i} \sim kF|_{D_i}$, and D_i is a \mathbb{P}^1 -bundle over a smooth elliptic curve. Hence $D_1 \cdot D_2 = 0$, and thus D_1 and D_2 are disjoint. Moreover, the support of a divisor $L \in |-K_X|$ is connected in codimension one by [Sha99, Lem. 2.3.9]. As A_v does not meet F and $A_v \cdot D_1 = A_v \cdot D_2 = 0$, we obtain that $A_v = 0$. Thus $A = A_h$.

Since D_i is a \mathbb{P}^1 -bundle over the smooth elliptic curve C_i , and the restriction of f to D_i induces an elliptic fibration with

$$h^0(D_i, \mathcal{O}_{D_i}(-K_{D_i})) = h^0(D_i, (f|_{D_i})^* \mathcal{O}_{\mathbb{P}^1}(k)) = k + 1 \geq 3,$$

we deduce $D_i \simeq C_i \times \mathbb{P}^1$ by Lemma 5.2.3, and thus $k = 2$. \square

7.1 Running the Minimal Model Program

Assume that X satisfies Setup 7.0.1. Consider a Mori contraction $\varphi: X \rightarrow Y$. Let Γ be an extremal ray contracted by φ . Let l be a rational curve such that $[l]$ generates Γ and that $-K_X \cdot l = l(\Gamma)$, where $l(\Gamma)$ is the length of the extremal ray Γ . In the birational case, denote the exceptional divisor of φ by E . In this section, we will suppose the following:

Assumption. $A_v = 0$, i.e. $A = A_h$.

Remark 7.1.1. *If $A_v = 0$, then for any f -vertical curve R contained in A , one has $A \cdot R = 0$. Indeed, by Lemma 7.0.3 we only need to prove for the case $A = 2D$. Suppose by contradiction that there exists a f -vertical curve $R \subset D$ such that $D \cdot R \neq 0$.*

- If $D \cdot R < 0$, then $-K_X \cdot R = A \cdot R + kF \cdot R < 0$ as $F \cdot R = 0$, which contradicts the fact that $-K_X$ is nef.
- if $D \cdot R > 0$, let F_0 be the special fibre of f which contains R . Since $D \cdot (F|_D) = 0$, we have that $D \cdot (F_0|_D) = 0$. Hence there exists a curve R' in $F_0|_D$ such that $D \cdot R' < 0$, this gives a contradiction by the previous case.

7.1.1 Birational contractions

Since X is a smooth threefold, the contraction φ is divisorial.

Case $A \cdot l = 0$. In this case, we have $F \cdot l = 1$ and $-K_X \cdot l = A \cdot l + kF \cdot l = k = 2$. Hence φ is the blow-up of Y at a smooth point, with exceptional divisor $E \simeq \mathbb{P}^2$. As E is not fibred, it is contained in a fibre of f and thus $F \cdot E = 0$. This contradicts the fact that $F \cdot l = 1$.

Case $A \cdot l < 0$. Since the contraction is divisorial, E is an irreducible component of A . Since every irreducible component of A is an elliptic fibration, we deduce from the classification of Mori (see Section 2.3) that E is a \mathbb{P}^1 -bundle over a smooth elliptic curve and φ contracts E to a smooth elliptic curve. Therefore,

$$\varphi^*(-K_Y) = -K_X + E.$$

Since $E \hookrightarrow A \hookrightarrow -K_X$, one has

$$\kappa(-K_X) \leq \kappa(-K_X + E) \leq \kappa(-2K_X) = \kappa(-K_X),$$

and similarly,

$$\nu(-K_X) \leq \nu(-K_X + E) \leq \nu(-2K_X) = \nu(-K_X).$$

Hence, $\kappa(-K_Y) = \kappa(-K_X) = 1$ and $\nu(-K_Y) = \nu(-K_X) = 2$.

- If $A = 2D$, then $E = D$ is a \mathbb{P}^1 -bundle over a smooth elliptic curve and φ contracts E to a smooth elliptic curve. This implies that Y is smooth with $-K_Y$ nef by Proposition 2.4.7. In this case, we have $D \cdot l = -1$, $-K_X \cdot l = 2D \cdot l + kF \cdot l = 1$, and thus $F \cdot l = 1$, $k = 3$.

As we contract the curves meeting F transversally, we conclude that $G := \varphi(F) \simeq F$. Since

$$-K_Y = \varphi_*(-K_X) = \varphi_*(A + 3F) = 3\varphi_*(F) = 3G,$$

we see that $| -K_Y | = |3G|$ has no fixed divisor. This contradicts the fact that $-K_Y$ is divisible by 2 in $\text{Pic}(Y)$ (see Theorem 3.0.1).

- If $A = D_1 + D_2$, then $E = D_i$ with $i = 1$ or 2 and φ contracts E to a smooth elliptic curve. This implies that Y is smooth with $-K_Y$ nef by Proposition 2.4.7. In this case, we have $D_i \cdot l = -1$, $-K_X \cdot l = D_i \cdot l + D_j \cdot l + 2F \cdot l = 1$ with $j \neq i$, and thus $F \cdot l = 1$.

As we contract the curves meeting F transversally, we conclude that $G := \varphi(F) \simeq F$. Let R be the intersection of two general members in $|G|$, then R is a smooth elliptic curve, and φ is the blow-up of Y along the curve R .

We have

$$-K_Y = \varphi_*(-K_X) = \varphi_*(A + 2F) = \varphi_*(D_j) + 2\varphi_*(F) = A_Y + 2G,$$

where $A_Y := \varphi_*(D_j)$.

Claim. A_Y is the fixed divisor of $|-K_Y|$. Suppose by contradiction that $|-K_Y|$ has no fixed divisor, then $-K_Y$ is divisible by two in $\text{Pic}(Y)$. As $-K_Y \sim A_Y + 2G$, this implies $A_Y \sim 2L$ for some $L \in \text{Pic}(Y)$. Hence

$$-K_G \sim (A_Y + G)|_G \sim (2L + G)|_{F_Y}.$$

Since $G \simeq F$, one has that G is a \mathbb{P}^1 -bundle over a smooth elliptic curve such that $-K_G \cdot g = 2$, where g is a fibre. Since $R \subset G$ is a smooth elliptic curve (i.e. a section of the \mathbb{P}^1 -bundle G), we deduce

$$G|_G \cdot g = 1.$$

This implies $2L|_G \cdot g = 1$, which contradicts the fact that $L|_G$ is a Cartier divisor. This proves the claim.

Claim. G is nef.

It is enough to show that $G \cdot R \geq 0$. Since R does not meet A_Y , we have $A_Y \cdot R = 0$. Hence

$$0 \leq -K_Y \cdot R = (2G + A_Y) \cdot R = 2G \cdot R.$$

This proves the claim.

Therefore, the anticanonical system $|-K_Y|$ has a non-zero fixed divisor, and its mobile part $|2G|$ is nef with $G^2 \neq 0$. This case cannot happen by Theorem 4.0.1.

Case $A \cdot l > 0$. In this case, $F \cdot l = 0$ since otherwise $-K_X \cdot l > 2$, which contradicts the classification of Mori (see Section 2.3). Furthermore, since either $A = 2D$ or $A = D_1 + D_2$ with D_1, D_2 disjoint, we know that E is not an irreducible component of A (otherwise $A \cdot l < 0$). Thus $A \cdot E$ is a non-zero effective 1-cycle.

By the classification of Mori (see Section 2.3), we are in one of the following cases:

- (1) If E is contracted to a point, then by the same arguments in Section 4.2.2 (1), one has that $F \cdot E = 0$. Hence E is contained in some fibre of f . Moreover, the intersection $A \cap E$ consists of rational curves contained in some fibre of f . We deduce from Remark 7.1.1 $A \cdot (A \cdot E) = 0$, and thus $A \cdot l = 0$. This contradicts $A \cdot l > 0$.
- (2) If E is contracted to a smooth curve, then $-K_X \cdot l = 1$ and $A \cdot l = 1$.
 - If $A = 2D$, then $D \cdot l = \frac{1}{2}$, which contradicts the fact that D is a Cartier divisor.

- If $A = D_1 + D_2$, then E is contained in some special fibre F_0 of f . Indeed, suppose by contradiction that $f|_E$ maps to \mathbb{P}^1 , then $F \cdot E$ is non-zero and in fact $F|_E \sim bl$ with $b > 0$. But F is a \mathbb{P}^1 -bundle over a smooth elliptic curve, we deduce that $F \cap E$ does not contain l or any of its deformations, since otherwise l moves on the surface F . This gives a contradiction.

Therefore, φ is an isomorphism outside F_0 , and $A \cap E = C_i$ with $i = 1$ or 2 . Hence $F_Y := \varphi(F) \simeq F$, $A_Y := \varphi(A) \simeq A$, and E is contracted to a smooth elliptic curve by φ , which implies that $-K_Y$ is nef by Proposition 2.4.7. Furthermore, the fibration f factors as $f = f' \circ \varphi$ and $f': Y \rightarrow \mathbb{P}^1$ gives Y the fibration structure. In this case, Y satisfies again Setup 5.0.1. Indeed, $-K_Y$ is not semi-ample, since otherwise $-K_{F_Y}$ is semi-ample; and $|-K_Y|$ has a non-zero fixed divisor (and hence A_Y is the fixed divisor), since otherwise A_Y is divisible by 2 in $\text{Pic}(Y)$ and thus $-K_{F_Y}$ is divisible by 2 in $\text{Pic}(F_Y)$, and this contradicts $-K_F = C_1 + C_2$ with $C_1 \not\sim C_2$.

To sum up, we have shown the following:

Proposition 7.1.2. *In Setup 7.0.1, assume that $A_v = 0$. Let $\varphi: X \rightarrow Y$ be a birational Mori contraction. Then $A = D_1 + D_2$ with D_1, D_2 disjoint and $\varphi: X \rightarrow Y$ factors as $f = f' \circ \varphi$. Furthermore, Y satisfies again Setup 5.0.1 with $|-K_Y| = A_Y + |2F_Y|$ such that $A_Y \simeq A$, $F_Y \simeq F$ and φ is the blow-up of Y along a smooth elliptic curve in some fibre of $f'|_{A_Y}$.*

By continuing the MMP, we will show that this above case cannot happen.

7.1.2 Non-birational cases

Case $\dim Y = 1$. In this case, $Y = \mathbb{P}^1$, $-K_X \cdot l = 1, 2$ or 3 , and a general fibre of φ is a del Pezzo surface. Since A is not a fibre of φ , one has $A \cdot l > 0$. If $-K_X \cdot l = 1$ or 2 , this implies $F \cdot l = 0$, and thus $F = \varphi^*(p)$ with $p \in Y$. Therefore, f and φ coincide. If $-K_X \cdot l = 3$, then φ is a \mathbb{P}^2 -bundle. As \mathbb{P}^2 is not fibred, F restricted to a \mathbb{P}^2 is trivial. Hence again, the two fibrations φ and f coincide.

By the classification of Mori (see Section 2.3), F is a del Pezzo surface, which contradicts the fact that $-K_F$ is not ample.

Case $\dim Y = 2$. In this case, $\varphi: X \rightarrow Y$ is a conic bundle with Y a smooth rational surface, and we have $-K_X \cdot l = 1$ or 2 .

- (i) If $F \cdot l = 1$, then φ is a \mathbb{P}^1 -bundle and induces a birational morphism from F to Y . This is impossible since $q(F) = 1$ and $q(Y) = 0$.
- (ii) If $F \cdot l = 0$, then we have $F = \varphi^*(R)$ with R an irreducible curve on Y , which gives a factorisation $f: X \xrightarrow{\varphi} Y \xrightarrow{\pi} \mathbb{P}^1$. On the other hand, as $F = \mathbb{P}(\mathcal{E})$ where \mathcal{E} is a rank-2 vector bundle over a smooth elliptic curve, we deduce that R is a smooth elliptic curve, and that the fibration $\varphi|_F$ coincides with the \mathbb{P}^1 -bundle structure $\mathbb{P}(\mathcal{E}) \rightarrow R$ on F . Let Δ be the discriminant locus of the conic bundle φ . Then Δ is contained in some special fibres of $\pi: Y \rightarrow \mathbb{P}^1$. As φ is an extremal contraction, by [Miy83, p. 83, Remark], every non singular rational curve in Δ must meet the other components of Δ in at least two points.

This implies that Δ is empty. Therefore, $\varphi: X \rightarrow Y$ is a \mathbb{P}^1 -bundle, and $-K_X \cdot l = A \cdot l = 2$. We can write $X \simeq \mathbb{P}(\mathcal{V})$, where \mathcal{V} is a rank-2 vector bundle over Y , and we have $\mathcal{V}|_R \simeq \mathcal{E}$.

- If $A = 2D$, then $D \cdot l = 1$. Since D is a rational section, we have an extension

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{V} \rightarrow \mathcal{J}_Z \otimes \det \mathcal{V} \rightarrow 0,$$

where \mathcal{J}_Z is the ideal sheaf of $c_2(\mathcal{V})$ points on Y and $D = \mathbb{P}(\mathcal{J}_Z \otimes \det \mathcal{V})$. We have that $-K_Y$ is nef by Proposition 2.4.7, and that $\pi: Y \rightarrow \mathbb{P}^1$ induces an elliptic fibration on Y . Hence, Y is isomorphic to \mathbb{P}^2 blown up at 9 points such that $-K_Y$ is nef and semi-ample (with some multiple of $-K_Y$ defining the elliptic fibration π), and thus $-K_Y \sim \alpha R$, where R is a general elliptic fibre of π and $\alpha \leq 1$. Hence, $(-K_Y)^2 = 0$. Now since D is isomorphic to $\text{Bl}_Z(Y)$ and $(-K_D)^2 = (D|_D + kF|_D)^2 = 0$ as $A^3 = A^2 \cdot F = 0$, we deduce that $Z = \emptyset$, $c_2(\mathcal{V}) = 0$, and $D \simeq Y$. Since

$$-K_X \sim \varphi^*(-K_Y - c_1(\mathcal{V})) + 2D,$$

and $-K_X \sim 2D + kF$, we deduce that $\varphi^*(c_1(\mathcal{V})) \sim -(k - \alpha)F$. By the Grothendieck relation, one has $D^2 \sim D \cdot \varphi^*(c_1(\mathcal{V})) \sim -(k - \alpha)D \cdot F$. Denote by e the smooth elliptic curve $D \cap F$. Then,

$$(-K_X)|_D \sim (2D + kF)|_D \sim (2\alpha - k)e.$$

Since $-K_X$ (and thus $-K_X|_D$) is nef, and $\alpha \leq 1$, $k \geq 2$, we deduce $\alpha = 1$ and $k = 2$. Therefore, $-K_X \sim 2D + 2F$ and \mathcal{V} is an extension

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{V} \rightarrow \mathcal{O}_Y(K_Y) \rightarrow 0.$$

- If $A = D_1 + D_2$, then $D_1 \cdot l = D_2 \cdot l = 1$ and D_1, D_2 are birational to Y . This contradicts the fact that Y is rational, whereas D_1, D_2 are not.

In conclusion, we have the following:

Proposition 7.1.3. *Let X be a smooth projective rationally connected threefold with anticanonical bundle $-K_X$ nef, $n(-K_X) = 3$ and $\nu(-K_X) = 2$. We suppose that the anticanonical system $|-K_X|$ has a non-zero fixed divisor A , and that its mobile part $|B|$ is nef so that $B^2 = 0$. Hence the mobile part induces a fibration $f: X \rightarrow \mathbb{P}^1$. Let F be a general fibre of f . Suppose that F is non-rational. Then $F = \mathbb{P}(\mathcal{E})$ is a \mathbb{P}^1 -bundle over a smooth elliptic curve, where \mathcal{E} is a rank-2 vector bundle over the elliptic curve which is a non-split extension*

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0.$$

Moreover, if the fixed divisor of $|-K_X|$ has no f -vertical component, then $X = \mathbb{P}(\mathcal{V})$ is a \mathbb{P}^1 -bundle over a surface Y and $|-K_X| = 2D + |2F|$, where Y is isomorphic to \mathbb{P}^2 blown up in 9 points such that $-K_Y$ is nef and base-point-free (thus induces an elliptic fibration $\pi: Y \rightarrow \mathbb{P}^1$), \mathcal{V} is a rank-2 vector bundle which is a non-split extension

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{V} \rightarrow \mathcal{O}_Y(K_Y) \rightarrow 0, \tag{7.1}$$

and $D = \mathbb{P}(\mathcal{O}_Y(K_Y))$. Furthermore, f factors as $X \xrightarrow{\varphi} Y \xrightarrow{\pi} \mathbb{P}^1$.

Proof. It remains to show that \mathcal{V} is indecomposable.

We first notice that

$$\mathrm{Ext}^1(\mathcal{O}_Y(K_Y), \mathcal{O}_Y) \simeq H^1(Y, \mathcal{O}_Y(-K_Y)) = \mathbb{C},$$

where the last equality follows from the Riemann-Roch formula.

Now suppose by contradiction that $\mathcal{V} = \mathcal{O}_Y \oplus \mathcal{O}_Y(K_Y)$. Then the quotient

$$\mathcal{V} \rightarrow \mathcal{O}_Y \rightarrow 0$$

gives a section D' of $\varphi: X = \mathbb{P}(\mathcal{V}) \rightarrow Y$ such that $D' \in |D - \varphi^*(K_Y)| = |D + F|$. Therefore, $-K_X \sim 2D'$ and $D' \neq D$, which contradicts the fact that $2D$ is the fixed divisor of $|-K_X|$. \square

Example 7.1.4. Conversely, let Y be \mathbb{P}^2 blown up at 9 points such that $-K_Y$ is nef, base-point-free and thus defines an elliptic fibration $\pi: Y \rightarrow \mathbb{P}^1$. Let \mathcal{V} be a rank-2 vector bundle which is a non-split extension

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{V} \rightarrow \mathcal{O}_Y(K_Y) \rightarrow 0, \quad (7.2)$$

and let $\varphi: X := \mathbb{P}(\mathcal{V}) \rightarrow Y$. Then $-K_X$ is nef and not semi-ample, and $|-K_X| = 2D + |2F|$, where $D := \mathbb{P}(\mathcal{O}_Y(K_Y))$ and F is a general fibre of $f := \pi \circ \varphi: X \rightarrow \mathbb{P}^1$. Moreover, $F = \mathbb{P}(\mathcal{E})$ is a \mathbb{P}^1 -bundle over a smooth elliptic curve, where \mathcal{E} is a rank-2 vector bundle over the elliptic curve which is a non-split extension

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0.$$

Proof. Let R be a general fibre of $\pi: Y \rightarrow \mathbb{P}^1$. Then $-K_Y \sim R$ and $F = \varphi^*(R)$. Since D is a tautological divisor of $\mathbb{P}(\mathcal{V}) = X$, we have

$$-K_X \sim 2D + \varphi^*(-K_Y - \det(\mathcal{V})) \sim 2D + 2F.$$

We first show that $F = \mathbb{P}(\mathcal{E})$, where \mathcal{E} is a rank-2 vector bundle over R , which is a non-split extension

$$0 \rightarrow \mathcal{O}_R \rightarrow \mathcal{E} \rightarrow \mathcal{O}_R \rightarrow 0. \quad (7.3)$$

Indeed, as $F = \varphi^*(R)$, we have $F = \mathbb{P}(\mathcal{E})$ with $\mathcal{E} \simeq \mathcal{V}|_R$. Restricting the short exact sequence (7.2) to R , we obtain

$$0 \rightarrow \mathcal{O}_R \rightarrow \mathcal{V}|_R \rightarrow \mathcal{O}_R \rightarrow 0,$$

as $\mathcal{O}_R(K_Y) \simeq \mathcal{O}_R$.

Let s be a non-zero element in $\mathrm{Ext}(\mathcal{O}_Y(K_Y), \mathcal{O}_Y) \simeq H^1(Y, \mathcal{O}_Y(-K_Y)) \simeq H^1(Y, \pi^*\mathcal{O}_{\mathbb{P}^1}(1))$. Since $H^1(\mathbb{P}^1, \pi_*(\pi^*\mathcal{O}_{\mathbb{P}^1}(1))) = 0$,

$$H^1(Y, \pi^*\mathcal{O}_{\mathbb{P}^1}(1)) \simeq H^0(Y, R^1\pi_*(\pi^*\mathcal{O}_{\mathbb{P}^1}(1)))$$

by the Leray spectral sequence. As $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \omega_{Y/\mathbb{P}^1}$, one has

$$R^1\pi_*(\pi^*\mathcal{O}_{\mathbb{P}^1}(1)) \simeq R^1\pi_*\omega_{Y/\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}$$

by [Kol86, Prop. 7.6]. Hence, $\text{Ext}(\mathcal{O}_Y(K_Y), \mathcal{O}_Y) \simeq H^0(\mathbb{P}^1, R^1\pi_*(\pi^*\mathcal{O}_{\mathbb{P}^1}(1))) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{C}$. Denote by $R_t \subset Y$ the fibre over $t \in \mathbb{P}^1$. Then the natural map

$$R^1\pi_*(\pi^*\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathbb{C}(t) \rightarrow H^1(R_t, \mathcal{O}_{R_t}) \simeq \text{Ext}(\mathcal{O}_{R_t}, \mathcal{O}_{R_t})$$

is an isomorphism (see for example [Har77, III, Cor. 12.9]). Therefore, the non-zero element $s \in \text{Ext}(\mathcal{O}_Y(K_Y), \mathcal{O}_Y)$ corresponds to a non-zero element $s_t \in \text{Ext}(\mathcal{O}_{R_t}, \mathcal{O}_{R_t})$. Thus \mathcal{E} is a non-split extension (7.3).

Now we show that $-K_X$ is nef. It is enough to check $-K_X \cdot C \geq 0$ for any integral curve $C \subset D$, as $-K_X \sim 2D + 2F$ and F is nef. Let $C \subset D$ be an integral curve. We have

$$D^2 \sim \varphi^*(c_1(\mathcal{V})) \cdot D \sim -D \cdot F$$

by the Grothendieck relation. Thus

$$-K_X \cdot C = (2D + 2F) \cdot C = (2D + 2F)|_D \cdot C = 0.$$

It remains to show that $-K_X$ is not semi-ample and that $2D$ is the fixed divisor of $|-K_X|$. Since

$$(-K_X)^2 \sim (2D + 2F)^2 \sim (-4D \cdot F + 8D \cdot F) = 4D \cdot F$$

is not numerically zero, and $(-K_X)^3 = 8(D + F) \cdot D \cdot F = 0$, one has $\nu(X, -K_X) = 2$.

Since $2D|_F \sim -K_F$ by the adjunction formula, and $\kappa(F, -K_F) = 0$, we deduce that

$$|-mK_X| = 2mD + |2mF|$$

for any integer $m \geq 1$. Thus, $2D$ is the fixed divisor of $|-K_X|$, and

$$\kappa(X, -K_X) = \kappa(X, F) = 1,$$

where the last equality follows from $\mathcal{O}_X(F) \simeq f^*\mathcal{O}_{\mathbb{P}^1}(1)$. □

Part II

Anticanonical system of the Fano model of \mathbb{P}^4 blown up at 8 points

Chapter 8

Introduction

Let $Y = M_{S, -K_S}$ be the moduli spaces of semi-stable rank-two torsion-free sheaves with $c_1 = -K_S$, $c_2 = 2$ on a polarised degree-one del Pezzo surface $(S, -K_S)$. The moduli spaces $Y = M_{S, -K_S}$ form a remarkable family of smooth Fano fourfolds with Picard number 9. The study of this family is motivated by two issues. Firstly, for Fano fourfolds with large Picard number (e.g. at least 7), only few examples which are not products of del Pezzo surfaces are known. As pointed out in [CCF19, Sect. 1, B], the family of Fano fourfolds Y is the only known example of Fano fourfolds with Picard number at least 9, which is not a product of surfaces. Secondly, it is delicate to find examples of Fano fourfolds whose anticanonical system has non-empty base locus, since most Fano fourfolds classified so far are toric, meaning that any ample line bundle on them is globally generated. Some examples are constructed in [Heu16, Chapter 6.3] as complete intersections of two hypersurfaces in weighted projective spaces. In [CCF19, Thm. 1.10], it is shown that the base locus of the anticanonical system $|-K_Y|$ has positive dimension. Therefore, the geometry of Y is worth detailed understanding.

The birational geometry of Y is related to the birational geometry of the blow-up X of \mathbb{P}^4 at 8 points. In [CCF19, Lem. 5.18], an explicit relation between X and Y is given: the Fano fourfold Y is obtained from X by flipping the strict transforms of the lines through all pairs of blown up points and of the quartics through 7 blown up points in \mathbb{P}^4 . Thanks to this relation between X and Y , it is shown in [CCF19, Lem. 7.5, Cor. 7.6] that the base locus of $|-K_Y|$ contains the strict transform R_Y of a smooth rational quintic curve through the 8 blown up points in \mathbb{P}^4 , and that $|-2K_Y|$ is base-point-free. We further our study of the anticanonical system and show more precisely that:

Proposition 8.0.1 (=Proposition 10.1.12). *The base scheme of $|-K_Y|$ is the smooth curve R_Y .*

As an application, we obtain the smoothness of a general member in the anticanonical system:

Corollary 8.0.2 (=Corollary 10.1.13). *Let $D \in |-K_Y|$ be a general divisor. Then D is smooth.*

Now we turn our attention to the automorphism group of Y . In [CCF19, Sect. 4], a group morphism ρ between the Picard groups of the surface S and of the moduli spaces $Y = M_{S, -K_S}$ is defined. This morphism ρ induces an isomorphism between the automorphism groups of S and of Y (see [CCF19, 6.15]). In particular, there is an involution ι_Y of Y which is induced by

the Bertini involution ι_S of S . By the analogy of Y and S , one expects that the action of ι_Y on Y has similar properties as the action of ι_S on S , where the latter is well understood (see for example [Dol12, 8.8.2]).

To understand the Bertini involution ι_Y , our approach would be analysing its behaviour on a special surface W_Y which is invariant by ι_Y . This surface W_Y is the strict transform of the cubic scroll swept out by the pencil of elliptic normal quintics in \mathbb{P}^4 through the 8 blown up points; in particular, it contains the curve R_Y . Inspired by the similarity with degree-one del Pezzo surfaces, we study the morphism defined by the restricted bianticanonical system of Y on W_Y , and we give the following description of ι_Y restricted to W_Y :

Proposition 8.0.3 (=Proposition 10.2.1). *The Bertini involution ι_Y preserves the surface W_Y , and its restriction $\iota_Y|_{W_Y}$ on W_Y is the biregular involution defined by the double covering*

$$\phi_{|-2K_Y|_{W_Y}} : W_Y \rightarrow V_{2,4} \subset \mathbb{P}^7,$$

where $V_{2,4} \simeq \mathbb{F}_2$ is a rational normal scroll of bidegree $(2, 4)$. In particular, the Bertini involution ι_Y is the identity on the curve R_Y and ι_Y induces an involution on each elliptic fibre F_Y of $W_Y \rightarrow \mathbb{P}^1$.

Furthermore, there exists a smooth curve $R' \sim 3(R_Y + F_Y)$ of genus 4 on the surface W_Y , such that R' is disjoint from R_Y and contained in the fixed locus of ι_Y .

As suggested in [CCF19, 7.15], further study is envisaged to describe completely the fixed locus of ι_Y , as well as the action of ι_Y on $|-K_Y|$ and on $|-2K_Y|$.

We mention here that another motivation behind the study of the Bertini involution ι_Y is the understanding of the corresponding birational involutions ι_X of X and $\iota_{\mathbb{P}^4}$ of \mathbb{P}^4 . These birational maps ι_X and $\iota_{\mathbb{P}^4}$ are classically known, as they can be defined via the Cremona action of the Weyl group $W(E_8)$ on sets of 8 points in \mathbb{P}^4 (see [DO88] and [DV81]). Nevertheless, the classical definitions of ι_X and $\iota_{\mathbb{P}^4}$ do not give a geometric description of these maps. In [CCF19, Prop. 8.9, Cor. 8.10], a factorization of these maps is given as smooth blow-ups and blow-downs using the interpretation of X as a moduli space of vector bundles on S . In view of the relation among Y , X and \mathbb{P}^4 , understanding one of the involutions helps describe the behaviour of the others.

Plan of Part II. In Chapter 9, we summarise some results in [CCF19], including the geometry of the Fano model Y , the connection between the blow-up X of \mathbb{P}^4 at 8 points and the degree-one del Pezzo surface S , and the relation between X and Y . We finish by recalling some basic properties of the Bertini involution of a degree-one del Pezzo surface.

In Chapter 10, we study the geometry of the Fano model Y . In Section 10.1, we investigate the anticanonical system $|-K_Y|$ and the bianticanonical system $|-2K_Y|$, and we prove Proposition 8.0.1 by an additional analysis on the simplicial facets of the cone of effective divisors on Y . In Section 10.2, we prove Proposition 8.0.3 by extensively using various properties of $|-K_Y|$ and $|-2K_Y|$.

In Appendix A, we include the code for several computations in Chapter 10 using the software system *Macaulay2*.

Chapter 9

Preliminaries

We fix S a general del Pezzo surface of degree one. Let $M_{S,L}$ be the moduli space of semi-stable (with respect to $L \in \text{Pic}(S)$ ample) rank-two torsion free sheaves \mathcal{F} on S with $c_1(\mathcal{F}) = -K_S$ and $c_2(\mathcal{F}) = 2$. Then it follows from the classical properties of the determinant line bundle that for the polarisation $L = -K_S$, the moduli space $Y := M_{S,-K_S}$ is Fano.

For the degree-one del Pezzo surface S , we introduce the following notions (see [CCF19, Sect. 2.1]). A conic on S is a smooth rational curve such that $-K_S \cdot C = 2$ and $C^2 = 0$. Every such conic yields a conic bundle $S \rightarrow \mathbb{P}^1$ having C as fibre. There are 2160 conics (as classes of a curve) in $H^2(S, \mathbb{Z})$. A big divisor h on S which realises S as the blow-up $\sigma : S \rightarrow \mathbb{P}^2$ at 8 distinct points is called a cubic. We have $h = \sigma^* \mathcal{O}_{\mathbb{P}^2}(1)$. There are 17280 cubics (as classes of a curve) in $H^2(S, \mathbb{Z})$.

Notation 9.0.1. *Given a cubic h , we use the following notation:*

- $\sigma_h : S \rightarrow \mathbb{P}^2$ is the birational map defined by h
- $q_1, \dots, q_8 \in \mathbb{P}^2$ are the points blown up by σ_h
- $e_i \subset S$ is the exceptional curve over q_i , for $i = 1, \dots, 8$
- $C_i \subset S$ is the transform of a general line through q_i , so that $C_i \sim h - e_i$, for $i = 1, \dots, 8$
- $\ell_{ij} \subset S$ is the transform of the line $\overline{q_i q_j} \subset \mathbb{P}^2$, so that $\ell_{ij} \sim h - e_i - e_j$ and ℓ_{ij} is a (-1) -curve, for $1 \leq i < j \leq 8$.

9.1 The Fano model Y

Proposition 9.1.1 ([CCF19], Prop.1.6). *The moduli space $Y := M_{S,-K_S}$ is a smooth, rational Fano 4-fold with index one and $b_2(Y) = 9$, $b_3(Y) = 0$, $h^{2,2}(Y) = b_4(Y) = 45$, $(-K_Y)^4 = 13$, $h^0(Y, -K_Y) = 6$, $h^0(Y, T_Y) = 0$, and $h^1(Y, T_Y) = 8$.*

For such a moduli space Y , the determinant map $\rho : H^2(S, \mathbb{R}) \rightarrow H^2(Y, \mathbb{R})$ is an isomorphism (see [CCF19, Thm.1.3]) and yields a completely explicit description of the relevant cones of divisors $\text{Eff}(Y)$, $\text{Mov}(Y)$ and $\text{Nef}(Y)$, as well as the cone of effective curves $\text{NE}(Y)$. We cite

the following statements for the cone of effective divisors $\text{Eff}(Y)$ and the cone of effective curves $\text{NE}(Y)$, and refer the readers to [CCF19, Sect. 6] for the description of the other relevant cones.

Proposition 9.1.2 ([CCF19], Sect.2.3, Cor.6.2). *The determinant map $\rho: H^2(S, \mathbb{R}) \rightarrow H^2(Y, \mathbb{R})$ yields an isomorphism between \mathcal{E} and $\text{Eff}(Y)$, where \mathcal{E} is the subcone of $\text{Nef}(S)$ generated by the conics:*

$$\mathcal{E} := \langle C \mid C \text{ a conic} \rangle \subset H^2(S, \mathbb{R}).$$

Hence, the cone $\text{Eff}(Y)$ has 2160 extremal rays, each generated by a fixed divisor E_C , where $C \subset S$ is a conic.

Moreover, given a cubic h , $(2h + K_S)^\perp \cap \mathcal{E}$ is a simplicial facet (i.e. a face of codimension one) of \mathcal{E} , generated by the conics C_i for $i = 1, \dots, 8$ (notations as in Notation 9.0.1). Hence, the fixed divisors E_{C_i} for $i = 1, \dots, 8$ generate a simplicial facet of $\text{Eff}(Y)$.

Proposition 9.1.3 ([CCF19], Prop.1.7). *The cone of effective curves $\text{NE}(Y)$ has 240 extremal rays, and is isomorphic to $\text{NE}(S)$. If ℓ is a (-1) -curve, the corresponding extremal ray of $\text{NE}(Y)$ is generated by the class of a line Γ_ℓ in $P_\ell \cong \mathbb{P}^2 \subset Y$. The corresponding elementary contraction is a small contraction, sending P_ℓ to a point.*

The determinant map ρ also relates the two automorphism groups $\text{Aut}(Y)$ and $\text{Aut}(S)$:

Theorem 9.1.4 ([CCF19], Thm.1.9). *The map $\psi: \text{Aut}(S) \rightarrow \text{Aut}(Y)$ given by $\psi(f)[\mathcal{F}] = [(f^{-1})^*\mathcal{F}]$, for $f \in \text{Aut}(S)$ and $[\mathcal{F}] \in Y$, is a group isomorphism. In particular $\text{Aut}(Y)$ is finite, and if S is general, then $\text{Aut}(Y) = \{\text{Id}_Y, \iota_Y\}$, where $\iota_Y: Y \rightarrow Y$ is induced by the Bertini involution of S .*

Definition 9.1.5 ([CCF19], Def.6.19). *The Bertini involution ι_S of S induces an involution $\iota_Y = \psi(\iota_S)$ of Y , that we still call the Bertini involution; explicitly $\iota_Y: Y \rightarrow Y$ is given by $\iota_Y([\mathcal{F}]) = [\iota_S^*\mathcal{F}]$. We have a commutative diagram:*

$$\begin{array}{ccc} H^2(S, \mathbb{R}) & \xrightarrow{\iota_S^*} & H^2(S, \mathbb{R}) \\ \rho \downarrow & & \downarrow \rho \\ H^2(Y, \mathbb{R}) & \xrightarrow{\iota_Y^*} & H^2(Y, \mathbb{R}). \end{array} \tag{9.1}$$

Finally, motivated by the analogy with del Pezzo surface of degree one, the study of the base loci of the anticanonical and the bianticanonical linear systems of Y gives the following:

Theorem 9.1.6 ([CCF19], Thm.1.10). *The linear system $| -K_Y |$ has a base locus of positive dimension, while the linear system $| -2K_Y |$ is base point free.*

9.2 The blow-up X of \mathbb{P}^4 at 8 general points

9.2.1 Degree-one del Pezzo surfaces and blow-ups of \mathbb{P}^4 in 8 points

For $S = \text{Bl}_{q_1, \dots, q_8} \mathbb{P}^2$ and $X = \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4$ the blow-ups respectively of \mathbb{P}^2 and \mathbb{P}^4 at 8 general points, there is a classical association between these two varieties due to Gale duality. The

following is summarised from [CCF19, 2.21]; for further details of the association, we refer to [CCF19, 2.18].

Let h be a cubic on S . We associate to (S, h) a blow-up X of \mathbb{P}^4 in 8 points in general linear position as follows.

Let $q_1, \dots, q_8 \in \mathbb{P}^2$ be the points blown up under the birational morphism $S \rightarrow \mathbb{P}^2$ defined by h (the points q_1, \dots, q_8 are in general linear position by [CCF19, Rem. 2.20]), and let $p_1, \dots, p_8 \in \mathbb{P}^4$ be the associated points to $q_1, \dots, q_8 \in \mathbb{P}^2$ (the points p_1, \dots, p_8 are in general linear position by [CCF19, Lem. 2.19]). Then we set

$$X = X_h = X_{(S,h)} := \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4.$$

We always assume that $q_1, \dots, q_8 \in \mathbb{P}^2$ and $p_1, \dots, p_8 \in \mathbb{P}^4$ are associated as ordered sets of point.

Conversely, let X be a blow-up of \mathbb{P}^4 in 8 general points. Differently from the case of surfaces, the blow-up map $X \rightarrow \mathbb{P}^4$ is unique and thus X determines $p_1, \dots, p_8 \in \mathbb{P}^4$ up to projective equivalence. The 8 points $p_1, \dots, p_8 \in \mathbb{P}^4$ in turn determine $q_1, \dots, q_8 \in \mathbb{P}^2$ up to projective equivalence, and thus a pair (S, h) such that $X \cong X_{(S,h)}$. The pair (S, h) is unique up to isomorphism, therefore S is determined up to isomorphism, and h is determined up to the action of the automorphism group $\text{Aut}(S)$ on cubics.

9.2.2 Notation for the blow-up X of \mathbb{P}^4 at 8 points

Let $p_1, \dots, p_8 \in \mathbb{P}^4$ be 8 points in general linear position, and set $X := \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4$. We use the following notation:

- $E_i \subset X$ is the exceptional divisor over $p_i \in \mathbb{P}^4$, for $i = 1, \dots, 8$
- $H \in \text{Pic}(X)$ is the pull-back of $\mathcal{O}_{\mathbb{P}^4}(1)$
- $L_{ij} \subset X$ is the transform of the line $\overline{p_i p_j} \subset \mathbb{P}^4$, for $1 \leq i < j \leq 8$
- $e_i \subset E_i$ is a line, for $i = 1, \dots, 8$
- $\gamma_i \subset \mathbb{P}^4$ is the rational normal quartic through $p_1, \dots, \check{p}_i, \dots, p_8$, for $i = 1, \dots, 8$
- $\Gamma_i \subset X$ is the transform of $\gamma_i \subset \mathbb{P}^4$, for $i = 1, \dots, 8$.

9.3 From the blow-up X to the Fano model Y

We recall the explicit relation between X and Y :

Lemma 9.3.1 ([CCF19], Lem.5.18). *The birational map $\xi: X \dashrightarrow Y$ is the composition of 36 (K -positive) flips: first the flips of L_{ij} for $1 \leq i < j \leq 8$, and then the flips of Γ_k for $k = 1, \dots, 8$. There is a commutative diagram:*

$$\begin{array}{ccc} & \widehat{X} & \\ & \swarrow & \searrow \\ X & \overset{\xi}{\dashrightarrow} & Y \end{array}$$

where $\widehat{X} \rightarrow X$ is the blow-up of the curves L_{ij} and Γ_k , with every exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1, -1)$, and $\widehat{X} \rightarrow Y$ is the blow-up of 36 pairwise disjoint smooth rational surfaces.

Notation 9.3.2. We use the following notation:

- $P_{\ell_{ij}} \subset Y$ is the flipped surface replacing $L_{ij} \subset X$, for $1 \leq i < j \leq 8$
- $P_{e_k} \subset Y$ is the flipped surface replacing $\Gamma_k \subset X$, for $k = 1, \dots, 8$.

We will sometimes write $\xi_h: X_h \dashrightarrow Y$ to stress that X_h and ξ_h depend on the chosen cubic h (while Y does not). Denote by η_h the composition map:

$$\begin{array}{ccc}
 & & \eta_h \\
 & \curvearrowright & \\
 Y & \xrightarrow{\xi_h^{-1}} & X_h \longrightarrow \mathbb{P}^4
 \end{array}$$

9.4 The Bertini involution of S

We recall some basic properties of the Bertini involution of a del Pezzo surface of degree one.

Proposition 9.4.1 ([Dol12], Thm.8.3.2). *Suppose that S is a del Pezzo surface of degree 1. Then*

- $| - K_S |$ is a pencil of genus 1 curves with smooth general member and one base point;
- $| - 2K_S |$ is base-point-free and defines a morphism $\phi_{|-2K_S|}: S \rightarrow \mathbb{P}^3$ which is finite of degree 2 with image Q a quadric cone.

The Bertini involution $\iota_S: S \rightarrow S$ is the biregular involution defined by the double covering

$$\phi_{|-2K_S|}: S \rightarrow Q.$$

For S general, ι_S is the unique non-trivial automorphism of S . The pull-back ι_S^* acts on $\text{Pic}(S)$ (and on $H^2(S, \mathbb{R})$) by fixing K_S and acting as -1 on K_S^\perp (see [Dol12, §8.8.2]). This yields:

$$\iota_S^* \gamma = 2(\gamma \cdot K_S)K_S - \gamma \quad \text{for every } \gamma \in H^2(S, \mathbb{R}). \quad (9.2)$$

The fixed locus of ι_S is a smooth irreducible curve of genus 4 isomorphic to the branch curve of the double cover and the base point of $| - K_S |$. The fixed curve belongs to the linear system $| - 3K_S |$.

Chapter 10

Geometry of the Fano model Y

10.1 Anticanonical and bianticanonical linear systems

To analyse the anticanonical linear system $|-K_Y|$, we introduce a special surface as follows.

Lemma 10.1.1 ([CCF19], Lem.7.2). *Let $p_1, \dots, p_8 \in \mathbb{P}^4$ be general points. Then there is a pencil of elliptic normal quintics in \mathbb{P}^4 through p_1, \dots, p_8 , which sweeps out a cubic scroll $W \subset \mathbb{P}^4$.*

Let moreover $q_1, \dots, q_8 \in \mathbb{P}^2$ be the associated points to $p_1, \dots, p_8 \in \mathbb{P}^4$. Then there is a birational map $\alpha: W \rightarrow \mathbb{P}^2$ such that $\alpha(p_i) = q_i$ for $i = 1, \dots, 8$, α sends the pencil of elliptic normal quintics to the pencil of plane cubics through q_1, \dots, q_8 , and α is the blow-up of the ninth base point $q_0 \in \mathbb{P}^2$ of the pencil of plane cubics.

Let $W' \subset X$ be the transform of the cubic scroll $W \subset \mathbb{P}^4$. By [CCF19, (7.3)], we have the following diagram:

$$\begin{array}{ccc}
 & W' \subset X & \\
 \eta \swarrow & & \searrow \\
 W \subset \mathbb{P}^4 & & S \\
 \alpha \searrow & & \swarrow \sigma \\
 & \mathbb{P}^2 &
 \end{array}
 \tag{10.1}$$

where $\eta: W' \rightarrow W$ is the blow-up of p_1, \dots, p_8 , so the composition $\alpha' := \alpha \circ \eta: W' \rightarrow \mathbb{P}^2$ is the blow-up of q_0, \dots, q_8 . Thus W' is isomorphic to the blow-up of S in the base point of $|-K_S|$. Hence, there is an elliptic fibration $\pi: W' \rightarrow \mathbb{P}^1$, where the smooth fibres are the transforms of the elliptic normal quintics through p_1, \dots, p_8 in \mathbb{P}^4 , and every fibre is integral.

Lemma 10.1.2 ([CCF19], Lem.7.4). *The surface $W' \subset X$ is disjoint from L_{ij} for $1 \leq i < j \leq 8$ and from Γ_k for $k = 1, \dots, 8$, and W' is contained in the open subset where $\xi: X \dashrightarrow Y$ is an isomorphism.*

We denote by W_Y the strict transform of W' in Y . Then $W_Y \simeq W'$.

Lemma 10.1.3 ([CCF19], Lem.7.5, Rem.7.10). *We have $(-K_X)|_{W'} = \mathcal{O}_{W'}(R + 2F)$ and $R = \text{Bs } |(-K_X)|_{W'}|$, where $F \subset W'$ is a fibre of the elliptic fibration, and $R \subset W'$ is a (-1) -curve*

and a section of the elliptic fibration. The curves R and F satisfy $-K_X \cdot R = -K_X \cdot F = 1$ and $E_i \cdot R = E_i \cdot F = 1$ for every $i = 1, \dots, 8$, so $R \equiv F$ in X and $\xi(R) \equiv \xi(F)$ in Y .

Remark 10.1.4 ([CCF19], Lem.7.7). Let $R_4 \subset \mathbb{P}^4$ and $R_2 \subset \mathbb{P}^2$ be the images of R under $\eta: W' \subset X \rightarrow W \subset \mathbb{P}^4$ and $\alpha': W' \rightarrow \mathbb{P}^2$ respectively (see (10.1)).

Then R_4 is a smooth rational quintic curve through p_1, \dots, p_8 , and R_2 is a rational plane quartic containing q_1, \dots, q_8 and having a triple point in q_0 .

Corollary 10.1.5 ([CCF19], Cor.7.6). The base locus of $|-K_X|$ contains the smooth rational curve R , and the base locus of $|-K_Y|$ contains the smooth rational curve $\xi(R)$.

We denote by R_Y the smooth rational curve $\xi(R)$ contained in the base locus of $|-K_Y|$, and F_Y a fibre of the elliptic fibration $W_Y \rightarrow \mathbb{P}^1$.

Lemma 10.1.6. The normal bundle $\mathcal{N}_{R_Y/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$.

Proof. Since R_4 is a rational quintic curve in \mathbb{P}^4 , one has

$$\mathcal{N}_{R_4/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$$

with $a \leq b \leq c$ and $a + b + c = 23$. Since $\mathcal{T}_{\mathbb{P}^4}|_{R_4} \rightarrow \mathcal{N}_{R_4/\mathbb{P}^4} \rightarrow 0$, one has that $\mathcal{N}_{R_4/\mathbb{P}^4}$ is ample. Hence, we deduce that $a, b, c > 0$. Moreover, by *Macaulay2* (see Listing A.2),

$$h^0(R_4, \mathcal{N}_{R_4/\mathbb{P}^4}^* \otimes \mathcal{O}_{\mathbb{P}^4}(1) \otimes \omega_{R_4}^*) = 1,$$

we deduce that $a = 7$ and $b, c > 7$. Hence, $b = c = 8$. Therefore, by [Ful98, B.6.10], one has

$$\mathcal{N}_{R/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}.$$

As R is disjoint from the indeterminacy locus of the map ξ_h , we deduce that

$$\mathcal{N}_{R_Y/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}.$$

□

Remark 10.1.7. (see also [CCF19, Remark 7.8]) In \mathbb{P}^4 , let \mathcal{M} be the linear system of quintic hypersurfaces with multiplicity at least 3 at 8 general points. Then by *Macaulay2* (see Listing A.1) the base ideal $\mathfrak{b}(\mathcal{M})$ is the intersection of the ideals of 28 line $\overline{p_i p_j}$ for $1 \leq i < j \leq 8$, the ideals of 8 rational normal quartic curves γ_k for $k = 1, \dots, 8$ and the ideal of the rational quintic curve R_4 . This shows that the base scheme of $|-K_Y|$, in the open subset of Y where $\eta_h: Y \dashrightarrow \mathbb{P}^4$ is an isomorphism, is R_Y (reduced) minus the 8 points of intersection with the 8 exceptional divisors $\xi(E_1), \dots, \xi(E_8)$. The base locus of $|-K_Y|$ is thus given by R_Y , possibly union some other components contained in $\xi(E_1), \dots, \xi(E_8)$.

Lemma 10.1.8. The base locus of the anticanonical system $|-K_Y|$ is disjoint from the surfaces $P_{\ell_{ij}}$ and P_{e_k} , for $1 \leq i < j \leq 8$ and $k = 1, \dots, 8$.

Proof. Consider the commutative diagram in Lemma 9.3.1:

$$\begin{array}{ccc} & \hat{X} & \\ p \swarrow & & \searrow q \\ X & \xrightarrow{\xi} & Y \end{array}$$

where $p: \hat{X} \rightarrow X$ is the blow-up of X along the curves L_{ij} and Γ_k with every exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, and $q: \hat{X} \rightarrow Y$ is the blow-up of 36 pairwise disjoint smooth rational surfaces $P_{\ell_{ij}}$ and P_{e_k} , for $1 \leq i < j \leq 8$ and $k = 1, \dots, 8$.

Suppose by contradiction that there exists a base point y of $|-K_Y|$ contained in some flipped surface that we denote by S (which is one of the surfaces $P_{\ell_{ij}}$ or P_{e_k}). Denote by $C \subset X$ the corresponding flipping curve (which is one of the curves L_{ij} or Γ_k).

Let E be the sum of exceptional divisors over L_{ij} for $1 \leq i < j \leq 8$ and over Γ_k for $k = 1, \dots, 8$. Since

$$p^*(-K_X) - 2E = -K_{\hat{X}} = q^*(-K_Y) - E,$$

one has

$$q^*(-K_Y) = p^*(-K_X) - E.$$

Let $E_y \simeq \mathbb{P}^1$ be the exceptional fibre in \hat{X} above y . Then E_y is contained in $\text{Bs}|q^*(-K_Y)| = \text{Bs}|p^*(-K_X) - E|$ and E_y is mapped surjectively onto C .

Since the blow-up of the 8 points $X = \text{Bl}_{p_1, \dots, p_8} \rightarrow \mathbb{P}^4$ is an isomorphism near a general point of C , the base scheme of $|-K_X|$ is generically reduced along C by Remark 10.1.7. Hence, the linear system $|p^*(-K_X) - E|$ is base-point-free above the generic point of C . This contradicts the fact that $\text{Bs}|p^*(-K_X) - E|$ contains a curve which is mapped surjectively onto C . \square

Remark 10.1.9. *More generally, the proof of Lemma 10.1.8 shows the following. Let X, Y be projective smooth fourfolds. Let $\xi: X \dashrightarrow Y$ be an (anti-)flip which flips a smooth curve $C \subset X$ by blowing up the curve and contracting the exceptional divisor to a smooth surface $S \subset Y$. If $\text{Bs}|-K_X|$ is reduced in the generic point of C , then $|-K_Y|$ is base-point-free on S .*

Corollary 10.1.10. *The curve R_Y is the unique base curve in $\text{Bs}|-K_Y|$ of anticanonical degree 1. Therefore, R_Y is independent of the choice of cubic h .*

Proof. Let $C \subset \text{Bs}|-K_Y|$ be a base curve contained in some exceptional divisor $\xi(E_i)$, for $i = 1, \dots, 8$. Let \tilde{C} be its strict transform in X . By Lemma 10.1.8, the curve C is disjoint from the indeterminacy locus of ξ^{-1} . Hence, one has $-K_Y \cdot C = -K_X \cdot \tilde{C}$ and $\tilde{C} \subset E_i$.

Since $-K_X = 5H - 3\sum_{j=1}^8 E_j$, $H \cdot \tilde{C} = 0$, $E_j \cdot \tilde{C} = 0$ for $j \neq i$, and $E_i \cdot \tilde{C} \leq -1$, one has

$$-K_Y \cdot C = -K_X \cdot \tilde{C} \geq 3.$$

Therefore, the curve R_Y is the unique base curve satisfying $-K_Y \cdot R_Y = -K_X \cdot R = 1$. \square

Corollary 10.1.11. *Let $B \subset Y$ be an irreducible component of the (set-theoretic) base locus of $|-K_Y|$, which is distinct from R_Y . Then for every simplicial facet $\langle E_{C_1}, \dots, E_{C_8} \rangle$ of $\text{Eff}(Y)$ (notation as in Notation 9.0.1 and Proposition 9.1.2), there exists a unique E_{C_i} for $i = 1, \dots, 8$ such that $B \subset E_i$.*

Proof. Given a cubic h , consider the simplicial facet $\langle E_{C_1}, \dots, E_{C_8} \rangle$ of $\text{Eff}(Y)$, where $C_i \sim h - e_i$ for $i = 1, \dots, 8$ (notation as in Notation 9.0.1). Then E_{C_i} are the strict transforms of the exceptional divisors $E_i \simeq \mathbb{P}^3 \subset X_h = X$ under $\xi_h = \xi: X \dashrightarrow Y$.

Since B is distinct from R_Y , we deduce that B is contained in some fixed divisor E_{C_i} by Remark 10.1.7. By the construction of the composition of flips ξ (see Lemma 9.3.1), the intersection of two fixed divisors E_{C_j} and E_{C_k} (for $k \neq j$) is the union of the flipped surfaces $P_{\ell_{jk}}$ and P_{e_l} for $l \neq j, k$. Hence, by Lemma 10.1.8, the fixed divisor E_i containing B is unique. \square

Proposition 10.1.12. *The base scheme of $| - K_Y |$ is the reduced smooth curve R_Y .*

Proof. We first show that R_Y is the unique irreducible component of the (set-theoretic) base locus of $| - K_Y |$.

Let h be a cubic. Let C_i be a conic such that $C_i \sim h - e_i$ for $i = 1, \dots, 8$ (notation as in Notation 9.0.1). Let $E_i := E_{C_i}$, where we use the notation of Proposition 9.1.2. By the same proposition, E_1, \dots, E_8 generate a simplicial facet of $\text{Eff}(Y)$. Suppose by contradiction that there exists another component B distinct from R_Y of the base locus of $| - K_Y |$. Then by Corollary 10.1.11, we may suppose that $B \subset E_1$ and $B \not\subset E_2, E_3, \dots, E_8$.

Let i, j, k, l be distinct indices in $\{1, \dots, 8\}$. Consider the conics C'_l such that $C'_l \sim 2h - e_i - e_j - e_k - e_l$ and the corresponding fixed divisors $F_{ijkl} := E_{2h - e_i - e_j - e_k - e_l}$.

Claim. The fixed divisors E_i, E_j, E_k and F_{ijkl} for $l \in \{1, \dots, 8\}$ distinct from i, j, k generate a simplicial facet of $\text{Eff}(Y)$.

Indeed, by Proposition 9.1.2, it is enough to find a cubic h' such that $2h' + K_S$ is orthogonal to the 8 conics C_i, C_j, C_k and C'_l for $l \in \{1, \dots, 8\}$ distinct from i, j, k .

We take $h' \sim 2h - e_i - e_j - e_k$. Then we can check that

$$\begin{aligned} C_i &\sim h' - \ell_{jk} \\ C_j &\sim h' - \ell_{ik} \\ C_k &\sim h' - \ell_{ij} \\ C'_l &\sim h' - e_l \end{aligned}$$

and $2h' + K_S$ is orthogonal to the above 8 conics. Moreover, h' is nef and big, and the corresponding birational map $\sigma_{h'}: S \rightarrow \mathbb{P}^2$ contracts the 8 pairwise disjoint (-1) -curves $\ell_{jk}, \ell_{ik}, \ell_{ij}, e_l$ for $l \neq i, j, k$. Hence, h' is a cubic. This proves the claim.

We will repeatedly use Corollary 10.1.11 in the following.

- Consider the simplicial facet generated by $E_1, E_2, E_3, F_{1234}, F_{1235}, F_{1236}, F_{1237}, F_{1238}$. Then $B \not\subset F_{1234}, F_{1235}, F_{1236}, F_{1237}, F_{1238}$.
- Consider the simplicial facet generated by $E_2, E_3, E_4, F_{1234}, F_{2345}, F_{2346}, F_{2347}, F_{2348}$. Then B is contained in one of the fixed divisors $F_{2345}, F_{2346}, F_{2347}, F_{2348}$. We may suppose that $B \subset F_{2345}$. Then $B \not\subset F_{2346}, F_{2347}, F_{2348}$.
- Consider the simplicial facet generated by $E_2, E_3, E_5, F_{1235}, F_{2345}, F_{2356}, F_{2357}, F_{2358}$. Then $B \not\subset F_{2356}, F_{2357}, F_{2358}$.

- Consider the simplicial facet generated by $E_2, E_3, E_6, F_{1236}, F_{2346}, F_{2356}, F_{2367}, F_{2368}$. Then by what precedes, we know that B is contained in one of the fixed divisors F_{2367}, F_{2368} . We may suppose that $B \subset F_{2367}$. Then $B \not\subset F_{2368}$.
- Consider the simplicial facet generated by $E_2, E_3, E_7, F_{1237}, F_{2347}, F_{2357}, F_{2367}, F_{2378}$. Then $B \not\subset F_{2378}$.
- Finally, consider the simplicial facet generated by $E_2, E_3, E_8, F_{1238}, F_{2348}, F_{2358}, F_{2368}, F_{2378}$. Then by what precedes, we know that B is contained in none of these 8 fixed divisors, which contradicts Corollary 10.1.11.

Therefore, the curve R_Y is the unique irreducible component of the base locus of $|-K_Y|$.

Now we show that the base scheme of $|-K_Y|$ is the reduced curve R_Y , i.e. there are no embedded points. Indeed, given a cubic h , consider the birational map $\eta_h: Y \dashrightarrow \mathbb{P}^4$. By Remark 10.1.7, the base scheme of $|-K_Y|$ is the reduced curve R_Y with possible embedded points which have support in the 8 points of intersection with the 8 exceptional divisors of η_h . By varying h , we may consider another map $\eta_{h'}: Y \dashrightarrow \mathbb{P}^4$ with other 8 exceptional divisors, so that we get 8 different points of intersection on R_Y . Such a cubic h' exists because otherwise, there is a base point y on R_Y such that for every simplicial facet $\langle E_{C_1}, \dots, E_{C_8} \rangle$ of $\text{Eff}(Y)$ the point y is contained in a unique E_{C_i} , and thus we obtain a contradiction by replacing B with y in the above paragraph. Hence, there is no embedded base point on R_Y . \square

Corollary 10.1.13. *Let $D \in |-K_Y|$ be a general divisor. Then D is smooth.*

Proof. Since the base scheme $\text{Bs}|-K_Y|$ is the smooth curve R_Y by Proposition 10.1.12, we can apply [MM86, Prop. 6.8] which implies that a general member in $|-K_Y|$ is smooth. \square

Lemma 10.1.14. *For a general point $x \in R_4$, there exists a unique divisor in \mathcal{M} which has multiplicity 3 at x : it is the secant variety of the elliptic normal quintic through the nine points p_1, \dots, p_8 and x .*

By varying x on R_4 , one obtains a one-dimensional family Sec of divisors in \mathcal{M} with scheme-theoretic intersection Bs Sec defined by the ideal $\mathfrak{b}(\text{Sec})$. Then

$$(\mathfrak{b}(\text{Sec}) : \mathfrak{b}(M)) : \mathfrak{I}_W = \mathfrak{I}_W,$$

where the scheme defined by the ideal \mathfrak{I}_W is the reduced surface W .

Proof. We choose a random point x on R_4 which is not one of the 8 blown up points. Let $\mathcal{M}_{x,3}$ be the linear subspace of divisors in \mathcal{M} having multiplicity at least 3 at the point x . Then $\dim \mathcal{M}_{x,3} = 0$ by *Macaulay2* (see Listing A.3) and thus the unique element in $\mathcal{M}_{x,3}$ is the secant variety $\text{Sec}(E_x)$, where E_x is the elliptic normal quintic in W passing through the point x and the 8 blown up points.

Let Sec be the family of secant varieties $\text{Sec}(E_x)$ for x varying on R_4 and $\mathfrak{b}(\text{Sec})$ be the ideal associated to the scheme-theoretic intersection Bs Sec of the family Sec . Let $\mathfrak{b}_3(\text{Sec})$ be the ideal associated to the scheme-theoretic intersection of three general secant varieties in Sec (obtained by choosing three distinct random points on R_4 and intersecting the corresponding secant varieties).

By *Macaulay2* (see Listing A.4), the quotient $\mathfrak{I}_S := (\mathfrak{b}_3(\text{Sec}) : \mathfrak{b}(\mathcal{M}))$ has degree 6 and dimension 2. Let \mathfrak{I}_W be the ideal of singular locus of the variety defined by \mathfrak{I}_S . Then by *Macaulay2* (see Listing A.4), \mathfrak{I}_W has dimension 2 and degree 3; moreover, the variety defined by \mathfrak{I}_W is smooth and one has $(\mathfrak{I}_S : \mathfrak{I}_W) = \mathfrak{I}_W$. Since each of these secant varieties in Sec contains the cubic scroll W , we deduce that the variety defined by \mathfrak{I}_W is indeed the surface W .

By *Macaulay2* (see Listing A.5),

$$h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{I}_{p_1, \dots, p_8}^3 \otimes \mathcal{I}_W) = 3.$$

Since Sec is a family of divisors in \mathcal{M} , we deduce that $\text{Bs}(\text{Sec})$ can be obtained by the scheme-intersection of three general members in Sec . Therefore, $\mathfrak{b}_3(\text{Sec}) = \mathfrak{b}(\text{Sec})$. \square

Lemma 10.1.15. *The surface W_Y is unique, i.e. W_Y is independent of the choice of cubic h . Therefore, W_Y is disjoint from every one of the loci P_ℓ of the small extremal rays of Y .*

Proof. Let Sec_Y be the family of the strict transforms in Y of the secant varieties in Sec . Let $\mathcal{M}_{Y,3}$ be the family of divisors in $| -K_Y |$ having multiplicity 3 at some point on R_Y . Then the two families $\mathcal{M}_{Y,3}$ and Sec_Y are equal, as $\dim \mathcal{M}_{x,3} = 0$ for a general point $x \in R_4$ by Lemma 10.1.14 and η_h is an isomorphism at the generic point of R_4 .

Suppose by contradiction that W_Y depends on h . Then there exist two distinct surfaces $W_{Y,h}$ and $W_{Y,h'}$. Let $\text{Bs } \mathcal{M}_{Y,3}$ be the scheme-theoretic intersection of the family $\mathcal{M}_{Y,3}$. Then by Lemma 10.1.14, one has the following set-theoretic inclusion:

$$\text{Bs } \mathcal{M}_{Y,3} \supset W_{Y,h} \cup W_{Y,h'}.$$

Since $W_{Y,h'}$ contains the curve R_Y which is generically in the locus where $\xi_h^{-1}: Y \dashrightarrow X_h$ is an isomorphism, we deduce that $W_{Y,h'}$ is not contracted by ξ_h^{-1} .

Since the surface $\xi_h^{-1}(W_{Y,h'})$ contains the curve R , this surface cannot be contained in any exceptional locus E_i , $i = 1, \dots, 8$ of $X_h \rightarrow \mathbb{P}^4$, and thus it cannot be contracted; we denote by $W_{h'}$ its image in \mathbb{P}^4 . Therefore, $\text{Bs } \text{Sec}$ contains two distinct surfaces W and $W_{h'}$, which contradicts Lemma 10.1.14.

Since by Lemma 10.1.2 the surface W_Y is disjoint from the indeterminacy locus of the map $\xi_h^{-1}: Y \dashrightarrow X_h$, which is a union of some of the loci P_ℓ (depending on h), and W_Y is the same for all h , we deduce that W_Y is disjoint from every one of the loci P_ℓ . \square

Lemma 10.1.16. (i) *We have $h^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = 3$. The restriction*

$$r_1: H^0(Y, \mathcal{O}_Y(-K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-K_Y))$$

is surjective.

(ii) *We have $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$. The restriction*

$$r_2: H^0(Y, \mathcal{O}_Y(-2K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y))$$

is surjective.

Proof. Since $-K_{W_Y} \sim F_Y$ and $-K_Y|_{W_Y} \sim R_Y + 2F_Y$ by Lemma 10.1.3, by the Riemann-Roch formula one has $\chi(W_Y, -K_Y|_{W_Y}) = 3$. Since $-K_Y|_{W_Y}$ is ample on W_Y and $-K_{W_Y}$ is nef, by Kodaira vanishing theorem one has

$$h^j(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = h^j(W_Y, \mathcal{O}_{W_Y}(K_{W_Y} - K_{W_Y} + (-K_Y))) = 0$$

for $j = 1, 2$. Therefore, $h^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = 3$. The same argument can be applied to obtain $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$.

(i) By *Macaulay2* (see Listing A.5),

$$h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{J}_{p_1, \dots, p_8}^3 \otimes \mathcal{J}_W) = 3.$$

Since $H^0(Y, \mathcal{O}_Y(-K_Y)) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{J}_{p_1, \dots, p_8}^3)$, and the surface W_Y is disjoint from the indeterminacy locus of η_h by Lemma 10.1.2 and W_Y is not contained in the exceptional locus of η_h , we deduce that

$$H^0(Y, \mathcal{O}_Y(-K_Y) \otimes \mathcal{J}_{W_Y}) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{J}_{p_1, \dots, p_8}^3 \otimes \mathcal{J}_W).$$

Hence,

$$h^0(Y, \mathcal{O}_Y(-K_Y) \otimes \mathcal{J}_{W_Y}) = 3.$$

As $h^0(Y, \mathcal{O}_Y(-K_Y)) = 6$ and $h^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = 3$, we deduce that the restriction morphism

$$H^0(Y, \mathcal{O}_Y(-K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-K_Y))$$

is surjective.

(ii) By *Macaulay2* (see Listing A.6),

$$h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10) \otimes \mathcal{J}_{p_1, \dots, p_8}^6 \otimes \mathcal{J}_W) = 21.$$

Since $H^0(Y, \mathcal{O}_Y(-2K_Y)) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10) \otimes \mathcal{J}_{p_1, \dots, p_8}^6)$ and by the same argument as above, we deduce that

$$H^0(Y, \mathcal{O}_Y(-2K_Y) \otimes \mathcal{J}_{W_Y}) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10) \otimes \mathcal{J}_{p_1, \dots, p_8}^6 \otimes \mathcal{J}_W).$$

Hence,

$$h^0(Y, \mathcal{O}_Y(-2K_Y) \otimes \mathcal{J}_{W_Y}) = 21.$$

As $h^0(Y, \mathcal{O}_Y(-2K_Y)) = 29$ and $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$, we deduce that the restriction morphism

$$H^0(Y, \mathcal{O}_Y(-2K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y))$$

is surjective. □

10.2 The Bertini involution

By the diagram (9.1) and the behaviour of the Bertini involution ι_S described in (9.2), the invariant part of $H^2(Y, \mathbb{R})$ by the action of the Bertini involution ι_Y is $\mathbb{R}K_Y$. In this section, we further our study of the involution ι_Y by looking at its action on the surface W_Y . The main result of this section is the following:

Proposition 10.2.1. *The Bertini involution ι_Y preserves the surface W_Y , and its restriction $\iota_Y|_{W_Y}$ on W_Y is the biregular involution defined by the double covering*

$$\phi|_{-2K_Y|_{W_Y}} : W_Y \rightarrow V_{2,4} \subset \mathbb{P}^7,$$

where $V_{2,4} \simeq \mathbb{F}_2$ is a rational normal scroll of bidegree $(2, 4)$. In particular, the Bertini involution ι_Y is the identity on the curve R_Y and ι_Y induces an involution on each elliptic fibre F_Y of $W_Y \rightarrow \mathbb{P}^1$.

Furthermore, there exists a smooth curve $R' \sim 3(R_Y + F_Y)$ of genus 4 on the surface W_Y , such that R' is disjoint from R_Y and contained in the fixed locus of ι_Y .

We start by showing that the surface W_Y is invariant by the Bertini involution ι_Y .

Lemma 10.2.2. *The Bertini involution ι_Y preserves the curve R_Y and the surface W_Y . Moreover, $(\iota_Y|_{W_Y})^*(e_i) \sim -2K_Y|_{W_Y} - e_i$ and $(\iota_Y|_{W_Y})^*(F_Y) \sim F_Y$, where e_i is the exceptional curve of $\eta_h|_{W_Y} : W_Y \rightarrow W$ for $i = 1, \dots, 8$.*

Proof. Since ι_Y preserves the family of divisors in the anticanonical system $|-K_Y|$, the involution ι_Y preserves the base locus of $|-K_Y|$. Thus $\iota_Y(R_Y) = R_Y$ by Proposition 10.1.12.

Let x be a general point in R_Y . Then by Lemma 10.1.14, there exists a unique divisor in $|-K_Y|$ having multiplicity 3 at x : it is the strict transform in Y of the secant variety of the elliptic normal quintic through p_1, \dots, p_8 and $\eta_h(x)$ in \mathbb{P}^4 . In particular, this divisor has multiplicity 3 along the elliptic fibre of W_Y through x . By varying x in R_Y , this gives a one-dimensional family $\mathcal{M}_{Y,3}$ of divisors in $|-K_Y|$, which is preserved by ι_Y . On the other hand, the intersection of these divisors is the surface W_Y , so W_Y is preserved by ι_Y . Let $D_1 \in \mathcal{M}_{Y,3}$ and $D_2 = \iota_Y(D_1) \in \mathcal{M}_{Y,3}$. Let F_1 (resp. F_2) be the elliptic fibre of W_Y along which D_1 (resp. D_2) has multiplicity 3. Then $\iota_Y(F_1) = F_2$, and thus ι_Y preserves the family of elliptic fibres of W_Y , i.e. $(\iota_Y|_{W_Y})^*(F_Y) \sim F_Y$.

By [CCF19, 7.12], one has $\iota_Y^*(\xi(E_i)) \sim -2K_Y - \xi(E_i)$. Hence, $(\iota_Y|_{W_Y})^*(e_i) \sim -2K_Y|_{W_Y} - e_i$. \square

Now we investigate the morphism defined by the linear system $|-2K_Y|_{W_Y}|$.

Proposition 10.2.3. *The linear system $|-2K_Y|_{W_Y}|$ defines a finite morphism $\phi : W_Y \rightarrow V \subset \mathbb{P}^7$ of degree 2, where $V = V_{2,4} \simeq \mathbb{F}_2$ is a rational normal scroll of bidegree $(2, 4)$. There is a non-trivial involution i of W_Y such that $\phi = \phi \circ i$. Moreover, i is the identity on R_Y and i induces an involution on each elliptic fibre of W_Y .*

Proof. Since $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$ (see Lemma 10.1.16), and $|-2K_Y|$ is base-point-free by Theorem 9.1.6, the linear system $|-2K_Y|_{W_Y}|$ defines a morphism $\phi : W_Y \rightarrow V \subset \mathbb{P}^7$, where V is the image of W_Y .

Claim. V is a surface of degree 6 in \mathbb{P}^7 , the image of an elliptic fibre F_Y by ϕ is a line and the image of R_Y by ϕ is a conic.

Since the restriction morphism $H^0(Y, \mathcal{O}_Y(-2K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y))$ is surjective by Lemma 10.1.16 (ii), the restriction of the morphism $\phi|_{-2K_Y}$ defined by $|-2K_Y|$ to the surface W_Y coincides with the morphism ϕ , i.e. $\phi = \phi|_{-2K_Y}|_{W_Y}$.

In \mathbb{P}^4 , let $2\mathcal{M}$ be the linear system of hypersurfaces of degree 10 with multiplicity at least 6 at the 8 general points p_1, \dots, p_8 . Consider the map $\phi_{2\mathcal{M}}$ defined by the linear system $2\mathcal{M}$. Then by *Macaulay2* (see Listing A.7), the image of the surface W by $\phi_{2\mathcal{M}}$ is a surface of degree 6, the image of an elliptic normal quintic through the 8 points by $\phi_{2\mathcal{M}}$ is a line and the image of the rational quintic R_4 through the 8 points by $\phi_{2\mathcal{M}}$ is a conic. This proves the claim.

Since $(-2K_Y|_{W_Y})^2 = 4(R_Y + 2F_Y)^2 = 12$, and the image of W_Y by ϕ is of degree 6, we deduce that ϕ is of degree 2. As $-K_Y$ is ample, the morphism ϕ does not contract any curve and thus it is a finite morphism of degree 2.

Since the linear system $|-2K_Y|_{W_Y}$ has no fixed divisor, the image V is not contained in any hyperplane of \mathbb{P}^7 (see for example [Bea96, II.6]), i.e. V is non-degenerate. Hence, V is a non-degenerate irreducible surface of degree 6 (variety of minimal degree) in \mathbb{P}^7 , and by [GH94, p. 525] we deduce that V is a rational normal scroll $V_{k,l}$ of bidegree (k, l) , with $0 \leq k \leq l$ and $k + l = 6$. In particular, V is isomorphic to one of the following: a cone over a rational normal curve of degree 6, $\mathbb{P}^1 \times \mathbb{P}^1$, or a Hirzebruch surface \mathbb{F}_{l-k} , where the minimal section is mapped to the rational normal curve of degree k , and the fibres are mapped to lines. Therefore, ϕ is a finite morphism between two normal surfaces and by [Fuj83, (2.3)], there is a non-trivial involution i of W_Y such that $\phi = \phi \circ i$ and $V \simeq W_Y/i$.

Since the restriction of ϕ to a general fibre F_Y induces a finite morphism from an elliptic curve to a line $l \subset V$, which cannot be an isomorphism, we deduce that $\phi^{-1}(l) = F_Y$ as ϕ is of degree 2. Hence, i induces an involution on F_Y .

Since $-2K_Y|_{W_Y}$ is i -invariant, one has $2(i^*(-K_Y|_{W_Y}) - (-K_Y|_{W_Y})) \sim 0$. As $\text{Pic}(W_Y)$ is torsion-free (this is because W_Y is isomorphic to \mathbb{P}^2 blown up at 9 points), we deduce that $i^*(-K_Y|_{W_Y}) \sim -K_Y|_{W_Y}$. Since R_Y is the base locus of $|-K_Y|_{W_Y}$, the curve R_Y is preserved by i . We claim that R_Y is contained in the ramification locus of ϕ . Indeed, suppose that R_Y is not contained in the ramification locus of ϕ . Then there exists a curve $C \subset V$ such that $R_Y = \phi^*(C)$. As R_Y is a (-1) -curve on W_Y , one has

$$-1 = R_Y^2 = (\phi^*(C))^2 = \deg \phi \cdot C^2,$$

i.e. $C^2 = -\frac{1}{2}$. Hence, C is not Cartier on V , i.e. V is singular. In view of the classification of minimal degree varieties, we see that V is a cone. But there is no curve with negative self-intersection number on a cone, which leads to a contradiction. Therefore, R_Y is in the ramification locus. As ϕ is a double cover, we deduce that i is the identity on R_Y .

Let $C = \phi(R_Y)$. Since R_Y is contained in the ramification locus of ϕ , and every point in R_Y has ramification index 2, one has

$$R_Y^2 = \left(\frac{1}{2}\phi^*(C)\right)^2 = \frac{1}{2}C^2.$$

Since R_Y is a (-1) -curve on W_Y , one has $C^2 = -2$. Therefore, $V = V_{2,4} \simeq \mathbb{F}_2$, and $\phi(R_Y)$ is minimal section of \mathbb{F}_2 which is a conic. \square

Remark 10.2.4. Since ϕ is a finite morphism of degree 2 between smooth surfaces, the ramification locus is a smooth divisor on W_Y (see [Fuj83, (2.5)]). Let e be the minimal section of $V \simeq \mathbb{F}_2$ and f be a fibre of V . Let D be the ramification divisor. Then

$$K_{W_Y} \sim \phi^*(K_S) + D.$$

As $K_{W_Y} \sim -F_Y = -\phi^*(f)$, and $K_S \sim -2e - 4f$, one has

$$D \sim \phi^*(2e + 3f).$$

Let $B \subset V$ be the branch locus. Then $D = \frac{1}{2}\phi^*B$ and thus $B \sim 4e + 6f$. As e is contained in the branch locus, we can write $B = e + B_1$, where B_1 is a smooth curve disjoint from e . Then $B_1 \sim 3e + 6f$. Notice that B_1 is irreducible. Indeed, suppose that B_1 has at least two disjoint irreducible components. Then we can decompose B_1 as

$$B_1 \sim (e + bf) + (2e + (6 - b)f)$$

with $0 \leq b \leq 6$ and $(e + bf) \cdot (2e + (6 - b)f) = 0$. Hence $b = -2$, which leads to a contradiction.

Hence $D = R_Y + R'$, where $R' \sim \frac{1}{2}\phi^*(3e + 6f) = 3(R_Y + F_Y)$ is a smooth curve of genus 4 which is disjoint from R_Y .

Finally, we compare the action of the two automorphisms i and $\iota_Y|_{W_Y}$ on W_Y .

Lemma 10.2.5. Let e_i be the exceptional curves of $\eta_h|_{W_Y}: W_Y \rightarrow W$ for $i = 1, \dots, 8$. Then

$$i^*(e_i) \sim -2K_Y|_{W_Y} - e_i.$$

Proof. For $i = 1, \dots, 8$, by *Macaulay2* (see Listing A.8), there exists a unique hypersurface of degree 10 with multiplicity at least 7 at the point p_i and multiplicity at least 6 at p_j for $j \neq i$. Moreover, this hypersurface does not contain the surface W . Therefore, the linear system $|-2K_Y|_{W_Y} - e_i|$ is non-empty.

Let $R_i \in |-2K_Y|_{W_Y} - e_i|$. Since $-K_Y|_{W_Y} \sim R_Y + 2F_Y$, and $R_Y \cdot e_i = F_Y \cdot e_i = 1$, one has $R_i^2 = -1$, and $R_i \cdot F_Y = R_i \cdot R_Y = 1$. Hence, R_i is a (-1) -curve on W_Y .

Since $e_i + R_i \in |-2K_Y|_{W_Y}| = \phi^*|\mathcal{O}_V(1)|$, one has $R_i \sim i^*(e_i)$. \square

Proposition 10.2.6. The involution i coincides with the restriction of the Bertini involution ι_Y on the surface W_Y , i.e. $\iota_Y|_{W_Y} = i$.

Proof. We first show that $(\iota_Y|_{W_Y})^* = i^*$. By Lemma 10.2.3, Lemma 10.2.5 and Lemma 10.2.2, it is enough to show that R_Y, F_Y and e_i for $i = 1, \dots, 8$ form a basis of $H^2(W_Y, \mathbb{R})$.

Since W' is disjoint from the indeterminacy locus of ξ_h , it is equivalent to show that R, F and e_i for $i = 1, \dots, 8$ form a basis of $H^2(W', \mathbb{R})$. We have the following diagram (see (10.1)):

$$\begin{array}{ccc}
 & W' \subset X & \\
 \eta \swarrow & & \downarrow \alpha' \\
 W \subset \mathbb{P}^4 & & \mathbb{P}^2 \\
 \searrow \alpha & &
 \end{array}$$

where α is the blow-up of \mathbb{P}^2 at one point and η is the blow-up of W at p_1, \dots, p_8 . Moreover, let $e_0 \subset W$ be the (-1) -curve and $f_0 \subset W$ be a fibre of the \mathbb{P}^1 -bundle on W , then by Lemma 10.1.1 and Remark 10.1.4, one has $F \sim \eta^*(2e_0 + 3f_0) - \sum_{i=1}^8 e_i$ and $R \sim \eta^*(e_0 + 4f_0) - \sum_{i=1}^8 e_i$. Therefore, R, F and e_i for $i = 1, \dots, 8$ form a basis of $H^2(W', \mathbb{R})$.

We have a group homomorphism $\rho_1: \text{Aut}(W_Y) \rightarrow \text{Aut}(H^2(W_Y, \mathbb{R}))$ given by $g \mapsto (g^{-1})^*$. Let $\text{Aut}(R_Y, W_Y)$ be the subgroup of automorphisms in $\text{Aut}(W_Y)$ fixing the curve R_Y . We show that the restriction $\rho_1: \text{Aut}(R_Y, W_Y) \rightarrow \text{Aut}(H^2(W_Y, \mathbb{R}))$ is injective, which implies that $\iota_Y|_{W_Y} = i$.

Since R_Y is a (-1) -curve on W_Y , by blowing down R_Y , we obtain a rational surface S' with $(-K_{S'})^2 = 1$, and the curve R_Y is contracted to a point $x_0 \in S'$. We denote by $\beta: W_Y \rightarrow S'$ the blow-up of S' at x_0 . Since $-K_{W_Y}$ is nef, we obtain that $-K_{S'}$ is nef by the projection formula (see for example [Har77, Appendix A, A4]). Moreover, since every fibre of $W_Y \rightarrow \mathbb{P}^1$ is integral, there is no $K_{S'}$ -trivial curve. Hence, S' is a del Pezzo surface of degree one. By [Dol12, Prop. 8.2.39], the homomorphism $\rho_2: \text{Aut}(S') \rightarrow \text{Aut}(H^2(S', \mathbb{R}))$ is injective.

Let $\text{Aut}(x_0, S')$ be the subgroup of automorphisms in $\text{Aut}(S')$ fixing the point x_0 . Then $\text{Aut}(x_0, S') \simeq \text{Aut}(R_Y, W_Y)$. Since $\text{Pic}(W_Y) \simeq \beta^* \text{Pic}(S') \oplus \mathbb{Z}[R_Y]$, the image $\rho_1(\text{Aut}(R_Y, W_Y))$ is contained in a subgroup G_1 of $\text{Aut}(H^2(W_Y, \mathbb{R}))$ such that $G_1 \simeq \text{Aut}(H^2(S', \mathbb{R}))$. Hence, we have the following diagram:

$$\begin{array}{ccc} \text{Aut}(R_Y, W_Y) & \xrightarrow{\rho_1} & G_1 \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Aut}(x_0, S') & \xrightarrow{\rho_2} & \text{Aut}(H^2(S', \mathbb{R})) \end{array}$$

Since ρ_2 is injective, the restriction $\rho_1: \text{Aut}(R_Y, W_Y) \rightarrow G_1 \subset \text{Aut}(H^2(W_Y, \mathbb{R}))$ is injective. \square

Proof of Proposition 10.2.1. The first paragraph follows from Lemma 10.2.2, Proposition 10.2.3 and Proposition 10.2.6. The second paragraph follows from Remark 10.2.4. \square

Appendix A

Computations by *Macaulay2*

```
restart
```

```
k = ZZ/67
```

We set up the projective space \mathbb{P}^4 :

```
R = k[x_0..x_4]
```

We choose 8 points in \mathbb{P}^4 :

```
I_0 = ideal(x_1, x_2, x_3, x_4)
```

```
I_1 = ideal(x_0, x_2, x_3, x_4)
```

```
I_2 = ideal(x_1, x_0, x_3, x_4)
```

```
I_3 = ideal(x_1, x_2, x_0, x_4)
```

```
I_4 = ideal(x_1, x_2, x_3, x_0)
```

```
I_5 = ideal(x_1-x_2, x_2-x_3, x_3-x_4, x_0-x_4)
```

```
I_6 = ideal(x_0-3*x_1, x_1-7*x_2, x_2-11*x_3, x_3-13*x_4)
```

```
I_7 = ideal(x_0-17*x_1, x_1-23*x_2, x_2-29*x_3, x_3-31*x_4)
```

We compute the ideal II defined by the 6 quintics through the 8 points with multiplicity at least 3:

```
J = I_0; for j from 1 to 7 do J = intersect(J, I_j);
```

```
H = saturate J^3;
```

```
G = gens (H);
```

```
beti G
```

```
G1 = submatrix (G, {0..5});
```

```
II = ideal(G1);
```

```
III = sheaf module II;
```

```
HH^0(III(5))
```

We check that II is the intersection of the ideal of the 28 lines, the ideal of the 8 quartics and the ideal I_5 of a smooth rational quintic curve:

```
LL = ideal(1_R);
```

```

for i from 0 to 7 do for j from 0 to i-1 do
LL = intersect(LL,ideal submatrix(gens intersect (I_i,I_j) ,{0..2}));
isSubset(II ,LL)
RN = ideal(1_R);
for i from 0 to 7 do
RN = intersect(RN,minors(2,submatrix((res (J:I_i)).dd_4,{3..6} ,{0..1})));
isSubset(II ,RN)
I5 = ((II:LL):RN);
degree I5 , genus I5 , ideal singularLocus variety I5
II = intersect(intersect(LL,RN) ,I5)

```

Listing A.1 – Base scheme

We compute the normal bundle of the smooth rational quintic curve:

```

RI5 = R/I5
N5 = (module I5)**RI5
PI5 = Proj RI5
SN5 = sheaf N5
HH^0(SN5)
HH^0(sheaf dual N5)
KI5 = Ext^3(R^1/I5 ,R^{-5})**RI5
HH^0(SN5**OO_PI5(1)**(sheaf dual KI5))

```

Listing A.2 – Normal bundle

We choose three points on the smooth rational quintic curve:

```

P1 = ideal(x_3-14*x_4,x_2-x_4,x_1+x_4,x_0-12*x_4)
P2 = ideal(x_3+17*x_4,x_2-22*x_4,x_1+20*x_4,x_0+2*x_4)
P3 = ideal(x_3-26*x_4,x_2+27*x_4,x_1-30*x_4,x_0+21*x_4)

```

We compute the quintic with multiplicity 3 at the 8 points and the point P1 (resp. P2 and resp. P3):

```

J13 = intersect(J^3,P1^3);
H13 = saturate J13;
G13 = gens(H13);
betti G13
GP1 = submatrix(G13,{0});
Q1 = ideal(GP1);

J23 = intersect(J^3,P2^3);
H23 = saturate J23;
G23 = gens(H23);
betti G23
GP2 = submatrix(G23,{0});
Q2 = ideal(GP2);

```

```

J33 = intersect(J^3,P3^3);
H33 = saturate J33;
G33 = gens(H33);
betti G33
GP3 = submatrix(G33,{0});
Q3 = ideal(GP3);

```

Listing A.3 – Three secant varieties

We compute the elliptic normal quintic curve along which Q1 is singular:

```

SingQ1 = ideal singularLocus variety Q1;
dim SingQ1, degree SingQ1
SSingQ1 = ideal singularLocus variety SingQ1;
dim SSingQ1, degree SSingQ1
E1 = (SingQ1:SSingQ1);
dim E1, degree E1, genus E1
ideal singularLocus variety E1

```

We compute the intersection of the three quintics and obtain the cubic scroll W:

```

SS3 = Q1 + Q2 + Q3;
SS = (SS3:II);
dim SS, degree SS
W = ideal singularLocus variety SS;
dim W, degree W, genus W, ideal singularLocus variety W
W == (SS:W)

```

Listing A.4 – Scheme-theoretic intersection of secant varieties

We compute the quintics through the 8 points with multiplicity at least 3 containing the surface W:

```

JW = intersect(J^3,W);
HW = saturate JW;
GW = gens(HW);
betti GW
GW1 = submatrix(GW,{0..2});
IIW = ideal(GW1);
IIIW = sheaf module IIW;
HH^0(IIIW(5))

```

Listing A.5 – Quintics containing W

We compute the hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```

JI5 = intersect(I_5^6,intersect(I_6^6,I_7^6));
HI5 = saturate JI5;
GI5 = gens HI5;

```

```

betti GI5
GI5 = submatrix(GI5, {0..105});
II5 = ideal(GI5);
JI4 = intersect(I_4^6, II5);
HI4 = saturate JI4;
GI4 = gens HI4;
betti GI4
GI14 = submatrix(GI4, {0..132});
II4 = ideal(GI14);
JI3 = intersect(I_3^6, II4);
HI3 = saturate JI3;
GI3 = gens HI3;
betti GI3
GI13 = submatrix(GI3, {0..154});
II3 = ideal(GI13);
JI2 = intersect(I_2^6, II3);
HI2 = saturate JI2;
GI2 = gens HI2;
betti GI2
GI12 = submatrix(GI2, {0..123});
II2 = ideal(GI12);
JI1 = intersect(I_1^6, II2);
HI1 = saturate JI1;
GI1 = gens HI1;
betti GI1
GI11 = submatrix(GI1, {0..136});
II1 = ideal(GI11);
JI0 = intersect(I_0^6, II1);
HI0 = saturate JI0;
GI0 = gens HI0;
betti GI0
GG = submatrix(GI0, {0..28});
IGG = ideal(GG);

```

We compute the hypersurfaces of degree 10 through the 8 points with multiplicity at least 6 containing the surface W :

```

JW2 = intersect(W, IGG);
HW2 = saturate JW2;
GW2 = gens HW2;
betti GW2
GGW = submatrix(GW2, {0..20});
IW2 = ideal(GGW);
IIW2 = sheaf module IW2;

```

$HH^0(IIW2(10))$

Listing A.6 – Decics containing W

We compute the image of the elliptic normal quintic $E1$ via the map defined by the linear system of hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```
S2 = k[u_0..u_28];  
ImE1 = ker map(R/E1, S2, GG);  
dim ImE1, degree ImE1
```

We compute the image of the rational quintic curve $I5$ via the map defined by the linear system of hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```
ImI5 = ker map(R/I5, S2, GG);  
dim ImI5, degree ImI5
```

We compute the image of the surface W via the map defined by the linear system of hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```
ImW = ker map(R/W, S2, GG);  
dim ImW, degree ImW
```

Listing A.7 – Image by bianticanonical map

We compute the hypersurfaces of degree 10 with multiplicity at least 7 at the point I_0 and multiplicity at least 6 at the other 7 points:

```
J100 = intersect(I_0^7, I1);  
H100 = saturate J100;  
G100 = gens H100;  
beti G100  
GG0 = submatrix(G100, {0});  
IGG0 = ideal(GG0);
```

And we obtain a unique such hypersurface of degree 10; now we check if this hypersurface contains the surface W :

```
JW0 = intersect(W, IGG0);  
HW0 = saturate JW0;  
GW0 = gens HW0;  
beti GW0
```

Listing A.8 – Special member in bianticanonical system

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