Partial differential equations on graphs: continuum limits on sparse graphs and applications

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THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHEMATIQUES

Préparée au sein de l'Université de Caen Normandie

Partial differential equations on graphs : Continuum limits on sparse graphs and applications

Présentée et soutenue par IMAD EL BOUCHAIRI

Thèse soutenue le 05/10/2021 devant le jury composé de		
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UNIVERSITÉ CAEN NORMANDIE

GREYC

To my family.

To the memory of my father.

1930-2016.

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Abstract

The nonlocal p-Laplacian operator, the associated evolution equation and boundary value problem, governed by a given kernel, have applications in various areas of science and engineering. In particular, they have become modern tools for massive data processing (including signals, images, geometry), and machine learning tasks such as semi-supervised learning.

In practice, these models are executed in discrete form (in space and time, or in space for the boundary value problem) as a numerical approximation to a continuous problem, where the kernel is replaced by an adjacency matrix of a graph. In this work, we first focus on the study of numerical approximations of these models. Combining tools from graph theory, convex analysis, Γ-convergence, nonlinear semigroup theory and evolution equations, we give a rigorous interpretation to the nonlocal continuous limit of the discrete nonlocal p-Laplacian evolution and boundary value problems on sparse graphs. Along the way, we provide consistency/error bounds. These results lead us to derive rate of convergence of solutions for the discrete models on K-random sparse graphs to the solution of the corresponding nonlocal problems on the continuum, as the number of vertices grows to infinity, and we highlight the influence of p, the sparsity of the graphon, and the regularity of initial/boundary data on the convergence rate.

In the context of image processing, we introduce a class of the analogue p-bilaplacian operators on graphs. We then turn to study regularized variational and boundary value problems associated to these operators on graphs. In the same vein, we introduce a general class of nonlocal discrete perimeters as well as mean curvature flow. These lead us to translate and establish an adaptation of the mean curvature level set equations on a general discrete domain.

Keywords: Nonlocal diffusion, evolution problem, boundary value problem, p-Laplacian, p-bilaplacian, nonlocal perimeter, mean curvature, graphs, sparse graph, L^q-graphon, graph limits, numerical approximation, error bound, convergence rate, convex analysis.

Résumé

L'opérateur du p-Laplacien nonlocal régi par un noyau donné, l'équation d'évolution et le problème aux limites associés régies par un noyau donné ont des applications dans divers domaines de la science et de l'ingénierie. En particulier, ils sont devenus des outils modernes pour le traitement des données massives (y compris les signaux, les images, la géométrie) et dans les tâches d'apprentissage automatique telles que l'apprentissage semi-supervisé.

En pratique, ces modèles sont implémentés sous forme discrète (en espace et en temps, ou en espace pour le problème aux limites) comme approximation numérique d'un problème continu, où le noyau est remplacé par la matrice d'adjacence d'un graphe. Dans ce travail, on se concentre dans un premier temps sur l'étude des approximations numériques de ces modèles. En combinant des outils de la théorie des graphes, de l'analyse convexe, Γ-convergence, de la théorie des semi-groupes nonlinéaires et des équations d'évolution, nous interprétons rigoureusement la limite continue du problème d'évolution et du problème aux limites du p-Laplacien discrets sur graphes parcimonieux. Ce faisant, on fournit des bornes d'erreur/consistance. Cela permit d'établir les taux de convergence nonasymptotiques en probabilité et en présentant le rôle de p, de la parcimonie du graphe, de la régularité des données initiales sur la vitesse de convergence.

Dans le cadre du traitement d'image, nous introduisons une classe d'opérateurs analogiques à l'opérateur p-bilaplacien sur graphes. Nous nous tournons ensuite vers l'étude du problème de régularisation variationnels et le problème aux limites associés à ces opérateurs sur graphes. Dans le même cadre, nous introduisons une classe générale de périmètres discrets non locaux ainsi que la courbure moyenne. Ceux-ci, nous amènent à transcrire et établir une adaptation des équations d'ensembles de niveaux de courbure moyenne sur un domaine discret général.

Mots-clés: Diffusion nonlocale, problème d'évolution, problème aux limites, p-Laplacien, p-bilaplacien, périmètre nonlocal, courbure moyenne, graphes, graphes parcimonieux, L^q-graphons, limites de graphes, approximation numérique, borne d'erreur, vitesse de convergence, analyse convexe.

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Chapter 1

Introduction

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1.1 Context and motivations

1.1.1 Context and main problems

Partial differential equations (PDEs) play an important role in mathematical modelling throughout applied and natural sciences. Indeed, several problems end up modelling and solving an evolution or a boundary value problem involving different kinds of operators depending on the tasks to carry out. The methods based on PDEs have also proven to provide very effective tools in various fields throughout science and engineering such as signal/image processing, machine learning, computer vision and biology [14, 52, 118, 15, 82, 11]. Such methods have the advantages of better mathematical modelling, connections with physics and better geometrical approximations. Differential operators involved in these methods are classically based on local derivatives that reflect local interactions in the support domain. The nonlocal counterparts have been introduced, in different settings e.g. [133, 106, 77, 22, 121], which are based on the integral form, particularly with respect to spatial variables, that reflect nonlocal interactions between the points in support domain. Recently, nonlocal models have been proposed in the context of image processing to design gradient-based regularization functionals and PDEs associated with their minimization [85] for many image processing tasks, such as denoising, deconvolution, segmentation, inpainting, optical-flow to name a few. A main advantage for image processing is the ability to process both structures (geometrical parts) and textures within the same framework. For instance, several work have been studied behaviour of nonlocal models under various classical perturbation limits, since they have similar properties as the local ones, it consists generally to replace the local operators in PDEs with newly defined nonlocal analogue operators converging to the local one in the continuum limit, see for example [133, 10, 150, 123]. Unlike classical PDE models, in the nonlocal setting the boundary conditions must be defined on a region with non-zero volume outside the surface [56, 65, 137, 2], in contrast to more traditional scenarios where boundary conditions are typically imposed on a sharp co-dimension one surface. The construction of such operators have been built on ideas developed in graph theory and nonlocal calculus of variations, e.g. nonlocal gradient, nonlocal divergence, nonlocal curl, and nonlocal Laplacian, see e.g. [85, 66, 3, 122], and references therein. Following these ideas, it has been shown that many PDE-based processes, minimizations and computation methods can be generalized to the nonlocal setting.

Among the operators introduced in this setting the nonlocal p-Laplacian operator, that has become more popular both in the setting of Euclidean domains and on discrete graphs, as the p-Laplacian problem possesses many important features shared by many practical problems in mathematics, physics, engineering, biology, and economy, such as continuum mechanics, phase transition phenomena, population dynamics [9, 19, 20, 47, 81, 148, 80] and references therein. Some closely related applications can be found in image processing, computer vision and machine learning [40, 71, 74, 100].

In the continuum case, this operator is defined on $L^p(\Omega)$ for a bounded set Ω, $p \in [1, +\infty[$, being a set-valued mapping for $p = 1$ and $p = \infty$, as follows

$$\Delta_p^K u(\boldsymbol{x}) = -\int_\Omega K(\boldsymbol{x},\boldsymbol{y})\big|u(\boldsymbol{y}) - u(\boldsymbol{x})\big|^{p-2}(u(\boldsymbol{y}) - u(\boldsymbol{x}))d\boldsymbol{y},$$

where Ω is a bounded set in $\mathbb{R}^d$ and $K(\cdot,\cdot)$ is a symmetric, non-negative measurable function on Ω^2. It can be seen as the nonlocal analogue of the p-Laplacian operator defined on $W^{1,p}(\Omega)$ for $p \in [1, +\infty[$, being also a set-valued mapping for $p = 1$ and $p = \infty$, as

$$\Delta_p u(\boldsymbol{x}) = \operatorname{div}\left(\big|\nabla u(\boldsymbol{x})\big|^{p-2}\nabla u(\boldsymbol{x})\right),$$

which occurs also in many mathematical models and physical processes such as nonlinear diffusion/filtration and non-Newtonian flows [27]. The nonlocal p-Laplacian operator is the negative gradient of the p-Dirichlet energy,

$$R(u,K) \overset{\text{def}}{=} \frac{1}{p}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left|u(\boldsymbol{y})-u(\boldsymbol{x})\right|^p d\boldsymbol{x}d\boldsymbol{y}, \tag{1.1.1}$$

which is the nonlocal analogue to the energy functional $\frac{1}{p}\int_\Omega |\nabla u|^p$ associated to the local p-Laplacian.

Nonlocal boundary value problem. The nonlinear nonlocal "elliptic" boundary value problem, known as the nonlocal p-Laplacian Dirichlet problem [93] associated to Δ_p^K is

$$\begin{cases} -\Delta_p^K u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in U, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma \overset{\text{def}}{=} \Omega \setminus U, \end{cases} \tag{$\mathcal{P}^D_{\text{nloc}}$}$$

where U is a bounded subset of Ω and Γ is a "collar" surrounding U which has nonzero volume with $\Omega = U \cup \Gamma$. The nonlocal boundary value problem shares many properties with the corresponding classical elliptic boundary value problem

$$\begin{cases} \Delta_p u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{$\mathcal{P}^D_{\text{loc}}$}$$

It has been shown in [88] that for $p = 2$ the problem ($\mathcal{P}^D_{\text{nloc}}$) provides a nonlocal equivalent of the problem ($\mathcal{P}^D_{\text{loc}}$). It has been shown also in [150] that the nonlocal Neumann-type boundary value problem governed by the nonlocal p-Laplacian recovers the classical Neumann problem as the nonlocal horizon parameter vanishes.

It has been proved in [93], using the Dirichlet principle, that the minimizers of the energy functional

$$\mathcal{F}(u) \overset{\text{def}}{=} \frac{1}{2p}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left|u(\boldsymbol{y})-u(\boldsymbol{x})\right|^p d\boldsymbol{x}d\boldsymbol{y} + \int_\Omega f(\boldsymbol{x})u(\boldsymbol{x})d\boldsymbol{x} + \iota_{L^p_g(\Omega,U)}(u) \tag{1.1.2}$$

satisfy ($\mathcal{P}^D_{\text{nloc}}$) and conversely, any solution of ($\mathcal{P}^D_{\text{nloc}}$) is a minimizer for $\mathcal{F}$,where $L^p_g(\Omega,U)$ is the space of functions in $L^p(\Omega)$ which coincide with g on Γ.

Nonlocal Cauchy problem. Another problem governed by this nonlocal operator is the nonlinear diffusion problem (Cauchy problem), known as the nonlocal p-Laplacian evolution problem with homogeneous Neumann boundary conditions [11]

$$\begin{cases} \frac{\partial}{\partial t}u(\boldsymbol{x},t) = -\Delta_p^K u(\boldsymbol{x}) + f(\boldsymbol{x},t), & \boldsymbol{x} \in \Omega, t > 0, \\ u(\boldsymbol{x},0) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{$\mathcal{P}_{\text{nloc}}$}$$

This nonlocal diffusion problem, in turn, shares many properties with the corresponding local one. If $f = 0$ and the kernel K is radially symmetric and properly rescaled with a parameter ε, it has been shown in [10] that the solutions to the nonlocal problems ($\mathcal{P}_{\text{nloc}}$) converge strongly in $L^\infty((0,T);L^p(\Omega))$, as ε goes to zero, to the solution of the well-known local p-Laplacian evolution problem

$$\begin{cases} u_t(\boldsymbol{x},t) = \frac{\partial}{\partial t}u(\boldsymbol{x},t) = \Delta_p(u(\boldsymbol{x},t)), & \boldsymbol{x} \in \Omega, t > 0, \\ u(\boldsymbol{x},0) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{$\mathcal{P}_{\text{loc}}$}$$

which corresponds for $p = 2$ to the heat equation $u_t(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t)$, while the extreme case, $p = 1$, corresponds to the total variation flow with homogeneous Neumann boundary conditions. The problem ($\mathcal{P}_{\text{loc}}$) occurs also in many applications such as physics, biology or economy [104, 62].

A particular case where $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$, with the kernel $J : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative continuous radial function with compact support verifying $J(0) > 0$ and $\int_{\mathbb{R}^d} J(\boldsymbol{x})d\boldsymbol{x} = 1$, nonlocal evolution equations of the form

$$u_t(\boldsymbol{x},t) = J * u(\boldsymbol{x},t) - u(\boldsymbol{x},t) = \int_{\mathbb{R}^d} J(\boldsymbol{x}-\boldsymbol{y})(u(\boldsymbol{y},t) - u(\boldsymbol{x},t))d\boldsymbol{y}, \tag{$\mathcal{P}^*_{\text{nloc}}$}$$

where $*$ stands for the convolution, have many applications in modeling diffusion processes [9, 19, 20, 47, 81, 148, 80]. As stated in [81], in modeling the dispersal of organisms in space when $u(\boldsymbol{x}, t)$ is their density at the point $\boldsymbol{x}$ at time t, $J(\boldsymbol{x}-\boldsymbol{y})$ is considered as the probability distribution of jumping from position $\boldsymbol{y}$ to position $\boldsymbol{x}$, then, the expression $J*u-u$ represents transport due to long-range dispersal mechanisms, that is the rate at which organisms are arriving to location $\boldsymbol{x}$ from any other place.

The nonlocal p-bilaplacian. In the context of the peridynamics, by iterating the nonlocal Laplacian, it has been introduced in [123] a nonlocal version of the bilaplacian operator, which can be generalized to $p \in]1, +\infty[$. This operator that we coin p-bilaplacian, is defined on $L^p(\Omega)$ for a bounded domain Ω, $p \in]1, +\infty[$ as follows

$$\Delta^2_{K,p} u(\boldsymbol{x}) \stackrel{\text{def}}{=} \Delta^K_2 \left(\left| \Delta^K_2 u \right|^{p-2} \Delta^2_K u \right)(\boldsymbol{x}), \tag{1.1.3}$$

where Δ^K_2 is the nonlocal Laplacian operator governed by the nonnegative symmetric measurable kernel K. It can be seen as the analogue of the local p-bilaplacian operator defined on $W^{2,p}(\Omega)$ as

$$\Delta^2_p u(\boldsymbol{x}) \stackrel{\text{def}}{=} \Delta \left(\left| \Delta u \right|^{p-2} \Delta_2 u \right)(\boldsymbol{x}), \tag{1.1.4}$$

which is a fourth-order operator, see e.g. [99]. The nonlocal p-bilaplacian operator can be interpreted as the gradient of the following energy functional

$$\mathcal{F}^{nloc}(u) = \frac{1}{2p} \int_\Omega \left| \Delta^K_2 u(\boldsymbol{x}) \right|^p d\boldsymbol{x},$$

which is the nonlocal analogue of the energy functional, associated to the usual p-bilaplacian operator,

$$\mathcal{F}^{loc}(u) = \frac{1}{p} \int_\Omega \left| \Delta u(\boldsymbol{x}) \right|^p d\boldsymbol{x}.$$

The nonlocal perimeter. Another notion was introduced, in the context of the nonlocal theory using the nonlocal 1-Dirichlet energy, called the nonlocal J-perimeter, see [33, 59]. The nonlocal J-perimeter of a set $E \in \mathbb{R}^d$ is defined by the following formula

$$\mathrm{Per}_J(E) \stackrel{\text{def}}{=} \int_E \int_{\mathbb{R}^d \setminus E} J(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(\boldsymbol{x}, \boldsymbol{y}) \left| \chi_E(\boldsymbol{y}) - \chi_E(\boldsymbol{x}) \right| d\boldsymbol{x} d\boldsymbol{y}, \tag{1.1.5}$$

where J is a nonnegative symmetric radial function in $L^1(\mathbb{R}^d)$ and χ_E the characteristic function of E. This definition of perimeter is nonlocal in the sense that it is determined by the behaviour of E in a neighborhood of the boundary ∂E. It can be seen as the nonlocal analogue of the usual perimeter,

$$\mathrm{Per}(E) = \int_{\mathbb{R}^d} \left| D\chi_E(\boldsymbol{x}) \right| d\boldsymbol{x}, \tag{1.1.6}$$

where $D\chi_E$ is the distributional derivatives of χ_E. The main idea of the nonlocal perimeter is that any point inside an Euclidean set "interact" with any point outside the set, given a functional whose minimization is taken account. This notion of the nonlocal J-perimeter, in turn, was used to introduce the concept of the J-mean curvature, which is defined at a point $\boldsymbol{x}$ for ∂E, with E a subset of $\mathbb{R}^d$, as follows

$$H^J_{\partial E}(\boldsymbol{x}) \stackrel{\text{def}}{=} -\int_{\mathbb{R}^d} J(\boldsymbol{x}-\boldsymbol{y}) \left(\chi_E(\boldsymbol{y}) - \chi_{\mathbb{R}^d \setminus E}(\boldsymbol{y}) \right) d\boldsymbol{y}. \tag{1.1.7}$$

1.1.2 Motivations

In many real-world problems, such as in mathematical data processing and machine learning, the data is discrete, and graphs constitute a natural structure suited to their representation. Each vertex of the graph corresponds to a datum, and the edges encode the pairwise relationships or similarities among the data. For the particular case of images, pixels (represented by nodes) have a specific organization expressed by their spatial connectivity. Therefore, a typical graph used to represent images is a grid graph. For the case of unorganized data such as point clouds, a graph can also be built by modelling neighbourhood relationships between the data elements.

Figure 1.1: Examples of images that can be represented by weighted graphs as their natural representation.

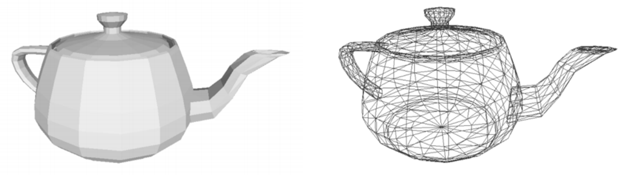


Figure 1.2: Examples of meshes that can be represented by weighted graphs as their natural representation.

Figure 1.3: Examples of networks that can be represented by weighted graphs as their natural representation.

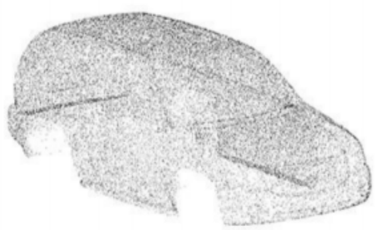


Figure 1.4: Example of point clouds/unorganized data that can be represented by weighted graphs.

For these reasons, there has been recently a wave of interest in adapting and solving nonlocal boundary value problems such as ($\mathcal{P}^D_{\mathrm{nloc}}$) and PDEs such as ($\mathcal{P}_{\mathrm{nloc}}$) on data which is represented by arbitrary graphs and networks. This requires translating their corresponding operators to the discrete setting. This principle also applies to other problems governed by many operators such as the nonlocal operator (1.1.3), the nonlocal perimeter (1.1.5). This in turn allows to attack nonlocal analogues of many problems such boundary value and variational problems, computing minimal surfaces, as well as Cheeger/Calibrable sets. Among the methods proposed to tackle such nonlocal problems in a discrete setting, we will focus here on that of partial difference equations (PdEs) on graphs. Using this framework, problems are directly expressed in a discrete setting where an appropriate discrete differential calculus have been proposed; see e.g. [73, 75] and references therein. Conceptually, the idea of introducing PdEs is to mimic continuum PDEs on graph structures by consistently adapting important mathematical concepts, e.g., integration and differentiation. This mimetic approach consists of replacing continuous differential operators, e.g., gradient or divergence, by reasonable discrete analogues, which makes it possible to transfer many important tools and results from the continuous setting. This way to proceed encompasses local and nonlocal methods in the same framework by using appropriate graphs topologies and edge weights depending on the data structure and the task to be performed. The demand for such methods is motivated by existing and potential future applications [75, 72, 143, 70, 84].

These practical considerations lead naturally to a discrete time and space approximation of ($\mathcal{P}_{\text{nloc}}$) and a space approximation of ($\mathcal{P}^{D}_{\text{nloc}}$) encoded by the structure of the graph. This can be extended also to cover regularization variational/boundary value problems governed by the p-bilapalcian (1.1.4), local/nonlocal mean curvature curvature flows and many other problems. So that these discrete problems can be applied in the same way to images, meshes or data of any size by simply adapting the topology of the graph and the weight function. The proposed framework works on any discrete data represented by weighted graphs which allows to take into account the nonlocal interactions in the data by explicitly introducing discrete nonlocal derivatives and functionals on graphs of arbitrary topologies, to transcribe the continuous setting.

Main goals of our work. The main goals of our work is twofold.

Our first is to design fully discretized problems (evolution, boundary value and variational problems) in space (graphs) and time, and show that they are provably consistent with respect to their continuum analogues. Indeed, the discrete nonlocal problems on graph are just approximations of the underlying nonlocal continuum problems. Thus, our objective is to rigorously the following legitimate questions for each problem:

- (**Q1**) Is there any (nonlocal) continuum limit as the number of vertices grows and time step vanishes ? If yes, in what sense ?
- (**Q2**) What is the rate of convergence to this limit and what is its relation to the solution of the continuum problem ?
- (**Q3**) What are the parameters involved in this convergence and what is their influence in the corresponding rate ?
- (**Q4**) Can this continuum limit help us get better insight into discrete models and their fundamental guarantees ?

In the literature, numerous works, that will review later, have been carried out in the recent years attempting to answer some of these questions. It is however important to stress that our focus will be nonlocal (in contrast to local ones) continuum limits. This is more in line with a numerical analysis standpoint. We will also be mostly interested in graph structures that show a sparsity behaviour and have applications such as in social networks. This will pose several challenges that we will solve properly.

The second objective of this work is to introduce a novel class of p-bilaplacian operators on weighted graphs, which can be seen as proper discretizations on graphs of the classical p-bilaplacian operators (1.1.4). Building upon this definition, we study the corresponding regularized variational problem as well as a boundary value problem. Finally we revisit the notions of the discrete perimeter and mean curvature. then, we propose a general adaptation and transcription of the level set mean curvature equation and variational curvature on the general discrete domain, weighted graphs.

1.2 Contributions and relation to prior work

1.2.1 The continuum limits of the evolution problem on sparse graphs

Our first main result, which is exposed in Chapter 3, is to revisit the nonlocal diffusion problem ($\mathcal{P}_{\text{nloc}}$) and extend the results of the work [90] to a much more general class of kernels and initial data. In particular, we are able to consider unbounded initial data, the case $p = 1$, and most importantly singular kernels, which in turn will allow to handle sparse graph sequences whose limit are the so-called L^q-graphons [31, 29]. On such graphs sequences, we will quantitatively analyse evolution problems and their continuum limit. We will also consider the case $p = 1$ which was not handled in [90].

More precisely, our study shows:

(i) Well-posedness of the Cauchy problem.

(ii) Error estimates to compare two trajectories, uniformly for $t \in [0, T]$, $T > 0$, corresponding to the p-Laplacian governed by two kernels, two second members and initial data.

(iii) Consistency and error estimates of the numerical solutions to the fully-discretized problem for both forward and backward discretization.

(iv) Error bound on fully discretized problems on sparse random graphs.

Let us summarize the main results of this Chapter.

Theorem 1.2.1. *Suppose that $p \in]1, +\infty[$. Let u be a solution of ($\mathcal{P}_{\text{nloc}}$) with kernel K and data (f, g). Let u_n be a sequence of solutions to ($\mathcal{P}_{\text{nloc}}$) with kernels K_n and data (f_n, g_n). Assume that K and K_n nonnegative symmetric kernels in $L^{\infty,2}(\Omega^2)$, and that either one of the following holds:*

(a) $p \in]1, 2[$, $g, g_n \in L^2(\Omega)$, and $f, f_n \in L^1([0, T]; L^2(\Omega))$;

(b) $p \geq 2$, $g, g_n \in L^{2(p-1)}(\Omega)$ and $f, f_n \in L^1([0, T]; L^{2(p-1)}(\Omega))$;

(c) $g, g_n \in L^\infty(\Omega)$ and $f, f_n \in L^1([0, T]; L^\infty(\Omega))$.

Then, we have the following error estimate

$$\begin{aligned}\|u_n - u\|_{C([0,T];L^2(\Omega))} \leq \|g_n - g\|_{L^2(\Omega)} + \|f_n - f\|_{L^1([0,T];L^2(\Omega))} \\ + CT \begin{cases} \|K_n - K\|_{L^{\infty,2}(\Omega^2)}, & \text{under (a) or (b)} \\ \|K_n - K\|_{L^2(\Omega^2)}, & \text{under (c)} \end{cases}\end{aligned} \tag{1.2.1}$$

where C is positive constant that may depend only on p, g and f.

$C(0, T; L^p(\Omega))$ denotes the space of uniformly time continuous functions with values in $L^p(\Omega)$ endowed with the norm $\|\cdot\| \stackrel{\text{def}}{=} \sup_{t\in[0,T]} \|\cdot\|_{L^p(\Omega)}$.

Under the same assumptions on the kernels, we obtain a similar error result as (1.2.1) for $p = 1$, with $C = 1$, here the data f and g satisfy the assumptions in (a). We also obtain convergence in $C(0, T; L^2(\Omega))$ for the totally discretized problems with both forward and backward Euler scheme in time for $p \in [1, 2]$ and $p \in]1, +\infty[$, respectively.

For networks on sparse graph sequences, where we assume that $\Omega = [0, 1]$ and let $0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$ be a partition (not necessarily equispaced) of $[0, T]$. Let $\tau_{k-1} \stackrel{\text{def}}{=} |t_k - t_{k-1}|$ and denote $\tau = \max_{k\in[N]} \tau_k$.

We prove non-asymptotic convergence and give the rate of convergence of the discrete solution to its continuous limit as the number of vertices $n \to \infty$. Some supplementary assumptions are added regarding the kernel K and the data g and f.

Theorem 1.2.2. *Suppose that $p \in]1, +\infty[$. Let u be a solution of ($\mathcal{P}_{\mathrm{nloc}}$) with kernel K and data (f, g). Assume that K is nonnegative symmetric kernel in $L^{\infty,2}(\Omega^2) \cap \mathrm{Lip}(s, L^2(\Omega^2))$, $g \in L^\infty(\Omega) \cap \mathrm{Lip}(s, L^2(\Omega))$, and $f \in L^1([0, T]; L^\infty(\Omega)) \cap \mathrm{Lip}(s, L^2(\Omega \times [0, T]))$. Then, for any $\delta \in]0, 1[$, with probability at least $1 - (\rho_n n)^{-(1-\beta)}$,*

$$\sup_{k\in[N],t\in]t_{k-1},t_k]} \left\|I_n \mathbf{u}^k - u(\cdot, t)\right\|_{L^2(\Omega)} \leq C \exp(T/2)\Bigg((1 + T^{1/2})n^{-s} + T^{1/2}\left\|(K - \rho_n^{-1})_+\right\|_{L^2(\Omega^2)} + T^{1/2}(\rho_n n)^{-\beta/2} + T^{1/2}\left(\begin{cases} \tau^{\min(s,1/(3-p))} & \text{when } p \in]1, 2] \\ \tau^s & \text{when } p \geq 2 \end{cases}\right)\Bigg). \quad (1.2.2)$$

for τ sufficiently small, where $\left\{\mathbf{u}^k\right\}_{k\in[N]}$ is the discrete solution, C is positive constant that depends only on p, g, f, K and s, and $\left\|(K - \rho_n^{-1})_+\right\|_{L^2(\Omega^2)} = o(1)$, see Section 2.1.5 for the definition of I_n.

Relation to prior work The kernels and initial data considered here are beyond reach of the approach developed in [90], and have not been considered in the literature to the best of our knowledge. Moreover, our error bounds are directly stated in $L^2(\Omega)$ and not in $L^p(\Omega)$ as done in this previous work. Our proof is also simpler, more elegant and the argument is made more transparent. This argument will allow us to handle the case $p = 1$. More importantly, some limiting assumptions on the kernel and the initial data made in [90] are removed and replaced by much less stringent ones. This allows in particular to cover a far larger class of kernels (including singular ones), and also sparse graph sequences that were not handled in that previous work.

Another related work is that in [98, 112]. In these papers, the authors focused on a nonlinear heat equation on sparse graphs, where Lipschitz-continuity of the operator is of paramount importance. This assumption was essential to prove well-posedness (existence and uniqueness follow immediately from the contraction principle), as well as to study the consistency in $L^2(\Omega)$ of the spatial semi-discrete approximation. The nonlocal p-Laplacian evolution problem considered here is much more general and cannot be covered by the approach of those previous papers because the lack of Lipschitzianity raises several challenges (including for well-posedness and error estimates). Unlike those previous works, we also consider both the semi-discrete and fully-discrete versions with both forward and backward Euler approximations, that we fully characterize, and develop novel proof techniques.

1.2.2 The continuum limits of the Dirichlet problem on sparse graphs

The main contribution at the heart of Chapter 4, is to establish general consistency estimates of the boundary value problem ($\mathcal{P}^D_{\mathrm{nloc}}$). Under mild conditions on the boundary data and an appropriate discretization, we give a priori estimates for the solution of this problem. We use these results to establish nonasymptotic convergence of solutions for the discrete model on K-random graphs to its continuum limit.

For these purposes, we consider a sequence of variational problems

$$\min\left\{F_n(\nabla^{\mathrm{NL}}_{K_n} u) + \int_\Omega f_n(\boldsymbol{x})u(\boldsymbol{x})d\boldsymbol{x} : \; u \in L^p_{g_n}(\Omega, U_n)\right\}, \qquad (\mathcal{VP}_n)$$

where F_n is an integral functional to be made precise later, $f_n \in L^q(\Omega)$,

$$L^p_{g_n}(\Omega, U_n) \overset{\mathrm{def}}{=} \left\{u \in L^p(\Omega) : \; u = g_n \text{ on } \Gamma_n \overset{\mathrm{def}}{=} \Omega \setminus U_n\right\},$$

and ∇_K^{NL} is the nonlocal gradient operator which is defined on $L^1(\Omega)$ as:

$$\nabla_K^{\mathrm{NL}} u(\boldsymbol{x},\boldsymbol{y}) = K(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{y})-u(\boldsymbol{x})), \qquad u \in L^p(\Omega),\, (\boldsymbol{x},\boldsymbol{y}) \in \Omega^2. \tag{1.2.3}$$

We start by showing well-posedness (existence and uniquess of the minimizers) of variational problems of the form

$$\min\left\{ F(\nabla_K^{\mathrm{NL}} u) + \int_\Omega f(\boldsymbol{x})u(\boldsymbol{x})d\boldsymbol{x} :\; u \in L_g^p(\Omega, U)\right\} \tag{$\mathcal{VP}$}$$

As the functional in ($\mathcal{VP}_n$) takes the form of the *sum of two* proper lower semicontinuous convex functionals, its Γ-limit is in general not the sum of the Γ-limits. This is the reason we turn to the concept of Mosco convergence, where stability of Mosco-convergence to the sum holds true under mild assumptions [17]. We thus study the Mosco-limit of a sequence of nonlocal integral functionals, as well as the Mosco-limit of a sequence of geometric constraints such as those in ($\mathcal{VP}_n$). In turn, under some mild conditions on F_n, K_n, g_n and U_n, we show that the sequence of minimizers of the variational problems ($\mathcal{VP}_n$) converges to the minimizer of ($\mathcal{VP}$), with respect to the weakly topology of $L^p(\Omega)$. As an immediate consequence of these results, and thanks to the Dirichlet variational principle, one obtains that the sequence of solutions of

$$\begin{cases} -\Delta_p^{K_n} u(\boldsymbol{x}) = f_n(\boldsymbol{x}), & \boldsymbol{x} \in U_n, \\ u(\boldsymbol{x}) = g_n(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma_n \stackrel{\text{def}}{=} \Omega \setminus U_n, \end{cases} \tag{$\mathcal{P}_{\mathrm{nloc}}^{D,n}$}$$

converges weakly in $L^p(\Omega)$ to the solution of ($\mathcal{P}_{\mathrm{nloc}}^D$). By similar arguments, we extend these result to the case of discrete p-Laplacian boundary value problems, under some mild conditions on the sequence of kernels, data and the sequence of geometric constraints. We finally apply these results to establish nonasymptotic rate of convergence of solutions for the discrete model on K-random graphs to its continuum limit with high probability. We also provide a primal-dual algorithm to solve the p-Laplacian boundary value problem on graphs and report some numerical results.

The type of theorems one can find in Chapter 4 take the following forms

Theorem 1.2.3. *Let K_n, $K \in L^{\infty,1}(\Omega^2)$ be nonnegative symmetric and measurable functions, g_n, $g \in L^p(\Omega)$, f_n, $f \in L^q(\Omega)$ and $U_n, U \subset\subset \Omega$ sub-domains, $n \in \mathbb{N}$. Assume that*

(1) the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to g in $L^p(\Omega)$.

(2) the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to f in $L^q(\Omega)$.

(3) the sequence $\{K^{\frac{1}{p}}, K_n^{\frac{1}{p}} : n \in \mathbb{N}\}$ satisfies some mild assumptions.

(4) the sequence $\{U_n, U,\ n \in \mathbb{N}\}$, of subdomains of Ω, satisfies that $U_n + \mathbb{B}(0,r) \subset \Omega$, $n \in \mathbb{N}$, and $|U_n \Delta U| \to 0$, as n tends to $+\infty$, where Δ is the symmetric difference between sets.

Then ($\mathcal{P}_{\mathrm{nloc}}^D$) and ($\mathcal{P}_{\mathrm{nloc}}^{D,n}$) have a unique solutions, respectively, u and u_n. Moreover, the sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to u in $L^p(\Omega)$.

Theorem 1.2.4. *Let $K \in L^{\infty,1}(\Omega^2)$ be nonnegative symmetric and measurable functions, $g = 0$ and $f \in L^q(\Omega)$. Let $\mathbf{K} = P_n K$, $\mathbf{g} = 0$ and $\mathbf{f} = P_n f$. Let $\mathbf{u}$ be a solution of the discrete Dirichlet problem of ($\mathcal{P}_{\mathrm{nloc}}^D$) with kernel $\mathbf{K}$ data $(\mathbf{f}, \mathbf{g})$ and the boundary set A_n^c, and u the solution of the continuous problem ($\mathcal{P}_{\mathrm{nloc}}^D$) with kernel K, data (f, g) and the boundary set Γ. Then,*

$$\begin{aligned}\|u - I_n\mathbf{u}\|_{L^p(\Omega)}^{p/\max(1,\frac{2}{p})} \leq C\Bigg(& \|K - I_nP_nK\|_{L^{\infty,1}(\Omega^2)}^{\max(2,\frac{p}{p-1})} + \|K - I_nP_nK\|_{L^{\infty,1}(\Omega^2)}\|I_nP_nu - u\|_{L^p(\Omega)} \\ & + \begin{cases} \|I_nP_nu - u\|_{L^p(\Omega)}^{\frac{p}{p-1}} & p \in [2, +\infty[, \\ \|I_nP_nu - u\|_{L^p(\Omega)}^{\frac{2}{3-p}} & p \in]1, 2]. \end{cases}\Bigg).\end{aligned} \tag{1.2.4}$$

where $C > 0$ independent of n. Moreover, if the kernel is such that $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$ and $J \in L^1(\Omega - \Omega)$, then

$$\lim_n \left\|u - u_n\right\|_{L^p(\Omega)} = 0.$$

See Section 2.1.5 for the definition of P_n.

Relation to prior work In [67], the authors obtained iterated pointwise convergence of rescaled graph p-Laplacian energies to the continuum (local) p-Laplacian as the fraction of labelled to unlabelled points is vanishingly small. The authors in [141] studied the consistency of rescaled total variation minimization on random point clouds in $\mathbb{R}^d$ for a clustering application. They considered the total variation on graphs with a radially symmetric and rescaled kernel $K(\boldsymbol{x},\boldsymbol{y}) = \varepsilon^{-N}J(|\boldsymbol{x}-\boldsymbol{y}|/\varepsilon)$, $\varepsilon > 0$. This corresponds to an instance of the functional energy in ($\mathcal{P}_n^d$) (see Section 4.4) for $d = 1$ and $p = 1$. Under some assumptions on J, and for an appropriate scaling of ε with respect to n, which makes the method become localised in the large data limit, they proved that the discrete total variation on graphs Γ-converges in an appropriate topology, as $n \to \infty$, to weighted local total variation, where the weight function is the density of the point cloud distribution. Motivated by the work of [67], the authors of [134] studied consistency of the graph p-Laplacian for semi-supervised learning in $\mathbb{R}^d$. They considered both constrained and penalized minimization of the functional energy of ($\mathcal{P}_n^d$) with a radially symmetric and rescaled kernel as explained before. They uncovered regimes of p and ranges on the scaling of ε with respect to n for the asymptotic consistency (in the sense of Γ-convergence) to hold. Continuing along the lines of [67], the work of [46] studies the consistency of Lipschitz semi-supervised learning (i.e., $p \to \infty$) on graphs in the same asymptotic limit. In all these works, however, the boundary condition is fixed. Moreover, our limit is of nonlocal type, while it is of local type in the existing literature. In this sense, our work is more in line with consistency/error bounds for discrete schemes in numerial analysis.

In the numerical analysis literature, consistent numerical approximations have also been studied for nonlocal models, focusing overwhelmingly on the nonlocal peridynamic model or nonlocal linear diffusion (i.e., ($\mathcal{P}^D_{\text{nloc}}$) for $p = 2$); see [63] for a recent overview. For instance, so-called asymptotically compatible schemes were proposed in [138, 139] as an abstract framework for the study of robust numerical methods for nonlocal models and their local limits. They studied in particular consistency of Galerkin finite element discretizations of ($\mathcal{P}^D_{\text{nloc}}$) with $p = 2$ (i.e., nonlocal linear diffusion) and $K(\boldsymbol{x},\boldsymbol{y}) = \varepsilon^{-N}J(|\boldsymbol{x}-\boldsymbol{y}|/\varepsilon)$, $\varepsilon > 0$, and established its continuum limit as both the mesh size and ε vanish. These results do not allow to cover the case of the p-Laplacian. For the latter, the authors in [90] established the continuum limit of the sequence of Cauchy problems governed by the p-Laplacian on graphs, and provided the corresponding rate of convergence, which will be extended to a large class of kernels and initial data in Chapter 3. The same authors in [91] studied the nonlocal continnum limit and the corresponding error bounds for a sequence of variational problems on graphs, which consisted of minimizing the sum of a quadratic data fidelity on $L^2(\Omega)$ and a regularization term corresponding to the L^p-norm of the nonlocal gradient. Their proof strongly relies on the Hilbertian structure and strong convexity, while none of these assumptions hold for problems of the form ($\mathcal{VP}_n$) we consider here.

As for numerical schemes to solve discrete problems of the form ($\mathcal{VP}_n$) (or equivalently ($\mathcal{P}^{D,n}_{\text{nloc}}$)), [75] propose a Jacobi iteration or gradient descent. It was suggested in [67] to use Newton's method. While an earlier draft of our work was finalized, we became aware of the recent but independent work of [127] who also considered a primal-dual splitting scheme to solve discrete problems of the form ($\mathcal{VP}_n$) on graphs. Capitalizing on [50], we propose here a flexible primal-dual scheme, in Chapter 7, that solves even more general problems beyond the p-Laplacian and provide their convergence guarantees.

1.2.3 Discrete p-bilaplacian operators on graphs

In Chapter 5, we introduce a novel class of p-bilaplacian operators on weighted graphs, which can be seen as proper discretization on graphs of the classical p-bilaplacian operators. Building upon this definition, we first study a corresponding regularized variational problem as well as a boundary value problem. For the last one, we establish a discrete version of the Poincaré inequality on connected graphs, which plays a key role in our study. The latter naturally gives rise to p-biharmonic functions on graphs and equivalent definitions of p-biharmonicity [95]. For these two problems, we start by establishing their well-posedness (existence and uniqueness). We then turn, in Chapter 7, to developing proximal splitting algorithms to solve them, appealing to sophisticated tools from non-smooth optimization. Numerical results are reported to support the viability of our approach.

1.2.4 Nonlocal perimeters and curvature flows on graphs

In this work, we revisit the notion of perimeters on graphs, introduced in [70], and we extend it to so-called inner and outer perimeters. Thanks to the co-area formula, we show that discrete total variations as well as several graph cut variants can be expressed through these perimeters. Then, we propose a novel class of curvature operators on graphs that unifies both local and nonlocal mean curvature on Euclidean domains. These lead us to translate and adapt the notion of the mean curvature flows on graphs as well as the level set mean curvature which can be seen as approximate schemes. Finally, we propose to use these methods for image processing, 3D-point clouds and high dimensional data classification.

1.3 Outline

The remainder of the thesis is organized as follows:

Chapter 2: This chapter collects the necessary mathematical material used throughout the manuscript.

Chapter 3: In this chapter, we present a consistency analysis for the nonlocal p-Laplacian evolution problem. Our results consist of four principle parts: well-posedness, consistency of the continuous-continuous problem, error bounds for the discrete problem and application of these results to the fully discretized problems on random graph models.

Chapter 4: In this chapter, we expose our consistency analysis of the nonlocal p-Laplacian Dirichlet problem: a general consistency for the discretized problem and an a priori estimate when the geometrical constraints is constant. We then use this error estimate to derive a rate of convergence for the discrete random model. We report some numerical results that are based on provably convergent primal-dual numerical scheme to solve discrete p-Laplacian boundary value problems.

Chapter 5: In this chapter, we introduce a new family of p-bilaplacian operators on graphs. We then turn to study the well-posedness of regularized variational and boundary value problems associated to these operators. We finish this chapter by showing some experiments related to data processing to illustrate the use of this operator.

Chapter 6: In this chapter, we introduce a large class of perimeters on graphs. We consider the curvatures related to these perimeters. We revisit some isoperimetry inequality from functional analysis point view. We illustrate these methods for applications on images/point clouds processing and high dimensional clustering.

Chapter 7: In this chapter, we develop a primal-dual proximal splitting algorithms to solve the discrete p-Laplacian boundary value problems considered in Chapter 4 as well as the regularized variational and boundary value problems governed by the p-bilaplacian operator of Chapter 5.

Chapter 8: This last chapter summarizes our contributions and draws important conclusions. It also discusses several interesting perspectives and open problems.

Chapter 2

Mathematical Background

Contents

In this chapter, we collect the necessary mathematical material used in the manuscript.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of nonnegative reals, $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ the extended real line and $\mathbb{R}^d$ the d-dimensional real Euclidean space. Vectors in $\mathbb{R}^m$, $m \geq 2$, will be denoted in bold small letters. We denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{N}^*$, the set of positive integers. We use the notation $[n] = \{1, \cdots, n\}$. For a set $\mathcal{C}$, $|\mathcal{C}|$ denotes its cardinality, $\chi_{\mathcal{C}}$ is its characteristic function (taking 1 on $\mathcal{C}$ and 0 otherwise).

2.1 Tools from analysis

2.1.1 Convex analysis

We here collect some important results from convex analysis which will be used in the up coming chapters. Throughout this section, (X, τ) is a locally convex topological vector space (LCTVS).

Definition 2.1.1 (Convex set). A set $\mathcal{S} \subset X$ is convex, if

$$\forall x,\, x' \in \mathcal{S}, \forall t \in]0,1[, \quad tx + (1-t)x' \in \mathcal{S}.$$

Let $\mathcal{S} \subset X$ be a nonempty set, function $f : \mathcal{S} \to \overline{\mathbb{R}}$ and $\lambda \in \mathbb{R}$. Consider:

$$\mathrm{dom}(f) \stackrel{\text{def}}{=} \{x \in X :\ f(x) < +\infty\},$$

$$\begin{aligned}\operatorname{epi}(f) &\stackrel{\text{def}}{=} \{(x,t)\in X\times\mathbb{R}:\ f(x)\le t\}\,,\\ [f\le\lambda] &\stackrel{\text{def}}{=} \{x\in X:\ f(x)\le\lambda\}\,,\\ [f<\lambda] &\stackrel{\text{def}}{=} \{x\in X:\ f(x)<\lambda\}\,.\end{aligned}$$

The set dom and epi are called the **domain** and **epigraph** of the function f, respectively, while the set $[f \le \lambda]$ and $[f < \lambda]$ are the **level set** and **strict level set** of f at height λ. One says that the function f is **proper** if $-\infty \notin f(\mathcal{S})$ and $\operatorname{dom}(f) \neq \emptyset$.

Definition 2.1.2 (Convex function). A function $f : X \to \overline{\mathbb{R}}$ is convex if

$$\forall x, x' \in X, \forall t \in [0,1], f(tx + (1-t)x') \le tf(x) + (1-t)f(x'),$$

With the conventions: $(+\infty) + (-\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$, $0 \cdot (-\infty) = 0$.

Theorem 2.1.3. *Let $f : X \to \overline{\mathbb{R}}$. The following statements are equivalent:*

(i) f is convex;

(ii) $\operatorname{dom}(f)$ *is convex and*

$$\forall x, x' \in \operatorname{dom}(f),\ \forall t \in]0,1[:\quad f(tx + (1-t)x') \le tf(x) + (1-t)f(x');$$

(iii) $\operatorname{epi}(f)$ *is a convex subset of $X \times \mathbb{R}$.*

Definition 2.1.4 (Lower semi-continuous function). Let $f : X \to \overline{\mathbb{R}}$ and $x \in X$. The function f is (τ-)*lower semi-continuous* at x if for every $\epsilon > 0$ there exists a τ-neighbourhood $\mathcal{U}_x$ of x such that

$$f(x) - \epsilon \le f(y), \quad \forall y \in \mathcal{U}_x,$$

One says that f is lower semi-continuous on X if it is lower semi-continuous at every point $x \in X$. The class of *proper*, *convex* and *lsc* functions on X is denoted by $\Gamma_0(X)$.

Theorem 2.1.5. *Let $f : X \to \overline{\mathbb{R}}$. The following statements are equivalent:*

(i) f is lower semi-continuous;

(ii) for all $x \in X$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ τ-converging to x,

$$\liminf_n f(x_n) \ge f(x).$$

(iii) $\operatorname{epi}(f)$ *is a closed subset of $X \times \mathbb{R}$ respect to the product topology of τ and the natural topology of* $\mathbb{R}$.

Theorem 2.1.6. *Let $f : X \to \overline{\mathbb{R}}$. The following statements are equivalent:*

(i) f is convex and lower semi-continuous;

(ii) f is convex and weakly-lower semi-continuous;

(iii) $\operatorname{epi}(f)$ *is convex and closed subset of $X \times \mathbb{R}$;*

(iv) $\operatorname{epi}(f)$ *is a convex and weakly-closed subset of $X \times \mathbb{R}$.*

Definition 2.1.7 (Indicator function). Let $\mathcal{S} \subseteq X$ be a non-empty set, *the indicator function* of $\mathcal{S}$, $\iota_{\mathcal{S}}$, is defined by

$$\iota_{\mathcal{S}} = \begin{cases} 0, & \text{if } x \in \mathcal{S}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

Observe that $\operatorname{dom}(\iota_{\mathcal{S}}) = \mathcal{S}$ hence $\iota_{\mathcal{S}}$ is lower semi-continuous (resp. convex) if and only if $\mathcal{S}$ is closed (resp. convex).

Definition 2.1.8. Assume that X is a normed space. A function $f : \mathcal{S} \to \mathbb{R}$ is said to be M-Lipschitz on $\mathcal{S}$, if

$$\forall x,\, x' \in \mathcal{S};\ \big|f(x) - f(x')\big| \leq M\big\|x - x'\big\|_X.$$

Proposition 2.1.9. *Assume that X be a normed space, $x_0 \in X$, $r > 0$, $\varepsilon \in (0, r)$, $m, M \in \mathbb{R}$. Let $f : \mathbb{B}(x_0, r) \to \mathbb{R}$ be a convex function.*

(i) If $f(x) \leq m$ on $\mathbb{B}(x_0, r)$, then $|f(x)| \leq |m| + 2|f(x_0)|$ on $\mathbb{B}(x_0, r)$.

(ii) If $|f(x)| \leq M$ on $\mathbb{B}(x_0, r)$, then f is $\left(\frac{2M}{\varepsilon}\right)$-Lipschitz on $\mathbb{B}(x_0, r - \varepsilon)$.

Here $\mathbb{B}(x_0, r)$ is the ball of the centre x_0 and radius $r > 0$.

Proof :

(i) See Proof of Theorem 3.9 in [55].

(ii) See [58, Proposition 5.11].

□

2.1.2 Γ-convergence

Γ-convergence was introduced by De Giorgi in 1970's to study limits of variational problems. We refer to [34, 58], for an in-depth introduction to Γ-convergence. In this subsection, we denote by (X, τ) a first countable topological space. For a sequential of equivalent definitions of Γ-convergence, we refer to [12, Proposition 1.14] and [58, Proposition 8.1].

Definition 2.1.10 (Γ-convergence). We say that a sequence of functions $f_n : X \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, Γ-converges in X to $f_\infty : X \to \overline{\mathbb{R}}$ if for all $x \in X$ we have

(i) (lim inf inequality) for every sequence $\{x_n\}_{n\in\mathbb{N}}$ τ-converging to x

$$f_\infty(x) \leq \liminf_n f_n(x_n). \tag{2.1.2}$$

(ii) (lim sup inequality) there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ τ-converging to x such that

$$f_\infty(x) \geq \limsup_n f_n(x_n). \tag{2.1.3}$$

The function f_∞ is called the Γ-limit of $\{f_n\}_{n\in\mathbb{N}}$, and we write $f_\infty = \Gamma\text{-}\lim_n f_n$.

It is clear that the lim sup inequality (2.1.3) in Definition 2.1.10 can be replaced by the equality $f_\infty(x) = \lim_n f_n(x_n)$.

Definition 2.1.11 (Equi-coercivity). A function $f : X \to \overline{\mathbb{R}}$ is (sequentially) coercive if for all $t \in \mathbb{R}$ the τ-closure of the sublevel set $\{x \in X :\ f(x) \leq t\}$ is sequentially compact. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-coercive on X if for every $t \in \mathbb{R}$ there exists a sequentially compact subset A_t of X such that $\{x \in X :\ f_n(x) \leq t\} \subset A_t$ for all $n \in \mathbb{N}$.

If X is a reflexive Banach space and $f \to +\infty$ as $\|x\| \to +\infty$, then f is coercive in the weak topology of X.

Proposition 2.1.12 ([58, Proposition 7.7]). *A sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-coercive if and only if there exists a lower semicontinuous coercive function $\psi : X \to \overline{\mathbb{R}}$ such that $f_n \geq \psi$ on X, for every n.*

The following theorem (fundamental theorem of Γ-convergence) concerns the convergence of the minimum values and minimizers of an equi-coercive sequence of functions.

Theorem 2.1.13. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of equi-coercive functions on X that Γ-converges to f_∞. Then,*

(i) f_∞ is coercive.

(ii) $\lim_n d_n = d$, where $d_n = \inf_{x\in X} f_n(x)$ and $d = \min_{x\in X} f_\infty(x)$, i.e. the minimal values converge.

(iii) If for every $n \in \mathbb{N}$, x_n is a minimizer of f_n on X, then every cluster point of $\{x_n\}_{n\in\mathbb{N}}$ is a minimizer of f_∞ on X.

Proof : (i) and (ii) follow from [58, Theorem 7.8]. To get (iii), combine Proposition 2.1.12, [58, Corollary 7.20] and claim (ii). □

2.1.3 Mosco-convergence

The concept of epi-convergence (Mosco-convergence) was first utilized by R. A. Wijsman [149]. U. Mosco [113] was responsible for bringing to the fore important relationships between Mosc-convergence and the convergence of solutions to variational inequalities (hence the name of the convergence), see [12] and references therein for more details. In this section, we assume that X is a reflexive Banach space. The corresponding ball centered at x and of radius r is denoted as $\mathbb{B}(x,r)$ and $\mathbb{B}_r$ when $x = 0$.

2.1.3.1 Functions

Definition 2.1.14. Let $\{F_n, F : X \to \overline{\mathbb{R}};\ n \in \mathbb{N}\}$ be a sequence of functions. The sequence $\{F_n\}_{n\in\mathbb{N}}$ is said to be Mosco-convergent to F, if for all $x \in X$:

(i) $M\text{-}\liminf_n F_n(x) \geq F(x)$; i.e., for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging weakly to x, $\liminf_n F_n(x_n) \geq F(x)$.

(ii) $M\text{-}\limsup_n F_n(x) \leq F(x)$; i.e., there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging strongly to x such that $\limsup_n F_n(x_n) \leq F(x)$.

The function F is called the Mosco-limit of $\{F_n\}_{n\in\mathbb{N}}$ and we then write $F_n \xrightarrow{M} F$.

Observe that by definition, Mosco-convergence implies Γ-convergence in the weak topology when X is a reflexive Banach space endowed with its weak topology.

Let us now recall this result which will be useful to prove the Mosco-convergence of the sequence of integral functionals in Chapter 4.

Theorem 2.1.15 ([130, Theorem 2]). *Let F, $F_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of closed convex functions such that $\{F_n\}_{n\in\mathbb{N}}$ converges pointwise to F on X. Then $F_n \xrightarrow{M} F$ if and only if the collection $\{F_n, F : X \to \overline{\mathbb{R}};\ n \in \mathbb{N}\}$ is equi-lower semi-continuous.*

In the context of the nonlocal p-Laplacian boundary value problem our consistency results will be derived, upon using Dirichlet's variational principle, from epi-convergence of the sequence of functionals ($\mathcal{VP}_n$). In this setting, these functionals take the form of the *sum of two* proper lsc convex functionals. Whether the Γ-limit (resp. Mosco-limit) of a sum is the sum of Γ-limits (resp. Mosco-limits) is a difficult issue in general. The claim is not true in general for the Γ-limit unless stringent assumptions are imposed (see e.g., [58, Proposition 6.20]). For Mosco convergence, the claim holds true for the sum of two lsc convex functions when X a reflexive Banach space, see [17]. The latter result generalizes that in [111] which is valid only in finite dimension under a simple domain qualification condition. The result of [17] will be instrumental in our consistency analysis and we recall it here.

Theorem 2.1.16 ([17, Theorem 4.1]). *Let $\{F_n\}_{n\in\mathbb{N}}$, F, $\{G_n\}_{n\in\mathbb{N}}$, G be lsc proper convex functions defined on X, such that $F_n \overset{M}{\to} F$ and $G_n \overset{M}{\to} G$. Assume that*

$$\begin{aligned}&\text{there exists } r>0 \text{ such that, for every } \zeta\in\mathbb{B}(0,r), \text{ there exist two}\\ &\text{sequences } \{x_n\}_{n\in\mathbb{N}} \text{ and } \{y_n\}_{n\in\mathbb{N}} \text{ of elements of } X \text{ verifying}:\\ &\quad \{x_n\}_{n\in\mathbb{N}} \text{ and } \{y_n\}_{n\in\mathbb{N}} \text{ are bounded with } \zeta = x_n - y_n,\\ &\quad \limsup_n F_n(x_n) < \infty \text{ and } \limsup_n G_n(y_n) < \infty.\end{aligned} \tag{2.1.4}$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$F_n + G_n \text{ is proper }, \forall n \geq n_0, \quad F_n + G_n \overset{M}{\to} F + G, \text{ and } F + G \text{ is proper.} \tag{2.1.5}$$

The following result gives a sufficient condition for (2.1.4) to hold.

Corollary 2.1.17 ([17, Remark 1]). *Let $\{F_n\}_{n\in\mathbb{N}}$, F, $\{G_n\}_{n\in\mathbb{N}}$, G and X as in Theorem 2.1.16 such that $F_n \overset{M}{\to} F$ and $G_n \overset{M}{\to} G$. Assume that there exist $x_0 \in \operatorname{dom} G \cap \operatorname{dom} F$ and $\rho > 0$ such that either F_n or G_n is uniformly bounded above on $\mathbb{B}(x_0, \rho)$. Then (2.1.5) holds.*

2.1.3.2 Sets

Definition 2.1.18. Let $\{A_n, A;\, n \in \mathbb{N}\}$ be a sequence of subsets of X. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to be Mosco-convergent to A if and only if the sequence (ι_{A_n}) Mosco-converges to ι_A on X.

From Definition 2.1.14, we immediately get the following equivalent characterization of Mosco convergence for sets.

Proposition 2.1.19. *Let $\{A_n, A;\, n \in \mathbb{N}\}$ be a sequence of subsets of X. Then, the sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to be Mosco-convergent to A if and only if*

(i) for any sequence $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in A_n$, converging weakly to x, implies $x \in A$.

(ii) for every $x \in A$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in A_n$, converging strongly to x.

2.1.4 Accretive operators and nonlinear semigroups

All the definitions and results with proofs can be found for instance in [11].

In this section we assume that $(X, \|\cdot\|)$ is a Banach space. Let $A : X \to 2^X$ be a set-valued operator. For notational convenience, the operator will be sometimes identified with its graph by denoting $(x, y) \in A$ for $y \in A(x)$. $\mathbf{Dom}(A) \overset{\text{def}}{=} \{x \in X :\ Ax \neq \emptyset\}$ is called the *domain* of A and $\mathbf{R}(A) \overset{\text{def}}{=} \{Ax :\ x \in \mathbf{Dom}(A)\}$ its range.

Definition 2.1.20 (Accretive operator). An operator A in X is *accretive* if

$$\|x - \hat{x}\| \leq \|x - \hat{x} + \lambda(y - \hat{y})\| \quad \text{whenever} \quad \lambda > 0 \quad \text{and} \quad (x, y), (\hat{x}, \hat{y}) \in A.$$

Definition 2.1.21 (Non-expansive operator). An operator $A : X \to X$ is called *non-expansive* if it is 1-Lipschitz continuous, *i.e.*

$$\|A(x) - A(\hat{x})\| \leq \|x - \hat{x}\|, \quad \forall x, \hat{x} \in X.$$

Definition 2.1.22 (Resolvent). Let $A : X \to 2^X$ and $\gamma > 0$. The *resolvent* of A is defined by

$$J_{\gamma A} \overset{\text{def}}{=} (\mathbf{I} + \gamma A)^{-1}.$$

We have the following equivalent characterization of accretivity, whose proof can be found in e.g., [126].

Lemma 2.1.23. *The operator A is accretive if and only if its resolvent is a single-valued non-expansive map on* $\mathbf{Dom}(J_{\lambda A})$ *for* $\lambda > 0$.

Definition 2.1.24 (m-accretive operator). An operator $A : X \to 2^X$ is m-accretive if it is accretive and $\mathbf{Dom}(J_{\lambda A}) = X$ for some (hence all) $\lambda > 0$.

In the Hilbertian case, the notion of m-accretivity coincides with maximal monotonicity which is the celebrated Minty theorem.

The accretive operators theory plays an important role for proving solution existence and uniqueness of the abstract Cauchy problem

$$\begin{cases} \dot{x} + Ax \ni f, \\ x(t_0) = x_0. \end{cases} \tag{2.1.6}$$

A particular case where $f = 0$, Crandall and Liggett proved in [57] that the following limit (semigroup):

$$S(t)x_0 = \lim_{n\to\infty} (J_{t/nA})^n$$

is the unique strong solution to the abstract Cauchy problem (2.1.6) under some closedness assumptions on the operator A. In the case where $f \neq 0$, Ph. Bénilan proved in [25] the existence and the uniqueness of a strong solution of the Cauchy problem in the price the exponential formula and some closedness assumptions on the function f and the initial data x_0. In the context of the nonlocal p-Laplacian evolution equation that will be at the heart of Chapter 3, this theory will be instrumental to prove not only for well-posedness, but also to establish Lipschitz continuity of the solution as a function of the initial data and the second member. A key step to prove this is to show that the nonlocal p-Laplacian operator belongs to a rich family of operators known as m-completely accretive operators. This family was introduced by Ph. Bénilan and M. G Crandall in [26].

Let S be an open set of $\mathbb{R}^d$ and let $\mathcal{M}(S)$ be the space of measurable functions from S into $\mathbb{R}$. For $u, v \in \mathcal{M}(S)$, we write

$$u \ll v \quad \text{if and only if} \quad \int_S j(u)d\boldsymbol{x} \leq \int_S j(v)d\boldsymbol{x}$$

for all $j \in \mathcal{J}_0 \stackrel{\text{def}}{=} \{j : \mathbb{R} \to [0, +\infty], j \text{ convex, lsc, } j(0) = 0\}$.

Definition 2.1.25 (Completely accretive operator). Let A be an operator in $\mathcal{M}(S)$. We say that A is *completely accretive* if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \quad \text{for all } \lambda > 0 \text{ and all } (u, v), (\hat{u}, \hat{v}) \in A.$$

The definition of completely accretive operators does not refer explicitly to topologies or norms. However, if A is completely accretive in $\mathcal{M}(S)$ and $A \subset L^p(S) \times L^p(S)$, $p \in [1, \infty]$ then A is accretive in $L^p(S)$.

Definition 2.1.26 (m-completely accretive operator). An operator A on X is completely accretive if it is completely accretive and $\mathbf{Dom}(J_A) = X$, A is said m-completely accretive.

2.1.5 Projector and injector operators

Let us recall some definition and properties of Lebesgue spaces. Let S to by a bounded subset of $\mathbb{R}^d$. For $q \in [1, +\infty]$, $L^q(S)$ is the standard Banach space of Lebesgue q-integrable functions on S. For a function $F : S \times S \to \mathbb{R}$, we define the $L^{\infty,q}(S^2)$-norm as

$$\|F\|_{L^{\infty,q}(S^2)} \stackrel{\text{def}}{=} \sup_{\boldsymbol{x}\in S} \|F(\boldsymbol{x}, \cdot)\|_{L^q(S)}.$$

If F is symmetric, then

$$\|F\|_{L^{\infty,q}(S^2)} = \sup_{\boldsymbol{y}\in S} \|F(\cdot,\boldsymbol{y})\|_{L^q(S)}.$$

$L^{\infty,q}(S^2)$ is the space of functions on S^2 of bounded $L^{\infty,q}(S^2)$-norm, which is of course a Banach space. Throughout this manuscript, we will often use Fubini's theorem without explicitly referring to it.

Let $n \in \mathbb{N}^*$ and denote the multi-index $\boldsymbol{i} = (\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots, \boldsymbol{i}_d) \in [n]^d$. Here we assume that $\Omega = [0,1]^d$, partition Ω into cells (hypercubes)

$$\mathcal{Q} \stackrel{\text{def}}{=} \left\{ \Omega_{\boldsymbol{i}}^{(n)} \stackrel{\text{def}}{=} \prod_{k=1}^{d}]\boldsymbol{x}_{\boldsymbol{i}_k-1}, \boldsymbol{x}_{\boldsymbol{i}_k}] : \boldsymbol{i} \in [n]^d \right\}$$

of size $h_{\boldsymbol{i}} \stackrel{\text{def}}{=} |\Omega_{\boldsymbol{i}}^{(n)}|$, and maximal mesh size

$$\delta(n) \stackrel{\text{def}}{=} \max_{\boldsymbol{i}\in[n]^d} \max_{k\in[d]} (|\boldsymbol{x}_{\boldsymbol{i}_k} - \boldsymbol{x}_{\boldsymbol{i}_k-1}|).$$

When the cells are equispaced, then $h_{\boldsymbol{i}} = 1/n^d$.

We consider the operator $P_n : L^1(\Omega) \to \mathbb{R}^{n^d}$

$$(P_n u)_{\boldsymbol{i}} \stackrel{\text{def}}{=} \frac{1}{h_{\boldsymbol{i}}} \int_{\Omega_{\boldsymbol{i}}^{(n)}} u(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.1.7}$$

This operator can be also seen as a piecewise constant projector of u on the space of discrete functions. For simplicity, and with a slight abuse of notation, we keep the same notation for the projector $P_n : L^1(\Omega^2) \to \mathbb{R}^{n^d \times n^d}$.

Our aim is to study the relationship between solutions of discrete approximations and the solution of the continuum model. It is then convenient to introduce an intermediate model which is the continuum extension of the discrete solution. Towards this goal, we consider the piecewise constant injector I_n of a vector $\mathbf{v} \in \mathbb{R}^{n^d}$ into $L^2(\Omega)$ defined as

$$I_n \mathbf{v}(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{i}\in[n]^d} \mathbf{v}_{\boldsymbol{i}} \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x}), \tag{2.1.8}$$

where we recall that $\chi_{\mathcal{C}}$ is the characteristic function of the set $\mathcal{C}$, i.e., takes 0 on $\mathcal{C}$ and 1 otherwise.

It is immediate to see that the operator $I_n P_n$ is the orthogonal projector on the subspace $\operatorname{Span}\left\{\chi_{\Omega_{\boldsymbol{i}}^{(n)}} : \boldsymbol{i} \in [n]^d\right\}$ of $L^1(\Omega)$. In turn, $I_n P_n u$ is the the piecewise constant approximation of u.

Lemma 2.1.27. *For a function $u \in L^q(\Omega)$, $q \in [1, +\infty]$, we have*

$$\|I_n P_n u\|_{L^q(\Omega)} \le \|u\|_{L^q(\Omega)}. \tag{2.1.9}$$

For a function $K \in L^{\infty,q}(\Omega^2)$, $q \in [1, +\infty]$, we have

$$\|I_n P_n K\|_{L^{\infty,q}(\Omega^2)} \le \|K\|_{L^{\infty,q}(\Omega)}. \tag{2.1.10}$$

Proof : We prove (2.1.10) as (2.1.9) is a consequence of it. Let $\mathbf{K} = P_n K$. We have, $\forall \boldsymbol{x} \in \Omega$,

$$\begin{aligned}
\int_\Omega I_n P_n |K(\boldsymbol{x},\boldsymbol{y})|^q d\boldsymbol{y} &= \int_\Omega \sum_{\boldsymbol{i},\boldsymbol{j}} |\mathbf{K}_{\boldsymbol{i},\boldsymbol{j}}|^q \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x}) \chi_{\Omega_{\boldsymbol{j}}^{(n)}}(\boldsymbol{y}) d\boldsymbol{y} \\
&= \sum_{\boldsymbol{i}} \left(\sum_{\boldsymbol{j}} \int_{\Omega_{\boldsymbol{j}}^{(n)}} |\mathbf{K}_{\boldsymbol{i},\boldsymbol{j}}|^q d\boldsymbol{y} \right) \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{i}} \left(\sum_{\boldsymbol{j}} h_{\boldsymbol{j}} \left| \frac{1}{h_{\boldsymbol{i}} h_{\boldsymbol{j}}} \int_{\Omega_{\boldsymbol{i}}^{(n)} \times \Omega_{\boldsymbol{j}}^{(n)}} K(\boldsymbol{x}', \boldsymbol{y}') d\boldsymbol{x}' d\boldsymbol{y}' \right|^q \right) \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x})
\end{aligned}$$

$$\leq \sum_{\boldsymbol{i}} \left(\sum_{\boldsymbol{j}} \frac{1}{h_{\boldsymbol{i}}} \int_{\Omega_{\boldsymbol{i}}^{(n)} \times \Omega_{\boldsymbol{j}}^{(n)}} |K(\boldsymbol{x}', \boldsymbol{y}')|^q d\boldsymbol{x}' d\boldsymbol{y}' \right) \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x})$$
$$= \sum_{\boldsymbol{i}} \left(\frac{1}{h_{\boldsymbol{i}}} \int_{\Omega_{\boldsymbol{i}}^{(n)}} \left(\sum_{\boldsymbol{j}} \int_{\Omega_{\boldsymbol{j}}^{(n)}} |K(\boldsymbol{x}', \boldsymbol{y}')|^q d\boldsymbol{y}' \right) d\boldsymbol{x}' \right) \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x})$$
$$= \sum_{\boldsymbol{i}} \left(\frac{1}{h_{\boldsymbol{i}}} \int_{\Omega_{\boldsymbol{i}}^{(n)}} \left(\int_{\Omega} |K(\boldsymbol{x}', \boldsymbol{y}')|^q d\boldsymbol{y}' \right) d\boldsymbol{x}' \right) \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x})$$
$$\leq \|K\|^q_{L^{\infty,q}(\Omega^2)} \sum_{\boldsymbol{i}} \chi_{\Omega_{\boldsymbol{i}}^{(n)}}(\boldsymbol{x}) = \|K\|^q_{L^{\infty,q}(\Omega^2)}.$$

Taking the supremum on the left-hand side yields the bound. □

Remark 2.1.28. Our exposition of the nonlocal p-Laplacian Dirichlet problem will be on a general bounded domain Ω of $\mathbb{R}^d$.

2.2 Tools from approximation theory

For $N \in \mathbb{N}^*$, let S be a compact subset of $\mathbb{R}^N$. We introduce the Lipschitz spaces $\mathrm{Lip}(s, L^q(S))$, $q \in [1, +\infty]$, which contain functions with, roughly speaking, s "derivatives" in $L^q(S)$ [61, Ch. 2, Section 9]. These spaces will be a key tool for us to study the full discretization as we will be able to get non-asymptotic error estimates for random graph model when adding the assumption of belonging to these spaces to the kernel $K(\cdot,\cdot)$, the second member $f(\cdot,t)$ and the initial condition $g(\cdot)$ in ($\mathcal{P}_{\mathrm{nloc}}$).

Definition 2.2.1. For $F \in L^q(S)$, $q \in [1, +\infty]$, we define the (first-order) $L^q(S)$ modulus of smoothness by

$$\omega(F, h)_q \stackrel{\text{def}}{=} \sup_{\boldsymbol{z} \in \mathbb{R}^d, |\boldsymbol{z}| < h} \left(\int_{\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{z} \in S} |F(\boldsymbol{x}+\boldsymbol{z}) - F(\boldsymbol{x})|^q \, d\boldsymbol{x} \right)^{1/q}. \tag{2.2.1}$$

The Lipschitz spaces $\mathrm{Lip}(s, L^q(S))$ consist of all functions F for which

$$|F|_{\mathrm{Lip}(s, L^q(S))} \stackrel{\text{def}}{=} \sup_{h>0} h^{-s} \omega(F, h)_q < +\infty.$$

We restrict ourselves to values $s \in]0, 1]$ since for $s > 1$, only constant functions are in $\mathrm{Lip}(s, L^q(S))$. It is easy to see that $|F|_{\mathrm{Lip}(s, L^q(S))}$ is a semi-norm. $\mathrm{Lip}(s, L^q(S))$ is endowed with the norm

$$\|F\|_{\mathrm{Lip}(s, L^q(S))} \stackrel{\text{def}}{=} \|F\|_{L^q(S)} + |F|_{\mathrm{Lip}(s, L^q(S))}.$$

The space $\mathrm{Lip}(s, L^q(S))$ is the Besov space $\mathrm{B}^s_{q,\infty}$ [61, Ch. 2, Section 10] which are very popular in approximation theory. In particular, $\mathrm{Lip}(s, L^{1/s}(S))$ contains the space $\mathrm{BV}(S)$ of functions of bounded variation on S; see [61, Ch. 2, Lemma 9.2]. Thus Lipschitz spaces are rich enough to contain functions with both discontinuities and fractal structure.

We now state the following approximation error bounds whose proofs use standard arguments from approximation theory; see [90, Section 6.2.1] for details.

Lemma 2.2.2. *There exists a positive constant C_s, depending only on s, such that for all $F \in \mathrm{Lip}(s, L^q(S))$, $s \in]0, 1]$, $q \in [1, +\infty]$,*

$$\|F - I_n P_n F\|_{L^q(S)} \leq C_s \delta(n)^s |F|_{\mathrm{Lip}(s, L^q(S))}. \tag{2.2.2}$$

We denote by $\mathrm{BV}([0,T];L^q(S))$ the Banach space of functions $f:\Omega\times[0,T]\to\mathbb{R}$ such that

$$\mathrm{Var}_q(f) \stackrel{\text{def}}{=} \sup_{0\le t_0<t_1<\cdots<t_N\le T} \sum_{i=1}^{N} \left\| f(\cdot,t_i)-f(\cdot,t_{i-1})\right\|_{L^q(S)} < +\infty,$$

endowed with the norm $\|f\|_{\mathrm{BV}([0,T];L^q(S))} \stackrel{\text{def}}{=} \|f(0)\|_{L^q(S)} + \mathrm{Var}_q(f)$.

2.3 Tools from graph limit theory

2.3.1 Preliminaries

A weighted graph $G=(V(G),E(G),\beta)$ consists of a finite set $V(G)$ of vertices (nodes, or points), a finite set $E(G)\subset V(G)\times V(G)$ of edges (or lines) and a weight function $\beta: V(G)\times V(G)\to\mathbb{R}^+$. Each node $i\in V(G)$ is an abstract representation of an element of the data structure represented by the graph. An edge $(i,j)\in E(G)$ is composed of a couple of vertices, which represents the connection between them and we write $i\sim j$. We say that G is connected graph if for all $i,j\in V(G)$, there exists a sequence $i_0,i_1,\cdots,i_m\in V(G)$ such that $i=i_0\sim i_1\sim\cdots\sim i_m=j$. In this manuscript, we consider undirected connected graphs without parallel edges in which case the edges are symmetric. We can therefore also define the set $E(G)$ such that:

$$E(G) \stackrel{\text{def}}{=} \{(i,j)\in V(G)\times V(G):\ i\sim j \ \text{ and } \ i\neq j\}.$$

Considering the symmetry of the edges, we can also note that if $(i,j)\in E(G)$, then $(i,j)\in E(G)$. The weight function represents a similarity measure between two vertices of the graph. Since we are dealing with undirected graphs, this function is symmetric: $\forall (i,j)\in V(G)^2,\ \beta(i,j)=\beta(i,j)$. The neighbourhood of a vertex i (i.e., the set of vertices adjacent to i) is denoted by $\mathcal{N}(i)$ and the degree of a vertex i is defined as $\deg_G(i)=\sum_{j\sim i}\beta(i,j)$. For two vertices $i,\ j\in V(G)$ with $i\not\sim j$ we set $\beta(i,j)=\beta(j,i)=0$ and thus the set of edges $E(G)$ can be characterized by the support of the weight function β, i.e. $E(G)=\{(i,j)\in V(G)\times V(G):\ \beta(i,j)>0\}$. A particular case where $\beta\in\{0,1\}$, these kind of graphs are called simple graphs. In order to simplify the writing, we will often use in the rest of this manuscript the condensed notation $\beta_{ij}=\beta(i,j)$.

The usual way to represent a graph is to draw a circle (or dot) for each vertex and join two of these circles with a line if the two corresponding vertices form an edge. It doesn't matter how these circles and lines are drawn: the importance is the information about the pairs of vertices that form an edge and those that do not. For a weighted graph, we add the weight next to the lines.

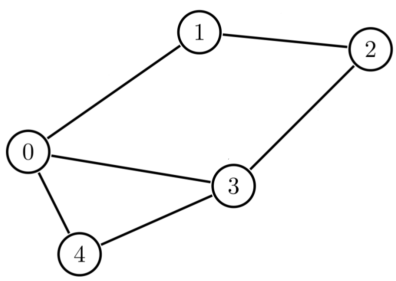


Figure 2.1: Example of an undirected simple graph G with $V(G)=\{0,\cdots,4\}$ nodes with edge set $E(G)=\{(0,4),\,(4,3),\,(3,0),\,(0,1),\,(1,2),\,(2,3)\}$.

The adjacency matrix of a graph G is a square $|V(G)|\times|V(G)|$ matrix A_G such that its elements indicate whether pairs of vertices are adjacent or not in the graph. In the case of a simple graph, this is a $(0,1)$-matrix. For a weighted graph, $(A_G)_{ij}$ represents the weight of the edge (i,j). If the graph is undirected, the adjacency matrix is symmetric. A non-standard way of visualizing graphs using another version of the adjacency matrix is the so-called pixel picture. On the left of Figure 2.3 we see

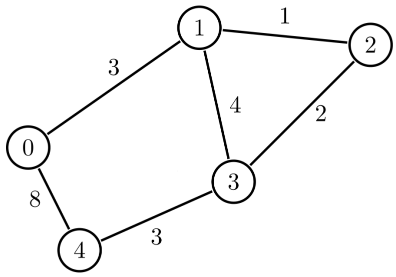
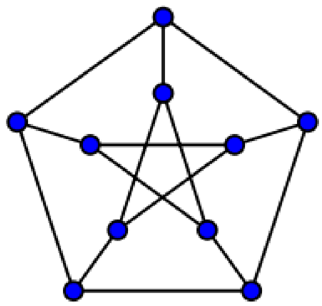


Figure 2.2: Example of a weighted graph G with $V(G) = \{0, \cdots, 4\}$ nodes with edge set $E(G) = \{(0,4), (4,3), (3,1), (1,0), (1,2), (2,3)\}$ and $\{8,3,4,3,1,2\}$ are weights assigned to edges.

a graph (the Petersen graph). In the middle, we see its adjacency matrix. On the right, we see another version of its adjacency matrix, where the 0's are replaced by white pixels and the 1's are replaced by black pixels.The whole picture is on the unit square.



$$\begin{matrix} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&0&0 \end{matrix}$$

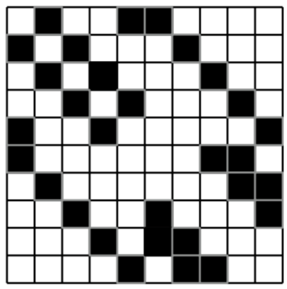


Figure 2.3: The Petersen graph, its adjacency matrix, and its pixel picture.

2.3.2 L^q graphons and graph limits

We now review some definitions and results of the L^q theory of sparse graphs developed in [31, 29]. This theory generalizes both existing theory of bounded graphons that are adapted to dense graph limits [105], and its extension in [28] to sparse graphs under a no dense spots assumption. Here, we follow considerably [31, 29], in which much more details can be found. We will be more interested in the random case, which plays a central role in our study.

Definition 2.3.1. Let $q \in [1, +\infty]$, an L^q *graphon* is a measurable, symmetric function $K \in L^q([0,1]^2)$. Here the symmetry means $K(x,y) = K(y,x)$ for all $x, y \in [0,1]$. If we do not specify q, we assume that K is in L^1 and call it simply a graphon, rather than an L^1 graphon.

Every finite weighted graph G_n such that $V(G_n) = [n]$, with edge weights $\{\beta_{ij}\}_{(i,j)\in[n]^2}$, can be represented by a measurable function $K_{G_n} : [0,1]^2 \to \mathbb{R}^+$. The construction is as follow: Let $\mathcal{Q}_n$ be a partition of $[0,1]$ to n equal intervals $\mathcal{Q}_n = \left\{\mathcal{I}_k^{(n)} : k \in [n]\right\}$, and for every $x \in \mathcal{I}_i^{(n)}$ and $y \in \mathcal{I}_j^{(n)}$ we set

$$K_{G_n} \stackrel{\text{def}}{=} \begin{cases} \beta_{ij}, & \text{if } (i,j) \in E(G_n), \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.1}$$

This construction is not unique, however given a graph, the set of kernels arising from (2.3.1) can be considered equivalent via the weakly isomorphic relation (2.3.4). Informally a graphon can be thought of as a generalization of the adjacency matrix of a (weighted) graph which has a continuum number of vertices.

Example 2.3.2 (Half graphs). Let $G_{n,n}$ denote the bipartite graph on $2n$ nodes $\{1, \cdots, n, 1', \cdots, n'\}$, where i is connected to j' if and only if $i \leq j$. It is easy to see that this sequence is convergent and its limit is the function

$$K(x,y) = \begin{cases} 1, & \text{if} \quad |x-y| \geq 1/2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.2}$$

Figure 2.4 shows an example of the half-graph for $n = 16$, its pixel picture and the corresponding graphon.

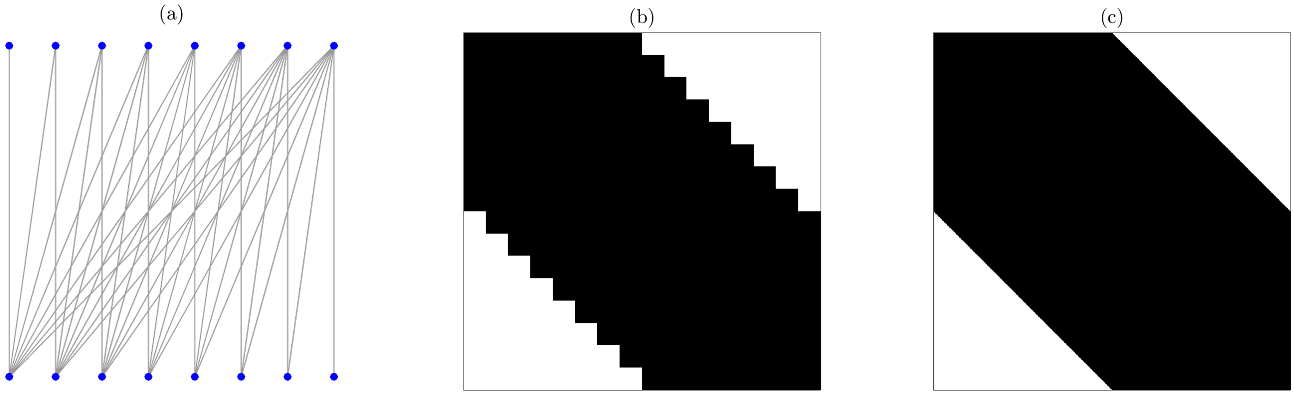


Figure 2.4: (a) A half-graph of 16 vertices. (b) The plot of its pixel picture. (c) The corresponding graphon.

Example 2.3.3 (Simple threshold graphs). These graphs are defined on the set $[n]$ by connecting i and j if and only if $i + j \leq n$. These graphs converge to the graphon defined by $K(x,y) = \mathbb{1}_{(x+y\leq 1)}$, which we call *the simple threshold graphon.*

Figure 2.5 displays an example of the threshold graph for $n = 16$ vertices, its pixel picture and the corresponding graphon.

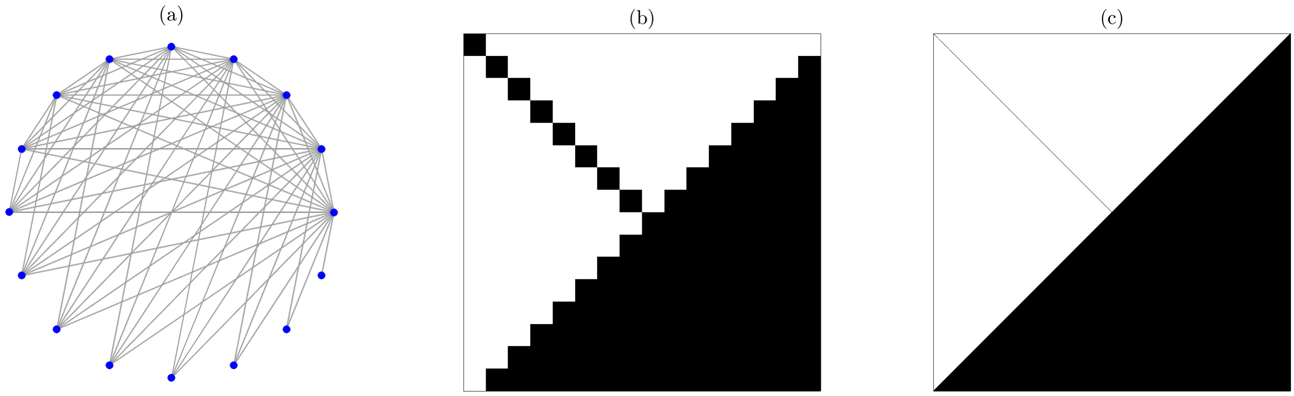


Figure 2.5: (a) A simple-threshold graph with 16 vertices. (b) The plot of its pixel picture. (c) The corresponding graphon.

Now, we introduce the most important metric on the space of graphons which is the *cut metric.* (Strictly speaking, it is merely a pseudometric, since two graphons with cut distance zero between them need not be equal.) It is defined in terms of the cut norm introduced in [83].

Definition 2.3.4 (Cut metric). For a graphon $K : [0,1]^2 \to \mathbb{R}$, define the *cut norm* by

$$\|K\|_{\square} \stackrel{\text{def}}{=} \sup_{S,T\subset[0,1]} \left| \int_{S\times S} K(x,y)dxdy \right|, \tag{2.3.3}$$

where S and T range over measurable subsets of $[0,1]$. Given two graphons K, $K' : [0,1]^2 \to \mathbb{R}$, define

$$d_\square(K,K') \stackrel{\text{def}}{=} \left\|K - K'\right\|_\square$$

and the *cut metric* (or *cut distance*) $\delta_\square$ by

$$\delta_\square(K,K') \stackrel{\text{def}}{=} \inf_\sigma d_\square(K^\sigma, K')$$

where σ range over all measure-preserving bijections $\sigma : [0,1] \to [0,1]$ and $K^\sigma(x,y) \stackrel{\text{def}}{=} K(\sigma(x),\sigma(y))$.

For a survey covering many properties of the cut metric, see [105, 29, 97] and reference therein. The notions d and δ extended to any norm on the spaces of graphons. In particular, for $1 \leq q \leq \infty$, by definition

$$d_p(K,K') \stackrel{\text{def}}{=} \left\|K - K'\right\|_{L^q([0,1]^2)} \qquad \text{and} \qquad \delta_p(K,K') \stackrel{\text{def}}{=} \inf_\sigma d_p(K^\sigma, K')$$

with σ ranging overall measure-preserving bijections $\sigma : [0,1] \to [0,1]$ as before.

We now introduce the weakly isomorphic relation, denoted $\approx$, which identifies sets of graphons which all have a cut distance of zero apart [105, Corollary 10.34]. Let K, K' be two graphons, we define the weakly isomorphic relation as follow

$$K \approx K' \Leftrightarrow \delta_\square(K,K') = 0. \tag{2.3.4}$$

Theorem 2.3.5 ([29, Theorem 2.13], Compactness of the L^q ball with respect to the cut metric). *Let $1 < q \leq \infty$ and $C > 0$, the ball $\mathcal{B}_{L^q([0,1]^2)}(C) \stackrel{\text{def}}{=} \left\{L^q \text{ graphons } K : \left\|K\right\|_{L^q([0,1]^2)} \leq C\right\}$ is compact with respect to the cut metric $\delta_\square$ (after identifying points of distance zero).*

2.3.3 Random graphs

The theory of random graphs was founded in the 50's-60's by Erdös and Rényi [76], who started the systematic study of the space of graphs with n labeled vertices and $M = M(n)$ edges, with all graphs equiprobable. The aim is to turn the set of all graphs with n vertices into a probability space. Intuitively we should be able to generate a sequence of graphs $\{G_n\}_{n\in\mathbb{N}}$ randomly as follows: for each edge $(i,j) \in [n]^2$, we decide by some random experiment whether or not (i,j) shall be an edge of G_n, these experiments are performed independently.

Example 2.3.6 (The Erdös-Renyi graphs.). Let $p \in]0,1[$ and consider the sequence of random graphs $G(n,p) = (V(G(n,p)), E(G(n,p)))$ such that $V(G(n,p)) = [n]$ and the probability $\mathbb{P}\{(i,j) \in E(G(n,p))\} = p$ for any $(i,j) \in [n]^2$. Then for any simple graph F, $t(F, G(n,p))$ converges almost surely to $p^{\left|E(F)\right|}$ as $n \to \infty$ [30] and $\{G(n,p)\}$ converges almost surely to the p-constant graphon.

Figure 2.6 shows a realization of the Erdös-Renyi graph model for $n = 16$, its pixel picture and the corresponding graphon.

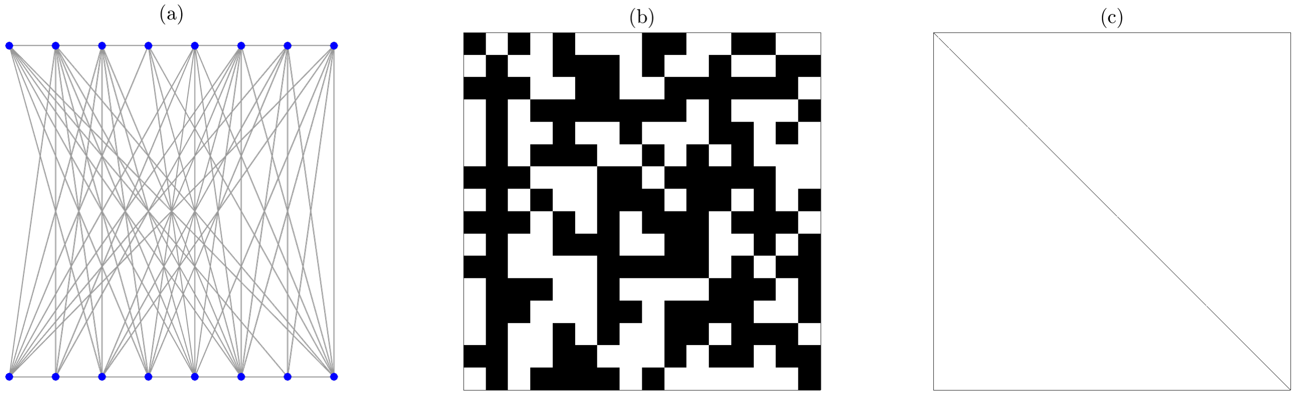


Figure 2.6: (a) A realization of the Erdös-Renyi random graph model with $p = 0.5$. (b) Its pixel picture. (c) The corresponding graphon.

2.3.4 Sparse K-random graph models

We consider weighted graphs, which include as a special case simple unweighted graphs. Let $G = (V(G), E(G))$, be a weighted graphs with vertex set $V(G)$ and edge set $E(G) \subseteq V(G)^2$, respectively. In G, every edge $(i,j) \in E(G)$ (allowing loops with $i = j$) is given a weight $\beta_{ij} \in \mathbb{R}^{+}$[1]. We set $\beta_{ij} = 0$ whenever $(i,j) \notin E(G)$.

The idea underlying the sparse K-random graph model proposed by [29] is that each L^q graphon K gives rise to a natural random graph model, which produces a sequence of sparse graphs converging to K in an appropriate metric. Inspired by their work, we propose the following construction.

Definition 2.3.7. Fix $n \in \mathbb{N}^*$, let K be an L^1 graphon and $\rho_n > 0$. Take the equispaced partition of $[0,1]$ in intervals $]x_{i-1}, x_i]$, $i \in [n]$, where $x_i = i/n$. Let $\mathbf{K} \in \mathbb{R}_+^{n \times n}$ be a weight matrix such that:

($\mathbf{H}_w$.1) $\left\| I_n\mathbf{K} - K \right\|_{L^1([0,1]^2)} \to 0$ as $n \to +\infty$.

($\mathbf{H}_w$.2) $\left\| I_n\mathbf{K}(x,\cdot) - K(x,\cdot) \right\|_{L^1([0,1])} \to 0$ uniformly in $x \in [0,1]$.

Generate the random graph

$$G_n = (V(G_n), E(G_n)) \stackrel{\text{def}}{=} \mathbf{G}(n, K, \rho_n)$$

as follows: join each pair $(i,j) \in [n]^2$ of vertices independently, with probability

$$\mathbb{P}\left((i,j) \in E(G_n) | \mathbf{X}\right) = \rho_n \overset{\wedge}{\mathbf{K}}_{ij}, \quad \text{where} \quad \overset{\wedge}{\mathbf{K}}_{ij} \stackrel{\text{def}}{=} \min\left(\mathbf{K}_{ij}, \rho_n^{-1}\right). \tag{2.3.5}$$

Remark 2.3.8. In the original sparse K-random graph model defined in [29], the x_i's are random iid samples drawn from the uniform distribution on $[0,1]$. Moreover, $\mathbf{K}_{ij} = K(x_i, x_j)$. In this case, it follows from [29, Theorem 2.14(a)] (which relies on [94, Theorem]) that assumptions **($\mathbf{H}_w$.1)** holds with probability 1.

Another interesting case is where $\mathbf{K} = P_n K$. Thanks to Lemma 2.1.27, $\left\| I_n P_n \mathbf{K} \right\|_{L^1(\Omega^2)} \leq \left\| K \right\|_{L^1(\Omega^2)}$ with probability 1. Thus, the Lebesgue differentiation theorem and the dominated convergence theorem allow to assert that $I_n P_n \mathbf{K}$ converges to K in $L^1(\Omega^2)$. In turn, assumption **($\mathbf{H}_w$.1)** holds.

For appropriate choices of ρ_n, the graph model constructed according to Definition 2.3.7 allows to sample both dense and sparse graphs from the graphon K. In particular, the sparsity assumption $\rho_n \to 0$ reflects the fact that ρ_n needs to be arbitrarily close to zero in order to see the unbounded/singular

[1] In [29], the weights are even allowed to be negative, but we will not consider this situation which is meaningless in our context.

part of K. The assumption that $n\rho_n \to +\infty$ means the average degree tends to infinity. To check this, the average number of edges in this graph model is

$$\begin{aligned}\mathbb{E}\left(E(\mathbf{G}(n,K,\rho_n))\right) &= \rho_n n^2 \left(n^{-2}\sum_{(i,j)\in[n]^2} \overset{\wedge}{\mathbf{K}}_{ij}\right)\\ &= \rho_n n^2 \left(\left\|I_n\mathbf{K}\right\|_{L^1([0,1]^2)} - \left\|\left(I_n\mathbf{K}-\rho_n^{-1}\right)_+\right\|_{L^1([0,1]^2)}\right).\end{aligned}$$

By assumption **(H$_w$.1)**, we have $\left\|I_n\mathbf{K}\right\|_{L^1([0,1]^2)} = \left\|K\right\|_{L^1([0,1]^2)} + o(1)$. Moreover, since $\rho_n \to 0$, we have from (2.3.6) that $\left\|\left(I_n\mathbf{K}-\rho_n^{-1}\right)_+\right\|_{L^1([0,1]^2)} = o(1)$. In turn,

$$\mathbb{E}\left(E(\mathbf{G}(n,K,\rho_n))\right) = \rho_n n^2\left(\left\|K\right\|_{L^1([0,1]^2)} + o(1)\right).$$

As expected, this gives rise to a sparse graph whose edge density is $\rho_n \to 0$. For the average degree of this graph model, arguing similarly to above, and using **(H$_w$.2)**, we have

$$\begin{aligned}\mathbb{E}\left(\deg_{G_n}(i)\right) &= \rho_n n\left(n^{-1}\sum_{j\in[n]} \overset{\wedge}{\mathbf{K}}_{ij}\right)\\ &= \rho_n n\left(\left\|I_n\mathbf{K}(x_i,\cdot)\right\|_{L^1([0,1])} - \left\|\left(I_n\mathbf{K}(x_i,\cdot)-\rho_n^{-1}\right)_+\right\|_{L^1([0,1])}\right)\\ &= \rho_n n\left(\int_0^1 K(x_i,y)dy + o(1)\right).\end{aligned}$$

As anticipated, the average degree is indeed unbounded since $\rho_n n \to +\infty$.

The above sequence of graphs generated also enjoys the following convergence result.

Proposition 2.3.9. *Let K be an L^1 graphon and $\mathbf{K}$ be a weight matrix such that* **(H$_w$.1)** *holds. If $\rho_n > 0$ with $\rho_n \to 0$ and $n\rho_n \to +\infty$ as $n \to +\infty$, then $\rho_n^{-1}\mathbf{G}(n,K,\rho_n)$ converges almost surely to K in the cut distance metric.*

PROOF : We essentially adapt the arguments of in the proof of [29, Theorem 2.14(b)]. More precisely, since **(H$_w$.1)** holds, one has to show [29, (7.1)]. For this, we invoke [29, Lemma 7.3] by checking the condition (7.3) therein. We have by sublinearity of $(\cdot)_+$ that

$$\begin{aligned}\frac{1}{n^2}\sum_{(i,j)\in[n]^2}\left(\mathbf{K}_{ij}-\rho_n^{-1}\right)_+ &= \int_{[0,1]^2}\left(I_n\mathbf{K}(x,y)-\rho_n^{-1}\right)_+ dxdy\\ &\leq \int_{[0,1]^2}\left(I_n\mathbf{K}(x,y)-K(x,y)\right)_+ dxdy + \int_{[0,1]^2}\left(K(x,y)-\rho_n^{-1}\right)_+ dxdy\\ &\leq \left\|I_n\mathbf{K}-K\right\|_{L^1([0,1]^2)} + \int_{[0,1]^2}\left(K(x,y)-\rho_n^{-1}\right)_+ dxdy.\end{aligned} \tag{2.3.6}$$

The right-hand side in the above display goes to 0 as $n \to +\infty$ by **(H$_w$.1)** and since $\rho_n \to 0$. Indeed, for every $L > 0$, the limit superior of the last term is bounded by $\left\|(K-L)_+\right\|_{L^1([0,1]^2)}$, and this can be made arbitrarily small by choosing L large. □

Example 2.3.1. For an example that cannot be handled using L^∞ graphons, and thus does not enter in the framework of [90, 89], consider a K-random graph model $\mathbf{G}(n,K,\rho_n)$ constructed according to Definition 2.3.7 with $\mathbf{K} = P_n K$, where $K(x,y) = J(x-y)$, $J : z \in [-1,1] \mapsto 2^{-1}(1-\beta)(2-\beta)|z|^{-\beta}$, $\beta \in]0,1[$. First, observe that the radially symmetric kernel J is singular but fulfills all assumptions, i.e. J is symmetric nonnegative function in $L^1(\Omega-\Omega)$.In addition, by virtue of Remark 2.3.8, ($\mathbf{H}_w$.1)-($\mathbf{H}_w$.2) also hold with

$$\|K\|_{L^1([0,1]^2)} = 1 \text{ and } \int_0^1 K(x,y)dy = 2^{-1}(2-\beta)\left(x^{1-\beta}+(1-x)^{1-\beta}\right) \in 2^{-1}(2-\beta)[1,2^\beta].$$

We also have the following convergence result in the $L^{\infty,1}$ norm that will be instrumental in Section 3.5. According to the construction in Definition 2.3.7, we let $\mathbf{\Lambda}_{ij}$, $(i,j)\in[n]^2, i\neq j$, be random variables such that $\rho_n\mathbf{\Lambda}_{ij}$ follows a Bernoulli distribution with parameter $\rho_n\overset{\wedge}{\mathbf{K}}_{ij}$. For each row $i\in[n]$, $(\mathbf{\Lambda}_{ij})_{j\in[n]}$ are independent.

Lemma 2.3.10. *Let K be a nonnegative $L^{\infty,1}$ graphon. Take the weight matrix $\mathbf{K}=P_nK$. Assume that $\rho_n\to 0$ and $n\rho_n=\omega\left((\log n)^\gamma\right)$ for some $\gamma>1$. Then with probability 1,*

$$\|I_n\mathbf{\Lambda}\|_{L^{\infty,1}([0,1]^2)} - \|I_n\overset{\wedge}{\mathbf{K}}\|_{L^{\infty,1}([0,1]^2)} \to 0.$$

If, moreover, ($\mathbf{H}_w$.2) holds, then

$$\|I_n\mathbf{\Lambda}\|_{L^{\infty,1}([0,1]^2)} \to \|K\|_{L^{\infty,1}([0,1]^2)}.$$

with probability 1.

Proof : For any $\varepsilon>0$, we have by the union bound

$$\begin{aligned}
&\mathbb{P}\left(\left|\|I_n\mathbf{\Lambda}\|_{L^{\infty,1}([0,1]^2)} - \|I_n\overset{\wedge}{\mathbf{K}}\|_{L^{\infty,1}([0,1]^2)}\right|\geq\varepsilon\right)\\
&=\mathbb{P}\left(\left|\max_i\sum_j\mathbf{\Lambda}_{ij}-\max_i\sum_j\overset{\wedge}{\mathbf{K}}_{ij}\right|\geq\varepsilon n\right)\\
&=\mathbb{P}\left(\left|\max_i\sum_j\rho_n\mathbf{\Lambda}_{ij}-\max_i\sum_j\rho_n\overset{\wedge}{\mathbf{K}}_{ij}\right|\geq\varepsilon\rho_n n\right)\\
&\leq\mathbb{P}\left(\max_i\left|\sum_j\rho_n(\mathbf{\Lambda}_{ij}-\overset{\wedge}{\mathbf{K}}_{ij})\right|\geq\varepsilon\rho_n n\right)\\
&\leq\sum_i\mathbb{P}\left(\left|\sum_j\rho_n(\mathbf{\Lambda}_{ij}-\overset{\wedge}{\mathbf{K}}_{ij})\right|\geq\varepsilon\rho_n n\right).
\end{aligned}$$

Since $(\rho_n\mathbf{\Lambda}_{ij})_j$ are independent Bernoulli variables with means $\left(\rho_n\overset{\wedge}{\mathbf{K}}_{ij}\right)_j$, it follows from the variant of the Chernoff bound in [29, Lemma 7.1], that for every $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\|I_n\mathbf{\Lambda}\|_{L^{\infty,1}([0,1]^2)} - \|I_n\overset{\wedge}{\mathbf{K}}\|_{L^{\infty,1}([0,1]^2)}\right|\geq\varepsilon\right)\\
&\leq 2\sum_i\exp\left(-\frac{1}{3}\min\left(\frac{\varepsilon\rho_n n}{\rho_n\sum_j\overset{\wedge}{\mathbf{K}}_{ij}},1\right)\varepsilon\rho_n n\right)\\
&\leq 2n\exp\left(-\frac{1}{3}\min\left(\frac{\varepsilon}{\|I_n\overset{\wedge}{\mathbf{K}}\|_{L^{\infty,1}([0,1]^2)}},1\right)\varepsilon\rho_n n\right)\\
&\leq 2n\exp\left(-\frac{1}{3}\min\left(\frac{\varepsilon}{\|K\|_{L^{\infty,1}([0,1]^2)}},1\right)\varepsilon\omega\left((\log n)^\gamma\right)\right)\\
&\leq 2n^{-\omega\left((\log n)^{\gamma-1}\right)},
\end{aligned}$$

since $\gamma > 1$, and where we used (2.3.5) and Lemma 2.1.27 to show that

$$\|I_n\hat{\mathbf{K}}\|_{L^{\infty,1}([0,1]^2)} \leq \|I_n\mathbf{K}\|_{L^{\infty,1}([0,1]^2)} = \|I_nP_nK\|_{L^{\infty,1}([0,1]^2)} \leq \|K\|_{L^{\infty,1}([0,1]^2)}.$$

Invoking the (first) Borel-Cantelli lemma, we have the first claim. On the other hand,

$$\begin{aligned}
&\left|\|I_n\hat{\mathbf{K}}\|_{L^{\infty,1}([0,1]^2)} - \|K\|_{L^{\infty,1}([0,1]^2)}\right| \leq \|I_n\hat{\mathbf{K}} - K\|_{L^{\infty,1}([0,1]^2)} \\
&\leq \|I_n\hat{\mathbf{K}} - I_nP_nK\|_{L^{\infty,1}([0,1]^2)} + \|I_nP_nK - K\|_{L^{\infty,1}([0,1]^2)} \\
&= \|(I_nP_nK - \rho_n^{-1})_+\|_{L^{\infty,1}([0,1]^2)} + \|I_nP_nK - K\|_{L^{\infty,1}([0,1]^2)} \\
&\leq \|(K - \rho_n^{-1})_+\|_{L^{\infty,1}([0,1]^2)} + \|(I_nP_nK - K)_+\|_{L^{\infty,1}([0,1]^2)} + \|I_nP_nK - K\|_{L^{\infty,1}([0,1]^2)} \\
&\leq \|(K - \rho_n^{-1})_+\|_{L^{\infty,1}([0,1]^2)} + 2\|I_nP_nK - K\|_{L^{\infty,1}([0,1]^2)}.
\end{aligned}$$

Since $\rho_n \to 0$ and in view of **($\mathbf{H}_w$.2)**, the right-hand side in the above display goes to 0 as $n \to +\infty$. Combined with the first claim we obtain the desired conclusion. □

2.4 Partial differences operators on graphs

Using the basic notation given in the above section, we recall the fundamental elements of the weighted partial difference operators on graphs on which we base our framework exposed in Chapters 5 and Chapter 6. For more detailed information on these operators we refer to [73, 87, 136, 72]. In order to use a consistent notation with the content of these references, we denote, in the present section and the chapters mentioned, the weight function of the graph considered by ω and the vertices by x, y, z.

Let $G = (V, E, \omega)$ be a weighted graph. We denote by $\mathcal{H}(V)$ the space of real-valued functions on the vertices of the graph, i.e., each function $u : V \to \mathbb{R}$ in $\mathcal{H}(V)$ assigns a real-value $u(x)$ to each vertex $x \in V$.

For a function $u \in \mathcal{H}(V)$ the $\ell^p(V)$-norm of u is given by

$$\begin{aligned}
\|u\|_p &= \left(\sum_{x\in V} |u(x)|^p\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \\
\|u\|_\infty &= \max_{x\in V} |u(x)|
\end{aligned}$$

The space $\mathcal{H}(V)$ endowed with the following inner product: $\langle u, v\rangle_{\mathcal{H}(V)} = \sum_{x\in V} u(x)v(x)$, u, $v \in \mathcal{H}(V)$, is a Hilbert space. Similarly, let $\mathcal{H}(E)$ be the space of real-valued functions defined on the edges of the graph, i.e., each function $H : E \to \mathbb{R}$ in $\mathcal{H}(E)$ assigns a real-valued $H(x, y)$ to each edge $(x, y) \in E$. The space $\mathcal{H}(E)$ endowed with the following inner product: $\langle H, G\rangle_{\mathcal{H}(E)} = \sum_{(x,y)\in E} H(x, y)G(x, y)$, H, $G \in \mathcal{H}(E)$, is a Hilbert space.

The *weighted finite difference operator* of a function $u \in \mathcal{H}(V)$, denoted by $\mathbf{d}_\omega : \mathcal{H}(V) \to \mathcal{H}(E)$, is defined on a pair of vertices $(x, y) \in E$ as :

$$\mathbf{d}_\omega u(x, y) = \sqrt{\omega(x, y)}(u(y) - u(x)).$$

Note that this difference operator is linear and antisymmetric.

The adjoint of the difference operator $\mathbf{d}_\omega$, denoted by $\mathbf{d}^*_\omega : \mathcal{H}(E) \to \mathcal{H}(V)$, is a linear operator which can be characterized by $\langle \mathbf{d}_\omega u, H\rangle_{\mathcal{H}(E)} = \langle u, \mathbf{d}^*_\omega H\rangle_{\mathcal{H}(V)}$ for all $u \in \mathcal{H}(V)$ and all $H \in \mathcal{H}(E)$. Using the definitions of the finite weighted difference operator and the inner products of $\mathcal{H}(V)$ and $\mathcal{H}(E)$, the adjoint operator $\mathbf{d}^*_\omega$ of a function $H \in \mathcal{H}(E)$ can be expressed at a vertex $x \in V$ by the following expression:

$$\mathbf{d}^*_\omega H(x) = \sum_{y\sim x} \sqrt{\omega(x, y)}(H(y, x) - H(x, y)).$$

The *divergence operator* defined by

$$\mathbf{div}_\omega = -\mathbf{d}^*_\omega,$$

measures the net outflow of a function of $\mathcal{H}(E)$ at each vertex of the graph. Each function $H \in \mathcal{H}(E)$ has a null divergence over the entire set of vertices. Indeed, from the previous definitions, it can be easily shown that $\sum_{x\in V}\sum_{y\sim x}\mathbf{d}_\omega u(x,y) = 0$, for all $u \in \mathcal{H}(V)$, and $\sum_{x\in V}\mathbf{div}_\omega H(x) = 0$, for all $H \in \mathcal{H}(E)$.

The *weighted directional finite difference* of u at a vertex x along the edge (x,y) is defined as:

$$\partial_y u(x) = \sqrt{\omega(x,y)}(u(y) - u(x)).$$

Similarly we define the upwind and downwind weighted directional finite differences of u at a vertex x along the edge (x,y) is defined as:

$$\partial_y^{\pm} u(x) = \sqrt{\omega(x,y)}\,(u(y) - u(x))^{\pm},$$

where $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$, $a \in \mathbb{R}$. Based on this definition, one can straightforwardly introduce the *weighted gradient* operator on graphs $\nabla_\omega : \mathcal{H}(V) \to \mathcal{H}(V)$, which is defined on a vertex $x \in V$ as the vector of all weighted finite differences with respect to the set of vertices V,i.e:

$$(\nabla_\omega u)(x) = (\partial_y u(x))_{y\in V}.$$

From the properties of the weighted partial difference above, it gets clear that the weighted gradient is linear and antisymmetric. Similarly we define the upwind downwind weighted gradient operators on graphs $\nabla^{\pm}_\omega : \mathcal{H}(V) \to \mathcal{H}(V)$

$$(\nabla^{\pm}_\omega u)(x) = (\partial^{\pm}_y u(x))_{y\in V}, \text{ for all } x \in V.$$

A family of gradient norm $\|\cdot\|_p \circ \nabla_\omega$, $\|\cdot\|_p \circ \nabla^{\pm}_\omega : \mathcal{H}(V) \to (\mathbb{R}^+)^{|V|}$ with $1 \leqslant p \leqslant \infty$ is given for a function $u \in \mathcal{H}(V)$ as:

$$\begin{aligned}
\|(\nabla_\omega u)(x)\|_p &= \left(\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}|u(y)-u(x)|^p\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty \\
\|(\nabla_\omega u)(x)\|_\infty &= \max_{y\sim x}(\sqrt{\omega(x,y)}|u(y)-u(x)|), \\
\|(\nabla^{\pm}_\omega u)(x)\|_p &= \left(\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}\left((u(y)-u(x))^{\pm}\right)^p\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty \\
\|(\nabla^{\pm}_\omega u)(x)\|_\infty &= \max_{y\sim x}(\sqrt{\omega(x,y)}\,(u(y)-u(x))^{\pm}).
\end{aligned}$$

The integral of a function u in $\mathcal{H}(V)$ (wrt to the empirical measure on V) is defined by:

$$E(u) = \sum_{x\in V} u(x).$$

The anisotropic graph p-Laplacian of a function $u \in \mathcal{H}(V)$, denoted by $\Delta_{\omega,p} : \mathcal{H}(V) \to \mathcal{H}(V)$ is defined as

$$\begin{aligned}
\Delta_{\omega,p} u(x) &\overset{\text{def}}{=} \mathbf{div}_\omega\left(|\mathbf{d}_\omega u|^{p-2}\mathbf{d}_\omega u\right)(x) \\
&= 2\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}|u(y)-u(x)|^{p-2}(u(y)-u(x)),
\end{aligned}$$

where $1 < p < \infty$ and $x \in V$.

Chapter 3

Continuum limits of the p-Laplacian evolution problem on sparse graphs

Main contributions of this chapter

- Well-posedness of the Cauchy problem.
- Error estimates to compare two trajectories corresponding to the p-Laplacian governed by two kernels, second member and initial data:
  - for $p \in]1, +\infty[$ (Theorem 3.3.1).
  - for $p = 1$ (Theorem 3.3.5).
- Consistency and error estimates of the numerical solutions to the fully-discretized problem:
  - forward discretization (Theorem 3.4.7 for $p \in]1, 2]$, Theorem 3.4.12 for $p = 1$).
  - backward discretization (Theorem 3.4.16 for $p \in]1, +\infty[$).
- Error bound on fully discretized problems on sparse random graphs. (Theorem 3.5.3)

The content of this chapter can be found in [69].

Contents

In this chapter, we present a consistency analysis for the nonlocal p-Laplacian evolution problem. Our results consist of four main parts: well-posedness, consistency of the continuous-continuous problem, error bounds for the discrete problem and application of these results to the fully discretized problems on random graph models. For the time-discrete problem, both the semi-discrete and fully-discrete versions with both forward and backward Euler approximations are exposed. We prove the convergence of these schemes before comparing their corresponding problems to the continuum one. The obtained error bounds will be used in the fourth part to analyse error bound on fully discretized problems on sparse random graphs.

3.1 Introduction

3.1.1 Problem formulation

Our main goal in this chapter is to study discretization of the following nonlocal p-Laplacian evolution problem with homogeneous Neumann boundary conditions:

$$\begin{cases} \frac{\partial}{\partial t}u(\boldsymbol{x},t) = \int_\Omega K(\boldsymbol{x},\boldsymbol{y})\left|u(\boldsymbol{y},t)-u(\boldsymbol{x},t)\right|^{p-2}(u(\boldsymbol{y},t)-u(\boldsymbol{x},t))d\boldsymbol{y} + f(\boldsymbol{x},t), & \boldsymbol{x}\in\Omega, t>0, \\ u(\boldsymbol{x},0) = g(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where $p \in [1, +\infty[$, $\Omega \subset \mathbb{R}^d$ is a bounded domain, $d \geq 1$, without loss of generality $\Omega = [0,1]^d$, and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the kernel function. In particular, in the setting of graphs, $d = 1$ and it will be seen that K is the limit object for some convergent graph sequence $\{G_n\}_n, n \in \mathbb{N}$, whose meaning and form will be specified in the sequel. Throughout, we assume that

(**H.1**) K is a nonnegative measurable function.

(**H.2**) K is symmetric, i.e., $K(\boldsymbol{x},\boldsymbol{y}) = K(\boldsymbol{y},\boldsymbol{x})$.

(**H.3**) $\sup_{\boldsymbol{x}\in\Omega}\int_\Omega K(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} < +\infty$.

By (**H.2**), it is straightforward to see that

$$\sup_{\boldsymbol{x}\in\Omega}\int_\Omega K(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} = \sup_{\boldsymbol{y}\in\Omega}\int_\Omega K(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x},$$

and thus, (**H.3**) is equivalent to

$$\sup_{\boldsymbol{y}\in\Omega}\int_\Omega K(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x} < +\infty.$$

When the kernel is such that $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$, where $J : \mathbb{R}^d \to \mathbb{R}$, then **(H.1)**, **(H.2)** and **(H.3)** read:

(H'.1) J is nonnegative and measurable.

(H'.2) J is symmetric, i.e., $J(-\boldsymbol{x}) = J(\boldsymbol{x})$.

(H'.3) $\int_{\Omega-\Omega} J(\boldsymbol{x})d\boldsymbol{x} < +\infty$.

Recall that $\Omega - \Omega$ is the Minkowski sum of Ω and $-\Omega$. In the case $\Omega = [0,1]^d$, we obviously have $\Omega - \Omega = [-1,1]^d$.

The main goal of this chapter is to revisit and extend the work of [90] by removing important limiting assumptions made there on the kernel K and the initial condition g. In turn, this will allow us to establish consistency estimates of the fully discretized p-Laplacian problem for singular kernels or on sparse graphs whose limits are known not to be bounded graphons.

3.1.2 Organization of the chapter

In Section 3.2, we study the well-posedness of the problem ($\mathcal{P}$). Section 3.3 is devoted to study stability of the problem ($\mathcal{P}$) with respect to sequences of kernels K, initial data g and second member f. Error bounds for the semi-discete (i.e., space discretization of (K, g, f)) problem are established in Section 3.4.1, and those for the fully discrete (time and space discretization) problem with forward and backward Euler time-discretization are provided in Section 3.4.2. Section 3.5 is devoted to applying these results to fully discretized problems on sparse random graph models.

3.2 Well-posedness

3.2.1 The case $p \in]1, +\infty[$

To lighten notation, for $1 < p < +\infty$, we define the function

$$\Psi : x \in \mathbb{R} \mapsto |x|^{p-2}x = \mathrm{sign}(x)|x|^{p-1},$$

where we take $\mathrm{sign}(0) = 0$. The next lemma summarizes key monotonicity and continuity properties of Ψ which will be instrumental to us.

Lemma 3.2.1. (i) *Monotonicity: assume that the constant β satisfies $\beta \in [\max(p,2), +\infty[$. Then for all $x, y \in \mathbb{R}$,*

$$(\Psi(y) - \Psi(x))(y - x) \geq C_1|y-x|^{\beta}(|y| + |x|)^{p-\beta}, \tag{3.2.1}$$

where the constant C_1 is sharp and given by

$$C_1 = 2^{2-p}\min(1, p-1). \tag{3.2.2}$$

In particular,

$$(\Psi(y) - \Psi(x))(y - x) \geq C_1 \begin{cases} |y-x|^p & p \in [2, +\infty[, \\ |y-x|^2(|y| + |x|)^{p-2} & p \in]1, 2]. \end{cases} \tag{3.2.3}$$

(ii) *Continuity: assume that the constant α satisfies $\alpha \in [0, \min(1, p-1)]$. Then for all $x, y \in \mathbb{R}$,*

$$|\Psi(y) - \Psi(x)| \leq C_2|y-x|^{\alpha}(|y| + |x|)^{p-1-\alpha}, \tag{3.2.4}$$

where the constant C_2 is sharp and given by

$$C_2 = \max(2^{2-p}, (p-1)2^{2-p}, 1). \tag{3.2.5}$$

In particular,

$$|\Psi(y) - \Psi(x)| \leq C_2 \begin{cases} |y-x|(|y| + |x|)^{p-2} & p \in [2, +\infty[, \\ |y-x|^{p-1} & p \in]1, 2], \end{cases} \tag{3.2.6}$$

PROOF :

(i) For (3.2.1), see [43, Theorem 2.2]. For (3.2.3), set $\beta = p$ for $p \geq 2$ and $\beta = 2$ otherwise in (3.2.1); see also the seminal results of [86, Lemma 5.1 and Lemma 5.2].

(ii) For (3.2.4), see [43, Theorem 2.1]. For (3.2.6), set $\alpha = 1$ for $p \geq 2$ and $\alpha = p-1$ otherwise in (3.2.4); see also the seminal results of [86, Lemma 5.3 and Lemma 5.4].

□

We now collect some preliminary properties of the nonlocal p-Laplacian, an operator on $L^1(\Omega)$ that we denote for short as

$$\Delta_p^K u(\boldsymbol{x},t) = -\int_\Omega K(\boldsymbol{x},\boldsymbol{y})\left|u(\boldsymbol{y},t)-u(\boldsymbol{x},t)\right|^{p-2}(u(\boldsymbol{y},t)-u(\boldsymbol{x},t))d\boldsymbol{y}.$$

Proposition 3.2.2. *Assume that K satisfies (H.1), (H.2) and (H.3).*

(i) *Δ_p^K is positively homogeneous of degree $p-1$.*

(ii) *If $p > 2$, $L^{p-1}(\Omega) \subset \mathrm{dom}(\Delta_p^K)$.*

(iii) *If $1 < p \leq 2$, $\mathrm{dom}(\Delta_p^K) = L^1(\Omega)$ and Δ_p^K is closed in $L^1(\Omega) \times L^1(\Omega)$.*

(iv) *Let $h : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Then for every $u, v \in L^p(\Omega)$,*

$$\begin{aligned} 0 &\leq \int_\Omega \left(\Delta_p^K u(\boldsymbol{x}) - \Delta_p^K v(\boldsymbol{x})\right) h(u(\boldsymbol{x})-v(\boldsymbol{x}))d\boldsymbol{x} \\ &= \frac{1}{2}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left(\Psi(u(\boldsymbol{y})-u(\boldsymbol{x})) - \Psi(v(\boldsymbol{y})-v(\boldsymbol{x}))\right)\left(h(u(\boldsymbol{y})-v(\boldsymbol{y})) - h(u(\boldsymbol{x})-v(\boldsymbol{x}))\right)d\boldsymbol{y}d\boldsymbol{x}. \end{aligned} \tag{3.2.7}$$

If h is bounded, then this holds for any $u, v \in \mathrm{dom}(\Delta_p^K)$.

(v) *For every $u, v \in L^p(\Omega)$,*

$$\int_\Omega \left(\Delta_p^K u(\boldsymbol{x}) - \Delta_p^K v(\boldsymbol{x}))(u(\boldsymbol{x})-v(\boldsymbol{x}))\right) d\boldsymbol{x} \geq \frac{C}{2}\left(\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left|(u(\boldsymbol{y})-u(\boldsymbol{x}))-(v(\boldsymbol{y})-v(\boldsymbol{x}))\right|^p d\boldsymbol{y}d\boldsymbol{x}\right)^{\max(1,2/p)}$$

where

$$C = \begin{cases} C_1 & p \in [2,+\infty[, \\ 2^{2p-5}C_1\|K\|_{L^{\infty,1}(\Omega^2)}^{1-2/p}\left(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\right)^{p-2} & p \in]1,2[. \end{cases}$$

and C_1 is the constant in (3.2.2). If $u, v \in L^\infty(\Omega)$, then

$$\int_\Omega \left(\Delta_p^K u(\boldsymbol{x}) - \Delta_p^K v(\boldsymbol{x}))(u(\boldsymbol{x})-v(\boldsymbol{x}))\right) d\boldsymbol{x} \geq \frac{C}{2}\left(\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left|(u(\boldsymbol{y})-u(\boldsymbol{x}))-(v(\boldsymbol{y})-v(\boldsymbol{x}))\right|^2 d\boldsymbol{y}d\boldsymbol{x}\right)^{\max(1,p/2)},$$

where

$$C = \begin{cases} C_1 & p \in [2,+\infty[, \\ 2^{p-2}C_1\left(\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}\right)^{p-2} & p \in]1,2[. \end{cases}$$

(vi) *For $p \in]1,2]$ and every $u, v \in L^2(\Omega)$,*

$$\int_\Omega \left(\Delta_p^K u(\boldsymbol{x}) - \Delta_p^K v(\boldsymbol{x}))(u(\boldsymbol{x})-v(\boldsymbol{x}))\right) d\boldsymbol{x} \geq C\left\|\Delta_p^K u - \Delta_p^K v\right\|_{L^2(\Omega)}^{p/(p-1)}$$

where

$$C = 2^{\frac{p-2}{2(p-1)}}\left(C_2^{1/2}\|K\|_{L^{\infty,1}(\Omega^2)}\right)^{\frac{1}{1-p}}(1-1/p), \quad \text{and } C_2 \text{ is the constant in (3.2.5).}$$

(vii) *For $p \in]1,+\infty[$, Δ_p^K is completely accretive and satisfies the range condition*

$$L^p(\Omega) \subset \mathrm{ran}(\mathbf{I} + \Delta_p^K). \tag{3.2.8}$$

Consequently, the resolvent $J_{\lambda\Delta_p^K} \stackrel{\text{def}}{=} \left(\mathbf{I} + \lambda\Delta_p^K\right)^{-1}$, $\lambda > 0$, is single-valued on $L^p(\Omega)$ and nonexpansive in $L^q(\Omega)$ for all $q \in [1,+\infty]$.

Proof : (i), (ii) and (iii) follow from [10, Remark 2.2] which still holds for our larger class of kernels K.

For (iv), see [90, Lemma A.2]. Monotonicity is immediate since h is non-decreasing.

The proof of (vii) is the same as that of [10, Theorem 2.4], where we invoke the monotonicity claim (i).

We now show (v)[1]. The case $p \in [2,+\infty[$ is immediate by inserting Lemma 3.2.1(i) into (3.2.7) with $h(x) = x$. For $p \in]1,2]$, to lighten notation, denote the nonlocal gradient $\nabla^{\mathrm{NL}} u(\boldsymbol{x},\boldsymbol{y}) = u(\boldsymbol{y}) - u(\boldsymbol{x})$. We then have by Lemma 3.2.1(i) that

$$\begin{aligned}
&C_1|\nabla^{\mathrm{NL}}(u-v)(\boldsymbol{x},\boldsymbol{y})|^2 \leq \\
&\left(\Psi(\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y})) - \Psi(\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y}))\right)\left(\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y}) - \nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y})\right)\left(|\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y})|\right)^{2-p}.
\end{aligned} \tag{3.2.9}$$

Taking the power $p/2$, multiplying by K and integrating, we get

$$\begin{aligned}
&C_1^{p/2}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})|\nabla^{\mathrm{NL}}(u-v)(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{x}d\boldsymbol{y} \leq \\
&\quad \int_{\Omega^2}\left(K(\boldsymbol{x},\boldsymbol{y})\left(\Psi(\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y})) - \Psi(\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y}))\right)\left(\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y}) - \nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y})\right)\right)^{p/2} \\
&\qquad\qquad \left(K(\boldsymbol{x},\boldsymbol{y})^{1/p}(|\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y})|)\right)^{(2-p)p/2} d\boldsymbol{x}d\boldsymbol{y}.
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
\left(K\cdot\left(\Psi(\nabla^{\mathrm{NL}}u) - \Psi(\nabla^{\mathrm{NL}}v)\right)\left(\nabla^{\mathrm{NL}}u - \nabla^{\mathrm{NL}}v\right)\right)^{p/2} &\in L^{2/p}(\Omega^2) \\
\left(K^{1/p}\cdot(|\nabla^{\mathrm{NL}}u| + |\nabla^{\mathrm{NL}}v|)\right)^{(2-p)p/2} &\in L^{2/(2-p)}(\Omega^2).
\end{aligned}$$

It then follows from Hölder inequality and (3.2.7) that

$$\begin{aligned}
&C_1^{p/2}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})|\nabla^{\mathrm{NL}}(u-v)(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{x}d\boldsymbol{y} \leq \\
&\quad 2\left(\int_\Omega \left(\Delta_p^K u(\boldsymbol{x}) - \Delta_p^K v(\boldsymbol{x})\right)(u(\boldsymbol{x}) - v(\boldsymbol{x}))d\boldsymbol{x}\right)^{p/2}\cdot \\
&\qquad \left(\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})(|\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y})|)^p d\boldsymbol{x}d\boldsymbol{y}\right)^{(2-p)/2}.
\end{aligned}$$

We have by Jensen's inequality

$$\begin{aligned}
&\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})(|\nabla^{\mathrm{NL}}u(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y})|)^p d\boldsymbol{x}d\boldsymbol{y} \\
&\leq 4^{p-1}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})(|u(\boldsymbol{x})|^p + |u(\boldsymbol{y})|^p + |v(\boldsymbol{x})|^p + |v(\boldsymbol{y})|^p)d\boldsymbol{x}d\boldsymbol{y} \\
&\leq 2^{2p-1}\|K\|_{L^{\infty,1}(\Omega^2)}\left(\|u\|_{L^p(\Omega)}^p + \|v\|_{L^p(\Omega)}^p\right),
\end{aligned}$$

whence we obtain

$$C_1\left(\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})|\nabla^{\mathrm{NL}}(u-v)(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{x}d\boldsymbol{y}\right)^{2/p} \leq$$

[1] This can be seen as a nonlocal analogue of [86, Proposition 5.1 and Proposition 5.2].

$$2^{5-2p}\left(\int_\Omega \left(\Delta_p^K u(\boldsymbol{x}) - \Delta_p^K v(\boldsymbol{x})\right)(u(\boldsymbol{x})-v(\boldsymbol{x}))d\boldsymbol{x}\right) \|K\|_{L^{\infty,1}(\Omega^2)}^{(2-p)/p}\left(\|u\|_{L^p(\Omega)}+\|v\|_{L^p(\Omega)}\right)^{2-p}.$$

Rearranging proves the bound. For $u,v \in L^\infty(\Omega)$ and $p \in [2,+\infty]$ we use that $L^p(\Omega) \subset L^2(\Omega)$. For $p \in]1,2]$, we embark from (3.2.9) and use that for all $(\boldsymbol{x},\boldsymbol{y}) \in \Omega^2$,

$$|\nabla^{\mathrm{NL}} u(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{\mathrm{NL}} v(\boldsymbol{x},\boldsymbol{y})| \leq 2\left(\|u\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}\right).$$

Multiplying (3.2.9) by K, integrating and using (3.2.7), we conclude.

To prove (vi), we start by showing that Δ_p^K is Hölder continuous with exponent $p-1$ on $L^2(\Omega)$. We have by Jensen inequality (twice) and (3.2.6),

$$\begin{aligned}
\|\Delta_p^K u - \Delta_p^K v\|_{L^2(\Omega)}^2 &= \int_\Omega \left|\int_\Omega K(\boldsymbol{x},\boldsymbol{y})\left(\Psi(\nabla^{\mathrm{NL}} u(\boldsymbol{x},\boldsymbol{y})) - \Psi(\nabla^{\mathrm{NL}} v(\boldsymbol{x},\boldsymbol{y}))\right)d\boldsymbol{y}\right|^2 d\boldsymbol{x}\\
&\leq \|K\|_{L^{\infty,1}(\Omega^2)}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left(\Psi(\nabla^{\mathrm{NL}} u(\boldsymbol{x},\boldsymbol{y})) - \Psi(\nabla^{\mathrm{NL}} v(\boldsymbol{x},\boldsymbol{y}))\right)^2 d\boldsymbol{x}d\boldsymbol{y}\\
&\leq C_2\|K\|_{L^{\infty,1}(\Omega^2)}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left(\nabla^{\mathrm{NL}}(u-v)(\boldsymbol{x},\boldsymbol{y})\right)^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\\
&\leq 2^p C_2\|K\|_{L^{\infty,1}(\Omega^2)}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left(u(\boldsymbol{x})-v(\boldsymbol{x})\right)^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\\
&\leq 2^p C_2\|K\|_{L^{\infty,1}(\Omega^2)}^2\left(\int_\Omega (u(\boldsymbol{x})-v(\boldsymbol{x}))^2 d\boldsymbol{x}\right)^{p-1}\\
&= 2^p C_2\|K\|_{L^{\infty,1}(\Omega^2)}^2\|u-v\|_{L^2(\Omega)}^{2(p-1)}.
\end{aligned} \tag{3.2.10}$$

We are now in position to invoke [21, Corollary 18.14(i)⇒(v)] to show that the claimed inequality holds. □

Solutions of ($\mathcal{P}$) will be understood in the following sense:

Definition 3.2.3. Let $p \in]1,+\infty[$. A solution of ($\mathcal{P}$) in $[0,T]$ is a function

$$u \in C([0,T];L^1(\Omega)) \cap W^{1,1}(]0,T[;L^1(\Omega)),$$

that satisfies $u(\boldsymbol{x},0) = g(\boldsymbol{x})$ a.e. $\boldsymbol{x} \in \Omega$ and

$$\frac{\partial}{\partial t}u(\boldsymbol{x},t) = -\Delta_p^K u(\boldsymbol{x},t) + f(\boldsymbol{x},t) \quad \text{a.e. in } \Omega\times]0,T[.$$

Such a solution is also a strong solution (see [11, Definition A.3]).

The main result of existence and uniqueness of a global solution, that is, a solution on $[0,T]$ for $T>0$ is stated in the following theorem.

Theorem 3.2.4. *Suppose that $p \in]1,+\infty[$ and assumptions **(H.1)**, **(H.2)** and **(H.3)** hold. Let $g \in L^p(\Omega)$ and $f \in L^1([0,T];L^p(\Omega))$.*

(i) For any $T>0$, there exists a unique strong solution in $[0,T]$ of ($\mathcal{P}$).

(ii) Moreover, for $q \in [1,+\infty]$, if $g_i \in L^q(\Omega)$ and $f_i \in L^1([0,T];L^q(\Omega))$, $i=1,2$, and u_i is the solution of ($\mathcal{P}$) with data (f_i,g_i), then

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|_{L^q(\Omega)} \leq \|g_1-g_2\|_{L^q(\Omega)} + \|f_1-f_2\|_{L^1([0,T];L^q(\Omega))}, \quad \forall t \in [0,T]. \tag{3.2.11}$$

Proof : The proof follows the same lines as that of [10, Theorem 1.2] extended to the case where $f \not\equiv 0$ thanks to the results of [25], where we invoke Proposition 3.2.2(ii), (iii) and (vii). □

Remark 3.2.5. In [10] (see also [11, Chapte 6]), the authors impose the following stringent assumptions: $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$, where J is nonnegative, continuous, radially symmetric, compactly supported, $J(0) > 0$ and $\int_{\mathbb{R}^d} J(\boldsymbol{x})d\boldsymbol{x} < +\infty$. Actually, these assumptions are not needed for existence and uniqueness. The particular form $J(\boldsymbol{x}-\boldsymbol{y})$ of the kernel is not needed. Continuity with radial symmetry and support compactness play a pivotal role to study convergence to the local p-Laplacian problem in [10, Theorem 1.5]. In addition, $J(0) > 0$ was mandatory to prove a Poincaré-type inequality in [10, Proposition 4.1]. Even for the form $J(\boldsymbol{x}-\boldsymbol{y})$, our assumptions **(H'.1)**, **(H'.2)** and **(H'.3)** are weaker than those of [10]. This discussion remains true also for the case $p = 1$.

3.2.2 The case $p = 1$

We will need to define subdifferential of the absolute value function on $\mathbb{R}$, which is the well-known set-valued mapping $\partial|\cdot| : \mathbb{R} \to 2^{\mathbb{R}}$,

$$\partial|\cdot|(x) = \begin{cases} 1 & x > 0 \\ [-1,1] & x = 0 \\ -1 & x < 0. \end{cases}$$

It will be convenient to denote the 1-Laplacian Δ_1^K. This is a set-valued operator in $L^1(\Omega) \times L^1(\Omega)$ such that $\eta \in \Delta_1^K u$ if and only if

$$\eta(\boldsymbol{x}) = -\int_\Omega K(\boldsymbol{x},\boldsymbol{y})w(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} \qquad \text{a.e. in } \Omega,$$

for a subgradient function w satisfying $\|w\|_{L^\infty(\Omega^2)} \leq 1$, $w(\boldsymbol{x},\boldsymbol{y}) = -w(\boldsymbol{y},\boldsymbol{x})$, and

$$w(\boldsymbol{x},\boldsymbol{y}) \in \partial|\cdot|(u(\boldsymbol{y}) - u(\boldsymbol{x})).$$

Solutions of $(\mathcal{P})$ will be understood in the following sense.

Definition 3.2.6. A solution of $(\mathcal{P})$ for $p = 1$ in $[0,T]$ is a function

$$u \in C([0,T]; L^1(\Omega)) \cap W^{1,1}(]0,T[; L^1(\Omega)),$$

that satisfies $u(\boldsymbol{x},0) = g(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$ and

$$\frac{\partial}{\partial t}u(\boldsymbol{x},t) = -\eta(\boldsymbol{x},t) + f(\boldsymbol{x},t) \qquad \text{a.e. in } \Omega\times]0,T[,$$

where $\eta(\cdot,t) \in \Delta_1^K u(\cdot,t)$.

Observe that for $p = 1$, the evolution problem $(\mathcal{P})$ reads

$$\begin{cases} \frac{\partial}{\partial t}u(\boldsymbol{x},t) = \int_\Omega K(\boldsymbol{x},\boldsymbol{y})\,\mathrm{sign}(u(\boldsymbol{y},t) - u(\boldsymbol{x},t))d\boldsymbol{y} + f(\boldsymbol{x},t), & \boldsymbol{x} \in \Omega, t > 0, \\ u(\boldsymbol{x},0) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \end{cases}$$

where

$$\mathrm{sign}(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Thus, it satisfies

$$\frac{\partial}{\partial t}u(\cdot,t) \in -\Delta_1^K u(\cdot,t).$$

In the same vein as Proposition 3.2.2, the 1-Laplacian enjoys the following properties.

Proposition 3.2.7. *Assume that K satisfies **(H.1)**, **(H.2)** and **(H.3)**.*

(i) $\mathrm{dom}(\Delta_1^K) = L^1(\Omega)$ *and (the graph of) Δ_1^K is closed in $L^1(\Omega) \times L^1(\Omega)$.*

(ii) *Let $h \in C^1(\mathbb{R})$ be a nondecreasing function. Then for every $u_i \in L^1(\Omega)$ and any $\eta_i \in \Delta_1^K u_i$, $i = 1, 2$,*

$$\begin{aligned} 0 &\leq \int_\Omega (\eta_1(\boldsymbol{x}) - \eta_2(\boldsymbol{x}))\,(h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x} \\ &= \frac{1}{2}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\,(w_1(\boldsymbol{x},\boldsymbol{y}) - w_2(\boldsymbol{x},\boldsymbol{y}))\,(h(u_1(\boldsymbol{y}) - u_2(\boldsymbol{y})) - h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x}d\boldsymbol{y}. \end{aligned} \tag{3.2.12}$$

where w_i are the corresponding subgradient functions defined above. In particular,

$$\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})w_i(\boldsymbol{x},\boldsymbol{y})u_i(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y} = -\frac{1}{2}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\big|u_i(\boldsymbol{y}) - u_i(\boldsymbol{x})\big|d\boldsymbol{x}d\boldsymbol{y}.$$

(iii) *Δ_1^K is completely accretive and satisfies the range condition*

$$L^\infty(\Omega) \subset \mathrm{ran}(\mathbf{I} + \Delta_1^K). \tag{3.2.13}$$

Proof : For (i), see [10, Remark 2.8] which still holds for our class of kernels K.

The proof of (iii) is again the same as that of [10, Theorem 2.9], where we invoke the monotonicity claim (ii) to which we turn now.

For any $v \in L^1(\Omega)$, we have the integration by parts formula

$$\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})w_i(\boldsymbol{x},\boldsymbol{y})(v(\boldsymbol{y}) - v(\boldsymbol{x}))d\boldsymbol{x}d\boldsymbol{y} \tag{3.2.14}$$

$$= -\int_{\Omega^2} K(\boldsymbol{y},\boldsymbol{x})w_i(\boldsymbol{y},\boldsymbol{x})v(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x} - \int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})w_i(\boldsymbol{x},\boldsymbol{y})v(\boldsymbol{x}))d\boldsymbol{x}d\boldsymbol{y} \tag{3.2.15}$$

$$= -2\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})w_i(\boldsymbol{x},\boldsymbol{y})v(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y}. \tag{3.2.16}$$

Taking $v(\boldsymbol{x}) = h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x}))$ in (3.2.16) with w_1 and w_2, and then taking the difference, we arrive at

$$\begin{aligned} &-2\int_\Omega \left(\int_\Omega K(\boldsymbol{x},\boldsymbol{y})(w_1(\boldsymbol{x},\boldsymbol{y}) - w_2(\boldsymbol{x},\boldsymbol{y}))d\boldsymbol{y}\right) h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x}))d\boldsymbol{x} \\ &= 2\int_\Omega (\eta_1(\boldsymbol{x}) - \eta_2(\boldsymbol{x}))\,(h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x} \\ &= \int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\,(w_1(\boldsymbol{x},\boldsymbol{y}) - w_2(\boldsymbol{x},\boldsymbol{y}))\,(h(u_1(\boldsymbol{y}) - u_2(\boldsymbol{y})) - h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x}d\boldsymbol{y}. \end{aligned}$$

By the mean-value theorem applied to h, we get

$$\begin{aligned} &= 2\int_\Omega (\eta_1(\boldsymbol{x}) - \eta_2(\boldsymbol{x}))\,(h(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x} \\ &= \int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\,(w_1(\boldsymbol{x},\boldsymbol{y}) - w_2(\boldsymbol{x},\boldsymbol{y}))\,h'(\zeta(\boldsymbol{x},\boldsymbol{y}))\,((u_1(\boldsymbol{y}) - u_2(\boldsymbol{y})) - (u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x}d\boldsymbol{y} \\ &= \int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})h'(\zeta(\boldsymbol{x},\boldsymbol{y}))\,(w_1(\boldsymbol{x},\boldsymbol{y}) - w_2(\boldsymbol{x},\boldsymbol{y}))\,((u_1(\boldsymbol{y}) - u_1(\boldsymbol{x})) - (u_2(\boldsymbol{y}) - u_2(\boldsymbol{x})))\,d\boldsymbol{x}d\boldsymbol{y}, \end{aligned}$$

where $\zeta(\boldsymbol{x},\boldsymbol{y})$ is an intermediate value between $u_1(\boldsymbol{y}) - u_2(\boldsymbol{y})$ and $u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})$. Since h is increasing, that $w_i(\boldsymbol{x},\boldsymbol{y}) \in \partial|\cdot|(u_i(\boldsymbol{y}) - u_i(\boldsymbol{x}))$, and $\partial|\cdot|$ is a monotone operator, we get the claimed monotonicity.

To get the particular identity, we specialize (3.2.16) by taking $v = u_i$, which entails

$$-\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})w_i(\boldsymbol{x},\boldsymbol{y})(u_i(\boldsymbol{y}) - u_i(\boldsymbol{x}))d\boldsymbol{x}d\boldsymbol{y} = 2\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})w_i(\boldsymbol{x},\boldsymbol{y})u_i(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y}.$$

We finally use the equivalent characterization of $\partial\big|\cdot\big|$, which originates from the Fenchel's identity since $\big|\cdot\big|$ is positively homogeneous,

$$\partial\big|\cdot\big|(x) = \left\{\xi \in \mathbb{R} : \ \big|\xi\big| \leq 1 \text{ and } \xi x = \big|x\big|\right\}.$$

Applying this identity with $x = u_i(\boldsymbol{y}) - u_i(\boldsymbol{x})$ and $\xi = w_i(\boldsymbol{x},\boldsymbol{y})$ gives the claim. □

Theorem 3.2.8. *Suppose that $p = 1$, and assumptions (**H.1**), (**H.2**) and (**H.3**) hold. Let $g \in L^1(\Omega)$ and $f \in L^1([0,T]; L^1(\Omega))$. For any $T > 0$, there exists a unique solution in $[0,T]$ of (𝒫) in the sense of Definition 3.2.6.*

Proof : The proof is an adaptation of [10, Theorem 1.4] to the case where $f \not\equiv 0$ thanks to the results of [25], where we invoke Proposition 3.2.7(i) and (iii). □

3.3 Continuous-continuous estimates

In this section, we provide an estimate that compares solutions of two p-Laplacian evolution problems of the form (𝒫) with two different kernels and initial data. This estimate will be instrumental to derive error bounds in the totally discrete case.

3.3.1 The case $p \in]1, +\infty[$

We have the following error bounds and convergence result.

Theorem 3.3.1. *Suppose that $p \in]1, +\infty[$. Let u be a solution of (𝒫) with kernel K and data (f, g). Let u_n be a sequence of solutions to (𝒫) with kernels K_n and data (f_n, g_n). Assume that K and K_n satisfy (**H.1**), (**H.2**) and $K, K_n \in L^{\infty,2}(\Omega^2)$, and that either one of the following holds:*

(a) $p \in]1, 2[$, $g, g_n \in L^2(\Omega)$, and $f, f_n \in L^1([0,T]; L^2(\Omega))$;

(b) $p \geq 2$, $g, g_n \in L^{2(p-1)}(\Omega)$ and $f, f_n \in L^1([0,T]; L^{2(p-1)}(\Omega))$;

(c) $g, g_n \in L^\infty(\Omega)$ and $f, f_n \in L^1([0,T]; L^\infty(\Omega))$.

Then, the following hold.

(i) u and u_n are the unique solutions of (𝒫) with respectively data (f, g) and (f_n, g_n).

(ii) We have the error estimate

$$\left\|u_n - u\right\|_{C([0,T];L^2(\Omega))} \leq \left\|g_n - g\right\|_{L^2(\Omega)} + \left\|f_n - f\right\|_{L^1([0,T];L^2(\Omega))} + CT \begin{cases} \left\|K_n - K\right\|_{L^{\infty,2}(\Omega^2)}, & \text{under (a) or (b)} \\ \left\|K_n - K\right\|_{L^2(\Omega^2)}, & \text{under (c)} \end{cases} \tag{3.3.1}$$

where C is positive constant that may depend only on p, g and f.

(iii) Moreover, if (c) holds, $\sup_{n\in\mathbb{N}} \left|g_n(\boldsymbol{x})\right| < +\infty$ a.e. on Ω and $g_n \to g$ pointwise a.e. on Ω, $\sup_{n\in\mathbb{N}} \left|f_n(\boldsymbol{x}, t)\right| < +\infty$ a.e. on $\Omega \times [0,T]$ and $f_n \to f$ pointwise a.e. on $\Omega \times [0,T]$, and the sequence $\{|K_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable over Ω^2 and $K_n \to K$ pointwise a.e. on Ω^2. Then

$$\lim_{n\to+\infty} \left\|u_n - u\right\|_{C([0,T];L^2(\Omega))} = 0.$$

Remark 3.3.2. Observe that since $L^\infty(\Omega) \subset L^2(\Omega)$ and $L^{2(p-1)}(\Omega) \subset L^2(\Omega)$ for $p \geq 2$, then the first two terms involved in (3.3.1) provide a non-trivial bound. Similarly, since $L^{\infty,2}(\Omega^2) \subset L^2(\Omega^2)$, the last term in the bound for case (c) is also non-trivial. In fact, both bounds in (3.3.1) can be summarized in one bound; the first one. However, the second bound for case (c) is obviously sharper.

Proof : In the proof, C is any positive constant that may depend solely on p, g and f.

(i) Since $L^{\infty,2}(\Omega^2) \subset L^{\infty,1}(\Omega^2)$, assumption (**H.3**) holds for both K and K_n. We also have the embeddings

- $L^2(\Omega) \subset L^p(\Omega)$ under (a),

- $L^{2(p-1)}(\Omega) \subset L^p(\Omega)$ under (b), and
- $L^\infty(\Omega) \subset L^p(\Omega)$ under (c) .

Thus $g, g_n \in L^p(\Omega)$ and $f, f_n \in L^1([0,T]; L^p(\Omega))$. Existence and uniqueness of the solutions u and u_n in the sense of Definition 3.2.3 is a consequence of Theorem 3.2.4.

(ii) Denote the error function $\xi_n(\boldsymbol{x},t) = u_n(\boldsymbol{x},t) - u(\boldsymbol{x},t)$, then from ($\mathcal{P}$), we have a.e.

$$\begin{aligned} \frac{\partial \xi_n(\boldsymbol{x},t)}{\partial t} &= -\left(\Delta_p^{K_n}(u_n(\boldsymbol{x},t)) - \Delta_p^{K}(u(\boldsymbol{x},t))\right) + f_n(\boldsymbol{x},t) - f(\boldsymbol{x},t) \\ &= -\left(\Delta_p^{K_n}(u_n(\boldsymbol{x},t)) - \Delta_p^{K_n}(u(\boldsymbol{x},t))\right) - \left(\Delta_p^{K_n}(u(\boldsymbol{x},t)) - \Delta_p^{K}(u(\boldsymbol{x},t))\right) \\ &\quad + f_n(\boldsymbol{x},t) - f(\boldsymbol{x},t). \end{aligned} \tag{3.3.2}$$

Multiplying both sides of (3.3.2) by $\xi_n(\boldsymbol{x},t)$ and integrating, we get

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} = &-\int_\Omega \left(\Delta_p^{K_n} u_n(\boldsymbol{x},t) - \Delta_p^{K_n} u(\boldsymbol{x},t)\right)(u_n(\boldsymbol{x},t) - u(\boldsymbol{x},t))d\boldsymbol{x} \\ &+ \int_{\Omega^2} (K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y}))\Psi(u(\boldsymbol{y},t) - u(\boldsymbol{x},t))\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y} \\ &+ \int_\Omega (f_n(\boldsymbol{x},t) - f(\boldsymbol{x},t))\,\xi_n(\boldsymbol{x},t)d\boldsymbol{x}. \end{aligned} \tag{3.3.3}$$

Since $g, g_n \in L^p(\Omega)$ and $f, f_n \in L^1([0,T]; L^p(\Omega))$, $u_n(\cdot,t), u(\cdot,t) \in L^p(\Omega)$ for any $t \in [0,T]$ thanks to (3.2.11). We can then apply Proposition 3.2.2(iv) with $h(x) = x$ to assert that the first term on the right-hand side of (3.3.3) is nonpositive. Let us now bound the second term.

- Case (c): in this case $\|u\|_{C([0,T];L^\infty(\Omega))} \le \|g\|_{L^\infty(\Omega)} + \|f\|_{L^1([0,T];L^\infty(\Omega))}$ thanks to (3.2.11), and we get from Cauchy-Schwartz inequality that

$$\begin{aligned} &\left|\int_{\Omega^2} (K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y}))\Psi(u(\boldsymbol{y},t) - u(\boldsymbol{x},t))\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\right| \\ &\le 2^{p-1}\|u(\cdot,t)\|^{p-1}_{L^\infty(\Omega)} \int_{\Omega^2} |K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})||\xi_n(\boldsymbol{x},t)|d\boldsymbol{x}d\boldsymbol{y} \\ &\le 2^{p-1}\left(\|g\|_{L^\infty(\Omega)} + \|f\|_{L^1([0,T];L^\infty(\Omega))}\right)^{p-1} \|K_n - K\|_{L^2(\Omega^2)}\|\xi_n(\cdot,t)\|_{L^2(\Omega)} \\ &= C\|K_n - K\|_{L^2(\Omega^2)}\|\xi_n(\cdot,t)\|_{L^2(\Omega)}. \end{aligned} \tag{3.3.4}$$

- Case (a) or (b): applying again Cauchy-Schwartz inequality we obtain

$$\begin{aligned} &\left|\int_{\Omega^2} (K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y}))\Psi(u(\boldsymbol{y},t) - u(\boldsymbol{x},t))\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\right| \\ &\le \left(\int_{\Omega^2} |u(\boldsymbol{y},t) - u(\boldsymbol{x},t)|^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\right)^{1/2} \left(\int_{\Omega^2} |K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})|^2|\xi_n(\boldsymbol{x},t)|^2 d\boldsymbol{x}d\boldsymbol{y}\right)^{1/2} \\ &= \left(\int_{\Omega^2} |u(\boldsymbol{y},t) - u(\boldsymbol{x},t)|^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\right)^{1/2} \left(\int_\Omega \left(\int_\Omega |K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})|^2 d\boldsymbol{y}\right)|\xi_n(\boldsymbol{x},t)|^2 d\boldsymbol{x}\right)^{1/2} \\ &= \left(\int_{\Omega^2} |u(\boldsymbol{y},t) - u(\boldsymbol{x},t)|^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\right)^{1/2} \|K_n - K\|_{L^{\infty,2}(\Omega^2)}\|\xi_n(\cdot,t)\|_{L^2(\Omega)}. \end{aligned}$$

On the one hand, under (a), Jensen's inequality applied to the concave function $x \in \mathbb{R}^+ \mapsto x^{p-1}$ entails

$$\begin{aligned} &\left(\int_{\Omega^2} |u(\boldsymbol{y},t) - u(\boldsymbol{x},t)|^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\right)^{1/2} \\ &\le \left(\int_{\Omega^2} |u(\boldsymbol{y},t) - u(\boldsymbol{x},t)|^{2} d\boldsymbol{x}d\boldsymbol{y}\right)^{(p-1)/2} \\ &\le 2^{p-1}\|u(\cdot,t)\|^{p-1}_{L^2(\Omega)} \le 2^{p-1}\left(\|g\|_{L^2(\Omega)} + \|f\|_{L^1([0,T];L^2(\Omega))}\right)^{p-1}, \end{aligned}$$

where we used (3.2.11) in the last inequality. On the other hand, under (b), we have

$$\left(\int_{\Omega^2} \left|u(\boldsymbol{y},t)-u(\boldsymbol{x},t)\right|^{2(p-1)} d\boldsymbol{x}d\boldsymbol{y}\right)^{1/2} \leq 2^{p-1}\left\|u(\cdot,t)\right\|^{p-1}_{L^{2(p-1)}(\Omega)}$$
$$\leq 2^{p-1}\left(\left\|g\right\|_{L^{2(p-1)}(\Omega)}+\left\|f\right\|_{L^1([0,T];L^{2(p-1)}(\Omega))}\right)^{p-1}.$$

In turn, under either (a) or (b), we have the bound

$$\begin{aligned}&\left|\int_{\Omega^2}(K_n(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x},\boldsymbol{y}))\Psi(u(\boldsymbol{y},t)-u(\boldsymbol{x},t))\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\right| \\ &\leq C\left\|K_n-K\right\|_{L^{\infty,2}(\Omega^2)}\left\|\xi_n(\cdot,t)\right\|_{L^2(\Omega)}.\end{aligned} \tag{3.3.5}$$

Inserting (3.3.4) and (3.3.5) into (3.3.3), ignoring the first term which is non-positive as argued above, and using Cauchy-Schwartz inequality on the last term, we obtain

$$\frac{\partial}{\partial t}\left\|\xi_n(\cdot,t)\right\|_{L^2(\Omega)} \leq \left\|f_n(\cdot,t)-f(\cdot,t)\right\|_{L^2(\Omega)} + \begin{cases} C\left\|K_n-K\right\|_{L^{\infty,2}(\Omega^2)}, & \text{under (a) or (b)} \\ C\left\|K_n-K\right\|_{L^2(\Omega^2)}, & \text{under (c).}\end{cases}$$

Integrating this inequality on $[0,t]$ and taking the supremum over $t\in[0,T]$, we get (3.3.1).

(iii) By assumptions on $\{K_n\}_{n\in\mathbb{N}}$, we are in position to apply the Vitali convergence theorem [129, p. 133] in $L^2(\Omega^2)$ to get that $\left\|K_n-K\right\|_{L^2(\Omega^2)}\to 0$ as $n\to+\infty$. We have by assumption that the sequence $\{g_n\}_{n\in\mathbb{N}}$ is dominated by a constant function. The latter is obviously in $L^2(\Omega)$ since $|\Omega|<+\infty$. It then follows from the dominated convergence theorem that $\left\|g_n-g\right\|_{L^2(\Omega)}\to 0$ as $n\to+\infty$. We now turn to the sequence f_n. We have

$$\left\|f_n-f\right\|_{L^1([0,T];L^2(\Omega))}\leq T^{1/2}\left\|f_n-f\right\|_{L^2([0,T];L^2(\Omega))}=T^{1/2}\left\|f_n-f\right\|_{L^2(\Omega\times[0,T])}.$$

Arguing as for g_n, using our assumptions, entails again that $\left\|f_n-f\right\|_{L^1([0,T];L^2(\Omega))}\to 0$ as $n\to+\infty$. Passing to the limit in the second inequality of (3.3.1), the claim follows.

□

In the case where the kernel takes the form $K(\boldsymbol{x},\boldsymbol{y})=J(\boldsymbol{x}-\boldsymbol{y})$, we have the following consequence of Theorem 3.3.1.

Corollary 3.3.3. *Suppose that $p\in]1,+\infty[$. Let u be a solution of $(\mathcal{P})$ with kernel $K(\boldsymbol{x},\boldsymbol{y})=J(\boldsymbol{x}-\boldsymbol{y})$ and data (f,g). Let u_n be a sequence of solutions to $(\mathcal{P})$ with kernels $K_n(\boldsymbol{x},\boldsymbol{y})=J_n(\boldsymbol{x}-\boldsymbol{y})$ and data (f_n,g_n). Assume that J and J_n satisfy **(H'.1)**, **(H'.2)** and $J,J_n\in L^2(\Omega-\Omega)$, and that either one of (a), (b) or (c) in Theorem 3.3.1 holds. Then, the following hold.*

(i) u and u_n are the unique solutions of the corresponding evolution problems.

(ii) We have the error estimate

$$\left\|u_n-u\right\|_{C([0,T];L^2(\Omega))}\leq\left\|g_n-g\right\|_{L^2(\Omega)}+\left\|f_n-f\right\|_{L^1([0,T];L^2(\Omega))}+CT\left\|J_n-J\right\|_{L^2(\Omega-\Omega)}, \tag{3.3.6}$$

where C is positive constant that may depend only on p, g and f.

(iii) Moreover, if the sequence $\{|J_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable over $\Omega-\Omega$, $J_n\to J$ pointwise a.e. on $\Omega-\Omega$, $g_n\to g$ pointwise a.e. on Ω, $f_n\to f$ pointwise a.e. on $\Omega\times[0,T]$, and either one of the following holds:

(a') $p\in]1,2[$, $\{|g_n|^2\}_{n\in\mathbb{N}}$ (resp. $\{|f_n|^2\}_{n\in\mathbb{N}}$) is uniformly integrable over Ω (resp. $\Omega\times[0,T]$);

(b') $p\geq 2$, $\{|g_n|^{2(p-1)}\}_{n\in\mathbb{N}}$ (resp. $\{|f_n|^{2(p-1)}\}_{n\in\mathbb{N}}$) is uniformly integrable over Ω (resp. $\Omega\times[0,T]$);

(c') $\sup_{n\in\mathbb{N}}\left|g_n(\boldsymbol{x})\right|<+\infty$ a.e. on Ω and $\sup_{n\in\mathbb{N}}\left|f_n(\boldsymbol{x},t)\right|<+\infty$ a.e. on $\Omega\times[0,T]$.

Then

$$\lim_{n\to+\infty} \|u_n - u\|_{C([0,T];L^2(\Omega))} = 0.$$

Proof :

(i) We argue in the same way as in the proof Theorem 3.3.1 since $L^2(\Omega-\Omega) \subset L^1(\Omega-\Omega)$ implies that assumption **(H'.3)** holds for both J and J_n.

(ii) The error bound (3.3.6) is a specialization of (3.3.1) since

$$\int_\Omega |K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})|^2 d\boldsymbol{y} = \int_{\Omega-\boldsymbol{x}} |J_n(\boldsymbol{z}) - J(\boldsymbol{z})|^2 d\boldsymbol{z} \leq \|J_n - J\|^2_{L^2(\Omega-\Omega)}.$$

Thus

$$\|K_n - K\|_{L^2(\Omega^2)} \leq \|K_n - K\|_{L^{\infty,2}(\Omega^2)} \leq \|J_n - J\|_{L^2(\Omega-\Omega)}.$$

(iii) Case (a') follows from the Vitali convergence theorem applied to J_n, g_n and f_n. The latter argument also applies to case (b') since $L^{2(p-1)}(\Omega-\Omega) \subset L^2(\Omega-\Omega)$, $L^{2(p-1)}(\Omega) \subset L^2(\Omega)$ and $L^{2(p-1)}(\Omega\times[0,T]) \subset L^1([0,T];L^2(\Omega))$. Case (c') uses the Vitali convergence theorem on J_n and the dominated convergence theorem on g_n and f_n as argued in the proof of Theorem 3.3.1(iii).

□

Remark 3.3.4. At this stage, we only relied on the monotonicity property of Δ_p^K in Proposition 3.2.2(iv) to get our bounds. One may then wonder if the stronger notions of monotonicity established in Proposition 3.2.2(v) can yield bounds better than (3.3.6). We answer this question positively by (slightly) improving the dependence on T for $p \in]1,2]$ but at the price of more stringent assumptions on J. For this, we embark from (3.3.3), bound all terms as in the proof of Theorem 3.3.1, use Proposition 3.2.2(v) and that $L^2(\Omega) \subset L^p(\Omega)$ in this case to get

$$\frac{1}{2}\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} + C_1 \int_{\Omega^2} J(\boldsymbol{x}-\boldsymbol{y})\left|\nabla^{\mathrm{NL}}\xi_n(\boldsymbol{x},\boldsymbol{y})\right|^2 d\boldsymbol{y}d\boldsymbol{x} \leq \left(C\|J_n - J\|_{L^2(\Omega-\Omega)} + \|f_n(\cdot,t) - f(\cdot,t)\|_{L^2(\Omega)}\right)\|\xi_n(\cdot,t)\|_{L^2(\Omega)},$$

for two positive constants C, C_1 (in the following C_i is a positive constant). Assume in addition that J is compactly supported and $J(0) > 0$. One can then invoke the Poincaré inequality [10, Proposition 4.1] to show that

$$C_2\left\|\xi_n(\cdot,t) - \int_\Omega \xi_n(\boldsymbol{x},t)d\boldsymbol{x}\right\|^2_{L^2(\Omega)} \leq \int_{\Omega^2} J(\boldsymbol{x}-\boldsymbol{y})\left|\nabla^{\mathrm{NL}}\xi_n(\boldsymbol{x},\boldsymbol{y})\right|^2 d\boldsymbol{y}d\boldsymbol{x}.$$

Thus

$$\frac{1}{2}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} \leq \left\|\xi_n(\cdot,t) - \int_\Omega \xi_n(\boldsymbol{x},t)d\boldsymbol{x}\right\|^2_{L^2(\Omega)} + \left(\int_\Omega \xi_n(\boldsymbol{x},t)d\boldsymbol{x}\right)^2.$$

Altogether, we arrive at

$$\frac{1}{2}\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} + \frac{C_1C_2}{2}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} \leq \left(C\|J_n - J\|_{L^2(\Omega-\Omega)} + \|f_n(\cdot,t) - f(\cdot,t)\|_{L^2(\Omega)}\right)\|\xi_n(\cdot,t)\|_{L^2(\Omega)} + C_1C_2\left(\int_\Omega \xi_n(\boldsymbol{x},t)d\boldsymbol{x}\right)^2.$$

By integrating ($\mathcal{P}$), it is easy to see by applying Proposition 3.2.2(v) and (iv) with $h(x) = 1$ that the solution of ($\mathcal{P}$) preserves the total mass in Ω, whence we deduce

$$\int_\Omega \xi_n(\boldsymbol{x},t)d\boldsymbol{x} = \int_\Omega (g_n(\boldsymbol{x}) - g(\boldsymbol{x})) + \int_0^t \int_\Omega (f_n(\boldsymbol{x},s) - f(\boldsymbol{x},s))d\boldsymbol{x}ds.$$

If (f,g) and (f_n,g_n) have the same mass, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} + \frac{C_1C_2}{2}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} \leq \left(C\|J_n - J\|_{L^2(\Omega-\Omega)} + \|f_n(\cdot,t) - f(\cdot,t)\|_{L^2(\Omega)}\right)\|\xi_n(\cdot,t)\|_{L^2(\Omega)},$$

and therefore

$$\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|_{L^2(\Omega)} + \frac{C_1C_2}{2}\|\xi_n(\cdot,t)\|_{L^2(\Omega)} \leq \left(C\|J_n - J\|_{L^2(\Omega-\Omega)} + \|f_n(\cdot,t) - f(\cdot,t)\|_{L^2(\Omega)}\right).$$

Applying Gronwall's lemma yields the estimate

$$\|u_n(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \leq \|f_n - f\|_{L^1([0,T];L^2(\Omega))} + \exp(-C_1C_2t/2)\|g_n - g\|_{L^2(\Omega)} + \frac{2C}{C_1C_2}(1-\exp(-C_1C_2t/2))\|J_n - J\|_{L^2(\Omega-\Omega)}.$$

This bound is clearly better than (3.3.6). In turn,

$$\|u_n - u\|_{C([0,T];L^2(\Omega))} \leq \|f_n - f\|_{L^1([0,T];L^2(\Omega))} + \max\left(\|g_n - g\|_{L^2(\Omega)}, \frac{2C}{C_1C_2}\|J_n - J\|_{L^2(\Omega-\Omega)}\right).$$

The same reasoning as above can be carried out to sharpen the error bounds for the discrete problems in Section 3.4. Nevertheless, this will not be detailed further in this work.

3.3.2 The case $p = 1$

We now turn to the case $p = 1$.

Theorem 3.3.5. *Let u be a solution of ($\mathcal{P}$) for $p = 1$ with kernel K and data (f, g). Let u_n be a sequence of solutions to ($\mathcal{P}$) for $p = 1$ with kernels K_n and data (f_n, g_n). Assume that K and K_n satisfy* ***(H.1)*** *and* ***(H.2)****, that $K, K_n \in L^{\infty,2}(\Omega^2)$, $g, g_n \in L^2(\Omega)$ and $f, f_n \in L^1([0,T];L^2(\Omega))$. Then, the following hold.*

(i) u and u_n are the unique solutions in the sense of Definition 3.2.6 of the corresponding evolution problems.

(ii) We have the error estimate

$$\|u_n - u\|_{C([0,T];L^2(\Omega))} \leq \|g_n - g\|_{L^2(\Omega)} + \|f_n - f\|_{L^1([0,T];L^2(\Omega))} + T\|K_n - K\|_{L^2(\Omega^2)}. \tag{3.3.7}$$

(iii) Moreover, if $K_n \to K$ pointwise a.e. on Ω^2, $g_n \to g$ pointwise a.e. on Ω, $f_n \to f$ pointwise a.e. on $\Omega \times [0,T]$, and $\{|K_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable over Ω^2, $\{|g_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable on Ω, and $\{|f_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable on $\Omega \times [0,T]$. Then

$$\lim_{n\to+\infty}\|u_n - u\|_{C([0,T];L^2(\Omega))} = 0.$$

Proof :

(i) Existence and uniqueness of u and u_n follow from Theorem 3.2.8 where we argue as in Theorem 3.3.1(i) since $g, g_n \in L^2(\Omega) \subset L^1(\Omega)$ and $K, K_n \in L^{\infty,2}(\Omega^2) \subset L^{\infty,1}(\Omega^2)$.

(ii) Denote the error function $\xi_n(\boldsymbol{x},t) = u_n(\boldsymbol{x},t) - u(\boldsymbol{x},t)$, then from Definition 3.2.6, we have a.e.

$$\begin{aligned}\frac{\partial\xi_n(\boldsymbol{x},t)}{\partial t} &= \int_\Omega (K_n(\boldsymbol{x},\boldsymbol{y})w_n(\boldsymbol{x},\boldsymbol{y},t) - K(\boldsymbol{x},\boldsymbol{y})w(\boldsymbol{x},\boldsymbol{y},t))\,d\boldsymbol{y} + f_n(\boldsymbol{x},t) - f(\boldsymbol{x},t)\\ &= \int_\Omega K_n(\boldsymbol{x},\boldsymbol{y})\,(w_n(\boldsymbol{x},\boldsymbol{y},t) - w(\boldsymbol{x},\boldsymbol{y},t))\,d\boldsymbol{y} + \int_\Omega (K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y}))w(\boldsymbol{x},\boldsymbol{y},t)d\boldsymbol{y}\\ &\quad + f_n(\boldsymbol{x},t) - f(\boldsymbol{x},t),\end{aligned} \tag{3.3.8}$$

where w (resp. w_n) is the subgradient function associated to u (resp. u_n) as in Definition 3.2.6. Multiplying both sides of (3.3.8) by $\xi_n(\boldsymbol{x},t)$ and integrating, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|^2_{L^2(\Omega)} = &\int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\left(w_n(\boldsymbol{x},\boldsymbol{y},t) - w(\boldsymbol{x},\boldsymbol{y},t)\right)\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\\ &+\int_{\Omega^2}(K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y}))w(\boldsymbol{x},\boldsymbol{y},t)\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\\ &+\int_{\Omega}\left(f_n(\boldsymbol{x},t) - f(\boldsymbol{x},t)\right)\xi_n(\boldsymbol{x},t)d\boldsymbol{x}.\end{aligned} \tag{3.3.9}$$

In view of the monotonicity claim in Proposition 3.2.7(ii), we have

$$\int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\left(w_n(\boldsymbol{x},\boldsymbol{y},t) - w(\boldsymbol{x},\boldsymbol{y},t)\right)\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y} \le 0.$$

Let us turn to bounding the second term. We have by the Cauchy-Schwartz inequality and that $\|w\|_{L^\infty(\Omega^2\times]0,T[)} \le 1$,

$$\begin{aligned}&\left|\int_{\Omega^2}(K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y}))w(\boldsymbol{x},\boldsymbol{y},t)\xi_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\right|\\ &\le \int_{\Omega^2}\left|K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})\right|\left|\xi_n(\boldsymbol{x},t)\right|d\boldsymbol{x}d\boldsymbol{y}\\ &\le \|K_n - K\|_{L^2(\Omega^2)}\|\xi_n(\cdot,t)\|_{L^2(\Omega)}.\end{aligned} \tag{3.3.10}$$

Inserting (3.3.10) into (3.3.9), ignoring the first term which is non-positive as argued above, and using Cauchy-Schwartz inequality on the last term, we obtain

$$\frac{\partial}{\partial t}\|\xi_n(\cdot,t)\|_{L^2(\Omega)} \le \|f_n(\cdot,t) - f(\cdot,t)\|_{L^2(\Omega)} + \|K_n - K\|_{L^2(\Omega^2)}.$$

Integrating this inequality on $[0,t]$ and taking the supremum over $t \in [0,T]$, we get (3.3.7).

(iii) We argue again using the Vitali convergence theorem since $K, K_n \in L^{\infty,2}(\Omega^2) \subset L^2(\Omega^2)$ and $L^1([0,T];L^2(\Omega)) \subset L^2(\Omega\times[0,T])$.

□

The following corollary is immediate in the same vein as Corollary 3.3.3.

Corollary 3.3.6. *Let u be a solution of $(\mathcal{P})$ for $p=1$ with kernel $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$ and data (f,g). Let u_n be a sequence of solutions to $(\mathcal{P})$ for $p=1$ with kernels $K_n(\boldsymbol{x},\boldsymbol{y}) = J_n(\boldsymbol{x}-\boldsymbol{y})$ and data (f_n,g_n). Assume that J and J_n satisfy* ***(H'.1)***, ***(H'.2)*** *and $J, J_n \in L^2(\Omega-\Omega)$, that $g, g_n \in L^2(\Omega)$ and $f, f_n \in L^1([0,T];L^2(\Omega))$. Then, the following hold.*

(i) u and u_n are the unique solutions in the sense of Definition 3.2.6 of the corresponding evolution problems.

(ii) We have the error estimate

$$\|u_n - u\|_{C([0,T];L^2(\Omega))} \le \|g_n - g\|_{L^2(\Omega)} + \|f_n - f\|_{L^1([0,T];L^2(\Omega))} + T\|J_n - J\|_{L^2(\Omega-\Omega)}. \tag{3.3.11}$$

(iii) Moreover, if $J_n \to J$ pointwise a.e. on $\Omega-\Omega$, $g_n \to g$ pointwise a.e. on Ω, $f_n \to f$ pointwise a.e. on $\Omega\times[0,T]$, and $\{|J_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable over $\Omega-\Omega$, $\{|g_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable on Ω, and $\{|f_n|^2\}_{n\in\mathbb{N}}$ is uniformly integrable on $\Omega\times[0,T]$. Then

$$\lim_{n\to+\infty}\|u_n - u\|_{C([0,T];L^2(\Omega))} = 0.$$

3.4 Error bounds for the discrete problem

Let $\mathbf{K} \in \mathbb{R}^{n^d\times n^d}$ and $\mathbf{g} \in \mathbb{R}^{n^d}$ be discrete approximations of, respectively, the kernel K and initial data g in $(\mathcal{P})$, on a regular mesh of size $\delta(n)$. Typically, one can take $\mathbf{K} = P_nK$ and $\mathbf{g} = P_ng$. For

$1 < p < \infty$, the discrete p-Laplacian operator with kernel $\mathbf{K}$ is

$$\widehat{\mathbf{\Delta}}_p^{\mathbf{K}} : \mathbf{u} \in \mathbb{R}^{n^d} \mapsto -\sum_{\boldsymbol{j}\in[n]^d} h_{\boldsymbol{j}} \mathbf{K}_{\boldsymbol{ij}} \left|\mathbf{u}_{\boldsymbol{j}} - \mathbf{u}_{\boldsymbol{i}}\right|^{p-2}(\mathbf{u}_{\boldsymbol{j}} - \mathbf{u}_{\boldsymbol{i}}) = -\sum_{\boldsymbol{j}\in[n]^d} h_{\boldsymbol{j}} \mathbf{K}_{\boldsymbol{ij}} \Psi(\mathbf{u}_{\boldsymbol{j}} - \mathbf{u}_{\boldsymbol{i}}).$$

In the same way, we define the discrete 1-Laplacian operator as the set-valued operator $\widehat{\mathbf{\Delta}}_1^{\mathbf{K}} : \mathbb{R}^{n^d} \to 2^{\mathbb{R}^{n^d}}$ such that $\boldsymbol{\eta} \in \widehat{\mathbf{\Delta}}_1^{\mathbf{K}} \mathbf{u}$ if and only if

$$\boldsymbol{\eta}_{\boldsymbol{i}} = -\sum_{\boldsymbol{j}\in[n]^d} h_{\boldsymbol{j}} \mathbf{K}_{\boldsymbol{ij}} \mathbf{w}_{\boldsymbol{ij}},$$

where $\left\|\mathbf{w}\right\|_\infty \leq 1$, $\mathbf{w}_{\boldsymbol{ij}} = -\mathbf{w}_{\boldsymbol{ji}}$, and

$$\mathbf{w}_{\boldsymbol{ij}} \in \partial|\cdot|(\mathbf{u}_{\boldsymbol{j}} - \mathbf{u}_{\boldsymbol{i}}).$$

By construction, we have the following simple lemma whose proof is immediate.

Lemma 3.4.1. *For any $\mathbf{K} \in \mathbb{R}^{n^d \times n^d}$ and $\mathbf{u} \in \mathbb{R}^{n^d}$, the following holds:*

(i) If $1 < p < +\infty$,

$$I_n \widehat{\mathbf{\Delta}}_p^{\mathbf{K}}(\mathbf{u}) = \Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}).$$

(ii) If $p = 1$,

$$I_n\boldsymbol{\eta}(\boldsymbol{x}) = -\int_\Omega I_n K(\boldsymbol{x},\boldsymbol{y}) I_n \mathbf{w}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y}, \quad \textit{where} \quad I_n\mathbf{w}(\boldsymbol{x},\boldsymbol{y}) \in \partial|\cdot|(I_n\mathbf{u}(\boldsymbol{y}) - I_n\mathbf{u}(\boldsymbol{x})).$$

Moreover, $\left\|I_n\mathbf{w}\right\|_{L^\infty(\Omega^2)} \leq 1$ and $I_n\mathbf{w}(\boldsymbol{x},\boldsymbol{y}) = -I_n\mathbf{w}(\boldsymbol{y},\boldsymbol{x})$.

3.4.1 The semi-discrete problem

Case $p \in]1,+\infty[$: We start with the case $1 < p < +\infty$ and consider the space semi-discretization of $(\mathcal{P})$,

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u}(t) = -\widehat{\mathbf{\Delta}}_p^{\mathbf{K}}\mathbf{u}(t) + \mathbf{f}(t), & t > 0, \\ \mathbf{u}(0) = \mathbf{g}. \end{cases} \tag{$\mathcal{P}_p^{\mathrm{SD}}$}$$

where $\mathbf{u} : t \in \mathbb{R}^+ \mapsto \mathbf{u}(t) \in \mathbb{R}^{n^d}$ and similarly for $\mathbf{f}$.

Our aim is to compare the solutions of problems $(\mathcal{P})$ and $(\mathcal{P}_p^{\mathrm{SD}})$. The solution of $(\mathcal{P}_p^{\mathrm{SD}})$ being discrete in space, we consider its continuum space extensions of $\mathbf{u}$ and $\mathbf{f}$ on Ω for any $t > 0$ as

$$u_n(\boldsymbol{x},t) = (I_n\mathbf{u}(t))(\boldsymbol{x}) \text{ and } f_n(\boldsymbol{x},t) = (I_n\mathbf{f}(t))(\boldsymbol{x}). \tag{3.4.1}$$

Theorem 3.4.2. *Suppose that $p \in]1,+\infty[$. Let u be a solution of $(\mathcal{P})$ with kernel K and data (f,g), and $\mathbf{u}$ that of $(\mathcal{P}_p^{\mathrm{SD}})$ with $\mathbf{K} = P_nK$, $\mathbf{g} = P_ng$ and $\mathbf{f}(t) = P_nf(\cdot,t)$ for $t \in [0,T]$. Let u_n and f_n as defined in (3.4.1). Assume that K satisfies **(H.1)**, **(H.2)** and $K \in L^{\infty,2}(\Omega^2)$, and that g and f satisfy either one of the conditions (a), (b) or (c) in Theorem 3.3.1. Then, the following hold.*

(i) u and u_n are the unique solutions of $(\mathcal{P})$ with data respectively (f,g) and (f_n, I_nP_ng).

(ii) We have the error estimate

$$\left\|u_n - u\right\|_{C([0,T];L^2(\Omega))} \leq \left\|I_nP_ng - g\right\|_{L^2(\Omega)} + \left\|f_n - f\right\|_{L^1([0,T];L^2(\Omega))} + CT\begin{cases} \left\|I_nP_nK - K\right\|_{L^{\infty,2}(\Omega^2)}, & \textit{under (a)-(b)} \\ \left\|I_nP_nK - K\right\|_{L^2(\Omega^2)}, & \textit{under (c)} \end{cases} \tag{3.4.2}$$

where C is positive constant that depends only on p, g and f.

(iii) If, moreover, $g \in L^\infty(\Omega) \cap \mathrm{Lip}(s, L^2(\Omega))$, $K \in \mathrm{Lip}(s, L^2(\Omega^2))$ *and* $f(\cdot, t) \in L^\infty(\Omega) \cap \mathrm{Lip}(s, L^2(\Omega))$ *for every* $t \in [0, T]$, *then*

$$\|u_n - u\|_{C([0,T];L^2(\Omega))} \le C(1+T)\delta(n)^s, \tag{3.4.3}$$

where C is positive constant that depends only on p, g, f, K, s.

Proof :

(i) Existence and uniqueness of u were proved in Theorem 3.3.1(i). We also see that $I_n\mathbf{K}$ verifies **(H.1)** and **(H.2)**. Using Lemma 2.1.27, we have $I_n\mathbf{g} \in L^p(\Omega)$, $f_n \in L^1([0;T], L^p(\Omega))$ and $I_n\mathbf{K} \in L^{\infty,2}(\Omega^2) \subset L^{\infty,1}(\Omega^2)$, and thus $I_n\mathbf{K}$ fulfills **(H.3)**. In view of Lemma 3.4.1(i), it follows from ($\mathcal{P}_p^{\mathrm{SD}}$) that the function u_n satisfies ($\mathcal{P}$) with kernel $I_n\mathbf{K}$ and data $(f_n, I_n\mathbf{g})$. Existence and uniqueness of u_n then follow from Theorem 3.2.4.

(ii) The claim is a specialization of (3.3.1) in Theorem 3.3.1(ii).

(iii) As $K \in L^{\infty,2}(\Omega^2) \subset L^2(\Omega^2)$, we insert the estimate (2.2.2) (see Lemma 2.2.2) in the second bound of (3.4.2).

□

Case $p = 1$: We now turn to the case $p = 1$, and consider the evolution problem

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u}(t) = -\boldsymbol{\eta}(t) + \mathbf{f}(t), & t > 0, \\ \mathbf{u}(0) = \mathbf{g}, \end{cases} \tag{$\mathcal{P}_1^{\mathrm{SD}}$}$$

where

$$\boldsymbol{\eta}_{\boldsymbol{i}}(t) = -\sum_{\boldsymbol{j} \in [n]^d} h_{\boldsymbol{j}} \mathbf{K}_{\boldsymbol{ij}} \,\mathrm{sign}(\mathbf{u}_{\boldsymbol{j}} - \mathbf{u}_{\boldsymbol{i}}), \quad \text{and thus} \quad \boldsymbol{\eta}(t) \in \widehat{\boldsymbol{\Delta}}_1^{\mathbf{K}}\mathbf{u}(t).$$

Theorem 3.4.3. *Let u be a solution of ($\mathcal{P}$) for $p = 1$ with kernel K and data (f, g), and $\mathbf{u}$ is that of ($\mathcal{P}_1^{\mathrm{SD}}$) with $\mathbf{K} = P_nK$, $\mathbf{g} = P_n g$ and $\mathbf{f}(t) = P_n f(\cdot, t)$ for $t \in [0, T]$. Let u_n and f_n as defined in (3.4.1). Assume that K satisfies **(H.1)**, **(H.2)** and $K \in L^{\infty,2}(\Omega^2)$, and that $g \in L^2(\Omega)$ and $f \in L^1([0, T]; L^2(\Omega))$. Then, the following hold.*

(i) u and u_n are the unique solutions in the sense of Definition 3.2.6 of the corresponding evolution problems.

(ii) We have the error estimate

$$\|u_n - u\|_{C([0,T];L^2(\Omega))} \le \|I_nP_ng - g\|_{L^2(\Omega)} + \|f_n - f\|_{L^1([0,T];L^2(\Omega))} + T\|I_nP_nK - K\|_{L^2(\Omega^2)}. \tag{3.4.4}$$

(iii) If, moreover, $g \in \mathrm{Lip}(s, L^2(\Omega))$, $K \in \mathrm{Lip}(s, L^2(\Omega^2))$ *and* $f(\cdot, t) \in \mathrm{Lip}(s, L^2(\Omega))$ *for every* $t \in [0, T]$, *then*

$$\|u_n - u\|_{C([0,T];L^2(\Omega))} \le C(1+T)\delta(n)^s, \tag{3.4.5}$$

where C is positive constant that depends only on p, g, f, K and s.

Proof :

(i) Existence and uniqueness of u were proved in Theorem 3.3.5(i). In addition, $I_n\mathbf{K}$ verifies **(H.1)** and **(H.2)**. Using Lemma 2.1.27, $I_n\mathbf{g} \in L^2(\Omega) \subset L^1(\Omega)$, $f_n \in L^1([0, T]; L^2(\Omega)) \subset L^1([0, T]; L^1(\Omega))$ and $I_n\mathbf{K} \in L^{\infty,2}(\Omega^2) \subset L^{\infty,1}(\Omega^2)$, and thus $I_n\mathbf{K}$ fulfills **(H.3)**. By virtue of Lemma 3.4.1(ii), u_n, the space continuum extension of $\mathbf{u}$, will satisfy ($\mathcal{P}$) with kernel $I_n\mathbf{K}$ and data $(f_n, I_n\mathbf{g})$. Existence and uniqueness of u_n in the sense of Definition 3.2.6 follow from Theorem 3.2.8.

(ii) This claim is a specialization of (3.3.7) in Theorem 3.3.5(ii).

(iii) Insert the estimate (2.2.2) in (3.4.4).

□

3.4.2 The totally discrete problem

We establish in this section error bounds for fully discrete (in time and space) approximations of $(\mathcal{P})$. For that, let $0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$ be a partition (not necessarily equispaced) of $[0,T]$. Let $\tau_{k-1} \stackrel{\text{def}}{=} |t_k - t_{k-1}|$ and denote $\tau = \max_{k\in[N]} \tau_k$.

3.4.2.1 Forward/Explicit Euler discretization

Case $p \in]1,2]$: We start with $p \in]1,2]$ and consider a totally discrete problem with forward/explicit Euler scheme in time,

$$\begin{cases} \dfrac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_{k-1}} = -\widehat{\mathbf{\Delta}}_p^{\mathbf{K}} \mathbf{u}^{k-1} + \mathbf{f}, & k \in [N], \\ \mathbf{u}^0 = \mathbf{g}, \end{cases} \tag{$\mathcal{P}_p^{\text{TDF}}$}$$

where $\mathbf{u}^k, \mathbf{f} \in \mathbb{R}^{n^d}$. We have implicitly assumed that $\mathbf{f}$ does not depend on time, which is a standard assumption in the context of explicit discretization.

Since our aim is to compare the solutions of problems $(\mathcal{P})$ and $(\mathcal{P}_p^{\text{TDF}})$, we introduce the following continuum extensions in space and/or time of $\{\mathbf{u}^k\}_{k\in[N]}$ as

$$\begin{aligned} u_n^k &= I_n \mathbf{u}^k, k \in [N], \text{ and } f_n = I_n \mathbf{f}, \\ \breve{u}_n(\boldsymbol{x}, t) &= \frac{t_k - t}{\tau_{k-1}} u_n^{k-1}(\boldsymbol{x}) + \frac{t - t_{k-1}}{\tau_{k-1}} u_n^k(\boldsymbol{x}), \quad (\boldsymbol{x}, t) \in \Omega \times]t_{k-1}, t_k], k \in [N], \\ \bar{u}_n(\boldsymbol{x}, t) &= \sum_{k=1}^N u_n^{k-1}(\boldsymbol{x}) \chi_{]t_{k-1}, t_k]}(t), \quad (\boldsymbol{x}, t) \in \Omega \times]0, T]. \end{aligned}$$

Then, in the same vein as Lemma 3.4.1, it is easy to see that $(\mathcal{P}_p^{\text{TDF}})$ is equivalent to the following evolution problem

$$\begin{cases} \frac{\partial}{\partial t} \breve{u}_n(\boldsymbol{x}, t) = -\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\boldsymbol{x}, t) + f_n(\boldsymbol{x}), & (\boldsymbol{x}, t) \in \Omega \times]0, T], \\ \breve{u}_n(\boldsymbol{x}, 0) = I_n \mathbf{g}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{3.4.6}$$

Before turning to the consistency result, we collect some useful estimates.

Lemma 3.4.4. *Consider problem $(\mathcal{P}_p^{\text{TDF}})$ with kernel $\mathbf{K}$, data $(\mathbf{f}, \mathbf{g})$ and variable step-size $\tau_k \leq 2C \left\| \Delta_p^{I_n\mathbf{K}} u_n^k - f_n \right\|_{L^2(\Omega)}^{\frac{2-p}{p-1}}$, where C is the constant in Proposition 3.2.2(vi). Assume that $I_n \mathbf{g} \in L^2(\Omega)$ and $I_n\mathbf{K}$ satisfies **(H.1)**, **(H.2)** and **(H.3)**. Suppose also that for each $n \in \mathbb{N}$, $\mathbf{f}$ is such that $(\mathcal{P}_p^{\text{TDF}})$ has a stationary solution $\mathbf{u}^\star$ and that $\sup_{n\in\mathbb{N}} \left\| I_n\mathbf{g} - I_n\mathbf{u}^\star \right\|_{L^2(\Omega)} < +\infty$. Then*

$$\bar{u}_n(\cdot, t) \in L^2(\Omega), \forall t \in [0,T], \quad \textit{and} \quad \sup_{t\in[0,T], n\in\mathbb{N}} \left\| \bar{u}_n(\cdot, t) - I_n \mathbf{u}^\star \right\|_{L^2(\Omega)} < +\infty.$$

Remark 3.4.5.

(1) Condition on the time-step τ_k can be seen as an abstract nonlinear CFL condition. It is better than the one in [90] since we here exploit the Hölder continuity of $\Delta_p^{I_n\mathbf{K}}$ on $L^2(\Omega)$ for $p \in]1,2]$, see Proposition 3.2.2(vi). For $p = 2$, where $\Delta_2^{I_n\mathbf{K}}$ is linear Lipschitz continuous operator on $L^2(\Omega)$, the condition reads $\tau_k \leq 2C$. Such condition for explicit time-discretization of evolution problems with accretive and Lipschitz-continuous operators is known, see e.g., [116]. It is also

consistent with known convergence results for finding zeros of co-called co-coercive operators on Hilbert spaces [21].

(2) The assumption on $\mathbf{f}$ and $\mathbf{K}$ imply that $f_n \in L^2(\Omega)$. Indeed, (3.2.10) entails

$$\left\|f_n\right\|_{L^2(\Omega)} = \left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}^\star)\right\|_{L^2(\Omega)} \le 2^p C_2 \left\|K\right\|^2_{L^{\infty,1}(\Omega^2)} \left\|I_n\mathbf{u}^\star\right\|^{p-1}_{L^2(\Omega)}.$$

(3) The assumption made on $\mathbf{f}$ is trivially true when $\mathbf{f} = \mathbf{0}$ since $\mathbf{0}$ is a stationary solution in this case. In turn, using Lemma 2.1.27, one can see that the uniform boundedness conditions on $\mathbf{g}$ and $\mathbf{K}$ are fulfilled if $\mathbf{g} = P_n g$ and $\mathbf{K} = P_n K$, where $g \in L^2(\Omega)$ and K satisfies **(H.1)**-**(H.3)**.

Proof : We show the claim by an induction argument. Since $\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}^\star) = f_n$, we have

$$\begin{aligned}
&\left\|u_n^1 - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} \\
&= \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} - 2\tau_0 \int_\Omega \left(\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g})(\boldsymbol{x}) - f_n(\boldsymbol{x})\right)\left(I_n\mathbf{g}(\boldsymbol{x}) - I_n\mathbf{u}^\star\right) d\boldsymbol{x} \\
&\quad + \tau_0^2 \left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g}) - f_n\right\|^2_{L^2(\Omega)} \\
&= \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} - 2\tau_0 \int_\Omega \left(\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g})(\boldsymbol{x}) - \Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}^\star)(\boldsymbol{x})\right)\left(I_n\mathbf{g}(\boldsymbol{x}) - I_n\mathbf{u}^\star\right) d\boldsymbol{x} \\
&\quad + \tau_0^2 \left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g}) - f_n\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

By assumption on $\mathbf{g}$, $\mathbf{u}^\star$ and τ_k, we can invoke Proposition 3.2.2(vi) to get

$$\begin{aligned}
&\left\|u_n^1 - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} \\
&\le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} - 2C\tau_0 \left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g}) - f_n\right\|^{p/(p-1)}_{L^2(\Omega)} + \tau_0^2 \left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g}) - f_n\right\|^2_{L^2(\Omega)} \\
&\le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} - \tau_0 \left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g}) - f_n\right\|^2_{L^2(\Omega)} \left(2C\left\|\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g}) - f_n\right\|^{(2-p)/(p-1)}_{L^2(\Omega)} - \tau_0\right) \\
&\le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

Suppose now that, for any $k > 1$,

$$\left\|u_n^k - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} \le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)},$$

and thus $u_n^k \in L^2(\Omega)$. We can then use Proposition 3.2.2(vi) as above to see that

$$\begin{aligned}
&\left\|u_n^{k+1} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} \\
&\le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)} - \tau_k \left\|\Delta_p^{I_n\mathbf{K}}(u_n^k) - f_n\right\|^2_{L^2(\Omega)} \left(2C\left\|\Delta_p^{I_n\mathbf{K}}(u_n^k) - f_n\right\|^{(2-p)/(p-1)}_{L^2(\Omega)} - \tau_k\right) \\
&\le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus the sequence $\left\{\left\|u_n^k\right\|_{L^2(\Omega)}\right\}_{k\in[N]}$ is bounded, and so is $\left\|\bar{u}_n(\cdot,t)\right\|_{L^2(\Omega)}$ for $t \in [0,T]$ by its definition. We also have

$$\sup_{t\in[0,T],n\in\mathbb{N}} \left\|\bar{u}_n(\cdot,t) - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} = \sup_{(n,N)\in\mathbb{N}^2,k\in[N]} \left\|u_n^k - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} \le \sup_{n\in\mathbb{N}} \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} < +\infty.$$

□

Lemma 3.4.6. *In addition to the assumptions of Lemma 3.4.4, suppose that* $\sup_{n\in\mathbb{N}} \left\|I_n\mathbf{K}\right\|_{L^{\infty,1}(\Omega^2)} < +\infty$. *Then*

$$\sup_{t\in[0,T],n\in\mathbb{N}} \left\|\breve{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\right\|_{L^2(\Omega)} \le C\tau,$$

where C is a positive constant that does not depend on (n, N, T).

PROOF : It is easy to see that for $t \in]t_{k-1}, t_k]$, $k \in \mathbb{N}$,

$$\begin{aligned}
\|\check{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\|_{L^2(\Omega)} &= (t-t_{k-1})\|\frac{u_n^k - u_n^{k-1}}{\tau_{k-1}}\|_{L^2(\Omega)} \\
&= (t-t_{k-1})\|\Delta_p^{I_n\mathbf{K}} u_n^{k-1} - f_n\|_{L^2(\Omega)} \\
&= (t-t_{k-1})\|\Delta_p^{I_n\mathbf{K}} u_n^{k-1} - \Delta_p^{I_n\mathbf{K}} I_n\mathbf{u}^\star\|_{L^2(\Omega)} \\
&\le \tau\|\Delta_p^{I_n\mathbf{K}} u_n^{k-1} - \Delta_p^{I_n\mathbf{K}} I_n\mathbf{u}^\star\|_{L^2(\Omega)} = \tau\|\Delta_p^{I_n\mathbf{K}}\bar{u}_n(\cdot,t) - \Delta_p^{I_n\mathbf{K}} I_n\mathbf{u}^\star\|_{L^2(\Omega)}.
\end{aligned}$$

As $\Delta_p^{I_n\mathbf{K}}$ is Hölder continuous on $L^2(\Omega)$ with exponent $p-1$, see (3.2.10), we get

$$\|\check{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\|_{L^2(\Omega)} \le \tau 2^{p/2} C_2^{1/2}\|K\|_{L^{\infty,1}(\Omega^2)}\|\bar{u}_n(\cdot,t) - I_n\mathbf{u}^\star\|_{L^2(\Omega)}^{p-1}.$$

We then take the supremum over t and n, and use Lemma 3.4.4 to conclude. □

We are now in position to state the error bound for the totally discrete problem $(\mathcal{P}_p^{\mathrm{TDF}})$.

Theorem 3.4.7. *Suppose that $p \in]1,2]$. Let u be a solution of $(\mathcal{P})$ with kernel K and data (f,g) where f is time-independent, and $\{\mathbf{u}^k\}_{k\in[N]}$ is the sequence generated by $(\mathcal{P}_p^{\mathrm{TDF}})$ with $\mathbf{K} = P_nK$, $\mathbf{g} = P_ng$, $\mathbf{f} = P_nf$ and τ_k as prescribed in Lemma 3.4.4. Assume that K satisfies **(H.1)**, **(H.2)** and $K \in L^{\infty,2}(\Omega^2)$, and that f,g belong either to $L^2(\Omega)$ or $L^\infty(\Omega)$. Then, the following hold.*

(i) u is the unique solution of $(\mathcal{P})$, $\{\mathbf{u}^k\}_{k\in[N]}$ is uniquely defined and $\left\{\|I_n\mathbf{u}^k\|_{L^2(\Omega)}\right\}_{k\in[N]}$ is bounded (uniformly in n when $\mathbf{f} = \mathbf{0}$).

(ii) We have the error estimate

$$\sup_{k\in[N],t\in]t_{k-1},t_k]}\|I_n\mathbf{u}^{k-1} - u(\cdot,t)\|_{L^2(\Omega)} \le \exp(T/2)\Bigg(\|I_nP_ng - g\|_{L^2(\Omega)} + CT^{1/2}\Bigg(\tau^{1/(3-p)} + \|f_n - f\|_{L^2(\Omega)} + \begin{cases}\|I_nP_nK - K\|_{L^{\infty,2}(\Omega^2)} & g\in L^2(\Omega)\\ \|I_nP_nK - K\|_{L^2(\Omega^2)}, & g \in L^\infty(\Omega)\end{cases}\Bigg)\Bigg). \quad (3.4.7)$$

for τ sufficiently small, where C is positive constant that depends only on p, g, f and K.

(iii) If, moreover, $f,g \in L^\infty(\Omega)\cap \mathrm{Lip}(s, L^2(\Omega))$ and $K \in \mathrm{Lip}(s, L^2(\Omega^2))$, then

$$\sup_{k\in[N],t\in]t_{k-1},t_k]}\|I_n\mathbf{u}^{k-1} - u(\cdot,t)\|_{L^2(\Omega)} \le C\exp(T/2)\left((1+T^{1/2})\delta(n)^s + T^{1/2}\tau^{1/(3-p)}\right), \quad (3.4.8)$$

for τ sufficiently small, where C is positive constant that depends only on p, g, f, K and s.

PROOF : In the proof, C is any positive constant that may depend only on p, g, f, K and/or s, and that may be different at each line.

(i) Existence and uniqueness of u were proved in Theorem 3.3.1(i). The claimed well-posedness of the sequence $\{\mathbf{u}^k\}_{k\in[N]}$ is a consequence of Lemma 3.4.4 and Remark 3.4.5(3).

(ii) Denote $\check{\xi}_n(\boldsymbol{x},t) = \check{u}_n(\boldsymbol{x},t) - u(\boldsymbol{x},t)$, $\bar{\xi}_n(\boldsymbol{x},t) = \bar{u}_n(\boldsymbol{x},t) - u(x,t)$, $g_n = I_nP_ng$ and $K_n = I_nP_nK$. We thus have a.e.

$$\begin{aligned}
\frac{\partial\check{\xi}_n(\boldsymbol{x},t)}{\partial t} &= -\left(\Delta_p^{K_n}(\bar{u}_n(\boldsymbol{x},t)) - \Delta_p^{K}(u(\boldsymbol{x},t))\right) + (f_n(\boldsymbol{x}) - f(\boldsymbol{x})) \\
&= -\left(\Delta_p^{K_n}(\bar{u}_n(\boldsymbol{x},t)) - \Delta_p^{K_n}(u(\boldsymbol{x},t))\right) - \left(\Delta_p^{K_n}(u(\boldsymbol{x},t)) - \Delta_p^{K}(u(\boldsymbol{x},t))\right) + (f_n(\boldsymbol{x}) - f(\boldsymbol{x})).
\end{aligned}$$

Multiplying both sides by $\check{\xi}_n(\boldsymbol{x},t)$, integrating and rearranging the terms, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\check{\xi}_n(\cdot,t)\|^2_{L^2(\Omega)} = &-\int_\Omega \left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t) - \Delta_p^{K_n}u(\boldsymbol{x},t)\right)(\bar{u}_n(\boldsymbol{x},t) - u(\boldsymbol{x},t))d\boldsymbol{x} \\ &- \int_\Omega \left(\Delta_p^{K_n}u(\boldsymbol{x},t) - \Delta_p^{K}u(\boldsymbol{x},t)\right)\check{\xi}_n(\boldsymbol{x},t)d\boldsymbol{x} \\ &- \int_\Omega \left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t) - \Delta_p^{K_n}u(\boldsymbol{x},t)\right)(\check{u}_n(\boldsymbol{x},t) - \bar{u}_n(\boldsymbol{x},t))\,d\boldsymbol{x} \\ &+ \int_\Omega (f_n(\boldsymbol{x}) - f(\boldsymbol{x}))\,\check{\xi}_n(\boldsymbol{x},t)d\boldsymbol{x}.\end{aligned} \tag{3.4.9}$$

Since $f, g \in L^p(\Omega)$ in both cases, so is $u(\cdot,t)$ thanks to (3.2.11). We also have $\bar{u}_n(\cdot,t) \in L^2(\Omega) \subset L^p(\Omega)$ by Lemma 3.4.4. We are then in position to use Proposition 3.2.2(iv) with $h(x) = x$ to assert that the first term on the right-hand side of (3.4.9) is nonpositive. Let us now bound the second term.

Similarly to the estimates (3.3.5) and (3.3.4) in the proof of Theorem 3.3.1, and using Young inequality, we have

$$\begin{aligned}&\left|\int_\Omega \left(\Delta_p^{K_n}u(\boldsymbol{x},t) - \Delta_p^{K}u(\boldsymbol{x},t)\right)\check{\xi}_n(\boldsymbol{x},t)d\boldsymbol{x}\right| \\ &\le \begin{cases} C\|K_n - K\|_{L^{\infty,2}(\Omega^2)}\|\check{\xi}_n(\cdot,t)\|_{L^2(\Omega)}, & g \in L^2(\Omega) \\ C\|K_n - K\|_{L^2(\Omega^2)}\|\check{\xi}_n(\cdot,t)\|_{L^2(\Omega)}, & g \in L^\infty(\Omega),\end{cases} \\ &\le \frac{1}{6}\|\check{\xi}_n(\cdot,t)\|^2_{L^2(\Omega)} + \begin{cases} C\|K_n - K\|^2_{L^{\infty,2}(\Omega^2)}, & g \in L^2(\Omega) \\ C\|K_n - K\|^2_{L^2(\Omega^2)}, & g \in L^\infty(\Omega).\end{cases}\end{aligned}$$

For the third term in (3.4.9), we invoke Lemma 3.4.6 to get

$$\begin{aligned}&\left|\int_\Omega \left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t) - \Delta_p^{K_n}u(\boldsymbol{x},t)\right)(\check{u}_n(\boldsymbol{x},t) - \bar{u}_n(\boldsymbol{x},t))\,d\boldsymbol{x}\right| \\ &\le \|\Delta_p^{K_n}\bar{u}_n(\cdot,t) - \Delta_p^{K_n}u(\cdot,t)\|_{L^2(\Omega)}\|\check{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\|_{L^2(\Omega)} \\ &\le C\|\Delta_p^{K_n}\bar{u}_n(\cdot,t) - \Delta_p^{K_n}u(\cdot,t)\|_{L^2(\Omega)}\tau.\end{aligned}$$

We then use the fact that $\Delta_p^{I_n\mathbf{K}}$ is Hölder continuous on $L^2(\Omega)$ with exponent $p-1$, see (3.2.10), to obtain

$$\|\Delta_p^{K_n}\bar{u}_n(\cdot,t) - \Delta_p^{K_n}u(\cdot,t)\|_{L^2(\Omega)} \le C\|\bar{\xi}_n(\cdot,t)\|^{p-1}_{L^2(\Omega)} \le C\left(\|\check{\xi}_n(\cdot,t)\|^{p-1}_{L^2(\Omega)} + \tau^{p-1}\right),$$

where we used Lemma 3.4.6 to go from $\bar{\xi}_n$ to $\check{\xi}_n$, and that $p \in]1,2]$. It then follows by Cauchy-Schwartz inequality that

$$\begin{aligned}&\left|\int_\Omega \left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t) - \Delta_p^{K_n}u(\boldsymbol{x},t)\right)(\check{u}_n(\boldsymbol{x},t) - \bar{u}_n(\boldsymbol{x},t))\,d\boldsymbol{x}\right| \\ &\le C\left(\|\check{\xi}_n(\cdot,t)\|^{p-1}_{L^2(\Omega)}\tau + \tau^p\right) \\ &\le \frac{1}{6}\|\check{\xi}_n(\cdot,t)\|^2_{L^2(\Omega)} + C(\tau^{2/(3-p)} + \tau^p).\end{aligned}$$

Using Young inequality to bound the last term in (3.4.9), and combining the bounds on the three other terms, we have shown that

$$\begin{aligned}\frac{\partial}{\partial t}\|\check{\xi}_n(\cdot,t)\|^2_{L^2(\Omega)} \le \|\check{\xi}_n(\cdot,t)\|^2_{L^2(\Omega)} + C\Bigg(&\tau^{2/(3-p)} + \tau^p + \|f_n - f\|^2_{L^2(\Omega)} \\ &+ \begin{cases}\|K_n - K\|^2_{L^{\infty,2}(\Omega^2)}, & g \in L^2(\Omega) \\ \|K_n - K\|^2_{L^2(\Omega^2)}, & g \in L^\infty(\Omega)\end{cases}\Bigg).\end{aligned}$$

Using the Gronwall's lemma and taking the square-root, we get

$$\|\check{u}_n - u\|_{C([0,T];L^2(\Omega))} \leq \exp(T/2)\Bigg(\|I_nP_ng - g\|_{L^2(\Omega)}$$
$$+ CT^{1/2}\left(\tau^{1/(3-p)} + \tau^{p/2} + \|f_n - f\|_{L^2(\Omega)} + \begin{cases} \|I_nP_nK - K\|_{L^{\infty,2}(\Omega^2)} & g \in L^2(\Omega) \\ \|I_nP_nK - K\|_{L^2(\Omega^2)}, & g \in L^\infty(\Omega)\end{cases}\right)\Bigg). \tag{3.4.10}$$

Since $1/2 < 1/(3-p) \leq p/2$ for $p \in]1,2]$ the dependence on τ scales as $O(\tau^{1/(3-p)})$ for τ sufficiently small (or N large enough). Inserting (3.4.10) into

$$\sup_{k\in[N], t\in]t_{k-1},t_k]} \|u_n^{k-1} - u(\cdot,t)\|_{L^2(\Omega)} = \|\bar{u}_n - u\|_{C([0,T];L^2(\Omega))} \leq \|\check{u}_n - u\|_{C([0,T];L^2(\Omega))} + C\tau, \tag{3.4.11}$$

completes the proof of the error bound.

(iii) Plug (2.2.2) into (3.4.7).

□

Remark 3.4.8. Error bounds in $L^p(\Omega)$ were derived in [90] for forward Euler discretization. Their rate is better than ours and is provided for the range $p \in]1,+\infty[$. Unfortunately, we believe that their proof contains invalid arguments that can be fixed but only for $p \in]1,2]$.

Case $p = 1$: We now turn to the case $p = 1$, and consider the discrete system

$$\begin{cases} \dfrac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_{k-1}} = -\boldsymbol{\eta}^{k-1} + \mathbf{f}, & k \in [N], \\ \mathbf{u}^0 = \mathbf{g}. \end{cases} \tag{$\mathcal{P}_1^{\mathrm{TDF}}$}$$

where

$$\boldsymbol{\eta}^k = -\sum_{\boldsymbol{j}\in[n]^d} h_{\boldsymbol{j}} \mathbf{K}_{\boldsymbol{ij}} \operatorname{sign}(\mathbf{u}_{\boldsymbol{j}}^k - \mathbf{u}_{\boldsymbol{i}}^k), \quad \text{and thus} \quad \boldsymbol{\eta}^k \in \widehat{\boldsymbol{\Delta}}_1^{\mathbf{K}} \mathbf{u}^k.$$

We consider the continuum extensions in space and/or time of $\{\mathbf{u}^k\}_{k\in[N]}$ as before, namely u_n^k, $\check{u}_n$ and $\bar{u}_n$, $f_n = I_n\mathbf{f}$, and the space-time continuum extension of $\{\boldsymbol{\eta}^k\}_{k\in[N]}$

$$\bar{\eta}_n(\boldsymbol{x},t) = \sum_{k=1}^N (I_n\boldsymbol{\eta}^{k-1})(\boldsymbol{x})\chi_{]t_{k-1},t_k]}(t) = -\int_\Omega I_n\mathbf{K}(\boldsymbol{x},\boldsymbol{y}) \operatorname{sign}(\bar{u}_n(\boldsymbol{y},t) - \bar{u}_n(\boldsymbol{x},t))d\boldsymbol{y}, \quad (\boldsymbol{x},t) \in \Omega\times]0,T].$$

In view of Lemma 3.4.1, these extensions satisfy the evolution problem

$$\begin{cases} \frac{\partial}{\partial t}\check{u}_n(\boldsymbol{x},t) = -\bar{\eta}_n(\boldsymbol{x},t) + f_n(\boldsymbol{x}), & (\boldsymbol{x},t) \in \Omega\times]0,T], \\ \check{u}_n(\boldsymbol{x},0) = I_n\mathbf{g}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{3.4.12}$$

and

$$\bar{\eta}_n(\boldsymbol{x},t) \in \Delta_1^{I_n\mathbf{K}} \bar{u}_n(\boldsymbol{x},t).$$

We have the following counterpart estimates of Lemma 3.4.4.

Lemma 3.4.9. *Consider problem ($\mathcal{P}_1^{\mathrm{TDF}}$) with kernel $\mathbf{K}$, data $(\mathbf{f},\mathbf{g})$ and variable step-size*

$$\tau_k = \frac{\alpha_k}{\max\left(\|I_n\boldsymbol{\eta}^k - f_n\|_{L^2(\Omega)}, 1\right)}, \quad \textit{where} \quad \sum_{k\in\mathbb{N}} \alpha_k^2 < +\infty.$$

Assume that $I_n\mathbf{g} \in L^2(\Omega)$ *and* $I_n\mathbf{K}$ *satisfies (**H.1**)-(**H.2**) and (**H.3**). Suppose also that for each* $n \in \mathbb{N}$, $\mathbf{f}$ *is such that* $(\mathcal{P}_p^{\mathrm{TDF}})$ *has a stationary solution* $\mathbf{u}^\star$ *and that* $\sup_{n\in\mathbb{N}} \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} < +\infty$. *Then*

$$\bar{u}_n(\cdot,t) \in L^2(\Omega), \forall t \in [0,T], \quad \textit{and} \quad \sup_{t\in[0,T],n\in\mathbb{N}} \left\|\bar{u}_n(\cdot,t) - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} < +\infty.$$

Remark 3.4.10. The condition on the time-step τ_k is reminiscent of subgradient descent and has been used in [90]. The assumptions on $(\mathbf{f},\mathbf{g},\mathbf{K})$ are again verified when $\mathbf{f} = 0$, $\mathbf{g} = P_n g$ and $\mathbf{K} = P_n K$, where $g \in L^2(\Omega)$ and K satisfies (**H.1**)-(**H.3**).

Proof : Define the series $s_k \stackrel{\text{def}}{=} \sum_{i=0}^k \alpha_i^2$. As in Lemma 3.4.4, we proceed by induction using the monotonicity of the 1-Laplacian (Proposition 3.2.7(ii)). Indeed, since $f_n \in \Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}^\star)$, we have

$$\begin{aligned}\left\|u_n^1 - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 &= \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 \\ &\quad - 2\tau_0 \int_\Omega \left(\Delta_p^{I_n\mathbf{K}}(I_n\mathbf{g})(\boldsymbol{x}) - \Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}^\star)(\boldsymbol{x})\right)\left(I_n\mathbf{g}(\boldsymbol{x}) - I_n\mathbf{u}^\star\right) d\boldsymbol{x} + \alpha_0^2.\end{aligned}$$

By assumption on $\mathbf{g}$, $\mathbf{u}^\star$, we can invoke Proposition 3.2.7(ii) to get

$$\left\|u_n^1 - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 \le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 + s_0.$$

Suppose now that, for any $k > 1$,

$$\left\|u_n^k - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 \le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 + s_{k-1},$$

and thus $u_n^k \in L^2(\Omega)$. We can then invoke again Proposition 3.2.7(ii) to see that

$$\begin{aligned}&\left\|u_n^{k+1} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 \\ &= \left\|u_n^k - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 - 2\tau_k \int_\Omega \left(\Delta_p^{I_n\mathbf{K}}(u_n^k)(\boldsymbol{x}) - \Delta_p^{I_n\mathbf{K}}(I_n\mathbf{u}^\star)(\boldsymbol{x})\right)\left(u_n^k(\boldsymbol{x}) - I_n\mathbf{u}^\star\right) d\boldsymbol{x} + \alpha_k^2 \\ &\le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 + s_k.\end{aligned}$$

This shows that for all $k \in \mathbb{N}$,

$$\left\|u_n^k - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 \le \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)}^2 + s_\infty,$$

and thus $\left\{\left\|I_n\mathbf{u}^k\right\|_{L^2(\Omega)}\right\}_{k\in[N]}$ is bounded. In turn, so is $\left\|\bar{u}_n(\cdot,t)\right\|_{L^2(\Omega)}$ for $t \in [0,T]$ by its definition. Moreover,

$$\begin{aligned}\sup_{t\in[0,T],n\in\mathbb{N}} \left\|\bar{u}_n(\cdot,t) - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} &= \sup_{(n,N)\in\mathbb{N}^2,k\in[N]} \left\|u_n^k - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} \\ &\le \sup_{n\in\mathbb{N}} \left\|I_n\mathbf{g} - I_n\mathbf{u}^\star\right\|_{L^2(\Omega)} + s_\infty^{1/2} < +\infty.\end{aligned}$$

□

We also have the following analogue of Lemma 3.4.6.

Lemma 3.4.11. *In addition to the assumptions of Lemma 3.4.9, suppose that* $\sup_{n\in\mathbb{N}} \left\|I_n\mathbf{K}\right\|_{L^{\infty,1}(\Omega^2)} < +\infty$. *Then*

$$\sup_{t\in[0,T],n\in\mathbb{N}} \left\|\breve{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\right\|_{L^2(\Omega)} \le C\tau,$$

where C *is a positive constant that does not depend on* (n,N,T).

Proof : Arguing as the beginning of Lemma 3.4.6, we get for any $t \in]t_{k-1},t_k]$, $k \in \mathbb{N}$,

$$\left\|\breve{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\right\|_{L^2(\Omega)} \le \tau\left\|\bar{\eta}_n(\boldsymbol{x},t) - f_n\right\|_{L^2(\Omega)}.$$

By Hölder inequality, we have

$$\|\bar{\eta}_n(\boldsymbol{x},t)\|^2_{L^2(\Omega)} = \int_\Omega \left| \int_\Omega I_n \mathbf{K}(\boldsymbol{x},\boldsymbol{y}) \operatorname{sign}(\bar{u}_n(\boldsymbol{y},t) - \bar{u}_n(\boldsymbol{x},t)) d\boldsymbol{y} \right|^2 d\boldsymbol{x}$$
$$\leq \int_\Omega \left(\int_\Omega I_n \mathbf{K}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} \right)^2 d\boldsymbol{x} \leq \|I_n \mathbf{K}\|^2_{L^{\infty,1}(\Omega^2)}.$$

The same bound also holds on $\|f_n\|_{L^2(\Omega)}$. We then take the supremum over t and n to conclude. □

Theorem 3.4.12. *Let u be a solution of ($\mathcal{P}$) with kernel K and data (f,g) where f is time-independent, and $\{\mathbf{u}^k\}_{k\in[N]}$ is the sequence generated by ($\mathcal{P}_1^{\mathrm{TDF}}$) with $\mathbf{K} = P_n K$, $\mathbf{g} = P_n g$, $\mathbf{f} = P_n f$ and τ_k as prescribed in Lemma 3.4.9. Assume that K satisfies **(H.1)**, **(H.2)** and $K \in L^{\infty,2}(\Omega^2)$, and that $f, g \in L^2(\Omega)$. Then, the following hold.*

(i) u is the unique solution of ($\mathcal{P}$), $\{\mathbf{u}^k\}_{k\in[N]}$ is uniquely defined and $\left\{\|I_n \mathbf{u}^k\|_{L^2(\Omega)}\right\}_{k\in[N]}$ is bounded (uniformly in n when $\mathbf{f} = \mathbf{0}$).

(ii) We have the error estimate

$$\sup_{k\in[N], t\in]t_{k-1},t_k]} \|I_n \mathbf{u}^{k-1} - u(\cdot,t)\|_{L^2(\Omega)} \leq \exp(T/2) \Big(\|I_n P_n g - g\|_{L^2(\Omega)}$$
$$+ C T^{1/2} \left(\tau^{1/2} + \|f_n - f\|_{L^2(\Omega)} + \|I_n P_n K - K\|_{L^2(\Omega^2)} \right) \Big) \quad (3.4.13)$$

where C is positive constant that depends only on K.

(iii) If, moreover, $f, g \in \operatorname{Lip}(s, L^2(\Omega))$ and $K \in \operatorname{Lip}(s, L^2(\Omega^2))$, then

$$\sup_{k\in[N], t\in]t_{k-1},t_k]} \|I_n \mathbf{u}^{k-1} - u(\cdot,t)\|_{L^2(\Omega)} \leq C \exp(T/2) \left((1 + T^{1/2}) \delta(n)^s + T^{1/2} \tau^{1/2} \right), \quad (3.4.14)$$

where C is positive constant that depends only on g, f, K and s.

Proof : C is any positive constant that may depend only on g, f, K and s, and that may be different at each line. We use the same notation as in the proof of Theorem 3.4.7.

(i) Existence and uniqueness of u were proved in Theorem 3.3.1(i). Well-posedness of $\{\mathbf{u}^k\}_{k\in[N]}$ follows from Lemma 3.4.9 and Remark 3.4.5(3).

(ii) We have

$$\frac{\partial \check{\xi}_n(\boldsymbol{x},t)}{\partial t} = \int_\Omega K_n(\boldsymbol{x},\boldsymbol{y}) \left(\bar{w}_n(\boldsymbol{x},\boldsymbol{y},t) - w(\boldsymbol{x},\boldsymbol{y},t) \right) d\boldsymbol{y}$$
$$+ \int_\Omega (K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})) w(\boldsymbol{x},\boldsymbol{y},t) d\boldsymbol{y} + (f_n(\boldsymbol{x}) - f(\boldsymbol{x})),$$

where w is the subgradient function associated to u (see Definition 3.2.6), and $\bar{w}_n(\boldsymbol{x},\boldsymbol{y},t) = \operatorname{sign}(\bar{u}_n(\boldsymbol{y},t) - \bar{u}_n(\boldsymbol{x},t))$. Multiplying both sides by $\check{\xi}_n(\boldsymbol{x},t)$, integrating and rearranging the terms, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\check{\xi}_n(\cdot,t)\|^2_{L^2(\Omega)} &= \int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y}) \left(\bar{w}_n(\boldsymbol{x},\boldsymbol{y},t) - w(\boldsymbol{x},\boldsymbol{y},t) \right) (\bar{u}_n(\boldsymbol{x},t) - u(\boldsymbol{x},t)) d\boldsymbol{x} d\boldsymbol{y} \\ &+ \int_{\Omega^2} (K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})) w(\boldsymbol{x},\boldsymbol{y},t) \check{\xi}_n(\boldsymbol{x},t) d\boldsymbol{x} d\boldsymbol{y} \\ &+ \int_\Omega K_n(\boldsymbol{x},\boldsymbol{y}) \left(\bar{w}_n(\boldsymbol{x},\boldsymbol{y},t) - w(\boldsymbol{x},\boldsymbol{y},t) \right) (\check{u}_n(\boldsymbol{x},t) - \bar{u}_n(\boldsymbol{x},t)) d\boldsymbol{x} d\boldsymbol{y} \\ &+ \int_\Omega (f_n(\boldsymbol{x}) - f(\boldsymbol{x})) \check{\xi}_n(\boldsymbol{x},t) d\boldsymbol{x}.\end{aligned} \quad (3.4.15)$$

As $u(\cdot,t) \in L^1$ and $\bar{u}_n(\cdot,t) \in L^2(\Omega) \subset L^1(\Omega)$ by Lemma 3.4.9, the monotonicity claim in Proposition 3.2.7(ii) yields that the first term in (3.4.15) is nonpositive. The second and third terms can be easily bounded as

$$\Big|\int_{\Omega^2}(K_n(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x},\boldsymbol{y}))w(\boldsymbol{x},\boldsymbol{y},t)\check{\xi}_n(\boldsymbol{x},t)d\boldsymbol{x}d\boldsymbol{y}\Big| \leq \left\|K_n-K\right\|_{L^2(\Omega^2)}\left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}$$
$$\leq \frac{1}{4}\left\|\check{\xi}_n(\cdot,t)\right\|^2_{L^2(\Omega)}+\left\|K_n-K\right\|^2_{L^2(\Omega^2)}.$$

and the third term using Lemma 3.4.11

$$\Big|\int_{\Omega}K_n(\boldsymbol{x},\boldsymbol{y})\left(\bar{w}_n(\boldsymbol{x},\boldsymbol{y},t)-w(\boldsymbol{x},\boldsymbol{y},t)\right)\left(\check{u}_n(\boldsymbol{x},t)-\bar{u}_n(\boldsymbol{x},t)\right)d\boldsymbol{x}d\boldsymbol{y}\Big| \leq 2\left\|K\right\|^2_{L^{\infty,2}(\Omega^2)}\tau.$$

Bounding the last term by Young inequality, we obtain

$$\frac{\partial}{\partial t}\left\|\check{\xi}_n(\cdot,t)\right\|^2_{L^2(\Omega)} \leq \left\|\check{\xi}_n(\cdot,t)\right\|^2_{L^2(\Omega)}+2\left\|f_n-f\right\|^2_{L^2(\Omega)}+2\left\|K_n-K\right\|^2_{L^2(\Omega^2)}+C\tau.$$

Using the Gronwall's lemma and (3.4.11), we get the claimed bound.

(iii) Insert (2.2.2) into (3.4.13).

□

3.4.2.2 Backward/Implicit Euler discretization

Forward Euler discretization was able to deal only with $p \in [1,2]$. For backward Euler discretization, we will tackle $p \in]1,+\infty[$.

We consider the fully discrete problem with backward Euler time scheme

$$\begin{cases} \dfrac{\mathbf{u}^k-\mathbf{u}^{k-1}}{\tau_{k-1}} = -\widehat{\boldsymbol{\Delta}}_p^{\mathbf{K}}\mathbf{u}^k+\mathbf{f}^k, & k \in [N], \\ \mathbf{u}^0 = \mathbf{g}, \end{cases} \tag{$\mathcal{P}_p^{\mathrm{TDB}}$}$$

where $\mathbf{u}^k, \mathbf{f}^k \in \mathbb{R}^{n^d}$. This can also be written equivalently as

$$\mathbf{u}^k = J_{\tau_{k-1}\widehat{\boldsymbol{\Delta}}_p^{\mathbf{K}}}(\mathbf{u}^{k-1}+\tau_{k-1}\mathbf{f}^k).$$

This is known as the proximal iteration, and is at the heart of so-called mild solutions as well as existence and uniqueness of solutions to ($\mathcal{P}$) through the nonlinear semigroups theory [57, 25, 101, 26]. Denoting as before $u_n^k = I_n\mathbf{u}^k$ and $f_n^k = I_n\mathbf{f}^k$ the space continuum extensions of $\mathbf{u}^k$ and $\mathbf{f}^k$, we also have

$$u_n^k = J_{\tau_{k-1}\Delta_p^{I_n\mathbf{K}}}(u_n^{k-1}+\tau_{k-1}f_n^k).$$

We also let the time-space continuum extensions

$$\check{u}_n(\boldsymbol{x},t) = \frac{t_k-t}{\tau_{k-1}}u_n^{k-1}(\boldsymbol{x})+\frac{t-t_{k-1}}{\tau_{k-1}}u_n^k(\boldsymbol{x}), \quad (\boldsymbol{x},t) \in \Omega\times]t_{k-1},t_k], k \in [N],$$
$$\bar{u}_n(\boldsymbol{x},t) = \sum_{k=1}^{N}u_n^k(\boldsymbol{x})\chi_{]t_{k-1},t_k]}(t) \text{ and } \bar{f}_n(\boldsymbol{x},t) = \sum_{k=1}^{N}f_n^k(\boldsymbol{x})\chi_{]t_{k-1},t_k]}(t), \quad (\boldsymbol{x},t) \in \Omega\times]0,T].$$

Observe that the difference with the explicit Euler case lies in the definition of $\bar{u}_n$. From ($\mathcal{P}_p^{\mathrm{TDB}}$) one clearly sees that $\check{u}_n$ and $\bar{u}_n$ then satisfy again (3.4.6) with $\bar{f}_n(\boldsymbol{x},t)$ replacing $f_n(\boldsymbol{x})$.

The following estimates holds.

Lemma 3.4.13. *Consider problem ($\mathcal{P}_p^{\mathrm{TDB}}$) with kernel $\mathbf{K}$ and data $(\mathbf{f},\mathbf{g})$ and step-sizes $\tau_k > 0$ for all k. Assume that $I_n\mathbf{K}$ satisfies **(H.1)**-**(H.2)** and **(H.3)**, that $I_n\mathbf{g} \in L^{\max(p,q)}(\Omega)$, for some $q \in [1,+\infty]$, and*

$\sup_{n\in\mathbb{N}} \|I_n\mathbf{g}\|_{L^q(\Omega)} < +\infty$, *and that* $\bar{f}_n \in L^1([0,T]; L^{\max(p,q)}(\Omega))$ *and* $\sup_{n\in\mathbb{N}} \|\bar{f}_n\|_{L^1([0,T];L^q(\Omega))} < +\infty$. *Then*

$$\bar{u}_n(\cdot,t) \in L^{\max(p,q)}(\Omega), \forall t \in [0,T], \quad \textit{and} \quad \sup_{t\in[0,T],n\in\mathbb{N}} \|\bar{u}_n(\cdot,t)\|_{L^q(\Omega)} < +\infty.$$

Proof : Recall from Proposition 3.2.2(vii) that $J_{\lambda\Delta_p^{I_n\mathbf{K}}}$, $\lambda > 0$, is single-valued on $L^p(\Omega)$ and nonexpansive on $L^q(\Omega)$ for all $q \in [1,+\infty]$. Therefore, by induction, we have that for any $k \in [N]$,

$$\begin{aligned}\|u_n^k\|_{L^p(\Omega)} \leq \|I_n\mathbf{g}\|_{L^p(\Omega)} + \sum_{i=0}^{k} \tau_i \|f_n^i\|_{L^p(\Omega)} &\leq \|I_n\mathbf{g}\|_{L^p(\Omega)} + \sum_{i=0}^{N} \tau_i \|f_n^i\|_{L^p(\Omega)} \\ &= \|I_n\mathbf{g}\|_{L^p(\Omega)} + \|\bar{f}_n\|_{L^1([0,T];L^p(\Omega))}.\end{aligned}$$

Thus $u_n^k \in L^p(\Omega)$, for all $k \in [N]$. In turn, $J_{\tau_k\Delta_p^{I_n\mathbf{K}}}(u_n^k)$ is single-valued for all k, and arguing as above, its nonexpansiveness yields

$$\|u_n^k\|_{L^q(\Omega)} \leq \|I_n\mathbf{g}\|_{L^q(\Omega)} + \|\bar{f}_n\|_{L^1([0,T];L^q(\Omega))}.$$

Taking the supremum over k and n and using the definition of $\bar{u}_n$ and the assumptions on $\mathbf{g}$ and $\mathbf{f}$, we conclude. □

Lemma 3.4.14. *Suppose that the assumptions of Lemma 3.4.13 are satisfied with* $q = 2$ *when* $p \in]1,2]$, $q = 2(p-1)$ *when* $p \geq 2$. *Assume in addition that* $\sup_{n\in\mathbb{N}} \|I_n\mathbf{K}\|_{L^{\infty,1}(\Omega^2)} < +\infty$ *and* $\sup_{n\in\mathbb{N}} \|\bar{f}_n\|_{\mathrm{BV}([0,T];L^2(\Omega))} < +\infty$. *Then*

$$\sup_{t\in[0,T],n\in\mathbb{N}} \|\check{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\|_{L^2(\Omega)} \leq C\tau,$$

where C *is a positive constant that does not depend on* (n, N, T).

Proof : For $t \in]t_{k-1}, t_k]$, $k \in \mathbb{N}$, we have

$$\begin{aligned}\|\check{u}_n(\cdot,t) - \bar{u}_n(\cdot,t)\|_{L^2(\Omega)} &= (t_k - t)\Big\|\frac{u_n^{k-1} - u_n^k}{\tau_{k-1}}\Big\|_{L^2(\Omega)} \\ &= (t_k - t)\|\Delta_p^{I_n\mathbf{K}} u_n^k - f_n^k\|_{L^2(\Omega)} \\ &\leq \tau \|\Delta_p^{I_n\mathbf{K}} u_n^k - f_n^k\|_{L^2(\Omega)} \\ &= \tau \|\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\cdot,t_k) - \bar{f}_n(\cdot,t_k)\|_{L^2(\Omega)} \\ &\leq \tau \left(\|\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\cdot,t_k)\|_{L^2(\Omega)} + \sum_{i=1}^{k} \|\bar{f}_n(\cdot,t_i) - \bar{f}_n(\cdot,t_{i-1})\|_{L^2(\Omega)} + \|\bar{f}_n(\cdot,0)\|_{L^2(\Omega)} \right) \\ &\leq \tau \left(\|\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\cdot,t_k)\|_{L^2(\Omega)} + \mathrm{Var}_2(\bar{f}_n) + \|\bar{f}_n(\cdot,0)\|_{L^2(\Omega)} \right) \\ &= \tau \left(\|\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\cdot,t_k)\|_{L^2(\Omega)} + \|\bar{f}_n\|_{\mathrm{BV}([0,T];L^2(\Omega))} \right). \end{aligned} \tag{3.4.16}$$

For $p \in]1,2]$, we have from (3.2.10) that

$$\|\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\cdot,t_k)\|_{L^2(\Omega)} \leq 2^{p/2} C_2^{1/2} \|K\|_{L^{\infty,1}(\Omega^2)} \|\bar{u}_n(\cdot,t)\|_{L^2(\Omega)}^{p-1}.$$

For $p \geq 2$, it is easy to to show with simple arguments as before that

$$\|\Delta_p^{I_n\mathbf{K}} \bar{u}_n(\cdot,t_k)\|_{L^2(\Omega)} \leq 2^{p-3/2} \|K\|_{L^{\infty,1}(\Omega^2)} \|\bar{u}_n(\cdot,t)\|_{L^{2(p-1)}(\Omega)}^{p-1}.$$

Inserting the last two estimates in (3.4.16), taking the supremum over t and n over both sides, and applying Lemma 3.4.13, we conclude. □

Remark 3.4.15. As observed in the case of explicit time-discretization the uniform (over n) boundedness assumption made in the last two lemmas hold true if $\mathbf{g} = P_n g$, $\mathbf{K} = P_n K$ and $\mathbf{f}^k = \tau_k^{-1} \int_{t_{k-1}}^{t_k} P_n f(\cdot, t) dt$, where g, f and K verify simple assumptions. Indeed, in this case, we have thanks to Lemma 2.1.27 that for any $q \in [1, +\infty]$,

$$\sup_{n \in \mathbb{N}} \|I_n \mathbf{g}\|_{L^q(\Omega)} \leq \|g\|_{L^q(\Omega)}, \quad \sup_{n \in \mathbb{N}} \|I_n \mathbf{K}\|_{L^{\infty,q}(\Omega^2)} \leq \|K\|_{L^{\infty,q}(\Omega^2)},$$
$$\sup_{n \in \mathbb{N}} \|\bar{f}_n\|_{L^1([0,T];L^q(\Omega))} \leq \|f\|_{L^1([0,T];L^q(\Omega))} \text{ and } \sup_{n \in \mathbb{N}} \|\bar{f}_n\|_{\mathrm{BV}([0,T];L^q(\Omega))} \leq \|f\|_{\mathrm{BV}([0,T];L^q(\Omega))}.$$

In fact, the condition $f \in \mathrm{BV}([0, T]; L^q(\Omega))$ is sufficient to ensure that

$$\sup_{n \in \mathbb{N}} \|\bar{f}_n\|_{L^1([0,T];L^q(\Omega))} < +\infty \text{ and } \sup_{n \in \mathbb{N}} \|\bar{f}_n\|_{\mathrm{BV}([0,T];L^q(\Omega))} < +\infty.$$

Indeed, arguing as in [37, Lemma A.1], this conditions implies $f \in L^\infty([0, T]; L^q(\Omega))$. In turn, using Lemma 2.1.27, we get

$$\begin{aligned} \|\bar{f}_n\|_{L^1([0,T];L^q(\Omega))} \leq \|f\|_{L^1([0,T];L^q(\Omega))} &\leq \|f\|_{L^\infty([0,T];L^q(\Omega))} \\ &\leq \|f(\cdot, 0)\|_{L^q(\Omega)} + \mathrm{Var}_q(f) = \|f\|_{\mathrm{BV}([0,T];L^q(\Omega))}. \end{aligned}$$

We are now in position to state the error bound for the fully discrete problem with backward/implicit Euler time discretization.

Theorem 3.4.16. *Suppose that $p \in]1, +\infty[$. Let u be a solution of ($\mathcal{P}$) with kernel K and data (f, g), and $\{\mathbf{u}^k\}_{k\in[N]}$ is the sequence generated by ($\mathcal{P}_p^{\mathrm{TDB}}$) with $\mathbf{K} = P_n K$, $\mathbf{g} = P_n g$, $\mathbf{f}^k = \tau_k^{-1} \int_{t_{k-1}}^{t_k} P_n f(\cdot, t) dt$. Assume that K satisfies **(H.1)**, **(H.2)** and $K \in L^{\infty,2}(\Omega^2)$, and that f, g satisfy either one of the conditions (a), (b) or (c) in Theorem 3.3.1, and that $f \in \mathrm{BV}([0, T]; L^2(\Omega))$. Then, the following hold.*

(i) u is the unique solution of ($\mathcal{P}$), $\{\mathbf{u}^k\}_{k\in[N]}$ is uniquely defined and $\left\{\|I_n \mathbf{u}^k\|_{L^2(\Omega)}\right\}_{k\in[N]}$ is bounded uniformly in n.

(ii) We have the error estimate

$$\begin{aligned} \sup_{k\in[N], t\in]t_{k-1}, t_k]} \|I_n \mathbf{u}^k - u(\cdot, t)\|_{L^2(\Omega)} \leq \exp(T/2) \Bigg(&\|I_n P_n g - g\|_{L^2(\Omega)} + \|\bar{f}_n - f\|_{L^1([0,T];L^2(\Omega))} \\ &+ C T^{1/2} \begin{cases} \tau^{1/(3-p)} + \|I_n P_n K - K\|_{L^{\infty,2}(\Omega^2)} & \text{under (a)} \\ \tau^{p/(2p-1)} + \|I_n P_n K - K\|_{L^{\infty,2}(\Omega^2)} & \text{under (b)} \\ \tau^{1/(3-p)} + \|I_n P_n K - K\|_{L^2(\Omega^2)} & \text{under (c) when } p \in]1, 2] \\ \tau + \|I_n P_n K - K\|_{L^2(\Omega^2)} & \text{under (c) when } p \geq 2. \end{cases} \Bigg), \end{aligned} \tag{3.4.17}$$

for τ sufficiently small, where C is positive constant that depends only on p, g, f and K.

(iii) If, moreover, $g \in L^\infty(\Omega) \cap \mathrm{Lip}(s, L^2(\Omega))$, $K \in \mathrm{Lip}(s, L^2(\Omega^2))$, and $f \in L^1([0, T]; L^\infty(\Omega)) \cap \mathrm{Lip}(s, L^2(\Omega \times [0, T]))$ then

$$\begin{aligned} \sup_{k\in[N], t\in]t_{k-1}, t_k]} \|I_n \mathbf{u}^k - u(\cdot, t)\|_{L^2(\Omega)} \leq C \exp(T/2) \Bigg(&(1 + T^{1/2}) \delta(n)^s \\ &+ T^{1/2} \left(\begin{cases} \tau^{\min(s, 1/(3-p))} & \text{when } p \in]1, 2] \\ \tau^s & \text{when } p \geq 2 \end{cases} \right) \Bigg). \end{aligned} \tag{3.4.18}$$

for τ sufficiently small, where C is positive constant that depends only on p, g, f, K and s. The term τ^s in the dependence on τ disappears when f is time-independent.

PROOF : In the proof, C is any positive constant that may depend solely on p, g, f, K and/or s, and that may be different at each line.

(i) Existence and uniqueness of u were proved in Theorem 3.3.1(i). Well-posedness of the sequence $\{\mathbf{u}^k\}_{k\in[N]}$ is a consequence of Lemma 3.4.13 and Remark 3.4.15.

(ii) For $p \in]1,2]$, the proof of the error bound is exactly the same as that of (3.4.7) in Theorem 3.4.7 using the modified definition of $\bar{u}_n$ and that now f is time-dependent, and thus we replace f_n there by $\bar{f}_n$. We also denote $g_n = I_n P_n g$ and $K_n = I_n P_n K$.

For the $p \geq 2$, the argument is also similar, and the main change consists in bounding appropriately the third term in (3.4.9). We then invoke Lemma 3.4.14 to show that

$$|\int_\Omega \left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t) - \Delta_p^{K_n}u(\boldsymbol{x},t)\right)\left(\check{u}_n(\boldsymbol{x},t) - \bar{u}_n(\boldsymbol{x},t)\right) d\boldsymbol{x}| \leq C\left\|\Delta_p^{K_n}\bar{u}_n(\cdot,t) - \Delta_p^{K_n}u(\cdot,t)\right\|_{L^2(\Omega)}\tau,$$

where C is indeed a finite constant owing to the assumption on f and Remark 3.4.15. We now use Lemma 3.2.1(ii) to get the bound

$$\begin{aligned}
&\left\|\Delta_p^{K_n}\bar{u}_n(\cdot,t) - \Delta_p^{K_n}u(\cdot,t)\right\|_{L^2(\Omega)}^2 \\
&= \int_\Omega |\int_\Omega K_n(\boldsymbol{x},\boldsymbol{y})\left(\Psi(\bar{u}_n(\boldsymbol{y},t) - \bar{u}_n(\boldsymbol{x},t)) - \Psi(u(\boldsymbol{y},t) - u(\boldsymbol{x},t))\right) d\boldsymbol{y}|^2 d\boldsymbol{x} \\
&\leq \int_\Omega \left(\int_\Omega K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{y},t) - \bar{\xi}_n(\boldsymbol{x},t)\right|\left(|\bar{u}_n(\boldsymbol{y},t) - \bar{u}_n(\boldsymbol{x},t)| + |u(\boldsymbol{y},t) - u(\boldsymbol{x},t)|\right)^{p-2} d\boldsymbol{y}\right)^2 d\boldsymbol{x}.
\end{aligned} \tag{3.4.19}$$

For case (c), we infer from Lemma 3.4.13 (with $q = +\infty$) and Lemma 2.1.27 that

$$\begin{aligned}
&\left\|\Delta_p^{K_n}\bar{u}_n(\cdot,t) - \Delta_p^{K_n}u(\cdot,t)\right\|_{L^2(\Omega)}^2 \\
&\leq \left(4\left(\|g\|_{L^\infty(\Omega)} + \|f\|_{L^1([0,T];L^\infty(\Omega))}\right)\right)^{2(p-2)} \int_\Omega \left(\int_\Omega K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{y},t) - \bar{\xi}_n(\boldsymbol{x},t)\right| d\boldsymbol{y}\right)^2 d\boldsymbol{x} \\
&\leq \left(4\left(\|g\|_{L^\infty(\Omega)}\|f\|_{L^1([0,T];L^\infty(\Omega))}\right)\right)^{2(p-2)} \|K\|_{L^{\infty,2}(\Omega^2)} \int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{y},t) - \bar{\xi}_n(\boldsymbol{x},t)\right|^2 d\boldsymbol{x}d\boldsymbol{y} \\
&= 4\left(4\left(\|g\|_{L^\infty(\Omega)}\|f\|_{L^1([0,T];L^\infty(\Omega))}\right)\right)^{2(p-2)} \|K\|_{L^{\infty,2}(\Omega^2)} \int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{x},t)\right|^2 d\boldsymbol{x}d\boldsymbol{y} \\
&\leq 4\left(4\left(\|g\|_{L^\infty(\Omega)}\|f\|_{L^1([0,T];L^\infty(\Omega))}\right)\right)^{2(p-2)} \|K\|_{L^{\infty,2}(\Omega^2)}^2 \left\|\bar{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.4.20}$$

It then follows by Cauchy-Schwartz inequality that

$$\begin{aligned}
&|\int_\Omega \left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t) - \Delta_p^{K_n}u(\boldsymbol{x},t)\right)\left(\check{u}_n(\boldsymbol{x},t) - \bar{u}_n(\boldsymbol{x},t)\right) d\boldsymbol{x}| \\
&\leq C\left\|\bar{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}\tau \\
&\leq C\left(\left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}\tau + \tau^2\right) \\
&\leq \frac{1}{6}\left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}^2 + C\tau^2.
\end{aligned}$$

Inserting this in (3.4.9), using again Young inequality for the last term, we have shown that when $p \geq 2$ and (c) holds,

$$\frac{\partial}{\partial t}\left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}^2 \leq \left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}^2 + C\left(\tau^2 + \left\|\bar{f}_n(\cdot,t) - f(\cdot,t)\right\|_{L^2(\Omega)}^2 + \|K_n - K\|_{L^2(\Omega^2)}^2\right).$$

Using the Gronwall's lemma, taking the square-root and using (3.4.11), we get the error bound in this case.

It remains to consider the case (b), when $p \geq 2$. For this, we embark from (3.4.19), and use the continuity of Ψ in Lemma 3.2.1 (i) (see (3.2.4)) with $\alpha = 1/p$. Combining this with Jensen and

Hölder inequalities, we get

$$\begin{aligned}
&\left\|\Delta_p^{K_n}\bar{u}_n(\cdot,t)-\Delta_p^{K_n}u(\cdot,t)\right\|_{L^2(\Omega)}^2\\
&\le \|K\|_{L^{\infty,1}(\Omega^2)}\int_{\Omega^2}\left(K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{y},t)-\bar{\xi}_n(\boldsymbol{x},t)\right|^{2/p}\right)\\
&\qquad (|\bar{u}_n(\boldsymbol{y},t)-\bar{u}_n(\boldsymbol{x},t)|+|u(\boldsymbol{y},t)-u(\boldsymbol{x},t)|)^{2(p-1)-2/p}\,d\boldsymbol{x}d\boldsymbol{y}\\
&\le \|K\|_{L^{\infty,1}(\Omega^2)}\int_{\Omega^2}\left(K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{y},t)-\bar{\xi}_n(\boldsymbol{x},t)\right|^{2}\right)^{1/p}\\
&\qquad \left((K_n(\boldsymbol{x},\boldsymbol{y}))^{(p-1)/p}(|\bar{u}_n(\boldsymbol{y},t)-\bar{u}_n(\boldsymbol{x},t)|+|u(\boldsymbol{y},t)-u(\boldsymbol{x},t)|)^{2(p-1)-2/p}\right)d\boldsymbol{x}d\boldsymbol{y}\\
&\le \|K\|_{L^{\infty,1}(\Omega^2)}\left(\int_{\Omega^2}K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{y},t)-\bar{\xi}_n(\boldsymbol{x},t)\right|^{2}d\boldsymbol{x}d\boldsymbol{y}\right)^{1/p}\\
&\qquad \left(\int_{\Omega^2}K_n(\boldsymbol{x},\boldsymbol{y})\,(|\bar{u}_n(\boldsymbol{y},t)-\bar{u}_n(\boldsymbol{x},t)|+|u(\boldsymbol{y},t)-u(\boldsymbol{x},t)|)^{2p-2/(p-1)}\,d\boldsymbol{x}d\boldsymbol{y}\right)^{(p-1)/p}\\
&\le \|K\|_{L^{\infty,1}(\Omega^2)}\left(4\int_{\Omega^2}K_n(\boldsymbol{x},\boldsymbol{y})\left|\bar{\xi}_n(\boldsymbol{x},t)\right|^{2}d\boldsymbol{x}d\boldsymbol{y}\right)^{1/p}\\
&\qquad \left(2^{2p-2/(p-1)}\int_{\Omega^2}K_n(\boldsymbol{x},\boldsymbol{y})\,\left(\left|\bar{u}_n(\boldsymbol{x},t)\right|+\left|u(\boldsymbol{x},t)\right|\right)^{2p-2/(p-1)}\,d\boldsymbol{x}d\boldsymbol{y}\right)^{(p-1)/p}.\\
&\le 4\|K\|_{L^{\infty,1}(\Omega^2)}^2\left(\int_{\Omega^2}\left|\bar{\xi}_n(\boldsymbol{x},t)\right|^{2}d\boldsymbol{x}d\boldsymbol{y}\right)^{1/p}\\
&\qquad \left(\int_{\Omega^2}\left(\left|\bar{u}_n(\boldsymbol{x},t)\right|+\left|u(\boldsymbol{x},t)\right|\right)^{2p-2/(p-1)}\,d\boldsymbol{x}d\boldsymbol{y}\right)^{(p-1)/p}.
\end{aligned}$$

Observe that $L^{2p-2/(p-1)}(\Omega)\subset L^{2(p-1)}(\Omega)$, hence by Hölder inequality and Lemma 3.4.13 with $q=2(p-1)$ and Lemma 2.1.27, the last term in the above display can be bounded as

$$\begin{aligned}
&\left(\int_{\Omega^2}\left(\left|\bar{u}_n(\boldsymbol{x},t)\right|+\left|u(\boldsymbol{x},t)\right|\right)^{2p-2/(p-1)}\,d\boldsymbol{x}d\boldsymbol{y}\right)^{(p-1)/p}\\
&\le \left\|\left|\bar{u}_n(\boldsymbol{x},t)\right|+\left|u(\boldsymbol{x},t)\right|\right\|_{L^{2(p-1)}(\Omega)}^{2(p-1)-2/p}\\
&\le \left(\|g\|_{L^{2(p-1)}(\Omega)}+\|f\|_{L^1([0,T];L^{2(p-1)}(\Omega))}\right)^{2(p-1)-2/p}.
\end{aligned}$$

We then arrive at

$$\left\|\Delta_p^{K_n}\bar{u}_n(\cdot,t)-\Delta_p^{K_n}u(\cdot,t)\right\|_{L^2(\Omega)}^2\le C\|K\|_{L^{\infty,1}(\Omega^2)}^2\|\bar{\xi}_n\|_{L^2(\Omega)}^{2/p}.$$

Hence

$$\begin{aligned}
&\left|\int_{\Omega}\left(\Delta_p^{K_n}\bar{u}_n(\boldsymbol{x},t)-\Delta_p^{K_n}u(\boldsymbol{x},t)\right)(\check{u}_n(\boldsymbol{x},t)-\bar{u}_n(\boldsymbol{x},t))\,d\boldsymbol{x}\right|\\
&\le C\|\bar{\xi}_n(\cdot,t)\|_{L^2(\Omega)}^{1/p}\tau\\
&\le C\left(\|\check{\xi}_n(\cdot,t)\|_{L^2(\Omega)}^{1/p}\tau+\tau^{(p+1)/p}\right)\\
&\le \frac{1}{6}\|\check{\xi}_n(\cdot,t)\|_{L^2(\Omega)}^{2}+C(\tau^{2p/(2p-1)}+\tau^{(p+1)/p}).
\end{aligned}$$

Inserting this into (3.4.9), using again Young inequality for the last term,

$$\begin{aligned}
\frac{\partial}{\partial t}\|\check{\xi}_n(\cdot,t)\|_{L^2(\Omega)}^2\le \|\check{\xi}_n(\cdot,t)\|_{L^2(\Omega)}^2+C\Big(&\tau^{2p/(2p-1)}+\tau^{(p+1)/p}+\|\bar{f}_n(\cdot,t)-f(\cdot,t)\|_{L^2(\Omega)}^2\\
&+\|K_n-K\|_{L^2(\Omega^2)}^2\Big).
\end{aligned}$$

Hence, using the Gronwall's lemma, taking the square-root and using (3.4.11), we get the error bound in this case, after observing that the dependence on τ scales as $O(\tau^{p/(2p-1)})$ for τ sufficiently small (or N large enough) since $1/2 < p/(2p-1) \leq (p+1)/(2p)$ for $p \geq 2$.

(iii) Plug (2.2.2) into (3.4.17) after observing that

$$\|\bar{f}_n - f\|_{L^1([0,T];L^2(\Omega))} \leq T^{1/2}\|\bar{f}_n - f\|_{L^2([0,T];L^2(\Omega))} = T^{1/2}\|\bar{f}_n - f\|_{L^2(\Omega\times[0,T])} \leq CT^{1/2}\max(\tau^s, \delta(n)^s).$$

For the scaling in τ, we use that $s \in]0,1]$.

□

Another way to derive error bounds for $(\mathcal{P}_p^{\mathrm{TDB}})$ is as follows. To lighten notation, denote $g_n = I_n P_n g$, $f_n(\cdot,t) = I_n P_n f(\cdot,t)$ for $t \in [0,T]$, and $K_n = I_n P_n \mathbf{K}$. Let u_n be a solution to $(\mathcal{P})$ with data (f_n, g_n) and kernel K_n. Under the assumptions of Theorem 3.4.16 on (f,g,K), u_n is unique. Then one has

$$\|\check{u}_n - u\|_{C([0,T];L^2(\Omega))} \leq \|\check{u}_n - u_n\|_{C([0,T];L^2(\Omega))} + \|u_n - u\|_{C([0,T];L^2(\Omega))}.$$

Theorem 3.3.1 provides a bound on the last term of the right-hand side in the above display, which captures the space-discretization error. Bounds for the first term, which corresponds to the time-discretization error, were derived in $C([0,T];L^p(\Omega))$ by Crandall and Liggett in their seminal paper [57] for constant time step-size and $f = 0$, and then extended to non-uniform time partitions in [101], see also [116]. More precisely, using [116, Theorem 1] and the fact that $\check{u}_n(\cdot,0) = u_n(\cdot,0) = g_n$, the following bound holds

$$\|\check{u}_n - u_n\|_{C([0,T];L^p(\Omega))} \leq \|\bar{f}_n - f_n\|_{L^1([0,T];L^p(\Omega))} + 2T^{1/2}\left(\|f_n^1 - \Delta_p^{K_n} g_n\|_{L^p(\Omega)} + \mathrm{Var}_p(\bar{f}_n)\right)\tau^{1/2}.$$

The first term can be bounded as follows (for constant step-size to simplify)

$$\begin{aligned}
\|\bar{f}_n - f_n\|_{L^1([0,T];L^p(\Omega))} &= \sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\|\tau_k^{-1}\int_{t_{k-1}}^{t_k} f_n(\cdot,s)ds - f_n(\cdot,t)\|_{L^p(\Omega)}dt \\
&\leq \tau^{-1}\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k}\|f(\cdot,s) - f(\cdot,t)\|_{L^p(\Omega)}dsdt \\
&\leq \tau^{-1}\int_{-\tau}^{\tau}\left(\int_0^T\|f(\cdot,t+s) - f(\cdot,t)\|_{L^p(\Omega)}dt\right)ds \\
&\leq \tau^{-1}\int_{-\tau}^{\tau} s\mathrm{Var}_p(f)ds = \tau\mathrm{Var}_p(f),
\end{aligned}$$

where we used Lemma 2.1.27 in the first inequality and [37, Lemma A.1] in the last one. Overall, this shows that the time discretization error $\|\check{u}_n - u_n\|_{C([0,T];L^p(\Omega))}$ scales as $O\left((T\tau)^{1/2}\right)$ for τ sufficiently small. The rate $O(\tau^{1/2})$ is known to be optimal for general accretive operators in Banach spaces (see [116]). In turn, by standard comparisons of $L^q(\Omega)$ norms (assuming that (c) holds so that boundedness of $\check{u}_n$ and u_n is in force), this strategy gives us a bound which scales as

$$\|\check{u}_n - u_n\|_{C([0,T];L^p(\Omega))} = \begin{cases} O\left(\tau^{1/2}\right) & p \geq 2, \\ O\left(\tau^{p/4}\right) & p \in]1,2]. \end{cases}$$

This is strictly worse than the rates in τ obtained from (3.4.17). There is however no contradiction in this and the reason is that the strategy outlined above is too general and does not exploit all properties of the operator Δ_p^K among which its continuity that was a key to derive better rates in τ. In this sense, our present results are optimal. We also remark that our rates are consistent with those in [90] for $p \geq 2$.

3.5 Application to random graph sequences

In this section, we study continuum limits of fully discrete problems on the random graph model of Definition 2.3.7 with backward/implicit Euler time discretization. Explicit discretization can also be treated following our results in Section 3.4.2.1, but we will not elaborate further on it for the sake of brevity.

Recall the notations in Section 2.3.4, in which case we now set $\Omega = [0,1]$. Recall also the the construction of the random graph model in Definition 2.3.7 where each edge (i,j) is independently set to 1 with probability (2.3.5). This entails that the random matrix $\mathbf{\Lambda}$ is symmetric. However, it is worth emphasizing that the entries of $\mathbf{\Lambda}$ are not independent, but only the entries in each row are mutually independent[2]. This observation will be instrumental in deducing our error bound.

We consider the fully discrete on K-random graphs $\mathbf{G}(n,K,\rho_n)$ with backward Euler time scheme

$$\begin{cases} \dfrac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_{k-1}} = \dfrac{1}{\rho_n n} \displaystyle\sum_{j:(i,j)\in E(\mathbf{G}(n,K,\rho_n))} \Psi(\mathbf{u}_j - \mathbf{u}_i) + \mathbf{f}^k, \quad k \in [N], \\ \mathbf{u}^0 = \mathbf{g}, \end{cases} \tag{$\mathcal{P}_p^{\mathrm{TDB},\mathbf{G}}$}$$

where $\mathbf{u}^k, \mathbf{f}^k \in \mathbb{R}^n$. It is important to keep in mind that, since $\mathbf{G}(n,K,\rho_n)$ is a random variable taking values in the set of simple graphs, the evolution problem ($\mathcal{P}_p^{\mathrm{TDB},\mathbf{G}}$) must be understood in this sense. Observe that the normalization in ($\mathcal{P}_p^{\mathrm{TDB},\mathbf{G}}$) by $\rho_n n$ corresponds to the average degree (see Section 2.3.4 for details).

Problem ($\mathcal{P}_p^{\mathrm{TDB},\mathbf{G}}$) can be equivalently written as

$$\begin{cases} \dfrac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_{k-1}} = -\widehat{\mathbf{\Delta}}_p^{\mathbf{\Lambda}} \mathbf{u}^k + \mathbf{f}^k, \quad k \in [N], \\ \mathbf{u}^0 = \mathbf{g}. \end{cases}$$

We define the time-space continuum extensions $\check{u}_n$ and $\bar{u}_n$ and as in Section 3.4.2.2. One then sees that they satisfy

$$\begin{cases} \frac{\partial}{\partial t}\check{u}_n(x,t) = -\Delta_p^{I_n\mathbf{\Lambda}}\bar{u}_n(x,t) + \bar{f}_n(x,t), \quad (x,t) \in \Omega\times]0,T], \\ \check{u}_n(x,0) = I_n\mathbf{g}(x), \quad x \in \Omega. \end{cases} \tag{3.5.1}$$

Toward our goal of establishing error bounds, we define $\mathbf{v}$ as the solution of the fully discrete problem ($\mathcal{P}_p^{\mathrm{TDB}}$) with data $(\mathbf{f},\mathbf{g})$ and discrete kernel $\hat{\mathbf{K}}$. Its time-space continuum extensions, $\check{v}_n$ and $\bar{v}_n$, defined similarly as above, fulfill

$$\begin{cases} \frac{\partial}{\partial t}\check{v}_n(x,t) = -\Delta_p^{I_n\hat{\mathbf{K}}}\bar{v}_n(x,t) + \bar{f}_n(x,t), \quad (x,t) \in \Omega\times]0,T], \\ \check{v}_n(x,0) = I_n\mathbf{g}(x), \quad x \in \Omega. \end{cases} \tag{3.5.2}$$

We have

$$\|\check{u}_n - u\|_{C([0,T];L^2(\Omega))} \leq \|\check{u}_n - \check{v}_n\|_{C([0,T];L^2(\Omega))} + \|\check{v}_n - u\|_{C([0,T];L^2(\Omega))}. \tag{3.5.3}$$

This bound is composed of two terms: the first one captures the error of random sampling, and the second that of (space and time) discretization. We start by bounding the first term by comparing (3.5.1) and (3.5.2).

Lemma 3.5.1. *Assume that $(\mathbf{f}^k, \mathbf{g}, \mathbf{K}, f, g, K)$ verify the assumptions of Theorem 3.4.16. Assume also that $\rho_n \to 0$ and $n\rho_n = \omega\left((\log n)^\gamma\right)$ for some $\gamma > 1$. Then, for any $\beta \in]0,1[$,*

$$\|\check{u}_n - \check{v}_n\|_{C([0,T];L^2(\Omega))} \leq C\exp\left(T/2\right)T^{1/2}\left((\rho_n n)^{-\beta/2} + \begin{cases} \tau^{1/(3-p)} & p \in]1,2], \\ \tau & p \geq 2. \end{cases}\right). \tag{3.5.4}$$

[2] This feature was already used in the proof of Lemma 2.3.10

with probability at least $1-(\rho_n n)^{-(1-\beta)}$. *In particular,*

$$\left\|\check{u}_n-\check{v}_n\right\|_{C([0,T];L^2(\Omega))} \leq C\exp\left(T/2\right)T^{1/2}\left(o\left((\log n)^{-\gamma\beta/2}\right)+\begin{cases}\tau^{1/(3-p)} & p\in]1,2],\\ \tau & p\geq 2.\end{cases}\right). \quad (3.5.5)$$

with probability at least $1-o\left((\log n)^{-\gamma(1-\beta)}\right)$.

Proof : Denote $\check{\xi}_n(x,t)=\check{v}_n(x,t)-\check{u}_n(x,t)$, $\bar{\xi}_n(x,t)=\bar{v}_n(x,t)-\bar{u}_n(x,t)$, $g_n=I_nP_ng$, $\hat{K}_n=I_n\overset{\wedge}{\mathbf{K}}$ and $\Lambda_n=I_n\mathbf{\Lambda}$. We thus have from (3.5.1) and (3.5.2) that a.e.

$$\begin{aligned}\frac{\partial\check{\xi}_n(x,t)}{\partial t} &= -\left(\Delta_p^{\hat{K}_n}(\bar{v}_n(x,t))-\Delta_p^{\Lambda_n}(\bar{u}_n(x,t))\right)\\ &= -\left(\Delta_p^{\hat{K}_n}(\bar{v}_n(x,t))-\Delta_p^{\Lambda_n}(\bar{v}_n(x,t))\right)-\left(\Delta_p^{\Lambda_n}(\bar{v}_n(x,t))-\Delta_p^{\Lambda_n}(\bar{u}_n(x,t))\right).\end{aligned}$$

Multiplying both sides by $\check{\xi}_n(x,t)$, integrating and rearranging the terms, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left\|\check{\xi}_n(\cdot,t)\right\|^2_{L^2(\Omega)} &= -\int_\Omega\left(\Delta_p^{\Lambda_n}\bar{v}_n(x,t)-\Delta_p^{\Lambda_n}\bar{u}_n(x,t)\right)(\bar{v}_n(x,t)-\bar{u}_n(x,t))dx\\ &-\int_\Omega\left(\Delta_p^{\hat{K}_n}\bar{v}_n(x,t)-\Delta_p^{\Lambda_n}\bar{v}_n(x,t)\right)\check{\xi}_n(x,t)dx \quad (3.5.6)\\ &-\int_\Omega\left(\Delta_p^{\Lambda_n}\bar{v}_n(x,t)-\Delta_p^{\Lambda_n}\bar{u}_n(x,t)\right)\left((\check{v}_n(x,t)-\bar{v}_n(x,t))-(\check{u}_n(x,t)-\bar{u}_n(x,t))\right)dx.\end{aligned}$$

Under our condition on $n\rho_n$, Lemma 2.3.10 tells us that with probability 1,

$$\left\|\Lambda_n\right\|_{L^{\infty,1}(\Omega^2)}=\left\|\hat{K}_n\right\|_{L^{\infty,1}(\Omega^2)}+o(1)\leq\left\|I_nP_nK\right\|_{L^{\infty,1}(\Omega^2)}+o(1)\leq\left\|K\right\|_{L^{\infty,1}(\Omega^2)}+o(1),$$

so in particular $\left\|\Lambda_n\right\|_{L^{\infty,1}(\Omega^2)}$ is uniformly bounded with probability 1. Λ_n is also positive and symmetric. Since $g\in L^q(\Omega)$ and $f\in L^1([0,T];L^q(\Omega))\cap \mathrm{BV}([0,T];L^2(\Omega))$, $q\in\{2,2(p-1),+\infty\}$, the conclusions of Lemma 3.4.13 and Lemma 3.4.14 remain true which shows that with probability 1,

$$\sup_{t\in[0,T],n\in\mathbb{N}}\left\|\bar{u}_n(\cdot,t)\right\|_{L^q(\Omega)}<+\infty \quad\text{and}\quad \sup_{t\in[0,T],n\in\mathbb{N}}\left\|\check{u}_n(\cdot,t)-\bar{u}_n(\cdot,t)\right\|_{L^2(\Omega)}\leq C\tau.$$

The same claim holds for $\check{v}_n$ and $\bar{v}_n$ since $\left\|\hat{K}_n\right\|_{L^{\infty,1}(\Omega^2)}\leq\left\|K\right\|_{L^{\infty,1}(\Omega^2)}<+\infty$ and $\hat{K}_n$ is positive and symmetric, i.e. $\hat{K}_n$ obeys **(H.1)**-**(H.3)**. Thus Proposition 3.2.2(iv) entails that the first term on the right-hand side of (3.5.6) is nonpositive with probability 1. Let us now bound the second term. Denote the random variables $\mathbf{Z}_i\overset{\text{def}}{=}\frac{1}{n}\sum_{j\in[n]}\left(\mathbf{\Lambda}_{ij}-\overset{\wedge}{\mathbf{K}}_{ij}\right)\Psi(\mathbf{v}_j-\mathbf{v}_i)$. By Cauchy-Schwartz inequality, we have

$$\left|\int_\Omega\left(\Delta_p^{\Lambda_n}\bar{u}_n(x,t)-\Delta_p^{\hat{K}_n}\bar{u}_n(x,t)\right)\check{\xi}_n(x,t)dx\right|\leq C\left\|I_n\mathbf{Z}\right\|_{L^2(\Omega)}\left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}.$$

For the last term in (3.5.6), we argue as in the proof of Theorem 3.4.16 to show that, with probability 1,

$$\begin{aligned}&\left|\int_\Omega\left(\Delta_p^{\Lambda_n}\bar{u}_n(x,t)-\Delta_p^{\Lambda_n}\bar{v}_n(x,t)\right)\left((\check{u}_n(x,t)-\bar{u}_n(x,t))-(\check{v}_n(x,t)-\bar{v}_n(x,t))\right)dx\right|\\ &\qquad\leq C\begin{cases}\left\|\check{\xi}_n(\cdot,t)\right\|^{p-1}_{L^2(\Omega)}\tau+\tau^p & p\in]1,2],\\ \left\|\check{\xi}_n(\cdot,t)\right\|_{L^2(\Omega)}\tau+\tau^2 & p\geq 2.\end{cases}\end{aligned}$$

Collecting all these bounds, after using Young inequality, we have shown that (again with probability 1),

$$\frac{\partial}{\partial t}\left\|\check{\xi}_n(\cdot,t)\right\|^2_{L^2(\Omega)}\leq\left\|\check{\xi}_n(\cdot,t)\right\|^2_{L^2(\Omega)}+C\left(\left\|I_n\mathbf{Z}\right\|^2_{L^2(\Omega)}+\begin{cases}\tau^{2/(3-p)}+\tau^p & p\in]1,2],\\ \tau^2 & p\geq 2.\end{cases}\right)$$

Using the Gronwall's lemma and taking the square-root, we get for τ sufficiently small

$$\|\breve{u}_n - \breve{v}_n\|_{C([0,T];L^2(\Omega))} \leq C \exp(T/2)\, T^{1/2} \left(\|I_n \mathbf{Z}\|_{L^2(\Omega)} + \begin{cases} \tau^{1/(3-p)} & p \in]1,2], \\ \tau & p \geq 2. \end{cases} \right). \tag{3.5.7}$$

It remains to bound the random variable $\|I_n \mathbf{Z}\|_{L^2(\Omega)}$. For this purpose, we have by Markov inequality that for $\varepsilon > 0$

$$\mathbb{P}\left(\|I_n \mathbf{Z}\|_{L^2(\Omega)} \geq \varepsilon\right) = \mathbb{P}\left(n^{-1} \sum_i \mathbf{Z}_i^2 \geq \varepsilon^2\right) \leq \varepsilon^{-2} n^{-1} \sum_i \mathbb{E}\left(\mathbf{Z}_i^2\right).$$

By independence of $(\mathbf{\Lambda}_{ij})_{j\in[n]}$, for each $i \in [n]$, we get

$$\begin{aligned}
\mathbb{E}\left(\mathbf{Z}_i^2\right) &= (\rho_n n)^{-2} \sum_{j\in[n]} \mathcal{V}\left(\rho_n \mathbf{\Lambda}_{ij}\right) \left(\Psi(\mathbf{v}_j - \mathbf{v}_i)\right)^2 = (\rho_n n)^{-2} \sum_{j\in[n]} \rho_n \overset{\wedge}{\mathbf{K}}_{ij} (1 - \rho_n \overset{\wedge}{\mathbf{K}}_{ij}) \left(\Psi(\mathbf{v}_j - \mathbf{v}_i)\right)^2 \\
&\leq (\rho_n n^2)^{-1} \sum_{j\in[n]} \overset{\wedge}{\mathbf{K}}_{ij} |\mathbf{v}_j - \mathbf{v}_i|^{2(p-1)}.
\end{aligned}$$

In turn,

$$\begin{aligned}
\mathbb{P}\left(\|I_n \mathbf{Z}\|_{L^2(\Omega)} \geq \varepsilon\right) &\leq (\varepsilon^2 \rho_n n)^{-1} \frac{1}{n^2} \sum_{i,j\in[n]} \overset{\wedge}{\mathbf{K}}_{ij} |\mathbf{v}_j - \mathbf{v}_i|^{2(p-1)} \\
&= (\varepsilon^2 \rho_n n)^{-1} \int_{\Omega^2} \hat{K}_n(\boldsymbol{x}, \boldsymbol{y}) |\bar{v}_n(\boldsymbol{y}) - \bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{y} d\boldsymbol{x}.
\end{aligned}$$

If the condition (a) holds, then by the symmetry of the kernel, Jensen inequality and Hölder inequality, one gets

$$\begin{aligned}
\int_{\Omega^2} \hat{K}_n(\boldsymbol{x}, \boldsymbol{y}) |\bar{v}_n(\boldsymbol{y}) - \bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{y} d\boldsymbol{x} &\leq 4 \int_{\Omega^2} \hat{K}_n(\boldsymbol{x}, \boldsymbol{y}) |\bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{y} d\boldsymbol{x} \\
&\leq 4 \|\hat{K}_n\|_{L^{\infty,1}(\Omega^2)} \int_{\Omega} |\bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{x} \\
&\leq 4 \|\hat{K}_n\|_{L^{\infty,1}(\Omega^2)} \|\bar{v}_n\|_{L^2(\Omega)}^{2(p-1)}.
\end{aligned}$$

Under the condition (b), by the symmetry of the kernel and Jensen inequality again, we have

$$\begin{aligned}
\int_{\Omega^2} \hat{K}_n(\boldsymbol{x}, \boldsymbol{y}) |\bar{v}_n(\boldsymbol{y}) - \bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{y} d\boldsymbol{x} &\leq 2^{2(p-1)} \int_{\Omega^2} \hat{K}_n(\boldsymbol{x}, \boldsymbol{y}) |\bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{y} d\boldsymbol{x} \\
&\leq 2^{2(p-1)} \|\hat{K}_n\|_{L^{\infty,1}(\Omega^2)} \|\bar{v}_n\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}.
\end{aligned}$$

Similarly, under condition (c), we have

$$\begin{aligned}
\int_{\Omega^2} \hat{K}_n(\boldsymbol{x}, \boldsymbol{y}) |\bar{v}_n(\boldsymbol{y}) - \bar{v}_n(\boldsymbol{x})|^{2(p-1)} d\boldsymbol{y} d\boldsymbol{x} &\leq 2^{2(p-1)} \|\bar{v}_n\|_{L^\infty(\Omega)}^{2(p-1)} \|\hat{K}_n\|_{L^1(\Omega^2)} \\
&\leq 2^{2(p-1)} \|\hat{K}_n\|_{L^{\infty,1}(\Omega^2)} \|\bar{v}_n\|_{L^\infty(\Omega)}^{2(p-1)}.
\end{aligned}$$

Since $\|\hat{K}_n\|_{L^{\infty,1}(\Omega^2)} \leq \|K\|_{L^{\infty,1}(\Omega^2)}$ (see (2.1.10) in Lemma 2.1.27), we have

$$\mathbb{P}\left(\|I_n \mathbf{Z}\|_{L^2(\Omega)} \geq \varepsilon\right) \leq C(\varepsilon^2 \rho_n n)^{-1} \|K\|_{L^{\infty,1}(\Omega^2)},$$

where

$$C = \begin{cases} 4 \sup_n \|\bar{v}_n\|_{L^2(\Omega)}^{2(p-1)}, & \text{under (a)}, \\ 2^{2(p-1)} \sup_n \|\bar{v}_n\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}, & \text{under (b)}, \\ 2^{2(p-1)} \sup_n \|\bar{v}_n\|_{L^\infty(\Omega)}^{2(p-1)}, & \text{under (c)}, \end{cases}$$

and $C < +\infty$ thanks to Lemma 3.4.13. Taking $\varepsilon = \left(\frac{C\|K\|_{L^{\infty,1}(\Omega^2)}}{(\rho_n n)^\beta}\right)^{1/2}$, we get

$$\mathbb{P}\left(\|I_n \mathbf{Z}\|_{L^2(\Omega)} \geq \varepsilon\right) \leq \frac{1}{(\rho_n n)^{1-\beta}}.$$

Plugging the latter into (3.5.7) completes the proof. □

Remark 3.5.2. Lemma 4.5.1 gives a deviation bound which holds with a controlled probability. On may ask if a claim with probability 1 could be afforded. A naive and straightforward approach would be to invoke the Borel-Cantelli lemma as done in [89, Remark 3.4(iv)] for the case of graphons. But this argument does not apply to the more complex setting of L^q-graphons given that the probability of success in the statement Lemma 4.5.1 does not converge sufficiently fast. This is not even possible to make faster as ρ_n has to converge to 0. Thus, it is not clear at this stage whether this is even possible to achieve or not. We leave this to a future research.

We finally obtain the following error bound on fully discretized problems on sparse random graphs.

Theorem 3.5.3. *Suppose that $p \in]1, +\infty[$. Let u be a solution of ($\mathcal{P}$) with kernel K and data (f, g), and $\{\mathbf{u}^k\}_{k\in[N]}$ is the sequence generated by ($\mathcal{P}_p^{\mathrm{TDB},\mathbf{G}}$) with $\mathbf{K} = P_n K$, $\mathbf{g} = P_n g$, $\mathbf{f}^k = \tau_k^{-1}\int_{t_{k-1}}^{t_k} P_n f(\cdot, t)dt$. Assume that (f, g, K) satisfy the assumptions of Theorem 3.4.16, and that those of Lemma 4.5.1 also hold.*

1. *For any $\beta \in]0, 1[$, with probability at least $1 - (\rho_n n)^{-(1-\beta)}$,*

$$\sup_{k\in[N],t\in]t_{k-1},t_k]} \|I_n\mathbf{u}^k - u(\cdot,t)\|_{L^2(\Omega)} \leq \exp(T/2)\Bigg(\|I_n P_n g - g\|_{L^2(\Omega)} + \|\bar{f}_n - f\|_{L^1([0,T];L^2(\Omega))} + CT^{1/2}(\rho_n n)^{-\beta/2}$$
$$+CT^{1/2}\begin{cases} \tau^{1/(3-p)} + \|(K-\rho_n^{-1})_+\|_{L^{\infty,2}(\Omega^2)} + \|I_n P_n K - K\|_{L^{\infty,2}(\Omega^2)} & \text{under (a)} \\ \tau^{p/(2p-1)} + \|(K-\rho_n^{-1})_+\|_{L^{\infty,2}(\Omega^2)} + \|I_n P_n K - K\|_{L^{\infty,2}(\Omega^2)} & \text{under (b)} \\ \tau^{1/(3-p)} + \|(K-\rho_n^{-1})_+\|_{L^{2}(\Omega^2)} + \|I_n P_n K - K\|_{L^{2}(\Omega^2)} & \text{under (c) when } p \in]1,2] \\ \tau + \|(K-\rho_n^{-1})_+\|_{L^{2}(\Omega^2)} + \|I_n P_n K - K\|_{L^{2}(\Omega^2)} & \text{under (c) when } p \geq 2. \end{cases}\Bigg). \tag{3.5.8}$$

   for τ sufficiently small, where C is positive constant that depends only on p, g, f and K.

2. *If, moreover, $g \in L^\infty(\Omega) \cap \mathrm{Lip}(s, L^2(\Omega))$, $K \in \mathrm{Lip}(s, L^2(\Omega^2))$, and $f \in L^1([0,T]; L^\infty(\Omega)) \cap \mathrm{Lip}(s, L^2(\Omega \times [0,T]))$ then, for any $\delta \in]0, 1[$, with probability at least $1 - (\rho_n n)^{-(1-\beta)}$,*

$$\sup_{k\in[N],t\in]t_{k-1},t_k]} \|I_n\mathbf{u}^k - u(\cdot,t)\|_{L^2(\Omega)} \leq C\exp(T/2)\Bigg((1+T^{1/2})n^{-s} + T^{1/2}\|(K-\rho_n^{-1})_+\|_{L^2(\Omega^2)}$$
$$+ T^{1/2}(\rho_n n)^{-\beta/2} + T^{1/2}\left(\begin{cases} \tau^{\min(s,1/(3-p))} & \text{when } p \in]1,2] \\ \tau^s & \text{when } p \geq 2 \end{cases}\right)\Bigg). \tag{3.5.9}$$

   for τ sufficiently small, where C is positive constant that depends only on p, g, f, K and s, and $\|(K-\rho_n^{-1})_+\|_{L^2(\Omega^2)} = o(1)$. The term τ^s in the dependence on τ disappears when f is time-independent.

Proof : In view of (3.5.3), we shall use Theorem 3.4.16 to bound the second term, and a bound on the first term is provided by Lemma 4.5.1. Since $I_n\overset{\wedge}{\mathbf{K}}(x,y) \leq I_n\mathbf{K}(x,y) = I_n P_n K(x,y)$, the assumptions on $\mathbf{K}$ transfer to $\overset{\wedge}{\mathbf{K}}$, and the second term of (3.5.3) can then be bounded using (3.4.17), replacing $I_n P_n K$ there by $I_n \overset{\wedge}{\mathbf{K}}$. Observing that

$$\begin{aligned}\left\|I_n\overset{\wedge}{\mathbf{K}} - K\right\|_{L^2(\Omega^2)} &= \left\|\min(I_nP_nK, \rho_n^{-1}) - K\right\|_{L^2(\Omega^2)}\\ &\le \left\|\min(I_nP_nK, \rho_n^{-1}) - I_nP_nK\right\|_{L^2(\Omega^2)} + \left\|I_nP_nK - K\right\|_{L^2(\Omega^2)}\\ &= \left\|(I_nP_nK - \rho_n^{-1})_+\right\|_{L^2(\Omega^2)} + \left\|I_nP_nK - K\right\|_{L^2(\Omega^2)}\\ &\le \left\|(K - \rho_n^{-1})_+\right\|_{L^2(\Omega^2)} + 2\left\|I_nP_nK - K\right\|_{L^2(\Omega^2)},\end{aligned}$$

and similarly for the $L^{\infty,2}$ norm. The fact that $\left\|(K-\rho_n^{-1})_+\right\|_{L^2(\Omega^2)} = o(1)$ is because $\rho_n \to 0$ by the same argument as the end of the proof of Proposition 2.3.9. This completes the proof. □

Chapter 4

Continuum limit of the p-Laplacian Dirichlet problem on sparse graphs

Main contributions of this chapter

- Convergence of minima of integral functionals
- The continuum limits of a sequence of p-Laplacian Dirichlet problems.
- Consistency and error estimates of the Dirichlet problem.
- Applications to random graph sequences.

A paper with the content of this chapter is under preparation for submission to a journal.

Contents

In this chapter, we present a consistency analysis for the nonlocal p-Laplacian Dirichlet problem. We start by studying the Mosco-convergence and Γ-convergence of sequences of integral functions and geometric constraints. We prove the convergence of the minimizers of the nonlocal Dirichlet energies under affine geometric constraints. Thanks to the Dirichlet principle, these results are used to study sequence of nonlocal p-Laplacian boundary value problems. We study also consistency and error estimates of this problem. Relying on these error estimates, we establish nonasymptotic rate of convergence of solutions for the discrete model on sparse random graphs to the solution of the nonlocal Dirichlet problem on the continuum.

4.1 Introduction

4.1.1 Problem statement

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 1$ and $p \in]1, +\infty[$. For $n \in \mathbb{N}$, we consider the following boundary value problem

$$\begin{cases} -\Delta_p^{K_n} u = f_n, & \text{in } U_n, \\ u = g_n, & \text{on } \Gamma_n = \Omega \setminus U_n, \end{cases} \tag{$\mathcal{P}_D^n$}$$

where U_n is a subdomain of Ω, K_n is a non-negative symmetric measurable function on Ω^2, $g_n \in L^p(\Omega)$ and $f_n \in L^q(\Omega)$, whith q is the Hölder conjugate of p. Recall that, for a kernel K, Δ_p^K is the nonlocal p-Laplacian operator governed by K,

$$\Delta_p^K u(\boldsymbol{x}) \stackrel{\text{def}}{=} -\int_\Omega K(\boldsymbol{x}, \boldsymbol{y}) \left| u(\boldsymbol{y}) - u(\boldsymbol{x}) \right|^{p-2} (u(\boldsymbol{y}) - u(\boldsymbol{x})) d\boldsymbol{y}. \tag{4.1.1}$$

Recall that, our chief goal in this chapter is to study the asymptotic behaviour and the corresponding continuum limit of the sequence problems ($\mathcal{P}_D^n$).

Studying the limit of solutions to ($\mathcal{P}_D^n$) will allow us to establish consistency of numerical approximations of the nonlocal p-Laplacian Dirichlet problem

$$\begin{cases} -\Delta_p^K u = f, & \text{in } U, \\ u = g, & \text{on } \Gamma. \end{cases} \tag{$\mathcal{P}_D$}$$

K, U and Γ represent some limit objects whose meaning and form will be specified in the sequel, separately for every class of problems that we consider below. As a major illustrative example, we will study the case where ($\mathcal{P}_D^n$) are defined on a sequence of convergent K-random graphs; see Section 4.5.

We impose the following assumptions on the kernel we consider, which will be useful in order to make the statements of our result brief and clear, (for some results only a subset will be necessary):

(**A.1**) K is a nonnegative measurable function.

(**A.2**) K is symmetric, i.e., $K(\boldsymbol{x},\boldsymbol{y}) = K(\boldsymbol{y},\boldsymbol{x})$.

(**A.3**) There exist m, $r_0 > 0$ such that $\left|K(\boldsymbol{x},\boldsymbol{y})\right| \geq m\chi_{[0,r_0[}(\left\|\boldsymbol{x}-\boldsymbol{y}\right\|)$ for all $\boldsymbol{x}$, $\boldsymbol{y} \in \Omega$.

(**A.4**) K belongs to $L^\infty(\Omega^2)$.

(**A.5**) $\sup_{\boldsymbol{x}\in\Omega} \int_\Omega \left|K(\boldsymbol{x},\boldsymbol{y})\right| d\boldsymbol{y} < +\infty$.

When the kernel is such that $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$, where $J : \mathbb{R}^N \to \mathbb{R}$, then (**A.1**)-(**A.5**) read:

(**A'.1**) J is a nonnegative measurable function.

(**A'.2**) J is symmetric, i.e., $J(\boldsymbol{x}) = J(-\boldsymbol{x})$.

(**A'.3**) There exist m, $r_0 > 0$ such that $\left|J(\boldsymbol{x})\right| \geq m\chi_{[0,r_0[}(\left\|\boldsymbol{x}\right\|)$ for all $\boldsymbol{x} \in \Omega$.

(**A'.4**) J belongs to $L^\infty(\Omega - \Omega)$.

(**A'.5**) $\int_{\Omega-\Omega} \left|J(\boldsymbol{x})\right| d\boldsymbol{x} < +\infty$.

Recall that $\Omega - \Omega$ is the Minkowski sum of Ω and $-\Omega$.

Let K_n, K be a sequence of measurable functions in $L^{\infty,p}(\Omega^2)$. We say that the sequence $\{K_n,\ K : n \in \mathbb{N}\}$ satisfies hypothesis ($\mathcal{A}_{ker}$) if the following hold.

($\mathcal{A}_{ker}$) The functions K_n, $n \in \mathbb{N}$ and K are symmetric (i.e. satisfy (**A.2**)), and $\{K_n\}_{n\in\mathbb{N}}$ converges pointwise to K almost everywhere on Ω^2 and $\{K_n\}_{n\in\mathbb{N}}$ converges strongly to K in $L^{\infty,p}(\Omega^2)$.

We say that the sequence $\{K_n,\ K :\ n \in \mathbb{N}\}$ satisfies hypothesis ($\mathcal{B}_{ker}$) if the following hold.

($\mathcal{B}_{ker}$) The sequence $\{K_n,\ K :\ n \in \mathbb{N}\}$ is uniformly bounded in $L^\infty(\Omega^2)$, and $\{K_n\}_{n\in\mathbb{N}}$ converges pointwise to K almost everywhere on Ω^2.

If the kernels K, K_n, $n \in \mathbb{N}$ are such that $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$, $K_n(\boldsymbol{x},\boldsymbol{y}) = J_n(\boldsymbol{x}-\boldsymbol{y})$, where J, $J_n : \mathbb{R}^N \to \mathbb{R}$, $n \in \mathbb{N}$, the assumptions ($\mathcal{A}_{ker}$) and ($\mathcal{B}_{ker}$) read respectively

($\mathcal{A}'_{rad}$) The functions J_n, $n \in \mathbb{N}$ and J are symmetric (i.e. satisfy (**A'.2**)), and $\{J_n\}_{n\in\mathbb{N}}$ converges pointwise to J almost everywhere on $\Omega - \Omega$ and $\{J_n\}_{n\in\mathbb{N}}$ converges strongly to J in $L^p(\Omega - \Omega)$.

($\mathcal{B}'_{rad}$) The sequence $\{J_n,\ J :\ n \in \mathbb{N}\}$ is uniformly bounded in $L^\infty(\Omega - \Omega)$, and $\{J_n\}_{n\in\mathbb{N}}$ converges pointwise to J almost everywhere on $\Omega - \Omega$.

Recall that, for a kernel K, the nonlocal gradient operator ∇_K^{NL} is given by (1.2.3). A key intermediate step to achieve our goal is to use the Dirichlet principle and transform ($\mathcal{P}_D^n$) into an equivalent sequence of variational problems

$$\min_{u\in L^p_{g_n}(\Omega,U_n)} F_n(\nabla_{K_n}^{\mathrm{NL}} u) + \int_\Omega f_n(\boldsymbol{x})u(\boldsymbol{x})d\boldsymbol{x}, \tag{$\mathcal{VP}_n$}$$

where F_n is an integral functional to be made precise later, $f \in L^q(\Omega)$ and

$$L^p_{g_n}(\Omega,U_n) \stackrel{\text{def}}{=} \left\{u \in L^p(\Omega) :\ u = g_n \text{ on } \Gamma_n \stackrel{\text{def}}{=} \Omega \setminus U_n\right\}.$$

To make this asymptotic analysis precise, we use the notation and methods of Γ-convergence and Mosco-convergence of De Giorgi and Mosco respectively (see Chapter 2, [58, 34, 24, 17, 32]). We will in particular show that ($\mathcal{VP}_n$) has a Γ-limit which takes the form

$$\min_{u\in L^p_g(\Omega,U)} F(\nabla^{\rm NL}_K u) + \int_\Omega f(\boldsymbol{x})u(\boldsymbol{x})d\boldsymbol{x} \tag{$\mathcal{VP}$}$$

where $g \in L^p(\Omega)$, $f \in L^q(\Omega)$ and F is an appropriate integral functional.

4.1.2 Organization of the chapter

The remainder of the chapter is organised as follows. In Section 4.2, we study the Mosco-convergence of sequences of integral functions and convex sets. In this section, we prove our first main results on the convergence of the minimizers to ($\mathcal{VP}_n$) under affine geometric constraints. These result is then used to study sequences of nonlocal p-Laplacian Dirichlet problems ($\mathcal{P}^n_D$) in Section 4.3. In Section 4.4, we study the relation between the solution of problem ($\mathcal{P}_D$) and the solution of its discretization. We dedicate Section 4.5 to study the continuum limit of the discrete random model. The primal-dual splitting algorithm to solve the discretized problems in graphs is described in Section 4.6 (see also Chapter 7 for a general treatment) and some numerical results are reported to illustrate our theoretical findings.

4.2 Convergence of minima of integral functionals

This section is devoted to study the well-posedness of the problem ($\mathcal{VP}$), we study also the Mosco-convergence and Γ-convergence of the sequence of the energy functionals and the sequence of geometry constraints given in ($\mathcal{VP}_n$).

4.2.1 Mosco-convergence of convex functionals

Let $\mathcal{U}$ be a bounded domain of $\mathbb{R}^N$, $N \geq 1$ and $p \in]1, +\infty[$. We consider the integral functional

$$\begin{aligned} F:\ \ & L^p(\mathcal{U}) \to \mathbb{R} \\ & v \mapsto \int_{\mathcal{U}} f(\boldsymbol{x}, v(\boldsymbol{x}))d\boldsymbol{x}, \end{aligned} \tag{4.2.1}$$

where $f : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the following requirements:

(H.1) for every $s \in \mathbb{R}$, the function $f(\cdot, s)$ is Lebesgue measurable on $\mathcal{U}$.

(H.2) for a.e. $\boldsymbol{x} \in \mathcal{U}$, the function $f(\boldsymbol{x}, \cdot)$ is convex on $\mathbb{R}$.

(H.3) there exist $C_2 \geq C_1 > 0$ and a positive function $a \in L^1(\mathcal{U})$ such that

$$-a(\boldsymbol{x}) + C_1|s|^p \leq f(\boldsymbol{x}, s) \leq a(\boldsymbol{x}) + C_2(|s|^p + 1), \tag{4.2.2}$$

for a.e $\boldsymbol{x} \in \mathcal{U}$ and for all $s \in \mathbb{R}$.

We denote by $\mathcal{F}(a, C_1, C_2, p)$ the set of all functional F of the form (4.2.1) where the corresponding integrands satisfy assumptions (H.1), (H.2) and (H.3) for the same function $a \in L^1(\mathcal{U})$ and same constant $C_2 \geq C_1 > 0$.

We denote by $\mathcal{F}'(a, C_1, C_2, p)$ the set of all functionals $F \in \mathcal{F}(a, C_1, C_2, p)$ such that assumption (H.2) of the corresponding integrand is replaced by

(H.2') for a.e. $\boldsymbol{x} \in \mathcal{U}$, the function $f(\boldsymbol{x}, \cdot)$ is strictly convex on $\mathbb{R}$.

Let F, F_n, $n \in \mathbb{N}$ be integral functionals taking the form of (4.2.1) with f, f_n, $n \in \mathbb{N}$ the corresponding integrands. We say that the sequence $\{F,\ F_n :\ n \in \mathbb{N}\}$ satisfies hypothesis ($\mathcal{H}_{seq}$) if the following holds.

($\mathcal{H}_{seq}$) There exist a positive function $a \in L^1(\mathcal{U})$ and $C_2 \geq C_1 > 0$ such that F, F_n, $\in \mathcal{F}(a, C_1, C_2, p)$, $n \in \mathbb{N}$, and for every $s \in \mathbb{R}$, $\{f_n(\cdot, s))\}_{n\in\mathbb{N}}$ converges to $f(\cdot, s)$ pointwise a.e. on $\mathcal{U}$.

By standard convex analysis arguments, we obtain the following properties of functions of these classes of functional.

Proposition 4.2.1. *Let $F \in \mathcal{F}(a, C_1, C_2, p)$ (reps. $F \in \mathcal{F}'(a, C_1, C_2, p)$) where $C_2 \geq C_1 > 0$ and a positive function $a \in L^1(\mathcal{U})$. Then F satisfies the following properties:*

(i) F is convex (resp. strictly convex) and continuous on $L^p(\mathcal{U})$;

(ii) F is weakly lower semicontinuous on $L^p(\mathcal{U})$;

(iii) F is coercive on $L^p(\mathcal{U})$ induced by the weak topology.

Let us announce a first result Mosco-convergence for sequences in these classes.

Theorem 4.2.2. *Let $\{F,\ F_n :\ n \in \mathbb{N}\}$ be a sequence of integral functionals given by (4.2.1) which satisfies $(\mathcal{H}_{seq})$. Then*

(i) $\{F_n\}$ Mosco-converges to F on $L^p(\mathcal{U})$;

(ii)

$$\min_{L^p(\mathcal{U})} F = \lim_n \min_{L^p(\mathcal{U})} F_n.$$

Moreover, if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence such that $\lim_n F_n(u_n) = \lim_n \inf_{L^p(\mathcal{U})} F_n$, then $\{u_n\}_{n\in\mathbb{N}}$ is weakly precompact in $L^p(\mathcal{U})$ and every weak cluster point of $\{u_n\}_{n\in\mathbb{N}}$ is a minimum point for F.

Proof : It is enough to show the point (i). The point (ii) is a consequence of (i), Theorem 2.1.13, Proposition 4.2.1 and the equi-coercivity of the sequence $\{F_n\}_{n\in\mathbb{N}}$ which is a trivial result of Proposition 2.1.12 and the growth condition (H.3) of the integrands.
By the dominated convergence theorem the sequence $\{F_n(u)\}_{n\in\mathbb{N}}$ converges to $F(u)$ for every $u \in L^p(\mathcal{U})$. The conclusion is achieved if we prove (ii) of Definition 2.1.14.

On the other hand, we have F, F_n, $n \in \mathbb{N}$ are closed convex functions, then by Theorem 2.1.15, the proof of the Mosco-convergence amounts to prove that the sequence $\{F_n\}_{n\in\mathbb{N}}$ is equi-lsc. By the assumption $(\mathcal{H}_{seq})$ we have that

$$|F_n(u)| \leq \|a\|_{L^1(\mathcal{U})} + C_2(R^p + |\mathcal{U}|),$$

for all $u \in \mathbb{B}_R \subset L^p(\mathcal{U})$ and all $n \in \mathbb{N}$. Then the sequence $\{F_n\}_{n\in\mathbb{N}}$ is locally uniformly Lipschitz, thanks to Proposition 2.1.9, in particular it is equi-lsc.

□

4.2.2 Mosco-convergence of convex sets

Throughout the chapter. We denote by

$$L^p_g(\Omega, U) = \{u \in L^p(\Omega) :\ u = g \text{ on } \Gamma = \Omega \setminus U\}, \tag{4.2.3}$$

where $g \in L^p(\Omega)$ and $U \subset\subset \Omega$ be a sub-domain such that $U + \mathbb{B}(0, r) \subset \Omega$ for some $r > 0$.

Proposition 4.2.3. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^p(\Omega)$ and $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of sub-domains of Ω. Assume that*

(i) the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to g in $L^p(\Omega)$.

(ii) the sequence $\{|U_n \Delta U|\}_{n\in\mathbb{N}}$ tends to 0 as $n \to +\infty$; where $U_n \Delta U$ is the symmetric difference of U_n and U.

Then $L^p_{g_n}(\Omega, U_n)$ Mosco-converges to $L^p_g(\Omega, U)$ in $L^p(\Omega)$.

PROOF : Let $f_n \in L^p_{g_n}(\Omega, U_n)$, $n \in \mathbb{N}$, such that $f_n \rightharpoonup f$ in $L^p(\Omega)$ for some $f \in L^p(\Omega)$. It is easy to see that

$$\int_\Omega |f_n\chi_{\Gamma_n} - g\chi_\Gamma| d\boldsymbol{x} \le \int_\Omega \left[|g_n - g| + |g|\chi_{\Gamma\Delta\Gamma_n}\right] d\boldsymbol{x}.$$

By assumption the terms in the right-hand side of the above inequality tend to 0 and so $f_n\chi_{\Gamma_n} \to g\chi_\Gamma$ in $L^1(\Omega)$, hence $f = g$ on Γ, i.e. $f \in L^p_g(\Omega, U)$.
Now, let $f \in L^p_g(\Omega, U)$. For all $n \in \mathbb{N}$ we set

$$f_n = \begin{cases} f, & \text{on } U_n, \\ g_n, & \text{on } \Gamma_n. \end{cases}$$

By construction, we have $f_n \in L^p_{g_n}(\Omega, U_n)$, $n \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned} |f_n - f| &= |(f_n - f)\cdot\chi_{\Gamma_n}| \\ &\le |(g_n - g)\cdot\chi_{\Gamma_n} + (g - f)\cdot\chi_{\Gamma_n\Delta\Gamma}|. \end{aligned}$$

And so

$$\|f_n - f\|_{L^p(\Omega)} \le \|g_n - g\|_{L^p(\Omega)} + \|(g - f)\cdot\chi_{\Gamma_n\Delta\Gamma}\|_{L^p(\Omega)}.$$

The terms in the right-hand side of the above inequality tend to zero, and thus $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to f in $L^p(\Omega)$. □

Now, we will approximate $L^p_g(\Omega, U)$ by a sequence of finite affine subspaces.
Throughout this chapter, we denote, for $n \in \mathbb{N}^\star$,

$$V_n(\Omega) = \{\boldsymbol{x} \in \frac{1}{n}\mathbb{Z}^d : \Omega^{(n)}_{\boldsymbol{x}} \subset \Omega\}, \tag{4.2.4}$$

where $\Omega^{(n)}_{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{n}[0,1[^d$, and

$$\mathbb{A}(V_n(\Omega)) = \{u \in L^1(\Omega) : u = \sum_{\boldsymbol{x}\in V_n(\Omega)} \alpha_{\boldsymbol{x}}\chi_{\Omega^{(n)}_{\boldsymbol{x}}},\ \alpha_{\boldsymbol{x}} \in \mathbb{R}\}.$$

We fix $n \in \mathbb{N}^\star$. Set

$$U_n = \bigcup_{\Omega^{(n)}_{\boldsymbol{x}} \subset U :\, \boldsymbol{x}\in V_n(\Omega)} \Omega^{(n)}_{\boldsymbol{x}} \ \textit{and}\ \Gamma_n = \Omega \setminus U_n,$$

$$U'_n = \bigcup_{\Omega^{(n)}_{\boldsymbol{x}} \cap U \neq \emptyset :\, \boldsymbol{x}\in V_n(\Omega)} \Omega^{(n)}_{\boldsymbol{x}} \ \textit{and}\ \Gamma'_n = \Omega \setminus U'_n.$$

For $g_n \in \mathbb{A}(V_n(\Omega))$, we denote

$$\mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n) \stackrel{\text{def}}{=} \{u \in \mathbb{A}(V_n(\Omega)) : u = g_n \text{ on } \Gamma_n\}, \tag{4.2.5}$$

$$\mathbb{A}'_{g_n}(V_n(\Omega), \Gamma'_n) \stackrel{\text{def}}{=} \{u \in \mathbb{A}(V_n(\Omega)) : u = g_n \text{ on } \Gamma'_n\}. \tag{4.2.6}$$

We have the following approximation of $L^p_g(\Omega, U)$.

Proposition 4.2.4. *Let g_n, Γ_n, Γ'_n, $\mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n)$, $\mathbb{A}'_{g_n}(V_n(\Omega), \Gamma'_n)$, as above. Assume that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to some function g in $L^p(\Omega)$. Then the sequences $\{\mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n)\}_{n\in\mathbb{N}}$ and $\{\mathbb{A}'_{g_n}(V_n(\Omega), \Gamma'_n)\}_{n\in\mathbb{N}}$ Mosco-converge to $L^p_g(\Omega, U)$ in $L^p(\Omega)$.*

PROOF : We show that $\{\mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n)\}_{n\in\mathbb{N}}$ Mosco-converges to $L^p_g(\Omega, U)$ in $L^p(\Omega)$, by similar argument we show the Mosco convergence of $\{\mathbb{A}'_{g_n}(V_n(\Omega), \Gamma'_n)\}_{n\in\mathbb{N}}$.

It is easy to see that $|U_n \Delta U| \to 0$. Thus, as in the proof of Proposition 4.2.3 any weakly convergent sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega)$, with $f_n \in \mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n)$, is such that its limit belongs to $L^p_g(\Omega, U)$.
Now, let $f \in L^p_g(\Omega, U)$. For $n \in \mathbb{N}^\star$, we consider

$$\hat{f}_n = I_n P_n f$$

$$= \sum_{\boldsymbol{x}\in V_n(\Omega)} \frac{1}{|\Omega_{\boldsymbol{x}}^{(n)}|} \int_{\Omega_{\boldsymbol{x}}^{(n)}} f(\boldsymbol{y})d\boldsymbol{y} \cdot \chi_{\Omega_{\boldsymbol{x}}^{(n)}},$$

and define

$$f_n(\boldsymbol{y}) = \begin{cases} g_n(\boldsymbol{y}), & \textit{if } \boldsymbol{y} \in \Gamma_n \\ \hat{f}_n(\boldsymbol{y}), & \textit{otherwise.} \end{cases}$$

By Lebesgue differentiation Theorem we have $\{\hat{f}_n\}_{n\in\mathbb{N}}$ converges pointwise to f a.e. on Ω, since $\|\hat{f}_n\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}$ for all $n \in \mathbb{N}$ we conclude by Riesz-Scheffé Lemma [102, Lemma 2] that $\{\hat{f}_n\}_{n\in\mathbb{N}}$ converges strongly to f in $L^p(\Omega)$. By construction, we have $f_n \in \mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n)$, for all $n \in \mathbb{N}$, and (f_n) converges strongly to f in $L^p(\Omega)$. Indeed, we have

$$\begin{aligned} |f - f_n| &\leq |f - \hat{f}_n| + |\hat{f}_n - f_n| \\ &\leq |f - \hat{f}_n| + |(\hat{f}_n - f_n) \cdot \chi_{\Gamma_n}| \\ &\leq |f - \hat{f}_n| + |(\hat{f}_n - g_n) \cdot \chi_{\Gamma_n}| \\ &\leq |f - \hat{f}_n| + |(\hat{f}_n - f) \cdot \chi_{\Gamma_n}| + |(f-g) \cdot \chi_{\Gamma_n}| + |(g - g_n) \cdot \chi_{\Gamma_n}| \\ &\leq 2|f - \hat{f}_n| + |g - g_n| + |(f-g) \cdot \chi_{\Gamma_n \Delta \Gamma}|, \end{aligned}$$

and so

$$\|f - f_n\|_{L^p(\Omega)} \leq 2\|f - \hat{f}_n\|_{L^p(\Omega)} + \|g - g_n\|_{L^p(\Omega)} + \|(f-g) \cdot \chi_{\Gamma_n \Delta \Gamma}\|_{L^p(\Omega)}.$$

Hence we get the result since the terms in the right-hand side of the above inequality tend to zero. □

4.2.3 Convergence of minimum problems

In the rest of the chapter, we assume that $N = 2d$ and $\mathcal{U} = \Omega \times \Omega$. Let $L_g^p(\Omega, U)$ the affine space given by (4.2.3), where $g \in L^p(\Omega)$ and $U \subset\subset \Omega$ a sub-domain such that $U + \mathbb{B}(0, r) \subset \Omega$ for some $r > 0$.

Lemma 4.2.5. *Let $K \in L^1(\Omega^2)$ satisfies (A.1)-(A.3) and let $g \in L^p(\Omega)$.*

(i) *We have the following nonlocal Poincaré's inequality. There exists $\lambda = \lambda(K, \Omega, \Gamma, r_0) > 0$ such that*

$$\lambda \int_\Omega |u(\boldsymbol{x})|^p d\boldsymbol{x} \leq \int_{\Omega^2} K(\boldsymbol{x}, \boldsymbol{y}) |u(\boldsymbol{y}) - u(\boldsymbol{x})|^p d\boldsymbol{x}d\boldsymbol{y} + \int_\Gamma |g(\boldsymbol{x})|^p d\boldsymbol{x}, \tag{4.2.7}$$

for all $u \in L_g^p(\Omega, U)$.

(ii) *Moreover, if $K \in L^{\infty,1}(\Omega^2)$, then there exists a positive constant $C > 0$ such that*

$$\|v - u\|_{L^p(\Omega)}^{p/\max(1,\frac{2}{p})} \leq C \int_\Omega \left(\Delta_p^K v - \Delta_p^K u\right)(\boldsymbol{x})\,(v-u)\,(\boldsymbol{x})d\boldsymbol{x}, \tag{4.2.8}$$

for all $u,\ v \in L_g^p(\Omega, U)$,

Proof :

(i) See [93, Lemma 3.5].

(ii) Combining Proposition 3.2.2 (v) and the nonlocal Poincaré's inequality, we get the desired result.

□

Proposition 4.2.6. *Let $F \in \mathcal{F}(a, C_1, C_2, p)$, $g \in L^p(\Omega)$ and let K be a function in $L^{\infty,p}(\Omega^2)$. We assume that K satisfies (A.2) or (A.4).*

Consider the following function $\mathcal{G} : L^p(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{G} = F \circ \nabla_K^{\mathrm{NL}}.$$

Then

(i) *$\mathcal{G}$ is convex and continuous on $L^p(\Omega)$. In particular, $\mathcal{G}$ is weakly lower semicontinuous on $L^p(\Omega)$.*
Moreover, if K satisfies (A.3). Then

(ii) *$\mathcal{G}$ is coercive on $L^p_g(\Omega, U)$ induced by the weakly topology.*

(iii) *$\mathcal{G}$ attains its minimizer on $L^p_g(\Omega, U)$.*

(iv) *if $F \in \mathcal{F}'(a, C_1, C_2, p)$, then $\mathcal{G}$ is strictly convex on $L^p_g(\Omega, U)$. In particular, $\mathcal{G}$ has a unique minimizer on $L^p_g(\Omega, U)$.*

Proof : The convexity of the point (i) is evident, therefore, by Proposition 2.1.9, it is enough to show that $\mathcal{G}$ is locally bounded below on $L^p(\Omega)$, since it is proper. Let $R > 0$, for all $u \in L^p(\Omega)$ such that $\|u\|_{L^p(\Omega)} \leq R$, we have

$$\|\nabla^{\mathrm{NL}}_K u\|_{L^p(\Omega^2)} \leq \begin{cases} 2\|K\|_{L^{\infty,p}(\Omega^2)} R, & \textit{if } K \textit{ satisfies } (\boldsymbol{A.2}), \\ 2\|K\|_{L^{\infty}(\Omega^2)} R, & \textit{if } K \textit{ satisfies } (\boldsymbol{A.4}). \end{cases}$$

On the other hand, by assumption (H.3) and the definition of $\mathcal{G}$, we get that

$$\mathcal{G}(u) = F(\nabla^{\mathrm{NL}}_K u) \leq \begin{cases} \|a\|_{L^1(\Omega^2)} + C_2\left(|\Omega^2| + 2^p\|K\|^p_{L^{\infty,p}(\Omega^2)} R^p\right), & \textit{if } K \textit{ satisfies } (\boldsymbol{A.2}), \\ \|a\|_{L^1(\Omega^2)} + C_2\left(|\Omega^2| + 2^p|\Omega|\|K\|^p_{L^{\infty}(\Omega^2)} R^p\right), & \textit{if } K \textit{ satisfies } (\boldsymbol{A.4}). \end{cases}$$

For (ii), it is enough to show that for all $t > 0$ the set $\mathcal{A}_t = \{\mathcal{G} \leq t\} \cap L^p_g(\Omega, U)$ is bounded in $L^p(\Omega)$, since $\mathcal{G}$ is weakly lsc, thanks the point (i). Let $u \in \mathcal{A}_t$, there exists $n \in \mathbb{N}$ such that $u \in L^p_g(\Omega, U)$, hence

$$\begin{aligned} t &\geq \mathcal{G}(u) \\ &\geq C_1 \int_{\Omega^2} |K(\boldsymbol{x}, \boldsymbol{y})|^p |u(\boldsymbol{x}) - u(\boldsymbol{y})|^p d\boldsymbol{x} d\boldsymbol{y} - \int_{\Omega^2} a(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} \\ &\geq C_1 m^p \int_{\Omega^2} \chi_{[0,r_0[}(\|\boldsymbol{x} - \boldsymbol{y}\|) |u(\boldsymbol{x}) - u(\boldsymbol{y})|^p d\boldsymbol{x} d\boldsymbol{y} - \|a\|_{L^1(\Omega^2)} \end{aligned}$$

By Poincaré inequality (see Lemma 4.2.5) there exists $\lambda(r, r_0, \Omega) > 0$ such that

$$\lambda \int_{\Omega} |v(\boldsymbol{x})|^p d\boldsymbol{x} \leq \int_{\Omega^2} \chi_{[0,r_0[}(\|\boldsymbol{x} - \boldsymbol{y}\|) |v(\boldsymbol{x}) - v(\boldsymbol{y})|^p d\boldsymbol{x} d\boldsymbol{y} + \int_{\Omega} |g(\boldsymbol{x})|^p d\boldsymbol{x} \tag{4.2.9}$$

for all $v \in L^p_g(\Omega, U)$. Then, we have

$$\int_{\Omega} |u(\boldsymbol{y})|^p d\boldsymbol{y} \leq \frac{\lambda^{-1}}{C_1 m^p}(t + \|a\|_{L^1(\Omega^2)}) + \lambda^{-1} \int_{\Gamma} |g(\boldsymbol{x})|^p d\boldsymbol{x}.$$

Now, we show (iii). Let $\iota_{L^p_g(\Omega,U)}$ be the indicator function of $L^p_g(\Omega, U) \subset L^p(\Omega)$. By the point (ii) we get that $\mathcal{G}(\cdot) + \iota_{L^p_g(\Omega,U)}(\cdot)$ is is coercive. Since this function is weakly lower semicontinuous by closedness of $L^p_g(\Omega, U)$ and weakly lower semi-continuous of $\mathcal{G}$, hence $\mathcal{G}$ has a minimizer in $L^p_g(\Omega, U)$. For (iv), it is enough to show that $\mathcal{G}$ is strictly convex. Assume that $\mathcal{G}$ is not. Let $u, v \in L^p_g(\Omega, U)$ with $u \neq v$ such that

$$\mathcal{G}(tu + (1-t)v) = t\mathcal{G}(u) + (1-t)\mathcal{G}(v), \quad \textit{for } t \in]0, 1[$$

since F is strictly convex we have $\nabla^{\mathrm{NL}}_K u = \nabla^{\mathrm{NL}}_K v$, and so $u - v \in \mathrm{Ker}(\nabla^{\mathrm{NL}}_K) \cap L^p_0(\Omega, U) = \{0\}$. Contradiction. □

Theorem 4.2.7. *Let F, F_n, $n \in \mathbb{N}$ be a sequence of integral functionals given by (4.2.1) which satisfy ($\mathcal{H}_{seq}$). Let K_n, K be a sequence functions in $L^{\infty,p}(\Omega^2)$ such that $\{K_n, K : n \in \mathbb{N}\}$ satisfy ($\mathcal{A}_{ker}$) or ($\mathcal{B}_{ker}$). Let $\mathcal{G}$, $\mathcal{G}_n : L^p(\Omega) \to \mathbb{R}$ be a sequence of functions defined by*

$$\mathcal{G} = F \circ \nabla^{\mathrm{NL}}_K \quad \textit{and} \quad \mathcal{G}_n = F_n \circ \nabla^{\mathrm{NL}}_{K_n}, \quad n \in \mathbb{N}.$$

Then, the sequence $(\mathcal{G}_n)$ Mosco-converges to $\mathcal{G}$ on $L^p(\Omega)$.

Let us first state the following result that will be useful for proving the above theorem. This result is an immediate consequence of Proposition 2.1.9 and Arzelà–Ascoli theorem.

Lemma 4.2.8. *Let $\{F,\ F_n : n \in \mathbb{N}\}$ as in Theorem 4.2.7, and let $\{f,\ f_n : n \in \mathbb{N}\}$ the corresponding integrands. Then, for almost all $(\boldsymbol{x}, \boldsymbol{y}) \in \Omega^2$, the sequence $\{f_n((\boldsymbol{x}, \boldsymbol{y}), \cdot)\}_{n\in\mathbb{N}}$ converges uniformly to $f((\boldsymbol{x}, \boldsymbol{y}), \cdot)$ on every compact subset of $\mathbb{R}$. In particular, the sequence $\{f_n(\cdot, s_n)\}_{n\in\mathbb{N}}$ converges to $f(\cdot, s)$ for every sequence (s_n) converging to s and almost everywhere on Ω^2.*

Proof : [of Theorem 4.2.7] First, we prove the pointwise convergence of $\mathcal{G}_n$ to $\mathcal{G}$ on $L^p(\Omega)$, under the both conditions on the sequence of kernels. Let $u \in L^p(\Omega)$, we have $\nabla^{\mathrm{NL}}_{K_n} u$ converge pointwise to $\nabla^{\mathrm{NL}}_{K} u$ almost everywhere on Ω^2, since K_n are (under the two conditions). By Lemma 4.2.8, we have that $\{f_n((\boldsymbol{x}, \boldsymbol{y}), \nabla^{\mathrm{NL}}_{K_n}(\boldsymbol{x}, \boldsymbol{y}))\}_{n\in\mathbb{N}}$ converges to $f_n((\boldsymbol{x}, \boldsymbol{y}), \nabla^{\mathrm{NL}}_{K}(\boldsymbol{x}, \boldsymbol{y}))$ for almost all $(\boldsymbol{x}, \boldsymbol{y})$ in Ω^2. On the other hand, we have

$$\left|f_n((\boldsymbol{x}, \boldsymbol{y}), \nabla^{\mathrm{NL}}_{K_n} u(\boldsymbol{x}, \boldsymbol{y}))\right| \le a(\boldsymbol{x}, \boldsymbol{y}) + C_2 \left(\sup_n \left\|K_n\right\|^p_{L^\infty(\Omega^2)} \left|\nabla^{\mathrm{NL}} u(\boldsymbol{x}, \boldsymbol{y})\right|^p + 1 \right),$$

under the condition $(\mathcal{B}_{ker})$, hence $\{f_n(\cdot, \nabla^{\mathrm{NL}}_{K_n} u \cdot)\}_{n\in\mathbb{N}}$ is equi-integrable, and by Vitali theorem we get the convergence of $\mathcal{G}_n(u)$ to $\mathcal{G}(u)$. Let us turn to the second case i.e. when the kernels satisfy condition $(\mathcal{A}_{ker})$. We have $\{\nabla^{\mathrm{NL}}_{K_n} u\}_{n\in\mathbb{N}}$ converges strongly to $\nabla^{\mathrm{NL}}_{K_n} u$ in $L^p(\Omega^2)$. Indeed, by

$$\begin{aligned}
\left\|\nabla^{\mathrm{NL}}_{K_n} u - \nabla^{\mathrm{NL}}_{K} u\right\|^p_{L^p(\Omega^2)} &\le 2^{p-1} \int_{\Omega^2} \left|K_n(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{y})\right|^p (\left|u(\boldsymbol{y})\right|^p + \left|u(\boldsymbol{x})\right|^p) d\boldsymbol{x} d\boldsymbol{y} \\
&\overset{\text{symmetry}}{\le} 2^p \int_\Omega \left|u(\boldsymbol{x})\right|^p \left(\int_\Omega \left|K_n(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{y})\right|^p d\boldsymbol{y} \right) d\boldsymbol{x} \\
&\le 2^p \left\|u\right\|^p_{L^p(\Omega)} \left\|K_n - K\right\|^p_{L^{\infty,p}(\Omega^2)},
\end{aligned}$$

we get the convergence by assumption on the kernels. Hence the sequence $\{\left|\nabla^{\mathrm{NL}}_{K_n} u\right|^p\}_{n\in\mathbb{N}}$ is equi-integrable on $L^p(\Omega^2)$, and so is $\{a + C_2 \left(\left|\nabla^{\mathrm{NL}}_{K_n} u\right|^p + 1\right)\}_{n\in\mathbb{N}}$. Therefore, $\{f_n(\cdot, \nabla^{\mathrm{NL}}_{K_n} u \cdot)\}_{n\in\mathbb{N}}$ is equi-integrable. By Vitali convergence again, we get the convergence of $(\mathcal{G}_n(u))$ to $\mathcal{G}(u)$ under condition $(\mathcal{A}_{ker})$.

Now, assume that $\{K_n,\ K : n \in \mathbb{N}\}$ satisfies $(\mathcal{B}_{ker})$, by arguments similar to those in the proof of Theorem 4.2.2. We have that $\mathcal{G}$, $\mathcal{G}_n$, $n \in \mathbb{N}$, are closed convex function in $L^p(\Omega)$ and that

$$\mathcal{G}_n(u) = F_n(\nabla^{\mathrm{NL}}_{K_n} u) \le \left\|a\right\|_{L^1(\Omega^2)} + C_2 \left(\left|\Omega^2\right| + 2^p \left|\Omega\right| \sup_n \left\|K_n\right\|^p_{L^\infty(\Omega^2)} R^p \right),$$

for all $u \in L^p(\Omega)$ such that $\left\|u\right\|_{L^p(\Omega)} \le R$, $R > 0$. Hence the sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is locally uniformly Lipschitz, and thanks again to Proposition 2.1.9, in particular it is equi-lower semi-continuous. Invoking Theorem 2.1.15, we get the Mosco-convergence.

Assume that $\{K_n,\ K : n \in \mathbb{N}\}$ satisfies $(\mathcal{A}_{ker})$. Since $\mathcal{G}_n$ converge pointwise to $\mathcal{G}$, by definition of Mosco-convergence, it is enough to show the first point (i) of Definition 2.1.14. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^p(\Omega)$ weakly converging to u in $L^p(\Omega)$. Since $\{F_n\}_{n\in\mathbb{N}}$ Mosco-converges to F in $L^p(\Omega^2)$, thanks to Theorem 2.1.15, it amounts to showing that $\{\nabla^{\mathrm{NL}}_{K_n} u_n\}_{n\in\mathbb{N}}$ converges weakly to $\nabla^{\mathrm{NL}}_{K} u$ in $L^p(\Omega^2)$. For $v \in L^q(\Omega^2)$, by Hölder inequality and symmetry of the kernels, we obtain

$$\begin{aligned}
\left|\int_{\Omega^2} v\left(\nabla^{\mathrm{NL}}_{K_n} u_n\right) - v\left(\nabla^{\mathrm{NL}}_{K} u\right)\right| &\le \left|\int_{\Omega^2} (v(\boldsymbol{x}, \boldsymbol{y}) + v(\boldsymbol{y}, \boldsymbol{x}))\left(K_n(\boldsymbol{x}, \boldsymbol{y}) u_n(\boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y})\right) d\boldsymbol{x} d\boldsymbol{y}\right| \\
&\le \left|\int_\Omega \left(\int_\Omega (v(\boldsymbol{x}, \boldsymbol{y}) + v(\boldsymbol{y}, \boldsymbol{x})) K(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} \right) (u_n(\boldsymbol{y}) - u(\boldsymbol{y}))\, d\boldsymbol{y}\right| \\
&\quad + \left|\int_{\Omega^2} (v(\boldsymbol{x}, \boldsymbol{y}) + v(\boldsymbol{y}, \boldsymbol{x}))\, u_n(\boldsymbol{y}) \left(K_n(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{y})\right) d\boldsymbol{x} d\boldsymbol{y}\right|.
\end{aligned}$$

hence $\left|\int_{\Omega^2} v\left(\nabla^{\mathrm{NL}}_{K_n} u_n\right) - v\left(\nabla^{\mathrm{NL}}_{K} u\right)\right| \to 0$ as n tends to ∞. Indeed, It is easy to see that the function $\boldsymbol{y} \to \int_\Omega (v(\boldsymbol{x}, \boldsymbol{y}) + v(\boldsymbol{y}, \boldsymbol{x})) K(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x}$ belongs in $L^q(\Omega)$, by weakly convergence of (u_n), the first term in the last line of the above inequalities tends to 0, for the second term, we have that

$$\begin{aligned}
&\left|\int_{\Omega^2} (v(\boldsymbol{x},\boldsymbol{y})+v(\boldsymbol{y},\boldsymbol{x}))\, u_n(\boldsymbol{y})\, (K_n(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x},\boldsymbol{y}))\, d\boldsymbol{x}d\boldsymbol{y}\right| \\
&\leq \|u_n\|_{L^p(\Omega)} \left|\int_\Omega \left|\int_\Omega (v(\boldsymbol{x},\boldsymbol{y})+v(\boldsymbol{y},\boldsymbol{x}))\, (K_n(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x},\boldsymbol{y}))\, d\boldsymbol{x}\right|^q d\boldsymbol{y}\right|^{1/q} \\
&\leq \|u_n\|_{L^p(\Omega)} \left(\int_\Omega \left|\int_\Omega |v(\boldsymbol{x},\boldsymbol{y})+v(\boldsymbol{y},\boldsymbol{x})|^q d\boldsymbol{x}\right| \left|\int_\Omega |K_n(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{x}\right|^{q/p} d\boldsymbol{y}\right)^{1/q} \\
&\leq C\|K_n-K\|_{L^{\infty,p}(\Omega^2)}.
\end{aligned} \tag{4.2.10}$$

where $C = 2\|v\|_{L^q(\Omega^2)} \sup_n \|u_n\|_{L^p(\Omega)} < \infty$, since (u_n) is weakly convergent. By assumption, we obtain $\left|\int_{\Omega^2} (v(\boldsymbol{x},\boldsymbol{y})+v(\boldsymbol{y},\boldsymbol{x}))\, u_n(\boldsymbol{y})\, (K_n(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x},\boldsymbol{y}))\, d\boldsymbol{x}d\boldsymbol{y}\right| \to 0$ as n tends to ∞. □

Theorem 4.2.9. *Let F, F_n, $n\in\mathbb{N}$ be a sequence of integral functionals given by (4.2.1) which satisfy $(\mathcal{H}_{seq})$. Let K_n, $K \in L^{\infty,p}(\Omega^2)$, g_n, $g\in L^p(\Omega)$, $n\in\mathbb{N}$, and $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of open subsets of Ω. We define $\mathcal{F}$, $\mathcal{F}_n : L^p(\Omega)\to\overline{\mathbb{R}}$ by*

$$\begin{aligned}
\mathcal{F}_n(u) &= F_n(\nabla^{\mathrm{NL}}_{K_n}u) + \iota_{L^p_{g_n}(\Omega,U_n)}(u) \\
\mathcal{F}(u) &= F(\nabla^{\mathrm{NL}}_{K}u) + \iota_{L^p_{g}(\Omega,U)}(u)
\end{aligned}$$

Assume that

(i) *the sequence $\{K, K_n : n\in\mathbb{N}\}$ satisfies $(\mathcal{A}_{ker})$ or $(\mathcal{B}_{ker})$, and (K_n) satisfies **(A.3)** uniformly for some $m>0$ and $r_0>0$.*

(ii) *the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to g in $L^p(\Omega)$.*

(iii) *the sequence $\{U_n, U,\ n\in\mathbb{N}\}$ satisfy $U_n+\mathbb{B}_r\subset\Omega$ and $|U_n\Delta U|\to 0$, as n tends to $+\infty$.*

Then,

(i) *the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ Mosco-converges to $\mathcal{F}$.*

(ii) *the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is equi-coervive on $L^p(\Omega)$ endowed with by the weak topology.*

(iii) *the functional $\mathcal{F}$ attains its minimum. Moreover*

$$\min_{L^p(\Omega)} \mathcal{F} = \lim_n \inf_{L^p(\Omega)} \mathcal{F}_n.$$

(iv) *every sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega)$ such that $\lim_n \mathcal{F}_n(u_n) = \lim_n \inf_{L^p(\Omega)} \mathcal{F}_n$, has a subsequence converging weakly in $L^p(\Omega)$ and its limit is a minimum point for $\mathcal{F}$.*

Proof : The point (i) is a consequence of Corollary 2.1.17, Theorem 4.2.7 and Proposition 4.2.3. The points (iii) and (iv) are an immediate result of (i), (ii) and Theorem 2.1.13. For (ii), it amounts to showing that for all $t>0$ the set $\mathcal{A}_t = \cup_n\{\mathcal{F}_n\leq t\}$ is bounded in $L^p(\Omega)$. Let $u\in\mathcal{A}_t$, there exists $n\in\mathbb{N}$ such that $u\in L^p_{g_n}(\Omega,U_n)$, hence

$$\begin{aligned}
t &\geq \mathcal{F}_n(u) \\
&\geq C_1\int_{\Omega^2} |K_n(\boldsymbol{x},\boldsymbol{y})|^p |u(\boldsymbol{x})-u(\boldsymbol{y})|^p d\boldsymbol{x}d\boldsymbol{y} - \int_{\Omega^2} a(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{x}d\boldsymbol{y} \\
&\geq C_1 m^p \int_{\Omega^2} \chi_{[0,r_0[}(\|\boldsymbol{x}-\boldsymbol{y}\|)|u(\boldsymbol{x})-u(\boldsymbol{y})|^p d\boldsymbol{x}d\boldsymbol{y} - \|a\|_{L^1(\Omega^2)} \\
&\geq C_1 m^p \left(\lambda\int_\Omega |u(\boldsymbol{x})|^p d\boldsymbol{x} - \int_\Omega |g_n(\boldsymbol{x})|^p d\boldsymbol{x}\right) - \|a\|_{L^1(\Omega^2)}
\end{aligned}$$

where $\lambda = \lambda(r, r_0, \Omega) > 0$, hence

$$\int_\Omega |u(\boldsymbol{y})|^p d\boldsymbol{y} \leq \frac{\lambda^{-1}}{C_1 m^p}(t+\|a\|_{L^1(\Omega^2)}) + \lambda^{-1}\sup_n \|g_n\|^p_{L^p(\Omega)}.$$

□

Theorem 4.2.10. *Let g_n, Γ_n, Γ'_n, $\mathbb{A}_{g_n}(V_n(\Omega), \Gamma_n)$, $\mathbb{A}'_{g_n}(V_n(\Omega), \Gamma'_n)$ as in Lemma 4.2.4, and let F, F_n, $n \in \mathbb{N}$ be a sequence of integral functionals given by (4.2.1) which satisfy $(\mathcal{H}_{seq})$. Let K_n, $K \in L^{\infty,p}(\Omega^2)$. We define $\mathcal{F}$, $\mathcal{F}_n : L^p(\Omega) \to \overline{\mathbb{R}}$ by*

$$\begin{aligned}\mathcal{F}_n(u) &= F_n(\nabla^{\mathrm{NL}}_{K_n} u) + \iota_{\mathbb{A}_{g_n}(V_n(\Omega),\Gamma_n)}(u)\\ \mathcal{F}(u) &= F(\nabla^{\mathrm{NL}}_{K} u) + \iota_{L^p_g(\Omega,U)}(u)\end{aligned}$$

Assume that

(i) *the sequence $\{K, K_n : n \in \mathbb{N}\}$ satisfies $(\mathcal{A}_{ker})$ or $(\mathcal{B}_{ker})$, and (K_n) satisfies **(A.3)** uniformly for some $m > 0$ and $r_0 > 0$.*

(ii) *the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to g in $L^p(\Omega)$.*

Then

(i) *the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ Mosco-converges to $\mathcal{F}$.*

(ii) *the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is equi-coervive on $L^p(\Omega)$ endowed with the weak topology.*

(iii) *the functional $\mathcal{F}$ attains its minimizer. Moreover*

$$\min_{L^p(\Omega)} \mathcal{F} = \lim_n \inf_{L^p(\Omega)} \mathcal{F}_n.$$

(iv) *every sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega)$ such that $\lim_n \mathcal{F}_n(u_n) = \lim_n \inf_{L^p(\Omega)} \mathcal{F}_n$, has a subsequence converging weakly in $L^p(\Omega)$ and its limit is a minimum point for $\mathcal{F}$.*

PROOF : We obtain the result by the same arguments as in the proof of Theorem 4.2.9, where we invoke Proposition 4.2.4 instead Proposition 4.2.3. □

Remark 4.2.11. All the results of this subsection remain true, when the kernels are such that $K(\boldsymbol{x}, \boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$ and $K_n(\boldsymbol{x}, \boldsymbol{y}) = J_n(\boldsymbol{x}-\boldsymbol{y})$, if we replace the hypotheses on the kernels K, K_n, $n \in \mathbb{N}$ by the equivalent ones on the kernels J, J_n, $n \in \mathbb{N}$.

4.3 Application to a sequence of Dirichlet problems

Now, we are in position to attack our main problem of this chapter, which consists in studying the asymptotic behaviour of the sequence of the problems $(\mathcal{P}^n_D)$. Before this, we start by establishing the well-posedness of the nonlocal boundary value problem $(\mathcal{P}_D)$. We keep the same assumptions as in Section 4.2.3, and in particular on U, Ω, i.e. U is a sub-domain of Ω such that $U + \mathbb{B}(0, r) \subset \Omega$ for some $r > 0$.

4.3.1 Existence and uniqueness of the solution

Define the energy functional $\mathcal{F}(u) : L^p(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(u) := \frac{1}{2p} \int_{\Omega^2} K(\boldsymbol{x}, \boldsymbol{y}) \big|u(\boldsymbol{y}) - u(\boldsymbol{x})\big|^p d\boldsymbol{y} d\boldsymbol{x} + \int_{\Omega} f(\boldsymbol{x}) u(\boldsymbol{x}) d\boldsymbol{x}. \tag{4.3.1}$$

We have the following "integration by parts" identity.

Lemma 4.3.1 ([93, Lemma 2.2]). *Let $K \in L^1(\Omega^2)$ satisfies **(A.2)**. For every u, $v \in L^p(\Omega)$ we have*

$$\int_{\Omega} (\Delta^K_p u) v d\boldsymbol{x} = \frac{1}{2} \int_{\Omega^2} K(\boldsymbol{x}, \boldsymbol{y}) \big|u(\boldsymbol{y}) - u(\boldsymbol{x})\big|^{p-2} (u(\boldsymbol{y}) - u(\boldsymbol{x}))(v(\boldsymbol{y}) - v(\boldsymbol{x})) d\boldsymbol{y} d\boldsymbol{x}. \tag{4.3.2}$$

Due to the nonlocal property of the operator Δ_p^K, classical boundary conditions (imposed on boundaries of zero volume) will not yield well posed systems. The authors in [65] solved this issue and showed well-posedness of ($\mathcal{P}_D$) for $p = 2$ in the scalar case by using a variational approach based on the Lax-Milgram lemma (see also [64]). Since the Lax-Milgram lemma is not applicable for the nonlinear problem ($\mathcal{P}_D$), [93] proved Dirichlet's principle for the nonlocal setting of ($\mathcal{P}_D$). Adapted to our setting, this can be stated as follows.

Proposition 4.3.2 (Dirichlet's principle). *Let $K \in L^1(\Omega^2)$ be a function satisfying **(A.1)** and **(A.2)**. Consider the functional $\mathcal{F} : L^p(\Omega) \to \mathbb{R}$ given in (4.3.1). Then, the following holds.*

(i) Assume u solves the Dirichlet problem ($\mathcal{P}_D$). Then

$$\mathcal{F}(u) \leq \mathcal{F}(v) \tag{4.3.3}$$

for every $v \in L^p_g(\Omega, U)$.

(ii) Conversely, if u satisfies (4.3.3) for every $v \in L^p_g(\Omega, U)$, then u solves the Dirichlet problem ($\mathcal{P}_D$).

Theorem 4.3.3. *Let $g \in L^p(\Omega)$, $f \in L^q(\Omega)$, with $1 < p, q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $K \in L^1(\Omega^2)$ satisfies **(A.1)**-**(A.3)**. Then the functional $\mathcal{F}$ has a unique minimizer in $L^p_g(\Omega, U)$, i.e. the problem ($\mathcal{P}_D$) has a unique solution.*

PROOF : See [93, Theorem 3.11]. □

4.3.2 The continuum limit

Let $n \in \mathbb{N}$. Define the energy functional $\mathcal{F}_n(u) : L^p(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}_n(u) := \frac{1}{2p} \int_{\Omega^2} K_n(\boldsymbol{x}, \boldsymbol{y}) \left| u(\boldsymbol{y}) - u(\boldsymbol{x}) \right|^p d\boldsymbol{y} d\boldsymbol{x} + \int_\Omega f_n(\boldsymbol{x}) u(\boldsymbol{x}) d\boldsymbol{x}, \tag{4.3.4}$$

where $K_n \in L^1(\Omega^2)$ satisfies **(A.1)**-**(A.2)** and $f_n \in L^q(\Omega)$, q is the Hölder conjugate of p. We have that, thanks to Dirichlet's principle, solving the problem ($\mathcal{P}_D^n$) is equivalent to minimizing the above functional $\mathcal{F}_n$ on $L^p_{g_n}(\Omega, U_n)$, with the function $g_n \in L^p(\Omega)$.

Theorem 4.3.4. *Let K_n, $K \in L^{\infty,1}(\Omega^2)$ satisfy **(A.1)** and **(A.2)**, g_n, $g \in L^p(\Omega)$, f_n, $f \in L^q(\Omega)$ and $U_n, U \subset\subset \Omega$ sub-domains, $n \in \mathbb{N}$. Assume that*

(1) the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to g in $L^p(\Omega)$.

(2) the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to f in $L^q(\Omega)$.

(3) the sequence $\{K^{\frac{1}{p}}, K_n^{\frac{1}{p}} : n \in \mathbb{N}\}$ satisfies ($\mathcal{A}_{ker}$) or ($\mathcal{B}_{ker}$), and there exist $m > 0$ and $r_0 > 0$ such that

$$m\chi_{[0,r_0[}(\|\boldsymbol{x} - \boldsymbol{y}\|) \leq K_n(\boldsymbol{x}, \boldsymbol{y}), \quad \text{for a.e. } \boldsymbol{x}, \boldsymbol{y} \in \Omega, \text{ all } n \in \mathbb{N}.$$

(4) the sequence $\{U_n, U,\ n \in \mathbb{N}\}$, of subdomains of Ω, satisfies that $U_n + \mathbb{B}(0, r) \subset \Omega$, $n \in \mathbb{N}$, and $|U_n \Delta U| \to 0$, as n tends to $+\infty$, where r is given at the head of this section.

Then ($\mathcal{P}_D$) and ($\mathcal{P}_D^n$) have unique solutions, respectively, u and u_n. Moreover, the sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to u in $L^p(\Omega)$.

PROOF : The existence and the uniqueness of the solutions u and u_n are a consequence of Theorem 4.3.3. On the other hand, since $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to f on $L^p(\Omega)$, we get that the sequence

$\{u \mapsto \int_\Omega f_n u\}_{n\in\mathbb{N}}$ is Mosco-convergent to $u \mapsto \int_\Omega f_n u$ in $L^p(\Omega)$. Then by Theorem 4.2.9 and Corollary 2.1.17, we have $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ Mosco-convergences to $\mathcal{E}$ in $L^p(\Omega)$, where

$$\begin{aligned}\mathcal{E}(u) &= \frac{1}{2p}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\big|u(\boldsymbol{y})-u(\boldsymbol{x})\big|^p d\boldsymbol{y}d\boldsymbol{x} + \int_\Omega fud\boldsymbol{x} + \iota_{L^p_g(\Omega,U)}(u);\\ \mathcal{E}_n(u) &= \frac{1}{2p}\int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\big|u(\boldsymbol{y})-u(\boldsymbol{x})\big|^p d\boldsymbol{y}d\boldsymbol{x} + \int_\Omega f_n ud\boldsymbol{x} + \iota_{L^p_{g_n}(\Omega,U_n)}(u).\end{aligned}$$

Hence by Theorem 2.1.13, if we show the equi-coercivity of the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, we get the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u. As in the proof of Theorem 4.2.9, we are going to show that $\mathcal{A}_t = \cup_n\{\mathcal{E}_n \leq t\}$ is bounded in $L^p(\Omega)$ for all $t\in\mathbb{R}$. Let $t\in\mathbb{R}$ and $u\in\mathcal{A}_t$, then there exists n such that $u \in L^p_{g_n}(\Omega,U_n)$, hence

$$\begin{aligned} t &\geq \mathcal{E}_n(u)\\ &= \frac{1}{2p}\int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\big|u(\boldsymbol{y})-u(\boldsymbol{x})\big|^p d\boldsymbol{y}d\boldsymbol{x} + \int_\Omega f_n ud\boldsymbol{x}\\ &= \frac{1}{2p}\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\big|u(\boldsymbol{y})-u(\boldsymbol{x})\big|^p d\boldsymbol{y}d\boldsymbol{x} + \int_\Omega f_n ud\boldsymbol{x}\\ &\geq \frac{m}{2p}\int_{\Omega^2} \chi_{[0,r_0[}(\|\boldsymbol{x}-\boldsymbol{y}\|)\big|u(\boldsymbol{y})-u(\boldsymbol{x})\big|^p d\boldsymbol{y}d\boldsymbol{x} - \int_\Omega |f_n u|d\boldsymbol{x}\\ &\geq \frac{m}{2p}\Big(\lambda\|u\|^p_{L^p(\Omega)} - \|g_n\|^p_{L^p(\Omega)}\Big) - \frac{\varepsilon}{2p}\|u\|^p_{L^p(\Omega)} - \frac{2}{q\varepsilon}\|f_n\|^q_{L^q(\Omega)}\end{aligned}$$

furthermore

$$\begin{aligned}\frac{1}{2p}(m\lambda-\varepsilon)\|u\|^p_{L^p(\Omega)} &\leq t + \frac{m}{2p}\|g_n\|^p_{L^p(\Omega)} + \frac{2}{q\varepsilon}\|f_n\|^q_{L^q(\Omega)}\\ &\leq t + \frac{m}{2p}\sup_n \|g_n\|^p_{L^p(\Omega)} + \frac{2}{q\varepsilon}\sup_n\|f_n\|^q_{L^q(\Omega)}\end{aligned}$$

where $\lambda(r,r_0,\Omega) > 0$. We choose ε small enough such that $m\lambda - \varepsilon > 0$, we obtain the equi-coercivity. □

Theorem 4.3.5. *Let K_n, $K \in L^{\infty,1}(\Omega^2)$ satisfy (A.1) and (A.2), g_n, $f_n \in \mathbb{A}(V_n(\Omega))$, $g\in L^p(\Omega)$ and $f\in L^q(\Omega)$, and let $U_n = \bigcup_{\Omega^{(n)}_{\boldsymbol{x}}\subset U}\Omega^{(n)}_{\boldsymbol{x}}$ and $\Gamma_n = \Omega\setminus U_n$ as in Proposition 4.2.4, $n\in\mathbb{N}$. Assume that*

(1) the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges strongly to g in $L^p(\Omega)$.

(2) the sequence $\{K^{\frac{1}{p}}, K_n^{\frac{1}{p}} : n\in\mathbb{N}\}$ satisfies ($\mathcal{A}_{ker}$) or ($\mathcal{B}_{ker}$), and there exist $m>0$ and $r_0>0$ such that

$$m\chi_{[0,r_0[}(\|\boldsymbol{x}-\boldsymbol{y}\|) \leq K_n(\boldsymbol{x},\boldsymbol{y}), \quad \textit{for a.e. } \boldsymbol{x},\boldsymbol{y}\in\Omega, \textit{ all } n\in\mathbb{N}.$$

(3) the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to f in $L^q(\Omega)$.

Then, for all $n\in\mathbb{N}$ the problem ($\mathcal{P}^n_D$) has a unique solution $u_n \in \mathbb{A}_{g_n}(V_n(\Omega),\Gamma_n)$. Moreover, the sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to the solution of the problem ($\mathcal{P}_D$) in $L^p(\Omega)$.

Proof : We obtain the result by the same arguments as in the proof of Theorem 4.3.4, where we use the result of Theorem 4.2.10 instead of that of Theorem 4.2.9. □

Remark 4.3.6. All the results of this section remain true, when the kernels are such that $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$ and $K_n(\boldsymbol{x},\boldsymbol{y}) = J_n(\boldsymbol{x}-\boldsymbol{y})$, if we replace the hypotheses on the kernels K, K_n, $n\in\mathbb{N}$ by the equivalent ones of the kernels J, J_n, $n\in\mathbb{N}$.

In this section, all kernels considered are positive, which is not the case in Section 4.2.3. In this case a sufficient condition for the strong convergence of the sequence $\{K_n^{\frac{1}{p}}\}_{n\in\mathbb{N}}$ converges to $K^{\frac{1}{p}}$ in $L^{\infty,p}(\Omega^2)$, under the assumption $(\mathcal{A}_{ker})$, to hold, is that $\{K_n\}_{n\in\mathbb{N}}$ to K in $L^{\infty,1}(\Omega^2)$. Indeed, for all $s,t\geq 0$ and $\alpha\in]0,1]$, we have

$$\left|s^\alpha - t^\alpha\right| \leq \left|s-t\right|^\alpha,$$

thanks to the concavity of the function : $t\to\left|t\right|^\alpha$. Applying this inequality, we get

$$\left|K_n^{\frac{1}{p}}(\boldsymbol{x},\boldsymbol{y}) - K^{\frac{1}{p}}(\boldsymbol{x},\boldsymbol{y})\right| \leq \left|K_n(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\boldsymbol{y})\right|^{\frac{1}{p}}, \quad \text{ for a.e. } \boldsymbol{x},\, \boldsymbol{y}\in\Omega,$$

hence

$$\left\|K_n^{\frac{1}{p}} - K^{\frac{1}{p}}\right\|_{L^{\infty,p}(\Omega^2)} \leq \left\|K_n - K\right\|^{\frac{1}{p}}_{L^{\infty,1}(\Omega^2)}.$$

4.4 Consistency and error estimates of Dirichlet problem

4.4.1 General consistency estimates

Fix $n\in\mathbb{N}$. We denote by $\mathbb{H}(V_n(\Omega))$ the set of real functions defined on $V_n(\Omega)$ where $V_n(\Omega)$ is given by (4.2.4), and similarly for $\mathbb{H}(V_n(\Omega)\times V_n(\Omega))$. Let K a nonnegative symmetric function in $L^{\infty,1}(\Omega^2)$, $g\in L^p(\Omega)$ and $f\in L^q(\Omega)$, with $\frac{1}{p}+\frac{1}{q}=1$.

We consider the discrete Dirichlet problem

$$\begin{cases} \frac{1}{n^d}\sum_{\boldsymbol{y}\in V_n(\Omega)} \mathbf{K}_{\boldsymbol{x}\boldsymbol{y}}\left|\mathbf{u}_{\boldsymbol{y}} - \mathbf{u}_{\boldsymbol{x}}\right|^{p-2}(\mathbf{u}_{\boldsymbol{y}} - \mathbf{u}_{\boldsymbol{x}}) = \mathbf{f}_{\boldsymbol{x}}, & \boldsymbol{x}\in A_n \\ \mathbf{u}_{\boldsymbol{x}} = \mathbf{g}_{\boldsymbol{x}}, & \boldsymbol{x}\in A_n^c, \end{cases} \tag{$\mathcal{P}_n^d$}$$

where $\mathbf{g}=P_n g$, $\mathbf{f}=P_n f$, $\mathbf{K}=P_n K$, $A_n=\{\boldsymbol{x}\in V_n(\Omega) : \Omega_{\boldsymbol{x}}^{(n)}\subset U\}$ and $A_n^c = V_n(\Omega)\setminus A_n$.

The problem $(\mathcal{P}_n^d)$ is equivalent to solving the following minimization problem

$$\min\{\mathcal{G}_n(\mathbf{u}),\ \textit{on}\ \mathbb{H}_{\mathbf{g}}(V_n(\Omega),A_n)\}, \tag{$\mathcal{VP}_n^d$}$$

where

$$\mathcal{G}_n(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2p}\sum_{\boldsymbol{x},\,\boldsymbol{y}\in V_n(\Omega)} \mathbf{K}_{\boldsymbol{x}\boldsymbol{y}}\left|\mathbf{u}_{\boldsymbol{y}} - \mathbf{u}_{\boldsymbol{x}}\right|^p + \sum_{\boldsymbol{x}\in V_n(\Omega)} \mathbf{u}_{\boldsymbol{x}}\mathbf{f}_{\boldsymbol{x}}$$

and

$$\mathbb{H}_{\mathbf{g}}(V_n(\Omega),A_n) = \{\mathbf{u}\in\mathbb{H}(V_n(\Omega)) :\ \mathbf{u}=\mathbf{g} \text{ on } A_n^c\}.$$

Our aim is to compare the solutions of problems $(\mathcal{P}_D)$ and $(\mathcal{P}_n^d)$. The solution of $(\mathcal{P}_n^d)$ being discrete, we consider its continuum extensions of $\mathbf{f}$, $\mathbf{g}$ $\mathbf{u}$ on Ω and $\mathbf{K}$ on Ω^2 as

$$f_n = I_n\mathbf{f}, \quad g_n = I_n\mathbf{g}, \quad u_n = I_n\mathbf{u} \quad \text{and} \quad K_n = I_n\mathbf{K}.$$

it is immediate to see that a function is a solution of the problem $(\mathcal{P}_n^d)$ if, and only if its continuous extension is a solution of the following problem

$$\begin{cases} -\Delta_p^{K_n} u_n = f_n, & \text{in } U_n \\ u_n = g_n, & \text{on } \Gamma_n = \Omega\setminus U_n, \end{cases} \tag{$\mathcal{P}_n^c$}$$

where $U_n = \bigcup_{\boldsymbol{x}\in A_n}\Omega_{\boldsymbol{x}}^{(n)}$, which in turn is equivalent to minimizing the functional

$$v \mapsto \frac{1}{2p}\int_{\Omega^2} K_n(\boldsymbol{x},\boldsymbol{y})\left|v(\boldsymbol{y}) - v(\boldsymbol{x})\right|^p d\boldsymbol{y}d\boldsymbol{x} + \int_\Omega v(\boldsymbol{x})f_n(\boldsymbol{x})d\boldsymbol{x}$$

on $\mathbb{A}_{g_n}(V_n(\Omega),\Gamma_n)$, see (4.2.5) for the definition of this set.

Corollary 4.4.1. *Let $g\in L^p(\Omega)$, $f\in L^q(\Omega)$ and $K\in L^\infty(\Omega^2)$. Assume that K satisfies **(A.1)**-**(A.3)**. Then, for all $n\in\mathbb{N}$ the problem $(\mathcal{P}_n^d)$ has a unique solution $\mathbf{u}\in\mathbb{H}(V_n(\Omega))$. Moreover, if u is the solution of the problem $(\mathcal{P}_D)$ then $\{I_n\mathbf{u}\}_{n\in\mathbb{N}}$ converges weakly to u in $L^p(\Omega)$.*

PROOF : In view of the definition of the functions f_n, g_n, K_n, by the Lebesgue differentiation theorem (see [119, Theorem 3.4.4]), we have that g_n, f_n and K_n converge pointwise to respectively f, g, and K a.e. on Ω and Ω^2 respectively. Combining this with Fatou's lemma and (2.1.9), we get

$$\lim_n \|g_n\|_{L^p(\Omega)} = \|g\|_{L^p(\Omega)} \quad \text{and} \quad \lim_n \|f_n\|_{L^q(\Omega)} = \|f\|_{L^q(\Omega)}.$$

Hence, by the Riesz-Scheffé lemma, we have that $I_nP_ng \to g$ strongly in $L^p(\Omega)$ and $I_nP_nf \to f$ strongly in $L^q(\Omega)$. On the other hand, by the assumption **(A.3)**, we have $m\chi_{[0,r_0[}(\|\boldsymbol{x}-\boldsymbol{y}\|) \leq I_n\mathbf{K}(\boldsymbol{x},\boldsymbol{y}) \leq \|K\|_{L^\infty(\Omega^2)}$ for a.e. on Ω^2, whence $\{K_n,\, K : n \in \mathbb{N}\}$ satisfies the condition $(\mathcal{A}_{ker})$. By Theorem 4.3.5 we get the result. □

Corollary 4.4.2. *Let $g \in L^p(\Omega)$, $f \in L^q(\Omega)$ and let $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$ where $J \in L^1(\Omega-\Omega)$ satisfies **(A'.1)**-**(A'.3)**. Then, for all $n \in \mathbb{N}$ the problem $(\mathcal{P}_n^d)$ has a unique solution $\mathbf{u} \in \mathbb{H}(V_n(\Omega))$. Moreover, if u is the solution of the problem $(\mathcal{P}_D)$ then $\{I_n\mathbf{u}\}_{n\in\mathbb{N}}$ converges weakly to u in $L^p(\Omega)$.*

PROOF : By the same arguments as the proof of the above corollary, we get that

$$g_n \to g \text{ and } f_n \to f,$$

strongly on $L^p(\Omega)$ and $L^q(\Omega)$, respectively, and

$$J_n \to J,$$

strongly on $L^1(\Omega-\Omega)$ and also pointwise almost everywhere on $\Omega-\Omega$, where $J_n = I_nP_nJ$. Moreover, by the assumption **(A'.3)**, we have $m\chi_{[0,r_0[}(\|\boldsymbol{x}-\boldsymbol{y}\|) \leq I_n\mathbf{K}(\boldsymbol{x},\boldsymbol{y}) = J_n(\boldsymbol{x}-\boldsymbol{y})$ a.e. on Ω^2, whence $\{J^{1/p},\, J_n^{1/p},\, n \in \mathbb{N}\}$ satisfies the condition $(\mathcal{A}'_{rad})$, thanks to Remark 4.3.6. By Theorem 4.3.5 again, we get the desired result. □

Remark 4.4.3. the results of Corollary 4.4.1 and Corollary 4.4.2 remain true if we replace A_n and A_n^c by $A'_n = \{\boldsymbol{x} \in V_n(\Omega) : \Omega_{\boldsymbol{x}}^{(n)} \cap U \neq \emptyset\}$ and $(A'_n)^c$, respectively, in $(\mathcal{P}_n^d)$.

4.4.2 A priori estimates

In this subsection we give an a priori estimate for the Dirichlet problem. The reason we separate this section from the previous one is that here, we restrict ourselves to the case where the datum g is constant, without loss of generality we take $g = 0$. The second reason is due to the choice of the boundary set $\Gamma_n \subset \Gamma$ as the set A_n^c defined in $(\mathcal{P}_n^d)$. These conditions feasibility of the constraint that $\mathbb{A}_{g_n}(V_n(\Omega),\Gamma_n) \subset L^p_{g_n}(V_n(\Omega),\Gamma_n)$, which plays a key role to get our estimation.

Theorem 4.4.4. *Let $K \in L^{\infty,1}(\Omega^2)$ satisfies **(A.1)**-**(A.3)**, $g = 0$ and $f \in L^q(\Omega)$. Let $\mathbf{K} = P_nK$, $\mathbf{g} = 0$ and $\mathbf{f} = P_nf$. Let $\mathbf{u}$ be a solution of the discrete problem $(\mathcal{P}_n^d)$ with kernel $\mathbf{K}$ data $(\mathbf{f},\mathbf{g})$ and the boundary set A_n^c, and u the solution of the continuous problem $(\mathcal{P}_D)$ with kernel K, data (f,g) and the boundary set Γ. Then,*

$$\|u-u_n\|_{L^p(\Omega)}^{p/\max(1,\frac{2}{p})} \leq C\Bigg(\|K-K_n\|_{L^{\infty,1}(\Omega^2)}^{\max(2,\frac{p}{p-1})} + \|K-K_n\|_{L^{\infty,1}(\Omega^2)}\|I_nP_nu-u\|_{L^p(\Omega)} + \begin{cases} \|I_nP_nu-u\|_{L^p(\Omega)}^{\frac{p}{p-1}} & p \in [2,+\infty[, \\ \|I_nP_nu-u\|_{L^p(\Omega)}^{\frac{2}{3-p}} & p \in]1,2]. \end{cases}\Bigg). \tag{4.4.1}$$

where $C > 0$ independent of n and K_n, g_n, f_n and u_n are the continuous extensions of the functions $\mathbf{K}$, $\mathbf{g}$, $\mathbf{f}$ and $\mathbf{u}$ respectively.

PROOF : We denote by $\langle\cdot,\cdot\rangle$ the usual inner product in $L^2(\Omega)$ and $\nabla^{\mathrm{NL}}v(\boldsymbol{x},\boldsymbol{y}) = v(\boldsymbol{y}) - v(\boldsymbol{x})$, $v \in L^1(\Omega)$. We have

$$\langle \Delta_p^K u, u_n - u\rangle = \langle -f, u_n - u\rangle, \tag{4.4.2}$$

$$\langle \Delta_p^{K_n} u_n, v - u_n\rangle = \langle -f_n, v - u_n\rangle \quad \textit{for all } v \in \mathbb{A}_{g_n}(V_n(\Omega),\Gamma_n). \tag{4.4.3}$$

By summing, we deduce from (4.4.2) and (4.4.3) :

$$\begin{aligned}\langle \Delta_p^{K_n} u_n - \Delta_p^K u, u_n - u\rangle &= \langle \Delta_p^{K_n} u_n, v - u\rangle - \langle -f, u_n - u\rangle - \langle -f_n, v - u_n\rangle \\ &= \langle \Delta_p^{K_n} u_n, v - u\rangle - \langle -f, v - u\rangle\end{aligned}$$

for all $v \in \mathbb{A}_{g_n}(V_n(\Omega),\Gamma_n)$. The last equality comes from the fact that $\langle f, v - u_n\rangle = \langle f_n, v - u_n\rangle$. Since $\Delta_p^K u = -f$ on U and $v = u_n$ on Γ_n, we get

$$\langle \Delta_p^{K_n} u_n - \Delta_p^K u, u_n - u\rangle = \langle \Delta_p^{K_n} u_n - \Delta_p^K u, v - u\rangle,$$

in turn,

$$\begin{aligned}\langle \Delta_p^K u_n - \Delta_p^K u, u_n - u\rangle = \; &\langle \Delta_p^{K_n} u_n - \Delta_p^K u_n, v - u\rangle + \langle \Delta_p^K u_n - \Delta_p^K u, v - u\rangle \\ &-\langle \Delta_p^{K_n} u_n - \Delta_p^K u_n, u_n - u\rangle.\end{aligned} \tag{4.4.4}$$

For the term in the left-hand side, we have by Lemma 4.2.5 (ii), there exists $C > 0$ independent of n such that

$$\|u - u_n\|_{L^p(\Omega)}^{p/\max(1,\frac{2}{p})} \leq C\langle \Delta_p^K u_n - \Delta_p^K u, u_n - u\rangle. \tag{4.4.5}$$

On the other hand, we start with the last term in the right-hand side of the equation (4.4.4). By by Hölder inequality, we have

$$\langle \Delta_p^{K_n} u_n - \Delta_p^K u_n, u_n - u\rangle \leq \|\Delta_p^{K_n} u_n - \Delta_p^K u_n\|_{L^q(\Omega)} \|u - u_n\|_{L^p(\Omega)}. \tag{4.4.6}$$

Applying Jensen inequality on the first term in the right-hand side of of the above inequality, we get

$$\begin{aligned}&\|\Delta_p^{K_n} u_n - \Delta_p^K u_n\|_{L^q(\Omega)} \\ &\leq \left(\int_\Omega \left(\int_\Omega |K - K_n|(\boldsymbol{x},\boldsymbol{y})|\nabla^{\mathrm{NL}} u_n(\boldsymbol{x},\boldsymbol{y})|^{p-1} d\boldsymbol{y}\right)^q d\boldsymbol{x}\right)^{1/q} \\ &\overset{Jensen}{\leq} \left(\int_\Omega \left(\int_\Omega |K - K_n|(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y}\right)^{q-1} \int_\Omega |K - K_n|(\boldsymbol{x},\boldsymbol{y})|\nabla^{\mathrm{NL}} u_n(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{y} d\boldsymbol{x}\right)^{1/q} \\ &\leq 2^p \|K - K_n\|_{L^{\infty,1}(\Omega^2)}^{\frac{q-1}{q}} \left(\int_{\Omega^2} |K - K_n|(\boldsymbol{x},\boldsymbol{y})|u_n(\boldsymbol{y})|^p d\boldsymbol{y} d\boldsymbol{x}\right)^{1/q} \\ &\leq C_1 \|K - K_n\|_{L^{\infty,1}(\Omega^2)},\end{aligned} \tag{4.4.7}$$

where $C_1 = 2^p|\Omega|^{\frac{p-1}{p}} \sup_n \|u_n\|_{L^p(\Omega)}^{p-1} < \infty$. Plugging (4.4.7) in (4.4.6), we get

$$\langle \Delta_p^{K_n} u_n - \Delta_p^K u_n, u_n - u\rangle \leq C_1 \|K - K_n\|_{L^{\infty,1}(\Omega^2)} \|u - u_n\|_{L^p(\Omega)}, \tag{4.4.8}$$

Similarly for the second term in the right-hand side of the equation (4.4.4), we have

$$|\langle \Delta_p^{K_n} u_n - \Delta_p^K u_n, v - u\rangle| \leq C_1 \|K - K_n\|_{L^{\infty,1}(\Omega^2)} \|v - u\|_{L^p(\Omega)}. \tag{4.4.9}$$

Let's turn to the first term in the right-hand side of the equation (4.4.4), we have

$$|\langle \Delta_p^K u_n - \Delta_p^K u, v - u\rangle| \leq \|\Delta_p^K u_n - \Delta_p^K u\|_{L^q(\Omega)} \|v - u\|_{L^p(\Omega)}. \tag{4.4.10}$$

For the case when $p \in]1,2]$, using inequality (3.2.6) and Jensen inequality, we get

$$\|\Delta_p^K u_n - \Delta_p^K u\|_{L^q(\Omega)} \leq \left(\int_\Omega \left(\int_\Omega K(\boldsymbol{x},\boldsymbol{y})|(u - u_n)(\boldsymbol{x}) - (u - u_n)(\boldsymbol{y})|^{p-1} d\boldsymbol{y}\right)^q d\boldsymbol{x}\right)^{\frac{1}{q}}$$

$$\leq \quad C\|K\|_{L^{\infty,1}(\Omega^2)}\|u-u_n\|^{p-1}_{L^p(\Omega)}, \tag{4.4.11}$$

where $C = 2^{p-1}|\Omega|^{\frac{p-1}{p}}$. In the case when $p > 2$, we apply inequality (3.2.6) and Hölder inequality twice, we obtain

$$\begin{aligned}
&\|\Delta_p^K u_n - \Delta_p^K u\|_{L^q(\Omega)}\\
&\leq \left(\int_\Omega \left(\int_\Omega K(\boldsymbol{x},\boldsymbol{y})\left(|\nabla^{NL}u_n(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{NL}u(\boldsymbol{x},\boldsymbol{y})|\right)^{p-2}|\nabla^{NL}(u-u_n)(\boldsymbol{x},\boldsymbol{y})|d\boldsymbol{y}\right)^q d\boldsymbol{x}\right)^{\frac{1}{q}}\\
&\leq \left(\int_\Omega \left(\int_\Omega K(\boldsymbol{x},\boldsymbol{y})\left(|\nabla^{NL}u_n(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{NL}u(\boldsymbol{x},\boldsymbol{y})|\right)^{p} d\boldsymbol{y}\right)^{\frac{p-2}{p-1}}\cdot\right.\\
&\qquad\left.\left(\int_\Omega K(\boldsymbol{x},\boldsymbol{y})|\nabla^{NL}(u-u_n)(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{y}\right)^{\frac{1}{p-1}}\left(\int_\Omega K(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}\right)^{q/p} d\boldsymbol{x}\right)^{1/q} \qquad (4.4.12)\\
&\leq \|K\|^{1/p}_{L^{\infty,1}(\Omega^2)}\left(\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})\left(|\nabla^{NL}u_n(\boldsymbol{x},\boldsymbol{y})| + |\nabla^{NL}u(\boldsymbol{x},\boldsymbol{y})|\right)^{p} d\boldsymbol{y}d\boldsymbol{x}\right)^{\frac{p-2}{p}}\\
&\qquad\left(\int_{\Omega^2} K(\boldsymbol{x},\boldsymbol{y})|\nabla^{NL}(u-u_n)(\boldsymbol{x},\boldsymbol{y})|^p d\boldsymbol{y}d\boldsymbol{x}\right)^{\frac{1}{p}}\\
&\leq C\|K\|_{L^{\infty,1}(\Omega^2)}\|u-u_n\|_{L^p(\Omega)}
\end{aligned}$$

where $C = 4^p(\sup_n \|u_n\|^{p-2}_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}) < +\infty$. Plugging the inequalities (4.4.11) and (4.4.12) in (4.4.10), one gets

$$|\langle \Delta_p^K u_n - \Delta_p^K u, v-u\rangle| \leq C\begin{cases}\|u_n-u\|_{L^p(\Omega)}\|v-u\|_{L^p(\Omega)} & p\in[2,+\infty[,\\ \|u_n-u\|^{p-1}_{L^p(\Omega)}\|v-u\|_{L^p(\Omega)} & p\in]1,2].\end{cases} \tag{4.4.13}$$

Assembling the above iqualities and inqualities (4.4.4), (4.4.5) (4.4.8), (4.4.9), and (4.4.13), we obtain

$$\begin{aligned}\|u-u_n\|^{p/\max(1,\frac{2}{p})}_{L^p(\Omega)} \leq C\Bigg(&\|K-K_n\|_{L^{\infty,1}(\Omega^2)}\|u_n-u\|_{L^p(\Omega)} + \|K-K_n\|_{L^{\infty,1}(\Omega^2)}\|v-u\|_{L^p(\Omega)}\\ &+\begin{cases}\|u_n-u\|_{L^p(\Omega)}\|v-u\|_{L^p(\Omega)} & p\in[2,+\infty[,\\ \|u_n-u\|^{p-1}_{L^p(\Omega)}\|v-u\|_{L^p(\Omega)} & p\in]1,2].\end{cases}\Bigg).\end{aligned} \tag{4.4.14}$$

Now, we use the Young inequality and take $v = I_nP_nu$, we obtain the desired result

$$\begin{aligned}&\|u-u_n\|^{p/\max(1,\frac{2}{p})}_{L^p(\Omega)} \leq\\ &C\begin{cases}\|K-K_n\|^{\frac{p}{p-1}}_{L^{\infty,1}(\Omega^2)} + \|K-K_n\|_{L^{\infty,1}(\Omega^2)}\|I_nP_nu-u\|_{L^p(\Omega)} + \|I_nP_nu-u\|^{\frac{p}{p-1}}_{L^p(\Omega)} & p\in[2,+\infty[,\\ \|K-K_n\|^{2}_{L^{\infty,1}(\Omega^2)} + \|K-K_n\|_{L^{\infty,1}(\Omega^2)}\|I_nP_nu-u\|_{L^p(\Omega)} + \|I_nP_nu-u\|^{\frac{2}{3-p}}_{L^p(\Omega)} & p\in]1,2].\end{cases}\end{aligned} \tag{4.4.15}$$

□

Corollary 4.4.5. *Let K be a kernel such that $K(\boldsymbol{x},\boldsymbol{y}) = J(\boldsymbol{x}-\boldsymbol{y})$ where $J\in L^1(\Omega-\Omega)$ satisfies (A'.1)-(A'.3) and let $\mathbf{K}$, g, $\mathbf{g}$, f, $\mathbf{f}$, $\mathbf{u}$, u as in Theorem 4.4.4. Then, there exists a positive constant C such that*

$$\begin{aligned}\|u-u_n\|^{p/\max(1,\frac{2}{p})}_{L^p(\Omega)} \leq C\Bigg(&\|J-J_n\|^{\max(2,\frac{p}{p-1})}_{L^1(\Omega-\Omega)} + \|J-J_n\|_{L^1(\Omega-\Omega)}\|I_nP_nu-u\|_{L^p(\Omega)}\\ &+\begin{cases}\|I_nP_nu-u\|^{\frac{p}{p-1}}_{L^p(\Omega)} & p\in[2,+\infty[,\\ \|I_nP_nu-u\|^{\frac{2}{3-p}}_{L^p(\Omega)} & p\in]1,2].\end{cases}\Bigg).\end{aligned} \tag{4.4.16}$$

where $J_n = I_n P_n J$ *and* g_n, f_n *and* u_n *are the continuous extensions of the functions* $\mathbf{g}$, $\mathbf{f}$ *and* $\mathbf{u}$ *respectively. In particuler,*

$$\lim_n \|u - u_n\|_{L^p(\Omega)} = 0. \tag{4.4.17}$$

PROOF : The inequality (4.4.16) is immediate result of Theorem 4.4.4. For (4.4.17), we apply the same arguments of the proof of Corollary 4.4.1, we get that

$$\lim_n \|J - J_n\|_{L^1(\Omega-\Omega)} = 0 \quad \text{and} \quad \lim_n \|I_n P_n u - u\|_{L^p(\Omega)} = 0.$$

Let n tends to $+\infty$ in (4.4.16), we get the desired result. □

4.5 Application to random graph sequences

In this section, we study continuum limits of the discrete problem on the random graph model of Definition 2.3.7.

Throughout this section, we suppose that $p \in]1, 2]$. Let $\Omega = [0, 1]$, recall the notation of Section 2.3.3, we define the boundary set $\Gamma = \Omega \setminus U$ where $U =]r, 1-r[$, $r \in]0, 1/2[$. Recall also the the construction of the random graph model in Definition 2.3.7 where each edge (i, j) is independently set to 1 with probability (2.3.5). This entails that the random matrix $\mathbf{\Lambda}$ is symmetric. However, it is worth emphasizing that the entries of $\mathbf{\Lambda}$ are not independent, but only the entries in each row are mutually independent.

We consider the discrete problem on K-random graphs $\mathbf{G}(n, K, \rho_n)$

$$\begin{cases} \frac{1}{\rho_n n} \displaystyle\sum_{j:(i,j)\in E(\mathbf{G}(n,K,\rho_n))} \Psi(\mathbf{u_j} - \mathbf{u_i}) = \mathbf{f}_i, & x_i \in A_n, \\ \mathbf{u}_i = 0, & \textit{otherwise}. \end{cases} \tag{$\mathcal{P}_n^{d,\mathbf{G}}$}$$

where $\mathbf{u}, \mathbf{f} = P_n f \in \mathbb{R}^n$ and $A_n = \{x_i : \ [x_i, x_{i+1}[\subset U\}$ with $x_i \stackrel{\text{def}}{=} \frac{i}{n}$, $i = 0, 1, \cdots, n$. It is important to keep in mind that, since $\mathbf{G}(n, K, \rho_n)$ is a random variable taking values in the set of simple graphs, the boundary value problem ($\mathcal{P}_n^{d,\mathbf{G}}$) must be understood in this sense. Observe that the normalization in ($\mathcal{P}_n^{d,\mathbf{G}}$) by $\rho_n n$ corresponds to the average degree (see Section 2.3.4 for details).

Problem ($\mathcal{P}_n^{d,\mathbf{G}}$) can be equivalently written as

$$\begin{cases} -\widehat{\mathbf{\Delta}}_p^{\mathbf{\Lambda}} \mathbf{u} = \mathbf{f}, & \textit{on } A_n, \\ \mathbf{u} = 0, & \textit{on } A_n^c. \end{cases}$$

We define the continuum extension u_n as in the above section. We then see that they satisfy

$$\begin{cases} -\Delta_p^{I_n \mathbf{\Lambda}} u_n(x) = I_n \mathbf{f}(x), & x \in U_n, \\ u_n(x) = 0, & x \in \Gamma_n. \end{cases} \tag{4.5.1}$$

where $U_n = [r_n, 1 - r_n]$ and $\Gamma_n = [0, 1] \setminus U_n$, with $r_n = \min\{x_i : \ r \le x_i, \ i = 0, 1, \cdots, n\}$.

Toward our goal of establishing error bounds, we define $\mathbf{v}$ as the solution of the discrete problem ($\mathcal{P}_n^d$) with data $(\mathbf{f}, 0)$, boundary set A_n^c and discrete kernel $\hat{\mathbf{K}}$. Its continuum extension v_n defined similarly as above, fullfill

$$\begin{cases} -\Delta_p^{I_n \hat{\mathbf{K}}} v_n(x) = I_n \mathbf{f}(x), & x \in U_n, \\ v_n(x) = 0, & x \in \Gamma_n. \end{cases} \tag{4.5.2}$$

We have

$$\|u_n - u\|_{L^p(\Omega)} \le \|u_n - v_n\|_{L^p(\Omega)} + \|v_n - u\|_{L^p(\Omega)}. \tag{4.5.3}$$

This bound is composed of two terms: the first one captures the error of random sampling, and the second that of discretization. Assume that $(\mathbf{f}, \mathbf{K}, f, K)$ verify the assumptions of Theorem 4.4.4. Since $I_n\hat{\mathbf{K}}(x,y) \leq I_n\mathbf{K}(x,y) = I_nP_nK(x,y)$, the assumptions on $\mathbf{K}$ transfer to $\hat{\mathbf{K}}$, and the second term can be bounded using (4.4.1), replacing I_nP_nK by $I_n\hat{\mathbf{K}}$. It remains to bound the first term by comparing (4.5.1) and (4.5.2).

Lemma 4.5.1. *Assume that $(J, \mathbf{g}, \mathbf{K}, , \mathbf{f}, g, K, f)$ verify the assumptions of Corollary 4.4.5. Assume also that $\rho_n \to 0$ and $\rho_n n = \omega\left((\log n)^\gamma\right)$ for some $\gamma > 1$. Then, for any $\beta \in]0,1[$,*

$$\mathbb{E}(\left\|u_n - v_n\right\|_{L^p(\Omega)}) \leq C\left(\rho_n n\right)^{1/2}, \tag{4.5.4}$$

in turn,

$$\left\|u_n - v_n\right\|_{L^p(\Omega)} \leq C(\rho_n n)^{-\beta/2} \tag{4.5.5}$$

with probability at least $1 - (\rho_n n)^{-(1-\beta)/2}$. In particular,

$$\left\|u_n - v_n\right\|_{L^p(\Omega)} \leq o\left((\log n)^{-\gamma\beta/2}\right) \tag{4.5.6}$$

with probability at least $1 - o\left((\log n)^{-\gamma(1-\beta)/2}\right)$.

To prove this lemma, we need the following deviation inequality that we include for the reader convenience.

Lemma 4.5.2 (Rosenthal's inequality, [96]). *Let m be a positive integer, $\gamma \geq 2$ and $\xi_1, \cdots, \xi_m$, be m zero mean independent random variables such that $\sup_i \mathbb{E}\left(|\xi_i|^\gamma\right) < \infty$. Then there exists a positive constant C such that*

$$\mathbb{E}\left(\left|\sum_i \xi_i\right|^\gamma\right) \leq C\max\left(\sum_i \mathbb{E}\left(|\xi_i|^\gamma\right), \left(\sum_i \mathbb{E}\left(|\xi_i|^2\right)\right)^{\gamma/2}\right).$$

Proof : Denote by $f_n = I_n\mathbf{f}$, $\hat{K}_n = I_n\hat{\mathbf{K}}$ and $\Lambda_n = I_n\mathbf{\Lambda}$. We thus have from (4.5.1) and (4.5.2) that a.e.

$$\langle \Delta_p^{\Lambda_n} u_n - \Delta_p^{\Lambda_n} v_n, u_n - v_n\rangle = -\langle \Delta_p^{\Lambda_n} v_n - \Delta_p^{\hat{K}_n} v_n, u_n - v_n\rangle,$$

since $\langle \Delta_p^{\Lambda_n} u_n - \Delta_p^{\hat{K}_n} v_n, u_n - v_n\rangle = 0$. Since $p \in]1,2]$ and $m\chi_{[0,r_0[}(\left\|\boldsymbol{x} - \boldsymbol{y}\right\|) \leq \Lambda_n(\boldsymbol{x}, \boldsymbol{y})$ almost surely, we have

$$\left\|u_n - v_n\right\|^2_{L^p(\Omega)} \leq C\langle \Delta_p^{\Lambda_n} u_n - \Delta_p^{\Lambda_n} v_n, u_n - v_n\rangle,$$

almost surely, thanks to Lemma 4.2.5 (ii).

On the other hand, let $\mathbf{Z}_i = \frac{1}{n}\sum_j (\hat{\mathbf{K}}_{ij} - \mathbf{\Lambda}_{ij})\Psi(\mathbf{v}_j - \mathbf{v}_i)$, by Hölder inequality, we have

$$\langle \Delta_p^{\hat{K}_n} v_n - \Delta_p^{\Lambda_n} v_n, u_n - v_n\rangle \leq \left\|I_n\mathbf{Z}\right\|_{L^q(\Omega)}\left\|u_n - v_n\right\|_{L^p(\Omega)}. \tag{4.5.7}$$

where q is the Hölder conjugate of p. In turn

$$\left\|u_n - v_n\right\|_{L^p(\Omega)} \leq C\left\|I_n\mathbf{Z}\right\|_{L^q(\Omega)}, \tag{4.5.8}$$

so, it remains to bound the random variable $\left\|I_n\mathbf{Z}\right\|_{L^q(\Omega)}$. For this purpose, we have by Jensen inequality that

$$\mathbb{E}\left(\left\|I_n\mathbf{Z}\right\|_{L^q(\Omega)}\right) \leq \left(n^{-1}\sum_i \mathbb{E}\left(|\mathbf{Z}_i|^q\right)\right)^{1/q}. \tag{4.5.9}$$

By independence of $(\mathbf{\Lambda}_{ij})_j$ and the fact that $\mathbb{E}(\mathbf{Z}_i)=0$, for each i, we are then in the position to apply Rosenthal's inequality, whence there exists a positive constant $C_1>0$ such that

$$\mathbb{E}\left(\left|\mathbf{Z}_i\right|^q\right) \leq C \max\Bigg(\sum_j \left(\frac{1}{\rho_n n}\right)^q \mathbb{E}(\left|\rho_n \mathbf{\Lambda}_{ij} - \rho_n \overset{\wedge}{\mathbf{K}}_{ij}\right|^q)\left|\mathbf{v}_j-\mathbf{v}_i\right|^p, \left(\sum_j \left(\frac{1}{\rho_n n}\right)^2 \mathcal{V}(\rho_n \mathbf{\Lambda}_{ij})\left|\mathbf{v}_j-\mathbf{v}_i\right|^{2(p-1)}\right)^{q/2}\Bigg).$$

Since $\rho_n^{-1}\mathbb{E}(\left|\rho_n \mathbf{\Lambda}_{ij} - \rho_n \overset{\wedge}{\mathbf{K}}_{ij}\right|^s) \leq \overset{\wedge}{\mathbf{K}}_{ij}$, for all $s\geq 2$, then we get

$$\mathbb{E}\left(\left\|I_n \mathbf{Z}\right\|_{L^q(\Omega)}\right) \leq C_1 \left(n^{-1}\sum_i \max\left(\left(\frac{1}{\rho_n n}\right)^{q-1} \frac{1}{n}\sum_j \overset{\wedge}{\mathbf{K}}_{ij}\left|\mathbf{v}_j-\mathbf{v}_i\right|^p, \left(\frac{1}{\rho_n n}\right)^{q/2}\left(\frac{1}{n}\sum_j \overset{\wedge}{\mathbf{K}}_{ij}\left|\mathbf{v}_j-\mathbf{v}_i\right|^{2(p-1)}\right)^{q/2}\right)\right)^{1/q}. \tag{4.5.10}$$

Let's start with the first term in the right-hand side of the above inequality, we have

$$\begin{aligned}
\left(\frac{1}{\rho_n n}\right)^{q-1} \frac{1}{n^2}\sum_i\sum_j \overset{\wedge}{\mathbf{K}}_{ij}\left|\mathbf{v}_j-\mathbf{v}_i\right|^p &\leq 2\left(\frac{1}{\rho_n n}\right)^{q-1}\int_{\Omega^2} \hat{K}_n(\boldsymbol{x},\boldsymbol{y})\left|v_n(\boldsymbol{y})-v_n(\boldsymbol{x})\right|^p d\boldsymbol{y}d\boldsymbol{x} \\
&\leq \left(\frac{1}{\rho_n n}\right)^{q-1}\int_{\Omega^2} \hat{K}_n(\boldsymbol{x},\boldsymbol{y})\left|v_n(\boldsymbol{y})-v_n(\boldsymbol{x})\right|^p d\boldsymbol{y}d\boldsymbol{x} \\
&\leq 2^{p-1}\left(\frac{1}{\rho_n n}\right)^{q-1}\left\|\hat{K}_n\right\|_{L^{\infty,1}(\Omega^2)}\left\|v_n\right\|^p_{L^p(\Omega)} \\
&\leq 2^{p-1}\left(\frac{1}{\rho_n n}\right)^{q-1}\left\|K\right\|_{L^{\infty,1}(\Omega^2)}\left\|v_n\right\|^p_{L^p(\Omega)},
\end{aligned} \tag{4.5.11}$$

we get the last inequality by applying Lemma 2.1.27. For the second term, we have

$$\begin{aligned}
&\left(\frac{1}{\rho_n n}\right)^{q/2} n^{-1}\sum_i\left(n^{-1}\sum_j \overset{\wedge}{\mathbf{K}}_{ij}\left|\mathbf{v}_j-\mathbf{v}_i\right|^{2(p-1)}\right)^{q/2} \\
&\leq \left(\frac{1}{\rho_n n}\right)^{q/2}\int_\Omega\left(\int_\Omega \hat{K}_n(\boldsymbol{x},\boldsymbol{y})\left|v_n(\boldsymbol{y})-v_n(\boldsymbol{x})\right|^{2(p-1)}d\boldsymbol{y}\right)^{q/2}d\boldsymbol{x} \\
&\leq \left(\frac{1}{\rho_n n}\right)^{q/2}\int_\Omega\left(\int_\Omega \hat{K}_n(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}\right)^{q/2}\left(\int_\Omega \frac{\hat{K}_n(\boldsymbol{x},\boldsymbol{y})\left|v_n(\boldsymbol{y})-v_n(\boldsymbol{x})\right|^{2(p-1)}}{\int_\Omega \hat{K}_n(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}}d\boldsymbol{y}\right)^{q/2}d\boldsymbol{x} \\
&\overset{\text{Jensen}}{\leq} \left(\frac{1}{\rho_n n}\right)^{q/2}\int_\Omega\left(\int_\Omega \hat{K}_n(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}\right)^{q/2-1}\left(\int_\Omega \hat{K}_n(\boldsymbol{x},\boldsymbol{y})\left|v_n(\boldsymbol{y})-v_n(\boldsymbol{x})\right|^p d\boldsymbol{y}\right)d\boldsymbol{x} \\
&\leq \left(\frac{1}{\rho_n n}\right)^{q/2}\left\|\hat{K}_n\right\|^{q/2-1}_{L^{\infty,1}(\Omega^2)}\int_\Omega\int_\Omega \hat{K}_n(\boldsymbol{x},\boldsymbol{y})\left|v_n(\boldsymbol{y})-v_n(\boldsymbol{x})\right|^p d\boldsymbol{y}d\boldsymbol{x} \\
&\leq 2^{p-1}\left(\frac{1}{\rho_n n}\right)^{q/2}\left\|\hat{K}_n\right\|^{q/2}_{L^{\infty,1}(\Omega^2)}\left\|v_n\right\|^p_{L^p(\Omega)} \\
&\leq 2^{p-1}\left(\frac{1}{\rho_n n}\right)^{q/2}\left\|K\right\|^{q/2}_{L^{\infty,1}(\Omega^2)}\left\|v_n\right\|^p_{L^p(\Omega)},
\end{aligned} \tag{4.5.12}$$

the last inequality follows from Lemma 2.1.27.

Plugging (4.5.11) and (4.5.12) into (4.5.10), and assembling the last with (4.5.9), we get

$$\mathbb{E}\left(\left\|I_n u_n - v_n\right\|_{L^p(\Omega)}\right) \leq C_2\left(\frac{1}{\rho_n n}\right)^{1/2}, \tag{4.5.13}$$

where

$$C_2 = C_1 2^p \sup_n \|v_n\|^{p-1}_{L^p(\Omega)} \max\left(\|K\|^{1/2}_{L^{\infty,1}(\Omega^2)}, \|K\|^{(p-1)/p}_{L^{\infty,1}(\Omega^2)}\right),$$

and $C_2 < +\infty$, thanks to Corollary 4.4.5. Now let $\varepsilon > 0$, using Markov inequality, we have

$$\begin{aligned}\mathbb{P}\left(\|I_n u_n - v_n\|_{L^p(\Omega)} \geq \varepsilon\right) &\leq \varepsilon^{-1}\mathbb{E}\left(\|I_n u_n - v_n\|_{L^p(\Omega)}\right) \\ &\leq \varepsilon^{-1} C_2 (\rho_n n)^{-1/2}.\end{aligned} \tag{4.5.14}$$

Taking $\varepsilon = \frac{C_2}{(\rho_n n)^{\beta/2}}$, we get the desired result. □

Theorem 4.5.3. *Suppose that $p \in]1,2]$. Let u be a solution of $(\mathcal{P}_D)$ with kernel J, data (f,g) and the boundary set Γ, and let $\{\mathbf{u}\}_{n\in\mathbb{N}}$ is the sequence generated by $(\mathcal{P}_n^{d,\mathbf{G}})$ with $\mathbf{K} = P_n K$, $\mathbf{f} = P_n f$, $\mathbf{g} = 0$ and the boundary set A_n. Assume that $(J, \mathbf{g}, \mathbf{K}, , \mathbf{f}, g, K, f)$ verify the assumptions of Corollary 4.4.5. Then, for any $\beta \in]0,1[$, we have*

$$\begin{aligned}\mathbb{E}\left(\|u-u_n\|_{L^p(\Omega)}\right) \leq C\Bigg(&\|J-J_n\|_{L^1(\Omega-\Omega)} + \|J-J_n\|^{\frac{1}{2}}_{L^1(\Omega-\Omega)}\|I_nP_nu-u\|^{\frac{1}{2}}_{L^p(\Omega)} \\ &+ \|I_nP_nu-u\|^{\frac{1}{3-p}}_{L^p(\Omega)} + (\rho_n n)^{-1/2}\Bigg),\end{aligned} \tag{4.5.15}$$

in turn, with probability at least $1-(\rho_n n)^{-(1-\beta)/2}$

$$\begin{aligned}\|u-u_n\|_{L^p(\Omega)} \leq C\Bigg(&\|J-J_n\|_{L^1(\Omega-\Omega)} + \|J-J_n\|^{\frac{1}{2}}_{L^1(\Omega-\Omega)}\|I_nP_nu-u\|^{\frac{1}{2}}_{L^p(\Omega)} \\ &+ \|I_nP_nu-u\|^{\frac{1}{3-p}}_{L^p(\Omega)} + (\rho_n n)^{-\beta/2}\Bigg),\end{aligned} \tag{4.5.16}$$

where $J_n = I_n P_n J$ and g_n, f_n and u_n are the continuous extensions of the functions $\mathbf{g}$, $\mathbf{f}$ and $\mathbf{u}$ respectively.

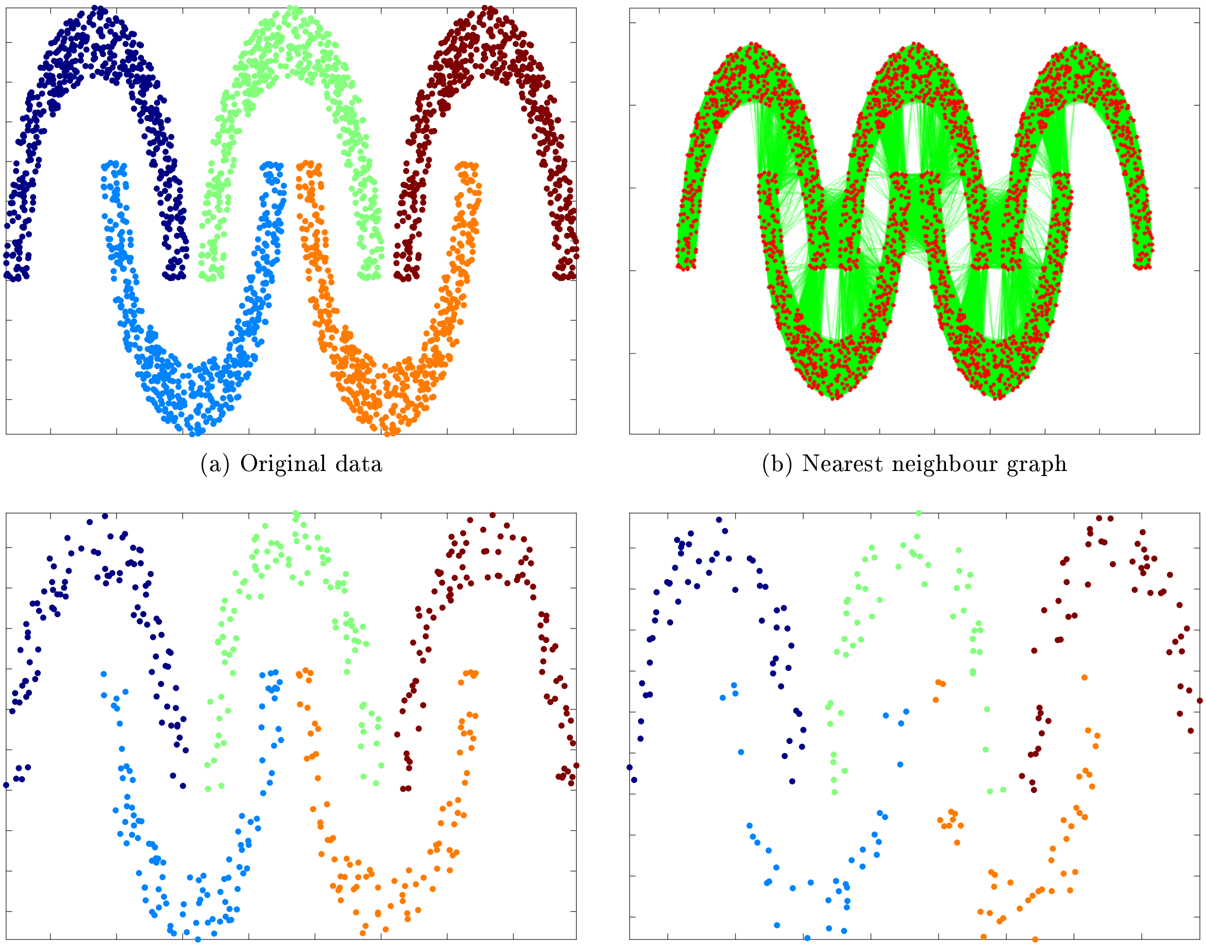
Proof : Embarking from (4.5.3), for the first term in the right-hand side, we apply the result of Lemma 4.5.1 and for the second we use the result of Corollary 4.4.5 on which we apply Jensen's inequality, we get the desired result. □

4.6 Numerical results

We apply a primal-dual proximal splitting scheme to solve $(\mathcal{VP}_n^d)$ (see Chapter 7 for details), in a semi-supervised classification problem which amounts to finding the missing labels of a label function $\mathbf{g}$ defined on a $2D/3D$ point cloud. The nodes of the graph are the points in the cloud and $\mathbf{u}_{\boldsymbol{x}}$ is the value of point/vertex $\boldsymbol{x}$. We chose the nearest neighbour graph with the standard weighting kernel $\exp^{-|x-y|}$ when $|x-y| \leq \delta$ and 0 otherwise, where x and y are the $2D/3D$ spatial coordinates of the points for the point cloud[1].In our numerical experiments, we will illustrate our results on five examples of cloud points, three in $2D$ and two in $3D$. For each point cloud, the boundary vertices (i.e. A_n^c) are chosen uniformly at random from the whole N points/vertices with two cardinalities: $|A_n^c| = N/5$ and $|A_n^c| = N/10$. Obviously, the label function $\mathbf{u}$ to be recovered agrees with $\mathbf{g}$ on A_n^c according to $(\mathcal{VP}_n^d)$. In our experiments, for each point cloud and each A_n^c, we solve $(\mathcal{VP}_n^d)$ with $\mathbf{f} = 0$ and $p \in \{1, 2, 10\}$. Although the case $p = 1$ was not covered by our study, we report the corresponding results as $(\mathcal{VP}_n^d)$

[1]For the $2D$ case, (x,y) are not to be confused with the spatial coordinates $(\boldsymbol{x},\boldsymbol{y})$ of the graph kernel on the continuum, though there is a bijection from one to another.

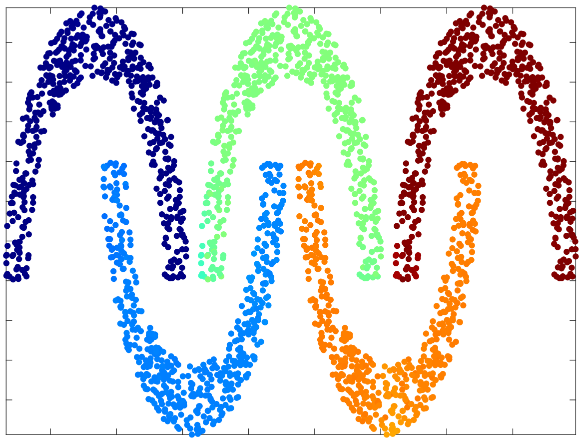
can be easily solved by our proximal splitting framework for $p = 1$ and even $p = +\infty$ (see Chapter 7). One can clearly see that the best performance is obtained (at least visually) for $p = 1$, which comes at no surprise since the underlying label function is "piecewise constant". The classification is also more accurate as the number of labeled points increases.



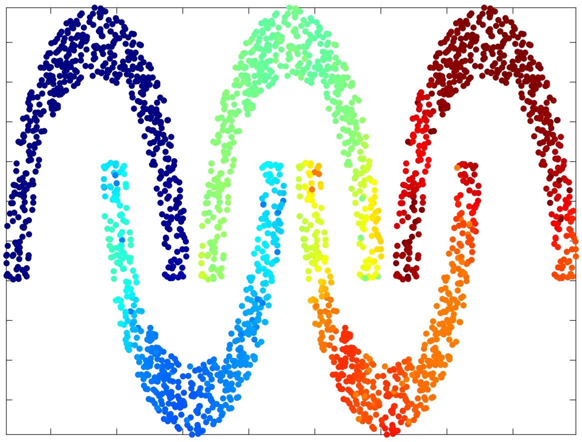
(a) Original data (b) Nearest neighbour graph

(c) Subsampled data (d) Subsampled data

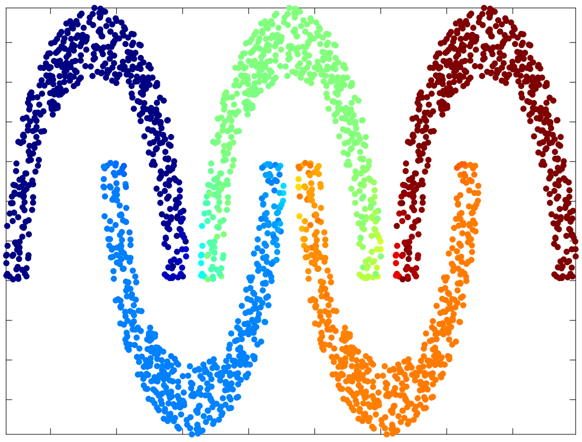
Figure 4.1: (a)The original data with $N = 2500$ points. (b) Graph considered. (c) Subsampled (boundary) data with $N/5 = 500$ points. (d) Subsampled (boundary) data with $N/10 = 250$ points.



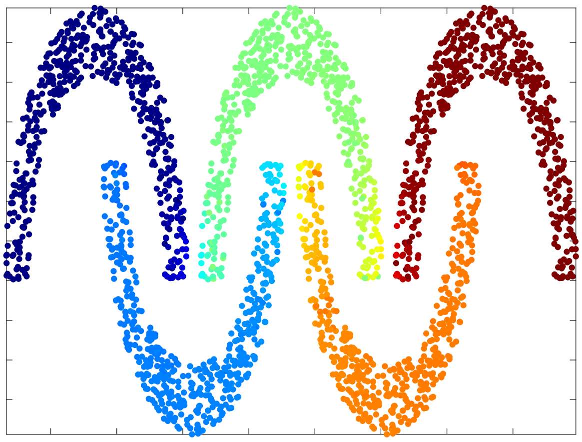
(a) $p = 1$



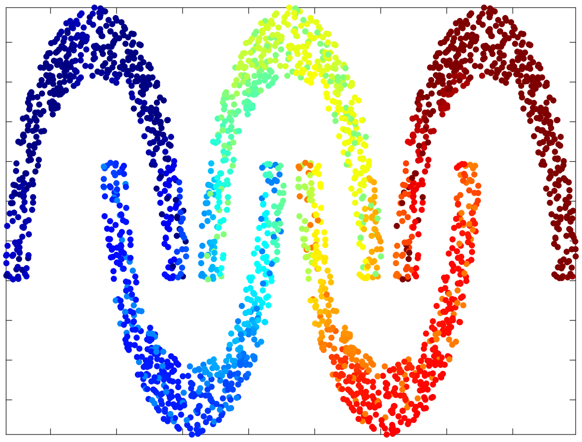
(b) $p = 1$



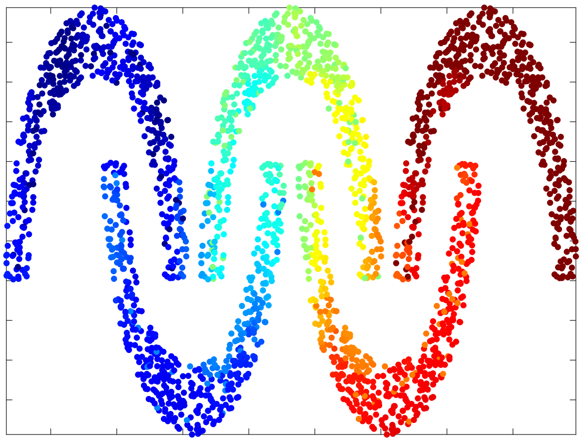
(c) $p = 2$



(d) $p = 2$

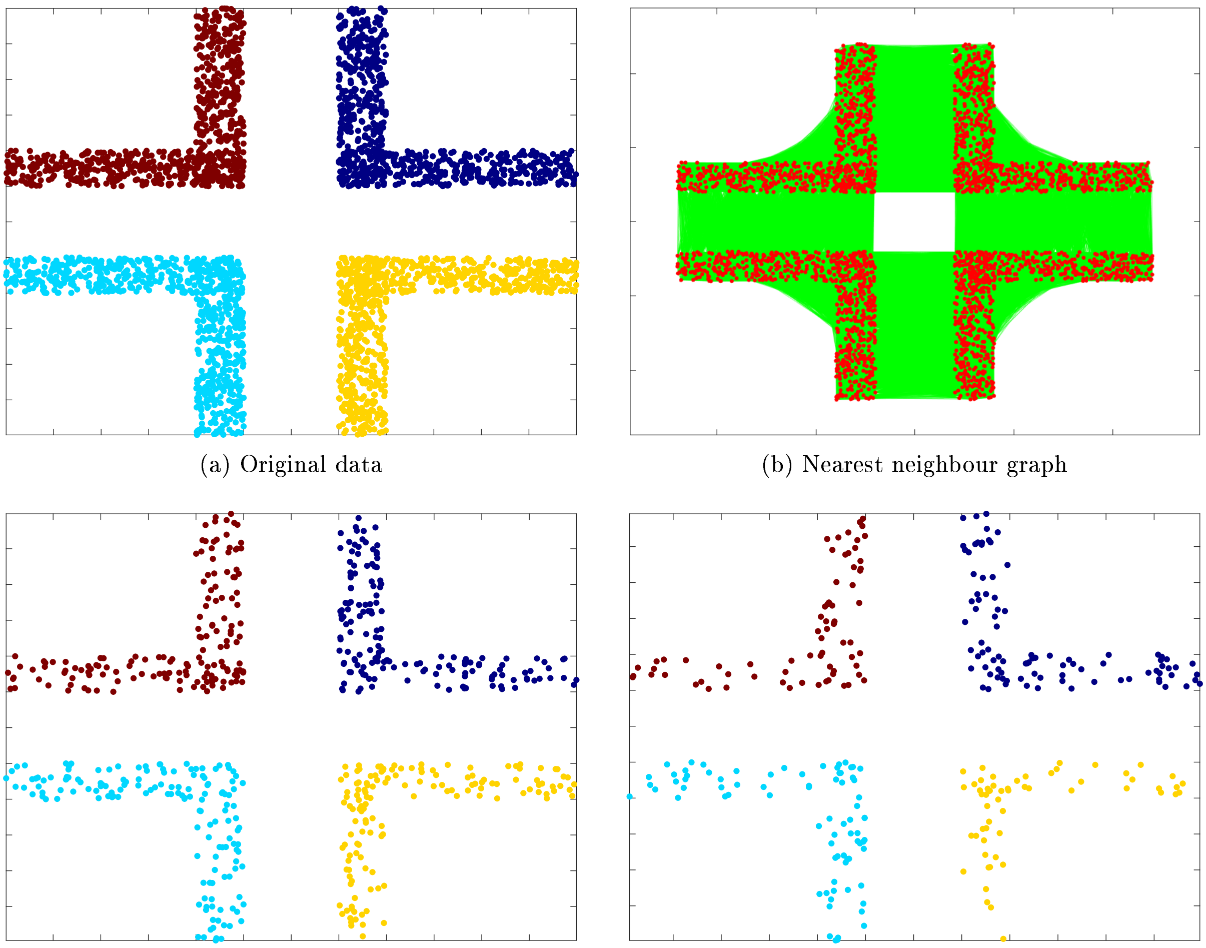


(e) $p = 10$



(f) $p = 10$

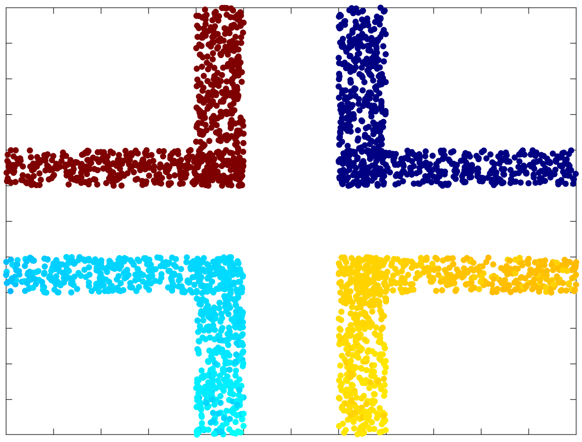
Figure 4.2: In the left-hand side, results obtained from the boundary data (c) Figure 4.1. In the right-hand side, results obtained from the boundary data (d) Figure 4.1.



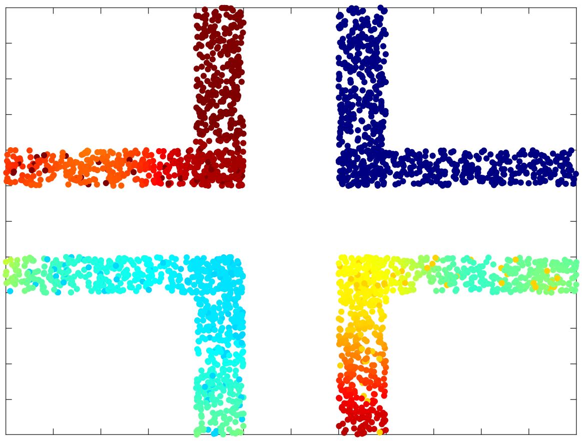
(a) Original data (b) Nearest neighbour graph

(c) Subsampled data (d) Subsampled data

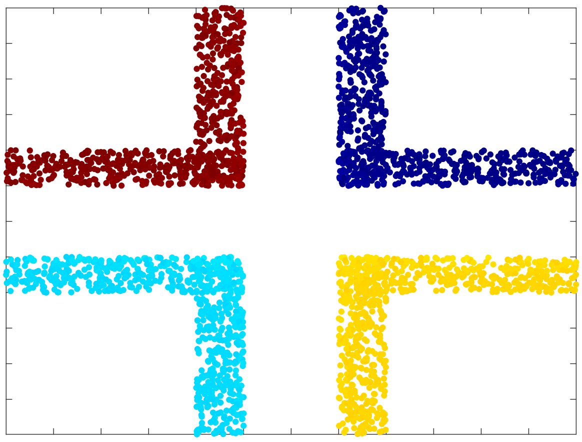
Figure 4.3: (a) The original data with $N = 3000$ points. (b) Graph considered. (c) Subsampled (boundary) data with $N/5 = 600$ points. (d) Subsampled (boundary) data with $N/10 = 300$ points.



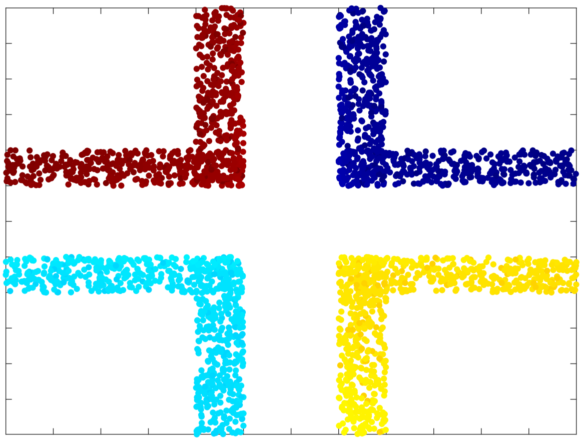
(a) $p = 1$



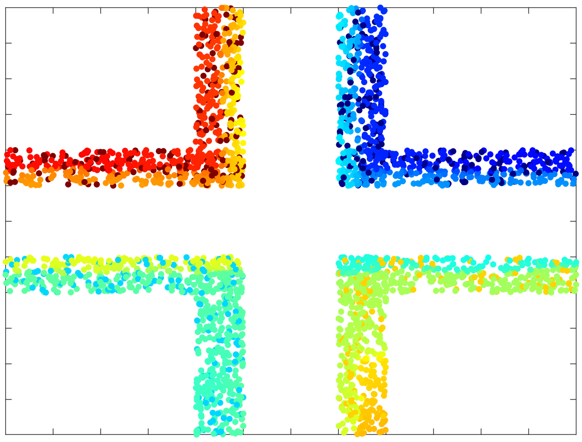
(b) $p = 1$



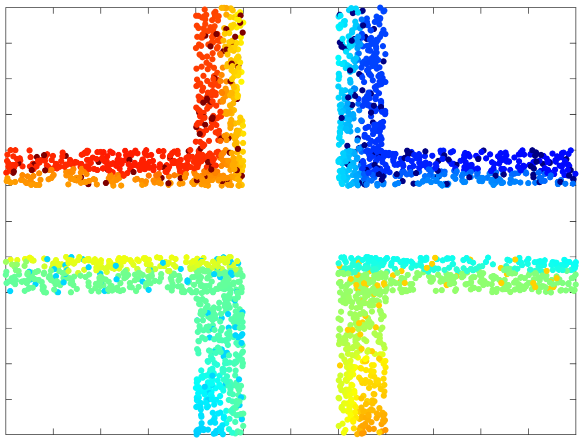
(c) $p = 2$



(d) $p = 2$

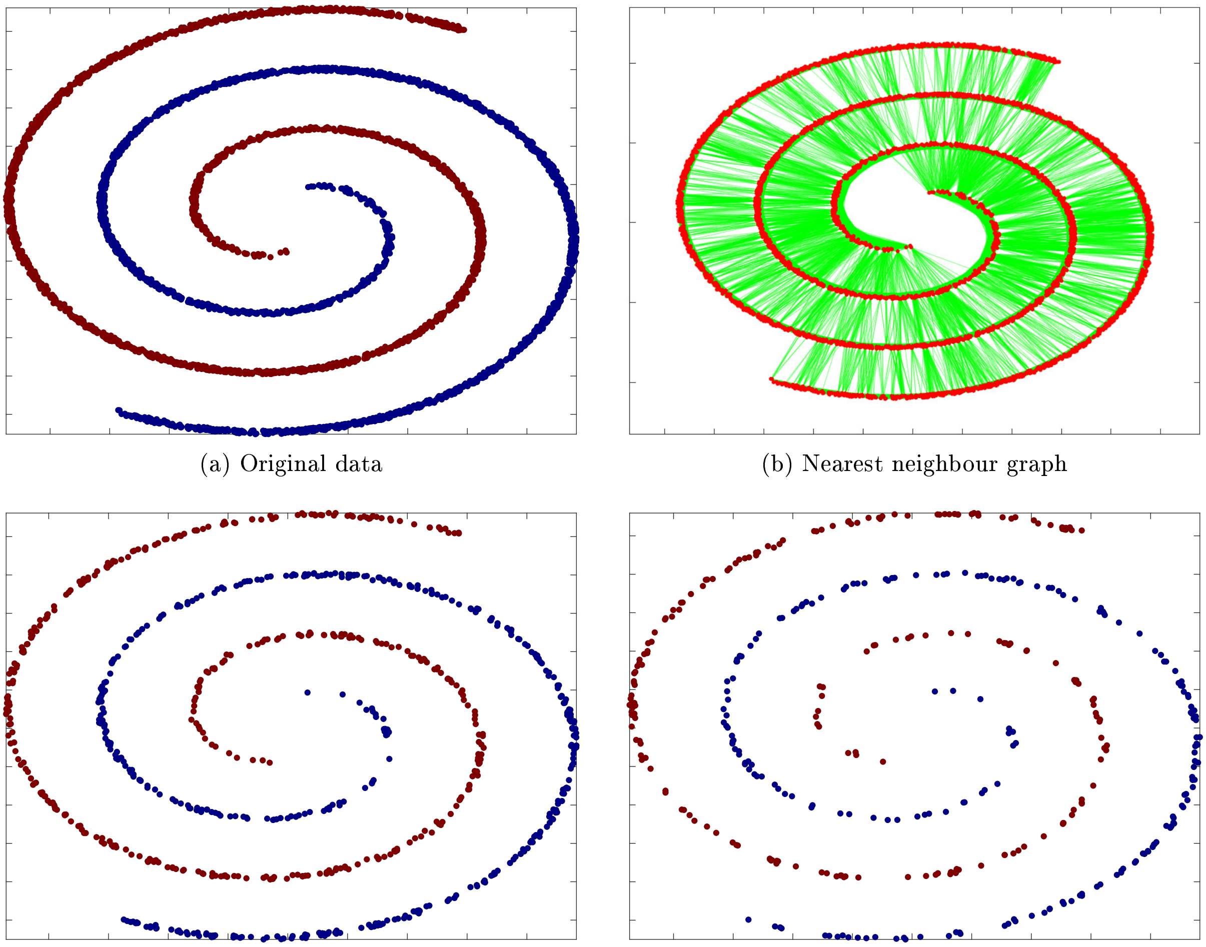


(e) $p = 10$



(f) $p = 10$

Figure 4.4: In the left-hand side, results obtained from the boundary data (c) Figure 4.3, and in the right-hand side, results obtained from the boundary data (d) Figure 4.3.



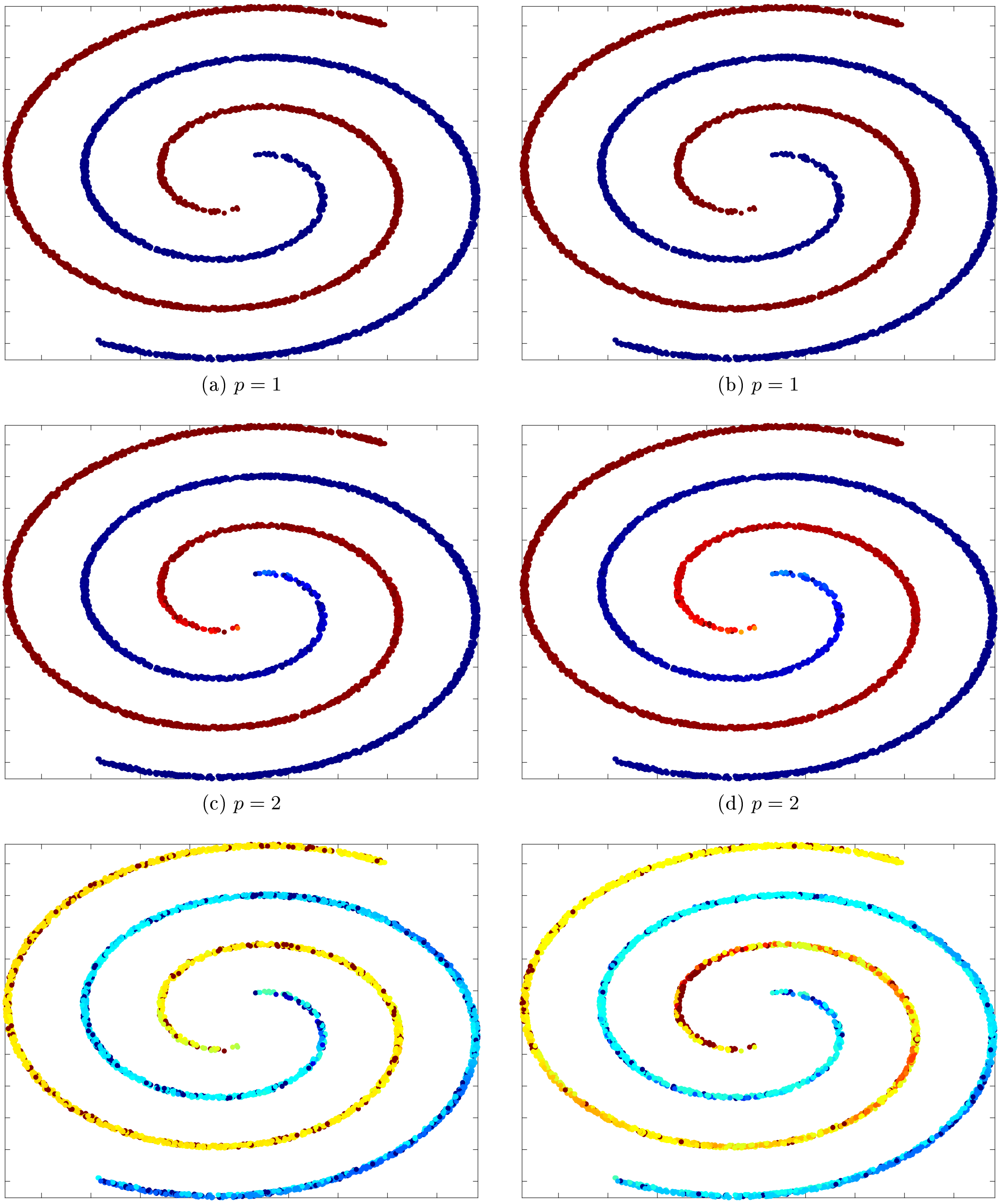
(a) Original data

(b) Nearest neighbour graph

(c) Subsampled data

(d) Subsampled data

Figure 4.5: (a) The original data with $N = 4000$ points. (b) Graph considered. (c) Subsampled (boundary) data with $N/5 = 800$ points. (d) Subsampled (boundary) data with $N/10 = 400$ points.

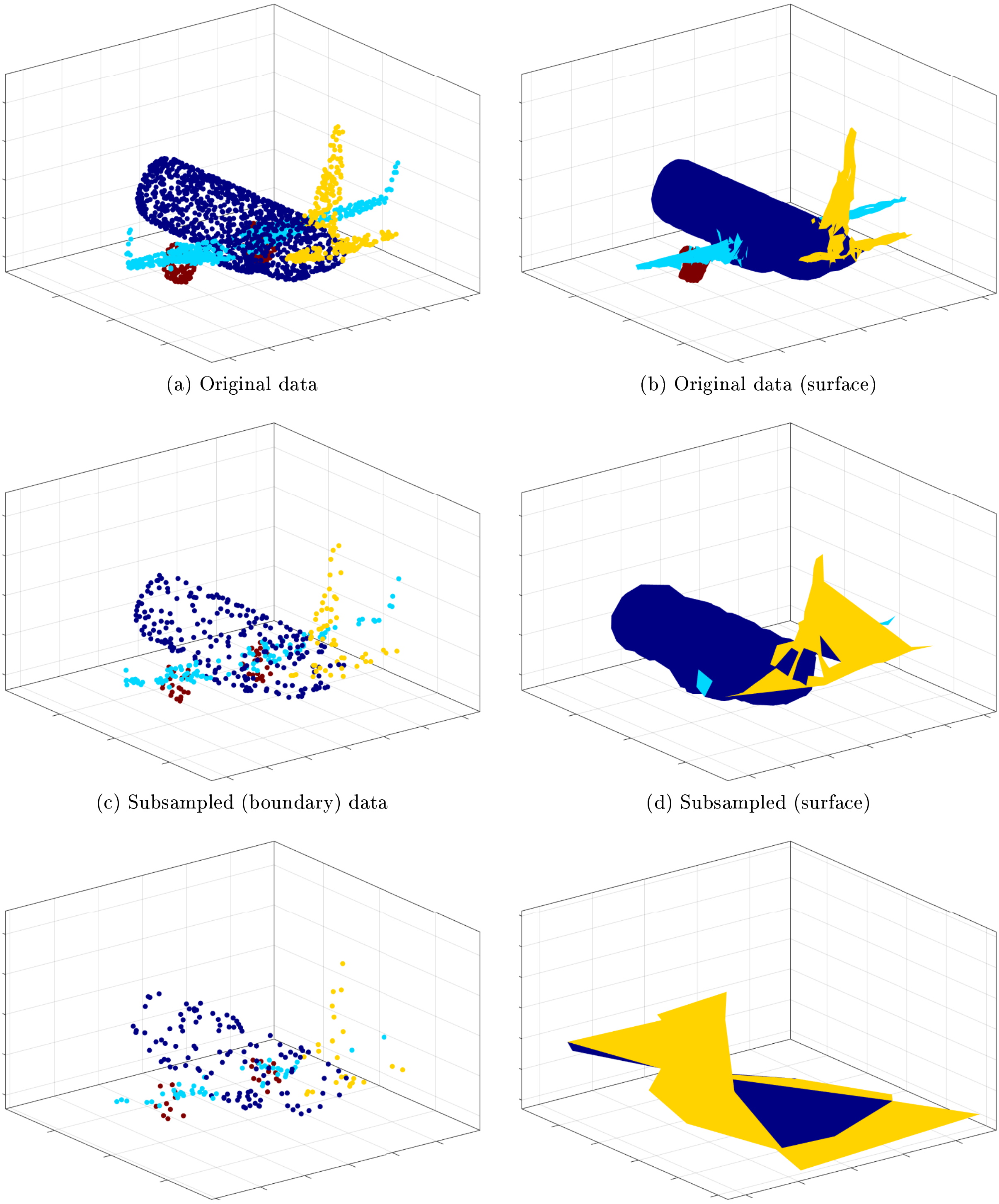


(a) $p = 1$ (b) $p = 1$

(c) $p = 2$ (d) $p = 2$

(e) $p = 10$ (f) $p = 10$

Figure 4.6: In the left-hand side, results obtained from the boundary data (c) Figure 4.5, and in the right-hand side, results obtained from the boundary data (d) Figure 4.5.



(a) Original data

(b) Original data (surface)

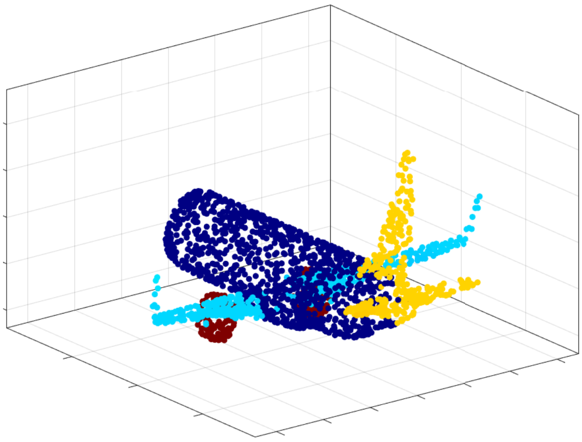
(c) Subsampled (boundary) data

(d) Subsampled (surface)

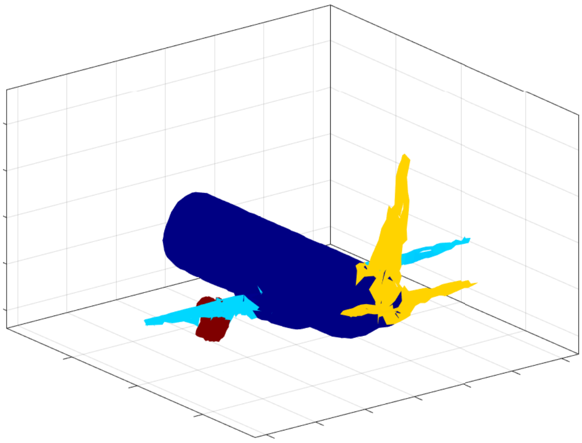
(e) Subsampled (boundary) data

(f) Subsampled data (surface)

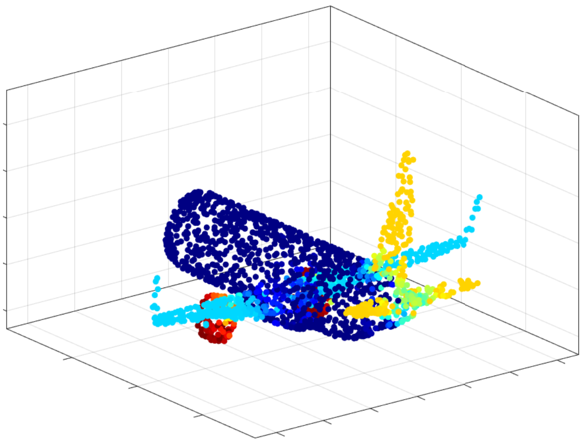
Figure 4.7: In the first line, the original data with $N = 2048$ points. In the second line, a subsampled (boundary) data with $N/5 = 409$ points. In the last line, a subsampled (boundary) data with $N/10 = 204$ points.



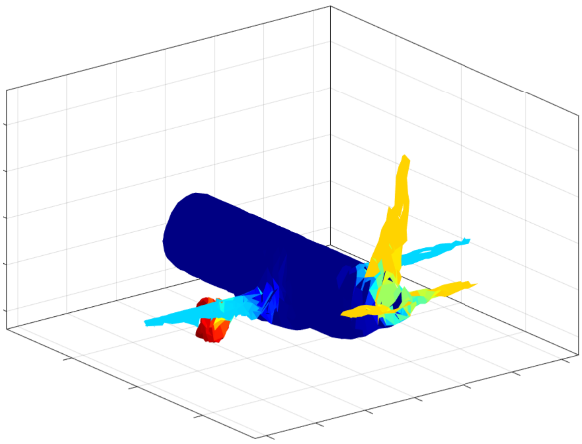
(a) $p = 1$



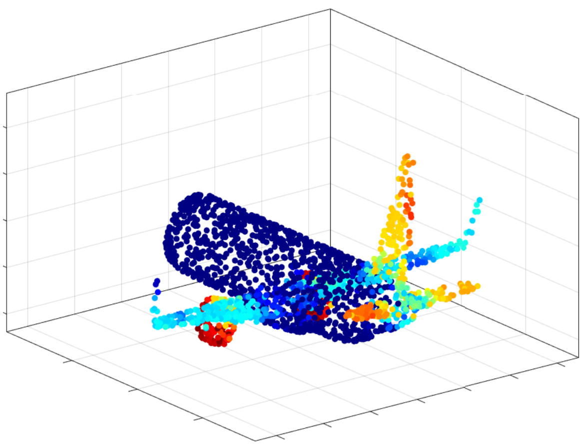
(b) $p = 1$ (surface)



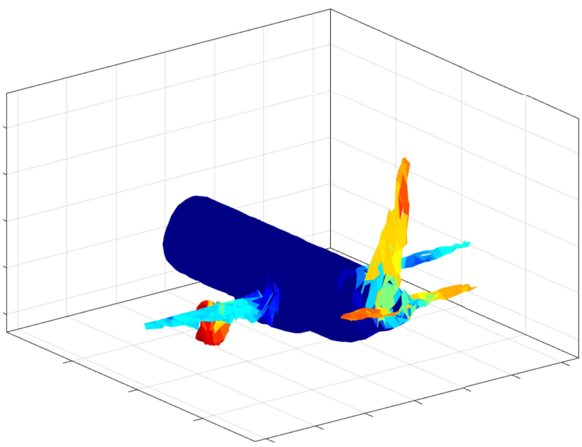
(c) $p = 2$



(d) $p = 2$ (surface)

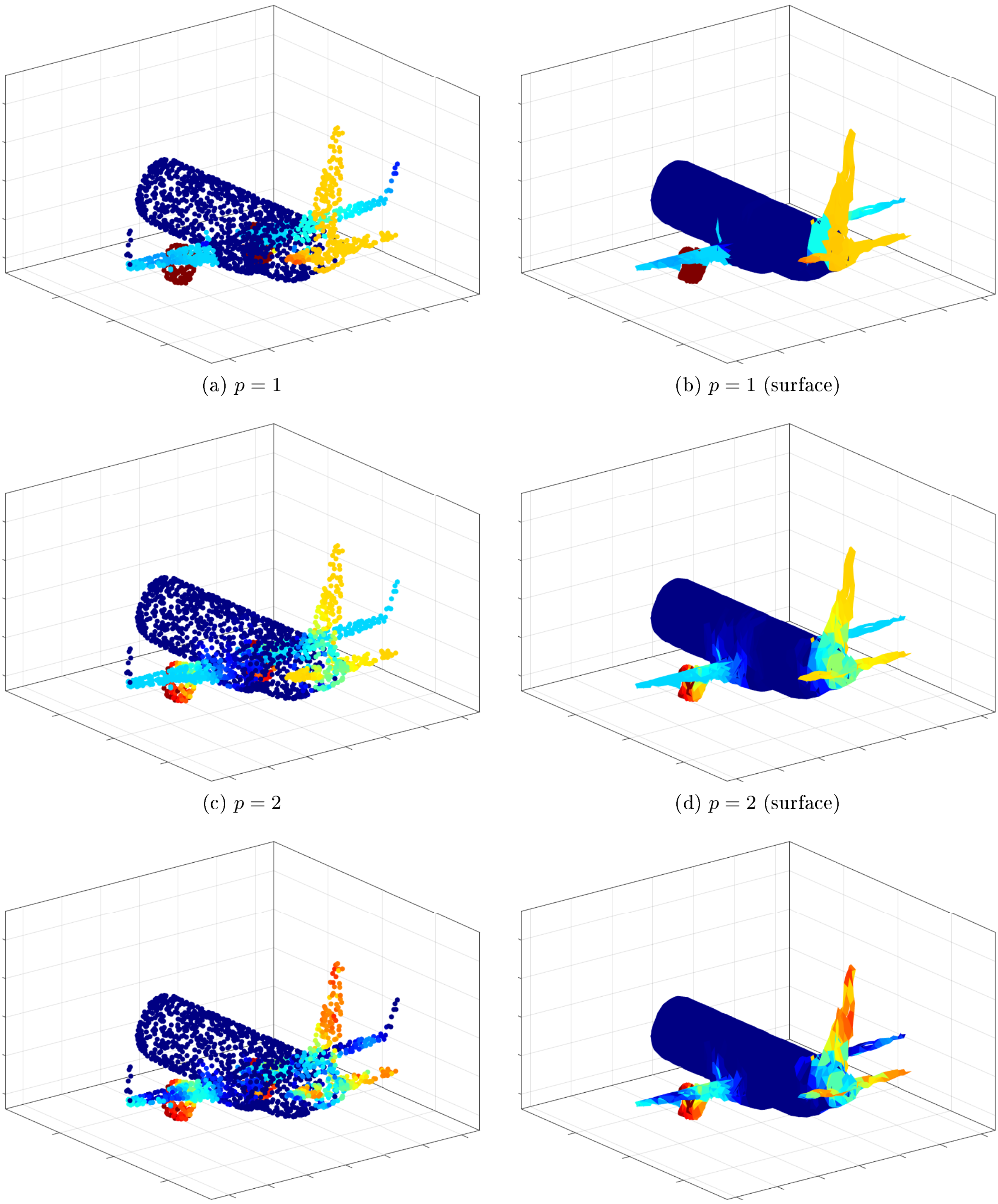


(e) $p = 10$



(f) $p = 10$ (surface)

Figure 4.8: Results obtained from the boundary data in the second line of Figure 4.7.



(a) $p = 1$

(b) $p = 1$ (surface)

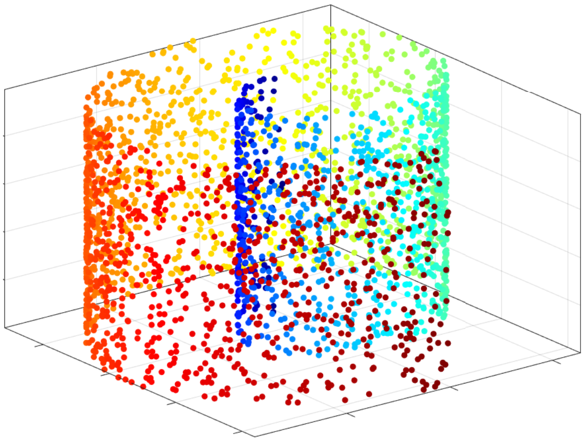
(c) $p = 2$

(d) $p = 2$ (surface)

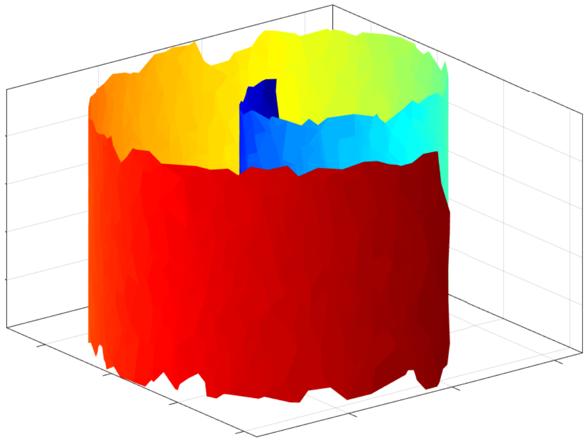
(e) $p = 10$

(f) $p = 10$ (surface)

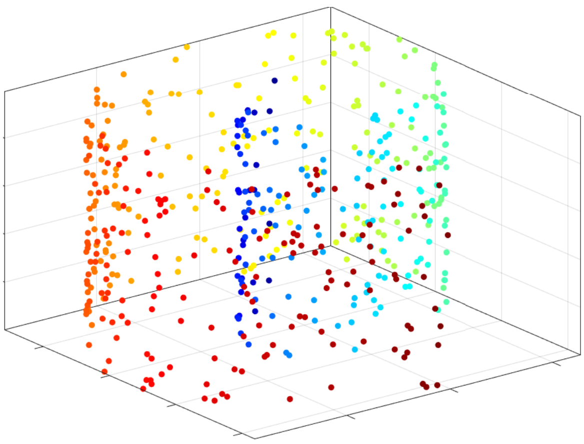
Figure 4.9: Results obtained from the boundary data in the last line of Figure 4.7.



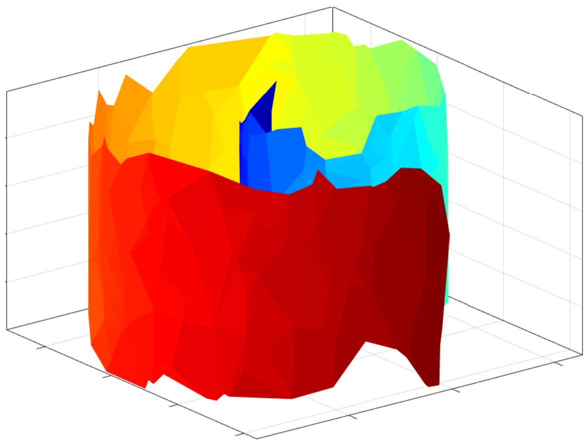
(a) Original data



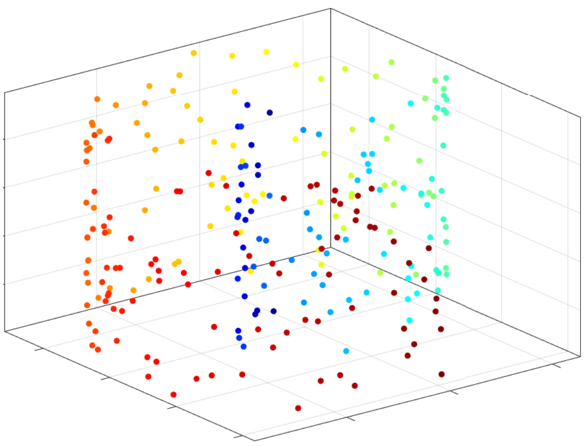
(b) Original data (surface)



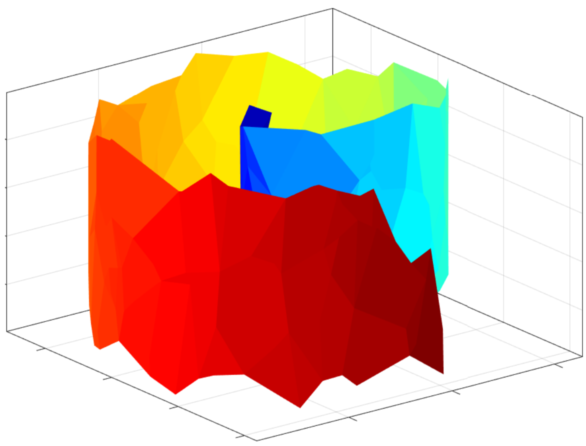
(c) Subsampled (boundary) data



(d) Subsampled data (surface)

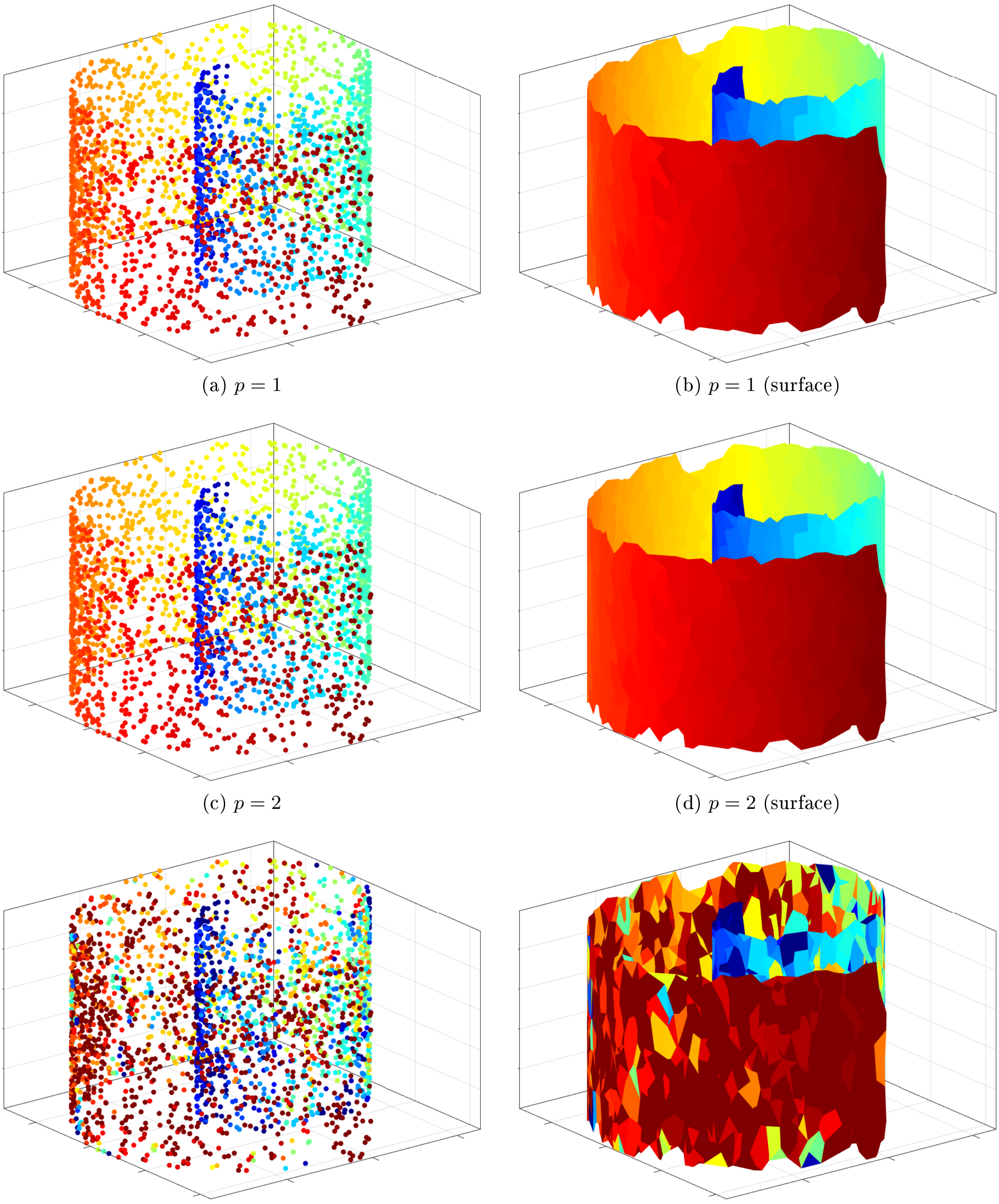


(e) Subsampled (boundary) data



(f) Subsampled data (surface)

Figure 4.10: In the first line, original data with $N = 2048$ points. In the second line, a subsampled (boundary) data with $N/5 = 409$ points. In the last line, a subsampled (boundary) data with $N/10 = 204$ points.

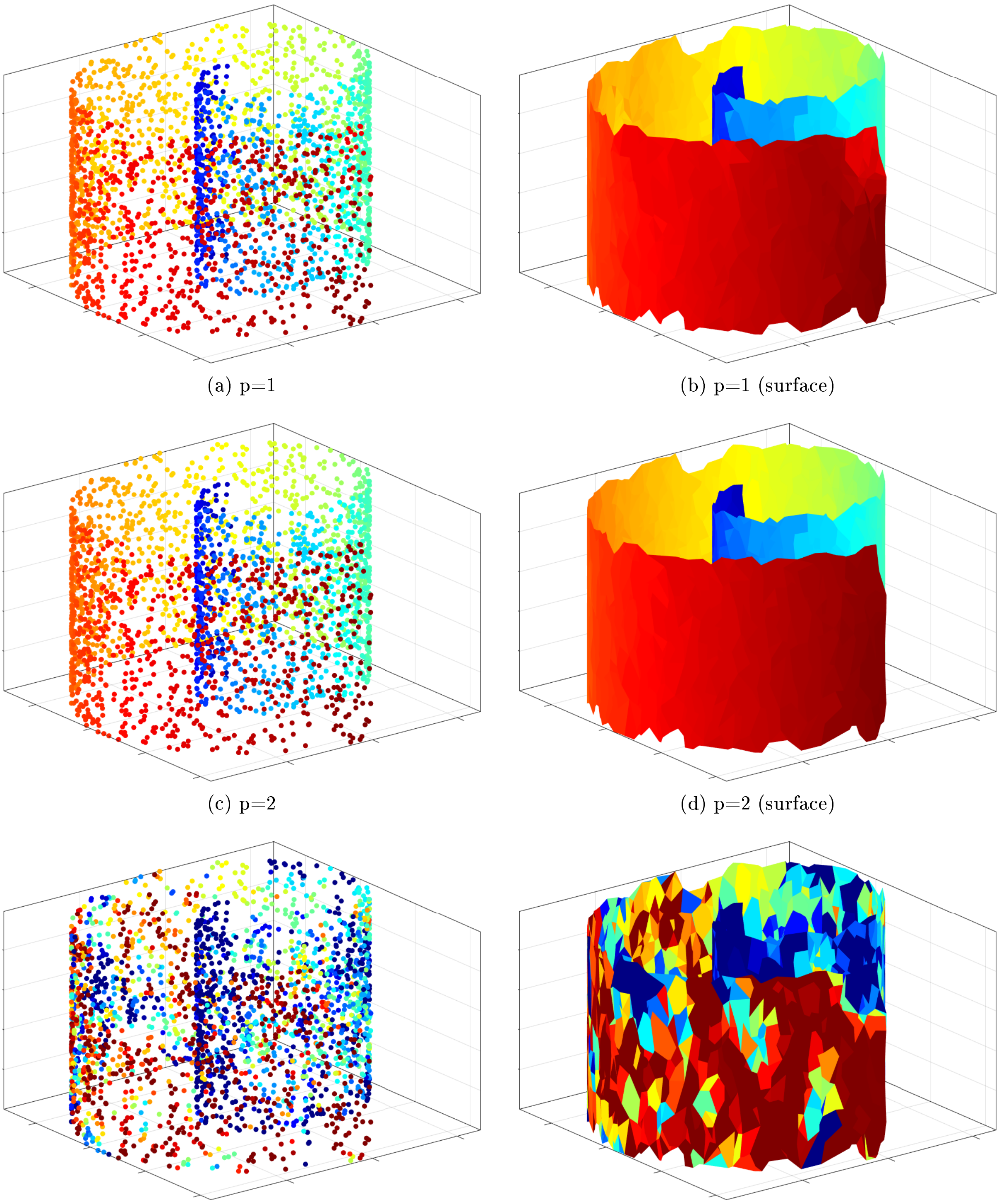


(a) $p = 1$ (b) $p = 1$ (surface)

(c) $p = 2$ (d) $p = 2$ (surface)

(e) $p = 10$ (f) $p = 10$ (surface)

Figure 4.11: Results obtained from the boundary data in the second line of Figure 4.10.



(a) p=1

(b) p=1 (surface)

(c) p=2

(d) p=2 (surface)

(e) p=10

(f) p=10 (surface)

Figure 4.12: Results obtained from the boundary data in the last line of Figure 4.10.

Chapter 5

On the discrete p-bilaplacian operator on graphs

Main contributions of this chapter

- We introduce a new family of operators on weighted graphs called p-bilaplacian operators.
- We study regularized variational problem associated to these operators.
- We study also boundary value problems associated to these operators.

The content of this chapter appeared in [68].

Contents

In this chapter, we introduce a new family of operators on weighted graphs called p-bilaplacian operators, which are the analogue on graphs of the continuous p-bilaplacian operators. We then turn to study regularized variational and boundary value problems associated to these operators. We study their well-posedness, we prove the existence and the uniqueness of the solutions. We finally report some numerical experiments to support our findings.

5.1 p-biharmonic functions on graphs

In this section, we introduce p-bilaplacian operator on weighted graphs, inspired by the way p-harmonic functions were introduced in [95] for networks. Throughout this chapter, we adopt the notation of Section 2.4. As in the continuous case [99], let's consider the energy functional

$$\mathcal{F}_d(u;p) \stackrel{\text{def}}{=} \frac{1}{p}\left\|\Delta_{\omega,2}u\right\|_p^p = \frac{1}{p}\sum_{x\in V}\left|\Delta_{\omega,2}u(x)\right|^p. \tag{5.1.1}$$

Definition 5.1.1. We define the p-bilaplacian operator for a function $u \in \mathcal{H}(V)$ as

$$\Delta^2_{\omega,p}u(x) \stackrel{\text{def}}{=} \Delta_{\omega,2}\left(\left|\Delta_{\omega,2}u\right|^{p-2}\Delta_{\omega,2}u\right)(x), \quad x \in V.$$

Definition 5.1.2. Let $\mathcal{A} \subset V$. We say that a function u is p-biharmonic in $\mathcal{A}$ if it is a minimiser of the functional $\mathcal{F}_d(\cdot;p)$ among functions in V with the same values in $\mathcal{A}^c = V \setminus \mathcal{A}$, that is, if

$$\mathcal{F}_d(u;p) \leqslant \mathcal{F}_d(v;p)$$

for every function $v \in \mathcal{H}(V)$, with $u = v$ in $\mathcal{A}^c$.

As a first result, we prove the following characterization of the p-biharmonic functions

Theorem 5.1.3. *Let $\mathcal{A} \subset V$. A function u is p-biharmonic in $\mathcal{A}$ if and only if*

$$\sum_{x\in V}\left|\Delta_{\omega,2}u(x)\right|^{p-2}\Delta_{\omega,2}u(x)\Delta_{\omega,2}w(x) = 0, \qquad x \in \mathcal{A}, \tag{5.1.2}$$

for every function $w \in \mathcal{H}(V)$, with $w = 0$ in $\mathcal{A}^c$.

Proof : Suppose that u is p-biharmonic function in $\mathcal{A}$ and let $u_t = u + tw$, where $t \in \mathbb{R}$ and w is a test function, that is, $w \in \mathcal{H}(V)$ with $w = 0$ in $\mathcal{A}^c$. Since the function u minimizes the energy functional, then

$$0 = \frac{d}{dt}(\mathcal{F}_d(u_t;p))_{|t=0} = \sum_{x\in V}\left|\Delta_{\omega,2}u(x)\right|^{p-2}\Delta_{\omega,2}u(x)\Delta_{\omega,2}w(x).$$

Assume that u satisfies (5.1.2) for all test functions. Let $v \in \mathcal{H}(V)$ with $u = v$ in $\mathcal{A}^c$. The equation (5.1.2) applied to $w = v - u$ and Young's inequality yield

$$\sum_{x\in V}\left|\Delta_{\omega,2}u(x)\right|^p \quad = \quad \sum_{x\in V}\left|\Delta_{\omega,2}u(x)\right|^{p-2}\Delta_{\omega,2}u(x)\Delta_{\omega,2}v(x)$$

$$\leqslant \sum_{x\in V} \left|\Delta_{\omega,2}u(x)\right|^{p-1}\left|\Delta_{\omega,2}v(x)\right|$$
$$\leqslant \sum_{x\in V} \left[\frac{p-1}{p}\left|\Delta_{\omega,2}u(x)\right|^{p} + \frac{1}{p}\left|\Delta_{\omega,2}v(x)\right|^{p}\right].$$

Hence $\mathcal{F}_d(u;p) \leqslant \mathcal{F}_d(v;p)$, and so that u is p-biharmonic. □

Observe that $\Delta_{\omega,2}$ the standard Laplacian on graphs is a self-adjoint operator, i.e., for all $u, v \in \mathcal{H}(V)$

$$\sum_{x\in V} u(x)\Delta_{\omega,2}v(x) = \sum_{x\in V} v(x)\Delta_{\omega,2}u(x).$$

Now, let w be an arbitrary test function in $\mathcal{H}(V)$. We can write $w = \sum_{y\in V} w_y$, where $w_y(x) = 0$ for all $x \in V \setminus \{y\}$. Then

$$\begin{aligned}\sum_{x\in V} \left|\Delta_{\omega,2}u(x)\right|^{p-2}\Delta_{\omega,2}u(x)\Delta_{\omega,2}w(x) &= \sum_{x\in V} w(x)\Delta_{\omega,2}\left(\left|\Delta_{\omega,2}u\right|^{p-2}\Delta_{\omega,2}u\right)(x) \\ &= \sum_{x\in V} w_x(x)\Delta_{\omega,2}\left(\left|\Delta_{\omega,2}u\right|^{p-2}\Delta_{\omega,2}u\right)(x).\end{aligned} \tag{5.1.3}$$

From this, we get the following theorem.

Theorem 5.1.4. *Let $\mathcal{A} \subset V$. A function u is p-biharmonic in $\mathcal{A}$ if and only if*

$$\Delta_{\omega,2}\left(\left|\Delta_{\omega,2}u\right|^{p-2}\Delta_{\omega,2}u\right)(x) = 0, \qquad \textit{fo all } x \in \mathcal{A}.$$

PROOF : Suppose that u is p-biharmonic function in $\mathcal{A}$. Fixed $x \in \mathcal{A}$ and let $w(x) = 1$ and $w = 0$ elsewhere. Then (5.1.2) is true for w and we have

$$\Delta^2_{\omega,p}u(x) = \sum_{y\in V} \left|\Delta_{\omega,2}u(y)\right|^{p-2}\Delta_{\omega,2}u(y)\Delta_{\omega,2}w(y) = 0$$

by (5.1.3). Conversely, it follows from (5.1.3) that (5.1.2) holds for all test function w if $\Delta^2_{\omega,p}u(x) = 0$ for all $x \in \mathcal{A}$. □

5.2 p-bilaplacian variational problem on graphs

In this section, we consider the following minimization problem, which is valid for any $p \in [1, +\infty]$[1],

$$\min_{u\in\mathcal{H}(V)} \left\{E(u;p) \stackrel{\text{def}}{=} \frac{1}{2}\left\|f - Au\right\|_2^2 + \lambda\mathcal{F}_d(u;p)\right\}, \tag{5.2.1}$$

where $A : \mathcal{H}(V) \to \mathcal{H}(V)$ is a linear operator, $f \in \mathcal{H}(V)$, $\lambda > 0$ is the regularization parameter, and $\mathcal{F}_d(\cdot;p)$ is given by (5.1.1). Problems of the form (5.2.1) can be of great interest for graph-based regularization in machine learning and inverse problems in imaging; see [91] and references therein. Problem 5.2.1 is well-posed under standard assumptions.

Theorem 5.2.1. *The set of minimizers of $E(\cdot;p)$ is non-empty and compact if and only if* $\mathrm{Ker}(A) \cap \mathrm{Ker}(\Delta_{\omega,2}) = \{0\}$. *If, moreover, either A is injective or $p \in]1, +\infty[$, then $E(\cdot;p)$ has a unique minimizer.*

PROOF : For any proper lsc convex function f, recall its recession function from [16, Chapter 2], denoted f_∞. We have from the calculus rules in [16, Chapter 2] that

$$E_\infty(d;p) = \lambda\left(\tfrac{1}{p}\|\cdot\|_p^p\right)_\infty(\Delta_{\omega,2}d) + \frac{1}{2}\left(\|f-\cdot\|_2^2\right)_\infty(Ad).$$

[1] Obviously $\lim_{p\to+\infty} \frac{1}{p}\|\cdot\|_p^p = \iota_{\|u\|_\infty \leq 1}$.

Since $\frac{1}{p}\|\cdot\|_p^p$ and $\|f-\cdot\|_2^2$ are non-negative and coercive, we have from [16, Proposition 3.1.2] that their recession functions are positive for any non-zero argument. Equivalently,

$$E_\infty(d;p) > 0, \quad \forall d \notin \mathrm{Ker}(A) \cap \mathrm{Ker}(\Delta_{\omega,2}).$$

Thus $E_\infty(d;p) > 0$ for all $d \neq 0$ if and only if $\mathrm{Ker}(A) \cap \mathrm{Ker}(\Delta_{\omega,2}) = \{0\}$. Equivalence with the existence and compactness assertion follows from [16, Proposition 3.1.3].

Let's turn to uniqueness. When A is injective, the claim follows from strict (in fact strong convexity) of the data fidelity term. Suppose now that $p \in]1,+\infty[$. By strict convexity of $\frac{1}{p}\|\cdot\|_p^p$ and $\|f-\cdot\|_2^2$, a standard contradiction argument shows that for any pair of minimizers $u^\star$ and $v^\star$, we have $u^\star - v^\star \in \mathrm{Ker}(A) \cap \mathrm{Ker}(\Delta_{\omega,2})$. This yields the uniqueness claim under the stated assumption. □

5.3 p-bilaplacian Dirichlet problem on graphs

Let us consider the following boundary value problem

$$\begin{cases} -\Delta^2_{\omega,p} u = f, & \text{on } \mathcal{A} \\ u = g, & \text{on } \mathcal{A}^c, \end{cases} \tag{5.3.1}$$

where $f, g \in \mathcal{H}(V)$, $p \in]1,+\infty[$, $\Delta^2_{\omega,p}$ is the p-bilaplacian operator and $\mathcal{A} \subset V$. Observe that since the graph G is connected, there always exists a path connecting any pair vertices in $\mathcal{A} \times \mathcal{A}^c$. Denoted

$$\begin{aligned} \mathcal{H}_0(V;\mathcal{A}) &= \{u \in \mathcal{H}(V) : \quad u = 0 \text{ on } \mathcal{A}^c\} \\ \mathcal{H}_g(V;\mathcal{A}) &= \{u \in \mathcal{H}(V) : \quad u = g \text{ on } \mathcal{A}^c\} \\ &= g + \mathcal{H}_0(V;\mathcal{A}). \end{aligned}$$

The main objective of this section is to study the boundary value problem (5.3.1). For this purpose, let us consider the following functional defined in $\mathcal{H}(V)$ as

$$\mathcal{F}(u) = \frac{1}{p}\sum_{x\in V} \left|\Delta_{\omega,2} u(x)\right|^p + \sum_{x \in V} u(x) f(x), \qquad u \in \mathcal{H}(V).$$

We have the Dirichlet's principle formulation associated to the p-bilaplacian Dirichlet problem :

Theorem 5.3.1. *1. Assume $u \in \mathcal{H}_g(V;\mathcal{A})$ solves the problem (5.3.1). Then*

$$\mathcal{F}(u) \leqslant \mathcal{F}(v), \tag{5.3.2}$$

for all $v \in \mathcal{H}_g(V;\mathcal{A})$.

2. Conversely, if $u \in \mathcal{H}_g(V;\mathcal{A})$ satisfies (5.3.2) for every $v \in \mathcal{H}_g(V;\mathcal{A})$, then u solves the problem (5.3.1).

Proof : Assume $u \in \mathcal{H}_g(V;\mathcal{A})$ solves the problem (5.3.1). Let $v \in \mathcal{H}_g(V;\mathcal{A})$ and set $w = v - u$. We have

$$\begin{aligned} 0 &= \sum_{x\in V} \Delta^2_{\omega,p} u w + \sum_{x\in V} f(x) w(x) \\ &= \sum_{x\in V} \Delta_{\omega,2}(\left|\Delta_{\omega,2} u\right|^{p-2} \Delta_{\omega,2} u) w + \sum_{x\in V} f(x) w(x) \\ &= \sum_{x\in V} \left|\Delta_{\omega,2} u\right|^{p-2} \Delta_{\omega,2} u \Delta_{\omega,2} w + \sum_{x\in V} f(x) w(x). \end{aligned}$$

Thus

$$\sum_{x\in V} |\Delta_{\omega,2} u|^p + \sum_{x\in V} f(x) u(x) \quad = \quad \sum_{x\in V} \left|\Delta_{\omega,2} u\right|^{p-2} \Delta_{\omega,2} u \Delta_{\omega,2} v + \sum_{x\in V} f(x) v(x)$$

$$\begin{aligned} &\leqslant \sum_{x\in V} \left|\Delta_{\omega,2}u\right|^{p-1}\left|\Delta_{\omega,2}v\right| + \sum_{x\in V} f(x)v(x) \\ &\overset{\text{Young}}{\leqslant} \sum_{x\in V}\left(\frac{p-1}{p}\left|\Delta_{\omega,2}u\right|^p + \frac{1}{p}\left|\Delta_{\omega,2}v\right|^p\right) + \sum_{x\in V} f(x)v(x). \end{aligned}$$

Hence $\mathcal{F}(u) \leqslant \mathcal{F}(v)$.

Conversely, assume $u \in \mathcal{H}_g(V;\mathcal{A})$ satisfies (5.3.2) for every $v \in \mathcal{H}_g(V;\mathcal{A})$. Fixed $w \in \mathcal{H}_0(V;\mathcal{A})$ and set $i(t) \overset{\text{def}}{=\!=} \mathcal{F}(u+tw)$, $t \in \mathbb{R}$. Then $i(\cdot)$ attains its minimun at $t=0$. By usual calculus, we obtain

$$\begin{aligned} 0 &= i'(0) \\ &= \sum_{x\in V} \left|\Delta_{\omega,2}u\right|^{p-2}\Delta_{\omega,2}u\Delta_{\omega,2}w + \sum_{x\in V} f(x)w(x) \\ &= \sum_{x\in V} \Delta_{\omega,2}(\left|\Delta_{\omega,2}u\right|^{p-2}\Delta_{\omega,2}u)w + \sum_{x\in V} f(x)w(x). \end{aligned}$$

Since w is an arbitrary function in $\mathcal{H}_0(V;\mathcal{A})$, u is a solution of the problem (5.3.1). □

The coming result presents a Poincaré-type inequality, which plays a key role to prove the coercivity of the energy functional $\mathcal{F}(\cdot)$. It can be seen as a discrete version of that one exposed in Chapter 4. It can also allow us to revisit and extend the result concerning the discrete p-Lapalcian Dirichlet problem on general connected weighted graphs.

Lemma 5.3.2 (Poincaré inequality on graph). *There is $\lambda = \lambda(\omega, V, \mathcal{A}, p) > 0$ such that*

$$\lambda \sum_{x\in\mathcal{A}} |u(x)|^p \leqslant \sum_{x\in V}\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}\left|u(y)-u(x)\right|^p + \sum_{x\in\mathcal{A}^c}\left|g(x)\right|^p, \tag{5.3.3}$$

for all $u \in \mathcal{H}_g(V;\mathcal{A})$.

Proof : Let

$$\begin{aligned} S_0 &= \mathcal{A}^c; \\ S_1 &= \{x \in V \setminus S_0 : \exists y \in S_0;\, y \sim x\}, \\ S_{j+1} &= \{x \in V \setminus (\cup_{k=0}^j S_k) : \exists y \in S_j;\, y\sim x\}, \quad j = 1,2,\cdots \end{aligned}$$

Since the graph G is connected, there is $l \in \mathbb{N}$ such that $\{S_j\}_{j=0}^l$ forms a partition of V. Now, we have

$$\sum_{x\in V}\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}|u(y)-u(x)|^p \geqslant \sum_{x\in S_j}\sum_{y\in S_{j-1}}(\omega(x,y))^{\frac{p}{2}}\left|u(y)-u(x)\right|^p,$$

$j = 1,\cdots,l$, and

$$\begin{aligned} &\sum_{x\in S_j}\sum_{y\in S_{j-1}}(\omega(x,y))^{\frac{p}{2}}\left|u(y)-u(x)\right|^p \\ &\geqslant \frac{1}{2^p}\sum_{x\in S_j}\sum_{y\in S_{j-1}}(\omega(x,y))^{\frac{p}{2}}\left|u(x)\right|^p - \sum_{x\in S_j}\sum_{y\in S_{j-1}}(\omega(x,y))^{\frac{p}{2}}\left|u(y)\right|^p \\ &\geqslant \frac{1}{2^p}\alpha_0\sum_{x\in S_j}\left|u(x)\right|^p - \beta\sum_{y\in S_{j-1}}\left|u(y)\right|^p, \end{aligned}$$

where $\alpha_0 = \min\{(\omega(x,y))^{\frac{p}{2}} : (x,y) \in E\}$ and $\beta = \sum_{x,y\in V}(\omega(x,y))^{\frac{p}{2}}$. Hence

$$\sum_{x\in V}\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}\left|u(y)-u(x)\right|^p \geqslant \alpha\sum_{x\in S_j}\left|u(x)\right|^p - \beta\sum_{y\in S_{j-1}}\left|u(y)\right|^p,$$

where

$$\alpha = \frac{1}{2^p}\alpha_0 > 0.$$

Therefore, since $u = g$ in $S_0 = \mathcal{A}^c$ and $\{S_j\}_{j=0}^{l}$ forms a partition of V, it is easy to see that there exists $\hat{\lambda} = \hat{\lambda}(\omega, V, \mathcal{A}, p) > 0$ such that

$$\sum_{x\in A} |u(x)|^p \leqslant \hat{\lambda} \sum_{x\in V}\sum_{y\sim x}(\omega(x,y))^{\frac{p}{2}}\big|u(y)-u(x)\big|^p + \hat{\lambda}\sum_{x\in A^c}\big|g(x)\big|^p.$$

We arrive at the coercivity result by taking $\lambda = \hat{\lambda}^{-1}$. □

Lemma 5.3.3. *The functional $\mathcal{F}(\cdot)$ is coercive and strictly convex on $\mathcal{H}_g(V;\mathcal{A})$.*

PROOF : For the coercivity, we distinguish three cases when $p > 2$, $p = 2$ and $1 < p < 2$. Let q be the Hölder conjugate of p.
By Lemma 5.3.2, there exists $\lambda > 0$ such that

$$\lambda\sum_{x\in V}|u(x)|^2 \leqslant \sum_{x\in V}\sum_{y\sim x}\omega(x,y)\big|u(y)-u(x)\big|^2 + C_{f_0}, \quad \textit{for all } u\in\mathcal{H}_g(V;\mathcal{A}), \tag{5.3.4}$$

where $C_g \geqslant 0$ depend only on g.
By Young's inequality, for $\epsilon \in (0,1)$ we have

$$\begin{aligned}\sum_{x\in V} f(x)u(x) &\geqslant -\sum_{x\in V}\big|u(x)f(x)u(x)\big| \\ &\overset{\text{Young}}{\geqslant} -\frac{1}{2\epsilon}\sum_{x\in V}\big|f(x)\big|^2 - \frac{\epsilon}{2}\sum_{x\in V}\big|u(x)\big|^2.\end{aligned} \tag{5.3.5}$$

- Case $p > 2$, ($i.e$ $q \in]1,2[$) : We have

$$\begin{aligned}\sum_{x\in V}\sum_{y\sim x}\omega(x,y)\big|u(y)-u(x)\big|^2 &= -\sum_{x\in V}u(x)(\Delta_{\omega,2}u)(x) \\ &\overset{\text{Young}}{\leqslant} \frac{1}{p}\sum_{x\in V}\big|\Delta_{\omega,2}u(x)\big|^p + \frac{1}{q}\sum_{x\in V}\big|u(x)\big|^q.\end{aligned} \tag{5.3.6}$$

  By the inequalities (5.3.4), (5.3.5) and (5.3.6), we have

$$\begin{aligned}\mathcal{F}(u) &= \frac{1}{p}\sum_{x\in V}\big|\Delta_{\omega,2}u(x)\big|^p + \sum_{x\in V}f(x)u(x) \\ &\geqslant \sum_{x\in V}\sum_{y\sim x}\omega(x,y)\big|u(y)-u(x)\big|^2 - \frac{1}{q}\sum_{x\in V}\big|u(x)\big|^q \\ &\quad -\frac{1}{2\epsilon}\sum_{x\in V}\big|f(x)\big|^2 - \frac{\epsilon}{2}\sum_{x\in V}\big|u(x)\big|^2 \\ &\geqslant (\lambda-\frac{\epsilon}{2})\sum_{x\in V}\big|u(x)\big|^2 - \frac{1}{q}\sum_{x\in V}\big|u(x)\big|^q - \frac{1}{2\epsilon}\sum_{x\in V}\big|f(x)\big|^2 - C_g.\end{aligned}$$

  Since $q < 2$ and for ϵ small enough $(\lambda - \frac{\epsilon}{2} > 0)$, we obtain

$$\lim_{\substack{\|u\|\to\infty \\ u\in\mathcal{H}_g(V;\mathcal{A})}} \mathcal{F}(u) = +\infty.$$

- Case $p = 2$, ($i.e$ $q = 2$) :

$$\begin{aligned}\sum_{x\in V}\sum_{y\sim x}\omega(x,y)\big|u(y)-u(x)\big|^2 &= -\sum_{x\in V}u(x)(\Delta_{\omega,2}u)(x) \\ &\overset{\text{Young}}{\leqslant} \frac{1}{2\epsilon_1}\sum_{x\in V}\big|\Delta_{\omega,2}u(x)\big|^2 + \frac{\epsilon_1}{2}\sum_{x\in V}\big|u(x)\big|^2,\end{aligned} \tag{5.3.7}$$

For all $\epsilon_1 \in (0,1)$.

Applying (5.3.7) in (5.3.4), we obtain

$$\epsilon_1(\lambda - \frac{\epsilon_1}{2}) \sum_{x\in V} |u(x)|^2 - \epsilon_1 C_{f_0} \leqslant \frac{1}{2} \sum_{x\in V} |\Delta_{\omega,2} u(x)|^2. \tag{5.3.8}$$

We sum the inequalities (5.3.8) and (5.3.5), we obtain

$$(\epsilon_1(\lambda - \frac{\epsilon_1}{2}) - \frac{\epsilon}{2}) \sum_{x\in V} |u(x)|^2 - \epsilon_1 C_{f_0} - \frac{1}{2\epsilon} \sum_{x\in V} |f(x)|^2 \leqslant \mathcal{F}(u). \tag{5.3.9}$$

For ϵ, $\epsilon_1 \in (0,1)$ fixed such that $\epsilon_1(\lambda - \frac{\epsilon_1}{2}) - \frac{\epsilon}{2} > 0$, we have

$$\lim_{\substack{\|u\|\to\infty \\ u\in\mathcal{H}_g(V;\mathcal{A})}} \mathcal{F}(u) = +\infty.$$

- Case $1 < p < 2$, ($i.e$ $q > 2$) : By the inequality (5.3.4), we have

$$\begin{aligned} \lambda \sum_{x\in V} |u(x)|^2 &\leqslant \sum_{x\in V} \sum_{y\sim x} \omega(x,y) |u(y) - u(x)|^2 + C_g \\ &= -\sum_{x\in V} u(x) \Delta_{\omega,2} u(x) + C_g \\ &\leqslant \left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}} \left(\sum_{x\in V} |\Delta_{\omega,2} u(x)|^2\right)^{\frac{1}{2}} + C_g \end{aligned}$$

Since the norms are equavalent in finite dimension vector space, there exists $C(n) > 0$, recall that $n = \mathbf{card}(V)$,

$$\lambda \sum_{x\in V} |u(x)|^2 \leqslant C(n) \left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}} \left(\sum_{x\in V} |\Delta_{\omega,2} u(x)|^p\right)^{\frac{1}{p}} + C_g.$$

Thus

$$\frac{1}{p} \left(\frac{\lambda \sum_{x\in V} |u(x)|^2 - C_g}{1 + C(n) \left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}}} \right)^p \leqslant \frac{1}{p} \sum_{x\in V} |\Delta_{\omega,2} u(x)|^p \tag{5.3.10}$$

On the other hand, by the Cauchy–Schwarz inequality, we have

$$-\left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}} \left(\sum_{x\in V} |f(x)|^2\right)^{\frac{1}{2}} \leqslant \sum_{x\in V} f(x) u(x). \tag{5.3.11}$$

We sum the inequalities (5.3.10) and (5.3.11), we obtain

$$\frac{1}{p} \left(\frac{\lambda \sum_{x\in V} |u(x)|^2 - C_g}{1 + C(n) \left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}}} \right)^p - \left(\sum_{x\in V} |u(x)|^2\right)^{\frac{1}{2}} \left(\sum_{x\in V} |f(x)|^2\right)^{\frac{1}{2}} \leqslant \mathcal{F}(u)$$

Hence

$$\lim_{\substack{\|u\|\to\infty \\ u\in\mathcal{H}_g(V;\mathcal{A})}} \mathcal{F}(u) = +\infty.$$

Now, we show the strict convexity of the functional $\mathcal{F}$ on the set $\mathcal{H}_g(V;\mathcal{A})$. Assume that $\mathcal{F}_d(\cdot)$ is not strictly convex on $\mathcal{H}_g(V;\mathcal{A})$. Then there exist $u, v \in \mathcal{H}_g(V;\mathcal{A})$ with $u \neq v$ such that $\tau\mathcal{F}_d(u) + (1-\tau)\mathcal{F}_d(v) = \mathcal{F}_d(\tau u + (1-\tau)v)$ for all $\tau \in]0,1[$. But since the function $t \mapsto t^p$ is strictly convex on $\mathbb{R}^+$ for $p \in]1, +\infty[$, this equality entails that $\Delta_{\omega,2} u = \Delta_{\omega,2} v$ on V, hence on $\mathcal{A}$. Clearly $w = u - v$ satisfies

$$\begin{cases} \Delta_{\omega,2} w = 0, & on\ \mathcal{A} \\ w = 0, & on\ \mathcal{A}^c. \end{cases}$$

But we know the only solution of the above problem is the null function $w = 0$ on V, see [95, Corollary 3.16.]. Hence $u = v$ on V, leading to a contradiction. □

Now, we have the tools to announce our main result in this section

Theorem 5.3.4. *The problem (5.3.1) has a unique solution in $\mathcal{H}_g(V;\mathcal{A})$.*

Proof : By Theorem 5.3.1, the problem (5.3.1) is equivalent to solve the minimization problem

$$\min\{\mathcal{F}_d(u) : u \in \mathcal{H}_g(V;\mathcal{A})\}. \tag{5.3.12}$$

Let $\iota_{\mathcal{H}_g(V;\mathcal{A})}$ be the indicator function of $\mathcal{H}_g(V;\mathcal{A})$. By Lemma 5.3.3, we get that $\mathcal{F}_d(\cdot) + \iota_{\mathcal{H}_g(V;\mathcal{A})}$ is coercive and strictly convex. Since this objective is lower semicontinuous (lsc) by closedness of $\mathcal{H}_g(V;\mathcal{A})$ and continuity of $\mathcal{F}_d(\cdot)$, (5.3.12) has a unique minimizer. In particular, our problem (5.3.1) has a unique solution. □

5.4 Numerical results

5.4.1 p-bilaplacian denoising on graphs

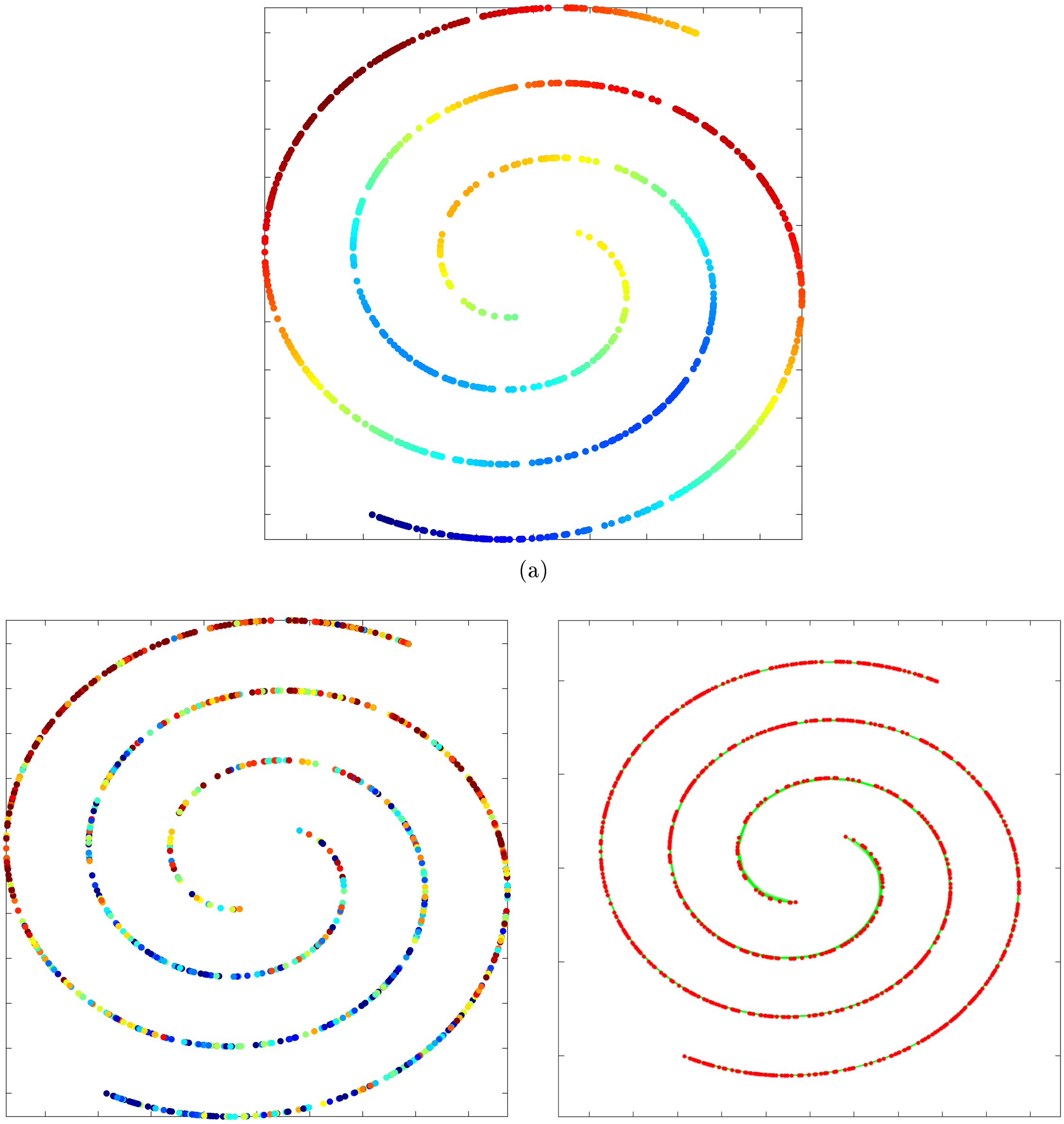
We apply the accelerated forward-backward proximal splitting scheme, see Chapter 7, to solve the dual problem of (5.2.1) for a denoising problem, i.e., A is the identity (5.2.1). Denoising of two types of datasets is considered: the first is a $2D$ point cloud, and the second is a 1D equispaced signal. In the first setting, the nodes of the graph are the points in the cloud and $\mathbf{u}_x$ the value of point/vertex index x. For signal denoising, each graph node correspond to a signal sample i, and $\mathbf{u}_x$ is the signal value at node/sample index i. We choose the nearest neighbour graph with the standard weighting kernel $\exp(-|\boldsymbol{x}-\boldsymbol{y}|)$ when$|\boldsymbol{x}-\boldsymbol{y}| \leq \delta$ and 0 otherwise, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the $2D$ spatial coordinates of the points for the point cloud, and sample index for the signal case.

Application to point cloud denoising The original point cloud used in our numerical experiments is shown in Figure 5.1(a). It consists of $N = 1000$ points that are not on a regular grid. The function on this point cloud, denoted u_0, was synthesized to be piecewise linear on the $2D$ point cloud. For the $1D$ signal case, the function is piecewise polynomial. A noisy observation f (see Figure 5.1(b)) is then generated by adding a white Gaussian noise noise of standard deviation 0.5 to u_0. Figure 5.2 displays the results by solving (5.2.1) using different values $p \in \{1, 2, 20\}$.

Application to signal denoising In this experiment, we choose a piecewise-polynomial signal u_0 shown in Figure 5.3(a) for $N = 1000$ together with its noisy version f with additive white Gaussian noise of standard deviation 0.05. Figure 5.3(b) depicts the denoised signal $u^\star$ by solving (7.4.1) with $p = 1$ and hand-tuned λ. Figure 5.3(c) also confirms the $o(1/k)$ rate predicted above on $\|u_k - u^\star\|_2$.

5.4.2 p-bilaplacian semi-supervised classification

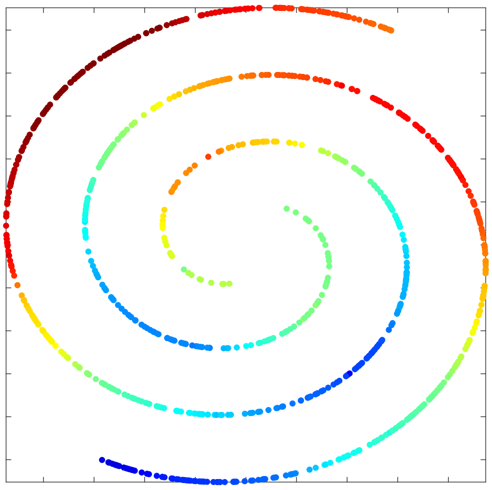
We apply the primal-dual proximal splitting scheme, see Chapter 7, to solve (5.3.1) in the setting of a semi-supervised classification problem. The latter amounts to finding the missing labels of a label function g defined on a $2D$ point cloud observing g only on some vertices $\mathcal{A}^c$. The nodes of the graph are the points in the cloud and u_x is the value of point/vertex x which agrees with the original label function g on $\mathcal{A}^c$. We chose the nearest neighbour graph with the standard weighting kernel $\exp^{-|\boldsymbol{x}-\boldsymbol{y}|}$ when $|\boldsymbol{x}-\boldsymbol{y}| \leq \delta$ and 0 otherwise, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the $2D$ spatial coordinates of the points for the point cloud. In the same vein as in Section 4.6, in our numerical experiments, the boundary vertices



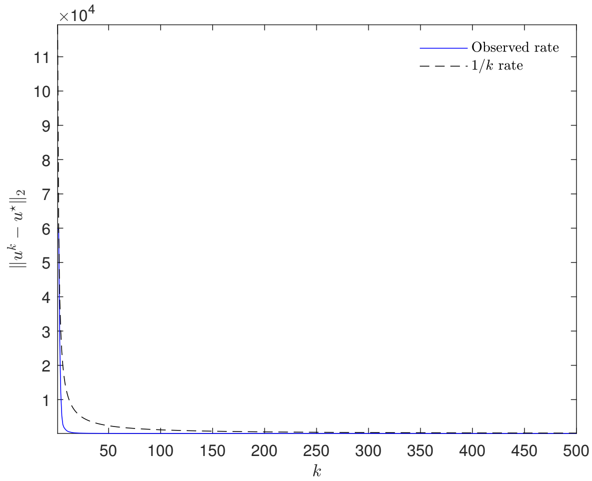
(a) (b) (c)

Figure 5.1: (a) Original data with $N = 1000$ points. (b) Noisy data. (c) Graph considered.

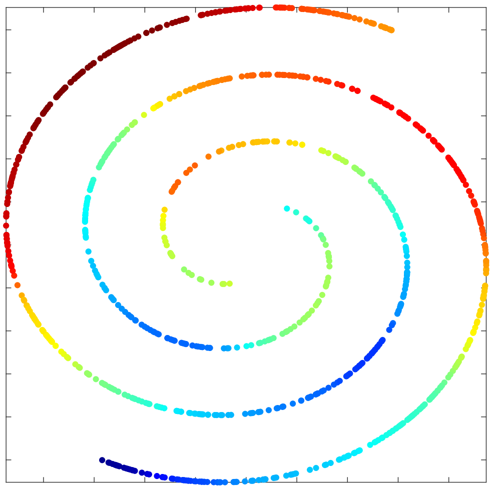
(i.e. A_n^c) are chosen uniformly at random from the whole N points/vertices with two cardinalities: $|\mathcal{A}^c| = N/5$ and $|\mathcal{A}^c| = N/10$. For each $\mathcal{A}^c$, we solve ($\mathcal{VP}_n^d$) with $f = 0$ and $p \in \{1, 2, 10\}$. Although the case $p = 1$ was not covered by our study, we report the corresponding results as the splitting algorithm readily handles $p = 1$ and even $p = +\infty$ just as well (see Chapter 7 for details).



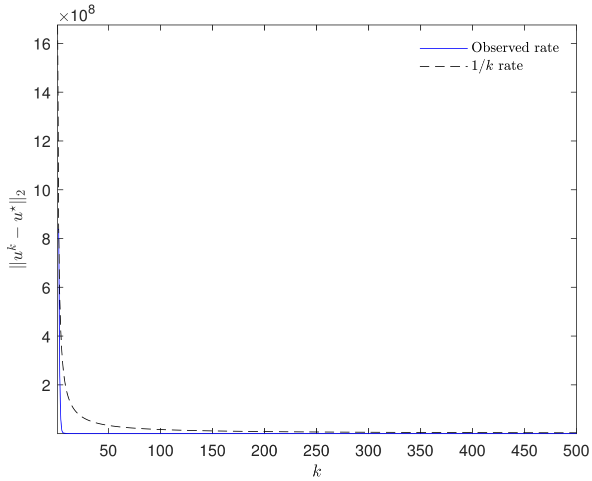
(a) $p = 1$



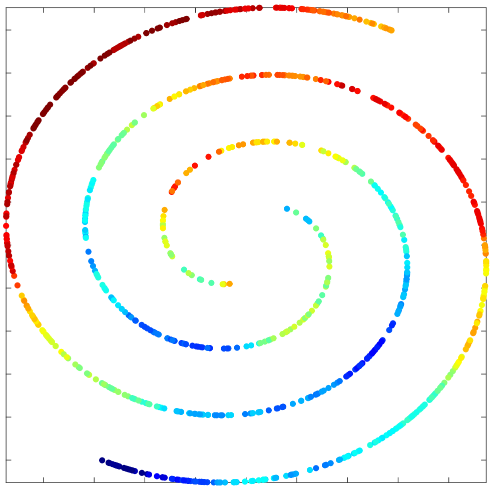
(b) $p = 1$



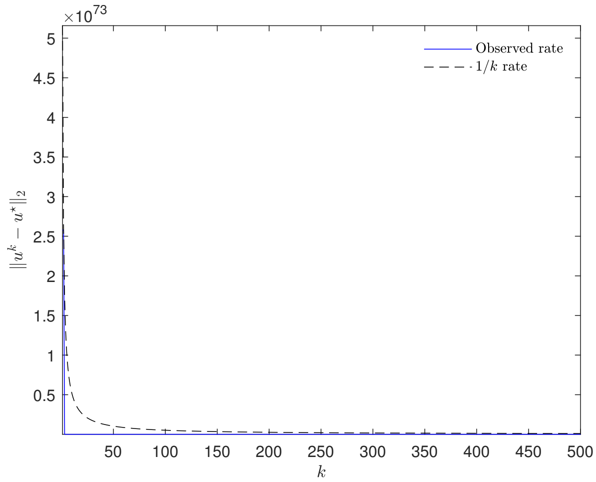
(c) $p = 2$



(d) $p = 2$



(e) $p = 20$



(f) $p = 20$

Figure 5.2: In the left-hand side column, the results obtained. In the right-hand side column, primal convergence criterion $\left\|u_k - u^\star\right\|_2$ as a function of the iteration counter k.

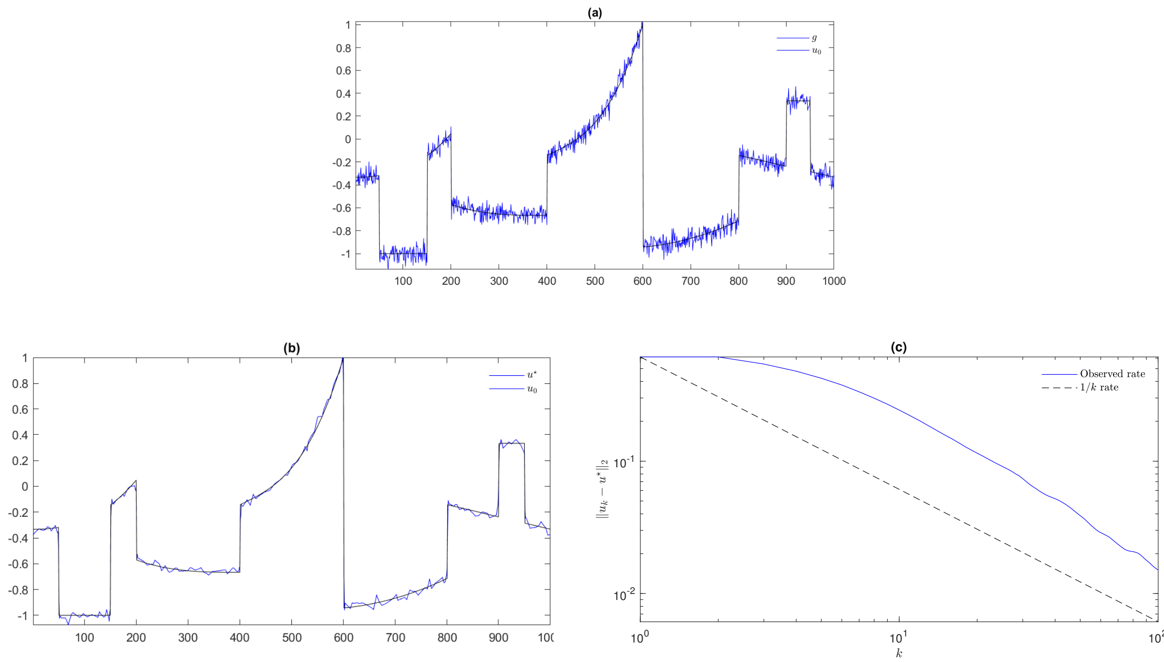
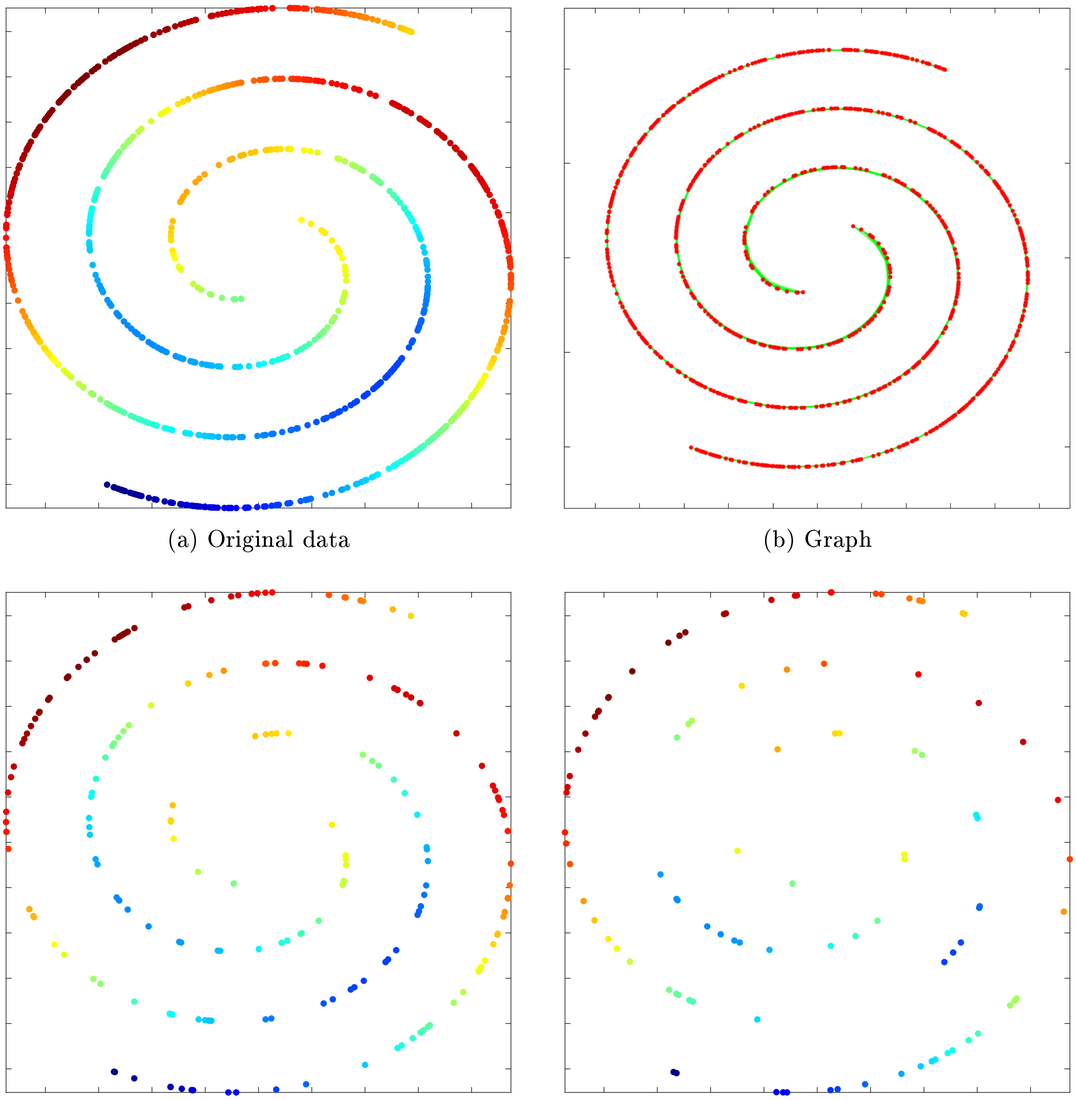


Figure 5.3: Results for signal denoising with $p = 1$. (a) Noisy and original signal. (b) Denoised and original signal. (c) Primal convergence criterion $\left\|u_k - u^\star\right\|_2$ as a function of the iteration counter k.



(a) Original data (b) Graph

(c) Subsampled data (d) Subsampled data

Figure 5.4: (a) The original data with $N = 1000$ points. (b) Graph considered. (c) Subsampled (boundary) data with $N/5 = 200$ points. (d) Subsampled (boundary) data with $N/10 = 100$ points.

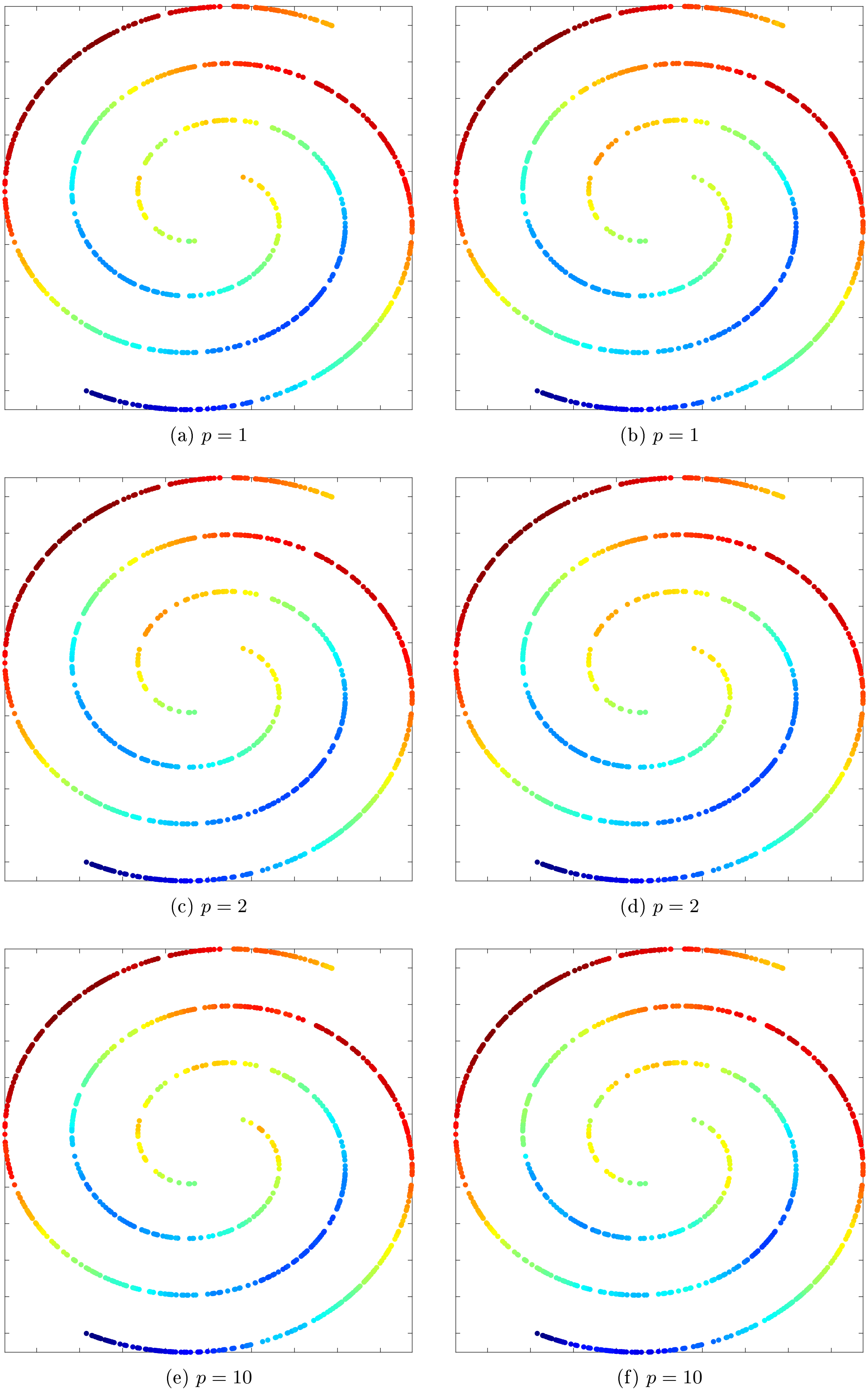


Figure 5.5: In the left-hand side, results obtained from the boundary data (c) Figure 5.4, and in the right-hand side, results obtained from the boundary data (d) Figure 5.4.

Chapter 6

Nonlocal perimeters and curvatures flows on graphs

Main contributions of this chapter

- ▶ General class of perimeters on graphs.
- ▶ Mean curvature co-area formula and total variation
- ▶ Level set formulation of nonlocal mean curvature flows on graphs and applications.

A paper with the content of this chapter is under preparation for submission to a journal.

Contents

The objectives of this chapter are as follows. We revisit the notion of perimeters on graphs, introduced in [70], we extend it to so-called inner and outer perimeters. Thanks to the co-area formula, we show that discrete total variations as well as several graph cuts can be expressed through these perimeters. Then, we propose a novel class of curvature operators on graphs that unifies both local and nonlocal mean curvature on Euclidean domains. These lead us to translate and adapt the notion of the mean curvature flows on graphs as well as the level set mean curvature which can be seen as approximate schemes. Finaly, we propose to use these methods for image processing, $3D$-point cloud and high dimensional data classification.

6.1 Introduction

Context and motivations

Partial Differential Equations (PDEs) and variational methods involving the notion of perimeters and curvatures have and still generate a lot of interest in both continuous and discrete domains. These operators under their different local or nonlocal forms, arise not only from subfields within mathematics such as differential geometry and analysis, but also in numerous PDEs and objective functionals related to many applications fields in sciences and engineering.

For instance, in mathematical image processing and computer vision, the notion of perimeter is a key idea for the regularization of many inverse ill-posed problems such as denoising, restoration, inpainting, segmentation, etc. Regularizing such problems is often used to find suitable clusters among data, to obtain image partitions for segmentation purposes, to denoise or to inpaint images while preserving sharp boundaries. It is worth noting that perimeters appear in the two most popular variational models for image processing and segmentation, namely the total variation and the Mumford-Shah models [51, 128, 114]

Motion by mean curvature and flows involving mean curvature in general play an important role in geometry and analysis. Many continuous models, involving a front propagation with a velocity depending on the mean curvature and their simulations by level set methods, are used in different application fields such as data processing, computer vision, fluid mechanics . For an overview and applications see the books [131, 117, 42] and references therein.

In recent literature, an intense mathematical study has been performed on nonlocal counterpart of the classical perimeters and curvature flows. A notion of fractional perimeters and nonlocal curvature was first introduced bay Caffarelli, Roquejoffre and Savin in [44] . The main idea of fractional perimeters is that any point inside an Euclidean set "interact" with any point outside the set, given a functional whose minimization is taken account. Then many works have been proposed to study functional involving nonlocal perimeters or nonlocal curvature flows, e.g [1, 49]. See also the recent monograph Mazon et al. [108]. We can notice that recently Mazon et al. have introduced a large class of perimeters and curvature flows on random metric graphs which embedded local and nonlocal perimeters on Euclidean domains and graphs [110, 109].

On the other hand, graphs and networks have been successfully used in a variety of fields such as machine learning, data mining, image analysis and social sciences that are confronted with the analysis and modelling high dimensional datasets. In machine learning, image analysis many tasks, such as classification, clustering or segmentation , can be often given in term of minimizing the graph perimeter (graph cut) or a related functional (normalized cut, ratio cut, balanced cut, etc). The cut size is, in this case, generally defined as the sum of the weight of edges between the considered set and its complement, which is closely related to the notion of the perimeter of a set. Such graph problems are traditionally solved by methods from combinatorial, graph theory or spectral analysis [92, 132, 135, 147, 40]. In recent years, there has been increasing interest in applying the models and techniques from variational methods and PDEs to solve problems in data science, see [143, 39, 73, 75] and references herein. The demand and the interest for such methods is motivated by existing and potential future applications in data science. PDEs analysis tools originally developed for Euclidean spaces and regular lattices are now being extended to general settings of graphs in order to analyse geometric and topological structures, as well as data measured on them.

In order to translate and to solve PDEs on graphs, Elmoataz et al. have adopted nonlocal calculus on weighted graphs [73, 72, 75], which consists in replacing continuous partial differential operators (e.g. gradient, divergence), with a reasonable discrete analogue. It allows to transfer many important tools and results from the continuous setting to the discrete one. It also allows graph theory to have new connections to analysis. Based on this framework, we revisit and extend the discrete notions of perimeters, mean curvatures, Cheeger cut and total variation, which lead us to adapt and transcribe level set equations on weighted graphs.

Outline of this chapter

In Section 6.2, we recall the notion of the boundary set on graph, as well as the discrete perimeters on graphs and we show its link with the local and nonlocal continuous perimeters. In Section 6.3, we prove an analogue version of the co-area formula on weighted graphs which allows us to derive relation with total variation as well as Cheeger inequality on graphs with discrete perimeters. In Section 6.4, we introduced a family of the mean curvature flows. We propose an adaptation and a transcription of the mean curvature level set equations on the general discrete domain, a weighted graph, in Section 6.5. Finally, we show some applications in image and data processing to illustrate the potential and the behaviour of this mean curvature formulation.

6.2 Generalized perimeters on graphs

In this section, we first define the notion of boundaries on graph and explicit the relations with the discrete gradients defined in Section 2.4. Based on these framework, we recall the family of the perimeters introduced in [70]. Next, we recall the definitions of the continuous local and nonlocal perimeters, we rigorously show their relations with the discrete ones.

Throughout the chapter $G = (V, E, \omega)$ is a connected weighted graph. Denoted by $\mathcal{A}$ a subset of V, $\mathcal{A}^c$ is its complement, and we recall that $\chi_{\mathcal{A}}$ is the characteristic function of $\mathcal{A}$.

Definition 6.2.1. The *outer* and *inner vertex boundaries*, and the *vertex boundary*, of a subset $\mathcal{A} \subset V$, are respectively defined by:

$$\partial^+\mathcal{A} \stackrel{\text{def}}{=} \{u \in \mathcal{A}^c : \; \exists v \in \mathcal{A},\, v \sim u\}, \tag{6.2.1}$$

$$\partial^-\mathcal{A} \stackrel{\text{def}}{=} \{u \in \mathcal{A} : \; \exists v \in \mathcal{A}^c,\, v \sim u\}, \tag{6.2.2}$$

$$\partial\mathcal{A} \stackrel{\text{def}}{=} \partial^+\mathcal{A} \cup \partial^-\mathcal{A}. \tag{6.2.3}$$

Note that $\partial^+\mathcal{A}^c = \partial^-\mathcal{A}$, $\partial\mathcal{A} = \partial\mathcal{A}^c$ and $\partial^+\mathcal{A} \cap \partial^-\mathcal{A} = \emptyset$.

The following proposition gives the relationships between the discrete gradients and the above boundary sets, which will be useful to define the discrete perimeters on graphs. The proof takes of by a simple computation of the p-norm of the characteristic function.

Proposition 6.2.2. *Let $\mathcal{A} \subset V$,*

(i) For $1 \le p < \infty$, we have the following relations:

$$\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\|_p = \left(\sum_{v\in\mathcal{A}} (\omega_{uv})^{\frac{p}{2}}\right)^{\frac{1}{p}} \chi_{\partial^+\mathcal{A}}(u), \tag{6.2.4}$$

$$\left\|(\nabla^-_\omega \chi_{\mathcal{A}})(u)\right\|_p = \left(\sum_{v\in\mathcal{A}^c} (\omega_{uv})^{\frac{p}{2}}\right)^{\frac{1}{p}} \chi_{\partial^-\mathcal{A}}(u), \tag{6.2.5}$$

$$\left\|(\nabla_\omega \chi_{\mathcal{A}})(u)\right\|_p = \left\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\right\|_p + \left\|(\nabla^-_\omega \chi_{\mathcal{A}})(u)\right\|_p. \tag{6.2.6}$$

(ii) For $p = \infty$, we have the following relations:

$$\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\|_\infty = \left(\max_{v\in\mathcal{A}} (\sqrt{\omega_{uv}})\right) \cdot \chi_{\partial^+\mathcal{A}}(u),$$

$$\left\|(\nabla^-_\omega \chi_{\mathcal{A}})(u)\right\|_\infty = \left(\max_{v\in\mathcal{A}^c} (\sqrt{\omega_{uv}})\right) \cdot \chi_{\partial^-\mathcal{A}}(u),$$

$$\left\|(\nabla_\omega \chi_{\mathcal{A}})(u)\right\|_\infty = \left\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\right\|_\infty + \left\|(\nabla^-_\omega \chi_{\mathcal{A}})(u)\right\|_\infty.$$

(iii) For $p \in [1, +\infty]$, we have the following relations:

$$\left\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\right\|_p = \left\|(\nabla^-_\omega \chi_{\mathcal{A}^c})(u)\right\|_p$$

$$\left\|(\nabla_\omega \chi_{\mathcal{A}})(u)\right\|_p = \left\|(\nabla_\omega \chi_{\mathcal{A}^c})(u)\right\|_p.$$

Remark 6.2.3. For unweighted graphs i.e. $\omega_{uv} \in \{0, 1\}$, we have that:

- $\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\|_1$ corresponds to the number of edges connecting the vertex $u \in \mathcal{A}^c$ with the vertices in $\mathcal{A}$. Therefore $\sum_{u\in V} \left\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\right\|_1$ is just the size of the usual edge boundary of $\mathcal{A}$.
- $\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\|_\infty$ is the indicator of $\partial^+\mathcal{A}$, and so $\sum_{u\in V} \left\|(\nabla^+_\omega \chi_{\mathcal{A}})(u)\right\|_\infty$ is the size of the outer vertex boundary of $\mathcal{A}$, while $\sum_{u\in V} \left\|(\nabla^-_\omega \chi_{\mathcal{A}})(u)\right\|_\infty$ is the size of the inner vertex boundary.

For weighted graphs i.e. $\omega_{uv} \in [0, 1]$, we observe that:

- $\|(\nabla_\omega^+\chi_\mathcal{A})(u)\|_p$ and $\left\|(\nabla_\omega^-\chi_\mathcal{A})(u)\right\|_p$ are the weighted sizes of edge boundaries of $\mathcal{A}$, $\partial^+\mathcal{A}\times\partial^-\mathcal{A}$ and $\partial^-\mathcal{A}\times\partial^+\mathcal{A}$ respectively.
- $\sum\limits_{u\in V}\left\|(\nabla_\omega^+\chi_\mathcal{A})(u)\right\|_\infty$ is the weighted size of the outer vertex boundary of $\mathcal{A}$ while $\sum\limits_{u\in V}\left\|(\nabla_\omega^-\chi_\mathcal{A})(u)\right\|_\infty$ is the weighted size of the inner vertex boundary of $\mathcal{A}$.

Remark 6.2.4. The *outer* and *inner vertex boundaries*, and the *vertex boundary* can be expressed through the characteristic function of $\mathcal{A}$ as:

$$\begin{aligned}\partial^+\mathcal{A} &= \left\{u\in V:\ \left\|(\nabla_\omega^+\chi_\mathcal{A})(u)\right\|_p>0\right\},\\ \partial^-\mathcal{A} &= \left\{u\in V:\ \left\|(\nabla_\omega^-\chi_\mathcal{A})(u)\right\|_p>0\right\},\\ \partial\mathcal{A} &= \left\{u\in V:\ \left\|(\nabla_\omega\chi_\mathcal{A})(u)\right\|_p>0\right\}.\end{aligned}$$

6.2.1 Discrete perimeters on graphs

Based on the interpretation of Proposition 6.2.2, we recall the definition of the family of weighted perimeters on graphs introduced in [70].

Definition 6.2.5. For $1\le p<\infty$ and $\mathcal{A}\subset V$, the family of weighted perimeters of $\mathcal{A}$ is defined as follows:

$$\begin{aligned}\mathrm{Per}^+_{\omega,p}(\mathcal{A}) &\overset{\text{def}}{=} E(\left\|\nabla_w^+\chi_\mathcal{A}\right\|_p)=\sum_{u\in\mathcal{A}^c}\left(\sum_{v\in\mathcal{A}}\omega_{uv}^{\frac{p}{2}}\right)^{\frac{1}{p}},\\ \mathrm{Per}^-_{\omega,p}(\mathcal{A}) &\overset{\text{def}}{=} E(\left\|\nabla_\omega^-\chi_\mathcal{A}\right\|_p)=\sum_{u\in\mathcal{A}}\left(\sum_{v\in\mathcal{A}^c}\omega_{uv}^{\frac{p}{2}}\right)^{\frac{1}{p}},\\ \mathrm{Per}_{\omega,p}(\mathcal{A}) &\overset{\text{def}}{=} E(\left\|\nabla_\omega\chi_\mathcal{A}\right\|_p)=\sum_{u\in\mathcal{A}^c}\left(\sum_{v\in\mathcal{A}}\omega_{uv}^{\frac{p}{2}}\right)^{\frac{1}{p}}+\sum_{u\in\mathcal{A}}\left(\sum_{v\in\mathcal{A}^c}\omega_{uv}^{\frac{p}{2}}\right)^{\frac{1}{p}}.\end{aligned}$$

For $p=\infty$, the family of weighted perimeters of $\mathcal{A}$ is defined as follows:

$$\begin{aligned}\mathrm{Per}^+_{\omega,\infty}(\mathcal{A}) &\overset{\text{def}}{=} E(\left\|\nabla_\omega^+\chi_\mathcal{A}\right\|_\infty)=\sum_{u\in\mathcal{A}^c}\left(\max_{v\in\mathcal{A}}\sqrt{\omega_{uv}}\right),\\ \mathrm{Per}^-_{\omega,\infty}(\mathcal{A}) &\overset{\text{def}}{=} E(\left\|\nabla_\omega^-\chi_\mathcal{A}\right\|_\infty)=\sum_{u\in\mathcal{A}}\left(\max_{v\in\mathcal{A}^c}\sqrt{\omega_{uv}}\right),\\ \mathrm{Per}_{\omega,\infty}(\mathcal{A}) &\overset{\text{def}}{=} E(\left\|\nabla_\omega\chi_\mathcal{A}\right\|_\infty)=\sum_{u\in\mathcal{A}^c}\left(\max_{v\in\mathcal{A}}\sqrt{\omega_{uv}}\right)+\sum_{u\in\mathcal{A}}\left(\max_{v\in\mathcal{A}^c}\sqrt{\omega_{uv}}\right).\end{aligned}$$

By definition we have, for $1\le p\le\infty$, the following relations:

$$\begin{aligned}\mathrm{Per}_{\omega,p}(\mathcal{A}) &= \mathrm{Per}^+_{\omega,p}(\mathcal{A})+\mathrm{Per}^-_{\omega,p}(\mathcal{A}),\\ \mathrm{Per}^+_{\omega,p}(\mathcal{A}) &= \mathrm{Per}^-_{\omega,p}(\mathcal{A}^c),\\ \mathrm{Per}_{\omega,p}(\mathcal{A}) &= \mathrm{Per}_{\omega,p}(\mathcal{A}^c),\\ \mathrm{Per}_{\omega,1}(\mathcal{A}) &= 2\,\mathrm{Per}^+_{\omega,1}(\mathcal{A}).\end{aligned}$$

Proposition 6.2.6. *Let $P_{\omega,p}$ belongs to $\{\mathrm{Per}^\pm_{\omega,1},\ \mathrm{Per}^\pm_{\omega,\infty},;\mathrm{Per}_{\omega,1},\ \mathrm{Per}_{\omega,\infty}\}$, for $p=\infty$ the weight function ω is a $\{0,1\}$-value. Then we have the following properties:*

(i) $P_{\omega,p}(\emptyset)=0$;

(ii) $P_{\omega,p}(V)=0$;

(iii) $P_{\omega,p}$ *is submodular, i.e. for all* $\mathcal{A}$, $\mathcal{B} \subset V$ *we have*

$$P_{\omega,p}(\mathcal{A} \cup \mathcal{B}) + P_{\omega,p}(\mathcal{A} \cap \mathcal{B}) \leq P_{\omega,p}(\mathcal{A}) + P_{\omega,p}(\mathcal{B}).$$

PROOF : Claims (i) and (ii) are straightforward. We thus focus on claim (iii). For $p = 1$, it is enough to prove the inequality for $\mathrm{Per}^+_{\omega,1}$ since $\mathrm{Per}_{\omega,1} = 2\,\mathrm{Per}^+_{\omega,1} = 2\,\mathrm{Per}^+_{\omega,1}$. We have

$$\begin{aligned}\mathrm{Per}^+_{\omega,1}(\mathcal{A} \cup \mathcal{B}) &= \sum_{u \in \mathcal{A}\cup\mathcal{B}} \sum_{v \in (\mathcal{A}\cup\mathcal{B})^c} \omega_{uv} \\ &= \sum_{u\in\mathcal{A}}\sum_{v\in\mathcal{A}^c} \omega_{uv} + \sum_{u\in\mathcal{B}}\sum_{v\in\mathcal{B}^c} \omega_{uv} - \sum_{u\in\mathcal{A}\cap\mathcal{B}} \sum_{v} \in (\mathcal{A}\cup\mathcal{B})^c \omega_{uv} \\ &\quad - \sum_{u\in\mathcal{A}} \sum_{v\in\mathcal{B}\setminus(\mathcal{A}\cup\mathcal{B})^c} \omega_{uv} - \sum_{u\in\mathcal{B}} \sum_{v\in\mathcal{A}\setminus(\mathcal{A}\cup\mathcal{B})^c} \omega_{uv}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Per}^+_{\omega,1}(\mathcal{A} \cap \mathcal{B}) &= \sum_{u \in \mathcal{A}\cap\mathcal{B}} \sum_{v \in (\mathcal{A}\cap\mathcal{B})^c} \omega_{uv} \\ &= \sum_{u \in \mathcal{A}\cap\mathcal{B}} \sum_{v \in (\mathcal{A}\cup\mathcal{B})^c} \omega_{uv} + \sum_{u \in \mathcal{A}\cap\mathcal{B}} \sum_{v \in \mathcal{A}\setminus(\mathcal{A}\cup\mathcal{B})^c} \omega_{uv} + \sum_{u \in \mathcal{A}\cap\mathcal{B}} \sum_{v \in \mathcal{B}\setminus(\mathcal{A}\cup\mathcal{B})^c} \omega_{uv}.\end{aligned}$$

For $\mathrm{Per}^{\pm}_{\omega,\infty}$, claim (iii) is a consequence of the following inequality, which is easy to verify,

$$\begin{aligned}&\max_{v\sim u} \left(\chi_{\mathcal{A}\cup\mathcal{B}}(v) - \chi_{\mathcal{A}\cup\mathcal{B}}(u)\right)^{\pm} + \max_{v\sim u} \left(\chi_{\mathcal{A}\cap\mathcal{B}}(v) - \chi_{\mathcal{A}\cap\mathcal{B}}(u)\right)^{\pm} \\ &\qquad \leq \max_{v\sim u} \left(\chi_{\mathcal{A}}(v) - \chi_{\mathcal{A}}(u)\right)^{\pm} + \max_{v\sim u} \left(\chi_{\mathcal{B}}(v) - \chi_{\mathcal{B}}(u)\right)^{\pm},\end{aligned}$$

for all $u \in V$. For $\mathrm{Per}_{\omega,\infty}$, the result holds from the following equality $\mathrm{Per}_{\omega,\infty} = \mathrm{Per}^+_{\omega,\infty} + \mathrm{Per}^-_{\omega,\infty}$. □

As a consequence, we have the following result for $p = 1$.

Corollary 6.2.7. *Let* $\mathcal{A}$, $\mathcal{B} \subset V$ *with* $\mathcal{A} \cap \mathcal{B} = \emptyset$, *then*

$$\begin{aligned}\mathrm{Per}^{\pm}_{\omega,1}(\mathcal{A} \cup \mathcal{B}) &= \mathrm{Per}^{\pm}_{\omega,1}(\mathcal{A}) + \mathrm{Per}^{\pm}_{\omega,1}(\mathcal{B}) - 2\sum_{\mathcal{A}}\sum_{\mathcal{B}} \sqrt{\omega_{uv}}, \\ \mathrm{Per}_{\omega,1}(\mathcal{A} \cup \mathcal{B}) &= \mathrm{Per}_{\omega,1}(\mathcal{A}) + \mathrm{Per}_{\omega,1}(\mathcal{B}) - 4\sum_{\mathcal{A}}\sum_{\mathcal{B}} \sqrt{\omega_{uv}}.\end{aligned}$$

If moreover, there are no edges between $\mathcal{A}$ *and* $\mathcal{B}$, *i.e.,* $\partial\mathcal{A} \cap \mathcal{B} = \emptyset$ *or equivalently* $\partial\mathcal{B} \cap \mathcal{A} = \emptyset$, *then*

$$\begin{aligned}\mathrm{Per}^{\pm}_{\omega,1}(\mathcal{A} \cup \mathcal{B}) &= \mathrm{Per}^{\pm}_{\omega,1}(\mathcal{A}) + \mathrm{Per}^{\pm}_{\omega,1}(\mathcal{B}), \\ \mathrm{Per}_{\omega,1}(\mathcal{A} \cup \mathcal{B}) &= \mathrm{Per}_{\omega,1}(\mathcal{A}) + \mathrm{Per}_{\omega,1}(\mathcal{B}).\end{aligned}$$

PROOF : By definition, we have

$$\begin{aligned}\mathrm{Per}_{\omega,1}(\mathcal{A} \cup \mathcal{B}) &= \sum_{u\in V}\sum_{v\in V} \sqrt{\omega_{uv}} \left(\chi_{\mathcal{A}\cup\mathcal{B}}(v) - \chi_{\mathcal{A}\cup\mathcal{B}}(u)\right)^2 \\ &= \sum_{u\in V}\sum_{v\in V} \sqrt{\omega_{uv}} \left(\chi_{\mathcal{A}}(v) + \chi_{\mathcal{B}}(v) - \chi_{\mathcal{A}}(u) - \chi_{\mathcal{B}}(u)\right)^2 \\ &= \sum_{u\in V}\sum_{v\in V} \sqrt{\omega_{uv}} \left(\chi_{\mathcal{A}}(v) - \chi_{\mathcal{A}}(u)\right)^2 + \sum_{u\in V}\sum_{v\in V} \sqrt{\omega_{uv}} \left(\chi_{\mathcal{B}}(v) - \chi_{\mathcal{B}}(u)\right)^2 \\ &\qquad + 2 \cdot \sum_{u\in V}\sum_{v\in V} \sqrt{\omega_{uv}} \left(\chi_{\mathcal{A}}(v) - \chi_{\mathcal{A}}(u)\right) \cdot \left(\chi_{\mathcal{B}}(v) - \chi_{\mathcal{B}}(u)\right) \\ &= \mathrm{Per}_{\omega,1}(\mathcal{A}) + \mathrm{Per}_{\omega,1}(\mathcal{B}) - 4 \cdot \sum_{\mathcal{A}}\sum_{\mathcal{B}} \sqrt{\omega_{uv}}.\end{aligned}$$

We obtain the result for $\mathrm{Per}^{\pm}_{\omega,1}$ immediately from the following relation $\mathrm{Per}^{\pm}_{\omega,1} = \frac{1}{2}\,\mathrm{Per}_{\omega,1}$. □

6.2.2 Relations to continuous perimeters

The concept of nonlocal perimeter was introduced in [33, 59] and was thoroughly studied in [38, 108]. For singular kernels of the form $\frac{1}{|\boldsymbol{x}|^{n+s}}$, $0 < s < 1$, the nonlocal perimeter reappeared in [44, 145, 146], where some functionals of this type were analyzed in connection with fractal dimensions. The nonlocal s-perimeter of a subset $\mathcal{A} \subset \mathbb{R}^n$ is defined (formally) as

$$\mathrm{Per}_s(\mathcal{A}) = \int_{\mathcal{A}} \int_{\mathcal{A}^c} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{n+s}} d\boldsymbol{y} d\boldsymbol{x}.$$

The main idea of the nonlocal s-perimeter is that any point inside $\mathcal{A}$ interacts with any outside. The usuel notion of perimeter is recovered by the limit

$$\lim_{s\to 1}(1-s)\,\mathrm{Per}_s(\mathcal{A}) = \mathrm{Per}(\mathcal{A}) = \int_{\mathbb{R}^N} |D\chi_{\mathcal{A}}|,$$

see [7, 38, 45, 59].

Now, we will show that the definition of the (s-)perimeter can be recovered by our definition. Indeed, let $J : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{R}^+$ defined as

$$J(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|^{n+s}}, \quad \forall \boldsymbol{x} \neq 0. \tag{6.2.7}$$

Let $\{J_k\}_k$ be a sequence of symmetric positive functions in $L^1(\mathbb{R}^n)$ satisfying:

(i) for all k, J_k of compact support and $J_k = \sum_{\boldsymbol{x}\in\frac{1}{k}\mathbb{Z}^n} \alpha_{\boldsymbol{x}} \chi_{Q^k_{\boldsymbol{x}}}$, where $\alpha_{\boldsymbol{x}} \in \mathbb{R}^+$ and $Q^k_{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{k^n}[0,1[^n$.

(ii) $\{J_k\}_k$ converges to J strongly in $L^1(\mathbb{R}^n)$.

Fix $k \in \mathbb{N}^\star$. Consider $G_k = (V_k, E_k, \omega^k)$ where $V_k = \frac{1}{k}\mathbb{Z}^n$ and

$$\omega^k(\boldsymbol{x},\boldsymbol{y}) = (k^{2n} J_k(\boldsymbol{x}-\boldsymbol{y}))^2, \quad \forall\, \boldsymbol{x},\, \boldsymbol{y} \in V_k.$$

For all $\mathcal{A} \subset \mathbb{R}^n$, we set $\mathcal{A}^d_k = \left\{\boldsymbol{x} \in V_k :\ Q^k_{\boldsymbol{x}} \cap \mathcal{A} \neq \emptyset\right\}$ and $\mathcal{A}_k = \bigcup_{\boldsymbol{x}\in\mathcal{A}^d_k} Q^k_{\boldsymbol{x}}$. Then

$$\mathrm{Per}_{\omega^k,1}(\mathcal{A}^d_k) = \sum_{\boldsymbol{y}\in\mathcal{A}^d_k} \sum_{\boldsymbol{x}\in(\mathcal{A}^d_k)c} \sqrt{\omega^k(\boldsymbol{x},\boldsymbol{y})} = \int_{\mathcal{A}_k}\int_{(\mathcal{A}_k)^c} J_k(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y}.$$

By construction, we can easy check that

$$\lim_k \int_{\mathcal{A}_k}\int_{(\mathcal{A}_k)^c} J_k(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} = \int_{\mathcal{A}}\int_{\mathcal{A}^c} J_k(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y}.$$

Hence

$$\begin{aligned} \lim_k \mathrm{Per}_{\omega^k,1}(\mathcal{A}^d_k) &= \int_{\mathcal{A}}\int_{\mathcal{A}^c} J(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} \\ &= \mathrm{Per}_s(\mathcal{A}). \end{aligned} \tag{6.2.8}$$

The pre-Minkowski content. Let $r > 0$, the pre-Minkowski content of a set ∂E is given by

$$\mathcal{M}_r(E) \overset{\text{def}}{=} \frac{1}{2r} \Big| \bigcup_{x\in\partial E} B_r(\boldsymbol{x}) \Big|$$

It is well-known that, under mild regularity assumption on E (see for instance [8]) we have

$$\lim_{r\to 0} \mathcal{M}_r(E) = \mathrm{Per}(E) = \int |D\chi_E| d\boldsymbol{x},$$

An issue with definition of $\mathcal{M}_r$ is that it depends on the choice of the representative within the Lebesgue equivalence class of the set E. For this reason, the authors of [49] have introduced the following variant:

$$J_r(E) = \frac{1}{2r}\int \mathrm{osc}_{B_r(\boldsymbol{x})}(\chi_E)d\boldsymbol{x},$$

where $osc_{\mathcal{A}}(\phi)$ denotes the essential oscillation of the measurable function ϕ over a measurable set $\mathcal{A}$, defined by

$$\mathrm{osc}_{\mathcal{A}}(\phi) = \operatorname*{ess\,sup}_{\mathcal{A}}(\phi) - \operatorname*{ess\,inf}_{\mathcal{A}}(\phi).$$

See [49] and references therein for more details. We have that $J_r(E)$ coincides with the pre-Minkowski content of the essential boundary of E. As a consequence, we have the following result

$$\lim_{r\to 0} J_r(E) = \mathrm{Per}(E) = \int |D\chi_E|d\boldsymbol{x}.$$

Let $r > 0$ and fix $k \in \mathbb{N}^\star$. We consider the following weighted graph $G_k = (V_k, E_k, \omega^k)$ where $V_k = \frac{1}{k}\mathbb{Z}^n$ and the weight function is given as

$$\omega^k(\boldsymbol{x},\boldsymbol{y}) = \frac{k^{-2n}}{(2r)^2}\begin{cases} 1, & \text{if } \boldsymbol{y} \in (B_r(\boldsymbol{x}))^d_k, \\ 0, & \text{otherwise,}\end{cases}$$

For all $\mathcal{A} \subset \mathbb{R}^n$, we consider $\mathcal{A}^d_k$ and $\mathcal{A}_k$ as above. Then, we have

$$\begin{aligned}
\mathrm{Per}_{\omega^k,\infty}(\mathcal{A}^d_k) &= \sum_{\boldsymbol{x}\in\frac{1}{k}\mathbb{Z}^n} \frac{k^{-n}}{2r}\left(\max_{\boldsymbol{y}\sim\boldsymbol{x}}\left(\chi_{\mathcal{A}^d_k}(\boldsymbol{y}) - \chi_{\mathcal{A}^d_k}(\boldsymbol{x})\right) + \max_{\boldsymbol{y}\sim\boldsymbol{x}}\left(\chi_{\mathcal{A}^d_k}(\boldsymbol{x}) - \chi_{\mathcal{A}^d_k}(\boldsymbol{y})\right)\right)\\
&= \sum_{\boldsymbol{x}\in\frac{1}{k}\mathbb{Z}^n} \frac{k^{-n}}{2r}\left(\max_{\boldsymbol{y}\sim\boldsymbol{x}}\chi_{\mathcal{A}^d_k}(\boldsymbol{y}) - \min_{\boldsymbol{y}\sim\boldsymbol{x}}\chi_{\mathcal{A}^d_k}(\boldsymbol{y})\right)\\
&= \sum_{\boldsymbol{x}\in\frac{1}{k}\mathbb{Z}^n} \frac{k^{-n}}{2r}\,\mathrm{osc}_{(B_r(\boldsymbol{x}))^d_k}(\chi_{\mathcal{A}^d_k}),
\end{aligned}$$

where $\mathrm{osc}_{(B_r(\boldsymbol{x}))^d_k}(\chi_{\mathcal{A}^d_k}) = \max_{(B_r(\boldsymbol{x}))^d_k}\chi_{\mathcal{A}^d_k} - \min_{(B_r(\boldsymbol{x}))^d_k}\chi_{\mathcal{A}^d_k}$. We conjecture that

$$\lim_k \mathrm{Per}_{\omega^k,\infty}(\mathcal{A}^d_k) = \frac{1}{2r}\int \mathrm{osc}_{B_r(\boldsymbol{x})}(\chi_{\mathcal{A}})d\boldsymbol{x}.$$

We leave this as an open problem for the future.

One can see that it is easy to transpose local and nonlocal continuum perimeters into the graph-based framework. Notice that these formulations are indeed special cases ($p = 1$) of a more general family of weighted perimeters that defined above.

6.3 Total variations and Cheeger inequality on graphs

In this section, we extend the notion of total variations, for $p = 1$, on graphs to upwind and downwind total variations and also for $p \in]1,\infty]$. We show that the result of the co-area formula provided in [70, 143] still true for $p = \infty$ on unweighted graphs. We jump to expose an extension of the Cheeger inequality.

6.3.1 Total variation on graphs

Let us extend the definition of the total variation introduced in [70] to $p \in [1,+\infty]$, see [143] also a similar definition for the anisotropic total variation.

Definition 6.3.1. For $1 \le p < \infty$, the total variation on graphs is defined as follows:

$$\mathrm{TV}_{\omega,p}(f) = E\left(\|\nabla_\omega f\|_p\right) = \sum_{u\in V}\left(\sum_{v\in V}\omega_{uv}^{\frac{p}{2}}|f(v)-f(u)|^p\right)^{\frac{1}{p}}$$

$$\mathrm{TV}^{\pm}_{\omega,p}(f) = E\left(\left\|\nabla^{\pm}_{\omega} f\right\|_p\right) = \sum_{u\in V}\left(\sum_{v\in V}\omega_{uv}^{\frac{p}{2}}\left((f(v)-f(u))^{\pm}\right)^p\right)^{\frac{1}{p}}.$$

Similarly we define the total variations for $p = \infty$:

$$\mathrm{TV}_{\omega,\infty}(f) = E\left(\left\|\nabla_{\omega} f\right\|_{\infty}\right) = \sum_{u\in V}\left(\max_{v\in V}\sqrt{\omega_{uv}}\left|f(v)-f(u)\right|\right)$$

$$\mathrm{TV}^{\pm}_{\omega,\infty}(f) = E\left(\left\|\nabla^{\pm}_{\omega} f\right\|_{\infty}\right) = \sum_{u\in V}\left(\max_{v\in V}\sqrt{\omega_{uv}}(f(v)-f(u))^{\pm}\right).$$

It is known that in the continuous case the perimeter is linked to the total variation via co-area formula. A similar results has been exposed in [70, 143] for the discrete case. For the reader's convenience, we recall this result and their extension to the upwind and downwind total variations.

Proposition 6.3.2. *For any function $f : V \to \mathbb{R}$, we have:*

$$\mathrm{TV}^{\pm}_{\omega,1}(f) = \int_{-\infty}^{+\infty} \mathrm{TV}^{\pm}_{\omega,1}(\chi_{\{f>t\}})dt, \tag{6.3.1}$$

$$\mathrm{TV}_{\omega,1}(f) = \int_{-\infty}^{+\infty} \mathrm{TV}_{\omega,1}(\chi_{\{f>t\}})dt. \tag{6.3.2}$$

In particular, for all $\mathcal{A} \subset V$ we have

$$\mathrm{TV}^{\pm}_{\omega,1}(\chi_{\mathcal{A}}) = \mathrm{Per}^{\pm}_{\omega,1}(\mathcal{A}) \text{ and } \mathrm{TV}_{\omega,1}(\chi_{\mathcal{A}}) = \mathrm{Per}_{\omega,1}(\mathcal{A}).$$

Proof : See [143] for a detailed proof of (6.3.2). The proof of (6.3.1) holds from (6.3.2) and the following relationship:

$$\mathrm{TV}^{\pm}_{\omega,1}(f) = \frac{1}{2}\mathrm{TV}_{\omega,1}(f), \text{ for every function } f \in \mathcal{H}(V).$$

□

For $p = \infty$, the co-area formula holds for unweighted graphs, as the following proposition shows. To remove confusion on the notation, we denote $\omega = 1$ to signify that the considered graph is unweighted.

Proposition 6.3.3. *For any function $f : V \to \mathbb{R}$, we have:*

$$\mathrm{TV}^{\pm}_{\omega=1,\infty}(f) = \int_{-\infty}^{+\infty} \mathrm{TV}^{\pm}_{\omega=1,\infty}(\chi_{\{f>t\}})dt,$$

$$\mathrm{TV}_{\omega=1,\infty}(f) = \int_{-\infty}^{+\infty} \mathrm{TV}_{\omega=1,\infty}(\chi_{\{f>t\}})dt.$$

Proof : Let $u \in V$ and let $v_u \in \mathcal{N}(u)$ such that $\left\|\nabla^{\pm}_{\omega=1} f(u)\right\|_{\infty} = (f(v_u) - f(u))^{\pm}$, we can easy to see that $\left\|\nabla^{\pm}_{\omega=1}\chi_{\{f>t\}}(u)\right\|_{\infty} = \left(\chi_{\{f>t\}}(v_u) - \chi_{\{f>t\}}(u)\right)^{\pm}$ for all $t \in \mathbb{R}$. Then

$$\begin{aligned}
\|\nabla^{\pm}_{\omega=1} f(u)\|_{\infty} &= (f(v_u) - f(u))^{\pm} \\
&= \int_{-\infty}^{+\infty} \left(\chi_{\{f>t\}}(v_u) - \chi_{\{f>t\}}(u)\right)^{\pm} dt \\
&= \int_{-\infty}^{+\infty} \|\nabla^{\pm}_{\omega=1}\chi_{\{f>t\}}(u)\|_{\infty} dt
\end{aligned}$$

Hence,

$$\mathrm{TV}^{\pm}_{\omega=1,\infty}(f) = E(\|\nabla^{\pm}_{\omega=1} f\|_\infty) = \int_{-\infty}^{+\infty} E\left(\|\nabla^{\pm}_{\omega=1}\chi_{\{f>t\}}\|_\infty\right) dt = \int_{-\infty}^{+\infty} \mathrm{TV}^{\pm}_{\omega=1,\infty}(\chi_{\{f>t\}})dt.$$

Using the fact that $\mathrm{TV}_{\omega=1,\infty}(f) = \mathrm{TV}^{+}_{\omega=1,\infty}(f) + \mathrm{TV}^{-}_{\omega=1,\infty}(f)$, to get the last equality. □

Remark 6.3.4. The co-area doesn't hold for a general weighted graphs, for $p = \infty$. Indeed, let G be a weighted graph with the vertex set $V = \{1, 2, 3\}$ and the weight function is given by

$$\omega_{ij}^2 = \begin{cases} 1, & \text{if } (i,j) = (1,2), \\ 1/4, & \text{if } (i,j) = (1,3), \\ 1/3, & \text{if } (i,j) = (2,3). \end{cases}$$

Consider the following function defined on V by $f(1) = 0$, $f(2) = 1$, $f(3) = 4$. By a simple computations one gets that

$$\mathrm{TV}^{\pm}_{\omega,\infty}(f) = 2 < \frac{11}{4} = \int_{-\infty}^{+\infty} \mathrm{TV}^{\pm}_{\omega,\infty}(\chi_{\{f>t\}})dt,$$
$$\mathrm{TV}_{\omega,\infty}(f) = 3 < 5 = \int_{-\infty}^{+\infty} \mathrm{TV}_{\omega,\infty}(\chi_{\{f>t\}})dt.$$

We close this subsection with an application of co-area formulas to an equivalent result on functional inequalities.

Let $\mathcal{G}$ be a non-empty set of pairs (g_1, g_2) functions on V an let $\mathcal{L}$ be a functional generated by $\mathcal{G}$ as follow:

$$\mathcal{L}(f) = \sup_{(g_1,g_2)\in\mathcal{G}} E(f^+ g_1 + f^- g_2). \tag{6.3.3}$$

We say that the functional $\mathcal{L}$ admits a quasi-linear representations. As noted in [140], many functionals have this representation, for example:

$$\mathcal{L}(f) = \left(E\left(|f|^p\right)\right)^{1/p}, \text{ for } 1 \le p \le \infty,$$
$$\mathcal{L}(f) = \left(E\left(|f - E(f)|^p\right)\right)^{1/p}, \text{ for } 1 \le p \le \infty,$$
$$\mathcal{L}(f) = \inf_{a\in\mathbb{R}} \left(E\left(|f - a|^p\right)\right)^{1/p} \text{ for } 1 \le p \le \infty.$$

The co-area formula implies the following equivalence.

Proposition 6.3.5. *Let $\lambda > 0$, and either $p = 1$ or $p = \infty$ with $\omega \in \{0, 1\}$, the following are equivalent:*

(i) $\mathcal{L}(f) \le \lambda E(\|\nabla^{\pm}_\omega f\|_p)$ for all $f : V \to \mathbb{R}$.

(ii) $\mathcal{L}(\chi_\mathcal{A}) \le \lambda E(\|\nabla^{\pm}_\omega \chi_\mathcal{A}\|_p)$ and $\mathcal{L}(-\chi_\mathcal{A}) \le \lambda E(\|\nabla^{\pm}_\omega(-\chi_\mathcal{A})\|_p)$, for all $\mathcal{A} \subset V$.

Proof : The implication (i) $\Longrightarrow$ (ii) is straightforward, it is enough to apply (i) to $f = \chi_A$ and $f = -\chi_\mathcal{A}$. Conversely, let $g_1, g_2 \in \mathcal{G}$, it is easy to see $E(\|\nabla^{\pm}_\omega \chi_\mathcal{A}\|_p) = E(\|\nabla^{\pm}_\omega(-\chi_{\mathcal{A}^c})\|_p)$ for all $\mathcal{A} \subset V$. Therefore

$$\begin{aligned} E(\|\nabla^{\pm}_\omega f\|_p) &= \int_0^{+\infty} E(\|\nabla^{\pm}_\omega \chi_{\{f>t\}}\|_p)dt + \int_{-\infty}^0 E(\|\nabla^{\pm}_\omega \chi_{\{f>t\}}\|_p)dt \\ &= \int_0^{+\infty} E(\|\nabla^{\pm}_\omega \chi_{\{f>t\}}\|_p)dt + \int_{-\infty}^0 E(\|\nabla^{\pm}_\omega(-\chi_{\{f\le t\}})\|_p)dt \\ &\ge \lambda^{-1}\int_0^{\infty} E[g_1 \cdot \chi_{\{f>t\}}]dt + \lambda^{-1}\cdot\int_{-\infty}^0 E\left(\chi_{\{f\le t\}} g_2\right) dt \end{aligned}$$

$$= \lambda^{-1} E(g_1 f^+) + \lambda^{-1} \cdot E(f^- g_2).$$

We get the desired inequality by taking the supremum over all function $g_1, g_2 \in \mathcal{G}$.

□

6.3.2 Extension of Cheeger inequality on graphs

Partitioning the set of vertices of a graph into two or more disjoints subsets, is a fundamental problem in graph theory. It is also a very powerful tool in data clustering with applications in image analysis and machine learning [35, 41, 142] . A popular criterion to partition the graph is to minimize the perimeter $\mathrm{cut}(A, A^c)$ defined as

$$\mathrm{cut}(\mathcal{A}, \mathcal{A}^c) \stackrel{\text{def}}{=} \sum_{u \in \mathcal{A}} \sum_{v \in A^c} \sqrt{\omega_{uv}}.$$

Direct minimization of the cut leads typically to unbalanced partitions. To solve this issue one can introduce a balance term. There exist several kinds of balanced cuts, among which a popular one is the Cheeger cut [54] defined as

$$\min_{\mathcal{A} \subsetneq V} \frac{\mathrm{cut}(\mathcal{A}, \mathcal{A}^c)}{\min(|\mathcal{A}|, |\mathcal{A}^c|)}, \qquad \text{with } |\mathcal{A}| \text{ the size of } \mathcal{A}.$$

The Cheeger constants can be rewritten as

$$h^+ = \min_{\substack{\mathcal{A} \subset V \\ 0 < |\mathcal{A}| < |V|}} \frac{E\left(\left\|\nabla_\omega^+ \chi_\mathcal{A}\right\|_1\right)}{\min\left(E(\chi_\mathcal{A}), E(\chi_{\mathcal{A}^c})\right)} = \min_{\substack{\mathcal{A} \subset V \\ 0 < |\mathcal{A}| < \frac{|V|}{2}}} \frac{E\left(\left\|\nabla_\omega^+ \chi_\mathcal{A}\right\|_1\right)}{E(\chi_\mathcal{A})},$$

we can then extend the isoperimetric and related constant associated to the discrete gradient defined above, for any $1 \leq p \leq \infty$, as follows:

$$h_p^\pm = \min_{0 < |\mathcal{A}| \leq \frac{|V|}{2}} \frac{E\left(\left\|\nabla_\omega^\pm \chi_\mathcal{A}\right\|_p\right)}{|\mathcal{A}|}, \quad h_p = \min_{0 < |\mathcal{A}| \leq \frac{|V|}{2}} \frac{E\left(\left\|\nabla_\omega \chi_\mathcal{A}\right\|_p\right)}{|\mathcal{A}|}.$$

By definition of the perimeters we have:

$$2h_1^+ = 2h_1^- = h_1 = \min_{0 < |\mathcal{A}| \leq \frac{|V|}{2}} \frac{\mathrm{Per}_{\omega,1}(\mathcal{A})}{|\mathcal{A}|}.$$

Definition 6.3.6. Let $p \in [1, \infty]$, we set:

$$k_p^\pm = \inf_{f \neq const} \frac{E\left(\left\|\nabla_\omega^\pm f\right\|_p\right)}{E\left(\left|f - m(f)\right|\right)},$$

$$k_p = \inf_{f \neq const} \frac{E\left(\left\|\nabla_\omega f\right\|_p\right)}{E\left(\left|f - m(f)\right|\right)},$$

where $m(f)$ is the median of f.

Proposition 6.3.7. *For either $p = 1$ or $p = \infty$ with $\omega \in \{0, 1\}$. We have the following relations:*

$$h_p^\pm = k_p^\pm \qquad \textit{and} \qquad h_p = k_p. \tag{6.3.4}$$

PROOF : Let $\mathcal{A} \subset V$ such that $0 < |\mathcal{A}| \leq |V|/2$, then we have $m(\chi_\mathcal{A}) = 0$ and

$$k_p^\pm \cdot |\mathcal{A}| = k_p^\pm \cdot E\left(\left|\chi_\mathcal{A} - m(\chi_\mathcal{A})\right|\right) \leq E\left(\left\|\nabla_\omega^\pm \chi_\mathcal{A}\right\|_p\right).$$

Similarly, one gets

$$k_p \cdot |\mathcal{A}| = k_p \cdot E\left(\left|\chi_\mathcal{A} - m(\chi_\mathcal{A})\right|\right) \leq E\left(\left\|\nabla_\omega \chi_\mathcal{A}\right\|_p\right).$$

$$k_p^{\pm} \leq h_p^{\pm} \qquad \text{and} \qquad k_p \leq h_p.$$

Let us prove the other inequality. Observe that

$$\min\left(E(\chi_{\{f-m(f)>t\}}), E(\chi_{\{f-m(f)\leq t\}})\right) = \begin{cases} E(\chi_{\{f-m(f)>t\}}), & \text{if } t>0, \\ E(\chi_{\{f-m(f)\leq t\}}), & \text{if } t\leq 0, \end{cases} \tag{6.3.5}$$

By the co-area formula and (6.3.5) , we obtain

$$\begin{aligned}
E\left(\left\|\nabla_\omega^{\pm} f\right\|_p\right) &= \int_{-\infty}^{+\infty} E\left(\left\|\nabla_\omega^{\pm}\chi_{\{f>t\}}\right\|_1\right) dt \\
&= \int_{-\infty}^{+\infty} E\left(\left\|\nabla_\omega^{\pm}\chi_{\{f-m(f)>t\}}\right\|_1\right) dt \\
&\geq h_p^{\pm}\cdot \int_{-\infty}^{+\infty} \min\left(E\left(\chi_{\{f-m(f)>t\}}\right), E\left(\chi_{\{f-m(f)\leq t\}}\right)\right) dt \\
&= h_p^{\pm}\cdot \int_{-\infty}^{0} E\left(\chi_{\{f-m(f)\leq t\}}\right) dt + h_p^{\pm}\cdot \int_{0}^{+\infty} E\left(\chi_{\{f-m(f)>t\}}\right) dt \\
&= h_p^{\pm}\cdot E\left((f-m(f))^-\right) + h_p^{\pm}\cdot E\left((f-m(f))^+\right) \\
&= h_p^{\pm}\cdot E\left(\left|f-m(f)\right|\right).
\end{aligned}$$

Similarly, we obtain

$$E\left(\left\|\nabla_\omega f\right\|_p\right) \geq h_p\cdot E\left(\left|f-m(f)\right|\right).$$

Hence, we get the inverse inequality. □

6.4 Discrete mean curvature flows on graphs

In this section, we expose a large class of mean curvature on graphs based on the definition of the nonlocal perimeters on graphs defined above. As in the nonlocal continuum case [49]. We define the mean curvature as the first variation of the perimeter. We denote by $\delta(u)$ the degree of a vertex $u\in V$ which is given by $\delta(u)=\sum_{v\sim u}\sqrt{\omega_{uv}}$.

Definition 6.4.1. Let $\mathcal{A}\subset V$, and $u_0\in V$. We define the upwind and downwind mean curvature as follow:

$$\begin{aligned}
\kappa_{\omega,1}^{+}(u_0,\mathcal{A}) &\overset{\text{def}}{=} \frac{\mathrm{Per}_{\omega,1}^{+}(\mathcal{A}\cup\{u_0\}) - \mathrm{Per}_{\omega,1}^{+}(\mathcal{A})}{\delta(u_0)}, \\
\kappa_{\omega,1}^{-}(u_0,\mathcal{A}) &\overset{\text{def}}{=} \frac{\mathrm{Per}_{\omega,1}^{-}(\mathcal{A}) - \mathrm{Per}_{\omega,1}^{-}(\mathcal{A}\setminus\{u_0\})}{\delta(u_0)}.
\end{aligned}$$

Finally, we define then the mean curvature for $u_0\in V$ as:

$$\kappa_{\omega,1}(u_0,\mathcal{A}) \overset{\text{def}}{=} \begin{cases} \kappa_{\omega,1}^{+}(u_0,\mathcal{A}), & \text{if } u_0\in\mathcal{A}^c, \\ \kappa_{\omega,1}^{-}(u_0,\mathcal{A}), & \text{if } u_0\in\mathcal{A}. \end{cases}$$

Observe that by a simple development of the definition of the perimeters, we show that

$$\mathrm{Per}_{\omega,1}^{+}(\mathcal{A}\cup\{u_0\}) - \mathrm{Per}_{\omega,1}^{+}(\mathcal{A}) = \begin{cases} \sum_{v\in\mathcal{A}^c}\sqrt{\omega_{u_0v}} - \sum_{v\in\mathcal{A}}\sqrt{\omega_{u_0v}}, & \text{if } u_0\in\mathcal{A}^c, \\ 0, \quad \text{if } u_0\in\mathcal{A}, \end{cases}$$

and

$$\mathrm{Per}_{\omega,1}^{-}(\mathcal{A}) - \mathrm{Per}_{\omega,1}^{-}(\mathcal{A}\setminus\{u_0\}) = \begin{cases} \sum_{v\in\mathcal{A}^c}\sqrt{\omega_{u_0v}} - \sum_{v\in\mathcal{A}}\sqrt{\omega_{u_0v}}, & \text{if } u_0\in\mathcal{A}, \\ 0, \quad \text{if } u_0\in\mathcal{A}^c. \end{cases}$$

Therefore, one gets an explicit formula of the discrete mean curvature.

Proposition 6.4.2. *For all $\mathcal{A} \subset V$ and all $u_0 \in V$, we have:*

$$\begin{aligned}\kappa_{\omega,1}(u_0, \mathcal{A}) &= \frac{\sum_{v\in\mathcal{A}^c}\sqrt{\omega_{u_0v}} - \sum_{v\in\mathcal{A}}\sqrt{\omega_{u_0v}}}{\delta(u_0)} \\ &= -\frac{\sum_{v\in V}\sqrt{\omega_{u_0v}}(\chi_{\mathcal{A}} - \chi_{\mathcal{A}^c})}{\delta(u_0)}.\end{aligned} \tag{6.4.1}$$

Remark 6.4.3. (i) We can interpreted the formula (6.4.1) as a discrete version of the nonlocal J-mean curvature introduced in [108, Definition 3.2], which is given by

$$H^J_{\partial E}(x) \stackrel{\text{def}}{=} -\int_{\mathbb{R}^n} J(\boldsymbol{x}-\boldsymbol{y})(\chi_E(\boldsymbol{y}) - \chi_E(\boldsymbol{x}))d\boldsymbol{y}, \qquad x \in \mathbb{R}^n,$$

where $E \subset \mathbb{R}^n$ measurable set and J is a nonnegative radial measurable function in $L^1(\mathbb{R}^n)$.

(ii) Based on the equation (6.4.1), we can extend the notion of the mean curvature to any function f on graphs by considering its level sets. Indeed, let $f : V \to \mathbb{R}$ and $u_0 \in V$. The mean curvature $\kappa_{\omega,1}$ (we keep the same notion) of f at u_0 on a graph is defined as

$$\begin{aligned}\kappa_{\omega,1}(u_0, f) &\stackrel{\text{def}}{=} \kappa_{\omega,1}(u_0, \{f \geq f(u_0)\}) \\ &= \frac{\sum_{v\in\{f\geq f(u_0)\}}\sqrt{\omega_{u_0v}} - \sum_{v\in\{f< f(u_0)\}}\sqrt{\omega_{u_0v}}}{\delta(u_0)} \\ &= \frac{\sum_{v\in V}\sqrt{\omega_{u_0v}}\,\text{sign}(f(v) - f(u_0))}{\delta(u_0)},\end{aligned}$$

where

$$\text{sign}(r) = \begin{cases} 1, & \text{if } r \geq 0, \\ -1, & \text{if } r < 0. \end{cases}$$

(iii) In the continuum (local) setting, the mean curvature, for a given smooth hypersurface $\Gamma \subset \mathbb{R}^N$, at a point $\boldsymbol{x}$ of Γ is given by the following formula

$$\kappa(\boldsymbol{x}) = -\text{div}(n_{\boldsymbol{x}}), \tag{6.4.2}$$

where $n_{\boldsymbol{x}}$, $\boldsymbol{x} \in \Gamma$, is the unit normal vector field.
As in the continuous case, we are going to expose a discrete version of (6.4.2) on graphs, introduced in [143]. Let $G = (V, E, \omega)$ be a weighted graph. For a nonempty set $\mathcal{A} \subset V$, the analogue of (6.4.2) on graph is given as follow:

$$\kappa^{loc}_{\omega,1}(u, \mathcal{A}) = \text{div}_w(n_{\mathcal{A}})(u) = \begin{cases} \sum_{v\in\mathcal{A}^c}\sqrt{\omega_{uv}}, & \text{if } u \in \mathcal{A}, \\ -\sum_{v\in\mathcal{A}}\sqrt{\omega_{uv}}, & \text{if } u \in \mathcal{A}^c, \end{cases} \tag{6.4.3}$$

where $n_{\mathcal{A}}$ is the discrete normal vector which is defined as

$$n_{\mathcal{A}}(u,v) = \begin{cases} 1 & \text{if } u \sim v \text{ and } (u,v) \in \mathcal{A} \times \mathcal{A}^c, \\ -1 & \text{if } u \sim v \text{ and } (u,v) \in \mathcal{A}^c \times \mathcal{A}, \\ 0 & \text{else.} \end{cases}$$

The formula given in [143] of the mean curvature is a little different to this one, this difference returns to the definition of divergences considered. Observe that, the sign of the mean curvature, given by (6.4.3), depends only on the side that contains the vertex u and not on the weights function, while it is not in the case of the mean curvature considered in Definition 6.4.1, which makes a difference in the study of the data processing especially the nonlocal ones. In the rest of this work, we adopt Definition 6.4.1 for the discrete mean curvature.

6.4.1 Variational curvature on graphs

Almgren, Taylor and Wang [5] proposed an implicit time discretization to study some geometric variational evolutions, in particular curvature-based motions. Following the formal consideration that curvature can be seen as the variation of the perimeter, they minimized iteratively the following incremental minimum problem

$$\min_{E}\{\mathcal{F}_h(E) \stackrel{\text{def}}{=} \text{Per}(E) + \frac{1}{h}\int_E d_{E^h_{k-1}}(\boldsymbol{x})d\boldsymbol{x}\}, \tag{6.4.4}$$

where E range over all measurable sets in $\mathbb{R}^n$, Per is the usual perimeter, h is the time step, $d_A(\cdot) = \text{dist}(\cdot, A) - \text{dist}(\cdot, A^c)$ with $\text{dist}(\cdot, A)$ is the Euclidean distance from the set A. An equivalent definition of the energy-functional of the problem (6.4.4) is given by

$$\mathcal{F}_h(E) = \text{Per}(E) + \frac{1}{h}\int_{E\Delta E_{k-1}} \text{dist}(x, \partial E_{k-1})d\boldsymbol{x}.$$

The equivalence comes from the following equality

$$\int_E d_E(\boldsymbol{x})d\boldsymbol{x} - \int_F d_F(\boldsymbol{x})d\boldsymbol{x} = \int_{E\Delta F} \text{dist}(\boldsymbol{x}, \partial F)d\boldsymbol{x}.$$

Thanks to the co-area formula, it can be shown that instead of minimization over sets, we can relax the problem by iteratively minimizing the following functional

$$\text{TV}(\phi) + \frac{1}{h}\int_{\mathbb{R}^n} \phi(\boldsymbol{x})d_{E_{k-1}}(\boldsymbol{x})d\boldsymbol{x}, \tag{6.4.5}$$

where ϕ range over all increasing functions in $L^\infty(\mathbb{R}^n; [0,1])$.

Following the approach in [49], we have any level set of a minimiser of (6.4.5) is a solution of the problem (6.4.4).

Many algorithms have been proposed to solve this problem, using different methods, e.g. dual projection [120], graph cut [125], etc.

Now, we are going to translate this problem to the discrete setting. For a graph $G = (V, E, \omega)$, let $(\text{Per}_\omega, \text{TV}_\omega)$ be one of the following pair of perimeter and their corresponding total variation $(\text{Per}_{\omega,p}, \text{TV}_{\omega,p})$ or $\left(\text{Per}^{\pm}_{\omega,p}, \text{TV}^{\pm}_{\omega,p}\right)$, where either $p = 1$ or $p = \infty$ with $\omega \in \{0, 1\}$. We consider corresponding following minimum problem on G.

$$\min_{\hat{\mathcal{A}}}\{\text{Per}_\omega(\hat{\mathcal{A}}) + \frac{1}{h}\sum_{v\in\hat{\mathcal{A}}} sd^G_{\mathcal{A}}(v)\}, \tag{6.4.6}$$

where $\mathcal{A} \subset V$ non-empty set, $h > 0$ corresponds to the time step and $sd^G_{\mathcal{A}}(\cdot)$ is the signed graph distance defined as follows

$$sd^G_{\mathcal{A}}(u) \stackrel{\text{def}}{=} d^G_{\mathcal{A}}(u) - d^G_{\mathcal{A}^c}(u), \qquad u \in V,$$

with $d^G_{\mathcal{B}}$ is the solution of the Eikonal equation (6.5.10) with respect to the boundary set $\mathcal{B}$.

We extend (6.4.6) to the binary function as

$$\min_{\phi:V\to\{0,1\}}\{\text{TV}_\omega(\phi) + \frac{1}{h}\sum_{v\in V} \phi(v) sd^G_{\mathcal{A}}(v)\}. \tag{6.4.7}$$

Using the co-area formula, (6.4.6) is equivalent to

$$\min_{\phi}\{\text{TV}_\omega(\phi) + \frac{1}{h}\sum_{v\in V} \phi(v) sd^G_{\mathcal{A}}(v)\}. \tag{6.4.8}$$

where ϕ range over all increasing functions $\phi : V \to [0,1]$. Since TV_ω is convex, it is easy to check that (6.4.8) has a solution. Observe that, given a solution ϕ of (6.4.8), we have

$$\mathrm{TV}_\omega(\phi) + \frac{1}{h}\sum_{v\in V}\phi(v)sd^G_{\mathcal{A}_0}(v) = \int_0^1 \mathrm{Per}(\{\phi > \xi\})d\xi + \frac{1}{h}\int_0^1 \sum_{v\in\{\phi>\xi\}} sd^G_{\mathcal{A}_0}(v)d\xi.$$

Therefore for almost everywhere $\xi \in [0,1]$, $\{\phi > \xi\}$ is a solution to (6.4.6).

Using the transcription above, we define the mean curvature flow, $\mathcal{A}_n, n \in \mathbb{N}$, with discrete time step $h > 0$ for an initial set $\mathcal{A}_0 \subset V$, recursively as

$$\mathcal{A}_n \in \arg\min_{\mathcal{A}\subset V}\{\mathrm{Per}_\omega(\mathcal{A}) + \frac{1}{h}\sum_{v\in\mathcal{A}} sd^G_{\mathcal{A}_{n-1}}(v)\}. \tag{6.4.9}$$

6.5 Level set formulation of nonlocal mean curvature flows on graphs and applications

Based in a discretization of the gradients and curvatures on a general domain, graph, we can adapt a large PDEs models on graphs involving mean curvature or variants of mean curvature. In this section we consider two general models used extensively to solve several tasks in image processing and computer vision. The level power mean curvature flows for image denoising, enhancement or simplification and the PDEs level set active contours for image segmentation and object detection. We will show that the transposition of these models on graphs leads to partial differences equations with coefficients that are data dependant and their applications are naturally extend to the processing of any data and for data classification.

6.5.1 Level set power mean curvature flow on Euclidean domain

We recall the level set approach for front propagation on Euclidean domains. The level set method for front propagation has been used with great success in both pure and applications and in different applications in image processing, computer vision and computer graphics. Given an initial front on surface Γ_0 a boundary of repere $\Gamma_0 \subset \mathbb{R}^n(\partial\Omega_0 = \Gamma_0)$, see [131, 117, 42]. The level set is used to analyse its subsequent motion under a normal velocity $c(\boldsymbol{x}, t)$. The idea is to represent the evolving front as a level set of a function $\phi(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^n$ and t is the time. The initial front is given by $\Gamma_0 = \{\boldsymbol{x} : \ \phi(\boldsymbol{x}, 0) = 0 = \phi_0\}$, where ϕ_0 is a smooth function defined on $\mathbb{R}^n$, and the evolving front is described for all later time as $\Gamma_t = \{\boldsymbol{x} : \ \phi(\boldsymbol{x}, t) = 0\}$. The evolving front is governed by the equation :

$$\begin{cases} \frac{\partial\phi}{\partial t}(\boldsymbol{x}, t) & = c(\boldsymbol{x}, t)\big\|\nabla\phi(\boldsymbol{x}, t)\big\|_2 \quad (\boldsymbol{x}, t) \in \mathbb{R}^n \times (0, T) \\ \phi(\boldsymbol{x}, 0) & = \phi_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n. \end{cases} \tag{6.5.1}$$

In particular, when $c(\boldsymbol{x}, t) = \big|\kappa(\boldsymbol{x}, t)\big|^{\alpha-1}\kappa(\boldsymbol{x}, t)$ where κ presents the usual mean curvature, we have the level set power mean curvature equation, and (6.5.1) reads

$$\begin{cases} \frac{\partial\phi}{\partial t}(\boldsymbol{x}, t) & = \Big|\mathrm{div}\left(\frac{\nabla\phi(\boldsymbol{x},t)}{\|\nabla\phi(\boldsymbol{x},t)\|_2}\right)\Big|^{\alpha-1}\mathrm{div}\left(\frac{\nabla\phi(\boldsymbol{x},t)}{\|\nabla\phi(\boldsymbol{x},t)\|_2}\right)\big\|\nabla\phi(\boldsymbol{x}, t)\big\|_2, \\ \phi(\boldsymbol{x}, 0) & = \phi_0(\boldsymbol{x}). \end{cases} \tag{6.5.2}$$

For $\alpha = 1$ this equation corresponds to the mean curvature flow filter which finds important applications in image processing [131]. A variant for positive/negative curvature flows are used in [107] for image enhancement in addition to noise removal.

When $\alpha \to 0$, we obtain so called erosion/dilatation used in mathematical morphology. In this case the equation (6.5.2) is given by:

$$\frac{\partial \phi}{\partial t}(\boldsymbol{x}, t) = \text{sign}\left(\text{div}\left(\frac{\nabla \phi(\boldsymbol{x}, t)}{\|\nabla \phi(\boldsymbol{x}, t)\|_2}\right)\right) \|\nabla \phi(\boldsymbol{x}, t)\|_2.$$

In the case, where ϕ_0 is an implicit representation of a front (surface), we get the active contour/snake model which is one of the most successful variational models in image segmentation. It consists of evolving a contour in images toward the boundaries of objects. Its success is based on strong mathematical properties and efficient numerical schemes via the level sets method. We consider the following curve evolution equation

$$\begin{cases} \frac{\partial \phi}{\partial t}(\boldsymbol{x}, t) & = \left(\alpha\, \text{div}\left(\frac{\nabla \phi(\boldsymbol{x}, t)}{\|\nabla \phi(\boldsymbol{x}, t)\|_2}\right) + \beta\, F(I, \phi(\boldsymbol{x}, t))\right) \|\nabla \phi(\boldsymbol{x}, t)\|_2, \\ \phi(\boldsymbol{x}, 0) & = \phi_0(\boldsymbol{x}), \end{cases} \tag{6.5.3}$$

where $I : \Omega \to \mathbb{R}$ is the initial image and F is a halting function of the active contour model.

Chan-Vese model for active contours [53, 144] is a powerful and flexible method which detects objects whose boundaries are not necessarily detected by the gradient. This model is based on an energy minimization problem, which can be reformulated in the level set formulation, leading to an easier way to solve the problem. Chan-Vese model has achieved good performance in image segmentation task due to its ability of obtaining a larger convergence range and handling topological changes naturally.

$$\begin{cases} \frac{\partial \phi}{\partial t}(\boldsymbol{x}, t) & = \left(\alpha\, \text{div}\left(\frac{\nabla \phi(\boldsymbol{x}, t)}{}\|\nabla \phi(\boldsymbol{x}, t)\|_2\right) - \lambda_1 (I - c_1)^2 + \lambda_2 (I - c_2)^2\right) \|\nabla \phi(\boldsymbol{x}, t)\|_2, \\ \phi(\boldsymbol{x}, 0) & = \phi_0(\boldsymbol{x}). \end{cases} \tag{6.5.4}$$

where α, λ_1, $\lambda_2 > 0$ are the fitting parameters, I corresponds to the initial image, ϕ_0 is a smooth function, c_1 the average of I on $\phi(\boldsymbol{x}, t) \geq 0$, and c_1 the average of I on $\phi(\boldsymbol{x}, t) \leq 0$.

6.5.2 Transcription of power mean curvature flow on graphs

We are interested in translating on graphs two PDEs models involving mean curvature. Let $G = (V, E, \omega)$ be a weighted graph, based on the definition of discrete gradient and the boundary set which are given above, our formulation for (6.5.2) on graphs can be expressed as follows:

$$\begin{cases} \frac{\partial \phi}{\partial t}(u, t) & = \left(|\kappa_\omega(\phi(u, t))|^{\alpha - 1} \kappa_\omega(\phi(u, t))\right)^+ \|\nabla_\omega^+ \phi(u, t)\|_p \\ & \qquad - \left(|\kappa_\omega(\phi(u, t))|^{\alpha - 1} \kappa_\omega(\phi(u, t))\right)^- \|\nabla_\omega^- \phi(u, t)\|_p, \\ \phi(u, 0) & = \phi_0(u), \end{cases} \tag{6.5.5}$$

where $\phi_0(\cdot, t) \in \mathcal{H}(V)$, $\alpha \in [0, 1]$, $p \in [1, +\infty]$ and

$$\kappa_\omega(\phi(u, t)) = \kappa_{\omega,1}\left(u, \{y \mid \phi(v, t) \geq \phi(u, t)\}\right).$$

We use the forward/explicit Euler scheme in the time to approximate the above problem, for that let $0 < t_1 < t_2 < \cdots < t_\ell = T$ be an equispaced partition of $[0, T]$, $T > 0$. i.e. $t_i = \frac{i}{\ell} T$, $i \in [\ell]$.

$$\frac{\partial \phi}{\partial t}(u, t) = \frac{\phi^{i+1}(u) - \phi^i(u)}{\Delta t},$$

where $\phi^i(u) = \phi(u, i\Delta t)$ with $\Delta t = \frac{T}{\ell}$ and the equation (6.5.5) can be rewritten as the following iterative equation:

$$\phi^{i+1}(u) - \phi^i(u) = \Delta t\Bigg(\Big(\big|\kappa_\omega(\phi^i(u))\big|^{\alpha-1}\kappa_\omega(\phi^i(u))\Big)^+ \big\|\nabla^+_\omega \phi^i(u)\big\|_p - \Big(\big|\kappa_\omega(\phi^i(u))\big|^{\alpha-1}\kappa_\omega(\phi^i(u))\Big)^- \big\|\nabla^-_\omega \phi^i(u)\big\|_p\Bigg).$$

In particular, for $\alpha = 1$ the equation (6.5.5) can rewritten as the following iterative equation:

$$\begin{cases} \phi^{i+1}(u) &= \phi^i(u) + \Delta t\Bigg(\big(\kappa_\omega(\phi^i(u))\big)^+ \big\|\nabla^+_\omega \phi^i(u)\big\|_p - \big(\kappa_\omega(\phi^i(u))\big)^- \big\|\nabla^-_\omega \phi^i(u)\big\|_p\Bigg), \\ \phi^0(u) &= \phi_0(u). \end{cases} \tag{6.5.6}$$

When $p = \infty$, we across the scheme considered in [70, Section 3.3]. Now, let us consider the case that when $\alpha \to 0$ and $p = \infty$. Similarly by using the explicit Euler method as above, one gets the following iterative equation :

$$\begin{cases} \phi^{i+1}(u) &= \phi^i(u) + \Delta t\Bigg(\text{sign}\,\big(\kappa_\omega(\phi^i(u))\big) \big\|\nabla^+_\omega \phi^i(u)\big\|_\infty + \text{sign}\,\big(\kappa_\omega(\phi^i(u))\big) \big\|\nabla^-_\omega \phi^i(u)\big\|_\infty\Bigg), \\ \phi^0(u) &= \phi_0(u). \end{cases} \tag{6.5.7}$$

In the case where $\Delta t = 1$, this previous equation can be interpreted as

$$\phi^{i+1}(u) = \begin{cases} \phi^i(u) + \big\|\nabla^+_\omega \phi^i(u)\big\|_\infty, & \text{if } \kappa_\omega(\phi^i(u)) \geq 0, \\ \phi^i(u) - \big\|\nabla^-_\omega \phi^i(u)\big\|_\infty, & \text{if } \kappa_\omega(\phi^i(u)) < 0. \end{cases} \tag{6.5.8}$$

6.5.3 Transcription of the active contour on graphs

In this section, we present a transcription of geometric PDEs on weighted graphs of arbitrary topology. A front evolving on G is defined as a subset $\mathcal{A}_0 \subset V$, and is implicitly represented by a level set function $\phi_0 = \chi_{\mathcal{A}_0} - \chi_{\mathcal{A}_0^c}$. In other word ϕ_0 equal 1 in $\mathcal{A}_0$ and -1 on its complementary. From the general equation (6.5.3) transposed on graph, the front propagation can be expressed in general by

$$\begin{cases} \frac{\partial \phi}{\partial t}(u,t) &= c(u,t) \cdot \big\|\nabla_\omega \phi(u,t)\big\|_p \quad (u,t) \in V \times [0,T) \\ \phi(u,0) &= \phi_0(u), \end{cases} \tag{6.5.9}$$

with $c(\cdot, t) \in \mathcal{H}(V)$. Based on the previous definition of discrete dilation and erosion on graphs, the front propagation can be expressed as a morphological process with the following sum of dilation and erosion.

$$\begin{cases} \frac{\partial \phi}{\partial t}(u,t) &= (c(u,t))^+ \cdot \big\|\nabla^+_\omega \phi(u,t)\big\|_p - (c(u,t))^- \cdot \big\|\nabla^-_\omega \phi(u,t)\big\|_p \\ \phi(u,0) &= \phi_0(u). \end{cases}$$

To solve this dilation and erosion process, on the contrary to the PDEs case, no spatial discretization is needed thanks to derivatives directly expressed in a discrete form. Then, the general iterative scheme to compute ϕ at time $t+1$ for all $u \in V$ is given by:

$$\phi^{i+1}(u) = \phi^i(u) + \Delta t\Bigg((c(u,t))^+ \big\|(\nabla^+_\omega \phi^i)(u)\big\| - (c(u,t))^- \big\|(\nabla^-_\omega \phi^i)(u)\big\|\Bigg).$$

At each time $i+1$, the new value at a vertex u only depends on its value at time i and the existing values in its neighborhood. This equation can be split in two independent equations, in function of the sign of $c(\cdot,\cdot)$:

$$\phi^{i+1}(u) = \begin{cases} \phi^i(u) + \Delta t(c(u,t))\big\|(\nabla^+_\omega \phi^i)(u)\big\|, & \text{if } c(u,t) > 0, \\ \phi^i(u) + \Delta t(c(u,t))\big\|(\nabla^+_\omega \phi^i)(u)\big\|, & \text{if } c(u,t) < 0, \end{cases}$$

Such decomposition of the process in two independent equations for erosion and dilation processes enhances the computation of the solution because one only has to compute one morphological gradient at each iteration, for a given vertex. Moreover, one can remark that at initialization both two gradients are zero everywhere, except for vertices which lies in the inner and outer boundaries of $\mathcal{A}_0$. Then, the set of vertices to be updated at each iteration can be restricted to two inner and outer narrow bands, initialized respectively with $\partial^-\mathcal{A}_0$ and $\partial^+\mathcal{A}_0$ and updated over time with neighbours of vertices already in. The narrow bands growth follows the fronts evolution and to avoid them to become too large, the narrow bands are reinitialized periodically. Thus, each τ iterations,which correspond to a step k, the front is given by the set $\mathcal{A}_k = \{u \in V : \phi^{k\tau}(u) > 0\}$ and the associated level set function is also reinitialized as $\phi_k(u) = \mathcal{U}_k = \chi_{\mathcal{A}_k}(u) - \chi_{\mathcal{A}_k^c}(u)$. Then, the inner and outer narrow bands are respectively reinitialized as $\partial^-\mathcal{A}_k$ and $\partial^+\mathcal{A}_k$.

Remark 6.5.1. Using previous definitions of morphological evolution equations, one can formulate the same relation and obtain a PdEs-based version of the Eikonal equation, defined on weighted graphs of arbitrary topology. Indeed, let $c = 1$ and $\phi(\cdot, t) = t - \varphi(\cdot)$ on the whole domain V, with $\varphi \in \mathcal{H}(V)$. We obtain a discrete adaptation of the Eikonal equation on graph, which describes a morphological erosion process, and defined as

$$\begin{cases} \| (\nabla_\omega^- \varphi)(u) \| = 1, & u \in V_0, \\ \varphi(u) = 0, & u \in V \setminus V_0, \end{cases} \tag{6.5.10}$$

where $V_0 \subset V$. Numerical schemes and algorithms to solve such equation have provided in [60]. These shemes allow to compute weighted geodesic distances, see [60, Section 5.2].

6.6 Numerical experiments

In this section, we present our numerical experiments to illustrate the potentialities of our formulations of the level set power mean curvature equation, through two models: power mean curvature flows and Chan-Vese model for active contour. These allow us to process both images and $3D$-point clouds. Different graph structures and weight functions are also used to show the flexibility of our approach.

6.6.1 Weighted graph construction

There exist several popular methods to transform discrete data $\{u_1, \cdots, u_n\}$ into a weighted graph structure. Considering a set of vertices V such that the data are embedded by functions of $\mathcal{H}(V)$, the construction of such a graph consists in modeling the neighborhood relationships between the data through the definition of a set of edges E and using a pairwise distance measure $\mu : V \times V \to \mathbb{R}^+$. In the particular case of images, graph construction methods based on geometric neighborhoods are particularly well-adapted to represent the geometry of the space, as well as the geometry of the function defined on that space. We distinguish the following types of graphs:

- *Grid graphs*, which are the most natural structures to describe an image with a graph. Each pixel is connected by an edge to its adjacent pixels. Classical grid graphs are 4-adjacency grid graphs and 8-adjacency grid graphs. Larger adjacency can be used to obtain nonlocal grid graphs.
- *Region adjacency graphs* (RAGs), which provide very useful ways of describing the structure of a picture: vertices represent regions and edges represent region adjacency relationship.
- *k-nearest neighborhood graphs* (k-NNGs), where each vertex u is connected with its k-nearest neighbors according to the distance measure μ. Such construction implies building a directed graph as the neighborhood relationship is not symmetric. Nevertheless, an undirected graph can

be obtained by adding an edge between two vertices u and v if u is among the k-nearest neighbors of v or if v is among the k-nearest neighbors of u.

- *k-extended RAGs* (k-ERAGs), which are RAGs extended by a k-NNG. Each vertex is connected to adjacent regions vertices and to its k most similar vertices of V.

The similarity between two vertices is computed with respect to an appropriate measure $s : E \to \mathbb{R}^+$, which satisfies

$$\omega_{uv} = \begin{cases} s(u,v), & \text{if } (u,v) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Examples for common similarity functions are as follows:

$$\begin{aligned} s_0(u,v) &= 1, \\ s_1(u,v) &= \exp(-\mu(f^0(u), f^0(v))/\sigma^2), \text{ with } \sigma > 0, \end{aligned}$$

for which σ depends on the variation of the function μ and controls the similarity scale.
Several choices can be considered as feature vectors computed from the given data, depending on the nature of the features to be used for graph processing. In the context of image processing one can use the simple grayscale or color feature vector F_u, or a patch feature vector $F_u^\tau = \bigcup_{v \in \mathcal{W}^\tau(u)} F_v$ (i.e., the set of values F_v, where v is in a square window $\mathcal{W}^\tau(u)$ of size $(2\tau+1) \times (2\tau+1)$ centered at a vertex pixel u) incorporating nonlocal features such as texture.

6.6.2 Power mean curvature flow

This paragraph illustrates the potentialities of the power mean curvature through an example of image filtering and an other one of 3D point cloud. Figure 6.1 presents filtering results of an image using the formulation of power mean curvature flows (6.5.5) for $\alpha = 0, 1$, with $p = 2$, on local weighted graph structures. In this example, we construct 4-adjacency grid graphs with $\omega_{uv} = \exp(-d(\phi_0(u), \phi_0(v))/\sigma^2)$. Figure 6.2 presents filtering results obtaining of $3D$ point cloud using the same formulation for $\alpha = 0, 1$, with $p = 2$. In this example, we apply our formulation using two graphs. Both built from the same $3D$-points clouds using k-NNGs, with $k = 8$ and $\omega = 1$ for the first one, which their correspond results are given by (b) and (c). For the second graph, we take $k = 20$ and $\omega_{uv} = \exp(-d(\phi_0(u), \phi_0(v))/10^2)$, which their correspond results are given by (e) and (f). Both examples the function ϕ_0 represents the initial data (image of the first example and $3D$ point cloud for the second one) and d represents the Euclidean distance between $\phi_0(u)$ and $\phi_0(v)$ RGB color vectors.

(a) Original image

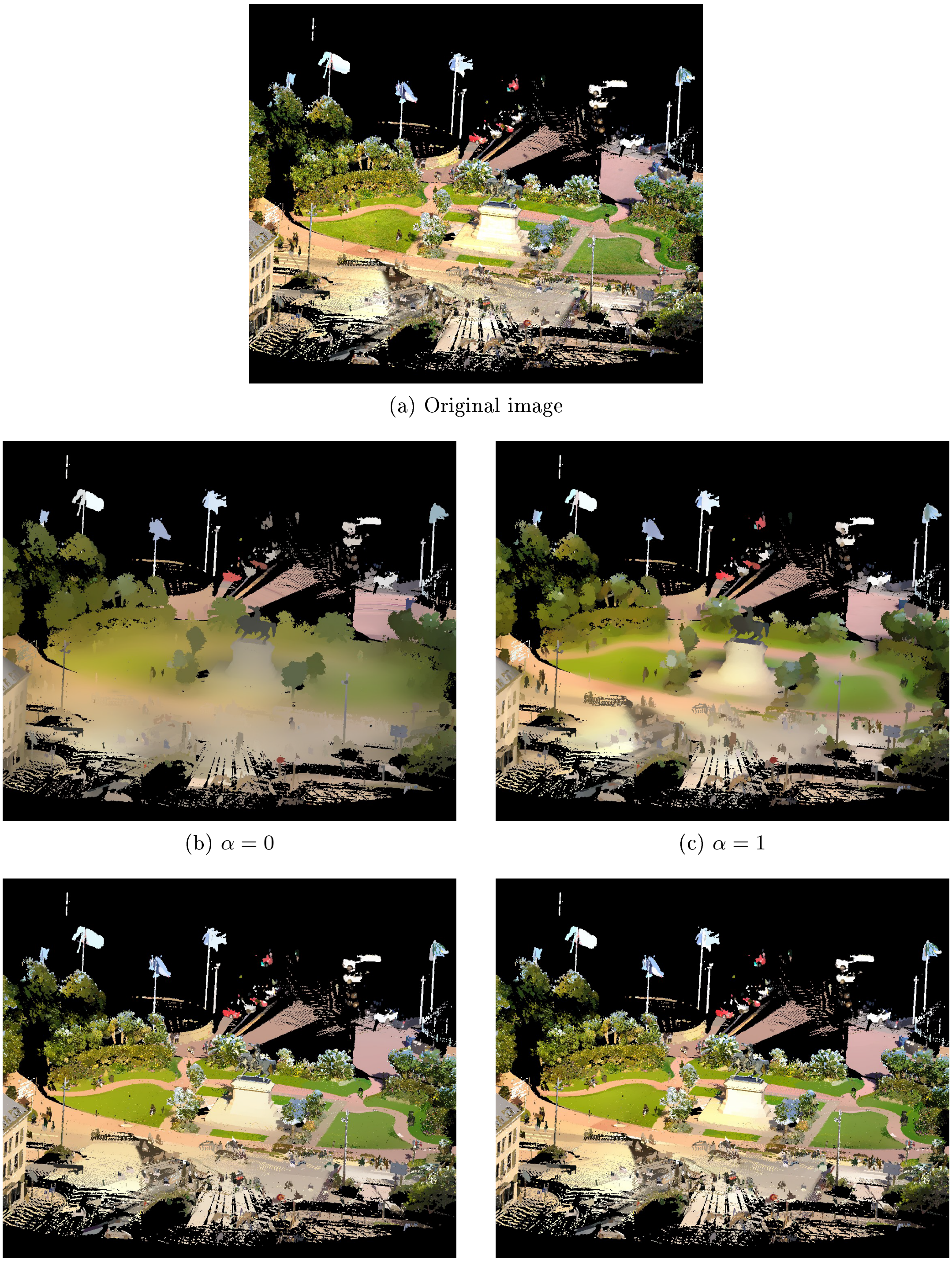
(b) $\alpha = 0$,

(c) $\alpha = 1$

Figure 6.1: Colored image filtering with power mean curvature flows. (a) Original image. (b) and (c) present results with 4-adjacency grid graph and ω =colour, which depends on the colour similarity between different pixels.

6.6.3 Active contour model on graphs

In this paragraph, we illustrate the behaviour of the Chan-Vese model (6.5.4). An advantage of our graph-based formulation is that the proposed formula can be applied to any graph, and therefore any graph representing images. To illustrate such an adaptive behaviour, we propose to use other image structures, such as regions maps, instead of pixels grids to build the graph for image segmentation. In the following, we propose three examples, where we use two graphs. The first one is a classic 4-adjacency grid graph where each pixel is connected to adjacent pixels and represented by its RGB color vector. The second on is a Region Adjacency Graph (RAG) built from an initial partition of the image, where each region (represented by a vertex) is connected to adjacent regions and represented by the mean RGB colour vector inside the region. The partition is computed using the multi-label approach presented in [60], that preserve image's strong boundaries. In both cases, the weight function is defined as: $\omega_{uv} = \frac{\exp^{-(d(u,v))^2}}{\sigma^2}$, where $d(u,v)$ is the Euclidean distance between u and v RGB colour vectors. Figure 6.3 and Figure 6.4 present several steps of the motion and the final position of the front using the first graph (4-grid) with two and three phases, respectively. Figure 6.5 presents different steps of the contour evolution using a RAG.



(a) Original image

(b) $\alpha = 0$

(c) $\alpha = 1$

(d) $\alpha = 0$

(e) $\alpha = 1$

Figure 6.2: Colored point cloud filtering with power mean curvature flows. In the middle line presents results obtained using local k-NNGs ($k = 8$ and $\omega = 1$). The last line presents results under the same configuration but with different similarity function ($\omega =$colour, which depends on the colour similarity between different $3D$-points). obtained using 20-NNGs with .

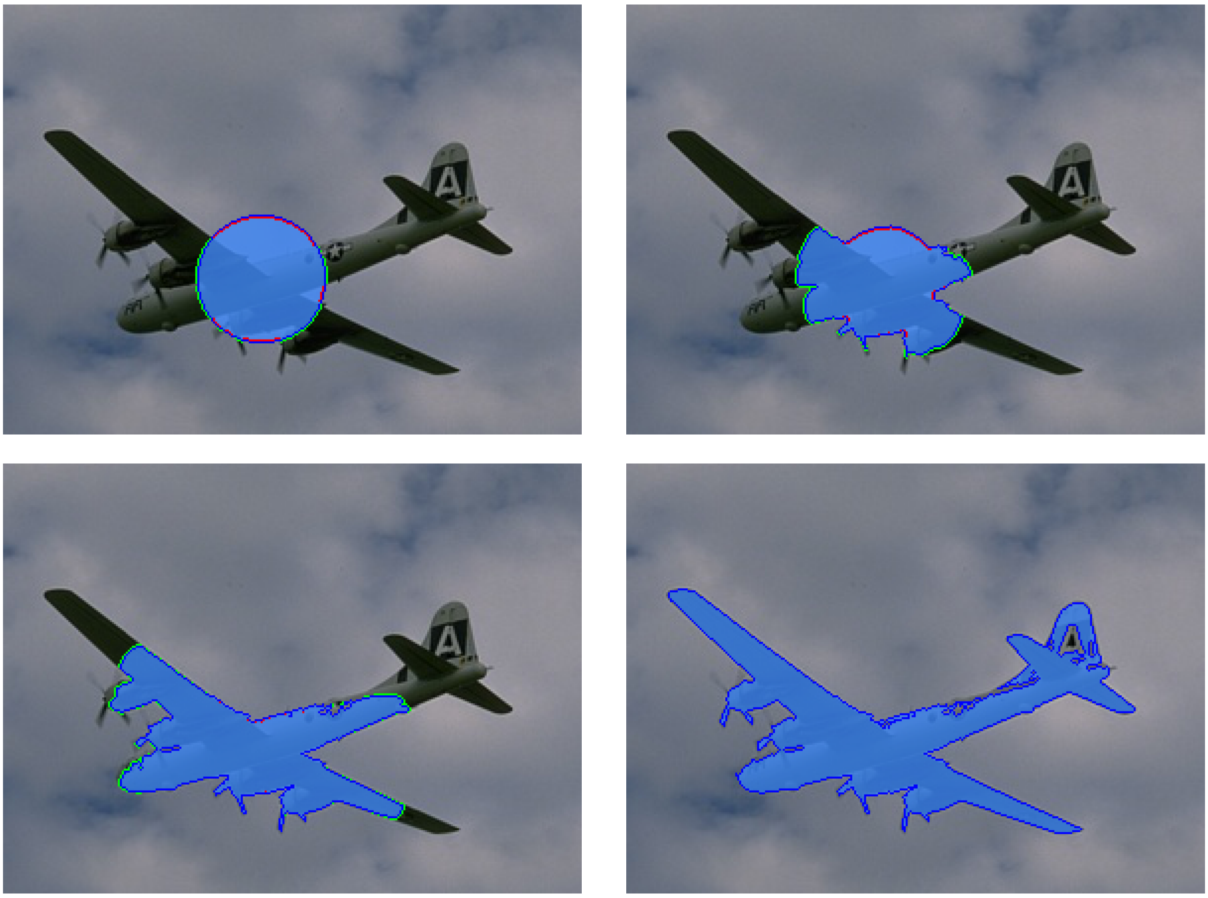


Figure 6.3: Illustration of active contour on a 4-grid graph. The weighted and velocity functions are computed from each pixel RGB color. In blue, the front and the area inside the front. In red and the green, the inner and outer candidate bands respectively.

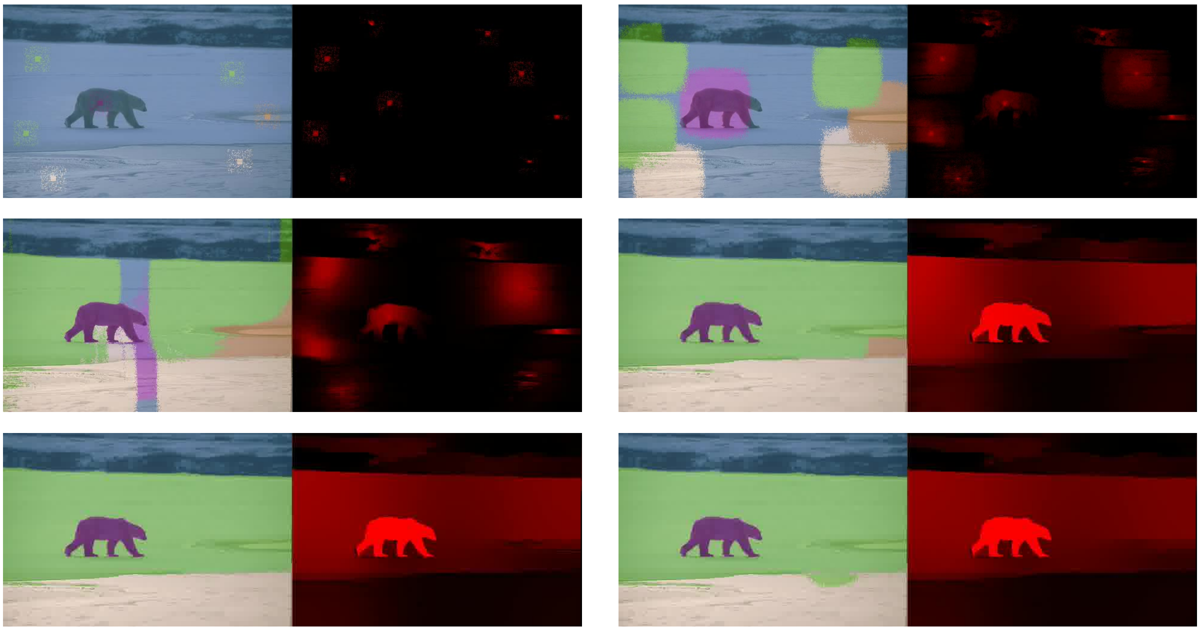
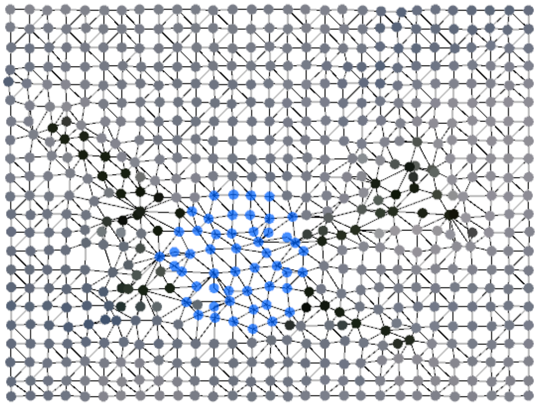
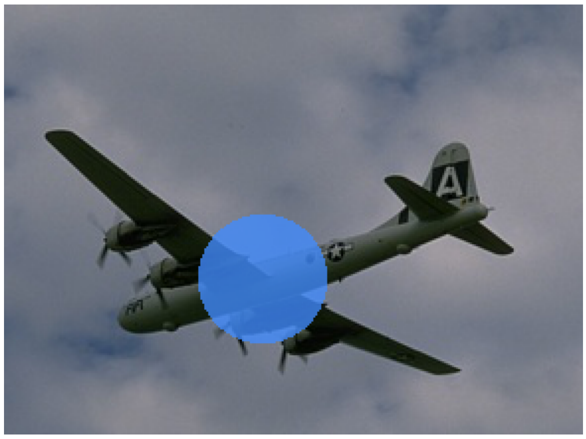


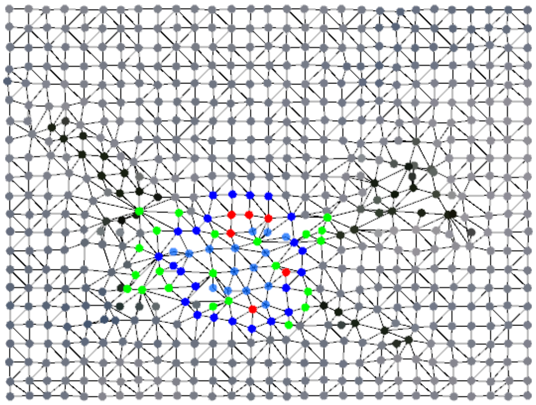
Figure 6.4: Illustration of Chan-Vese segmentation with three phases.Results of different steps on 4-grid graph representation.



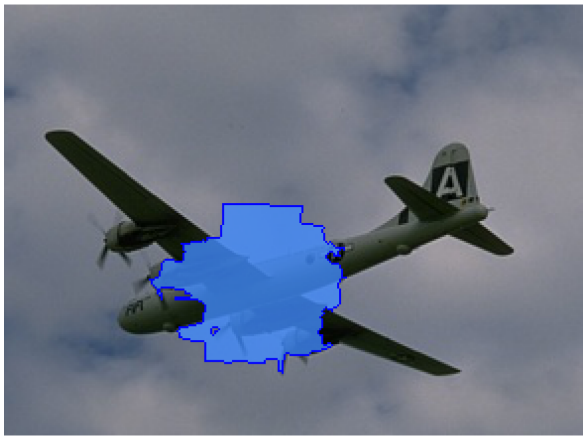
(a) RAG, front and bands



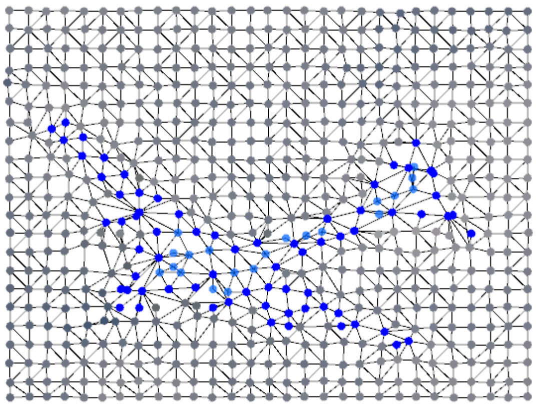
(b) Contour reported on image



(c) After one iteration



(d) After one iterations



(e) After 82 iterations

(f) After 82 iterations

Figure 6.5: Illustration of active contour on a region adjacency graph (RAG). The RAG is built from a superpixel decomposition of the initial image, where each region is connected to its adjacent regions. The weight and velocity functions are computed from the mean color inside regions. Left column shows the RAG, with the front in blue and candidate bands in red (inner) an green (outer). Right column shows the initial image with the front transposed from the RAG (using the superpixels boundaries).

6.6.4 Classification

Performance for Data classification Finally, we have tested the performance of our proposed framework when applied to semi-supervised classification on three standard databases from the literature: MNIST [103], OPTDIGITS [6], and PENDIGITS [4]. We compare two kinds of velocities. The first one is the level set mean curvature flow on the graph-based curvature. We denote it as LSM. The second one is propagation using the evolution eikonal equation but constant in time and based on the characteristic of graph vertices. We denote it as FM. For these databases we merged both the training and the test sets (as performed in [36]), resulting in datasets of 70000 instances, 5620 instances, and 10992 instances, for MNIST, OPTDIGITS, and PENDIGITS, respectively. In our tests, we propose also, to refine the classification results of FM with LSM algorithm (i.e., FM is used as seeds for LSM), and we denote it as FM + LSM.

Comparison with state-of-the-art method: We compare the proposed method with p-Laplacian on weighted graphs (pLPL) (case $p = 2$) [75]. To do so, we vary the amount of initial seeds from 1% to 10%, and compute the average classification rate over 10 runs of each algorithm. The result comparison is shown in Table 6.1. As it can be seen for MNIST and PENDIGITS datasets there is always one of our methods that outperforms the state-of-the-art, while for OPTDIGITS our methods perform equally well.

seeds	datasets	pLPL	FM	LSM	FM+LSM
1%	MNIST	97.84%	97.45%	98.20%	98.24%
	OPTDIGITS	95%	95.22%	96.82%	97.10%
	PENDIGITS	94.97%	95.75%	95.71%	96.25%
2%	MNIST	97.88%	97.64%	98.24%	98.29%
	OPTDIGITS	97.53%	97.41%	97.88%	97.92%
	PENDIGITS	96.81%	97.38%	97.06%	98.56%
5%	MNIST	97.99%	97.95%	98.33%	98.37%
	OPTDIGITS	98.12%	98.09%	98.38%	98.35%
	PENDIGITS	97.95%	98.25%	98.30%	98.56%
10%	MNIST	98.02%	98.19%	98.39%	98.45%
	OPTDIGITS	98.05%	98.41%	98.64%	98.51%
	PENDIGITS	98.61%	98.94%	98.92%	99.10%

Table 6.1: Classification rates on the three datasets we used. FM+LSM works better in general.

Chapter 7

Proximal splitting schemes

Main contributions of this chapter

- We adopt proximal splitting to solve regularization and boundary value problems

Contents

In this chapter, we adopt a primal-dual proximal splitting (PDS) to solve the regularization problem (5.2.1) and the boundary value problems ($\mathcal{P}_n^d$) and (5.3.1). We start by recall the primal dual hybrid gradient scheme developed in [50]. Second, we recall the fast iterative Shrinkage-Thresholding algorithm developed in [23]. Finally, we expose our adaptation and we present a calculus of the proximal mapping proposed in our framework.

7.1 Algorithm for the Dirichlet problem on graphs

In this section, we adopt primal-dual hybrid gradient developed in [50] to solve the discrete boundary value problems ($\mathcal{P}_n^d$) and (5.3.1). In order to avoid redundancy, we expose a general problem for the both problems, which we will adapt to each of these two problems .

7.1.1 Primal-dual splitting

Let X, Y two finite-dimensional real vector spaces equipped with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. The general primal problem is given as

$$\min_{x \in X} \; F(Tx) + G(x), \tag{7.1.1}$$

where $T : X \to Y$ is a continuous linear operator, $G : X \to]-\infty, +\infty]$ and $F : Y \to]-\infty, +\infty]$ are proper, convex lower-semicontinuous functions. The Fenchel-Rockafellar dual problem of (7.1.1) reads

$$\min_{y \in Y} \; (G^\star(-(T^\star y))) + F^\star(y), \tag{7.1.2}$$

where $F^\star$ and $G^\star$ and the Legendre-Fenchel conjugates of F and G, and $T^\star$ is the adjoint operator of T.

Notice that the primal-dual gap, given by

$$\mathcal{G}(x, y) \stackrel{\text{def}}{=} F(Tx) + G(x) + G^\star(-(T^\star y)) + F^\star(y), \tag{7.1.3}$$

is a measure of optimality. If it vanishes at $(x^\star, y^\star) \in X \times Y$ (i.e., strong duality holds), then $(x^\star, y^\star)$ is a saddle point of the Lagrangian

$$\mathcal{L}(x, y) \stackrel{\text{def}}{=} \langle Tx, y \rangle + G(x) - F^\star(y) \tag{7.1.4}$$

as one has

$$\mathcal{L}(x^\star, y) \le \mathcal{L}(x^\star, y^\star) \le \mathcal{L}(x, y^\star), \tag{7.1.5}$$

for all $x \in X$ and $y \in Y$.

The KKT equations, translating the primal-dual optimality conditions, then read

$$\begin{aligned} -T^\star y^\star &\in \partial G(x^\star), \\ Tx^\star &\in \partial F^\star(y^\star), \end{aligned}$$

which may be written

$$0 \in \begin{pmatrix} \partial G(x) \\ \partial F^\star(y) \end{pmatrix} + \begin{pmatrix} 0 & T^\star \\ -T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7.1.6}$$

meaning the solution is found by finding the zeros of the sum of two monotone operators. The latter can be solved with the following PDHG iterative scheme

$$\begin{aligned} x^{k+1} &= \operatorname{prox}_{\tau G}(x^k - \tau T^\star y^k) \\ y^{k+1} &= \operatorname{prox}_{\sigma F^\star}(y^k + \sigma T(2x^{k+1} - x^k)), \end{aligned} \tag{7.1.7}$$

where $\tau, \sigma > 0$ and the proximal operators are given by

$$\operatorname{prox}_{\tau G}(x) \stackrel{\text{def}}{=} \arg\min_{x' \in X} \tau G(x') + \frac{1}{2}\|x - x'\|^2, \tag{7.1.8}$$

$$\operatorname{prox}_{\sigma F^\star}(y) \stackrel{\text{def}}{=} \arg\min_{y' \in Y} \sigma F^\star(y') + \frac{1}{2}\|y - y'\|^2. \tag{7.1.9}$$

The convergence guarantees of (7.1.7) are summarized in the following proposition.

Proposition 7.1.1 ([50]). *Let $L = \|T\|$. choose $0 < \tau\sigma L^2 < 1$ and $((x^k, y^k))_k$ the sequence generated by (7.1.7). Then $((x^k, y^k))_k$ converges to a saddle point $(x^\star, y^\star)$.*

7.1.2 Application to the p-Laplacian Dirichlet problem

We adapt the primal-dual algorithm (7.1.7) with appropriate functions and linear operators to solve problem ($\mathcal{P}_n^d$). We keep the notation of Section 4.4.1. Without loss of generality, we assume that $V_n(\Omega) = \{0, 1, \cdots, n\}$ and $A_n \subset V_n(\Omega)$. Set $V_n = V_n(\Omega)$, $\mathbb{R}^{V_n} = \mathbb{H}(V_n)$ and $\mathbb{R}^{V_n \times V_n} = \mathbb{H}(V_n \times V_n)$.

The problem ($\mathcal{P}_n^d$) is equivalent to ($\mathcal{VP}_n^d$), where the latter takes the form

$$\min_{\mathbb{H}(V_n(\Omega))} F_{d,n}(\boldsymbol{\nabla}_{\mathbf{K}}\mathbf{u}) + G_{d,n}(\mathbf{u}), \tag{7.1.10}$$

where $F_{d,n} : \mathbb{R}^{V_n \times V_n} \to \mathbb{R}^+$ and $G_{d,n} : \mathbb{R}^{V_n} \to \mathbb{R}$, which are defined by

$$F_{d,n}(\mathbf{U}) = \frac{1}{p}\|\mathbf{U}\|^p_{p,\mathbb{R}^{V_n \times V_n}}, \quad \text{and} \quad G_{d,n}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{f}\rangle_{\mathbb{R}^{V_n}} + \iota_{\mathbf{C_g}}(\mathbf{u}), \tag{7.1.11}$$

with $p \in [1, +\infty]$[1], $\mathbf{C_g} = \{\mathbf{u} \in \mathbb{R}^{V_n} : \mathbf{u} = \mathbf{g} \text{ on } \mathcal{A}_n^c\}$, and $\boldsymbol{\nabla}_{\mathbf{K}}$ is the (nonlocal) weighted gradient operator

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{K}} :& \mathbb{R}^{V_n} \to \mathbb{R}^{V_n \times V_n} \\ & \mathbf{u} \to \boldsymbol{\nabla}_{\mathbf{K}}\mathbf{u}, \quad (\boldsymbol{\nabla}_{\mathbf{K}}\mathbf{u})_{\boldsymbol{xy}} = \mathbf{K}_{\boldsymbol{xy}}^{\frac{1}{p}}(\mathbf{u}_{\boldsymbol{y}} - \mathbf{u}_{\boldsymbol{x}}). \end{aligned}$$

This is a linear operator, its norm defined by

$$\|\boldsymbol{\nabla}_{\mathbf{K}}\| \stackrel{\text{def}}{=} \sup_{\|\mathbf{u}\|_{\mathbb{R}^{V_n}}=1} \|\boldsymbol{\nabla}_{\mathbf{K}}\mathbf{u}\|_{\mathbb{R}^{V_n \times V_n}},$$

and whose adjoint, the (nonlocal) weighted divergence operator denoted $\mathbf{div}_{\mathbf{K}}$ and given as

$$\begin{aligned} \mathbf{div}_{\mathbf{K}} :& \mathbb{R}^{V_n \times V_n} \to \mathbb{R}^{V_n} \\ & \mathbf{U} \to \mathbf{div}_{\mathbf{K}}\mathbf{U}, \quad (\mathbf{div}_{\mathbf{K}}\mathbf{U})_{\boldsymbol{x}} = \sum_{\boldsymbol{y} \in V_n} \mathbf{K}_{\boldsymbol{xy}}^{\frac{1}{p}}(\mathbf{U}_{\boldsymbol{yx}} - \mathbf{U}_{\boldsymbol{xy}}). \end{aligned}$$

As mentioned above, we adapt the PDHG iterative scheme (7.1.7) to solve the problem (7.1.10), which reads in this case as

$$\begin{aligned} \mathbf{u}^{k+1} &= \mathbf{prox}_{\tau G_{d,n}}(\mathbf{u}^k - \tau \mathbf{div}_{\mathbf{K}}\mathbf{U}^k) \\ \mathbf{U}^{k+1} &= \mathbf{prox}_{\sigma F^\star_{d,n}}(\mathbf{U}^k + \sigma \boldsymbol{\nabla}_{\mathbf{K}}(2\mathbf{u}^{k+1} - \mathbf{u}^k)), \end{aligned} \tag{7.1.12}$$

where $\tau, \sigma > 0$. The convergence guarantees of (7.1.12) are summarized in the following corollary which is an immediate consequence of Prposition 7.1.1.

[1]The case $p = +\infty$ has to be understood as $\lim_{p \to +\infty} \frac{1}{p}\|\cdot\|^p_{p,\mathbb{R}^{V_n \times V_n}} = \iota_{\|\mathbf{U}\|_{\infty,\mathbb{R}^{V_n \times V_n}} \leq 1}$.

Corollary 7.1.2. *If* $0 < \tau\sigma\|\nabla_{\mathbf{K}}\|^2 < 1$*, then* $(\mathbf{u}^k)_k$ *generated by* (7.1.12) *converges to a solution of* (7.1.10).

7.1.3 Application to the p-bilaplacian Dirichlet problem

Let $G = (V, E, \omega)$ be a weighted graph. We keep the same notation used in Section 5.3. Recall that the boundary value problem, governed by the (nonlocal) p-bilaplacian operator (5.3.1), is equivalent to solve the following minimization problem

$$\min_{\mathbf{u}\in\mathcal{H}(V)} F(\Delta_{\omega,2}\mathbf{u}) + G(\mathbf{u}), \tag{7.1.13}$$

where $\Delta_{\omega,2}$ is the (nonlocal) weighted Laplacian operator and F, $G : \mathcal{H}(V) \to \mathbb{R}$ are defined by

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{p}\|\mathbf{u}\|_p^p \\ G(\mathbf{u}) &= \langle \mathbf{f}, \mathbf{u}\rangle_{\mathcal{H}(V)} + \iota_{\mathcal{H}_{\mathbf{g}}(V;\mathcal{A})}(\mathbf{u}), \end{aligned}$$

with $p \in [1, +\infty]$ (with the appropriate meaning for $p = +\infty$ recalled above), and $\mathbf{u} \in \mathcal{H}(V)$ and $\mathcal{A} \subset V$. Recall that, the operator $\Delta_{\omega,2}$ is self-adjoint and its norm is given by

$$\|\Delta_{\omega,2}\| \stackrel{\text{def}}{=} \sup_{\|\mathbf{u}\|_2=1} \|\Delta_{\omega,2}\|_2.$$

We adapt the minimization problem (7.1.7) to solve the problem (7.1.13). In this case the scheme reads as

$$\begin{aligned} \mathbf{u}^{k+1} &= \mathbf{prox}_{\tau G}(\mathbf{u}^k - \tau\Delta_{\omega,2}\mathbf{v}^k) \\ \mathbf{v}^{k+1} &= \mathbf{prox}_{\sigma F^\star}(\mathbf{v}^k + \sigma\Delta_{\omega,2}(2\mathbf{u}^{k+1} - \mathbf{u}^k)), \end{aligned} \tag{7.1.14}$$

where $\tau, \sigma > 0$. The convergence guarantees of (7.1.14) are summarized in the following corollary which is a consequence of Prposition 7.1.1.

Corollary 7.1.3. *If* $0 < \tau\sigma\|\Delta_{\omega,2}\|^2 < 1$*, then* $(u^k)_k$ *generated by* (7.1.14) *converges to a solution of* (7.1.13).

7.2 Algorithm for the p-bilaplacian variational problem on graphs

In this section, we are going to describe an algorithm to solve the regularization problem (5.2.1). For this purpose, we keep the notation used in Section 5.2. This algorithm is valid for any $p \in [1, +\infty]$, and for simplicity, we restrict ourselves here to the case where A is the identity operator (i.e. with a denoising-type application in mind). We rewrite the problem (5.2.1) as follow

$$\min\{E^p_\omega(\mathbf{u}; \mathbf{f}, \lambda) \stackrel{\text{def}}{=} \frac{\lambda}{p}\|\Delta_{\omega,2}\mathbf{u}\|_p^p + \frac{1}{2}\|\mathbf{u} - \mathbf{f}\|_2^2 : \ \mathbf{u} \in \mathcal{H}(V)\}. \tag{7.2.1}$$

Problem (7.2.1) can be easily solved using standard duality-based first-order algorithms. For this we follow [79]. By standard conjugacy calculus, the Fenchel-Rockafellar dual problem of (7.2.1) reads

$$\min\{\frac{\lambda}{q}\|\mathbf{v}/\lambda\|_p^p + \frac{1}{2}\|\Delta_{\omega,2}\mathbf{v} - \mathbf{f}\|_2^2 : \ \mathbf{v} \in \mathcal{H}(V)\}, \tag{7.2.2}$$

where $p \in [1, +\infty]$ (with the appropriate meaning for $p = +\infty$, see above), and q is the Hölder conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.
Applying Theorem 5.2.1 to the dual problem (7.2.2) has a convex compact set of minimizers for any $p \in]1, +\infty[$. Moreover, the unique solution $\mathbf{u}^\star$ to the primal problem (7.2.1) can be recovered from any dual solution $\mathbf{v}^\star$ as

$$\mathbf{u}^\star = \mathbf{f} - \Delta_{\omega,2}\mathbf{v}^\star.$$

It remains now to solve (7.2.2). The latter can be solved with the (accelerated) FISTA iterative scheme [115, 23, 48] which reads in this case

$$\begin{aligned} \mathbf{W}^k &= \mathbf{v}^k + \frac{k-1}{k+b}(\mathbf{v}^k - \mathbf{v}^{k-1}), \\ \mathbf{v}^{k+1} &= \mathbf{prox}_{\gamma \frac{\lambda}{q}\left\|\cdot/\lambda\right\|_q^q}\left(\mathbf{W}^k + \gamma \Delta_{\omega,2}(\mathbf{f} - \Delta_{\omega,2}\mathbf{W}^k)\right), \\ \mathbf{u}^{k+1} &= \mathbf{f} - \Delta_{\omega,2}\mathbf{v}^{k+1}, \end{aligned} \tag{7.2.3}$$

where $\gamma \in]0, \left(\left\|\Delta_{\omega,2}\right\|_2\right)^{-1}[$, and $b > 2$. Combining Theorem 5.2.1, [79, Theorem 2], [13, Theorem 1.1], the scheme (7.2.3) has the following convergence guarantees.

Proposition 7.2.1. *The primal iterates $\mathbf{u}^k$ converge to $\mathbf{u}^\star$, the unique minimizer of (7.2.1), at the rate*

$$\left\|\mathbf{u}^k - \mathbf{u}^\star\right\|_2 = o(1/k).$$

7.3 Computing the proximal operators

Let us turn to the computation of the proximal mapping $\mathbf{prox}_{\sigma\frac{1}{q}\left\|\cdot/\lambda\right\|_q^q}$ for more detail see [91]. Since $\|\cdot\|_q^q$ is separable, one has that

$$\mathbf{prox}_{\sigma\frac{1}{q}\left\|\cdot/\lambda\right\|_q^q}(\mathbf{v}) = \left(\mathbf{prox}_{\sigma\frac{1}{q}\left|\cdot/\lambda\right|^q}(\mathbf{v}_x)\right)_{x \in V}.$$

Moreover, as $\left|\cdot\right|^q$ is an even function on $\mathbb{R}$, $\mathbf{prox}_{\sigma\frac{1}{q}\left|\cdot\right|^q}$ is an odd mapping on $\mathbb{R}$, that is,

$$\mathbf{prox}_{\sigma\frac{1}{q}\left|\cdot\right|^q}(\mathbf{v}_x) = \mathbf{prox}_{\sigma\frac{1}{q}\left|\cdot\right|^q}(\left|\mathbf{v}_x\right|)\,\mathrm{sign}(\mathbf{v}_x).$$

Hence, one has to compute $\mathrm{prox}_{\sigma\frac{1}{q}\left|\cdot\right|^q}(t)$[2] for $t \in \mathbb{R}^+$. We distinguish different situations depending on the value of q:

- $q = +\infty$ (i.e., $p = 1$): this case amounts to computing the orthogonal projector on $[-\lambda, \lambda]$ which reads
$$t \in \mathbb{R}^+ \to \mathbf{P}_{[-\lambda,\lambda]}(t) = \min(t, \lambda).$$
- $q = 1$ (i.e., $p = +\infty$): this case corresponds to the well-known soft-thresholding operator, which is given by
$$t \in \mathbb{R}^+ \to \mathbf{prox}_{\sigma\left|\cdot\right|}(t) = \max(t - \sigma, 0).$$
- $q = 2$ (i.e.,$p = 2$): it is immediate to see that
$$\mathbf{prox}_{\sigma\frac{1}{2}\left|\cdot\right|^2}(t) = \frac{t}{1+\sigma}$$
- $q \in]1, +\infty[$: in this case, as $\left|\cdot\right|^q$ is differentiable, the proximal point $\mathrm{prox}_{\sigma\frac{1}{q}\left|\cdot\right|^q}(t)$ is the unique solution $\alpha^\star$ on $\mathbb{R}^+$ of the nonlinear equation:
$$\alpha - t + \sigma\alpha^{q-1} = 0.$$

[2]Recall that $\lim_{q\to\infty} \frac{1}{q}\left|\cdot\right|^q = \iota_{[-1,1]}(\cdot)$.

Chapter 8

Conclusion and Perspectives

8.1 Summary and take-away messages

In this manuscript, we have provided new results on consistency of nonlocal p-Laplacian evolution and boundary value problems on sparse graphs. In particular, we have established general error bounds comparing the continuum problems and their discrete approximations on graphs and global convergence rates, for the evolution problem. Regarding the boundary value problem, we have studied the asymptotic behaviour of a sequence of this problem. Moreover, we have shown a general consistency as well as as a priori estimate for the homogeneous problem on graphs. Based on these error estimates of both problem, we have established nonasymptotic rate of convergence of solutions for the discrete models on sparse random graphs.

In addition, we have introduced two new concepts on graphs, based on nonlocal calculus on weighted graphs. The first one concerns the discrete p-bilaplacian operator on graphs, which allowed us to study its corresponding variational and boundary value problems on weighted graphs. The second one is a general class of perimeters on graphs, which allowed us to redefine and extend other notions on graphs. These led us to transcribe and adapt the notion of the mean curvature flows on graphs as well as the level set mean curvature.

We summarize the main conclusions to be drawn from our work:

(i) We extended the results of [90] to a far more general class of kernels and L^q-graphons sequences. More precisely, we established a bound on the distance between two continuous-in-time trajectories defined by two different evolution systems, without any boundedness assumptions on the kernels, second member and initial data. Similarly, we provided a bound in the case that one of the trajectories is discrete-in-time and the other is continuous. In turn, these results led us to establish error estimates of the full discretization of the p-Laplacian problem on sparse random graphs. In particular, we provided rate of convergence of solutions for the discrete models to the solution of the nonlocal problem on the continuum as the number of vertices grows.

(ii) For the boundary value problem ($\mathcal{P}^D_{\mathrm{nloc}}$), we established continuum limits of a sequence of discrete (e.g., on sparse graphs) nonlocal p-Laplacian boundary value problems. Using the Dirichlet principle, we showed the problem amounts to studying limits of nonlocal variational problems consisting in minimizing a sequence of convex lower-semicontinuous functional in $L^p(\Omega)$. We first established well-posedness of these problems. Then, using the notions of Mosco and Γ-convergence, and under mild conditions, we established convergence of this sequence of variational problems and provided the form of the limit variational problem. In turn, this allowed us to provide consistent estimates of the discretisation of the nonlocal p-Laplacian Dirichlet problem on graphs. In particular, under some mild conditions we provided a priori estimates for solutions of this problem. These results led us to derive rate of convergence for the discrete model defined on K-random sparse graphs.

(iii) We introduced a new class of operators, called p-bilaplacian operators. We studied the well-posedness (existence and uniqueness) of the variational problem as well as the boundary value problem associated to these operators on weighted graphs. We adapted also primal-dual algorithms to solve these problems. Some numerical results were reported to illustrate our findings.

(iv) For the nonlocal discrete perimeter, we revisited [70], where we proposed a general class of perimeters on weighted graphs. We introduced the curvatures related to these perimeters. we proposed also an adaptation and a transcription of the power mean curvature level set equation on a general discrete domain, represented via weighted graphs. Employing these models, we reported some numerical results, on image processing and $3D$-point clouds and high dimensional data classification, to exemplify the potential impact of our framework.

8.2 Future work

Our work uncovers several interesting problems which will be very interesting to investigate in the future.

8.2.1 The nonlocal p-Laplacian operator.

The limiting cases $p = 1$ and $p = +\infty$ Starting with the study of the well-posedness and going through the study of the consistency of ($\mathcal{P}^D_{\text{nloc}}$) and ($\mathcal{P}_{\text{nloc}}$), excluding the value $p = 1$ and $p = \infty$ for the boundary value problem and $p = \infty$ for the evolution problem, were crucial to get our results. The existence of a solution can be concluded by a simple argument of convex analysis, for the nonlocal 1-Laplacian Dirichlet problem thanks to the Dirichlet principle, but there is no uniqueness in general since the functional of the variational problem lacks strict convexity. On the other hand, to get our estimate for the problem Dirichlet problem, Theorem 4.2.7 was fundamental. It would by interesting to find a way to overcome these difficulties and establish the consistency for $p = 1$. For $p = \infty$, the definition of the operator Δ^K_p becomes completely different, many challenges arise in addition to well-posedness for both problems.

8.2.2 Other nonlocal evolution problems: beyond ($\mathcal{P}_{\text{nloc}}$)

It would be also very interesting to extend our results to analyze the consistency of other nonlocal evolution problems such as the nonlocal Hamilton-Jacobi equation; see e.g., [18, 78]. Extension to other nonlocal operators that are of importance in practice would be also an interesting research avenue. One may think of the case of the normalized p-Laplacian, the case when p is spatially varying, or the case where the kernel is unknown.

8.2.3 The discrete p-bilapalcian operator

The limiting cases $p = 1$ and $p = +\infty$ We studied the well-posedness of the boundary value problem on graphs, excluding the values $p = 1$ and $p = \infty$ was crucial to get our results. For $p = 1$ and $p = \infty$, the definition of the p-bilaplacian operator becomes completely different, many challenges arise to prove well-posedness.

Continuum limit for the p-bilaplacian on graphs In Chapter 5, we have focused on the discrete setting. A natural continuum counterpart of the discrete p-bilaplacian operator we have introduced would be

$$\Delta^2_{K,p} u \stackrel{\text{def}}{=} \Delta^K_p \left(\left| \Delta^K_p u \right|^{p-2} \Delta^K_p u \right),$$

where $u \in L^p(\Omega)$, $p \in]1, +\infty[$. This operator coincides, for $p = 2$, with the nonlocal bilaplacian operator introduced in [124]. It would be very interesting to study the well-posedness of different problems governed by this family of operators e.g. evolution, variational and boundary value problems. In this manuscript, the natural question of continuum limits of the problems governed by this family of operators is left completely open. It would then be interesting to study such limits and establish consistency/error bounds of the corresponding discretizations on (sparse) graphs. The idea will be to extend and adapt our arguments and results to this family of operators.

8.2.4 Continuum limits of the the mean curvature flow on graphs

Studying continuum limits in the context of Chapter 6, with the discrete definitions provided there, is a challenging question worth investigating in the future.

8.2.5 Other continuum limits

We have focused in this work on nonlocal-type limits of discrete problems on graphs. This allowed to get deeper understanding of the behaviour and guarantees of such models as the number of vertices grows. This is not the only possible framework for deriving continuum limits. For instance, one can view some of these discrete models as individual-based models based on "particle"-like assumptions, which can be connected to hydrodynamics/macroscopic descriptions via kinetic theory. The kinetic viewpoint can be very enlightening both in the modelling, and in the derivation of continuum models.

List of Publications

In preparation

- El Bouchairi, I., Fadili, J., & Elmoataz, A., Continuum limit of the nonlocal p-Laplacian Dirichlet problem on graphs: L^q graphons and sparse graphs.
- El Bouchairi, I., Elmoataz, A., & Fadili, J., Nonlocal perimeters and curvature flows on graphs with application in data processing and clustering.

Preprints

- El Bouchairi, I., Fadili, J., & Elmoataz, A. (2020). Continuum limit of p-Laplacian evolution problems on graphs: L^q graphons and sparse graphs. arXiv preprint arXiv:2010.08697. submitted to Numerische Mathematik.

Conference Proceedings

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List of Notations

General definitions

$\mathbb{R}$: the set of real numbers
$\mathbb{R}^+$: positive real numbers
$\overline{\mathbb{R}}$: $]-\infty, +\infty[\cup\{+\infty\}$, the extended real value
$\mathbb{N}$: set of non-negative integers
$\mathbb{N}^*$: set of positive integers
$\mathbb{R}^d, \mathbb{R}^m$: finite dimensional real Euclidean spaces

Spaces related

$\mathcal{H}$: real Hilbert space
$\mathcal{X}$: Banach space
$\Gamma_0(\mathcal{H})$: the set of proper convex and lower semicontinuous functions on $\mathcal{H}$
$L^p(\Omega)$: the Banach space of p-integrable functions on Ω, $p \in [1, +\infty]$
$C(0, T; \mathcal{X})$: the space of functions on $\mathcal{X} \times [0, T]$ which are continuous in the time variable

Sets related

$\iota_{\mathcal{S}}$: indicator function of a set $\mathcal{S}$
$\chi_{\mathcal{S}}$: charactetistic function of a set $\mathcal{S}$
$N_{\mathcal{S}}$: normal cone of a set $\mathcal{S}$
$\mathbf{P}_{\mathcal{S}}$: projection operator onto $\mathcal{S}$
$\mathrm{int}(\mathcal{S})$: interior of $\mathcal{S}$
$\mathrm{bd}(\mathcal{S})$: boundary of $\mathcal{S}$
$\overline{\mathcal{S}}$: closure of $\mathcal{S}$
$\mathrm{span}(\mathcal{S})$: smallest linear subspace that contains $\mathcal{S}$

Functions related

$\mathrm{dom}(F)$: domain of a function F
∇F: gradient of F
$\nabla^{\mathrm{NL}} F$: nonlocal gradient of F
$\mathrm{prox}_{\gamma F}$: proximity operator of F with $\gamma > 0$
∂F: subdifferential of function F
$\mathrm{supp}(F)$: support of a function F

Operators

$\mathbf{Dom}(A)$: domain of the operator A
$\mathbf{R}(A)$: range of the operator A
J_A: resolvent of the operator A
$\mathbf{I}$: identity operator on a space to be understood from the context

Norms

$\|\cdot\|_{L^p(\Omega)}$: the norm of functions on $L^p(\Omega)$

$\|\cdot\|_p$: the p-norm of a vector in $\mathbb{R}^n$, $p \in [1, +\infty]$

$\|\cdot\|_{p,n}$: the normalized p-norm of a vector in $\mathbb{R}^n$, $p \in [1, +\infty]$

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