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THÈSE DE DOCTORAT

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET
D'INFORMATIQUE

PAR **Rolando PEREZ III**

SOUS LA DIRECTION DE **Philippe JAMING** ET **Karim KELLAY**

POUR OBTENIR LE GRADE DE **DOCTEUR**

SPÉCIALITÉ : MATHÉMATIQUES PURES (ANALYSE)

APPLICATIONS DE L'ANALYSE COMPLEXE AU PROBLÈME DE LA PHASE

Soutenue le 08 juillet 2021 à l'Institut de Mathématiques de Bordeaux
devant le jury composé de :

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Applications de l'Analyse Complexe au Problème de la Phase

Résumé

Dans cette thèse, notre objectif est d'appliquer des outils d'analyse complexe pour déterminer les solutions et étudier la stabilité de certains problèmes de reconstruction de phase. Nous étudions d'abord le problème de reconstruction de phase pour les signaux à large bande, des fonctions dont les transformées de Fourier sont lentement décroissantes. Nous ramenons ce problème, via une transformation conforme, à un problème de phase dans l'espace de Hardy, le théorème de factorisation de Beurling nous permet de résoudre complètement ce problème. Nous donnons ensuite des résultats d'unicité sur le problème de reconstruction de phase dans l'espace de Hardy. Plus précisément, nous montrons que certaines fonctions holomorphes sont déterminées de manière unique par leurs modules sur deux segments qui se croisent et dont l'angle n'est pas un multiple rationnel de π ou sur deux cercles concentriques. Enfin, nous étudions l'effet de "zero-flipping" sur la stabilité du problème de reconstruction de phase pour les fonctions de la classe de Paley-Wiener, le "zero-flipping" fait référence ici au remplacement des zéros par leurs conjugués complexes. Nous utilisons les propriétés analytiques de l'opérateur "zero-flipping" ainsi que sa transformées de Fourier pour obtenir les résultats sur la stabilité.

Mots-clés: *reconstruction de phase, espace de Hardy, espace de Paley-Wiener, zero-flipping, stabilité*

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Applications of Complex Analysis to the Phase Retrieval Problem

Abstract

The study of phase retrieval involves the recovery of a function f in some function space from given data about the magnitude of $|f|$ (phaseless information) and other assumptions on f , where these other assumptions can be in terms of some transform of f . Phase retrieval problems are widely studied because of their physical applications in fields of science and engineering.

In this thesis, our central objective is to apply complex analytic tools to determine the solutions and investigate the stability of certain phase retrieval problems. Firstly, we solve the phase retrieval problem for wide band signals, which are functions with mildly decreasing Fourier transforms. To do so, we first translate the problem to functions in the Hardy spaces on the disc via a conformal bijection, and take advantage of the inner-outer factorization. We also consider the same problem coupled with additional magnitude constraints, and determine if these constraints force uniqueness of the solution. Secondly, we extend some uniqueness results on the phase retrieval problem on the Hardy space on the disc to more general situations. More precisely, we show that certain holomorphic functions are uniquely determined by their moduli on two intersecting segments or on two concentric circles. Finally, we investigate the effect of zero-flipping on the stability of the phase retrieval problem for functions in the Paley-Wiener class, where zero-flipping refers to the replacement of zeros by their complex conjugates. We represent zero-flipping as an operator, and use its Fourier analytic properties to show our stability results.

Keywords: *phase retrieval, Hardy spaces, Paley-Wiener class, zero-flipping, stability*

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*“Elevate yourself through the power of your mind, and not degrade yourself,
for the mind can be the friend and also the enemy of the self.”*

– from the *Bhagavad Gita*, Chapter 06, Verse 05

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Chapter 1

Introduction en Français

Le thème central de cette thèse est le *problème de reconstruction de phase*, qui se réfère à la reconstruction de la phase d'une fonction f à partir de données connues sur son module $|f|$ ainsi que d'autres hypothèses sur f ou sur sa transformée de Fourier. Les problèmes de reconstruction de phase sont des problèmes largement étudiés en raison de leurs applications en physique et ingénierie. L'application la plus notable est l'imagerie par diffraction, où la diffraction désigne le phénomène par lequel une onde se propage autour d'obstacles ou d'ouvertures. En imagerie par diffraction, un objet représenté par une fonction f est exposé à une onde électromagnétique cohérente par l'intermédiaire d'un laser, qui produit un front d'onde donné par la transformée de Fourier \widehat{f} . Le dispositif de détection permet de ne mesurer que les amplitudes, il reste alors à récupérer l'information de phase perdue pour caractériser l'objet. Cette perte d'information de phase donne lieu au célèbre problème de reconstruction de phase de Fourier, qui consiste à reconstruire f à partir de données connues $|\widehat{f}|$. D'autres applications physiques des problèmes de reconstruction de phase incluent des travaux liés à la cristallographie [55], à l'imagerie diffractive cohérente [49, 54], à l'astronomie [22], conception de lentilles [23], mécanique quantique [20, 21, 37, 38], diffusion inverse [63], microscopie [24, 72, 73], radar [38], et optique [55, 65]. Quelques exemples classiques peuvent également être trouvés dans les articles de synthèse de Klivanov et. al. [44] et Grohs et. al. [33], et dans le livre de Hurt [36].

De manière générale, étant donné une fonction à valeur réelle ou complexe f d'un espace vectoriel X , le but de la reconstruction de phase est de retrouver toutes les fonctions $g \in X$ telles que $|g| = |f|$. Les premiers travaux sur ce problème se sont concentrés sur la description de l'ensemble des solutions. Cependant, les problèmes de reconstruction de phase ont généralement plusieurs solutions, ceci nous amène à étudier l'*unicité* de la solution sous contraintes supplémentaires pour réduire de manière significative l'ensemble de solutions, et ainsi forcer par conséquent une solution unique. Ici l'unicité signifie que les seules solutions sont de la forme $g = cf$ pour un certain c de module 1. Nous renvoyons le lecteur aux travaux de Klivanov et. al. [44] et Jaming [38, 39] pour des exemples concrets de contraintes supplémentaires.

La *stabilité* du problème de reconstruction de phase a également fait l'objet de plusieurs travaux comme celle de l'unicité de la solution. La stabilité fait référence à la dépendance continue de la solution par rapport aux données du module de la

fonction. Plus précisément, étant donné $f, g \in X$, si $|g|$ est proche de $|f|$, est-ce que g est proche de f ? Afin d'examiner la stabilité, il est nécessaire de quantifier la "proximité" en attribuant une mesure appropriée. Certains travaux antérieurs sur la stabilité des problèmes de reconstruction de phase (voir par exemple [3, 32]) utilisaient la mesure classique $\inf_{|c|=1} \|f - cg\|$ telle que

$$\inf_{|c|=1} \|f - cg\|_{\mathcal{B}} \leq C \| |f| - |g| \|_{\mathcal{B}'}, \quad (1.0.1)$$

où $\mathcal{B}, \mathcal{B}'$ sont des espaces de Banach ou de Hilbert appropriés. Certains articles (voir par exemple [4, 9, 15, 33]) ont fourni des discussions sur les propriétés de stabilité généralisées en utilisant des espaces de Banach et des "frames" en dimensions finies et infinies. Le lecteur peut consulter l'article de synthèse de Grohs et al. [33] pour une discussion sur la formulation générale du problème de reconstruction de phase (unicité et stabilité) ainsi que plusieurs exemples.

Dans ce travail, notre objectif est d'appliquer les outils de l'analyse complexe pour (i) déterminer toutes les solutions ; (ii) trouver des contraintes supplémentaires qui permettent d'avoir l'unicité de la solution ; et (iii) étudier la stabilité de certains problèmes de reconstruction de phase. L'ensemble du travail est séparé en trois parties principales.

Dans le Chapitre 4 nous étudions le problème de reconstruction de phase pour les signaux à *large bande*. Ce sont les fonctions dont le module de leurs transformées de Fourier est lentement décroissante.

Avant d'énoncer nos résultats, rappelons d'abord le problème de reconstruction de phase dans ce cadre pour les signaux à *bande limitée*. Les signaux/fonctions à bande limitée sont des fonctions dans $L^2(\mathbb{R})$ avec des transformées de Fourier à support compact. Étant donné une fonction à bande limitée f , l'objectif est de trouver toutes les fonctions à bande limitée g telles que

$$|f(x)| = |g(x)|, \quad x \in \mathbb{R}. \quad (1.0.2)$$

Ce problème a été résolu indépendamment par Akutowicz [1, 2], Walther [77] et Hofstetter [34]. Pour résoudre ce problème, ils ont d'abord utilisé le théorème de Paley-Wiener qui stipule que f et g se prolongent en fonctions holomorphes dans le plan complexe d'ordre fini 1, c'est-à-dire que f et g croissent comme $e^{a|z|}$ pour un certain nombre réel positif a . Ensuite, ils ont montré que (1.0.2) est alors équivalent à

$$f(z)\overline{f(\bar{z})} = g(z)\overline{g(\bar{z})}, \quad z \in \mathbb{C}. \quad (1.0.3)$$

Notons que (1.0.3) est une reformulation de (1.0.2) lorsque z est réel et est une égalité entre deux fonctions holomorphes de sorte qu'elle est valable pour tout $z \in \mathbb{C}$. Ils ont ensuite utilisé le théorème de factorisation de Hadamard : si f est une fonction holomorphe de type exponentiel, alors

$$f(z) = ce^{az} z^k \prod_{\alpha \in \mathcal{Z}(f)} \left(1 - \frac{z}{\alpha}\right) e^{z/\alpha}, \quad z \in \mathbb{C}$$

avec $c, a \in \mathbb{C}$, $k \in \mathbb{N}$ et où $\mathcal{Z}(f)$ est l'ensemble des zéros (excluant 0) de f . Remarquons aussi que (1.0.3) implique que chaque zéro de g est soit un zéro de f , soit

un conjugué complexe d'un tel zéro. Ainsi, il s'ensuit que g peut être obtenu en changeant un nombre arbitraire de zéros de f en leurs conjugués complexes dans la factorisation de Hadamard de g , et ceci a été appelé *zero-flipping* depuis Walther. Il s'ensuit que tous ces g ont une factorisation de Hadamard donnée par

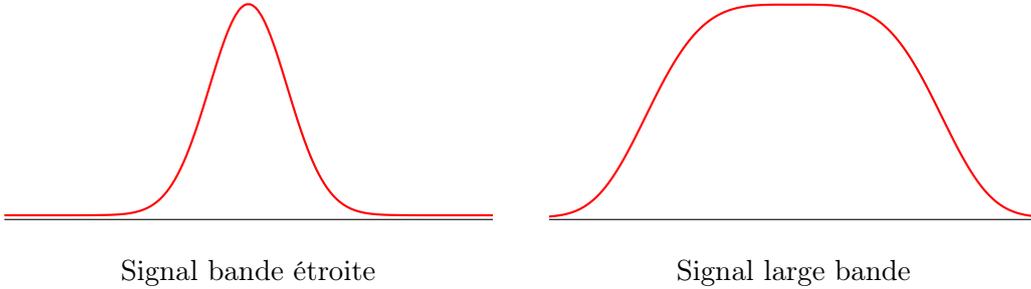
$$g(z) = \tilde{c}e^{(a+i\gamma)z} z^k \prod_{\alpha \in A} \left(1 - \frac{z}{\alpha}\right) e^{z/\alpha} \prod_{\alpha \in \mathcal{Z}(f) \setminus A} \left(1 - \frac{z}{\bar{\alpha}}\right) e^{z/\bar{\alpha}}, \quad z \in \mathbb{C}$$

où $A \subseteq \mathcal{Z}(f)$, $|\tilde{c}| = |c|$, et $\gamma \in \mathbb{R}$. On peut consulter le livre de Hurt [36, Section 3.17] sur une étude de zero-flipping dans ce contexte.

McDonald [51] a étendu cette démonstration aux fonctions dont le module de leurs transformées de Fourier décroît très rapide à l'infini. Par exemple, dans le cas d'une décroissance gaussienne, si $|\hat{f}(\xi)|, |\hat{g}(\xi)| \lesssim e^{-a|\xi|^2}$, $a > 0$, alors f et g s'étendent à des fonctions holomorphes d'ordre fini 2. Grâce à la factorisation de Hadamard, les solutions peuvent aussi être obtenues par zero-flipping. De plus, cette démonstration s'étend aux fonctions qui satisfont une condition de décroissance exponentielle de la forme

$$|\hat{f}(\xi)|, |\hat{g}(\xi)| \lesssim e^{-a|\xi|^\alpha}, \quad \xi \in \mathbb{R} \tag{1.0.4}$$

avec $a > 0$ et $\alpha > 1$ mais n'est plus valable lorsque $\alpha = 1$. Nous nous intéressons alors au problème de reconstruction de phase pour les fonctions satisfaisant (1.0.4) lorsque $\alpha = 1$, c'est-à-dire $|\hat{f}(\xi)|, |\hat{g}(\xi)| \lesssim e^{-a|\xi|}$. Les fonctions ayant une telle décroissance sont parfois appelées signaux *large bande*, tandis que celles présentant une décroissance du type (1.0.4) pour $\alpha > 1$ sont appelées *bande étroite*.



Nous donnons la solution au problème de reconstruction de phase suivant : étant donné $f \in L^2(\mathbb{R})$ tel que $\hat{f} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$, trouver toutes les fonctions $g \in L^2(\mathbb{R})$ telles que $\hat{g} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ et $|g(x)| = |f(x)|$ pour tous $x \in \mathbb{R}$.

Les fonctions \hat{f} et \hat{g} appartiennent à $L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ si et seulement si f et g appartiennent à un espace de Hardy sur la bande $H^2_\tau(\mathcal{S}_c)$ [8, 43], l'espace de fonctions holomorphes sur \mathcal{S}_c telles que

$$\sup_{|y| < c} \int_{\mathbb{R}} |f(t + iy)|^2 dt < +\infty.$$

On ramène ensuite le problème à l'espace de Hardy sur le disque unité \mathbb{D} , noté $H^2(\mathbb{D})$, en utilisant une bijection conforme entre \mathcal{S}_c et \mathbb{D} . Il est intéressant de noter que les espaces de Hardy ont été étudiés de manière approfondie lorsqu'ils sont définis sur

\mathbb{D} et le demi-plan supérieur (voir par exemple : [25, 29, 45, 50, 56, 69]). Bien qu'il soit possible de travailler directement sur $H^2_\tau(\mathcal{S})$, l'expression du noyau de Poisson de \mathbb{D} est assez simple pour être utilisée. En ramenant le problème à l'espace de Hardy du disque unité, grâce au théorème de factorisation de Beurling (l'analogie du théorème de Hadamard pour les espace de Hardy), on peut ainsi écrire $f \in H^2(\mathbb{D})$ sous la forme

$$f = BSO$$

où B est le produit de Blaschke formé par les zéros, S est la fonction intérieure singulière associée à une mesure singulière positive sur le cercle unité \mathbb{T} , et O est la fonction extérieure déterminée par le module de sa limite radial sur \mathbb{T} . Nous obtenons la caractérisation suivante.

Théorème. *Soit $f, g \in H^2_\tau(\mathcal{S})$ où $\mathcal{S} = \mathcal{S}_1$. Alors $|f| = |g|$ sur \mathbb{R} si et seulement si*

$$f = e^{i\gamma} W^{-1/2} B_f S_f O_f \quad \text{et} \quad g = e^{i\gamma'} W^{-1/2} B_g S_g O_g$$

où

1. $W(z) = 4 \cosh^2(\frac{\pi}{4}z)$ pour tout $z \in \mathcal{S}$, B_f est le produit de Blaschke, S_f est la fonction intérieure singulière, et O_f est la fonction extérieure ;
2. B_g est le produit de Blaschke associé à l'ensemble des zéros $A \cup \{\bar{z} : z \in Z(f) \setminus A\}$ avec $A \subset Z(f)$ où $Z(f)$ est l'ensemble des zéros de f en comptant leurs multiplicités ;
3. S_g est la fonction intérieure singulière associée à la mesure singulière positive $\mu_g = \mu_f + \sigma$, où μ_f est la mesure singulière associée à f et σ est une mesure singulière réelle impaire sur $\partial\mathcal{S}$; et
4. O_g est la fonction extérieure de uO_f où u est une fonction extérieure appartenant à la classe de Smirnov sur la bande et $u = 1/u^*$ sur \mathcal{S} .

De plus, en utilisant ces factorisations pour f et g , on peut écrire $f = uv$ et $g = uv^*$ où u, v sont des fonctions holomorphes sur \mathcal{S} .

En couplant à un autre problème de reconstruction de phase avec notre problème original, on peut réduire l'ensemble des solutions. Ces problèmes couplés sont de la forme

$$|g| = |f| \quad \text{et} \quad |Tg| = |Tf|, \quad f, g \in H^2_\tau(\mathcal{S})$$

où T est une transformation quelconque. Pour notre premier problème couplé, nous ajoutons une condition impliquant un signal de référence fixe h :

$$|g| = |f| \quad \text{et} \quad |g - h| = |f - h|, \quad \text{sur } \mathbb{R}$$

où h est une fonction à valeur complexe non nulle avec $e^{i \arg h}$ analytique sur \mathbb{R} . Nous utilisons l'interprétation géométrique de ces contraintes pour montrer que ce problème a exactement deux solutions.

Nous examinons également le problème avec une condition supplémentaire impliquant des conditions sur la transformé de Fourier :

$$|g| = |f| \quad \text{et} \quad |\widehat{g}| = |\widehat{f}|, \quad \text{sur } \mathbb{R}. \quad (1.0.5)$$

Ce problème couplé est également connu sous le nom de *problème de Pauli* [58, 38, 39]. Jaming [38] et Ismagilov [37] ont indépendamment considéré ce problème dans le cas de fonction à bande limitée, où ils ont explicitement construit une famille de solutions en utilisant les produits de Riesz. A fortiori, nous n'obtenons pas non plus l'unicité pour le cas à large bande, en fait nous pouvons trouver $f \in H^2_\tau(\mathcal{S})$ tel qu'il existe une infinité de $g \in H^2_\tau(\mathcal{S})$ qui satisfait (1.0.5). Notre construction s'inspire de la démonstration de Jaming et Ismagilov.

Ensuite, nous examinons le problème couplé avec la condition impliquant un opérateur de dérivation D

$$|g| = |f| \quad \text{et} \quad |Dg| = |Df|, \quad \text{sur } \mathbb{R}.$$

Ce problème couplé a été inspiré par le travail de McDonald sur le problème de reconstruction de phase pour les fonctions entières dans [51]. En utilisant les propriétés spéciales de D, f , et g et une des conséquences de notre résultat principal, nous montrons que notre problème couplé a exactement deux solutions.

Enfin, pour notre dernier problème couplé, Nous montrons que si $f, g \in H^2_\tau(\mathcal{S})$ avec

$$|g| = |f| \quad \text{sur} \quad \mathbb{R} \cup (-e^{i\alpha} + a, e^{i\alpha} + a), \quad a \in \mathbb{R} \text{ et } \alpha \notin \pi\mathbb{Q} \quad (1.0.6)$$

alors g est déterminé de façon unique par f .

Dans le Chapitre 5, nous généralisons certains résultats d'unicité dans la reconstruction de phase de fonctions de $H^2(\mathbb{D})$. Une de nos motivations provient du problème de la reconstruction de phase couplée sur \mathbb{D} avec des contraintes similaires à celle mentionnée dans (1.0.6). Nous étendons ce résultat d'unicité aux fonctions holomorphes sur des domaines connexes ouverts.

Théorème. *Soit Ω un domaine connexe ouvert. Soit $f, g \in \text{Hol}(\Omega)$, on suppose que*

$$|g(z)| = |f(z)|, \quad z \in I \cup I_\alpha$$

où I et I_α sont des segments à l'intérieur de Ω , I_α est la rotation α de I autour du point central de I , et $\alpha \notin \pi\mathbb{Q}$. Alors $g(z) = cf(z)$ pour tout $z \in \Omega$ et pour un certain $c \in \mathbb{T}$.

Nous réduisons ce problème couplé ci-dessus dans une situation similaire à celle de (1.0.6), mais avec une démonstration différente mais plus simple utilisant les séries de Fourier.

Un autre problème de même nature et celui Boche et al. [12, Theorem 3]. Ils montrent que si f et g sont deux fonctions dans l'espace de Hardy du disque sans facteur intérieur singulier et si $|f| = |g|$ sur le cercle unité et sur un cercle plus petit à l'intérieur du cercle unité, alors g peut être déterminé de manière unique à partir de f . Nous étendons ce résultat d'unicité à toutes les fonctions de la classe de Nevanlinna \mathcal{N} , l'espace des fonctions holomorphes sur \mathbb{D} qui s'écrit comme un quotient de deux fonctions holomorphes bornées sur le disque. Nous avons :

Théorème. *Soit $f, g \in \mathcal{N}$ et soit $\rho \in (0, 1)$. Si*

$$|g_*(\zeta)| = |f_*(\zeta)|, \quad p.p. \quad \zeta \in \mathbb{T} \quad \text{et} \quad |g(z)| = |f(z)|, \quad z \in \rho\mathbb{T}$$

alors $g(z) = cf(z)$ pour tout $z \in \mathbb{D}$ et pour un certain $c \in \mathbb{T}$.

Alors que la première partie a abordé le problème de reconstruction de phase pour les fonctions de la classe de Paley-Wiener, le Chapitre 6 étudie l'effet du zero-flipping sur la stabilité de ce problème. L'espace de Paley-Wiener PW_L , est l'ensemble de toutes les fonctions à carré sommable dont les transformées de Fourier sont supportées dans $[-L, L]$ pour un certain $L > 0$.

Soit $f \in PW_L$ et soit $a \in \mathbb{C} \setminus \mathbb{R}$. On définit l'opérateur de flipping, désigné par \mathfrak{F}_a , comme

$$\mathfrak{F}_a : f \mapsto \frac{(1 - x/\bar{a}) e^{x/\bar{a}}}{(1 - x/a) e^{x/a}} \cdot f, \quad x \in \mathbb{R}.$$

Cet opérateur représente le "zero-flipping" de f en a . En effet, la division de f par le facteur $(1 - x/a)e^{x/a}$ annule le facteur canonique associé à a , tandis que la multiplication du résultat par $(1 - x/\bar{a})e^{x/\bar{a}}$ complète le processus de retournement (flipping). Si $f(a) = 0$, alors $\mathfrak{F}_a f$ est dans la classe de Paley-Wiener et est toujours une solution du problème de reconstruction de phase. Par contre, lorsque $f(a) \neq 0$, $\mathfrak{F}_a f$ n'appartient plus à la classe de Paley-Wiener. Cependant nous montrerons que $\mathfrak{F}_a f$ est à large bande, c'est-à-dire que sa transformée de Fourier satisfait une condition d'intégrabilité avec un poids exponentiel.

La principale question que nous abordons dans ce chapitre est celle de la stabilité du zero-flipping. Rappelons que la stabilité a été démontrée par une inégalité de la forme dans (1.0.1). Certains termes d'erreur peuvent éventuellement être ajoutés. Dans notre cas, la stabilité du problème de reconstruction de phase dans une certaine sous-classe X de la classe de Paley-Wiener équivaut à

$$\inf_{|c|=1} \|f - cg\|_2 \leq C \| |f| - |g| \|_2 + (\text{terme d'erreur})$$

pour chaque $f \in X$ et chaque solution $g \in X$ du problème de reconstruction de phase. En particulier, pour $g = \mathfrak{F}_a f$ nous devrions reconstruire uniquement le terme d'erreur et la stabilité impliquerait que ce terme d'erreur soit petit. Notre objectif est alors d'estimer

$$\inf_{|c|=1} \|f - c\mathfrak{F}_a f\|_2.$$

Il s'avère que lorsque a se trouve dans une petite région proche de l'origine, cette quantité est en fait grande (proche de $2\|f\|_2$). Nous obtenons le résultat suivant :

Théorème. Soit $f \in PW_L$ pour un certain $L > 0$. Soit $a \in \mathbb{C}$ tel que $\text{Im } a > 0$ et $\beta_a = \frac{2 \text{Im } a}{|a|^2}$. Alors

$$\left| \inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 - 2\|f\|_2^2 \right| \leq 30(L \text{Im } a) \|f\|_2^2, \quad \text{si } \beta_a > 2L$$

et

$$\inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 \leq 2\omega_2(\hat{f}; \beta_a) \|f\|_2 + 8\sqrt{L \text{Im } a} \|f\|_2^2, \quad \text{si } \beta_a \leq 2L$$

où ω_2 est le module de continuité de \hat{f} dans L^2 .

Dans un deuxième étape, nous comparons l'effet de deux zero-flipping, c'est-à-dire nous étudions

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2. \quad (1.0.7)$$

Par exemple, si $a, b \in \mathbb{C} \setminus \mathbb{R}$ sont tels que $f(a) \neq 0$ et $f(b) = 0$, alors nous comparons une solution du problème de reconstruction de phase dans la classe de Paley-Wiener avec une solution obtenue après avoir fait une erreur sur la localisation du zéro. Notons que $\mathfrak{F}_a f, \mathfrak{F}_b f$ sont des solutions du problème de reconstruction de phase. Ainsi, si le problème de reconstruction de phase était stable, alors cette quantité devrait être un terme d'erreur puisqu'elle est bornée par

$$\inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2 + \inf_{|c|=1} \|\mathfrak{F}_b f - cf\|_2.$$

Nous obtiendrons effectivement une borne supérieure de (1.0.7) de la forme $C(f) \text{dist}(a, b)$ où $C(f)$ est une constante positive dépendant de f , et $\text{dist}(a, b)$ est une certaine fonction de distance dépendant de a et b . Nous avons le résultat de stabilité suivant :

Théorème. Soit $f \in PW_L$ pour un certain $L > 0$. Soit $a, b \in \mathbb{C}$ tel que $\text{Im } a, \text{Im } b > 0$, et $|a - b| \leq \frac{|b|}{2}$. Soit $\beta_a = \frac{2 \text{Im } a}{|a|^2}$ et $\beta_b = \frac{2 \text{Im } b}{|b|^2}$.

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2^2 \leq C_1(b) \omega_2(\hat{f}; \beta_a - \beta_b) \|f\|_2 + C_2(a, b) \|f\|_2^2.$$

où $C_1(b)$ et $C_2(a, b)$ sont des constantes positives dépendant de a et b , avec $C_2(a, b) \rightarrow 0$ comme $a \rightarrow b$.

Avec cette mesure de stabilité, ce théorème implique que le zero-flipping devient stable lorsque a et b sont convenablement proches l'un de l'autre. De plus, si nous supposons que $f(a) \neq 0$ et $f(b) = 0$, alors le théorème précédent implique que la solution à large bande $\mathfrak{F}_a f$ se rapproche d'une solution à bande limitée $c\mathfrak{F}_b f$ lorsque a se rapproche de b .

Chapter 2

Introduction

2.1 The Phase Retrieval Problem

The central theme of this thesis is the *phase retrieval problem*, which refers to the recovery of the phase of a function f from known data on its magnitude $|f|$ and a priori assumptions on f usually expressed in terms of some transform of f . We first provide a brief overview of developments on the theory and applications of phase retrieval problems.

2.1.1 Applications of Phase Retrieval

Phase retrieval problems are widely studied because of their physical applications in which the quantities involved are identified by their magnitude and phase, where the phase is difficult to measure while the magnitude is easily obtainable. The most notable application of phase retrieval problems is *diffraction imaging*, where diffraction refers to the phenomena when a wave spreads around obstacles or openings. In diffraction imaging, an object represented by a function f is exposed to a coherent electromagnetic wave through a laser, which produces a wave front given by the Fourier transform \hat{f} . Unfortunately, the detection device can only measure the magnitudes, so it remains to recover the lost phase information to characterize the object. This loss of phase information gives rise to the well-known Fourier phase retrieval problem, that is, to recover f from known data $|\hat{f}|$. In this case, once \hat{f} has been recovered, the last step is to apply inverse Fourier transform in order to fully recover f .

In 1912, it was discovered by von Laue that diffraction occurs when X-rays interact with crystals [26]. This very significant discovery had him receive a Nobel Prize in physics in 1914, and eventually gave rise to one very important application of the phase retrieval problem—*X-ray crystallography*. The goal of X-ray crystallography is to determine the position of atoms in a molecule (molecular structure) of a crystal, which in chemistry refers to a material whose atoms are arranged in an ordered microscopic structure. For this experiment, it would suffice to use a crystalline sample since molecules are very small. X-rays are scattered to the sample, and the resulting diffraction patterns are approximated by the Fourier transform of the electron density represented by a nonnegative function. In this case, only the

intensity of the patterns can be measured, and so we have another phase retrieval problem. The electron density f is represented as a function in \mathbb{R}^3 and is given by the convolution $f = e * \ell$ where e is the electron density in a single unit cell and ℓ is the lattice function, in terms of the Dirac delta function and the lattice points. By recovering the electron density, it will then recover the distribution of electrons, and thus fully obtaining the molecular structure. In case the molecules do not form crystals (e.g. polymers), the usual crystallography methods might not work. However, if these molecules behave like fibers, their structures can still be determined by X-ray diffraction, more specifically called *fiber diffraction*. This was mainly used for the discovery of the double helix structure of DNA during the 1950's [78]. All of the information stated above on crystallography and fiber diffraction can be found in the paper of Millane [55] with complete details.

Going forward to 1980, Sayre [64] proposed to apply the methods of X-ray crystallography to non-crystalline objects. Eventually in 1999, Sayre et. al. [53] were able to demonstrate the first successful image reconstruction of an array of gold dots, and this generalized method is now known as *coherent diffraction imaging* (CDI). This paved the way for various developments on lensless imaging and its applications on different fields of science (see e.g. [35, 46, 57, 61, 66] for more details and other references). The underlying process in CDI is the same as the previously mentioned imaging methods, and iterative phase retrieval algorithms are used to completely recover the phase. We refer the reader to [49, 54, 67] for more information on phase retrieval methods on optical imaging and CDI. Other physical applications of phase retrieval problems include work related to astronomy [22], lens design [23], quantum mechanics [20, 21, 37, 38], inverse scattering [63], microscopy [24, 72, 73], radar [38], and optics [55, 65]. Some classical examples can also be found in the survey articles of Klibanov et. al. [44] and Grohs et. al. [33], and in the book of Hurt [36].

2.1.2 Mathematics of Phase Retrieval

Generally speaking, given a real or complex-valued function f inside a vector space X (which can be finite or infinite dimensional), the aim of phase retrieval is to recover all $g \in X$ such that $|g| = |f|$. Note that if we have such a g , then cg with $c \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is also a solution. Early work on this problem centered on describing the set of solutions. However, phase retrieval problems generally have large solution sets, so various studies have introduced restrictions and coupled additional constraints to significantly reduce the solution set, or to consequently force the *uniqueness* of the solution. Here, we say that the solution g is unique if all such g 's are only of the form $g = cf$ for some $c \in \mathbb{T}$. For instance, one can impose some restrictions by solving the problem exclusively to a proper subset of X , or by limiting the support of the function. It is also necessary to assume nonnegativity for describing some physical quantities. On the other hand, some examples of coupled constraints include magnitudes on another portion of the domain, magnitudes involving certain operators, or magnitudes involving an additional reference. We refer the reader to the work by Klibanov et. al. [44] and Jaming [38, 39] for concrete examples of additional constraints.

Aside from uniqueness, the *stability* of the phase retrieval problem has also been

a subject of several work. Stability refers to continuous dependence of the solution from the given magnitude data. More precisely, given $f, g \in X$, if $|g|$ is close to $|f|$, then g is close to f up to the multiplication of a unimodular constant. In order to examine the stability, it is necessary to quantify the ‘closeness’ by assigning a proper measure. Some previous work on the stability of phase retrieval problems (see e.g. [3, 32]) used the classical measure $\inf_{|c|=1} \|f - cg\|$, and stability was shown by finding (in some cases) a positive constant C such that

$$\inf_{|c|=1} \|f - cg\|_{\mathcal{B}} \leq C \| |f| - |g| \|_{\mathcal{B}'}, \quad (2.1.1)$$

where $\mathcal{B}, \mathcal{B}'$ are suitable Banach or Hilbert spaces. Some papers (see e.g. [4, 9, 15, 33]) provided discussions on generalized stability properties using Banach spaces and frames in finite and infinite dimensions. They showed that phase retrieval problems situated in finite dimensions are stable. However, uniform instability occurs for the infinite dimensional case. In particular, as the dimension grows, the stability deteriorates severely [4, 15] (an example of this phenomenon can be found in [4]). Meanwhile, Alafari et. al. [3] proposed a new idea of stability in infinite dimensions by looking at the reconstruction of a function up to a non-global phase factor which may differ for some disjoint subsets of the domain. The reader may consult the survey article by Grohs et. al. [33] for a more technical discussion on the general formulation of the phase retrieval problem (uniqueness and stability) together with several examples.

Phase retrieval problems have recently been given more interest because of the progress in the discrete (finite-dimensional) case through the use of algorithms and numerical methods. Pioneering examples of iterative phase retrieval algorithms include the Gerchberg-Saxton algorithm [28] and the Fienup algorithm [27] which used alternating projections between the spatial and frequency domains via the Fourier transform. However, the main problem with these projection algorithms is that they sometimes diverge or give erroneous solutions [17, 47]. Other examples of numerical approaches are optimization methods which include the work of Candès et. al. [16, 17, 18] and of Waldspurger et. al. [75], though the investigation of optimization tools in phase retrieval algorithms seems to go back at least to the work of Luke et. al. [11, 14, 47]. Candès et. al. introduced the *PhaseLift* algorithm, which approximates the phase retrieval problem as a convex optimization problem through a matrix form. Waldspurger et. al. then considered an algorithm called *PhaseCut* which is similar to *PhaseLift* but with a different matrix reformulation. One may refer to [75] for a more precise comparison between these two algorithms. Even though convex optimization methods achieve accurate results, they have issues on high computational complexity when the signal dimension is large. For instance, the work of Arik et. al. [7] explored on how convex optimization-based phase retrieval algorithms can be implemented with low complexity. Furthermore, nonconvex and stochastic methods have also been developed (see [79] and references therein).

On the other hand, phase retrieval problems devoted to the continuous (infinite-dimensional) case have been solved in various settings. In some cases, the problem shifts to the complex analytic scenario by holomorphic extensions or by integral transforms. For instance, consider f and g to be two band-limited L^2 functions, i.e.

functions with compactly supported Fourier transforms. Then from the Paley-Wiener theorem, they extend to entire functions of finite type. Complex analysis was the key tool used by Akutowicz [1, 2], and a few years later independently by Walther [77] and Hofstetter [34], to determine all compactly-supported functions g such that $|g| = |f|$. They used the Hadamard factorization to obtain all possible ambiguities. This problem was also studied in two variables. In this case, the functions involved can be extended to \mathbb{C}^2 which is of exponential type, by the Plancherel-Polya theorem [59]. This \mathbb{C}^2 -entire function can be written as a product of globally irreducible factors. Barakat et. al. [10] were able to find necessary conditions for the nonuniqueness of the solution. A thorough discussion on the two-dimensional phase retrieval was provided in the book of Hurt [36].

We now enumerate some further work which used complex analytic tools. McDonald [51] has extended the work of Akutowicz to cover entire functions of finite genus. The solutions were characterized with the help of Hadamard factorization. He was also able to show the reduction of the solution set by imposing additional constraints related to some derivation operators. In [39], Jaming also used Hadamard factorization to show that an entire function of finite order can be reconstructed from its modulus on two lines, where these lines intersect at an angle which is an irrational multiple of π . This result then became the key to show several uniqueness results on the phase retrieval problem for the fractional Fourier transform. Bodmann et. al. [13] also used Jaming's result together with conformal mappings to show that a polynomial of degree at most $n - 1$ can be determined by its magnitude at $4n - 4$ well-chosen points lying in certain circles. Thakur [70] used a sampling result in the Paley-Wiener class to show that a real-valued band-limited function can be uniquely determined if it is sampled at more than twice its Nyquist rate. Grohs et. al. [32] solved the recovery of a function in a modulation space from phaseless Gabor measurements, where they considered the short-time Fourier transform and used the Poisson-Jensen formula in their estimates. Finally, Gröchenig [31] solved a phase retrieval problem in the class of shift-invariant spaces generated by a Gaussian using holomorphic extensions of Fourier series and the factorization of periodic entire functions.

Phase retrieval problems were also studied in the setting of holomorphic function spaces. McDonald [52] proved some uniqueness results for a phase retrieval problem in the Hardy space on the upper half-plane using a factorization which gives information on the zeros and boundary behavior [48]. Waldspurger et. al. [76] also used this factorization to show that a function belonging to the Hardy space on the upper half-plane is uniquely determined from its given moduli on \mathbb{R} and on another line in the upper half-plane parallel to \mathbb{R} . This result was then used to solve a continuous case of the recovery of an L^2 -function from the modulus of its wavelet transform. On the other hand, phase retrieval problems have also been situated on the Hardy space on the unit disc. For instance, Boche et. al. [12] showed that a function on the Hardy space on the unit disc without a singular part is uniquely determined by its modulus on two concentric circles. They also proved a special case for this result on complex polynomials, akin to the one by Bodmann stated above. More information on phase retrieval problems from holomorphic measurements can be found in [33].

2.2 Overview of the Thesis

In this dissertation, our unifying objective is to employ tools from complex analysis to (i) determine all the solutions; (ii) find additional constraints that yield significant reductions of the solution set; and (iii) investigate the stability of certain phase retrieval problems. The entire work is separated into three major parts.

Chapter 4 investigates the phase retrieval problem for *wide band* functions, namely functions with mildly decreasing Fourier transforms. Before we summarize our results, we provide an overview of the phase retrieval problem for *band-limited* signals, which are functions in $L^2(\mathbb{R})$ with compactly supported Fourier transforms. In this phase retrieval problem, given a band-limited function f , the goal is to find all band-limited functions g such that

$$|f(x)| = |g(x)|, \quad x \in \mathbb{R}. \quad (2.2.1)$$

This problem has been independently solved by Akutowicz, Walther and Hofstetter as mentioned earlier. In solving the problem, they first used the Paley-Wiener theorem which states that f and g extend into holomorphic functions in the complex plane of finite order 1, that is, f and g grow like $e^{a|z|}$ for some positive real number a . Next, they showed that (2.2.1) is then equivalent to

$$f(z)\overline{f(\bar{z})} = g(z)\overline{g(\bar{z})}, \quad z \in \mathbb{C}. \quad (2.2.2)$$

Observe that (2.2.2) is a reformulation of (2.2.1) when z is real and is an equality between two holomorphic functions so that it is valid for all $z \in \mathbb{C}$. Finally, they used the Hadamard factorization theorem which states that holomorphic functions of exponential type are characterized by their zeros. Here, we can write f as

$$f(z) = ce^{az}z^k \prod_{\alpha \in \mathcal{Z}(f)} \left(1 - \frac{z}{\alpha}\right) e^{z/\alpha}, \quad z \in \mathbb{C}$$

for some $c, a \in \mathbb{C}$ and $k \in \mathbb{N}$, where $\mathcal{Z}(f)$ is the zero set (excluding 0) of f . Now, (2.2.2) implies that each zero of g is either a zero of f or a complex conjugate of such a zero. Thus, it follows that g can be obtained by changing arbitrarily many zeros of f into their complex conjugates in the Hadamard factorization of g , and this was called *zero-flipping* by Walther. With this, it follows that all such g 's have Hadamard factorization given by

$$g(z) = \tilde{c}e^{(a+i\gamma)z}z^k \prod_{\alpha \in A} \left(1 - \frac{z}{\alpha}\right) e^{z/\alpha} \prod_{\alpha \in \mathcal{Z}(f) \setminus A} \left(1 - \frac{z}{\bar{\alpha}}\right) e^{z/\bar{\alpha}}, \quad z \in \mathbb{C}$$

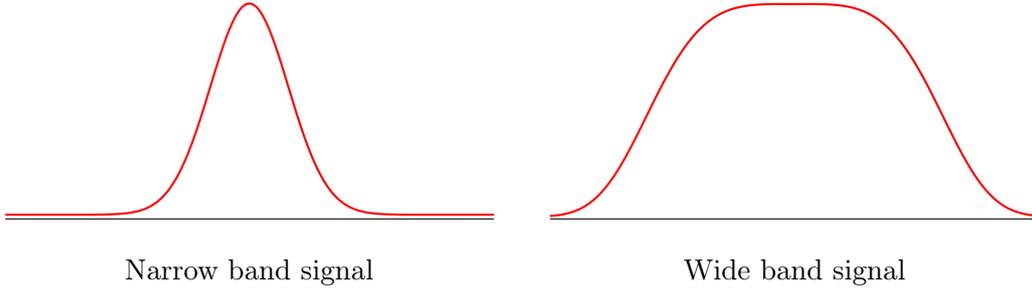
where $A \subseteq \mathcal{Z}(f)$, $|\tilde{c}| = |c|$, and $\gamma \in \mathbb{R}$. A more technical treatment of zero-flipping in this context can be found in the book of Hurt [36, Section 3.17].

McDonald [51] extended this proof to functions that have Fourier transforms with very fast decrease at infinity. For instance, in the case of Gaussian decrease, if $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^2}$, $a > 0$, then f and g extend to holomorphic functions of finite order 2 so that Hadamard factorization can still be used. Thus, the solutions

can also be obtained by zero-flipping. Furthermore, this proof extends to functions which satisfy an exponential decay condition of the form

$$|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^\alpha}, \quad \xi \in \mathbb{R} \quad (2.2.3)$$

for some $a > 0$ and $\alpha > 1$, but breaks down at $\alpha = 1$. Following these observations, it is then worthwhile to investigate the phase retrieval problem for functions satisfying (2.2.3) but for $\alpha = 1$, i.e. $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|}$. Functions with such decay are sometimes referred to as *wide band* signals in the engineering community, while those with a decay like (2.2.3) for $\alpha > 1$ are referred to as *narrow-banded*.



We thus solve the following phase retrieval problem: given $f \in L^2(\mathbb{R})$ such that $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$, find all $g \in L^2(\mathbb{R})$ such that $\widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ and $|g(x)| = |f(x)|$ for all $x \in \mathbb{R}$.

We shift to the complex analytic scenario by using the fact that f and g only extend holomorphically to an horizontal strip $\mathcal{S}_c = \{z \in \mathbb{C} : |\text{Im}z| < c\}$ in the complex plane so that (2.2.2) only holds for $z \in \mathcal{S}_c$, which implies that Hadamard factorization cannot be used. More precisely, \widehat{f} and \widehat{g} belong to $L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ if and only if f and g belong to a Hardy space on the strip $H^2_\tau(\mathcal{S}_c)$ [8, 43], which is the collection of holomorphic functions on \mathcal{S}_c such that

$$\sup_{|y|<c} \int_{\mathbb{R}} |f(t+iy)|^2 dt < +\infty.$$

To overcome this difficulty, we first reduce the problem to the Hardy space on the disc, denoted by $H^2(\mathbb{D})$, using a conformal bijection between \mathcal{S}_c and \mathbb{D} . It is worth noting that the Hardy spaces have been thoroughly studied when defined on the unit disc and the upper half plane (see for example [25, 29, 45, 50, 56, 69]). While it is possible to work directly on $H^2_\tau(\mathcal{S})$, we have decided that the proofs can be made more familiar on the unit disc. One of the reasons for working on a more familiar Hardy space is that the expression for the Poisson kernel is sufficiently simple to work with. Due to the conformal mapping, the expressions of the Poisson kernel in the other domains can be more complicated even in simpler situations like the strip. By transferring the problem to the Hardy space on the unit disc, we are able to exploit the fact that every function $f \in H^2(\mathbb{D})$ can be written as

$$f = BSO$$

where B is the Blaschke product formed by the zeros, S is the singular inner function associated to a positive singular measure on the unit circle \mathbb{T} , and O is the outer

2. Introduction

function determined by the boundary values on \mathbb{T} . The solution is now more evolved than the band-limited case as, aside from zero-flipping, the boundary behavior via the singular inner function and the outer function also take part in the solution. We then go back to the strip to solve our initial problem using an equivalent factorization on the strip to obtain the following characterization.

Theorem. *Let $f, g \in H_\tau^2(\mathcal{S})$ where $\mathcal{S} = \mathcal{S}_1$. Then $|f| = |g|$ on \mathbb{R} if and only if the inner-outer decomposition of f and g are given by*

$$f = e^{i\gamma} W^{-1/2} B_f S_f O_f \quad \text{and} \quad g = e^{i\gamma'} W^{-1/2} B_g S_g O_g$$

where

1. $W(z) = 4 \cosh^2(\frac{\pi}{4}z)$ for all $z \in \mathcal{S}$, B_f is the Blaschke product formed by the zeros of f , S_f is the singular inner part and O_f is the outer part;
2. B_g is the Blaschke product formed by the set of zeros $A \cup \{\bar{z} : z \in Z(f) \setminus A\}$ with $A \subset Z(f)$ where $Z(f)$ is the set of zeros of f counting multiplicities;
3. S_g is the singular inner function associated with the positive singular measure $\mu_g = \mu_f + \sigma$, where μ_f is the singular measure associated to f and σ is an odd real singular measure on $\partial\mathcal{S}$; and
4. O_g is the outer part of uO_f where u is an outer function belonging to the Smirnov class on the strip and $u = 1/u^*$ on \mathcal{S} .

Furthermore, using these factorizations for f and g , we can write $f = uv$ and $g = uv^*$ where u, v are holomorphic functions on \mathcal{S} .

Our next aim is to reduce the set of solutions by coupling another phase retrieval problem with our original one. These coupled problems are of the form

$$|g| = |f| \quad \text{and} \quad |Tg| = |Tf|, \quad f, g \in H_\tau^2(\mathcal{S})$$

where T is some transform. For our first coupled problem, we add a condition involving a fixed reference signal h :

$$|g| = |f| \quad \text{and} \quad |g - h| = |f - h| \quad \text{on } \mathbb{R}.$$

Here h is a nonzero complex-valued function with $e^{i \arg h}$ analytic on \mathbb{R} . We use the geometric interpretation of these constraints to show that this problem has exactly two solutions.

We also look at the problem with an additional condition involving the Fourier transforms:

$$|g| = |f| \quad \text{and} \quad |\widehat{g}| = |\widehat{f}| \quad \text{on } \mathbb{R}. \quad (2.2.4)$$

This coupled problem is also known as the *Pauli problem* [58, 38, 39]. Jaming [38] and Ismagilov [37] independently solved this in the band-limited case, where they explicitly constructed an uncountable family of solutions using *Riesz products*. Riesz products are infinite products that can be written as a well-defined trigonometric series [43]. We also do not get uniqueness for the wide band case, in fact we can find

a nondenumerable infinity of pairs $(f, g) \in H^2_\tau(\mathcal{S}) \times H^2_\tau(\mathcal{S})$ that satisfy (2.2.4) and are not constant multiples of one another. We construct these pairs by adapting the proof of Jaming and Ismagilov.

Next, we look at the coupled problem with the condition involving a derivation operator

$$|g| = |f| \quad \text{and} \quad |Dg| = |Df| \quad \text{on } \mathbb{R}$$

where D is a derivation operator given by one of the following:

- $\frac{\partial}{\partial z}$ where $\frac{\partial}{\partial z} f(z) = f'(z)$;
- δ given by $\delta(f)(z) = f(z + b) - f(z)$ for $b > 0$; and
- γ given by $\gamma(f)(z) = f(qz) - f(z)$ for $|q| < 1$.

This coupled problem was inspired by the work of McDonald on the phase retrieval problem for entire functions in [51]. Using the special properties of D , f and g and one of the consequences of our main result, we show that our coupled problem has exactly two solutions.

Finally, for our last coupled problem, we add the condition $|g| = |f|$ on a segment on the strip. More precisely, we prove that if $f, g \in H^2_\tau(\mathcal{S})$ with

$$|g| = |f| \quad \text{on} \quad \mathbb{R} \cup (-e^{i\alpha} + a, e^{i\alpha} + a), \quad a \in \mathbb{R} \text{ and } \alpha \notin \pi\mathbb{Q}$$

then g is uniquely determined by f . To show this, we first use our main results to show a uniqueness result for $f, g \in H^2(\mathbb{D})$ with similar constraints to the problem above, that is,

$$|g| = |f| \quad \text{on} \quad (-1, 1) \cup e^{i\alpha}(-1, 1), \quad \alpha \notin \pi\mathbb{Q}. \quad (2.2.5)$$

After which, the uniqueness result for our last coupled problem will shortly follow.

In Chapter 5, we generalize some uniqueness results in the phase retrieval of functions on $H^2(\mathbb{D})$. One of our motivation stems from the coupled phase retrieval problems on the unit disc mentioned above in (2.2.5). We extend this uniqueness result to holomorphic functions on open connected domains.

Theorem. *Let Ω be an open connected domain. Let $f, g \in \text{Hol}(\Omega)$ and suppose that*

$$|g(z)| = |f(z)|, \quad z \in I \cup I_\alpha$$

where I and I_α are segments inside Ω , I_α is the α -rotation of I about the midpoint of I , and $\alpha \notin \pi\mathbb{Q}$. Then $g(z) = cf(z)$ for all $z \in \Omega$ and for some $c \in \mathbb{T}$.

We reduce this coupled problem above in a similar situation as in (2.2.5), but with a different yet simpler proof using Fourier series. On the other hand, our next goal is to improve the result of Boche et. al. [12, Theorem 3], which states that given functions f and g in the Hardy space on the disc without singular inner part, if $|f| = |g|$ on the unit circle and on a smaller circle inside the unit circle, then g can be uniquely determined from f . Moreover, they also showed by construction that the Blaschke product associated with g can be uniquely recovered by its modulus on a

smaller circle inside the unit circle. We extend this uniqueness result to all functions in the Nevanlinna class \mathcal{N} , which is the collection of holomorphic functions on the unit disc which can be written as a quotient of two bounded holomorphic functions on the disc. The following theorem holds regardless of the presence of the singular parts in contrast to [12, Theorem 3].

Theorem. *Let $f, g \in \mathcal{N}$ and let $\rho \in (0, 1)$. If*

$$|g_*(\zeta)| = |f_*(\zeta)|, \quad \text{a.e. } \zeta \in \mathbb{T} \quad \text{and} \quad |g(z)| = |f(z)|, \quad z \in \rho\mathbb{T}$$

then $g(z) = cf(z)$ for all $z \in \mathbb{D}$ and for some $c \in \mathbb{T}$.

We stress that the proof of this result does not make use of the inner-outer factorization in the Nevanlinna class. Furthermore, we use this result to show that a meromorphic function in \mathbb{C} with no pole at 0 is uniquely determined by its modulus on two concentric circles.

While the first part touched on the phase retrieval problem for functions in the Paley-Wiener class, Chapter 6 investigates the effect of zero-flipping on this problem's stability. We recall that the Paley-Wiener class, often denoted as PW_L , is the set of all L^2 functions whose Fourier transforms are supported in $[-L, L]$ for some $L > 0$.

We introduce the major tool which we use to investigate the stability. Let $f \in PW_L$ and let $a \in \mathbb{C} \setminus \mathbb{R}$. Define the *flipping operator*, denoted by \mathfrak{F}_a where

$$\mathfrak{F}_a : f \mapsto \frac{(1 - x/\bar{a}) e^{x/\bar{a}}}{(1 - x/a) e^{x/a}} \cdot f, \quad x \in \mathbb{R}.$$

This operator exhibits the zero-flipping of f at a . Indeed, dividing f by the factor $(1 - x/a)e^{x/a}$ cancels the canonical factor associated to a while multiplying the result by $(1 - x/\bar{a})e^{x/\bar{a}}$ completes the flipping process. Whenever $f(a) = 0$, $\mathfrak{F}_a f$ is still inside the Paley-Wiener class and is always a solution of the phase retrieval problem. On the other hand, when $f(a) \neq 0$, $\mathfrak{F}_a f$ no longer belongs to the Paley-Wiener class. However we will show that $\mathfrak{F}_a f$ is wide-banded, that is, its Fourier transform satisfies a square integrability condition with an exponential weight.

The main question we address in this chapter is that of the stability of zero-flipping. Recall that stability was shown by an inequality of the form in (2.1.1). Some error terms may eventually be added. In our case, stability of the phase retrieval problem in some subclass X of the Paley-Wiener class would mean that

$$\inf_{|c|=1} \|f - cg\|_2 \leq C \| |f| - |g| \|_2 + (\text{error term})$$

for every $f \in X$ and every solution $g \in X$ of the phase retrieval problem. In particular, for $g = \mathfrak{F}_a f$ we should recover the error term only and stability would imply that this error term be small. Our aim here is to investigate this issue, namely to get an estimate of

$$\inf_{|c|=1} \|f - c\mathfrak{F}_a f\|_2.$$

It turns out that when a is in some small region near the origin, then this quantity is actually large (close to $2\|f\|_2$) so that zero-flipping of such a zero leads to instabilities.

On the other hand, we will show that the error term is small when we flip a zero that is large and close to the real axis so that such a flipping does not lead to instabilities. More precisely, we have the following:

Theorem. *Let $f \in PW_L$ for some $L > 0$. Let $a \in \mathbb{C}$ such that $\text{Im } a > 0$ and $\beta_a = \frac{2 \text{Im } a}{|a|^2}$. Then*

$$\left| \inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 - 2\|f\|_2^2 \right| \leq 30(L \text{Im } a) \|f\|_2^2, \quad \text{if } \beta_a > 2L$$

and

$$\inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 \leq 2\omega_2(\hat{f}; \beta_a) \|f\|_2 + 8\sqrt{L \text{Im } a} \|f\|_2^2, \quad \text{if } \beta_a \leq 2L$$

where ω_2 is the L^2 -modulus of continuity of \hat{f} .

In a second stage, we compare the effect of two zero-flipping, that is, we investigate

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2. \quad (2.2.6)$$

For instance, if $a, b \in \mathbb{C} \setminus \mathbb{R}$ are such that $f(a) \neq 0$ and $f(b) = 0$, then we are comparing a genuine solution of the phase retrieval problem in the Paley-Wiener class with a solution obtained after having made a mistake on the location of the zero. Note that $\mathfrak{F}_a f, \mathfrak{F}_b f$ are solutions of the phase retrieval problem. Thus, if the phase retrieval problem were stable, then this quantity should be an error term since it is bounded by

$$\inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2 + \inf_{|c|=1} \|\mathfrak{F}_b f - cf\|_2$$

and should thus be an error term. We will indeed obtain an upper bound of (2.2.6) of the form $C(f) \text{dist}(a, b)$ where $C(f)$ is a positive constant depending on f , and $\text{dist}(a, b)$ is some distance function depending on a and b . We have the following stability result:

Theorem. *Let $f \in PW_L$ for some $L > 0$. Let $a, b \in \mathbb{C}$ such that $\text{Im } a, \text{Im } b > 0$, and $|a - b| \leq \frac{|b|}{2}$. Let $\beta_a = \frac{2 \text{Im } a}{|a|^2}$ and $\beta_b = \frac{2 \text{Im } b}{|b|^2}$.*

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2^2 \leq C_1(b) \omega_2(\hat{f}; \beta_a - \beta_b) \|f\|_2 + C_2(a, b) \|f\|_2^2.$$

where $C_1(b)$ and $C_2(a, b)$ are positive constants depending on a and b , with $C_2(a, b) \rightarrow 0$ as $a \rightarrow b$.

With this stability measure, this theorem implies that zero-flipping becomes stable when a and b are suitably close to each other. In addition, if we suppose that $f(a) \neq 0$ and $f(b) = 0$, then the previous theorem implies that the wide-banded solution $\mathfrak{F}_a f$ goes close to a band-limited solution $c\mathfrak{F}_b f$ as a goes close to b .

Chapter 3

Preliminaries

This dissertation will use various concepts from complex and Fourier analysis. In this chapter, we recall all the necessary definitions and results which will be considered in this study.

3.1 Holomorphic Functions and their Properties

We first recall some properties of holomorphic functions. The information here can be found in most complex analysis books (see e.g. [19, 62, 68]).

Let Ω be an open subset of \mathbb{C} and f be a complex-valued function defined on Ω . Recall that a function f is said to be **holomorphic** on Ω if f is continuously \mathbb{C} -differentiable at every point in Ω . We denote by $\text{Hol}(\Omega)$ the set of holomorphic functions on Ω . An important fact about holomorphic functions is that they can be represented as a power series, which directly implies that holomorphic functions are infinitely differentiable. The function f is said to be **entire** if it is holomorphic on all of \mathbb{C} .

It is also necessary to recall that the zeros and poles of a nontrivial holomorphic function are isolated. Using this important property, it can easily be shown that a holomorphic function defined on a compact set has a finite number of zeros. Moreover, the isolation of zeros was used to show that if two holomorphic functions agree on a subset of Ω which has accumulation points (e.g. has positive measure), then the two functions agree everywhere on Ω . This mentioned fact now gives an insight on the extension of the domain of a holomorphic function. Another classical result about holomorphic extensions is given by the Schwarz Reflection Principle, which says that if a holomorphic function is continuous on some open set in the upper half plane and has real values on the real line, then it can be extended by reflecting across the real line.

We now recall the Poisson-Jensen formula, which gives the relationship of the modulus of a holomorphic function on a circle with the moduli of the zeros inside this circle. We first consider the following notation which will be used several times: denote by $D(a, r)$ the open disc centered at $a \in \mathbb{C}$ of positive radius r , $r\mathbb{D} = D(0, r)$ the disc centered at the origin of radius r , $r\mathbb{T}$ the boundary of $r\mathbb{D}$, and $r\overline{\mathbb{D}}$ the closed disc given by $r\mathbb{D} \cup r\mathbb{T}$. Whenever $r = 1$, we simply use \mathbb{D} and \mathbb{T} .

Theorem 3.1.1 (Poisson-Jensen Formula). *Let f be holomorphic on a region that contains $r\overline{\mathbb{D}}$ and suppose that a_1, a_2, \dots, a_n are the zeros of f in $r\mathbb{D}$ counted with multiplicities. If $z \in r\mathbb{D}$ and $f(z) \neq 0$, then*

$$\log |f(z)| = \sum_{k=1}^n \log \left| \frac{r(z - a_k)}{r^2 - \bar{a}_k z} \right| + \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \left(\frac{re^{i\theta} + z}{re^{i\theta} - z} \right) \log |f(re^{i\theta})| d\theta.$$

When $z = 0$, we obtain a special case of the Poisson-Jensen formula known as **Jensen's formula**:

$$\log |f(0)| = \sum_{k=1}^n \log \left(\frac{|a_k|}{r} \right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta.$$

This formula has various applications in the theory of entire functions and holomorphic function spaces.

We also recall the Weierstrass Factorization Theorem, which says that every entire function can be expressed as an infinite product of **canonical factors** formed by the zeros, multiplied by a zero-free factor. These canonical factors are given by

$$E_0(z, \zeta) = 1 - \frac{z}{\zeta} \quad \text{and}$$

$$E_p(z, \zeta) = \left(1 - \frac{z}{\zeta} \right) \exp \left(\sum_{k=1}^p \frac{1}{k} \cdot \frac{z^k}{\zeta^k} \right), \quad p \in \mathbb{N}$$

with $0 \neq \zeta \in \mathbb{C}$, where p is said to be the degree of the canonical factor.

Theorem 3.1.2 (Weierstrass Factorization Theorem). *Let f be an entire function and let $\{z_k\}_{k \in \mathbb{N}}$ be the nonzero zeros of f counted with multiplicities. Suppose that f has a zero at 0 of multiplicity m . Then there is an entire function g and a sequence $\{p_k\}$ of nonnegative integers such that for all $z \in \mathbb{C}$,*

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} E_{p_k}(z, z_k), \quad z \in \mathbb{C}. \quad (3.1.1)$$

The sequence $\{p_k\}_{k \in \mathbb{N}}$ in this theorem satisfies

$$\sum_{k \in \mathbb{N}} \left(\frac{r}{|z_k|} \right)^{p_k+1} < \infty, \quad r > 0$$

so that the infinite product in (3.1.1) converges uniformly on compact subsets of \mathbb{C} . Moreover, the factor e^g is present as the quotient of f and the factor formed by the zeros gives a nowhere vanishing entire function.

Now, we recall a condition one can possibly impose on an entire function f to get a special form for the Weierstrass factorization.

Definition 3.1.3. An entire function f is of **finite order** if there exists a $\rho > 0$ and constants $A, B > 0$ such that

$$|f(z)| \leq A e^{B|z|^\rho}, \quad z \in \mathbb{C}.$$

The infimum of such ρ is called the **order** (of growth) of f .

By using some consequences of Jensen's formula and properties of the canonical factors, it can be shown that if f is of finite order, then we get a refinement of the Weierstrass product in (3.1.1) where g is a polynomial and all of the p_k 's are equal.

Theorem 3.1.4 (Hadamard Factorization Theorem). *Let f be an entire function and let $\{z_k\}_{k \in \mathbb{N}}$ be the nonzero zeros of f counted with multiplicities. Suppose that f has a zero at 0 of multiplicity m . If f is of finite order ρ , then f can be written as*

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} E_{[\rho]}(z, z_k), \quad z \in \mathbb{C}$$

where g is a polynomial of degree at most ρ , $[\rho]$ is the integer part of ρ , and for all $\varepsilon > 0$

$$\sum_{k=1}^{\infty} |z_k|^{-(\rho+\varepsilon)} < \infty.$$

The Hadamard factorization shows that every entire function of finite order is essentially determined by its zeros. Moreover, one may replace ρ by $\rho' \geq \rho$. Let us look at a classic example to illustrate this factorization theorem.

Example 3.1.5. For all $z \in \mathbb{C}$, consider the entire function $f(z) = \sin(\pi z)$. Then its Hadamard factorization is given by

$$f(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \quad (3.1.2)$$

Solution. It is easy to see that $|f(z)| \leq e^{\pi|z|}$ for all $z \in \mathbb{C}$, and so f is of order ≤ 1 . Consider $z = ix$ for $x \in \mathbb{R}$. As $f(ix) = \sinh(\pi x) \sim \frac{e^{\pi x}}{2}$ at infinity, the order is exactly 1. Now, note that the zero set of f is \mathbb{Z} , all of which have multiplicity 1. Moreover, for all $\varepsilon > 0$,

$$\sum_{0 \neq n \in \mathbb{Z}} \frac{1}{|n|^{1+\varepsilon}} < \infty.$$

Thus by the Hadamard factorization theorem,

$$f(z) = \sin(\pi z) = z e^{Az+B} \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right) e^{z/n}, \quad z \in \mathbb{C}$$

and for some constants $A, B \in \mathbb{R}$. Dividing this equation both sides by z and letting $z \rightarrow 0$, we get that $e^B = \pi$. As $\sin(\pi z)$ and πz are odd and the infinite product is even, e^{Az} must be even, and so $A = 0$. Therefore f has the factorization given in (3.1.2). \square

3.2 Fourier Transform and the Paley-Wiener Theorems

In this section, we recall the Fourier transform and some of its most important properties. For more detailed discussions about the facts stated in this section, we

refer the reader to the books by Grafakos [30], Katznelson [43], and by Stein and Weiss [69].

Recall that for $f \in L^1(\mathbb{R})$, the **Fourier transform** of f , denoted by \widehat{f} or $\mathcal{F}[f]$, is defined as

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}.$$

It can easily be shown that the Fourier transform satisfies the following properties: for all $x, \xi \in \mathbb{R}$ and some nonzero $a \in \mathbb{R}$,

- (translation) if $\tau_a f(x) = f(x - a)$, then

$$\widehat{\tau_a f}(\xi) = e^{ia\xi} \widehat{f}(\xi) = M_{-a} \widehat{f}(\xi);$$

- (dilation) if $\delta_a f(x) = f(ax)$, then

$$\widehat{\delta_a f}(\xi) = \frac{1}{|a|} \widehat{f}\left(\frac{\xi}{a}\right) = \frac{1}{|a|} \delta_{1/a} \widehat{f}(\xi);$$

- (modulation) if $M_a f(x) = e^{iax} f(x)$, then

$$\widehat{M_a f}(\xi) = \widehat{f}(\xi - a) = \tau_a \widehat{f}(\xi); \text{ and}$$

- (convolution) if $f, g \in L^2(\mathbb{R})$, then $\widehat{fg} = \widehat{f} * \widehat{g}$ where

$$(\widehat{f} * \widehat{g})(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(s) \widehat{g}(\xi - s) ds.$$

Now, if the Fourier transform \widehat{f} is known provided that it is in $L^1(\mathbb{R})$, then f can be recovered by using the **Fourier inversion** formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} d\xi, \quad \text{a.e. } x \in \mathbb{R}.$$

In the context of this dissertation, the Fourier transform on the Hilbert space $L^2(\mathbb{R})$ will be very important, as it has various useful properties and consequences. To link the $L^1(\mathbb{R})$ case to the $L^2(\mathbb{R})$ case, we recall Plancherel's theorem which states that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\widehat{f} \in L^2(\mathbb{R})$ and $\|\widehat{f}\|_2 = \|f\|_2$. Using this result and the fact that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the Fourier transform extends to a unitary operator on $L^2(\mathbb{R})$.

Now, we recall some **Paley-Wiener theorems**, which describes the relationship between the holomorphicity and growth of a function f , and the growth of its Fourier transform \widehat{f} . The following Paley-Wiener theorem characterizes functions with compactly supported Fourier transforms.

Theorem 3.2.1. *Let F be entire and $a > 0$. The following conditions are equivalent:*

1. $|F(z)| \leq Ce^{a|z|}$ for all $z \in \mathbb{C}$ and some $C > 0$, and $F \in L^2(\mathbb{R})$.

2. There exists a function $f \in L^2(\mathbb{R})$ supported in $[-a, a]$ such that

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{iz\xi} d\xi, \quad z \in \mathbb{C}.$$

This result says that a function in $L^2(\mathbb{R})$ with a compactly-supported Fourier transform extends to an entire function of exponential type. Recall also that whenever $f \in L^2(\mathbb{R})$ with $\text{supp } \widehat{f} \subseteq [-a, a]$, f is said to be **band-limited** and the collection of all such f 's is called the **Paley-Wiener class** denoted by PW_a . The space PW_a is a closed linear subspace of $L^2(\mathbb{R})$ and is a Hilbert space itself with the same inner product as in $L^2(\mathbb{R})$.

We also recall another Paley-Wiener theorem which deals with the connection of the growth of the Fourier transform with certain holomorphic functions on a horizontal strip:

Theorem 3.2.2. *Let $f \in L^2(\mathbb{R})$ and $a > 0$. The following are equivalent:*

1. f is the restriction to \mathbb{R} of a function F holomorphic on the strip $\{z \in \mathbb{C} : |\text{Im}z| < a\}$ such that

$$\sup_{|y| < a} \int_{\mathbb{R}} |F(x + iy)|^2 dx < +\infty.$$

2. $\widehat{f} \in L^2(\mathbb{R}, e^{2a|\xi|} d\xi)$, where

$$L^2(\mathbb{R}, e^{2a|\xi|} d\xi) = \left\{ f \text{ is measurable} : \int_{\mathbb{R}} |f(\xi)|^2 e^{2a|\xi|} d\xi < +\infty \right\}.$$

3.3 Hardy Spaces on the Unit Disc

In this section, we recall a class of holomorphic functions on the disc called the **Hardy spaces**, together with some definitions and properties relevant to our study. We refer the reader to the books [25, 29, 45, 50, 56, 62] for the complete proofs of the results presented here, as well as other topics related to Hardy spaces.

Recall that the Hardy spaces on the unit disc, denoted by $H^p(\mathbb{D})$ for $0 < p \leq \infty$, is the set of holomorphic functions on \mathbb{D} which satisfies

$$\|f\|_{H^p(\mathbb{D})}^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty \quad (3.3.1)$$

where $0 < p < \infty$ and

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{0 \leq r < 1} |f(re^{i\theta})| < \infty \quad (3.3.2)$$

where $p = \infty$. It can be shown by Hölder's inequality that $H^\infty(\mathbb{D}) \subset H^q(\mathbb{D}) \subset H^p(\mathbb{D})$ for $0 < p < q < \infty$. For $1 \leq p \leq \infty$, $H^p(\mathbb{D})$ is a Banach space endowed with the norm given in (3.3.1) and (3.3.2), and is isomorphic to a closed subspace of $L^p(\mathbb{T})$ given by

$$\left\{ f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for } n = -1, -2, \dots \right\},$$

where

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}$$

is the n -th **Fourier coefficient** of f . As a matter of fact, every $f \in H^p(\mathbb{D})$, $1 \leq p \leq \infty$, can be written as a Fourier series given by

$$f(z) = \sum_{n \geq 0} \widehat{f}(n) z^n, \quad z \in \mathbb{D}.$$

Whenever $p = 2$, $H^2(\mathbb{D})$ is a Hilbert space, and by Parseval's identity it follows that

$$\|f\|_{H^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2.$$

In other words, $f \in H^2(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$.

We also recall Fatou's theorem, which guarantees the existence of the boundary values of functions in the Hardy space.

Theorem 3.3.1 (Fatou's Theorem). *For $0 < p \leq \infty$, every nonzero $f \in H^p(\mathbb{D})$ admits a radial limit given by*

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = f_*(e^{i\theta})$$

for almost every $e^{i\theta} \in \mathbb{T}$.

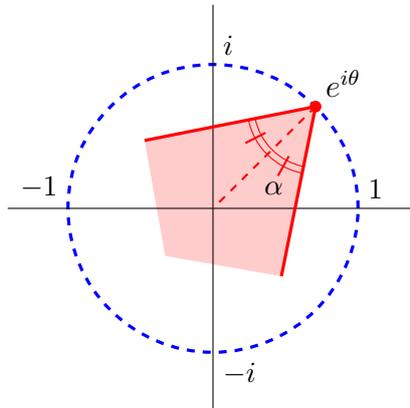
In addition, the nontangential limit of f given by

$$\lim_{\substack{z \rightarrow e^{i\theta} \\ z \in S_\alpha(\theta)}} f(z) = f_*(e^{i\theta})$$

exists for almost every $e^{i\theta} \in \mathbb{T}$ and for every $\alpha \in (0, \frac{\pi}{2})$. Here, $S_\alpha(\theta)$ is called the **Stolz domain** defined as

$$S_\alpha(\theta) = \left\{ z \in \mathbb{D} : |z - e^{i\theta}| \leq C_\alpha(1 - |z|) \right\}$$

with $C_\alpha > 1$.



The Stolz domain $S_\alpha(\theta)$.

Using Fatou's theorem, it can be shown that $f_* \in L^p(\mathbb{T})$, $\log |f_*| \in L^1(\mathbb{T})$, and $\|f\|_{H^p(\mathbb{D})} = \|f_*\|_{L^p(\mathbb{T})}$.

Given the boundary values f_* of a function $f \in H^p(\mathbb{D})$, it is possible to recover f using the integrals given by

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} f_*(e^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_*(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}.$$

3.3.1 Inner-Outer Factorization in $H^p(\mathbb{D})$

Let $\alpha \in \mathbb{D}$. Recall that a **Blaschke factor** is defined as

$$b_\alpha(z) = \begin{cases} \frac{|\alpha|}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha}z}, & \text{if } \alpha \neq 0 \\ z, & \text{if } \alpha = 0 \end{cases}$$

for all $z \in \mathbb{D}$. These Blaschke factors are conformal maps from \mathbb{D} to itself. If $\{z_n\}_{n=1}^\infty$ is a sequence of complex numbers in the unit disc \mathbb{D} , recall that for all $z \in \mathbb{D}$,

$$B_N(z) = \prod_{n=1}^N b_{z_n}(z) \tag{3.3.3}$$

is called a **finite Blaschke product**. Under certain conditions, the partial products in (3.3.3) converges uniformly to an **infinite Blaschke product** given by

$$B(z) = \prod_{n=1}^{\infty} b_{z_n}(z), \quad z \in \mathbb{D}.$$

Theorem 3.3.2. *Let $\{z_n\}_{n=1}^\infty$ be a sequence in \mathbb{D} such that $\lim_{n \rightarrow \infty} |z_n| = 1$. Then the partial products B_N given by (3.3.3) converges uniformly to B on compact subsets of \mathbb{D} (and even on compact subsets of $\mathbb{C} \setminus \text{clos}\{1/z_n : n \in \mathbb{N}\}$) if and only if*

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

The convergence criterion stated above is called the **Blaschke condition**. Using this theorem, it can be shown that $B \in H^\infty(\mathbb{D})$ with $|B(z)| \leq 1$ for all $z \in \mathbb{D}$ and $|B_*(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in \mathbb{T}$.

Blaschke products are examples of **inner functions**, which are bounded holomorphic functions on \mathbb{D} with modulus 1 almost everywhere on \mathbb{T} . However, unlike Blaschke products, there are other inner functions which do not have any zeros on \mathbb{D} . Such functions are called **singular inner functions** and they are defined as

$$S_\mu(z) = \exp \left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right), \quad z \in \mathbb{D}$$

where μ is a finite, positive measure on \mathbb{T} singular with respect to the Lebesgue measure. It can also be verified that $0 \neq |S_\mu(z)| \leq 1$ for all $z \in \mathbb{D}$, $S_\mu(0) > 0$, and $|(S_\mu)_*(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. We recall a classical example of a singular inner function.

Example 3.3.3. Let $a > 0$. The function $S_{a\delta_1} \in \text{Hol}(\mathbb{D})$ given by

$$S_{a\delta_1}(z) = \exp\left(-a \frac{1+z}{1-z}\right), \quad z \in \mathbb{D}$$

is a singular inner function associated to the Dirac measure $a\delta_1$.

Observe also that the product of a Blaschke product and a singular inner function is also an inner function. In fact, it actually turns out that every inner function can be written in this way.

Theorem 3.3.4. *Let I be inner and B be the Blaschke product formed by the zeros of I . Then there exists a unimodular constant $e^{i\gamma}$ and a unique, finite, positive, and singular measure μ on \mathbb{T} such that $I = e^{i\gamma}BS_\mu$.*

Now, let φ be a nonzero function on \mathbb{T} with $\log |\varphi| \in L^1(\mathbb{T})$. For all $z \in \mathbb{D}$, recall that an **outer function** is defined as

$$O(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |\varphi(e^{i\theta})| d\theta\right).$$

The outer function satisfies

$$|O_*(e^{i\theta})| = \lim_{r \rightarrow 1} |O(re^{i\theta})| = |\varphi(e^{i\theta})| \quad (3.3.4)$$

for almost every $e^{i\theta} \in \mathbb{T}$, and $O \in H^p(\mathbb{D})$ if and only if $\varphi \in L^p(\mathbb{T})$. Furthermore, every outer function can be written as a quotient of two bounded outer functions.

Given a function $f \in H^p(\mathbb{D})$, its **outer part** O_f defined by

$$O_f(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f_*(e^{i\theta})| d\theta\right), \quad z \in \mathbb{D} \quad (3.3.5)$$

satisfies

$$|f(z)| \leq |O_f(z)|, \quad z \in \mathbb{D} \quad (3.3.6)$$

and $\|f\|_{H^p(\mathbb{D})} = \|O_f\|_{H^p(\mathbb{D})}$. On the other hand, consider the function $I_f = f/O_f$. Note that $I_f \in \text{Hol}(\mathbb{D})$ as O_f is zero-free on \mathbb{D} . Then (3.3.4) and (3.3.6) imply that I_f is an inner function, called the **inner part** of f . Finally, by decomposing the inner part I_f as in Theorem 3.3.4, we finally get the complete **inner-outer factorization** of $f \in H^p(\mathbb{D})$ given by

$$f = e^{i\gamma}BS_\mu O_f,$$

where $e^{i\gamma} \in \mathbb{T}$, B is the Blaschke product formed by the zeros of f , and S_μ is the singular inner function associated with the finite positive singular measure μ . This factorization is unique up to the multiplication of a unimodular constant.

3.3.2 Nevanlinna and Smirnov Classes

Let us now recall a larger class of holomorphic functions on the disc called the **Nevanlinna class**, denoted by \mathcal{N} , which is defined as

$$\mathcal{N} = \left\{ f \in \text{Hol}(\mathbb{D}) : f = \frac{f_1}{f_2}, \text{ where } f_1, f_2 \in H^\infty(\mathbb{D}) \right\}. \quad (3.3.7)$$

3. Preliminaries

With this representation, it is immediate that $H^p(\mathbb{D}) \subset \mathcal{N}$. Moreover, the Nevanlinna class and the Hardy spaces share the following properties: for every $f \in \mathcal{N}$,

- the zeros of f also satisfies the Blaschke condition;
- the radial limit f_* also exists almost everywhere and is finite;
- $\log |f_*| \in L^1(\mathbb{T})$ whenever $f \not\equiv 0$.

Unlike that of $H^p(\mathbb{D})$, the radial limit of a function in \mathcal{N} is not necessarily in $L^p(\mathbb{T})$. Furthermore, (3.3.7) implies that every $f \in \mathcal{N}$ can be decomposed as

$$f = \frac{e^{i\gamma} B S_{\mu_1} O_f}{S_{\mu_2}}.$$

where $e^{i\gamma} \in \mathbb{T}$, B is the Blaschke product formed by the zeros of f , S_{μ_1} and S_{μ_2} are singular inner functions associated to the finite positive singular measures μ_1 and μ_2 respectively, and O_f is the outer part as defined in (3.3.5). Notice that $S_{\mu_1} (S_{\mu_2})^{-1} = S_{\mu_1 - \mu_2}$ where $\mu_1 - \mu_2$ is a signed singular measure on \mathbb{T} . This indicates that the singular measure associated to the factorization in \mathcal{N} is not necessarily positive. If in case that $\mu_2 \equiv 0$, then f belongs to a subclass of \mathcal{N} called the **Smirnov class**, denoted by \mathcal{N}^+ . With this, it easily follows that

$$\mathcal{N}^+ = \left\{ f \in \text{Hol}(\mathbb{D}) : f = \frac{f_1}{f_2}, \text{ where } f_1, f_2 \in H^\infty(\mathbb{D}) \text{ and } f_2 \text{ is outer} \right\}.$$

The classes \mathcal{N}^+ and \mathcal{N} together with $H^p(\mathbb{D})$ are related by the inclusion

$$H^p(\mathbb{D}) \subset \mathcal{N}^+ \subset \mathcal{N}.$$

Finally, we recall the Generalized Maximum Principle by Smirnov which gives the main advantage of \mathcal{N}^+ over \mathcal{N} . This also gives a characterization between \mathcal{N}^+ and $H^p(\mathbb{D})$.

Theorem 3.3.5 (Generalized Maximum Principle). *Let $f \in \mathcal{N}^+$ and let g be an outer function. If $|f_*| \leq |g_*|$ on \mathbb{T} , then $|f| \leq |g|$ on \mathbb{D} . In particular, $f \in H^p(\mathbb{D})$, $0 \leq p \leq \infty$, if and only if the radial limit of f is in $L^p(\mathbb{T})$, i.e.*

$$H^p(\mathbb{D}) = \mathcal{N}^+ \cap L^p(\mathbb{T}).$$

The latter equivalence, however, may not hold if \mathcal{N}^+ is replaced by \mathcal{N} . For instance, if we consider $f = (S_{\delta_1})^{-1}$, clearly $f \in \mathcal{N}$ with $f_* \in L^\infty(\mathbb{T})$ but $f \notin H^\infty(\mathbb{D})$.

Chapter 4

Phase Retrieval for Wide Band Signals

This chapter is devoted to the phase retrieval problem for wide band signals, which was presented in the papers [40, 41]. We solve the following problem: given $f \in L^2(\mathbb{R})$ with $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|}d\xi)$, find all $g \in L^2(\mathbb{R})$ with $\widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|}d\xi)$ such that $|f(x)| = |g(x)|$ for all $x \in \mathbb{R}$. We first reduce the problem to an equivalent problem on the unit disc using a conformal bijection and some properties of holomorphic functions on the strip. Then, we use results on Hardy spaces on the disc to solve the reduced problem. Using this same conformal bijection, we then construct the solutions of the initial problem. Furthermore, we look at this phase retrieval problem coupled with various kinds of additional constraints, and check whether these constraints make it possible to obtain uniqueness of the solution.

The work here is organized as follows: Section 4.1 is a summary of the phase retrieval problem in the compactly-supported and narrow band cases, with their major connection to our problem. Section 4.2 is a quick review of the Hardy spaces on the strip. Section 4.3 is devoted to the solution of the phase retrieval problem in the wide band case. We look at the phase retrieval problem on the unit disc and on the strip. Finally, Section 4.4 is devoted to the coupled phase retrieval problems. We also provide more details on some of the results announced in [40, 41].

Throughout this chapter, we consider the following notations: for $F \in \text{Hol}(\Omega)$ we denote by $Z(F)$ the set of zeros of F , counted with multiplicity. Write $\overline{\Omega} = \{\bar{z} : z \in \Omega\}$ and if $F \in \text{Hol}(\Omega)$, we denote by F^* the function in $\text{Hol}(\overline{\Omega})$ defined by $F^*(z) = \overline{F(\bar{z})}$. It will be convenient to denote the conjugation function by C , where $C(z) = \bar{z}$ for all $z \in \mathbb{C}$. Now, consider a measure space $(X_1, \mathcal{A}_1, \mu)$, a measurable space (X_2, \mathcal{A}_2) , and a measurable map $\psi : X_1 \rightarrow X_2$. Recall that the pushforward measure of μ by ψ is given by

$$\psi_*\mu(A) = \mu(\psi^{-1}(A))$$

for $A \in \mathcal{A}_2$. Equivalently, if h is a function such that $h \circ \psi$ is integrable on X_1 with respect to μ , then we have the change of variables formula

$$\int_{X_2} h d(\psi_*\mu) = \int_{X_1} (h \circ \psi) d\mu.$$

4.1 Phase Retrieval for Band-Limited Signals

Before we solve our phase retrieval problem, we first recall the phase retrieval problem in the band-limited case. Akutowicz [1, 2] in 1956-57, and independently Walther [77] in 1963 and Hofstetter [34] in 1964, solved a phase retrieval problem for band-limited functions: given a band-limited $f \in L^2(\mathbb{R})$, find all band-limited $g \in L^2(\mathbb{R})$ such that

$$|g(x)| = |f(x)|, \quad x \in \mathbb{R}. \quad (4.1.1)$$

We shall give the proof of their solution. The first step is to use the Paley-Wiener theorem to extend f and g to entire functions which are of finite order less than or equal to 1. Thus, (4.1.1) can be written as $|f(x)|^2 = |g(x)|^2$ and so we have

$$g(x)\overline{g(\bar{x})} = f(x)\overline{f(\bar{x})}, \quad x \in \mathbb{R}$$

and we see that their extensions satisfy

$$g(z)g^*(z) = f(z)f^*(z), \quad z \in \mathbb{C}. \quad (4.1.2)$$

Since f and g are of finite order, f and g can be written as their Hadamard factorizations given by

$$f(z) = ce^{az}z^k \prod_{\alpha \in \mathcal{Z}(f)} E_1(z, \alpha) \quad \text{and} \quad g(z) = \tilde{c}e^{bz}z^\ell \prod_{\beta \in \mathcal{Z}(g)} E_1(z, \beta),$$

for all $z \in \mathbb{C}$, where $c, \tilde{c}, a, b \in \mathbb{C}$, $k, \ell \in \mathbb{N}$, and $\mathcal{Z}(f), \mathcal{Z}(g)$ are the zero sets (excluding 0) of f and g respectively. By substituting the factorizations of f and g to (4.1.2), it follows that $k = \ell$ by (4.1.1), and

$$\mathcal{Z}(g) \cup \overline{\mathcal{Z}(g)} = \mathcal{Z}(f) \cup \overline{\mathcal{Z}(f)},$$

i.e. for every $\beta \in \mathcal{Z}(g)$ there are two possibilities: either $\beta = \alpha$ or $\beta = \bar{\alpha}$ for some $\alpha \in \mathcal{Z}(f)$. This process of replacement by the conjugates is called **zero-flipping** by Walther, where an arbitrary set of zeros is flipped across the real line. In other words,

$$\mathcal{Z}(g) = A \cup \overline{(\mathcal{Z}(f) \setminus A)}, \quad \text{for some } A \subset \mathcal{Z}(f).$$

Furthermore, the convergence of the infinite product

$$\prod_{\alpha \in A} E_1(z, \alpha) \prod_{\alpha \in \mathcal{Z}(f) \setminus A} E_1(z, \bar{\alpha}), \quad z \in \mathbb{C}$$

is guaranteed by a result of Titchmarsh [71]. Now, it trivially follows that $\tilde{c} = \eta c$ where $\eta \in \mathbb{T}$ and $b = a + i\gamma$ for some $\gamma \in \mathbb{R}$. Moreover, we see that $g \in L^2(\mathbb{R})$ since $|g| = |f|$. Therefore, we have the following:

Theorem 4.1.1 (Akutowicz-Walther-Hofstetter). *Let $f, g \in L^2(\mathbb{R})$ be band-limited functions such that $|f(x)| = |g(x)|$ for all $x \in \mathbb{R}$. Write f as its Hadamard factorization given by*

$$f(z) = ce^{az}z^k \prod_{\alpha \in \mathcal{Z}(f)} E_1(z, \alpha), \quad z \in \mathbb{C}$$

for some $c, a \in \mathbb{C}$ and $k \in \mathbb{N}$. Then for all $z \in \mathbb{C}$, the Hadamard factorization of g is given by

$$g(z) = \eta c e^{(a+i\gamma)z} z^k \prod_{\alpha \in A} E_1(z, \alpha) \prod_{\alpha \in \mathcal{Z}(f) \setminus A} E_1(z, \bar{\alpha})$$

for some $\eta \in \mathbb{T}$, $\gamma \in \mathbb{R}$, and $A \subset \mathcal{Z}(f)$.

It was noted in [10], however, that zero-flipping may be impossible for higher dimensions.

McDonald [51] then solved a more general phase retrieval problem for entire functions: given an entire function f of finite genus, find all entire functions g of finite genus such that $|g(x)| = |f(x)|$ for all $x \in \mathbb{R}$. Since f and g are of finite genus, f and g are of finite order and so Hadamard factorization also applies. By using the same strategy as what was done in the previous case, all such g 's can also be obtained by zero-flipping. It was also shown equivalently that f and g can be decomposed as $f = uv$ and $g = uv^*$ for some entire functions u and v . A very important thing to note is that the proof of McDonald also applies when f and g are **narrow-banded**, that is, when f and g have Fourier transforms which decrease rapidly at infinity:

$$|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^p}$$

for all $\xi \in \mathbb{R}$, for some $a > 0$ and $p > 1$. Indeed, with this decay condition, f and g extend holomorphically to \mathbb{C} with

$$|f(z)|, |g(z)| \lesssim e^{B|z|^{p/p-1}}$$

for all $z \in \mathbb{C}$ where $B > 0$. These imply that f and g are of finite order and so by the proof of McDonald, the solutions can be obtained by zero-flipping as well.

However, when $p = 1$, i.e. $|\widehat{f}(\xi)| \lesssim e^{-a|\xi|}$ for all $\xi \in \mathbb{R}$, the previous proof using the Hadamard factorization does not apply anymore as f only extends to a holomorphic function on a strip $\mathcal{S}_a = \{z \in \mathbb{C} : |\operatorname{Im} z| < a\}$. A function with this kind of exponential decay on its Fourier transform is sometimes called a **wide band** signal. Note also that if \widehat{f} has this decay, then $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ for $0 < c < a$. Therefore, in this chapter, our main objective is to solve the following problem:

Problem (Phase Retrieval Problem for Wide Band Signals). Given $f \in L^2(\mathbb{R})$ such that $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$, find all $g \in L^2(\mathbb{R})$ such that $\widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ and $|g(x)| = |f(x)|$ for all $x \in \mathbb{R}$.

Furthermore, as shown in Theorem 3.2.2, for $f \in L^2(\mathbb{R})$, $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|} d\xi)$ if and only if f extends holomorphically to \mathcal{S}_c with

$$\sup_{|y| < c} \int_{\mathbb{R}} |f(x + iy)|^2 dx < +\infty.$$

This equivalence enables us to work on our problem using tools from complex analysis. Holomorphic functions on the strip which satisfy the above supremum condition will be discussed further in the next section.

4.2 Hardy Spaces on the Strip

The Hardy spaces on the strip will play a major role in solving our phase retrieval problem. In this section, we summarize the most relevant results and derive an analogous inner-outer factorization for the strip. We refer the reader to the work of Bakan et. al. [8] for the complete details about the facts stated in this section.

There are essentially two ways of defining the Hardy space on the strip \mathcal{S} , where $\mathcal{S} = \mathcal{S}_1 = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$. To start, let us define the conformal bijection $\phi : \mathcal{S} \rightarrow \mathbb{D}$ given by

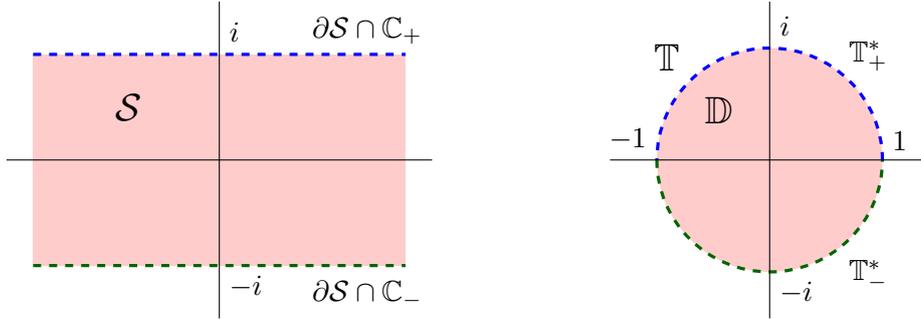
$$\phi(z) := \tanh\left(\frac{\pi}{4}z\right), \quad z \in \mathcal{S}.$$

The inverse mapping is given by the function

$$\phi^{-1}(w) = \frac{2}{\pi} \ln \frac{1+w}{1-w}, \quad w \in \mathbb{D} \tag{4.2.1}$$

where $\ln(\cdot)$ is the principal branch of the logarithm defined on $\mathbb{C} \setminus \mathbb{R}_-$. Observe that ϕ has the following basic properties:

- $\phi^* = \phi$;
- $\phi(\mathbb{R}) =]-1, 1[$;
- $\phi : \partial\mathcal{S} \rightarrow \mathbb{T} \setminus \{-1, 1\}$ is a bijection;
- $\phi(\partial\mathcal{S} \cap \mathbb{C}_\pm) = \mathbb{T}_\pm^*$, where $\mathbb{C}_+, \mathbb{C}_-$ denote the upper and lower halves of \mathbb{C} respectively, and $\mathbb{T}_+^*, \mathbb{T}_-^*$ denote the upper and lower halves of $\mathbb{T} \setminus \{-1, 1\}$ respectively; and
- $\lim_{x \rightarrow +\infty} \phi(x \pm i) = 1$ and $\lim_{x \rightarrow -\infty} \phi(x \pm i) = -1$.



The strip \mathcal{S} and the unit disc \mathbb{D} .

On one hand we shall consider the following Hardy spaces defined as

$$H^2(\mathcal{S}) = \left\{ f \in \operatorname{Hol}(\mathcal{S}) : f \circ \phi^{-1} \in H^2(\mathbb{D}) \right\},$$

and $\|f\|_{H^2(\mathcal{S})} = \|f \circ \phi^{-1}\|_{H^2(\mathbb{D})}$. It can then be shown [8, Theorem 2.2] that $H^2(\mathcal{S}) = H_{\mathcal{W}}^2(\mathcal{S})$ and $\|f\|_{H^2(\mathcal{S})} = \|f\|_{H_{\mathcal{W}}^2(\mathcal{S})}$ for all $f \in H^2(\mathcal{S})$, where

$$H_{\mathcal{W}}^2(\mathcal{S}) = \left\{ f \in \operatorname{Hol}(\mathcal{S}) : \|f\|_{H_{\mathcal{W}}^2(\mathcal{S})}^2 = \sup_{|y| < 1} \int_{\mathbb{R}} \frac{|f(t+iy)|^2 + |f(t-iy)|^2}{2|W(t+iy)|} dt < +\infty \right\},$$

$W(z) = 4 \cosh^2(\frac{\pi}{4}z) = \frac{\pi}{\phi'(z)}$ and $|W(x + iy)| = 2(\cosh \frac{\pi}{2}x + \cos \frac{\pi}{2}y)$ for all $z = x + iy \in \mathcal{S}$.

Now this last space can be identified to the natural analogue of the Hardy space on the disc:

$$H_\tau^2(\mathcal{S}) = \left\{ f \in \text{Hol}(\mathcal{S}) : \|f\|_{H_\tau^2(\mathcal{S})}^2 = \sup_{|y| < 1} \int_{\mathbb{R}} |f(t + iy)|^2 dt < +\infty \right\}. \quad (4.2.2)$$

More precisely $f \in H_\tau^2(\mathcal{S})$ if and only if $W^{1/2}f \in H_W^2(\mathcal{S})$.

Since $F = (W^{1/2}f) \circ \phi^{-1} \in H^2(\mathbb{D})$, then we may write $F = e^{i\gamma} B_F S_F O_F$ where $e^{i\gamma} \in \mathbb{T}$, B_F is the Blaschke product formed from the zeros of F , S_F is a singular inner function, and O_F is the outer part of F . In order to have an easier reference for the succeeding sections, we restate the formulas of each of these factors. The Blaschke product is defined as

$$B_F(w) = \prod_{\alpha \in Z(F)} b_\alpha(w), \quad w \in \mathbb{D}. \quad (4.2.3)$$

The singular part of F is given by

$$S_F(w) = \exp \left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} d\nu_F(e^{i\theta}) \right), \quad w \in \mathbb{D} \quad (4.2.4)$$

where ν_F is a finite positive singular measure, while the outer part is determined by the modulus of the radial limit of F ,

$$O_F(w) = \exp \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} \log |F_*(e^{i\theta})| d\theta \right), \quad w \in \mathbb{D}. \quad (4.2.5)$$

Finally, using these formulas and the conformal map ϕ , we obtain the factorization on $H_\tau^2(\mathcal{S})$.

Lemma 4.2.1. *Let $f \in H_\tau^2(\mathcal{S})$. Then the unique inner-outer factorization of f is given by*

$$f(z) = e^{i\gamma} \frac{B_F(\phi(z)) S_F(\phi(z)) O_F(\phi(z))}{W(z)^{1/2}}, \quad z \in \mathcal{S}$$

where $F = (W^{1/2}f) \circ \phi^{-1} \in H^2(\mathbb{D})$ and for some $\gamma \in \mathbb{R}$. For all $z \in \mathcal{S}$, the Blaschke product $B_f(z) = B_F(\phi(z))$ is given by

$$B_f(z) = \prod_{\beta \in Z(f)} b_{\phi(\beta)}(\phi(z)) \quad (4.2.6)$$

while the singular inner function $S_f(z) = S_F(\phi(z))$ is given by

$$S_f(z) = \exp \left(-a_{\{+1\}} e^{\frac{\pi}{2}z} - a_{\{-1\}} e^{-\frac{\pi}{2}z} + \int_{\partial\mathcal{S}} \frac{\phi(z) + \phi(\zeta)}{\phi(z) - \phi(\zeta)} d\mu_f(\zeta) \right), \quad (4.2.7)$$

for some positive constants $a_{\{\pm 1\}} \geq 0$, where $\mu_f = \phi^{-1}_* \nu_F$ is the pushforward measure of ν_F on $\partial\mathcal{S}$, and the outer function $O_f(z) = O_F(\phi(z))$ is given by

$$O_f(z) = \exp \left(\frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z) + \phi(x+i)}{\phi(z) - \phi(x+i)} \frac{\phi'(x+i)}{\phi(x+i)} \log |W(x+i)^{1/2} f(x+i)| dx \right. \\ \left. + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z) + \phi(x-i)}{\phi(z) - \phi(x-i)} \frac{\phi'(x-i)}{\phi(x-i)} \log |W(x-i)^{1/2} f(x-i)| dx \right). \quad (4.2.8)$$

Proof. For $F \in H^2(\mathbb{D})$ and $z \in \mathcal{S}$, according to the above connection between Hardy spaces, we have $F(\phi(z)) = W^{1/2}(z)f(z)$ and equivalently,

$$f(z) = \frac{F(\phi(z))}{W(z)^{1/2}} = e^{i\gamma} \frac{B_F(\phi(z))S_F(\phi(z))O_F(\phi(z))}{W(z)^{1/2}}, \quad z \in \mathcal{S}.$$

Here B_F, S_F and O_F are given by (4.2.3), (4.2.4) and (4.2.5), respectively. Note that this is well-defined on \mathcal{S} since $W(z) = \frac{\pi}{\phi'(z)} \neq 0$ for any $z \in \mathcal{S}$.

The formula for the Blaschke product easily follows from the properties of the bijection ϕ and the fact that

$$B_f(z) = B_F(\phi(z)) = \prod_{\alpha \in Z(F)} b_\alpha(\phi(z)), \quad z \in \mathcal{S}.$$

For the singular inner part, since $\phi : \partial\mathcal{S} \rightarrow \mathbb{T} \setminus \{-1, 1\}$ is a bijection, we have for $z \in \mathcal{S}$

$$S_f(z) = S_F(\phi(z)) \\ = \exp \left(\int_{\mathbb{T}} \frac{\phi(z) + e^{i\theta}}{\phi(z) - e^{i\theta}} d\nu_F(e^{i\theta}) \right) \\ = \exp \left(\nu_F(\{1\}) \frac{\phi(z) + 1}{\phi(z) - 1} + \nu_F(\{-1\}) \frac{\phi(z) - 1}{\phi(z) + 1} + \int_{\mathbb{T} \setminus \{1, -1\}} \frac{\phi(z) + e^{i\theta}}{\phi(z) - e^{i\theta}} d\nu_F(e^{i\theta}) \right) \\ = \exp \left(-\nu_F(\{1\})e^{\frac{\pi}{2}z} - \nu_F(\{-1\})e^{-\frac{\pi}{2}z} + \int_{\partial\mathcal{S}} \frac{\phi(z) + \phi(\zeta)}{\phi(z) - \phi(\zeta)} d\mu_f(\zeta) \right),$$

where $\mu_f = \phi^{-1}_* \nu_F$ is the pushforward measure of ν_F on $\partial\mathcal{S}$. Since ν_F is a positive singular measure, $\nu_F(\{\pm 1\}) = 0$ if ν_F has no mass at ± 1 , otherwise $\nu_F(\{\pm 1\}) > 0$. For the outer function, we also need to split the integral since $\phi(\partial\mathcal{S} \cap \mathbb{C}_\pm) = \mathbb{T}_\pm^*$. Hence, the outer part is given by

$$O_f(z) = O_F(\phi(z)) = \exp \left(-\frac{1}{2\pi} \int_0^\pi \frac{\phi(z) + e^{i\theta}}{\phi(z) - e^{i\theta}} \log |F(e^{i\theta})| d\theta \right. \\ \left. + \frac{1}{2\pi} \int_{-\pi}^0 \frac{\phi(z) + e^{i\theta}}{\phi(z) - e^{i\theta}} \log |F(e^{i\theta})| d\theta \right)$$

for all $z \in \mathcal{S}$. By applying the substitutions $e^{i\theta} = \phi(x + i)$, $\theta \in]0, \pi[$ on \mathbb{T}_+^* and $e^{i\theta} = \phi(x - i)$, $\theta \in]-\pi, 0[$ on \mathbb{T}_-^* , we get (4.2.8). \square

Remark 4.2.2. Since $W^{-1/2}(z) = 1/(2 \cosh(\frac{\pi}{4}z))$ for $z \in \mathcal{S}$, by (4.2.1), we get $W^{-1/2}(\phi^{-1}(w)) = \sqrt{1 - w^2}/2$ for $w \in \mathbb{D}$, and so $W^{-1/2} \circ \phi^{-1}$ is a bounded outer function on the disc.

To further illustrate the factorization in $H_\tau^2(\mathcal{S})$, let us consider the following example.

Example 4.2.3. For all $z \in \mathcal{S}$, consider

$$f(z) = \exp\left(-e^{\frac{\pi}{2}z}\right) W^{-1/2}(z).$$

Observe that $f \in H_\tau^2(\mathcal{S})$ and $\exp\left(-e^{\frac{\pi}{2}z}\right)$ is a singular inner function in $H_\tau^2(\mathcal{S})$. Moreover, we have $((W^{1/2}f) \circ \phi^{-1})(w) = \exp\left(-\frac{1+w}{1-w}\right)$ for $w \in \mathbb{D}$, and $(W^{1/2}f) \circ \phi^{-1}$ is the singular inner function $S_{\delta_1} \in H^2(\mathbb{D})$ associated with the Dirac measure at 1, as stated in Example 3.3.3.

4.3 Phase Retrieval in $H_\tau^2(\mathcal{S})$

In this section, we formally solve our phase retrieval problem. We divide the content here into three major steps. First, we reduce the problem to the disc by using the previously defined conformal bijection ϕ . Next, we construct the solutions in the disc using known results on Hardy spaces. Lastly, we use the factorization on Lemma 4.2.1 to go back to the strip and ultimately solve our initial problem. We also apply our main results to show some immediate consequences.

4.3.1 Reduction of the Problem

In this part, we consider $f, g \in L^2(\mathbb{R})$ with $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|}d\xi)$ such that $|f(x)| = |g(x)|$ for every $x \in \mathbb{R}$. Our main goal is to determine, for a given f , all possible g 's. To do so, let us write $f_c(x) = f(cx)$ and $g_c(x) = g(cx)$ so that $f_c, g_c \in L^2(\mathbb{R})$ with $\widehat{f}_c, \widehat{g}_c \in L^2(\mathbb{R}, e^{2|\xi|}d\xi)$ and $|f_c(x)| = |g_c(x)|$ for every $x \in \mathbb{R}$ so that it is enough to consider the case $c = 1$.

From (4.2.2) and Theorem 3.2.2, it follows that $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}, e^{2|\xi|}d\xi)$ if and only if $f, g \in H_\tau^2(\mathcal{S})$. Thus, f and g extend holomorphically to \mathcal{S} and $|f(x)| = |g(x)|$ for every $x \in \mathbb{R}$ can be written as

$$f(x)\overline{f(\bar{x})} = g(x)\overline{g(\bar{x})}, \quad x \in \mathbb{R}.$$

But now, this is an equality between two holomorphic functions on \mathbb{R} so that it is valid also for all $x \in \mathcal{S}$ by the Schwarz Reflection Principle. In other words, we are now trying to solve the following problem: given $f \in H_\tau^2(\mathcal{S})$, find all $g \in H_\tau^2(\mathcal{S})$ such that

$$f(z)f^*(z) = g(z)g^*(z), \quad z \in \mathcal{S}.$$

It turns out that this problem is easier to solve when transferring the problem to the disc. Multiplying by $W^{1/2}(z)$, $\overline{W^{1/2}(\bar{z})}$ both sides of the previous equation we obtain

$$(W^{1/2}f)(z)\overline{(W^{1/2}f)(\bar{z})} = (W^{1/2}g)(z)\overline{(W^{1/2}g)(\bar{z})}$$

for all $z \in \mathcal{S}$. Observe that the functions $F = W^{1/2}f \circ \phi^{-1}$ and $G = W^{1/2}g \circ \phi^{-1}$ are in $H^2(\mathbb{D})$. Hence, by applying the substitution $z = \phi^{-1}(w)$ and $\bar{z} = \phi^{-1}(\bar{w})$ to the previous equation, we get

$$F(w)F^*(w) = G(w)G^*(w), \quad w \in \mathbb{D}$$

which is equivalent to $|F(w)|^2 = |G(w)|^2$ for $w \in (-1, 1)$. Thus, we have translated the equality on the strip to an equivalent equality on the disc. Finally, we are now trying to solve the following problem:

Problem (Reduced Problem on the Disc). Given $F \in H^2(\mathbb{D})$, find all $G \in H^2(\mathbb{D})$ such that $|F(w)|^2 = |G(w)|^2$ for $w \in (-1, 1)$.

Although the proofs to obtain the solution can be done directly on the strip by using Lemma 4.2.1, it is better to work on $H^2(\mathbb{D})$ because the formulas involved in the inner-outer factorization in $H^2(\mathbb{D})$ are relatively easier to handle compared with their counterparts in the factorization on $H^2_\tau(\mathcal{S})$. Observe that even with a simple change of variable, the formulas for the corresponding singular inner and outer parts break apart into multiple factors and thus making it more complicated to work with.

4.3.2 The Phase Retrieval Problem on the Disc

In this part, we look at the equivalent phase retrieval problem on the disc. From our reduced problem, we will be needing the factorization of F^* from $F \in H^2(\mathbb{D})$ to characterize the solutions on the disc. We write $F = B_F S_F O_F$ where B_F , S_F , and O_F are given in (4.2.3), (4.2.4) and (4.2.5), respectively. The factorization of F^* is given by

$$F^* = e^{i\lambda} B_{F^*} S_{F^*} O_{F^*} = e^{i\lambda} B_F^* S_F^* O_F^*.$$

Since the factorization in $H^2(\mathbb{D})$ is unique, we have $B_{F^*} = B_F^*$, $S_{F^*} = S_F^*$, and $O_{F^*} = O_F^*$. Hence, for all $w \in \mathbb{D}$, the Blaschke product formed from the zeros of F^* is given by

$$B_{F^*}(w) = B_F^*(w) = \prod_{\alpha \in Z(F)} b_{\bar{\alpha}}(w) = \prod_{\alpha \in \overline{Z(F)}} b_{\alpha}(w). \quad (4.3.1)$$

The singular part of F^* is given by

$$S_{F^*}(w) = S_F^*(w) = \exp \left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} d(C_* \nu_F)(e^{i\theta}) \right), \quad (4.3.2)$$

for all $w \in \mathbb{D}$, where $C_* \nu_F$ is the pushforward measure of \mathbb{T} by the conjugation function C . Finally, for all $w \in \mathbb{D}$, the outer part of F^* is given by

$$O_{F^*}(w) = O_F^*(w) = \exp \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} \log |F_*(e^{-i\theta})| d\theta \right). \quad (4.3.3)$$

For us to construct all possible G 's, it is enough to characterize the zero set of G , the singular measure associated to G , and the radial limit of G based on the factorization of the given function F . We use all of the facts above to prove the following lemma.

Lemma 4.3.1. *Let $F, G \in H^2(\mathbb{D})$. Then*

$$|F(w)|^2 = |G(w)|^2, \quad w \in (-1, 1)$$

if and only if

i. the zero sets of F and G satisfy

$$Z(F) \cup \overline{Z(F)} = Z(G) \cup \overline{Z(G)};$$

ii. the singular measures ν_F and ν_G , associated with F and G respectively, satisfy

$$\nu_F + C_*\nu_F = \nu_G + C_*\nu_G \quad \text{on } \mathbb{T};$$

iii. the radial limits satisfy

$$|F_*(e^{i\theta})F_*(e^{-i\theta})| = |G_*(e^{i\theta})G_*(e^{-i\theta})| \quad \text{a.e. on } \mathbb{T}.$$

Proof. Let $F, G \in H^2(\mathbb{D})$. Note that FF^* and GG^* have decompositions given by

$$FF^* = B_F B_{F^*} S_F S_{F^*} O_F O_{F^*} \quad \text{and} \quad GG^* = B_G B_{G^*} S_G S_{G^*} O_G O_{G^*}.$$

Notice that $B_F B_{F^*}$ is again a Blaschke product, $S_F S_{F^*}$ is again a singular inner function, and $O_F O_{F^*}$ is again an outer function. Indeed, for all $w \in \mathbb{D}$, (4.3.1) implies that

$$B_F(w)B_{F^*}(w) = \prod_{\alpha \in Z(F) \cup \overline{Z(F)}} b_\alpha(w),$$

while (4.3.2) implies that

$$S_F(w)S_{F^*}(w) = \exp\left(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} d(\nu_F + C_*\nu_F)(e^{i\theta})\right),$$

and finally, (4.3.3) implies that

$$O_F(w)O_{F^*}(w) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} \log |F_*(e^{i\theta})F_*(e^{-i\theta})| d\theta\right).$$

Thus, writing the same for GG^* and using the uniqueness of the decomposition, $FF^* = GG^*$ implies that

$$B_F B_{F^*} = B_G B_{G^*}, \tag{4.3.4}$$

which in turn implies that

$$Z(F) \cup \overline{Z(F)} = Z(G) \cup \overline{Z(G)}.$$

Furthermore, $FF^* = GG^*$ also implies that

$$S_F S_{F^*} = S_G S_{G^*} \quad \text{and} \quad O_F O_{F^*} = O_G O_{G^*}.$$

Thus, by [50, Corollary 2.12] we get

$$\nu_F + C_* \nu_F = \nu_G + C_* \nu_G$$

on \mathbb{T} , and by (3.3.4), we have for almost every $\theta \in \mathbb{R}$

$$\lim_{r \rightarrow 1} (O_F(r e^{i\theta}) O_{F^*}(r e^{i\theta})) = \lim_{r \rightarrow 1} (O_G(r e^{i\theta}) O_{G^*}(r e^{i\theta})),$$

which in turn implies that

$$|F_*(e^{i\theta}) F_*(e^{-i\theta})| = |G_*(e^{i\theta}) G_*(e^{-i\theta})|$$

almost everywhere on \mathbb{T} . □

Observe that (4.3.4) corresponds to zero-flipping. Thus, the zero set of G can be obtained in the same way as in the band-limited case. We can now construct such G 's to solve the problem on the disc. The following corollary immediately follows from Lemma 4.3.1.

Corollary 4.3.2. *Let $F, G \in H^2(\mathbb{D})$. Then $|F| = |G|$ on $(-1, 1)$ if and only if the inner-outer decomposition of F and G are given by*

$$F = e^{i\gamma} B_F S_F O_F \quad \text{and} \quad G = e^{i\gamma'} B_G S_G O_G$$

where

1. B_F, S_F, O_F are given by (4.2.3), (4.2.4), (4.2.5) respectively;
2. B_G is the Blaschke product associated with the set $A \cup \overline{(Z(F) \setminus A)}$ for some $A \subset Z(F)$;
3. S_G is the singular inner function associated with the positive singular measure $\nu_G = \nu_F + \rho$, where ρ is an odd real singular measure; and
4. $O_G = U O_F$ where $U \in \mathcal{N}^+$ is an outer function and $U = 1/U^*$ on \mathbb{D} .

Remark 4.3.3. We can make the condition on ρ a bit more explicit so as to be constructive. We write $\rho = \rho_+ - \rho_-$, where ρ_+ is the positive part while ρ_- is the negative part. In particular, ρ_+ and ρ_- have disjoint supports. The fact that ρ is an odd measure, that is, $C_* \rho = -\rho$ or

$$d(C_* \rho)(e^{i\theta}) = d\rho(e^{-i\theta}) = -d\rho(e^{i\theta}), \quad e^{i\theta} \in \mathbb{T}$$

is equivalent to $\rho_- = C_* \rho_+$. The fact that ν_G is positive, is equivalent to the condition $C_* \rho_+ \leq \nu_F$, or equivalently, $\rho_+ \leq C_* \nu_F$. Conversely, take a set $E \subset \mathbb{T}$ such that $E \cap \overline{E} = \emptyset$ and a positive singular measure ρ_+ supported in E and such that $\rho_+ \leq C_* \nu_F$. Then we can take $\rho = \rho_+ - C_* \rho_+$.

Proof of Corollary 4.3.2. Let $F, G \in H^2(\mathbb{D})$ with inner-outer decompositions as defined on Corollary 4.3.2. Note that the property of the Blaschke product B_G immediately follows from Lemma 4.3.1 and convergence of B_G is guaranteed by the Blaschke condition.

For the singular part, Lemma 4.3.1 implies that $\nu_G - \nu_F = \rho$ where ρ is an odd real singular measure on \mathbb{T} , not necessarily positive.

Finally for the outer function, by Lemma 4.3.1, we have

$$|O_F(e^{i\theta})O_F(e^{-i\theta})| = |O_G(e^{i\theta})O_G(e^{-i\theta})| \quad (4.3.5)$$

almost everywhere on \mathbb{T} . Hence,

$$\log |O_G(e^{i\theta})| = \log |O_F(e^{i\theta})| + \log |U(e^{i\theta})|$$

almost everywhere on \mathbb{T} , where $\log |U(e^{i\theta})|$ is an odd real-valued function of θ and $\log |U| \in L^1(\mathbb{T})$. Since $|O_G(e^{i\theta})| = |O_F(e^{i\theta})U(e^{i\theta})|$ almost everywhere on \mathbb{T} and O_G and O_F are outer functions, we get

$$O_G(z) = O_F(z)O_U(z), \quad z \in \mathbb{D}.$$

Hence $O_U = O_G/O_F \in \mathcal{N}^+$, since every outer function can be written as a quotient of two bounded outer functions. Moreover, (4.3.5) implies that $|O_U(e^{i\theta})O_U(e^{-i\theta})| = 1$ almost everywhere on \mathbb{T} , and so $O_U(z)O_U^*(z) = 1$ on \mathbb{D} .

The sufficiency immediately follows from Lemma 4.3.1. \square

We can actually identify the solutions of the phase retrieval problem on the disc in terms of a factorization. Let us consider an analog of the result of McDonald [51, Proposition 1]. We further decompose the factorizations given in Corollary 4.3.2 to prove the following consequence:

Corollary 4.3.4. *Let $F, G \in H^2(\mathbb{D})$. Then $|F| = |G|$ on $(-1, 1)$ if and only if there exist $u, v \in \text{Hol}(\mathbb{D})$ such that $F = uv$ and $G = uv^*$.*

Proof. Let $F, G \in H^2(\mathbb{D})$. By Corollary 4.3.2, we have the factorizations

$$F = B_F S_F O_F \quad \text{and} \quad G = B_G S_G O_G.$$

First, observe that we can write the Blaschke products B_F and B_G as

$$B_F = B_1 B_2 \quad \text{and} \quad B_G = B_1 B_2^*$$

where B_1 is the Blaschke product associated with $A \subset Z(F)$ and B_2 is the Blaschke product associated with $Z(F) \setminus A$. On the other hand, we can write the singular measures ν_F and ν_G as

$$\nu_F = \nu_1 + \nu_2 \quad \text{and} \quad \nu_G = \nu_1 + C_* \nu_2$$

where

$$\nu_1 = \nu_F + \frac{\rho_+ - C_* \rho_+}{2} \quad \text{and} \quad \nu_2 = \frac{C_* \rho_+ - \rho_+}{2},$$

so that $S_F = S_{\nu_1} S_{\nu_2}$ and $S_G = S_{\nu_1} S_{\nu_2}^*$.

Since $O_G = U O_F$ where U is an outer function, we have $U \in \mathcal{N}^+$ and $U U^* = 1$ on \mathbb{D} . We write

$$O_F = O_F U^{1/2} U^{-1/2}$$

and

$$O_G = O_F U^{1/2} U^{1/2} = O_F U^{1/2} (U^{-1/2})^*$$

Therefore, we take

$$u = B_1 S_{\nu_1} O_F U^{1/2} \quad \text{and} \quad v = B_2 S_{\nu_2} U^{-1/2}.$$

□

4.3.3 Back to the Strip

In this section, we now solve the phase retrieval problem on the strip. Using Lemma 4.2.1, we see that Lemma 4.3.1 translates to functions in $H_\tau^2(\mathcal{S})$. By a change of variable and by applying the inner-outer factorization on $H_\tau^2(\mathcal{S})$, we have:

Lemma 4.3.5. *Let $f, g \in H_\tau^2(\mathcal{S})$. Then*

$$|f(z)|^2 = |g(z)|^2, \quad z \in \mathbb{R}$$

if and only if

i. the zero sets of f and g satisfy

$$Z(f) \cup \overline{Z(f)} = Z(g) \cup \overline{Z(g)};$$

ii. the singular measures μ_f and μ_g , associated with f and g respectively, satisfy

$$\mu_f + C_* \mu_f = \mu_g + C_* \mu_g \quad \text{on } \partial \mathcal{S};$$

iii. the boundary values satisfy

$$|f(x+i)f(x-i)| = |g(x+i)g(x-i)| \quad \text{a.e. on } \mathbb{R}.$$

We now construct the solutions of the problem on the strip. Let $\mathcal{N}_\tau^+(\mathcal{S})$ be the Smirnov class of holomorphic functions in \mathcal{S} such that every $f \in \mathcal{N}_\tau^+(\mathcal{S})$ can be written as $f = (F \circ \phi)/W^{1/2}$ where $F \in \mathcal{N}^+$. We have the following relation between outer functions in \mathcal{N}^+ and in $\mathcal{N}_\tau^+(\mathcal{S})$:

Remark 4.3.6. If $O \in \mathcal{N}^+$ is outer, then $O \circ \phi \in \mathcal{N}_\tau^+(\mathcal{S})$ is outer. Indeed, $O \circ \phi$ is an outer function on \mathcal{S} by Lemma 4.2.1. Moreover, by writing

$$O \circ \phi = \frac{\left(\frac{O}{W^{-1/2} \circ \phi^{-1}} \right) \circ \phi}{W^{1/2}},$$

we have $O \circ \phi \in \mathcal{N}_\tau^+(\mathcal{S})$, as $W^{-1/2} \circ \phi^{-1}$ is a bounded outer function on the disc by Remark 4.2.2.

The following result immediately follows from Lemma 4.3.5.

Theorem 4.3.7. *Let $f, g \in H_\tau^2(\mathcal{S})$. Then $|f| = |g|$ on \mathbb{R} if and only if the inner-outer decomposition of f and g are given by*

$$f = e^{i\gamma} W^{-1/2} B_f S_f O_f \quad \text{and} \quad g = e^{i\gamma'} W^{-1/2} B_g S_g O_g$$

where

1. B_f, S_f, O_f are given by (4.2.6), (4.2.7), (4.2.8) respectively;
2. B_g is the Blaschke product associated with the set $A \cup \overline{(Z(f) \setminus A)}$ with $A \subset Z(f)$;
3. S_g is the singular inner function associated with the positive singular measure $\mu_g = \mu_f + \sigma$, where σ is an odd real singular measure, given by $\sigma = \sigma_+ - C_*\sigma_+$, satisfying $C_*\sigma = -\sigma$ and $0 \leq \sigma_+ \leq C_*\mu_f$; and
4. O_g is the outer part of uO_f where $u \in \mathcal{N}_\tau^+(\mathcal{S})$ is an outer function and $u = 1/u^*$ on \mathcal{S} .

Proof. We only prove the result on the outer part since the expressions for B_g and S_g can be obtained similarly as how it was done on the disc in Corollary 4.3.2.

By Lemma 4.3.5, $|f(x+i)f(x-i)| = |g(x+i)g(x-i)|$ a.e. on \mathbb{R} if and only if $O_f O_{f^*} = O_g O_{g^*}$. Furthermore, this is equivalent to $O_F O_{F^*} = O_G O_{G^*}$ where $F = (W^{1/2}f) \circ \phi^{-1}$ and $G = (W^{1/2}g) \circ \phi^{-1}$ with $F, G \in H^2(\mathbb{D})$. Then $O_G = U O_F$ where $U \in \mathcal{N}^+$ is an outer function and $U = 1/U^*$ on \mathbb{D} by Corollary 4.3.2. This implies that

$$O_g = O_G \circ \phi = [U \circ \phi] \cdot [O_F \circ \phi] = [U \circ \phi] \cdot O_f.$$

Let $u = U \circ \phi$. By the previous remark, we get that $u \in \mathcal{N}_\tau^+(\mathcal{S})$ is outer. Furthermore, for all $z \in \mathcal{S}$, we have

$$u(z)u^*(z) = U(\phi(z))\overline{U(\phi(\bar{z}))} = U(\phi(z))U^*(\phi(z)) = 1$$

as stated. □

Remark 4.3.8. To form S_g , we may choose the same nonnegative constants $a_{\{\pm 1\}}$ from S_f given in (4.2.7) since $a_{\{\pm 1\}} = \nu_F(\{\pm 1\}) = \nu_G(\{\pm 1\})$ where $F = (W^{1/2}f) \circ \phi^{-1}$ and $G = (W^{1/2}g) \circ \phi^{-1}$. Furthermore, the measure σ is constructed the same way as we have done in Remark 4.3.3.

Remark 4.3.9. Observe that possible trivial solutions to the problem on the strip are given by:

$$(1) \quad g(z) = ce^{i\eta z} f(z) \quad \text{and} \quad (2) \quad g(z) = ce^{i\eta z} f^*(z)$$

with $|c| = 1$ and $\eta \in \mathbb{R}$. These trivial solutions are retrieved as follows:

1. the factor $e^{i\eta z}$ is the factor u of the outer part as $e^{i\eta z}(e^{i\eta z})^* = 1$
2. the replacement of f by f^* is obtained by taking $A = \emptyset$ for the Blaschke part, $\sigma = C_*\mu_f - \mu_f$ so that $\mu_g = C_*\mu_f$ for the inner part and finally $u = O_{f^*}/O_f$ so that the outer part of g is $O_g = uO_f = O_{f^*}$

Corollary 4.3.4 also translates to a result on the strip via a change of variable using the conformal map ϕ :

Corollary 4.3.10. *Let $f, g \in H_\tau^2(\mathcal{S})$. Then $|f| = |g|$ on \mathbb{R} if and only if there exist $u, v \in \text{Hol}(\mathcal{S})$ such that $f = uv$ and $g = uv^*$.*

Finally, we go back to our initial phase retrieval problem. The following result directly follows from Paley-Wiener theorem on the strip (Theorem 3.2.2) and Theorem 4.3.7.

Corollary 4.3.11. *Let $f \in L^2(\mathbb{R})$ and $\hat{f} \in L^2(\mathbb{R}, e^{2|\xi|} d\xi)$. For all $g \in L^2(\mathbb{R})$ such that $\hat{g} \in L^2(\mathbb{R}, e^{2|\xi|} d\xi)$ with $|f(x)| = |g(x)|$ for all $x \in \mathbb{R}$, g can be written as*

$$g = e^{i\kappa} W^{-1/2} B_g S_g O_g$$

where

1. B_g is the Blaschke product associated with the set $A \cup \overline{(Z(f) \setminus A)}$ with $A \subset Z(f)$;
2. S_g is the singular inner function associated with the positive singular measure $\mu_g = \mu_f + \sigma$, where σ is an odd real singular measure, given by $\sigma = \sigma_+ - C_* \sigma_+$, satisfying $C_* \sigma = -\sigma$ and $0 \leq \sigma_+ \leq C_* \mu_f$; and
3. O_g is the outer part of $u O_f$ where $u \in \mathcal{N}_\tau^+(\mathcal{S})$ is an outer function and $u = 1/u^*$ on \mathcal{S} .

4.4 Coupled Phase Retrieval Problems

In this section, we are investigating coupled phase retrieval problems, i.e. problems of the form $|u| = |v|$, $|Tu| = |Tv|$ where T is some transform. This additional assumption involving T may either lead to uniqueness or at least to the reduction of the set of solutions.

4.4.1 Adding a Fixed Reference

Klibanov et. al. [44, Proposition 6.5] considered the following constrained problem:

$$|g| = |f| \quad \text{and} \quad |g - h| = |f - h|$$

where h is a fixed reference signal. They were able to show that there are at most two solutions of this problem. For the following result, we look at a similar problem. It turns out that for the wide band case, we also obtain two solutions. The corresponding result for the wide band case immediately follows from the following coupled phase retrieval problem.

Theorem 4.4.1. *Let f, g be functions holomorphic in a connected neighborhood of the real axis, and h be a nonzero complex-valued function such that $\Phi = e^{i \arg h}$ is analytic on \mathbb{R} . Suppose that $|g(x)| = |f(x)|$ and $|g(x) - h(x)| = |f(x) - h(x)|$ for (a.e.) $x \in \mathbb{R}$. Then there exists two solutions of this problem, namely $g(x) = f(x)$ or $g(x) = \overline{f(x)\Phi(x)^2}$, for $x \in \mathbb{R}$.*

Proof. Consider the two circles on \mathbb{C} : $\mathcal{C}(0, |f(x)|)$ and $\mathcal{C}(h(x), |f(x) - h(x)|)$. For (a.e.) $x \in \mathbb{R}$, writing $h(x) = |h(x)|e^{i\theta(x)}$ for some real-valued function $\theta \not\equiv 0$, we have

$$\begin{aligned} \left|g(x) - |h(x)|e^{i\theta(x)}\right|^2 &= \left|f(x) - |h(x)|e^{i\theta(x)}\right|^2 \\ 2|h(x)|\operatorname{Re}(g(x)e^{-i\theta(x)}) &= 2|h(x)|\operatorname{Re}(f(x)e^{-i\theta(x)}) \end{aligned}$$

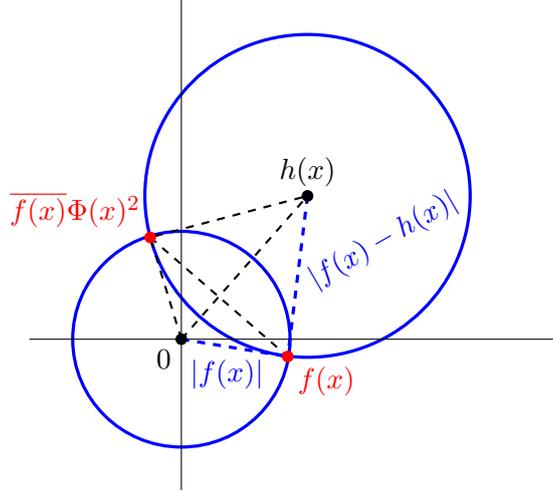
and so $\operatorname{Re}((g(x) - f(x))e^{-i\theta(x)}) = 0$. It follows that $f - g \perp e^{i\theta}$. Clearly, $g = f$ is a solution. If $g \neq f$, then there exists a real-valued function φ such that

$$f(x) = |f(x)|e^{i(\theta(x)-\varphi(x))} \quad \text{and} \quad g(x) = |f(x)|e^{i(\theta(x)+\varphi(x))}$$

for (a.e.) $x \in \mathbb{R}$. Thus for (a.e.) $x \in \mathbb{R}$, we write

$$g(x) = \overline{|f(x)|e^{i(\theta(x)-\varphi(x))}} \left(e^{i\theta(x)}\right)^2 = \overline{f(x)}\Phi(x)^2.$$

Hence, these two circles have two intersection points, one being $f(x)$, the other being $\overline{f(x)}\Phi(x)^2$ (eventually being the same as the first one).



The circles $\mathcal{C}(0, |f(x)|)$ and $\mathcal{C}(h(x), |f(x) - h(x)|)$.

Therefore, for each $x \in \mathbb{R}$, either $g(x) = f(x)$ or $g(x) = \overline{f(x)}\Phi(x)^2$. By the pigeon-hole principle, one of these two alternatives is valid on a set of positive measure. Indeed, we can write $\mathbb{R} = E_1 \cup E_2$ where

$$E_1 = \{x \in \mathbb{R} : g(x) = f(x)\} \quad \text{and} \quad E_2 = \{x \in \mathbb{R} : g(x) = \overline{f(x)}\Phi(x)^2\},$$

and so E_1, E_2 have positive measure. But f, g and $\overline{f}\Phi^2$ are all analytic so that if $g = f$ on a set of positive measure, then $g = f$ everywhere, otherwise if $g = \overline{f}\Phi^2$ on a set of positive measure, then $g = \overline{f}\Phi^2$ everywhere as well. \square

Remark 4.4.2. If we do not assume Φ to be analytic, then $\overline{f}\Phi^2$ may not be analytic and would therefore not be a solution.

4.4.2 Pauli's Problem

For our next result, we add a constraint involving the Fourier transforms:

$$|g| = |f| \quad \text{and} \quad |\widehat{g}| = |\widehat{f}| \quad (4.4.1)$$

This problem is due to Pauli [58], who speculated that (4.4.1) would imply $g = cf$ for some $c \in \mathbb{T}$. Originally, this problem was studied in quantum mechanics, where it was asked if a wave function can be determined by the probability densities of position and momentum. However, one may construct many pairs (f, g) satisfying (4.4.1) for which this is not the case (see e.g. Vogt [74], Corbett and Hurst [20, 21]). Such pairs are now called **Pauli partners**. In the band-limited case, Ismagilov [37] and Jaming [38] have independently shown that the set of the Pauli partners may be arbitrarily large. However, although this is not explicitly stated in [37, 38], for a given band-limited f only finitely band-limited partners (up to trivial solutions) are constructed. In their proof, the construction of the Pauli partners involved **Riesz products** which are of the following form:

$$\prod_{n=1}^{\infty} (1 + i\alpha_n \sin(\lambda_n x + \varphi_n)), \quad x \in \mathbb{R}$$

where $\alpha_n \in \mathbb{R}$ satisfies $\sum_{n \in \mathbb{N}} |\alpha_n|^2 < \infty$, $\lambda_n > 0$ and $\varphi_n \in \mathbb{T}$, for all $n \in \mathbb{N}$. This product can be expanded as a well-defined trigonometric series. We refer the reader to the book of Katznelson [43, Chapter V, Section 1.3] for more information on Riesz products.

Furthermore, the Pauli problem has been extended in terms of the fractional Fourier transform in the work of Jaming [39]. Recall that the fractional Fourier transform (see e.g. [5]) is defined for all $f \in L^1(\mathbb{R})$ as

$$\mathcal{F}_\alpha f(\xi) = c_\alpha e^{i\pi|\xi|^2 \cot \alpha} \mathcal{F}[e^{-i\pi|\cdot|^2 \cot \alpha} f](\xi/\sin \alpha), \quad \xi \in \mathbb{R}$$

where $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$ and c_α is a square root of $1 - i \cot \alpha$, while for $k \in \mathbb{Z}$, $\mathcal{F}_{2k\pi} f(\xi) = f(\xi)$ and $\mathcal{F}_{(2k+1)\pi} f(\xi) = f(-\xi)$. Jaming showed several uniqueness results from suitably chosen fractional Fourier transform intensities. In a succeeding paper, Andreys and Jaming [6] showed a nonuniqueness case: there exist $f, g \in L^2(\mathbb{R})$ which are not constant multiples of each other with $|\mathcal{F}_{\alpha_i} f| = |\mathcal{F}_{\alpha_i} g|$ for some well-chosen angles $\{\alpha_i\}_{i=1}^N$.

The following result shows that the solution set of the Pauli problem in the wide band case may be arbitrarily large as well and even uncountable.

Theorem 4.4.3. *There exists $f \in H_\tau^2(\mathcal{S})$ which has a nondenumerable infinity of Pauli partners which are not constant multiples of one another.*

Proof. The proof is a direct adaptation of [37, 38].

Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of non-zero real numbers such that $\sum_{n \in \mathbb{N}} |\alpha_n|^2 < \infty$ and consider the associated Riesz product

$$R_\alpha(x) = \prod_{n=1}^{\infty} (1 + 2i\alpha_n \sin(2\pi 3^n x)), \quad x \in \mathbb{R}.$$

We may write this Riesz product as a Fourier series

$$R_\alpha(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}. \quad (4.4.2)$$

Next, let $\varphi \in L^2(\mathbb{R})$ be such that $\widehat{\varphi}$ is supported on $[0, 1]$ and bounded. For all $x \in \mathbb{R}$, take $f = R_\alpha \varphi$. As

$$f(x) = \left(\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x} \right) \varphi(x),$$

we get

$$\widehat{f}(\xi) = \sum_{k \in \mathbb{Z}} a_k \widehat{\varphi}(\xi - k).$$

Now, observe that $a_k = 0$ unless there exists an integer N and $\eta_1, \dots, \eta_N \in \{-1, 0, 1\}$ with $\eta_N \neq 0$ such that $k = \sum_{j=1}^N \eta_j 3^j$. Further, N and the η_j 's are uniquely determined by k . In this case, a simple computation shows that $3^{N-1} \leq |k| \leq 3^{N+1}$ and that

$$|a_k| = \prod_{j=1, \eta_j \neq 0}^N |\alpha_j|. \quad (4.4.3)$$

Therefore, if we choose $0 < |\alpha_j| \leq e^{-2 \cdot 3^{j+1}}$, we get

$$|a_k| \leq |\alpha_N| \leq e^{-2 \cdot 3^{N+1}} \leq e^{-2|k|}.$$

As a consequence, for $k \leq |\xi| \leq k + 1$,

$$|\widehat{f}(\xi)| = |a_k| |\widehat{\varphi}(\xi - k)| \leq e^{-2|k|} \|\widehat{\varphi}\|_\infty \leq C e^{-2|\xi|}.$$

It follows that $f \in H_\tau^2(\mathcal{S})$.

Next, let $\varepsilon = \{\varepsilon_n\}_{n=1}^\infty \in \{-1, 1\}^\mathbb{N}$ and $\alpha(\varepsilon) = \{\alpha_n \varepsilon_n\}_{n=1}^\infty$. In particular, for $\varepsilon = \mathbf{1} = (1, 1, \dots)$, $\alpha(\mathbf{1}) = \alpha$. Observe that the associated Riesz product

$$R_{\alpha(\varepsilon)}(x) = \prod_{n=1}^\infty (1 + 2i\alpha_n \varepsilon_n \sin(2\pi 3^n x)) = \sum_{k \in \mathbb{Z}} a_k(\varepsilon) e^{2\pi i k x}$$

has the following properties:

- for every $x \in \mathbb{R}$, $|R_{\alpha(\varepsilon)}(x)| = |R_\alpha(x)|$; and
- for every $k \in \mathbb{Z}$, $|a_k(\varepsilon)| = |a_k|$.

This last property follows directly from (4.4.3). Note also that $R_{\alpha(\varepsilon)}$ is not a constant multiple of $R_{\alpha(\varepsilon')}$ if $\varepsilon \neq \varepsilon'$.

It remains to define $f_\varepsilon = R_{\alpha(\varepsilon)} \varphi$. Then f_ε has the following properties:

- $f_\varepsilon \in H_\tau^2(\mathcal{S})$ and f_ε is not a constant multiple of $f_{\varepsilon'}$ if $\varepsilon \neq \varepsilon'$;
- $|f_\varepsilon(x)| = |f_{\varepsilon'}(x)|$ for all $x \in \mathbb{R}$; and
- $|\widehat{f}_\varepsilon(\xi)| = |\widehat{f}_{\varepsilon'}(\xi)|$ since for $k \leq |\xi| \leq k + 1$, $k \in \mathbb{Z}$, we have

$$|\widehat{f}_\varepsilon(\xi)| = |a_k(\varepsilon)| |\widehat{\varphi}(\xi - k)| = |a_k(\varepsilon')| |\widehat{\varphi}(\xi - k)| = |\widehat{f}_{\varepsilon'}(\xi)|.$$

□

4.4.3 Derivation Operator

We now look at a direct consequence of Corollary 4.3.10. Let $b, q \in \mathbb{R}$ with $|q| < 1$. For all $z \in \mathcal{S}$ and $f \in H_\tau^2(\mathcal{S})$, consider the operator $\frac{\partial}{\partial z}$ where $\frac{\partial}{\partial z}f(z) = f'(z)$, the operator δ given by

$$\delta(f)(z) = f(z + b) - f(z),$$

and the operator γ given by

$$\gamma(f)(z) = f(qz) - f(z).$$

Let D be one of $\frac{\partial}{\partial z}, \delta$ or γ . McDonald [51, Theorem 1] considered the coupled phase retrieval problem: f, g entire, $|g(x)| = |f(x)|$ with the additional constraint $|Dg(x)| = |Df(x)|$ for $x \in \mathbb{R}$. McDonald showed that if $f = uv$ and $g = uv^*$, then $|Dg| = |Df|$ is equivalent to

$$\left(\frac{Dv}{v} - \frac{Dv^*}{v^*}\right)\left(\frac{Du^*}{u^*} - \frac{Du}{u}\right) = \frac{DfDf^* - DgDg^*}{ff^*} = 0, \quad (4.4.4)$$

which imposes strong restrictions on either u or v . With these, McDonald was able to significantly reduce the solution set into two solutions. As a consequence of Corollary 4.3.10, McDonald's result directly extends to the wide band case.

Corollary 4.4.4. *Let $f, g \in H_\tau^2(\mathcal{S})$, $\frac{d}{dx}$ be the operator where $\frac{d}{dx}f(x) = f'(x)$ for all $x \in \mathbb{R}$, and D be one of the operators $\frac{d}{dx}, \delta$ or γ . Suppose that $|g(x)| = |f(x)|$ and $|Dg(x)| = |Df(x)|$ for $x \in \mathbb{R}$. Then:*

- i. *For the cases $D = \frac{d}{dx}$ and $D = \gamma$, either $g = \beta f$ or $g = \beta f^*$ for some constant $\beta \in \mathbb{C}$.*
- ii. *For the case $D = \delta$, either $g = Vf$ or $g = Vf^*$ where V is a meromorphic function that has period b and continuous and unimodular on \mathbb{R} .*

Proof. The proof is *mutatis mutandis* the one provided by McDonald in [51].

Since $|g| = |f|$ on \mathbb{R} , Corollary 4.3.10 implies that there exists $u, v \in \text{Hol}(\mathcal{S})$ such that $f = uv$ and $g = uv^*$. As $|Dg| = |Df|$ implies (4.4.4), we then obtain

$$\frac{Dv}{v} - \frac{Dv^*}{v^*} = 0 \quad \text{or} \quad \frac{Du^*}{u^*} - \frac{Du}{u} = 0. \quad (4.4.5)$$

When $D = \frac{d}{dx}$, (4.4.5) implies that either $v = \beta v^*$ or $u = \beta u^*$ for some $\beta \in \mathbb{C}$. It follows that $g = \beta uv = \beta f$ or $g = \beta u^*v^* = \beta f^*$.

Now for $D = \delta$, (4.4.5) implies that

$$u^*(z)u(z + b) = u(z)u^*(z + b) \quad \text{or} \quad v^*(z)v(z + b) = v(z)v^*(z + b), \quad z \in \mathcal{S}.$$

In the first case, $V = \frac{u}{u^*}$ is meromorphic on \mathcal{S} and has period b . Moreover, it is easy to see that V is continuous and unimodular on \mathbb{R} . Hence, we get $g = Vu^*v^* = Vf^*$. In the second case, a similar argument shows that $g = Vf$.

Finally for $D = \gamma$, we follow the proof from the previous case so we obtain $g = Vf$ or $g = Vf^*$ where V is a meromorphic function on \mathcal{S} which is continuous on \mathbb{R} , with $V(z) = V(qz)$ for all $z \in \mathcal{S}$. Observe that $V(z) = V(qz) = \dots = V(q^n z)$ for all $n \in \mathbb{N}$. Letting $n \rightarrow +\infty$, we obtain $V(z) = V(0)$ since $|q| < 1$. Therefore, $g = V(0)f$ or $g = V(0)f^*$. \square

4.4.4 Modulus on a Segment on \mathcal{S}

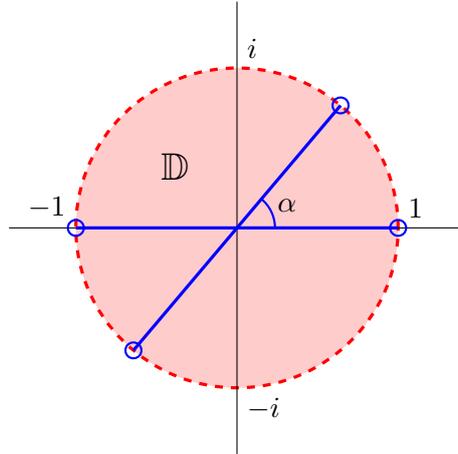
In the spirit of what was done by Boche et. al. [12], we now consider that $|g(z)| = |f(z)|$ for z in a curve on \mathcal{S} . For this coupled problem, we add the fact that $|g(z)| = |f(z)|$ for every z on a segment lying on the strip \mathcal{S} . We first look at this additional constraint on the phase retrieval problem on the disc.

Lemma 4.4.5. *Let $f, g \in H^2(\mathbb{D})$ such that $|g(x)| = |f(x)|$ for $x \in (-1, 1)$ and*

$$|g(z)| = |f(z)|, \quad z \in e^{i\alpha}(-1, 1) \quad (4.4.6)$$

where $\alpha \notin \pi\mathbb{Q}$. Then $g = cf$ for some $c \in \mathbb{T}$.

Proof. Let $f, g \in H^2(\mathbb{D})$ and $\mathcal{Z} = Z(f) \triangle Z(g)$. Since $|g(x)| = |f(x)|$ for all $x \in (-1, 1)$, we have $\mathcal{Z} = \overline{\mathcal{Z}}$. It clearly follows that $\mathcal{Z} \cap \mathbb{R} = \emptyset$.



The disc \mathbb{D} with the segments $(-1, 1)$ and $e^{i\alpha}(-1, 1)$.

Since $|g(x)| = |f(x)|$ for all $x \in e^{i\alpha}(-1, 1)$, we have $\mathcal{Z} = \text{Ref}_\alpha \mathcal{Z}$ where Ref_α refers to a reflection with respect to the segment $e^{i\alpha}(-1, 1)$. Hence, by composing $\mathcal{Z} = \overline{\mathcal{Z}}$ and $\mathcal{Z} = \text{Ref}_\alpha \mathcal{Z}$, we get that $\mathcal{Z} = \text{Rot}_{2\alpha} \mathcal{Z}$, where $\text{Rot}_{2\alpha}$ refers to a counterclockwise 2α -rotation with respect to 0. Now, since $\alpha \notin \pi\mathbb{Q}$, either $\mathcal{Z} = \emptyset$ or \mathcal{Z} is uncountable. Since the zero set is discrete, \mathcal{Z} cannot be uncountable, and so $\mathcal{Z} = \emptyset$. Hence, $Z(f) = Z(g)$, which implies that the Blaschke products formed by the zeros of f and g given by B_f and B_g respectively, are equal.

Now, observe that since $|g(x)| = |f(x)|$ for all $x \in (-1, 1)$, Lemma 4.3.1 implies that for $e^{i\theta} \in \mathbb{T}$,

$$\nu_f(e^{i\theta}) + \nu_f(e^{-i\theta}) = \nu_g(e^{i\theta}) + \nu_g(e^{-i\theta}).$$

Using this equation, the Fourier coefficients of ν_f and ν_g satisfy

$$\widehat{\nu}_f(n) + \widehat{\nu}_f(-n) = \widehat{\nu}_g(n) + \widehat{\nu}_g(-n), \quad n \in \mathbb{N}. \quad (4.4.7)$$

On the other hand, $|g(x)| = |f(x)|$ for all $x \in e^{i\alpha}(-1, 1)$ implies that $|f(e^{i\alpha}x)| = |g(e^{i\alpha}x)|$ for all $x \in (-1, 1)$. For $z \in \mathbb{D}$, we now write $F(z) = f(e^{i\alpha}z)$ and $G(z) = g(e^{i\alpha}z)$ so that $F, G \in H^2(\mathbb{D})$ and $|F(w)| = |G(w)|$ for all $w \in (-1, 1)$. Note that for $w, z \in \mathbb{D}$ such that $w = ze^{i\alpha}$, we have

$$S_F(w) = \exp\left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} d\nu_F(e^{i\theta})\right) = \exp\left(\int_{\mathbb{T}} \frac{ze^{i\alpha} + e^{i\theta}}{ze^{i\alpha} - e^{i\theta}} d\nu_F(e^{i\theta})\right),$$

and so by letting $u = \theta - \alpha$, we get

$$S_F(w) = \exp\left(\int_{\mathbb{T}} \frac{z + e^{iu}}{z - e^{iu}} d\nu_f(e^{i(u+\alpha)})\right).$$

Thus by Lemma 4.3.1, we have for $e^{i\alpha} \in \mathbb{T}$,

$$\nu_f(e^{i(\alpha+\theta)}) + \nu_f(e^{i(\alpha-\theta)}) = \nu_g(e^{i(\alpha+\theta)}) + \nu_g(e^{i(\alpha-\theta)}). \quad (4.4.8)$$

Next, define the measure μ on \mathbb{T} by $\mu(e^{i\theta}) = \nu_f(e^{i(\theta+\alpha)})$ for $e^{i\theta} \in \mathbb{T}$, with Fourier coefficients given by

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-in\theta} d\nu_f(e^{i(\theta+\alpha)}) = e^{in\alpha} \widehat{\nu}_f(n)$$

for $n \in \mathbb{N}$. Hence, the previous equation and (4.4.8) imply that for $n \in \mathbb{N}$,

$$e^{in\alpha} \widehat{\nu}_f(n) + e^{-in\alpha} \widehat{\nu}_f(-n) = e^{in\alpha} \widehat{\nu}_g(n) + e^{-in\alpha} \widehat{\nu}_g(-n).$$

Now this equation together with (4.4.7) imply that

$$\widehat{\nu}_g(n) = \frac{e^{-in\alpha} \widehat{\nu}_f(n) - e^{in\alpha} \widehat{\nu}_f(-n)}{e^{-in\alpha} - e^{in\alpha}} = \widehat{\nu}_f(n)$$

and $\widehat{\nu}_g(-n) = \widehat{\nu}_f(-n)$, for all $n \in \mathbb{N}$. Since ν_f and ν_g are uniquely determined by their Fourier coefficients, it follows that $\nu_f = \nu_g$ and so $S_f = S_g$.

We now prove the same for the outer part. Since $|g(x)| = |f(x)|$ for all $x \in (-1, 1)$, Lemma 4.3.1 again implies that for a.e. $e^{i\theta} \in \mathbb{T}$,

$$\log |f_*(e^{i\theta})| + \log |f_*(e^{-i\theta})| = \log |g_*(e^{i\theta})| + \log |g_*(e^{-i\theta})|.$$

For $e^{i\theta} \in \mathbb{T}$, letting $h_f(e^{i\theta}) = \log |f_*(e^{i\theta})|$ implies that the Fourier coefficients of h_f and h_g satisfy

$$\widehat{h}_f(n) + \widehat{h}_f(-n) = \widehat{h}_g(n) + \widehat{h}_g(-n), \quad n \in \mathbb{N}. \quad (4.4.9)$$

On the other hand, by definition of F and G , we have for a.e. $e^{i\theta} \in \mathbb{T}$,

$$\log |f_*(e^{i(\alpha+\theta)})| + \log |f_*(e^{i(\alpha-\theta)})| = \log |g_*(e^{i(\alpha+\theta)})| + \log |g_*(e^{i(\alpha-\theta)})|.$$

Using this equation and a similar argument to the one for the Fourier coefficients of the singular measures, we get that for $n \in \mathbb{N}$,

$$e^{in\alpha}\widehat{h}_f(n) + e^{-in\alpha}\widehat{h}_f(-n) = e^{in\alpha}\widehat{h}_g(n) + e^{-in\alpha}\widehat{h}_g(-n).$$

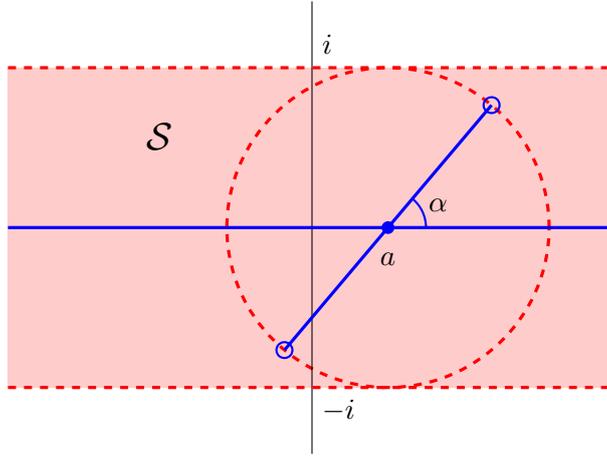
Hence, by this equation and (4.4.9) we get that $\widehat{h}_g(n) = \widehat{h}_f(n)$ for all $n \in \mathbb{Z}$. Therefore $h_f = h_g$, and so $O_f = O_g$.

Finally, since $B_f = B_g$, $S_f = S_g$ and $O_f = O_g$, we have $g = cf$ for some $c \in \mathbb{T}$. \square

Remark 4.4.6. Observe that we can simply extend the previous lemma by having $|g| = |f|$ on $e^{i\alpha_1}(-1, 1) \cup e^{i\alpha_2}(-1, 1)$ where $\alpha_1, \alpha_2 \in [0, 2\pi)$ such that $\alpha_1 - \alpha_2 \notin \pi\mathbb{Q}$. Indeed, without loss of generality, we apply a rotation about the origin such that $\alpha_1 \notin \pi\mathbb{Q}$ and $\alpha_2 = 0$ so that the previous lemma immediately applies.

Remark 4.4.7. If in case $\alpha \in \pi\mathbb{Q}$, then \mathcal{L} is bounded by a polygon inside the unit disc, so it is finite. It follows that $B_f = PB$ and $B_g = QB$, for some finite Blaschke products P, Q while B is the Blaschke product associated with $Z(f) \cap Z(g)$. Moreover, the singular inner part and the outer part are not necessarily equal since the corresponding Fourier coefficients are possibly not equal when αn becomes an integer multiple of π .

We now consider the coupled phase retrieval problem on the strip that includes a more general form of the constraint given in (4.4.6). Using the previous lemma, we establish the uniqueness of the solution of the following problem.



The strip \mathcal{S} with \mathbb{R} and the segment $(-e^{i\alpha} + a, e^{i\alpha} + a)$.

Theorem 4.4.8. Let $f, g \in H^2_\tau(\mathcal{S})$ such that $|g(x)| = |f(x)|$ for $x \in \mathbb{R}$ and

$$|g(z)| = |f(z)|, \quad z \in (-e^{i\alpha} + a, e^{i\alpha} + a)$$

where $a \in \mathbb{R}$ and $\alpha \notin \pi\mathbb{Q}$. Then $g = cf$ for some $c \in \mathbb{T}$.

Proof. Without loss of generality, we let $a = 0$ so that the segment intersects the real line at the origin. Consider $f_{1/2}(z) = f(\frac{1}{2}z)$, $g_{1/2}(z) = g(\frac{1}{2}z)$ for all $z \in \mathbb{D}$.

Observe that $f_{1/2}, g_{1/2} \in H^2(\mathbb{D})$, and $|g_{1/2}| = |f_{1/2}|$ on $(-1, 1)$ and on $e^{i\alpha}(-1, 1)$. Hence, $g_{1/2} = cf_{1/2}$ on \mathbb{D} for some $c \in \mathbb{T}$ by the Lemma 4.4.5, and so $g = cf$ on $\frac{1}{2}\mathbb{D}$. Therefore, since $f, g \in \text{Hol}(\mathcal{S})$ and $g = cf$ on $\frac{1}{2}\mathbb{D}$ so we have $g = cf$ on \mathcal{S} . \square

Chapter 5

Notes on the Phase Retrieval of Holomorphic Functions

This chapter is inspired by two uniqueness results from coupled phase retrieval problems on the Hardy space on the unit disc. The first one is Lemma 4.4.5, which states that a function $f \in H^2(\mathbb{D})$ is uniquely determined by its moduli on two segments on \mathbb{D} , one of which is $(-1, 1)$ whereas the other is a well-chosen segment intersecting $(-1, 1)$ at the origin. The other one is a result by Boche et. al. [12], which states that if the singular part of $f \in H^2(\mathbb{D})$ is absent, then f is uniquely determined by its moduli on two concentric circles, \mathbb{T} and $\rho\mathbb{T}$ for some $\rho < 1$. Our main goal here is to generalize these uniqueness results with simpler proofs without the help of the inner-outer factorization. The contents of this chapter is based on our paper in [60].

In Section 5.1, we prove that if f and g are holomorphic functions on an open connected domain, with the same moduli on two intersecting segments inside the domain, then $f = g$ up to the multiplication of a unimodular constant. For our next result, we show in Section 5.2 that if f and g are functions in the Nevanlinna class, and if $|f| = |g|$ on the unit circle and on a circle inside the unit disc, then $f = g$ up to the multiplication of a unimodular constant.

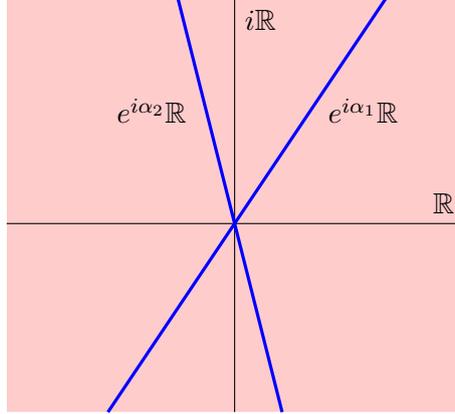
5.1 Recovery from the Modulus on Two Segments

We begin with a simple observation. Let ω, Ω be open connected sets such that $\omega \subset \Omega$, let $f, g \in \text{Hol}(\Omega)$ and suppose that $|f| = |g|$ on ω . Then, for some $c \in \mathbb{T}$, $g = cf$ on Ω . Indeed, we can assume that $|f| = |g|$ on a closed disc \overline{D} , hence f and g have the same zeros with the same multiplicities on \overline{D} . Consequently, $F = f/g$ is a holomorphic function on D and $|F| = 1$. Therefore $0 = \Delta|F|^2 = |F'|^2$ on D and hence $F = c$ for some $c \in \mathbb{T}$.

In the same spirit when ω is not open, it was shown in the paper of Jaming [39, Theorem 3.3] that if f, g are entire functions of finite order such that

$$|f(z)| = |g(z)|, \quad z \in e^{i\alpha_1}\mathbb{R} \cup e^{i\alpha_2}\mathbb{R}$$

where $\alpha_1, \alpha_2 \in [0, 2\pi)$ with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Q}$, then f and g are equal up to the multiplication of a unimodular constant.



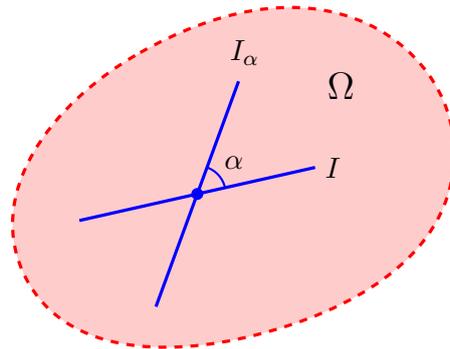
The complex plane with the two intersecting lines $e^{i\alpha_1}\mathbb{R}$ and $e^{i\alpha_2}\mathbb{R}$.

We then considered a similar situation on $H^2(\mathbb{D})$ in Lemma 4.4.5, with the modulus known on two intersecting segments chosen in a similar fashion to the lines in the figure above. Here, we adapted the proof of Jaming to show that the Blaschke products associated to f and g are equal, and we used Fourier coefficients to show that the singular inner parts and outer parts of f and g are equal. The following result consists in showing that this is true for arbitrary holomorphic functions in an open connected domain.

Theorem 5.1.1. *Let Ω be an open connected domain. Let $f, g \in \text{Hol}(\Omega)$ and suppose that*

$$|g(z)| = |f(z)|, \quad z \in I \cup I_\alpha \tag{5.1.1}$$

where I and I_α are segments inside Ω , I_α is the α -rotation of I about the midpoint of I , and $\alpha \notin \pi\mathbb{Q}$. Then $g(z) = cf(z)$ for all $z \in \Omega$ and for some $c \in \mathbb{T}$.



The domain Ω with the segments I and I_α .

Proof. Observe that replacing $f(z)$ by $f(z_0 + rz e^{i\beta})$ with z_0, r, β appropriately chosen, we may assume that

- $(1 + \varepsilon)\overline{\mathbb{D}} \subset \Omega$ for $\varepsilon > 0$,

- $I = (-1, 1)$, $I_\alpha = e^{i\alpha}(-1, 1)$.

Note that now $f, g \in H^2(\mathbb{D})$ so that we could apply Lemma 4.4.5 and obtain $g = cf$, for some $c \in \mathbb{T}$. We will give an alternative simpler proof.

Note that, as the zeros of f and g are isolated, by choosing r small enough, we can assume that they have at most one zero in $(1 + \varepsilon)\overline{\mathbb{D}}$ which is at 0. We can write

$$f(z) = z^k e^{\varphi(z)} \quad \text{and} \quad g(z) = z^l e^{\psi(z)}, \quad z \in \overline{\mathbb{D}} \quad (5.1.2)$$

where $\varphi, \psi \in \text{Hol}((1 + \varepsilon)\overline{\mathbb{D}})$ and k, l are nonnegative integers. As $|f(x)| = |g(x)|$ for $x \in (-1, 1)$ we conclude that $k = l$.

It remains to show that the zero-free factors of f and g are equal up to a unimodular constant. First, we note that (5.1.1) is equivalent to

$$\text{Re } \varphi(t) = \text{Re } \psi(t) \quad \text{and} \quad \text{Re } \varphi(te^{i\alpha}) = \text{Re } \psi(te^{i\alpha}), \quad t \in (-1, 1).$$

Since $\varphi, \psi \in \text{Hol}(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}})$, $\text{Re } \varphi$ and $\text{Re } \psi$ are harmonic and

$$\varphi(z) = \sum_{n \geq 0} |z|^n \widehat{\varphi}(n) e^{in\theta} \quad \text{and} \quad \psi(z) = \sum_{n \geq 0} |z|^n \widehat{\psi}(n) e^{in\theta} \quad (5.1.3)$$

for $z = |z|e^{i\theta} \in \mathbb{D}$. It follows that

$$\text{Re } \varphi(z) = \text{Re } \widehat{\varphi}(0) + \sum_{n \in \mathbb{N}} |z|^n \frac{\widehat{\varphi}(n) e^{in\theta} + \overline{\widehat{\varphi}(n)} e^{-in\theta}}{2}$$

and

$$\text{Re } \psi(z) = \text{Re } \widehat{\psi}(0) + \sum_{n \in \mathbb{N}} |z|^n \frac{\widehat{\psi}(n) e^{in\theta} + \overline{\widehat{\psi}(n)} e^{-in\theta}}{2}.$$

Thus (5.1.1) and (5.1.2) together with the sums above imply that $\text{Re } \varphi(t) = \text{Re } \psi(t)$ for $t \in (-1, 1)$, if and only if

$$\text{Re } \widehat{\varphi}(n) = \text{Re } \widehat{\psi}(n), \quad n \in \mathbb{N}$$

and $\text{Re } \varphi(te^{i\alpha}) = \text{Re } \psi(te^{i\alpha})$ if and only if

$$\text{Re } \widehat{\varphi}(0) = \text{Re } \widehat{\psi}(0) \quad \text{and} \quad \text{Re } (\widehat{\varphi}(n) e^{in\alpha}) = \text{Re } (\widehat{\psi}(n) e^{in\alpha})$$

for all $n \in \mathbb{N}$. In other words

$$\begin{cases} \text{Re } (\widehat{\varphi}(n) - \widehat{\psi}(n)) = \langle \widehat{\varphi}(n) - \widehat{\psi}(n), 1 \rangle_{\mathbb{C}} = 0, \\ \text{Re } ((\widehat{\varphi}(n) - \widehat{\psi}(n)) e^{in\alpha}) = \langle \widehat{\varphi}(n) - \widehat{\psi}(n), e^{-in\alpha} \rangle_{\mathbb{C}} = 0, \end{cases} \quad (5.1.4)$$

for all $n \in \mathbb{N}$ where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is just the inner product in \mathbb{C} . Since $\alpha \notin \pi\mathbb{Q}$, $\{1, e^{-in\alpha}\}$ is a basis for \mathbb{C} when $n \neq 0$ so by (5.1.4), we have $\widehat{\varphi}(n) = \widehat{\psi}(n)$. On the other hand, as $\text{Re } \widehat{\varphi}(0) = \text{Re } \widehat{\psi}(0)$ there exists $\lambda \in \mathbb{R}$ such that $\widehat{\psi}(0) = \widehat{\varphi}(0) + i\lambda$. It follows from (5.1.3) that $\psi = \varphi + i\lambda$ thus $g(z) = e^{i\lambda} f(z)$ for all $z \in \mathbb{D} \subset \Omega$. As Ω is connected and $f, g \in \text{Hol}(\Omega)$, this implies that $g(z) = e^{i\lambda} f(z)$ also holds on Ω . \square

Note that the result of Jaming [39, Theorem 3.3] mentioned earlier easily follows from Theorem 5.1.1. Moreover, as an immediate corollary, we get the following probably well-known fact stated at the beginning of this section.

Corollary 5.1.2. *Let ω, Ω be open connected sets such that $\omega \subset \Omega$. Let $f, g \in \text{Hol}(\Omega)$ and suppose that $|g(z)| = |f(z)|$ for all $z \in \omega$. Then $g(z) = cf(z)$ for all $z \in \Omega$ and for some $c \in \mathbb{T}$.*

Indeed, note that ω contains two segments of the form appearing in Theorem 5.1.1, thus $g = cf$ on ω and thus on Ω by unique continuation of holomorphic functions.

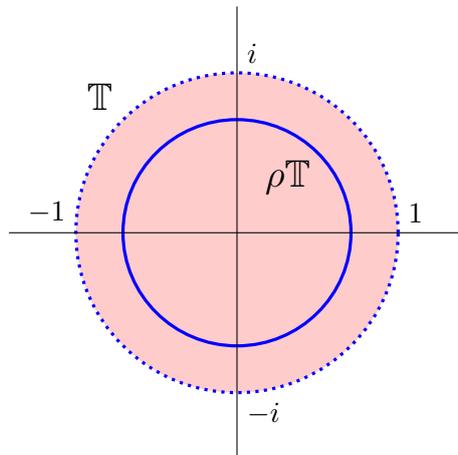
5.2 Recovery from the Modulus on Two Circles

Let us now see what is happening if segments are replaced by circles. To do so, recall that if f and g are outer functions in \mathcal{N} such that $|f| = |g|$ almost everywhere on \mathbb{T} , then f is equal to g up to the multiplication of a unimodular constant. With this, it is easy to see that only the outer part of a function in \mathcal{N} can be fully recovered from its modulus on the unit circle. Now, Boche et. al. [12, Theorem 3] solved a more general problem: if $f, g \in H^1(\mathbb{D})$ have no singular parts (i.e. $f = B_f O_f$, $g = B_g O_g$) and $|f| = |g|$ almost everywhere on \mathbb{T} and $|f| = |g|$ on $\rho\mathbb{T}$ for some $0 < \rho < 1$, then g is uniquely determined by f . The heart of their proof is the explicit construction of the Blaschke product associated to g , as the equality of the outer parts immediately follow. For our next result, with the same equalities of the moduli on the aforementioned circles, we improve the result by Boche et. al. by showing that uniqueness holds for all functions in \mathcal{N} . We emphasize that in this result, we may either have the presence or the absence of the singular inner part.

Theorem 5.2.1. *Let $f, g \in \mathcal{N}$ and let $\rho \in (0, 1)$. If*

$$|g_*(\zeta)| = |f_*(\zeta)|, \quad \text{a.e. } \zeta \in \mathbb{T} \quad \text{and} \quad |g(z)| = |f(z)|, \quad z \in \rho\mathbb{T} \quad (5.2.1)$$

then $g(z) = cf(z)$ for all $z \in \mathbb{D}$ and for some $c \in \mathbb{T}$.



The disc \mathbb{D} with the circles \mathbb{T} and $\rho\mathbb{T}$.

Proof. We begin with a simple observation. If $z_0 \in \rho\mathbb{T}$ is a zero of f , we write $f(z) = (z - z_0)^k \tilde{f}(z)$ and $g(z) = (z - z_0)^j \tilde{g}(z)$ with nonnegative integers j, k and $\tilde{f}, \tilde{g} \in \mathcal{N}$. Then $|f| = |g|$ on $\rho\mathbb{T}$ reads

$$|(z - z_0)^k \tilde{f}(z)| = |(z - z_0)^j \tilde{g}(z)|, \quad z \in \rho\mathbb{T}$$

and this implies that $k = j$. Therefore, f and g have the same zeros on $\rho\mathbb{T}$ with the same multiplicities. We may thus write $f = Pf_1$ and $g = Pg_1$ with P a polynomial which has all the zeros in $\rho\mathbb{T}$ and $f_1, g_1 \in \mathcal{N}$ nonvanishing on $\rho\mathbb{T}$. Then $|f_1| = |g_1|$ on $\rho\mathbb{T} \cup \mathbb{T}$. In other words, we may assume that f and g do not vanish on $\rho\mathbb{T}$.

Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ be the zeros of f and g on $\rho\mathbb{D}$ respectively, counted with multiplicities. For all $z \in \mathbb{C}$, write

$$P_f(z) = \prod_{i=1}^n \frac{\rho(z - a_i)}{\rho^2 - \bar{a}_i z} \quad \text{and} \quad P_g(z) = \prod_{i=1}^m \frac{\rho(z - b_i)}{\rho^2 - \bar{b}_i z}.$$

Notice also that if $z \in \rho\mathbb{T}$, $|P_f(z)| = |P_g(z)| = 1$. Further $\frac{f}{P_f}$ and $\frac{g}{P_g}$ do not vanish in $\rho\mathbb{D}$. By the Poisson-Jensen formula, for all $z \in \rho\mathbb{D}$ we have

$$\begin{aligned} \log \left| \frac{f(z)}{P_f(z)} \right| &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \right) \log |f(\rho e^{i\theta})| \, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \right) \log |g(\rho e^{i\theta})| \, d\theta \\ &= \log \left| \frac{g(z)}{P_g(z)} \right|. \end{aligned} \tag{5.2.2}$$

But then $\left| \frac{g(z)}{P_g(z)} \right| = \left| \frac{f(z)}{P_f(z)} \right|$ for all $z \in \rho\mathbb{D}$. From Corollary 5.1.2 we get that there is some $\tilde{c} \in \mathbb{T}$ such that $\frac{g(z)}{P_g(z)} = \tilde{c} \frac{f(z)}{P_f(z)}$ for all $z \in \rho\mathbb{D}$. Finally, as an identity between holomorphic functions on \mathbb{D} , this is valid for all $z \in \mathbb{D}$. In particular, taking $z = re^{i\theta}$ and letting $r \rightarrow 1$, we get $\frac{g_*(z)}{P_g(z)} = \tilde{c} \frac{f_*(z)}{P_f(z)}$ for almost every $z \in \mathbb{T}$. By (5.2.1), we have $\frac{1}{|P_f|} = \frac{1}{|P_g|}$ on \mathbb{T} as well. Hence, by the following lemma, we get $\frac{1}{P_f} = \frac{c}{P_g}$ for some $c \in \mathbb{T}$, which implies $g = c\tilde{c}f$. \square

Lemma 5.2.2. *Let F and G be meromorphic on \mathbb{C} without poles on $\mathbb{T} \cup \rho\mathbb{T}$ for $0 < \rho \neq 1$. Suppose*

$$|G(\zeta)| = |F(\zeta)| \quad \text{and} \quad |G(\rho\zeta)| = |F(\rho\zeta)|, \quad \zeta \in \mathbb{T}. \tag{5.2.3}$$

Then F and G have the same zeros and poles in $\mathbb{C} \setminus \{0\}$, with the same multiplicities. In particular, if F and G are rational functions that satisfy (5.2.3), then $G = cF$ outside the poles with $c \in \mathbb{T}$.

Proof. Let F be meromorphic on \mathbb{C} and $z_0 \in \mathbb{C}$. Write $F(z) = (z - z_0)^k \tilde{F}(z)$ and for some $k \in \mathbb{Z}$ with $\tilde{F}(z_0) \neq 0$. Define the multiplicity $k := m_F(z_0)$, where $k > 0$ if z_0 is a zero of F , and $k < 0$ if z_0 is a pole of F . In particular, for meromorphic functions F_1, F_2 on \mathbb{C} , we have $m_{F_1 F_2} = m_{F_1} + m_{F_2}$, $m_{F_1/F_2} = m_{F_1} - m_{F_2}$ and if $F_1 = F_2$ in the neighborhood of a pole z_0 , then $m_{F_1} = m_{F_2}$.

First, note that if $\rho > 1$, we replace F and G by $f(z/\rho)$ and $g(z/\rho)$, and replace ρ by $1/\rho < 1$. We thus assume that $\rho < 1$. Observe that (5.2.3) is equivalent to

$$F(z) \overline{F\left(\frac{1}{\bar{z}}\right)} = G(z) \overline{G\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{T}$$

and

$$F(z) \overline{F\left(\frac{\rho^2}{\bar{z}}\right)} = G(z) \overline{G\left(\frac{\rho^2}{\bar{z}}\right)}, \quad z \in \rho\mathbb{T}.$$

As an identity between meromorphic functions in \mathbb{C} , these equations are also valid for $z \in \mathbb{C}$ not a pole of any of the functions involved. As poles are isolated, we have for $z \neq 0$,

$$m_F(z) + m_F\left(\frac{1}{\bar{z}}\right) = m_G(z) + m_G\left(\frac{1}{\bar{z}}\right) \quad (5.2.4)$$

and

$$m_F(z) + m_F\left(\frac{\rho^2}{\bar{z}}\right) = m_G(z) + m_G\left(\frac{\rho^2}{\bar{z}}\right). \quad (5.2.5)$$

Now, (5.2.4) gives

$$m_F(z) - m_G(z) = m_G\left(\frac{1}{\bar{z}}\right) - m_F\left(\frac{1}{\bar{z}}\right) = m_F(\rho^2 z) - m_G(\rho^2 z)$$

with (5.2.5) applied to $1/\bar{z}$, for $z \in \mathbb{C} \setminus \{0\}$. If $z_0 \neq 0$ is such that $m_F(z_0) \neq m_G(z_0)$ then

$$\begin{aligned} 0 \neq m_F(z_0) - m_G(z_0) &= m_F(\rho^2 z_0) - m_G(\rho^2 z_0) = \dots \\ &= m_F(\rho^{2k} z_0) - m_G(\rho^{2k} z_0) = m_{F/G}(\rho^{2k} z_0) \end{aligned}$$

for all $k \in \mathbb{N}$. But then F/G is meromorphic and either has $\rho^{2k} z_0$ as a zero ($m_{F/G}(\rho^{2k} z_0) > 0$) or as a pole ($m_{F/G}(\rho^{2k} z_0) < 0$) for every k . Letting $k \rightarrow \infty$ we have $\rho^{2k} z_0 \rightarrow 0$. As $z_0 \neq 0$, this contradicts the fact that zeros and poles of F/G are isolated. Hence, F and G have the same nonzero zeros and poles with the same multiplicities.

Furthermore, if F and G are rational functions, then they have same zeros and poles in $\mathbb{C} \setminus \{0\}$, thus there exists $c \in \mathbb{T}$ and $m \in \mathbb{Z}$ such that $G = cz^m F$. But then (5.2.3) implies that

- on one hand $|F(\rho e^{it})| = |c| \rho^m |F(e^{it})|$ for all t thus $|c| \rho^m = 1$;
- on the other hand $|F(e^{it})| = |c| |F(e^{it})|$ thus $|c| = 1$.

As $\rho < 1$ this implies $m = 0$ and then $G = cF$. □

We may generalize the previous uniqueness result as follows:

Corollary 5.2.3. *Let F, G be two meromorphic functions in \mathbb{C} with no pole at 0 such that*

$$|G(\zeta)| = |F(\zeta)| \quad \text{and} \quad |G(\rho\zeta)| = |F(\rho\zeta)|, \quad \zeta \in \mathbb{T}, 0 < \rho \neq 1.$$

Then there exists $c \in \mathbb{T}$ such that $G = cF$.

Proof. As seen in the previous proof, we can assume that $\rho < 1$.

The previous lemma then shows that F and G have the same nonzero poles in \mathbb{C} . Note that there is at most a finite number of such poles in a neighborhood of \mathbb{D} . We can factor them out and write $F = \tilde{F}/P$, $G = \tilde{G}/P$ with \tilde{F}, \tilde{G} having no pole in a neighborhood of \mathbb{D} and P a polynomial. But then $\tilde{F}, \tilde{G} \in \mathcal{N}$ and $|\tilde{F}| = |\tilde{G}|$ on $\mathbb{T} \cup \rho\mathbb{T}$. The previous theorem shows that there exists $c \in \mathbb{T}$ such that $\tilde{G} = c\tilde{F}$ thus $G = cF$ on \mathbb{D} thus on \mathbb{C} . \square

Chapter 6

On the Effect of Zero-Flipping on the Stability of the Phase Retrieval Problem in the Paley-Wiener Class

In Section 4.1, we discussed the phase retrieval problem in the Paley-Wiener class PW_L for $L > 0$: given $f \in PW_L$, find all $g \in PW_L$ such that $|g| = |f|$ on \mathbb{R} . We provided a complete proof on how the solutions were obtained through zero-flipping, which refers to the replacement of an arbitrary set of zeros by the set of their complex conjugates. In this chapter, we are now interested on the effect of zero-flipping to the stability of this phase retrieval problem. To examine the stability, we shall introduce a bounded operator \mathfrak{F}_a which exhibits the zero-flipping process at a complex number a , regardless whether a is a genuine complex zero of f or not.

In particular, we show the stability of the phase retrieval problem in PW_L on some subclass $X \subset PW_L$ by finding a positive constant C such that for every $f \in X$ and for every solution $g \in X$, we have

$$\inf_{|c|=1} \|f - cg\|_2 \leq C \||f| - |g|\|_2 + (\text{error term}) \quad (6.0.1)$$

where this error term can possibly be dependent on f . We first investigate the effect of \mathfrak{F}_a on the stability of phase retrieval by estimating the quantity

$$\inf_{|c|=1} \|f - c\mathfrak{F}_a f\|_2.$$

We prove that this is in general not well-suited to investigate stability as in the criteria in (6.0.1), and so we introduce the quantity

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2. \quad (6.0.2)$$

We show that this quantity is dominated by the distance between a and b . The contents of this chapter have been announced in [42].

The work here is organized as follows. Section 6.1 introduces the operator \mathfrak{F}_a and its important properties. Sections 6.2 and 6.3 are devoted to our stability results.

We shall consider the following notations for this chapter: we denote the reflection of a function f with respect to the y -axis by Rf given by

$$Rf(x) = f(-x), \quad x \in \mathbb{R}.$$

We also recall the L^2 -modulus of continuity of $F \in L^2(\mathbb{R})$, denoted by $\omega_2(F; h)$ for some $h > 0$, given by

$$\omega_2(F; h) = \sup_{|\eta| \leq h} \left(\int_{\mathbb{R}} |F(x - \eta) - F(x)|^2 dx \right)^{1/2} = \sup_{|\eta| \leq h} \|\tau_\eta F - F\|_2.$$

We use the normalized Fourier transform of $f \in L^1(\mathbb{R})$ given by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}$$

and the convolution of $f, g \in L^2(\mathbb{R})$ is given by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s)g(x - s) ds, \quad \xi \in \mathbb{R}.$$

We use the notation $C(\alpha_1, \dots, \alpha_n)$ to denote a positive constant that depends only on $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. The constant may change from one line to the next.

6.1 The Operator \mathfrak{F}_a

Let f belong to PW_L and let $a \in \mathbb{C}$ such that $\text{Im } a > 0$. Define the *flipping operator* which we denote by \mathfrak{F}_a where

$$(\mathfrak{F}_a f)(x) = \frac{1 - x/\bar{a}}{1 - x/a} \cdot \frac{e^{x/\bar{a}}}{e^{x/a}} f(x), \quad x \in \mathbb{R}. \quad (6.1.1)$$

It is easy to verify that $|\mathfrak{F}_a f| = |f|$ on \mathbb{R} and so $\|\mathfrak{F}_a f\|_2 = \|f\|_2$, and thus also $\|\widehat{\mathfrak{F}_a f}\|_2 = \|\widehat{f}\|_2$. Note that it will suffice to analyze the stability for \mathfrak{F}_a when $\text{Im } a > 0$ since we can cover the case $\text{Im } a < 0$ by looking at $\mathfrak{F}_{\bar{a}}$ since for $x \in \mathbb{R}$, $\mathfrak{F}_{\bar{a}} f(x) = \overline{\mathfrak{F}_a \bar{f}(x)}$.

Observe that $\mathfrak{F}_a f$ extends into a meromorphic function and that if $f(a) \neq 0$, then $\mathfrak{F}_a f$ has a pole at a and so that $\mathfrak{F}_a f \notin PW_L$. On the other hand, if $f(a) = 0$, from the Hadamard factorization of f we see that $\mathfrak{F}_a f$ has the effect of replacing the zero at $z = a$ by a zero at $z = \bar{a}$, and that $\mathfrak{F}_a f$ is still holomorphic. From the Paley-Wiener theorem, we conclude that $\mathfrak{F}_a f \in PW_L$. However, when $f(a) \neq 0$, \mathfrak{F}_a extends to a holomorphic function on a strip. More precisely:

Lemma 6.1.1. *Let $f \in PW_L$ and let $a \in \mathbb{C}$ such that $\text{Im } a > 0$ and $f(a) \neq 0$. Then the operator $\mathfrak{F}_a : PW_L \rightarrow H_\tau^2(\mathcal{S}_\lambda)$ is bounded with*

$$\|\mathfrak{F}_a f\|_{H_\tau^2(\mathcal{S}_\lambda)} < \left[1 + \frac{2 \text{Im } a}{\text{Im } a - \lambda} \right] e^{\frac{2(\text{Im } a)^2}{|a|^2} L} e^{L\lambda} \|f\|_2$$

where $\mathcal{S}_\lambda = \{z \in \mathbb{C} : |\text{Im } z| < \lambda\}$ and $\lambda < \text{Im } a$. In particular, $\widehat{\mathfrak{F}_a f} \in L^2(\mathbb{R}, e^{2\lambda|x|} dx)$.

Proof. For $x, y \in \mathbb{R}$ with $|y| < \lambda < \operatorname{Im} a$, observe that if $z = x + iy$,

$$\begin{aligned} \left| \frac{(x + iy) - \bar{a}}{(x + iy) - a} \right| &= \left| 1 - \frac{2i \operatorname{Im} a}{z - a} \right| \\ &\leq 1 + \frac{2 \operatorname{Im} a}{\sqrt{(x - \operatorname{Re} a)^2 + (y - \operatorname{Im} a)^2}} \\ &< 1 + \frac{2 \operatorname{Im} a}{|y - \operatorname{Im} a|} \\ &< 1 + \frac{2 \operatorname{Im} a}{\operatorname{Im} a - \lambda} \end{aligned}$$

and

$$\left| \frac{e^{(x+iy)/\bar{a}}}{e^{(x+iy)/a}} \right| = e^{-\frac{2 \operatorname{Im} a}{|a|^2} y} < e^{\frac{2(\operatorname{Im} a)^2}{|a|^2}}.$$

Moreover, since $\widehat{\tau_{-iy} f}(\xi) = \widehat{f}(\xi) e^{\xi y}$ for $y \in \mathbb{R}$ such that $|y| < \lambda < \operatorname{Im} a$ and for $\xi \in \mathbb{R}$, Parseval's identity implies that

$$\int_{\mathbb{R}} |f(x + iy)|^2 dx = \int_{-L}^L |\widehat{f}(\xi)|^2 e^{2\xi y} d\xi \leq e^{2L\lambda} \|f\|_2^2.$$

Thus, if $y \in \mathbb{R}$ such that $|y| < \lambda < \operatorname{Im} a$, we have

$$\begin{aligned} \int_{\mathbb{R}} |(\mathfrak{F}_a f)(x + iy)|^2 dx &= \int_{\mathbb{R}} \left| \frac{(x + iy) - \bar{a}}{(x + iy) - a} \cdot \frac{e^{(x+iy)/\bar{a}}}{e^{(x+iy)/a}} \right|^2 |f(x + iy)|^2 dx \\ &< \left[1 + \frac{2 \operatorname{Im} a}{\operatorname{Im} a - \lambda} \right]^2 e^{\frac{4(\operatorname{Im} a)^2}{|a|^2}} \int_{\mathbb{R}} |f(x + iy)|^2 dx \\ &< \left[1 + \frac{2 \operatorname{Im} a}{\operatorname{Im} a - \lambda} \right]^2 e^{\frac{4(\operatorname{Im} a)^2}{|a|^2}} e^{2L\lambda} \|f\|_2^2 < +\infty. \end{aligned}$$

Taking the supremum for all y such that $|y| < \lambda < \operatorname{Im} a$ yields the first result. The second result then follows from the Paley-Wiener theorem on the strip. \square

We now compute the explicit form of the Fourier transform of $\mathfrak{F}_a f$ which we will need for our results.

Lemma 6.1.2. *Let $f \in PW_L$ for some $L > 0$ and let $a \in \mathbb{C}$ such that $\operatorname{Im} a > 0$. For all $x \in \mathbb{R}$,*

$$(\widehat{\mathfrak{F}_a f})(x) = \frac{a}{\bar{a}} \left[\widehat{f}(x - \beta_a) - (2 \operatorname{Im} a) \int_0^{+\infty} e^{ias} \widehat{f}(x - \beta_a + s) ds \right] \quad (6.1.2)$$

where $\beta_a = \frac{2 \operatorname{Im} a}{|a|^2}$.

Proof. Consider the function γ_a defined by

$$\gamma_a(x) = -\frac{\sqrt{2\pi}i}{2} [1 + \operatorname{sgn}(x)] e^{iax}, \quad x \in \mathbb{R}. \quad (6.1.3)$$

It is easy to check that $\gamma_a \in L^1(\mathbb{R})$ with $\|\gamma_a\|_1 = \frac{\sqrt{2\pi}}{\operatorname{Im} a}$. Then, for all $w \in \mathbb{R}$,

$$\begin{aligned}\widehat{\gamma}_a(w) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} -\frac{\sqrt{2\pi}i}{2} [1 + \operatorname{sgn}(x)] e^{i(a-w)x} dx \\ &= -i \int_0^{+\infty} e^{i(a-w)x} dx \\ &= \frac{1}{a-w}.\end{aligned}$$

Now, for all $x \in \mathbb{R}$, write (6.1.1) as

$$\begin{aligned}(\mathfrak{F}_a f)(x) &= \frac{1 - x/\bar{a}}{1 - x/a} \cdot e^{i\beta_a x} f(x) \\ &= \frac{a}{\bar{a}} \left[1 - \frac{a - \bar{a}}{a - x} \right] e^{i\beta_a x} f(x) \\ &= \frac{a}{\bar{a}} \left[e^{i\beta_a x} f(x) - (2i \operatorname{Im} a) e^{i\beta_a x} f(x) \widehat{\gamma}_a(x) \right].\end{aligned}$$

Then

$$(\widehat{\mathfrak{F}_a f})(x) = \frac{a}{\bar{a}} \left[\tau_{\beta_a} \widehat{f}(x) - (2i \operatorname{Im} a) (R\gamma_a * \tau_{\beta_a} \widehat{f})(x) \right], \quad x \in \mathbb{R}. \quad (6.1.4)$$

Expanding this equation, we get

$$\begin{aligned}(\widehat{\mathfrak{F}_a f})(x) &= \frac{a}{\bar{a}} \left[\widehat{f}(x - \beta_a) - \frac{2i \operatorname{Im} a}{\sqrt{2\pi}} \int_{\mathbb{R}} \gamma_a(-s) \widehat{f}(x - \beta_a - s) ds \right] \\ &= \frac{a}{\bar{a}} \left[\widehat{f}(x - \beta_a) - \frac{2i \operatorname{Im} a}{\sqrt{2\pi}} \int_{\mathbb{R}} \gamma_a(s) \widehat{f}(x - \beta_a + s) ds \right] \\ &= \frac{a}{\bar{a}} \left[\widehat{f}(x - \beta_a) - 2 \operatorname{Im} a \int_0^{+\infty} e^{ias} \widehat{f}(x - \beta_a + s) ds \right]\end{aligned}$$

as claimed. \square

6.2 Stability between $\mathfrak{F}_a f$ and f

In this section, we will give an estimate of

$$\inf_{|c|=1} \|f - c\mathfrak{F}_a f\|_2.$$

This is a classical measure of stability for the phase retrieval problem. We are here investigating how far zero-flipping drives us from the original function (up to the trivial solution $f \rightarrow cf$). Recall that $\mathfrak{F}_a f$ is bandlimited only if $f(a) = 0$, this will however play no role here, that is we allow the solution $\mathfrak{F}_a f$ to be wide-banded. This can also be considered as a simpler case of (6.0.2), where b is real so that $\mathfrak{F}_b f = f$. Our result here is the following:

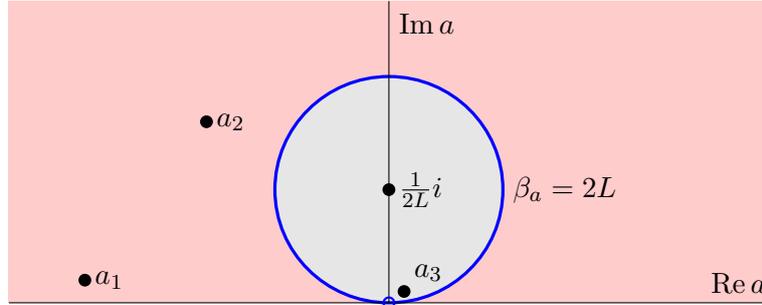
Theorem 6.2.1. Let $f \in PW_L$ for some $L > 0$. Let $a \in \mathbb{C}$ such that $\text{Im } a > 0$ and $\beta_a = \frac{2 \text{Im } a}{|a|^2}$. Then

$$\left| \inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 - 2\|f\|_2^2 \right| \leq 30(L \text{Im } a)\|f\|_2^2, \quad \text{if } \beta_a > 2L \quad (6.2.1)$$

and

$$\inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 \leq 2\omega_2(\widehat{f}; \beta_a)\|f\|_2 + 8\sqrt{L \text{Im } a}\|f\|_2^2, \quad \text{if } \beta_a \leq 2L. \quad (6.2.2)$$

Remark 6.2.2. The actual bounds are a bit more precise, see (6.2.6) and (6.2.10) in the proof below. Note that $\beta_a = \frac{2 \text{Im } a}{|a|^2} = 2L$ is equivalent to $(\text{Re } a)^2 + (\text{Im } a - \frac{1}{2L})^2 = \frac{1}{4L^2}$ which represents a circle, with a hole at the origin, illustrated below (in blue).



The stability region

From Theorem 6.2.1, we have stability when $\beta_a \leq 2L$ (in red), whereas we have instability when $\beta_a > 2L$ (in gray). Consider $a_1, a_2, a_3 \in \mathbb{C}$ which have positive imaginary parts as plotted in the figure above. Note that the zero-flipping is ‘more stable’ at a_1 as it is farther from the origin and has a smaller imaginary part than of a_2 , and the zero-flipping at a_3 is unstable as it is very close to the origin.

This result says that zero-flipping becomes unstable (for this criteria) when a approaches the real axis inside this disc. On the other hand, if a approaches the real line while staying away from the origin, we have stability. Indeed, if $|a| \geq \alpha > 0$ and $\text{Im } a \rightarrow 0$ then $\beta_a \rightarrow 0$ so that $\omega_2(\widehat{f}; \beta_a) \rightarrow 0$.

Proof of Theorem 6.2.1. Observe first that for $|c| = 1$,

$$\begin{aligned} \|\mathfrak{F}_a f - cf\|_2^2 &= \|\widehat{\mathfrak{F}_a f} - c\widehat{f}\|_2^2 \\ &= \|\widehat{f}\|_2^2 - 2 \text{Re} \left(\bar{c} \langle \widehat{\mathfrak{F}_a f}, \widehat{f} \rangle \right) + \|\widehat{\mathfrak{F}_a f}\|_2^2 \\ &= 2 \left[\|f\|_2^2 - \text{Re} \left(\bar{c} \langle \widehat{\mathfrak{F}_a f}, \widehat{f} \rangle \right) \right], \end{aligned}$$

and thus

$$\inf_{|c|=1} \|\mathfrak{F}_a f - cf\|_2^2 = 2 \left[\|f\|_2^2 - |\langle \widehat{\mathfrak{F}_a f}, \widehat{f} \rangle| \right]. \quad (6.2.3)$$

For our calculations, we notice from (6.1.2) that

$$\frac{\bar{a}}{a} \langle \widehat{\mathfrak{F}_a f}, \widehat{f} \rangle = \sqrt{2\pi} \left(R\widehat{f} * \widehat{\bar{f}} \right) (\beta_a) - 2 \operatorname{Im} a \int_{-L}^L \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) \overline{\widehat{f}(x)} \, ds \, dx \quad (6.2.4)$$

with $R\widehat{f}(x) = \widehat{f}(-x)$ for $x \in \mathbb{R}$.

Case 1 $\beta_a > 2L$.

Observe that if $\beta_a > 2L$, then

$$\left(R\widehat{f} * \widehat{\bar{f}} \right) (\beta_a) = 0. \quad (6.2.5)$$

For the second term, if $\beta_a > 2L$, then $x - \beta_a \leq L - \beta_a < -L$ for any $x \in [-L, L]$, thus

$$\begin{aligned} \left| \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) \, ds \right| &= \left| e^{-ia(x-\beta_a)} \int_{-L}^L e^{iat} \widehat{f}(t) \, dt \right| \\ &\leq e^{\operatorname{Im} a (x-\beta_a+L)} \|\widehat{f}\|_1 \\ &\leq e^{\operatorname{Im} a (x-\beta_a+L)} \sqrt{2L} \|f\|_2 \end{aligned}$$

and so

$$\begin{aligned} |\langle \widehat{\mathfrak{F}_a f}, \widehat{f} \rangle| &= \left| 2 \operatorname{Im} a \int_{-L}^L \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) \overline{\widehat{f}(x)} \, ds \, dx \right| \\ &\leq 2 \operatorname{Im} a \left[\sqrt{2L} \|f\|_2 \int_{-L}^L |\widehat{f}(x)| e^{\operatorname{Im} a (x-\beta_a+L)} \, dx \right] \\ &\leq 2\sqrt{2L} \operatorname{Im} a \|f\|_2^2 \left(\int_{-L}^L e^{2\operatorname{Im} a (x-\beta_a+L)} \, dx \right)^{1/2} \\ &= 2\sqrt{2L \operatorname{Im} a \cdot \sinh(2L \operatorname{Im} a)} e^{L \operatorname{Im} a} e^{-\beta_a \operatorname{Im} a} \|f\|_2^2. \end{aligned} \quad (6.2.6)$$

Note that $\beta_a \operatorname{Im} a = \frac{2(\operatorname{Im} a)^2}{|a|^2} \leq 2$ so that $e^{-\beta_a \operatorname{Im} a}$ plays no role and we just bound it by 1. Further, if $\beta_a > 2L$ then $L \operatorname{Im} a < 1$ thus $\sinh(L \operatorname{Im} a) \leq \sinh(1)L \operatorname{Im} a$. As $2\sqrt{2 \sinh(2)}e^1 \leq 15$ we get

$$2\sqrt{2L \operatorname{Im} a \cdot \sinh(2L \operatorname{Im} a)} e^{L \operatorname{Im} a} \leq 15L \operatorname{Im} a$$

and finally $|\langle \widehat{\mathfrak{F}_a f}, \widehat{f} \rangle| \leq 15(L \operatorname{Im} a) \|f\|_2^2$. Together with (6.2.5), we see that (6.2.3) implies (6.2.1).

Case 2 $\beta_a \leq 2L$.

Observe first that

$$\begin{aligned} \|f\|_2^2 - \sqrt{2\pi} \left(R\widehat{f} * \widehat{\bar{f}} \right) (\beta_a) &\leq \left| \sqrt{2\pi} \left(R\widehat{f} * \widehat{\bar{f}} \right) (\beta_a) - \|f\|_2^2 \right| \\ &= \left| \int_{-L}^L \overline{\widehat{f}(\xi)} \left(\widehat{f}(\xi - \beta_a) - \widehat{f}(\xi) \right) \, d\xi \right| \end{aligned}$$

$$\begin{aligned} &\leq \|f\|_2 \|\widehat{f} - \tau_{\beta_a} \widehat{f}\|_2 \\ &\leq \|f\|_2 \omega_2(\widehat{f}; \beta_a). \end{aligned} \quad (6.2.7)$$

It remains to bound the second term in (6.2.4), that is, to show that

$$\left| 2 \operatorname{Im} a \int_{-L}^L \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) ds \overline{\widehat{f}(x)} dx \right| \leq C(a) \|f\|_2^2$$

with $C(a) \leq \sqrt{4 \operatorname{Im} a}$.

We first want to bound

$$\int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) ds = e^{-ia(x-\beta_a)} \int_{x-\beta_a}^L e^{iat} \widehat{f}(t) dt \quad (6.2.8)$$

when $x \in [-L, L]$. To do so, assume first that $-L \leq x \leq \beta_a - L$. Then

$$\begin{aligned} \left| \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) ds \right| &\leq e^{\operatorname{Im} a(x-\beta_a)} \left| \int_{x-\beta_a}^L e^{iat} \widehat{f}(t) dt \right| \\ &\leq e^{\operatorname{Im} a(x-\beta_a)} \int_{-L}^L e^{-\operatorname{Im} at} |\widehat{f}(t)| dt \\ &\leq e^{\operatorname{Im} a(x-\beta_a+L)} \sqrt{2L} \|f\|_2. \end{aligned}$$

On the other hand, if $\beta_a - L < x \leq L$, then by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left| \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) ds \right| &= \left| e^{-ia(x-\beta_a)} \int_{x-\beta_a}^L e^{iat} \widehat{f}(t) dt \right| \\ &\leq e^{\operatorname{Im} a(x-\beta_a)} \|f\|_2 \left[\int_{x-\beta_a}^L e^{-2\operatorname{Im} at} dt \right]^{1/2} \\ &= \frac{e^{\operatorname{Im} a(x-\beta_a)} \|f\|_2}{\sqrt{2 \operatorname{Im} a}} \left[-e^{-2L \operatorname{Im} a} + e^{-2 \operatorname{Im} a(x-\beta_a)} \right]^{1/2} \\ &= \frac{\|f\|_2}{\sqrt{2 \operatorname{Im} a}} \left[1 - e^{2 \operatorname{Im} a(x-\beta_a-L)} \right]^{1/2}. \end{aligned}$$

Hence, combining these two bounds, we have

$$\begin{aligned} &\left| 2 \operatorname{Im} a \int_{-L}^L \int_0^{+\infty} e^{ias} \widehat{f}(x+s-\beta_a) \overline{\widehat{f}(x)} ds dx \right| \\ &\leq 2 \operatorname{Im} a \left[\sqrt{2L} \|f\|_2 \int_{-L}^{\beta_a-L} |\widehat{f}(x)| e^{\operatorname{Im} a(x-\beta_a+L)} dx \right. \\ &\quad \left. + \frac{\|f\|_2}{\sqrt{2 \operatorname{Im} a}} \int_{\beta_a-L}^L |\widehat{f}(x)| \left(1 - e^{2 \operatorname{Im} a(x-\beta_a-L)} \right)^{1/2} dx \right] \\ &\leq \sqrt{2 \operatorname{Im} a} \|f\|_2^2 \left[2\sqrt{L \operatorname{Im} a} \left(\int_{-L}^{\beta_a-L} e^{2 \operatorname{Im} a(x-\beta_a+L)} dx \right)^{1/2} \right. \end{aligned}$$

$$\begin{aligned}
 & + \left(\int_{\beta_a - L}^L \left(1 - e^{2\operatorname{Im} a (x - \beta_a - L)} \right) dx \right)^{1/2} \Big] \\
 & = \sqrt{2\operatorname{Im} a} \|f\|_2^2 \left[\sqrt{2L} \left(1 - e^{-2\beta_a \operatorname{Im} a} \right)^{1/2} \right. \\
 & \quad \left. + \left(2L - \beta_a - \frac{e^{-2\beta_a \operatorname{Im} a}}{2\operatorname{Im} a} \left(1 - e^{2(\beta_a - 2L)\operatorname{Im} a} \right) \right)^{1/2} \right].
 \end{aligned} \tag{6.2.9}$$

This gives the desired bound with

$$C(a) = \sqrt{2\operatorname{Im} a} \left[\sqrt{2L} \left(1 - e^{-2\beta_a \operatorname{Im} a} \right)^{1/2} + \left(2L - \beta_a - \frac{e^{-2\beta_a \operatorname{Im} a}}{2\operatorname{Im} a} \left(1 - e^{2(\beta_a - 2L)\operatorname{Im} a} \right) \right)^{1/2} \right] \tag{6.2.10}$$

and it is easy to see that $0 < C(a) \leq 4\sqrt{L\operatorname{Im} a}$. Finally, by (6.2.7) and (6.2.9), we obtain the estimate in (6.2.2). \square

6.3 Stability between $\mathfrak{F}_a f$ and $\mathfrak{F}_b f$

In this section, we now compare two nontrivial solutions $\mathfrak{F}_a f$ and $c\mathfrak{F}_b f$ of the phase retrieval problem. To do this, we introduce the following stability measure given by

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2.$$

Note that we also allow these solutions to be either bandlimited or wide-banded. By using a similar computation we did to obtain (6.2.3), note that

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c\mathfrak{F}_b f\|_2^2 = 2 \left[\|f\|_2^2 - |\langle \widehat{\mathfrak{F}_a f}, \widehat{\mathfrak{F}_b f} \rangle| \right]. \tag{6.3.1}$$

Before we look at the next stability result, we first prove some technical lemmas which we will need.

Lemma 6.3.1. *Let $f \in L^2(\mathbb{R})$ and let $a, b \in \mathbb{C}$ with $\operatorname{Im} a, \operatorname{Im} b > 0$ and $|a - b| \leq \frac{|b|}{2}$. Let $\beta_a = \frac{2\operatorname{Im} a}{|a|^2}$ and $\beta_b = \frac{2\operatorname{Im} b}{|b|^2}$. Consider $\gamma_a, \gamma_b \in L^1(\mathbb{R})$ as defined in (6.1.3). Then*

$$\|\operatorname{Im} a (R\gamma_a * \tau_{\beta_a} \widehat{f}) - \operatorname{Im} b (R\gamma_b * \tau_{\beta_b} \widehat{f})\|_2 \leq C(a, b) \|f\|_2 + \sqrt{2\pi} \omega_2(\widehat{f}; \beta_a - \beta_b)$$

where $C(a, b) \leq 14 \frac{|a - b|}{\operatorname{Im} b}$.

Remark 6.3.2. The actual value of $C(a, b)$ is a bit more precise and given in (6.3.2) below.

Proof. First, observe that for all $x \geq 0$,

$$\begin{aligned} \left| \sin\left(\frac{a-b}{2}x\right) \right| &\leq \left| \sin\left(\operatorname{Re}\left(\frac{a-b}{2}x\right)\right) \right| \cosh\left(\operatorname{Im}\left(\frac{a-b}{2}x\right)\right) + \left| \sinh\left(\operatorname{Im}\left(\frac{a-b}{2}x\right)\right) \right| \\ &\leq \left| \operatorname{Re}\left(\frac{a-b}{2}\right) \right| x \cdot e^{\frac{\operatorname{Im}(a-b)}{2}x} + \left| \sinh\left(\operatorname{Im}\left(\frac{a-b}{2}x\right)\right) \right|. \end{aligned}$$

Hence, with this bound, we get

$$\begin{aligned} \|R\gamma_a - R\gamma_b\|_1 &= \sqrt{2\pi} \int_0^{+\infty} |e^{iax} - e^{ibx}| \, dx \\ &= \sqrt{2\pi} \int_0^{+\infty} |e^{i\frac{a+b}{2}x}| |e^{i\frac{a-b}{2}x} - e^{-i\frac{a-b}{2}x}| \, dx \\ &= 2\sqrt{2\pi} \int_0^{+\infty} e^{-\frac{\operatorname{Im}(a+b)}{2}x} \left| \sin\left(\frac{a-b}{2}x\right) \right| \, dx \\ &\leq 2\sqrt{2\pi} \left[\int_0^{+\infty} \left| \operatorname{Re}\left(\frac{a-b}{2}\right) \right| x e^{-\operatorname{Im}bx} \, dx \right. \\ &\quad \left. + \int_0^{+\infty} e^{-\frac{\operatorname{Im}(a+b)}{2}x} \left| \sinh\left(\operatorname{Im}\left(\frac{a-b}{2}x\right)\right) \right| \, dx \right] \\ &= \sqrt{2\pi} \left[\frac{|\operatorname{Re}a - \operatorname{Re}b|}{(\operatorname{Im}b)^2} + \left| \frac{1}{\operatorname{Im}a} - \frac{1}{\operatorname{Im}b} \right| \right]. \end{aligned}$$

Note also that $|a - b| \leq \frac{|b|}{2}$ implies that $\operatorname{Im}a \leq \frac{3}{2}\operatorname{Im}b$. Using this, the previous norm estimate, and Young's convolution inequality, we then have

$$\begin{aligned} &\| \operatorname{Im}a (R\gamma_a * \tau_{\beta_a} \widehat{f}) - \operatorname{Im}b (R\gamma_b * \tau_{\beta_b} \widehat{f}) \|_2 \\ &\leq |\operatorname{Im}a - \operatorname{Im}b| \cdot \|R\gamma_b * \tau_{\beta_b} \widehat{f}\|_2 + \operatorname{Im}a \| (R\gamma_a - R\gamma_b) * \tau_{\beta_b} \widehat{f} \|_2 \\ &\quad + \operatorname{Im}a \| R\gamma_a * (\tau_{\beta_a} \widehat{f} - \tau_{\beta_b} \widehat{f}) \|_2 \\ &\leq |\operatorname{Im}a - \operatorname{Im}b| \cdot \|R\gamma_b\|_1 \|f\|_2 + \operatorname{Im}a \|R\gamma_a - R\gamma_b\|_1 \|f\|_2 \\ &\quad + \operatorname{Im}a \|R\gamma_a\|_1 \omega_2(\widehat{f}; \beta_a - \beta_b) \\ &= \sqrt{2\pi} \frac{|\operatorname{Im}a - \operatorname{Im}b|}{\operatorname{Im}b} \|f\|_2 + \operatorname{Im}a \|R\gamma_a - R\gamma_b\|_1 \|f\|_2 \\ &\quad + \sqrt{2\pi} \omega_2(\widehat{f}; \beta_a - \beta_b) \\ &\leq \left[\sqrt{2\pi} \frac{|\operatorname{Im}a - \operatorname{Im}b|}{\operatorname{Im}b} + \frac{3\sqrt{2\pi}}{2} \operatorname{Im}b \left[\frac{|\operatorname{Re}a - \operatorname{Re}b|}{(\operatorname{Im}b)^2} + \left| \frac{1}{\operatorname{Im}a} - \frac{1}{\operatorname{Im}b} \right| \right] \right] \|f\|_2 \\ &\quad + \sqrt{2\pi} \omega_2(\widehat{f}; \beta_a - \beta_b). \end{aligned}$$

We thus obtain the lemma with

$$C(a, b) = \sqrt{2\pi} \frac{|\operatorname{Im}a - \operatorname{Im}b|}{\operatorname{Im}b} + \frac{3\sqrt{2\pi}}{2} \operatorname{Im}b \left[\frac{|\operatorname{Re}a - \operatorname{Re}b|}{(\operatorname{Im}b)^2} + \left| \frac{1}{\operatorname{Im}a} - \frac{1}{\operatorname{Im}b} \right| \right]. \quad (6.3.2)$$

Note that if $|a - b| \leq \frac{|b|}{2}$ then $\operatorname{Im}a \geq \frac{1}{2}\operatorname{Im}b$ thus

$$\left| \frac{1}{\operatorname{Im}a} - \frac{1}{\operatorname{Im}b} \right| \leq 2 \frac{|\operatorname{Im}a - \operatorname{Im}b|}{(\operatorname{Im}b)^2}$$

from which the bound $C(a, b) \leq 14 \frac{|a-b|}{\operatorname{Im} b}$ immediately follows. \square

Lemma 6.3.3. *Let $f \in PW_L$ and let $b \in \mathbb{C}$ with $\operatorname{Im} b > 0$ and $\beta_b = \frac{2 \operatorname{Im} b}{|b|^2}$. Then*

$$\left[\int_{\mathbb{R}} \left(e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy \right)^2 dx \right]^{1/2} \leq \|f\|_2 C(b)$$

with

$$C(b) = \frac{1}{\sqrt{2 \operatorname{Im} b}} \left[2L + 1 + \frac{e^{-4L \operatorname{Im} b} - 1}{2 \operatorname{Im} b} \right]^{1/2}.$$

Proof. Firstly, if $x \geq L + \beta_b$, then

$$\int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy = 0.$$

Secondly, if $-L + \beta_b \leq x \leq L + \beta_b$, Cauchy-Schwarz inequality implies that

$$\begin{aligned} e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy &\leq e^{\operatorname{Im} b(x-\beta_b)} \|f\|_2 \left(\int_{x-\beta_b}^L e^{-2 \operatorname{Im} b y} dy \right)^{1/2} \\ &= e^{\operatorname{Im} b(x-\beta_b)} \|f\|_2 \left(\frac{e^{-2 \operatorname{Im} b(x-\beta_b)} - e^{-2 \operatorname{Im} b L}}{2 \operatorname{Im} b} \right)^{1/2} \\ &= \frac{\|f\|_2}{\sqrt{2 \operatorname{Im} b}} \left(1 - e^{2 \operatorname{Im} b x} e^{-2 \operatorname{Im} b(L+\beta_b)} \right)^{1/2} \end{aligned}$$

and so

$$\begin{aligned} \int_{-L+\beta_b}^{L+\beta_b} \left(e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy \right)^2 dx &= \frac{\|f\|_2^2}{2 \operatorname{Im} b} \int_{-L+\beta_b}^{L+\beta_b} \left(1 - e^{2 \operatorname{Im} b x} e^{-2 \operatorname{Im} b(L+\beta_b)} \right) dx \\ &= \frac{\|f\|_2^2}{2 \operatorname{Im} b} \left[2L + \frac{e^{-4L \operatorname{Im} b} - 1}{2 \operatorname{Im} b} \right]. \end{aligned}$$

Lastly, if $x \leq -L + \beta_b$,

$$\begin{aligned} e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy &= e^{\operatorname{Im} b(x-\beta_b)} \int_{-L}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy \\ &\leq e^{2 \operatorname{Im} b(x-\beta_b)} e^{2 \operatorname{Im} b L} \|f\|_2^2 \end{aligned}$$

and thus,

$$\begin{aligned} \int_{-\infty}^{-L+\beta_b} \left(e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| dy \right)^2 dx &\leq \|f\|_2^2 e^{2 \operatorname{Im} b(L-\beta_b)} \int_{-\infty}^{-L+\beta_b} e^{2 \operatorname{Im} b x} dx \\ &= \frac{\|f\|_2^2}{2 \operatorname{Im} b}. \end{aligned}$$

Combining all these cases, we obtain

$$\int_{\mathbb{R}} \left(e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} b y} |\widehat{f}(y)| \, dy \right)^2 dx \leq \|f\|_2^2 C(b)^2$$

where

$$C(b) = \frac{1}{\sqrt{2 \operatorname{Im} b}} \left[2L + 1 + \frac{e^{-4L \operatorname{Im} b} - 1}{2 \operatorname{Im} b} \right]^{1/2}$$

as announced. \square

With these lemmas, we now state and prove our next stability result.

Theorem 6.3.4. *Let $f \in PW_L$ for some $L > 0$. Let $a, b \in \mathbb{C}$ such that $\operatorname{Im} a, \operatorname{Im} b > 0$, and $|a - b| \leq \frac{|b|}{2}$. Let $\beta_a = \frac{2 \operatorname{Im} a}{|a|^2}$ and $\beta_b = \frac{2 \operatorname{Im} b}{|b|^2}$. Then*

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c \mathfrak{F}_b f\|_2^2 \leq C_1(b) \omega_2(\widehat{f}; \beta_b - \beta_a) \|f\|_2 + C_2(a, b) \|f\|_2^2.$$

where $C_1(b)$ and $C_2(a, b)$ are positive constants depending on a and b , with $C_2(a, b) \rightarrow 0$ as $a \rightarrow b$.

Remark 6.3.5. The constants $C_1(b)$ and $C_2(a, b)$ are given in (6.3.4)-(6.3.5) and depend on the quantities $C(a, b)$ and $C(b)$ given in Lemmas 6.3.1 and 6.3.3, respectively.

Proof. We use the formula for $\mathfrak{F}_a f$ from (6.1.4). Write

$$\frac{\bar{a}b}{ab} \langle \widehat{\mathfrak{F}_a f}, \widehat{\mathfrak{F}_b f} \rangle = \int_{\mathbb{R}} (\mathbf{A}_{a,b}(x) + \mathbf{B}_{a,b}(x) + \mathbf{C}_{a,b}(x) + \mathbf{D}_{a,b}(x)) \, dx$$

where

$$\begin{aligned} \mathbf{A}_{a,b}(x) &= \tau_{\beta_b} \widehat{f}(x) \overline{\tau_{\beta_a} \widehat{f}(x)} \\ \mathbf{B}_{a,b}(x) &= -\tau_{\beta_b} \widehat{f}(x) (2i \operatorname{Im} a) \overline{(R\gamma_a * \tau_{\beta_a} \widehat{f})(x)} \\ \mathbf{C}_{a,b}(x) &= -\tau_{\beta_a} \widehat{f}(x) (2i \operatorname{Im} b) \overline{(R\gamma_b * \tau_{\beta_b} \widehat{f})(x)} \\ \mathbf{D}_{a,b}(x) &= (4 \operatorname{Im} a \operatorname{Im} b) (R\gamma_b * \tau_{\beta_b} \widehat{f})(x) \overline{(R\gamma_a * \tau_{\beta_a} \widehat{f})(x)} \end{aligned}$$

for all $x \in \mathbb{R}$. Since $\int_{\mathbb{R}} \mathbf{A}_{b,b}(x) \, dx = \|f\|_2^2$ and

$$\|f\|_2^2 = \langle \widehat{\mathfrak{F}_b f}, \widehat{\mathfrak{F}_b f} \rangle = \int_{\mathbb{R}} (\mathbf{A}_{b,b}(x) + \mathbf{B}_{b,b}(x) + \mathbf{C}_{b,b}(x) + \mathbf{D}_{b,b}(x)) \, dx,$$

we have

$$\int_{\mathbb{R}} (\mathbf{B}_{b,b}(x) + \mathbf{C}_{b,b}(x) + \mathbf{D}_{b,b}(x)) \, dx = 0.$$

Hence,

$$\frac{\bar{a}b}{ab} \langle \widehat{\mathfrak{F}_a f}, \widehat{\mathfrak{F}_b f} \rangle = \int_{\mathbb{R}} \left[\mathbf{A}_{a,b}(x) + (\mathbf{B}_{a,b} - \mathbf{B}_{b,b})(x) + (\mathbf{C}_{a,b} - \mathbf{C}_{b,b})(x) + (\mathbf{D}_{a,b} - \mathbf{D}_{b,b})(x) \right] dx. \quad (6.3.3)$$

We will show the result by estimating each term of this integral.

We first look at $\mathbf{A}_{a,b}$. Observe that

$$\begin{aligned} \|f\|_2^2 - \int_{\mathbb{R}} \mathbf{A}_{a,b}(x) \, dx &\leq \left| \int_{\mathbb{R}} \mathbf{A}_{a,b}(x) \, dx - \|f\|_2^2 \right| \\ &\leq \int_{\mathbb{R}} |\widehat{f}(x - \beta_b)| |\tau_{\beta_a} \widehat{f}(x) - \tau_{\beta_b} \widehat{f}(x)| \, dx \\ &\leq \|f\|_2 \|\tau_{\beta_a} \widehat{f} - \tau_{\beta_b} \widehat{f}\|_2 \\ &\leq \|f\|_2 \omega_2(\widehat{f}; \beta_a - \beta_b). \end{aligned}$$

For $\mathbf{B}_{a,b} - \mathbf{B}_{b,b}$, we use the bounds from Lemma 6.3.1 to obtain

$$\begin{aligned} \int_{\mathbb{R}} |(\mathbf{B}_{a,b} - \mathbf{B}_{b,b})(x)| \, dx &= 2 \int_{\mathbb{R}} |\tau_{\beta_b} \widehat{f}(x)| |\operatorname{Im} a(R\gamma_a * \tau_{\beta_a} \widehat{f})(x) - \operatorname{Im} b(R\gamma_b * \tau_{\beta_b} \widehat{f})(x)| \, dx \\ &\leq 2\|f\|_2 \|\operatorname{Im} a(R\gamma_a * \tau_{\beta_a} \widehat{f}) - \operatorname{Im} b(R\gamma_b * \tau_{\beta_b} \widehat{f})\|_2 \\ &\leq 2C(a, b) \|f\|_2^2 + 2\sqrt{2\pi} \omega_2(\widehat{f}; \beta_a - \beta_b) \|f\|_2. \end{aligned}$$

Next, we use the bounds from Lemma 6.3.3 so that

$$\begin{aligned} \int_{\mathbb{R}} |(\mathbf{C}_{a,b} - \mathbf{C}_{b,b})(x)| \, dx &= 2 \operatorname{Im} b \int_{\mathbb{R}} |\tau_{\beta_a} \widehat{f}(x) - \tau_{\beta_b} \widehat{f}(x)| |(R\gamma_b * \tau_{\beta_b} \widehat{f})(x)| \, dx \\ &= 2 \operatorname{Im} b \int_{\mathbb{R}} |\tau_{\beta_a} \widehat{f}(x) - \tau_{\beta_b} \widehat{f}(x)| \left| \int_0^{+\infty} e^{ibs} \widehat{f}(x - \beta_b + s) \, ds \right| \, dx \\ &= 2 \operatorname{Im} b \int_{\mathbb{R}} |\tau_{\beta_a} \widehat{f}(x) - \tau_{\beta_b} \widehat{f}(x)| \left| e^{-ib(x-\beta_b)} \int_{x-\beta_b}^L e^{iby} \widehat{f}(y) \, dy \right| \, dx \\ &\leq 2 \operatorname{Im} b \int_{\mathbb{R}} |\tau_{\beta_a} \widehat{f}(x) - \tau_{\beta_b} \widehat{f}(x)| \left[e^{\operatorname{Im} b(x-\beta_b)} \int_{x-\beta_b}^L e^{-\operatorname{Im} by} |\widehat{f}(y)| \, dy \right] \, dx \\ &\leq 2 \operatorname{Im} b C(b) \cdot \omega_2(\widehat{f}; \beta_a - \beta_b) \|f\|_2. \end{aligned}$$

For the last term, the bounds from Lemmas 6.3.1 and 6.3.3 imply that

$$\begin{aligned} \int_{\mathbb{R}} |(\mathbf{D}_{a,b} - \mathbf{D}_{b,b})(x)| \, dx &= 4 \operatorname{Im} b \int_{\mathbb{R}} |(R\gamma_b * \tau_{\beta_b} \widehat{f})(x)| |\operatorname{Im} a(R\gamma_a * \tau_{\beta_a} \widehat{f})(x) - \operatorname{Im} b(R\gamma_b * \tau_{\beta_b} \widehat{f})(x)| \, dx \\ &\leq 4 \operatorname{Im} b C(b) \|f\|_2 \|\operatorname{Im} a(R\gamma_a * \tau_{\beta_a} \widehat{f}) - \operatorname{Im} b(R\gamma_b * \tau_{\beta_b} \widehat{f})\|_2 \\ &\leq 4 \operatorname{Im} b C(b) C(a, b) \|f\|_2^2 + 4\sqrt{2\pi} \operatorname{Im} b C(b) \cdot \omega_2(\widehat{f}; \beta_a - \beta_b) \|f\|_2. \end{aligned}$$

Combining these three estimates from above, we get

$$\begin{aligned} &\left| \int_{\mathbb{R}} [(\mathbf{B}_{a,b} - \mathbf{B}_{b,b})(x) + (\mathbf{C}_{a,b} - \mathbf{C}_{b,b})(x) + (\mathbf{D}_{a,b} - \mathbf{D}_{b,b})(x)] \, dx \right| \\ &\leq \int_{\mathbb{R}} [|(\mathbf{B}_{a,b} - \mathbf{B}_{b,b})(x)| + |(\mathbf{C}_{a,b} - \mathbf{C}_{b,b})(x)| + |(\mathbf{D}_{a,b} - \mathbf{D}_{b,b})(x)|] \, dx \end{aligned}$$

$$\begin{aligned} &\leq \left[2\sqrt{2\pi} + (2 + 4\sqrt{2\pi}) \operatorname{Im} b C(b) \right] \omega_2(\widehat{f}; \beta_a - \beta_b) \|f\|_2 \\ &\quad + \left[2C(a, b) + 4 \operatorname{Im} b C(b) C(a, b) \right] \|f\|_2^2. \end{aligned}$$

Finally, from (6.3.1), we obtain

$$\begin{aligned} \inf_{|c|=1} \|\mathfrak{F}_a f - c \mathfrak{F}_b f\|_2^2 &\leq 2 \left| \frac{\bar{a}b}{ab} \langle \widehat{\mathfrak{F}_a f}, \widehat{\mathfrak{F}_b f} \rangle - \|f\|_2^2 \right| \\ &\leq 2 \left[2 + 2\sqrt{2\pi} + (2 + 4\sqrt{2\pi}) \operatorname{Im} b C(b) \right] \omega_2(\widehat{f}; \beta_a - \beta_b) \|f\|_2 \\ &\quad + 2 \left[2C(a, b) + 4 \operatorname{Im} b C(b) C(a, b) \right] \|f\|_2^2. \end{aligned}$$

Setting

$$C_1(b) = 4 + 4\sqrt{2\pi} + (4 + 8\sqrt{2\pi}) \operatorname{Im} b C(b) \quad (6.3.4)$$

and

$$C_2(a, b) = 4C(a, b) + 8 \operatorname{Im} b C(b) C(a, b), \quad (6.3.5)$$

so that $C_2(a, b) \rightarrow 0$ as $a \rightarrow b$, we obtain the theorem. \square

In this corollary, we see that if a false complex zero a goes close to a genuine complex zero, then the corresponding wide-banded solution $\mathfrak{F}_a f$ goes close to a genuine solution in the Paley-Wiener class.

Corollary 6.3.6. *Let $f \in PW_L$ for some $L > 0$. Fix $b \in \mathbb{C}$, a simple zero of f with $\operatorname{Im} b > 0$. Suppose $a \in \mathbb{C}$ with $\operatorname{Im} a > 0$, $f(a) \neq 0$ and $|a - b| \leq \frac{|b|}{2}$. Then*

$$\inf_{|c|=1} \|\mathfrak{F}_a f - c \mathfrak{F}_b f\|_2^2 \rightarrow 0 \text{ as } a \rightarrow b.$$

Remark 6.3.7. Since zeros are isolated, f does not vanish on $V \setminus \{b\}$ where V is a neighborhood of b . Without loss of generality, $V \subset \left\{ a \in \mathbb{C} : |a - b| \leq \frac{|b|}{2} \right\}$.

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