HAL open science

Landau–Ginzburg/Calabi–Yau correspondence for a complete intersection via matrix factorizations

Yizhen Zhao

To cite this version:

Yizhen Zhao. Landau–Ginzburg/Calabi–Yau correspondence for a complete intersection via matrix factorizations. Algebraic Geometry [math.AG]. Sorbonne Université, 2020. English. NNT : 2020SORUS357 . tel-03406127

HAL Id: tel-03406127

https://theses.hal.science/tel-03406127v1

Submitted on 27 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sorbonne Université

École Doctorale de Science Mathématiques de Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Yizhen Zhao

Landau–Ginzburg/Calabi–Yau correspondence for a complete intersection via matrix factorizations

---

dirigée par Alessandro Chiodo

Institut de Mathématiques de Jussieu
4 Place Jussieu
75 005 Paris

Ecole Doctorale Paris Centre Case 188
4 place Jussieu
75252 Paris Cedex 05

Acknowledgements

I wish to express my sincere appreciation to my supervisor, Alessandro Chiodo, for the countless conversations, support, and advice.

I would like to thank Emily Clader, Jérémy Guéré, Dustin Ross and Mark Shoemaker for helpful conversations and suggestions.

A special thank to Alexander Polishchuk for pointing out to us references to Isik [32], Segal [41] and Shipman [42], together with a potential relation to Bridgeland conjecture [6].

I would also like to thank Tom Coates for suggesting to look into a general approach via Fourier–Mukai transformation in [18].

Résumé

En généralisant la correspondance de Landau–Ginzburg/Calabi–Yau pour les hypersurfaces, nous pouvons relier une intersection complète de Calabi–Yau à un modèle hybride de Landau–Ginzburg : une famille de singularités isolées au-dessus d'une droite projective. Pendant ces dernières années, Fan, Jarvis et Ruan ont défini des invariants quantiques pour les singularités de ce type, et Clader et Clader–Ross ont fourni une équivalence entre ces invariants et les invariants de Gromov–Witten d'intersections complètes. De cette manière, la cohomologie quantique donne un identification des groupes de cohomologie de la intersection complète de Calabi–Yau et du modèle hybride de Landau–Ginzburg. Il n'est pas clair comment relier cela à l'isomorphisme connu qui découle de certaines équivalences dérivées dues à Segal, Shipman, Orlov et Isik. Nous répondons à cette question pour les intersections complètes de Calabi–Yau de deux cubiques.

Mots-clefs

Géométrie algébrique ; symétrie miroir ; théorie de Gromov–Witten ; théorie de Fan–Jarvis–Ruan–Witten ; correspondance de Landau–Ginzburg/Calabi–Yau ; factorisation matricielles ; équivalences d'Orlov.

Abstract

By generalizing the Landau–Ginzburg/Calabi–Yau correspondence for hypersurfaces, we can relate a Calabi–Yau complete intersection to a hybrid Landau–Ginzburg model: a family of isolated singularities fibered over a projective line. In recent years, Fan, Jarvis and Ruan have defined quantum invariants for singularities of this type, and Clader and Clader–Ross have provided an equivalence between these invariants and Gromov–Witten invariants of complete intersections. In this way, quantum cohomology yields an identification of the cohomology groups of the Calabi–Yau and of the hybrid Landau–Ginzburg model. It is not clear how to relate this to the known isomorphism descending from certain derived equivalences (due to Segal, Shipman, Orlov and Isik). We answer this question for Calabi–Yau complete intersections of two cubics.

Keywords

Algebraic geometry; mirror symmetry; Gromov–Witten theory; Fan–Jarvis–Ruan–Witten theory; Landau–Ginzburg/Calabi–Yau correspondence; matrix factorization; Orlov equivalence.

Contents

Introduction **9**

Notations **15**

Preliminaries **17**

0.1 Moduli spaces curves 17
0.2 Twisted curve 18
0.3 Geometric invariant theory 20
0.4 Orbifold Chen–Ruan cohomology 22
0.5 Structure of the thesis 24

1 Enumerative theories **25**

1.1 Gromov–Witten theory 25
1.2 Fan–Jarvis–Ruan–Witten theory 27
1.3 Relate GW theory and FJRW theory via GIT 32
1.4 Givental's formalism and computation of I-functions 37
1.5 Picard–Fuchs equation 41

2 Analytic continuation **43**

2.1 Gradings on the state spaces 43
2.2 The $\mathfrak{H}$-functions 44
2.3 Linear maps relating the $\mathfrak{H}$-functions 46

3 Orlov-type functors **53**

3.1 Graded matrix factorizations 53
3.2 Koszul matrix factorizations 55
3.3 Grade restriction rule 56
3.4 Strategy to fit the window 58
3.5 Orlov's functor for complete intersection 63

4 Matching analytic continuation and categorical equivalences **65**

4.1 Image of $\mathcal{K}_-(q)[m]$ under Orlov functor 65

4.2 Chern character of $\mathcal{K}_-(q)[m]$ 71

4.3 Image of $\mathrm{inv}^*\left(\mathrm{ch}(\mathcal{K}_-(q)[m])\right)$ under $\mathbb{U}_l$ 72

4.4 Main result 75

4.5 Application to monodromy 77

4.6 Relation with the crepant transformation conjecture 78

Bibliography **83**

Introduction

The Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence in string theory describes a relationship between the sigma model on a Calabi–Yau hypersurface and the Landau–Ginzburg model whose potential is the defining equation of the Calabi–Yau variety. Following Witten [46], we can present the LG/CY correspondence in a purely algebro-geometric way starting from a variation of stability conditions in geometric invariant theory (GIT). From this point of view, we can naturally generalize the LG/CY correspondence to the Calabi–Yau complete intersections.

The variation of stability conditions leads to two different curve-counting theories. Analytic continuation naturally allows us to compare them. A natural question arises: what is the interpretation of the linear transformation matching the generating functions encoding the two theories? The answer given in this thesis for Calabi–Yau complete intersections of two cubics is an equivalence of triangulated categories known as Orlov equivalence applied to the two GIT quotients. More precisely GIT quotients are classically interpreted as chambers and the transition between them is usually phrased in terms of window-transitions. Each of these transitions is related to a specialization of Orlov's functor (see §3.3). This is a mathematical object of independent interest not directly related to curve-counting theories. In particular it sheds new light on the LG/CY correspondence.

The isomorphism of cohomology groups

We start from r homogeneous polynomials $W_1, \dots, W_r$ of the same degree d defining a smooth complete intersection

$$i\colon X_{d,\dots,d} \hookrightarrow \mathbb{P}^{N-1}.$$

The complete intersection $X_{d,\dots,d}$ is Calabi–Yau [1] as soon as dr equals N. Following a standard procedure (see Witten [46], we also refer to Herbst–Hori–Page [29]) we can cast this setup

1. Here the Calabi–Yau condition is meant in the weak sense: the canonical bundle ω is trivial.

within a representation of $\mathbb{C}^*$ as follows. Consider a $\mathbb{C}^*$-action on the vector space

$$V = \mathbb{C}^N \times \mathbb{C}^r = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_N, p_1, \dots, p_r]$$

with weight 1 on the first N variables, and weight $-d$ on the following r variables:

$$\lambda \cdot (x_1, \dots, x_N, p_1, \dots, p_r) = (\lambda x_1, \dots, \lambda x_N, \lambda^{-d} p_1, \dots, \lambda^{-d} p_r).$$

GIT provides a systematic description of the geometric quotients that can be obtained from the action $\mathbb{C}^* \times V \to V$. Indeed we can choose two different GIT stability conditions to identify two maximal open sets within $\Omega \subset V$ whose quotient $[\Omega/\mathbb{C}^*]$ is a smooth Deligne–Mumford stack. For

$$\Omega_+ = V \setminus (x_1 = \cdots = x_N = 0) \qquad \text{and} \qquad \Omega_- = V \setminus (p_1 = \cdots = p_r = 0),$$

we obtain the total space X_+ of the vector bundle $\mathcal{O}(-d)^{\oplus r}$ on $\mathbb{P}^{N-1}$ and the total space X_- of the vector bundle $\mathcal{O}(-1)^{\oplus N}$ on $\mathbb{P}(d, \dots, d)$ (the weighted projective stack with an overall stabilizer $\boldsymbol{\mu}_d$, whose coarse space equals $\mathbb{P}^{r-1}$).

The Calabi–Yau complete intersection $X_{d,\dots,d}$, or rather its cohomology, arises as the cohomology of X_+ relative to the generic fiber of

$$W = \sum_{j=1}^{r} p_j W_j \colon X_+ \longrightarrow \mathbb{C}. \tag{1}$$

In fact, we have isomorphisms

$$H^{*-2r}(X_+, W^{-1}(1)) \cong H^{*-2r}(\mathbb{P}^{N-1}, \mathbb{P}^{N-1} \setminus X_{d,\dots,d}) \cong H^*(X_{d,\dots,d}),$$

where the first isomorphism comes from retraction, the second isomorphism is the Thom isomorphism.

For X_-, the same procedure yields a family of isolated singularities over $\mathbb{P}(d, \dots, d)$:

$$\begin{tikzcd} X_- \arrow[r, "W = \sum_{j=1}^{r} p_j W_j"] \arrow[d] & \mathbb{C}\,. \\ \mathbb{P}(d, \dots, d) & \end{tikzcd} \tag{2}$$

We call it a Landau–Ginzburg model (X_-, W). It is the analogue of the $\boldsymbol{\mu}_d$-invariant polynomial

$$\begin{tikzcd} {[\mathbb{C}^N/\boldsymbol{\mu}_d]} \arrow[r, "W"] \arrow[d] & \mathbb{C} \\ B\boldsymbol{\mu}_d = \mathbb{P}(d) & \end{tikzcd}$$

defining the Landau–Ginzburg singularity model of Calabi–Yau hypersurface in [9, 11].

Since X_- is an orbifold, we consider its orbifold Chen–Ruan cohomology (see 0.4) relative to the generic fiber of W. It is isomorphic to the the cohomology group of $X_{d,\dots,d}$ via the isomorphism

$$H^*_{\mathrm{CR}}(X_-, W^{-1}(1)) \cong H^*(X_+, W^{-1}(1))$$

coming from the variation of GIT stability condition (see Chiodo–Nagel [10] in higher generality).

The isomorphism

$$H^*(X_{d,\dots,d}) \cong H^{*-2r}_{\mathrm{CR}}(X_-, W^{-1}(1))$$

is an isomorphism of graded vector spaces. It is not an isomorphism of rings. The subject of this thesis is an enhanced correspondence in terms of curve-counting theories.

Curve-counting theories

For the Calabi–Yau complete intersection $X_{d,\dots,d}$, we consider the quantum product, a deformation of the cup product on $H^*(X_{d,\dots,d})$. It can be defined via the Gromov–Witten (GW) theory (see [27]).

For the Landau–Ginzburg model (X_-, W), a curve-counting theory was constructed by Fan–Jarvis–Ruan in [23, 24, 25, 26]. We will use their definition, which we will refer to as FJRW theory, see §1.2 and §1.3. Since the first definition of FJRW theory [23], several alternative constructions have been provided: Polishchuk–Vaintrob [40], Chang–Li–Li [8], Ciocan-Fontanine–Favero–Guéré–Kim–Shoemaker [12].

According to the LG/CY correspondence, it is natural to conjecture that the genus-0 GW theory of the Calabi–Yau complete intersection and the genus-0 FJRW theory of (X_-, W) are equivalent in the following sense.

The genus-0 GW theory of the Calabi–Yau complete intersection is determined by a function I_{GW} taking values in the subspace

$$H_{\mathrm{GW}} := i^* H(\mathbb{P}^{r-1}) \subset H^*(X_{d,\dots,d})$$

of classes coming from the ambient space $\mathbb{P}^{r-1}$. Similarly a subspace

$$H_{\mathrm{FJRW}} \subset H^{*-2r}_{\mathrm{CR}}(X_-, W^{-1}(1)).$$

is defined (see §1.2.3). The genus-0 FJRW theory of (X_-, W) is determined by a function I_{FJRW} taking values in H_{FJRW}. The two theories are equivalent in the sense that I_{GW} matches I_{FJRW} up to an analytic continuation and a linear map.

The above conjecture was proven in the hypersurface case (*i.e.* $r = 1$) by Chiodo–Ruan [11], Chiodo–Iritani–Ruan [9] and Lee–Priddis–Shoemaker [36], and generalized to certain complete intersections (*i.e.* $r > 1$) by Clader [13] and Clader–Ross [15]. Chiodo–Iritani–Ruan [9] provided a geometric interpretation in the hypersurface case in terms of Orlov functor. We focus on the complete intersection case.

In this thesis, we simplify the I-functions with the help of Γ-classes introduced by Iritani [31], and compute the explicit form of a discrete family of linear maps

$$\mathbb{U}_l \colon H_{\mathrm{FJRW}} \to H_{\mathrm{GW}} \tag{3}$$

indexed by $l \in \mathbb{Z}$ relating I_{FJRW} (1.24) and I_{GW} (1.23). Then, we can relate $\mathbb{U}_l$ to categorical equivalences.

Equivalences of categories

A matrix factorization of a polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ is the datum of $\mathbb{C}[x_1, \ldots, x_n]$-module homomorphism f_1 and f_2

$$E_1 \underset{f_1}{\overset{f_2}{\leftrightarrows}} E_2$$

such that $f_1 \circ f_2 = P \cdot \mathrm{Id}_{E_2}$ and $f_2 \circ f_1 = P \cdot \mathrm{Id}_{E_1}$. A generalized version is a matrix factorization of a function over a stack; it can be found in [41, 42]. The graded matrix factorizations of a given function form a triangulated category, we call it the derived category of graded matrix factorizations [39, 41, 42].

Orlov [39] proved that there is a discrete family of equivalences of triangulated categories indexed by $\mathbb{Z}$ between the derived category of graded matrix factorizations of a homogeneous polynomial P and the bounded derived category of coherent sheaves on the Calabi–Yau hypersurface defined by $P = 0$.

We use the generalization of Orlov's result to the complete intersections. There is a discrete family of equivalences

$$\mathrm{Orl}_t \colon \mathrm{DMF}(X_-, W) \to \mathrm{D}^{\mathrm{b}}(X_{d,\ldots,d}) \tag{4}$$

between the derived category of graded matrix factorizations of the function W on X_- and the bounded derived category of coherent sheaves on the Calabi–Yau complete intersection $X_{d,\ldots,d}$. These equivalences are constructed by composing two functors. One of them is due to Segal [41]; the other one is due to Shipman [42], see also Isik [32] for an alternative construction.

In this thesis, we find a method to compute the functor Orl_t explicitly. The strategy is stated in §3.4.

Main result

A physics paper [29] by Herbst–Hori–Page predicts that the LG/CY correspondence is related to equivalences of categories. In the hypersurface case, it was verified by Chiodo–Iritani–Ruan in [9]. We study the case of complete intersection of two cubics in $\mathbb{P}^5$, *i.e.* the case $N = 6, d = 3, r = 2$.

Let $X_{3,3}$ be the complete intersection of two cubics in $\mathbb{P}^5$. We define $\mathrm{D}^{\mathrm{b}}(X_{3,3})_{\mathrm{amb}}$ to be the subcategory

$$\mathrm{D}^{\mathrm{b}}(X_{3,3})_{\mathrm{amb}} := i^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}^5) \subset \mathrm{D}^{\mathrm{b}}(X_{3,3}).$$

Then, the Chern character on $\mathrm{D}^{\mathrm{b}}(X_{3,3})_{\mathrm{amb}}$ takes values in H_{GW}.

We also define a subcategory (see §4.4)

$$\mathrm{DMF}(X_-, W)_{\mathrm{nar}} \subset \mathrm{DMF}(X_-, W)$$

such that for every $t \in \mathbb{Z}$, the image of $\mathrm{DMF}(X_-, W)_{\mathrm{nar}}$ under the functor Orl_t lies within $\mathrm{D}^{\mathrm{b}}(X_{3,3})_{\mathrm{amb}}$. Moreover, we define a Chern character on $\mathrm{DMF}(X_-, W)_{\mathrm{nar}}$ taking values in H_{FJRW}.

Our main result states that the categorical equivalences (4) match the linear maps (3) relating the two curve-counting theories.

Theorem 0.1 (Theorem 4.10)**.** *For every integer t, the following diagram commutes*

$$\begin{tikzcd}
DMF(X_-, W)_{nar} \arrow[r, "\mathrm{Orl}_t"] \arrow[d, "inv^* \mathrm{ch}"'] & D^b(X_{3,3})_{amb} \arrow[d, "inv^* \mathrm{ch}"] \\
H_{\mathrm{FJRW}} \arrow[r, "\mathbb{U}_t"] & H_{\mathrm{GW}},
\end{tikzcd}$$

where inv^* on the right/left side is the involution on $H_{\mathrm{GW}/\mathrm{FJRW}}$ induced by the canonical involution of the inertia stack of $X_{3,3}/X_-$ (see §0.4).

Our method provides an effective algorithm for any N, d, r. Howerer, the complexity of the analytic continuation prevents us from giving a general proof in terms of closed formulae. So we restrict ourselves to the case $N = 6, d = 3, r = 2$, the simplest case where the complete intersection is a Calabi–Yau 3-fold but not a hypersurface.

In the light of Theorem 0.1, we interprete the monodromies of the I-function in terms of auto-equivalences of the derived category. It is interesting to see that, in the case of Calabi–Yau complete intersection of four conics, we have maximally unipotent type monodromy on both LG and CY side. This was also observed by Joshi and Klemm [33].

Notations

— $\boldsymbol{\mu}_r$ is the group of rth roots of unit. We denote $e^{\frac{2\pi i}{r}}$ by ζ_r. We write ζ_3 as ζ when there is no ambiguity.

— $\mathbb{P}(w_1, \ldots, w_k)$ is the weighted projective stack with weights $w_1, \ldots, w_k$. It is the quotient stack $[(\mathbb{C}^k - \{\mathbf{0}\})/\mathbb{C}^*]$, where the $\mathbb{C}^*$-action on $\mathbb{C}^k - \{\mathbf{0}\}$ is given by

$$\lambda \cdot (x_1, \ldots, x_k) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_k} x_k).$$

— $X_{d,\ldots,d}$ is a Calabi–Yau complete intersection in the weighted projective stack $\mathbb{P}(w_1, \ldots, w_N)$, see §1.2.1; in particular, $X_{3,3}$ is the complete intersection of two cubics in $\mathbb{P}^5$.

— $\overline{\mathcal{M}}_{g,n}^{\frac{1}{d}}(\mathbb{P}^{r-1}, \beta)$ is the moduli space of Landau–Ginzburg stable map introduced in Definition 1.2.

— $H^*(X)$ is the singular cohomology of X with coefficients in $\mathbb{C}$.

— $H^{*+k}(X)$ is the graded vector space obtained by shifting the grading on $H^*(X)$ by k; *i.e.* the degree-n part of $H^{*+k}(X)$ is isomorphic to the degree-$(n-k)$ part of $H^*(X)$.

— $H^*(X, \mathbb{Z})$ is the singular cohomology of X with coefficients in $\mathbb{Z}$.

— $H^*_{\mathrm{CR}}(X)$ is the Chen–Ruan cohomology of X with coefficients in $\mathbb{C}$, see §0.4.

— X_+ and X_- are two GIT quotients introduced in example 0.11.

— $\mathcal{O}(k)[l]$ are equivariant line bundles over both X_+ and X_-; they are introduced in §3.2.

— $\mathcal{K}_+$ and $\mathcal{K}_-$ are two Koszul matrix factorizations over both X_+ and X_-; they are introduced in §3.2.

— H_{GW} is the ambient part of the state space of GW theory of $X_{3,3}$, see §1.1.

— H_{FJRW} is the narrow part of the state space of FJRW theory of the case of interest, see §1.2.3.

— $\mathbb{1}^{(1)}$, $\mathbb{1}^{(2)}$, $H^{(1)}$ and $H^{(2)}$ are elements of H_{FJRW}; $\mathbb{1}^{(0)}$ and $H^{(0)}$ are formally added elements, see Remark 2.7.

— $H^{(k)}$ and $H^{(l)}$ represent the same element if $k \equiv l \mod 3$; same for $\mathbb{1}^{(k)}$.

— $\mathbb{U}_l$ is a discrete family of linear maps between H_{FJRW} and H_{GW}; they are introduced in the definition 2.3.

— $\mathrm{DMF}^{\mathbb{C}^*_R}(X, F)$ is the derived category of graded matrix factorizations of a function F

over X, see Definition 3.4.

— Orl_k is a discrete family of equivalences of triangulated categories between $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$ and $\mathrm{DMF}^{\mathbb{C}^*_R}(X_+, W)$; they are introduced in §3.3.

— $\mathrm{Orl}_k^{\mathrm{mod}}$ is a modified family of Orl_k, see §4.4.

Preliminaries

0.1 Moduli spaces curves

We work over the field of complex numbers $\mathbb{C}$.

Definition 0.2. A *nodal curve* is a proper and connected curve for which all singularities have a neighbourhood isomorphic to a neighbourhood of the origin in

$$\{(x, y) \in \mathbb{C}^2 : xy = 0\}.$$

The singularities are called nodes.

Definition 0.3. A *genus-g n-pointed nodal curve* (over a scheme S) is the data of a proper and flat morphism $\pi\colon C \to S$ with sections $\sigma_i\colon S \to C$ for $i \in \{1, \dots, n\}$, satisfying the following conditions:

1. each geometric fiber C_x of π is a projective, connected, reduced nodal curve of arithmetic genus g;
2. the images of σ_i are disjoint and lies within the smooth locus of π.

If $\pi'\colon C' \to S', \sigma_i'\colon S' \to C', i = 1, \dots, n$ is another n-pointed genus-g nodal curve, a *morphism* between them is a pull-back square

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, shift right, "\pi"'] & C' \arrow[d, shift right, "\pi'"'] \\ S \arrow[r, "f"] \arrow[u, shift right, bend right, "\sigma_i"'] & S' \arrow[u, shift right, bend right, "\sigma_i'"'] \end{tikzcd}$$

such that $F \circ \sigma_i = \sigma_i' \circ f$ for all $i = 1, \dots, n$.

We say the n-pointed curve $\pi\colon C \to S, \sigma_i\colon S \to C, i = 1, \dots, n$ is *stable* if each geometric fiber $(C_x, \sigma_1(x), \dots, \sigma_n(x))$ has finitely many automorphisms; or equivalently, the *log-canonical sheaf*

$$\omega_{\log, C_x} := \omega_{C_x}(\sigma_1(x) + \cdots + \sigma_n(x))$$

is ample.

We can define a contravariant functor $\overline{\mathcal{M}}_{g,n}$ from the category of algebraic schemes over $\mathbb{C}$ to the category of sets. For a scheme S, let $\overline{\mathcal{M}}_{g,n}(S)$ be the set of isomorphism classes of stable n-pointed genus-g nodal curve over S; for a morphism of scheme $f\colon S \to S'$, a morphism of set $\overline{\mathcal{M}}_{g,n}(S') \to \overline{\mathcal{M}}_{g,n}(S)$ can be defined by pulling back $\pi\colon C \to S'$ to S. It is proven by Deligne and Mumford [22] that $\overline{\mathcal{M}}_{g,n}$ can be represented by a proper Deligne–Mumford stack (see [3] for a complete definition). We call it the moduli of stable curves.

Definition 0.4. Let X be an algebraic scheme, a *map from a genus-g n-pointed nodal curve to* X is the data of a genus-g n-pointed nodal curve $\pi\colon C \to S$, $\sigma_i\colon S \to C, i = 1, \ldots, n$, together with a *morphism* $\mu\colon C \to X$. A morphism between two maps from genus-g n-pointed nodal curve to X is given by the following commutative diagram:

$$\begin{tikzcd}
 & & X \\
C \arrow[r, "F"] \arrow[urr, "\mu"] \arrow[d, shift right, "\pi"'] & C' \arrow[ur, "\mu'"'] \arrow[d, shift right, "\pi'"'] \\
S \arrow[r, "f"] \arrow[u, bend left, "\sigma_i"'] & S' \arrow[u, bend left, "\sigma'_i"']
\end{tikzcd}$$

A map from a genus-g n-pointed nodal curve to X is *stable* if each geometric fiber has finitely many automorphisms; or equivalently, the sheaf

$$\omega_{\log, C_x} \otimes f^* M$$

is ample for any sufficiently ample sheaf M on X. We say a map from genus-g n-pointed nodal curves to X is of degree $\beta \in H_2(X, \mathbb{Z})$ if $\mu_*([C_x]) = \beta$ for every geometric fiber C_x.

Fix an homology class $\beta \in H_2(X, \mathbb{Z})$, we can define a contravariant funtor $\overline{\mathcal{M}}_{g,n}(X, \beta)$ sending an algebraic scheme S to the set of isomorphism classes of stable maps from genus-g n-pointed nodal curves over S to X of degree β. The functor $\overline{\mathcal{M}}_{g,n}(X, \beta)$ can be also represented by a proper Deligne–Mumford stack. We call it the moduli of stable maps; it is constructed by Kontsevich [35], see also [3, 27].

Remark 0.5. If X is a point, then we have $\overline{\mathcal{M}}_{g,n}(X, 0) = \overline{\mathcal{M}}_{g,n}$.

0.2 Twisted curve

In this thesis, we need to consider moduli problems of rth roots of certain line bundles. However, these moduli spaces are not proper if we only work on nodal curves. To fix this, we give the definition of twisted nodal curve, which is interesting on its own right.

Definition 0.6. [1] A *twisted genus-g n-pointed nodal curve* (over a scheme S) is a proper and flat morphism of Deligne–Mumford stacks $\mathcal{C} \to S$ with n closed substacks $\Sigma_i \subset \mathcal{C}$, such that

1. the fibers are purely 1-dimensional with at most nodal singularities;
2. the substacks Σ_i are étale gerbes banded by $\boldsymbol{\mu}_r$ over S (see Remark 0.7);
3. the coarse moduli space (C, Σ_i) forms a genus-g n-pointed nodal curve over S;
4. the morphism to coarse moduli space $\mathcal{C} \to C$ is an isomorphism away from the nodes and the substacks Σ_i;
5. the local picture at a node is given by $[U/\boldsymbol{\mu}_r] \to T$, where
   — $T = \operatorname{Spec} A$,
   — $U = \operatorname{Spec} A[z, w]/(zw - t)$ for some $t \in A$,
   — the action of $\boldsymbol{\mu}_r$ is given by $(z, w) \mapsto (\xi_r z, \xi_r^{-1} w)$;
6. the local picture at a marked point is given by $[U/\boldsymbol{\mu}_r] \to T$, where
   — $T = \operatorname{Spec} A$,
   — $U = \operatorname{Spec} A[x]$,
   — the action of $\boldsymbol{\mu}_r$ is given by $x \mapsto \xi_r x$.

Remark 0.7. Recall that an étale gerbe banded by $\boldsymbol{\mu}_r$ is a stack which locally looks like $X \times B\boldsymbol{\mu}_r$, where $B\boldsymbol{\mu}_r$ is the classifying stack of principal $\boldsymbol{\mu}_r$-bundles. The stack $B\boldsymbol{\mu}_r$ can be represented as the quotient stack of the $\boldsymbol{\mu}_r$-action on a point $*$, *i.e.*

$$B\boldsymbol{\mu}_r = [*/\boldsymbol{\mu}_r].$$

Let $\mathcal{L}$ be a line bundle over a twisted curve $\mathcal{C}$. Then there exist $m_1, \ldots, m_n \in \mathbb{Z}_r$, such that local picture of $\mathcal{L}$ at the ith marked point is given by $[V/\boldsymbol{\mu}_r] \to [U/\boldsymbol{\mu}_r]$, where
— $U = \operatorname{Spec} A[x]$,
— $V = \operatorname{Spec} A[x, y]$
— the action of $\boldsymbol{\mu}_r$ is given by $(x, y) \mapsto (\xi_r x, \xi_r^{m_i} y)$.
We call m_i the *multiplicity* of $\mathcal{L}$ at the ith marked point.

Similarly, the local picture of $\mathcal{L}$ at a node is given by $[V/\boldsymbol{\mu}_r] \to [U/\boldsymbol{\mu}_r]$, where
— $U = \operatorname{Spec} A[z, w]/(zw - t)$ for some $t \in A$,
— $V = \operatorname{Spec} A[z, w, y]/(zw - t)$,
— the action of $\boldsymbol{\mu}_r$ is given by $(z, w, y) \mapsto (\xi_r z, \xi_r^{-1} w, \xi_r^m y)$.
We call $m \in \mathbb{Z}_r$ the *multiplicity* of $\mathcal{L}$ at this node. Note that if we change the order of z and w, the multiplicity changes from m to $-m$ in $\mathbb{Z}_r$.

0.3 Geometric invariant theory

Definition 0.8. Let L be a line bundle on X with the projection $\pi\colon L \to X$, and let G be a reductive algebraic group with an action σ on X . A *G-linearisation* of L is an extension of the action σ on X to an action $\overline{\sigma}$ on L such that the diagram

$$\begin{tikzcd} G \times L \arrow[r, "\overline{\sigma}"] \arrow[d, "\mathrm{id} \times \pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

commutes, and G acts linearly on each fiber. We denote the line bundle L with a G-linearisation χ by L_χ.

Definition 0.9 ([38])**.** Let L_χ be a G-linearized line bundle on X. We define the following conditions.

1. Semistability. A geometric point $x \in X$ is said to be *semistable* if there exists a section $s \in H^0(X, L_\chi^{\otimes n})$ for some $n > 0$, such that $s(x) \neq 0$, $X_s = \{s \neq 0\}$ is affine, and s is G-invariant.
2. Stability. A geometric point $x \in X$ is said to be stable if it is *semistable* and the action of G on X_s is closed.
3. Unstability. A geometric point $x \in X$ is said to be *unstable* if it is not semistable.

The sets of semistable, stable and unstable points are denoted by $X_G^{\mathrm{ss}}(L_\chi)$, $X_G^{\mathrm{s}}(L_\chi)$ and $X_G^{\mathrm{us}}(L_\chi)$ respectively.

Definition 0.10. The GIT quotient stack $[X /\!\!/_\chi G]$ is defined to be the stack

$$[X /\!\!/_\chi G] := [X_G^{\mathrm{ss}}(L_\chi)/G].$$

Example 0.11. Let $G = \mathbb{C}^*$, consider a G-action on the vector space

$$V = \mathbb{C}^N \times \mathbb{C}^r = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_N, p_1, \ldots, p_r]$$

with weights w_i on the first N variables x_i, and weights $-d_j$ on the following r variables p_j:

$$\lambda \cdot (x_1, \ldots, x_N, p_1, \ldots, p_r) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N, \lambda^{-d_1} p_1, \ldots, \lambda^{-d_r} p_r),$$

where $w_1, \ldots, w_N, d_1, \ldots, d_r$ are positive integers. A character $\theta\colon G \to \mathbb{C}^*$ determines a G-linearisation of the trivial line bundle L over V.

Positive phase. Consider the character $\theta_1 \colon G \to \mathbb{C}^*$ defined by

$$\theta_1(\lambda) = \lambda^k, \quad k > 0.$$

We claim that if $\boldsymbol{\alpha} \in (\mathbb{C}^N \backslash \{\mathbf{0}\}) \times \mathbb{C}^r$, then $\boldsymbol{\alpha}$ is semistable. Without loss of generality, we can assume the first component of $\boldsymbol{\alpha}$ is not zero. Consider the section $s \in H^0(V, L_{\theta_1}^{\otimes w_1})$ given by

$$(x_1, \ldots, x_N, p_1, \ldots, p_r) \mapsto x_1^k.$$

The section s satisfies $s(\boldsymbol{\alpha}) \neq 0$, and s is G-invariant in $H^0(V, L_{\theta_1}^{\otimes w_1})$; so $\boldsymbol{\alpha}$ is semistable by definition.

On the other hand, if $\boldsymbol{\alpha} \notin (\mathbb{C}^N \backslash \{\mathbf{0}\}) \times \mathbb{C}^r$, *i.e.* $x_1 = \cdots = x_N = 0$, then $\boldsymbol{\alpha}$ is unstable. This happens because, for any positive l, every G-invariant section of $L_{\theta_1}^{\otimes l}$ vanishes at $\boldsymbol{\alpha}$. Every section of $L_{\theta_1}^{\otimes l}$ is a polynomial on $\mathbb{C}^{N+r}$. The sections s satisfying $s(\boldsymbol{\alpha}) \neq 0$ contains $p_1^{n_1} \ldots p_r^{n_r}$ for some nonnegative integers $n_1, \ldots, n_r$. Such sections are not G-invariant in $H^0(V, L_{\theta_1}^{\otimes l})$ for any $l > 0$. Therefore we get

$$V_G^{\mathrm{ss}}(L_{\theta_1}) = (\mathbb{C}^N \backslash \{\mathbf{0}\}) \times \mathbb{C}^r$$

and

$$[V /\!\!/_{\theta_1} \mathbb{C}^*] = [(\mathbb{C}^N \backslash \{\mathbf{0}\}) \times \mathbb{C}^r / \mathbb{C}^*].$$

This is the total space of the vector bundle

$$\bigoplus_{i=1}^{r} \mathcal{O}(-d_i)$$

over the weighted projective space $\mathbb{P}(w_1, \ldots, w_N)$. We denote it by X_+.

Negative phase. Consider another character $\theta_2 \colon G \to \mathbb{C}^*$ defined by

$$\theta_2(\lambda) = \lambda^{-k}, \quad k > 0.$$

By the same argument as above, we get

$$V_G^{\mathrm{ss}}(L_{\theta_2}) = \mathbb{C}^N \times (\mathbb{C}^r \backslash \{\mathbf{0}\}).$$

and

$$[V /\!\!/_{\theta_2} \mathbb{C}^*] = [\mathbb{C}^N \times (\mathbb{C}^r \backslash \{\mathbf{0}\}) / \mathbb{C}^*].$$

This is isomorphic to the total space of the vector bundle

$$\bigoplus_{i=1}^{N} \mathcal{O}(-w_i)$$

over the weighted projective space $\mathbb{P}(d_1, \ldots, d_r)$. We denote it by X_-.

Remark 0.12. Note that there is a change of sign in the identification of X_- with the total space of the vector bundle

$$\bigoplus_{i=1}^{N} \mathcal{O}(-w_i)$$

over $\mathbb{P}(d_1, \ldots, d_r)$. It is somehow better to understand X_- as the total space of

$$\bigoplus_{i=1}^{N} \mathcal{O}(w_i)$$

over "$\mathbb{P}(-d_1, \ldots, -d_r)$".

0.4 Orbifold Chen–Ruan cohomology

Let $\mathcal{X}$ be a smooth Deligne–Mumford stack over $\mathbb{C}$, and let $I\mathcal{X}$ be the inertia stack of $\mathcal{X}$. A point on $I\mathcal{X}$ is given by a pair (x, g) of a point $x \in \mathcal{X}$ and $g \in \mathrm{Aut}(x)$. Let T be the index set of components of $I\mathcal{X}$, then we can write

$$I\mathcal{X} = \bigsqcup_{v \in T} \mathcal{X}_v.$$

There is a canonical involution

$$\mathrm{inv}\colon I\mathcal{X} \to I\mathcal{X}$$

defined by

$$(x, g) \mapsto (x, g^{-1}).$$

Take a point $(x, g) \in I\mathcal{X}$ and let

$$T_x\mathcal{X} = \bigoplus_{0 \le f < 1} (T_x\mathcal{X})_f$$

be the eigenvalue decomposition of $T_x\mathcal{X}$ with respect to the action given by g, where g acts on $(T_x\mathcal{X})_f$ by $e^{2\pi i f}$. We define

$$a_{(x,g)} := \sum_{0 \le f < 1} f \dim (T_x\mathcal{X})_f.$$

This number is independent of the choice of $(x, g) \in \mathcal{X}_v$, so we can associate a rational number a_v to each connected component $\mathcal{X}_v$ of $I\mathcal{X}$. This is called age-shifting number.

Definition 0.13. The *Chen–Ruan cohomology group* of $\mathcal{X}$ is the direct sum of the singular cohomology of $\mathcal{X}_v, v \in T$ with coefficients in $\mathbb{C}$, together with the age-shift in gradings:

$$H^*_{\mathrm{CR}}(\mathcal{X}, \mathbb{C}) := \bigoplus_{v \in T} H^{*+2a_v}(\mathcal{X}_v, \mathbb{C}).$$

We will omit $\mathbb{C}$ in the notation of cohomology groups with coefficients in $\mathbb{C}$.

The poincaré pairing for $\alpha, \beta \in H^*_{\mathrm{CR}}(\mathcal{X})$ is defined by

$$(\alpha, \beta) := \int_{I\mathcal{X}} \alpha \wedge \mathrm{inv}^* \beta.$$

For an orbifold vector bundle $\tilde{E}$ on the inertia stack $I\mathcal{X}$, we have an eigenbundle decomposition of $\tilde{E}|_{\mathcal{X}_v}$

$$\tilde{E}|_{\mathcal{X}_v} = \bigoplus_{0 \le f < 1} \tilde{E}_{v,f}$$

with respect to the action of the stabilizer of $\mathcal{X}_v$, where $\tilde{E}_{v,f}$ is the subbundle with eigenvalue $e^{2\pi i f}$. Let $\mathrm{pr}\colon I\mathcal{X} \to \mathcal{X}$ be the projection. For an orbifold vector bundle E on $\mathcal{X}$, let $\{\delta_{v,f,i}\}_{1 \le i \le l_{v,f}}$ be the Chern roots of $(\mathrm{pr}^* E)_{v,f}$, where $l_{v,f}$ is the dimension of $(\mathrm{pr}^* E)_{v,f}$.

Definition 0.14. We define some $H^*_{\mathrm{CR}}(\mathcal{X})$-value characteristic classes of an orbifold vector bundle E on $\mathcal{X}$:

— The *Chern character* of E is defined by

$$\mathrm{ch}(E) := \bigoplus_{v \in T} \sum_{0 \le f < 1} e^{2\pi i f} \,\mathrm{ch}\left((\mathrm{pr}^* E)_{v,f}\right).$$

— The *Todd class* of E is defined by

$$\mathrm{Td}(E) := \bigoplus_{v \in T} \prod_{\substack{0 < f < 1 \\ 1 \le i \le l_{v,f}}} \frac{1}{1 - e^{-2\pi i f} e^{-\delta_{v,f,i}}} \prod_{1 \le i \le l_{v,0}} \frac{\delta_{v,0,i}}{1 - e^{-\delta_{v,0,i}}}.$$

— The *Gamma class* of E is defined in [31], which is

$$\Gamma(E) := \bigoplus_{v \in T} \prod_{0 \le f < 1} \prod_{i=1}^{l_{v,f}} \Gamma(1 - f + \delta_{v,f,i}).$$

The Gamma function on the right-hand side should be expanded in series at $1 - f$, *i.e.*

$$\Gamma(1 - f + \delta_{v,f,i}) = \Gamma(1 - f) + \Gamma'(1 - f)\delta_{v,f,i} + \frac{1}{2}\Gamma''(1 - f)\delta^2_{v,f,i} + \dots.$$

Proposition 0.15 (Grothendieck–Riemann–Roth formula [45])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Deligne–Mumford stacks with quasiprojective coarse moduli spaces and $f\colon \mathcal{X} \to \mathcal{Y}$ a proper morphism which factors as*

$$f = g \circ i,$$

where $i\colon \mathcal{X} \to \mathcal{P}$ is a closed regular immersion and $g\colon \mathcal{P} \to \mathcal{Y}$ is a smooth morphism. Assume that every coherent sheaf on $\mathcal{X}$ and $\mathcal{Y}$ is a quotient of a vector bundle. Let $E \in K^0(\mathcal{X})$, then

$$\mathrm{ch}(f_* E) = If_*(\mathrm{ch}(E)\,\mathrm{Td}(T_f)),$$

where f_* is the K-theoretic pushforward and $If\colon I\mathcal{X} \to I\mathcal{Y}$ is the map induced by f.

0.5 Structure of the thesis

In §1, we introduce GW theory and FJRW theory; we show these two theories are related via GIT; we state Clader's result [13] on relating these two theories by analytic continuation.

In §2, we carry out the analytic continuation explicitly; we find linear maps relating GW theory and FJRW theory.

In §3, we introduce the notion of matrix factorizations; we introduce categorical equivalences between derived categories of matrix factorizations constructed by Segal [41] and Shipman [42]; in particular, we find a way to compute these equivalences explicitly.

In §4, we prove our main result stating the linear maps find in §2 are compatible with the categorical equivalences introduced in §3; we apply our result to study the monodromy of a local system; we also discuss the relation between our main result and the result of Coates, Iritani and Jiang [18] on crepant transformation conjecture.

Chapter 1

Enumerative theories

We introduce the Gromov–Witten (GW) theory and the Fan–Jarvis–Ruan–Witten theory in this chapter. We are particularly interested in two special cases; one of them is the genus-0 Gromov–Witten theory of the complete intersection $X_{3,3}$ of two cubics in $\mathbb{P}^5$, the other one is a genus-0 Fan–Jarvis–Ruan–Witten (FJRW) theory constructed by Clader in [13]. We recall Clader's result stating that these two theories are related to each other in §1.5. A more general theory constructed by Fan, Jarvis and Ruan is introduced in §1.3. We explain how the above two theories can be regarded as special cases of the general theory by taking different GIT stability conditions of the same group action.

1.1 Gromov–Witten theory

Let X be a smooth projective variety, the Gromov–Witten theory for X is aimed to "count" the number of curves inside X satisfying certain restrictions. The main idea of the theory to achieve this by considering the intersection theory for the moduli space of stable maps into X introduced in Definition 0.4.

The moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is usually singular and its components have different dimensions. However, by using Hirzebruch–Riemann–Roch formula, it has an *expected dimension* (see [21] for example)

$$\mathrm{vdim}\overline{\mathcal{M}}_{g,n}(X,\beta) = (\dim X - 3)(1-g) + \int_\beta c_1(T_X) + n. \tag{1.1}$$

For each $s = 1, \ldots, n$, there exists a natural evaluation map

$$\mathrm{ev}_s \colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$$

sending $f\colon (C, x_1, \ldots, x_n) \to X$ to $f(x_s)$. If we are in an ideal situation (when $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is

smooth, and all of its components have expected dimension), for any choice of $\phi_1, \dots, \phi_n \in H^*(X)$, the integral

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} \prod_{s=1}^{n} (\mathrm{ev}_s^* \phi_s) \tag{1.2}$$

should count the number of genus-g curves C in X with homology class β such that the intersections of C with the Poincaré duals of ϕ_s are nonzero.

However, we are usually not in an ideal situation. To solve this, we replace the fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]$ by the *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$. It is a homology class of expected dimension, and behaves like the ordinary fundamental class (see [5]).

We can also insert some canonical cohomology classes of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ into the integral (1.2). Let

$$\psi_s \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

be the first Chern class of the universal cotangent line bundle at the sth marked point. For any choice of $\phi_1, \dots, \phi_n \in H^*(X)$, $a_1, \dots, a_n \in \mathbb{Z}^{\geq 0}$ and $d \in \mathbb{Z}$, the corresponding Gromov–Witten invariant is defined as

$$\langle \phi_1 \psi_1^{a_1}, \dots, \phi_{n-1}\psi_{n-1}^{a_{n-1}}, \phi_n \psi_n^{a_n} \rangle_{g,n,\beta}^{\mathrm{GW},X} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{s=1}^{n} (\psi_s^{a_s} \, \mathrm{ev}_s^* \phi_s). \tag{1.3}$$

The graded vector space $H^*(X)$ is called the *state space* of the Gromov–Witten theory for X; the invariants defined above are also called correlators.

We can define a generating function for genus-0 Gromov–Witten invariants as follows:

$$\mathcal{F}^0_{\mathrm{GW},X}(\boldsymbol{t}) := \sum_{n,\beta} \frac{Q^\beta}{n!} \langle \boldsymbol{t}(\psi_1), \dots, \boldsymbol{t}(\psi_{n-1}), \boldsymbol{t}(\psi_n) \rangle_{0,n,\beta}^{\mathrm{GW},X},$$

where

$$\boldsymbol{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H^*(X) \otimes \mathbb{C}[z]$$

is a polynomial taking values in $H^*(X)$, and Q^β is the representative of β in the group ring of $H_2(X,\mathbb{Z})$.

Let $X_{3,3}$ be a complete intersection of two cubics inside $\mathbb{P}^5$. We are particularly interested in the genus-0 Gromov–Witten theory of $X_{3,3}$. It is related to the genus-0 Gromov–Witten theory of $\mathbb{P}^5$. In fact, let $i\colon X_{3,3} \to \mathbb{P}^5$ be the inclusion morphism, it induces a morphism of moduli space

$$i\colon \overline{\mathcal{M}}_{0,n}(X_{3,3},\beta) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^5, i_*\beta).$$

By Lefschetz hyperplane theorem, we have an isomorphism

$$i_*\colon H_2(X_{3,3},\mathbb{Z}) \cong H_2(\mathbb{P}^5,\mathbb{Z});$$

we identify both of them with $\mathbb{Z}$, and represent d times the hyperplane class p by the integer d for simplicity.

Let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^5, d)$ be the moduli spaces of genus-0 degree-d n-marked stable maps to $\mathbb{P}^5$, $\overline{\mathcal{C}}_{0,n}(\mathbb{P}^5, d)$ be the universal curve over it, and ev be the evaluation map as in the following diagram.

$$\begin{tikzcd} \overline{\mathcal{C}}_{0,n}(\mathbb{P}^5, d) \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"] & \mathbb{P}^5 . \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^5, d) & \end{tikzcd}$$

Then $\overline{\mathcal{M}}_{0,n}(X_{3,3}, d)$ is the intersection of the zero locus of two sections of the vector bundle $\pi_* \operatorname{ev}^* \mathcal{O}_{\mathbb{P}^5}(3)$. For any choice of $\phi_1, \ldots, \phi_n \in i^* H^*(X) \subset H^*(X_{3,3})$, we can rewrite the GW invariants as follows:

$$\langle \phi_1 \psi_1^{a_1}, \ldots, \phi_{n-1} \psi_{n-1}^{a_{n-1}}, \phi_n \psi_n^{a_n} \rangle_{0,n,d}^{\mathrm{GW}, X_{3,3}} = \int_{i^*[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^5, d)]} c_{\mathrm{top}} \left((\pi_* \operatorname{ev}^* \mathcal{O}_{\mathbb{P}^5}(3))^{\oplus 2} \right) \prod_{s=1}^{n} (\psi_s^{a_s} \operatorname{ev}_s^* \phi_s). \tag{1.4}$$

Note that in the above definition, $\phi_1, \ldots, \phi_n$ are taken from $i^* H^*(X)$, which is a subspace of the state space $H^*(X_{3,3})$. This subspace is the *ambient part* of $H^*(X_{3,3})$; the theory we defined is the genus-0 Gromov–Witten theory for $X_{3,3}$ restricted to the ambient part. The restricted theory is what we are interested in. We omit the superscript $X_{3,3}$ and set

$$H_{\mathrm{GW}} := i^* H^*(X);$$

we can define a generating function for the restricted theory:

$$\mathcal{F}_{\mathrm{GW}}^0(\boldsymbol{t}) := \sum_{n,d} \frac{Q^d}{n!} \langle \boldsymbol{t}(\psi_1), \ldots, \boldsymbol{t}(\psi_{n-1}), \boldsymbol{t}(\psi_n) \rangle_{0,n,d}^{\mathrm{GW}}, \tag{1.5}$$

where $\boldsymbol{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H_{\mathrm{GW}} \otimes \mathbb{C}[z]$.

Remark 1.1. Let $p \in H^2(X_{3,3})$ be the hyperplane class. Denote the degree-2 part of t_0 by $t_0^2 p$. We can take $t_0^2 p$ out of the bracket by repeatedly applying the divisor equation (see [2] for example). Then Q and $t_0^2 p$ always appear together in the form $Q e^{t_0^2}$. So we can set $Q = 1$, and denote $e^{t_0^2}$ by v without losing any information. This is a standard procedure, see [16].

1.2 Fan–Jarvis–Ruan–Witten theory

Fan–Jarvis–Ruan–Witten (FJRW) theory was introduced in the series of papers [23, 24, 25, 26]. In this section, we focus on a version of FJRW theory defined by Clader [13].

1.2.1 Input data

Let $W_1, \ldots, W_r$ be a collection of degree-d quasihomogeneous polynomials in the variables $x_1, \ldots, x_N$, where x_i has weight w_i. The weights $w_1, \ldots, w_N$ are coprime. We require the *nondegenerate condition* be satisfied, *i.e.* the forms $dW_1, \ldots, dW_r$ are linearly independent at the common 0-locus of the polynomials W_i, except at the point $x_1 = \cdots = x_N = 0$. Then the equation

$$W_1 = \cdots = W_r = 0$$

defines a complete intersection $X_{d,\ldots,d}$ in the weighted projective stack $\mathbb{P}(w_1, \ldots, w_N)$. The weights $w_1, \ldots, w_N$ satisfy the *Calabi–Yau condition*

$$\sum_{i=1}^{N} w_i = rd.$$

By the adjunction formula, $X_{d,\ldots,d}$ is Calabi–Yau in the sense that its canonical sheaf is trivial. We further require the *Gorenstein condition* be satisfied:

$$w_i | d, \quad 1 \leq i \leq N.$$

Clader defined a enumerative theory for above data in [13]. We are particularly interested in the case $N = 6$, $r = 2$, $w_1 = \cdots = w_6 = 1$, $d = 3$.

1.2.2 Moduli space

Definition 1.2. [13] A genus-g n-pointed degree-β Landau–Ginzburg stable map (over a scheme S) is given by the following objects.

$$\mathcal{L} \longrightarrow (\mathcal{C}, \Sigma_1, \ldots, \Sigma_n) \xrightarrow{f} \mathbb{P}^{r-1},$$

together with an isomorphism

$$\varphi \colon \mathcal{L}^{\otimes d} \xrightarrow{\sim} \omega_{\log,\mathcal{C}} \otimes f^*\mathcal{O}(-1),$$

where

1. $(\mathcal{C}, \Sigma_1, \ldots, \Sigma_n)$ is a twisted genus-g n-pointed curve;
2. f is a morphism whose induced map between coarse moduli spaces is a genus-g n-pointed stable map of degree β;
3. $\mathcal{L}$ is a representable orbifold line bundle on $\mathcal{C}$ and φ is an isomorphism of line bundles. ($\mathcal{L}$ is representable means for any $p \in \mathcal{C}$, the representation $\rho_p \colon G_p \to \mathbb{C}^*$ given by the action of isotropy group on the fiber of $\mathcal{L}$ is faithful.)

The morphism between two Landau–Ginzburg stable maps is defined in an obvious way. These objects forms a category. In [13], Clader proves that it can be represented by a proper Deligne–Munford stack $\overline{\mathcal{M}}^{\frac{1}{d}}_{g,n}(\mathbb{P}^{r-1},\beta)$.

We have a natural decomposition

$$\overline{\mathcal{M}}^{\frac{1}{d}}_{g,n}(\mathbb{P}^{r-1},\beta) = \bigsqcup_{m_1,\dots,m_n\in\mathbb{Z}_d} \overline{\mathcal{M}}^{\frac{1}{d}}_{g,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta),$$

where $\overline{\mathcal{M}}^{\frac{1}{d}}_{g,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)$ is the substack in which the multiplicity of $\mathcal{L}$ at the ith marked point is m_i (see §0.2).

Definition 1.3. A component $\overline{\mathcal{M}}^{\frac{1}{d}}_{g,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)$ is called *narrow* if all of the line bundles $\mathcal{L}^{\otimes w_1},\dots,\mathcal{L}^{\otimes w_N}$ have nonzero multiplicities at all marked points.

Using the cosection technique developed in [34, 7, 8], a virtual fundamental class is defined on each narrow component in [13]. In particular, in genus zero, under the *Gorenstein condition*

$$w_j|d, \quad \forall 1\le j\le N, \tag{1.6}$$

the virtual fundamental class is easy to define. Let

$$\pi\colon \overline{\mathcal{C}}^{\frac{1}{d}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)\to\overline{\mathcal{M}}^{\frac{1}{d}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)$$

be the universal curve and $\mathcal{T}$ be the universal line bundle over $\overline{\mathcal{C}}^{\frac{1}{d}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)$. Let

$$\rho\colon \overline{\mathcal{M}}^{\frac{1}{d}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)\to\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r-1},\beta)$$

be the morphism defined by forgetting $\mathcal{L}$. The virtual fundamental class of the narrow component $\overline{\mathcal{M}}^{\frac{1}{d}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)$ can be defined by the following formula:

$$[\overline{\mathcal{M}}^{\frac{1}{d}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta)]^{\mathrm{vir}} = c_{\mathrm{top}}(R^1\pi_*(\mathcal{T}^{\otimes w_1}\oplus\cdots\oplus\mathcal{T}^{\otimes w_N}))\cap\rho^*[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r-1},\beta)]. \tag{1.7}$$

Remark 1.4. We can take the top Chern class of $R^1\pi_*(\mathcal{T}^{\otimes w_1}\oplus\cdots\oplus\mathcal{T}^{\otimes w_N})$ because it is a vector bundle. To see this, we show that the bundles $\mathcal{L}^{\otimes w_i}$ have no nonzero global sections for $\mathcal{L}$ in Definition 1.2. Let $p\colon\mathcal{C}_x\to C_x$ be the morphism between be a geometric $\mathcal{C}_x$ fiber of $\mathcal{C}$ in Definition 1.2 and its coarse moduli space C_x. Let Z be an irreducible component C_x. Then we have the following formula:

$$p_*(\mathcal{L}^{\otimes w_i})|_Z^{\otimes d/w_i} = \omega_{\log,C_x}\otimes f^*\mathcal{O}(-1)\otimes\mathcal{O}(-\Sigma_j m_{i,j}x_j), \tag{1.8}$$

where x_j are special points (marked points and nodes) on Z and $m_{i,j}$ is the multiplicity of $\mathcal{L}^{\otimes w_i}$ at x_j. Since we are working on the narrow components, the multiplicities at marked points are all nonzero. Thus the degree of the right-hand side of (1.8) is less than the number of points where Z meets the rest of the C_x minus 1. From this we can show that the sections of $p_*(\mathcal{L}^{\otimes w_i})|_Z^{\otimes d/w_i}$ must be zero on each components inductively; therefore $\mathcal{L}^{\otimes w_i}$ have no nonzero global sections.

1.2.3 State space

In order to describe the state space of FJRW theory, we define a new polynomial

$$W := p_1 W_1 + \cdots + p_r W_r.$$

This is a polynomial on

$$V = \mathbb{C}^N \times \mathbb{C}^r = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_N, p_1, \ldots, p_r];$$

it is invariant under the $\mathbb{C}^*$ action with weights $(w_1, \ldots, w_N, -d, \ldots, -d)$. Then we can regard W as a function on the GIT quotient

$$X_- = [V /\!\!/_{\theta_2} \mathbb{C}^*].$$

(See Example 0.11 for details, where we take $d_1 = \cdots = d_r = d$.)

The state space of the FJRW theory is defined to be the Chen–Ruan cohomology of X_- relative to a generic fiber of W, with an additional shift $-2r$ in degree; *i.e.*

$$H^{*-2r}_{\mathrm{CR}}(X_-, W^{-1}(1)).$$

Case of interest

We are particularly interested in the case $N = 6$, $r = 2$, $w_1 = \cdots = w_6 = 1$, $d = 3$. In this case,

$$X_- = [\mathbb{C}^6 \times (\mathbb{C}^2 \backslash \{\mathbf{0}\})/\mathbb{C}^*].$$

This is the total space of the vector bundle

$$\mathcal{O}(-1)^{\oplus 6}$$

over the weighted projective space $\mathbb{P}(3,3)$. The state space in this case can be computed as

$$H^{*-4}_{\mathrm{CR}}(X_-, W^{-1}(1)) = H^{*-4}(X_-, W^{-1}(1)) \oplus H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)).$$

The subspace $H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)$ is called the *narrow part* of the FJRW state space. We denote it by H_{FJRW}. It is the analogue of the ambient part of the GW state space. Let $H^{(1)}$ and $H^{(2)}$ be the hyperplane classes in the first and second $\mathbb{P}(3,3)$, then we can write

$$H_{\mathrm{FJRW}} = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)) = \mathbb{C}\mathbb{1}^{(1)} \oplus \mathbb{C}H^{(1)} \oplus \mathbb{C}\mathbb{1}^{(2)} \oplus \mathbb{C}H^{(2)}. \tag{1.9}$$

Remark 1.5. We will discuss the relation between the state spaces of GW theory and FJRW theory in §1.3.1.

1.2.4 Correlators

For $\phi_1, \dots, \phi_n \in H_{\mathrm{FJRW}}$, we want to define the FJRW correators

$$\langle \phi_1 \psi_1^{a_1}, \dots, \phi_{n-1}\psi_{n-1}^{a_{n-1}}, \phi_n \psi_n^{a_n} \rangle_{0,n,\beta}^{\mathrm{FJRW}}.$$

In order to define them in a similar way to equation (1.3), we need to define the maps

$$\mathrm{ev}_i^* \colon H_{\mathrm{FJRW}} \to H^*(\overline{\mathcal{M}}_{g,n}^{\frac{1}{d}}(\mathbb{P}^{r-1}, \beta)).$$

We can do this in two equivalent ways. For simplicity, we focus on the case of interest, *i.e.* $N = 6$, $r = 2$, $w_1 = \cdots = w_6 = 1$, $d = 3$.

1. The log-canonical sheaf is trivial when restricted to the ith marked point Σ_i. Then, by Definition 1.2, the restriction of the line bundle $\mathcal{L}$ to Σ_i satisfies

   $$\mathcal{L}|_{\Sigma_i}^{\otimes 3} \cong f^*\mathcal{O}(-1).$$

   This datum is equivalent to a morphism from Σ_i to the weighted projective space $\mathbb{P}(3,3)$. It induces a morphism

   $$\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}^{\frac{1}{3}}(\mathbb{P}^1, \beta) \to \bar{I}\mathbb{P}(3,3),$$

   where $\bar{I}\mathbb{P}(3,3)$ is the *rigidified inertia stack* of $\mathbb{P}(3,3)$ (see [1]). The cohomology group $H^*(\bar{I}\mathbb{P}(3,3))$ is isomorphic to the Chen–Ruan cohomology of $\mathbb{P}(3,3)$, which is

   $$H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)) \oplus H^{*+8}(\mathbb{P}(3,3)). \tag{1.10}$$

   We identify

   $$H_{\mathrm{FJRW}} = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3))$$

   with the second and third direct summands of (1.10); then the map

   $$\mathrm{ev}_i^* \colon H_{\mathrm{FJRW}} \to H^*(\overline{\mathcal{M}}_{g,n}^{\frac{1}{3}}(\mathbb{P}^1, \beta))$$

   is defined.

2. The map ev_i^* can be defined in a more direct way. Let ϕ be an element of the kth direct summand of

   $$H_{\mathrm{FJRW}} = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)),$$

   where $k \in \{1,2\}$; we regard k as an element of $\mathbb{Z}_3$.
   Let $\overline{\mathcal{M}}_{g,m_i=k}^{\frac{1}{3}}(\mathbb{P}^1, \beta)$ be the component of $\overline{\mathcal{M}}_{g,n}^{\frac{1}{3}}(\mathbb{P}^1, \beta)$ in which the multiplicity of $\mathcal{L}$ at the ith marked point is k. Let

   $$\mathrm{ev} \colon \overline{\mathcal{M}}_{g,m_i=k}^{\frac{1}{3}}(\mathbb{P}^1, \beta) \to \mathbb{P}^1$$

be the evaluation map defined by passing through the forgetful morphism

$$\rho\colon \overline{\mathcal{M}}^{\frac{1}{3}}_{g,m_i=k}(\mathbb{P}^1,\beta) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,\beta).$$

We regard ϕ as an element of $H^*(\mathbb{P}^1)$ after identifying $H^*(\mathbb{P}(3,3))$ with $H^*(\mathbb{P}^1)$. Then we define the class

$$\mathrm{ev}_i^*(\phi) := \mathrm{ev}^*(\phi).$$

This is a class in $H^*(\overline{\mathcal{M}}^{\frac{1}{3}}_{g,n}(\mathbb{P}^1,\beta))$ supported on the component $\overline{\mathcal{M}}^{\frac{1}{3}}_{g,m_i=k}(\mathbb{P}^1,\beta)$.

For $\phi_1,\dots,\phi_n \in H_{\mathrm{FJRW}}$, the genus-0 FJRW correlators for the case of interest are defined by the following formula:

$$\langle \phi_1\psi_1^{a_1},\dots,\phi_{n-1}\psi_{n-1}^{a_{n-1}},\phi_n\psi_n^{a_n}\rangle^{\mathrm{FJRW}}_{0,n,\beta} := 3\int_{\rho^*[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,\beta)]} c_{\mathrm{top}}\left((R^1\pi_*(\mathcal{T}^{\oplus 6}))^\vee\right)\prod_{s=1}^n(\psi_s^{a_s}\,\mathrm{ev}_s^*\,\phi_s). \tag{1.11}$$

Remark 1.6. The classes $\phi_1,\dots,\phi_n$ are taken from H_{FJRW}; from the second description of the map ev_i^*, the product

$$\prod_{s=1}^n \mathrm{ev}_s^*\,\phi_s$$

is supported on the narrow components of $\overline{\mathcal{M}}^{\frac{1}{3}}_{g,n}(\mathbb{P}^1,\beta)$. Recall that the virtual fundamental class (1.7) is only defined on the narrow components.

Remark 1.7. The factor 3 in (1.11) is the multiplicative inverse of the degree of the forgetful morphism

$$\rho\colon \overline{\mathcal{M}}^{\frac{1}{3}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^1,\beta) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,\beta)$$

when $\overline{\mathcal{M}}^{\frac{1}{3}}_{0,(m_1,\dots,m_n)}(\mathbb{P}^1,\beta)$ is not empty.

Similarly to equation (1.5), we define the genus-0 generating function for the FJRW theory:

$$\mathcal{F}^0_{\mathrm{FJRW}}(\boldsymbol{t}) := \sum_{n,d}\frac{Q^d}{n!}\langle \boldsymbol{t}(\psi_1),\dots,\boldsymbol{t}(\psi_{n-1}),\boldsymbol{t}(\psi_n)\rangle^{\mathrm{FJRW}}_{0,n,d}, \tag{1.12}$$

where $\boldsymbol{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H_{\mathrm{GW}}\otimes\mathbb{C}[z]$.

1.3 Relate GW theory and FJRW theory via GIT

In this section, we show that the FJRW theory introduced in §1.2 is related to the GW theory of $X_{d,\dots,d}$ via the variation of GIT stability condition in Example 0.11. Recall that $X_{d,\dots,d}$ is the Calabi–Yau complete intersection defined by

$$W_1 = \cdots = W_r = 0$$

in the weighted projective stack $\mathbb{P}(w_1,\dots,w_N)$ as in §1.2.1.

1.3.1 Relate the state spaces

We start by showing the above two theories have isomorphic state spaces, *i.e.*

$$H_{\mathrm{CR}}^{*-2r}(X_-, W^{-1}(1)) \cong H_{\mathrm{CR}}^{*}(X_{d,\dots,d}). \tag{1.13}$$

This can be understood as a result of the variation of GIT stability condition in Example 0.11. In fact, the polynomial

$$W = p_1 W_1 + \cdots + p_r W_r$$

can also be regarded as a function on

$$X_+ = [(\mathbb{C}^N \backslash \{\mathbf{0}\}) \times \mathbb{C}^r / \mathbb{C}^*].$$

We have an isomorphism of graded vector spaces

$$H_{\mathrm{CR}}^{*-2r}(X_+, W^{-1}(1)) \cong H_{\mathrm{CR}}^{*}(X_{d,\dots,d}). \tag{1.14}$$

We prove (1.14) for the case of interest (a complete proof can be found in [10]). In this case X_+ is the total space of $\mathcal{O}(-3)^{\oplus 2}$ over $\mathbb{P}^5$; both X_+ and $X_{3,3}$ are smooth schemes so their Chen–Ruan cohomology groups are just their singular cohomology groups. The isomorphism (1.14) can be deduced from two successive isomorphisms

$$H^{*-4}(X_+, W^{-1}(1)) \cong H^{*-4}(\mathbb{P}^5, \mathbb{P}^5 \backslash X_{3,3}) \cong H^*(X_{3,3}), \tag{1.15}$$

where the first isomorphism comes from retraction, the second isomorphism is the Thom isomorphism. Then (1.13) is a consequence of the isomorphism

$$H_{\mathrm{CR}}^{*}(X_+, W^{-1}(1)) \cong H_{\mathrm{CR}}^{*}(X_-, W^{-1}(1)). \tag{1.16}$$

The equation (1.16) is proven in [10]; it follows from its nonrelative version

$$H_{\mathrm{CR}}^{*}(X_+) \cong H_{\mathrm{CR}}^{*}(X_-). \tag{1.17}$$

In the case of interest, a direct computation shows that

$$H_{\mathrm{CR}}^{*}(X_+) = H^*(\mathbb{P}^5)$$

and

$$H_{\mathrm{CR}}^{*}(X_-) = H_{\mathrm{CR}}^{*}(\mathbb{P}(3,3)) = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)) \oplus H^{*+8}(\mathbb{P}(3,3))$$

are isomorphic as graded vector spaces.

We can also say that the ambient part H_{GW} and the narrow part H_{FJRW} are related by the variation of GIT stability condition. In fact, they are the *subspaces of compact type* of

respective entire state spaces in the sense of [14]. Since both X_- and X_+ are total space of vector bundles, We denote the corresponding zero sections by X_-^{cp} and X_+^{cp}. The *subspaces of compact type* are defined to be the image of the morphisms

$$H_{\mathrm{CR}}^{*-2r}(X_\pm, X_\pm \backslash X_\pm^{\mathrm{cp}}) \to H_{\mathrm{CR}}^{*-2r}(X_\pm, W^{-1}(1)) \tag{1.18}$$

defined by restricting the relative cohomology classes.

We compute the subspaces of compact type for the cases of interest.

— The state space of the FJRW theory of case of interest is

$$H_{\mathrm{CR}}^{*-4}(X_-, W^{-1}(1)) = H^{*-4}(X_-, W^{-1}(1)) \oplus H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)).$$

The morphism

$$H_{\mathrm{CR}}^{*-4}(X_-, X_- \backslash X_-^{\mathrm{cp}}) \to H_{\mathrm{CR}}^{*-4}(X_-, W^{-1}(1))$$

are isomorphisms when restricted to the last two direct summands. It is shown in [10] that $H_{\mathrm{CR}}^k(X_-, W^{-1}(1)) = 0$ if $k \neq 7$. So it is a zero morphism when restricted on the first direct summand. Therefore, the subspace of compact type is

$$H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)).$$

It coincides with the narrow part H_{FJRW}.

— We identify the state space of the GW theory of $X_{3,3}$ with $H_{\mathrm{CR}}^{*-4}(X_+, W^{-1}(1))$ via equation (1.14). We can also identify $H_{\mathrm{CR}}^{*-4}(X_+, X_+ \backslash X_+^{\mathrm{cp}})$ with $H^*(\mathbb{P}^5)$ via Thom isomorphism. Then (1.18) induces a morphism

$$u\colon H^*(\mathbb{P}^5) \to H^*(X_{3,3}).$$

After tracking the Thom classes carefully, we can prove that u coincides with the restriction

$$i^*\colon H^*(\mathbb{P}^5) \to H^*(X_{3,3}).$$

Therefore, the subspace of compact type coincides with H_{GW}.

1.3.2 Relate the moduli space

Fan, Jarvis and Ruan defined a new theory in [26]. Both the GW theory of $X_{d,\dots,d}$ and the FJRW theory introduced in §1.2 can be obtained from this new theory by taking different GIT stability conditions.

To define the new theory, we need some more input data of group actions in addition to §1.2.1. Let $G = \mathbb{C}^*$, the first action we need is the G-action on the vector space

$$V = \mathbb{C}^N \times \mathbb{C}^r = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_N, p_1, \dots, p_r]$$

as in Example 0.11, with weights w_i on the first N variables x_i, and weight $-d$ on the following r variables:

$$\lambda \cdot (x_1, \ldots, x_N, p_1, \ldots, p_r) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N, \lambda^{-d} p_1, \ldots, \lambda^{-d} p_r).$$

Since $w_1, \ldots, w_N$ are coprime, we can regard G as a subgroup of $\mathrm{GL}(V)$.

We introduce another $\mathbb{C}^*$-action. We denote this $\mathbb{C}^*$ by $\mathbb{C}^*_R$. The group $\mathbb{C}^*_R$ acts on V with weight 0 on the first N variables, and weight 1 on the following r variables:

$$\mu \cdot (x_1, \ldots, x_N, p_1, \ldots, p_r) = (x_1, \ldots, x_N, \mu p_1, \ldots, \mu p_r). \tag{1.19}$$

We can also regard $\mathbb{C}^*_R$ as a subgroup of $\mathrm{GL}(V)$. Let Γ be the subgroup of $\mathrm{GL}(V)$ generated by G and $\mathbb{C}^*_R$. Then we have an isomorphism

$$\Gamma = G\mathbb{C}^*_R \cong G \times \mathbb{C}^*_R.$$

Denote by $\xi \colon \Gamma \to G$ and $\zeta \colon \Gamma \to \mathbb{C}^*$ the first and second projections.

We need the concept of good lifts.

Definition 1.8. We say a Γ-character $\hat{\theta}$ is a *good lift* of a G-character θ if it is compatible with the inclusion $G \leq \Gamma$, and satisfies

$$V^{\mathrm{ss}}_{\Gamma,\hat{\theta}} = V^{\mathrm{ss}}_{G,\theta}.$$

Example 1.9. When we choose a positive G-character

$$\theta_1(\lambda) = \lambda^k$$

for $k > 0$, we claim that the character of $\Gamma = G \times \mathbb{C}^*_R$

$$\hat{\theta}_1(\lambda, \mu) = \lambda^k$$

is a good lift of θ_1. Since $V^{\mathrm{ss}}_{\Gamma,\hat{\theta}_1} \subset V^{\mathrm{ss}}_{G,\theta_1}$, it is sufficient to check that the section of $L^{\otimes w_1}_{\hat{\theta}_1}$

$$(x_1, \ldots, x_N, p_1, \ldots, p_r) \mapsto x_1^k$$

is invariant under the Γ-action (*see* Eample 0.11).

When we choose a negative G-character

$$\theta_2(\lambda) = \lambda^k$$

for $k < 0$, we claim that the Γ-character

$$\hat{\theta}_2(\lambda, \mu) = \lambda^k \mu^{-\frac{k}{d}} \tag{1.20}$$

is a good lift if $d|k$. It is sufficient to check that the section of $L^{\otimes d}_{\hat{\theta}_2}$

$$(x_1, \ldots, x_N, p_1, \ldots, p_r) \mapsto p_1^{-k}$$

is invariant under the Γ-action.

Let $\mathrm{Crit}(W)$ denote the critical locus of the polynomial

$$W = p_1 W_1 + \cdots + p_r W_r.$$

The moduli space in the theory constructed in [26] is the moduli space of the following objects.

Definition 1.10 ([26])**.** Let $\hat{\theta}$ be a good lift of θ, an ∞-stable, k-pointed, genus-g LG-quasimaps to $[\mathrm{Crit}(W)/\!\!/_\theta G]$ consists the following data:

— A twisted, k-pointed curve $(\mathcal{C}, \Sigma_1, \ldots, \Sigma_k)$ of genus g.

— A representable principal orbifold Γ-bundle $\mathcal{P}\colon \mathcal{C} \to B\Gamma$ over $\mathcal{C}$.

— A global section $\sigma\colon \mathcal{C} \to \mathcal{P} \times_\Gamma V$.

— An isomorphism $\kappa\colon \zeta_*\mathcal{P} \to \mathring{\omega}_{\log,\mathcal{C}}$ of principal $\mathbb{C}^*$ bundles, where $\mathring{\omega}_{\log,\mathcal{C}}$ is the principal bundle associated to the line bundle $\omega_{\log,\mathcal{C}}$.

such that the following conditions are satisfied:

1. The image of the induced map $[\sigma]\colon \mathcal{C} \to [V/\Gamma]$ lies within $[(V^{\mathrm{ss}}_{G,\theta} \cap \mathrm{Crit}(W))/\Gamma]$.
2. The line bundle $\omega_{\log,\mathcal{C}} \otimes \sigma^*(\mathcal{N})^\epsilon$ is ample for all sufficiently large ϵ, where $\mathcal{N}$ is the line bundles over $[V/\Gamma]$ determined by the Γ-character $\hat{\theta}$.

Remark 1.11. It is more natural to start from quasihomogeneous polynomials $W_1, \ldots, W_r$ with different degrees $d_1, \ldots, d_r$. However, we do not have a well-defined enumerative theory due to the lack of a good lift when we choose a negative character. In fact, when $d_1, \ldots, d_r$ are not all the same, we can not find the Γ-character (1.20) as in Example 1.9.

Since $\Gamma \cong G \times \mathbb{C}^*_R$, and ζ is just the second projection, giving a principal Γ bundle $\mathcal{P}$ is the same as giving a line bundle $\mathcal{L}$ such that $\mathcal{P} \cong \mathring{\mathcal{L}} \times \mathring{\omega}_{\log,\mathcal{C}}$. Then we can write

$$\mathcal{P} \times_\Gamma V \cong \bigoplus_{i=1}^{N} \mathcal{L}^{\otimes w_i} \oplus \left(\omega_{\log,\mathcal{C}} \otimes L^{\otimes -d}\right)^{\oplus r}.$$

Thus giving a section of $\mathcal{P} \times_\Gamma V$ is the same as giving sections $s_i \in \Gamma(\mathcal{C}, \mathcal{L}^{\otimes w_i})$ and $t_j \in \Gamma(\mathcal{C}, \omega_{\log,\mathcal{C}} \otimes \mathcal{L}^{\otimes -d})$ for $1 \le i \le N$ and $1 \le j \le r$.

In order to determine the semistable locus of the critical locus of W, we write

$$dW = \sum_{i=1}^{r} (p_i dW_i + W_i dp_i).$$

According to the nondegeneracy condition, the critical locus of W is

$$\{x_1 = \cdots = x_N = 0\} \cup \{p_1 = \cdots = p_r = W_1 = \cdots = W_r = 0\}.$$

— When we choose a positive G-character, condition 1 in Definition 1.10 implies

$$p_1 = \cdots = p_r = 0$$

and

$$W_1(s_1, \ldots, s_N) = \cdots = W_r(s_1, \ldots, s_N) = 0.$$

In this case the above datum is equivalent to a stable map to the complete intersection $X_{d,\ldots,d}$. The theory is only defined on the classes of compact type. It coincides with the classic Gromov–Witten theory for $X_{d,\ldots,d}$ restricted to the hyperplane section classes.

— When we choose a negative G-character, condition 1 in Definition 1.10 implies

$$x_1 = \cdots = x_N = 0.$$

In this case the above datum is equivalent to a map $f\colon \mathcal{C} \to \mathbb{P}^{r-1}$ together with an isomorphism $\phi\colon \mathcal{L}^{\otimes d} \cong \omega_{\log,\mathcal{C}} \otimes f^*\mathcal{O}(-1)$. This coincides with the FJRW theory constructed in §1.2.

1.4 Givental's formalism and computation of I-functions

In this section, we explain how the genus-0 GW/FJRW theory introduced above can be encoded by a $H_{\mathrm{GW}/\mathrm{FJRW}}$ valued function $I_{\mathrm{GW}/\mathrm{FJRW}}$. This is a standard procedure in GW theory (see [28] for example); the argument for FJRW theory can be found in [13].

The following construction works for both GW and FJRW theories. In this section, this construction is applied to the the following four special cases:

— the GW theory of $X_{3,3}$ restricted to the ambient part;

— the GW theory of $\mathbb{P}^5$;

— the FJRW theory of the case of interest (restricted to the narrow part) introduced in §1.2;

— the GW theory of $\mathbb{P}^1$.

Let H be the state space of one of the above theories. Introduce the vector space of infinite dimension

$$\mathcal{H} = H \otimes \mathbb{C}((z^{-1}))$$

of H-valued Laurent series in z^{-1}. We define a symplectic form on $\mathcal{H}$ by

$$\Omega(f, g) = \mathrm{Res}_{z=0}\left(f(-z), g(z)\right),$$

where $(\cdot, \cdot)$ is the Poincaré paring on H. In this way $\mathcal{H}$ is decomposed as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

with $\mathcal{H}_+ = H \otimes \mathbb{C}[z]$ and $\mathcal{H}_- = z^{-1}H \otimes \mathbb{C}[[z^{-1}]]$. Then $\mathcal{H}$ can be regarded as the total cotangent space of $\mathcal{H}_+$. An element of $\mathcal{H}$ can be expressed in Darboux coordinates $\{q_k^\alpha, p_{l,\beta}\}$ as

$$\sum_{\substack{\alpha \\ k\geq 0}} q_k^\alpha \phi_\alpha z^k + \sum_{\substack{\beta \\ l\geq 0}} p_{l,\beta}\phi^\beta(-z)^{-l-1},$$

where $\{\phi_\alpha\}$ is a bass for H and $\{\phi^\beta\}$ is its dual bass under Poincaré duality. Set

$$\boldsymbol{q} = \sum_{\substack{\alpha \\ k\geq 0}} q_k^\alpha \phi_\alpha z^k;$$

we regard the genus-0 generating function $\mathcal{F}^0$ (see (1.5), (1.12), $\mathcal{F}^0$ of the GW theories of $\mathbb{P}^1$ and $\mathbb{P}^5$ can be defined in the same way) as a function on $\mathcal{H}_+$ after a *dilaton shift* relating $\boldsymbol{q}$ and $\boldsymbol{t}$ in $\mathcal{H}_+ = H \otimes \mathbb{C}[z]$. In the GW case, the dilaton shift is

$$\boldsymbol{q} = \boldsymbol{t} - \mathbb{1}z,$$

where $\mathbb{1}$ denotes the constant function in H^0; in the FJRW case, the dilaton shift is

$$\boldsymbol{q} = \boldsymbol{t} - \mathbb{1}^{(1)}z,$$

where $\mathbb{1}^{(1)}$ denotes the constant function from the first summand of the state space (see (1.9)). In this way, the genus-0 theory is encoded by a Lagrangian cone

$$\mathcal{L} = \{(\boldsymbol{p}, \boldsymbol{q}) \colon \boldsymbol{p} = d_{\boldsymbol{q}}F^0\} \subset T^*\mathcal{H}_+ \cong \mathcal{H}.$$

At every point $f \in \mathcal{L}$, the tangent space $T_f\mathcal{L}$ satisfies Givental's geometric condition [17, 13]

$$zT_f\mathcal{L} = \mathcal{L} \cap T_f\mathcal{L}.$$

Therefore $\mathcal{L}$ is ruled by a family of subspaces

$$\{zT \colon T \text{ is a tangent space to } \mathcal{L}\}.$$

Let the J-function J be the $\mathcal{H}$-valued function of $\tau \in H$ defined by

$$J(\tau, -z) = -z + \tau + \sum_{n,d,\alpha} \frac{Q^d}{n!} \left\langle \tau, \ldots, \tau, \frac{\phi_\alpha}{-z - \psi_{n+1}} \right\rangle_{0,n+1,d} \phi^\alpha \in -z + \tau + \mathcal{H}_-,$$

where ϕ_α ranges over a basis for H with dual basis ϕ^α, and $\frac{\phi_\alpha}{-z-\psi_{n+1}}$ should be expanded as Laurent series in z, *i.e.*

$$\frac{\phi_\alpha}{-z - \psi_{n+1}} = \phi_\alpha \cdot \left(-\frac{1}{z} + \frac{\psi_{n+1}}{z^2} - \frac{\psi_{n+1}^2}{z^3} + \ldots \right).$$

The J-function can be interpreted as the intersection of $\mathcal{L}$ with the hyperplane $\{-z+\tau+\mathcal{H}_-\}$. According to [17], the partial derivatives of $J(\tau, -z)$ in directions in H generate the tangent space $T_{J(\tau,-z)}\mathcal{L}$; also, the cone $\mathcal{L}$ is ruled by the family of subspaces

$$\{zT_{J(\tau,-z)}\mathcal{L} \colon \tau \in H\}.$$

In this sense, the J-function $J(\tau, -z)$ determines the cone $\mathcal{L}$.

Twisted theory

The genus-0 GW theory of $\mathbb{P}^5$ well-known. We can compute the genus-0 GW theory for $X_{3,3}$ via its relation to the genus-0 GW theory of $\mathbb{P}^5$ (see (1.4)). Similarly, we can compute the FJRW theory of the case of interest via its relation to the genus-0 GW theory of $\mathbb{P}^1$ and (see (1.11)).

Recall that in §1.1 the correlators of genus-0 GW theory for $X_{3,3}$ restricted to the ambient part can be written as

$$\langle \phi_1\psi_1^{a_1},\dots,\phi_{n-1}\psi_{n-1}^{a_{n-1}},\phi_n\psi_n^{a_n}\rangle_{0,n,d}^{\mathrm{GW},X_{3,3}} = \int_{i^*[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^5,d)]} c_{\mathrm{top}}\left((\pi_*\operatorname{ev}^*\mathcal{O}_{\mathbb{P}^5}(3))^{\oplus 2}\right)\prod_{s=1}^{n}(\psi_s^{a_s}\operatorname{ev}_s^*\phi_s), \tag{1.21}$$

where $\pi\colon \overline{\mathcal{C}}_{0,n}(\mathbb{P}^5,d)\to\overline{\mathcal{M}}_{0,n}(\mathbb{P}^5,d)$ is the universal curve, and $\operatorname{ev}\colon\overline{\mathcal{C}}_{0,n}(\mathbb{P}^5,d)\to\mathbb{P}^5$ is the evaluation map.

We can define a new theory by replacing the top Chern class in (1.21) with any multiplicative characteristic class. In particular, if we replace the top Chern class with trivial characteristic class, which is identical $\mathbb{1}$, we obtain the Gromov–Witten theory of $\mathbb{P}^5$.

By applying Grothendieck–Riemann–Roth formula to $\pi\colon\overline{\mathcal{C}}_{0,n}(\mathbb{P}^5,d)\to\overline{\mathcal{M}}_{0,n}(\mathbb{P}^5,d)$, we compute the Chern characters of $\pi_*\operatorname{ev}^*\mathcal{O}_{\mathbb{P}^5}(3)$ and find a modification I_{GW} of the J-function of the GW theory of $\mathbb{P}^5$, which lies on the Lagrangian cone of the GW theory of $X_{3,3}$! Similar to the J function, due to Givental's geometric condition, the function I_{GW} also determines the whole Lagrangian cone of the GW theory of $X_{3,3}$ restricted to the ambient part.

Remark 1.12. In fact we need to work via the equivariant cohomology. We take the multiplicative characteristic class to be the equivariant top Chern class and take nonequivariant limit at last. See [17] for details.

The same story happens for the genus-0 FJRW theory of the case of interest. In fact, its correlators can be written as

$$\langle \phi_1\psi_1^{a_1},\dots,\phi_{n-1}\psi_{n-1}^{a_{n-1}},\phi_n\psi_n^{a_n}\rangle_{0,n,\beta}^{\mathrm{FJRW}} = 3\int_{\rho^*[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,\beta)]} c_{\mathrm{top}}\left((R^1\pi_*(\mathcal{T}^{\oplus 6}))^\vee\right)\prod_{s=1}^{n}(\psi_s^{a_s}\operatorname{ev}_s^*\phi_s). \tag{1.22}$$

If we replace the top Chern class with trivial characteristic class, we get the GW theory of $\mathbb{P}^1$ essentially (the factor 3 is canceled since it is the inverse of the degree of ρ, see Remark 1.7). Then we can find a modification I_{FJRW} of the J-function of the GW theory of $\mathbb{P}^1$. The whole Lagrangian cone of the FJRW theory of the case of interest is determined by I_{FJRW}.

Furthermore, in the both genus-0 GW theory of $X_{3,3}$ and the genus-0 FJRW of the case of interest, the I-function is determined by its restriction to the degree-2 part of the state space

H. This is because in these two cases, the corrector

$$\langle \phi_1\psi_1^{a_1}, \dots, \phi_{n-1}\psi_{n-1}^{a_{n-1}}, \phi_n\psi_n^{a_n}\rangle_{0,n,d}$$

vanishes unless

$$\sum_{i=1}^{n} \deg \phi_i + 2\sum_{i=1}^{n} a_i = 2n$$

for the degree reason. By applying string equation and dilaton equation repeatedly (see [2]), we can show that all the correctors can be deduced from the correctors of the form

$$\langle \phi_1, \dots, \phi_{n-1}, \phi_n\rangle_{0,n,d}$$

where $\phi_1, \dots, \phi_n$ come from the degree-2 part of the state space. In this way all information of these two theories is encoded by their I-functions restricted to the degree-2 part of their state space.

For the GW theory of $X_{3,3}$ restricted to the ambient part, the I-function I_{GW} was computed in [19]:

$$I_{\mathrm{GW}}(tp, z) = ze^{\frac{tp}{z}} \sum_{n\geq 0} e^{nt} \frac{\left(\prod_{0<b\leq 3n}(3p+bz)\right)^2}{\left(\prod_{0<b\leq n}(p+bz)\right)^6} Q^n,$$

where $p \in H_{\mathrm{GW}}$ is the hyperplane class. We can regard it as a function of $v = e^t$ and set $Q = 1$ (see Remark 1.1), then we rewrite

$$I_{\mathrm{GW}}(v, z) = zv^{\frac{p}{z}} \sum_{n\geq 0} v^n \frac{\left(\prod_{0<b\leq 3n}(3p+bz)\right)^2}{\left(\prod_{0<b\leq n}(p+bz)\right)^6}. \tag{1.23}$$

This is a multivalued function in v taking values in H_{GW}. It is analytic on $|v| < 3^{-6}$.

For the FJRW theory of the case of interest, the I-function I_{FJRW} was computed in [13]:

$$I_{\mathrm{FJRW}}(tH^{(1)}, z) = \sum_{\substack{d\geq 0\\ d\not\equiv -1 \mod 3}} \frac{ze^{(d+1)t+\frac{tH^{(d+1)}}{z}}}{3^{6[\frac{d}{3}]}} \frac{\prod_{\substack{0<b\leq d\\ b\equiv d+1 \mod 3}} (H^{(d+1)}+bz)^6}{\prod_{0<b\leq d}(H^{(d+1)}+bz)^2} \mathbb{1}^{(d+1)} Q^d,$$

where $H^{(h)} \in H_{\mathrm{FJRW}}$, $H^{(h)} = H^{(h \mod 3)}$ if $h \geq 3$. We can regard it as a function of $u = e^t$ and set $Q = 1$, then we rewrite

$$I_{\mathrm{FJRW}}(u, z) = \sum_{\substack{d\geq 0\\ d\not\equiv -1 \mod 3}} \frac{zu^{d+1+\frac{H^{(d+1)}}{z}}}{3^{6[\frac{d}{3}]}} \frac{\prod_{\substack{0<b\leq d\\ b\equiv d+1 \mod 3}} (H^{(d+1)}+bz)^6}{\prod_{0<b\leq d}(H^{(d+1)}+bz)^2} \mathbb{1}^{(d+1)}. \tag{1.24}$$

This is a multivalued function in u taking values in H_{FJRW}. It is analytic on $|u| < 3^2$.

1.5 Picard–Fuchs equation

Let V_λ be given in $\mathbb{P}^5$ by

$$Q_1 = x_1^3 + x_2^3 + x_3^3 - 3\lambda x_4 x_5 x_6 = 0$$
$$Q_2 = x_4^3 + x_5^3 + x_6^3 - 3\lambda x_1 x_2 x_3 = 0$$

This is a Calabi–Yau complete intersection of dimension 3 for $\lambda \in \mathbb{C}\backslash\{0, e^{\frac{2k\pi i}{6}}, k = 0, 1, \ldots, 5\}$. Let G_{81} be a group with elements of form $g_{\alpha,\beta,\delta,\epsilon,\mu}$, where $\alpha, \beta, \delta, \epsilon \in \mathbb{Z}_3$, $\mu \in \mathbb{Z}_9$, and $3\mu = \alpha + \beta = \delta + \epsilon \mod 3$. The action of G_{81} on $\mathbb{P}^5$ is given by

$$g_{\alpha,\beta,\delta,\epsilon,\mu} \colon (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (\zeta_3^\alpha \zeta_9^\mu x_1, \zeta_3^\beta \zeta_9^\mu x_2, \zeta_9^\mu x_3, \zeta_9^{-\delta} \zeta_9^{-\mu} x_4, \zeta_3^{-\epsilon} \zeta_9^{-\mu} x_5, \zeta_9^{-\mu} x_6),$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$. Note that V_λ is preserved by the G_{81} action. In [37], Libgober deduce the Picard–Fuchs equation of the family $[V_\lambda/G_{81}]$. Under the change of variable $v = 3^{-6}\lambda^{-6}$ and set

$$\Theta = v\frac{d}{dv},$$

the Picard–Fuchs equation is

$$\left(\Theta^4 - 3^6 v \left(\Theta + \frac{1}{3}\right)^2 \left(\Theta + \frac{2}{3}\right)^2\right) F = 0. \tag{1.25}$$

Clader showed in [13] that $I_{GW}(v, z)$ satisfies equation (1.25); moreover, $I_{\mathrm{FJRW}}(u, z)$ satisfies equation (1.25) after a change of variables $v = u^{-3}$. Since equation (1.25) is a degree-4 differential equation, and $I_{GW}, I_{\mathrm{FJRW}}$ are vector valued functions taking values in 4-dimensional vector spaces, Clader deduced the following result.

Theorem 1.13 (Clader [13]). *There is a $\mathbb{C}[z, z^{-1}]$-valued degree-preserving linear transformation mapping I_{FJRW} to the analytic continuation of I_{GW}.*

Clader only showed the existence of such linear map. In the next section, we will simplify the I-functions, and get a family of explicit $\mathbb{C}$-valued linear maps relating the simplified I-functions by a different method. This will allow us to relate these linear maps to equivalences of certain categories in §4.

Chapter 2

Analytic continuation

In this chapter, we introduce the $\mathfrak{H}$-functions, which are constant linear transform of the I-functions. Then we compute the analytic continuation of $\mathfrak{H}_{\mathrm{GW}}$ and compare it with $\mathfrak{H}_{\mathrm{FJRW}}$. In this way we find a linear map

$$\mathbb{U}\colon H_{\mathrm{FJRW}} \to H_{\mathrm{GW}},$$

which identifies $\mathfrak{H}_{\mathrm{FJRW}}$ with the analytic continuation $\mathfrak{H}_{\mathrm{GW}}$.

2.1 Gradings on the state spaces

To define the $\mathfrak{H}$-functions, we need two different gradings on the state spaces H_{FJRW} and H_{GW}.

The first grading Gr is the standard grading inherited from the Chen–Ruan cohomology groups in (1.13). More precisely, the standard grading Gr on H_{FJRW} is given by

$$\mathrm{Gr}(\mathbb{1}^{(1)}) = 0, \mathrm{Gr}(H^{(1)}) = 2, \mathrm{Gr}(\mathbb{1}^{(2)}) = 4, \mathrm{Gr}(H^{(2)}) = 6;$$

the standard grading Gr on H_{GW} is given by

$$\mathrm{Gr}(p^n) = 2n.$$

The second grading $\deg_0$ is called *bare degree* in [9]. It is the degree without age-shift in the definition of Chen–Ruan cohomology (see §0.4). The bare degree $\deg_0$ on H_{FJRW} is given by

$$\deg_0(\mathbb{1}^{(1)}) = \deg_0(\mathbb{1}^{(2)}) = -4, \deg_0(H^{(1)}) = \deg_0(H^{(2)}) = -2;$$

the bare degree $\deg_0$ on H_{GW} is given by

$$\deg_0(p^n) = 2n.$$

Remark 2.1. The bare degree on H_{GW} agrees with the standard grading since $X_{3,3}$ is a smooth scheme and the age-shift number is zero.

2.2 The $\mathfrak{H}$-functions

We introduce the $\mathfrak{H}$-functions as in [9]. In both GW theory and FJRW theory, the $\mathfrak{H}$-function is defined by the formula

$$I = z^{-\frac{\mathrm{Gr}}{2}} \left(\Gamma \cdot (2\pi i)^{\frac{\deg_0}{2}} \mathfrak{H} \right), \tag{2.1}$$

where I, Gr and $\deg_0$ are the I-function, standard grading and bare degree in the corresponding theory; Γ is a chosen class in the corresponding state space. The notation $(2\pi i)^{\frac{\deg_0}{2}}$ represents the linear endomorphism of the state space given by

$$\alpha \mapsto (2\pi i)^{\frac{\deg_0(\alpha)}{2}} \cdot \alpha.$$

The notation $z^{-\frac{\mathrm{Gr}}{2}}$ is defined similarly.

2.2.1 Computation of $\mathfrak{H}_{\mathrm{GW}}$

The class Γ_{GW} is chosen to be the Gamma class (see §0.4) of the tangent bundle of $X_{3,3}$. Using the exact sequence

$$0 \to i^*\mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2} \to i^*\Omega_{\mathbb{P}^5} \to \Omega_{X_{3,3}} \to 0$$

and the Euler sequence

$$0 \to \Omega_{\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^5} \to 0,$$

we get

$$\Gamma_{\mathrm{GW}} = \frac{\Gamma(1+p)^6}{\Gamma(1+3p)^2}.$$

We rewrite (1.23) as

$$I_{\mathrm{GW}}(v,z) = \sum_{n\geq 0} z v^{\frac{p}{z}+n} \frac{\Gamma(\frac{p}{z}+1)^6 \Gamma(\frac{3p}{z}+3n+1)^2}{\Gamma(\frac{3p}{z}+1)^2 \Gamma(\frac{p}{z}+n+1)^6}.$$

Then we have

$$\begin{aligned} I_{\mathrm{GW}}(v,z) &= z^{-\frac{\mathrm{Gr}}{2}} \sum_{n\geq 0} z v^{p+n} \frac{\Gamma(p+1)^6}{\Gamma(3p+1)^2} \frac{\Gamma(3p+3n+1)^2}{\Gamma(p+n+1)^6} \\ &= z^{-\frac{\mathrm{Gr}}{2}} \left(\Gamma_{\mathrm{GW}} \cdot (2\pi i)^{\frac{\deg_0}{2}} \sum_{d\geq 0} z v^{\frac{p}{2\pi i}+n} \frac{\Gamma(3\frac{p}{2\pi i}+3n+1)^2}{\Gamma(\frac{p}{2\pi i}+n+1)^6} \right). \end{aligned} \tag{2.2}$$

Compare (2.2) with (2.1), we have

$$\mathfrak{H}_{\mathrm{GW}}(v,z) = \sum_{n\geq 0} z v^{\frac{p}{2\pi i}+n} \frac{\Gamma(3\frac{p}{2\pi i}+3n+1)^2}{\Gamma(\frac{p}{2\pi i}+n+1)^6}. \tag{2.3}$$

2.2.2 Computation of $\mathfrak{H}_{\mathrm{FJRW}}$

The class Γ_{FJRW} is chosen to be the narrow part of the Gamma class of the tangent bundle of X_-, which is

$$\Gamma_{\mathrm{FJRW}} = \Gamma\left(\frac{2}{3} - \frac{H^{(1)}}{3}\right)^6 \Gamma(1+H^{(1)})^2 \mathbb{1}^{(1)} + \Gamma\left(\frac{1}{3} - \frac{H^{(2)}}{3}\right)^6 \Gamma(1+H^{(2)})^2 \mathbb{1}^{(2)}. \tag{2.4}$$

Remark 2.2. Note that there is a sign issue; see Remark 0.12. In fact, although we can identify X_- with total space of $\mathcal{O}(-1)^{\oplus 6}$ over $\mathbb{P}(3,3)$, there is a change of sign. Under this identification, the line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(3,3)$ has eigenvalue $e^{\frac{2\pi i}{3}}$ instead of $e^{\frac{4\pi i}{3}}$ with respect to the stabilizer $e^{\frac{2\pi i}{3}} \in \mathbb{C}^*$.

We can rewrite (1.24) as

$$\begin{aligned}
I_{\mathrm{FJRW}}(u,z) =& z \sum_{\substack{d\geq 0\\ d\not\equiv -1 \mod 3}} u^{d+1+\frac{H^{(d+1)}}{z}} z^{-6\langle \frac{d}{3}\rangle} \frac{\Gamma(\frac{H^{(d+1)}}{3z}+\frac{d}{3}+\frac{1}{3})^6 \Gamma(\frac{H^{(d+1)}}{z}+1)^2}{\Gamma(\frac{H^{(d+1)}}{3z}+\langle\frac{d}{3}\rangle+\frac{1}{3})^6 \Gamma(\frac{H^{(d+1)}}{z}+d+1)^2} \mathbb{1}^{(d+1)} \\
=& z \sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} u^{d+1+\frac{H^{(1)}}{z}} \frac{\Gamma(\frac{H^{(1)}}{3z}+\frac{d}{3}+\frac{1}{3})^6 \Gamma(\frac{H^{(1)}}{z}+1)^2}{\Gamma(\frac{H^{(1)}}{3z}+\frac{1}{3})^6 \Gamma(\frac{H^{(1)}}{z}+d+1)^2} \mathbb{1}^{(1)} \\
&+ z \sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} u^{d+1+\frac{H^{(2)}}{z}} z^{-2} \frac{\Gamma(\frac{H^{(2)}}{3z}+\frac{d}{3}+\frac{2}{3})^6 \Gamma(\frac{H^{(2)}}{z}+1)^2}{\Gamma(\frac{H^{(2)}}{3z}+\frac{1}{3})^6 \Gamma(\frac{H^{(2)}}{z}+d+1)^2} \mathbb{1}^{(2)}.
\end{aligned} \tag{2.5}$$

By (2.1) we have

$$\begin{aligned}
(2\pi i)^{-2}\mathfrak{H}_{\mathrm{FJRW}}(u,z) =& z \sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} u^{d+1+\frac{H^{(1)}}{2\pi i}} \frac{\Gamma(1+\frac{H^{(1)}}{2\pi i})^2}{\Gamma(1+\frac{H^{(1)}}{2\pi i})^2} \\
&\cdot \frac{\Gamma(\frac{d}{3}+\frac{1}{3}+\frac{1}{3}\frac{H^{(1)}}{2\pi i})^6}{\Gamma(\frac{1}{3}+\frac{1}{3}\frac{H^{(1)}}{2\pi i})^6 \Gamma(\frac{2}{3}-\frac{1}{3}\frac{H^{(1)}}{2\pi i})^6 \Gamma(d+1+\frac{H^{(1)}}{2\pi i})^2} \mathbb{1}^{(1)} \\
&+ z \sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} u^{d+1+\frac{H^{(2)}}{2\pi i}} \frac{\Gamma(1+\frac{H^{(2)}}{2\pi i})^2}{\Gamma(1+\frac{H^{(2)}}{2\pi i})^2} \\
&\cdot \frac{\Gamma(\frac{d}{3}+\frac{1}{3}+\frac{1}{3}\frac{H^{(2)}}{2\pi i})^6}{\Gamma(\frac{2}{3}+\frac{1}{3}\frac{H^{(2)}}{2\pi i})^6 \Gamma(\frac{1}{3}-\frac{1}{3}\frac{H^{(2)}}{2\pi i})^6 \Gamma(d+1+\frac{H^{(2)}}{2\pi i})^2} \mathbb{1}^{(2)} \\
=& z \sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} u^{d+1+\frac{H^{(1)}}{2\pi i}} \frac{\Gamma(\frac{d}{3}+\frac{1}{3}+\frac{1}{3}\frac{H^{(1)}}{2\pi i})^6}{\Gamma(\frac{1}{3}+\frac{1}{3}\frac{H^{(1)}}{2\pi i})^6 \Gamma(\frac{2}{3}-\frac{1}{3}\frac{H^{(1)}}{2\pi i})^6 \Gamma(d+1+\frac{H^{(1)}}{2\pi i})^2} \mathbb{1}^{(1)} \\
&+ z \sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} u^{d+1+\frac{H^{(2)}}{2\pi i}} \frac{\Gamma(\frac{d}{3}+\frac{1}{3}+\frac{1}{3}\frac{H^{(2)}}{2\pi i})^6}{\Gamma(\frac{2}{3}+\frac{1}{3}\frac{H^{(2)}}{2\pi i})^6 \Gamma(\frac{1}{3}-\frac{1}{3}\frac{H^{(2)}}{2\pi i})^6 \Gamma(d+1+\frac{H^{(2)}}{2\pi i})^2} \mathbb{1}^{(2)}.
\end{aligned} \tag{2.6}$$

2.3 Linear maps relating the $\mathfrak{H}$-functions

We can regard $\mathfrak{H}_{\mathrm{GW}}$ as a function of $\log v$ by writing v as $e^{\log v}$. Then $\mathfrak{H}_{\mathrm{GW}}$ is analytic on $\Re(\log v) < -6\log 3$. In the same way we can regard $\mathfrak{H}_{\mathrm{FJRW}}$ as a function of $\log u$. Then $\mathfrak{H}_{\mathrm{FJRW}}$ is analytic on $\Re(\log v) > -6\log 3$ after a change of variable $\log v = -3\log u$. We can extend $\mathfrak{H}_{\mathrm{GW}}$ analytically to the right side of the line $\Re(\log v) = -6\log 3$ along a path passing through the window w_l as in figure 2.1, and compare it with $\mathfrak{H}_{\mathrm{FJRW}}$. In fact, they are related by the following linear maps.

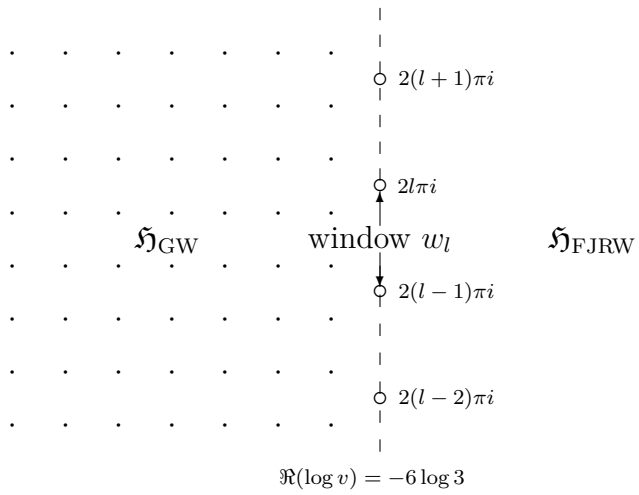

Figure 2.1 – The $(\log v)$-plane.

Definition 2.3. For each $l \in \mathbb{Z}$, the linear map $\mathbb{U}_l \colon H_{\mathrm{FJRW}} \to H_{\mathrm{GW}}$ is defined by

$$\begin{aligned}
\mathbb{1}^{(1)} &\mapsto \frac{l}{9}\frac{(\zeta e^p)^l}{1-\zeta e^p} + \frac{1}{9}\frac{(\zeta e^p)^{l+1}}{(1-\zeta e^p)^2} \\
H^{(1)} &\mapsto \frac{1}{3}\frac{(\zeta e^p)^l}{1-\zeta e^p} \\
\mathbb{1}^{(2)} &\mapsto \frac{l}{9}\frac{(\zeta^2 e^p)^l}{1-\zeta^2 e^p} + \frac{1}{9}\frac{(\zeta^2 e^p)^{l+1}}{(1-\zeta^2 e^p)^2} \\
H^{(2)} &\mapsto \frac{1}{3}\frac{(\zeta^2 e^p)^l}{1-\zeta^2 e^p}
\end{aligned} \tag{2.7}$$

where $\zeta = e^{\frac{2\pi i}{3}}$.

Remark 2.4. The right-hand side of (2.7) should be understood as elements of H_{GW} in the following way. We regard p as a small complex number and expand the right-hand side of (2.7) at $p = 0$. Then we set $p^n = 0$ for $n \geq 4$. Or equivalently, we only take the first four terms in the Taylor expansion. The remaining part is a linear combination of $1, p, p^2$ and p^3; it can be regarded as an element of H_{GW}.

Theorem 2.5. *For every $l \in \mathbb{Z}$, $\mathbb{U}_l(\mathfrak{H}_{\mathrm{FJRW}}(u,z))$ coincides with the analytic continuation of $\mathfrak{H}_{\mathrm{GW}}(v,z)$ along a path passing through the window w_l after the change of variable*

$$\log v = -3 \log u.$$

Remark 2.6. We can write down the explicit linear map in Theorem 1.13 if we recover the I-functions from the $\mathfrak{H}$-functions by (2.1).

Remark 2.7. We regard e^p as complex number in the unit disk. By using

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^3 + \dots,$$

and adding formal elements $\mathbb{1}^{(0)}$ and $H^{(0)}$, we can rewrite (2.7) as

$$\begin{aligned}
\begin{pmatrix} \mathbb{1}^{(0)} & \mathbb{1}^{(1)} & \mathbb{1}^{(2)} \end{pmatrix} &\mapsto \frac{1}{9} e^{pl} \begin{pmatrix} 1 & e^p & e^{2p} & e^{3p} \dots \end{pmatrix} \begin{pmatrix} l & l & l \\ l+1 & (l+1)\zeta & (l+1)\zeta^2 \\ l+2 & (l+2)\zeta^2 & (l+2)\zeta \\ l+3 & l+3 & l+3 \\ l+4 & (l+4)\zeta & (l+4)\zeta^2 \\ l+5 & (l+5)\zeta^2 & (l+5)\zeta \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & & \\ & \zeta^l & \\ & & \zeta^{2l} \end{pmatrix} \\
\begin{pmatrix} H^{(0)} & H^{(1)} & H^{(2)} \end{pmatrix} &\mapsto \frac{1}{3} e^{pl} \begin{pmatrix} 1 & e^p & e^{2p} & e^{3p} \dots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \\ 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & & \\ & \zeta^l & \\ & & \zeta^{2l} \end{pmatrix}.
\end{aligned} \tag{2.8}$$

When we regard the right-hand side of (2.7) as functions of a complex number p, the two expressions (2.7) and (2.8) coincide in the region

$$\Omega = \{p \in \mathbb{C} \colon\ |e^p| < 1\}.$$

The origin $p = 0$ of the complex plane is contained in the closure of Ω; the right hand side of (2.7) are locally bounded near $p = 0$. This allows us to take the Taylor expansion of the right-hand side of (2.8) (except the image of $\mathbb{1}^{(0)}$ and $H^{(0)}$) at $p = 0$ and regard them as elements of H_{GW} (see Remark 2.4).

Note that the above argument does not work for the formally added elements $\mathbb{1}^{(0)}$ and $H^{(0)}$. This does not matter because they are formal and will always produce 0-terms in practice (see the proof of Proposition 4.7 for example).

Proof of Theorem 2.5. For $l \in \mathbb{Z}$, consider the function

$$F_l(s) = z e^{(\frac{p}{2\pi i}+s)\log v} \cdot \frac{\Gamma(3\frac{p}{2\pi i}+3s+1)^2}{\Gamma(\frac{p}{2\pi i}+s+1)^6} \cdot \frac{\pi}{\sin(\pi s)} \cdot e^{-(2l-1)\pi i s}. \tag{2.9}$$

The poles of $F_l(s)$ are of the forms

$$s = k \in \mathbb{Z}$$

or

$$3\frac{p}{2\pi i} + 3s + 1 = -d \in \mathbb{Z}^{\leq 0}, \quad d \equiv 0, 1 \mod 3.$$

They are represented by the black dots in figure 2.2. Consider the contour integral $\int_C F_l(s)ds$

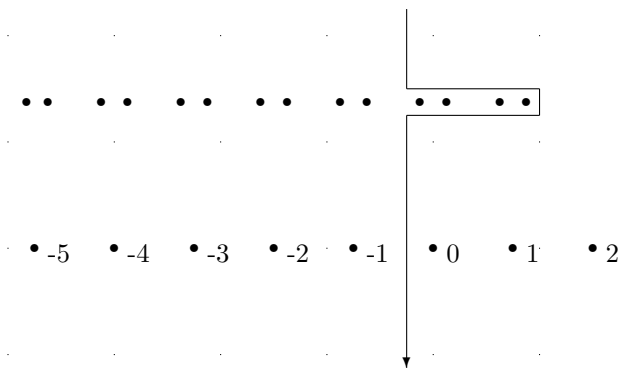

Figure 2.2 – The s-plane.

along the path of figure 2.2. According to Lemma 3.3 in [30], the integral is absolutely convergent (and defines an analytic function of v) if

$$|\Im(\log v) - (2l-1)\pi| < \pi.$$

Moreover, the integral is equal to the sum of of the residues on the right of the contour for $\Re(\log v) < -6\log 3$, and to the opposite of the sum of the residues on the left of the contour for $\Re(\log v) > -6\log 3$.

Near the poles $s = k \in \mathbb{Z}$ we have

$$\frac{\pi}{\sin(\pi s)} \cdot e^{-(2l-1)\pi i s} = \frac{1}{s-k} + O(1),$$

therefore

$$\begin{aligned}\mathfrak{H}_{\mathrm{GW}}(v,z) &= \sum_{n\geq 0} z v^{\frac{p}{2\pi i}+n} \frac{\Gamma(3\frac{p}{2\pi i}+3n+1)^2}{\Gamma(\frac{p}{2\pi i}+n+1)^6} \\ &= \sum_{n\geq 0} \mathrm{Res}_{s=n} F_l(s)ds \\ &= \int_C F_l(s)ds\end{aligned} \tag{2.10}$$

for $\Re(\log v) < -6\log 3$. Then the opposite of the sum of the residues on the left of the contour gives the analytic continuation of $\mathfrak{H}_{\mathrm{GW}}$ along a path passing through the windows w_l.

In order to compute the residues, we introduce ψ, the logarithmic derivative of the gamma function. It is often called the digamma function, and defined by

$$\psi(z) = \frac{d}{dz}\log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Near the nonpositive integer $-k$ we have the Laurent expansion

$$\Gamma(z) = \frac{(-1)^k}{k!}\left(\frac{1}{z+k} + \psi(k+1)\right) + O(z+k). \tag{2.11}$$

Thus for a negative integer poles $s = n < 0$,

$$\mathrm{Res}_{s=n} F_l(s)ds = z v^{\frac{p}{2\pi i}+n} \frac{\Gamma(3\frac{p}{2\pi i}+3n+1)^2}{\Gamma(\frac{p}{2\pi i}+n+1)^6} = 0$$

since

$$p^4 \Big| \frac{\Gamma(3\frac{p}{2\pi i}+3n+1)^2}{\Gamma(\frac{p}{2\pi i}+n+1)^6}$$

and $p^4 = 0$ in H_{GW}.

The other poles of $F_l(s)ds$ are of the form $3\frac{p}{2\pi i} + 3s + 1 = -d$ for $d \geq 0$, $d \equiv 0, 1 \mod 3$. We calculate the residue at these poles. Near $s = -\frac{p}{2\pi i} - \frac{d}{3} - \frac{1}{3}$, set $s = \Delta s + \frac{p}{2\pi i} + \frac{d}{3} + \frac{1}{3}$, we have

$$\begin{aligned}F_l(s) =& z v^{-\frac{d}{3}-\frac{1}{3}}\left(1 + (\log v)\Delta s + O((\Delta s)^2)\right) \\ &\cdot \frac{1}{(d!)^2}\left(\frac{1}{9}\frac{1}{(\Delta s)^2} + \frac{2\psi(d+1)}{3}\frac{1}{\Delta s} + O(1)\right) \\ &\cdot \frac{1}{\Gamma(-\frac{d}{3}+\frac{2}{3})^6}\left(1 - 6\psi(-\frac{d}{3}+\frac{2}{3})\Delta s + O((\Delta s)^2)\right) \\ &\cdot \frac{\pi}{\sin^2(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi}\left(-\sin(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi - \pi\cos(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi\Delta s + O((\Delta s)^2)\right) \\ &\cdot e^{(2l-1)\pi i(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})}\left(1 - (2l-1)\pi i\Delta s + O((\Delta s)^2)\right).\end{aligned} \tag{2.12}$$

Thus

$$\begin{aligned}
\mathrm{Res}_{s=-\frac{p}{2\pi i}-\frac{d}{3}-\frac{1}{3}} F_l =& zv^{-\frac{d}{3}-\frac{1}{3}} \frac{1}{(d!)^2} \frac{1}{\Gamma(-\frac{d}{3}+\frac{2}{3})^6} \frac{\pi}{\sin^2(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi} e^{(2l-1)\pi i(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})} \\
&\cdot \left[\sin(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi \cdot \left(\frac{2}{3}\psi(-\frac{d}{3}+\frac{2}{3}) - \frac{2}{3}\psi(d+1) - \frac{1}{9}\log v + \frac{2l-1}{9}\pi i\right)\right. \\
&\left. - \cos(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi \cdot \frac{\pi}{9}\right] \\
=& zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2} \frac{\left(e^{p+(\frac{d}{3}+\frac{1}{3})2\pi i}\right)^l}{e^{p+(\frac{d}{3}+\frac{1}{3})2\pi i}-1} \\
&\cdot \frac{1}{3}\left(2\psi(\frac{d}{3}+\frac{1}{3}) - 2\psi(d+1) + 2\psi(-\frac{d}{3}+\frac{2}{3}) - 2\psi(\frac{d}{3}+\frac{1}{3}) - \frac{1}{3}\log v\right) \\
&+ zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)^2\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2} \frac{\left(e^{p+(\frac{d}{3}+\frac{1}{3})2\pi i}\right)^l}{e^{p+(\frac{d}{3}+\frac{1}{3})2\pi i}-1} \cdot \frac{l}{9} \\
&- zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)^2\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2} \frac{\left(e^{p+(\frac{d}{3}+\frac{1}{3})2\pi i}\right)^{l+1}}{\left(e^{p+(\frac{d}{3}+\frac{1}{3})2\pi i}-1\right)^2} \cdot \frac{1}{9}.
\end{aligned} \tag{2.13}$$

We used

$$\Gamma(-\frac{d}{3}+\frac{2}{3})^6\Gamma(\frac{d}{3}+\frac{1}{3})^6 = \left(\frac{\pi}{\sin(\frac{d}{3}+\frac{1}{3})\pi}\right)^6 = \left(\frac{2\pi}{\sqrt{3}}\right)^6$$

and

$$\sin(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi = \frac{e^{(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})2\pi i}-1}{2ie^{(\frac{p}{2\pi i}+\frac{d}{3}+\frac{1}{3})\pi i}}.$$

Then we get the analytic continuation of $\mathfrak{H}_{\mathrm{GW}}$ along a path passing through the windows w_l, which is

$$\begin{aligned}
&\sum_{\substack{d\geq 0\\ d\not\equiv -1 \mod 3}} -\mathrm{Res}_{s=-\frac{p}{2\pi i}-\frac{d}{3}-\frac{1}{3}} F_l(s) \\
=& \sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2} \frac{\left(e^{p+\frac{2\pi i}{3}}\right)^l}{1-e^{p+\frac{2\pi i}{3}}} \\
&\cdot \frac{1}{3}\left(2\psi(\frac{d}{3}+\frac{1}{3}) - 2\psi(d+1) + 2\psi(\frac{2}{3}) - 2\psi(\frac{1}{3}) - \frac{1}{3}\log v\right) \\
&+ \sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)^2\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2} \\
&\cdot \left(\frac{\left(e^{p+\frac{2\pi i}{3}}\right)^l}{1-e^{p+\frac{2\pi i}{3}}} \cdot \frac{l}{9} + \frac{\left(e^{p+\frac{2\pi i}{3}}\right)^{l+1}}{\left(1-e^{p+\frac{2\pi i}{3}}\right)^2} \cdot \frac{1}{9}\right)
\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&+ \sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2}\frac{\left(e^{p+\frac{4\pi i}{3}}\right)^l}{1-e^{p+\frac{4\pi i}{3}}}\\
&\cdot \frac{1}{3}\left(2\psi(\frac{d}{3}+\frac{1}{3})-2\psi(d+1)+2\psi(\frac{1}{3})-2\psi(\frac{2}{3})-\frac{1}{3}\log v\right)\\
&+ \sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} zv^{-\frac{d}{3}-\frac{1}{3}}(2\pi i)^2\left(\frac{\sqrt{3}}{2\pi}\right)^6 \frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2}\\
&\cdot\left(\frac{\left(e^{p+\frac{4\pi i}{3}}\right)^l}{1-e^{p+\frac{4\pi i}{3}}}\cdot\frac{l}{9}+\frac{\left(e^{p+\frac{4\pi i}{3}}\right)^{l+1}}{\left(1-e^{p+\frac{4\pi i}{3}}\right)^2}\cdot\frac{1}{9}\right),
\end{aligned}
$$

where we used

$$
\psi(-\frac{d}{3}+\frac{2}{3})-\psi(\frac{d}{3}+\frac{1}{3})=\frac{\pi\cos(\frac{d}{3}-\frac{1}{3})\pi}{\sin(\frac{d}{3}+\frac{1}{3})\pi}=\begin{cases}\psi(\frac{2}{3})-\psi(\frac{1}{3}) & d\equiv 0 \mod 3\\ \psi(\frac{1}{3})-\psi(\frac{2}{3}) & d\equiv 1 \mod 3.\end{cases}
$$

On the other hand, we can expand $\mathfrak{H}_{\mathrm{FJRW}}$ with respect to $H^{(1)}, H^{(2)}$ by differentiating (2.6):

$$
\begin{aligned}
\mathfrak{H}_{\mathrm{FJRW}}(u,z) = z &\sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} u^{d+1}\cdot(2\pi i)^2\cdot\left(\frac{\sqrt{3}}{2\pi}\right)^6\frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2}\mathbb{1}^{(1)}\\
&+z\sum_{\substack{d\geq 0\\ d\equiv 0 \mod 3}} u^{d+1}\cdot 2\pi i\cdot\left(\frac{\sqrt{3}}{2\pi}\right)^6\frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2}\\
&\quad\cdot\left(2\psi(\frac{d}{3}+\frac{1}{3})-2\psi(d+1)+2\psi(\frac{2}{3})-2\psi(\frac{1}{3})+\log u\right)H^{(1)}\\
&+z\sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} u^{d+1}\cdot(2\pi i)^2\cdot\left(\frac{\sqrt{3}}{2\pi}\right)^6\frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2}\mathbb{1}^{(2)}\\
&+z\sum_{\substack{d\geq 0\\ d\equiv 1 \mod 3}} u^{d+1}\cdot 2\pi i\cdot\left(\frac{\sqrt{3}}{2\pi}\right)^6\frac{\Gamma(\frac{d}{3}+\frac{1}{3})^6}{\Gamma(d+1)^2}\\
&\quad\cdot\left(2\psi(\frac{d}{3}+\frac{1}{3})-2\psi(d+1)+2\psi(\frac{1}{3})-2\psi(\frac{2}{3})+\log u\right)H^{(2)}.
\end{aligned}
\tag{2.15}
$$

We complete the proof by comparing (2.14) with (2.15). Indeed, we can obtain (2.14) by replacing $\mathbb{1}^{(1)}$, $\mathbb{1}^{(2)}$, $H^{(1)}$ and $H^{(2)}$ in (2.15) with their image under $\mathbb{U}_l$ in Definition 2.3. $\square$

Chapter 3

Orlov-type functors

In this chapter, we introduce the categories of graded matrix factorizations. We introduce two functors from "grade restriction rule". An equivalent functor between the derived category of graded matrix factorizations and the derived category of $X_{3,3}$ is defined as a generalization of Orlov functor.

3.1 Graded matrix factorizations

Definition 3.1. A *Landau–Ginzburg (LG) model* is the datum of a stack X with a $\mathbb{C}^*_R$-action, together with a regular function F on X, where $-1 \in \mathbb{C}^*_R$ acts trivially on X, and F has $\mathbb{C}^*_R$-weight 2, *i.e.* for all $\lambda \in \mathbb{C}^*_R$ and $x \in X$, we have

$$F(\lambda \cdot x) = \lambda^2 F(x).$$

Example 3.2. As in the Example 0.11, we consider a vector space

$$V = \mathbb{C}^8 = \mathrm{Spec}[x_1, \ldots, x_6, p_1, p_2]$$

with a $\mathbb{C}^*$-action of weights $(1, 1, 1, 1, 1, 1, -3, -3)$, then there are two different GIT quotients:

$$X_+ := [(\mathbb{C}^6 \backslash \{\mathbf{0}\}) \times \mathbb{C}^2 / \mathbb{C}^*] = \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2}$$

and

$$X_- := [\mathbb{C}^6 \times (\mathbb{C}^2 \backslash \{\mathbf{0}\}) / \mathbb{C}^*] = \mathcal{O}_{\mathbb{P}(3,3)}(-1)^{\oplus 6}.$$

We define a $\mathbb{C}^*_R$-action on V with weights $(0, 0, 0, 0, 0, 0, 2, 2)$, then it induces the $\mathbb{C}^*_R$-action on both X_+ and X_-. Let W_1 and W_2 be two homogeneous polynomials of degree 3 as in §1.2.1, then the function $W := p_1 W_1 + p_2 W_2$ on V is invariant under $\mathbb{C}^*$, thus we can regard W as a function on X_+ and X_-. We get two LG models (X_-, W) and (X_+, W) in this way.

Remark 3.3. The $\mathbb{C}_R^*$-action defined above is different from the one define by (1.19) in §1.3. However, the images of $\mathbb{C}_R^*$ in $GL(V)$ induced by these two actions are the same. These two actions are equivalent in the sense that the theory defined in §1.3 only depends on the groups Γ and G.

Definition 3.4. A *graded matrix factorization* on a LG model (X, F) is a finite rank vector bundle E, equivariant with respect to $\mathbb{C}_R^*$, equipped with an endomorphism d_E of $\mathbb{C}_R^*$-degree 1, *i.e.* for all $\lambda \in \mathbb{C}_R^*$ and all points (x, v) in the total space of E, we have

$$\lambda^{-1} d_E(\lambda \cdot (x, v)) = \lambda \cdot d_E((x, v)),$$

where the λ^{-1} on the left-hand side is a scalar multiplication. We also require d_E satisfy

$$d_E^2 = F \cdot \mathrm{Id}_E\,.$$

We call E the underlying vector bundle of the graded matrix factorization.

A dg-category $\mathcal{MF}^{\mathbb{C}_R^*}(X, F)$ is constructed by Segal [41] and Shipman[42]. Its objects are graded matrix factorizations over (X, F). We define $\mathrm{DMF}^{\mathbb{C}_R^*}(X, F)$ to be the homotopy category of $\mathcal{MF}^{\mathbb{C}_R^*}(X, F)$, which is a trianglated category.

Now we describe how $\mathrm{DMF}^{\mathbb{C}_R^*}(X, F)$ naturally possess the structure of a triangulated category. The shift functor on $\mathrm{DMF}^{\mathbb{C}_R^*}(X, F)$ is given by

$$(E, d)[1] = (E \otimes \mathcal{O}[1], -d \otimes \mathrm{Id}),$$

where $\mathcal{O}[1]$ is the trivial line bundle endowed with a $\mathbb{C}_R^*$-action of weight 1 on fiber direction. Let $f\colon E_1 \to E_2$ be a $\mathbb{C}_R^*$-equivariant morphism that intertwines the differentials, then we define the cone by

$$\mathrm{cone}(f\colon E_1 \to E_2) = C_f := \left(E_1[1] \oplus E_2, \begin{pmatrix} d_1[1] & 0 \\ f & d_2 \end{pmatrix}\right).$$

A distinguished triangle is a triangle isomorphic to a triangle of the form

$$E_1 \xrightarrow{f} E_2 \to C_f \to E_1[1] \to \dots.$$

Remark 3.5. If we replace the $\mathbb{C}_R^*$-equivariant vector bundle in Definition 3.4 with a $\mathbb{C}_R^*$-equivariant quasicoherent sheaf, we get the an object called *curved graded quasicoherent sheaf* in [42]. All of these objects form a trianglated category $\mathrm{DQcoh}^{\mathbb{C}_R^*}(X, F)$. Shipman [42] realizes $\mathrm{DMF}^{\mathbb{C}_R^*}(X_\pm, W)$ as a full triangulated subcategory of $\mathrm{DQcoh}^{\mathbb{C}_R^*}(X_\pm, W)$.

3.2 Koszul matrix factorizations

Koszul matrix factorizations are important examples of on graded matrix factorizations on $(X_\pm, W)$. Before describing these graded matrix factorizations, we need to introduce line bundles over $X_\pm$.

Consider a $\mathbb{C}^*$-action over $\mathbb{C}^6 \times (\mathbb{C}^2 - \{0\}) \times \mathbb{C}$ with weights $(1,1,1,1,1,1,-3,-3,k)$, together with a $\mathbb{C}^*_R$-action with weights $(0,0,0,0,0,0,2,2,l)$, then $[\mathbb{C}^6 \times (\mathbb{C}^2 - \{0\}) \times \mathbb{C}/\mathbb{C}^*]$ is a $\mathbb{C}^*_R$-equivariant line bundle over X_-. We denote this line bundle by $\mathcal{O}(k)[l]$. We can define $\mathcal{O}(k)[l]$ over X_+ in the same way.

Given a vector bundle E over $X_\pm$, we denote the vector bundle $E \otimes \mathcal{O}(k)[l]$ by $E(k)[l]$. Given a matrix factorization $\mathcal{M} = (E, d)$ over $(X_\pm, W)$, we denote the matrix factorization $(E \otimes \mathcal{O}(k)[l], (-1)^l d \otimes \mathrm{Id})$ by $\mathcal{M}(k)[l]$.

Now we describe the Koszul matrix factorizations $\mathcal{K}_-$ and $\mathcal{K}_+$ over $(X_\pm, W)$.

The underlying $\mathbb{C}^*_R$-equivariant vector bundle of the Koszul matrix factorization $\mathcal{K}_-$ is

$$\bigwedge^\bullet \mathcal{O}(1)[-1]^{\oplus 6}. \tag{3.1}$$

In order to describe its differential, we take

$$f_{ij} := \frac{1}{3}\partial_{x_j} W_i;$$

then $f_{11}, \ldots, f_{16}$ and $f_{21}, \ldots, f_{26}$ are homogeneous polynomials of degree 2 such that

$$W_1 = x_1 f_{11} + \cdots + x_6 f_{16}$$

and

$$W_2 = x_1 f_{21} + \cdots + x_6 f_{26}.$$

Then $s_x := (x_1, \ldots, x_6)$ is a section of $\mathcal{O}(1)[0]^{\oplus 6}$, and $s_{pf} := (p_1 f_{11} + p_2 f_{21}, \ldots, p_1 f_{16} + p_2 f_{26})$ is a cosection of $\mathcal{O}(1)[-2]^{\oplus 6}$. Then the differential of $\mathcal{K}_-$ is given by

$$d_-(-) = s_x \wedge (-) + s_{pf} \vee (-),$$

where $\wedge$ denote the wedge product and $\vee$ denote the contraction.

Similarly, the underlying $\mathbb{C}^*_R$-equivariant vector bundle of the other Koszul matrix factorization $\mathcal{K}_+$ is

$$\bigwedge^\bullet \mathcal{O}(-3)[1]^{\oplus 2}, \tag{3.2}$$

and the differential is given by

$$d_+(-) = s_p \wedge (-) + s_W \vee (-),$$

where $s_p := (p_1, p_2)$ and $s_W := (W_1, W_2)$.

Remark 3.6. Note that $\mathcal{K}_+ = 0$ in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$. This happens because the complex

$$0 \longrightarrow \mathcal{O}(0)[0] \xrightarrow{s_p} \mathcal{O}(-3)[1]^{\oplus 2} \xrightarrow{s_p} \mathcal{O}(-6)[2] \longrightarrow 0$$

is exact on X_- (see Lemma 3.15).

Similarly, we have $\mathcal{K}_- = 0$ in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_+, W)$.

Given a graded matrix factorization (E, d), if E can be written as direct sum of subbundles and d can be written as sum of the zero extension of morphisms between those subbundles, then we can represent (E, d) by a diagram whose vertices are the subbundles, and whose arrows are morphisms between them. For example, we can represent $\mathcal{K}_+$ by the diagram

$$\mathcal{O}(0)[0] \underset{s_p}{\overset{s_W}{\rightleftarrows}} \mathcal{O}(-3)[1]^{\oplus 2} \underset{s_p}{\overset{s_W}{\rightleftarrows}} \mathcal{O}(-6)[2].$$

Remark 3.7. Let A be a vector bundle over $X_\pm$, we define the graded matrix factorization $A \otimes \mathcal{K}_+(q)[m]$ to be $(A(q)[m] \otimes \bigwedge^\bullet \mathcal{O}(-3)[1]^{\oplus 2}, (-1)^m \operatorname{Id} \otimes d_+)$, by an abuse of notation, it can be represented by

$$A(q)[m] \underset{(-1)^m s_p}{\overset{(-1)^m s_W}{\rightleftarrows}} A(q-3)[m+1]^{\oplus 2} \underset{(-1)^m s_p}{\overset{(-1)^m s_W}{\rightleftarrows}} A(q-6)[m+2]. \tag{3.3}$$

The notation s_W and s_p can also be understood as morphisms with $\mathbb{C}^*_R$-weights different from 1; for example, the morphisms in the following diagram are also denoted by s_W and s_p

$$A(q)[i] \underset{s_p}{\overset{s_W}{\rightleftarrows}} A(q-3)[j]^{\oplus 2} \underset{s_p}{\overset{s_W}{\rightleftarrows}} A(q-6)[k]$$

for arbitrary i, j and k; they are the same morphisms as in (3.3) after forgetting the $\mathbb{C}^*_R$-action and the $(-1)^m$ coming from sign convention. The notations s_x and s_{pf} will be used in the same way.

3.3 Grade restriction rule

For each $t \in \mathbb{Z}$, Segal [41] constructs an equivalent functor between the bounded derived categories of coherent sheaves

$$\widetilde{\Phi_t} \colon \mathrm{D^b}(X_-) \to \mathrm{D^b}(X_+)$$

and an equivalent functor between the derived categories of graded matrix factorizations

$$\Phi_t \colon \mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W) \to \mathrm{DMF}^{\mathbb{C}^*_R}(X_+, W)$$

from "grade restriction rule".

Let X_0 denote the Artin stack $[\mathbb{C}^6 \times \mathbb{C}^2/\mathbb{C}^*]$, where the group $\mathbb{C}^*$ acts on $\mathbb{C}^6 \times \mathbb{C}^2$ with weights $(1,1,1,1,1,1,-3,-3)$. Both X_+ and X_- are open substacks of X_0. We denote by

$$i_\pm \colon X_\pm \to X_0$$

the inclusions. Let

$$\widetilde{\mathcal{G}}_t \subset \mathrm{D^b}(X_0)$$

be the triangulated subcategory generated by the line bundles $\mathcal{O}(k)$ for $t \leq k < t+6$. It is proven in [41, 4] that both i_-^* and i_+^* restrict to give equivalences

$$\mathrm{D^b}(X_-) \xleftarrow{\sim} \widetilde{\mathcal{G}}_t \xrightarrow{\sim} \mathrm{D^b}(X_+).$$

then for each $t \in \mathbb{Z}$, we have an equivalent functor

$$\widetilde{\Phi_t} \colon \mathrm{D^b}(X_-) \to \mathrm{D^b}(X_+)$$

passing through $\widetilde{\mathcal{G}}_t$.

For the derived categories of graded matrix factorizations, Segal [41] proves the following theorem.

Theorem 3.8 (Segal [41])**.** *There is a family of quasiequivalences*

$$\Phi_t \colon \mathcal{MF}^{\mathbb{C}^*_R}(X_-, W) \xrightarrow{\sim} \mathcal{MF}^{\mathbb{C}^*_R}(X_+, W)$$

indexed by $t \in \mathbb{Z}$. When passing to homotopy category, we get a family of equivalences of trianglated category

$$\Phi_t \colon DMF^{\mathbb{C}^*_R}(X_-, W) \xrightarrow{\sim} DMF^{\mathbb{C}^*_R}(X_+, W).$$

Similar to the derived category case, the functor Φ_t is constructed by passing through a triangulated subcategory

$$\mathcal{G}_t \subset \mathrm{DMF}^{\mathbb{C}^*_R}(X_0, W).$$

The construction of $\mathcal{G}_t$ is slightly different from $\widetilde{\mathcal{G}}_t$ in the derived category of coherent sheaves case since we can not regard the line bundles $\mathcal{O}(k)[l]$ as objects of $\mathrm{DMF}^{\mathbb{C}^*_R}(X_0, W)$. Instead, $\mathcal{G}_t$ is the full triangulated subcategory of $\mathrm{DMF}^{\mathbb{C}^*_R}(X_0, W)$ consisting of graded matrix factorizations (E, d) where E is a direct sum of $\mathcal{O}(k)[l]$ for $t \leq k < t+6$.

Given a graded matrix factorization (E, d) over (X_-, W), we can find its image under Segal's functor Φ_t in two steps.

1. Find a graded matrix factorization (E', d') which is isomorphic to (E, d) in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$, where E' is a direct sum of $\mathcal{O}(k)[l]$ for $t \leq k < t+6$.

2. Since $\mathcal{O}(k)[l]$ also stand for line bundles over X_+, we take $\Phi_t((E,d))$ to be the graded matrix factorization over (X_+, W) with the same direct summands and endomorphisms as (E', d').

The interval $[t, t+6)$ is called a *window*. In order to apply Segal's functor, we need to find (E', d') in step 1 which fit the window. We explain the strategy in the next section.

3.4 Strategy to fit the window

We start by computing the image of an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_-)$ under the functor

$$\widetilde{\Phi_t} \colon \mathrm{D}^{\mathrm{b}}(X_-) \to \mathrm{D}^{\mathrm{b}}(X_+).$$

Since X_- is quasiprojective, we can represent $\mathcal{E}$ by objects from the set $\{\mathcal{O}(k), k \in \mathbb{Z}\}$ after shifting, taking direct sums and taking cones. Since over X_- we have an exact sequence

$$0 \longrightarrow \mathcal{O}(i+6) \xrightarrow{s_p} \mathcal{O}(i+3)^{\oplus 2} \xrightarrow{s_p} \mathcal{O}(i) \longrightarrow 0,$$

for $k < t$, we replace $\mathcal{O}(k)$ with the complex

$$0 \longrightarrow \mathcal{O}(k+6) \xrightarrow{s_p} \mathcal{O}(k+3)^{\oplus 2} \longrightarrow 0\,;$$

for $k \geq t+6$, we replace $\mathcal{O}(i)$ with the complex

$$0 \longrightarrow \mathcal{O}(k-3)^{\oplus 2} \xrightarrow{s_p} \mathcal{O}(k-6) \longrightarrow 0\,.$$

After replacing objects not fitting the window $[t, t+6)$ repeatedly, we can represent $\mathcal{E}$ by objects from the set $\{\mathcal{O}(k), t \leq k < t+6\}$. Then the object in $\mathrm{D}^{\mathrm{b}}(X_+)$ with the same representation is the image of $\mathcal{E}$ under $\widetilde{\Phi_t}$.

Our strategy for the computation of the functor

$$\Phi_t \colon \mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W) \xrightarrow{\sim} \mathrm{DMF}^{\mathbb{C}^*_R}(X_+, W)$$

is similar. Let (E, d) be a graded matrix factorization, we want to modify it to make it fit the window $[t, t+6)$. If E has a direct summand $\mathcal{O}(k)^{\oplus m}$ which does not fit the window, assume $k < t$, we want to replace it by $\mathcal{O}(k+6)^{\oplus m} \oplus \mathcal{O}(k+3)^{\oplus 2m}$. If we can do this repeatedly, then we can kill all direct summands not fitting the window, and finally get a graded matrix factorization fitting the window.

However, the replacement is not as easy as in the derived category case. The following lemma and proposition show when and how we can replace a direct summand.

Definition 3.9. Let $\mathcal{M}$ be a graded matrix factorization over $(X_\pm, W)$, and A be a direct summand of the underlying vector bundle of $\mathcal{M}$. We say A is *replaceable* in $\mathcal{M}$ if $\mathcal{M}$ can be represented by the diagram

$$\begin{tikzcd} A \arrow[r, shift right, "d_{BA}+p_1\delta^1_{BA}+p_2\delta^2_{BA}"'] & B \arrow[l, shift right, "p_1\delta^1_{AB}+p_2\delta^2_{AB}"'] \arrow[r, shift right, "d_{CB}+p_1\delta^1_{CB}+p_2\delta^2_{CB}"'] & C \arrow[l, shift right, "d_{BC}+p_1\delta^1_{BC}+p_2\delta^2_{BC}"'] \arrow[loop right, "d_{CC}+p_1\delta^1_{CC}+p_2\delta^2_{CC}"] \end{tikzcd}$$

such that

1. if we write A, B and C as direct sums of $\mathcal{O}(k)[l]$, then all morphisms d and δ with some indexes can be represented by matrices with entries in $\mathbb{C}[x_1, \ldots, x_6]$;
2. the following equations hold

$$\delta^2_{AB}\delta^1_{BA} = \delta^1_{AB}\delta^2_{BA} = 0.$$

Lemma 3.10. *Assume A is replaceable in $\mathcal{M}$ as in definition 3.9. Then, with the notation s_W and s_p in Remark 3.7, the diagram*

$$\begin{tikzcd} A(3)[-2]^{\oplus 2} \arrow[r, shift right, "-s_W"'] \arrow[d, shift right, "-(d_{BA}+p_1\delta^1_{BA}+p_2\delta^2_{BA})\circ s_p"'] & A(6)[-3] \arrow[l, shift right, "-s_p"'] \\ B \arrow[r, shift right, "d_{CB}+p_1\delta^1_{CB}+p_2\delta^2_{CB}"'] \arrow[u, shift right, "-(\delta^1_{AB},\delta^2_{AB})"'] & C \arrow[l, shift right, "d_{BC}+p_1\delta^1_{BC}+p_2\delta^2_{BC}"'] \arrow[u, "\delta^1_{AB}\delta^2_{BC}"'] \arrow[loop right, "d_{CC}+p_1\delta^1_{CC}+p_2\delta^2_{CC}"] \end{tikzcd}$$

represents a graded matrix factorization over $(X_\pm, W)$. We denote the new graded matrix factorization by $\mathcal{M}\backslash A$.

Proof. It is easy to check the morphisms in the diagram have $\mathbb{C}^*_R$-weight 1. We need to prove the square of sum of them equals to $W \cdot \mathrm{Id}$, *i.e.*

1. For each vertex, the sum of arrows going out composed with their reverse equals $W \cdot \mathrm{Id}$. Because $\mathcal{M}$ and $A \otimes K_+(6)[-3]$ are graded matrix factorizations, this is true at the vertices $A(6)[-3]$ and C. Note that we have

$$(p_1\delta^1_{AB} + p_2\delta^2_{AB})(d_{BA} + p_1\delta^1_{BA} + p_2\delta^2_{BA}) = p_1W_1 + p_2W_2;$$

since the sections $p_1, p_2, x_1, \ldots, x_6$ are algebraically independent, we deduce

$$(\delta^1_{AB}, \delta^2_{AB}) \circ (d_{BA} + p_1\delta^1_{BA} + p_2\delta^2_{BA}) = s_W.$$

It follows that the property holds at $A(3)[-2]$. At B, it follows from

$$s_p \circ (\delta^1_{AB}, \delta^2_{AB}) = p_1\delta^1_{AB} + p_2\delta^2_{AB}.$$

2. By composing two successive arrows, we get morphisms from one vertex to another. If we fix a pair of different source and target, the sum of those morphisms should be zero. The morphism from $A(6)[-3]$ to B is zero because $s_p \circ s_p = 0$ in $A \otimes \mathcal{K}_+(6)[-3]$. Similarly the morphism froms $A(3)[-2]$ to C, from B to C, and from C to B are zero. Since $\mathcal{M}$ is a graded matrix factorization, we have

$$\begin{aligned}&(d_{BC} + p_1\delta^1_{BC} + p_2\delta^2_{BC})(d_{CB} + p_1\delta^1_{CB} + p_2\delta^2_{CB}) + (d_{BA} + p_1\delta^1_{BA} + p_2\delta^2_{BA})(p_1\delta^1_{AB} + p_2\delta^2_{AB})\\ =&p_1W_1 + p_2W_2\end{aligned}$$

and

$$(p_1\delta^1_{AB} + p_2\delta^2_{AB})(d_{BC} + p_1\delta^1_{BC} + p_2\delta^2_{BC}) = 0.$$

Hence we have

$$\begin{aligned}s_W \circ (\delta^1_{AB}, \delta^2_{AB}) &= -W_2\delta^1_{AB} + W_1\delta^2_{AB}\\ &= -\delta^2_{AB}d_{BA}\delta^1_{AB} + \delta^1_{AB}d_{BA}\delta^2_{AB}\\ &= -\delta^2_{AB}(W_1 - \delta^1_{BC}d_{CB} - d_{BC}\delta^1_{CB}) + \delta^1_{AB}((W_2 - \delta^2_{BC}d_{CB} - d_{BC}\delta^2_{CB})\\ &= W_2\delta^1_{AB} - W_1\delta^2_{AB} - 2\delta^1_{AB}\delta^2_{BC}d_{CB}.\end{aligned}$$

So we get

$$\begin{aligned}&s_W \circ (\delta^1_{AB}, \delta^2_{AB}) + \delta^1_{AB}\delta^2_{BC}(d_{CB} + p_1\delta^1_{CB} + p_2\delta^2_{CB})\\ =&(-W_2\delta^1_{AB} + W_1\delta^2_{AB} + \delta^1_{AB}\delta^2_{BC}d_{CB}) - p_1\delta^2_{AB}\delta^1_{BC}\delta^1_{CB} + p_2\delta^1_{AB}\delta^2_{BC}\delta^2_{CB}\\ =&p_1\delta^2_{AB}\delta^1_{BA}\delta^1_{AB} - p_2\delta^1_{AB}\delta^2_{BA}\delta^2_{AB} = 0.\end{aligned}$$

This proves that the sum of morphisms from B to $A(6)[-3]$ is zero.

We also have

$$\begin{aligned}&(\delta^1_{AB}, \delta^2_{AB}) \circ (d_{BC} + p_1\delta^1_{BC} + p_2\delta^2_{BC}) + s_p\delta^1_{AB}\delta^2_{BC}\\ =&(p_1\delta^1_{AB}\delta^1_{BC} + p_2\delta^1_{AB}\delta^2_{BC} - p_2\delta^1_{AB}\delta^2_{BC}, p_1\delta^2_{AB}\delta^1_{BC} + p_2\delta^2_{AB}\delta^2_{BC} + p_1\delta^1_{AB}\delta^2_{BC})\\ =&(0, 0).\end{aligned}$$

This proves that the sum morphisms from C to $A(3)[-2]$ is zero.

Finally, since

$$(d_{BC} + p_1\delta^1_{BC} + p_2\delta^2_{BC})(d_{CC} + p_1\delta^1_{CC} + p_2\delta^2_{CC}) = 0,$$

we have

$$\begin{aligned}&\delta^1_{AB}\delta^2_{BC}(d_{CC} + p_1\delta^1_{CC} + p_2\delta^2_{CC})\\ =&\delta^1_{AB}\delta^2_{BC}d_{CC} - p_1\delta^2_{AB}\delta^1_{BC}\delta^1_{CC} + p_2\delta^1_{AB}\delta^2_{BC}\delta^2_{CC}\\ =&-\delta^1_{AB}d_{BC}\delta^2_{CC} = 0.\end{aligned}$$

This proves that the morphism from C to $A(6)[-3]$ is zero.

□

Proposition 3.11. *If A is replaceable in $\mathcal{M}$, then there exists a morphism of graded matrix factorization*

$$f\colon \mathcal{M} \to A \otimes \mathcal{K}_+(6)[-2].$$

Moreover, the cone C_f is isomorphic to $(\mathcal{M}\backslash A)[1]$ in $DMF^{\mathbb{C}^*_R}(X_\pm, W)$.

Proof. The morphism between the underlying vector bundles $A \oplus B \oplus C$ and $A \oplus A(3)[-1]^{\oplus 2} \oplus A(6)[-2]$ is given by the matrix

$$\begin{pmatrix} \mathrm{Id}_A & 0 & 0 \\ 0 & (\delta^1_{AB}, \delta^2_{AB}) & 0 \\ 0 & 0 & -\delta^1_{AB}\delta^2_{BC} \end{pmatrix}.$$

We can check this is indeed a morphism of graded matrix factorization by the method used in the proof of Lemma 3.10. The cone C_f of f is given by

$$\begin{tikzcd}
A \arrow[r, shift right, "s_W"'] & A(3)[-1]^{\oplus 2} \arrow[l, shift right, "s_p"'] \arrow[r, shift right, "s_W"'] & A(6)[-2] \arrow[l, shift right, "s_p"'] \\
A[1] \arrow[u, "\mathrm{Id}_A"] \arrow[r, shift right, "-(d_{BA}+p_1\delta^1_{BA}+p_2\delta^2_{BA})"'] & B[1] \arrow[u, "(\delta^1_{AB},\delta^2_{AB})"'] \arrow[l, shift right, "-(p_1\delta^1_{AB}+p_2\delta^2_{AB})"'] \arrow[r, shift right, "-(d_{CB}+p_1\delta^1_{CB}+p_2\delta^2_{CB})"'] & C[1] \arrow[u, "-\delta^1_{AB}\delta^2_{BC}"'] \arrow[l, shift right, "-(d_{BC}+p_1\delta^1_{BC}+p_2\delta^2_{BC})"'] \arrow[loop right, "-(d_{CC}+p_1\delta^1_{CC}+p_2\delta^2_{CC})"]
\end{tikzcd} \tag{3.4}$$

We know that $(\mathcal{M}\backslash A)[1]$ is given by

$$\begin{tikzcd}
A(3)[-1]^{\oplus 2} \arrow[r, shift right, "s_W"'] \arrow[d, shift right, "(d_{BA}+p_1\delta^1_{BA}+p_2\delta^2_{BA})\circ s_p"'] & A(6)[-2] \arrow[l, shift right, "s_p"'] \\
B[1] \arrow[u, shift right, "(\delta^1_{AB},\delta^2_{AB})"'] \arrow[r, shift right, "-(d_{CB}+p_1\delta^1_{CB}+p_2\delta^2_{CB})"'] & C[1] \arrow[u, "-\delta^1_{AB}\delta^2_{BC}"'] \arrow[l, shift right, "-(d_{BC}+p_1\delta^1_{BC}+p_2\delta^2_{BC})"'] \arrow[loop right, "-(d_{CC}+p_1\delta^1_{CC}+p_2\delta^2_{CC})"]
\end{tikzcd} \tag{3.5}$$

We write the underlying vector bundles of (3.4) and (3.5) as direct sums

$$A(6)[-2] \oplus A(3)[-1]^{\oplus 2} \oplus C[1] \oplus B[1] \oplus A \oplus A[1]$$

and

$$A(6)[-2] \oplus A(3)[-1]^{\oplus 2} \oplus C[1] \oplus B[1].$$

Under the order of direct summands above, we define a morphism of graded matrix factorization

$$F\colon C_f \to (\mathcal{M}\backslash A)[1]$$

by the matrix

$$\begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & d_{BA} + p_1\delta^1_{BA} + p_2\delta^2_{BA} & 0 \end{pmatrix}$$

and we define

$$G\colon (\mathcal{M}\backslash A)[1] \to C_f$$

by

$$\begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ 0 & 0 & 0 & 0 \\ 0 & -s_p & 0 & 0 \end{pmatrix}.$$

We have

$$F \circ G = \mathrm{Id}_{(\mathcal{M}\backslash A)[1]}$$

and

$$G \circ F = \mathrm{Id}_{C_f} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{BA} + p_1\delta^1_{BA} + p_2\delta^2_{BA} & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{Id} & 0 \\ 0 & -s_p & 0 & 0 & 0 & -\mathrm{Id} \end{pmatrix}.$$

Define

$$H\colon C_f \to C_f$$

as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{Id} & 0 \end{pmatrix}.$$

Then, we have

$$G \circ F = \mathrm{Id}_{C_f} + H \circ d_{C_f} + d_{C_f} \circ H,$$

which means $G \circ F$ is homotopy to Id_{C_f}. Hence we get $C_f = (\mathcal{M}\backslash A)[1]$ in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_\pm, W)$. $\square$

Corollary 3.12. *In $DMF^{\mathbb{C}^*_R}(X_-, W)$, we have $\mathcal{M} = \mathcal{M}\backslash A$. In this way we replace A by $A(6)[-3] \oplus A(3)[-2]$ as claimed in the beginning of this section.*

Proof. In $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$, we have

$$\mathcal{K}_+ = 0,$$

so

$$\mathcal{M} = \mathrm{cone}\,(f\colon \mathcal{M} \to A \otimes \mathcal{K}_+(6)[-2])\,[-1] = \mathcal{M}\backslash A.$$

□

3.5 Orlov's functor for complete intersection

Orolv [39] constructed a family of equivalences between a category of matrix factorization and the derived category of a Calabi–Yau hypersurface in projective space. We want to generalize it and get a family of equivalences between $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$ and $\mathrm{D^b}(X_{3,3})$. We can do it by composing the family of Segal's functors Ψ_i introduced in §3.3 with a functor constructed by Shipman [42].

Let

$$p : X_+ = \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2} \longrightarrow \mathbb{P}^5$$

be the bundle projection, and let

$$i : \mathcal{O}_{X_{3,3}}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2}$$

be the inclusion of the total space. Since p and i are $\mathbb{C}^*_R$-equivariant and W vanishes on $\mathcal{O}_{X_{3,3}}(-3)^{\oplus 2}$, there are functors between the derived categories of curved graded quasicoherent sheaves (see Remark 3.5)

$$p^*\colon \mathrm{DQcoh}^{\mathbb{C}^*_R}(X_{3,3}, 0) \to \mathrm{DQcoh}^{\mathbb{C}^*_R}(\mathcal{O}_{X_{3,3}}(-3)^{\oplus 2}, 0)$$

and

$$i_*\colon \mathrm{DQcoh}^{\mathbb{C}^*_R}(\mathcal{O}_{X_{3,3}}(-3)^{\oplus 2}, 0) \to \mathrm{DQcoh}^{\mathbb{C}^*_R}(X_+, W).$$

Since the $\mathbb{C}^*_R$-action is trivial on $X_{3,3}$, we can regard $\mathrm{D^b}(X_{3,3})$ as a subcategory of $\mathrm{DQcoh}^{\mathbb{C}^*_R}(X_{3,3}, 0)$.

Theorem 3.13 (Shipman [42])**.** *The functor $i_* \circ p^*$ sends $D^b(X_{3,3})$ to $DMF^{\mathbb{C}^*_R}(X_+, W)$. Moreover, the restriction of $i_* \circ p^*$ to $D^b(X_{3,3})$*

$$Shi := i_* \circ p^*\colon D^b(X_{3,3}) \longrightarrow DMF^{\mathbb{C}^*_R}(X_+, W)$$

is an equivalence of trianglated category.

We define Orl_t to be the composition $(\mathrm{Shi})^{-1} \circ \Phi_t$. Then we obtain a family of equivalences

$$\mathrm{Orl}_t\colon \mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W) \xrightarrow{\sim} \mathrm{D^b}(X_{3,3}).$$

Description of Shipman's functor

Shipman's functor Shi can be characterized by the following proposition.

Proposition 3.14 (Shipman [42]). *The image of $\mathcal{O}(k)[l] \in D^b(X_{3,3})$ under the functor Shi in Theorem 3.13 is $\mathcal{K}_+(k)[l]$.*

This proposition is a consequence of the following lemma.

Lemma 3.15 (Shipman [42]). *Let $\mathcal{S}$ be an object in $DQcoh^{\mathbb{C}_R^*}(X_\pm, W)$ represented by the digram*

$$S_1 \underset{\alpha_1}{\overset{\beta_1}{\leftrightarrows}} S_2 \underset{\alpha_2}{\overset{\beta_2}{\leftrightarrows}} \dots \underset{\alpha_n}{\overset{\beta_n}{\leftrightarrows}} S_{n+1}.$$

If the complex

$$0 \longrightarrow S_1 \xrightarrow{\alpha_1} S_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} S_{n+1} \longrightarrow 0$$

is exact, then $\mathcal{S}$ is zero in $DQcoh^{\mathbb{C}_R^*}(X_\pm, W)$.

Remark 3.16. Let

$$j\colon \mathbb{P}(3,3) \to X_-$$

be the inclusion as zero section. It is equivariant under the $\mathbb{C}_R^*$-action. Since the sheaves $\mathcal{O}_{\mathbb{P}(3,3)}(k)$ are invariant under the $\mathbb{C}_R^*$-action on $\mathbb{P}(3,3)$, we can regard them as objects of $\mathrm{DQcoh}^{\mathbb{C}_R^*}(\mathbb{P}(3,3),0)$. Then $j_*\mathcal{O}_{\mathbb{P}(3,3)}(k)[l]$ are objects in $\mathrm{DQcoh}^{\mathbb{C}_R^*}(X_-, W)$ with

$$j_*\colon \mathrm{DQcoh}^{\mathbb{C}_R^*}(\mathbb{P}(3,3),0) \to \mathrm{DQcoh}^{\mathbb{C}_R^*}(X_-, W)$$

the pushforward functor.

Using Lemma 3.15, we can show that $\mathcal{K}_-(q)[m]$ is isomorphic to $j_*\mathcal{O}_{\mathbb{P}(3,3)}(-q-6)[m-6]$ in $\mathrm{DQcoh}^{\mathbb{C}_R^*}(X_-, W)$. This is because the complex

$$0 \longrightarrow \mathcal{O}(q) \xrightarrow{s_x} \mathcal{O}(q+1)^{\oplus 6} \xrightarrow{s_x} \dots \xrightarrow{s_x} \mathcal{O}(q+6) \longrightarrow j_*\mathcal{O}_{\mathbb{P}(3,3)}(-q-6) \longrightarrow 0 \tag{3.6}$$

is exact on X_-.

Since $D^b(\mathbb{P}(3,3))$ is generated by the set of objects $\{\mathcal{O}_{\mathbb{P}(3,3)}(k), k \in \mathbb{Z}\}$, we can get a functor

$$j_*\colon \mathrm{D^b}(\mathbb{P}(3,3)) \to \mathrm{DMF}^{\mathbb{C}_R^*}(X_-, W).$$

Notice that the last term in (3.6) is $j_*\mathcal{O}_{\mathbb{P}(3,3)}(-q-6)$ instead of $j_*\mathcal{O}_{\mathbb{P}(3,3)}(q+6)$. This is due to the sign issue discussed in Remark 0.12. Although we can identify X_- with total space of $\mathcal{O}(-1)^{\oplus 6}$ over $\mathbb{P}(3,3)$, there is a change of sign. In fact, when we restrict the line bundle $\mathcal{O}(k)$ over X_- (see §3.11 for the definition of $\mathcal{O}(k)$) to the zero section $\mathbb{P}(3,3)$, we get the line bundle $\mathcal{O}(-k)$ over $\mathbb{P}(3,3)$.

Chapter 4

Matching analytic continuation and categorical equivalences

In this chapter we compute the image of $\mathcal{K}_-(q)[m]$ under the Orlov functors using the strategy from the previous chapter. Then we show that the Orlov functors coincide with the linear maps gotten from analytic continuation in the sense that

$$\mathrm{inv}^* \operatorname{ch}\left(\mathrm{Orl}_{t-3}(\mathcal{K}_-(q)[m]) \otimes \mathcal{O}(-3)\right) = \mathbb{U}_t\left(\mathrm{inv}^* \operatorname{ch}(\mathcal{K}_-(q)[m])\right).$$

We apply our result to study the monodromy of a local system. We also discuss the relation between our main result and the result of Coates, Iritani and Jiang [18] on crepant transformation conjecture.

4.1 Image of $\mathcal{K}_-(q)[m]$ under Orlov functor

We compute $\operatorname{ch}\left(\mathrm{Orl}_t(\mathcal{K}_-(q)[m])\right)$ in this section.

Since $\mathrm{Orl}_t = \mathrm{Shi}^{-1} \circ \Phi_t$, we need to compute $\Phi_t(\mathcal{K}_-(q)[m])$ first. The following Lemma shows that it is sufficient to compute $\Phi_t(\mathcal{K}_-)$.

Lemma 4.1. *For any object $\mathcal{F}$ in $DMF^{\mathbb{C}^*_R}(X_-, W)$ and any integer q, m, t, we have*

$$\Phi_t(\mathcal{F}(q)[m]) = \Phi_{t-q}(\mathcal{F})(q)[m].$$

Proof. If we can find a graded matrix factorization $\mathcal{E}$ such that $\mathcal{E} = \mathcal{F}$ in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_\pm, W)$ and the underlying vector bundle of $\mathcal{E}$ is a direct sum of $\mathcal{O}(i)[j]$ for $t \leq i < t+6$, then $\mathcal{E}(q)[m] = \mathcal{F}(q)[m]$ in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_\pm, W)$ and the underlying vector bundle of $\mathcal{E}(q)[m]$ is a direct sum of $\mathcal{O}(i)[j]$ for $t+q \leq i < t+q+6$. By the construction of Segal's functor Φ_i, we have

$$\Phi_t(\mathcal{F})(q)[m] = \Phi_{t+q}(\mathcal{F}(q)[m]).$$

□

4.1.1 Introductory examples: $\mathrm{Orl}_1(\mathcal{K}_-)$ and $\mathrm{Orl}_2(\mathcal{K}_-)$

We show how to use the strategy in §3.4 to compute $\mathrm{Orl}_1(\mathcal{K}_-)$ and $\mathrm{Orl}_2(\mathcal{K}_-)$.

By (3.1) we can represent $\mathcal{K}_-$ by

$$\begin{tikzcd} \mathcal{O} \arrow[r, shift left, "s_{pf}"] & \mathcal{O}(1)[-1]^{\oplus 6} \arrow[l, shift left, "s_x"] \arrow[r, shift left, "s_{pf}"] & C \arrow[l, shift left, "s_x"] \arrow[loop right, "s_x+s_{pf}"] \end{tikzcd},$$

where

$$C = \bigwedge^{\geq 2} \mathcal{O}(1)[-1]^{\oplus 6}.$$

We compute $\mathrm{Orl}_1(\mathcal{K}_-)$ first. The direct summand $\mathcal{O}$ of the underlying vector bundle of $\mathcal{K}_-$ does not fit the window $1 \leq k \leq 6$. We can decompose s_{pf} as

$$s_{pf} = p_1 s_{f_1} + p_2 s_{f_2},$$

so $\mathcal{O}$ is replaceable in $\mathcal{K}_-$. Then, by Corollary 3.12, in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$ we have an isomorphism between $\mathcal{K}_-$ and $\mathcal{K}_-^{(1)} := \mathcal{K}_- \backslash \mathcal{O}$, where $\mathcal{K}_-^{(1)}$ can be represented by

$$\begin{tikzcd} \mathcal{O}(3)[-2]^{\oplus 2} \arrow[d, shift left, "-(s_{f_1}, s_{f_2})"] & \mathcal{O}(6)[-3] \arrow[l, shift right, "-s_p"'] \arrow[l, shift left, "-s_W"] \\ \mathcal{O}(1)[-1]^{\oplus 6} \arrow[u, shift left, "-s_x \circ s_p"] \arrow[r, shift right, "s_{pf}"'] & C \arrow[u, "0"'] \arrow[l, shift right, "s_x"'] \arrow[loop right, "s_x+s_{pf}"] \end{tikzcd}$$

Note that the underlying vector bundle of $\mathcal{K}_-^{(1)}$ is a direct sum of $\mathcal{O}(k)[l]$ for $1 \leq k \leq 6$, which fit the window, thus in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_+, W)$ we have

$$\Phi_1(\mathcal{K}_-) = \mathcal{K}_-^{(1)}.$$

By Proposition 3.11 we know in $\mathrm{DMF}^{\mathbb{C}^*_R}(X_\pm, W)$

$$\mathcal{K}_-^{(1)} = \mathrm{cone}(\mathcal{K}_- \to \mathcal{K}_+(6)[-2])[-1].$$

In $\mathrm{DMF}^{\mathbb{C}^*_R}(X_+, W)$ we have $\mathcal{K}_- = 0$ hence

$$\mathcal{K}_-^{(1)} = \mathcal{K}_+(6)[-3] = \mathrm{Shi}(\mathcal{O}(6)[-3]),$$

so by Proposition 3.14 we have

$$\mathrm{Orl}_1(\mathcal{K}_-) = \mathcal{O}(6)[-3].$$

Next we compute $\mathrm{Orl}_2(\mathcal{K}_-)$. The window becomes $2 \le k \le 7$. The direct summand $\mathcal{O}(1)[-1]^{\oplus 6}$ of the underlying vector bundle of $\mathcal{K}_-^{(1)}$ does not fit the window. Again we can check that $\mathcal{O}(1)[-1]^{\oplus 6}$ is replaceable in $\mathcal{O}(1)[-1]^{\oplus 6}$. Let

$$\mathcal{K}_-^{(2)} := \mathcal{K}_-^{(1)} \backslash \mathcal{O}(1)[-1]^{\oplus 6},$$

then in $\mathrm{DMF}^{\mathbb{C}_R^*}(X_-, W)$ we have

$$\mathcal{K}_-^{(2)} = \mathcal{K}_-^{(1)} = \mathcal{K}_-$$

and all the direct summands of the underlying vector bundle of $\mathcal{K}_-^{(2)}$ fit the window. Thus in $\mathrm{DMF}^{\mathbb{C}_R^*}(X_+, W)$ we have

$$\Phi_2(\mathcal{K}_-) = K_-^{(2)}.$$

By Proposition 3.11,

$$\mathcal{K}_-^{(2)} = \mathrm{cone}\left(\mathcal{K}_-^{(1)} \to \mathcal{K}_+(7)[-3]^{\oplus 6}\right)[-1],$$

so

$$\mathrm{Orl}_2(\mathcal{K}_-) = \mathrm{cone}\left(\mathcal{O}(6)[-4] \to \mathcal{O}(7)[-4]^{\oplus 6}\right).$$

4.1.2 Replaceablity of $\mathcal{K}_-$

In order to continue the procedure in §4.1.1 to compute $\mathrm{Orl}_k(\mathcal{K}_-)$ for any integer $k > 0$, we need the following proposition stating that we can always use the strategy in §3.4 to make $\mathcal{K}_-$ fit the window $[t, t+6)$ for any integer $t > 0$.

Proposition 4.2. *There exists a sequence of graded matrix factorizations $\{\mathcal{K}_-^{(1)}, \mathcal{K}_-^{(2)}, \mathcal{K}_-^{(3)}, \ldots\}$ in $DMF^{\mathbb{C}_R^*}(X_\pm, W)$ such that*

1. *in $DMF^{\mathbb{C}_R^*}(X_-, W)$ we have $\mathcal{K}_-^{(1)} = \mathcal{K}_-^{(2)} = \cdots = \mathcal{K}_-$;*
2. *the underlying vector bundle of $\mathcal{K}_-^{(t)}$ is a direct sum of $\mathcal{O}(k)[l]$ for $t \le k < t+6$.*

Moreover, let the direct summand of underlying vector bundle of $\mathcal{K}_-^{(t)}$ consisting of line bundles in $\{\mathcal{O}(t)[l], l \in \mathbb{Z}\}$ be

$$\bigoplus_{i=1}^{s} \mathcal{O}(t)[n_i]^{\oplus m_i} \text{ for } n_1 > n_2 > \cdots > n_s,$$

then there exists a sequence of graded matrix factorizations

$$\{\mathcal{K}_-^{(t)(0)} = \mathcal{K}_-^{(t)}, \mathcal{K}_-^{(t)(1)}, \mathcal{K}_-^{(t)(2)}, \ldots, \mathcal{K}_-^{(t)(s)} = \mathcal{K}_-^{(t+1)}\}$$

such that

$$\mathcal{K}_-^{(t)(i+1)} = \mathrm{cone}\left(\mathcal{K}_-^{(t)(i)} \to \mathcal{K}_+(t+6)[n_i - 2]^{\oplus m_i}\right)[-1]$$

in $DMF^{\mathbb{C}_R^*}(X_\pm, W)$ for a suitable morphism $\mathcal{K}_-^{(t)(i)} \to \mathcal{K}_+(t+6)[n_i-2]^{\oplus m_i}$. The underlying vector bundle of $\mathcal{K}_-^{(t)(i+1)}$ is obtained by replacing the direct summand $\mathcal{O}(t)[n_i]^{\oplus m_i}$ in the underlying vector bundle of $\mathcal{K}_-^{(t)(i)}$ by $\mathcal{O}(t+6)[n_i - 3]^{\oplus m_i} \oplus \mathcal{O}(t+3)[n_i - 2]^{\oplus 2m_i}$.

Proof. If we can show that $\mathcal{O}(t)[n_i]^{\oplus m_i}$ is replaceable in $\mathcal{K}_-^{(t)(i)}$, then we can define

$$\mathcal{K}_-^{(t)(i+1)} := \mathcal{K}_-^{(t)(i)} \backslash \mathcal{O}(t)[n_i]^{\oplus m_i}.$$

We show that this is always possible. We have written down $\mathcal{K}_-^{(1)}$ and $\mathcal{K}_-^{(2)}$ already. Note that $\mathcal{K}_-^{(1)}$ and $\mathcal{K}_-^{(2)}$ satisfy the following property: their differentials are the sum of following types of morphisms:

1. the zero extension of the morphism $\mathcal{O}(k)[l]^{\oplus m} \xrightarrow{f} \mathcal{O}(k+i)[l-1]^{\oplus n}$, where
$$i \in \mathbb{Z}^{\geq 0},$$
and f can be represented by a matrix with entries in $\mathbb{C}[x_1, \ldots, x_6]$;
2. the zero extension of the morphism $\mathcal{O}(k)[l]^{\oplus m} \xrightarrow{p_1 f_1 + p_2 f_2} \mathcal{O}(k+i)[l+1]^{\oplus n}$, where
$$i \in \mathbb{Z}^{\geq -3},$$
and f_1, f_2 can be represented by matrices with entries in $\mathbb{C}[x_1, \ldots, x_6]$.

If $\mathcal{K}_-^{(t)(i)}$ satisfies the property, since all the direct summands of $\mathcal{K}_-^{(t)(i)}$ are of the form $\mathcal{O}(s)[j]$ with $s > t$ or $s = t, j \leq n_i$, all the nonzero arrow with target $\mathcal{O}(t)[n_i]^{\oplus m_i}$ are of type 2. Moreover, we can not find two successive nonzero type-2 arrows such that the first start from $\mathcal{O}(t)[n_i]^{\oplus m_i}$ and the second goes back to $\mathcal{O}(t)[n_i]^{\oplus m_i}$. This means $\mathcal{O}(t)[n_i]^{\oplus m_i}$ is replaceable in $\mathcal{K}_-^{(t)(i)}$. By construction in Lemma 3.10,

$$\mathcal{K}_-^{(t)(i+1)} := \mathcal{K}_-^{(t)(i)} \backslash \mathcal{O}(t)[n_i]^{\oplus m_i}$$

also satisfies above property, thus we can define all $\mathcal{K}_-^{(t)(i)}$ inductively. □

Remark 4.3. We can get a completely similar sequence $\{\mathcal{K}_-^{(0)}, \mathcal{K}_-^{(-1)}, \mathcal{K}_-^{(-2)}, \ldots\}$ if we replace the direct summand

$$\mathcal{O}(-t+5)[l]^{\oplus m}$$

in the underlying vector bundle of $\mathcal{K}_-^{(-t)}$ with

$$\mathcal{O}(-t+2)[l+2]^{\oplus 2m} \oplus \mathcal{O}(-t-1)[l+3]^{\oplus m}$$

to get $\mathcal{K}_-^{(-t-1)}$. In this way we can compute $\mathrm{Orl}_k(\mathcal{K}_-)$ for any $k \leq 0$.

Corollary 4.4. *For any integer q, m and t, we have*

$$\mathrm{Orl}_t(\mathcal{K}_-(q)[m]) = \mathrm{Orl}_{t-q}(\mathcal{K}_-)(q)[m].$$

Proof. According to Proposition 4.2, $\Phi_t(\mathcal{K}_-(q)[m])$ lies within the triangulated subcategory of $\mathrm{DMF}^{\mathbb{C}_R^*}(X_+, W)$ generated by the graded matrix factorizations $\mathcal{K}_+(i)[j]$. Then we can use Lemma 4.1 and Proposition 3.14 to conclude. □

4.1.3 Computation of $\mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-(q)[m]))$

From (1) and (2) in Proposition 4.2, we have

$$\Phi_t(\mathcal{K}_-) = \mathcal{K}_-^{(t)}.$$

Since $\mathcal{K}_-^{(t)}$ can be obtained by taking cones of morphism to $\mathcal{K}_+(k)[n]^{\oplus m}$ repeatedly, after applying the functor $(\mathrm{Shi})^{-1}$, we know $\mathrm{Orl}_t(\mathcal{K}_-)$ can be obtained by taking cones of morphism to $\mathcal{O}(k)[n]^{\oplus m}$ repeatedly. In particular, we can compute $\mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-))$.

Theorem 4.5. *For all integers t, q and m we have*

$$\mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-(q)[m])) = (-1)^m \sum_{\substack{t-3\le s\le t+2\\ s\equiv q \mod 3}} \frac{s-q}{3} \sum_{k=0}^{t-s+2} (-1)^{k+1} \binom{6}{k} e^{(k+s+3)p}.$$

Proof. It is computed in §4.1.1 that

$$\mathrm{Orl}_1(\mathcal{K}_-) = \mathcal{O}(6)[-3],$$

so we have

$$\mathrm{ch}(\mathrm{Orl}_1(\mathcal{K}_-)) = -e^{6p}.$$

Now we compute

$$\mathrm{ch}(\mathrm{Orl}_{t+1}(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-)).$$

We write the direct summand of underlying vector bundle of $\mathcal{K}_-^{(t)}$ consisting of line bundles in $\{\mathcal{O}(t)[l], l \in \mathbb{Z}\}$ as

$$\bigoplus_{i=1}^{s} \mathcal{O}(t)[n_i]^{\oplus m_i}, \text{ for } n_1 > n_2 > \cdots > n_s,$$

by Proposition 4.2 there exists a sequence of graded matrix factorizations

$$\{\mathcal{K}_-^{(t)(0)} = \mathcal{K}_-^{(t)}, \mathcal{K}_-^{(t)(1)}, \mathcal{K}_-^{(t)(2)}, \ldots, \mathcal{K}_-^{(t)(s)} = \mathcal{K}_-^{(t+1)}\}$$

such that

$$\mathcal{K}_-^{(t)(i+1)} = \mathrm{cone}\left(\mathcal{K}_-^{(t)(i)} \to \mathcal{K}_+(t+6)[n_i-2]^{\oplus m_i}\right)[-1].$$

Therefore,

$$\begin{aligned}\mathrm{ch}(\mathrm{Orl}_{t+1}(\mathcal{K}_-)) - ch(\mathrm{Orl}_t(\mathcal{K}_-)) &= \mathrm{ch}\left((\mathrm{Shi})^{-1}(\mathcal{K}_-^{(t)(s)})\right) - \mathrm{ch}\left((\mathrm{Shi})^{-1}(\mathcal{K}_-^{(t)(0)})\right)\\ &= -\sum_{i=1}^{s} \mathrm{ch}\left(\mathcal{O}(t+6)[n_i-2]^{\oplus m_i}\right).\end{aligned}$$

When $t \geq 7$, the direct summand $\mathcal{O}(t)[n_i]^{\oplus m_i}$ in the underlying vector bundle of $\mathcal{K}_-^{(t)}$ comes from $\mathcal{O}(t-3)[n_i+2]^{\oplus a_i}$ in the underlying vector bundle of $\mathcal{K}_-^{(t-3)}$ and $\mathcal{O}(t-6)[n_i+3]^{\oplus b_i}$ in the underlying vector bundle of $\mathcal{K}_-^{(t-6)}$, with $2a_i + b_i = n_i$. Therefore when $t \geq 7$ we have

$$\begin{aligned}\mathrm{ch}(\mathrm{Orl}_{t+1}(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-)) =& 2e^{3p}\left(\mathrm{ch}(\mathrm{Orl}_{t-2}(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_{t-3}(\mathcal{K}_-))\right)\\ &- e^{6p}\left(\mathrm{ch}(\mathrm{Orl}_{t-5}(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_{t-6}(\mathcal{K}_-))\right).\end{aligned}$$

We can compute directly that

$$\begin{aligned}
\mathrm{ch}(\mathrm{Orl}_2(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_1(\mathcal{K}_-)) &= 6e^{7p},\\
\mathrm{ch}(\mathrm{Orl}_3(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_2(\mathcal{K}_-)) &= -15e^{8p},\\
\mathrm{ch}(\mathrm{Orl}_4(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_3(\mathcal{K}_-)) &= 20e^{9p} - 2\cdot e^{9p},\\
\mathrm{ch}(\mathrm{Orl}_5(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_4(\mathcal{K}_-)) &= -15e^{10p} + 2\cdot 6e^{10p},\\
\mathrm{ch}(\mathrm{Orl}_6(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_5(\mathcal{K}_-)) &= 6e^{11p} - 2\cdot 15e^{11p},\\
\mathrm{ch}(\mathrm{Orl}_7(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_6(\mathcal{K}_-)) &= -e^{12p} + 2\cdot 20e^{12p} - 3\cdot e^{12p}.
\end{aligned}$$

Then we can check that for all $t \geq 1$ we have

$$\begin{aligned}&\mathrm{ch}(\mathrm{Orl}_{t+1}(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-))\\ =&(-1)^{s+1}\lceil\frac{t+1}{3}\rceil\binom{6}{s}e^{p+6} + (-1)^s\lceil\frac{t+1}{3}-1\rceil\binom{6}{s+3}e^{p+6} + (-1)^{s+1}\lceil\frac{t+1}{3}-2\rceil\binom{6}{s+6}e^{p+6},\end{aligned}$$

where $s \in \{0,1,2\}$, $s \equiv t \mod 3$, and we set $\binom{6}{k} = 0$ if $k > 6$. Using the fact that in H_{GW}

$$\sum_{i=0}^{6}(-1)^k\binom{6}{k}e^{ip} = (1-e^p)^6 = 0$$

for dimension reason, we compute

$$\begin{aligned}\mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-)) &= \mathrm{ch}(\mathrm{Orl}_1(\mathcal{K}_-)) + \sum_{i=1}^{t-1}\left(\mathrm{ch}(\mathrm{Orl}_i(\mathcal{K}_-)) - \mathrm{ch}(\mathrm{Orl}_{i-1}(\mathcal{K}_-))\right)\\
&= \sum_{n=1}^{\lceil\frac{t}{3}\rceil} n \sum_{\substack{0\leq k\leq 6\\ 3n+k-3\leq t}} (-1)^{k+1}\binom{6}{k}e^{(3n+k+3)p}\\
&= \sum_{\substack{t-3\leq s\leq t+2\\ s\equiv 0 \mod 3}} \frac{s}{3}\sum_{k=0}^{t-s+2}(-1)^{k+1}\binom{6}{k}e^{(s+k+3)p}\end{aligned}$$

Then use Corollary 4.4, we get

$$\begin{aligned}\mathrm{ch}(\mathrm{Orl}_t(\mathcal{K}_-(q)[m])) &= (-1)^m e^{qp}\,\mathrm{ch}(\mathrm{Orl}_{t-q}(\mathcal{K}_-))\\
&= (-1)^m \sum_{\substack{t-q-3\leq s\leq t-q+2\\ s\equiv 0 \mod 3}} \frac{s}{3}\sum_{k=0}^{t-q-s+2}(-1)^{k+1}\binom{6}{k}e^{(k+s+q+3)p}\\
&= (-1)^m \sum_{\substack{t-3\leq s\leq t+2\\ s\equiv q \mod 3}} \frac{s-q}{3}\sum_{k=0}^{t-s+2}(-1)^{k+1}\binom{6}{k}e^{(k+s+3)p}.\end{aligned}$$

□

4.2 Chern character of $\mathcal{K}_-(q)[m]$

The Chern character on $\mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W)$ (more precisely, on $\mathcal{MF}^{\mathbb{C}^*_R}(X_-, W)$) takes values in the Hochschild cohomology $HH(\mathcal{MF}^{\mathbb{C}^*_R}(X_-, W))$. We do not have an isomorphism between $HH(\mathcal{MF}^{\mathbb{C}^*_R}(X_-, W))$ and H_{FJRW} in the complete intersection case currently. But since all Chern characters satisfy Grothendieck–Riemann–Roth, we can define a H_{FJRW}-valued Chern character for objects coming from the pushforward functor as follows.

Consider the inclusion

$$j\colon \mathbb{P}(3,3) \longrightarrow X_- = \mathcal{O}_{\mathbb{P}(3,3)}(-1)^{\oplus 6}$$

as zero section. As discussed in Remark 3.16, we have a pushforward functor

$$j_*\colon \mathrm{D}^{\mathrm{b}}(\mathbb{P}(3,3)) \to \mathrm{DMF}^{\mathbb{C}^*_R}(X_-, W).$$

We identify

$$H_{\mathrm{FJRW}} = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3))$$

with the second and third direct summands of

$$H^*_{\mathrm{CR}}(X_-) = H^*_{\mathrm{CR}}(\mathbb{P}(3,3)) = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)) \oplus H^{*+8}(\mathbb{P}(3,3)).$$

Then we use Grothendieck–Riemann–Roth to define a H_{FJRW}-valued Chern character

$$\mathrm{ch}\left(j_*(\mathcal{E})\right) := \left(\mathrm{ch}(\mathcal{E}) \cdot \frac{1}{\mathrm{Td}(O_{\mathbb{P}(3,3)}(-1)^{\oplus 6})}\right)\Bigg|_{\mathrm{nar}},$$

where the notation $|_{\mathrm{nar}}$ means we take the part coming from the second and third direct summands of $H^*_{\mathrm{CR}}(\mathbb{P}(3,3))$.

The Todd class (see §0.4) of the bundle $O_{\mathbb{P}(3,3)}(-1)^{\oplus 6}$ is computed as

$$\begin{aligned}\mathrm{Td}(O_{\mathbb{P}(3,3)}(-1)^{\oplus 6})|_{\mathrm{nar}} &= \left(\mathrm{Td}(O_{\mathbb{P}(3,3)}(-1))\right)^6 |_{\mathrm{nar}} \\ &= \left(\frac{1}{1-\zeta^2 e^{\frac{H^{(1)}}{3}}}\mathbb{1}^{(1)} + \frac{1}{1-\zeta e^{\frac{H^{(2)}}{3}}}\mathbb{1}^{(2)}\right)^6,\end{aligned} \tag{4.1}$$

hence we have

$$\begin{aligned}\left(\frac{1}{\mathrm{Td}(O_{\mathbb{P}(3,3)}(-1)^{\oplus 6})}\right)\Bigg|_{\mathrm{nar}} &= \left(\mathbb{1}^{(1)} - \zeta^2 e^{\frac{H^{(1)}}{3}}\right)^6 + \left(\mathbb{1}^{(2)} - \zeta e^{\frac{H^{(2)}}{3}}\right)^6 \\ &= \sum_{k=1}^{2}\left((1-\zeta^{-k})\mathbb{1}^{(k)} - \frac{1}{3}\zeta^{-k}H^{(k)}\right)^6.\end{aligned}$$

On the other hand, as mentioned in Remark 3.16,

$$\mathcal{K}_-(q)[m] = j_*\left(\mathcal{O}_{\mathbb{P}(3,3)}(-q-6)[m-6]\right),$$

and

$$\begin{aligned}\operatorname{ch}\left(\mathcal{O}_{\mathbb{P}(3,3)}(-q-6)[m-6]\right)|_{\mathrm{nar}} &= (-1)^{m-6}\operatorname{ch}\left(\mathcal{O}_{\mathbb{P}(3,3)}(-1)\right)^{q+6}|_{\mathrm{nar}} \\ &= (-1)^m\sum_{k=1}^{2}\left(\zeta^k\mathbb{1}^{(k)} - \frac{1}{3}\zeta^k H^{(k)}\right)^{q+6}.\end{aligned} \tag{4.2}$$

Then we get

$$\operatorname{inv}^*\left(\operatorname{ch}(\mathcal{K}_-(q)[m])\right) = (-1)^m\sum_{k=1}^{2}\left(\zeta^{-k}\mathbb{1}^{(k)} - \frac{1}{3}\zeta^{-k}H^{(k)}\right)^{q+6}\left((1-\zeta^k)\mathbb{1}^{(k)} - \frac{1}{3}\zeta^k H^{(k)}\right)^6.$$

Where

$$\operatorname{inv}^*\colon H_{\mathrm{FJRW}} \to H_{\mathrm{FJRW}}.$$

is induced by the canonical involution (see §0.4)

$$\operatorname{inv}\colon IX_- \to IX_-.$$

It maps $\mathbb{1}^{(k)}$ and $\mathbb{H}^{(k)}$ to $\mathbb{1}^{(3-k)}$ and $\mathbb{H}^{(3-k)}$.

Remark 4.6. The exponents of ζ in (4.1) and (4.2) are different from the standard ones. This is due to the sign issue discussed in Remark 2.2.

4.3 Image of $\operatorname{inv}^*\left(\operatorname{ch}(\mathcal{K}_-(q)[m])\right)$ under $\mathbb{U}_l$

We can expand $\operatorname{inv}^*\left(\operatorname{ch}(\mathcal{K}_-(q)[m])\right)$ as

$$\begin{aligned}\operatorname{inv}^*\left(\operatorname{ch}(\mathcal{K}_-(q)[m])\right) =&(-1)^m\sum_{k=1}^{2}\zeta^{-k(q+6)}(1-\zeta^k)^6\mathbb{1}^{(k)} \\ &- 2\cdot(-1)^m\sum_{k=1}^{2}\zeta^{-k(q+5)}(1-\zeta^k)^5 H^{(k)} \\ &- \frac{q+6}{3}(-1)^m\sum_{k=1}^{2}\zeta^{-k(q+6)}(1-\zeta^k)^6 H^{(k)}.\end{aligned}$$

There are three types of elements in the expansion, we compute their images under $\mathbb{U}_l$.

Proposition 4.7. *For any integer l and q, the following three equations hold.*

$$\begin{aligned}\mathbb{U}_l\left(\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6\mathbb{1}^{(k)}\right)=&\frac{1}{3}\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}} s\sum_{k=0}^{s-l}(-1)^{l+k+6-s}\binom{6}{l+k+6-s}e^{(k+l)p}\\ &-2\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}\sum_{k=0}^{s-l-1}(-1)^{s-l-k}\binom{5}{s-l-k-1}e^{(k+l)p};\end{aligned}\tag{4.3}$$

$$\mathbb{U}_l\left(\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^5H^{(k)}\right)=\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+4}}\sum_{k=0}^{s-l}(-1)^{s-l-k}\binom{5}{s-l-k}e^{(k+l)p};\tag{4.4}$$

$$\mathbb{U}_l\left(\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6H^{(k)}\right)=\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}\sum_{k=0}^{s-l}(-1)^{s-l-k}\binom{6}{s-l-k}e^{(k+l)p}.\tag{4.5}$$

Proof. We only prove equation (4.3), the other two can be proven in the same way. We add formal element $\mathbb{1}^{(0)}$ then we can write

$$\begin{aligned}&\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6\mathbb{1}^{(k)}\\ =&\sum_{k=0}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6\mathbb{1}^{(k)}\\ =&\begin{pmatrix}\mathbb{1}^{(0)} & \mathbb{1}^{(1)} & \mathbb{1}^{(2)}\end{pmatrix}\begin{pmatrix}1 & & \\ & \zeta^{-q} & \\ & & \zeta^{-2q}\end{pmatrix}\begin{pmatrix}1&1&1&1&1&1&1\\1&\zeta&\zeta^2&1&\zeta&\zeta^2&1\\1&\zeta^2&\zeta&1&\zeta^2&\zeta&1\end{pmatrix}\begin{pmatrix}1\\-6\\15\\-20\\15\\-6\\1\end{pmatrix}.\end{aligned}\tag{4.6}$$

By (2.8) we have

$$
\begin{aligned}
&\mathbb{U}_l\left(\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6\mathbb{1}^{(k)}\right)\\
=&\frac{1}{9}e^{pl}\begin{pmatrix}1 & e^p & e^{2p} & e^{3p}\dots\end{pmatrix}\begin{pmatrix} l & l & l\\ l+1 & (l+1)\zeta & (l+1)\zeta^2\\ l+2 & (l+2)\zeta^2 & (l+2)\zeta\\ l+3 & l+3 & l+3\\ l+4 & (l+4)\zeta & (l+4)\zeta^2\\ l+5 & (l+5)\zeta^2 & (l+5)\zeta\\ \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1 & & \\ & \zeta^l & \\ & & \zeta^{2l}\end{pmatrix}\\
&\cdot\begin{pmatrix}1 & & \\ & \zeta^{-q} & \\ & & \zeta^{-2q}\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & \zeta & \zeta^2 & 1 & \zeta & \zeta^2 & 1\\ 1 & \zeta^2 & \zeta & 1 & \zeta^2 & \zeta & 1\end{pmatrix}\begin{pmatrix}1\\ -6\\ 15\\ -20\\ 15\\ -6\\ 1\end{pmatrix}\\
=&\frac{1}{3}e^{pl}\begin{pmatrix}1 & e^p & e^{2p} & e^{3p}\dots\end{pmatrix}\begin{pmatrix} & & & l & & & l & & \\ & & \iddots & & & \iddots & & & \\ m & & & & m & & & & m\\ & & & m+1 & & & m+1 & & \\ & & m+2 & & & m+2 & & & \\ m+3 & & & & m+3 & & & & m+3\\ & & & \iddots & & & \iddots & & \\ & & \iddots & & & \iddots & & & \end{pmatrix}\begin{pmatrix}1\\ -6\\ 15\\ -20\\ 15\\ -6\\ 1\end{pmatrix},
\end{aligned}
\tag{4.7}
$$

where $m\in\{l,l+1,l+2\}$, and $m\equiv q \mod 3$. Note that the centre matrix consists of slope-1 lines. Start from the third line, the contribution of each line to $\mathbb{U}_l\left(\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6\mathbb{1}^{(k)}\right)$ is

$$
\begin{aligned}
&\frac{1}{3}e^{pl}\sum_{k=0}^{6}(m+3t+k)(-1)^{6-k}\binom{6}{6-k}e^{kp}\\
=&\frac{1}{3}e^{pl}(m+3t)(1-e^p)^6+\frac{1}{3}e^{pl}\sum_{k=1}^{6}(-1)^{6-k}\cdot 6\cdot\binom{5}{k-1}e^{kp}\\
=&\frac{1}{3}e^{pl}(m+3t)(1-e^p)^6-2e^{(l+1)p}(1-e^p)^5=0.
\end{aligned}
$$

Thus only the first two lines contribute. We have

$$
\begin{aligned}
&\mathbb{U}_l\left(\sum_{k=1}^{2}\zeta^{-qk}\left(1-\zeta^k\right)^6\mathbb{1}^{(k)}\right)\\
=&\frac{1}{3}\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}\sum_{k=0}^{s-l}(l+k)(-1)^{s-l-k}\binom{6}{s-l-k}e^{(k+l)p}\\
=&\frac{1}{3}\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}s\sum_{k=0}^{s-l}(-1)^{l+k+6-s}\binom{6}{l+k+6-s}e^{(k+l)p}\\
&-\frac{1}{3}\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}\sum_{k=0}^{s-l}(s-l-k)(-1)^{s-l-k}\binom{6}{s-l-k}e^{(k+l)p}\\
=&\frac{1}{3}\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}s\sum_{k=0}^{s-l}(-1)^{l+k+6-s}\binom{6}{l+k+6-s}e^{(k+l)p}\\
&-2\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}\sum_{k=0}^{s-l-1}(-1)^{s-l-k}\binom{5}{s-l-k-1}e^{(k+l)p}.
\end{aligned}
\tag{4.8}
$$

□

Corollary 4.8. *The image of* $inv^*\left(\mathrm{ch}(\mathcal{K}_-(q)[m])\right)$ *under* $\mathbb{U}_l$ *is*

$$
\mathbb{U}_l\left(inv^*\left(\mathrm{ch}(\mathcal{K}_-(q)[m])\right)\right)=(-1)^m\sum_{\substack{s\equiv q \mod 3\\ l\le s\le l+5}}\frac{s-q-6}{3}\sum_{k=0}^{s-l}(-1)^{l+k+6-s}\binom{6}{l+k+6-s}e^{(k+l)p}. \tag{4.9}
$$

4.4 Main result

Proposition 4.9. *For any integers* q, m *and* t, *we have*

$$
\mathbb{U}_t\left(inv^*\left(\mathrm{ch}(\mathcal{K}_-(q)[m])\right)\right)=\mathrm{ch}(\mathrm{Orl}_{t-3}(\mathcal{K}_-(q)[m]))\cdot e^{-3p}. \tag{4.10}
$$

Proof. Using the fact that

$$
0=(1-e^p)^6=\sum_{k=0}^{6}(-1)^k\binom{6}{k}e^{kp},
$$

we compute

$$\begin{aligned}
\mathbb{U}_t\left(\mathrm{inv}^*\left(\mathrm{ch}(\mathcal{K}_-(q)[m])\right)\right) &= (-1)^m \sum_{\substack{s\equiv q \mod 3\\ t\le s\le t+5}} \frac{s-q-6}{3} \sum_{k=0}^{s-t} (-1)^{t+k+6-s} \binom{6}{t+k+6-s} e^{(k+t)p} \\
&= (-1)^m \sum_{\substack{s\equiv q \mod 3\\ t\le s\le t+5}} \frac{s-q-6}{3} \sum_{k=s-t-6}^{-1} (-1)^{t+k+5-s} \binom{6}{t+k+6-s} e^{(k+t)p} \\
&= (-1)^m \sum_{\substack{s\equiv q \mod 3\\ t\le s\le t+5}} \frac{s-q-6}{3} \sum_{k=0}^{t-s+5} (-1)^{k+1} \binom{6}{k} e^{(k+s-6)p} \\
&= (-1)^m \sum_{\substack{s\equiv q \mod 3\\ t-6\le s\le t-1}} \frac{s-q}{3} \sum_{k=0}^{t-s-1} (-1)^{k+1} \binom{6}{k} e^{(k+s)p} \\
&= \mathrm{ch}(\mathrm{Orl}_{t-3}(\mathcal{K}_-(q)[m]))\cdot e^{-3p}.
\end{aligned}$$

□

The functor

$$\otimes\mathcal{O}(-3)\colon \mathrm{D^b}(X_{3,3}) \to \mathrm{D^b}(X_{3,3})$$

given by

$$\mathcal{E} \mapsto \mathcal{E}\otimes\mathcal{O}(-3)$$

is an equivalence of triangulated category. We define the modified Orlov functors by

$$\mathrm{Orl}_t^{\mathrm{mod}} := \otimes\mathcal{O}(-3)\circ \mathrm{Orl}_{t-3}\colon \mathrm{DMF}^{\mathbb{C}_R^*}(X_-,W) \to \mathrm{D^b}(X_{3,3}),$$

they are still equivalences of triangulated category.

The Proposition 4.9 can be interpreted as the following theorem.

Theorem 4.10. *Let $\mathcal{G}$ be the subcategory of $DMF^{\mathbb{C}_R^*}(X_-,W)$ generated by $\{\mathcal{K}_-(q)[m]\}_{q,m\in\mathbb{Z}}$, then for any $t\in\mathbb{Z}$, we have the following commutative diagram*

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathrm{Orl}_t^{mod}"] \arrow[d, "inv^*\,\mathrm{ch}"'] & D^b(X_{3,3}) \arrow[d, "inv^*\,\mathrm{ch}"] \\
H_{\mathrm{FJRW}} \arrow[r, "\mathbb{U}_t"] & H_{\mathrm{GW}}.
\end{tikzcd}$$

Remark 4.11. The canonical involution (see §0.4)

$$\mathrm{inv}\colon IX_{3,3} \to IX_{3,3}$$

is identical since $X_{3,3}$ has only one sector. So the inv^* in the right arrow is is the identical map of H_{GW}.

4.5 Application to monodromy

The Picard-Fuchs equation (1.25) determines a local system over $\mathbb{C}_v\backslash\{0,3^{-6}\}=\mathbb{P}^1_v\backslash\{0,3^{-6},\infty\}$. We study the monodromy around the three points in this section.

Theorem 4.12. *Let $\gamma_{CY},\gamma_{LG},\gamma_{con}$ be the paths in $\mathbb{P}^1_v\backslash\{0,3^{-6},\infty\}$ as in figure 4.1.*

1. *The action on H_{GW} induced by the circle γ_{CY} around 0 is given by multiplying*

$$inv^*\,\mathrm{ch}\left(\mathcal{O}_{X_{3,3}}(1)\right)=e^p.$$

2. *The action on H_{FJRW} induced by the circle γ_{LG} around ∞ is given by multiplying the narrow part of $inv^*\,\mathrm{ch}(\mathcal{O}_{X_-}(-1))$, which is (see Remark 2.2 for the sign issue)*

$$\zeta_3 e^{\frac{H^{(1)}}{3}}\mathbb{1}^{(1)}+\zeta_3^2 e^{\frac{H^{(2)}}{3}}\mathbb{1}^{(2)}.$$

3. *The action on H_{GW} induced by the circle γ_{con} around 3^{-6} coincides (up to conjugation) with the one induced by a spherical twist of $D^b(X_{3,3})$.*

Proof. 1. After the monodromy along γ_{CY}, $\log v$ turns into $\log v+2\pi i$. According to equation (2.3), we have

$$\begin{aligned}\mathfrak{H}_{\mathrm{GW}}(\log v+2\pi i,z)&=\sum_{n\geq 0} z e^{(\frac{p}{2\pi i}+n)(\log v+2\pi i)}\frac{\Gamma(3\frac{p}{2\pi i}+3n+1)^2}{\Gamma(\frac{p}{2\pi i}+n+1)^6}\\&=\mathfrak{H}_{\mathrm{GW}}(\log v,z)\cdot e^p.\end{aligned}$$

2. After the monodromy along γ_{LG}, $\log v$ turns into $\log v-2\pi i$, and $\log u$ turns into $\log u+\frac{2\pi i}{3}$. According to equation (2.6), we have

$$\mathfrak{H}_{\mathrm{FJRW}}(\log u+\frac{2\pi i}{3},z)=\mathfrak{H}_{\mathrm{FJRW}}(\log u,z)\cdot\left(\zeta_3 e^{\frac{H^{(1)}}{3}}\mathbb{1}^{(1)}+\zeta_3^2 e^{\frac{H^{(2)}}{3}}\mathbb{1}^{(2)}\right).$$

3. According to Theorem 4.10, go along γ_{con} is the composition

$$\begin{aligned}&(\otimes\mathcal{O}(-3)\circ\mathrm{Orl}_{t-2})\circ(\otimes\mathcal{O}(-3)\circ\mathrm{Orl}_{t-3})^{-1}\\=&\otimes\mathcal{O}(-3)\circ(\mathrm{Shi})^{-1}\circ\Phi_{t-2}\circ\Phi_{t-3}^{-1}\circ\mathrm{Shi}\circ\otimes\mathcal{O}(3),\end{aligned}$$

which is conjugate to

$$\Phi_{t-2}\circ\Phi_{t-3}^{-1}.$$

Then we can apply Theorem 3.13 in [41] to conclude.

□

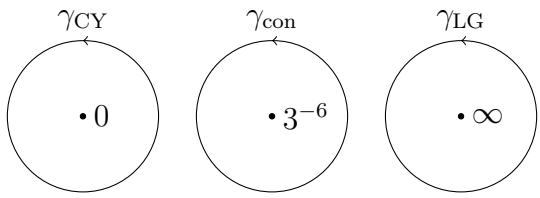

Figure 4.1 – Paths in $\mathbb{P}^1_v \backslash \{0, 3^{-6}, \infty\}$.

Remark 4.13. Under the basis $H^{(1)}, \mathbb{1}^{(1)}, H^{(2)}, \mathbb{1}^{(2)}$, the action induced by γ_{LG} can be represented by

$$\begin{pmatrix} \zeta_3 & \frac{\zeta_3}{3} & & \\ & \zeta_3 & & \\ & & \zeta_3^2 & \frac{\zeta_3^2}{3} \\ & & & \zeta_3^2 \end{pmatrix}.$$

It has infinite order, so the point ∞ is no longer a orbifold point as in the hypersurface cases [9]. It is called "K-type point" in, for example, [44].

Remark 4.14. We can use the same method to determine the monodromy around ∞ in the case of complete intersection of four conics in $\mathbb{P}^7$. Up to conjugation, the matrix representation of the monodromy is

$$\begin{pmatrix} \zeta_2 & \frac{\zeta_2}{2} & \frac{\zeta_2}{8} & \frac{\zeta_2}{48} \\ & \zeta_2 & \frac{\zeta_2}{2} & \frac{\zeta_2}{8} \\ & & \zeta_2 & \frac{\zeta_2}{2} \\ & & & \zeta_2 \end{pmatrix}.$$

Its square is maximally unipotent. Since the monodromy around 0 is also maximally unipotent, it is a model with two maximally unipotent type points. This model was studied by Joshi and Klemm in [33], and we check this property by FJRW theory.

4.6 Relation with the crepant transformation conjecture

In [18], Coates, Iritani and Jiang prove a theorem relating the equivariant GW theory of X_+ and X_-. We discuss the relation between their result and Theorem 4.10.

4.6.1 Coates, Iritani and Jiang's result

Theorem 4.15 ([18]). *Let $T = (C^*)^8$ be a torus with standard action on $\mathbb{C}^{6+2}$, and induced action on X_+ and X_-. We have*

1. *The analytic continuation (along a chosen path) of the $\mathfrak{H}$-function of the T-equivariant GW theory of X_+ coincide with the $\mathfrak{H}$-function of the T-equivariant GW theory of X_- after the linear map*
$$\widetilde{\mathbb{U}_T}\colon H^*_{CR,T}(X_-) \to H^*_{CR,T}(X_+)$$

2. *There exists a common T-equivariant blowup $\widetilde{X}$ of $X_\pm$. The Fourier–Mukai transformation with kernel $\mathcal{O}_{\widetilde{X}}$:*
$$\mathbb{FM}_T\colon D^b_T(X_-) \to D^b_T(X_+)$$
coincides with $\widetilde{\mathbb{U}_T}$ via Chern character, i.e. the following diagram commutes
$$\begin{tikzcd} D^b_T(X_-) \arrow[r, "\mathbb{FM}_T"] \arrow[d, "inv^*\,\mathrm{ch}"'] & D^b_T(X_+) \arrow[d, "inv^*\,\mathrm{ch}"] \\ H^*_{CR,T}(X_-) \arrow[r, "\widetilde{\mathbb{U}_T}"] & H^*_{CR,T}(X_+). \end{tikzcd}$$

Remark 4.16. Since we can define a non-equivariant Fourier–Mukai transformation
$$\mathbb{FM}\colon \mathrm{D^b}(X_-) \to \mathrm{D^b}(X_+),$$
the linear map $\widetilde{\mathbb{U}_T}$ admits a non-equivariant limit $\widetilde{\mathbb{U}}$.

4.6.2 A circle of ideas

Now we discuss the relation between Theorem 4.10 and Theorem 4.15.

In fact, we can relate $\widetilde{\mathbb{U}}$ to $\mathbb{U}_t$, and relate $\mathbb{FM}$ to Orl_t. The circle of ideas is as follows. The analogue version for for Calabi–Yau hypersurface case can be found in [43].

In [20], the Fourier–Mukai functor $\mathbb{FM}$ is proven to be equivalent to the functor $\widetilde{\Phi_0}$ defined from graded restriction rule in §3.3.

Proposition 4.17 ([20])**.** *The Fourier–Mukai functor*
$$\mathbb{FM}\colon D^b(X_-) \to D^b(X_+)$$
and the functor
$$\widetilde{\Phi_0}\colon D^b(X_-) \to D^b(X_+)$$
are isomorphic.

We decompose
$$H^*_{\mathrm{CR}}(X_-) = H^*(\mathbb{P}(3,3)) \oplus H^{*+4}(\mathbb{P}(3,3)) \oplus H^{*+8}(\mathbb{P}(3,3))$$

as

$$\mathbb{C}\mathbb{1}^{(0)} \oplus \mathbb{C}H^{(0)} \oplus \mathbb{C}\mathbb{1}^{(1)} \oplus \mathbb{C}H^{(1)} \oplus \mathbb{C}\mathbb{1}^{(2)} \oplus \mathbb{C}H^{(2)}.$$

Consider the linear maps

$$\widetilde{\mathbb{U}_l} \colon H^*_{\mathrm{CR}}(X_-) \to H^*_{\mathrm{CR}}(X_+)$$

given by

$$\begin{aligned} \mathbb{1}^{(k)} &\mapsto \left(\frac{l+3}{9} \frac{\left(\zeta^k e^p\right)^{l+6}}{1-\zeta^k e^p} + \frac{1}{9} \frac{\left(\zeta^k e^p\right)^{l+7}}{\left(1-\zeta^k e^p\right)^2} \right) \cdot (1-e^{-3p})^2 \\ H^{(k)} &\mapsto \frac{1}{3} \frac{\left(\zeta^k e^p\right)^{l+6}}{1-\zeta^k e^p} \cdot (1-e^{-3p})^2 \end{aligned} \tag{4.11}$$

where $\zeta = e^{\frac{2\pi i}{3}}$. Note that the non-equivariant limit $\widetilde{\mathbb{U}}$ of $\widetilde{\mathbb{U}_T}$ coincides with $\widetilde{\mathbb{U}_0}$.

A direct computation shows that for all $\mathcal{E} \in \mathrm{D^b}(X_-)$, we have

$$\widetilde{\mathbb{U}_l}\left(\mathrm{inv}^* \operatorname{ch}(\mathcal{E})\right) = \mathrm{inv}^* \operatorname{ch}\left(\widetilde{\Phi_l}(\mathcal{E})\right). \tag{4.12}$$

After identifying

$$\mathbb{C}\mathbb{1}^{(1)} \oplus \mathbb{C}H^{(1)} \oplus \mathbb{C}\mathbb{1}^{(2)} \oplus \mathbb{C}H^{(2)} \subset H^*_{\mathrm{CR}}(X_-)$$

with H_{FJRW}, we can compare $\mathbb{U}_t$ in Theorem 4.10 with $\widetilde{\mathbb{U}_t}$. In fact, the following proposition holds.

Proposition 4.18. *For all $\alpha \in H_{\mathrm{FJRW}}$, we have*

$$\mathbb{U}_t(\alpha) \cdot (1-e^{3p})^2 \cdot e^{-6p} = i^* \left(\widetilde{\mathbb{U}_{t-3}}(\alpha) \cdot e^{-3p}\right), \tag{4.13}$$

where $i \colon X_{3,3} \to X_+$ is the inclusion.

Proof. We can conclude by comparing (2.7) and (4.11) directly. □

On the other hand, we can compare the functors Orl_t with $\widetilde{\Phi_l}$. In fact, we have a K-theoretic commutative diagram.

Proposition 4.19. *The diagram*

$$\begin{tikzcd} & K\left(D^b(X_-)\right) \arrow[r, "\widetilde{\Phi_t}"] & K\left(D^b(X_+)\right) \arrow[d, "\pi_*"'] \\ K\left(D^b(\mathbb{P}(3,3))\right) \arrow[ur, "i_*"] \arrow[dr, "i_*"'] & & K\left(D^b(\mathbb{P}^5)\right) \arrow[d, "i^*"'] \\ & K\left(DMF^{\mathbb{C}^*_R}(X_-,W)\right) \arrow[r, "\otimes\mathcal{O}(-6)[2]\circ\mathrm{Orl}_t"] & K\left(D^b(X_{3,3})\right) \end{tikzcd} \tag{4.14}$$

commutes, where $K(D)$ represents the K-group of the triangulated category D.

Proof. In K-theory, the computation of $\widetilde{\Phi_t}$ is essentially the same as the computation of Φ_t in §4.1; where the Koszul matrix factorizations $\mathcal{K}_-$ and $\mathcal{K}_+$ are replaced by Koszul complexes. The proof of the commutativity of diagram (4.14) is a tautological check. □

Remark 4.20. Note that we can deduce Proposition 4.18 from Proposition 4.19 according to Theorem 4.10 and equation (4.12).

Remark 4.21. The method of analytic continuation used by Coates, Iritani and Jiang in [18] is similar to the one we used in §2. The complexity of our computation is cause by the multiplicity of poles of (2.9). Coates, Iritani and Jiang work in equivariant cohonology so that all the poles are simple. We try to make our result into the equivariant setting to split the poles. A candidate of the group action on the FJRW theory is the $(\mathbb{C}^*)^2$-action on X_- induced by the standard $(\mathbb{C}^*)^2$-action on the last two coordinates of $\mathbb{C}^6 \times \mathbb{C}^2$. However, this $(\mathbb{C}^*)^2$-action does not fix $\mathbb{P}(3,3) \subset X_-$. Then the Grothendieck–Riemann–Roth computation for the theory twisted by the $(\mathbb{C}^*)^2$-equivariant top Chern class (see §1.4) can not be done as in the standard cases, where the group only acts on the fiber direction.

Bibliography

[1] Dan Abramovich. "Lectures on Gromov-Witten invariants of orbifolds". *Enumerative invariants in algebraic geometry and string theory.* Vol. 1947. Lecture Notes in Math. Springer, Berlin, 2008, 1–48.

[2] Dan Abramovich, Tom Graber, and Angelo Vistoli. "Gromov-Witten theory of Deligne-Mumford stacks". *Amer. J. Math.* 130.5 (2008), 1337–1398.

[3] Dan Abramovich and Angelo Vistoli. "Compactifying the space of stable maps". *J. Amer. Math. Soc.* 15.1 (2002), 27–75.

[4] Matthew Ballard, David Favero, and Ludmil Katzarkov. "Variation of geometric invariant theory quotients and derived categories". *J. Reine Angew. Math.* 746 (2019), 235–303.

[5] Kai Behrend and Barbara Fantechi. "The intrinsic normal cone". *Invent. Math.* 128.1 (1997), 45–88.

[6] Tom Bridgeland. "Stability conditions on $K3$ surfaces". *Duke Math. J.* 141.2 (2008), 241–291.

[7] Huai-Liang Chang and Jun Li. "Gromov-Witten invariants of stable maps with fields". *Int. Math. Res. Not. IMRN* 18 (2012), 4163–4217.

[8] Huai-Liang Chang, Jun Li, and Wei-Ping Li. "Witten's top Chern class via cosection localization". *Invent. Math.* 200.3 (2015), 1015–1063.

[9] Alessandro Chiodo, Hiroshi Iritani, and Yongbin Ruan. "Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence". *Publ. Math. Inst. Hautes Études Sci.* 119 (2014), 127–216.

[10] Alessandro Chiodo and Jan Nagel. "The hybrid Landau-Ginzburg models of Calabi-Yau complete intersections". *Topological Recursion and its Influence in Analysis, Geometry, and Topology.* Vol. 100. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2018, 103–118.

[11] Alessandro Chiodo and Yongbin Ruan. "Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations". *Invent. Math.* 182.1 (2010), 117–165.

[12] Ionut Ciocan-Fontanine, David Favero, Jérémy Guéré, Bumsig Kim, and Mark Shoemaker. "Fundamental Factorization of a GLSM, Part I: Construction". *ArXiv e-prints* (2018). arXiv: 1802.05247.

[13] Emily Clader. "Landau-Ginzburg/Calabi-Yau correspondence for the complete intersections $X_{3,3}$ and $X_{2,2,2,2}$". *Adv. Math.* 307 (2017), 1–52.

[14] Emily Clader, Felix Janda, and Yongbin Ruan. "Higher-genus wall-crossing in the gauged linear sigma model". *ArXiv e-prints* (2017). arXiv: 1706.05038.

[15] Emily Clader and Dustin Ross. "Sigma models and phase transitions for complete intersections". *Int. Math. Res. Not. IMRN* 15 (2018), 4799–4851.

[16] Tom Coates. "On the crepant resolution conjecture in the local case". *Comm. Math. Phys.* 287.3 (2009), 1071–1108.

[17] Tom Coates and Alexander Givental. "Quantum Riemann-Roch, Lefschetz and Serre". *Ann. of Math. (2)* 165.1 (2007), 15–53.

[18] Tom Coates, Hiroshi Iritani, and Yunfeng Jiang. "The crepant transformation conjecture for toric complete intersections". *Adv. Math.* 329 (2018), 1002–1087.

[19] Tom Coates, Yuan-Pin Lee, Alessio Corti, and Hsian-Hua Tseng. "The quantum orbifold cohomology of weighted projective spaces". *Acta Math.* 202.2 (2009), 139–193.

[20] Tom Coates, Hiroshi Iritani, Yunfeng Jiang, and Ed Segal. "K-theoretic and categorical properties of toric Deligne-Mumford stacks". *Pure Appl. Math. Q.* 11.2 (2015), 239–266.

[21] David A Cox and Sheldon Katz. "Mirror symmetry and algebraic geometry". *Mathematical surveys and Monographs* 68 (2000), 190.

[22] Pierre Deligne and David Mumford. "The irreducibility of the space of curves of given genus". *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), 75–109.

[23] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan. "The Witten equation and its virtual fundamental cycle". *ArXiv e-prints* (2007). arXiv: 0712.4025.

[24] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan. "Geometry and analysis of spin equations". *Comm. Pure Appl. Math.* 61.6 (2008), 745–788.

[25] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan. "The Witten equation, mirror symmetry, and quantum singularity theory". *Ann. of Math. (2)* 178.1 (2013), 1–106.

[26] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan. "A mathematical theory of the gauged linear sigma model". *Geom. Topol.* 22.1 (2018), 235–303.

[27] William Fulton and Rahul Pandharipande. "Notes on stable maps and quantum cohomology". *Algebraic geometry—Santa Cruz 1995.* Vol. 62. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1997, 45–96.

[28] Alexander B. Givental. "Equivariant Gromov-Witten invariants". *Internat. Math. Res. Notices* 13 (1996), 613–663.

[29] Manfred Herbst, Kentaro Hori, and David Page. "Phases of $N = 2$ theories in $1 + 1$ dimensions with boundary". *ArXiv e-prints* (2008). arXiv: 0803.2045.

[30] Richard Paul Horja. *Hypergeometric functions and mirror symmetry in toric varieties.* Thesis (Ph.D.)–Duke University. ProQuest LLC, Ann Arbor, MI, 1999, 120.

[31] Hiroshi Iritani. "An integral structure in quantum cohomology and mirror symmetry for toric orbifolds". *Adv. Math.* 222.3 (2009), 1016–1079.

[32] Mehmet Umut Isik. "Equivalence of the derived category of a variety with a singularity category". *Int. Math. Res. Not. IMRN* 12 (2013), 2787–2808.

[33] Abhinav Joshi and Albrecht Klemm. "Swampland distance conjecture for one-parameter Calabi-Yau threefolds". *J. High Energy Phys.* 2019.8 (2019), 86.

[34] Young-Hoon Kiem and Jun Li. "Localizing virtual cycles by cosections". *J. Amer. Math. Soc.* 26.4 (2013), 1025–1050.

[35] Maxim Kontsevich and Yu Manin. "Gromov-Witten classes, quantum cohomology, and enumerative geometry". *Comm. Math. Phys.* 164.3 (1994), 525–562.

[36] Yuan-Pin Lee, Nathan Priddis, and Mark Shoemaker. "A proof of the Landau–Ginzburg/Calabi–Yau correspondence via the crepant transformation conjecture". *Ann. Sci. Éc. Norm. Supér. (4)* 49.6 (2016), 1403–1443.

[37] Anatoly Libgober and Jeremy Teitelbaum. "Lines on Calabi-Yau complete intersections, mirror symmetry, and Picard-Fuchs equations". *Int. Math. Res. Not. IMRN* 1 (1993), 29–39.

[38] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.* 1994.

[39] Dmitri Orlov. "Derived categories of coherent sheaves and triangulated categories of singularities". Progr. Math. 270 (2009), 503–531.

[40] Alexander Polishchuk and Arkady Vaintrob. "Matrix factorizations and cohomological field theories". *J. Reine Angew. Math.* 714 (2016), 1–122.

[41] Ed Segal. "Equivalence between GIT quotients of Landau-Ginzburg B-models". *Comm. Math. Phys.* 304.2 (2011), 411–432.

[42] Ian Shipman. "A geometric approach to Orlov's theorem". *Compos. Math.* 148.5 (2012), 1365–1389.

[43] Mark Shoemaker. "Integral transforms and quantum correspondences". *arXiv preprint arXiv:1811.01879* (2018).

[44] Duco van Straten. "Calabi–Yau Operators". *ArXiv e-prints* (2017). arXiv: 1704.00164.

[45] Bertrand Toën. "Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford". *K-Theory* 18.1 (1999), 33–76.

[46] Edward Witten. "Phases of $N = 2$ theories in two dimensions". *Nuclear Phys. B* 403.1-2 (1993), 159–222.