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Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Yu Min

Integral p-adic Hodge theory of formal schemes in low ramification

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dirigée par Matthew Morrow

Soutenue le 14 octobre 2020 devant le jury composé de :

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Like the first human being
to express in words everything you see
and experience and love and lose.

—Rainer Maria Rilke

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Résumé

Nous prouvons que pour tout schéma formel propre et lisse $\mathfrak{X}$ sur $\mathcal{O}_K$, où $\mathcal{O}_K$ est l'anneau d'entiers dans une extension non archimédienne complète de valuation discrère K de $\mathbb{Q}_p$ avec corps résiduel parfait k et degré de ramification e, le i-ème groupe de cohomologie de Breuil-Kisin et sa spécialisation de Hodge-Tate admettent de belles décompositions lorsque $ie < p - 1$. Grâce aux théorèmes de comparaison issus des travaux récents de Bhatt, Morrow and Scholze [BMS18], [BMS19], nous pouvons alors obtenir un théorème de comparaison entier pour des schémas formels, qui généralise le cas des schémas prouvé par Fontaine et Messing dans [FM87] et Caruso dans [Car08].

Mots-clé

Cohomologie de A_{inf}, cohomologie de Breuil-Kisin, théorie de Hodge p-adique.

Abstract

We prove that for any proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e, the i-th Breuil-Kisin cohomology group and its Hodge-Tate specialization admit nice decompositions when $ie < p - 1$. Thanks to the comparison theorems in the recent works of Bhatt, Morrow and Scholze [BMS18], [BMS19], we can then get an integral comparison theorem for formal schemes, which generalizes the case of schemes proven by Fontaine and Messing in [FM87] and Caruso in [Car08].

Keywords

A_{inf}-cohomology, Breuil-Kisin cohomology, p-adic Hodge theory.

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Introduction

Fix a prime number p. Given a proper smooth scheme X over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e, there are several classical cohomology theories one can define, such as p-adic étale cohomology $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p)$ of the geometric generic fibre $X_{\bar{K}}$, crystalline cohomology $H^*_{\text{crys}}(X_k/W(k))$ of the special fibre X_k and de Rham cohomology $H^*_{\text{dR}}(X/\mathcal{O}_K)$. These cohomology theories are not unrelated. In fact, there are very deep connections among their structures. p-adic Hodge theory, roughly speaking, studies the relations of these cohomology theories. It was Fontaine who formulated the so-called crystalline conjecture making precise these relations with p inverted, via his period ring B_{crys}. Note that p is invertible in B_{crys}. This conjecture now has been proved after many people's works (cf. [Fal88], [Tsu99], [Niz08], [Bei12]) :

Theorem 0.0.1 (Crystalline conjecture)**.** *Let X be a proper smooth scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k. There is a natural isomorphism compatible with Galois and Frobenius actions*

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}} \cong H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} B_{\text{crys}}.$$

for each $i \geq 0$.

The crystalline conjecture is very beautiful but it requires p to be invertible. There are many cases where we would like to study the cohomology theories integrally. In particular, the integral structure contains the information about the torsion. So what happens without inverting p ? Fontaine and Messing have proved in [FM87] that there is an abstract isomorphism of $W(k)$-modules $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\text{crys}}(X_k/W(k))$ in the unramified case (i.e. the ramification degree e of the field K is 1) under the restriction $i < p-1$. In the ramified case, a similar result has been obtained by Caruso under the restriction $(i+1)e < p-1$ in [Car08]. It is until very recently that Bhatt, Morrow and Scholze have made great breakthroughs in integral p-adic Hodge theory. They have constructed a new cohomology theory called A_{inf}-cohomology in [BMS18]. With the help of A_{inf}-cohomology, they have generalized the rational comparison theorem to the case of formal schemes and

obtained a very general result comparing the integral structure of p-adic étale cohomology to crystalline cohomology without any restrictions on the ramification degree and the degree of the cohomology groups.

The goal of this thesis is to present some results generalizing the works of Fontaine-Messing and Caruso to the case of formal schemes and reproving some of their results, by studying the new cohomology theories introduced by Bhatt, Morrow and Scholze. Before we state our main theorems, we will discuss some history of integral p-adic Hodge theory.

Integral p-adic Hodge theory. As we have said, integral p-adic Hodge theory studies the relations of different cohomology theories without inverting p. It also tries to understand Galois stable $\mathbb{Z}_p$-lattices in crystalline (semi-stable) p-adic representations and their links with integral p-adic cohomology theories via semi-linear algebra data.

The first result concerning integral comparison was given by Fontaine and Messing in [FM87].

Theorem 0.0.2 ([FM87])**.** *Let X be a proper smooth scheme over $W(k)$ and $X_n = X \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(W_n(k))$, where k is a perfect field of characteristic p. Let G_{K_0} denote the absolute Galois group of $K_0 = W(k)[\frac{1}{p}]$. Then for any integer i such that $0 \leq i \leq p-2$, there exists a natural isomorphism of G_{K_0}-modules*

$$T_{\mathrm{crys}}(H^i_{\mathrm{dR}}(X_n)) \simeq H^i_{\text{ét}}(X_{\bar{K}_0}, \mathbb{Z}_p/p^n)$$

where T_{crys} is a functor from the category of torsion Fontaine-Laffaille modules to the category of $\mathbb{Z}_p[G_{K_0}]$-modules, which preserves invariant factors.

Note that $H^r_{\mathrm{dR}}(X_n) \cong H^r_{\mathrm{crys}}(X_k/W_n(k))$. Here we have used implicitly that $H^r_{\mathrm{dR}}(X_n)$ is in the category of torsion Fontaine-Laffaille modules, which is actually one of the main difficulties. The proof of Fontaine-Messing's theorem relies on syntomic cohomology which acts as a bridge connecting p-adic étale cohomology and crystalline cohomology.

Recall that rational p-adic Hodge theory provides an equivalence between the category of crystalline representations and the category of (weakly) admissible filtered φ-modules. The idea of Fontaine-Laffaille's theory is to try to classify G_{K_0}-stable $\mathbb{Z}_p$-lattices in a crystalline representation V by φ-stable $W(k)$-lattices in D satisfying some conditions, where D is the corresponding admissible filtered φ-module.

In order to generalize Fontaine-Laffaille's theory to the semi-stable case, Breuil introduced the ring S (see Definition 6.0.7) and related categories of S-modules in order to add a monodromy operator. He has also obtained an integral comparison result in the unramified case when $i < p-1$ in [Bre98a]. Later, this result was generalized to the case that $e(i+1) < p-1$ by Caruso in [Car08].

Theorem 0.0.3 ([Bre98a] [Car08])**.** *Let X be a proper and semi-stable scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension*

K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let X_n be $X \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}(\mathcal{O}_K/p^n)$. Fix a non-negative integer r such that $er < p-1$. Then there exists a canonical isomorphism of Galois modules

$$H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})(r) \cong T_{\mathrm{st}*}(H^i_{\log-\mathrm{crys}}(X_n/(S/p^nS)))$$

for any $i < r$.

$T_{\mathrm{st}*}$ is a functor from the category $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$ (see Definition 6.0.10) to the category of $\mathbb{Z}_p[G_K]$-modules, which preserves invariant factors. The proof also relies on the use of syntomic cohomology. One of the main difficulties in their proof is to show $H^i_{\log-\mathrm{crys}}(X_n/(S/p^nS))$ is in the category $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$, in particular, to show $H^i_{\log-\mathrm{crys}}(X_n/(S/pS))$ is finite free over S/pS.

Remark 0.0.4. One crucial point of Breuil's theory is that it highly depends on the restriction $r \leq p-1$ which is rooted in the fact that the inclusion $\varphi(\mathrm{Fil}^r S) \subset p^r S$ is true only when $r \leq p-1$. One way to remove this restriction is to consider Breuil-Kisin modules. In fact, one of the main motivations of A_{inf}-cohomology theory is to give a cohomological construction of Breuil-Kisin modules. The techniques in [BMS18] can not directly give the desired Breuil-Kisin cohomology. However, this goal is achieved in [BMS19] by using topological cyclic homology and in [BS19] by defining the prismatic site in a more general setting.

Recently, Bhatt, Morrow and Scholze have obtained a more general result about the relation between p-adic étale cohomology and crystalline cohomology in [BMS18] by using A_{inf}-cohomology. Their result is about formal schemes and does not impose any restriction on the ramification degree, roughly saying that the crystalline cohomology is a degeneration of the p-adic étale cohomology. Throughout the thesis, formal schemes always mean p-adic formal schemes.

Theorem 0.0.5 ([BMS18]Theorem 1.1)**.** *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k. Let C be a completed algebraic closure of K and write $\mathfrak{X}_C$ for the geometric rigid analytic fiber of $\mathfrak{X}$. Fix some $i \geq 0$.*

1. *There is a comparison isomorphism*

$$H^i_{\acute{e}t}(\mathfrak{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}} \cong H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{\mathrm{crys}},$$

   compatible with the Galois and Frobenius actions, and filtration.

2. *For all $n \geq 0$, we have the inequality*

$$\mathrm{length}_{W(k)}(H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}/p^n) \geq \mathrm{length}_{\mathbb{Z}_p}(H^i_{\acute{e}t}(\mathfrak{X}_C, \mathbb{Z}_p)_{\mathrm{tor}}/p^n).$$

3. *Assume that $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ and $H^{i+1}_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ are p-torsion-free. Then one can recover $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ with its φ-action from $H^i_{\text{ét}}(\mathfrak{X}_C, \mathbb{Z}_p)$ with its G_K-action.*

Main theorems. In this part, we will state our main theorems and briefly explain our strategies of proofs.

Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k. Define $\mathfrak{S} := W(k)[[u]]$ and fix a uniformizer π of $\mathcal{O}_K$. There is a natural $W(k)$-linear surjective morphism $\beta : \mathfrak{S} \to \mathcal{O}_K$ sending u to π, whose kernel is generated by an Eisenstein polynomial $E = E(u)$ for π. We can define the Breuil-Kisin cohomology $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ of $\mathfrak{X}$ by using topological cyclic homology or define it to be the prismatic cohomology of $\mathfrak{X}$ associated to the prism $(\mathfrak{S}, (E))$ (see Theorem 1.6.6). Moreover we can also define its Hodge-Tate specialization : $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) := R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\beta} \mathcal{O}_K$. We simply call $R\Gamma_{\mathrm{HT}}(\mathfrak{X})$ the Hodge-Tate cohomology of $\mathfrak{X}$ (this may not be a standard notion).

Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed nonarchimedean extension C of K and $\bar{\mathfrak{X}}$ denote $\mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Define $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$ where $\mathcal{O}_C^\flat := \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$. Then we can define the A_{inf}-cohomology of $\bar{\mathfrak{X}}$ as : $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, A\Omega_{\bar{\mathfrak{X}}})$, where $A\Omega_{\bar{\mathfrak{X}}}$ is a certain complex of sheaves of A_{inf}-modules on the Zariski site of $\bar{\mathfrak{X}}$. There is also a Hodge-Tate specialization of A_{inf}-cohomology : $R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) = R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C$ (for the definition of $\widetilde{\theta}$, see Definition 1.4.1).

For more precise definitions about these cohomology theories, see Chapter 1. Now we can state our first main result which is about the structure of the A_{inf}-cohomology groups and the Breuil-Kisin cohomology groups in low ramification.

Theorem 0.0.6 (Theorem 3.2.5, Theorem 5.2.1)**.** *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed nonarchimedean extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Assume $ie < p-1$. Then there is an isomorphism of $\mathfrak{S} = W(k)[[u]]$-modules*

$$H^i_{\mathfrak{S}}(\mathfrak{X}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}$$

where $H^i_{\mathfrak{S}}(\mathfrak{X}) := H^i(R\Gamma_{\mathfrak{S}}(\mathfrak{X}))$ is the Breuil-Kisin cohomology of $\mathfrak{X}$. Consequently, we also have

$$H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}},$$

where $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := H^i(R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}))$ is the A_{inf}-cohomology of $\bar{\mathfrak{X}}$. Similarly under the same assumption $ie < p-1$, there is an isomorphism of $\mathcal{O}_K$-modules

$$H^i_{\mathrm{HT}}(\mathfrak{X}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_K,$$

and an isomorphism of $\mathcal{O}_C$-modules

$$H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C,$$

where $H^i_{\mathrm{HT}}(\mathfrak{X}) := H^i(R\Gamma_{\mathrm{HT}}(\mathfrak{X}))$(resp. $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) := R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}})$) is the Hodge-Tate cohomology of $\mathfrak{X}$ (resp. $\bar{\mathfrak{X}}$).

Remark 0.0.7. The A_{inf}-cohomology is usually difficult to calculate and its structure can be very subtle. In [CDN19], the authors have calculated the A_{inf}-cohomology of Drinfeld symmetric spaces and given an explicit description. In particular, they have also shown that the A_{inf}-cohomology groups of Drinfeld symmetric spaces are $\widetilde{\xi}$-torsion-free (for the definition of $\widetilde{\xi}$, see Definition 1.4.1) by using different regulator maps.

Remark 0.0.8. Note that the definition of Breuil-Kisin modules (see Definition 1.5.1) in [BMS18], [BMS19] is slightly more general than the original definition given by Kisin in [Kis06]. The difference lies in the existence of u-torsion (note that $\mathfrak{S} = W(k)[[u]]$ is a two dimensional regular local ring). However, the theorem above shows that the Breuil-Kisin cohomology theory constructed by Bhatt, Morrow and Schloze does take values in the category of Breuil-Kisin modules in a traditional sense, at least when $ie < p-1$.

Unfortunately, we can not give any canonical isomorphisms between these modules. Our method only enables us to compare the module structure. The proof of this theorem relies essentially on the existence of Breuil-Kisin cohomology and the construction of A_{inf}-cohomology in [BMS18] by using the $L\eta$-functor and the pro-étale site. In fact, this construction presents a close relation between A_{inf}-cohomology and p-adic étale cohomology. The $L\eta$-functor provides us with two morphisms between $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ (resp. $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$) and $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ (resp. $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$), whose composition in both direction is μ^i (resp. $(\zeta_p - 1)^i$). For the definitions of μ and ζ_p, see Definition 1.4.1.

Note that $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$ is just the base change of $H^i_{\mathrm{HT}}(\mathfrak{X})$ along the canonical injection $\mathcal{O}_K \to \mathcal{O}_C$. We can then directly obtain the statement about the Hodge-Tate cohomology groups in Theorem 0.0.6 by studying the two morphisms provided by the $L\eta$-functor and the lemmas of commutative algebra in Chapter 2.

For the part concerning the Breuil-Kisin cohomology groups, we need to prove some torsion-free results. Namely, when $ie < p-1$, the Breuil-Kisin cohomology group $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$ is E-torsion-free (equivalently u-torsion-free by Corollary 5.1.4). Moreover for any positive integer n, we have $H^i_{\mathfrak{S}}(\mathfrak{X})/p^n$ is also $E(u)$-torsion-free when $ie < p-1$.

As we have said, by studying A_{inf}-cohomology and its descent Breuil-Kisin cohomology, we can generalize the results of Fontaine-Messing, Breuil and Caruso to the case of formal schemes, at least in the good reduction case. This is actually the main motivation of this work.

Theorem 0.0.9 (Theorem 4.3.6, Theorem 5.2.3)**.** *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let C be a complete algebraically closed nonarchimedean extension of K and $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Write X for the adic generic fiber of $\bar{\mathfrak{X}}$. Then when $ie < p-1$, there is an isomorphism of $W(k)$-modules $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$.*

We will study unramified case and ramified case in different ways. For the proof in the unramified case, we need the following theorem :

Theorem 0.0.10 (Theorem 4.3.5)**.** *With the same assumptions as the theorem above, when $e = 1$, for any $n < p-1$, we have*

$$\mathrm{length}_{\mathbb{Z}_p}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}/p^m)) \geq \mathrm{length}_{W(k)}(H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}/p^m)$$

for all positive integer m.

In fact, we first compare Hodge-Tate cohomology to Hodge cohomology by proving that the truncated Hodge-Tate complex of sheaves $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal in this case, i.e. there is an isomorphism $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \bigoplus_{i=0}^{p-1} \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})[-i]$. We then study the Hodge-to-de Rham spectral sequence to relate Hodge cohomology to de Rham cohomology. By Theorem 0.0.6, we can finally relate de Rham (or crystalline) cohomology to p-adic étale cohomology. Note that the theorem above gives a converse to Theorem 1.4.7 in [BMS18], which implies that $H^n_{\text{ét}}(X, \mathbb{Z}_p)$ and $H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ have the same invariant factors.

In the ramified case, the integral comparison theorem follows directly from Theorem 0.0.6 and Theorem 1.4.5.

Remark 0.0.11. The A_{inf}-cohomology theory in the semi-stable case has been studied in [CK19]. The Breuil-Kisin cohomology might be also generalized to the semi-stable case by using prismatic site. Then one could also hope to generalize Theorem 0.0.6 and Theorem 0.0.9 to the semi-stable case.

We also remark that although the result in the ramified case can recover that in the unramified case, the method used in the unramified case can lead to the following theorem concerning Hodge-to-de Rham spectral sequence and integral comparison result for all cohomological degrees..

Theorem 0.0.12 (Theorem 4.4.3, Corollary 4.4.4)**.** *Assume $d = \dim\mathfrak{X} < p-1$.*

1. *There is an isomorphism of $W(k)$-modules for all n*

$$H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

2. *The (integral) Hodge-to-de Rham spectral sequence degenerates at E_1-page.*

Outline. We explain the content of each chapter.

In Chapter 1, we recall the construction of A_{inf}-cohomology and recollect some important theorems that we need.

In Chapter 2, we prove the key lemmas of commutative algebra that lies in the heart of our proof of the comparison isomorphism and collect some useful results about derived Hom functor and derived completion.

In Chapter 3, we study the Hodge-Tate cohomology and prove the part concerning the Hodge-Tate cohomology in Theorem 0.0.6.

In Chapter 4, we prove the comparison isomorphism in the unramified case. We will prove a decomposition of the Hodge-Tate cohomology groups and study the Hodge-to-de Rham spectral sequence.

In Chapter 5, we turn to study the comparison isomorphism in the ramified case.

In Chapter 6, we will prove some results about Breuil-Kisin modules which might be already known to experts. This will provide another (partial) proof of the comparison isomorphism in Chapter 5.

Chapitre 1

Recollections on prismatic cohomology

In this chapter, we simply recall the necessary ingredients for defining the A_{inf}-cohomology theory in [BMS18]. We will stick to the method using the pro-étale site and the décalage functor $L\eta$ for defining the A_{inf}-cohomology, which will provide us with some useful morphisms between A_{inf}-cohomology groups and p-adic étale cohomology groups for later use. We will also briefly introduce the prismatic cohomology that provides a site-theoretical construction of the A_{inf}-cohomology.

1.1 Adic spaces

In this section, we briefly introduce the theory of adic spaces, which was introduced by Huber. Our basic references are [Hub13] and [SW20].

There are several theories of non-archimedean geometry, including the theory of rigid-analytic spaces due to Tate and the theory of Berkovich spaces. But both of them have some defects (cf. [Con18], [Wei17]), which are solved by the theory of adic spaces. The category of adic spaces contain both the category of locally noetherian formal schemes and the category of rigid-analytic varieties as full subcategories. More recently, in his Phd thesis [Sch12], Scholze used the theory of adic spaces as the basic language of his theory of perfectoid spaces.

Like rigid-analytic spaces are built out of affinoid spaces associated to affinoid algebras, adic spaces are built out of affinoid adic spaces, which are associated to pairs of certain topological rings (A, A^+).

Definition 1.1.1. *A topological ring A is called a Huber ring if it admits an open subring $A_0 \subset A$ which is adic with respect to a finitely generated idael of definition. Any such A_0 is called a ring of definition of A.*

Definition 1.1.2. *Let A be a Huber ring. An element $x \in A$ is power-bounded if the set $\{x^n | n \geq 0\}$ is bounded. Let A° denote the subring of power-bounded elements.*

Definition 1.1.3. *Let A be a Huber ring. A subring $A^+ \subset A$ is called a ring of integral elements if it is open and integrally closed in A and $A^+ \subset A^\circ$.*

Definition 1.1.4. *A Huber pair is a pair (A, A^+), where A is a Huber ring and $A^+ \subset A$ is a ring of integral elements.*

Example 1.1.5. If $A = \mathbb{Q}_p\langle T\rangle$, then $A^\circ = \mathbb{Z}_p\langle T\rangle$. We can take $A^+ = A^\circ$, i.e. the pair (A, A°) is a Huber pair.

Now we show how to construct topological spaces out of Huber pairs by considering continuous valuations as the points.

Definition 1.1.6. *A continuous valuation on a topological ring A is a map $|\cdot| : A \to \Gamma \cup \{0\}$ into a totally ordered abelian group Γ such that*

1. $|ab| = |a||b|$
2. $|a + b| \leq \max(|a|, |b|)$
3. $|1| = 1$
4. $|0| = 0$
5. *For all $\gamma \in \Gamma$ lying in the image of $|\cdot|$, the set $\{a \in A \mid |a| < \gamma\}$ is open in A.*

We say two continuous valuations $|\cdot|_1$, $|\cdot|_2$ valued in Γ_1 resp. Γ_2 are equivalent when it is true that $|a|_1 \leq |b|_1$ if and only if $|a|_2 \leq |b|_2$.

Definition 1.1.7. *Let (A, A^+) be a Huber pair. The adic spectrum $\mathrm{Spa}(A, A^+)$ is the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$. We write $f \mapsto |f(x)|$ for a choice of valuation corresponding to $x \in \mathrm{Spa}(A, A^+)$. The topology on $\mathrm{Spa}(A, A^+)$ is generated by open subsets of the form $\{x \mid |f(x)| \leq |g(x)| \neq 0\}$ with $f, g \in A$.*

In order to define the "structure sheaf" on the adic spectrum $X = \mathrm{Spa}(A, A^+)$, we need to introduce the rational subsets.

Definition 1.1.8. *Let $s \in A$ and $T \subset A$ be a finite subset such that $TA \subset A$ is open. We define*

$$U(\frac{T}{s}) = \{x \in X \mid |t(x)| \leq |s(x)| \neq 0, \text{for all } t \in T\}.$$

We call subsets of this form rational subsets.

Now we state a theorem saying that rational subsets are adic spectra.

Theorem 1.1.9 ([Hub94] Proposition 1.3)**.** *Let $U \subset X = \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ and a morphism of Huber pairs $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the induced map*

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$$

factors through U and is universal for such maps. This map is a homeomorphism onto U.

Definition 1.1.10. *We define a presheaf $\mathcal{O}_X$ of topological rings on $X = \mathrm{Spa}(A, A^+)$: if $U \subset X$ is a rational subset, $\mathcal{O}_X(U)$ is as in the theorem above. For an open subset $W \subset X$, we define*

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \, rational} \mathcal{O}_X(U)$$

We can define a presheaf $\mathcal{O}_X^+$ of topological rings in a similar way.

Definition 1.1.11. *A Huber pair (A, A^+) is sheafy if $\mathcal{O}_X$ is a sheaf of topological rings.*

Remark 1.1.12. The presheaf $\mathcal{O}_X$ needs not to be a sheaf. There are some examples in [Mih16], [BV18].

Now we come to define adic spaces.

Definition 1.1.13. *Let V be the category as follows. The objects are triples $(X, \mathcal{O}_X, (|\cdot(x)|)_{x \in X})$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$. The morphisms are maps of topologically ringed spaces$f : X \to Y$ that make the following diagram commute up to equivalence for all $x \in X$:*

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\}. \end{array}$$

Then an adic space is an object $(X, \mathcal{O}_X, (|\cdot(x)|)_{x \in X})$ of V, which admits a covering by spaces U_i such that the triple $(U_i, \mathcal{O}_X|_{U_i}, (|\cdot(x)|)_{x \in U_i})$ is isomorphic to $\mathrm{Spa}(A_i, A_i^+)$ for a sheafy Huber pair (A_i, A_i^+). For sheafy (A, A^+), we call the topological space $\mathrm{Spa}(A, A^+)$ together with its structure sheaf and continuous valuations an affinoid adic space.

Remark 1.1.14. For every rigid analytic variety X, there is an associated adic space $r(X)$. And there is an equivalence between the étale topos of $r(X)$ and the étale topos of X (cf. [Hub13, Proposition 2.1.4]). In particular, the étale cohomology of X is the same as the étale cohomology of $r(X)$.

We end this section by introducing the notion of perfectoid Tate rings which will appear in the study of pro-étale site in next section.

Definition 1.1.15. *A Huber ring A is Tate if it contains a topologically nilpotent unit $g \in A$. We also call topological nilpotent units in A pseudo-uniformizers.*

Definition 1.1.16. *A complete Tate ring A is perfectoid if A° is a ring of definition and there exists a pseudo-uniformize $\pi \in A$ such that π^p divides p in A° and the p-th power Frobenius map $\Phi : A^\circ/\pi \to A^\circ/\pi^p$ is an isomorphism.*

Remark 1.1.17. We remark that there is another notion of integral perfectoid rings (cf. [BMS18, Section 3]). These two notions are closely related. In fact, if A is a perfectoid Tate ring with a ring of integral elements A^+, then A^+ is an integral perfectoid ring ([BMS18, Lemma 3.20]).

1.2 The pro-étale site

The first ingredient for defining the A_{inf}-cohomology is the pro-étale site. The pro-étale site was introduced by Scholze in [Sch13] in order to study p-adic Hodge theory of rigid-analytic varieties. The local structure of rigid-analytic varieties is often complicated. The introduction of the pro-étale site makes the local study much simpler : it is "locally perfectoid".

In this subsection, let C be a complete and algebraically closed nonarchimedean extension of $\mathbb{Q}_p$ and X be a smooth rigid-analytic variety over C, viewed as an adic space.

Definition 1.2.1. *Let pro-$X_{\text{ét}}$ be the category of pro-objects associated to the category $X_{\text{ét}}$. The objects are functors from a small cofiltered category I to $X_{\text{ét}}$ sending i to U_i, which we denote "$\varprojlim$"$_{i\in I}U_i$. The underlying topological space of "$\varprojlim$"$_{i\in I}U_i$ is defined to be $\varprojlim_{i\in I}|U_i|$, where $|U_i|$ is the underlying topological space of U_i. The morphisms in pro-$X_{\text{ét}}$ are given by*

$$\mathrm{Hom}(F,G) = \varprojlim_{i_2\in I_2}\varinjlim_{i_1\in I_1}\mathrm{Hom}(F(i_1),G(i_2))$$

for any objects $F : I_1 \to X_{\text{ét}}$ and $G : I_2 \to X_{\text{ét}}$.

Definition 1.2.2 ([Sch13] Definition 3.9)**.** *The category $X_{\text{proét}}$ is defined to be the full subcategory of pro-$X_{\text{ét}}$ consisting of those objects pro-étale over X. We say that a pro-object in pro-$X_{\text{ét}}$ is pro-étale over X if it is isomorphic to an object $U =$ "$\varprojlim$"$_{i\in I}U_i$ with $U_i \in X_{\text{ét}}$ such that all maps $U_i \to U_j$ are finite étale and surjective.*

Definition 1.2.3 ([Sch13] Definition 3.9)**.** *We say a collection of maps $\{f_i : U_i \to U\}$ in $X_{\text{proét}}$ is a covering if and only if the following conditions are satisfied :*

1. *The collection of the maps of the underlying topological spaces $\{f_i : |U_i| \to |U|\}$ is a pointwise covering.*

2. *Each f_i satisfies the condition : One can write U_i as an inverse limit $U_i = \varprojlim_{\mu<\lambda} U_\mu$ of $U_\mu \in X_{\text{proét}}$ along some ordinal λ such that for all $\mu > 0$, the map $U_\mu \to U_{<\mu} := \varprojlim_{\mu'<\mu} U_{\mu'}$ is the pullback of a finite étale and surjective map in $X_{\text{ét}}$. And there is an étale map $U_0 \to U$, i.e. the pullback of an étale map in $X_{\text{ét}}$, such that $f_i : U_i \to U$ is the composite of the projection $U_i \to U_0$ and the étale map $U_0 \to U$.*

The category $X_{\text{proét}}$ together with the coverings defined above forms a site.

Definition 1.2.4 ([Sch13] Definition 4.3)**.** *An object $U \in X_{\text{proét}}$ is said to be affinoid perfectoid if and only if it is isomorphic in $X_{\text{proét}}$ to an object "$\varprojlim$"$_{i\in I} U_i$ which has the following properties :*

1. *The transition maps $U_j \to U_i$ are finite étale surjective for $j \geq i$;*
2. *$U_i = \mathrm{Spa}(R_i, R_i^+)$ is affinoid for each i ;*
3. *The complete Tate ring $R := (\varinjlim_i R_i^+)_p^\wedge \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is perfectoid, where $()_p^\wedge$ means p-adic completion.*

The next proposition says that the site $X_{\text{proét}}$ is locally perfectoid.

Proposition 1.2.5 ([Sch13] Proposition 4.8)**.** *The set $U \in X_{\text{proét}}$ which are affinoid perfectoid form a basis for the topology.*

Now we want to define some sheaves on $X_{\text{proét}}$. Consider the natural projection map of sites

$$\omega : X_{\text{proét}} \to X_{\text{ét}}.$$

which is defined by pulling back $U \in X_{\text{ét}}$ to the constant tower $(\cdots \to U \to U \to X)$ in $X_{\text{proét}}$. This just reflects the fact that an étale morphism is pro-étale.

Definition 1.2.6 ([Sch13] Section 6)**.** *Consider the following sheaves on $X_{\text{proét}}$.*

1. *The integral structure sheaf $\mathcal{O}_X^+ := \omega^* \mathcal{O}_{X_{\text{ét}}}^+$.*
2. *The structure sheaf $\mathcal{O}_X := \omega^* \mathcal{O}_{X_{\text{ét}}}$.*
3. *The completed integral structure sheaf $\widehat{\mathcal{O}}_X^+ := \varprojlim_n \mathcal{O}_X^+/p^n$.*
4. *The completed structure sheaf $\widehat{\mathcal{O}}_X := \widehat{\mathcal{O}}_X^+[\frac{1}{p}]$.*
5. *The tilted completed integral structure sheaf $\widehat{\mathcal{O}}_{X^\flat}^+ := \varprojlim_\varphi \mathcal{O}_X^+/p$.*
6. *Fontaine's period sheaf $\mathbb{A}_{\text{inf},X}$, which is the derived p-adic completion of $W(\widehat{\mathcal{O}}_{X^\flat}^+)$ (for the definition of derived p-adic completion, see Section 2.4).*

Remark 1.2.7. 1. In [BMS18, Remark 5.5], it has been pointed out that it is not clear whether $W(\widehat{\mathcal{O}}_{X^\flat}^+)$ is derived p-adic complete. So in order to make the A_{inf}-cohomology theory work well, we need to define $\mathbb{A}_{\text{inf},X}$ as the derived p-adic completion of $W(\widehat{\mathcal{O}}_{X^\flat}^+)$, which is actually a complex of sheaves.

2. Let $U = \text{“}\varprojlim\text{”}_i U_i \in X_{\text{proét}}$ be affinoid perfectoid with $U_i = \mathrm{Spa}(R_i, R_i^+)$. Then $\mathcal{O}_X^+(U) = \varinjlim_i R_i^+$, $\mathcal{O}_X(U) = \varinjlim_i R_i$, $\widehat{\mathcal{O}}_X^+(U) = R^+$, $\widehat{\mathcal{O}}_X(U) = R$, $\widehat{\mathcal{O}}_{X^\flat}^+(U) = R^{+\flat}(:= \varprojlim_{x \mapsto x^p} R^+/p)$, $H^0(U, \mathbb{A}_{\inf,X}) = \mathbb{A}_{\inf}(R^+)$ (cf. [Sch13, Lemma 4.10, Lemma 5.10, Thoerem 6.5]).

1.3 The $L\eta$-functor

The other important ingredient for defining the $A_{\inf}$-cohomology is the décalage functor, which functions as a tool to get rid of “junk torsion”. The “junk torsion” exists already in Faltings’ approach to p-adic Hodge theory in [Fal88]. The introduction of the décalage functor is actually the main novelty of [BMS18] to deal with this “junk torsion”.

Definition 1.3.1 (The $L\eta$-functor, [BMS18] Section 6)**.** *Let $(T, \mathcal{O}_T)$ be a ringed topos and $\mathcal{I} \subset \mathcal{O}_T$ be an invertible ideal sheaf. For any $\mathcal{I}$-torsion-free complex $C^\bullet \in K(\mathcal{O}_T)$ [1], we can define a new complex $\eta_{\mathcal{I}} C^\bullet = (\eta_{\mathcal{I}} C)^\bullet \in K(\mathcal{O}_T)$ with terms*

$$(\eta_{\mathcal{I}} C)^i := \{x \in C^i | dx \in \mathcal{I} \cdot C^{i+1}\} \otimes_{\mathcal{O}_T} \mathcal{I}^{\otimes i}$$

For every complex $D^\bullet \in K(\mathcal{O}_T)$, there exists a strongly K-flat complex $C^\bullet \in K(\mathcal{O}_T)$ and a quasi-isomorphism $C^\bullet \to D^\bullet$. By saying strongly K-flat, we mean that each C^i is a flat $\mathcal{O}_T$-module and for every acyclic complex $P^\bullet \in K(\mathcal{O}_T)$, the total complex $\mathrm{Tot}(C^\bullet \otimes P^\bullet)$ is acyclic. In particular, $C^\bullet$ is $\mathcal{I}$-torsion free. Then we can define

$$L\eta_{\mathcal{I}} : D(\mathcal{O}_T) \to D(\mathcal{O}_T)$$

$$L\eta_{\mathcal{I}}(D^\bullet) := \eta_{\mathcal{I}}(C^\bullet)$$

A concrete example is to consider a ring A and a non-zero-divisor $a \in A$. If C is a cochain complex of a-torsion free A-modules, we can define the subcomplex $\eta_a C$ of $C[\frac{1}{a}]$ as

$$(\eta_a C)^i := \{x \in a^i C^i : dx \in a^{i+1} \cdot C^{i+1}\}$$

and this induces the corresponding functor $L\eta_a : D(A) \to D(A)$.

Lemma 1.3.2. *The map $f : Z(C^i) \to (\eta_a C)^i$ defined by sending m to $a^i m$ induces a natural isomorphism*

$$H^i(C)/H^i(C)[a] \xrightarrow{\simeq} H^i(\eta_a C).$$

Proof. Let $m \in C^i$ be a cocycle, i.e. $dm = 0$. Then $a^i m \in (\eta_a C)^i$ is also a cocycle. Let $n \in C^i$ be a coboundary, i.e. $n = dx$ for some $x \in C^{i-1}$, then $a^i n \in (\eta_a C)^i$ is also a

1. The category $K(\mathcal{O}_T)$ is the category whose objects are complexes of $\mathcal{O}_T$-modules and morphisms are maps of complexes modulo homotopy equivalence.

coboundary as $a^i n = d(a^i x)$ and $a^i x$ is in $(\eta_a C)^{i-1}$. So the map sending m to $a^i m$ induces a map $f : H^i(C) \to H^i(\eta_a C)$.

It is easy to see that the induced map is surjective. Since for any cocycle n in $(\eta_a C)^i$, we can write $n = a^i y$ for some $y \in C^i$. Then $dn = 0$ implies $dy = 0$ since C is a-torsion free. This means y is a cocycle in C^i.

The kernel of the induced map corresponds to those $x \in C^i$ such that $dx = 0$ and $ax \in d(C^{i-1})$, i.e. $H^i(C)[a]$. □

This lemma justifies that the $L\eta$-functor can kill "junk torsion" in some sense.

Remark 1.3.3.

1. The $L\eta$-functor is not an exact functor between derived categories. For example, consider the distinguished triangle $\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p$ where the first map is induced by multiplication by p on $\mathbb{Z}$ and the second map is modulo p. It is easy to see that $L\eta_p(\mathbb{Z}/p) = 0$ and $L\eta_p(\mathbb{Z}/p^2) \neq 0$.
2. By [BMS18, Proposition 6.7], the $L\eta$-functor is lax symmetric monoidal.
3. The $L\eta$-functor was first introduced by Berthelot-Ogus in [BO15] following a suggestion of Deligne. They used it to study the crystalline cohomology of a proper smooth scheme over a perfect field of characteristic p and the relation between the associated Newton and Hodge polygons.

1.4 The A_{inf}-cohomology

From now on to the end of this chapter, let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed nonarchimedean extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Let $\mathfrak{X}_k$ denote the special fiber of $\mathfrak{X}$ and $\mathfrak{X}_{\bar{k}}$ denote its base change to $\bar{k}$ which is the residue field of $\mathcal{O}_C$ (note that $\bar{k}$ is not necessarily the algebraic closure of k).

We first recall some basic definitions in p-adic Hodge theory.

Definition 1.4.1 ([Fon94]).

1. *Define $\mathcal{O}_C^\flat := \varprojlim_{x \to x^p} \mathcal{O}_C/p$ which is called the tilt of $\mathcal{O}_C$ and $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$, the Witt vector ring of $\mathcal{O}_C^\flat$. Note that $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p and A_{inf} is equipped with a natural Frobenius map φ, which is an isomorphism of rings.*
2. *Fix a compatible system of primitive p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{N}}$ in $\mathcal{O}_C$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Under the isomorphism of multiplicative monoids $\mathcal{O}_C^\flat \cong \varprojlim_{x \to x^p} \mathcal{O}_C$, we define $\epsilon := (1, \zeta_p, \zeta_{p^2}, \cdots, \zeta_{p^n}, \cdots) \in \mathcal{O}_C^\flat$ and $\mu := [\epsilon] - 1 \in A_{\mathrm{inf}}$.*
3. *There is a map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$ defined by Fontaine. The map θ is surjective and $\ker(\theta)$ is generated by $\xi = \mu/\varphi^{-1}(\mu)$. After twisting with the Frobenius map, we get $\widetilde{\theta} := \theta \circ \varphi^{-1} : A_{\mathrm{inf}} \to \mathcal{O}_C$, whose kernel is generated by $\widetilde{\xi} := \varphi(\xi) = \varphi(\mu)/\mu$.*

Now we are ready to define the A_{inf}-cohomology theory. We consider the natural projection $\nu : X_{\text{proét}} \to \bar{\mathfrak{X}}_{\mathrm{zar}}$, which is actually the composite $X_{\text{proét}} \xrightarrow{\omega} X_{\text{ét}} \to \bar{\mathfrak{X}}_{\text{ét}} \to \bar{\mathfrak{X}}_{\mathrm{zar}}$.

Definition 1.4.2 ([BMS18] Definition 9.1)**.** *Define $A\Omega_{\bar{\mathfrak{X}}} := L\eta_\mu R\nu_*(\mathbb{A}_{\mathrm{inf},X})$ and $\widetilde{\Omega}_{\bar{\mathfrak{X}}} := L\eta_{\zeta_{p-1}} R\nu_*(\widehat{\mathcal{O}}_X^+)$. The A_{inf}-cohomology is defined to be the Zariski hypercohomology of the complex of sheaves $A\Omega_{\bar{\mathfrak{X}}}$, i.e. $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, A\Omega_{\bar{\mathfrak{X}}})$. We can also define the Hodge-Tate cohomology $R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) := R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$.*

Remark 1.4.3. As both $R\nu_*$ and the $L\eta$-functor are lax symmetric monoidal, the complex $\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is a commutative $\mathcal{O}_{\bar{\mathfrak{X}}}$-algebra object in the derived category of $\mathcal{O}_{\bar{\mathfrak{X}}}$-modules $D(\mathcal{O}_{\bar{\mathfrak{X}}})$. For the same reason, the complex $A\Omega_{\bar{\mathfrak{X}}}$ is a commutative ring in the derived category $D(\bar{\mathfrak{X}}_{\mathrm{zar}}, \mathbb{Z})$ of abelian sheaves.

The A_{inf}-cohomology takes values in the category of what we call Breuil-Kisin-Fargues modules.

Definition 1.4.4 ([BMS18] Definition 4.22)**.** *A Breuil-Kisin-Fargues module is a finitely presented A_{inf}-module M which becomes free over $A_{\mathrm{inf}}[\frac{1}{p}]$ after inverting p and is equipped with an isomorphism*

$$\varphi_M : M \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}}[\frac{1}{\widetilde{\xi}}] \xrightarrow{\simeq} M[\frac{1}{\widetilde{\xi}}].$$

The main theorem about the A_{inf}-cohomology theory is the following :

Theorem 1.4.5 ([BMS18] Theorem 14.3)**.** *The complex $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is a perfect complex of A_{inf}-modules with a φ-linear map $\varphi : R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \to R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ which becomes an isomorphism after inverting ξ resp. $\widetilde{\xi}$. The cohomology groups $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := H^i(R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}))$ are Breuil-Kisin-Fargues modules. Moreover, there are several comparison results :*

1. *With étale cohomology : $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$.*
2. *With crystalline cohomology : $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(\bar{k}) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k}))$, where the map $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to W(\bar{k})$ is induced by the natural projection $\mathcal{O}_C^\flat \to \bar{k}$ (in fact, $\mathcal{O}_C^\flat$ is a valuation ring with residue field $\bar{k}$).*
3. *With de Rham cohomology : $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \simeq R\Gamma_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$.*
4. *With Hodge-Tate cohomology : $\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq A\Omega_{\bar{\mathfrak{X}}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C$ and $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \simeq R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}})$.*

Corollary 1.4.6. *For all $i \geq 0$, we have isomorphisms and short exact sequences*

1. $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$.
2. $0 \to H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}}} W(\bar{k}) \to H^i_{\mathrm{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})) \to Tor_1^{A_{\mathrm{inf}}}(H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}), W(\bar{k})) \to 0$.
3. $0 \to H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \to H^i_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \to H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\xi] \to 0$.
4. $0 \to H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \to H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \to H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}] \to 0$.

One of the most important applications of the A_{inf}-cohomology theory is to enable us to compare étale cohomology to crystalline cohomology integrally without any restrictions on the degree of cohomology groups and the ramification degree of the base field. More precisely, it can be showed that the torsion in the crystalline cohomology gives an upper bound for the torsion in the étale cohomology.

Theorem 1.4.7 ([BMS18] Theorem 14.5)**.** *For any $n, i \geq 0$, we have the inequality*

$$\mathrm{length}_{W(k)}(H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}/p^n) \geq \mathrm{length}_{\mathbb{Z}_p}(H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}/p^n)$$

where $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}$ is the torsion submodule of $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ and $H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}$ is the torsion submodule of $H^i_{\text{ét}}(X, \mathbb{Z}_p)$.

1.5 Breuil-Kisin cohomology

As we have mentioned, there is a refinement of the A_{inf}-cohomology, i.e. the Breuil-Kisin cohomology, which recovers A_{inf}-cohomology after base change along a faithfully flat map $\alpha : \mathfrak{S} = W(k)[[u]] \to A_{\mathrm{inf}}$. In fact, there is a commutative diagram

$$\begin{tikzcd} \mathfrak{S} \arrow[r, "\alpha"] \arrow[d, "\beta"] & A_{\mathrm{inf}} \arrow[d, "\widetilde{\theta}"] \\ \mathcal{O}_K \arrow[r, "i"] & \mathcal{O}_C \end{tikzcd}$$

where β is the natural projection sending u to π and i is the natural injection. To define the map α, we fix a uniformizer π of K and a compatible system of p-power roots $\pi^{1/p^n} \in C$, which defines an element $\pi^\flat = (\pi, \pi^{1/p}, \pi^{1/p^2}, \cdots) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \mathcal{O}_C^\flat$. Then α is defined to send u to $[\pi^\flat]^p$ and be the Frobenius on $W(k)$. In particular, we have $(\alpha(E)) = (\widetilde{\xi})$ where E is a fixed Eisenstein polynomial for π. To see this, recall that an element $x \in \ker(\theta)$ is a generator if and only if the second term x_1 of the Witt vector expansion $x = (x_0, x_1, \cdots)$ is a unit in $\mathcal{O}_C^\flat$ (cf. [Fon82, Proposition 2.4]). So an element $y \in \ker(\widetilde{\theta})$ is a generator if and only if the second term y_1 of $y = (y_0, y_1, \cdots)$ is a unit in $\mathcal{O}_C^\flat$. Note that $E = \sum_{i=0}^e a_i u^i$ for some $a_i \in W(k)$ such that $a_e = 1$, $a_i \in pW(k)$ for $0 \leq i < e$ and $a_0 \notin p^2 W(k)$. Then $\alpha(E) = \sum_{i=0}^e \varphi(a_i)[\pi^\flat]^{pi}$. Let $x = (x_0, x_1, \cdots), y = (y_0, y_1, \cdots)$ be two elements in $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$. Recall the following formulas

$$x + y = (x_0 + y_0, x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0 + y_0)^p}{p}, \cdots)$$

$$px = (0, x_0^p, x_1^p, \cdots)$$

Then it is easy to check that the second item of $\alpha(E)$ is a unit in $\mathcal{O}_C^\flat$. For the flatness of the map α, see [BMS18, Lemma 4.30].

The Breuil-Kisin cohomology takes value in the category of Breuil-Kisin modules, which dates back to the work of Breuil [Bre] and the work of Kisin [Kis06]. It has shown great power in integral p-adic Hodge theory. For instance, it has been used by Kisin to give an alternative proof of Clomez-Fontaine's fundamental result in [CF00] that weakly admissibility implies admissibility and also to give a classification of p-divisible groups and finite flat group schemes. In [Kis09], Kisin has used the Breuil-Kisin modules with coefficients to prove some modularity lifting theorems. More recently, Emerton and Gee constructed a moduli stack of Breuil-Kisin modules and used it to study Galois representations in [EG19].

The first attempt to find a cohomological construction of Breuil-Kisin modules dates back to the PhD thesis [BÏ2] of Ojeda Bär, who used crystalline cohomology to construct certain perfect complexes. Later, after the birth of A_{inf}-cohomology theory, Cais and Liu, in their paper [CL19], also used crystalline cohomology and A_{inf}-cohomology to give a cohomological construction of Breuil-Kisin modules under some restrictions on the ramification degree.

The first unconditional construction of the Breuil-Kisin cohomology is given in [BMS19] by using topological cyclic homology. Another construction is given in [BS19] by using the prismatic site. We will not say anything about the construction of Breuil-Kisin cohomology theory here but choose to state a similar comparison theorem as in the A_{inf} case.

We give the definition of Breuil-Kisin module which is slightly more general than the original definition due to Kisin.

Definition 1.5.1 ([BMS18] Definition 4.1)**.** *A Breuil-Kisin module is a finitely generated $\mathfrak{S}$-module M together with an isomorphism*

$$\varphi_M : M \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \to M[\frac{1}{E}].$$

Theorem 1.5.2 ([BMS19] Theorem 1.2)**.** *For any proper smooth formal scheme $\mathfrak{X}/\mathcal{O}_K$, there is a $\mathfrak{S}$-linear cohomology theory $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ which is a perfect complex of $\mathfrak{S}$-modules. Moreover, it is equipped with a φ-linear map $\varphi : R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \to R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ which induces an isomorphism*

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X})[\frac{1}{E}]$$

The cohomology groups $H^i_{\mathfrak{S}}(\mathfrak{X}) := H^i(R\Gamma_{\mathfrak{S}}(\mathfrak{X}))$ are Breuil-Kisin modules. There are several specializations that recover other p-adic cohomology theories :

1. *With A_{inf}-cohomology : after base change along $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$, it recovers A_{inf}-cohomology : $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\alpha} A_{\mathrm{inf}} \simeq R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$.*
2. *With étale cohomology : $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\widetilde{\alpha}} W(C^\flat) \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$, where $\widetilde{\alpha}$ is the composition $\mathfrak{S} \xrightarrow{\alpha} A_{\mathrm{inf}} \hookrightarrow W(C^\flat)$.*
3. *With de Rham cohomology : $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\widetilde{\beta}} \mathcal{O}_K \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)$, where $\widetilde{\beta} := \beta \circ \varphi : \mathfrak{S} \to \mathcal{O}_K$.*

4. *With crystalline cohomology : after base change along the map $\mathfrak{S} \to W(k)$ which is the Frobenius on $W(k)$ and sends u to 0, it recovers the crystalline cohomology of the special fiber :* $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$.

For later convenience, we define $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) := R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\beta} \mathcal{O}_K$ and call it the Hodge-Tate cohomology of $\mathfrak{X}$. Note that there is an isomorphism : $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_C \simeq R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}})$.

Remark 1.5.3. Note that there is a Frobenius twist appearing in the specializations above. As explained in [BMS19, Remark 1.4], this is not artificial but contains some information about the torsion in the de Rham cohomology.

1.6 Prismatic cohomology

The goal of this subsection is to recall how A_{inf}-cohomology and Breuil-Kisin cohomology can be unified by prismatic cohomology. We hope that the results in this thesis can be generalized to the semi-stable case by applying the prismatic formalism. But we will not address this question here.

The basic object for defining the prismatic site is called a prism, which can be viewed as a deperfection of a perfectoid ring. Before introducing prisms, we need to define δ-rings.

Definition 1.6.1 ([BS19] Definition 2.1)**.** *A δ-ring is a pair (R, δ) where R is a commutative ring and $\delta : R \to R$ is a map of sets such that $\delta(0) = \delta(1) = 0$ and satisfies the following identities :*

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

and

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$

The identities on δ enables us to construct a ring map $\varphi_R : R \to R$ by defining $\varphi_R(x) = x^p + p\delta(x)$. Then φ_R can be regarded as a Frobenius lift. Now we introduce the notion of a prism.

Definition 1.6.2 ([BS19] Definition 1.1)**.** *A prism is a pair (A, I) where A is δ-ring and $I \subset A$ is an ideal defining a Cartier divisor in* $\mathrm{Spec}(A)$*, and satisfying the following two conditions :*

1. *The ring A is derived (p, I)-adically complete (for the definition of derived completion, see Section 2.4).*
2. *The ideal $I + \varphi_A(I)$ contains p, where φ_A is the Frobenius lift induced by the δ-structure on A.*

A map of prisms $(A, I) \to (B, J)$ is given by a map of $A \to B$ of δ-rings sending I into J.

Remark 1.6.3. There are two examples which are related to the rings A_{inf} and $\mathfrak{S}$.

1. One is $(A_{\inf}, \ker(\theta))$, where θ is Fontaine's map. In fact, for any perfectoid ring R, there is a corresponding perfect prism $(A_{\inf}(R), \ker(\theta))$. Here perfect means that $\varphi_R : R \to R$ is an isomorphism.
2. The other is $(\mathfrak{S}, (E))$. The Frobenius lift on $\mathfrak{S}$ extends the Frobenius on $W(k)$ and sends u to u^p.

Definition 1.6.4. *A prism (A, I) is bounded if there exists some integer n such that $A/I[p^\infty] = A/I[p^n]$.*

In general, given a bounded prism (A, I) and a smooth p-adic formal scheme X over A/I, one can define the prismatic site $((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$. The construction of the prismatic site is similar to that of the crystalline site, but considering prisms instead of nilpotent thickenings.

Definition 1.6.5. *The prismatic site $(X/A)_{\mathbb{\Delta}}$ is the opposite category of prisms (B, J) with a map $(A, I) \to (B, J)$ and a map $\mathrm{Spf}(B/J) \to X$ over $\mathrm{Spf}(A/I)$, equipped with the faithfully flat covers. We say a map $(A, I) \to (B, J)$ of prisms is faithfully flat if $A \to B$ is (p, I)-completely faithfully flat which means that the derived tensor product $M \otimes^{\mathbb{L}}_A N$ is concentrated in degree 0 for any (p, I)-torsion A-module N and $B \otimes^{\mathbb{L}}_A A/(p, I)$ is a faithfully flat $A/(p, I)$-module.*

The structure sheaf $\mathcal{O}_{\mathbb{\Delta}}$ on $(X/A)_{\mathbb{\Delta}}$ is defined to be the sheaf taking a pair (B, J) to B. And there is another sheaf $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ of rings on $(X/A)_{\mathbb{\Delta}}$ which is defined to take a pair (B, J) to B/J. This is a sheaf of $\mathcal{O}(X)$-algebras.

Now we state the theorem showing that prismatic cohomology can recover $A_{\inf}$-cohomology and Breuil-Kisin cohomology.

Theorem 1.6.6 ([BS19] Theorem 1.8)**.** *Let (A, I) be a bounded prism, and let $\mathfrak{X}$ be a smooth p-adic formal scheme over A/I. We consider the sheaf cohomology of the structure sheaf*

$$R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A) := R\Gamma((\mathfrak{X}/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}),$$

which is a commutative algebra in the derived category $D(A)$ of A-modules and is equipped with a φ_A-linear map φ.

1. *(Base change) Let $(A, I) \to (B, J)$ be a map of bounded prisms, and let $\mathfrak{Y} = \mathfrak{X} \times_{\mathrm{Spf}(A/I)} \mathrm{Spf}(B/J)$. Then there is a canonical isomorphsim*

$$R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A) \widehat{\otimes}^{\mathbb{L}}_A B \simeq R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/B),$$

where the completion on the left is the derived (p, J)-adic completion.

2. *(A_{inf}-cohomology) Let (A, I) be the perfect prism $(A_{\mathrm{inf}}, \ker(\theta))$, then there exists a canonical φ-equivariant isomorphism*

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \simeq \varphi_A^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A_{\mathrm{inf}}) = R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A_{\mathrm{inf}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}}.$$

3. *(Breuil-Kisin cohomology) Let (A, I) to be the prism $(\mathfrak{S}, (E))$. Then there is a canonical φ-equivariant isomorpshim*

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \simeq R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}).$$

4. *(Crystalline cohomology) If $I = (p)$, then there is a canonical φ-equivariant isomorphism*

$$R\Gamma_{\mathrm{crys}}(X/A) \simeq \varphi_A^* R\Gamma_{\mathbb{\Delta}}(X/A) = R\Gamma_{\mathbb{\Delta}}(X/A) \widehat{\otimes}^{\mathbb{L}}_{A,\varphi_A} A.$$

of commutative algebras in $D(A)$.

Chapitre 2

Preliminaries on commutative algebra

In this chapter, we will recollect some results on commutative algebra and prove some key lemmas that are frequently used later. We fix a complete, algebraically closed nonarchimedean extension C of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_C$.

2.1 Finitely presented modules over valuation rings

In this section, we study finitely presented modules over $\mathcal{O}_C$.

Lemma 2.1.1 ([Sta19]Lemma 0ASN)**.** *Let R be a ring. The following are equivalent :*

1. *For $a, b \in R$, either a divides b or b divides a.*
2. *Every finitely generated ideal is principal and R is local.*
3. *The set of ideals of R are linearly ordered by inclusion.*

In particular, all valuation rings satisfy these equivalent conditions. The module structure of finitely presented modules over valuation rings is similar to that of finitely generated modules over principal ideal domains as the following lemma shows.

Lemma 2.1.2 ([Sta19]Lemma 0ASP)**.** *Let R be a ring satisfying the equivalent conditions above, then every finitely presented R-module is isomorphic to a finite direct sum of modules of the form R/aR where $a \in R$.*

Corollary 2.1.3. *Any finitely presented module over $\mathcal{O}_C$ is of the form $\bigoplus_{i=1}^{n} \mathcal{O}_C/\pi_i$ for some $\pi_i \in \mathcal{O}_C$.*

We will need to study finitely presented torsion $\mathcal{O}_C$-modules later. The main tool to deal with these modules is the length function $l_{\mathcal{O}_C}$, as used in [CK19], see also [Bha17a]. In particular, this length function behaves additively under short exact sequences. Usually, we use the normalized length function, i.e. $l_{\mathcal{O}_C}(\mathcal{O}_C/p) = 1$.

Lemma 2.1.4. *Let A and B be base changes to $\mathcal{O}_C$ of finitely presented torsion $W(k)$-modules. If for each $m > 0$, we have*

$$l_{\mathcal{O}_C}(A/p^m) = l_{\mathcal{O}_C}(B/p^m)$$

then A is isomorphic to B as $\mathcal{O}_C$-modules.

Proof. Note that $2l_{\mathcal{O}_C}(A/p) - l_{\mathcal{O}_C}(A/p^2)$ is the number of copies of $\mathcal{O}_C/p$ in A. This implies that the number of copies of $\mathcal{O}_C/p$ in A is equal to that of B. By induction on m, it is easy to prove $A \cong B$ as $\mathcal{O}_C$-modules.

□

2.2 Key lemmas

In this section, we want to prove the following key lemma which will be used in the comparison of Hodge-Tate cohomology and p-adic étale cohomology.

Lemma 2.2.1. *Let $M = \bigoplus_{i=1}^m \mathcal{O}_C/\beta^{m_i}$ and $N = \bigoplus_{j=1}^n \mathcal{O}_C/\beta^{n_j}$, where $\beta \neq 0$ is in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_C$ and all m_i, n_j are positive integers. Suppose there are two $\mathcal{O}_C$-linear morphisms $f : M \to N$ and $g : N \to M$ such that $g \circ f = \alpha$ and $f \circ g = \alpha$, where $\alpha \in \mathcal{O}_C$ and $v(\beta) > v(\alpha)$. Then $m = n$ and the multi-sets $\{m_i\}$ and $\{n_j\}$ are the same, i.e., $M \cong N$.*

In order to prove this lemma, we consider all finitely presented torsion modules over $\mathcal{O}_C$. As we have mentioned, any such module looks like $\bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ for some non-zero $\pi_i \in \mathfrak{m}$. We call $\mathrm{trk}(N) := n$ the torsion-rank of N. Note that the torsion-rank of N is equal to the dimension of N base changed to the residue field of $\mathcal{O}_C$. So it is well-defined. We will also use the normalized length function $l_{\mathcal{O}_C}$ for finitely presented torsion $\mathcal{O}_C$-modules.

Now we prove a lemma concerning the torsion-rank :

Lemma 2.2.2. *Let $N \hookrightarrow M$ be an injection of finitely presented torsion $\mathcal{O}_C$-modules. Then $\mathrm{trk}(N) \leq \mathrm{trk}(M)$. Dually if $N \twoheadrightarrow M$ is a surjection of finitely presented torsion $\mathcal{O}_C$-modules, then $\mathrm{trk}(N) \geq \mathrm{trk}(M)$.*

Proof. Write $N = \bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ and $M = \bigoplus_{i=1}^m \mathcal{O}_C/\varpi_i$. Let π be the smallest of the π_i (i.e., the one with the smallest valuation), and let ϖ be the largest of ϖ_i. Then

$$(\mathcal{O}_C/\pi)^n \subset N \hookrightarrow M \subset (\mathcal{O}_C/\varpi)^m,$$

which shows that $\varpi \in \pi\mathcal{O}_C$; write $\varpi = \pi x$. The composition of these maps lands in the π-torsion of $(\mathcal{O}_C/\varpi)^m$, which is isomorphic to $(x\mathcal{O}_C/\varpi\mathcal{O}_C)^m \cong (\mathcal{O}_C/\pi\mathcal{O}_C)^m$. So now we have an injection $(\mathcal{O}_C/\pi)^n \hookrightarrow (\mathcal{O}_C/\pi)^m$. Taking length shows that $n \leq m$.

If $N \twoheadrightarrow M$ is a surjection of finitely presented torsion $\mathcal{O}_C$-modules, we can consider the injection $\mathrm{Hom}(M, \mathcal{O}_C/t) \hookrightarrow \mathrm{Hom}(N, \mathcal{O}_C/t)$ where t is any non-zero element in $\mathfrak{m}$. Then we have $\mathrm{trk}(M) = \mathrm{trk}(\mathrm{Hom}(M, \mathcal{O}_C/t)) \leq \mathrm{trk}(\mathrm{Hom}(N, \mathcal{O}_C/t)) = \mathrm{trk}(N)$. □

Lemma 2.2.3. *Let $g : N \to M$ be a morphism of finitely presented torsion $\mathcal{O}_C$-modules; write $N = \bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ and $M = \bigoplus_{i=1}^m \mathcal{O}_C/\varpi_i$. Assume that $\ker(g)$ is killed by some element $\alpha \in \mathcal{O}_C$ whose valuation is strictly smaller that all of the π_i. Then $\mathrm{trk}(N) \leq \mathrm{trk}(M)$.*

Proof. By assumption $\ker(g)$ is contained in the α-torsion $N[\alpha]$ of N, which is given by $N[\alpha] \cong \bigoplus_{i=1}^n \frac{\pi_i}{\alpha}\mathcal{O}_C/\pi_i\mathcal{O}_C$. So

$$N \twoheadrightarrow N/\ker(g) \twoheadrightarrow N/N[\alpha] \cong \bigoplus_{i=1}^n \frac{\pi_i}{\alpha}\mathcal{O}_C/\pi_i\mathcal{O}_C.$$

Taking torsion-ranks, Lemma 2.2.2 for surjections shows that $\mathrm{trk}(N/\ker(g)) = \mathrm{trk}(N)$. But $N/\ker(g) \hookrightarrow M$, so Lemma 2.2.2 also shows that $\mathrm{trk}(N/\ker(g)) \leq \mathrm{trk}(M)$. □

Now we are ready to prove Lemma 2.2.1.

Proof of Lemma 2.2.1. Note that the number of invariant factor β^k in M is equal to $\mathrm{trk}(\beta^{k-1}M) - \mathrm{trk}(\beta^k M)$. By Lemma 2.2.3 applied to $f|_{\beta^k M} : \beta^k M \to \beta^k N$ and $g|_{\beta^k N} : \beta^k N \to \beta^k M$, we have $\mathrm{trk}(\beta^k M) = \mathrm{trk}(\beta^k N)$ for any k. This means that the number of invariant factor β^k in M and that in N are equal for any k. So we must have $M \cong N$. □

Lemma 2.2.4. *Let $M = \mathcal{O}_C^r \oplus (\bigoplus_{i=1}^m \mathcal{O}_C/\beta^{m_i})$ and $N = \mathcal{O}_C^s \oplus (\bigoplus_{j=1}^n \mathcal{O}_C/\beta^{n_j})$. Suppose there are two $\mathcal{O}_C$-linear morphisms $f : M \to N$ and $g : N \to M$ such that $g \circ f = \alpha$ and $f \circ g = \alpha$, where $\alpha \in \mathcal{O}_C$ and $v(\beta) > v(\alpha)$. Then $M \cong N$. In particular, if $M = 0$, then $N = 0$.*

Proof. According to Lemma 2.2.1, M/β^k and N/β^k are isomorphic for all k. For large enough k, this means the torsion submodule M_{tor} of M is isomorphic to the torsion submodule N_{tor} of N and also the rank of the free part of M is equal to that of N, i.e. $r = s$. We are done. □

2.3 Derived Hom

In this section, we collect some results about the derived Hom functor. These results will be used in Chapter 3 when we deal with derived category. For the proofs of these results, we refer to [Sta19, Section 0A5W].

Let R be a ring. The derived Hom is a functor

$$D(R)^{\mathrm{opp}} \times D(R) \to D(R)$$

$$(K, L) \mapsto \mathrm{RHom}_R(K, L)$$

More explicitly, choose a K-injective complex I of R-modules representing L, then $\mathrm{RHom}_R(K, L) \simeq \mathrm{Hom}^\bullet(K, I)$, where the Hom complex $\mathrm{Hom}^\bullet(K, I)$ is defined by $\mathrm{Hom}^n(K, I) = \prod_{n=p+q} \mathrm{Hom}(K^{-q}, I^p)$ with differential $d(f) = d_I \circ f - (-1)^n f \circ d_K$ for $f \in \mathrm{Hom}^n(K, I)$.

Note that the derived Hom is right adjoint to the derived tensor product, i.e.

$$\mathrm{Hom}_{D(R)}(K, \mathrm{RHom}_R(L, M)) = \mathrm{Hom}_{D(R)}(K \otimes_R^{\mathbb{L}} L, M)$$

for $K, L, M \in D(R)$.

Lemma 2.3.1 ([Sta19]Lemma 0A65)**.** *Let R be a ring and $K, L, M \in D(R)$. Then there is a canonical isomorphsim*

$$\mathrm{RHom}_R(K, \mathrm{RHom}_R(L, M)) \simeq \mathrm{RHom}_R(K \otimes_R^{\mathbb{L}} L, M)$$

in $D(R)$ functorial in K, L, M. In particular, if we take the 0-th cohomology groups, this gives back

$$\mathrm{Hom}_{D(R)}(K, \mathrm{RHom}_R(L, M)) = \mathrm{Hom}_{D(R)}(K \otimes_R^{\mathbb{L}} L, M)$$

Lemma 2.3.2 ([Sta19]Lemma 0A66)**.** *Let R be a ring. Let P be a bounded above complex of projective R-modules and L be a complex of R-modules. Then $\mathrm{RHom}_R(P, L)$ is represented by the complex $\mathrm{Hom}^\bullet(P, L)$.*

Lemma 2.3.3 ([Sta19]Lemma 07VI)**.** *Let R be a ring. Let $K \in D(R)$ be perfect. Then $K^\vee = \mathrm{RHom}_R(K, R)$ is a perfect complex and $K \simeq (K^\vee)^\vee$.*

2.4 Derived completion

In this section, we collect some basic facts about derived completion. We will focus on the derived category of A-modules for some ring A. The basic reference is [Sta19, Section 091N]. One can also consider the more general case about ringed topoi but the results in this section remain true in that general case. We refer to [BS15] and [Sta19, Section 0995] for discussions in the case of ringed topoi.

Definition 2.4.1. *Let A be a ring and $f \in A$. Let $K \in D(A)$. We say K is derived f-adically complete if the derived limit $T(K, f)$ of the system*

$$\cdots \xrightarrow{f} K \xrightarrow{f} K \xrightarrow{f} K$$

vanishes. Let I be an ideal of A. We say K is derived I-adically complete if $T(K, f)$ vanishes for all $f \in I$.

Proposition 2.4.2 ([Sta19][Proposition 091T, [BS15] Proposition 3.4.2)**.** *Let I be a finitely generated ideal of a ring A. Let M be an A-module. Then M is I-adically complete if and only if M is derived I-adically complete and I-adically separated.*

There is a useful criterion to tell if a complex of A-modules is derived I-adically complete :

Lemma 2.4.3 ([Sta19]Lemma 091P)**.** *Let I be a finitely generated ideal of a ring A. An object $K \in D(A)$ is derived I-adically complete if and only if each $H^i(K)$ is so.*

Lemma 2.4.4 ([Sta19]Lemma 091V)**.** *Let I be a finitely generated ideal of a ring A. Let $D_{\mathrm{comp}}(A, I)$ be the full subcategory consisting of deirved I-adically complete objects of $D(A)$. The inclusion functor $D_{\mathrm{comp}}(A, I) \to D(A)$ has a left adjoint, i.e. given any object $K \in D(A)$, there exists a map $K \to \widehat{K}$, where $\widehat{K}$ is derived I-adically complete, such that the map*

$$\mathrm{Hom}_{D(A)}(\widehat{K}, E) \to \mathrm{Hom}_{D(A)}(K, E)$$

is bijective whenever E is a derived I-adically complete object of $D(A)$. In fact, if I is generated by $f_1, f_2, \cdots, f_r \in A$, then we have

$$\widehat{K} = \mathrm{RHom}((A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \cdots \to A_{f_1 f_2 \cdots f_r}), K)$$

functorially in K.

Remark 2.4.5. Since $\mathrm{RHom}_{D(A)}(L, (-))$ is an exact functor from $D(A)$ to $D(A)$ for any $L \in D(A)$, the derived completion functor defined above is also exact.

Lemma 2.4.6 (Derived Nakayama lemma, [Sta19]Lemma 0G1U)**.** *Let I be a finitely generated ideal of a ring A. Let K be a derived I-adically complete object of $D(A)$. Then $K = 0$ if and only if $K \otimes^{\mathbb{L}}_A A/I \simeq 0$.*

Lemma 2.4.7 ([Sta19]Lemma 0A6E)**.** *Let I be a finitely generated ideal of a ring A. Let $K, L \in D(A)$. Then*

$$\widehat{\mathrm{RHom}_A(K, L)} \simeq \mathrm{RHom}_A(K, \widehat{L}) \simeq \mathrm{RHom}_A(\widehat{K}, \widehat{L}).$$

Lemma 2.4.8. *Let I be a finitely generated ideal of a ring A and $K \in D(A)$. Then $\widehat{K} \otimes^{\mathbb{L}}_A A/I \simeq K \otimes^{\mathbb{L}}_A A/I$.*

Proof. First note that any A/I-complex is derived I-adically complete. In fact, all the cohomology groups of a A/I-complex is classically I-complete, then also derived I-adically complete by Proposition 2.4.2. Then this statement follows from Lemma 2.4.3. So $\widehat{K} \otimes^{\mathbb{L}}_A A/I$ and $K \otimes^{\mathbb{L}}_A A/I$ are both derived I-adically complete. Let M be any derived I-adically

complete object in $D(A)$. By the universal property of the derived completion, there exists a unique map $f : \widehat{K} \to K \otimes^{\mathbb{L}}_A A/I$ which makes the following diagram commute :

$$\begin{tikzcd} K \arrow[r, "s_1"] \arrow[d, "h_1"] & \widehat{K} \arrow[d, "h_2"] \arrow[dl, dashed, "f"] \\ K \otimes^{\mathbb{L}}_A A/I \arrow[r, "s_2"] & \widehat{K} \otimes^{\mathbb{L}}_A A/I \end{tikzcd}$$

Next we consider the composition of maps $A/I \to A/I \otimes^{\mathbb{L}}_A A/I \to A/I$ where the first map is induced by $A \to A/I$ and the second map is the multiplication map (one can choose the resolution $A^{\oplus r} \to A$ of A/I induced by the generators $i_1, \cdots, i_r$ of I to make the maps explicit). This composite is in fact the identity map. Now we can get a commutative diagram induced by the composite $A/I \to A/I \otimes^{\mathbb{L}}_A A/I \to A/I$ as follows :

$$\begin{tikzcd} K \otimes^{\mathbb{L}}_A A/I \arrow[r, "s_2"] \arrow[d, "\bar{h}_1"] & \widehat{K} \otimes^{\mathbb{L}}_A A/I \arrow[d, "\bar{h}_2"] \arrow[dl, dashed, "\bar{f}"] \\ K \otimes^{\mathbb{L}}_A A/I \otimes^{\mathbb{L}}_A A/I \arrow[r, "\bar{s}_2"] \arrow[d] & \widehat{K} \otimes^{\mathbb{L}}_A A/I \otimes^{\mathbb{L}}_A A/I \arrow[d] \\ K \otimes^{\mathbb{L}}_A A/I \arrow[r, "s_2"] & \widehat{K} \otimes^{\mathbb{L}}_A A/I \end{tikzcd}$$

in particular, the compositions of the vertical maps are the identity map. The commutativity of the diagram implies $s_2 \circ (p_1 \circ \bar{f}) = (p_1 \circ \bar{f}) \circ s_2 = id$ (where $p_1 : (K \otimes^{\mathbb{L}}_A A/I) \otimes^{\mathbb{L}}_A A/I \to K \otimes^{\mathbb{L}}_A A/I$), which shows that s_2 is an isomorphism and $p_1 \circ \bar{f}$ is its inverse. So we are done.

□

Corollary 2.4.9. *Let I be a finitely generated ideal of a ring A and $K, M \in D(A)$. Then we have*

$$\widehat{K \otimes^{\mathbb{L}}_A M} \simeq \widehat{\widehat{K} \otimes^{\mathbb{L}}_A M}.$$

Proof. Recall that there is a natural map $i : K \to \widehat{K}$. By the exactness of derived completion, we have the distinguished triangle $\widehat{K \otimes^{\mathbb{L}}_A M} \to \widehat{\widehat{K} \otimes^{\mathbb{L}}_A M} \to \widehat{cone(i) \otimes^{\mathbb{L}}_A M}$. By Lemma 2.4.6, we just need to check that $A/I \otimes^{\mathbb{L}}_A (\widehat{cone(i) \otimes^{\mathbb{L}}_A M}) \simeq 0$ or equivalently $A/I \otimes^{\mathbb{L}}_A (\widehat{K \otimes^{\mathbb{L}}_A M}) \simeq A/I \otimes^{\mathbb{L}}_A (\widehat{\widehat{K} \otimes^{\mathbb{L}}_A M})$. By Lemma 2.4.8, this is equivalent to proving $A/I \otimes^{\mathbb{L}}_A (K \otimes^{\mathbb{L}}_A M) \simeq A/I \otimes^{\mathbb{L}}_A (\widehat{K} \otimes^{\mathbb{L}}_A M)$. Again, this follows from Lemma 2.4.8. □

Chapitre 3

Hodge-Tate cohomology

In this chapter, we study the Hodge-Tate specialization of Breuil-Kisin cohomology of a proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field and ramification degree e. We will prove the isomorphism concerning Hodge-Tate cohomology groups in Theorem 0.0.6 under the restriction : $ie < p - 1$.

Our strategy is to first study the Hodge-Tate specialization of A_{inf}-cohomology. Then we can take advantage of the $L\eta$-construction of A_{inf}-cohomology and its Hodge-Tate specializaton. This will provide us with two morphisms which enable us to use Lemma 2.2.4.

3.1 Almost mathematics

In order to make our strategy more precise, we need to introduce the framework of almost mathematics (derived category version) following [Bha18, Section 3]. All results in this subsection can be found there.

Throughout this section, let C be a complete, algebraically closed nonarchimedean extension of K and $\mathcal{O}_C$ be its ring of integers. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}_C$ and k denote the residue field of $\mathcal{O}_C$. Recall that $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$ where $\mathcal{O}_C^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$.

Definition 3.1.1 (The pair $(\mathcal{O}_C, \mathfrak{m})$)**.** *We say an $\mathcal{O}_C$-module M is almost zero if $\mathfrak{m} \cdot M = 0$. A map $f : K \to L$ in $D(\mathcal{O}_C)$ is an almost isomorphism if the cohomology groups of the mapping cone of f are almost zero.*

Example 3.1.2 ([Sch13] Lemma 4.10(v))**.** Let X be a locally notherian adic space over $\mathrm{Spa}(\mathrm{C}, \mathcal{O}_\mathrm{C})$. Then for any $U \in X_{\text{proét}}$ which is affinoid perfectoid, the cohomology groups $H^i(U, \widehat{\mathcal{O}}_X^+)$ are almost zero for $i > 0$.

Now we consider the almost derived category of $\mathcal{O}_C$-modules. Firstly, the restriction of scalar functor $Res : D(k) \to D(\mathcal{O}_C)$ is in fact fully faithful. To see this, note that

the restriction of scalar functor $Res : \mathrm{Mod}(k) \to \mathrm{Mod}(\mathcal{O}_C)$ admits a left adjoint functor given by $(-) \otimes_{\mathcal{O}_C} k$. By [Sta19, Lemma 09T5], $Res : D(k) \to D(\mathcal{O}_C)$ has a left adjoint functor given by $(-) \otimes^{\mathbb{L}}_{\mathcal{O}_C} k : D(\mathcal{O}_C) \to D(k)$. For any $M, N \in D(k)$, we have $\mathrm{Hom}_{D(k)}(Res(M) \otimes^{\mathbb{L}}_{\mathcal{O}_C} k, N) \cong \mathrm{Hom}_{D(\mathcal{O}_C)}(Res(M), Res(N))$ by the adjunction. Next we will construct a natural isomorphism $(- \otimes^{\mathbb{L}}_{\mathcal{O}_C} k) \circ Res \to id$ and then the restriction of scalar functor Res is fully faithful by Lemma [Sta19, Lemma 07RB].

To prove there is a natural isomorphism $Res(M) \otimes^{\mathbb{L}}_{\mathcal{O}_C} k \simeq M$ in $D(k)$, note that $Res(M) \otimes^{\mathbb{L}}_{\mathcal{O}_C} k \simeq M \otimes^{\mathbb{L}}_{k} k \otimes^{\mathbb{L}}_{\mathcal{O}_C} k$. So we just need to prove there is an isomorphism $k \otimes^{\mathbb{L}}_{\mathcal{O}_C} k \simeq k$ in $D(k)$. To prove this, we consider the short exact sequence $0 \to \mathfrak{m} \to \mathcal{O}_C \to k \to 0$. In fact, $\mathfrak{m} \to \mathcal{O}_C$ is a flat resolution of k in $D(\mathcal{O}_C)$ as $\mathfrak{m}$ can be regarded as a filtered colimit of flat (free) $\mathcal{O}_C$-modules and filtered colimit of flat modules is flat. So $k \otimes^{\mathbb{L}}_{\mathcal{O}_C} k \simeq (k \otimes_{\mathcal{O}_C} \mathfrak{m} \to k \otimes_{\mathcal{O}_C} \mathcal{O}_C)$. Since $\mathfrak{m} = \mathfrak{m}^2$, we have $k \otimes_{\mathcal{O}_C} \mathfrak{m} = 0$. So we are done and the restriction of scalar functor $Res : D(k) \to D(\mathcal{O}_C)$ is fully faithful.

Now we can consider the following two functors :

$$D(\mathcal{O}_C) \xrightarrow{()^a} D(\mathcal{O}_C)^a := D(\mathcal{O}_C)/D(k), \qquad L \mapsto L^a$$

$$D(\mathcal{O}_C)^a \xrightarrow{()_*} D(\mathcal{O}_C), \qquad L^a \mapsto (L^a)_* := \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, L)$$

where the Verdier quotient $D(\mathcal{O}_C)/D(k)$, i.e. the derived category of almost $\mathcal{O}_C$-modules, is actually the localization of $D(\mathcal{O}_C)$ with respect to almost isomorphisms. For the definitions of Verdier quotient and localization of triangulated categories, we refer to [Nee01].

Next we state a lemma which enables us to move freely between the real world and the almost world.

Lemma 3.1.3. *Let C be spherically complete, i.e. any decreasing sequence of discs in C has nonempty intersection. For any perfect complex $L \in D(\mathcal{O}_C)$, the natural map $L = \mathrm{RHom}_{\mathcal{O}_C}(\mathcal{O}_C, L) \to \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, L)$ induced by the natural injection $\mathfrak{m} \to \mathcal{O}_C$ is an isomorphism.*

Proof. This is [Bha18, Lemma 3.4]. For readers' convenience, we give the proof here.

We first prove that if this lemma is true when $L = \mathcal{O}_C$, then it is true for any perfect complex $L \in D(\mathcal{O}_C)$. Let $L^\vee$ denote $\mathrm{RHom}_{\mathcal{O}_C}(L, \mathcal{O}_C)$. By Lemma 2.3.3, we have

$$\mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathcal{O}_C) \simeq L.$$

So in order to prove that $\mathrm{RHom}_{\mathcal{O}_C}(\mathcal{O}_C, L) \to \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, L)$ is an isomorphism, we just need to prove $\mathrm{RHom}_{\mathcal{O}_C}(\mathcal{O}_C, \mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathcal{O}_C)) \to \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, \mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathcal{O}_C))$ is an isomorphism. Next by Lemma 2.3.1, we have

$$\mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, \mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathcal{O}_C)) \simeq \mathrm{RHom}_{\mathcal{O}_C}(L^\vee \otimes^{\mathbb{L}}_{\mathcal{O}_C} \mathfrak{m}, \mathcal{O}_C)$$

and

$$\mathrm{RHom}_{\mathcal{O}_C}(\mathcal{O}_C, \mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathcal{O}_C)) \simeq \mathrm{RHom}_{\mathcal{O}_C}(L^\vee \otimes^{\mathbb{L}}_{\mathcal{O}_C} \mathcal{O}_C, \mathcal{O}_C).$$

So now we just need to prove that the map $\mathrm{RHom}_{\mathcal{O}_C}(L^\vee \otimes^{\mathbb{L}}_{\mathcal{O}_C} \mathcal{O}_C, \mathcal{O}_C) \to \mathrm{RHom}_{\mathcal{O}_C}(L^\vee \otimes^{\mathbb{L}}_{\mathcal{O}_C} \mathfrak{m}, \mathcal{O}_C)$ is an isomorphism. Again by Lemma 2.3.1, we see that this is equivalent to requiring the map

$$\mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathrm{RHom}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C)) \to \mathrm{RHom}_{\mathcal{O}_C}(L^\vee, \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C))$$

to be an isomorphism. By our assumption that the natural map $\mathrm{RHom}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C)$ is an isomorphism, we are done.

Now we assume $L = \mathcal{O}_C$. We have to show that the natural map $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Hom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C)$ induced by the injection $\mathfrak{m} \to \mathcal{O}_C$ is an isomorphism and all the higher cohomology groups of $\mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C)$ vanish. Write $\mathfrak{m} = \cup_n \mathfrak{m}_n$ with $\mathfrak{m}_n = (a^{\frac{1}{n}})$, where a is any non-zero element in $\mathfrak{m}$ and $a^{\frac{1}{n}}$ is a fixed n-th root of a.

1. For the first one. Note that for any $f \in \mathrm{Hom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C)$, we have $f(a) = f(a^{\frac{n-1}{n}} a^{\frac{1}{n}}) = a^{\frac{n-1}{n}} f(a^{\frac{1}{n}})$ for all n. Then $v(f(a)) = v(a^{\frac{n-1}{n}}) + v(f(a^{\frac{1}{n}}))$ for all n. This implies that $v(f(a)) \geq v(a)$. So we can find $x \in \mathcal{O}_C$ such that $f(a) = ax$. This shows that the natural map $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Hom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C)$ is an isomorphism.

2. For the second one. Note that $\mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, \mathcal{O}_C) = \mathrm{RHom}_{\mathcal{O}_C}(\cup_n \mathfrak{m}_n, \mathcal{O}_C)$. Choose a K-injectve resolution I of $\mathcal{O}_C$ and then $\mathrm{RHom}_{\mathcal{O}_C}(\cup_n \mathfrak{m}_n, \mathcal{O}_C) \simeq \mathrm{Hom}^\bullet_{\mathcal{O}_C}(\cup_n \mathfrak{m}_n, I) \simeq \varprojlim_n \mathrm{Hom}(\mathfrak{m}_n, I)$. Let $\mathfrak{m}_n^\vee$ denote $\mathrm{Hom}_{\mathcal{O}_C}(\mathfrak{m}_n, \mathcal{O}_C)$ and $\mathrm{Mod}(\mathcal{O}_C, \mathbb{N})$ denote the abelian category of inverse systems of $\mathcal{O}_C$-modules. Then the inverse system $(\mathfrak{m}_n^\vee)$ is in $\mathrm{Mod}(\mathcal{O}_C, \mathbb{N})$. Since $\mathfrak{m}_n$ is principal for all n, the complex of inverse system $\mathrm{Hom}(\mathfrak{m}_n, I)$ gives an injective resolution of $(\mathfrak{m}_n^\vee)$. So we can write $\varprojlim_n \mathrm{Hom}(\mathfrak{m}_n, I) = \mathrm{Rlim}(\mathfrak{m}_n^\vee)$. By [Sta19, Lemma 091D], we have $\mathrm{R}^i\mathrm{lim}(\mathfrak{m}_n^\vee) = H^i(\mathrm{Rlim}(\mathfrak{m}_n^\vee)) = 0$ for $i > 1$. So now we just need to prove $\mathrm{R}^1\mathrm{lim}(\mathfrak{m}_n^\vee) = 0$. For each n, consider the map $\mathfrak{m}_n^\vee \to C$ defined by $f \mapsto f(a^{\frac{1}{n}})a^{-\frac{1}{n}}$. This map is injective and its image is the $\mathcal{O}_C$-submodule of C generated by $a^{-\frac{1}{n}}$. Now we will see $\mathfrak{m}_n^\vee$ as an $\mathcal{O}_C$-submodule of C via this injection. Note that the canonical map $\alpha : C \to \varprojlim C/\mathfrak{m}_n^\vee$ is surjective. Indeed, an element $(\overline{x_n}) \in \varprojlim C/\mathfrak{m}_n^\vee$ gives a descending sequence $\{x_n + \mathfrak{m}_n^\vee\}$ of open discs in C, where x_n is any lift of $\overline{x_n} \in C/\mathfrak{m}_n^\vee$. By spherical completeness of C, there exists some $x \in C$ such that $x \in x_n + \mathfrak{m}_n^\vee$ for all n. Then $x \in C$ maps to $(\overline{x_n}) \in \varprojlim C/\mathfrak{m}_n^\vee$ under the map α. This shows the surjectivity of α. Now we apply the long exact sequence for $\mathrm{R}^i\mathrm{lim}$ to the short exact sequence of inverse systems of $\mathcal{O}_C$-modules :

$$0 \to \{\mathfrak{m}_n^\vee\} \to \{C\} \to \{C/\mathfrak{m}_n^\vee\} \to 0$$

and we get the sequence

$$0 \to \varprojlim \mathfrak{m}_n^\vee \to C \xrightarrow{\alpha} \varprojlim C/\mathfrak{m}_n^\vee \to \mathrm{R}^1\lim(\mathfrak{m}_n^\vee) \to 0.$$

As the map α is surjective, we see that $\mathrm{R}^1\lim(\mathfrak{m}_n^\vee) = 0$. So we are done.

□

Remark 3.1.4. As we see in the proof above, the spherical completeness is necessary. There are some examples which are not spherical complete. For instance, $C = \widehat{\overline{\mathbb{Q}_p}}$ is not spherical complete.

There are similar constructions and results in the setting of A_{inf}-modules.

Definition 3.1.5 (The pair $(A_{\mathrm{inf}}, W(\mathfrak{m}^\flat))$)**.** *An A_{inf}-module M is called almost zero if $W(\mathfrak{m}^\flat) \cdot M = 0$, where $W(\mathfrak{m}^\flat) = \mathrm{Ker}(A_{\mathrm{inf}} \to W(k))$. A map $f : K \to L$ in $D(A_{\mathrm{inf}})$ is called an almost isomorphism if the cohomology groups of the mapping cone of f are almost zero.*

Similarly, we consider the almost derived category of A_{inf}-modules. But we can not simply repeat the preceding definition as $W(\mathfrak{m}^\flat)^2$ might not be equal to $W(\mathfrak{m}^\flat)$ as ideals in A_{inf} (see [Ked16, Remark 1.4]). The solution is to consider the derived p-adically complete complexes.

Let $D_{\mathrm{comp}}(A_{\mathrm{inf}}) \subset D(A_{\mathrm{inf}})$ be the full subcategory consisting of all derived p-adically complete complexes. Again, we first show that the restriction of scalar functor $Res : D_{\mathrm{comp}}(W(k)) \to D_{\mathrm{comp}}(A_{\mathrm{inf}})$ is fully faithful. For any $N \in D_{\mathrm{comp}}(A_{\mathrm{inf}})$ and $M \in D_{\mathrm{comp}}(W(k))$, we have

$$\mathrm{Hom}_{D(W(k))}(N \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k), M) \simeq \mathrm{Hom}_{D(A_{\mathrm{inf}})}(N, Res(M)).$$

By Lemma 2.4.4 , we have

$$\mathrm{Hom}_{D(W(k))}(N \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k), M) \simeq \mathrm{Hom}_{D_{\mathrm{comp}}(W(k))}(N \widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}} W(k), M).$$

This means that the restriction of scalar functor $Res : D_{\mathrm{comp}}(W(k)) \to D_{\mathrm{comp}}(A_{\mathrm{inf}})$ is right adjoint to the functor $(-) \widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}} W(k) : D_{\mathrm{comp}}(A_{\mathrm{inf}}) \to D_{\mathrm{comp}}(W(k))$.

Now we want to prove $Res(M) \widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}} W(k) \simeq M$. Note that

$$Res(M) \widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}} W(k) \simeq M \otimes^{\mathbb{L}}_{W(k)} \widehat{W(k)} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k).$$

Considering the short exact sequence $0 \to W(\mathfrak{m}^\flat) \to A_{\mathrm{inf}} \to W(k) \to 0$, we get a distinguished triangle

$$W(k) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(\mathfrak{m}^\flat) \to W(k) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}} \to W(k) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k).$$

Since derived p-adic completion is exact by Remark 2.4.5, we get another distinguished triangle

$$W(k)\widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}W(\mathfrak{m}^\flat)\to W(k)\widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}A_{\mathrm{inf}}\to W(k)\widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}W(k).$$

Note that we have $A_{\mathrm{inf}}/p\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}(W(k)\widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}W(\mathfrak{m}^\flat))\simeq A_{\mathrm{inf}}/p\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}W(k)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}W(\mathfrak{m}^\flat)$ by Lemma 2.4.8. Moreover, $A_{\mathrm{inf}}/p\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}W(k)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}W(\mathfrak{m}^\flat)\simeq k\otimes^{\mathbb{L}}_{\mathcal{O}^\flat_C}\mathfrak{m}^\flat$ where $\mathfrak{m}^\flat$ is the maximal ideal in $\mathcal{O}^\flat_C$. Since $\mathfrak{m}^\flat$ is a flat $\mathcal{O}^\flat_C$-module and $(\mathfrak{m}^\flat)^2=\mathfrak{m}^\flat$, we have $k\otimes^{\mathbb{L}}_{\mathcal{O}^\flat_C}\mathfrak{m}^\flat\simeq k\otimes_{\mathcal{O}^\flat_C}\mathfrak{m}^\flat=0$. Then by Lemma 2.4.6, the derived Nakayama lemma, we have $W(k)\widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}W(\mathfrak{m}^\flat)=0$. This implies $W(k)\widehat{\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}W(k)\simeq W(k)$. Finally, by Corollary 2.4.9 we have $M\otimes^{\mathbb{L}}_{W(k)}\widehat{W(k)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}}W(k)\simeq M\otimes^{\mathbb{L}}_{W(k)}\widehat{W(k)}\simeq M$. So the restriction of scalar functor is fully faithful by Lemma [Sta19, Lemma 07RB].

Now, we can consider the following functors :

$$D_{\mathrm{comp}}(A_{\mathrm{inf}})\xrightarrow{()^a}D_{\mathrm{comp}}(A_{\mathrm{inf}})^a:=D_{\mathrm{comp}}(A_{\mathrm{inf}})/D_{\mathrm{comp}}(W(k)),\quad L\mapsto L^a$$

$$D_{\mathrm{comp}}(A_{\mathrm{inf}})^a\xrightarrow{()_*}D_{\mathrm{comp}}(A_{\mathrm{inf}}),\quad L^a\to(L^a)_*:=\mathrm{RHom}_{A_{\mathrm{inf}}}(W(\mathfrak{m}^\flat),L)$$

where the Verdier quotient $D_{\mathrm{comp}}(A_{\mathrm{inf}})^a:=D_{\mathrm{comp}}(A_{\mathrm{inf}})/D_{\mathrm{comp}}(W(k))$ is actually the localization of $D_{\mathrm{comp}}(A_{\mathrm{inf}})$ with respect to almost isomorphisms.

Lemma 3.1.6. *Let C be spherically complete. If $L\in D_{\mathrm{comp}}(A_{\mathrm{inf}})$ is perfect, then the natural map $L=\mathrm{RHom}_{A_{\mathrm{inf}}}(A_{\mathrm{inf}},L)\to(L^a)_*=\mathrm{RHom}_{A_{\mathrm{inf}}}(W(\mathfrak{m}^\flat),L)$ induced by the injection $W(\mathfrak{m}^\flat)\to A_{\mathrm{inf}}$ is an isomorphism.*

Proof. This is [Bha18, Lemma 3.10]. For readers' convenience, we give the proof here.

By the same argument as in Lemma 3.1.3, we may assume $L=A_{\mathrm{inf}}$. So we need to check that $A_{\mathrm{inf}}=\mathrm{RHom}_{A_{\mathrm{inf}}}(A_{\mathrm{inf}},A_{\mathrm{inf}})\simeq\mathrm{RHom}_{A_{\mathrm{inf}}}(W(\mathfrak{m}^\flat),A_{\mathrm{inf}})$. By Lemma 2.4.7, both sides are derived ξ-adically complete (for the definition of ξ, see Definition 1.4.1). So we just need to check the isomorphism after applying $(-)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{inf}}/\xi$ by derived Nakayama lemma.

By choosing a projective (free) resolution P of $W(\mathfrak{m}^\flat)$ and Lemma 2.3.2, we can see that

$$\mathrm{RHom}_{A_{\mathrm{inf}}}(W(\mathfrak{m}^\flat),A_{\mathrm{inf}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{inf}}/\xi\simeq\mathrm{Hom}^\bullet(P,A_{\mathrm{inf}})\otimes_{A_{\mathrm{inf}}}A_{\mathrm{inf}}/\xi\simeq\mathrm{Hom}^\bullet(P,\mathcal{O}_C)$$

and moreover, we have an isomorphism of complexes of $\mathcal{O}_C$-modules,

$$\mathrm{Hom}^\bullet(P,\mathcal{O}_C)\simeq\mathrm{Hom}^\bullet(P\otimes_{A_{\mathrm{inf}}}\mathcal{O}_C,\mathcal{O}_C).$$

As $P\otimes_{A_{\mathrm{inf}}}\mathcal{O}_C$ is also a representative of $W(\mathfrak{m}^\flat)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}\mathcal{O}_C$, we have

$$\mathrm{Hom}^\bullet(P\otimes_{A_{\mathrm{inf}}}\mathcal{O}_C,\mathcal{O}_C)\simeq\mathrm{RHom}_{\mathcal{O}_C}(W(\mathfrak{m}^\flat)\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}\mathcal{O}_C,\mathcal{O}_C).$$

Using the projective resolution $A_{\inf} \xrightarrow{\times\xi} A_{\inf}$ of $\mathcal{O}_C$, we can get that $W(\mathfrak{m}^\flat) \otimes^{\mathbb{L}}_{A_{\inf}} \mathcal{O}_C \simeq W(\mathfrak{m}^\flat)/\xi$. Then it is enough to check $\mathrm{RHom}_{\mathcal{O}_C}(A_{\inf}/\xi, \mathcal{O}_C) \simeq \mathrm{RHom}_{\mathcal{O}_C}(W(\mathfrak{m}^\flat)/\xi, \mathcal{O}_C)$.

Recall that there is a short exact sequence of $A_{\inf}$-modules $0 \to W(\mathfrak{m}^\flat) \to A_{\inf} \to W(k) \to 0$. Then we can get a long exact sequence

$$W(k)[\xi] \to W(\mathfrak{m}^\flat)/\xi \to A_{\inf}/\xi \to W(k)/\xi \to 0$$

Since ξ is sent to p under the canonical map $A_{\inf} \to W(k)$, we have $W(k)[\xi] = 0$ and $W(\mathfrak{m}^\flat)/\xi \cong \mathfrak{m}$. Then by Lemma 3.1.3, we are done. □

3.2 Structure of the Hodge-Tate cohomology groups

Now we are ready to study the structure of the Hodge-Tate cohomology groups. We first state a lemma about the $L\eta$-functor, which will give us two important maps connecting Hodge-Tate cohomology and p-adic étale cohomology.

Lemma 3.2.1. *Let A be a commutative ring and $a \in A$ be a non-zero divisor. Assume $K \in D^{[0,s]}(A)$ with $H^0(K)$ being a-torsion free. Then there are natural maps $L\eta_a(K) \to K$ and $K \to L\eta_a(K)$ whose composition in either direction is a^s.*

Proof. This is [BMS18, Lemma 6.9]. We also give the proof here.

Firstly, we choose a representative L of K such that L is a-torsion free. Then we apply the truncation functor $\tau^{\leq s}$ and $\tau^{\geq 0}$ to L, i.e. $\tau^{\leq s}\tau^{\geq 0}L = (\cdots \to 0 \to L^0/\mathrm{Im}(d^{-1}) \to L^1 \to \cdots \to L^{s-1} \to \ker(d^s) \to 0 \cdots)$. Since $K \in D^{[0,s]}(A)$, $\tau^{\leq s}\tau^{\geq 0}L$ is still isomorphic to K. We now prove $\tau^{\leq s}\tau^{\geq 0}L$ is still a-torsion-free. It is easy to see that $\ker(d^s)$ is a-torsion free. For $L^0/\mathrm{Im}(d^{-1})$, suppose $\bar{x} \in L^0/\mathrm{Im}(d^{-1})$ is killed by a, then $ax \in \mathrm{Im}(d^{-1})$ for any lifting $x \in L^0$ of $\bar{x}$ and $d^0(ax) = ad^0(x) = 0$. As L^0 is a-torsion free, $d^0(x)$ must be 0, which implies that $x \in \ker(d^0)$. But this also means that $H^0(L) = H^0(K)$ has a-torsion. So $\tau^{\leq s}\tau^{\geq 0}L$ is still a-torsion free and we can apply η-functor to it.

There is a natural inclusion $\eta_a(\tau^{\leq s}\tau^{\geq 0}L) \to \tau^{\leq s}\tau^{\geq 0}L$. We can define another map $\tau^{\leq s}\tau^{\geq 0}L \to \eta_a(\tau^{\leq s}\tau^{\geq 0}L)$ by multiplying by a^s. Then the composition of these two maps in either direction is a^s. □

Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed nonarchimedean extension C of K and $\bar{\mathfrak{X}}$ denote $\mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. We may apply Lemma 3.2.1 to $A = \mathcal{O}_{\bar{\mathfrak{X}}}$, $a = \zeta_p - 1$ and $K = \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$. In fact $\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$ is in $D^{[0,i]}(\mathcal{O}_{\bar{\mathfrak{X}}})$ with $H^0(\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+)$ being $(\zeta_p - 1)$-torsion free. By the same argument in the proof of Lemma 3.2.1 we can always find a representative L of $\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$ such that L is $(\zeta_p - 1)$-torsion free and $L^s = 0$ for any $s \notin [0, i]$. Then there are two natural maps which we denote by f and g,

$$f : L\eta_{\zeta_p - 1}(\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+) \simeq \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \to \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$$

$$g : \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+ \to \tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}}$$

whose composition in either direction is $(\zeta_p - 1)^i$. The isomorphism $L\eta_{\zeta_p - 1}(\tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+) \simeq \tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is due to the commutativity of the $L\eta$ functor and the canonical truncation functor $\tau^{\leq i}$ (see [BMS18, Corollary 6.5]). Recall that for any $K \in D(\mathcal{O}_{\bar{\mathfrak{X}}})$, $\tau^{\leq i} K := (\cdots \to K^{i-1} \xrightarrow{d^{i-1}} \ker(d^i) \to 0 \to \cdots)$.

Passing to sheaf cohomology, we get two natural maps

$$f : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+)$$

$$g : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+) \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$. Since there is an isomorphsim $\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \simeq \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\mathfrak{X}})$ which is induced by the canonical morphism $\tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}} \to \widetilde{\Omega}_{\bar{\mathfrak{X}}}$, we get two maps

$$f : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+)$$

$$g : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+) \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$.

Note that there is an isomorphism $R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+) \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \widehat{\mathcal{O}}_X^+)$. What we want to study at the end is the p-adic étale cohomology but not pro-étale cohomology. But actually we get almost what we want. Recall the primitive comparison theorem due to Scholze.

Theorem 3.2.2 ([Sch13, Theorem 8.4])**.** *For any proper smooth adic space X over C, there are natural isomorphisms in $D(\mathcal{O}_C)^a$ and $D(A_{\mathrm{inf}})^a$ respectively,*

$$R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathcal{O}_C \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \widehat{\mathcal{O}}_X^+)$$

and

$$R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}} \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{A}_{\mathrm{inf},X}).$$

Then by passing to the world of almost mathematics, we get two natural maps in $D(\mathcal{O}_C)^a$:

$$f^a : (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}))^a \to (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+))^a \simeq (\tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a$$

$$g^a : (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+))^a \simeq (\tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a \to (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i} \widetilde{\Omega}_{\bar{\mathfrak{X}}}))^a$$

where we have used the following lemma.

Lemma 3.2.3. *If $h : K \to L$ is an almost isomorphism in $D(\mathcal{O}_C)$ (resp. $D(A_{\mathrm{inf}})$), then so is $\bar{h} : \tau^{\leq i} K \to \tau^{\leq i} L$.*

Proof. Let M denote $cone(h)$ and $\bar{M}$ denote $cone(\bar{h})$. Then there is a morphism of distinguished triangle

$$\begin{tikzcd} \tau^{\leq i}K \arrow[r] \arrow[d] & \tau^{\leq i}L \arrow[r] \arrow[d] & \bar{M} \arrow[d] \\ K \arrow[r] & L \arrow[r] & M. \end{tikzcd}$$

We then get a morphism of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & H^{i-1}(K) \arrow[r] \arrow[d] & H^{i-1}(L) \arrow[r] \arrow[d] & H^{i-1}(\bar{M}) \arrow[r] \arrow[d] & H^i(K) \arrow[r] \arrow[d] & H^i(L) \arrow[r] \arrow[d] & H^i(\bar{M}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[d] \\ \cdots \arrow[r] & H^{i-1}(K) \arrow[r] & H^{i-1}(L) \arrow[r] & H^{i-1}(M) \arrow[r] & H^i(K) \arrow[r] & H^i(L) \arrow[r] & H^i(M) \arrow[r] & H^{i+1}(K) \arrow[r] & \cdots \end{tikzcd}$$

By using five lemma, it is easy to deduce that when $n < i$, we have $H^n(\bar{M}) \cong H^n(M)$; when $n = i$, the map $H^i(\bar{M}) \to H^i(M)$ is injective; when $n > i$, we have $H^n(\bar{M}) = 0$. Then we conclude that the cohomology groups of $cone(\bar{h})$ are almost zero, i.e. $\bar{h} : \tau^{\leq i}K \to \tau^{\leq i}L$ is an almost isomorphism. □

Lemma 3.2.4. *The complex $\tau^{\leq i}R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) = \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$ (resp. $\tau^{\leq i}R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$) is a perfect complex of $\mathcal{O}_C$-modules (resp. A_{inf}-modules).*

Proof. By Theorem 1.4.5.4 and Theorem 1.5.2.1, we have

$$R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\alpha} A_{\mathrm{inf}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\widetilde{\xi} \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\beta} \mathcal{O}_K \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_C.$$

Since $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ is a perfect complex of $\mathfrak{S}$-modules and $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) := R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\beta} \mathcal{O}_K$, the Hodge-Tate cohomology $R\Gamma_{\mathrm{HT}}(\mathfrak{X})$ of $\mathfrak{X}$ is a perfect complex of $\mathcal{O}_K$-modules by [Sta19, Lemma 066W]. Moreover as $\mathcal{O}_K$ is a Notherian local ring, the cohomology groups $H^i_{\mathrm{HT}}(\mathfrak{X})$ are finitely generated $\mathcal{O}_K$-modules and so finitely presented $\mathcal{O}_K$-modules. So we see that every Hodge-Tate cohomology group $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$ is also finitely presented over $\mathcal{O}_C$. By Lemma 2.1.3, this means $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \cong \bigoplus_{j=1}^n \mathcal{O}_C/\pi_j$ for some $\pi_j \in \mathcal{O}_C$. So $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$ is perfect. The lemma hence follows from [Sta19, Lemma 066U]. For $\tau^{\leq i}R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$, this follows from [BMS18, Lemma 4.9] stating that each $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is perfect. □

As $\tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p)$ and $\tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$ are perfect complexes, then Lemma 3.1.3 tells us that if C is spherically complete, then $(\tau^{\leq i}R\Gamma(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}))^a_* \simeq \tau^{\leq i}R\Gamma(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}})$ and $(\tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a_* \simeq \tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$. By moving back to the real world, we have two maps

$$(f^a)_* : \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to \tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$(g^a)_* : \tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$. These two maps induce maps between

cohomology groups for any $n \leq i$.

$$f : H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$g : H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$.

Now we come to the following key theorem :

Theorem 3.2.5. *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete and algebraically closed nonarchimedean extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Assuming $ie < p - 1$, then there is an isomorphism of $\mathcal{O}_C$-modules between Hodge-Tate cohomology group and p-adic étale cohomology group,*

$$H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) := H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Proof. Note that replacing C by its spherical completion C' will not make any difference to this theorem. The spherical completion always exists (see [Rob13, Chapter 3]), which is still complete and algebraically closed. On one hand, p-adic étale cohomology is insensitive to such extensions in the rigid-analytic setting (see [Hub13, Section 0.3.2]). On the other hand, by the base change of prismatic cohomology, we have $H^i_{\mathrm{HT}}(\overline{\mathfrak{X}} \otimes_{\mathcal{O}_C} \mathcal{O}_{C'}) \cong H^i_{\mathrm{HT}}(\overline{\mathfrak{X}}) \otimes_{\mathcal{O}_C} \mathcal{O}_{C'}$ and the natural injection $\mathcal{O}_C \to \mathcal{O}_{C'}$ is flat.

So now we assume C is spherically complete. We have seen in the proof of Lemma 3.2.4 that $H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$ has a decomposition as $\mathcal{O}_C^n \oplus (\bigoplus_{j=1}^m \mathcal{O}_C/\pi^{m_j})$. By requiring $ie < p - 1$, we have $v((\zeta_p - 1)^i) < v(\pi)$ in $\mathcal{O}_C$ as $v((\zeta_p - 1)^{p-1}) = v(p)$ and $v(p) = v(\pi^e)$. Now the theorem follows from Lemma 2.2.4 and the existence of maps

$$f : H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$g : H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in both directions is $(\zeta_p - 1)^i$. $\square$

Chapitre 4

The unramified case : comparison theorem

In this chapter, let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a fixed complete algebraically closed nonarchimedean extension C of K. We will study the relation between the p-adic étale cohomology group $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ of the adic generic fiber X of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$ and the crystalline cohomology group $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ of the special fiber $\mathfrak{X}_k$ in the unramifed case, i.e. the ramification degree $e = 1$ and $\mathcal{O}_K = W(k)$. Note that in the unramified case, the crystalline cohomology $R\Gamma_{\text{crys}}(\mathfrak{X}_k/W(k))$ is canonically isomorphic to the de Rham cohomology $R\Gamma_{\text{dR}}(\mathfrak{X}/W(k))$ (cf. [Ber06]).

In order to prove our integral comparison theorem, we first relate Hodge-Tate cohomology to Hodge cohomology. And then we can use Theorem 3.2.5 to get a link between Hodge-Tate cohomology and p-adic étale cohomology. The last step is to study the Hodge-to-de Rham spectral sequence and we can prove the converse to [BMS18, Theorem 14.5], which results in the final comparison theorem.

4.1 The cotangent complex

Before we move forward, we briefly recall the construction and basic properties of the cotangent complex that will be used later.

Definition 4.1.1 (Quillen)**.** *For any map of commutative rings $A \to B$, we define the cotangent complex $\mathbb{L}_{B/A} := \Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$, where $P^\bullet \xrightarrow{\sim} B$ is a simplicial resolution of B by polynomial A-algebras and the tensor product is componentwise. Note that $\mathbb{L}_{B/A}$ is a simplicial B-module, which is unique up to homotopy (since $P^\bullet$ is unique up to homotopy). So it can be viewed as a complex of B-modules in $D(B)$ via the Dold-Kan correspondence.*

Remark 4.1.2. let A be a commutative ring and S be a set. One can define $A[S]$, the polynomial A-algebra with variables indexed by S. Consider an A-algebra B. By thinking of B as a set, we get a natural A-linear map $\gamma_B : A[B] \to B$. Moreover, there are two maps $A[A[B]] \rightrightarrows A[B]$, one of which is $A[\gamma_B]$ and the other is $\gamma_{A[B]}$. By iterating this process, we can define the canonical A-algebra resolution $P^{\bullet}_{B/A}$ of B as follows :

$$P^{\bullet}_{B/A} := (\cdots\ A[A[A[B]]] \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} A[A[B]] \rightrightarrows A[B]) \longrightarrow B$$

Using this canonical resolution can make the definition of cotangent complex functorial.

Remark 4.1.3. For any commutative ring map $A \to B$, there is a canonical isomorphism $H^0(\mathbb{L}_{B/A}) \cong \Omega^1_{B/A}$. If $A \to B$ is surjective with kernel I, then $H^0(\mathbb{L}_{B/A}) = 0$ and $H^{-1}(\mathbb{L}_{B/A}) = I/I^2$. In some sense, the cotangent complex may be regarded as a "left derived functor" of taking Kähler differentials. In fact, for any sequence of commutative rings $R \to S \to T$, there is an exact sequence

$$\Omega^1_{S/R} \otimes_S T \to \Omega^1_{T/R} \to \Omega^1_{T/S} \to 0.$$

If $S \to T$ is surjective with kernel J, then the sequence above can be extended to

$$J/J^2 \to \Omega^1_{S/R} \otimes_S S/J \to \Omega^1_{T/R} \to 0.$$

The cotangent complex extends these two exact sequences further to the left. Before the definition of the cotangent complex, there are already some efforts devoted to extending these two exact sequences further to the left, such as the Lichtenbaum-Schlessinger functors [LS67].

Proposition 4.1.4 ([Ill06])**.** *Let R be a commutative ring and A be a commutative R-algebra.*

1. *Künneth formula : For any commutative R-algebra B such that $\mathrm{Tor}_i^R(A, B) = 0$ for all $i > 0$, then*
$$\mathbb{L}_{A\otimes_R B/R} \simeq (\mathbb{L}_{A/R} \otimes^{\mathbb{L}}_R B) \oplus (\mathbb{L}_{B/R} \otimes^{\mathbb{L}}_R A)$$
2. *Transitivity triangle : Let B be a commutative A-algebra, there is a natural distinguished triangle*
$$\mathbb{L}_{A/R} \otimes^{\mathbb{L}}_A B \to \mathbb{L}_{B/R} \to \mathbb{L}_{B/A}$$
3. *Base change : For any commutative R-algebra B such that $\mathrm{Tor}_i^R(A, B) = 0$ for all $i > 0$, then*
$$\mathbb{L}_{A/R} \otimes^{\mathbb{L}}_R B \simeq \mathbb{L}_{A\otimes_R B/B}.$$

Remark 4.1.5. The cotangent complex is difficult to compute in general. But there are some cases that the cotangent complex behaves well.

1. If a commutative ring map $R \to A$ is smooth, then $\mathbb{L}_{A/R} \simeq \Omega^1_{A/R}[0]$. For the proof, see [Sta19, Lemma 08R5].
2. Let R be a commutative ring of characteristic p and A be a R-algebra such that the relative Frobenius $F_{A/R} : A^{(1)} := A \otimes_{R,F_R} R \to A$ is an isomorphism, then $\mathbb{L}_{A/R}$ vanishes. In fact, if A is a polynomial R-algebra, then the relative Frobenius induces the zero map $\mathbb{L}_{F_{A/R}} : \mathbb{L}_{A^{(1)}/R} \to \mathbb{L}_{A/R}$ as $d(x^p) = 0$. As the relative Frobenius $F_{A/R}$ is an isomorphism, then $\mathbb{L}_{F_{A/R}}$ is an isomorphism by functoriality. This implies that $\mathbb{L}_{A/R} \simeq 0$. For more details, see [Bha17b, Proposition 6.1.4].
3. Let A be a commutative ring and $I \subset A$ be an ideal generated by a regular sequence. Then $\mathbb{L}_{(A/I)/A} \simeq I/I^2[1]$. To see this, we first consider the case $A = \mathbb{Z}[x_1, \cdots, x_r]$ and $I = (x_1, \cdots, x_r)$. The transitivity triangle for $\mathbb{Z} \to A \to A/I$ is $\mathbb{L}_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A A/I \to \mathbb{L}_{(A/I)/\mathbb{Z}} \to \mathbb{L}_{(A/I)/A}$. So $\mathbb{L}_{(A/I)/A} \simeq \Omega^1_{A/\mathbb{Z}} \otimes_A A/I[1] \simeq I/I^2[1]$. For the general case, we choose a regular sequence $f_1, \cdots, f_r$ generating I, which induces $A/I \simeq \mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}[x_1,\cdots,x_r],(x_i \mapsto f_i)} A$. Base change for the cotangent complex then implies that $\mathbb{L}_{(A/I)/A} \simeq \mathbb{L}_{\mathbb{Z}/\mathbb{Z}[x_1,\cdots,x_r]} \otimes^{\mathbb{L}}_{\mathbb{Z}[x_1,\cdots,x_r]} A \simeq I/I^2[1]$. For more details, see [Bha17a, Example 3.1.3].
4. Let R be a perfect ring of characteristic p, then $\mathbb{L}_{\widehat{W(R)/\mathbb{Z}_p}}$, the derived p-adic completion of $\mathbb{L}_{W(R)/\mathbb{Z}_p}$, vanishes. This results from Lemma 2.4.6, Lemma 2.4.8 and the vanishing of the base change $\mathbb{L}_{W(R)/\mathbb{Z}_p} \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p \simeq \mathbb{L}_{R/\mathbb{F}_p} \simeq 0$.

As an application of the cotangent complex, one can give a description of the Breuil-Kisin twist as defined in [BMS18, Definition 8.2].

Definition 4.1.6 (Breuil-Kisin twist)**.** *Define $\mathcal{O}_C\{1\} := T_p(\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p})$, the p-adic Tate module of $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$. For any $\mathcal{O}_C$-module M, define the i-th Breuil-Kisin twist of M as*

$$M\{i\} := M \otimes_{\mathcal{O}_C} \mathcal{O}_C\{1\}^{\otimes i}.$$

We remark that $\mathcal{O}_C\{1\}$ is a free $\mathcal{O}_C$-module of rank 1.

Now we will explain how to use cotangent complex to describe Breuil-Kisin twist. Let $\overline{W(k)}$ be the integral closure of $W(k)$ in $\mathcal{O}_C$.

Theorem 4.1.7 ([Bei12] Section 1.3)**.** *The map $W(k) \to \overline{W(k)}$ has a discrete cotangent complex, i.e., $\mathbb{L}_{\overline{W(k)}/W(k)} \simeq \Omega^1_{\overline{W(k)}/W(k)}$.*

Note that by base change of cotangent complex, we have

$$\mathbb{L}_{\widehat{\overline{W(k)}}/W(k)} \otimes^{\mathbb{L}}_{W(k)} k \simeq \mathbb{L}_{(\overline{W(k)}/p)/k} \simeq \mathbb{L}_{\overline{W(k)}/W(k)} \otimes^{\mathbb{L}}_{W(k)} k.$$

Then by Lemma 2.4.6 and Lemma 2.4.8 , there is an isomorphism

$$\mathbb{L}_{\widehat{\widehat{\overline{W(k)}}}/W(k)} \simeq \mathbb{L}_{\widehat{\overline{W(k)}}/W(k)}.$$

On the other hand by Theorem 4.1.7, we have

$$\mathbb{L}_{\widehat{\overline{W(k)}/W(k)}} \simeq \Omega^1_{\widehat{\overline{W(k)}/W(k)}} = \mathrm{Rlim}(\Omega^1_{\overline{W(k)}/W(k)} \otimes^{\mathbb{L}}_{W(k)} W(k)/p^n)$$

and since every element of $\overline{W(k)}$ admits a p^n-th root for all n, the map $\Omega^1_{\overline{W(k)}/W(k)} \xrightarrow{\times p^n} \Omega^1_{\overline{W(k)}/W(k)}$ is surjective for all n. So we have

$$\Omega^1_{\overline{W(k)}/W(k)} \otimes^{\mathbb{L}}_{W(k)} W(k)/p^n \simeq \Omega^1_{\overline{W(k)}/W(k)}[p^n][1].$$

As $\Omega^1_{\overline{W(k)}/W(k)}[p^{n+1}] \xrightarrow{\times p} \Omega^1_{\overline{W(k)}/W(k)}[p^n]$ is surjective for all n, the inverse system $(\Omega^1_{\overline{W(k)}/W(k)}[p^n])$ is Mittag-Leffler. Hence by [Sta19, Lemma 091D]

$$\mathrm{Rlim}(\Omega^1_{\overline{W(k)}/W(k)} \otimes^{\mathbb{L}}_{W(k)} W(k)/p^n) \simeq \varprojlim_n \Omega^1_{\overline{W(k)}/W(k)}[p^n][1] = T_p(\Omega^1_{\overline{W(k)}/W(k)})[1].$$

Finally we have

$$\mathbb{L}_{\widehat{\widehat{\overline{W(k)}}/W(k)}} \simeq T_p(\Omega^1_{\overline{W(k)}/W(k)})[1] = T_p(\Omega^1_{\widehat{\overline{W(k)}}/W(k)})[1].$$

Since $\Omega^1_{W(k)/\mathbb{Z}_p} = 0$, we have $\Omega^1_{\widehat{\overline{W(k)}}/W(k)} \cong \Omega^1_{\widehat{\overline{W(k)}}/\mathbb{Z}_p}$ by considering the exact sequence associated to $\mathbb{Z}_p \to W(k) \to \widehat{\overline{W(k)}}$. Similarly as $\mathbb{L}_{\widehat{W(k)/\mathbb{Z}_p}}$ vanishes by Remark 4.1.5.4, we have $\mathbb{L}_{\widehat{\overline{W(k)}/W(k)}} \simeq \mathbb{L}_{\widehat{\overline{W(k)}/\mathbb{Z}_p}}$ by Proposition 4.1.4.2 and the exacteness of derived completion (Remark 2.4.5). By the derived p-adic completion of the transitivity triangle associated to $\mathbb{Z}_p \to \widehat{\overline{W(k)}} \to \mathcal{O}_C$, we have $\mathbb{L}_{\widehat{\mathcal{O}_C/\mathbb{Z}_p}} \simeq \mathbb{L}_{\widehat{\widehat{\overline{W(k)}}/\mathbb{Z}_p}} \otimes^{\mathbb{L}}_{\widehat{\overline{W(k)}}} \mathcal{O}_C$ as $\mathbb{L}_{\widehat{\mathcal{O}_C/\widehat{\overline{W(k)}}}} = 0$ (which follows from derived Nakayama lemma, Proposition 4.1.4.3 and Remark 4.1.5.2).

So finally, we get $\mathbb{L}_{\widehat{\mathcal{O}_C/\mathbb{Z}_p}} \simeq T_p(\Omega^1_{\widehat{\overline{W(k)}}/\mathbb{Z}_p}[1]) \otimes_{\widehat{\overline{W(k)}}} \mathcal{O}_C \simeq T_p(\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p})[1]$. For passing from $\Omega^1_{\widehat{\overline{W(k)}}/\mathbb{Z}_p}$ to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$, we refer to [GR03, 6.5.20] [1].

4.2 Decomposition of Hodge-Tate cohomology groups

In this section, we explain how to relate Hodge-Tate cohomology to Hodge cohomology. In fact, we can show that the complex of sheaves $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal in the unramified case.

Theorem 4.2.1. *For any proper smooth formal scheme $\mathfrak{X}$ over $W(k)$ and $\bar{\mathfrak{X}} = \mathfrak{X} \times_{\mathrm{Spf}(W(k))}$*

1. One can conclude from [GR03, 6.5.20] that there is a short exact sequence $0 \to \Omega^1_{\widehat{\overline{W(k)}}/\mathbb{Z}_p} \otimes_{\widehat{\overline{W(k)}}} \mathcal{O}_C \to \Omega^1_{\mathcal{O}_C/\mathbb{Z}_p} \to \Omega^1_{\mathcal{O}_C/\widehat{\overline{W(k)}}} \to 0$ and $\Omega^1_{\mathcal{O}_C/\widehat{\overline{W(k)}}}$ is p-torsion-free. This implies that $\Omega^1_{\widehat{\overline{W(k)}}/\mathbb{Z}_p}[p^n] \otimes_{\widehat{\overline{W(k)}}} \mathcal{O}_C \cong \Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n]$ for all n.

$\mathrm{Spf}(\mathcal{O}_C)$, *the complex of sheaves* $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ *is formal, i.e. there is an isomorphism*

$$\gamma : \bigoplus_{i=0}^{p-1} \Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \simeq \tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}},$$

where $\Omega^i_{\bar{\mathfrak{X}}} := \varprojlim \Omega^i_{(\bar{\mathfrak{X}}/p^n)/(\mathcal{O}_C/p^n)}$ *is the* $\mathcal{O}_{\bar{\mathfrak{X}}}$*-module of continuous differentials and* $\Omega^i_{\bar{\mathfrak{X}}}\{-i\}$ *is the Breuil-Kisin twist of* $\Omega^i_{\bar{\mathfrak{X}}}$.

Proof. We proceed by first showing that $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal and then constructing the general isomorphism in the statement. In this proof, $\mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}$ and $\mathbb{L}_{\bar{\mathfrak{X}}/W(k)}$ always mean the derived p-adic complete cotangent complex.

By [BMS18, Proposition 8.15], there is an isomorphism $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}\{-1\}[-1]$. Considering the sequence of sheaves $\mathbb{Z}_p \to W(k) \to \mathcal{O}_{\bar{\mathfrak{X}}}$, there is an associated distinguished triangle

$$\widehat{\mathbb{L}_{W(k)/\mathbb{Z}_p}} \otimes^{\mathbb{L}}_{W(k)} \mathcal{O}_{\bar{\mathfrak{X}}} \to \mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p} \to \mathbb{L}_{\bar{\mathfrak{X}}/W(k)}.$$

By Remark 4.1.5, we know that $\widehat{\mathbb{L}_{W(k)/\mathbb{Z}_p}}$ vanishes. Therefore, we have

$$\mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}\{-1\}[-1] \simeq \mathbb{L}_{\bar{\mathfrak{X}}/W(k)}\{-1\}[-1].$$

For any affine open $\mathrm{Spf}(R) \subset \mathfrak{X}$, write $\bar{R}$ for the base change $R \otimes_{W(k)} \mathcal{O}_C$ and $\widehat{\bar{R}}$ for its p-adic completion. Then we have $\widehat{\mathbb{L}_{\widehat{\bar{R}}/W(k)}} \simeq \widehat{\mathbb{L}_{\bar{R}/W(k)}}$ which follows from the derived Nakayama lemma, Remark 2.4.5 and Lemma 2.4.8.

By the Künneth property of cotangent complex in Proposition 4.1.4, we have

$$\mathbb{L}_{\bar{R}/W(k)} \simeq (\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes^{\mathbb{L}}_{W(k)} R) \oplus (\mathbb{L}_{R/W(k)} \otimes^{\mathbb{L}}_{W(k)} \mathcal{O}_C).$$

Applying the derived p-adic completion functor (which is exact by Remark 2.4.5), we have

$$\widehat{\mathbb{L}_{R\otimes_{W(k)}\mathcal{O}_C/W(k)}} \simeq (\widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes^{\mathbb{L}}_{W(k)} R}) \oplus (\widehat{\mathbb{L}_{R/W(k)} \otimes^{\mathbb{L}}_{W(k)} \mathcal{O}_C}).$$

On one hand, by Corollary 2.4.9, we get

$$\widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes_{W(k)} R} \simeq \widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes_{W(k)} R} \simeq \widehat{\bar{R}\{1\}[1]}$$

where the last isomorphism follows from the discussion at the end of Section 4.1.

As $\widehat{\bar{R}}$ coincides with the derived p-adic completion of $\bar{R}$ (cf. [Sta19, Example 0BKG]), we have $\widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes_{W(k)} R} \simeq \widehat{\bar{R}}\{1\}[1]$. On the other hand, by the base change property in Proposition 4.1.4, we get $\mathbb{L}_{R/W(k)} \otimes_{W(k)} \mathcal{O}_C \simeq \mathbb{L}_{\bar{R}/\mathcal{O}_C}$. The derived p-adic completion $\widehat{\mathbb{L}_{\bar{R}/\mathcal{O}_C}}$ is isomorphic to $\varprojlim_n \Omega^1_{(\bar{R}/p^n)/(\mathcal{O}_C/p^n)}$. Indeed as $\bar{R}/p^n$ is a smooth $\mathcal{O}_C/p^n$-algebra

for all n, we have

$$\widehat{\mathbb{L}_{\bar{R}/\mathcal{O}_C}} \simeq \mathrm{Rlim}(\mathbb{L}_{\bar{R}/\mathcal{O}_C} \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{Z}_p/p^n) \simeq \mathrm{Rlim}(\mathbb{L}_{(\bar{R}/p^n)/(\mathcal{O}_C/p^n)}) \simeq \varprojlim_n \Omega^1_{(\bar{R}/p^n)/(\mathcal{O}_C/p^n)}.$$

So finally there is an isomorphism

$$\mathbb{L}_{\bar{\mathfrak{X}}/W(k)} \simeq \mathcal{O}_{\bar{\mathfrak{X}}}\{1\}[1] \oplus \Omega^1_{\bar{\mathfrak{X}}}$$

So we get a decomposition $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \mathcal{O}_{\bar{\mathfrak{X}}} \oplus \Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1]$. In particular, we have a map $\gamma_1 : \Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1] \to \widetilde{\Omega}_{\bar{\mathfrak{X}}}$ which gives the Hodge-Tate isomorphism $C^{-1} : \Omega^1_{\bar{\mathfrak{X}}/\mathcal{O}_C}\{-1\} \to \mathcal{H}^1(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$ (cf. [BMS18, Theorem 8.3]).

Now we consider the map for any $i \leq p-1$ given by

$$(\Omega^1_{\bar{\mathfrak{X}}})^{\otimes i} \to \Omega^i_{\bar{\mathfrak{X}}}, \quad \omega_1 \otimes \cdots \otimes \omega_i \mapsto \omega_1 \wedge \cdots \wedge \omega_i$$

It has an anti-symmetrization section a as shown in [DI87], given by

$$a(\omega_1 \wedge \cdots \wedge \omega_i) = (1/i!) \sum_{s \in Sym_i} \mathrm{sgn}(s)\omega_{s(1)} \otimes \cdots \otimes \omega_{s(i)}.$$

Then we define γ_i as the composition

$$\Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \xrightarrow{a} (\Omega^1_{\bar{\mathfrak{X}}}\{-1\})^{\otimes i}[-i] \simeq (\Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1])^{\otimes^{\mathbb{L}} i} \xrightarrow{\gamma_1^{\otimes^{\mathbb{L}} i}} (\widetilde{\Omega}_{\bar{\mathfrak{X}}})^{\otimes^{\mathbb{L}} i} \xrightarrow{\text{multi}} \widetilde{\Omega}_{\bar{\mathfrak{X}}}$$

where " $\otimes^{\mathbb{L}} i$" means i-fold derived tensor product. Note that $\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is a commutative $\mathcal{O}_{\overline{\mathfrak{X}}}$-algebra object in $D(\mathcal{O}_{\overline{\mathfrak{X}}})$ (see Remark 1.4.3). By applying $\mathcal{H}^i$, we have

$$\Omega^i_{\bar{\mathfrak{X}}}\{-i\} \xrightarrow{a} \mathcal{H}^i((\Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1])^{\otimes^{\mathbb{L}} i}) \cong (\mathcal{H}^1(\Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1]))^{\otimes i} \xrightarrow{\gamma_1^{\otimes^{\mathbb{L}} i}} (\mathcal{H}^1(\widetilde{\Omega}_{\bar{\mathfrak{X}}}))^{\otimes i} \to \mathcal{H}^i((\widetilde{\Omega}_{\bar{\mathfrak{X}}})^{\otimes^{\mathbb{L}} i}) \xrightarrow{\text{multi}} \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

Since the Hodge-Tate isomorphism is compatible with multiplication (cf. [BMS18, Corollary 8.13]), this composition is exactly the Hodge-Tate isomorphism $C^{-1} : \Omega^i_{\bar{\mathfrak{X}}}\{-i\} \simeq \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$. So we have the map $\gamma = \bigoplus_{i=0}^{p-1} \gamma_i : \bigoplus_{i=0}^{p-1} \Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \to \widetilde{\Omega}_{\bar{\mathfrak{X}}}$. Taking $(p-1)$-th truncation, we get the desired isomorphism $\gamma = \bigoplus_{i=0}^{p-1} \gamma_i : \bigoplus_{i=0}^{p-1} \Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \simeq \tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$. □

Remark 4.2.2. Note that the key input in the proof above is the Hodge-Tate isomorphism $C^{-1} : \Omega^i_{\bar{\mathfrak{X}}}\{-i\} \to \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$. In general, there is a Hodge-Tate isomorphism for any bounded prism (A, I) (cf. [BS19, Theorem 4.10]) and also a generalization of the isomorphism $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}\{-1\}[-1]$.

The map $\mathcal{O}_{\bar{\mathfrak{X}}} \to \tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ splits as an $\mathcal{O}_{\bar{\mathfrak{X}}}$-module map if and only if $\bar{\mathfrak{X}}$ lifts to $A_{\mathrm{inf}}/\widetilde{\xi}^2$ (cf. [BMS18, Remark 8.4]). In the ramified case, this seems to be hardly satisfied due to the

non-vanishing of the cotangent complex $\mathbb{L}_{\mathcal{O}_K/W(k)}$. Note that $H^0(\mathbb{L}_{\mathcal{O}_K/W(k)}) \simeq \Omega^1_{\mathcal{O}_K/W(k)}$ is generated by one element (cf. [Ser13, Chapter III, Proposition 14]).

Corollary 4.2.3. *There is a natural decomposition for any $n \leq p-1$,*

$$H^n_{\mathrm{HT}}(\bar{\mathfrak{X}}) = H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}\{-i\}).$$

4.3 Hodge-to-de Rham spectral sequence

In this section, we study the Hodge-to-de Rham spectral sequence and finish the proof of the integral comparison theorem in the unramified case. More precisely, we will prove the converse to Theorem 1.4.7 by analyzing the length of the torsion part of de Rham cohomology groups and that of p-adic étale cohomology groups.

Note that we have the Hodge-to-de Rham spectral sequence

$$E_1^{i,j} = H^j(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}) \Longrightarrow H^{i+j}(\bar{\mathfrak{X}}, \Omega^\bullet_{\bar{\mathfrak{X}}}) = H^{i+j}_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$$

As $\bar{\mathfrak{X}} = \mathfrak{X} \otimes_{W(k)} \mathcal{O}_C$, this spectral sequence can be seen as the flat base change to $\mathcal{O}_C$ of the Hodge-to-de Rham spectral sequence of $\mathfrak{X}$ over $W(k)$. This tells us $E_\infty^{i,j}$ is a finitely presented $\mathcal{O}_C$-module (note that $E_\infty^{i,j}$ is also a subquotient of $H^j(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})$).

For any integers i and n such that $0 \leq i \leq n$, we have the abutment filtration

$$0 = F^{n+1} \subset F^n \subset \cdots \subset F^0 = H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$$

and the short exact sequences

$$0 \to F^{i+1} \to F^i \to E_\infty^{i,n-i} \to 0.$$

Now we consider the normalized length $l_{\mathcal{O}_C}$ for finitely presented torsion $\mathcal{O}_C$-modules. Recall that this length behaves additively under short exact sequences and $l_{\mathcal{O}_C}(\mathcal{O}_C/p) = 1$. For any finitely presented $\mathcal{O}_C$-module M, one can deduce from Lemma 2.1.3 that M_{tor} is also a finitely presented $\mathcal{O}_C$-module and so is M_{tor}/p^m for any $m > 0$. Then we have the following lemma :

Lemma 4.3.1. *For any short exact sequence of finitely presented $\mathcal{O}_C$-modules*

$$0 \to A \to B \to C \to 0$$

we have $l_{\mathcal{O}_C}(B_{\mathrm{tor}}) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}) + l_{\mathcal{O}_C}(C_{\mathrm{tor}})$ and $l_{\mathcal{O}_C}(B_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}(C_{\mathrm{tor}}/p^m)$ for any $m > 0$.

Proof. For the first statement, it is easy to see that $M = B_{\mathrm{tor}}/A_{\mathrm{tor}}$ is a submodule of C_{tor}, so we have $l_{\mathcal{O}_C}(M) = l_{\mathcal{O}_C}(B_{\mathrm{tor}}) - l_{\mathcal{O}_C}(A_{\mathrm{tor}}) \leq l_{\mathcal{O}_C}(C_{\mathrm{tor}})$ by the additivity of the length. For the second one, we have an exact sequence

$$M[p^m] \to A_{\mathrm{tor}}/p^m \to B_{\mathrm{tor}}/p^m \to M/p^m \to 0$$

So we get $l_{\mathcal{O}_C}(B_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}(M/p^m)$. Then we need to prove $l_{\mathcal{O}_C}(M/p^m) \leq l_{\mathcal{O}_C}(C_{\mathrm{tor}}/p^m)$. More generally, given two finitely presented torsion $\mathcal{O}_C$ modules $N_1 \subset N_2$, there is an exact sequence

$$N[p^m] \to N_1/p^m \to N_2/p^m \to N/p^m \to 0$$

where $N = N_2/N_1$. Note that $l_{\mathcal{O}_C}(N[p^m]) = l_{\mathcal{O}_C}(N/p^m)$. In fact, this follows from the exact sequence

$$0 \to N[p^m] \to N \xrightarrow{p^m} N \to N/p^m \to 0$$

Hence $l_{\mathcal{O}_C}(N_2/p^m) \geq l_{\mathcal{O}_C}(N/p^m) + l_{\mathcal{O}_C}(N_1/p^m) - l_{\mathcal{O}_C}(N[p^m]) = l_{\mathcal{O}_C}(N_1/p^m)$. So finally we get $l_{\mathcal{O}_C}(B_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}(C_{\mathrm{tor}}/p^m)$. □

Corollary 4.3.2. *For any integers i and n such that $0 \leq i \leq n$ and any positive integer m, we have $l_{\mathcal{O}_C}(F^i_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(F^{i+1}_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m)$. In particular, $l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^n l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m)$.*

Recall that the rational Hodge-to-de Rham spectral sequence degenerates at E_1 page :

Theorem 4.3.3 ([Sch13, Corollary 1.8])**.** *For any proper smooth rigid analytic space X over C, the Hodge-to-de Rham spectral sequence*

$$E_1^{i,j} = H^j(X, \Omega^i_X) \Longrightarrow H^{i+j}_{\mathrm{dR}}(X/C)$$

degenerates at E_1. Moreover, for all $i \geq 0$,

$$\sum_{j=0}^{i} \dim_C H^{i-j}(X, \Omega^j_X) = \dim_C H^i_{\mathrm{dR}}(X/C) = \dim_{\mathbb{Q}_p} H^i_{\text{ét}}(X, \mathbb{Q}_p).$$

As a consequence, we have the following lemma :

Lemma 4.3.4. *For any $m > 0$, we have*

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

Proof. Theorem 4.3.3 tells us that the integral Hodge-to-de Rham spectral sequence degenerates at E_1 after inverting p. This means that the coboundaries $B_\infty^{i,n-i}$ must be a finitely

presented torsion $\mathcal{O}_C$-module. Consider the short exact sequence

$$0 \to B_\infty^{i,n-i} \to Z_\infty^{i,n-i} \to E_\infty^{i,n-i} \to 0.$$

For any $x \in {E_\infty^{i,n-i}}_{\mathrm{tor}}$, there exists $\widehat{x} \in Z_\infty^{i,n-i}$ whose image in $E_\infty^{i,n-i}$ is x. As ${E_\infty^{i,n-i}}_{\mathrm{tor}}$ is killed by p^N for some large enough N, we can see that $p^N\widehat{x}$ is in $B_\infty^{i,n-i} \subset {Z_\infty^{i,n-i}}_{\mathrm{tor}}$. So we have another short exact sequence

$$0 \to B_\infty^{i,n-i} \to {Z_\infty^{i,n-i}}_{\mathrm{tor}} \to {E_\infty^{i,n-i}}_{\mathrm{tor}} \to 0.$$

Then by the additivity of the length, we get that

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m),$$

and

$$l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}[p^m]) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}[p^m]) = l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m)$$

where the middle inequality results from the inclusion ${Z_\infty^{i,n-i}}_{\mathrm{tor}}[p^m] \hookrightarrow E_1^{i,n-i}[p^m] = H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})[p^m]$.

So we have $l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m)$. □

Now we prove the converse to Theorem 1.4.7.

Theorem 4.3.5. *For any positive integer m and any integer n such that $0 \leq n < p-1$, we have*

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m).$$

Proof. By Theorem 3.2.5 and Theorem 4.2.3, we have

$$H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \cong H^n_{\mathrm{HT}}(\bar{\mathfrak{X}}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}).$$

This implies that

$$\sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m)$$

Moreover, by Corollary 4.3.2 and Lemma 4.3.4, we have

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

So we get that

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^n_{\text{ét}}(X,\mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m)$$

□

Theorem 4.3.6. *For any $n < p-1$, there is an isomorphism of $W(k)$-modules*

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \cong H^n_{\text{ét}}(X,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k).$$

Proof. We first prove that there is an isomorphism of $\mathcal{O}_C$-modules

$$H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^n_{\text{ét}}(X,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Note that Theorem 1.4.7 tells us that for any positive integer m,

$$l_{\mathcal{O}_C}(H^n_{\text{ét}}(X,\mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m) \leq l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m)$$

So they must be equal by Theorem 4.3.5. This means that $H^n_{\text{ét}}(X,\mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C \cong H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}$ by Lemma 2.1.4. Furthermore by [BMS18, Theorem 1.1], the $\mathcal{O}_C$-modules $H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$ and $H^n_{\text{ét}}(X,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$ have the same rank. So we have $H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^n_{\text{ét}}(X,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$.

On the other hand, there is an isomorphism between de Rham cohomology and crystalline cohomology in the unramified case (cf. [Ber06])

$$H^n_{\mathrm{dR}}(\mathfrak{X}/W(k)) \cong H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

We also have

$$H^n_{\mathrm{dR}}(\mathfrak{X}/W(k)) \otimes_{W(k)} \mathcal{O}_C \cong H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$$

by base change of de Rham cohomology. So finally we get the isomorphism of $W(k)$-modules

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \cong H^n_{\text{ét}}(X,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k).$$

□

4.4 Degeneration of the Hodge-to-de Rham spectral sequence

In this section, we assume $d = \dim\mathfrak{X} < p-1$. We will improve Theorem 4.3.6 by considering all cohomological degrees and study the degeneration of the Hodge-to-de Rham spectral sequence. These will follow from improvements of Theorem 3.2.5 and Corollary 4.2.3.

We begin with an improvement of Corollary 4.2.3.

Lemma 4.4.1. *When $d = \dim\mathfrak{X} < p-1$, we have*

$$H^n_{\mathrm{HT}}(\bar{\mathfrak{X}}) = H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}\{-i\}).$$

for all n.

Proof. Recall the Hodge-Tate isomorphism : $H^i(\widetilde{\Omega}_{\overline{\mathfrak{X}}}) \cong \Omega^i_{\overline{\mathfrak{X}}}$ (cf. [BMS18, Theorem 8.3]). When $i \geq p-1 > d$, we have $\Omega^i_{\overline{\mathfrak{X}}} = 0$. This implies $\tau^{\leq p-2}\widetilde{\Omega}_{\overline{\mathfrak{X}}} \simeq \widetilde{\Omega}_{\overline{\mathfrak{X}}}$. In particular, the whole complex $\widetilde{\Omega}_{\overline{\mathfrak{X}}}$ is formal by Theorem 4.2.1, from which this lemma follows.

□

Next we think about the comparison between Hodge-Tate cohomology and p-adic étale cohomology. Recall that we have the following two maps (see Page 45)

$$f : \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}} \to \tau^{\leq d} R\nu_* \widehat{\mathcal{O}}^+_X$$

$$g : \tau^{\leq d} R\nu_* \widehat{\mathcal{O}}^+_X \to \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}}$$

whose composition in either direction is $(\zeta_p - 1)^d$.

We claim that $R\nu_*\widehat{\mathcal{O}}^+_X$ is almost supported in degrees $\leq d$, i.e. there is an almost isomorphism $\tau^{\leq d} R\nu_*\widehat{\mathcal{O}}^+_X \simeq R\nu_*\widehat{\mathcal{O}}^+_X$. We will check this locally.

Recall that an $\mathcal{O}_C$-algebra R is called formally smooth (as in [BMS18]) if it is a p-adically complete flat $\mathcal{O}_C$-algebra such that R/p is a smooth $\mathcal{O}_C/p$-algebra. And a formally smooth $\mathcal{O}_C$-algebra R is called small (cf. [BMS18, Definition 8.5]) if there is an étale map

$$\square : \mathrm{Spf} R \to \mathrm{Spf}\mathcal{O}_C\langle T_1^{\pm 1}, \cdots, T_d^{\pm 1}\rangle.$$

We call such étale map a framing. Given a framing, we can define

$$R_\infty := R\widehat{\otimes}_{\mathcal{O}_C\langle T_1^{\pm 1}, \cdots, T_d^{\pm 1}\rangle}\mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \cdots, T_d^{\pm 1/p^\infty}\rangle$$

which is an integral perfectoid ring. And there is an action of $\Gamma = \mathbb{Z}_p(1)^d$ on it. More precisely, choose a compatible system (ζ_{p^m}) of p-power roots of unity and let γ_i, $i = 1, \cdots, d$ be generators of Γ. Then γ_i acts by sending T_i^{1/p^m} to $\zeta_{p^m} T_i^{1/p^m}$ and sending T_j^{1/p^m} to T_j^{1/p^m} for $j \neq i$.

By Faltings' almost purity theorem (cf. [Fal88, Chapter 1, Section 3 and 4]) and [Sch13, Proposition 3.5, Proposition 3.7, Corollary 6.6], there is an almost isomorphism of complexes of $\mathcal{O}_C$-modules

$$R\Gamma(\Gamma, R_\infty) \to R\Gamma(Y_{\text{proét}}, \widehat{\mathcal{O}}^+_Y),$$

where $Y = \mathrm{Spa}(R[1/p], R)$. Moreover the continuous group cohomology on the left hand side can be calculated by the Koszul complex $K_{R_\infty}(\gamma_1 - 1, \cdots, \gamma_d - 1)$ by [BMS18, Lemma 7.3], which can be defined as

$$K_{R_\infty}(\gamma_1 - 1, \cdots, \gamma_d - 1) = R_\infty \otimes_{\mathbb{Z}[\gamma_1, \cdots, \gamma_d]} (\bigotimes_{i=1}^{d} (\mathbb{Z}[\gamma_1, \cdots, \gamma_d] \xrightarrow{\gamma_i - 1} \mathbb{Z}[\gamma_1, \cdots, \gamma_d])).$$

This complex sits in non-negative cohomological degrees $[0, d]$. On the other hand, since $\overline{\mathfrak{X}}$ is a proper smooth formal scheme over $\mathcal{O}_C$, there exists a basis of small affine opens (cf. [Ked03, Theorem 2], [Bha18, Lemma 4.9]). So when $i > d$, we get that $R^i\nu_*\widehat{\mathcal{O}}_X^+$ is almost zero.

So now we have an almost isomorphism : $\tau^{\leq d} R\nu_*\widehat{\mathcal{O}}_X^+ \to R\nu_*\widehat{\mathcal{O}}_X^+$. Taking cohomology, we then get an almost isomorphism : $R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} R\nu_*\widehat{\mathcal{O}}_X^+) \to R\Gamma(\overline{\mathfrak{X}}, R\nu_*\widehat{\mathcal{O}}_X^+)$. Again by Theorem 3.2.2, we get two maps in almost derived category $D(\mathcal{O}_C)^a$:

$$f : (R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}}))^a \to (R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a$$

$$g : (R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a \to (R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}}))^a$$

whose composition in either direction is $(\zeta_p - 1)^d$. Since both sides are perfect complexes of $\mathcal{O}_C$-modules, we get two maps in the derived category $D(\mathcal{O}_C)$:

$$f : R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}}) \to R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$g : R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^d$.

Now as $\tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}} \simeq \widetilde{\Omega}_{\overline{\mathfrak{X}}}$, we have $R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}}) \simeq R\Gamma(\overline{\mathfrak{X}}, \widetilde{\Omega}_{\overline{\mathfrak{X}}}) = R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}})$. So we get two maps

$$f : R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}}) \to R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$g : R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}})$$

whose composition in either direction is $(\zeta_p - 1)^d$.

Theorem 4.4.2. *There is an isomorphism of $\mathcal{O}_C$-modules for all n*

$$H^n_{\mathrm{HT}}(\overline{\mathfrak{X}}) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Proof. This follows from Lemma 2.2.4. □

Theorem 4.4.3. *Assume $d = \dim\mathfrak{X} < p - 1$. Then there is an isomorphism of $W(k)$-modules for all n*

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k).$$

Proof. Note that if Theorem 4.3.5 is true for all n, then Theorem 4.3.6 is true for all n. And if Theorem 3.2.5 and Corollary 4.2.3 are true for all cohomological degrees, then Theorem 4.3.5 is true for all cohomological degrees. So this theorem follows from Theorem 4.4.1 and Theorem 4.4.2.

□

Corollary 4.4.4. *If $d = \dim(\mathfrak{X}) < p-1$, the coboundaries $B_\infty^{i,n-i}$ vanish for all n. In particular the Hodge-to-de Rham spectral sequence degenerates at E_1-page.*

Proof. By Theorem 4.4.1 and Theorem 4.4.2, we see that

$$\sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m)$$

is true for all n.

Theorem 4.4.3 shows that for all n we have

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m).$$

So we conclude that

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) = \sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m)$$

holds for all n.

As we have seen in the proof of Lemma 4.3.4, there are inequalities for all n

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

Also by using the same argument as in the proof of Theorem 4.3.5, we have

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

holds for all n. But these inequalities are in fact equalities. This means that

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

In other words, the coboundaries $B_\infty^{i,n-i}$ vanish as we have $l_{\mathcal{O}_C}(B_\infty^{i,n-i}) = l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}) - l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}) = 0$. So the Hodge-to-de Rham spectral sequence degenerates at E_1-page.

□

Remark 4.4.5. We collect some other results about the degeneration of the (integral) Hodge-to-de Rham spectral sequence.

1. In [FM87, Corollary 2.7], Fontaine and Messing have proved that for any proper smooth (formal) schem $\mathfrak{X}$ whose special fiber has dimension strictly less than p, the Hodge-to-de Rham spectral sequence degenerates at E_1-page. Their proof makes use of the syntomic cohomology.
2. For any projective smooth scheme X over $W(k)$ where k is a perfect field of characteristic p, Kazuya Kato has proved that if $\dim(X) \leq p$, the Hodge-to-de Rham spectral sequence degenerates at E_1-page and the de Rham cohomology groups are Fontaine-Laffaille modules (cf. [K$^+$87, Proposition 2.5]).
3. For any proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathfrak{S}$ be $W(k)[[u]]$ and E be an Eisenstein polynomial for a uniformizer π of $\mathcal{O}_K$. Shizhang Li has proved that if $\mathfrak{X}$ can be lifted to $\mathfrak{S}/(E^2)$ and $\dim(\mathfrak{X}) \cdot e < p-1$, then the Hodge-to-de Rham spectral sequence is split degenerate (cf. [Li20, Theorem 1.1]). His proof uses Theorem 0.0.6.

Chapitre 5

The ramified case : comparison theorem

In this chapter, let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, which is the ring of integers of a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. We will get some properties about the torsion in the Breuil-Kisin cohomology groups $H^i_{\mathfrak{S}}(\mathfrak{X})$ when $ie < p-1$ and obtain an integral comparison theorem comparing the p-adic étale cohomology groups and the crystalline cohomology groups.

5.1 Torsion in Breuil-Kisin cohomology groups

We first fix a uniformaizer π in $\mathcal{O}_K$ and choose an Eisenstein polynomial E for π. Note that the ring $\mathfrak{S} = W(k)[[u]]$ is a two-dimensional regular local ring. The structure of $\mathfrak{S}$-modules is subtle in general (see Remark 5.2.2). In particular, it is difficult to study the u-torsion. But in our case, it turns out to be simpler.

Let C be a complete algebraically closed nonarchimedean extension of K and $\mathcal{O}_C$ be its ring of integers. Let $\overline{\mathfrak{X}}$ be $\mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Recall that we can define $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$ as in Definition 1.4.1. We start by studying the A_{inf}-cohomology groups of $\overline{\mathfrak{X}}$.

Lemma 5.1.1. *For any i such that $ie < p-1$, the cohomology group $H^{i+1}_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}})$ is $\widetilde{\xi}$-torsion-free.*

Proof. We assume that C is spherically complete. As in the proof of Theorem 3.2.5, we see that the spherical completion of C exists and is still complete and algebraically closed. Moreover since $R\Gamma_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}}) \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\alpha} A_{\mathrm{inf}}$ where $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ is the faithfully flat map taking (E) to $(\widetilde{\xi})$ (for the definition of the map α, see Section 1.5), we have $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \cong H^{i+1}_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\alpha} A_{\mathrm{inf}}$, in particular $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is $\widetilde{\xi}$-torsion-free if and only if $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$

is E-torsion-free as $(\alpha(E)) = (\widetilde{\xi})$. So it does not matter whether C is spherically complete or not.

As in Chapter 3, we apply Lemma 3.2.1 to the complex of sheaves of A_{inf}-modules $\tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X}$ and the element $\mu \in A_{\mathrm{inf}}$. Precisely, in the category $D^{[0,i]}(\overline{\mathfrak{X}}, A_{\mathrm{inf}})$, we get two natural maps

$$f : \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \to L\eta_\mu \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \simeq \tau^{\leq i} A\Omega_{\overline{\mathfrak{X}}}$$

$$g : \tau^{\leq i} A\Omega_{\overline{\mathfrak{X}}} \simeq L\eta_\mu \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \to \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X}$$

whose composition in either direction is μ^i.

We consider the the complex of sheaves $\tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+$ as in the category $D(\bar{\mathfrak{X}}, A_{\mathrm{inf}})$ via the map $A_{\mathrm{inf}} \xrightarrow{\widetilde{\theta}} \mathcal{O}_C \to \mathcal{O}_{\bar{\mathfrak{X}}}$. Moreover it is in the category $D^{[0,i]}(\bar{\mathfrak{X}}, A_{\mathrm{inf}})$.

There is a map $\tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf,X}} \to \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+$ induced by $\widetilde{\theta} : \mathbb{A}_{\mathrm{inf,X}} \to \widehat{\mathcal{O}}_X^+$. So we can get a commutative diagram

$$\begin{tikzcd}
L\eta_\mu \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \arrow[r, "s_1"] \arrow[d, shift left, "f_1"] & L\eta_\mu \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+ \arrow[d, shift left, "f_2"] \\
\tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \arrow[r, "s_2"] \arrow[u, shift left, "g_1"] & \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+ \arrow[u, shift left, "g_2"]
\end{tikzcd}$$

where the composition of f_j with g_j in either direction is μ^i for $j = 1, 2$. Note that $L\eta_{\zeta_p - 1} \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+$ is isomorphic to $L\eta_\mu \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+$ in $D(\bar{\mathfrak{X}}, A_{\mathrm{inf}})$.

Recall that $\tau^{\leq i} R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is a perfect complex of A_{inf}-modules according to Lemma 3.2.4. Then by the second almost isomorphism in Theorem 3.2.2 and Lemma 3.1.6, we can get two maps

$$f : \tau^{\leq i} R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \to \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}.$$

$$g : \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \to \tau^{\leq i} R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$$

whose composition in either direction is μ^i.

By taking cohomology, we can obtain another commutative diagram

$$\begin{tikzcd}
H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \arrow[r, "s_1"] \arrow[d, shift left, "f_1"] & H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \arrow[d, shift left, "f_2"] \\
H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \arrow[r, "s_2"] \arrow[u, shift left, "g_1"] & H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \arrow[u, shift left, "g_2"]
\end{tikzcd}$$

Note that $\mathrm{Coker}(s_1)$ is in fact $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}]$ and $\mathrm{Coker}(s_2) = 0$.

Therefore we get two induced maps

$$H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}] \underset{g_3}{\overset{f_3}{\rightleftarrows}} 0$$

where the composition of f_3 and g_3 in either direction is μ^i. Since $H^{i+1}_{A_{\inf}}(\bar{\mathfrak{X}})[\widetilde{\xi}] \simeq H^{i+1}_{\mathfrak{S}}(\mathfrak{X})[E]\otimes_{\mathcal{O}_K} \mathcal{O}_C$ as $\mathcal{O}_C$-modules, it has a decomposition as $\mathcal{O}_C^m \oplus (\bigoplus_{s=1}^n \mathcal{O}_C/\pi^{n_s})$. Note that the image of μ under the reduction $A_{\inf} \to A_{\inf}/\widetilde{\xi}$ is $\zeta_p - 1$ and $v((\zeta_p-1)^i) < v(\pi)$ when $ie < p-1$. We then can get $H^{i+1}_{A_{\inf}}(\bar{\mathfrak{X}})[\widetilde{\xi}] = 0$ by Lemma 2.2.4.

□

Remark 5.1.2. The previous version of this lemma covers the cohomological degree i such that $ie < p-1$. We want to thank Shizhang Li for pointing out that the previous proof can be improved slightly to include the cohomological degree $i+1$ such that $ie < p-1$.

In the next lemma, we give an equivalent statement to the $\widetilde{\xi}$-torsion-freeness for some special $A_{\inf}$-modules.

Lemma 5.1.3. *Let M be a finitely presented $A_{\inf}$-module such that $M[\frac{1}{p}]$ is finite projective over $A_{\inf}[\frac{1}{p}]$, and let $x \in \mathfrak{m}\backslash(p)$ where $\mathfrak{m}$ is the maximal ideal of $A_{\inf}$. Then M is $\widetilde{\xi}$-torsion-free if and only if it is x-torison-free.*

Proof. Note that the radical ideal of (p,x) is the maximal ideal. To see this, we take any $y \in \mathfrak{m}$. Let $\bar{y}$ be its image in $\mathcal{O}_C^\flat = A_{\inf}/p$. Then there exists a positive integer s and an element $a \in A_{\inf}$ with image $\bar{a} \in \mathcal{O}_C^\flat$ such that $(\bar{y})^s = \bar{x}\bar{a}$ where $\bar{x}$ is the image of x in $\mathcal{O}_C^\flat$. So we get y^s is in (p,x).

Now If there exists $b \in M$ such that $xb = 0$, then for any other $z \in m\backslash(p)$, we have $z^n b = 0$ for any sufficiently large n. This is because all torsion in M is killed by some power of p. Then this lemma follows. □

Corollary 5.1.4. *When $ie < p-1$, the $A_{\inf}$-cohomology group $H^{i+1}_{A_{\inf}}(\bar{\mathfrak{X}})$ is ξ-torsion-free and the Breuil-Kisin cohomology group $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$ is both E-torsion-free and u-torsion-free.*

Recall that for any finitely presented $A_{\inf}$-module M such that $M[\frac{1}{p}]$ is finite projective over $A_{\inf}[\frac{1}{p}]$, we have the following proposition :

Proposition 5.1.5 ([BMS18] Proposition 4.13)**.** *Let M be a finitely presented $A_{\inf}$-module such that $M[\frac{1}{p}]$ is finite projective over $A_{\inf}[\frac{1}{p}]$. Then there is a functorial exact sequence*

$$0 \to M_{\mathrm{tor}} \to M \to M_{\mathrm{free}} \to \overline{M} \to 0$$

satisfying :

1. *M_{tor}, the torsion submodule of M, is finitely presented and perfect as an $A_{\inf}$-module, and is killed by p^n for $n \gg 0$.*
2. *M_{free} is a finite free $A_{\inf}$-module.*
3. *$\overline{M}$ is finitely presented and perfect as an $A_{\inf}$-module, and is supported at the closed point $s \in \mathrm{Spec}(A_{\inf})$.*

Here we recall the construction of the free module M_{free}. Since M/M_{tor} is torsion-free, the quasi-coherent sheaf associated to it restricts to a vector bundle on $\text{Spec}(A_{\text{inf}})\backslash\{s\}$ by [BMS18, Lemma 4.10]. By [BMS18, Lemma 4.6], the global section of this vector bundle is a finite free A_{inf}-module, which gives M_{free}. In particular, if M/M_{tor} is free itself, then $M/M_{\text{tor}} = M_{\text{free}}$. For more details, see the proof of [BMS18, Proposition 4.13].

By applying this result to $H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}})$, we can obtain the following lemma saying that $H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}})$ is a direct sum of its torsion submodule and a free A_{inf}-module.

Lemma 5.1.6. *For any i such that $ie < p-1$, the term $\overline{M}$ in the functorial exact sequence*

$$0 \to M_{\text{tor}} \to M = H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}}) \to M_{\text{free}} \to \overline{M} \to 0$$

vanishes.

Proof. Let $N = H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\text{inf}}$, we have two maps $f : M \to N$ and $g : N \to M$, whose composition in either direction is μ^i. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\text{tor}} & \longrightarrow & M & \longrightarrow & M_{\text{free}} & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{\text{tor}} & \longrightarrow & N & \longrightarrow & N_{\text{free}} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

by functoriality.

On the other hand, the exact sequence associated to $H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}})$ is the flat base change of the canonical exact sequence associated to $H^i_{\mathfrak{S}}(\mathfrak{X})$ (see [BMS18, Proposition 4.3 and 4.13]). Hence $\overline{M} \cong \overline{H^i_{\mathfrak{S}}(\mathfrak{X})} \otimes_{\mathfrak{S}} A_{\text{inf}}$ and $\overline{M}/\widetilde{\xi} \cong (\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}/E) \otimes_{\mathfrak{S}} A_{\text{inf}}$ where $\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}$ is a torsion $\mathfrak{S}$-module and is killed by some power of (p, u). Again, by using the decomposition of $\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}/E$ and the fact that $v((\zeta_p - 1)^i) < v(\pi)$ when $ie < p-1$, we get $\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}/E = 0$ and $\overline{M}/\widetilde{\xi} = 0$ by Lemma 2.2.4. Then $\overline{M} = 0$ follows from Nakayama lemma.

□

Corollary 5.1.7. *For any i such that $ie < p-1$, the A_{inf}-cohomology group $H^i_{A_{\text{inf}}}(\overline{\mathfrak{X}})$ is a direct sum of a free A_{inf}-module and its torsion submodule. Also, the Breuil-Kisin cohomology group $H^i_{\mathfrak{S}}(\mathfrak{X})$ is a direct sum of a free $\mathfrak{S}$-module and its torsion submodule.*

In the following part, we consider the torsion submodules of the cohomology groups $H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}})$ and $H^i_{\mathfrak{S}}(\mathfrak{X})$, and let $H^i_{A-\text{tor}}$, $H^i_{\mathfrak{S}-\text{tor}}$ denote them respectively.

We first prove a key lemma which enables us to study the structure of $H^n_{\mathfrak{S}-\text{tor}}$.

Lemma 5.1.8. *For any i such that $ie < p-1$, the modules $(p^s H^i_{A-\text{tor}})/p^m$ (resp. $(p^s H^i_{\mathfrak{S}-\text{tor}})/p^m$) are $\widetilde{\xi}$-torsion-free (resp. E-torsion-free) for all non-negative integers m, s.*

Proof. Recall that we have two injective map $f : H^i_{A-\text{tor}} \to H^i_{\text{ét}-\text{tor}} \otimes_{\mathbb{Z}_p} A_{\text{inf}}$ and $g : H^i_{\text{ét}-\text{tor}} \otimes_{\mathbb{Z}_p} A_{\text{inf}} \to H^i_{A-\text{tor}}$ whose composition in either direction is μ^i. These induce

two new maps (we still denote f and g) between $((p^sH^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}]$ and $((p^sH^i_{\text{ét}-\mathrm{tor}})\otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/p^m)[\widetilde{\xi}]$ whose composition in either direction is μ^i. Note that $((p^sH^i_{\text{ét}-\mathrm{tor}})\otimes_{\mathbb{Z}_p}A_{\mathrm{inf}}/p^m)[\widetilde{\xi}] = 0$. This means $((p^sH^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}]$ is killed by μ^i. As $((p^sH^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}] \cong ((p^sH^i_{\mathfrak{S}-\mathrm{tor}})/p^m)[E]\otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ admits a decomposition as $\bigoplus_{t=1}^n \mathcal{O}_C/\pi^{n_t}$ and $v((\zeta_p-1)^i) < v(\pi)$, the module $((p^sH^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}]$ must be 0 by Lemma 2.2.4. Since $((p^sH^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}] \cong (p^sH^i_{\mathfrak{S}-\mathrm{tor}})/p^m)[E]\otimes_{\mathfrak{S},\alpha} A_{\mathrm{inf}}$ and the map $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ is faithfully flat, we also have $(p^sH^i_{\mathfrak{S}-\mathrm{tor}})/p^m)$ is E-torsion-free. □

In order to determine the module structure of $H^i_{\mathfrak{S}}(\mathfrak{X})$, we need the following lemma.

Lemma 5.1.9. *Let M be a finitely presented torsion $\mathfrak{S}$-module. If $M/p \cong (\mathfrak{S}/p)^n$ and $pM \cong \bigoplus_{i=1}^r \mathfrak{S}/p^{n_i}$, then we have an isomorphism of $\mathfrak{S}$-modules : $M \cong \bigoplus_{i=1}^n \mathfrak{S}/p^{m_i}$.*

Proof. The proof is just that of [Bre98b, Lemma 2.3.1.1], simply by replacing S by $\mathfrak{S}$. For readers' convenience, we give the proof here.

Choose $m \geq 0$ such that $p^mM = 0$. Let $(e_1, e_2, \cdots, e_n)$ be a basis of M/pM over $\mathfrak{S}/p$ and we choose their liftings $\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_n$ in M. By Nakayama lemma, we see that M is generated by $(\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_n)$ as a $\mathfrak{S}/p^m$-module. So $(p\widehat{e}_1, p\widehat{e}_2, \cdots, p\widehat{e}_n)$ generate the $\mathfrak{S}/p^m$-module pM.

After renumbering $(\widehat{e}_i)$, we can suppose that the images of $p\widehat{e}_1, p\widehat{e}_2, \cdots, p\widehat{e}_r$ in $pM\otimes_{\mathfrak{S}/p^m} k$ form a basis over k. Choose $f_1, \cdots, f_r \in pM$ such that $pM \cong \bigoplus_{i=1}^r \mathfrak{S}/p^{n_i}\mathfrak{S}\cdot f_i$. Then there exists a $r\times r$-matrix $A \in M_r(\mathfrak{S}/p^m\mathfrak{S})$ such that $(f_1, f_2, \cdots, f_r)A = (p\widehat{e}_1, p\widehat{e}_2, \cdots, p\widehat{e}_r)$. Since $A \mod (p,u) \in GL_r(k)$, we know that A is in $GL_r(\mathfrak{S}/p^m\mathfrak{S})$. So we can replace $(\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_r)$ by $(\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_r)A^{-1}$ and suppose $p\widehat{e}_i = f_i$ for $1 \leq i \leq r$.

For $r+1 \leq j \leq n$, there exist $a_{ij} \in \mathfrak{S}/p^m\mathfrak{S}$ for $1 \leq i \leq r$ such that $p\widehat{e}_j = \sum_{i=1}^r a_{ij}f_i = \sum_{i=1}^r a_{ij}p\widehat{e}_i$. Again, we can replace $\widehat{e}_j$ by $\widehat{e}_j - \sum_{i=1}^r a_{ij}\widehat{e}_i$ for $r+1 \leq j \leq n$. That means we can suppose $p\widehat{e}_j = 0$ for $r+1 \leq j \leq n$.

Finally, we can construct a surjective morphism of $\mathfrak{S}/p^m\mathfrak{S}$-module :

$$h : M' = (\bigoplus_{i=1}^r \mathfrak{S}/p^{n_i+1}\mathfrak{S}\times g_i)\bigoplus(\bigoplus_{i=r+1}^n \mathfrak{S}/p\mathfrak{S}\times g_i) \to M$$

$$g_i \mapsto \widehat{e}_i$$

Note that the morphism $h : M' \to M$ induces two isomorphisms : $h_1 : pM' \xrightarrow{\sim} pM$ and $h_2 : M'/pM' \xrightarrow{\sim} M/pM$ under the choice of $\widehat{e}_i$, $1 \leq i \leq n$. For any x such that $h(x) = 0$, if $x \in pM'$, then $x = 0$ since $h_2(x) = h(x) = 0$. If $x \notin pM'$, then $h_2(\bar{x}) = 0$ implies that $x \in pM'$ where $\bar{x}$ is the image of x in M'/pM'. So $h : M' \to M$ must be an isomorphism. We are done. □

Corollary 5.1.10. *Let M be a finitely presented torsion $\mathfrak{S}$-module which is killed by some power of p. If $(p^sM)/p$ is u-torsion-free for all $s \geq 0$, then M admits a decomposition as $M \cong \bigoplus_{i=1}^n \mathfrak{S}/p^{m_i}$.*

Proof. To prove this corollary, we want to apply Lemma 5.1.9 to M. Note that M/p is u-torsion-free by our assumption, therefore finite free as a $\mathfrak{S}/p = k[[u]]$-module. So we need to prove that pM admits a nice decomposition as in Lemma 5.1.9. Since the module $(pM)/p$ is also u-torsion-free by our assumption, we only need to prove that p^2M admits a nice decomposition as in Lemma 5.1.9. We can continue this process until that we need to prove p^mM admits a nice decomposition as in Lemma 5.1.9 for some m such that M is killed by p^{m+1}. As $p(p^mM) = 0$ and $(p^mM)/p = p^mM$ has no u-torsion, we see that p^mM is a free $\mathfrak{S}/p$-module by Lemma 5.1.9. So we are done.

□

5.2 Integral comparison theorem

Now we state our main theorem of this chapter comparing the module structure of Breuil-Kisin cohomology groups to that of p-adic étale cohomology groups.

Theorem 5.2.1. *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed nonarchimedean extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Assuming $ie < p-1$, there is an isomorphism of $\mathfrak{S}$-modules*

$$H^i_{\mathfrak{S}}(\mathfrak{X}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}.$$

In particular, we also have an isomorphism of A_{inf}-modules

$$H^i_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}.$$

Proof. Note that the torsion submodule $H^i_{\mathfrak{S}-\mathrm{tor}}$ of $H^i_{\mathfrak{S}}(\mathfrak{X})$ is killed by some power of p by the first statement in Proposition 5.1.5. Then by Lemma 5.1.8 and Corollary 5.1.10, we get a decomposition $H^i_{\mathfrak{S}-\mathrm{tor}} \cong \bigoplus_{t=1}^n \mathfrak{S}/p^{m_t}$.

Since $H^i_{\mathfrak{S}}(\mathfrak{X})$ is a direct sum of a free $\mathfrak{S}$-module and $H^i_{\mathfrak{S}-\mathrm{tor}}$ by Corollary 5.1.7, this theorem then follows from the A_{inf}-comparison of the Breuil-Kisin cohomology groups (see Theorem 1.5.2.1) and Corollary 1.4.6.1

$$H^n_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}[1/\mu] \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu].$$

where the map $\mathfrak{S} \to A_{\mathrm{inf}}[1/\mu]$ is the composition of the faithfully flat map $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ and the natural injection $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to A_{\mathrm{inf}}[1/\mu]$.

□

Remark 5.2.2. In general, for any finitely generated module M over $\mathfrak{S}$ (or any other two dimensional regular local ring), there is a pseudo-isomorphism between M and $\mathfrak{S}^r \oplus (\bigoplus_{i=1}^n \mathfrak{S}/\mathcal{P}_i)$ where each $\mathcal{P}_i$ is a prime ideal of height 1. Pseudo-isomorphism means its localization at all prime ideals of height 1 is in fact an isomorphism. Within the range $ie < p-1$, the theorem above tells us that the classical p-adic cohomology theories provide enough information to determine the structure of Breuil-Kisin cohomology groups. But beyond this range, the situation gets subtle.

Now we come to prove the integral comparison theorem in the ramified case.

Theorem 5.2.3. *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed nonarchimedean extension C of K with residue field $\bar{k}$. Let X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_C$ and $\mathfrak{X}_k$ be the special fiber of $\mathfrak{X}$. Then if $ie < p-1$, there is an isomorphism of $W(k)$-modules*

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

Proof. Assume $ie < p-1$. By Corollary 1.4.6.3 and Corollary 5.1.4, we have an isomorphism of $\mathcal{O}_C$-modules

$$H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}})/\xi \cong H^i_{\text{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C).$$

Since we also have $H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\text{inf}}$ by Theorem 5.2.1, we get an isomorphism of $\mathcal{O}_C$-modules

$$H^i_{\text{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Note that when $e < p$, we have an integral comparison isomorphism between de Rham cohomology and crystalline cohomology (cf. [Ber06])

$$H^i_{\text{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})) \otimes_{W(\bar{k})} \mathcal{O}_C$$

where $\mathfrak{X}_{\bar{k}} := \mathfrak{X}_k \otimes_k \bar{k}$.

So finally, we get the isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\bar{k}) \cong H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})).$$

By virtue of the base change of crystalline cohomology $H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} W(\bar{k})$, we also have

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

□

Remark 5.2.4. When $(i+1)e < p-1$, the proof of the integral comparison isomorphism for schemes in [Car08] depends on the fact that the crystalline cohomology groups $H^i_{\mathrm{crys}}(\mathfrak{X}_{\mathcal{O}_K/p}/S)$ admits a decomposition as $H^i_{\mathrm{crys}}(\mathfrak{X}_{\mathcal{O}_K/p}/S) \cong S^n \oplus (\bigoplus_{j=1}^m S/p^{a_j})$. This can also be deduced from Theorem 5.2.1 and the base change of prismatic cohomology along the map of prisms $(\mathfrak{S},(E)) \to (S,(p))$, which is the composition of the Frobenius map $\mathfrak{S} \to \mathfrak{S}$ and the natural injection $\mathfrak{S} \hookrightarrow S$.

Chapitre 6

Categories of Breuil-Kisin modules

In this chapter, let K be a complete nonarchimedean extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. Let k be its residue field and e be its ramification degree. We want to give a slightly more general result about the structure of torsion Breuil-Kisin modules of height r, under the restriction $er < p-1$. Namely, all torsion Breuil-Kisin modules in this case are isomorphic to $\bigoplus_{i=1}^{n} \mathfrak{S}/p^{a_i}$, where $\mathfrak{S} := W(k)[[u]]$. As a result, this gives another proof of Theorem 5.2.1 without using Lemma 5.1.8.

Let π be a fixed uniformizer of $\mathcal{O}_K$. There is a natural $W(k)$-linear surjection from $\mathfrak{S} = W(k)[[u]]$ to $\mathcal{O}_K$ by sending u to π. The kernel of this map is generated by an Eisenstein polynomial $E = E(u)$ for π. Fix a non-negative integer r. We first need to define some categories that we will study.

Definition 6.0.1 ($'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$)**.** *The objects of category* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *are defined to be* $\mathfrak{S}$*-modules* $\mathfrak{M}$ *equipped with a* φ*-linear endomorphism* $\varphi : \mathfrak{M} \to \mathfrak{M}$ *such that the cokernel of* $id \otimes \varphi : \varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ *is killed by* E^r*. Morphisms are homomorphisms of* $\mathfrak{S}$*-modules compatible with* φ*. We say that a short sequence* $0 \to \mathfrak{M}_1 \to \mathfrak{M}_2 \to \mathfrak{M}_3 \to 0$ *is exact if it is exact in the abelian category of* $\mathfrak{S}$*-modules.*

Definition 6.0.2 ($\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$)**.** *The category* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$ *is the full subcategory of* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *spanned by the objects which are finite free over* $\mathfrak{S}_1 := \mathfrak{S}/p = k[[u]]$*.*

Definition 6.0.3 ($\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$)**.** *We define* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *to be the smallest full subcategory of* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *which contains* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$ *and is stable under extensions.*

Remark 6.0.4. The category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$ first appeared in [Bre]. And the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$ is just the category $\mathrm{Mod}/\mathfrak{S}$ defined by Kisin in [Kis06].

The following lemma gives us some important descriptions of objects in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$.

Lemma 6.0.5. *1. For any* $\mathfrak{M}$ *in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, the morphism* $id \otimes \varphi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ *is injective.*

2. *An object* $\mathfrak{M}$ *in* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *is in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *if and only if it is of finite type over* $\mathfrak{S}$*, it has no u-torsion and it is killed by some power of p.*

Proof. See [Liu07, section 2.3]. □

Corollary 6.0.6. *The torsion submodule* $H^i_{\mathfrak{S}-\mathrm{tor}}$ *of the Breuil–Kisin cohomology groups of a proper smooth formal scheme over* $\mathcal{O}_K$ *is in the category* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *when* $i \leq r < \frac{p-1}{e}$.

Proof. This follow from Corollary 5.1.4 and [BS19, Theorem 1.8 (6)]. □

Next we introduce Breuil's ring S and define some related categories analogous to those associated with the ring $\mathfrak{S}$.

Definition 6.0.7 (Breuil's ring)**.** *Let* S *be the p-adic completion of the PD-envelope of* $W(k)[u]$ *with respect to the ideal* $(E) \subset W(k)[u]$*. The ring* S *is endowed with several additional structures :*

1. *a canonical (PD-)filtration :* $\mathrm{Fil}^{\mathrm{i}}S$ *is the p-adic completion of the ideal generated by elements* $(\frac{E^m}{m!})_{m\geq i}$.
2. *a Frobenius* φ *: it is the unique continuous map which is Frobenius semi-linear over* $W(k)$ *and sends* u *to* u^p.

For $r < p-1$, we have $\varphi(\mathrm{Fil}^{\mathrm{r}}S) \subset p^r S$ and we can define $\varphi_r = \frac{\varphi}{p^r} : \mathrm{Fil}^{\mathrm{r}}S \to S$. Set $S_n := S/p^n$.

Definition 6.0.8 ($'\mathrm{Mod}^{r,\varphi}_{/S}$)**.** *The objects of* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *are the following data :*

1. *an* S*-module ;*
2. *a submodule* $\mathrm{Fil}^{\mathrm{r}}M \subset M$ *such that* $\mathrm{Fil}^{\mathrm{r}}S \cdot M \subset \mathrm{Fil}^{\mathrm{r}}M$ *;*
3. *a* φ*-linear map* $\varphi_r : \mathrm{Fil}^{\mathrm{r}}M \to M$ *such that for all* $s \in \mathrm{Fil}^{\mathrm{r}}S$ *and* $x \in M$ *we have* $\varphi_r(sx) = c^{-r}\varphi_r(s)\varphi_r(E^r x)$*, where* $c = \varphi_1(E)$.

The morphisms are homomorphisms of S*-modules compatible with additional structures. We say a short sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *in* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *is exact if both sequences* $0 \to M_1 \to M_2 \to M_3 \to 0$ *and* $0 \to \mathrm{Fil}^{\mathrm{r}}M_1 \to \mathrm{Fil}^{\mathrm{r}}M_2 \to \mathrm{Fil}^{\mathrm{r}}M_3 \to 0$ *are exact in the abelian category of* S*-modules.*

Definition 6.0.9 ($\mathrm{Mod}^{r,\varphi}_{/S_1}$)**.** *The objects of* $\mathrm{Mod}^{r,\varphi}_{/S_1}$ *are* M *in* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *such that* M *is finite free over* S_1 *and the image of* φ_r *generates* M *as an* S*-module.*

Definition 6.0.10 ($\mathrm{Mod}^{r,\varphi}_{/S_\infty}$)**.** *The category* $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$ *is the smallest subcategory of* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *containing* $\mathrm{Mod}^{r,\varphi}_{/S_1}$ *and is stable under extensions.*

For any $r < p-1$, one can define a functor $M_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty} \to {}'\mathrm{Mod}^{r,\varphi}_{/S}$ as follows :

1. $M_{\mathfrak{S}_\infty}(\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Here $\varphi : \mathfrak{S} \to S$ is the composite $\mathfrak{S} \to \mathfrak{S} \to S$ where the first map is the Frobenius on $\mathfrak{S}$ and the second map is the canonical injection.

2. Submodule : The Frobenius on $\mathfrak{M}$ induces a S-linear map $id \otimes \varphi : S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to S \otimes_{\mathfrak{S}} \mathfrak{M}$. The submodule $\mathrm{Fil}^{\mathrm{r}} M_{\mathfrak{S}_\infty}(\mathfrak{M})$ is then defined by the following formula :

$$\mathrm{Fil}^{\mathrm{r}} M_{\mathfrak{S}_\infty}(\mathfrak{M}) := \{x \in M_{\mathfrak{S}_\infty}(\mathfrak{M}) \mid (id \otimes \varphi)(x) \in \mathrm{Fil}^{\mathrm{r}} S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M})\}$$

3. Frobenius : the map φ_r is the following composite :

$$\mathrm{Fil}^{\mathrm{r}} M_{\mathfrak{S}_\infty}(\mathfrak{M}) \xrightarrow{id\otimes\varphi} \mathrm{Fil}^{\mathrm{r}} S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_r\otimes id} M_{\mathfrak{S}_\infty}(\mathfrak{M}).$$

Remark 6.0.11. This functor was in fact defined by Breuil (see [Bre, Section 2.2]).

We state a theorem describing the functor $M_{\mathfrak{S}_\infty}$.

Theorem 6.0.12. *For any $r < p-1$, the functor $M_{\mathfrak{S}_\infty}$ takes value in $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$. The induced functor $M_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\varphi}_{/S_\infty}$ is exact and it is an equivalence of categories. Moreover, if we choose M_{S_∞} a quasi-inverse of $M_{\mathfrak{S}_\infty}$, then the functor M_{S_∞} is also exact.*

Proof. See [CL09, Proposition 2.1.2, Theorem 2.3.1, Proposition 2.3.2]. □

Theorem 6.0.13. *Assuming $er < p-1$, the category $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$ is an abelian category and every object is of the form $\bigoplus_{i=1}^n S/p^{a_i}$. For any morphism $f : (M_1, \mathrm{Fil}^{\mathrm{r}} M_1, \varphi_r) \to (M_2, \mathrm{Fil}^{\mathrm{r}} M_2, \varphi_r)$ in $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$, the underlying module of $\mathrm{Ker}(f)$ is the kernel of the morphism $f : M_1 \to M_2$ in the category of S-modules and the underlying module of $\mathrm{Fil}^{\mathrm{r}}\mathrm{Ker}(f)$ is the kernel of the morphism $f : \mathrm{Fil}^{\mathrm{r}} M_1 \to \mathrm{Fil}^{\mathrm{r}} M_2$ in the category of S-modules. A Similar statement is true for $\mathrm{Coker}(f)$.*

Proof. See [Car06, Section 3]. We remark that the category which Caruso used is different from ours but they can be proved to be equivalent by using a generalization of [Bre98a, Proposition 2.3.1.2], as mentioned in the proof of [Car08, Theorem 4.2.1]. □

Remark 6.0.14. This theorem is false without the restriction $er < p-1$.

From now on, we fix a non-negative integer r such that $er < p-1$. Then $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ is an abelian category.

Lemma 6.0.15. *For any morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$, the underlying module of $\mathrm{Ker}(f)$ is the kernel of the morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in the category of $\mathfrak{S}$-modules. A Similar statement is true for $\mathrm{Coker}(f)$.*

Proof. By Lemma 6.0.5, the kernel and the image of the underlying morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in the category of $\mathfrak{S}$-modules together with the induced Frobenius maps are objects of $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. It is easy to see that the kernel equipped with the induced Frobenius map is indeed $\mathrm{Ker}(f)$ in the category $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. So we can assume $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ is injective. Then $M_{\mathfrak{S}_\infty}(f)$ is also injective. In fact, let L be the kernel of $M_{\mathfrak{S}_\infty}(f)$ and we choose

a quasi-inverse functor M_{S_∞} of $M_{\mathfrak{S}_\infty}$. Let $h : \mathfrak{L} \to \mathfrak{M}_1$ be the image of the inclusion $L \to M_{\mathfrak{S}_\infty}(\mathfrak{M}_1)$ under M_{S_∞}. Then $f \circ h = 0$, which implies $h = 0$. In consequence, we have $L = 0$. Put $M = \mathrm{Coker}(f)$. By Theorem 6.0.12 and Theorem 6.0.13, we get an exact sequence $0 \to \mathfrak{M}_1 \to \mathfrak{M}_2 \to M_{S_\infty}(M) \to 0$ in the exact category $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ (where the class of the exact sequences is as defined in Definition 6.0.1). So we have $M_{S_\infty}(M)$ is isomorphic to $\mathfrak{M}_2/\mathfrak{M}_1$ as $\mathfrak{S}$-modules. In particular $\mathfrak{M}_2/\mathfrak{M}_1$ has no u-torsion. By Lemma 6.0.5, the module $\mathfrak{M}_2/\mathfrak{M}_1$ equipped with the induced Frobenius map is an object of $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. It is easy to check that $\mathrm{Coker}(f)$ is isomorphic to $\mathfrak{M}_2/\mathfrak{M}_1$ equipped with the induced Frobenius map. □

Corollary 6.0.16. *The full subcategory* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$ *of* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *is an abelian category.*

Proof. For any morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$, $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are then both killed by p. By Lemma 6.0.5, they are u-torsion free. So $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are in the category $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$. □

Let $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ denote the full subcategory of $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ spanned by the objects that are isomorphic to $\bigoplus_{i=1}^n \mathfrak{S}/p^{a_i}$ as $\mathfrak{S}$-modules. In particular, $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ contains $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$.

Lemma 6.0.17. *For any* $\mathfrak{M} \in \mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, the quotient* $\mathfrak{M}/p$ *is in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$.

Proof. Consider the morphism $\mathfrak{M} \xrightarrow{\times p} \mathfrak{M}$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Since $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ is an abelian category, we know that $\mathfrak{M}/p$ is also in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. It is killed by p and has no u-torsion by Lemma 6.0.5, therefore $\mathfrak{M}/p$ is in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$. □

We now reformulate Lemma 5.1.9 by using the categories we have defined.

Lemma 6.0.18. *Let* $\mathfrak{M}$ *be in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*. If* $p\mathfrak{M}$ *is in* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, so is* $\mathfrak{M}$.

Proof. By Lemma 6.0.17, we have $\mathfrak{M}/p \in \mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Then this lemma follows from Lemma 5.1.9. □

Lemma 6.0.19. *Let* $\mathfrak{L} \hookrightarrow \mathfrak{M}$ *be an injection in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*. If* $\mathfrak{M}$ *is in* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, so is* $\mathfrak{L}$.

Proof. We show that $p\mathfrak{L}$ is in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$, then this lemma follows from Lemma 6.0.18. Consider the map $p\mathfrak{L} \hookrightarrow p\mathfrak{M}$. We proceed by induction on the minimal integer such that $p^n\mathfrak{M} = 0$. If $n = 1$, this is easy. Assume that when $n < m$ this lemma is true. Then when $n = m$, $p\mathfrak{L}$ is also in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ as $p^{m-1}(p\mathfrak{M}) = 0$. We are done. □

Theorem 6.0.20. *The category* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *is an abelian category.*

Proof. For any morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$, we need to show $\mathfrak{L} = \mathrm{Ker}(f)$ and $\mathfrak{C} = \mathrm{Coker}(f)$ are also in the category $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. For the kernel $\mathfrak{L}$, this follows from Lemma 6.0.19. For the cokernel $\mathfrak{C}$, we proceed by induction on the minimal integer n such that $p^n\mathfrak{M}_2 = 0$. Without loss of generality, we can assume f is an injection.

When $n = 1$, we have $\mathfrak{M}_1, \mathfrak{M}_2$ are both in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$. Then by Corollary 6.0.16, we see that $\mathfrak{C}$ is also in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1} \subset \mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Now suppose the statement is true when $n < m$. When $n = m$, consider the sequence $p\mathfrak{M}_1 \to p\mathfrak{M}_2 \to p\mathfrak{C}$. Then there is a short exact sequence $0 \to \mathfrak{L}' \to p\mathfrak{M}_2/p\mathfrak{M}_1 \to p\mathfrak{C} \to 0$. Since $p^{m-1}(p\mathfrak{M}_2/p\mathfrak{M}_1) = 0$, by the assumption, we get $p\mathfrak{C}$ is in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Then by Lemma 6.0.18, we see that $\mathfrak{C}$ is also in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. This finishes the proof. □

Theorem 6.0.21. *There is an equivalence of categories :* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ =$\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$.

Proof. We just need to prove that every object $\mathfrak{M}$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ is also in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. To see this, we proceed by induction on the minimal integer n such that $p^n\mathfrak{M} = 0$.

When $n = 1$, this follows from Lemma 6.0.18. Now suppose the statement is true when $n < m$. Then when $n = m$, we know that $p\mathfrak{M}$ is killed by p^{m-1}. So by the assumption, we have $p\mathfrak{M} \in \mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. By Lemma 6.0.18, we can obtain that $\mathfrak{M} \in \mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. We are done. □

So Theorem 6.0.21 and Corollary 6.0.6 provide another proof of Theorem 5.2.1.

Theorem 6.0.22. *For any $i \leq r < \frac{p-1}{e}$, we have $H^i_{\mathfrak{S}-\mathrm{tor}}$, the torsion submodule of the Breuil–Kisin cohomology group of a proper smooth formal scheme over $\mathcal{O}_K$, is in the category* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, i.e.* $H^i_{\mathfrak{S}-\mathrm{tor}} \cong \bigoplus_{i=1}^n \mathfrak{S}/p^{a_i}$.

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